# <u>Nonlinear System of ODEs</u>

The second system of ODEs are given by:

$$dx/dt = x(\alpha 1 - \beta 1x - \gamma 1y),$$
$$dy/dt = y(\alpha 2 - \beta 2x - \gamma 2y) \qquad -(1)$$

where $\alpha 1, \alpha 2, \beta 1, \beta 2, \gamma 1, \gamma 2 > 0$.

Taking $u = \beta 1x$, $v = \gamma 2y$, $\gamma = \gamma 1/\gamma 2$, $\beta = \beta 2/\beta 1$, the above nonlinear equations can be deduced to linear system F(u,v)

$$u' = u(\alpha 1 - u - \gamma v),$$
$$v' = v(\alpha 2 - \beta u - v) \qquad -(2)$$

Linearization techniques can now be applied to the equations (2)

To find equilibrium points:

u'=0 and v'=0, we will get four lines in uv plane

L1: $u=0$

L2: $u=\alpha 1-\gamma v$

L3: $v=0$

L4: $v=\alpha 2-\beta u$

Total Derivative of F(u,v) i.e. Jacobian Matrix A:

$$DF(u,v)= A = [\ [(\alpha 1\text{-}2u\text{-}\gamma v) \quad (-\gamma u)],$$
$$[(-\beta v) \quad (\alpha 2-\beta u-2v)]]$$

Depending upon the values of α1,α2, β and γ the system will have a certain number of equilibria points and lines L2 and L4 will be differently oriented in the uv plane.

We have three possibilities:

1. L2 and L4 can be parallel to each other. ( $\beta\gamma = 1$ and $\beta != \alpha 1/\alpha 2$ )
2. L2 and L4 can coincide. ( $\beta\gamma = 1$ and $\beta = \alpha 1/\alpha 2$ )
3. L2 and L4 intersect each other at a point.

### <u>Case 1: L2 and L4 can be parallel to each other. ( βγ = 1 and β != α1/α2 )</u>

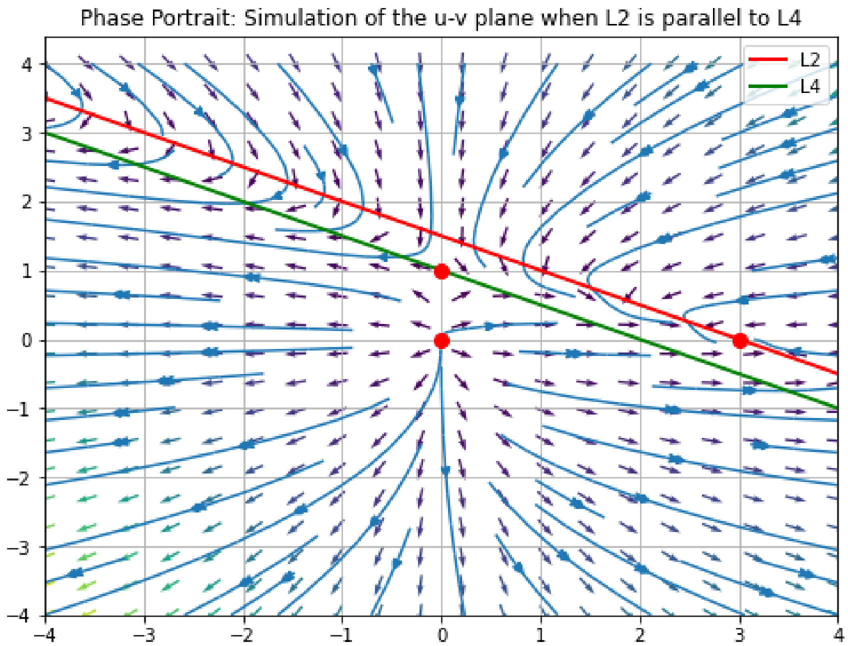
Considering following set of parameters : ($\alpha 1=3$, $\alpha 2=1$, $\beta=0.5$ , $\gamma=2$)

<u>We obtain 3 equilibrium points:</u> P1(0.0, 0.0), P2(0.0, 1.0) and P3(3.0, 0.0)

<u>Eigenvalues corresponding to the given equilibrium points are:</u>

| Eigenvalues | Equilibrium Point | Stability |
|---|---|---|
| 3.0 and 1.0 | P1(0.0, 0.0) : Source | Unstable |
| -1.0 and 1.0 | P2(0.0, 1.0) : Saddle Point | Unstable |
| -3.0 and -0.5 | P3(3.0, 0.0) : Sink | Asymptotically Stable |

Obtained Phase Portrait (using code):




