

Universidad del Valle de Guatemala

Departamento de Matemática

Licenciatura en Matemática Aplicada

Estudiante: Rudik Roberto Rompich

E-mail: rom19857@uvg.edu.gt

Carné: 19857

Ecuaciones Diferenciales 2 - Catedrático: Dorval Carías

5 de junio de 2021

Examen parcial final

1. Integrales en los complejos

1.1. Intervalo cerrado

f función compleja, $f(x) = u(x) + iv(x)$, $a \leq x \leq b$.

$$\int_a^b f(x)dx = \int_a^b u(x)dx + i \int_a^b v(x)dx.$$

1.2. Sobre una curva

f es función compleja, $\gamma(t)$, $a \leq t \leq b$,

$$\int_{\gamma} f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

$$\int_{\gamma} f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

$$z(t) = \gamma(t)$$

1.3. Reversing

Dado γ en $[a, b]$, $\phi(t) = \gamma(a + b - t)$, $a \leq t \leq b \implies \gamma(a) = \phi(b)$ y $\gamma(t) = \phi(a)$.

$$\implies \int_{-\gamma} = - \int_{\gamma} f(z)dz.$$

1.4. Teorema Fundamental del Cálculo en complejos

$f(z)$ es continua y γ es una curva suave en $[a, b]$.

$$\int_{\gamma} f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

1.5. Parametrizar

1.5.1. Segmento

$$z(t) = \alpha + t(\beta - \alpha), \quad 0 \leq t \leq 1.$$

1.5.2. Círculo

$$\gamma(t) = re^{it}, 0 \leq t \leq 2\pi.$$

1.6. Teorema de Cauchy

Si γ es una curva cerrada y simple, y si $f(z)$ es analítica sobre y en el interior de γ , entonces $\int_{\gamma} f = 0$.

1.7. Fórmula integral de Cauchy

$$\int_{\gamma} \frac{f(z)}{z-a} dz = 2\pi i \cdot f(a).$$
$$\int_{\gamma} \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} \cdot f^{(n)}(a).$$

2. Ecuaciones Diferenciales Ordinarias Comunes

12.5 Sturm–Liouville Problem

INTRODUCTION For convenience we present here a brief review of some of the ordinary differential equations that will be of importance in the sections and chapters that follow.

Linear equations

$$y' + \alpha y = 0,$$

$$y'' + \alpha^2 y = 0, \quad \alpha > 0$$

$$y'' - \alpha^2 y = 0, \quad \alpha > 0$$

General solutions

$$y = c_1 e^{-\alpha x}$$

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$\begin{cases} y = c_1 e^{-\alpha x} + c_2 e^{\alpha x}, & \text{or} \\ y = c_1 \cosh \alpha x + c_2 \sinh \alpha x \end{cases}$$

Cauchy–Euler equation

$$x^2 y'' + xy' - \alpha^2 y = 0, \quad \alpha \geq 0$$

General solutions, $x > 0$

$$\begin{cases} y = c_1 x^{-\alpha} + c_2 x^{\alpha}, & \alpha \neq 0 \\ y = c_1 + c_2 \ln x, & \alpha = 0 \end{cases}$$

Parametric Bessel equation ($\nu = 0$)

$$xy'' + y' + \alpha^2 xy = 0$$

General solution, $x > 0$

$$y = c_1 J_0(\alpha x) + c_2 Y_0(\alpha x)$$

Legendre's equation

($n = 0, 1, 2, \dots$)

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

Particular solutions

are polynomials

$$y = P_0(x) = 1,$$

$$y = P_1(x) = x,$$

$$y = P_2(x) = \frac{1}{2}(3x^2 - 1), \dots$$

TABLA 4.1 Soluciones particulares de prueba

$g(x)$	Forma de y_p
1. 1 (cualquier constante)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. e^{3x}	Ae^{3x}
8. $(9x - 2)e^{3x}$	$(Ax + B)e^{3x}$
9. $x^2 e^{3x}$	$(Ax^2 + Bx + C)e^{3x}$
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

