Universidad del Valle de Guatemala
Departamento de Matemática
Licenciatura en Matemática Aplicada

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Parcial 2 - Revisión

1. Problema 1

Funciones ortogonales

1. Compruebe que $f_1(x) = x$ y $f_2(x) = x^2$ son ortogonales en [-2, 2].

   Solución.

$$\begin{aligned} \langle f_1(x), f_2(x)\rangle &= \int_{-2}^{2} x\cdot x^2 \,\mathrm{d}x = \int_{-2}^{2} x^3 \,\mathrm{d}x = \frac{1}{4}x^4\Big|_{-2}^{2} \\ &= \frac{1}{4}\left[(2)^4 - (-2)^4\right] = 0 \end{aligned} \tag{1}$$

   $\therefore\ f_1(x),\ f_2(x)$ son ortogonales en el intervalo [-2,2]. □

2. Determine las constantes c_1 y c_2 tales que $f_3(x) = x + c_1x^2 + c_2x^3$ sea ortogonal a f_1 y f_2 en el mismo intervalo.

$$\begin{aligned} \langle f_3(x), f_1(x)\rangle &= \int_{-2}^{2} (x + c_1x^2 + c_2x^3)\cdot(x)\,\mathrm{d}x = \int_{-2}^{2}(x^2 + c_1x^3 + c_2x^4)\,\mathrm{d}x \\ &= \frac{1}{3}x^3 + \frac{c_1}{4}x^4 + \frac{c_2}{5}x^5\Big|_{-2}^{2} = \frac{1}{3}[(2)^3 - (-2)^3] + \frac{c_2}{5}[(2)^5 - (-2)^5] \\ &= \frac{1}{3}[2^4] + \frac{c_2}{5}[2^6] \end{aligned} \tag{1}$$

   Se sabe que $\langle f_3(x), f_1(x)\rangle = 0$, entonces:

$$\implies \frac{1}{3}[2^4] + \frac{c_2}{5}[2^6] = 0 \implies c_2 = -\frac{2^4\cdot 5}{3\cdot 2^6} = -\frac{5}{3\cdot 2^2} = -\frac{5}{12} \tag{2}$$

$$\begin{aligned} \langle f_3(x), f_1(x)\rangle &= \int_{-2}^{2}(x + c_1x^2 - \frac{5}{12}x^3)\cdot(x^2)\,\mathrm{d}x = \int_{-2}^{2}(x^3 + c_1x^4 - \frac{5}{12}x^5)\,\mathrm{d}x \\ &= \frac{1}{4}x^4 + c_1\frac{1}{5}x^5 - \frac{5}{60}x^6\Big|_{-2}^{2} = \frac{c_1}{5}[(2)^5 - (-2)^5] \end{aligned} \tag{3}$$

Nuevamente, se conoce que $\langle f_3(x), f_2(x)\rangle = 0$, entonces:

$$\implies \frac{c_1}{5}[(2)^5 - (-2)^5] = 0 \implies c_1 = 0 \tag{4}$$

$$\boxed{c_1 = 0 \text{ y } c_2 = -\tfrac{5}{12}}$$

2. Problema 2

Serie de Fourier

1. Encuentre la serie de Fourier de $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 \leq x < \pi \end{cases}$

$$\boxed{f(x) = \tfrac{a_0}{2} + \textstyle\sum_{n=1}^{\infty} \left[a_n \cos \tfrac{x\pi n}{L} + b_n \sin \tfrac{x\pi n}{L}\right]}$$

Para a_0:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,\mathrm{d}x + \int_0^{\pi} \sin x\,\mathrm{d}x\right] = \frac{1}{\pi}\left[-\cos x\Big|_0^{\pi}\right] \\ &= -\frac{1}{\pi}\left[\cos\pi - \cos 0\right] = -\frac{1}{\pi}\left[-1-1\right] = \frac{2}{\pi} \end{aligned} \tag{1}$$

