PDE - FOURIER

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June 2, 2021

Transformada de Fourier

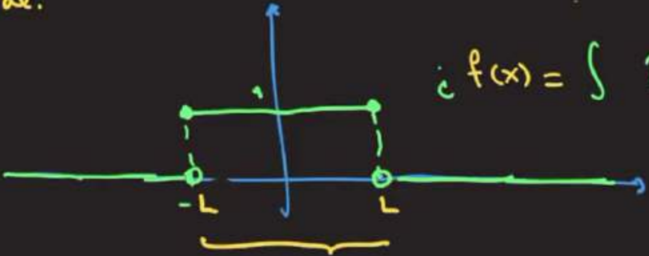
PVF - Resolver EDP (Particular)

Separación de Variables

Trans. de Fourier.

Transformada de Fourier

Integral de Fourier: Dada $f(x)$ que no necesariamente es periódica y que está definida en $-\infty < x < \infty$, encontrar una representación de f en términos de una integral.



¿ $f(x) = \int$?

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Suponemos $-L \leq x \leq L$ y encontramos su serie de Fourier en ese intervalo:

$$f(x) = \underbrace{\frac{1}{2L}}_{\frac{a_0}{2}} \int_{-L}^{L} f(\xi)\,d\xi + \sum_{n=1}^{\infty}\left[\underbrace{\left(\frac{1}{L}\int_{-L}^{L} f(\xi)\cos\frac{n\pi\xi}{L}\,d\xi\right)}_{a_n}\cos\frac{n\pi x}{L} + \underbrace{\left(\frac{1}{L}\int_{-L}^{L} f(\xi)\,\mathrm{sen}\,\frac{n\pi\xi}{L}\,d\xi\right)}_{b_n}\mathrm{sen}\,\frac{n\pi x}{L}\right]$$

Ahora, queremos hacer $L \to \infty$

¿cómo? Realizamos las sustituciones;

$$\omega_n = \frac{n\pi}{L} \Rightarrow \underbrace{\omega_n - \omega_{n-1}}_{\Delta_n \omega} = \frac{n\pi}{L} - \frac{(n-1)\pi}{L} = \frac{\pi}{L} = \Delta\omega$$

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$$\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\,d\xi\right)\Delta w \longrightarrow 0$$

$$\Rightarrow f(x) = \cancel{\left(\frac{1}{2\pi}\int_{-L}^{L} f(\xi)\,d\xi\right)\Delta w} +$$

$$+ \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\left(\int_{-L}^{L} f(\xi)\cos w_n\xi\,d\xi\right)\cos w_n x + \left(\int_{-L}^{L} f(\xi)\,\mathrm{sen}\,w_n\xi\,d\xi\right)\mathrm{sen}\,w_n x\right]\Delta w$$

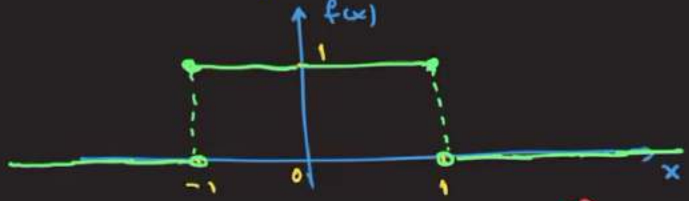
Hacemos $L \to \infty \Rightarrow \Delta w \to 0$

Integral de Fourier de $f(x)$

$$\Rightarrow f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[A_n \cos wx + B_n\,\mathrm{sen}\,wx\right]dw,$$

donde $A_n = \int_{-\infty}^{\infty} f(\xi)\cos(w\xi)\,d\xi$, $B_n = \int_{-\infty}^{\infty} f(\xi)\,\mathrm{sen}(w\xi)\,d\xi$

Ej: Sea $f(x) = \begin{cases} 1, & -1 \le x \le 1 \\ 0, & \text{en otro caso} \end{cases}$



$$\Rightarrow \quad f(x) = \frac{1}{\pi}\int_0^{\infty} \left[A_\omega \cos \omega x + \overbrace{B_\omega}^{0} \operatorname{sen} \omega x \right] d\omega$$

f es par

$$\Rightarrow \quad A_\omega = \int_{-\infty}^{\infty} f(\xi)\cos(\omega \xi)\, d\xi = \int_{-1}^{1} \cos(\omega\xi)\, d\xi$$

$$= 2\int^{1} \cos(\omega\xi)\, d\xi$$

$$= \frac{2}{\omega}\,\mathrm{sen}(\omega\xi)\Big|_0^1 = \frac{2}{\omega}\,\mathrm{sen}\,\omega$$

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^\infty \Big(\underbrace{\frac{2}{\omega}\,\mathrm{sen}\,\omega}_{A_\omega}\Big)\cos\omega x \; d\omega$$

$f(x)$

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \left[A_\omega \cos\omega x + B_\omega \operatorname{sen} \omega x\right] d\omega,$$

$$A_\omega = \int_{-\infty}^{\infty} f(t)\cos\omega t \, dt, \quad B_\omega = \int_{-\infty}^{\infty} f(t) \operatorname{sen}\omega t \, dt$$

Ej: Encuentre la integral de Fourier para

$$f(x) = \begin{cases} x, & -\pi \le x \le \pi \\ 0, & |x| > \pi \end{cases}$$

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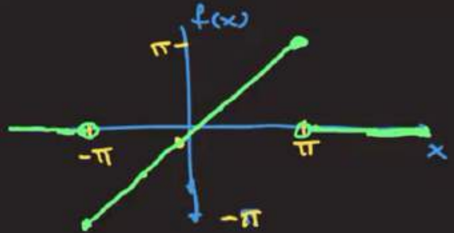


$$\Rightarrow f(x) = \frac{1}{\pi}\int_{0}^{\infty} \left[A_\omega \cos \omega x + B_\omega \,\mathrm{sen}\, \omega x\right] d\omega .$$

$$\Rightarrow A_\omega = \int_{-\infty}^{\infty} f(x) \cos \omega x \, dx = \int_{-\pi}^{\pi} \underbrace{\overset{\text{impar}}{x} \underbrace{\cos \omega x}_{\text{par}}}_{\text{impar}} dx = 0$$

$$B_\omega = \int_{-\infty}^{\infty} f(x) \,\mathrm{sen}\, \omega x \, dx = \int_{-\pi}^{\pi} \underbrace{x \,\mathrm{sen}\, \omega x}_{\text{par}} dx$$

$$= 2\int_0^{\pi} x\,\text{sen}\,\omega x\,dx = 2\left[-\frac{x}{\omega}\cos\omega x\Big|_0^{\pi} + \frac{1}{\omega}\int_0^{\pi}\cos\omega x\,dx\right]$$

$$\left.\begin{array}{ll} u = x & v = -\frac{1}{\omega}\cos\omega x \\ du = dx & dv = \text{sen}\,\omega x\,dx \end{array}\right| = 2\left[-\frac{\pi}{\omega}\cos\omega\pi + \frac{1}{\omega^2}\,\text{sen}\,\omega x\Big|_0^{\pi}\right]$$

$$= 2\left[-\frac{\pi}{\omega}\cos\omega\pi + \frac{1}{\omega^2}\left(\text{sen}\,\omega\pi - \overset{\to 0}{\cancel{\text{sen}\,0}}\right)\right]$$

$$\Rightarrow\ B_\omega = 2\left[\frac{1}{\omega^2}\,\text{sen}\,\omega\pi - \frac{\pi}{\omega}\cos\omega\pi\right]$$

$$\Rightarrow\ f(x) = \frac{2}{\pi}\int_0^{\infty}\left[\frac{1}{\omega^2}\,\text{sen}\,\omega\pi - \frac{\pi}{\omega}\cos\omega\pi\right]\text{sen}\,\omega x\,d\omega$$

<u>Nota</u>: 1) Si $f(x)$ es una función impar en $-\infty < x < \infty$, entonces

$$f(x) = \frac{1}{\pi}\int_0^\infty B_w \operatorname{sen} wx \, dw, \quad \text{donde}$$

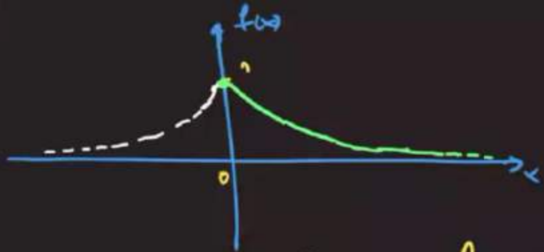
$$B_w = 2\int_0^\infty f(x) \operatorname{sen} wx \, dx$$

2) Si $f(x)$ es una función par en $-\infty < x < \infty$,

entonces $$f(x) = \frac{1}{\pi}\int_0^\infty A_w \cos wx \, dw, \quad \text{donde}$$

$$A_w = 2\int_0^\infty f(x) \cos wx \, dx$$

Ej: Sea $f(x) = e^{-kx}$, $x \geq 0$, $k \in \mathbb{R}^+$



$\Rightarrow$ Seleccionamos el supuesto que f es una función par en $(-\infty, \infty)$.

