Universidad del Valle de Guatemala
Departamento de Matemática
Licenciatura en Matemática Aplicada

Estudiante: Rudik Roberto Rompich
E-mail: rom19857@uvg.edu.gt
Carné: 19857

Ecuaciones Diferenciales 2 - Catedrático: Dorval Carías
4 de junio de 2021

Examen parcial final

1. Integrales en los complejos

1.1. Intervalo cerrado

f función compleja, $f(x) = u(x) + iv(x), a \leq x \leq b$.

$$\int_a^b f(x)dx = \int_a^b u(x)dx + i\int_a^b v(x)dx.$$

1.2. Sobre una curva

f es función compleja, $\gamma(t), a \leq t \leq b$,

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

$$z(t) = \gamma(t)$$

1.3. Reversing

Dado γ en $[a, b]$, $\phi(t) = \gamma(a + b - t), a \leq t \leq b \implies \gamma(a) = \phi(b)$ y $\gamma(t) = \phi(a)$.

$$\implies \int_{-\gamma} = -\int_\gamma f(z)dz.$$

1.4. Teorema Fundamental del Cálculo en complejos

$f(z)$ es continua y γ es una curva suave en $[a, b]$.

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

1.5. Parametrizar

1.5.1. Segmento

$$z(t) = \alpha + t(\beta - \alpha), \qquad 0 \leq t \leq 1.$$

1.5.2. Círculo

$$\gamma(t) = re^{it}, 0 \leq t \leq 2\pi.$$

1.6. Teorema de Cauchy

Si γ es una curva cerrada y simple, y si $f(z)$ es analítica sobre y en el interior de γ, entonces $\int_\gamma f = 0$.

1.7. Fórmula integral de Cauchy

$$\int_\gamma \frac{f(z)}{z-a} dz = 2\pi i \cdot f(a).$$

$$\int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} \cdot f^{(n)}(a).$$

2. Ecuaciones Diferenciales Ordinarias Comunes

12.5 Sturm–Liouville Problem

INTRODUCTION For convenience we present here a brief review of some of the ordinary differential equations that will be of importance in the sections and chapters that follow.

Linear equations	General solutions
$y' + \alpha y = 0,$	$y = c_1 e^{-\alpha x}$
$y'' + \alpha^2 y = 0, \quad \alpha > 0$	$y = c_1 \cos \alpha x + c_2 \sin \alpha x$
$y'' - \alpha^2 y = 0, \quad \alpha > 0$	$\begin{cases} y = c_1 e^{-\alpha x} + c_2 e^{\alpha x}, & \text{or} \\ y = c_1 \cosh \alpha x + c_2 \sinh \alpha x \end{cases}$

Cauchy–Euler equation	General solutions, $x > 0$
$x^2 y'' + xy' - \alpha^2 y = 0, \quad \alpha \geq 0$	$\begin{cases} y = c_1 x^{-\alpha} + c_2 x^{\alpha}, & \alpha \neq 0 \\ y = c_1 + c_2 \ln x, & \alpha = 0 \end{cases}$

Parametric Bessel equation ($\nu = 0$)	General solution, $x > 0$
$xy'' + y' + \alpha^2 xy = 0$	$y = c_1 J_0(\alpha x) + c_2 Y_0(\alpha x)$

Legendre's equation ($n = 0, 1, 2, \ldots$)	Particular solutions are polynomials
$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$	$y = P_0(x) = 1,$
	$y = P_1(x) = x,$
	$y = P_2(x) = \frac{1}{2}(3x^2 - 1), \ldots$

TABLA 4.1 Soluciones particulares de prueba

$g(x)$	Forma de y_p
1. 1 (cualquier constante)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. sen $4x$	$A \cos 4x + B$ sen $4x$
6. $\cos 4x$	$A \cos 4x + B$ sen $4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. $x^2 e^{5x}$	$(Ax^2 + Bx + C)e^{5x}$
10. e^{3x} sen $4x$	$Ae^{3x} \cos 4x + Be^{3x}$ sen $4x$
11. $5x^2$ sen $4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G)$ sen $4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x}$ sen $4x$

3. Series de Fourier

Definition 12.2.1 Fourier Series

The **Fourier series** of a function f defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right), \tag{8}$$

where

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx \tag{9}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx \tag{10}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx. \tag{11}$$

Definition 12.3.1 Fourier Cosine and Sine Series

(*i*) The Fourier series of an even function on the interval $(-p, p)$ is the **cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x, \tag{1}$$

where

$$a_0 = \frac{2}{p}\int_{0}^{p} f(x)\,dx \tag{2}$$

$$a_n = \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{n\pi}{p}x\,dx. \tag{3}$$