El caso base de a_n, i.e. $n = 1$:

$$\begin{aligned} a_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}\sin x\cos x\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{\pi}[\sin(x+x)]\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_0^{\pi}\sin(2x)\,\mathrm{d}x = \frac{1}{4\pi}\int_0^{2\pi}\sin u\,\mathrm{d}u = -\frac{1}{4\pi}\left[\cos u\Big|_0^{2\pi}\right] \\ &= -\frac{1}{4\pi}\left[\cos 2\pi - \cos 0\right] = 0 \end{aligned} \tag{2}$$

Para a_n:

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, \mathrm{d}x = \frac{1}{\pi}\left[\int_0^{\pi} \sin x \cos nx\right] \mathrm{d}x \\
&= \frac{1}{2\pi}\left\{\int_0^{\pi} [\sin(x-nx)+\sin(nx+x)] \, \mathrm{d}x\right\} \\
&= \frac{1}{2\pi}\left\{\int_0^{\pi} [\sin(1-n)x+\sin(n+1)x] \, \mathrm{d}x\right\} \\
&= \frac{1}{2\pi}\left\{-\frac{1}{1-n}\cos(1-n)x - \frac{1}{n+1}\cos(n+1)x\Big|_0^{\pi}\right\} \\
&= \frac{1}{2\pi}\left\{-\frac{1}{1-n}[\cos(1-n)\pi - \cos 0] - \frac{1}{n+1}[\cos(n+1)\pi - \cos 0]\right\} \\
&= -\frac{1}{2\pi}\left\{\frac{1}{1-n}[\cos(1-n)\pi - \cos 0] + \frac{1}{n+1}[\cos(n+1)\pi - \cos 0]\right\} \\
&= -\frac{1}{2\pi}\left\{\frac{1}{1-n}[(-1)^{n-1}-1] + \frac{1}{n+1}[(-1)^{n+1}-1]\right\} \\
&= -\frac{1}{2\pi}\left\{\frac{(-1)^{n-1}}{1-n} - \frac{1}{1-n} + \frac{(-1)^{n+1}}{n+1} - \frac{1}{n+1}\right\} \\
&= -\frac{1}{2\pi}\left\{\frac{(-1)(-1)^{n-1}}{n-1} + \frac{1}{n-1} + \frac{(-1)^{n+1}}{n+1} - \frac{1}{n+1}\right\} \\
&= -\frac{1}{2\pi}\left\{\frac{(-1)^n(n+1)+(-1)^{n+1}(n-1)}{(n-1)(n+1)} + \frac{(n+1)-(n-1)}{(n-1)(n+1)}\right\} \\
&= -\frac{1}{2\pi}\left\{\frac{(-1)^n(n+1)+(-1)^{n}(1-n)}{(n-1)(n+1)} + \frac{2}{(n-1)(n+1)}\right\} \\
&= -\frac{1}{2\pi}\left\{\frac{(-1)^n[n+1+1-n]}{(n-1)(n+1)} + \frac{2}{(n-1)(n+1)}\right\} \\
&= -\frac{1}{2\pi}\left\{\frac{2(-1)^n+2}{n^2-1}\right\} \\
&= -\frac{1}{\pi}\left\{\frac{(-1)^n+1}{n^2-1}\right\} \\
&= \frac{1+(-1)^n}{\pi(1-n^2)} = \begin{cases} 0 & n \text{ impar} \\ \frac{2}{\pi(1-4n^2)} & n \text{ par} \end{cases}
\end{aligned}
\tag{3}
$$

El caso base de b_n, i.e $n=1$:

$$
\begin{aligned}
b_1 &= \frac{1}{\pi}\int_0^{\pi} \sin x \sin x \, \mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} \sin^2 x \, \mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{\pi} [\cos(x-x)-\cos(x+x)] \, \mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{\pi} [\cos(x-x)-\cos(x+x)] \, \mathrm{d}x = \frac{1}{2\pi}\int_0^{\pi} [1-\cos(2x)] \, \mathrm{d}x \\
&= \frac{1}{2\pi}\left[x - \frac{1}{2}\sin 2x\Big|_0^{\pi}\right] = \frac{1}{2\pi}[\pi] = \frac{1}{2}
\end{aligned}
\tag{4}
$$

Para b_n:

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx \, \mathrm{d}x = \frac{1}{2\pi}\int_0^{\pi} [\cos(x-nx) - \cos(x+nx)] \, \mathrm{d}x \\ &= \frac{1}{2\pi}\int_0^{\pi} [\cos(1-n)x - \cos(1+n)x] \, \mathrm{d}x = 0 \end{aligned} \tag{5}$$

Por lo tanto, *la serie de Fourier* es:

$$\begin{aligned} f(x) &= \frac{1}{\pi} + \frac{1}{2}\sin x + \sum_{n=1}^{\infty} \frac{2}{\pi(1-4n^2)} \cos 2nx \\ &= \frac{1}{\pi} + \frac{1}{2}\sin x + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{(1-4n^2)} \\ &= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{(4n^2-1)} \end{aligned} \tag{6}$$

Se puede consultar en: https://www.desmos.com/calculator/nviq2wplpt

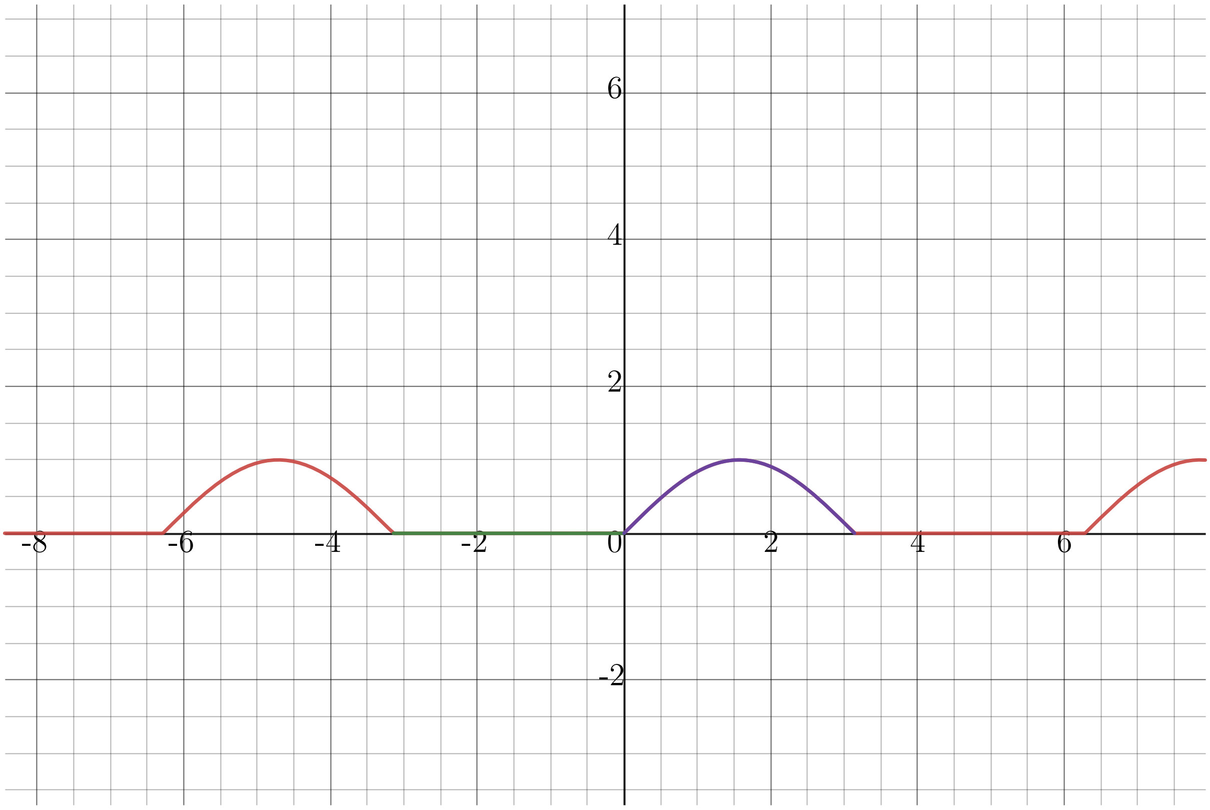

Figura 1: Serie de Fourier

2. Utilice el resultado del inciso anterior para deducir que

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots$$

Solución. Para la demostración de la serie de $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9}$ se tomará como referencia la demostración de (https://math.stackexchange.com/users/458544/fghj) en el caso de la serie positiva. Entonces, se tiene que:

$$\frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\frac{1}{7\cdot 9}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)(2n+1)}$$