-

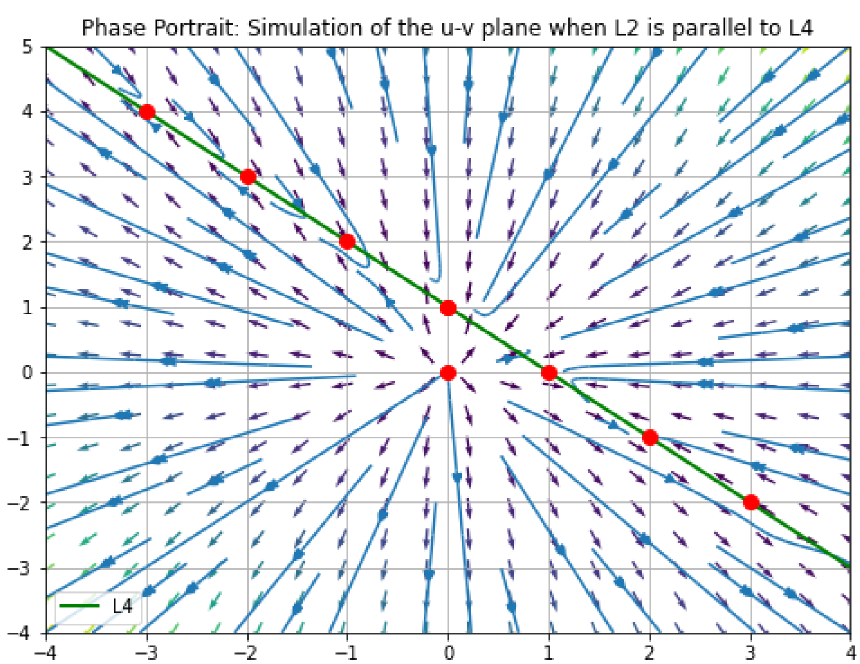
## **Case 2: L2 and L4 coincide (βγ = 1 and β = α1/α2 )**

Considering following set of parameters : (α1=1, α2=1, β=1 , γ=1)
Equilibrium points obtained: P1(0, 0) and at (u, 1 - u) [lies on L2=L4 line] for all u belongs to R

Eigenvalues corresponding to the given equilibrium points are:
Equilibrium Point:  P1(0, 0)   and corresponding Eigenvalues are :  1.0 and 1.0

| Eigenvalues | Equilibrium Point | Stability |
|---|---|---|
| 1.0 and 1.0 | P1(0.0, 0.0) : Source | Unstable |

Obtained Phase Portrait (using code):



## Case 3: L2 and L4 intersect each other at a point ($\beta\gamma < 1$ )

We will have three sub cases:

3.1 $c_1, c_2 > 0$

3.2. $c_1 < 0 < c_2$

3.3. $c_1 > 0 > c_2$

In all these sub cases we will get four equilibrium points given by:

P1(0,0), P2($\alpha_1$,0), P3(0,$\alpha_2$) and P4((($\alpha_1$-$\gamma$*$\alpha_2$)/(1-$\beta$*$\alpha$)), (($\alpha_2$-$\beta$*$\alpha_1$)/(1-$\beta$*$\gamma$)))

### 3.1. $c_1, c_2 > 0$

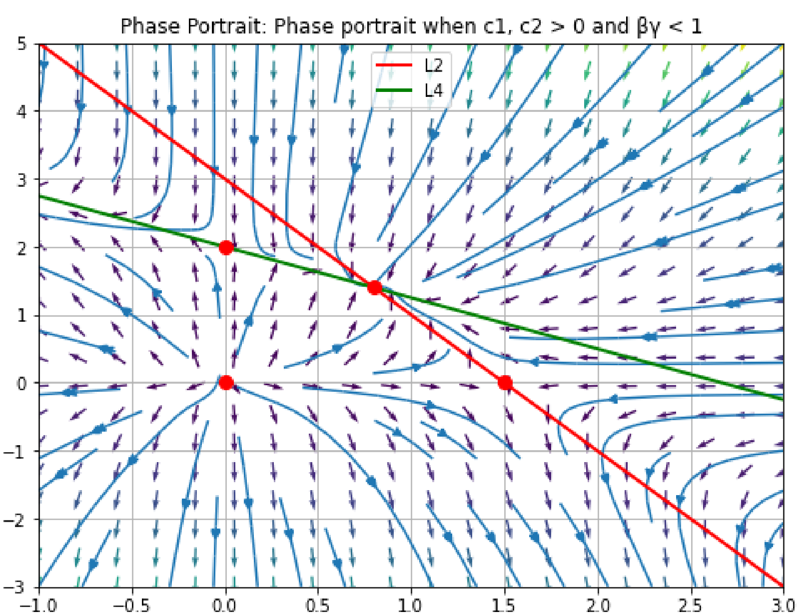
Considering following set of parameters : ($\alpha_1 = 1.5$, $\alpha_2 = 2$, $\beta_1 = 1$, $\beta_2 = 0.75$, $\gamma_1 = 0.5$, $\gamma_2 = 1.0$)
$\gamma = \gamma_1/\gamma_2 = 0.5$, $\beta = \beta_2/\beta_1 = 0.75$, $c_1 = \alpha_1 - \gamma * \alpha_2 = 0.5$, $c_2 = \alpha_2 - \beta * \alpha_1 = 0.875$, $\gamma\beta = 0.375$

