DYNAMICS OF FREQUENCYESTIMATION IN THE FREQUENCY DOMAIN

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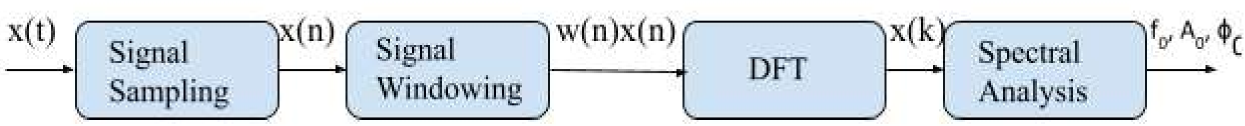
ABSTRACT

In this report we will try to show how much error we can reduce in frequency estimation by using multipoint interpolated DFT for Hanning window, Rectangular window, Blackman window. We also try to show a tradeoff between the reduction in systematic error of the frequency estimation and the uncertainty of the estimated results due to the interpolated algorithm.

IMPORTANT CONCEPTS



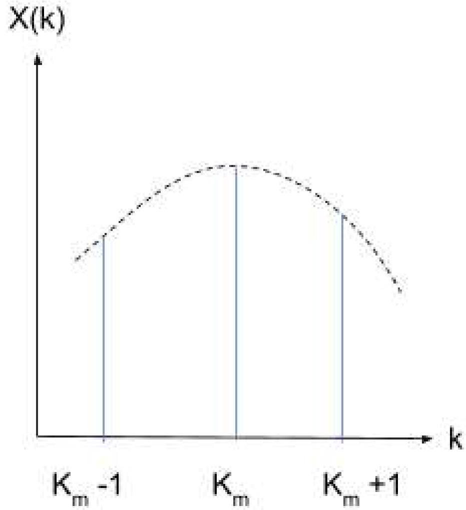
Our main motive is to find f_0 (main tone frequency) via Interpolated DFT.

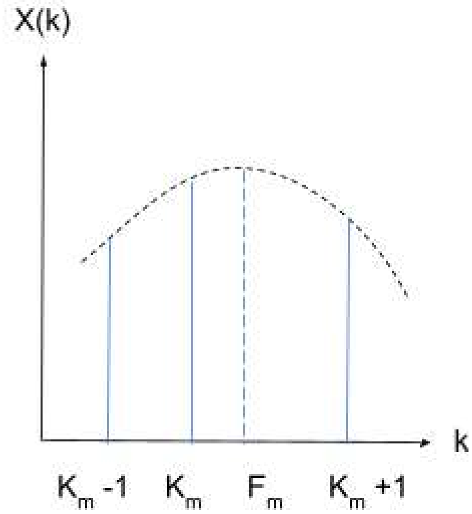
If we try to find main tone frequency without interpolation we will get

$$f_0 = k_m(\Delta f)$$

here $k_m = bin$ of a local DFT maximum. And$\Delta f = f_{max} - f_{min}$

And the accuracy of this solution for frequency depends in the accuracy of location of DFT peak. And there can be 2 cases either main tone frequency overlaps with the k_m and if it does overlap then we do not need to do interpolation because we already know main-tone frequency. But if does not overlap then we have to interpolate DFT.





So , now we can find maximum error in the frequency estimation. In particular the maximum error in the peak location is equal to half of the DFT frequency resolution $(\Delta f') = \frac{1}{T}$.

$$Max(err) = \frac{max(|k_m \Delta f - f_0|)}{f_0}$$

$$Max(err) = \frac{\Delta f'}{2 \cdot f_0} = \frac{1}{(2T \cdot f_0)} = \frac{F_s}{2N \cdot f_0}$$

From the above relation we can compute some points to increase the accuracy.

1. Decrease the sampling frequency. (Aliasing may arise)
2. DFT interpolation.

DFT INTERPOLATION

A more exact estimation of the main spectrum tone location can be given by calculating the abscissa of the maximum of an interpolation curve of the DFT spectrum.

