1) $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$; $u(x,0) = f(x)$ | $\hat{u}(w,0) = \hat{f}(w)$

$\dfrac{\partial u}{\partial t}(x,0) = g(x)$ | $\dfrac{\partial \hat{u}}{\partial t}(w,0) = \hat{g}(w)$

Fourier

$$(iw)^2 \hat{u}(w,t) = \frac{\partial^2 }{\partial t^2}\hat{u}(w,t)$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\hat{u}(w,t) + w^2 \hat{u}(w,t) = 0$$

$$\Rightarrow \hat{u}(w,t) = A_w \cos(wt) + B_w \operatorname{sen}(wt)$$

Condiciones

$$\hat{u}(w,0) = A_w = \hat{f}(w)$$

$$\frac{\partial}{\partial t}\hat{u}(w,t) = -A_w w \operatorname{sen}(wt) + B_w w \cos(wt)$$

$$\frac{\partial}{\partial t}\hat{u}(w,0) = B_w w = \hat{g}(w) \Rightarrow B_w = \frac{\hat{g}(w)}{w}$$

$$\Rightarrow \hat{u}(w,t) = \hat{f}(w)\cos(wt) + \frac{\hat{g}(w)}{w}\operatorname{sen}(wt)$$

$$\mathcal{F}^{-1}[\hat{u}(w,t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\hat{f}(w)\cos wt + \frac{\hat{g}(w)}{w}\operatorname{sen} wt\right] e^{-iwx}\,dw$$

Parte Real

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\hat{f}(w)\cos wt + \frac{\hat{g}(w)}{w}\operatorname{sen} wt\right)\cos wx\,dw$$

② $\frac{\partial^2 u}{\partial t \partial x} = \frac{\partial^2 u}{\partial x^2}$, sujeta $u(x,0) = f(x)$

~~$\frac{\partial u(x,0)}{\partial t} = g(x)$~~

Fourier

$$\frac{\partial u}{\partial t}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\partial}{\partial t}(iw)\hat{u}(w,t) = -\cancel{i}w^2\hat{u}(w,t)$$

$$\Rightarrow \frac{\partial}{\partial t}(iw)\hat{u}(w,t) + w^2\hat{u}(w,t) = 0 \;*\; \frac{i}{w}$$

$$\cancel{\Rightarrow \frac{\partial}{\partial t}\hat{u}(w,t) + \frac{w}{i}\hat{u}(w,t) = 0}$$

$$\Rightarrow \frac{\partial}{\partial t}(-1)\hat{u}(w,t) + iw\,\hat{u}(w,t) = 0 \;*\; -1$$

$$\frac{\partial}{\partial t}\hat{u}(w,t) - iw\,\hat{u}(w,t) = 0$$

$$\hat{u}(w,t) = A_w e^{-iwt}$$

Condition

$$\hat{u}(w,0) = A_w = \hat{f}(w)$$

$$\Rightarrow \hat{u}(w,t) = \hat{f}(w)e^{-iwt}$$

$$\mathcal{F}^{-1}[\hat{u}(w,t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(w)\,e^{-iwt}\,e^{-iwx}\,dw$$

③ $-c^2 \frac{\partial^4 u}{\partial x^4} = \frac{\partial^2 u}{\partial t^2}$ / $u(x,0) = f(x)$

$\frac{\partial \cancel{u}}{\partial t}(x,0) = g(x)$

$-c^2 (i\omega)^4 \hat{u}(\omega,t) = \frac{\partial^2}{\partial t^2} \hat{u}(\omega,t)$

$-c^2 (i^2 i^2 w^4) \hat{u}(\omega,t) = \frac{\partial^2}{\partial t^2} \hat{u}(\omega,t)$

$-c^2 w^4 \hat{u}(\omega,t) =$ — — — —

$\Rightarrow \frac{\partial^2}{\partial t^2} \hat{u}(\omega,t) + \underbrace{c^2 w^4}_{(cw^2)^2} \hat{u}(\omega,t) = 0$

$\Rightarrow \hat{u}(\omega,t) = A_w \cos(cw^2 t) + B_w \operatorname{sen}(cw^2 t)$

$\frac{\partial}{\partial t} \hat{u}(\omega,t) = -A_w c w^2 \operatorname{sen}(cw^2 t) + B_w c w^2 \cos(cw^2 t)$

Aplicando las condiciones

$\hat{u}(\omega,0) = A_w = \hat{f}(\omega)$

$\frac{\partial}{\partial t} \hat{u}(\omega,0) = B_w c w^2 = \hat{g}(\omega) \Rightarrow B_w = \frac{\hat{g}(\omega)}{cw^2}$

$\Rightarrow \hat{u}(\omega,t) = \hat{f}(\omega) \cos(cw^2 t) + \cancel{B} \frac{\hat{g}(\omega)}{cw^2} \operatorname{sen}(cw^2 t)$

$= \mathcal{F}^{-1}[\hat{u}(\omega,t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\hat{f}(\omega) \cos(cw^2 t) + \frac{\hat{g}(\omega)}{cw^2} \operatorname{sen}(cw^2 t) \right) e^{-i\omega x} \, d\omega$

(4) $t\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial t} = 0$, sujeta a

$u(x,0) = f(x)$

$$t(iw)\hat{u}(w,t) + \frac{\partial}{\partial t}\hat{u}(w,t) = 0$$

$$\frac{\partial}{\partial t}\hat{u}(w,t) + t\,iw\,\hat{u}(w,t) = 0$$

$$\Rightarrow \hat{u}(w,t) = A_w e^{-\left(\frac{wt^2}{2}\right)}$$

Aplicando condición

$$\hat{u}(w,0) = A_w = \hat{f}(w)$$

$$\hat{u}(w,t) = \hat{f}(w) e^{-\left(\frac{wt^2}{2}\right)}$$

$$\mathcal{F}^{-1}(\hat{u}(w,t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(w) e^{-\left(\frac{wt^2}{2}\right)} e^{-iwx}\, dw$$