Universidad del Valle de Guatemala
Departamento de Matemática
Licenciatura en Matemática Aplicada

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Parcial 2 - Revisión

1. Problema 1

Funciones ortogonales

1. Compruebe que $f_1(x) = x$ y $f_2(x) = x^2$ son ortogonales en [-2, 2].

   Solución.

$$\begin{aligned} \langle f_1(x), f_2(x)\rangle &= \int_{-2}^{2} x \cdot x^2 \, dx = \int_{-2}^{2} x^3 \, dx = \frac{1}{4}x^4\Big|_{-2}^{2} \\ &= \frac{1}{4}\left[(2)^4 - (-2)^4\right] = 0 \end{aligned} \tag{1}$$

   $\therefore \;\; f_1(x),\;\; f_2(x)$ son ortogonales en el intervalo [-2,2]. $\square$

2. Determine las constantes c_1 y c_2 tales que $f_3(x) = x + c_1x^2 + c_2x^3$ sea ortogonal a f_1 y f_2 en el mismo intervalo.

$$\begin{aligned} \langle f_3(x), f_1(x)\rangle &= \int_{-2}^{2} (x + c_1x^2 + c_2x^3)\cdot(x)\, dx = \int_{-2}^{2} (x^2 + c_1x^3 + c_2x^4)\, dx \\ &= \frac{1}{3}x^3 + \frac{c_1}{4}x^4 + \frac{c_2}{5}x^5\Big|_{-2}^{2} = \frac{1}{3}[(2)^3 - (-2)^3] + \frac{c_2}{5}[(2)^5 - (-2)^5] \\ &= \frac{1}{3}[2^4] + \frac{c_2}{5}[2^6] \end{aligned} \tag{1}$$

   Se sabe que $\langle f_3(x), f_1(x)\rangle = 0$, entonces:

$$\implies \frac{1}{3}[2^4] + \frac{c_2}{5}[2^6] = 0 \implies c_2 = -\frac{2^4\cdot 5}{3\cdot 2^6} = -\frac{5}{3\cdot 2^2} = -\frac{5}{12} \tag{2}$$

$$\begin{aligned} \langle f_3(x), f_1(x)\rangle &= \int_{-2}^{2} (x + c_1x^2 - \frac{5}{12}x^3)\cdot(x^2)\, dx = \int_{-2}^{2} (x^3 + c_1x^4 - \frac{5}{12}x^5)\, dx \\ &= \frac{1}{4}x^4 + c_1\frac{1}{5}x^5 - \frac{5}{60}x^6\Big|_{-2}^{2} = \frac{c_1}{5}[(2)^5 - (-2)^5] \end{aligned} \tag{3}$$

Nuevamente, se conoce que $\langle f_3(x), f_2(x)\rangle = 0$, entonces:

$$\implies \frac{c_1}{5}[(2)^5 - (-2)^5] = 0 \implies c_1 = 0 \tag{4}$$

$$\boxed{c_1 = 0 \text{ y } c_2 = -\tfrac{5}{12}}$$

2. Problema 2

Serie de Fourier

1. Encuentre la serie de Fourier de $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 \le x < \pi \end{cases}$

$$\boxed{f(x) = \tfrac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\tfrac{x\pi n}{L} + b_n \sin\tfrac{x\pi n}{L}\right]}$$

Para a_0:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,\mathrm{d}x + \int_0^{\pi}\sin x\,\mathrm{d}x\right] = \frac{1}{\pi}\left[-\cos x\Big|_0^{\pi}\right] \\ &= -\frac{1}{\pi}\left[\cos\pi - \cos 0\right] = -\frac{1}{\pi}\left[-1-1\right] = \frac{2}{\pi} \end{aligned} \tag{1}$$

El caso base de a_n, i.e. $n = 1$:

$$\begin{aligned} a_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}\sin x\cos x\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{\pi}[\sin(x+x)]\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_0^{\pi}\sin(2x)\,\mathrm{d}x = \frac{1}{4\pi}\int_0^{2\pi}\sin u\,\mathrm{d}u = -\frac{1}{4\pi}\left[\cos u\Big|_0^{2\pi}\right] \\ &= -\frac{1}{4\pi}\left[\cos 2\pi - \cos 0\right] = 0 \end{aligned} \tag{2}$$

Para a_n:

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \,\mathrm{d}x = \frac{1}{\pi}\left[\int_0^{\pi}\sin x\cos nx\right]\mathrm{d}x \\
&= \frac{1}{2\pi}\left\{\int_0^{\pi}[\sin(x-nx)+\sin(nx+x)]\,\mathrm{d}x\right\} \\
&= \frac{1}{2\pi}\left\{\int_0^{\pi}[\sin(1-n)x+\sin(n+1)x]\,\mathrm{d}x\right\} \\
&= \frac{1}{2\pi}\left\{-\frac{1}{1-n}\cos(1-n)x-\frac{1}{n+1}\cos(n+1)x\Big|_0^{\pi}\right\} \\
&= \frac{1}{2\pi}\left\{-\frac{1}{1-n}[\cos(1-n)\pi-\cos 0]-\frac{1}{n+1}[\cos(n+1)\pi-\cos 0]\right\} \\
&= -\frac{1}{2\pi}\left\{\frac{1}{1-n}[\cos(1-n)\pi-\cos 0]+\frac{1}{n+1}[\cos(n+1)\pi-\cos 0]\right\} \\
&= -\frac{1}{2\pi}\left\{\frac{1}{1-n}[(-1)^{n-1}-1]+\frac{1}{n+1}[(-1)^{n+1}-1]\right\} \\
&= -\frac{1}{2\pi}\left\{\frac{(-1)^{n-1}}{1-n}-\frac{1}{1-n}+\frac{(-1)^{n+1}}{n+1}-\frac{1}{n+1}\right\} \\
&= -\frac{1}{2\pi}\left\{\frac{(-1)(-1)^{n-1}}{n-1}+\frac{1}{n-1}+\frac{(-1)^{n+1}}{n+1}-\frac{1}{n+1}\right\} \\
&= -\frac{1}{2\pi}\left\{\frac{(-1)^n(n+1)+(-1)^{n+1}(n-1)}{(n-1)(n+1)}+\frac{(n+1)-(n-1)}{(n-1)(n+1)}\right\} \\
&= -\frac{1}{2\pi}\left\{\frac{(-1)^n(n+1)+(-1)^n(1-n)}{(n-1)(n+1)}+\frac{2}{(n-1)(n+1)}\right\} \\
&= -\frac{1}{2\pi}\left\{\frac{(-1)^n[n+1+1-n]}{(n-1)(n+1)}+\frac{2}{(n-1)(n+1)}\right\} \\
&= -\frac{1}{2\pi}\left\{\frac{2(-1)^n+2}{n^2-1}\right\} \\
&= -\frac{1}{\pi}\left\{\frac{(-1)^n+1}{n^2-1}\right\} \\
&= \frac{1+(-1)^n}{\pi(1-n^2)} = \begin{cases} 0 & n \text{ impar} \\ \frac{2}{\pi(1-4n^2)} & n \text{ par}\end{cases}
\end{aligned} \tag{3}
$$

El caso base de b_n, i.e $n=1$:

$$
\begin{aligned}
b_1 &= \frac{1}{\pi}\int_0^{\pi}\sin x\sin x\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}\sin^2 x\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{\pi}[\cos(x-x)-\cos(x+x)]\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{\pi}[\cos(x-x)-\cos(x+x)]\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{\pi}[1-\cos(2x)]\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left[x-\frac{1}{2}\sin 2x\Big|_0^{\pi}\right] = \frac{1}{2\pi}[\pi] = \frac{1}{2}
\end{aligned} \tag{4}
$$

Para b_n:

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx \, \mathrm{d}x = \frac{1}{2\pi}\int_0^{\pi} [\cos(x-nx) - \cos(x+nx)] \, \mathrm{d}x \\ &= \frac{1}{2\pi}\int_0^{\pi} [\cos(1-n)x - \cos(1+n)x] \, \mathrm{d}x = 0 \end{aligned} \tag{5}$$

Por lo tanto, *la serie de Fourier* es:

$$\begin{aligned} f(x) &= \frac{1}{\pi} + \frac{1}{2}\sin x + \sum_{n=1}^{\infty} \frac{2}{\pi(1-4n^2)} \cos 2nx \\ &= \frac{1}{\pi} + \frac{1}{2}\sin x + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{(1-4n^2)} \\ &= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{(4n^2-1)} \end{aligned} \tag{6}$$

