PDE - LAPLACE

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Solución de EDP con Transformada de Laplace

Dada la función $u(x,t)$, definida para $t>0$ y que suponemos acotada, aplicamos la transformada de Laplace en t, considerando a x como un parámetro.

$$\Rightarrow \mathcal{L}[u(x,t)] = \int_0^\infty e^{-st} u(x,t)\, dt = \check{u}(x,s)$$

Nota:

① $\mathcal{L}\left[\frac{\partial}{\partial t} u(x,t)\right] = \int_0^\infty e^{-st}\,\frac{\partial}{\partial t} u(x,t)\, dt =$

Por Partes:

$w = e^{-st}$ $\qquad v = u(x,t)$

$dw = -s\,e^{-st}\,dt$ $\qquad dv = \frac{\partial}{\partial t} u(x,t)\, dt$

$$= u(x,t)\, e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st} u(x,t)\, dt$$

$$= -u(x,0) + s\,\mathcal{L}[u(x,t)]$$

$$\Rightarrow \mathcal{L}\left[\frac{\partial}{\partial t} u(x,t)\right] = s\,\mathcal{L}[u(x,t)] - u(x,0)$$

② $$\mathcal{L}\left[\frac{\partial^2}{\partial t^2}u(x,t)\right] = \mathcal{L}\left[\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}u(x,t)\right)\right]$$

$$= s\,\mathcal{L}\left[\frac{\partial}{\partial t}u(x,t)\right] - \frac{\partial}{\partial t}u(x,0)$$

$$= s\left[s\,\mathcal{L}[u(x,t)] - u(x,0)\right] - \frac{\partial}{\partial t}u(x,0)$$

$$= s^2\,\mathcal{L}[u(x,t)] - s\,\underline{u(x,0)} - \frac{\partial}{\partial t}\underline{u(x,0)}$$

③ $$\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u(x,t)\right] = \int_0^\infty e^{-st}\,\underbrace{\frac{\partial}{\partial x^2}}\,u(x,t)\,d\underline{t}$$

$$= \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-st}u(x,t)\,dt = \frac{\partial^2}{\partial x^2}\tilde{u}(x,s)$$

Ejs: Resuelva los problemas con valores en la frontera:

1) $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = x, \quad x>0,\ t>0,\ u(0,t)=0,\ \boxed{u(x,0)=0}$$

$$\Rightarrow \mathcal{L}\left[\frac{\partial u}{\partial x}\right] + \underbrace{\mathcal{L}\left[\frac{\partial u}{\partial t}\right]} = \mathcal{L}[x] \rightarrow x\,\mathcal{L}[1]$$

$$\Rightarrow \frac{\partial}{\partial x}\check{u}(x,s) + \underbrace{s\,\check{u}(x,s) - \cancelto{0}{u(x,0)}} = \frac{x}{s}$$

$$\Rightarrow \frac{d}{dx}\check{u} + s\,\check{u} = \frac{x}{s} \leftarrow$$ Ecuación lineal de 1er orden, con $P(x) = s$

$$\Rightarrow \mu(x) = e^{\int P(x)\,dx} = e^{\int s\,dx} = e^{sx}$$

$$\Rightarrow \frac{d}{dx}\left[e^{sx} \cdot \check{u}\right] = \frac{x}{s} e^{sx}$$

$$\Rightarrow \int d\left[e^{sx} \cdot \check{u}\right] = \int \frac{x}{s} e^{sx}\, dx$$

$$\Rightarrow e^{sx}\check{u} = \frac{1}{s}\int \underbrace{x e^{sx}\, dx}_{\text{por partes}} + c$$

$$\Rightarrow e^{sx}\check{u} = \frac{1}{s}\left[\frac{x}{s} e^{sx} - \frac{1}{s^2} e^{sx}\right] + c$$

$$\Rightarrow \check{u}(x,s) = \frac{x}{s^2} - \frac{1}{s^3} + c e^{-sx} \qquad (*)$$

Como $u(0,t) = 0 \Rightarrow \mathcal{L}[u(0,t)] = \mathcal{L}[0]$

$$\Rightarrow \check{u}(0,s) = 0$$

$$\Rightarrow \tilde{u}(0,s) = -\frac{1}{s^3} + C = 0 \Rightarrow C = \frac{1}{s^3}$$

$$\Rightarrow \tilde{u}(x,s) = \frac{x}{s^2} - \frac{1}{s^3} + \frac{1}{s^3} e^{-sx} \quad (*)$$

$$\Rightarrow u(x,t) = \mathcal{L}^{-1}\left[\frac{x}{s^2}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^3}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^3} e^{-sx}\right]$$

$$\Rightarrow u(x,t) = xt - \frac{1}{2}t^2 + \frac{1}{2}(t-x)^2 u(t-x)$$

2) $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$, $\quad 0 < x < 2,\ t > 0$, sujeta a

