Kinematics of SwiftPro Arm on Turtlebot Base

Hands-on Intervetion

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1 Kinematics

1.1 Generalized Coordinates

The generalized coordinates of the mobile manipulator system can be defined as:

$$\mathbf{q} = \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{q}_{\text{arm}} \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \tag{1}$$

Where:

- $\boldsymbol{\eta} = \begin{bmatrix} x & y & \theta \end{bmatrix}^\top$: Generalized coordinates of the mobile base (TurtleBot 2).
- $\mathbf{q}_{\text{arm}} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^\top$: Joint configuration of the 4-DOF uArm Swift Pro.

1.2 Local Arm Kinematics (NED Frame)

The arm is modeled using geometric forward kinematics in the North-East-Down (NED) frame. The effective planar reach R is computed as:

$$R = 158.8\cos(q_3) - 142\sin(q_2) + 56.5 + 13.2 \tag{2}$$

Using this reach, the position of the end-effector in the local NED arm frame is:

$$\begin{aligned} AE_x &= R\cos(q_1) \\ AE_y &= R\sin(q_1) \\ AE_z &= 72.2 - 108 - 142\cos(q_2) - 158.8\sin(q_3) \end{aligned} \tag{3}$$

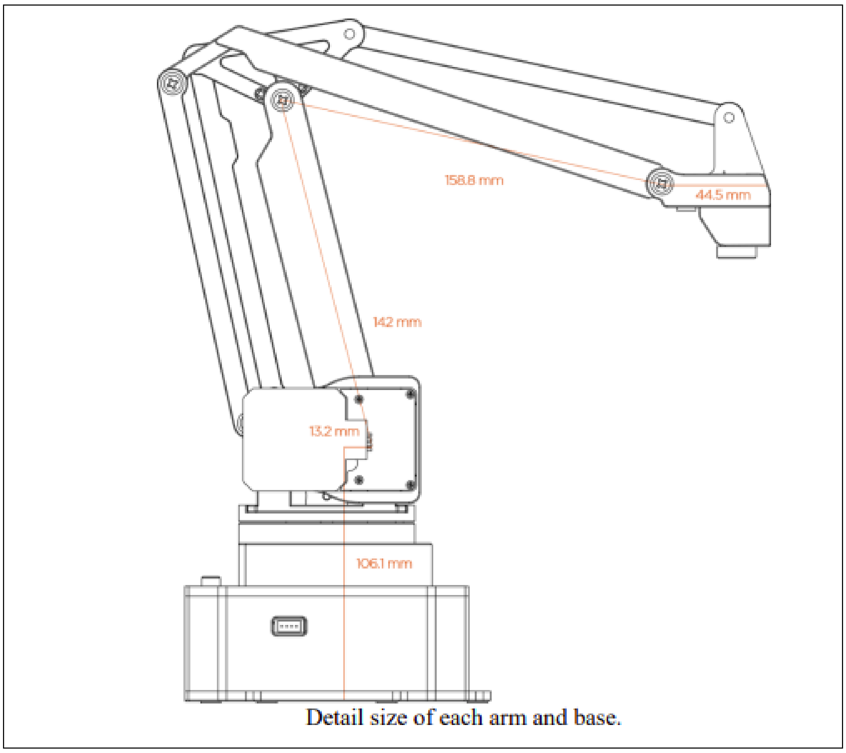

Figure 1: Arm Structure

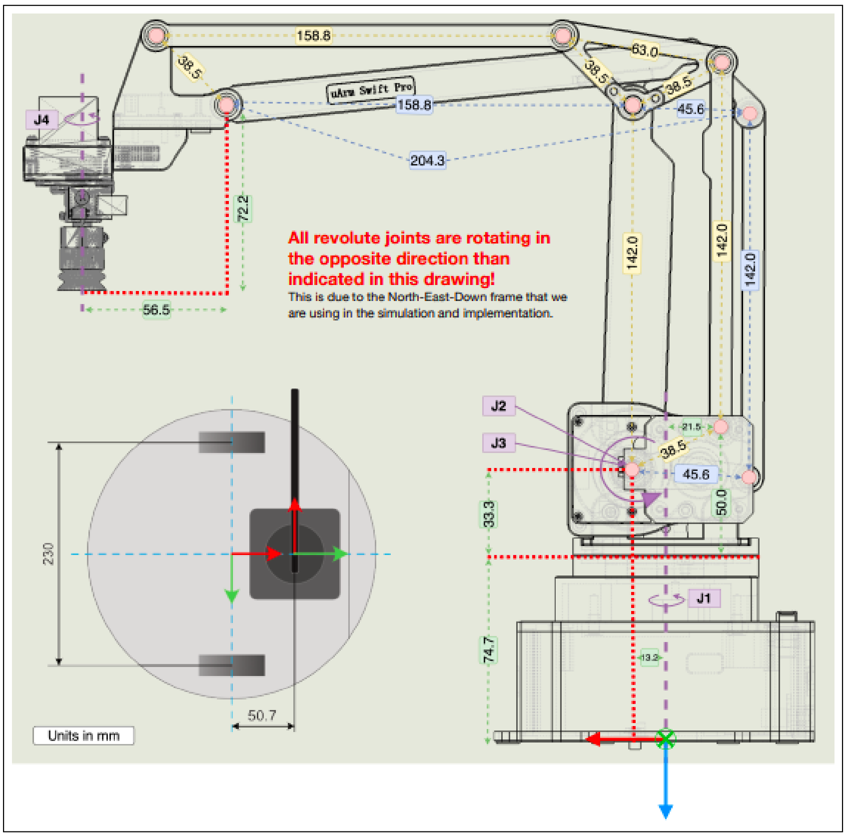

Figure 2: Detailed Description

1.3 World Frame Projection

To project the end-effector into the world frame, we account for the base rotation θ and the arm offset from the center of the base. The coordinates in the world frame are given by:

$$\begin{aligned} WE_x &= R\sin(q_1+\theta) + 50.7\cos(\theta) + x \\ WE_y &= -R\cos(q_1+\theta) + 50.7\sin(\theta) + y \\ WE_z &= AE_z - HR \end{aligned} \tag{4}$$

Here, HR is the vertical height from the TurtleBot base to the arm base (typically $HR = 108(74.7 + 33.3$ mm).

1.4 End-Effector Homogeneous Transformation

The full homogeneous transformation matrix from the world frame to the end-effector frame is:

$$T_{\text{world}}^{\text{EE}} = \begin{bmatrix} \cos(\theta+q_1+q_4) & -\sin(\theta+q_1+q_4) & 0 & WE_x \\ \sin(\theta+q_1+q_4) & \cos(\theta+q_1+q_4) & 0 & WE_y \\ 0 & 0 & 1 & WE_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

