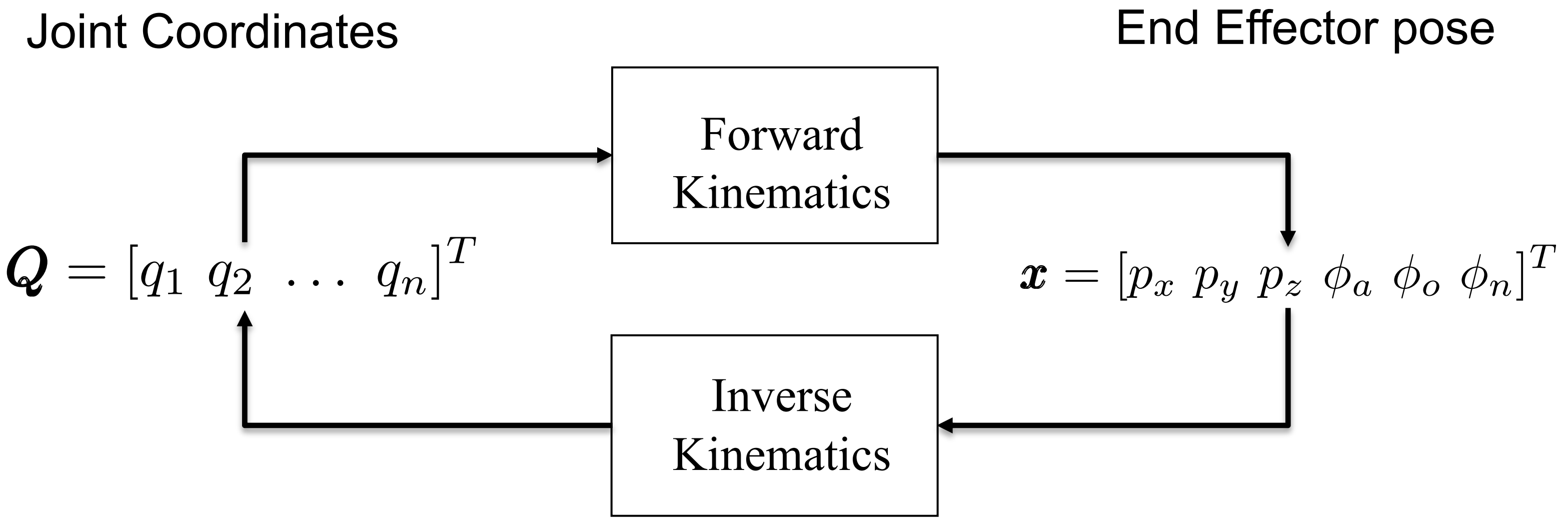
Chapter V: Jacobians



Joint Coordinates

End Effector pose

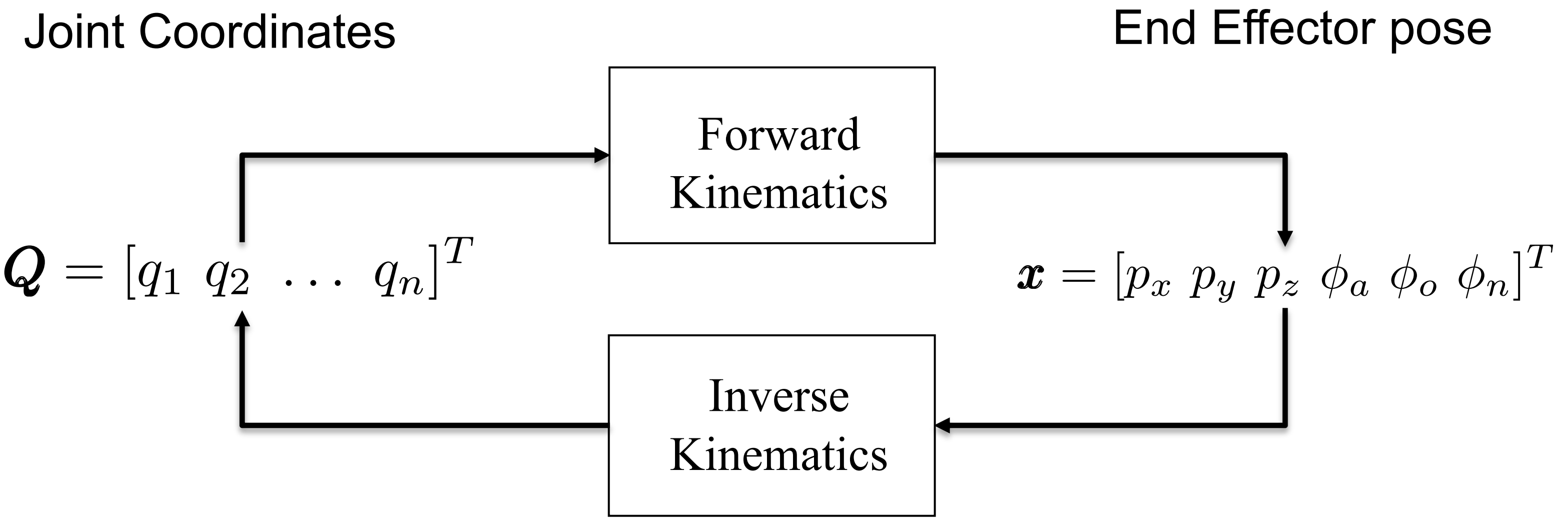
Forward Kinematics

Inverse Kinematics

$$\boldsymbol{Q} = [q_1\ q_2\ \dots\ q_n]^T$$

$$\boldsymbol{x} = [p_x\ p_y\ p_z\ \phi_a\ \phi_o\ \phi_n]^T$$

Chapter V: Jacobians



Joint Coordinates

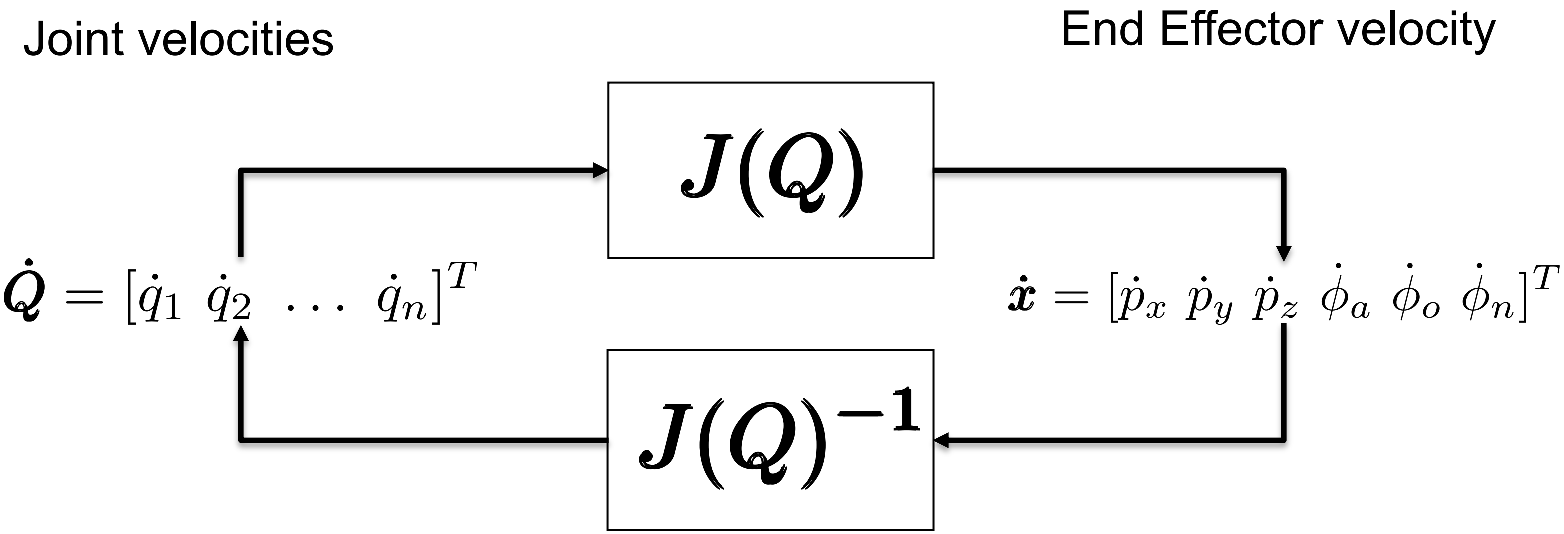