Now let we consider a very simple discrete-time signal $x(n)$ produced by a sampling process characterized by a sampling frequency F_s.

$$x(n) \ = \ A_0 \ cos(2\pi f_0 n T_s \ + \ \phi_0)$$

Being f_0 , A_0 and ϕ_0 the signal frequency, amplitude and phase respectively. As we know that the input signal is sliced in portions containing N samples.

Using a pre-selected windowing function (Hanning window). It’s DFT spectrum can be written as

$X(k) = \frac{(A_0 e^{j\phi_0} W(k - \lambda_0) + A_0 e^{-j\phi_0} W(k + \lambda_0))}{2}$

And $\lambda_0 = \frac{f_0}{\Delta f} = \frac{f_0 \cdot N}{Fs}$,λ_0 is a normalized frequency.

And $W(.)$ is Fourier transform of Hanning window.

As we already saw that the main tone of the signal is located between two consecutive DFT bins, it's location can therefore be expressed as:

$$K_{peak} = K_m + \delta$$

Being k_m the index of the DFT bin characterized by the highest magnitude

And $-0.5 < \delta < 0.5$ is a fractional correction term.

Based on the DFT spectrum $X(k)$ of the signal $x(n)$ analyzed with the known windowing function $w(n)$. Now, we have to find correction term δ that better approximates the exact location of the spectrum tone.

Assuming that the effects of leakage are properly compensated by windowing, we can neglect the long range spectral leakage produced by the $-ve$ spectrum image and express the 2 highest bins of the DFT. As a function of $+ve$ spectrum image only.

$X(K_m) \rightarrow$ Highest bin

$X(K_m + \varepsilon) \rightarrow$ Second highest bin with $\varepsilon = + - 1$.

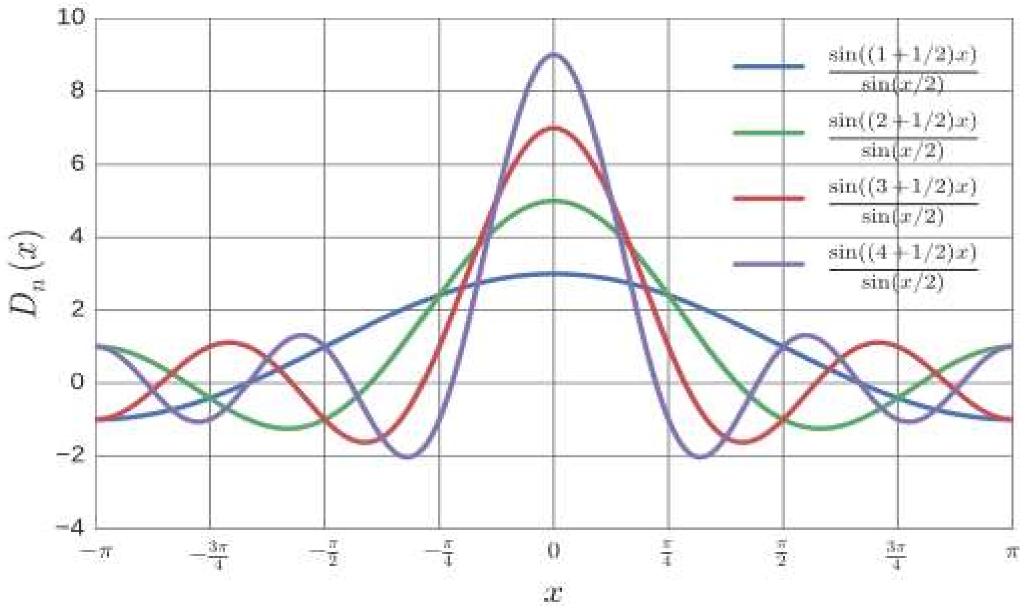
$$\frac{|X(K_m + \varepsilon)|}{|X(K_m)|} = \frac{|W(1 - \delta)|}{|W(\delta)|}$$