3. Series de Fourier

Definition 12.2.1 Fourier Series

The **Fourier series** of a function f defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right), \quad (8)$$

where

$$a_0 = \frac{1}{p} \int_{-p}^p f(x) dx \quad (9)$$

$$a_n = \frac{1}{p} \int_{-p}^p f(x) \cos \frac{n\pi}{p} x dx \quad (10)$$

$$b_n = \frac{1}{p} \int_{-p}^p f(x) \sin \frac{n\pi}{p} x dx. \quad (11)$$

Definition 12.3.1 Fourier Cosine and Sine Series

(i) The Fourier series of an even function on the interval $(-p, p)$ is the **cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x, \quad (1)$$

where

$$a_0 = \frac{2}{p} \int_0^p f(x) dx \quad (2)$$

$$a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x dx. \quad (3)$$

(ii) The Fourier series of an odd function on the interval $(-p, p)$ is the **sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x, \quad (4)$$

where

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x dx. \quad (5)$$

4. Integral de Fourier

Definition 15.3.1 Fourier Integral

The **Fourier integral** of a function f defined on the interval $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi} \int_0^{\infty} [A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x] d\alpha, \quad (4)$$

where

$$A(\alpha) = \int_{-\infty}^{\infty} f(x) \cos \alpha x dx \quad (5)$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x) \sin \alpha x dx. \quad (6)$$

Theorem 15.3.1 Conditions for Convergence

Let f and f' be piecewise continuous on every finite interval, and let f be absolutely integrable on $(-\infty, \infty)$.* Then the Fourier integral of f on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity, the Fourier integral will converge to the average

$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.

Definition 15.3.2 Fourier Cosine and Sine Integrals

(i) The Fourier integral of an even function on the interval $(-\infty, \infty)$ is the **cosine integral**

$$f(x) = \frac{2}{\pi} \int_0^{\infty} A(\alpha) \cos \alpha x d\alpha, \quad (8)$$

where

$$A(\alpha) = \int_0^{\infty} f(x) \cos \alpha x dx. \quad (9)$$

(ii) The Fourier integral of an odd function on the interval $(-\infty, \infty)$ is the **sine integral**

$$f(x) = \frac{2}{\pi} \int_0^{\infty} B(\alpha) \sin \alpha x d\alpha, \quad (10)$$

where

$$B(\alpha) = \int_0^{\infty} f(x) \sin \alpha x dx. \quad (11)$$

5. Transformada de Fourier

Definition 15.4.1 Fourier Transform Pairs

(i) Fourier transform: $\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx = F(\alpha)$ (5)

Inverse Fourier transform: $\mathcal{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha = f(x)$ (6)

(ii) Fourier sine transform: $\mathcal{F}_s\{f(x)\} = \int_0^{\infty} f(x) \sin \alpha x dx = F(\alpha)$ (7)

Inverse Fourier sine transform: $\mathcal{F}_s^{-1}\{F(\alpha)\} = \frac{2}{\pi} \int_0^{\infty} F(\alpha) \sin \alpha x d\alpha = f(x)$ (8)

(iii) Fourier cosine transform: $\mathcal{F}_c\{f(x)\} = \int_0^{\infty} f(x) \cos \alpha x dx = F(\alpha)$ (9)

Inverse Fourier cosine transform: $\mathcal{F}_c^{-1}\{F(\alpha)\} = \frac{2}{\pi} \int_0^{\infty} F(\alpha) \cos \alpha x d\alpha = f(x)$ (10)

Trigonometric identities

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \sin \beta \cos \alpha & \sin^2 t + \cos^2 t &= 1 \\ \cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \beta \sin \alpha & \sin^2 t &= \frac{1}{2} (1 - \cos(2t)) \\ \sin \alpha \cos \beta &= \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)] & \cos^2 t &= \frac{1}{2} (1 + \cos(2t)) \\ \cos \alpha \cos \beta &= \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)] & \sin(2t) &= 2 \sin t \cos t \\ \sin \alpha \sin \beta &= \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)], & \cos(2t) &= 2 \cos^2 t - 1 = 1 - 2 \sin^2 t \\ \int x \cos \omega x dx &= \frac{x \sin \omega x}{\omega} + \frac{\cos \omega x}{\omega^2}, & \int x \sin \omega x dx &= -\frac{x \cos \omega x}{\omega} + \frac{\sin \omega x}{\omega^2} \\ \int x^2 \sin \omega x dx &= -\frac{x^2 \cos \omega x}{\omega} + \frac{2x \sin \omega x}{\omega^2} + \frac{2 \cos \omega x}{\omega^3} \end{aligned}$$