$u(0,t) = 0,\quad u(2,t) = 0,$

$u(x,0) = 3\,\text{sen}(2\pi x)$

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$$\Rightarrow \mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right] = \mathcal{L}\left[\frac{\partial u}{\partial t}\right]$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\check{u}(x,s) = s\,\check{u}(x,s) - u(x,0)$$

?

$$\Rightarrow \frac{d^2}{dx^2}\check{u} - s\,\check{u} = -3\,\text{sen}\,2\pi x$$

$$\Rightarrow \check{u}(x,s) = \check{u}_h(x,s) + \check{u}_p(x,s)$$

$\underline{\check{u}_h(x,s)}$: $\frac{d^2}{dx^2}\check{u} - s\,\check{u} = 0$

$$\Rightarrow m^2 - s = 0 \Rightarrow m = \pm\sqrt{s}$$

$$\Rightarrow \underline{\check{u}_h(x,s) = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x}}$$

$\tilde{u}_p(x,s)$: Proponemos: $\tilde{u}_p(x,s) = A\cos 2\pi x + B \operatorname{sen} 2\pi x$

utilizando coeficientes indeterminados, se obtiene $A=0$ y $B = \dfrac{3}{4\pi^2+s}$

$$\Rightarrow \tilde{u}_p(x,s) = \frac{3}{4\pi^2+s} \operatorname{sen} 2\pi x$$

$$\Rightarrow \tilde{u}(x,s) = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} + \frac{3}{4\pi^2+s} \operatorname{sen} 2\pi x \quad (*)$$

condiciones: $u(0,t) = 0 \Rightarrow \tilde{u}(0,s) = 0$

$u(2,t) = 0 \Rightarrow \tilde{u}(2,s) = 0$

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$$\Rightarrow \tilde{u}(0,s) = c_1 + c_2 = 0$$

$$\tilde{u}(2,s) = c_1 e^{2\sqrt{s}} + c_2 e^{-2\sqrt{s}} = 0$$

$$\Rightarrow c_1 = c_2 = 0$$

$$\Rightarrow \tilde{u}(x,s) = \frac{3\,\text{sen}\,2\pi x}{4\pi^2 + s}$$

$$\Rightarrow u(x,s) = \mathcal{L}^{-1}\left[\frac{3\,\text{sen}\,2\pi x}{4\pi^2 + s}\right]$$

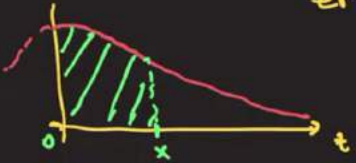
$$= (3\,\text{sen}\,2\pi x)\,\mathcal{L}^{-1}\left[\frac{1}{4\pi^2 + s}\right]$$

$$u(x,s) = (3\,\text{sen}\,2\pi x)\,e^{-4\pi^2 t}$$

Función ERROR (erf) (Función de Cramp)

Def: Se define la erf de la forma siguiente:

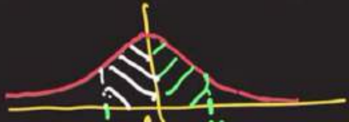
$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$



Nota: ① $\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,dt$

$$= \frac{2}{\sqrt{\pi}}\,\frac{\sqrt{\pi}}{2} = 1$$

② $\operatorname{erf}(-x) = \frac{2}{\sqrt{\pi}}\int_0^{-x} e^{-t^2}\,dt = -\frac{2}{\sqrt{\pi}}\int_{-x}^{0} e^{-t^2}\,dt$



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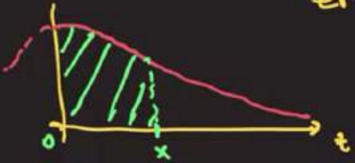

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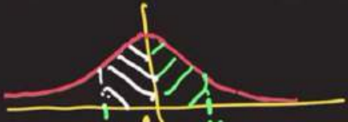
$$= -\operatorname{erf}(x)$$