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^\infty A_\omega \cos\omega x \, d\omega, \quad \text{donde}$$

$$A_\omega = 2\int_0^\infty f(x)\cos\omega x \, dx = 2\int_0^\infty e^{-kx}\cos\omega x \, dx$$

$$\Rightarrow \int_0^\infty e^{-kx}\cos wx\,dx = \boxed{\overbrace{\frac{e^{-kx}}{w}\sin wx\Big|_0^\infty}^{0}} + \frac{k}{w}\int_0^\infty e^{-kx}\sin wx\,dx$$

$u = e^{-kx}$ $\quad$ $v = \frac{1}{w}\sin wx$

$du = -k e^{-kx}\,dx$ $\quad$ $dv = \cos wx\,dx$

$u = e^{-kx}$ $\quad$ $v = -\frac{1}{w}\cos wx$

$du = -k e^{-kx}\,dx$ $\quad$ $dv = \sin wx\,dx$

$$= \frac{k}{w}\left[\left[-\frac{e^{-kx}}{w}\cos wx\right]_0^\infty - \frac{k}{w}\int_0^\infty e^{-kx}\cos wx\,dx\right]$$

$$= \underbrace{\frac{k}{w}\left[-\frac{e^{0}}{w}\cos 0\right]} - \frac{k^2}{w^2}\int_0^\infty e^{-kx}\cos wx\,dx$$

$$\Rightarrow \int_0^\infty e^{-kx}\cos\omega x\,dx + \frac{k^2}{\omega^2}\int_0^\infty e^{-kx}\cos\omega x\,dx = \frac{k}{\omega^2}$$

$$\Rightarrow \left(1+\frac{k^2}{\omega^2}\right)\int_0^\infty e^{-kx}\cos\omega x\,dx = \frac{k}{\omega^2}$$

$$\Rightarrow \frac{(\omega^2+k^2)}{\cancel{\omega^2}}\int_0^\infty e^{-kx}\cos\omega x\,dx = \frac{k}{\cancel{\omega^2}}$$

$$\Rightarrow \int_0^\infty e^{-kx}\cos\omega x\,dx = \frac{k}{\omega^2+k^2}$$

$$\Rightarrow A_\omega = \frac{2k}{\omega^2+k^2} \Rightarrow \boxed{f(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{k}{\omega^2+k^2}\right)\cos\omega x\,d\omega}$$

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Nota ① sabemos que, si $-\infty < x < \infty$, entonces

$$f(x) = \frac{1}{\pi}\int_0^{\infty}[A_w \cos wx + B_w \operatorname{sen} wx]\,dw$$

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^{\infty}\Big[\Big(\int_{-\infty}^{\infty} f(t)\cos wt\,dt\Big)\cos wx + \Big(\int_{-\infty}^{\infty} f(t)\operatorname{sen} wt\,dt\Big)\cdot \operatorname{sen} wx\Big]\,dw$$

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\,[\cos wt \cos wx + \operatorname{sen} wt \operatorname{sen} wx]\,dt\,dw$$

$$\Rightarrow \boxed{f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos w(t-x)\,dt\,dw}$$

② Recordemos:

$$\cos\omega(t-x) = \frac{e^{i\omega(t-x)} + e^{-i\omega(t-x)}}{2}$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\left[e^{i\omega(t-x)} + e^{-i\omega(t-x)}\right]dt\,d\omega$$

$$= \underbrace{\frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{i\omega(t-x)}\,dt\,d\omega}_{A} + \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{-i\omega(t-x)}\,dt\,d\omega$$

Ⓐ Hacemos: $v = -\omega \Leftrightarrow \omega = -v \Rightarrow d\omega = -dv$

$$\Rightarrow \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{-i\omega(t-x)}\,dt\,d\omega =$$

$$= -\frac{1}{2\pi}\int_{0}^{-\infty}\int_{-\infty}^{\infty} f(t)\, e^{-iv(t-x)}\, dt\, dv$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0}\int_{-\infty}^{\infty} f(t)\, e^{-iv(t-x)}\, dt\, dv$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0}\int_{-\infty}^{\infty} f(t)\cdot e^{-i\omega(t-x)}\, dt\, d\omega$$

Retornando:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{0}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega(t-x)}\, dt\, d\omega + \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega(t-x)}\, dt\, d\omega$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega(t-x)}\, dt\, d\omega$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\cdot e^{+i\omega x}\, dt\, d\omega$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt\right] e^{i\omega x}\, d\omega$$

La función $\hat{f}(\omega) = \mathcal{F}[f] := \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$

se llama <u>transformada de Fourier de f</u>

además,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwx}\, dw$$

es la transformada inversa de $\hat{f}(w)$
(de Fourier)

¿Para qué la Transformada de Fourier?

- Resolver ecs. Dif. parciales.

Dada

$$\begin{tikzcd}
f(x) \arrow[r, "\mathcal{F}"] \arrow[d, "\text{“Cálculo”}"'] & \hat{f}(w) \arrow[d, "\text{álgebra}"] \\
f(x) & \hat{f}(w) \arrow[l, "\mathcal{F}^{-1}"] \text{ (cambio de forma)}
\end{tikzcd}$$

Ej: calcule la transformada de Fourier de la función $f(t) = e^{-5t}u(t)$, $-\infty < t < \infty$

<u>Nota</u>:

Entonces,

$$f(t) = \begin{cases} e^{-5t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\Rightarrow \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt =$$

$$= \int_0^\infty e^{-5t} \cdot e^{-i\omega t}\, dt = \int_0^\infty e^{-(5+i\omega)t}\, dt$$

$$= -\frac{1}{5+i\omega}\, e^{-(5+i\omega)t} \Big|_0^a$$

$\boxed{a \to \infty}$

$$= -\frac{1}{5+i\omega}\left[\overset{\to 0}{e^{-(5+i\omega)a}} - \overset{\to 1}{e^{0}} \right]$$

$$e^{-5a} \cdot e^{-i\omega a} = e^{-5a}\left[\cos \omega a - i \sin \omega a\right]$$

$a \to \infty$

$$\Rightarrow \boxed{\mathfrak{F}[f(t)] = \frac{1}{5+i\omega}}$$

Parcial 3 : 7 de Mayo → - Integral de Fourier
- Transf de Fourier

Ej: Sea $f(t) = \begin{cases} e^{-ct}, & t \geq 0 \\ 0, & t < 0 \end{cases}$, $c > 0$

$$\Rightarrow \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \int_{0}^{\infty} e^{-ct} e^{-i\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(i\omega + c)t}\, dt = \lim_{a \to \infty} \int_{0}^{a} e^{-(i\omega + c)t}\, dt$$

$$= \lim_{a \to \infty} -\frac{1}{i\omega + c}\, e^{-(i\omega + c)t} \Big|_{0}^{a}$$

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Parcial 3 : 7 de Mayo → - Integral de Fourier
- Transf

Ej: Sea $f(t) = \begin{cases} e^{-ct}, & t \geq 0 \\ 0, & t < 0 \end{cases}$, $c > 0$

$$\Rightarrow \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = \int_{0}^{\infty} e^{-ct} e^{-i\omega t} dt$$

$$= \int_{0}^{\infty} e^{-(i\omega + c)t} dt = \lim_{a\to\infty} \int_{0}^{a} e^{-(i\omega + c)t} dt$$

$$= \lim_{a\to\infty} - \frac{1}{i\omega + c} e^{-(i\omega+c)t} \Big|_{0}^{a} =$$

$$= \lim_{a\to\infty} -\frac{1}{i\omega+c}\left[\overset{0}{e^{-(i\omega+c)a}} - \overset{1}{e^{-(i\omega+c)\cdot 0}}\right]$$