Equilibrium points obtained: P1(0, 0), P2(1.5, 0), P3(0, 2), P4(0.8, 1.4)

Eigenvalues corresponding to the given equilibrium points are:

| Eigenvalues | Equilibrium Point | Stability |
|---|---|---|
| 1.5 and 2.0 | P1(0, 0) : Source | Unstable |
| -1.5 and 0.875 | P2(1.5, 0) : Saddle Point | Unstable |
| -2.0 and 0.5 | P3(0, 2) : Saddle Point | Unstable |
| -0.385 and -1.814 | P4(0.8, 1.4) : Sink | Asymptotically Stable |

Obtained Phase Portrait (using code):



### 3.2. $c_1 < 0 < c_2$

Considering following set of parameters :

($\alpha_1 = 1$, $\alpha_2 = 4$, $\beta_1 = 4$, $\beta_2 = 1$, $\gamma_1 = 1.5$, $\gamma_2 = 1$)

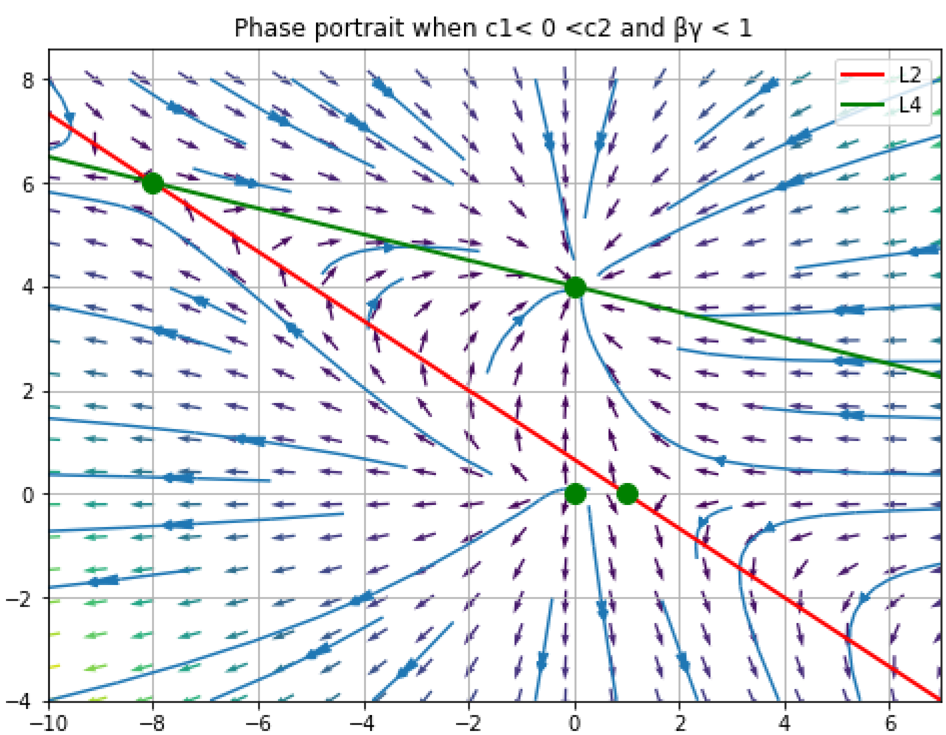
$\gamma = \gamma_1/\gamma_2 = 1.5$, $\beta = \beta_2/\beta_1 = 1$, $c_1 = \alpha_1 - \gamma^*\alpha_2 = -5$, $c_2 = \alpha_2 - \beta^*\alpha_1 = 3.75$, $\gamma\beta = 0.375$

Equilibrium points obtained: P1(0, 0), P2(1, 0), P3(0, 4), P4(-8, 6)

Eigenvalues corresponding to the given equilibrium points are:

| Eigenvalues | Equilibrium Point | Stability |
|---|---|---|
| 1.0 and 4.0 | P1(0, 0) : Source | Unstable |
| -1.0 and 3.75 | P2(1, 0) : Saddle Point | Unstable |
| -4.0 and -5.0 | P3(0, 4) : Sink | Asymptotically Stable |
| 6.567 and -4.567 | P4(-8, 6) : Saddle Point | Unstable |

Obtained Phase Portrait (using code):



