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23 de enero de 2021
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Ecuaciones Diferenciales 2 - Dorval Carías

Tarea 1

1. Sea $z \in \mathbb{C}, z \neq 0$. Encuentre las condiciones para que $z + \frac{1}{z} \in \mathbb{R}$.

Demostración. Demostración:

$$\text{Sea } z = x + iy \tag{1}$$

$$\Longrightarrow z + \frac{1}{z} = x + iy + \frac{1}{x+iy} \implies x + iy + \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} \tag{2}$$

$$\Longrightarrow x + iy + \frac{x-iy}{x^2 - xyi + xyi - i^2y^2} \implies x + iy + \frac{x-yi}{x^2+y^2} \tag{3}$$

$$\Longrightarrow x + \frac{x}{x^2+y^2} + iy - \frac{yi}{x^2+y^2} \implies \frac{x(x^2+y^2)+x}{x^2+y^2} + \frac{iy(x^2+y^2)-yi}{x^2+y^2} \tag{4}$$

$$\Longrightarrow \frac{x(x^2+y^2+1)}{x^2+y^2} + i\frac{y(x^2+y^2-1)}{x^2+y^2} \tag{5}$$

$$\Longrightarrow z + \frac{1}{z} \in \mathbb{R} \iff x^2 + y^2 = 1 \vee b = 0. \tag{6}$$

□

2. Muestre que una ecuación para el circulo $C(z_0, r)$ en el plano complejo es: $z\bar{z} - z_0\bar{z} - z\overline{z_0} + z_0\overline{z_0} = r^2$

Demostración.

$$\Longrightarrow \bar{z} - z_0\bar{z} - z\overline{z_0} + z_0\overline{z_0} = r^2 \tag{1}$$

$$\Longrightarrow z(\bar{z} - \bar{z}_0) - z_0(\bar{z} - \bar{z}_0) = r^2 \tag{2}$$

$$\Longrightarrow (z - z_0)(\bar{z} - \bar{z}_0) = r^2 \tag{3}$$

$$\Longrightarrow (z - z_0)(z \bar{-} z_0) = r^2 \tag{4}$$

$$\Longrightarrow |z - z_0|^2 = r^2, \text{ si } |z - z_0| = r \tag{5}$$

$$\Longrightarrow |z - z_0|^2 = r^2 \tag{6}$$

□

3 .

1. Encuentre las raíces quintas de la unidad.

$$\Longrightarrow z^5 = 1 \tag{1}$$

$$\Longrightarrow r = \sqrt{(1)^2 + (0)} = 1 \qquad \theta = arg(Z) = \tan\theta = \frac{0}{1} = 0 \tag{2}$$

$$\Longrightarrow t = 1 \qquad r = 1 \qquad arg(Z) = 1 \tag{3}$$

$$\Longrightarrow z = 1\cos\theta + i\sin\theta = 1\cos(0) + i(1)\sin(0) \tag{4}$$

Entonces:

$$\Longrightarrow w_k = (1)^{1/5}[\cos(\frac{0+2k\pi}{5}) + i\sin(\frac{0+2k\pi}{5})] \tag{5}$$

$$k = 0; \qquad w_0 = \cos(0) + i\sin(0) \tag{6}$$

$$k = 1; \qquad w_1 = \cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5}) \tag{7}$$

$$k = 2; \qquad w_2 = \cos(\frac{4\pi}{5}) + i\sin(\frac{4\pi}{5}) \tag{8}$$

$$k = 3; \qquad w_3 = \cos(\frac{6\pi}{5}) + i\sin(\frac{6\pi}{5}) \tag{9}$$

$$k = 4; \qquad w_4 = \cos(\frac{8\pi}{5}) + i\sin(\frac{8\pi}{5}) \tag{10}$$

2. Encuentre todas las soluciones complejas de $z^6 = -8i$.

$$x = 0 \quad y = -8 \tag{1}$$

$$\implies r = \sqrt{0^2 + (-8)^2} = 8 \quad arg(Z) = \theta = -\frac{\pi}{2} \tag{2}$$