End Effector pose

Forward Kinematics

Inverse Kinematics

$$\boldsymbol{Q} = [q_1 \; q_2 \; \dots \; q_n]^T$$

$$\boldsymbol{x} = [p_x \; p_y \; p_z \; \phi_a \; \phi_o \; \phi_n]^T$$



Joint velocities

End Effector velocity

$$J(Q)$$

$$J(Q)^{-1}$$

$$\dot{\boldsymbol{Q}} = [\dot{q}_1 \; \dot{q}_2 \; \dots \; \dot{q}_n]^T$$

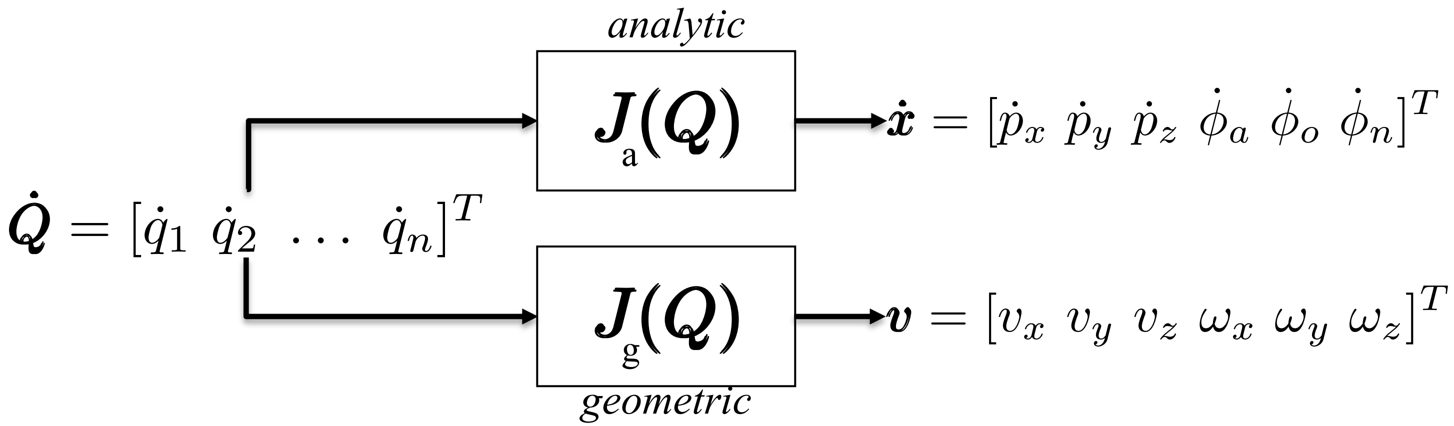
$$\dot{\boldsymbol{x}} = [\dot{p}_x \; \dot{p}_y \; \dot{p}_z \; \dot{\phi}_a \; \dot{\phi}_o \; \dot{\phi}_n]^T$$

Chapter V: Jacobians

Jacobian matrix of a robot:

- Relates the vector of joint velocities $\dot{\boldsymbol{Q}} = [\dot{q}_1 \ \dot{q}_2 \ \ldots \ \dot{q}_n]^T$ to another vector of velocities expressed in a different space.