(*ii*) The Fourier series of an odd function on the interval $(-p, p)$ is the **sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x, \tag{4}$$

where

$$b_n = \frac{2}{p}\int_{0}^{p} f(x)\sin\frac{n\pi}{p}x\,dx. \tag{5}$$

4. Integral de Fourier

Definition 15.3.1 Fourier Integral

The **Fourier integral** of a function f defined on the interval $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^{\infty} [A(\alpha)\cos \alpha x + B(\alpha)\sin \alpha x]\, d\alpha, \tag{4}$$

where

$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos \alpha x\, dx \tag{5}$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin \alpha x\, dx. \tag{6}$$

Theorem 15.3.1 Conditions for Convergence

Let f and f' be piecewise continuous on every finite interval, and let f be absolutely integrable on $(-\infty, \infty)$.* Then the Fourier integral of f on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity, the Fourier integral will converge to the average

$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.

Definition 15.3.2 Fourier Cosine and Sine Integrals

(*i*) The Fourier integral of an even function on the interval $(-\infty, \infty)$ is the **cosine integral**

$$f(x) = \frac{2}{\pi}\int_0^{\infty} A(\alpha)\cos \alpha x\, d\alpha, \tag{8}$$

where

$$A(\alpha) = \int_0^{\infty} f(x)\cos \alpha x\, dx. \tag{9}$$

(*ii*) The Fourier integral of an odd function on the interval $(-\infty, \infty)$ is the **sine integral**

$$f(x) = \frac{2}{\pi}\int_0^{\infty} B(\alpha)\sin \alpha x\, d\alpha, \tag{10}$$

where

$$B(\alpha) = \int_0^{\infty} f(x)\sin \alpha x\, dx. \tag{11}$$

5. Transformada de Fourier

Definition 15.4.1 Fourier Transform Pairs

(*i*) Fourier transform:
$$\mathscr{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx = F(\alpha) \tag{5}$$

Inverse Fourier transform:
$$\mathscr{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x}\,d\alpha = f(x) \tag{6}$$

(*ii*) Fourier sine transform:
$$\mathscr{F}_s\{f(x)\} = \int_{0}^{\infty} f(x)\sin\alpha x\,dx = F(\alpha) \tag{7}$$

Inverse Fourier sine transform:
$$\mathscr{F}_s^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_{0}^{\infty} F(\alpha)\sin\alpha x\,d\alpha = f(x) \tag{8}$$

(*iii*) Fourier cosine transform:
$$\mathscr{F}_c\{f(x)\} = \int_{0}^{\infty} f(x)\cos\alpha x\,dx = F(\alpha) \tag{9}$$

Inverse Fourier cosine transform:
$$\mathscr{F}_c^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_{0}^{\infty} F(\alpha)\cos\alpha x\,d\alpha = f(x) \tag{10}$$

Trigonometric identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha \qquad \sin^2 t + \cos^2 t = 1$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\beta\sin\alpha. \qquad \sin^2 t = \tfrac{1}{2}(1-\cos(2t))$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)] \qquad \cos^2 t = \tfrac{1}{2}(1+\cos(2t))$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \qquad \sin(2t) = 2\sin t\cos t$$
$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)], \quad \cos(2t) = 2\cos^2 t - 1 = 1 - 2\sin^2 t$$
$$\int x\cos\omega x\,dx = \frac{x\sin\omega x}{\omega} + \frac{\cos\omega x}{\omega^2}, \qquad \int x\sin\omega x\,dx = -\frac{x\cos\omega x}{\omega} + \frac{\sin\omega x}{\omega^2}$$
$$\int x^2\sin\omega x\,dx = -\frac{x^2\cos\omega x}{\omega} + \frac{2x\sin\omega x}{\omega^2} + \frac{2\cos\omega x}{\omega^3}$$

Table of Fourier Transform Pairs

Function, f(t)	**Fourier Transform, F(ω)**
Definition of Inverse Fourier Transform $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$	***Definition of Fourier Transform*** $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$
$f(t-t_0)$	$F(\omega)e^{-j\omega t_0}$
$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
$f(\alpha t)$	$\frac{1}{\lvert\alpha\rvert}F(\frac{\omega}{\alpha})$
$F(t)$	$2\pi f(-\omega)$
$\frac{d^n f(t)}{dt^n}$	$(j\omega)^n F(\omega)$
$(-jt)^n f(t)$	$\frac{d^n F(\omega)}{d\omega^n}$
$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$
$\delta(t)$	1
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
sgn *(t)*	$\frac{2}{j\omega}$