This transformation gives the full pose (position and orientation) of the end-effector in the world frame.

1.5 Transformation Matrices Step-by-Step

To compute the full forward kinematics, we chain the transformations from the world frame to the end-effector frame:

$$T_{\text{world}}^{\text{EE}} = T_0^1 \cdot T_1^2 \cdot T_2^3 \cdot T_3^4 \cdot T_4^5 \cdot T_5^{\text{EE}} \tag{6}$$

Where:

- T_0^1 is the transformation from world to base.
- T_1^2 is base to arm mount.
- T_2^3 is rotation about z by q_1 and lift along z.
- T_3^4 is rotation about y by q_2 and link 1 translation.
- T_4^5 is rotation about y by q_3 and link 2 translation.
- T_5^{EE} is rotation about y by q_4 and wrist translation.

1. **World to Base (Mobile Platform Pose):**

$$T_0^1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & x \\ \sin\theta & \cos\theta & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

2. Base to Arm Mount Offset:

$$T_1^2 = \begin{bmatrix} 1 & 0 & 0 & 0.0507 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.108 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

3. Joint 1 (Base rotation about Z):

$$T_2^3 = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 & 0 \\ \sin q_1 & \cos q_1 & 0 & 0 \\ 0 & 0 & 1 & 0.0132 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

4. Joint 2 (Shoulder, rotate about Y, link length 142 mm):

$$T_3^4 = \begin{bmatrix} \cos q_2 & 0 & \sin q_2 & 0.142\cos q_2 \\ 0 & 1 & 0 & 0 \\ -\sin q_2 & 0 & \cos q_2 & -0.142\sin q_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

5. Joint 3 (Elbow, rotate about Y, link length 158.8 mm):

$$T_4^5 = \begin{bmatrix} \cos q_3 & 0 & \sin q_3 & 0.1588\cos q_3 \\ 0 & 1 & 0 & 0 \\ -\sin q_3 & 0 & \cos q_3 & -0.1588\sin q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

6. Joint 4 (Wrist, rotate about Y, link length 44.5 mm):

$$T_5^{\text{EE}} = \begin{bmatrix} \cos q_4 & 0 & \sin q_4 & 0.0445\cos q_4 \\ 0 & 1 & 0 & 0 \\ -\sin q_4 & 0 & \cos q_4 & -0.0445\sin q_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

Final Transformation:

$$T_{\text{world}}^{\text{EE}} = T_0^1 T_1^2 T_2^3 T_3^4 T_4^5 T_5^{\text{EE}} \tag{13}$$

This cumulative product yields the end-effector pose in the world frame.