Es decir que el problema pide deducir:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)(2n+1)}=\frac{\pi}{4}-\frac{1}{2}=\frac{\pi-2}{4}$$

Es decir, expresado de otra forma:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}=\frac{\pi-2}{4}$$

Entonces, se tiene:

$$f(x)=\frac{1}{\pi}+\frac{1}{2}\sin x-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(4n^2-1)} \tag{1}$$

$$\sin(x)=\frac{1}{\pi}+\frac{1}{2}\sin x-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(4n^2-1)} \tag{2}$$

Se propone utilizar $x=\frac{\pi}{2}$:

$$\sin(\frac{\pi}{2})=\frac{1}{\pi}+\frac{1}{2}\sin\frac{\pi}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2n\frac{\pi}{2}}{(4n^2-1)} \tag{3}$$

$$1=\frac{1}{\pi}+\frac{1}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} \tag{4}$$

$$1-\frac{1}{\pi}-\frac{1}{2}=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1} \tag{5}$$

$$\frac{\pi(2\pi-2-\pi)}{4\pi}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1} \tag{6}$$

$$\frac{(2\pi-2-\pi)}{4}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1} \tag{7}$$

$$\frac{\pi-2}{4}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1} \tag{8}$$

□

3. Problema 3

3. Resuelva la ecuación de Laplace

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

para una placa rectangular y sujeta a las condiciones

$$\frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial x}(a,y) = 0$$
$$u(x,0) = x, \quad u(x,b) = 0$$

Solución. Comenzamos planteando una sustitución de variables:

$$u(x,y) = X(x) \cdot Y(y) = X \cdot Y$$

Es decir que el problema se puede plantear como:

$$X''Y + Y''X = 0 \implies X''Y = -Y''X \implies \boxed{\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda}$$

del cual se generan 2 EDOs.

Para la primera EDO:

$$\frac{X''}{X} = -\lambda \implies X'' + \lambda X = 0$$

Con las condiciones de frontera:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0$$

La solución para $\lambda = 0$ es:

$$X(x) = A + Bx$$
$$X'(x) = B$$

Aplicando las condiciones de frontera:

$$X'(0) = B = 0$$

$$X'(a) = B = 0$$

Por lo tanto:

$$\boxed{X_0(x) = 1}$$

La solución para $\lambda \neq 0$ es:

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

Derivando:

$$X'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$$

Aplicando las condiciones de frontera:

$$X'(0) = B = 0$$
$$X'(a) = -A\sqrt{\lambda}\sin\sqrt{\lambda}a = 0$$

Ahora es necesario analizar la relación de $\sqrt{\lambda}a$:

$$\sqrt{\lambda}a = \pi n \implies \sqrt{\lambda} = \frac{\pi n}{a}, \qquad n = 1, 2, 3, ...$$

Por lo tanto, la solución de la EDO es:

$$\boxed{X_n(x) = \cos \tfrac{\pi n}{a} x, \qquad n = 1, 2, 3, ...}$$

La segunda EDO:

$$-\frac{Y''}{Y} = -\lambda \implies Y'' - \lambda Y = 0$$

Con la condición de frontera:

$$u(x, b) = 0$$

La solución para $\lambda = 0$:

$$Y(y) = C + Dy$$

Aplicando las condiciones de frontera:

$$Y(b) = C + Db = 0 \implies C = -Db$$

Finalmente:

$$Y(y) = -Db + Dy$$

$$\boxed{Y_0(y) = (y - b)}$$

La solución para $\lambda \neq 0$ es:

$$Y(y) = C \cosh \sqrt{\lambda} y + D \sinh \sqrt{\lambda} y$$

Aplicando la condición inicial:

$$Y(b) = C \cosh \sqrt{\lambda} b + D \sinh \sqrt{\lambda} b = 0$$

Despejando para C:

$$C = -\frac{D \sinh \sqrt{\lambda} b}{\cosh \sqrt{\lambda} b} = -D \tanh \sqrt{\lambda} b$$