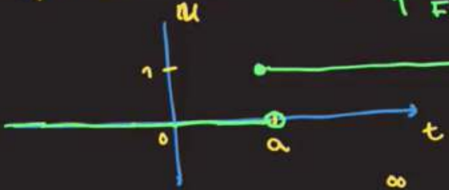
$$\left\lceil \lim_{a\to\infty} e^{-(i\omega+c)a} = \lim_{a\to\infty} \underset{*}{e^{-i\omega a}}\cdot e^{-ca} \to 0\right.$$

$$* \lim_{a\to\infty} e^{-ca}\cdot\left[\cos\omega a - i\,\mathrm{sen}\,\omega a\right] = 0 \quad \rfloor$$

$$= -\frac{1}{i\omega+c}(-1) = \frac{1}{i\omega+c}$$

Ej: Calcule $\mathcal{F}[u(t-a)]$

↑ Función de Heaviside



$$\Rightarrow \mathcal{F}[u(t-a)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \int_{a}^{\infty} e^{-i\omega t}\, dt$$

$$= \lim_{b\to\infty} \int_{a}^{b} e^{-i\omega t}\, dt = \underset{b\to\infty}{-\frac{1}{i\omega}\, e^{-i\omega t}}\Big|_{a}^{b} =$$

$$= -\frac{1}{i\omega}\Big[\underset{b\to\infty}{e^{-i\omega b}} - e^{-i\omega a}\Big]$$

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$$* \lim_{b\to\infty} e^{-i\omega b} = \lim_{b\to\infty} \left[\cos(\omega b) - i \,\mathrm{sen}(\omega b)\right],$$

el cual <u>NO</u> existe.

$\Rightarrow \mathcal{F}[u(t-a)]$ no existe.

Ej: Sea $f(t) = \begin{cases} 1-|t| & , -1 \le t \le 1 \\ 0 & , |t| > 1 \end{cases}$

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$$\Rightarrow \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\cdot e^{-i\omega t}\,dt =$$

$$= \int_{-\infty}^{\infty} f(t)\left[\cos\omega t - i\,\text{sen}\,\omega t\right]dt$$

$$= \int_{-\infty}^{\infty} f(t)\cos\omega t\,dt - i\underbrace{\int_{-\infty}^{\infty} \underbrace{\underbrace{f(t)}_{\text{par}}\;\underbrace{\text{sen}\,\omega t}_{\text{impar}}}_{\text{impar}}\,dt}_{0}$$

$$= 2\int_0^1 (-t+1)\cos\omega t\,dt$$

$$= -2\int_0^1 t\cos\omega t\,dt + 2\int_0^1 \cos\omega t\,dt$$

(A) $$\int_0^1 t\cos\omega t\,dt = \frac{t}{\omega}\,\mathrm{sen}\,\omega t\Big|_0^1 - \frac{1}{\omega}\int_0^1 \mathrm{sen}\,\omega t\,dt$$

$$\boxed{\begin{array}{ll} u = t & v = \frac{1}{\omega}\,\mathrm{sen}\,\omega t \\ du = dt & dv = \cos\omega t\,dt \end{array}}$$

$$= \frac{1}{\omega}\,\mathrm{sen}\,\omega - 0 + \frac{1}{\omega^2}\cos\omega t\Big|_0^1$$

$$= \frac{1}{\omega}\,\mathrm{sen}\,\omega + \frac{1}{\omega}(\cos\omega - 1)$$

$$\Rightarrow \mathcal{F}[f(t)] = -\frac{2}{\omega}\,\mathrm{sen}\,\omega - \frac{2}{\omega}(\cos\omega - 1) + \frac{2}{\omega^2}\,\mathrm{sen}\,\omega t\Big|_0^1$$

$$= -\cancel{\frac{2}{\omega}\,\mathrm{sen}\,\omega} - \frac{2}{\omega}(\cos\omega - 1) + \cancel{\frac{2}{\omega^2}}\,\mathrm{sen}\,\omega$$

$$= \frac{2}{\omega^2}(1 - \cos\omega)$$

Prop. $\mathcal{F}$ es una transformación lineal; i.e.

$$\mathcal{F}[f(t)+g(t)] = \mathcal{F}[f(t)] + \mathcal{F}[g(t)]$$

$$\mathcal{F}[k f(t)] = k\,\mathcal{F}[f(t)]$$

Nota: Si $\mathcal{F}[f(t)] = \hat{f}(\omega)$. Entonces

$$\mathcal{F}[f(t-t_0)] = \int_{-\infty}^{\infty} f(\underbrace{t-t_0}_{u})\, e^{-i\omega t}\, dt$$

$u = t - t_0$

$t = u + t_0$

$dt = du$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-i\omega(u+t_0)}\, dt$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-i\omega u} \cdot e^{-i\omega t_0}\, dt$$

$$= e^{-i\omega t_0}\int_{-\infty}^{\infty} f(u)\, e^{-i\omega u}\, du = e^{-i\omega t_0}\cdot \hat{f}(\omega)$$

$$\Rightarrow \boxed{\mathcal{F}[f(t-t_0)] = e^{-i\omega t_0}\,\hat{f}(\omega)} \leftarrow \text{PTT}$$

Prop: Si $\mathcal{F}[f(t)] = \hat{f}(\omega)$, entonces

$$\mathcal{F}[e^{i\omega_0 t}\cdot f(t)] = \hat{f}(\omega-\omega_0) \leftarrow \text{STT}$$

Prop: Si $\mathcal{F}[f(t)] = \hat{f}(\omega)$, entonces

$$\mathcal{F}[f(-t)] = \hat{f}(-\omega)$$

② Si c es un número real, $c \neq 0$, y si $\mathcal{F}[f(t)] = \hat{f}(\omega)$, entonces:

$$\mathcal{F}[f(ct)] = \frac{1}{|c|}\hat{f}\left(\frac{\omega}{c}\right)$$

Prop (*) Si: $\mathcal{F}[f(t)] = \hat{f}(\omega)$, entonces:

$$\mathcal{F}[f(t)\cdot\cos(\omega_0 t)] = \frac{1}{2}\left[\hat{f}(\omega+\omega_0) + \hat{f}(\omega-\omega_0)\right]$$

Dem: $\mathcal{F}[f(t)\cdot\cos(\omega_0 t)] = \int_{-\infty}^{\infty} f(t)\cdot\cos(\omega_0 t)\cdot e^{-i\omega t}\,dt$

$$\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}$$

$$= \int_{-\infty}^{\infty} f(t) \cdot \left[\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} \right] e^{-i\omega t} dt$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} f(t)\, e^{i\omega_0 t} \cdot e^{-i\omega t} dt + \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega_0 t} \cdot e^{-i\omega t} dt \right]$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} f(t)\, e^{-i(\omega-\omega_0)t} dt + \int_{-\infty}^{\infty} f(t)\, e^{-i(\omega+\omega_0)t} dt \right]$$

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$$\mathcal{F}[f] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \hat{f}(\omega)$$

<u>Prop</u>: Si $\mathcal{F}[f] = \hat{f}(\omega)$, entonces:

$$\mathcal{F}[f'(t)] = i\omega\, \hat{f}(\omega)$$

<u>Nota</u>: $\mathcal{F}[f^{(k)}(t)] = (i\omega)^k\, \hat{f}(\omega)$

$$\mathcal{L}[f'(t)] = s\,\mathcal{L}[f] - \underline{y(0)}$$

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$$\mathcal{F}[f] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \hat{f}(\omega)$$

Prop: Si $\mathcal{F}[f] = \hat{f}(\omega)$, entonces:

$$\mathcal{F}[f'(t)] = i\omega\, \hat{f}(\omega)$$

Nota: $\mathcal{F}[f^{(k)}(t)] = (i\omega)^k\, \hat{f}(\omega)$

Ej: Resuelva $y' - 4y = e^{-4t}u(t)$

$$\mathcal{F}[y' - 4y] = \mathcal{F}[e^{-4t}u(t)]$$

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$$\Rightarrow \mathcal{F}[y'] - 4\,\mathcal{F}[y] = \frac{1}{i\omega+4}$$

$$\Rightarrow i\omega\,\mathcal{F}[y] - 4\,\mathcal{F}[y] = \frac{1}{i\omega+4}$$

$$\Rightarrow \mathcal{F}[y]\cdot(i\omega-4) = \frac{1}{i\omega+4}$$

$$\Rightarrow \mathcal{F}[y] = \frac{1}{(i\omega+4)(i\omega-4)} = \frac{1}{-\omega^2-16}$$

$$\Rightarrow \mathcal{F}[y] = -\frac{1}{\omega^2+16}$$

$$\Rightarrow y(t) = \mathcal{F}^{-1}\left(-\frac{1}{\omega^2+16}\right)$$

Transformada de Fourier:

Funciones (señales)

- Descomponer en elementos más simples | Ensamblar funciones

↑ Análisis de Fourier

•• Funciones
- Periódicas ← Series de F
- no periódicas ← Transf. de F

••• Dada $f(t) \Rightarrow \mathcal{F}[f] = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$

$$\Rightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega = \mathcal{F}^{-1}[\hat{f}(\omega)]$$

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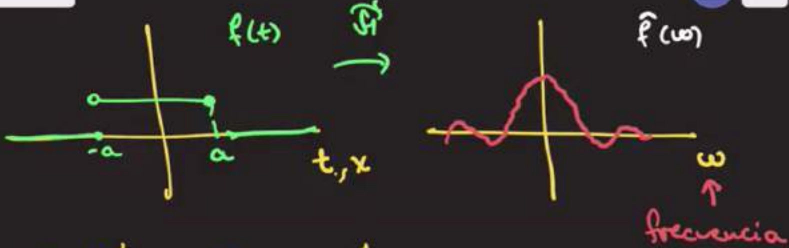


Ej's: ① $\mathcal{F}[e^{-at}u(t)] = \frac{1}{a+i\omega}$

② $f(t) = \begin{cases} K, & -a \le t \le a \\ 0, & |t| > a \end{cases} \Rightarrow \mathcal{F}[f] = \frac{2K \operatorname{sen} a\omega}{\omega}$

③ $f(t) = \begin{cases} 1-|t|, & -1 \le t \le 1 \\ 0, & |t| > 1 \end{cases} \Rightarrow \mathcal{F}[f] = \frac{2}{\omega^2}(1-\cos\omega)$

Prop: ① $\mathcal{F}[\alpha f + \beta g] = \alpha \mathcal{F}[f] + \beta \mathcal{F}[g], \quad \alpha, \beta \in \mathbb{R}$

② $\mathcal{F}[f(t-t_0)] = e^{-i\omega t_0} \mathcal{F}[f(t)]$

③ $\mathcal{F}[e^{-i\omega_0 t} f(t)] = \hat{f}(\omega - \omega_0)$

④ $\mathcal{F}[f(ct)] = \frac{1}{|c|}\hat{f}\left(\frac{\omega}{c}\right).$

⑤ $\mathcal{F}[f(t)\cos\omega_0 t] = \frac{1}{2}\left[\hat{f}(\omega+\omega_0) + \hat{f}(\omega-\omega_0)\right]$

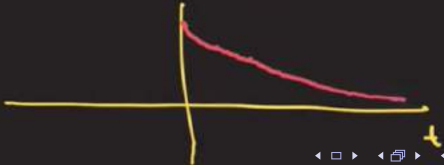
$\mathcal{F}[f(t)\,\text{sen}\,\omega_0 t] = \frac{1}{2}\left[\hat{f}(\omega+\omega_0) - \hat{f}(\omega-\omega_0)\right]$

⑥ $\mathcal{F}[f^{(k)}(t)] = (i\omega)^k\hat{f}(\omega).$

Ej: $y' - 4y = e^{-4t}u(t) \Rightarrow \hat{f}(\omega) = -\frac{1}{\omega^2+16}$

$\Rightarrow y = \mathcal{F}^{-1}\left[-\frac{1}{\omega^2+16}\right]$

Ej: Encuentre $\mathcal{F}[e^{-c|t|}]$, $c>0$



$$\Rightarrow f(t) = \begin{cases} e^{-ct}, & t \geq 0 \\ e^{ct} & t < 0 \end{cases}$$

$\Rightarrow$ Como f es par, entonces:

$$\Rightarrow \hat{f}(w) = \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt$$

$$= \int_{-\infty}^{\infty} f(t) \left[\cos wt - i \,\text{sen}\, wt\right] dt$$

$$= \int_{-\infty}^{\infty} \underbrace{\underbrace{f(t)}_{\text{par}}\ \underbrace{\cos wt}_{\text{par}}}_{\text{par}}\, dt - i \int_{-\infty}^{\infty} \underbrace{\underbrace{f(t)}_{\text{par}}\ \underbrace{\text{sen}\, wt}_{\text{impar}}}_{\text{impar}}\, dt$$

0

$$= 2\int_0^\infty f(t)\cos\omega t\,dt = 2\int_0^\infty e^{-ct}\cos\omega t\,dt$$

$$\Rightarrow \hat{f}(\omega) = \frac{2c}{\omega^2+c^2}$$

$$\Rightarrow \mathcal{F}^{-1}\left[\frac{2c}{\omega^2+c^2}\right] = e^{-c|t|} \quad (**)$$

Regresando a $*$ $\quad y = \mathcal{F}^{-1}\left[-\frac{1}{\omega^2+\underset{4^2}{16}}\right]$

$$\Rightarrow y(t) = \mathcal{F}^{-1}\left[-\frac{1}{\omega^2+16}\right] = -\mathcal{F}^{-1}\left[\frac{1}{\omega^2+\underset{c=4}{4^2}}\right]$$

$$= -\frac{1}{8}\mathcal{F}^{-1}\left[\frac{8}{\omega^2+4^2}\right] = -\frac{1}{8}e^{-4|t|}$$

N.B. Dada $f(t)$, sea $\mathcal{F}[f(t)] = \hat{f}(\omega)$

$$\Rightarrow \mathcal{F}^{-1}[\hat{f}(\omega)] = \frac{f(t^+) + f(t^-)}{2}$$

En los puntos de continuidad

$$\mathcal{F}^{-1}[\hat{f}(\omega)] = f(t)$$

Ej: i) Evalúe $\displaystyle\int_{-\infty}^{\infty} \frac{\operatorname{sen} a\omega \cos \omega t}{\omega}\, d\omega$

ii) Calcule $\displaystyle\int_{0}^{\infty} \frac{\operatorname{sen} u}{u}\, du$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega$$

Sol: ① Nótese que:

$$\mathcal{F}^{-1}\left[\frac{2\,\text{sen}\,a\omega}{\omega}\right] = \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty}\left(\frac{\cancel{2}\,\text{sen}\,a\omega}{\omega}\right)e^{i\omega t}\,d\omega$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\text{sen}\,a\omega}{\omega}\left[\cos\omega t + i\,\text{sen}\,\omega t\right]d\omega$$

$$= \frac{1}{\pi}\boxed{\int_{-\infty}^{\infty}\frac{\text{sen}\,a\omega\,\cos\omega t}{\omega}\,d\omega} + i\,\frac{1}{\pi}\int_{-\infty}^{\infty}\underbrace{\frac{\overset{i}{\text{sen}\,\omega}\cdot\overset{i}{\text{sen}\,\omega t}}{\underset{i}{\omega}}}_{\text{impar}}\,d\omega \to 0$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\text{sen}\,a\omega\cos\omega t}{\omega}\,d\omega = \begin{cases} 1, & -a<t<a \\ 1/2, & t=\pm a \\ 0, & |t|>a \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\operatorname{sen} a\omega \cos \omega t}{\omega}\, d\omega = \begin{cases} \pi, & -\overset{-1}{a} < \overset{0}{t} < \overset{1}{a} \\ \pi/2, & t = \pm a \\ 0, & |t| > a \end{cases}$$

ii) Hagamos $a = 1,\ t = 0.$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\operatorname{sen}\omega \cos 0}{\omega}\, d\omega = \int_{-\infty}^{\infty} \frac{\overset{\text{impar}}{\operatorname{sen}\omega}}{\underset{\text{impar}}{\omega}}\, d\omega =$$

$$= 2\int_{0}^{\infty} \frac{\operatorname{sen}\omega}{\omega}\, d\omega = \pi$$

$$\Rightarrow \int_{0}^{\infty} \frac{\operatorname{sen}\omega}{\omega}\, d\omega = \frac{\pi}{2}$$

Prop: Si $\mathcal{F}[f(t)] = \hat{f}(\omega)$, entonces,

$$\frac{d}{d\omega}\hat{f}(\omega) = -i\,\underbrace{\mathcal{F}[t f(t)]}$$

Dem: $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$

$$\Rightarrow \frac{d}{d\omega}\hat{f}(\omega) = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\cdot\left[\frac{\partial}{\partial\omega} e^{-i\omega t}\right] dt$$

$$= \int_{-\infty}^{\infty} f(t)\cdot(-it)\, e^{-i\omega t}\, dt =$$

$$= -i \int_{-\infty}^{\infty} (t f(t)) \cdot e^{-i\omega t}\, dt$$

$$= -i\, \mathcal{F}[t f(t)].$$

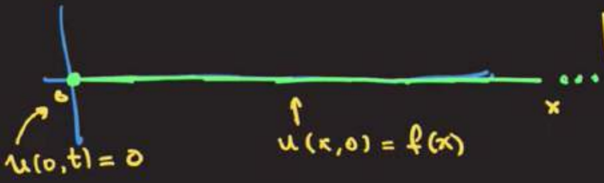
En general, $\dfrac{d^k}{d\omega^k} \hat{f}(\omega) = (-i)^k \underbrace{\mathcal{F}[t^k f(t)]}$

Ej: Resuelva $\kappa \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$, $\boxed{x > 0}$, $t > 0$

$u(x,0) = f(x)$

$u(0,t) = 0$

$\underbrace{|u(x,t)| < M}$



La temperatura $\forall x$, $\forall t$, es finita.