Table of Fourier Transform Pairs

Function, $f(t)$	Fourier Transform, $F(\omega)$
<i>Definition of Inverse Fourier Transform</i>	<i>Definition of Fourier Transform</i>
$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$	$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$
$f(t - t_0)$	$F(\omega) e^{-j\omega t_0}$
$f(t) e^{j\omega_0 t}$	$F(\omega - \omega_0)$
$f(\alpha t)$	$\frac{1}{ \alpha } F\left(\frac{\omega}{\alpha}\right)$
$F(t)$	$2\pi f(-\omega)$
$\frac{d^n f(t)}{dt^n}$	$(j\omega)^n F(\omega)$
$(-jt)^n f(t)$	$\frac{d^n F(\omega)}{d\omega^n}$
$\int_{-\infty}^t f(\tau) d\tau$	$\frac{F(\omega)}{j\omega} + \pi F(0) \delta(\omega)$
$\delta(t)$	1
$e^{j\omega_0 t}$	$2\pi \delta(\omega - \omega_0)$
$\text{sgn}(t)$	$\frac{2}{j\omega}$

$j \frac{1}{\pi t}$	$\text{sgn}(\omega)$
$u(t)$	$\pi \delta(\omega) + \frac{1}{j\omega}$
$\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$	$2\pi \sum_{n=-\infty}^{\infty} F_n \delta(\omega - n\omega_0)$
$\text{rect}(\frac{t}{\tau})$	$\tau \text{Sa}(\frac{\omega \tau}{2})$
$\frac{B}{2\pi} \text{Sa}(\frac{Bt}{2})$	$\text{rect}(\frac{\omega}{B})$
$\text{tri}(t)$	$\text{Sa}^2(\frac{\omega}{2})$
$A \cos(\frac{\pi t}{2\tau}) \text{rect}(\frac{t}{2\tau})$	$\frac{A\pi}{\tau} \frac{\cos(\omega \tau)}{(\pi/2\tau)^2 - \omega^2}$
$\cos(\omega_0 t)$	$\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$\sin(\omega_0 t)$	$\frac{\pi}{j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$u(t) \cos(\omega_0 t)$	$\frac{\pi}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$
$u(t) \sin(\omega_0 t)$	$\frac{\pi}{2j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega^2}{\omega_0^2 - \omega^2}$
$u(t) e^{-\alpha t} \cos(\omega_0 t)$	$\frac{(\alpha + j\omega)}{\omega_0^2 + (\alpha + j\omega)^2}$

$u(t)e^{-\alpha t} \sin(\omega_0 t)$	$\frac{\omega_0}{\omega_0^2 + (\alpha + j\omega)^2}$
$e^{-\alpha t }$	$\frac{2\alpha}{\alpha^2 + \omega^2}$
$e^{-t^2/(2\sigma^2)}$	$\sigma\sqrt{2\pi} e^{-\sigma^2\omega^2/2}$
$u(t)e^{-\alpha t}$	$\frac{1}{\alpha + j\omega}$
$u(t)te^{-\alpha t}$	$\frac{1}{(\alpha + j\omega)^2}$

➤ **Trigonometric Fourier Series**

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(\omega_0 nt) + b_n \sin(\omega_0 nt))$$

where

$$a_0 = \frac{1}{T} \int_0^T f(t) dt \quad , \quad a_n = \frac{2}{T} \int_0^T f(t) \cos(\omega_0 nt) dt \quad , \text{ and}$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin(\omega_0 nt) dt$$

➤ **Complex Exponential Fourier Series**

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{j\omega_0 nt} \quad , \text{ where } \quad F_n = \frac{1}{T} \int_0^T f(t) e^{-j\omega_0 nt} dt$$