$$\implies 8\cos(-\frac{\pi}{2}) + i(8\sin(-\frac{\pi}{2})) \tag{3}$$

$$w_k = (8)^{1/6}\left[\cos\left(\frac{-\frac{\pi}{2} + 2k\pi}{6}\right) + i\sin\left(\frac{-\frac{\pi}{2} + 2k\pi}{6}\right)\right] \tag{4}$$

$$w_0 = (8)^{1/6}\left[\cos\left(\frac{-\frac{\pi}{2}}{6}\right) + i\sin\left(\frac{-\frac{\pi}{2}}{6}\right)\right] \tag{5}$$

$$= (8)^{1/6}\left[\frac{\sqrt{6}+\sqrt{2}}{4} + i\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)\right] \tag{6}$$

$$w_1 = (8)^{1/6}\left[\cos\left(\frac{-\frac{\pi}{2} + 2\pi}{6}\right) + i\sin\left(\frac{-\frac{\pi}{2} + 2\pi}{6}\right)\right] \tag{7}$$

$$= (8)^{1/6}\left[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right] \tag{8}$$

$$w_2 = (8)^{1/6}\left[\cos\left(\frac{-\frac{\pi}{2} + 4\pi}{6}\right) + i\sin\left(\frac{-\frac{\pi}{2} + 4\pi}{6}\right)\right] \tag{9}$$

$$= (8)^{1/6}\left[\frac{-(\sqrt{3}-1)\sqrt{2}}{4} + i\frac{(\sqrt{3}+1)\sqrt{2}}{4}\right] \tag{10}$$

$$w_3 = (8)^{1/6}\left[\cos\left(\frac{-\frac{\pi}{2} + 6\pi}{6}\right) + i\sin\left(\frac{-\frac{\pi}{2} + 6\pi}{6}\right)\right] \tag{11}$$

$$= (8)^{1/6}\left[\frac{-(\sqrt{3}-1)\sqrt{2}}{4} + i\frac{(\sqrt{3}+1)\sqrt{2}}{4}\right] \tag{12}$$

$$w_4 = (8)^{1/6}\left[\cos\left(\frac{-\frac{\pi}{2} + 8\pi}{6}\right) + i\sin\left(\frac{-\frac{\pi}{2} + 8\pi}{6}\right)\right] \tag{13}$$

$$= (8)^{1/6}\left[-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right] \tag{14}$$

$$w_5 = (8)^{1/6}\left[\cos\left(\frac{-\frac{\pi}{2} + 10\pi}{6}\right) + i\sin\left(\frac{-\frac{\pi}{2} + 10\pi}{6}\right)\right] \tag{15}$$

$$= (8)^{1/6}\left[\frac{-(\sqrt{3}-1)\sqrt{2}}{4} + i\frac{(\sqrt{3}+1)\sqrt{2}}{4}\right] \tag{16}$$

4. Compruebe las identidades trigonométricas a continuación:

1. $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

$$\begin{aligned}
\cos 5\theta &= \cos(3\theta + 2\theta) && (1)\\
&= \cos 3\theta \cos 2\theta - \sin 3\theta \sin 2\theta && (2)\\
&= (4\cos^3\theta - 3\cos\theta)(2\cos^2\theta - 1) - (3\sin\theta - 4\sin^3\theta)(2\sin\theta\cos\theta) && (3)\\
&= \left(8\cos^5\theta - 10\cos^3\theta + 3\cos\theta\right) - \left(3 - 4\sin^2\theta\right)\left(2\sin^2\theta\cos\theta\right) && (4)\\
&= \left(8\cos^5\theta - 10\cos^3\theta + 3\cos\theta\right) - \left(3 - 4\left(1-\cos^2\theta\right)\right)\left(2\left(1-\cos^2\theta\right)\cos\theta\right) && (5)\\
&= \left(8\cos^5\theta - 10\cos^3\theta + 3\cos\theta\right) - \left(3 - 4 + 4\cos^2\theta\right)\left(2\cos\theta - 2\cos^3\theta\right) && (6)\\
&= \left(8\cos^5\theta - 10\cos^3\theta + 3\cos\theta\right) - \left(-8\cos^5\theta + 10\cos^3\theta - 2\cos\theta\right) && (7)\\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta && (8)
\end{aligned}$$