Por separación de variables : $u(x,t) = X\,T$

Sustituyendo:

$$k X'' T = T' X$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{k}\frac{T'}{T} = -\lambda^2$$

$\Rightarrow$ ① $X'' + \lambda^2 X = 0$ y ② $T' + k\lambda^2 T = 0$

① $X'' + \lambda^2 X = 0, \quad X(0) = 0$

$\lambda > 0$

$$\Rightarrow X(x) = c_1 \cos \lambda x + c_2 \operatorname{sen} \lambda x$$

$$\Rightarrow X(0) = c_1 = 0 \Rightarrow X_\lambda(x) = \operatorname{sen} \lambda x$$

$$X_n(x) = \operatorname{sen} \frac{n\pi x}{a}$$

Por separación de variables : $u(x,t) = X\,T$

Sustituyendo:

$$k X'' T = T' X$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{k}\frac{T'}{T} = -\lambda^2$$

$\Rightarrow$ ① $X'' + \lambda^2 X = 0$ y ② $T' + k\lambda^2 T = 0$

① $X'' + \lambda^2 X = 0$, $X(0) = 0$

$\lambda > 0$

$$\Rightarrow X(x) = c_1 \cos\lambda x + c_2 \operatorname{sen}\lambda x$$

$$\Rightarrow X(0) = c_1 = 0 \Rightarrow X_\lambda(x) = \operatorname{sen}\lambda x$$

② $T' + \lambda^2 k T = 0 \Rightarrow T(t) = e^{-\lambda^2 k t}$

$$\Rightarrow u_\lambda(x,t) = e^{-\lambda^2 \kappa t} \cdot \operatorname{sen} \lambda x$$

Por superposición (sobre $\lambda > 0$)

$$\Rightarrow u(x,t) = \int_0^\infty A_\lambda e^{-\lambda^2 \kappa t} \operatorname{sen} \lambda x \, d\lambda$$

$A(\lambda)$

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$$\Rightarrow u_\lambda(x,t) = e^{-\lambda^2 k t} \cdot \operatorname{sen} \lambda x$$

Por superposición (sobre $\lambda > 0$)

$$\Rightarrow u(x,t) = \int_0^\infty \underbrace{A_\lambda}_{A(\lambda)} e^{-\lambda^2 k t} \operatorname{sen} \lambda x \, d\lambda$$

<u>Def</u> ① Si $f(t)$ es una función impar, entonces la <u>transformada de Fourier en senos</u> de $f(t)$ es:

$$\mathcal{F}_s^+ [f(t)] := \int_0^\infty f(t) \operatorname{sen} \omega t \, dt$$

$$\Rightarrow \quad f(t) = \frac{2}{\pi}\int_0^{\infty} \left[\mathcal{F}_s(f) \right] \operatorname{sen} \omega t \, d\omega$$

(2) Si $f(t)$ es una función par, entonces la transformada de Fourier en cosenos de $f(t)$ es:

$$\mathcal{F}_c\left[f(t)\right] = \int_0^{\infty} f(t) \cos \omega t \, dt$$

$$\Rightarrow \quad f(t) = \frac{2}{\pi}\int_0^{\infty} \mathcal{F}_c\left[f\right] \cos \omega t \, d\omega$$

$\Rightarrow$ Aplicando la condición $u(x,0) = f(x)$

$$\Rightarrow u(x,0) = f(x) = \int_0^{\infty} A(\lambda) \operatorname{sen} \lambda x \, d\lambda$$

$$\Rightarrow A(\lambda) = \frac{2}{\pi} \int_0^{\infty} f(x) \operatorname{sen} \lambda x \, dx$$

Transformada inversa en senos

$\Rightarrow$ Aplicando la condición $u(x,0) = f(x)$

$$\Rightarrow u(x,0) = f(x) = \int_0^\infty A(\lambda) \operatorname{sen} \lambda x \, d\lambda$$

$$\Rightarrow A(\lambda) = \frac{2}{\pi}\int_0^\infty f(\overset{u}{\cancel{x}}) \operatorname{sen} \lambda \overset{u}{\cancel{x}} \, d\overset{u}{\cancel{x}}$$

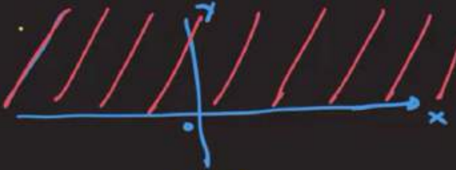
Transformada inversa en senos

$$\Rightarrow u(x,t) = \frac{2}{\pi}\int_0^\infty \left[\int_0^\infty f(u) \operatorname{sen} \lambda u \, du\right] e^{-\lambda^2 k} \operatorname{sen} \lambda x \, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \int_0^\infty f(u)(\operatorname{sen} \lambda u)\, e^{-\lambda^2 k} \operatorname{sen} \lambda x \, du \, d\lambda$$

Ej: Resuelva $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, sujeta a:

$u(x,0) = f(x)$, $y>0$, $|u(x,y)| < M$, $-\infty < x < \infty$.



Supongamos que $u(x,y) = XY$

$\Rightarrow$ Sustituyendo: $X''Y + XY'' = 0$

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$$

1. $X'' + \lambda^2 X = 0 \Rightarrow X_\lambda(x) = A_\lambda \cos \lambda x + B_\lambda \operatorname{sen} \lambda x$

$\boxed{\lambda > 0}$

2. $Y'' - \lambda^2 Y = 0 \Rightarrow Y_\lambda(y) = C_\lambda e^{\lambda y} + D_\lambda e^{-\lambda y}$

$$\Rightarrow u_\lambda(x,y) = X_\lambda(x) \cdot Y_\lambda(y)$$

$$\Rightarrow u_\lambda(x,y) = [A_\lambda \cos \lambda x + B_\lambda \operatorname{sen} \lambda x][C_\lambda e^{\lambda y} + D_\lambda e^{-\lambda y}]$$

Por $|u(x,y)| < M$

$$\Rightarrow C_\lambda = 0$$

$$\Rightarrow u_\lambda(x,y) = D_\lambda e^{-\lambda y} [A_\lambda \cos \lambda x + B_\lambda \operatorname{sen} \lambda x]$$

$$\Rightarrow u_\lambda(x,y) = e^{-\lambda y}\left[A_\lambda \cos \lambda x + B_\lambda \operatorname{sen} \lambda x\right]$$

Entonces, por superposición

$$\Rightarrow u(x,y) = \int_0^\infty e^{-\lambda y}\left[A_\lambda \cos \lambda x + B_\lambda \operatorname{sen} \lambda x\right] d\lambda$$

$$u(x,y) = \int_0^\infty e^{-\lambda y}\left[A(\lambda) \cos \lambda x + B(\lambda) \operatorname{sen} \lambda x\right] d\lambda$$

$$\Rightarrow u(x,0) = f(x) = \int_0^\infty \left[A(\lambda) \cos \lambda x + B(\lambda) \operatorname{sen} \lambda (x)\right] d\lambda$$

$$\Rightarrow f(x) = \int_0^\infty A(\lambda) \cos \lambda x \, d\lambda + \int_0^\infty B(\lambda) \operatorname{sen} \lambda (x) \, d\lambda$$

$$\Rightarrow u_\lambda(x,y) = e^{-\lambda y}\left[A_\lambda \cos \lambda x + B_\lambda \operatorname{sen} \lambda x\right]$$

