Teorema: Suponga que $g: I \to \mathbb{R}$ es diferenciable y que g' es integrable en I. Sea $g(I) = J$. Si $f: J \to \mathbb{R}$, entonces, $\forall\, a, b \in I$, se cumple.

$$\int_a^b f(\underline{g(x)})\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

TFC (2): Sean $f \in \mathcal{R}[a,b]$ y $F: [a,b] \to \mathbb{R}$ $\ni$ $F(x) = \int_a^x f(t)\, dt$.
$\Rightarrow$ F es continua en $[a,b]$. Si f es continua en $[a,b]$
$\Rightarrow F' = f$.

TFC (1): Si $F: [a,b] \to \mathbb{R}$ es continua sobre $[a,b]$ y diferenciable en (a,b), con $F' = f$ y $f \in \mathcal{R}[a,b]$, entonces:

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$$\int_a^b f = F(b) - F(a).$$

Dem: Sea $F(x) = \int_a^x f(u)\, du$. Como f es continua $\Rightarrow$ $F' = f$ (TFC 2). Entonces, nótese que:

$$(F \circ g)'(x) = F'(g(x)) \cdot g'(x) = f(g(x))\, g'(x)$$

$$\Rightarrow \int_a^b (F \circ g)'(x)\, dx = \int_a^b f(g(x))\, g'(x)\, dx$$

$$\Rightarrow (F \circ g)(b) - (F \circ g)(a) = \int_a^b f(g(x))\, g'(x)\, dx$$

$$\Rightarrow F(g(b)) - F(g(a)) = \int_a^b f(g(x))\, g'(x)\, dx$$

$$\Rightarrow \int_{g(a)}^{g(b)} F'(u)\, du = \int_{g(a)}^{g(b)} f(u)\, du = \int_a^b f(g(x))\, g'(x)\, dx \qquad \square$$

Nota: **Prop**: Suponga que $f_n : [a,b] \to \mathbb{R}$, donde $n \in \mathbb{Z}^+$ y $f_n \in \mathcal{R}[a,b]$, $\forall n$; y suponga que $f_n \xrightarrow[\text{unif}]{} f$ sobre $[a,b]$. Entonces

(i) $f \in \mathcal{R}[a,b]$; y

(ii) $\lim\limits_{n\to\infty} \int_a^b f_n = \int_a^b f$.

$[a,b]$ — cerrado, acotado

Integrales Impropias:

Def: Suponga que $f : (a,b] \to \mathbb{R}$ es Riemann integrable en $[c,b]$ $\ni$ $a<c<b$. Entonces, la integral impropia de f sobre $[a,b]$ es:

$$\int_a^b f = \lim_{\epsilon\to 0} \int_{a+\epsilon}^b f$$

La integral impropia converge si este límite existe. En otro caso, la integral diverge.

Ej: Sea $\int_0^1 \frac{dx}{x^p}$, cuando $x \in (0,1]$, $p>0$.

$\boxed{\text{Impropia}}$

$$\Rightarrow \int_0^1 \frac{dx}{x^p} = \lim_{\epsilon\to 0} \int_{0+\epsilon}^1 \frac{dx}{x^p} = \lim_{\epsilon\to 0} \frac{x^{-p+1}}{-p+1}\Big|_\epsilon^1 =$$

Integral de Riemann. $\qquad p \neq 1$

$$= \lim_{\epsilon\to 0}\left[\frac{1}{-p+1} - \frac{\epsilon^{-p+1}}{-p+1}\right] = \begin{cases} \text{convergente}, & 0<p<1 \\ \text{divergente}, & p>1 \end{cases}$$

Si $p=1 \Rightarrow \int_0^1 \frac{dx}{x} = \lim\limits_{\epsilon\to 0} \int_\epsilon^1 \frac{dx}{x} = \lim\limits_{\epsilon\to 0} \ln(x)\Big|_\epsilon^1 =$

$$= \lim_{\epsilon\to 0}[\ln 1 - \ln \epsilon] \to \text{diverge.}$$

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Def: Suponga que $f : [a,\infty) \to \mathbb{R}$ es integrable sobre $[a,r]$, $r>a$. Entonces, la integral impropia de f es:

$$\int_a^\infty f = \lim_{r\to\infty} \int_a^r f$$

Ej: Considere la integral de Frullani:

$$I = \int_0^\infty \frac{f(ax) - f(bx)}{x}\,dx, \quad a,b>0,$$

y donde $f : [0,\infty) \to \mathbb{R}$ es continua y es t.q.