Es decir que sustituyendo la C en la solución inicial:

$$Y(y) = -D \tanh \sqrt{\lambda} b \cdot \cosh \sqrt{\lambda} y + D \sinh \sqrt{\lambda} y = D \left[\sinh \sqrt{\lambda} y - \tanh \sqrt{\lambda} b \cdot \cosh \sqrt{\lambda} y \right]$$

Ahora bien, sustituyendo $\sqrt{\lambda}$, la solución es:

$$\boxed{Y_n(y) = \sinh \tfrac{\pi n}{a} y - \tanh \tfrac{\pi n}{a} b \cdot \cosh \tfrac{\pi n}{a} y, \qquad n = 1, 2, 3, ...}$$

La solución, entonces:

$$\boxed{u(x, y) = \tfrac{A_0}{2}(y - b) + \textstyle\sum_{n=1}^{\infty} A_n \cos \tfrac{\pi n}{a} x \cdot \left[\sinh \tfrac{\pi n}{a} y - \tanh \tfrac{\pi n}{a} b \cdot \cosh \tfrac{\pi n}{a} y \right]}$$

Sujeta a la condición inicial:

$$u(x, 0) = x$$

Entonces:

$$x = -\frac{A_0 b}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{\pi n}{a}x \cdot \left[-\tanh\frac{\pi n}{a}b\right]$$

Haciendo dos sustitución para simplificar:

$$x = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos\frac{\pi n}{a}x$$

Identificamos que se trata de una serie de cosenos. Comenzamos calculando C_0:

$$C_0 = \frac{2}{a}\int_0^a x\ dx = \frac{1}{a}x^2\Big|_0^a = \frac{1}{a}(a^2) = a$$

Para C_n:

$$C_n = \frac{2}{a}\int_0^a x\cos\frac{\pi n}{a}x\ dx$$

En donde:

$$\begin{array}{ccc} + & x & \cos\frac{\pi n}{a}x \\ - & 1 & \frac{a}{\pi n}\sin\frac{\pi n}{a}x \\ + & 0 & \frac{a^2}{\pi^2 n^2}\cos\frac{\pi n}{a}x \end{array}$$

Por lo cual:

$$\begin{aligned} C_n &= \frac{2}{a}\left[\frac{xa}{\pi n}\sin\frac{\pi n}{a}x + \frac{a^2}{\pi^2 n^2}\cos\frac{\pi n}{a}x\right]_0^a \\ &= \frac{2}{a}\left[\left(\frac{a^2}{\pi n}\sin\pi n + \frac{a^2}{\pi^2 n^2}\cos\pi n\right) - \left(\frac{a^2}{\pi^2 n^2}\right)\right] \\ &= \frac{2}{a}\left[\frac{a^2}{\pi^2 n^2}\cos\pi n - \frac{a^2}{\pi^2 n^2}\right] \\ &= \frac{2a}{\pi^2 n^2}\left[(-1)^n - 1\right] \end{aligned}$$

Volviendo a las expresiones originales de las primeras 2 substituciones:

$$C_0 = -A_0 b \implies a = -A_0 b \implies A_0 = -\frac{a}{b} \implies$$

Por otra parte:

$$C_n = -A_n\tanh\frac{\pi n}{a}b \implies \frac{2a}{\pi^2 n^2}\left[(-1)^n - 1\right] = -A_n\tanh\frac{\pi n}{a}b$$
$$\implies A_n = \frac{\frac{2a}{\pi^2 n^2}\left[(-1)^n - 1\right]}{-\tanh\frac{\pi n}{a}b} = \frac{2a[1-(-1)^n]}{\pi^2 n^2\tanh\frac{\pi n}{a}b} = \frac{2a[1+(-1)^{n+1}]}{\pi^2 n^2\tanh\frac{\pi n}{a}b}$$