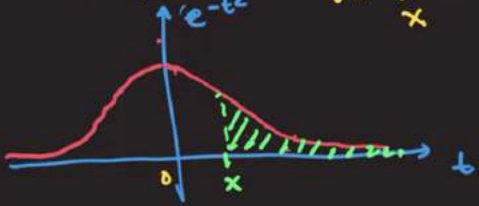


③ $$\operatorname{erf}(-\infty) = \frac{2}{\sqrt{\pi}} \int_0^{-\infty} e^{-t^2}\, dt = -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-t^2}\, dt$$

$$= -\operatorname{erf}(\infty) = -1$$

Def: La función complementaria de error, denotada erfc, se define:

$$\operatorname{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt = 1 - \operatorname{erf}(x)$$



$$\boxed{\operatorname{erf}(x) + \operatorname{erfc}(x) = 1}$$

(complementarias)

Nota: Recordemos que en ED$\genfrac{}{}{0pt}{}{1}{2}$ estudiamos:

$$\mathcal{L}\left[t^{-1/2}\right] = \mathcal{L}\left[\frac{1}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}, \quad s>0$$

i.e. $$\mathcal{L}\left[\frac{1}{\sqrt{t}}\right] = \sqrt{\pi}\,\frac{1}{\sqrt{s}}$$

$$\Rightarrow \frac{1}{\sqrt{t}} = \sqrt{\pi}\,\mathcal{L}^{-1}\left[\frac{1}{\sqrt{s}}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{\sqrt{s}}\right] = \frac{1}{\sqrt{\pi t}}$$

Problema: Encuentre $\mathcal{L}^{-1}\left[\dfrac{1}{\sqrt{s}\,(s-1)}\right]$

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$$\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{\sqrt{s}\,(s-1)}\right] = \mathcal{L}^{-1}\left[\left(\frac{1}{\sqrt{s}}\right)\cdot\left(\frac{1}{s-1}\right)\right]$$

$$= \int_0^t \frac{1}{\sqrt{\pi\tau}}\cdot e^{t-\tau}\,d\tau \leftarrow \text{convolución}$$

$$= \frac{e^t}{\sqrt{\pi}}\int_0^t \frac{e^{-\tau}}{\sqrt{\tau}}\,d\tau = \frac{2e^t}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du$$

$$u = \sqrt{\tau}$$

$$du = \frac{1}{2\sqrt{\tau}}\,d\tau$$

$$= e^t\cdot\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du$$

$$= e^t\cdot \operatorname{erf}(\sqrt{t}).$$

Teorema:

① $\mathcal{L}^{-1}\left[\dfrac{e^{-a\sqrt{s}}}{s}\right] = \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$

② $\mathcal{L}^{-1}\left[e^{-a\sqrt{s}}\right] = \dfrac{a}{2\sqrt{\pi t^3}}\, e^{-a^2/4t}$ $\quad a>0$

③ $\mathcal{L}^{-1}\left[\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}\right] = \dfrac{1}{\sqrt{\pi t}}\, e^{-a^2/4t}$

Ej: $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$, $\boxed{x>0}$, $\boxed{t>0}$, $u(x,0)=1$, $u(0,t)=0$

$\lim\limits_{x\to\infty} u(x,t) = 1$

Por Laplace:

$$\Rightarrow \quad \mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right] = \mathcal{L}\left[\frac{\partial u}{\partial t}\right]$$

$$\Rightarrow \quad \frac{\partial^2}{\partial x^2}\check{u}(x,s) = s\,\check{u}(x,s) - \cancel{u(x,0)}^{\,1}$$

$$* \leftarrow$$

$$\Rightarrow \quad \frac{d^2}{dx^2}\check{u} - s\,\check{u} = -1$$

$$\Rightarrow \quad D\cdot(D^2 - s)\,\check{u} = 0 \qquad \Rightarrow \quad m = \pm\sqrt{s}$$

$$m(m^2 - s) = 0 \quad \Rightarrow \quad m^2 - s = 0$$

$$m = 0$$

$$\Rightarrow \quad \underbrace{m_1 = \sqrt{s}\,,\quad m_2 = -\sqrt{s}}_{\check{u}_h}\,,\quad \underbrace{m_3 = 0}_{\check{u}_p}$$

$$\Rightarrow \quad \check{u}(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + c_3$$

