Universidad del Valle de Guatemala
Departamento de Matemática
Licenciatura en Matemática Aplicada

Estudiante: Rudik Roberto Rompich
E-mail: rom19857@uvg.edu.gt
Carné: 19857

MM2030 - Ecuaciones Diferenciales 2 - Catedrático: Dorval Carías
28 de mayo de 2021

Tarea 6

Resuelva los problemas con valores en la frontera que se presentan a continuación, utilizando el método a su elección.

1. Problema 1.

Resuelva:

$$\frac{\partial u}{\partial t^2} = a^2 \frac{\partial^2}{\partial x^2}, \qquad x > 0, t, 0$$

Sujeta a: $u(0,t) = f(t), \quad u(x,0) = 0, \lim_{x\to\infty} u(x,t) = 0, \frac{\partial u}{\partial t}(x,0) = 0.$

Solución. Aplicando la transformada de Laplace:

$$\begin{aligned} \mathcal{L}\left\{\frac{\partial u}{\partial t^2}\right\} &= a\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}\right\} \\ s^2\widehat{u}(x,s) - su(x,0) - u'(x,0) &= a^2\widehat{u}''(x,s) \\ s^2\widehat{u} - 0 - 0 &= a^2\widehat{u}''(x,s) \end{aligned}$$

Aplicando las condiciones:

$$\begin{aligned} s^2\widehat{u} &= a^2\widehat{u}'' \\ a^2\widehat{u}'' - s^2\widehat{u} &= 0 \\ \widehat{u}'' - \left(\frac{s}{a}\right)^2\widehat{u} &= 0 \end{aligned}$$

Resolviendo la EDO:

$$\widehat{u}(x,s) = Ae^{\frac{s}{a}x} + Be^{-\frac{s}{a}x}$$

Aplicando $\lim_{x\to\infty} \widehat{u}(x,s) = 0$, B se hace 0:

$$\lim_{x\to\infty} \widehat{u}(x,s) = \underbrace{\lim_{x\to\infty}\left(Ae^{\frac{s}{a}x} + Be^{-\frac{s}{a}x}\right)}_{\text{A=0, para que se cumpla la condición.}} = 0$$

Entonces

$$\widehat{u}(x,s) = Be^{\frac{s}{a}x}$$

Aplicando $\widehat{u}(0,s) = \widehat{f}(s)$, tenemos:

$$\widehat{u}(0,s) = B = \widehat{f}(s)$$

Por lo que tenemos:

$$\widehat{u}(x,s) = \widehat{f}(s)e^{-\frac{s}{a}x}$$

$$u(x,s) = \mathcal{L}^{-1}\left[\widehat{u}(x,s)\right] = \mathcal{L}^{-1}\left[\widehat{f}(s)e^{-\frac{s}{a}x}\right] = f\left(t-\frac{x}{a}\right)H\left(t-\frac{x}{a}\right).$$

$\square$

2. Problema 2.

Resuelva:

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, t > 0.$$

Sujeta a: $u(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = K$ y con $u(x,0) = 0$, $\frac{\partial u}{\partial t}(x,0) = 0$.

Solución.

$$\begin{aligned}
\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} &= a^2\mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\}\\
s^2\widehat{u}(x,s) - su(x,0) - u'(x,0) &= a^2\widehat{u}''(x,s)\\
s^2\widehat{u}(x,s) - 0 - 0 &= a^2\widehat{u}''(x,s)\\
a^2\widehat{u}''(x,s) - s^2\widehat{u}(x,s) &= 0\\
\widehat{u}''(x,s) - \left(\frac{s}{a}\right)^2\widehat{u}(x,s) &= 0
\end{aligned}$$

La solución de la EDO:

$$\widehat{u}(x,s) = Ae^{\frac{s}{a}x} + Be^{-\frac{s}{a}x}$$

Tenemos la condición $\widehat{u}(0,t) = 0$:

$$\widehat{u}(0,t) = Ae^{\frac{s}{a}x} - Ae^{-\frac{s}{a}x} = A\left(e^{\frac{s}{a}x} - e^{-\frac{s}{a}x}\right) = 2A\,\mathrm{senh}\left(\frac{s}{a}x\right)$$

Aplicando la condición $\frac{\partial u}{\partial x}(L,t) = K$:

$$\begin{aligned}
\widehat{u}'(x,t) &= A\left(\frac{s}{a}e^{\frac{s}{a}x} + \frac{s}{a}e^{-\frac{s}{a}x}\right)\\
&= \frac{2As}{a}\cosh\left(\frac{s}{a}x\right)\\
\widehat{u}'(L,t) &= \frac{2As}{a}\cosh\left(\frac{s}{a}L\right) = \frac{k}{s}
\end{aligned}$$

$$A = \frac{aK}{2s^2\cosh\left(\frac{s}{a}L\right)}$$

Entonces, tenemos:

$$\widehat{u}(x,s) = \frac{aK}{s^2\cosh\left(\frac{s}{a}L\right)}\mathrm{senh}\left(\frac{s}{a}x\right)$$

Entonces, la solución:

$$u(x,t) = \mathcal{L}^{-1}\left[\frac{aK\operatorname{senh}\left(\frac{s}{a}x\right)}{s^2\cosh\left(\frac{s}{a}L\right)}\right]$$

□

3. Problema 3.

Resuelva:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, t > 0$$

Sujeta a: $u(0,t) = 0, u(L,t) = 0, u(x,0) = 1$.