Entonces, por superposición

$$\Rightarrow u(x,y) = \int_0^\infty e^{-\lambda y}\left[A_\lambda \cos \lambda x + B_\lambda \operatorname{sen} \lambda x\right] d\lambda$$

$$u(x,y) = \int_0^\infty e^{-\lambda y}\left[A(\lambda) \cos \lambda x + B(\lambda) \operatorname{sen} \lambda x\right] d\lambda$$

$$\Rightarrow u(x,0) = f(x) = \int_0^\infty \left[A(\lambda) \cos \lambda x + B(\lambda) \operatorname{sen} \lambda (x)\right] d\lambda$$

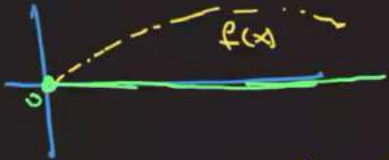
$$\Rightarrow f(x) = \int_0^\infty A(\lambda) \cos \lambda x \, d\lambda + \int_0^\infty B(\lambda) \operatorname{sen} \lambda(x) \, d\lambda$$

$$\left(\frac{e^{i\lambda x} + e^{-i\lambda x}}{2}\right) \qquad \left(\frac{e^{i\lambda x} - e^{-i\lambda x}}{2i}\right)$$

Ej. Resuelva $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 y}{\partial t^2}$, sujeta a

$y(0,t) = 0$; $y(x,0) = f(x)$, $\dfrac{\partial y}{\partial t}(x,0) = 0.$

$0 \le x < \infty$



$\Rightarrow$ Sea $y(x,t) = XT \Rightarrow$ sustituyendo:

$$X''T = \frac{1}{c^2} X T''$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -\omega^2$$

① $X'' + \omega^2 X = 0, \quad X(0) = 0$

$\Rightarrow X(x) = A\cos\omega x + B \,\text{sen}\, \omega x$

$X(0) = A = 0 \Rightarrow X_\omega(x) = B_\omega \,\text{sen}\, \omega x,$
$\omega > 0$

② $T'' + c^2\omega^2 T = 0, \quad T'(0) = 0$

$\Rightarrow T_\omega(t) = D_\omega \cos c\omega t + E_\omega \,\text{sen}\, c\omega t$

$\Rightarrow T_\omega'(t) = -c\omega D_\omega \,\text{sen}\, c\omega t + c\omega E_\omega \cos c\omega t$

$\Rightarrow T_\omega'(0) = c\omega E_\omega = 0 \Rightarrow E_\omega = 0, \ \forall \omega > 0$

$\Rightarrow T_\omega(t) = D_\omega \cos c\omega t$

⇒ por superposición:

$$y(x,t) = \int_0^\infty A_\omega \left(B_\omega \operatorname{sen} \omega x\right)\left(D_\omega \cos c\omega t\right) d\omega$$

$$\Rightarrow y(x,t) = \int_0^\infty A(\omega) \operatorname{sen} \omega x \cos c\omega t \, d\omega$$

⇒ Si aplicamos la condición $y(x,0) = f(x)$

$$\Rightarrow y(x,0) = f(x) = \int_0^\infty A(w) \operatorname{sen} wx \, dw$$

$$\Rightarrow A(\omega) = \frac{2}{\pi} \int_0^\infty f(u) \operatorname{sen} wu \, du$$

Nota: Considere $y = y(x,t)$; entonces:

$$\Rightarrow \mathcal{F}[y(x,t)] = \int_{-\infty}^{\infty} y(x,t)\, e^{-i\omega t}\, d\boxed{t}$$

$x^2 + t$

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$\mathcal{F}\left[\frac{\partial^2 y}{\partial x^2}\right] \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Solución de EDP por transformada de Fourier

- Recordemos: $\boxed{\mathcal{F}\left[f^{(k)}(t) \right] = (i\omega)^k \underbrace{\hat{f}(\omega)}_{\mathcal{F}[f(t)]}}$

- Convención: $\mathcal{F}\left[\dfrac{\partial^2 u}{\partial x^2} \right] = \displaystyle\int_{-\infty}^{\infty} \dfrac{\partial^2}{\partial x^2} u(x,t)\, e^{-i\omega x}\, dx$

$$= -\omega^2 \hat{u}(\omega,t)$$

$$\Rightarrow \mathcal{F}\left[\frac{\partial u}{\partial t} \right] = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} u(x,t)\, e^{-i\omega x}\, dx$$

$$= \frac{\partial}{\partial t} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx$$

$$= \frac{\partial}{\partial t} \hat{u}(\omega,t)$$

Ej: Resuelva: $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 y}{\partial t^2}$, sujeta a:

$$y(x,0) = f(x) \quad , \quad \frac{\partial y}{\partial t}(x,0) = g(x), \quad t>0,$$

$$-\infty < x < \infty.$$

Por transf de Fourier:

$$\mathcal{F}\left[\frac{\partial^2 y}{\partial x^2}\right] = \mathcal{F}\left[\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}\right]$$

$$\Rightarrow -w^2\,\hat{y}(\omega,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\hat{y}(\omega,t)$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\hat{y}(\omega,t) + \underbrace{c^2 w^2}\,\hat{y}(\omega,t) = 0 \leftarrow \text{¡1-variable! "}t\text{"}$$

$$\Rightarrow \hat{y}(w,t) = c_w \cos(wct) + d_w \operatorname{sen}(wct)$$

Aplicando las condiciones de frontera:

① $y(x,0) = f(x)$. $\Rightarrow \tilde{\mathcal{F}}[y(x,0)] = \tilde{\mathcal{F}}[f(x)]$

$$\Rightarrow \boxed{\hat{y}(w,0) = \hat{f}(w)}$$

② $\frac{\partial y}{\partial t}(x,0) = g(x) \Rightarrow \tilde{\mathcal{F}}\left[\frac{\partial}{\partial t} y(x,0)\right] = \tilde{\mathcal{F}}[g(x)]$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\hat{y}(w,0) = \hat{g}(w)}$$

$$\Rightarrow \hat{y}(w,0) = C_w = \hat{f}(w)$$

$$\frac{\partial \hat{y}}{\partial t}(w,t) = -wc \cdot C_w \operatorname{sen}(wct) + wc d_w \cos(wct)$$

$$\Rightarrow \frac{\partial}{\partial t}\hat{y}(w,0) = wc\, d_w = \hat{g}(w)$$

$$\Rightarrow d_w = \frac{\hat{g}(w)}{wc}$$

$$\Rightarrow \hat{y}(w,t) = \hat{f}(w)\cos(wct) + \frac{\hat{g}(w)}{wc}\,\text{sen}(wct)$$

$$\Rightarrow y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\hat{f}(w)\cos(wct) + \frac{\hat{g}(w)}{wc}\,\text{sen}(wct)\right]e^{iwx}\,dw$$

$$\mathcal{F}^{-1}[\hat{h}(w)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{h}(w)\,e^{iwx}\,dw$$

- ¿Qué sucede si $g(x) = 0$?

$$\Rightarrow y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\hat{f}(\omega)}_{\Phi} (\cos \omega c t)\, e^{i\omega x}\, d\omega$$

$$\Rightarrow y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi)\, e^{-i\omega\xi}\, d\xi \; e^{i\omega x}\, dx$$

$$\Rightarrow y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)\, e^{-i\omega(\xi - x)}\, d\xi\, dx$$

Ej: Resuelva: $\nabla^2 u = 0$, $-\infty < x < \infty$, $y > 0$,

$u(x,0) = f(x)$; $|u(x,y)| < M$

$-\infty < x < \infty$

Por transformada de Fourier:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 u}{\partial y^2}\right] = 0$$

$$\Rightarrow \quad -w^2 \hat{u}(w,y) + \frac{\partial^2}{\partial y^2}\hat{u}(w,y) = 0$$

$$\Rightarrow \quad \frac{\partial^2}{\partial y^2}\hat{u}(w,y) - w^2\hat{u}(w,y) = 0$$

$$\Rightarrow \quad \hat{u}(w,y) = a_w e^{wy} + b_w e^{-wy}$$

Si $y \to \infty \Rightarrow \begin{cases} w > 0 \Rightarrow a_w = 0 \\ w < 0 \Rightarrow b_w = 0 \end{cases}$