2 Jacobian Matrix

To relate joint velocities to the end-effector velocity, we compute the Jacobian matrix $J(q)$ such that:

$$\dot{\mathbf{x}} = J(q) \cdot \dot{\mathbf{q}}$$

We compute the partial derivatives of the end-effector position $\mathbf{AE} = [AE_x, AE_y, AE_z]^\top$ with respect to each joint.

Row 1: $\frac{\partial AE_x}{\partial q_i}$

$$\frac{\partial AE_x}{\partial q_1} = -R\sin(q_1)$$

$$\frac{\partial AE_x}{\partial q_2} = -142\cos(q_2)\cos(q_1)$$

$$\frac{\partial AE_x}{\partial q_3} = -158.8\sin(q_3)\cos(q_1)$$

$$\frac{\partial AE_x}{\partial q_4} = 0$$

Row 2: $\frac{\partial AE_y}{\partial q_i}$

$$\frac{\partial AE_y}{\partial q_1} = R\cos(q_1)$$

$$\frac{\partial AE_y}{\partial q_2} = -142\cos(q_2)\sin(q_1)$$

$$\frac{\partial AE_y}{\partial q_3} = -158.8\sin(q_3)\sin(q_1)$$

$$\frac{\partial AE_y}{\partial q_4} = 0$$

Row 3: $\frac{\partial AE_z}{\partial q_i}$

$$\frac{\partial AE_z}{\partial q_1} = 0$$

$$\frac{\partial AE_z}{\partial q_2} = 142\sin(q_2)$$

$$\frac{\partial AE_z}{\partial q_3} = -158.8\cos(q_3)$$

$$\frac{\partial AE_z}{\partial q_4} = 0$$

Each revolute joint contributes angular velocity around its axis:

- q_1: rotation about the Z-axis.
- q_2, q_3, q_4: rotation about the Y-axis.

Expressed as angular velocity unit vectors:

$$J_{\text{angular}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Final 6×4 Full Jacobian Matrix

$$J(q) = \begin{bmatrix} -R\sin(q_1) & -142\cos(q_2)\cos(q_1) & -158.8\sin(q_3)\cos(q_1) & 0 \\ R\cos(q_1) & -142\cos(q_2)\sin(q_1) & -158.8\sin(q_3)\sin(q_1) & 0 \\ 0 & 142\sin(q_2) & -158.8\cos(q_3) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

where

$$R = 158.8\cos(q_3) - 142\sin(q_2) + 56.5 + 13.2$$

This Jacobian can now be used in kinematic control for both position and orientation tracking of the end-effector.

3 Forward and Inverse Kinematics

3.1 Forward Kinematics (Symbolic)

Given the joint angles $q = [q_1, q_2, q_3, q_4]^\top$, the symbolic expressions for the end-effector position $[AE_x, AE_y, AE_z]^\top$ in the arm base frame are:

$$\begin{aligned} R &= L_3\cos(q_3) - L_2\sin(q_2) + L_{extgripper} + L_{extoffset} \\ AE_x &= R\cos(q_1) \\ AE_y &= R\sin(q_1) \\ AE_z &= L_1 - L_2\cos(q_2) - L_3\sin(q_3) \end{aligned}$$

Where:

- $L_1 = 72.2 - 108$
- $L_2 = 142$
- $L_3 = 158.8$
- $L_{\text{gripper}} = 56.5$
- $L_{\text{offset}} = 13.2$

3.2 Inverse Kinematics (Symbolic)

Given a desired end-effector position $[x_d, y_d, z_d]$, the inverse kinematics steps are:

1. Compute $R = \sqrt{x_d^2 + y_d^2}$
2. $q_1 = \text{atan2}(y_d, x_d)$
3. From z_d, solve:

$$\begin{aligned} AE_z &= L_1 - L_2 \cos(q_2) - L_3 \sin(q_3) \\ R &= L_3 \cos(q_3) - L_2 \sin(q_2) + L_{\text{gripper}} + L_{\text{offset}} \end{aligned}$$

4. Solve the system to find q_2 and q_3
5. If orientation is tracked, set q_4 such that the wrist aligns with desired angle