### 3.3. c1 >  0 > c2

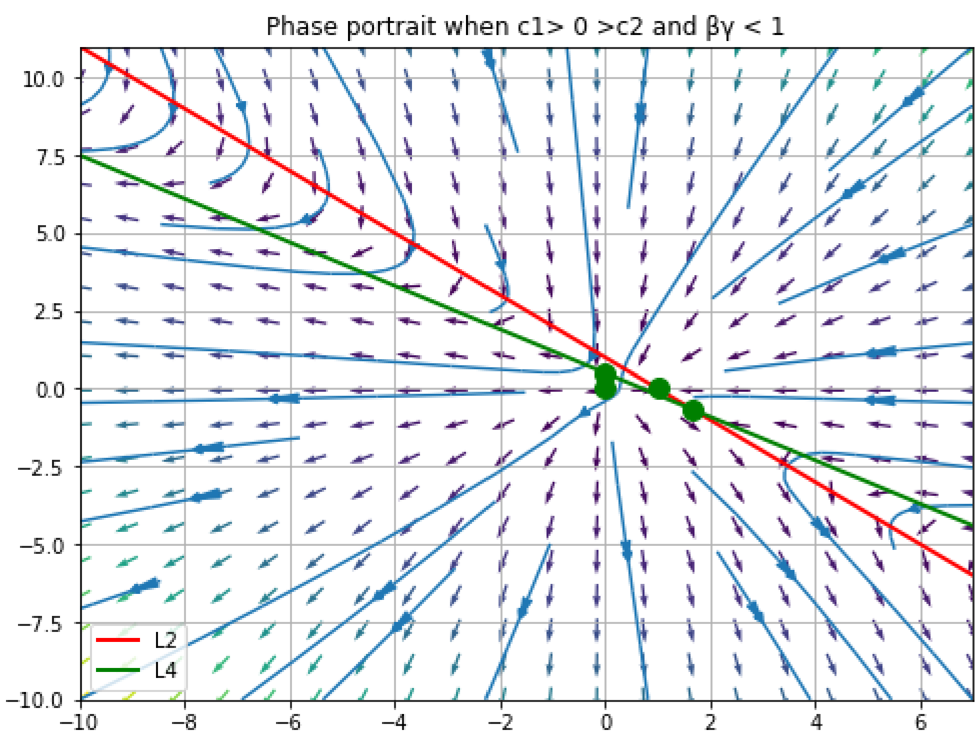
Considering following set of parameters :
($\alpha_1 = 1$, $\alpha_2 = 0.5$, $\beta_1 = 1$, $\beta_2 = 0.7$, $\gamma_1 = 1$, $\gamma_2 = 1$)
$\gamma = \gamma_1/\gamma_2 = 1$,    $\beta = \beta_2/\beta_1 = 0.7$, $c_1 = \alpha_1-\gamma*\alpha_2 = 0.5$,   $c_2 = \alpha_2-\beta*\alpha_1 = -0.199$,  $\gamma\beta = 0.7$

Equilibrium points obtained: P1(0, 0), P2(1, 0), P3(0, 0.5), P4(1.666, -0.664)

Eigenvalues corresponding to the given equilibrium points are:

| Eigenvalues | Equilibrium Point | Stability |
|---|---|---|
| 1.0 and  0.5 | P1(0, 0) : Source | Unstable |
| -1.0 and  -0.2 | P2(1, 0) : Sink | Asymptotically Stable |
| -0.5 and 0.5 | P3(0, 0.5) : Saddle Point | Unstable |
| -1.263 and 0.263 | P4(1.666, -0.664) : Saddle Point | Unstable |

Obtained Phase Portrait (using code):



## Case4: βγ != 1 and βγ > 1 (Lines L2 and L4 intersect at one point)

### 4.1. c1 < 0 < c2

Considering following set of parameters : c($\alpha1$ = 0.1, $\alpha2$ = 2.1, $\beta$ = 4, $\gamma$ = 0.5)
c1 = $\alpha1$-$\gamma$*$\alpha2$ = -0.95, c2 = $\alpha2$-$\beta$*$\alpha1$ = 1.70, $\gamma\beta$ = 2.0

Equilibrium points obtained: P1(0, 0), P2(0.1, 0), P3(0, 2.1), P4(0.951, -1.70)

Eigenvalues corresponding to the given equilibrium points are:

| Eigenvalues | Equilibrium Point | Stability |
|---|---|---|
| 0.1 and 2.1 | P1(0, 0) : Source | Unstable |
| -0.1 and 1.7 | P2(0.1, 0) : Saddle Point | Unstable |
| -2.1 and -0.95 | P3(0, 2.1) : Sink | Asymptotically Stable |
| 0.375+1.214j and 0.375-1.214j | P4(0.951, -1.70) : Inward Spiral | Asymptotically Stable |

Obtained Phase Portrait (using code):