Condiciones de frontera:

$$u(0,t) = 0 \Rightarrow \boxed{\check{u}(0,s) = 0}$$

$$\lim_{x\to\infty} u(x,t) = 1 \Rightarrow \mathcal{L}\left[\lim_{x\to\infty} u(x,t)\right] = \mathcal{L}[1]$$

$$\Rightarrow \lim_{x\to\infty} \mathcal{L}[u(x,t)] = \frac{1}{s}$$

$$\Rightarrow \boxed{\lim_{x\to\infty} \check{u}(x,s) = \frac{1}{s}}$$

$$\Rightarrow \check{u}(x,s) = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} + c_3$$

$$\Rightarrow \lim_{x\to\infty} \check{u}(x,s) = \lim_{x\to\infty}\left[c_1 e^{\sqrt{s}\,x} + \underbrace{c_2 e^{-\sqrt{s}\,x}}_{\to 0} + c_3\right] = \frac{1}{s}$$

$c_1 = 0$, para que se cumpla la condición.

$$\Rightarrow c_3 = \frac{1}{s}.$$

$$\Rightarrow \check{u}(x,s) = C_2 e^{-\sqrt{s}\,x} + \frac{1}{s}$$

$$\Rightarrow \check{u}(0,s) = C_2 + \frac{1}{s} = 0 \Rightarrow C_2 = -\frac{1}{s}$$

$$\Rightarrow \check{u}(x,s) = -\frac{1}{s} e^{-\sqrt{s}\,x} + \frac{1}{s}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\check{u}(x,s)\right] = \mathcal{L}^{-1}\left[-\frac{1}{s} e^{-\sqrt{s}\,x} + \frac{1}{s}\right]$$

$$\Rightarrow u(x,t) = -\mathcal{L}^{-1}\left[\frac{e^{-\sqrt{s}\,x}}{s}\right] + \mathcal{L}^{-1}\left[\frac{1}{s}\right]$$

$$\Rightarrow u(x,t) = -\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) + 1$$

$$\Rightarrow u(x,t) = 1 - \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{t}}^{\infty} e^{-w^2}\,dw$$

Ejercicio : $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $\boxed{0 < x < 1}$, $\boxed{t > 0}$,

$u(0,t) = 0$; $u(1,t) = u_0$

$\boxed{u(x,0) = 0}$

Ej: $a^2 \frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}$, $x > 0$, $t > 0$, $u(0,t) = 0$, $\lim\limits_{x\to\infty} \frac{\partial u}{\partial x}(x,t) = 0$

$\boxed{u(x,0) = 0}$, $\boxed{\frac{\partial u}{\partial t}(x,0) = 0}$.

$\Rightarrow \mathcal{L}\left[a^2 \frac{\partial^2 u}{\partial x^2}\right] - \mathcal{L}[g] = \mathcal{L}\left[\partial^2 u/\partial t^2\right]$

$\Rightarrow a^2 \frac{\partial^2}{\partial x^2} \check{u}(x,s) - \frac{g}{s} = s^2 \check{u}(x,s) - s\overset{0}{\cancel{u(x,0)}} - \overset{0}{\cancel{\frac{\partial}{\partial t} u(x,0)}}$

$$\Rightarrow a^2 \frac{\partial^2}{\partial x^2}\check{u} - \frac{g}{s} = s^2 \check{u}$$

$$\Rightarrow \frac{d^2}{dx^2}\check{u} - \frac{s^2}{a^2}\check{u} = \frac{g}{a^2 s} \quad (*)$$

$$\Rightarrow \left(D^2\check{u} - \frac{s^2}{a^2}\check{u}\right) = \frac{g}{a^2 s}$$

$$\Rightarrow D\left(D^2 - \frac{s^2}{a^2}\right)\check{u} = 0$$

$$m\left(m^2 - \frac{s^2}{a^2}\right) = 0 \Rightarrow m_1 = \frac{s}{a},\ m_2 = -\frac{s}{a},\ m_3 = 0$$

$$\Rightarrow \check{u}(x,s) = \underbrace{c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x}}_{\check{u}_h} + \underbrace{c_3}_{\check{u}_p}$$

Condiciones de f

Sea $\check{u}_p = c_3 \Rightarrow$ sustituyendo en (*)

$$0 - \frac{s^2}{\cancel{a^2}} c_3 = \frac{g}{\cancel{a^2}\, s}$$

$$\Rightarrow c_3 = -\frac{g}{s^3}$$

$$\Rightarrow \check{u}(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x} - \frac{g}{s^3}$$

$$x \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} + ay = bx^2, \quad x>0, \; t>0,$$

$$y(0,t) = y(x,0) = 0$$

1. Solución del ex. Recup.

Sea $\tilde{u}_p = c_3 \Rightarrow$ sustituyendo en (*)

$$0 - \frac{s^2}{\cancel{a^2}} c_3 = \frac{g}{\cancel{a^2} s}$$

$$\Rightarrow c_3 = -\frac{g}{s^3}$$

$$\Rightarrow \tilde{u}(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x} - \frac{g}{s^3}$$