Por lo tanto, la solución final:

$$\boxed{u(x,y) = -\frac{a}{2b}(y-b) + \sum_{n=1}^{\infty}\frac{2a[1+(-1)^{n+1}]}{\pi^2 n^2\tanh\frac{\pi n}{a}b}\cos\frac{\pi n}{a}x \cdot \left[\sinh\frac{\pi n}{a}y - \tanh\frac{\pi n}{a}b\cdot\cosh\frac{\pi n}{a}y\right]}$$

$\square$

4. Problema 4

Resuelva el problema con valores en la frontera:

$$\frac{\partial^2 u}{\partial t^2} + a^2 \frac{\partial^4 u}{\partial x^4} = 0$$

Sujeta a las condiciones:

$$\begin{array}{c} u(0,t) = u(l,t) = 0, t > 0 \\ \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(l,t) = 0, t > 0 \\ u(x,0) = f(x), \quad 0 \leq x \leq l \\ \frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 \leq x \leq l \end{array}$$

Solución. Considerando:

$$u(x,t) = X(x) \cdot T(t)$$

Entonces, substituyendo en la ecuación original:

$$XT'' + a^2 X^{(4)} T = 0 \implies a^2 X^{(4)} T = -XT'' \implies \frac{X^{(4)}}{-X} = \frac{T''}{a^2 T} = -\lambda^2$$

La primera EDO, se define como:

$$X^{(4)} - \lambda^2 X = 0.$$

Con las condiciones,

$$u(0,t) = u(l,t) = 0$$

$$\frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(l,t) = 0$$

$\boxed{\text{Caso } \lambda^2 > 0}$

Proponemos una substitución,

$$m^4 - \lambda^2 = 0 \implies (m^2 - \lambda)(m^2 + \lambda) = 0$$

$$m = \pm\sqrt{\lambda} \qquad m = \pm i\lambda$$

Por lo que la solución general de la EDO es,

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} + C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$$

Con las condiciones,

$$u(0,t) = u(l,t) = 0.$$

Entonces,

$$X(0) = A + B + C = 0 \tag{1}$$

$$X(l) = Ae^{\sqrt{\lambda}l} + Be^{-\sqrt{\lambda}l} + C\cos(\sqrt{\lambda}l) + D\sin(\sqrt{\lambda}l) = 0 \tag{2}$$

Su primera derivada,

$$X'(x) = \sqrt{\lambda}Ae^{\sqrt{\lambda}x} - \sqrt{\lambda}Be^{-\sqrt{\lambda}x} - \sqrt{\lambda}C\sin(\sqrt{\lambda}x) + \sqrt{\lambda}D\cos(\sqrt{\lambda}x)$$

Su segunda derivada,

$$X''(x) = \lambda Ae^{\sqrt{\lambda}x} + \lambda Be^{-\sqrt{\lambda}x} - \lambda C\cos(\sqrt{\lambda}x) - \lambda D\sin(\sqrt{\lambda}x)$$

Con las condiciones,

$$\frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(l,t) = 0$$

Entonces,

$$X''(0) = \lambda A + \lambda B - \lambda C = \lambda(A+B-C) = 0 \implies A+B-C = 0 \tag{3}$$

$$X''(l) = \lambda Ae^{\sqrt{\lambda}l} + \lambda Be^{-\sqrt{\lambda}l} - \lambda C\cos(\sqrt{\lambda}l) - \lambda D\sin(\sqrt{\lambda}l) = 0$$
$$\implies \lambda(Ae^{\sqrt{\lambda}l} + Be^{-\sqrt{\lambda}l} - C\cos(\sqrt{\lambda}l) - D\sin(\sqrt{\lambda}l)) = 0$$

$$\implies X''(l) = Ae^{\sqrt{\lambda}l} + Be^{-\sqrt{\lambda}l} - C\cos(\sqrt{\lambda}l) - D\sin(\sqrt{\lambda}l) = 0 \tag{4}$$

Entonces, por (1) y (3) tenemos que,

$$\begin{cases} A+B+C=0 \\ A+B-C=0 \end{cases} \implies -C-C=0 \implies -2C=0 \implies C=0.$$