And we can also represent $W(w)$ Fourier transform of Hanning window as

$W(w) = -0.25D(w - \frac{2\pi}{N}) + 0.5D(w) - 0.25D(w + \frac{2\pi}{N})$

Where $D(.)$ is so-called Dirichlet kernel, Having value:

$$D_n(x) = \frac{sin((n+\frac{1}{2})x)}{2\pi sin(\frac{x}{2})}$$



Substituting the Fourier transform of Hanning window in ratio of highest and second highest bin we get

$$\frac{|X(K_m+\varepsilon)|}{|X(K_m)|} = \frac{|W(1-\delta)|}{|W(\delta)|} = \frac{|\delta+\varepsilon|}{|\delta-2\varepsilon|}$$

And in the view of above approximated equality. We can express δ as

$$\delta = \varepsilon\frac{2|X(K_m+\varepsilon)| - |X(K_m)|}{|X(K_m)| - |X(K_m+\varepsilon)|}$$

And the main tone frequency can be represented as:

$$f_0 = (K_m+\delta)\Delta f$$

Similarly, Based on number of DFT bins used to perform the interpolation. And we can find 3-point , 4 -point , multipoint DFT interpolation.

The above experiment and result was for two- point interpolation. Using similar operations we can find for 3-point interpolation.

So,

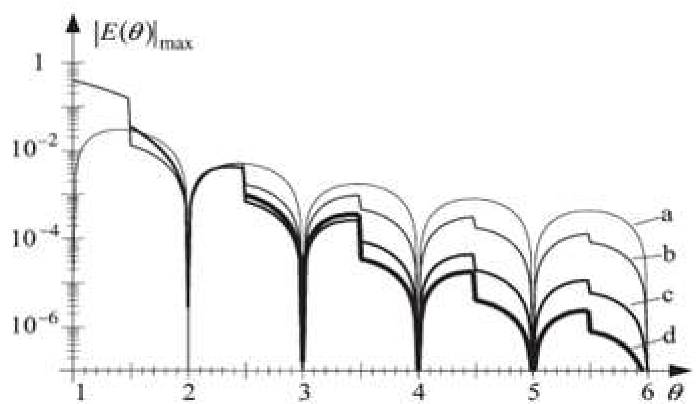
$$\delta = 2\frac{|X(K_m+1)| - |X(K_m-1)|}{|X(K_m-1)| + 2X(K_m) - |X(K_m+1)|}$$

As we found errors for multipoint interpolations we can say that the procedure for multipoint estimations reduces leakage tails and consequently reduces systematic errors. It is possible to increase the number of DFT coefficients in the estimations to decrease systematic errors, but the "main lobe of the estimation" spreads.

The systematic errors of the frequency estimations $E = \theta_m - \theta_0$ (where θ_0 is the true value of the frequency) are phase dependent. The error curves are very close to sine like functions. Estimation errors decrease with the increase in relative frequency.

UNCERTAINITY:

After seeing the graph between relative frequency and systematic errors of frequency estimations we can say that the distribution of errors of the largest amplitude DFT coefficients have very similar shapes, with almost similar standard deviations.



Maximal errors of frequency estimation with multipoint interpolations of the DFT for the Hanning window (a: two-point interpolation, b: three-point interpolation, c: five-point interpolation, d: seven-point interpolation)

$\Theta = Relative\ frequency = \frac{T_{meas}}{T}$

And $E = \theta_m - \theta_0$

where θ_m = Freq of particular component of signal and

θ_0 = True value of frequency

The standard deviations of other higher multipoint interpolations in comparison to that of the basic three-point interpolation increase as follows:

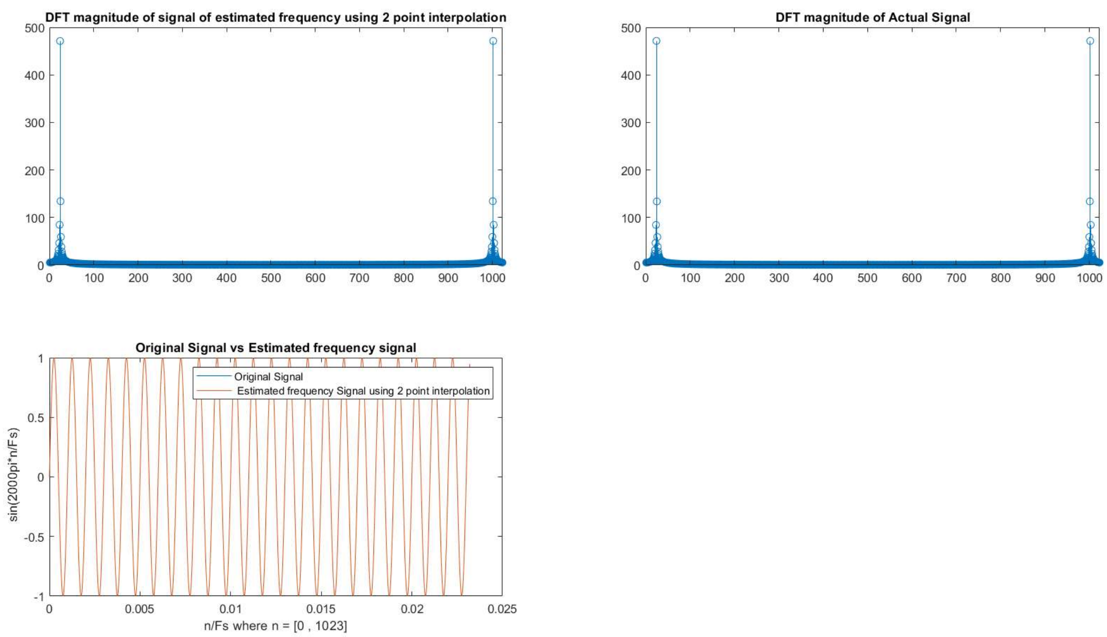
$$_3\sigma/_5\sigma/_7\sigma \approx 1/1.27/1.52$$

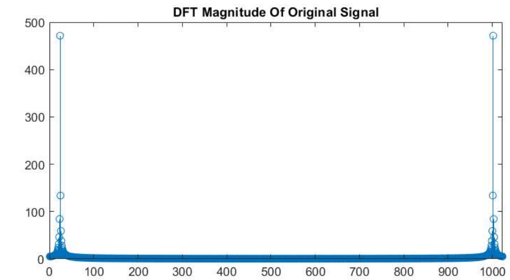
The relative frequency has been changed, show that the influence of noise on the estimations (σ_0) increases with the number of interpolation points.

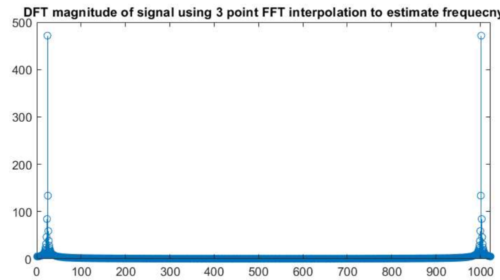
Interpolation is the weighted summation of the amplitude coefficients or, better, the symmetrical subtraction of the successive adjacent leakage parts of the window spectrum. If we reduce the leakage tails or systematic errors by interpolation, we apparently widen the estimation main lobe, and the noise in the estimation increases.

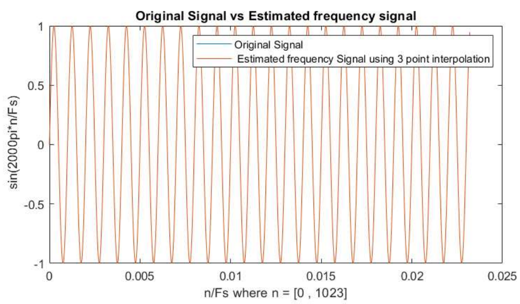
At the same time, the systematic errors $|E_0|_{max}$ decrease with increase in the number of points. Increasing the number of used DFT coefficients is reasonable until the systematic error decreases under the noise error. After this point, by increasing the relative frequency θ_m or with spacing between the two frequency components, it is logical to decrease the number of interpolation points.