condiciones de frontera:

$$\lim_{x\to\infty} \frac{\partial u}{\partial x}(x,t) = 0 \Rightarrow \lim_{x\to\infty} \frac{d\tilde{u}(x,s)}{dx} = 0$$

$$\Rightarrow \frac{\partial \tilde{u}}{\partial x}(x,s) = c_1 \frac{s}{a} e^{\frac{s}{a}x} - c_2 \frac{s}{a} e^{-s/a\, x}$$

$$\Rightarrow \lim_{x\to\infty} \frac{\partial \tilde{u}}{\partial x} = \lim_{x\to\infty}\left[c_1 \frac{s}{a} e^{\frac{s}{a}x} - c_2 \frac{s}{a} e^{-\frac{s}{a}x}\right] = 0$$

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$$\Rightarrow c_1 = 0 \Rightarrow \check{u}(x,s) = c_2 e^{-\frac{s}{a}x} - \frac{g}{s^3}$$

Como $u(0,t) = 0 \rightarrow \check{u}(0,s) = 0$

$$\Rightarrow \check{u}(0,s) = c_2 e^0 - \frac{g}{s^3} = c_2 - \frac{g}{s^3} = 0$$

$$\Rightarrow c_2 = \frac{g}{s^3}$$

$$\Rightarrow \check{u}(x,s) = \frac{g}{s^3} e^{-\frac{s}{a}x} - \frac{g}{s^3}$$

$$\Rightarrow \mathcal{L}^{-1}[\check{u}(x,s)] = \frac{g}{2}\mathcal{L}^{-1}\left[\frac{2}{s^3} e^{-s/a\, x}\right] - \frac{g}{2}\mathcal{L}^{-1}\left[\frac{2}{s^3}\right]$$

$$\Rightarrow u(x,t) = \frac{g}{2}\left(t - \frac{x}{a}\right)^2 H\left(t - \frac{x}{a}\right) - \frac{g}{2}t^2$$

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Ej: $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} + \sin \pi x$, $0 < x < 1$, $\boxed{t > 0}$,

$\boxed{u(x,0) = 0}$; $\boxed{\dfrac{\partial u}{\partial t}(x,0) = 0}$, $u(0,t) = 0$, $u(1,t) = 0$

$$\Rightarrow \mathcal{L}\left[\frac{\partial^2 u}{\partial t^2}\right] = \mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right] + \mathcal{L}\left[\sin \pi x\right]$$

$$\Rightarrow s^2 \check{u}(x,s) - s\,\cancelto{0}{u(x,0)} - \frac{\partial}{\partial t}\cancelto{0}{u(x,0)} = \frac{\partial^2}{\partial x^2}\check{u}(x,s) + \frac{\sin \pi x}{s}$$

$$\Rightarrow \frac{d^2 \check{u}}{dx^2} - s^2 \check{u} = -\frac{\sin \pi x}{s}$$

$$\Rightarrow \check{u}(x,s) = \check{u}_h + \check{u}_p$$

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$$\Rightarrow \check{u}_h: \quad \frac{d^2\check{u}}{dx} - s^2\check{u} = 0$$

$$m^2 - s^2 = 0 \Rightarrow m = \pm s$$

$$\Rightarrow \check{u}_h(x,s) = c_1 e^{sx} + c_2 e^{-sx}$$

$\underline{\check{u}_p}$: $\check{u}_p(x,s) = A\cos\pi x + B\,\text{sen}\,\pi x$

$$\check{u}_p' = -A\pi\,\text{sen}\,\pi x + B\pi\cos\pi x$$

$$\check{u}_p'' = -A\pi^2\cos\pi x - B\pi^2\,\text{sen}\,\pi x$$

Sustituyendo:

$$-A\pi^2\cos\pi x - B\pi^2\,\text{sen}\,\pi x - s^2 A\cos\pi x - s^2 B\,\text{sen}\,\pi x =$$

$$= -\frac{\text{sen}\,\pi x}{s}$$

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$$\Rightarrow \left(-A\pi^2 - s^2 A\right)\cos \pi x + \left(-\pi^2 B - s^2 B\right) \operatorname{sen} \pi x = -\frac{\operatorname{sen} \pi x}{s}$$

$$\Rightarrow -A(\pi^2 + s^2) = 0 \quad \Rightarrow A = 0$$

$$-B(\pi^2 + s^2) = -\frac{1}{s} \quad \Rightarrow \quad B = \frac{1}{s(s^2+\pi^2)}$$