Por lo que, sabemos que,

$$A+B=0 \implies B=-A.$$

Por otra parte, por (2), y (4) sabemos:

$$\begin{cases} Ae^{\sqrt{\lambda}l} + Be^{-\sqrt{\lambda}l} + C\cos(\sqrt{\lambda}l) + D\sin(\sqrt{\lambda}l) = 0 \\ Ae^{\sqrt{\lambda}l} + Be^{-\sqrt{\lambda}l} - C\cos(\sqrt{\lambda}l) - D\sin(\sqrt{\lambda}l) = 0 \end{cases}$$

Ahora bien, considerando que $C=0$ y $B-A$, entonces:

$$\begin{cases} Ae^{\sqrt{\lambda}l} + -Ae^{-\sqrt{\lambda}l} + D\sin(\sqrt{\lambda}l) = 0 \\ Ae^{\sqrt{\lambda}l} - Ae^{-\sqrt{\lambda}l} - D\sin(\sqrt{\lambda}l) = 0 \end{cases} \quad \text{Si aplicamos una resta, entonces:}$$

$$2D\sin(\sqrt{\lambda}l) = 0 \implies \sin(\sqrt{\lambda}l) = 0, \quad \text{en donde: } \sqrt{\lambda}l = \pi n \implies \sqrt{\lambda} = \frac{\pi n}{l}$$

Por lo tanto,

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} + C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$$

Como se mostró previamente, $C=0$ y $B=-A$, entonces:

$$X(x) = Ae^{\sqrt{\lambda}x} - Ae^{-\sqrt{\lambda}x} + D\sin(\sqrt{\lambda}x) = A\left(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}\right) + D\sin(\sqrt{\lambda}x)$$

$$\implies X(x) = 2A\sinh(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$$

Considerando que $\sqrt{\lambda} = \pi n/l$, entonces:

$$X_n(x) = A_n \sinh\left(\frac{\pi n}{l}x\right) + B_n \sin\left(\frac{\pi n}{l}x\right), \quad n = 1, 2, 3, ...$$

$\boxed{\text{Caso } \lambda^2 = 0}$

Es decir que tenemos,

$$X^{(4)} = 0.$$

Con la solución general,

$$X(x) = Ax^3 + Bx^2 + Cx + D$$

Con las condiciones,

$$u(0,t) = u(l,t) = 0.$$

Entonces,

$$X(0) = D = 0$$

$$X(l) = Al^3 + Bl^2 + Cl + D = Al^3 + Bl^2 + Cl = 0$$

Su primera derivada,

$$X'(x) = 3Ax^2 + 2Bx + C$$

Su segunda derivada,

$$X''(x) = 6Ax + 2B$$

Con las condiciones,

$$\frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(l,t) = 0$$

Entonces,

$$X''(0) = B = 0$$

$$X''(l) = 6Al + 2B = 6Al = 0 \implies A = 0.$$

Por lo tanto,

$$X_0(x) = 0.$$

La segunda EDO, se define como:

$$T'' + (a\lambda)^2 T = 0.$$

Su solución general es,

$$T(t) = A\cos(a\lambda t) + B\sin(a\lambda t)$$

En donde, $\sqrt{\lambda} = \pi n/l \implies \lambda = (\pi n/l)^2$, entonces:

$$T_n(t) = C_n \cos\left(a\frac{\pi n}{l}t\right) + D_n \sin\left(a\frac{\pi n}{l}t\right), \quad n = 1, 2, ...$$

Ahora bien, entonces,

$$u(x,t) = X_nT_n = \left[A_n \sinh\left(\frac{\pi n}{l}x\right) + B_n \sin\left(\frac{\pi n}{l}x\right)\right] \cdot \left[C_n \cos\left(a\frac{\pi n}{l}t\right) + D_n \sin\left(a\frac{\pi n}{l}t\right)\right]$$

Implica:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} X_n \cdot T_n \\ &= \sum_{n=1}^{\infty} \left[A_n \sinh\left(\frac{\pi n}{l}x\right) + B_n \sin\left(\frac{\pi n}{l}x\right)\right] \cdot \left[C_n \cos\left(a\frac{\pi n}{l}t\right) + D_n \sin\left(a\frac{\pi n}{l}t\right)\right] \end{aligned}$$

□

Referencias

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