$j\dfrac{1}{\pi t}$	$\operatorname{sgn}(\omega)$
$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$
$\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$	$2\pi\sum_{n=-\infty}^{\infty} F_n \delta(\omega-n\omega_0)$
$rect(\dfrac{t}{\tau})$	$\tau Sa(\dfrac{\omega\tau}{2})$
$\dfrac{B}{2\pi}Sa(\dfrac{Bt}{2})$	$rect(\dfrac{\omega}{B})$
$tri(t)$	$Sa^2(\dfrac{\omega}{2})$
$A\cos(\dfrac{\pi t}{2\tau})rect(\dfrac{t}{2\tau})$	$\dfrac{A\pi}{\tau}\dfrac{\cos(\omega\tau)}{(\pi/2\tau)^2-\omega^2}$
$\cos(\omega_0 t)$	$\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$
$\sin(\omega_0 t)$	$\dfrac{\pi}{j}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$
$u(t)\cos(\omega_0 t)$	$\dfrac{\pi}{2}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]+\dfrac{j\omega}{\omega_0^2-\omega^2}$
$u(t)\sin(\omega_0 t)$	$\dfrac{\pi}{2j}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]+\dfrac{\omega^2}{\omega_0^2-\omega^2}$
$u(t)e^{-\alpha t}\cos(\omega_0 t)$	$\dfrac{(\alpha+j\omega)}{\omega_0^2+(\alpha+j\omega)^2}$

$u(t)e^{-\alpha t}\sin(\omega_0 t)$	$\dfrac{\omega_0}{\omega_0^2+(\alpha+j\omega)^2}$
$e^{-\alpha\|t\|}$	$\dfrac{2\alpha}{\alpha^2+\omega^2}$
$e^{-t^2/(2\sigma^2)}$	$\sigma\sqrt{2\pi}\, e^{-\sigma^2\omega^2/2}$
$u(t)e^{-\alpha t}$	$\dfrac{1}{\alpha+j\omega}$
$u(t)te^{-\alpha t}$	$\dfrac{1}{(\alpha+j\omega)^2}$

- ***Trigonometric Fourier Series***

$$f(t)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos(\omega_0 nt)+b_n\sin(\omega_0 nt)\right)$$

where

$$a_0=\frac{1}{T}\int_0^T f(t)dt \quad , \quad a_n=\frac{2}{T}\int_0^T f(t)\cos(\omega_0 nt)dt \quad ,\text{and}$$

$$b_n=\frac{2}{T}\int_0^T f(t)\sin(\omega_0 nt)dt$$

- ***Complex Exponential Fourier Series***

$$f(t)=\sum_{n=-\infty}^{\infty}F_n e^{j\omega nt} \quad ,\text{where} \qquad F_n=\frac{1}{T}\int_0^T f(t)e^{-j\omega_0 nt}dt$$

6. Laplace

1. $\mathcal{L}[u'(x,t)] = s\mathcal{L}[u(x,t)] - u(x,0)$.
2. $\mathcal{L}[u'(x,t)] = s^2\mathcal{L}[u(x,t)] - su(x,0) - u'(x,0)$.
3. $\mathcal{L}\left[\frac{e^{-a\sqrt{s}}}{s}\right] = \operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$.
4. $\mathcal{L}\left[e^{-a\sqrt{s}}\right] = \frac{a}{2\sqrt{\pi t^3}}e^{-\alpha^2/4t}, \qquad a > 0.$
5. $\mathcal{L}\left[\frac{e^{-a\sqrt{s}}}{\sqrt{s}}\right] = \frac{1}{\sqrt{\pi t}}e^{-\alpha^2/4t}$

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$	2.	$\mathbf{e}^{at}$	$\dfrac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\dfrac{n!}{s^{n+1}}$	4.	$t^p,\ p>-1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\dfrac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\dfrac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\dfrac{a}{s^2+a^2}$	8.	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\dfrac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\dfrac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\dfrac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\dfrac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\dfrac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\dfrac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\dfrac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\dfrac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\dfrac{1}{t}f(t)$	$\int_s^{\infty}F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\dfrac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\dfrac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

Table Notes

1. This list is not a complete listing of Laplace transforms and only contains some of the more commonly used Laplace transforms and formulas.

2. Recall the definition of hyperbolic functions.

$$\cosh(t) = \frac{\mathbf{e}^{t} + \mathbf{e}^{-t}}{2} \qquad\qquad \sinh(t) = \frac{\mathbf{e}^{t} - \mathbf{e}^{-t}}{2}$$

3. Be careful when using "normal" trig function vs. hyperbolic functions. The only difference in the formulas is the "+ a^2" for the "normal" trig functions becomes a "- a^2" for the hyperbolic functions!

4. Formula #4 uses the Gamma function which is defined as

$$\Gamma(t) = \int_0^\infty \mathbf{e}^{-x} x^{t-1}\, dx$$

   If n is a positive integer then,

$$\Gamma(n+1) = n!$$

   The Gamma function is an extension of the normal factorial function. Here are a couple of quick facts for the Gamma function

$$\Gamma(p+1) = p\Gamma(p)$$

$$p(p+1)(p+2)\cdots(p+n-1) = \frac{\Gamma(p+n)}{\Gamma(p)}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$