$$\Rightarrow \check{u}_p(x, s) = \frac{\operatorname{sen} \pi x}{s(s^2+\pi^2)}$$

$$\Rightarrow \check{u}(x, s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x} + \frac{\operatorname{sen} \pi x}{s(s^2+\pi^2)}$$

condiciones de frontera:

$$u(0,t) = 0 \quad \Rightarrow \quad \check{u}(0, s) = 0$$

$$u(1,t) = 0 \quad \Rightarrow \quad \check{u}(1, s) = 0$$

$$\Rightarrow \quad \begin{cases} c_1 + c_2 = 0 \\ c_1 e^{\frac{s}{a}} + c_2 e^{-\frac{s}{a}} = 0 \end{cases} \quad \Rightarrow \det\begin{pmatrix} 1 & 1 \\ e^{s/a} & e^{-s/a} \end{pmatrix} =$$

$$= e^{-s/a} - e^{+s/a} \neq 0$$

$$\Rightarrow c_1 = c_2 = 0$$

$$\Rightarrow \tilde{u}(x,s) = \frac{\operatorname{sen} \pi x}{s(s^2+\pi^2)}$$

$$\Rightarrow u(x,t) = \mathcal{L}^{-1}\left[\frac{\operatorname{sen} \pi x}{s(s^2+\pi^2)}\right]$$

$$= (\operatorname{sen} \pi x)\, \mathcal{L}^{-1}\left[\frac{1}{s(s^2+\pi^2)}\right]$$

$$\Gamma \quad \frac{1}{s(s^2+\pi^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+\pi^2}$$

$$\Rightarrow 1 = As^2 + A\pi^2 + Bs^2 + Cs$$

$$\Rightarrow 1 = (A+B)s^2 + Cs + A\pi^2$$

$$\Rightarrow A+B=0 \quad ; \quad A\pi^2 = 1 \Rightarrow A = 1/\pi^2$$

$$C = 0 \qquad \Rightarrow B = -A = -1/\pi^2$$

$$\Rightarrow u(x,t) = (\sin \pi x)\, \mathcal{L}^{-1}\left[\frac{1/\pi^2}{s} - \frac{1/\pi^2\, s}{s^2+\pi^2}\right]$$

$$\Rightarrow u(x,t) = \frac{\sin \pi x}{\pi^2}\left[\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{s}{s^2+\pi^2}\right)\right]$$

$$\Rightarrow u(x,t) = \frac{\sin \pi x}{\pi^2}\left[1 - \cos \pi t\right]$$

Ej: $x\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial t} + au = bx^2$, $x>0$, $\boxed{t>0,}$

$a, b \in \mathbb{R}$, $u(0,t)=0$, $\boxed{u(x,0)=0}$

$$\Rightarrow \mathcal{L}\left[x\frac{\partial u}{\partial x}\right] + \mathcal{L}\left[\frac{\partial u}{\partial t}\right] + a\mathcal{L}[u] = b\mathcal{L}[x^2]$$

$$\Rightarrow x\frac{\partial}{\partial x}\check{u}(x,s) + s\check{u}(x,s) - \cancelto{0}{u(x,0)} + a\check{u}(x,s) = \frac{bx^2}{s}$$

EDO

$$\Rightarrow x\frac{d\check{u}}{dx} + (s+a)\check{u} = \frac{bx^2}{s}$$

$$\Rightarrow \frac{d\check{u}}{dx} + \overbrace{\boxed{\frac{(s+a)}{x}}}^{P(x)}\check{u} = \frac{bx}{s}$$

$\Rightarrow$ Factor integrante $\mu(x) = e^{\int P(x)\,dx}$

$$\Rightarrow \mu(x) = e^{\int \frac{n+a}{x}\,dx} = e^{(n+a)\ln x} =$$

$$= e^{\ln x^{n+a}} = x^{n+a}.$$ Entonces,

multiplicando la E.D. por el $\mu(x)$:

$$\Rightarrow \frac{d}{dx}\left[x^{n+a}\cdot \check{u}\right] = \frac{b}{n}x^{n+a+1}$$

$$\Rightarrow \int d\left[x^{n+a}\cdot \check{u}\right] = \int \frac{b}{n}x^{n+a+1}\,dx$$

$$\Rightarrow x^{n+a}\cdot \check{u} = \frac{b}{n(n+a+2)}x^{n+a+2} + C$$

$$\Rightarrow \check{u}(x,s) = \frac{bx^2}{s(s+a+2)} + \frac{c}{x^{s+a}}$$

Condición de frontera: $u(0,t) = 0 \Rightarrow \check{u}(0,s) = 0$

$x=0$

$\Rightarrow c = 0$ (en otro caso, hay división por cero).