Types of Jacobian matrices:



$$\dot{\boldsymbol{Q}} = [\dot{q}_1 \ \dot{q}_2 \ \ldots \ \dot{q}_n]^T$$

$$\boldsymbol{J}_{\mathrm{a}}(\boldsymbol{Q}) \rightarrow \dot{\boldsymbol{x}} = [\dot{p}_x \ \dot{p}_y \ \dot{p}_z \ \dot{\phi}_a \ \dot{\phi}_o \ \dot{\phi}_n]^T$$

$$\boldsymbol{J}_{\mathrm{g}}(\boldsymbol{Q}) \rightarrow \boldsymbol{v} = [v_x \ v_y \ v_z \ \omega_x \ \omega_y \ \omega_z]^T$$

Direct and inverse Jacobian matrix:

- **Direct**: to calculate EE velocities from joint velocities.

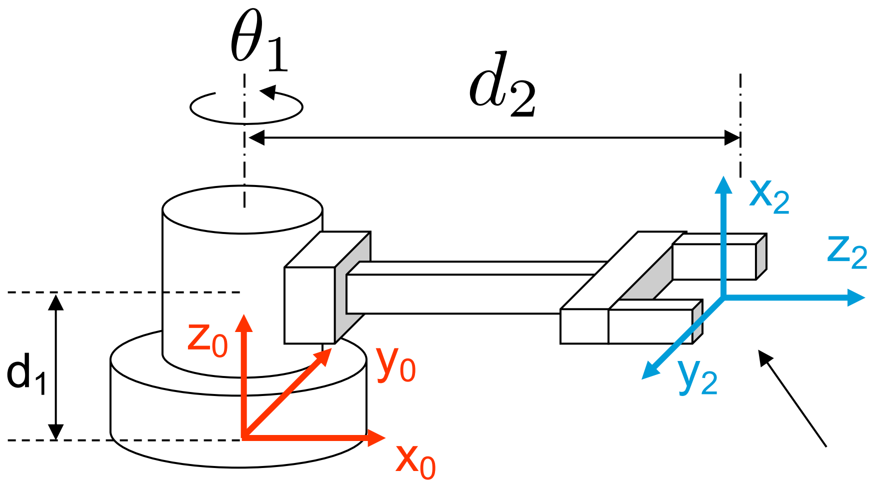
- **Inverse**: to calculate joint velocities from EE velocities.

Chapter V: Jacobians

5.1 Analytical Jacobian Matrix

$$\dot{\boldsymbol{Q}} = [\dot{q}_1 \;\; \dot{q}_2 \;\; \ldots \;\; \dot{q}_n]^T \longrightarrow \boxed{\boldsymbol{J(Q)}} \longrightarrow \dot{\boldsymbol{x}} = [\dot{p}_x \; \dot{p}_y \; \dot{p}_z \; \dot{\phi}_a \; \dot{\phi}_o \; \dot{\phi}_n]^T$$

From basic physics we know that: $\boldsymbol{v} = \dfrac{\boldsymbol{dx}}{dt}$



The speed of the EE depends on the speed of the joints!

Chapter V: Jacobians

5.1 Analytical Jacobian matrix

We know that:

$$\left\{ \begin{array}{l} \boldsymbol{v} = \dfrac{\boldsymbol{dx}}{dt} \\[2ex] {}^{\boldsymbol{R}}\boldsymbol{T}_{\boldsymbol{H}} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} \phi_a = atan2(n_y, n_x) \\ \phi_o = atan2(-n_z, (n_x c\phi_a + n_y s\phi_a)) \\ \phi_n = atan2((-a_y c\phi_a + a_x s\phi_a), (o_y c\phi_a - o_x s\phi_a)) \end{array} \end{array} \right.$$

5.1 Analytical Jacobian matrix

We know that:

$$\boldsymbol{v} = \frac{\boldsymbol{dx}}{dt}$$

$$\boldsymbol{{}^{R}T_{H}} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{aligned} \phi_a &= atan2(n_y, n_x) \\ \phi_o &= atan2(-n_z, (n_x c\phi_a + n_y s\phi_a)) \\ \phi_n &= atan2((-a_y c\phi_a + a_x s\phi_a), (o_y c\phi_a - o_x s\phi_a)) \end{aligned}$$