$$\lim_{x\to\infty} f(x) = f(\infty) < \infty.$$

- Sea $I = I_1 + I_2$

$$I_1 = \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \frac{f(ax)-f(bx)}{x}\,dx \;;\quad I_2 = \lim_{r \to \infty} \int_{1}^{r} \frac{f(ax)-f(bx)}{x}\,dx$$

Estudiemos I_1: SPG, hagamos: $0 < a < b$, y $u = ax$, $t = bx$

$$\Rightarrow I_1 = \lim_{\epsilon \to 0} \left[\int_{\epsilon a}^{a} \frac{f(u)}{u}\,du - \int_{\epsilon b}^{b} \frac{f(t)}{t}\,dt \right]$$

$$= \lim_{\epsilon \to 0} \left[-\int_{a}^{\epsilon a} \frac{f(u)}{u}\,du - \int_{\epsilon a}^{\epsilon b} \frac{f(u)}{u}\,du - \int_{\epsilon b}^{b} \frac{f(t)}{t}\,dt + \int_{\epsilon a}^{\epsilon b} \frac{f(t)}{t}\,dt \right]$$

$$= \lim_{\epsilon \to 0} \left[\int_{\epsilon a}^{\epsilon b} \frac{f(t)}{t}\,dt - \int_{a}^{b} \frac{f(t)}{t}\,dt \right].$$

$$\Rightarrow \lim_{\epsilon \to 0} \int_{\epsilon a}^{\epsilon b} \frac{f(t)}{t}\,dt = ?$$

$$\Rightarrow \int_{\epsilon a}^{\epsilon b} \frac{f(t)}{t}\,dt = \int_{\epsilon a}^{\epsilon b} \frac{f(t)}{t}\,dt - \int_{\epsilon a}^{\epsilon b} \frac{f(0)}{t}\,dt + \int_{\epsilon a}^{\epsilon b} \frac{f(0)}{t}\,dt =$$

$$= \int_{\epsilon a}^{\epsilon b} \left[\frac{f(t)-f(0)}{t} \right] dt + f(0) \int_{\epsilon a}^{\epsilon b} \frac{dt}{t}$$

$$= \int_{\epsilon a}^{\epsilon b} \left[\frac{f(t)-f(0)}{t} \right] dt + f(0) \cdot \ln t \Big|_{\epsilon a}^{\epsilon b}$$

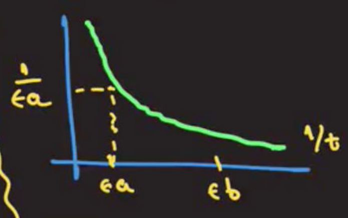
$$\ln(b/a)$$

$$\le \int_{\epsilon a}^{\epsilon b} \left| \frac{f(t)-f(0)}{t} \right| dt + f(0) \ln(b/a)$$

$$= \int_{\epsilon a}^{\epsilon b} |f(t)-f(0)| \cdot \frac{1}{t}\,dt + f(0) \ln(b/a)$$

$$\le \int_{\epsilon a}^{\epsilon b} \max |f(t)-f(0)| \cdot \frac{1}{t}\,dt + f(0) \ln(b/a)$$

$$= \max |f(t)-f(0)| \int_{\epsilon a}^{\epsilon b} \frac{1}{t}\,dt + f(0) \ln(b/a)$$



$$\le \max |f(t)-f(0)| \int_{\epsilon a}^{\epsilon b} \frac{1}{\epsilon a}\,dt + f(0) \cdot \ln(b/a)$$

$$= \max |f(t)-f(0)| \cdot \frac{1}{\cancel{\epsilon} a} (\cancel{\epsilon} b - \cancel{\epsilon} a) + f(0) \cdot \ln(b/a)$$

$$= \max |f(t)-f(0)| \cdot \left(\frac{b-a}{a} \right) + f(0) \cdot \ln(b/a) \to 0$$

$$\xrightarrow[\epsilon \to 0]{} \quad \Rightarrow I_1 = f(0) \ln(b/a) - \int_{a}^{b} \frac{f(t)}{t}\,dt$$