$$\Rightarrow \check{u}(x,s) = \frac{bx^2}{s(s+a+2)}$$

$$\Rightarrow u(x,t) = bx^2 \mathcal{L}^{-1}\left[\frac{1}{s(s+a+2)}\right]$$

aplicamos fracciones parciales.

$$\Rightarrow u(x,t) = bx^2 \mathcal{L}^{-1}\left[\frac{1/(a+2)}{s} - \frac{1/(a+2)}{s+a+2}\right]$$

$$\Rightarrow u(x,t) = \frac{bx^2}{a+2} \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+a+2}\right]$$

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$$\Rightarrow u(x,t) = \frac{bx^2}{a+2}\left[\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+a+2}\right]\right]$$

$$= \frac{bx^2}{a+2}\left[1 - e^{-(a+2)t}\right]$$

Ejemplo (****): $\frac{\partial^2 u}{\partial x^2} = \boxed{\frac{\partial u}{\partial t}}$, $0 < x < 1$, $\boxed{t > 0}$

$u(0,t) = 0$; $u(1,t) = u_0$

$\boxed{u(x,0) = 0}$.

$$\Rightarrow \mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right] = \mathcal{L}\left[\frac{\partial u}{\partial t}\right]$$

$$\Rightarrow \frac{d^2}{dx^2}\check{u}(x,s) = s\,\check{u}(x,s) - \cancel{u(x,0)}^{\,0}$$

$$\Rightarrow \frac{d^2}{dx^2}\check{u} - s\,\check{u} = 0 \quad \Rightarrow m^2 - s = 0$$

$$\Rightarrow m = \pm\sqrt{s}$$

$$\Rightarrow \check{u}(x,s) = c_1 \cosh\sqrt{s}\,x + c_2 \operatorname{senh}\sqrt{s}\,x$$

Condiciones de frontera:

$$u(0,t) = 0 \Rightarrow \check{u}(0,s) = 0$$

$$u(1,t) = u_0 \Rightarrow \check{u}(1,s) = \frac{u_0}{s}$$

$$\Rightarrow \check{u}(0,s) = c_1 = 0$$

$$\Rightarrow \check{u}(x,s) = c_2 \operatorname{senh}\sqrt{s}\,x$$

$$\Rightarrow \quad c_2 \operatorname{senh} \sqrt{s} = \frac{u_0}{s}$$

$$\Rightarrow \quad c_2 = \frac{u_0}{s(\operatorname{senh} \sqrt{s})}$$

$$\Rightarrow \quad \bar{u}(x, s) = \frac{u_0}{s(\operatorname{senh} \sqrt{s})} \operatorname{senh} \sqrt{s}\, x$$

$$\Rightarrow \quad u(x, t) = u_0 \, \mathcal{L}^{-1} \left[\frac{\operatorname{senh} \sqrt{s}\, x}{s(\operatorname{senh} \sqrt{s})} \right]$$

$$\left[\quad \frac{\operatorname{senh} \sqrt{s}\, x}{s(\operatorname{senh} \sqrt{s})} = \frac{e^{\sqrt{s}\, x} - e^{-\sqrt{s}\, x}}{s\left[e^{\sqrt{s}} - e^{-\sqrt{s}}\right]}$$

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$$= \frac{e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}}{\lambda e^{\sqrt{\lambda}}\left[1 - e^{-2\sqrt{\lambda}}\right]}$$; Nó tene que:

$$\frac{1}{1 - e^{-2\sqrt{\lambda}}} = \sum_{n=0}^{\infty} \left(e^{-2\sqrt{\lambda}}\right)^n$$ (Limite de serie geométrica

$$= \underbrace{\left(\frac{e^{\sqrt{\lambda}(x-1)} - e^{-\sqrt{\lambda}(x+1)}}{\lambda}\right)} \sum_{n=0}^{\infty} \left(e^{-2\sqrt{\lambda}}\right)^n \quad e^{-2n\sqrt{\lambda}}$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{\sqrt{\lambda}(x-2n-1)}}{\lambda} - \frac{e^{-\sqrt{\lambda}(x+2n+1)}}{\lambda}\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{e^{-\sqrt{s}(2n+1-x)}}{s} - \frac{e^{-\sqrt{s}(x+2n+1)}}{s}\right]$$

$$\Rightarrow u(x,t) = u_0 \sum_{n=0}^{\infty}\left[\mathcal{L}^{-1}\left(\frac{e^{-\sqrt{s}(2n+1-x)}}{s} - \frac{e^{-\sqrt{s}(2n+1+x)}}{s}\right)\right]$$

$$\Rightarrow u(x,t) = u_0 \sum_{n=0}^{\infty}\left[\operatorname{erfc}\left(\frac{2n+1-x}{2\sqrt{t}}\right) - \operatorname{erfc}\left(\frac{2n+1+x}{2\sqrt{t}}\right)\right]$$