In our case:

$$\begin{aligned} p_x &= f_x(\boldsymbol{Q}) \\ p_y &= f_y(\boldsymbol{Q}) \\ p_z &= f_z(\boldsymbol{Q}) \\ \phi_a &= f_{\phi_a}(\boldsymbol{Q}) \\ \phi_o &= f_{\phi_o}(\boldsymbol{Q}) \\ \phi_n &= f_{\phi_n}(\boldsymbol{Q}) \end{aligned}$$

Chain Rule

$$\begin{aligned} \dot{p}_x &= \frac{\partial f_x}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial f_x}{\partial q_2}\dot{q}_n \\ \dot{p}_y &= \frac{\partial f_y}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial f_y}{\partial q_2}\dot{q}_n \\ \dot{p}_z &= \frac{\partial f_z}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial f_z}{\partial q_2}\dot{q}_n \\ \dot{\phi}_a &= \frac{\partial f_{\phi_q}}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial f_{\phi_a}}{\partial q_2}\dot{q}_n \\ \dot{\phi}_o &= \frac{\partial f_{\phi_o}}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial f_{\phi_o}}{\partial q_2}\dot{q}_n \\ \dot{\phi}_n &= \frac{\partial f_{\phi_n}}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial f_{\phi_n}}{\partial q_2}\dot{q}_n \end{aligned}$$

Chapter V: Jacobians

5.1 Analytical Jacobian Matrix

We know that:

$$\left\{ \begin{array}{l} \boldsymbol{v} = \dfrac{\boldsymbol{dx}}{dt} \\[2ex] {}^{\boldsymbol{R}}\boldsymbol{T}_{\boldsymbol{H}} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} \phi_a = atan2(n_y, n_x) \\ \phi_o = atan2(-n_z, (n_x c\phi_a + n_y s\phi_a)) \\ \phi_n = atan2((-a_y c\phi_a + a_x s\phi_a), (o_y c\phi_a - o_x s\phi_a)) \end{array} \end{array} \right.$$

By rearranging the terms in matrix form we obtain the Jacobian (***J***):

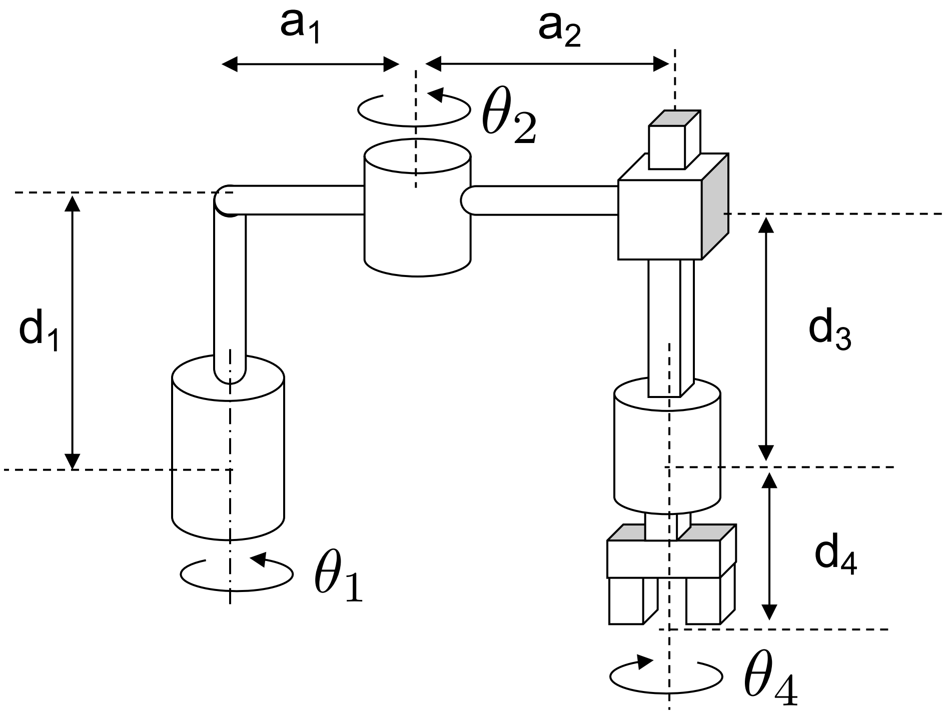
$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \\ \dot{\phi}_a \\ \dot{\phi}_o \\ \dot{\phi}_n \end{bmatrix} = \boldsymbol{J(Q)} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} = \overset{\xleftrightarrow{\textit{As many columns as robot DOFs}}}{\begin{bmatrix} \dfrac{\partial f_x}{\partial q_1} & \cdots & \dfrac{\partial f_x}{\partial q_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_{\phi_n}}{\partial q_1} & \cdots & \dfrac{\partial f_{\phi_n}}{\partial q_n} \end{bmatrix}} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