Solución.

$$\begin{aligned} \mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} &= \mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} \\ s\widehat{u}(x,s) - u(x,0) &= \widehat{u}''(x,s) \\ s\widehat{u}(x,s) - 1 &= \widehat{u}''(x,s) \\ \widehat{u}''(x,s) - s\widehat{u}(x,s) &= -1 \end{aligned}$$

$$\widehat{u}(x,s) = \widehat{u}_H(x,s) + \widehat{u}_P(x,s)$$

Caso homógeneo:

$$\widehat{u}''_H(x,s) - s\widehat{u}(x,s) = 0$$

Caso particular

$$\widehat{u}''_P(x,s) - s\widehat{u}_P(x,s) = -1$$

Se propocede con coeficientes indeterminados, se propone $\widehat{u}_P = c$

$$\frac{d}{dx^2}c - sc = -1 \implies 0 - sc = -1 \implies c = \frac{1}{s} \implies \widehat{u}_P(x,s) = \frac{1}{s}.$$

Solución general:

$$\widehat{u}(x,s) = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x} + \frac{1}{s}.$$

Aplicando $\widehat{u}(0,s) = 0$, entonces:

$$\widehat{u}(0,s) = A + B + \frac{1}{s} = 0 \implies B = -\left(\frac{As+1}{s}\right)$$

Implica que

$$\widehat{u}(x,s) = Ae^{\sqrt{s}x} - \left(\frac{As+1}{s}\right)e^{-\sqrt{s}x} + \frac{1}{s} = 2A\operatorname{senh}\left(\sqrt{s}x\right) + \frac{1}{s}\left(1 - e^{-\sqrt{s}x}\right).$$

Aplicando $\widehat{u}(L,s)=0$, entonces:

$$\widehat{u}(L,s) = 2A\operatorname{senh}\left(\sqrt{s}L\right) + \frac{1}{s}\left(1 - e^{-\sqrt{s}L}\right) \implies A = \frac{e^{-\sqrt{s}L}-1}{2s\operatorname{senh}\left(\sqrt{s}L\right)}.$$

Implica:

$$\widehat{u}(x,s) = \left(\frac{e^{-\sqrt{s}L}-1}{s\operatorname{senh}\left(\sqrt{s}L\right)}\right)\operatorname{senh}\left(\sqrt{s}x\right) + \frac{1}{s}\left(1 - e^{-\sqrt{s}x}\right).$$

Entonces, la solución es:

$$u(x,t) = \mathcal{L}^{-1}\left[\hat{u}(x,s)\right] = \mathcal{L}^{-1}\left[\left(\frac{e^{-\sqrt{s}L}-1}{s\operatorname{senh}\left(\sqrt{s}L\right)}\right)\operatorname{senh}\left(\sqrt{s}x\right) + \frac{1}{s}\left(1 - e^{-\sqrt{s}x}\right)\right]$$

□

4. Problema 4.

Resuelva:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, t > 0.$$

Sujeta a: $u(x,0) = e^{-|x|}$

Solución. Aplicando transformada de Fourier.

$$\begin{aligned}\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} &= \mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\}\\ \hat{u}(w,t) &= (iw)^2\hat{u}(w,t)\\ \hat{u}(w,t) &= -w^2\hat{u}(w,t)\end{aligned}$$

Entonces, tenemos:

$$\hat{u}'(w,t) + w^2\hat{u}(w,t) = 0.$$

La solución de la EDO:

$$\hat{u}(w,t) = A_w\cos(wt) + B_w\operatorname{sen}(wt)$$

Aplicando la condición $u(x,0) = e^{-|x|} \implies \hat{u}(w,0) = \frac{2}{w^2+1}$:

$$\hat{u}(w,0) = A_w = \frac{2}{w^2+1}.$$

Entonces, tenemos:

$$\hat{u}(w,t) = \left(\frac{2}{w^2+1}\right)\cos(wt) + B_w\operatorname{sen}(wt)$$

Implica que la solución:

$$u(w,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\left(\frac{2}{w^2+1}\right)\cos(wt) + B_w\operatorname{sen}(wt)\right]e^{iwx}dw.$$

□