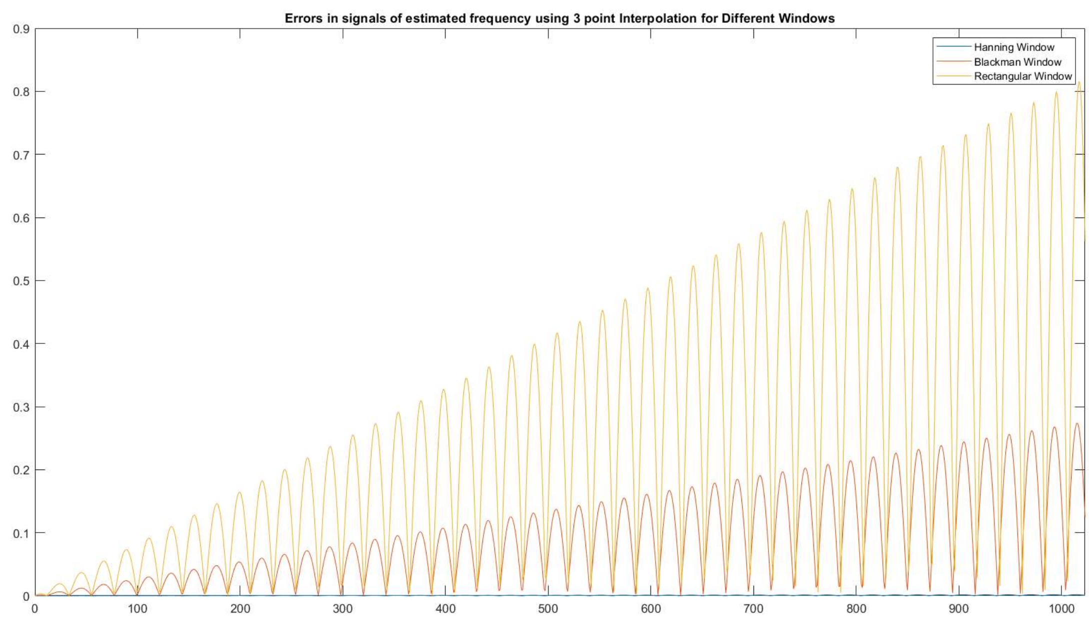
SIMULATIONS AND RESULTS











CONCLUSION

The frequency of the measurement component can be estimated by means of the interpolated DFT very effectively, however this procedure is more sensitive to noise than the best possible likelihood solution. Nevertheless, this is much faster option. Interpolations with a larger number of DFT coefficients decreased the systematic errors and increased the noise distortion in the results. If we use different optimal interpolation algorithms for the frequency estimation, we change the apparent window shape for the particular component.

The **two point interpolation is less accurate** as well **less noise sensitive** than the three point interpolation. Same trends are followed for subsequent 'n' point interpolations. While its true that the as we involve more and more bins we achieve more accuracy, it cant be denied that we are subjecting our results to be more and more Noise sensitive that is allowing them to be distorted easily by external noise. This is the tradeoff that our paper talks about.

Our paper draws similarity between the interpolation techniques and the windowing techniques. The analogy being as we involve more and more bins in our interpolation algorithm we are effectively attenuating the side lobes and widening the main lobe. While the side lobe attenuation brings accuracy the widening of the main lobe makes our results extremely distorted in presence of external noise. The same can be said for different windows as well, while the Hanning window provides better accuracy due to small side lobes the wide main lobe makes the output more prone to distortion by external noise.

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