6. Laplace

1. $\mathcal{L}[u'(x, t)] = s\mathcal{L}[u(x, t)] - u(x, 0).$
2. $\mathcal{L}[u'(x, t)] = s^2\mathcal{L}[u(x, t)] - su(x, 0) - u'(x, 0).$
3. $\mathcal{L}\left[\frac{e^{-a\sqrt{s}}}{s}\right] = \operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right).$
4. $\mathcal{L}\left[e^{-a\sqrt{s}}\right] = \frac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}, \quad a > 0.$
5. $\mathcal{L}\left[\frac{e^{-a\sqrt{s}}}{\sqrt{s}}\right] = \frac{1}{\sqrt{\pi t}}e^{-a^2/4t}$

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}$	2. e^{at}	$\frac{1}{s-a}$
3. $t^n, \quad n=1,2,3,\dots$	$\frac{n!}{s^{n+1}}$	4. $t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5. \sqrt{t}	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6. $t^{n-\frac{1}{2}}, \quad n=1,2,3,\dots$	$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7. $\sin(at)$	$\frac{a}{s^2+a^2}$	8. $\cos(at)$	$\frac{s}{s^2+a^2}$
9. $t \sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10. $t \cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11. $\sin(at) - at \cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12. $\sin(at) + at \cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13. $\cos(at) - at \sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14. $\cos(at) + at \sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15. $\sin(at+b)$	$\frac{s \sin(b) + a \cos(b)}{s^2+a^2}$	16. $\cos(at+b)$	$\frac{s \cos(b) - a \sin(b)}{s^2+a^2}$
17. $\sinh(at)$	$\frac{a}{s^2-a^2}$	18. $\cosh(at)$	$\frac{s}{s^2-a^2}$
19. $e^{at} \sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20. $e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21. $e^{at} \sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22. $e^{at} \cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23. $t^n e^{at}, \quad n=1,2,3,\dots$	$\frac{n!}{(s-a)^{n+1}}$	24. $f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$
25. $u_c(t) = u(t-c)$ Heaviside Function	$\frac{e^{-cs}}{s}$	26. $\delta(t-c)$ Dirac Delta Function	e^{-cs}
27. $u_c(t) f(t-c)$	$e^{-cs} F(s)$	28. $u_c(t) g(t)$	$e^{-cs} \mathcal{L}\{g(t+c)\}$
29. $e^{ct} f(t)$	$F(s-c)$	30. $t^n f(t), \quad n=1,2,3,\dots$	$(-1)^n F^{(n)}(s)$
31. $\frac{1}{t} f(t)$	$\int_s^\infty F(u) du$	32. $\int_0^t f(v) dv$	$\frac{F(s)}{s}$
33. $\int_0^t f(t-\tau) g(\tau) d\tau$	$F(s)G(s)$	34. $f(t+T) = f(t)$	$\frac{\int_0^T e^{-st} f(t) dt}{1-e^{-sT}}$
35. $f'(t)$	$sF(s) - f(0)$	36. $f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
37. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$		

Table Notes

1. This list is not a complete listing of Laplace transforms and only contains some of the more commonly used Laplace transforms and formulas.
2. Recall the definition of hyperbolic functions.

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \qquad \sinh(t) = \frac{e^t - e^{-t}}{2}$$

3. Be careful when using “normal” trig function vs. hyperbolic functions. The only difference in the formulas is the “+ a²” for the “normal” trig functions becomes a “- a²” for the hyperbolic functions!
4. Formula #4 uses the Gamma function which is defined as

$$\Gamma(t) = \int_0^{\infty} e^{-x} x^{t-1} dx$$

If n is a positive integer then,

$$\Gamma(n+1) = n!$$

The Gamma function is an extension of the normal factorial function. Here are a couple of quick facts for the Gamma function

$$\Gamma(p+1) = p\Gamma(p)$$

$$p(p+1)(p+2)\cdots(p+n-1) = \frac{\Gamma(p+n)}{\Gamma(p)}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$