2. $(\operatorname{sen} 5\theta)/(\operatorname{sen}\theta) = 16\cos^4\theta - 12\cos^2\theta + 1, \theta \neq 0, \pm\pi, \pm 2\pi, \cdots$

Primero se determina $\sin 5\theta$

$$\begin{aligned}
\sin 5\theta &= \sin(3\theta + 2\theta) && (1)\\
&= \sin 3\theta \cos 2\theta + \cos 3\theta \sin 2\theta && (2)\\
&= \left(3\sin\theta - 4\sin^3\theta\right)\left(1 - \sin^2\theta\right) + \left(4\cos^3\theta - 3\cos\theta\right)\left(2\sin\theta\cos\theta\right) && (3)\\
&= \sin\theta\left[\left(3 - 4\sin^2\theta\right)\left(2\cos^2\theta - 1\right) + \left(4\cos^3\theta - 3\cos\theta\right)\left(\cos\theta\right)\right] && (4)
\end{aligned}$$

Entonces:

$$\begin{aligned}
\frac{\sin 5\theta}{\sin\theta} &= \left[\left(3 - 4\left(1 - \cos^2\theta\right)\right]\left(2\cos^2\theta - 1\right) + \left(8\cos^4\theta - 6\cos^2\theta\right)\right] && (5)\\
&= \left[\left(3 - 4 + 4\cos^2\theta\right)\left(2\cos^2\theta - 1\right) + 8\cos^4\theta - 6\cos^2\theta\right) && (6)\\
&= \left[\left(-1 + 4\cos^2\theta\right)\left(2\cos^2\theta - 1\right) + 3\cos^4\theta - 6\cos^2\theta\right) && (7)\\
&= \left(-2\cos^2\theta + 8\cos^4\theta + 1 - 4\cos^2\theta + 8\cos^4\theta - 6\cos^2\theta\right) && (8)\\
&= 16\cos^4\theta - 12\cos^2\theta + 1, \theta \neq 0, \pm\pi, \pm 2\pi, \cdots && (9)
\end{aligned}$$

5. Realice las operaciones indicadas:

1. $(-1+i)^{1/3}$

Se puede representar como: $z^3 = (-1+i)$

Entonces $x = -1$ y $y = 1$. Por lo que $argZ = -\frac{\pi}{4}$ y $r = \sqrt{2}$

$$\begin{aligned}
&\Longrightarrow w_k = (\sqrt{2})^{1/3}[\cos(\frac{-\frac{\pi}{4} + 2k\pi}{3}) + i\sin(\frac{-\frac{\pi}{4} + 2k\pi}{3})] && (1)\\
&\Longrightarrow w_0 = (\sqrt{2})^{1/3}[\cos(\frac{-\frac{\pi}{4}}{3}) + i\sin(\frac{-\frac{\pi}{4}}{3})] && (2)\\
&\Longrightarrow w_1 = (\sqrt{2})^{1/3}[\cos(\frac{-\frac{\pi}{4} + 2\pi}{3}) + i\sin(\frac{-\frac{\pi}{4} + 2\pi}{3})] && (3)\\
&\Longrightarrow w_2 = (\sqrt{2})^{1/3}[\cos(\frac{-\frac{\pi}{4} + 4\pi}{3}) + i\sin(\frac{-\frac{\pi}{4} + 4\pi}{3})] && (4)
\end{aligned}$$