$\Rightarrow\!\!\!/$

Ej. $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$; $0 < x < l$, $t > 0$, $u(x,0) = u_0$,

$\longrightarrow \dfrac{\partial u}{\partial x}(0,t) = 0$; $u(l,t) = u_1$.

$$\Rightarrow \mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right] = \mathcal{L}\left[\frac{\partial u}{\partial t}\right]$$

u_0

$$\Rightarrow \frac{d^2}{dx^2}\check{u}(x,s) = s\,\check{u}(x,s) - u(\cancel{x},0)$$

$$\Rightarrow \frac{d^2\check{u}}{dx^2} - s\,\check{u} = -u_0$$

$\underline{\check{u}_h}$:
$$\frac{d^2\check{u}}{dx^2} - s\,\check{u} = 0$$

$$m^2 - s = 0 \Rightarrow m = \pm\sqrt{s}$$

$$\Rightarrow \check{u}_h(x,s) = c_1 \cosh\sqrt{s}\,x + c_2\,\mathrm{senh}\sqrt{s}\,x$$

$\underline{\check{u}_p}$:
$$\check{u}_p(x,s) = A \quad (\text{constante})$$

Sustituyendo en la ec. dif:

$$0 - s A = -u_0 \Rightarrow A = \frac{u_0}{s}$$

$$\Rightarrow \check{u}(x,s) = \check{u}_h + \check{u}_p$$

$$\Rightarrow \check{u}(x,s) = c_1 \cosh\sqrt{s}\,x + c_2 \operatorname{senh}\sqrt{s}\,x + \frac{u_0}{s} \quad (*)$$

Condiciones de frontera:

$$\frac{\partial u}{\partial x}(0,t) = 0 \Rightarrow \frac{d}{dx}\check{u}(0,s) = 0$$

$$u(l,t) = u_1 \Rightarrow \check{u}(l,s) = \frac{u_1}{s}$$

$$\Rightarrow \frac{d\check{u}}{dx}(x,s) = c_1 \operatorname{senh}\sqrt{s}\,x + c_2 \cosh\sqrt{s}\,x + 0$$

$$\Rightarrow \frac{d\check{u}}{dx}(0,s) = c_2 = 0.$$

$$\Rightarrow \tilde{u}(x,s) = c_1 \cosh \sqrt{s}\, x + \frac{u_0}{s}$$

$$\Rightarrow \tilde{u}(l,s) = c_1 \cosh \sqrt{s}\, l + \frac{u_0}{s} = \frac{u_1}{s}$$

$$\Rightarrow c_1 = \frac{u_1 - u_0}{s(\cosh \sqrt{s}\, l)}$$

$$\Rightarrow \tilde{u}(x,s) = \frac{(u_1 - u_0)\cosh \sqrt{s}\, x}{s(\cosh \sqrt{s}\, l)} + \frac{u_0}{s}$$

$$\Rightarrow u(x,t) = (u_1 - u_0)\, \mathcal{L}^{-1}\underbrace{\left[\frac{\cosh \sqrt{s}\, x}{s(\cosh \sqrt{s}\, l)}\right]}_{} + \underbrace{\mathcal{L}^{-1}\left(\frac{u_0}{s}\right)}_{u_0}$$

De las tablas* de transformadas:

$$\mathcal{L}^{-1}\left[\frac{\cosh\sqrt{s}\,x}{s\cosh\sqrt{s}\,a}\right] = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\, e^{-(2n-1)^2\pi^2 t/4a^2}\cdot\cos\left(\frac{2n-1}{2a}\right)\pi x$$

$$\mathcal{L}^{-1}\left[\frac{\operatorname{senh}\sqrt{s}\,x}{s\cosh\sqrt{s}\,a}\right] = \frac{x}{a} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\, e^{-n^2\pi^2 t/a^2}\cdot\operatorname{sen}\frac{n\pi x}{a}.$$