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$$\Rightarrow \hat{u}(\omega, y) = \begin{cases} b_\omega e^{-\omega y}, & \omega \geq 0 \\ a_\omega e^{\omega y}, & \omega < 0 \end{cases}$$

$$\Rightarrow \hat{u}(\omega, y) = c_\omega e^{-|\omega| y} \quad *$$

Aplicando la condición de frontera:

$$u(x,0) = f(x), \quad -\infty < x < \infty$$

$$\Rightarrow \mathcal{F}[u(x,0)] = \mathcal{F}[f(x)]$$

$$\Rightarrow \hat{u}(\omega, 0) = \hat{f}(\omega)$$

$$\Rightarrow \hat{u}(\omega, 0) = \hat{f}(\omega) = c_\omega \cdot \cancel{e^{0}}^{\,1} \Rightarrow c_\omega = \hat{f}(\omega)$$

$$\Rightarrow \quad \hat{u}(\omega, y) = \hat{f}(\omega)\, e^{-|\omega| y}$$

$\Rightarrow$ Aplicando transf. inversa:

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\hat{f}(\omega)\, e^{-|\omega| y}}\, e^{i\omega x}\, d\omega$$

Nota:

$$\mathfrak{F}_c\left[f''(x)\right] = -\omega^2 \hat{f}_c(\omega) - f'(0)$$

$$\mathfrak{F}_s\left[f''(x)\right] = -\omega^2 \hat{f}_s(\omega) + \omega f(0)$$

Ej: $c^2 \dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial^2 y}{\partial t^2}$, $\quad 0 < x < \infty$, $\quad t > 0$,

$y(0,t) = 0$, $t \geq 0$; $\quad y(x,0) = 0$; $\quad \dfrac{\partial y}{\partial x}(x,0) = g(x)$

$0 < x < \infty$

Por transf. de Fourier en $\boxed{\text{senos}}$

$$c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \mathcal{F}_s\left[c^2\, \partial^2 y / \partial x^2\right] = \mathcal{F}_s\left[\partial^2 y / \partial t^2\right]$$

$$\Rightarrow c^2\left[-w^2 \hat{y}_s(w,t) + w\, \cancel{y(0,t)}^{\,0}\right] = \frac{\partial^2}{\partial t}\hat{y}_s(w,t)$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\hat{y}_s(w,t) + c^2 w^2 \hat{y}_s(w,t) = 0$$

$$\Rightarrow \hat{y}_s(w,t) = a_w \cos(wct) + b_w \operatorname{sen}(wct)$$

Aplicando condici

$$y(x,0) = 0 \Rightarrow \hat{\underset{\sim}{y}}(w,0) = 0$$

$$\frac{\partial}{\partial t} y(x,0) = g(x) \Rightarrow \frac{\partial}{\partial t} \hat{\underset{\sim}{y}}(w,0) = \hat{\underset{\sim}{g}}(w)$$

$$\Rightarrow \hat{\underset{\sim}{y}}(w,0) = a_w = 0$$

$$\Rightarrow \hat{\underset{\sim}{y}}(w,t) = b_w \operatorname{sen}(wct)$$

$$\Rightarrow \frac{\partial}{\partial t} \hat{\underset{\sim}{y}}(w,t) = wc b_w \cos(wct)$$

$$\Rightarrow \frac{\partial}{\partial t} \hat{\underset{\sim}{y}}(w,0) = wc b_w = \hat{\underset{\sim}{g}}(w)$$

$$\Rightarrow b_w = \frac{\hat{\underset{\sim}{g}}(w)}{wc}$$

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$$\Rightarrow \hat{y}(\omega, t) = \frac{\hat{g}_x(\omega)}{\omega c} \operatorname{sen}(\omega c t)$$

$$\Rightarrow y(x, t) = \frac{2}{\pi} \int_0^{\infty} \frac{\hat{g}_x(\omega)}{\omega c} \operatorname{sen}(\omega c t)\, e^{i \omega x}\, d\omega$$

Temas adicionales TDF

(1) ¿Y el cálculo de las $\mathcal{F}^{-1}$?

Def: Sean $f(x)$ y $g(x)$ funciones definidas en $(-\infty, \infty)$. Entonces, la convolución de f con g se define:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-u)\, g(u)\, du$$

τ

Notas: ① $(f*g)(x) = \int_{-\infty}^{\infty} f(x-u)\, g(u)\, du$

$v = x - u$

$\Rightarrow u = x - v$

$$= -\int_{\infty}^{-\infty} f(v)\, g(x-v)\, dv = \int_{-\infty}^{\infty} g(x-v)\, f(v)\, dv$$

$$= (g * f)(x) \Rightarrow$$ La convolución es una operación conmutativa.

② $$\mathcal{F}[f * g] := \int_{-\infty}^{\infty} [f*g]\, e^{-i\omega x}\, dx =$$

$$= \int_{-\infty}^{\infty} \left[\int_{u=-\infty}^{u=\infty} f(x-u)\, g(u)\, du \right] e^{-i\omega x}\, dx$$

$$= \int_{u=-\infty}^{u=\infty} \int_{-\infty}^{\infty} f(x-u) \boxed{g(u)}\, e^{-i\omega x}\, dx\, du$$

$$= \int_{u=-\infty}^{\infty} g(u) \left[\int_{-\infty}^{\infty} f(x-u)\, e^{-i\omega x}\, dx \right] du \quad ; \quad \nu = x-u, \quad x = \nu + u, \quad dx = d\nu$$

$$= \int_{-\infty}^{\infty} g(u) \left[\int_{-\infty}^{\infty} f(\nu)\, e^{-i\omega(\nu+u)}\, d\nu \right] du$$

$$e^{-i\omega \nu} \cdot e^{-i\omega u}$$

$$= \int_{-\infty}^{\infty} g(u) \cdot \left[\int_{-\infty}^{\infty} f(\nu)\, e^{-i\omega \nu}\, d\nu \right] e^{-i\omega u}\, du$$

$$= \left(\int_{-\infty}^{\infty} f(\nu)\, e^{-i\omega \nu}\, d\nu \right) \left(\int_{-\infty}^{\infty} g(u)\, e^{-i\omega u}\, du \right)$$

$$= \hat{f}(\omega) \cdot \hat{g}(\omega)$$

$$\Rightarrow \quad \mathcal{F}\left[(f * g)(x)\right] = \hat{f}(w) \cdot \hat{g}(w).$$

(Teorema de convolución)

$$\Leftrightarrow \quad \mathcal{F}^{-1}\left[\hat{f}(w) \cdot \hat{g}(w)\right] = (f * g)(x)$$

$$\Leftrightarrow \quad \mathcal{F}^{-1}\left[\hat{f}(w) \cdot \hat{g}(w)\right] = \int_{-\infty}^{\infty} f(x-u)\, g(u)\, du$$

<u>Ej</u>: Calcule $\mathcal{F}^{-1}\left[\dfrac{1}{(4+w^2)(9+w^2)}\right]$.

<u>Ayuda</u>: $\mathcal{F}\left[e^{-a|x|}\right] = \dfrac{2a}{a^2+w^2}, \quad a>0.$

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$$\Rightarrow \mathcal{F}^{-1}\left[\frac{1}{(4+\omega^2)(9+\omega^2)}\right] = \mathcal{F}^{-1}\left[\left(\frac{1}{4+\omega^2}\right)\cdot\left(\frac{1}{9+\omega^2}\right)\right]$$

$$\frac{1}{4+\omega^2} = \frac{1}{2^2+\omega^2} = \frac{1}{4}\left(\frac{2(2)}{2^2+\omega^2}\right) = \frac{1}{4}\mathcal{F}\left[e^{-2|x|}\right]$$

$$\frac{1}{9+\omega^2} = \frac{1}{3^2+\omega^2} = \frac{1}{6}\left(\frac{2(3)}{3^2+\omega^2}\right) = \frac{1}{6}\mathcal{F}\left[e^{-3|x|}\right]$$

$$= \frac{1}{4}e^{-2|x|} * \frac{1}{6}e^{-3|x|}$$

$$= \int_{-\infty}^{\infty} \frac{1}{4}e^{-2|x-u|}\cdot\frac{1}{6}e^{-3|u|}\,du$$

$$= \frac{1}{24}\int_{-\infty}^{\infty} e^{-2|x-u|} \cdot e^{-3|u|}\, du \; ; \quad \text{casos} \begin{cases} x>0 \\ x=0 \\ x<0 \end{cases}$$