2. $(-2\sqrt{3}-2i)^{1/4}$

   Se puede representar como: $z^4 = (-2\sqrt{3}-2i)$

   Entonces $x = -2\sqrt{3}$ y $y = -2$. Por lo que $argZ = \frac{\pi}{6}$ y $r = 4$

$$\implies w_k = (\sqrt{4})^{1/4}[\cos(\frac{\frac{\pi}{6}+2k\pi}{4}) + i\sin(\frac{-\frac{\pi}{5}+2k\pi}{4})] \tag{1}$$
$$\implies w_0 = (\sqrt{4})^{1/4}[\cos(\frac{\frac{\pi}{6}}{4}) + i\sin(\frac{-\frac{\pi}{5}}{4})] \tag{2}$$
$$\implies w_1 = (\sqrt{4})^{1/4}[\cos(\frac{\frac{\pi}{6}+2\pi}{4}) + i\sin(\frac{-\frac{\pi}{5}+2\pi}{4})] \tag{3}$$
$$\implies w_2 = (\sqrt{4})^{1/4}[\cos(\frac{\frac{\pi}{6}+4\pi}{4}) + i\sin(\frac{-\frac{\pi}{5}+4\pi}{4})] \tag{4}$$
$$\implies w_3 = (\sqrt{4})^{1/4}[\cos(\frac{\frac{\pi}{6}+6\pi}{4}) + i\sin(\frac{-\frac{\pi}{5}+6\pi}{4})] \tag{5}$$

3. $\sqrt{-15-8i}$

   Se puede representar como: $z^2 = (-15-8i)$

   Entonces $x = -15\sqrt{3}$ y $y = -8$. Por lo que $argZ = 0{,}3$ y $r = 17$

$$\implies w_k = (\sqrt{17})^{1/2}[\cos(\frac{0,3+2k\pi}{2}) + i\sin(\frac{0,3+2k\pi}{2})] \tag{1}$$
$$\implies w_0 = (\sqrt{17})^{1/2}[\cos(\frac{0,3}{2}) + i\sin(\frac{0,3}{2})] \tag{2}$$
$$\implies w_1 = (\sqrt{17})^{1/2}[\cos(\frac{0,3+2\pi}{2}) + i\sin(\frac{0,3+2\pi}{2})] \tag{3}$$
$$\tag{4}$$

6. Resuelva las ecuaciones siguientes:

1. $z^2 + (2i-3)z + 5 - i = 0$

   Supóngase $a = 1, b = (2i-3), c = 5 - i$

$$\implies z = \frac{-(2i-3) \pm \sqrt{(2i-3)^2 - 4(5-i)}}{2} \tag{1}$$
$$= \frac{(-2i-3) \pm \sqrt{-12i+5-20+4i}}{2} \tag{2}$$
$$= \frac{3-2i \pm i\sqrt{8i+15}}{2} \tag{3}$$

   En donde $\sqrt{8i+15} = 16 + 8i - 1 = 16 + 8i + i^2 = (i+4)^2$

$$= \frac{3-2i \pm i(i+4)}{2} \tag{4}$$

Entonces:

$$z_1 = \frac{2+2i}{2} = 1+i \tag{5}$$
$$z_2 = \frac{4-2i}{2} = 2-i \tag{6}$$

2. $6z^4 - 25z^3 + 32z^2 + 3z - 10 = 0$

Luego de una larga factorización por división sintética...

$$6z^4 - 25z^3 + 32z^2 + 3z - 10 = (2z+1)(3z-2)(z^2-4z+5) \tag{1}$$

Se obtiene: $z_1 = -\frac{1}{2}$ y $z_2 = \frac{2}{3}$

$$z_3 = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2} = \tag{2}$$
$$= \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = \tag{3}$$
$$= 2 \pm i \tag{4}$$

3. $z^2\,(1-z^2) = 16$

$$z^2 - z^4 - 16 = 0 \tag{1}$$
$$-z^4 + z^2 - 16 = 0 \tag{2}$$
$$-z^4 + z^2 - 16 = 0 \tag{3}$$
$$z^4 - z^2 + 16 = 0 \tag{4}$$

Si asumimos que $z^2 = x$

$$x^2 - x + 16 = 0 \tag{5}$$

Aplicando la fórmula cuadrática:

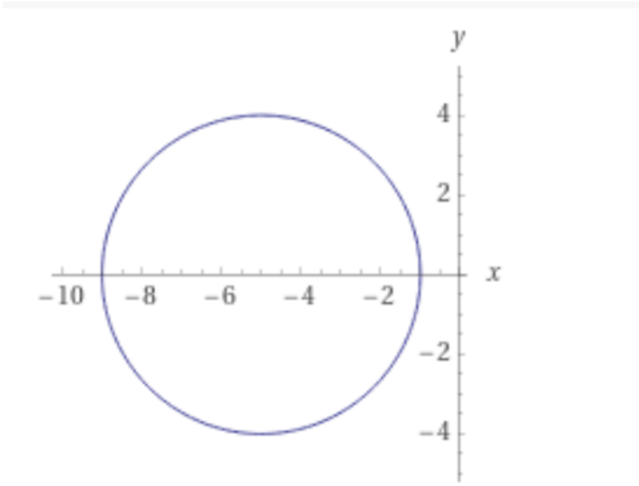
$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(16)}}{2} = \frac{1 \pm \sqrt{1-64}}{2} \tag{6}$$
$$= \frac{1 \pm \sqrt{-63}}{2} = \frac{1 \pm \sqrt{63}i}{2} \tag{7}$$
$$\implies x = \frac{1 \pm \sqrt{63}i}{2} \implies z^2 = \frac{1 \pm \sqrt{63}i}{2} \tag{8}$$
$$z = \pm\sqrt{\frac{1 \pm 3\sqrt{7}i}{2}} \tag{9}$$

7. Represente gráficamente el conjunto de valores de z para los cuales:

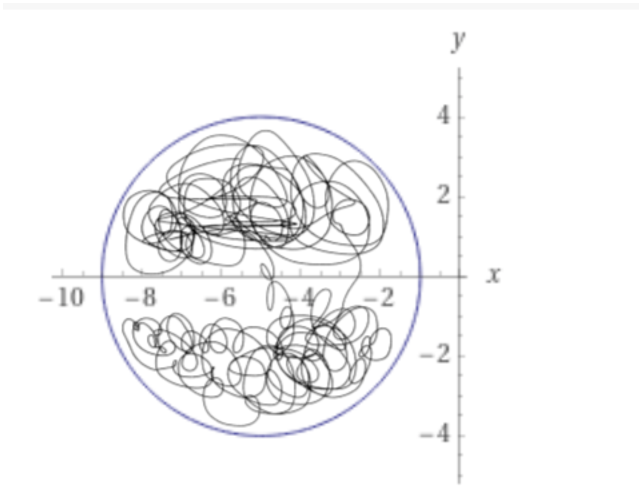
1. $\left|\frac{z-3}{z+3}\right| = 2$

$$z = x + iy \implies \left|\frac{x+iy-3}{x-iy+3}\right| = 2$$

$$\implies |x+iy-3| = 2|x+iy+3| \tag{1}$$
$$\implies (x-3)^2 + y^2 = 4[(x+3)^2 + y^2] \tag{2}$$



2. $\left|\frac{z-3}{z+3}\right| < 2$



8. Un número algebraico es un número que es solución de una ecuación polinómica de la forma:

$$a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0 = 0$$

donde $a_0, a_1, \cdots, a_n$ son enteros. Compruebe que los números siguientes son algebraicos.

1. $\sqrt{3} + \sqrt{2}$

Supóngase $x = \sqrt{3} + \sqrt{2}$

$$\begin{align}
&\implies (\sqrt{3} + \sqrt{2})^2 = 2 + \sqrt{3}\sqrt{2} + 2 \tag{1}\\
&\implies x^2 = 5 + 2\sqrt{6} \tag{2}\\
&\implies (x^2 - 5)^2 = (2\sqrt{6})^2 \tag{3}\\
&\implies (x^2 - 5)^2 = 24 \tag{4}
\end{align}$$

2. $\sqrt[3]{4} - 2i$

Supóngase $x = \sqrt[3]{4} + 2i$

$$\begin{align}
\implies x^2 &= (\sqrt[3]{4} + 2i)^2 \tag{1}\\
\implies x^2 &= (4^{2/3} + 4\sqrt[3]{4}i + 4i^2) \tag{2}\\
\implies x^2 &= (4^{2/3} - 4 + 4\sqrt[3]{4}) \tag{3}\\
\implies (x^2 + 4 - 4^{2/3})^2 &= 16 \cdot 4^{2/3} i \tag{4}
\end{align}$$