Se puede consultar en: https://www.desmos.com/calculator/nviq2wplpt

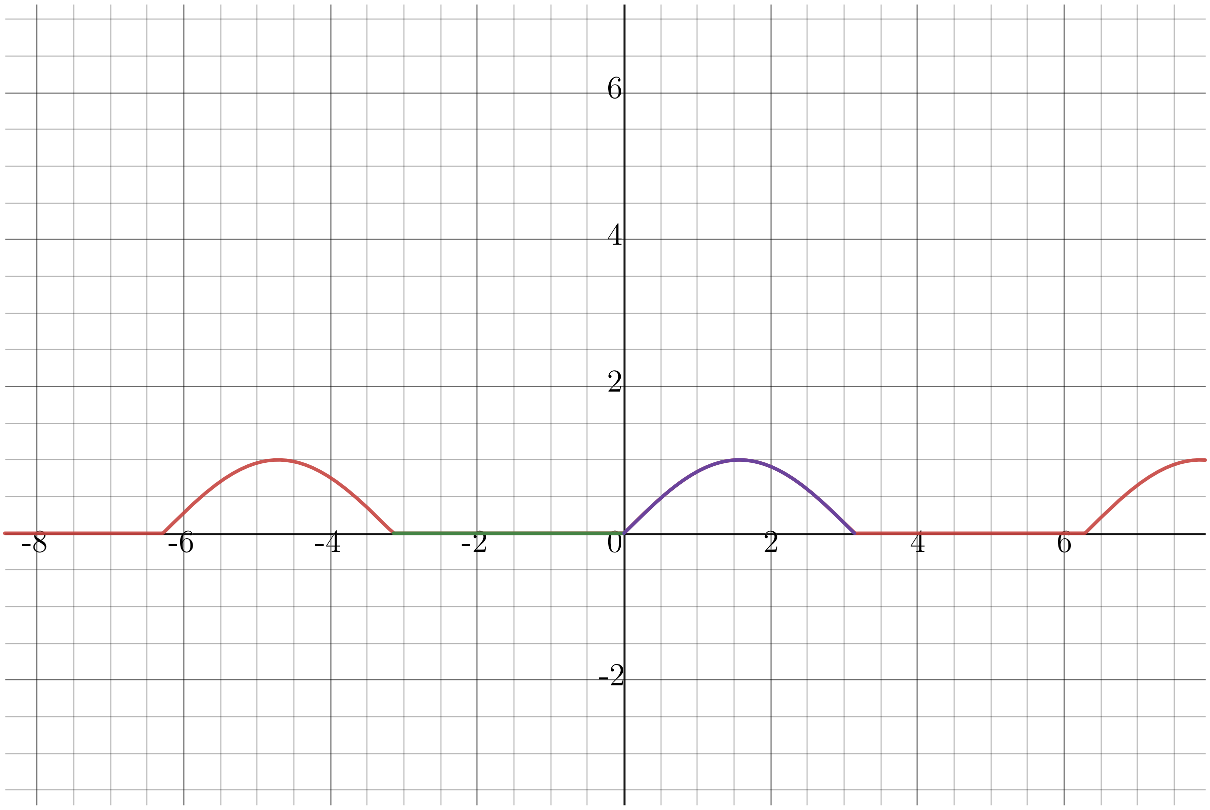

Figura 1: Serie de Fourier

2. Utilice el resultado del inciso anterior para deducir que

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots$$

Solución. Para la demostración de la serie de $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9}$ se tomará como referencia la demostración de (https://math.stackexchange.com/users/458544/fghj) en el caso de la serie positiva. Entonces, se tiene que:

$$\frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\frac{1}{7\cdot 9}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)(2n+1)}$$

Es decir que el problema pide deducir:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)(2n+1)}=\frac{\pi}{4}-\frac{1}{2}=\frac{\pi-2}{4}$$

Es decir, expresado de otra forma:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}=\frac{\pi-2}{4}$$

Entonces, se tiene:

$$f(x)=\frac{1}{\pi}+\frac{1}{2}\sin x-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(4n^2-1)} \tag{1}$$

$$\sin(x)=\frac{1}{\pi}+\frac{1}{2}\sin x-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(4n^2-1)} \tag{2}$$

Se propone utilizar $x=\frac{\pi}{2}$:

$$\sin(\frac{\pi}{2})=\frac{1}{\pi}+\frac{1}{2}\sin\frac{\pi}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2n\frac{\pi}{2}}{(4n^2-1)} \tag{3}$$

$$1=\frac{1}{\pi}+\frac{1}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} \tag{4}$$

$$1-\frac{1}{\pi}-\frac{1}{2}=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1} \tag{5}$$

$$\frac{\pi(2\pi-2-\pi)}{4\pi}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1} \tag{6}$$

$$\frac{(2\pi-2-\pi)}{4}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1} \tag{7}$$

$$\frac{\pi-2}{4}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1} \tag{8}$$

□

3. Problema 3

3. Resuelva la ecuación de Laplace

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

para una placa rectangular y sujeta a las condiciones

$$\begin{gathered}\frac{\partial u}{\partial x}(0,y)=\frac{\partial u}{\partial x}(a,y)=0\\ u(x,0)=x,\quad u(x,b)=0\end{gathered}$$

4. Problema 4

Resuelva el problema con valores en la frontera:

$$\frac{\partial^2 u}{\partial t^2} + a^2 \frac{\partial^4 u}{\partial x^4} = 0$$

Sujeta a las condiciones:

$$\begin{gathered} u(0,t) = u(l,t) = 0, t > 0 \\ \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(l,t) = 0, t > 0 \\ u(x,0) = f(x), \quad 0 \leq x \leq l \\ \frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 \leq x \leq l \end{gathered}$$

Referencias

(https://math.stackexchange.com/users/458544/fghj), F. Find the sum of the series of $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} + ...$ Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/2556675 (version: 2017-12-08).