<u>Caso $x>0$</u>

$$\Rightarrow \int_{-\infty}^{\infty} e^{-2|x-u|} \cdot e^{-3|u|}\, du =$$

$$= \int_{-\infty}^{0} e^{-2(x-u)} \cdot e^{3u}\, du + \int_{0}^{x} e^{-2(x-u)} \cdot e^{-3u}\, du +$$

$$+ \int_{x}^{\infty} e^{2(x-u)}\, e^{-3u}\, du =$$

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$$= e^{-2x}\int_{-\infty}^{0} e^{5u}\,du + e^{-2x}\int_{0}^{x} e^{-u}\,du + e^{2x}\int_{x}^{\infty} e^{-5u}\,du$$

$$= e^{-2x}\left[\frac{1}{5}e^{5u}\Big|_{-\infty}^{0}\right] + e^{-2x}\left[-e^{-u}\Big|_{0}^{x}\right] + e^{2x}\left[-\frac{1}{5}e^{-5u}\Big|_{x}^{\infty}\right]$$

$$= e^{-2x}\left[\frac{1}{5}(1-0)\right] + e^{-2x}\left[-(e^{-x}-1)\right] + e^{2x}\left[-\frac{1}{5}(0-e^{-5x})\right]$$

$$= \frac{1}{5}e^{-2x} - e^{-3x} + e^{-2x} + \frac{1}{5}e^{-3x}$$

$$= \frac{6}{5}e^{-2x} - \frac{4}{5}e^{-3x} \quad , \quad x>0$$

$$\Rightarrow \mathcal{F}^{-1}[\quad] = \begin{cases} \_\_ , & x>0 \\ \_\_ , & x=0 \\ \_\_ , & x<0 \end{cases}$$

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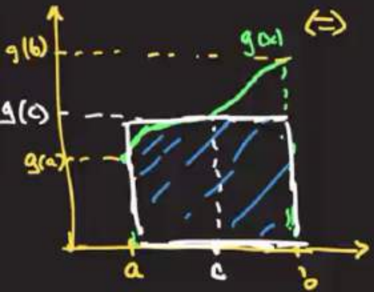
) Recordemos el Teorema del valor medio para integrales:

Si $g(x)$ es continua sobre $[a,b]$, entonces existe $c \in [a,b]$, tal que

$$\frac{1}{b-a}\int_a^b g(x)\,dx = g(c)$$

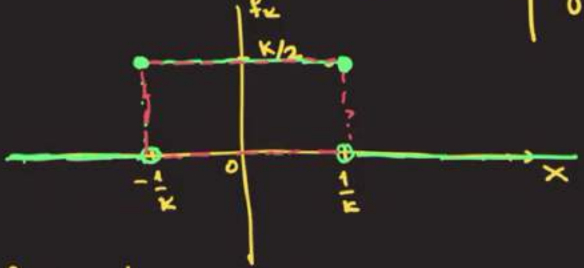
$$\Leftrightarrow \int_a^b g(x)\,dx = g(c)\cdot(b-a)$$



$/// : \int_a^b g(x)\,dx$

Delta de Dirac (Funciones Generalizadas)

Considere la función $f_k(x) = \begin{cases} \frac{k}{2}, & -\frac{1}{k} \leq x \leq \frac{1}{k} \\ 0, & |x| > \frac{1}{k} \end{cases}$



$$\Rightarrow \int_{-\infty}^{\infty} f_k(x)\,dx = \frac{2}{k} \cdot \frac{k}{2} = 1.$$

Def: La delta de Dirac en la función generalizada $\delta(x) = \lim_{k \to \infty} f_k(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$

Prop: $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

Nota: Sea $g(x)$ una función definida en $(-\infty, \infty)$. Entonces:

$$\int_{-\infty}^{\infty} g(x)\,\delta(x)\,dx = \int_{-1/k}^{1/k} g(x)\left[\lim_{k\to\infty} f_k(x)\right] dx =$$

$$= \lim_{k\to\infty} \int_{-1/k}^{1/k} g(x)\,\underbrace{\cancel{f_k}}_{\frac{k}{2}}(x)\,dx = \lim_{k\to\infty} \frac{k}{2} \underbrace{\int_{-1/k}^{1/k} g(x)\,dx}_{\text{Aplicamos teorema valor medio de integrales}}$$

$$= \lim_{k\to\infty} \cancel{\frac{k}{2}}\, g(c)\cdot\left(\cancel{\frac{2}{k}}\right) = g(0)$$

$-\frac{1}{k} \leq c \leq \frac{1}{k}$

Prop: ① $\int_{-\infty}^{\infty} g(x)\,\delta(x)\,dx = g(0).$

② $\int_{-\infty}^{\infty} g(x)\,\delta(x-a)\,dx = g(a)$

N.B.: ① $\mathcal{F}[\delta(x)] = \int_{-\infty}^{\infty} \delta(x) \cdot e^{-i\omega x}\,dx$

$= e^{-i\omega(0)} = 1$

② $\mathcal{F}^{-1}(1) = \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot e^{i\omega x}\,d\omega \quad \Rightarrow$

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$$\Rightarrow \int_{-\infty}^{\infty} e^{i\omega x}\, d\omega = 2\pi\, \delta(x)$$

e^{-x^2}

(4) El pulso Gaussiano:

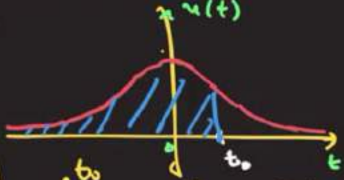
Considere $u(t) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-t^2/2\sigma^2}$ (Función del tipo Gaussiano)

$\Rightarrow$ Nótese que: $\displaystyle\int_{-\infty}^{\infty} u(t)\, dt = 1.$

Problema: $\mathcal{F}[u(t)] =$ Función Gaussiana

El pulso Gaussiano ; considere función densidad de probabilidades:

$$u(t) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-t^2/2\sigma^2}$$



Problema: Encuentre

$$\mathcal{F}[u(t)].$$

$$\Rightarrow \hat{u}(\omega) = \int_{-\infty}^{\infty} u(t)\, e^{-i\omega t}\, dt$$

- $\displaystyle\int_{-\infty}^{t_0} u(t)\,dt = \text{Prob.} = P(x \le t_0)$
- $\displaystyle\int_{-\infty}^{\infty} u(t)\,dt = 1$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-t^2/2\sigma^2} \cdot e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[\underbrace{t^2 + 2\sigma^2 i\omega t}_{\text{completamos}}\right]}\, dt$$

$$(\sigma^2 i\omega)^2$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[t^2 + 2\sigma^2 i\omega t + (\sigma^2 i\omega)^2 - (\sigma^2 i\omega)^2\right]}\, dt$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[(t+\sigma^2 i\omega)^2 - (\sigma^2 i\omega)^2\right]}\, dt$$

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$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(t+\sigma^2 i\omega)^2} \cdot \underbrace{e^{\frac{1}{2\sigma^2}(\sigma^2 i\omega)^2}} dt$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{1}{2\sigma^2}(\sigma^2 i\omega)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(t+\sigma^2 i\omega)^2} dt$$

Hacemos: $y = t + \sigma^2 i\omega \Rightarrow dy = dt$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{1}{2\sigma^2}(\sigma^2 i\omega)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}y^2} dy$$

$$e^{-\left(\frac{y}{\sqrt{2}\sigma}\right)^2}$$

$$= \frac{1}{\cancel{\sqrt{2}}\sqrt{\pi}\,\cancel{\sigma}} e^{(\sigma^2 i \omega)^2/2\sigma^2} \cdot \cancel{\sqrt{2}}\,\cancel{\sigma} \int_{-\infty}^{\infty} e^{-m^2}\, dm \to \sqrt{\pi}$$

$$= \frac{1}{\cancel{\sqrt{\pi}}} e^{(\sigma^2 i \omega)^2/2\sigma^2} \cdot \cancel{\sqrt{\pi}} = e^{-\sigma^4 \omega^2/2\sigma^2} = e^{-\frac{\sigma^2 \omega^2}{2}}$$

$$\Rightarrow \mathcal{F}\left[\frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/2\sigma^2}\right] = e^{-\frac{\sigma^2}{2}\omega^2}$$

- Separación de Variables
  - Series ortogonales (Fourier)
  - Integral de Fourier.
- Transformada de Fourier: $u = u(x,t)$ $\quad t \geq 0$

$-\infty < x < \infty$, $t > 0$