Lecture 20

Gambler’s Ruin Model on DTMC

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20.1 Introduction

What is Markov Chains

A Markov chain essentially consists of a set of transitions, which are determined by some probability distribution, that satisfy the Markov property.

Gambler’s Ruin Model

The term gambler’s ruin is a statistical concept, most commonly expressed as the fact that a gambler playing a game with negative expected value will eventually go broke, regardless of their betting system.

20.2 Motivational and IDEA

A Gambler is nothing but a man who makes his living out of false hope. State space of all possible wealth of Gambler.

$$S = 0, 1, ..., (i-1), i, (i-1), ...N$$

20.3 DTMC

Discrete-time Markov chain (DTMC) is a sequence of random variables, known as a stochastic process, in which the value of the next variable depends only on the value of the current variable, and not any variables in the past. For Example, a machine may have two states, A and E. When it is in state A, there is a 40

20.4 Initial condition

$$X_0 = i \Sigma S$$

$$\alpha_i = P[X_n = N | X_0 = i]$$

Probability that gambling winning betting game.

$$(1 - \alpha_i)$$

Probability that gambling losing betting game.

At each step of betting. P = P [Win that bet step]

$$P[X_{m+1} = (K + 1) | X_m = K]$$

At each step of betting. P = P [losing that bet step]

$$q = (1 - p) = P[X_{m+1} = (k_{+1}) | x_m = k]$$

ASSUME:- Time homogeneous DTMC are P and q are constant do not change with time.

METHOD: Set up recurrence relation in α_i by using DTMC, conditioning on 1^{st} step in Markov Property.

In second order recurrence relation after applying boundary conditions on the losing and winning states.

$$\alpha_i = \alpha_{i+1}P + \alpha_{i1}q$$

$$\alpha_i = P[X_n = N|X_0 = i]$$

$$\alpha_0 = P[X_n = N|X_0 = 0] = 0$$

$$\alpha_N = P[X_n = N|X_0 = N] = 1$$

$$\alpha_i = (1 - \delta^i)/(1 - \delta^N)$$

Therefore

$$\delta \neq 1$$

and

$$\delta = q/p$$

CASE (1) :- If $q > P \Rightarrow$loss probability $>$ win probability.

Since,

$$\delta > 1 and N \sim \infty$$

$$\delta^N \sim \infty$$

$$\alpha_i = (\delta^i - 1)/(\delta^N - 1)$$

$$(\delta^i - 1)/\delta^N = 0$$

CASE (2):- If $q < P$ $\Rightarrow$loss probability $<$ win probability.

$$\delta_i = (1 - \delta^i)/(1 - \delta^N = (1 = \delta^i) = 1$$

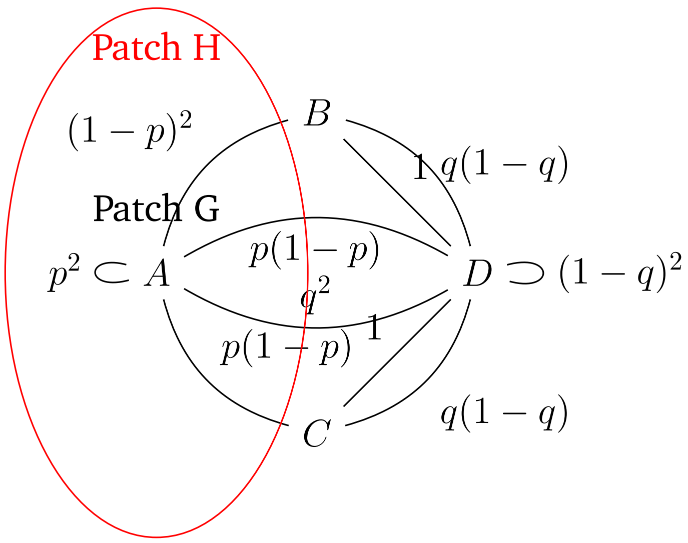
NOTE 1 :- $n = \infty$ mean n = large time

$$p(\infty)^T = [P_1(\infty), P_2(\infty)] = r^T$$

Suppose, $r^T = [0.6, 0.4] \Rightarrow P_1(\infty) = 0.6 and P_2(\infty) = 0.4$

NOTE 2 :- When certain DTMC is run for a long time $n \longrightarrow \infty$ the DTMC set into an equilibrium stationary distribution.

20.5 characterisation of states of DTMC



To check the characteristics of a particular state $i \sum S$ start DTMC at state $X_0 = i$. Run the DTMC long enough $n \to \infty$ and are if DTMC return to state $i \sum S$.

Definition 1 :- Transient state might be visited at the beginning but eventually DTMC stop visiting them as $n \to \infty$, since DTMC escapes from them.
Definition 2 :- Visits to recurrent state keep happening forever as $n \to$. For arbitary m, almost surely, $X_n = i \sum S$.

Definition 3:- If state $i \sum S$ in recurrent then $f_i = 1$. Since there is an escape probability of state i that is $= (1 - f_i) > 0$, there is a positive escape probability of never coming back to the transient state $i \sum S$.

20.6 Conclusion

The Gambler's Ruin Problem is a great example of can take a complex situation and derive a simple general structure from it using statistical tools. Given a fair game, the probability that someone will win enough games to claim the total wealth of both players is determined by their initial wealth and the total wealth. This is known not only at the beginning of the sequence, but also at each step. Using Markov chains, can determine the same probabilities between any sequences of games using the transition matrix and the probability vector at the initial state.

20.7 Practice problems

1. An irreducible homogeneous Markov chain on a finite state space is positive recurrent. Therefore, it always has a stationary distribution?
2. Gopi starts with 2 rupee, and $p = 0.6$. What is the probability that Gopi obtains a fortune of N = 4 without going broke?
3. What is the probability that Gopi will become infinitely rich?
4. Recurrence and transience are class properties, i.e. in a communication class C either all states are recurrent or all states are transient?
5. Suppose T is a stopping time of a DTMC $(X_n)n > 0$. Then conditioned on $T <$ and $X_T = i$, the sequence $(X_{T+n})n > 0$.