Note that the speed will depend on the instantaneous value of each of the joint variables (angular or prismatic) !

5.1 Analytical Jacobian matrix

Example

SCARA Robot



5.1 Analytical Jacobian matrix

Example
SCARA Robot

Form the forward kinematics:

$$^{R}T_{H} = \begin{bmatrix} c_{1+2-4} & s_{1+2-4} & 0 & a_1c_1 + a_2c_{1+2} \\ s_{1+2-4} & -c_{1+2-4} & 0 & a_1s_1 + a_2s_{1+2} \\ 0 & 0 & -1 & d_1 - q_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore

$$\begin{cases} p_x = a_1c_1 + a_2c_{1+2} \\ p_y = a_1s_1 + a_2s_{1+2} \\ p_z = d_1 - q_3 - d_4 \end{cases}$$

What about roll, pitch & yaw?

Chapter V: Jacobians

5.1 Analytical Jacobian matrix

Example

SCARA Robot

$$^{R}\boldsymbol{T_H} = \begin{bmatrix} c_{1+2-4} & s_{1+2-4} & 0 & a_1c_1 + a_2c_{1+2} \\ s_{1+2-4} & -c_{1+2-4} & 0 & a_1s_1 + a_2s_{1+2} \\ 0 & 0 & -1 & d_1 - q_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As we studied earlier, we can get the roll, pitch and yaw from the numerical $^{R}\boldsymbol{T_H}$ matrix:

$$\phi_a = atan2(n_y, n_x)$$

$$\phi_o = atan2(-n_z, (n_x c\phi_a + n_y s\phi_a))$$

$$\phi_n = atan2((-a_y c\phi_a + a_x s\phi_a), (o_y c\phi_a - o_x s\phi_a))$$

Chapter V: Jacobians

5.1 Analytical Jacobian matrix

Example

SCARA Robot

$$
{}^{\boldsymbol{R}}\boldsymbol{T}_{\boldsymbol{H}} = \begin{bmatrix} c_{1+2-4} & s_{1+2-4} & 0 & a_1c_1 + a_2c_{1+2} \\ s_{1+2-4} & -c_{1+2-4} & 0 & a_1s_1 + a_2s_{1+2} \\ 0 & 0 & -1 & d_1 - q_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

As we studied earlier, we can get the roll, pitch and yaw from the numerical${}^{\boldsymbol{R}}\,\boldsymbol{T}_{\boldsymbol{H}}$ matrix:

$$
\begin{aligned}
\phi_a &= atan2(n_y, n_x) \\
&= atan2(s_{1+2-4}, c_{1+2-4}) = q_1 + q_2 - q_4 \\
\phi_o &= atan2(-n_z, (n_x c\phi_a + n_y s\phi_a)) \\
&= atan2(0, c_{1+2-4}c\phi_a + s_{1+2-4}s\phi_a) = atan2(0, c^2_{1+2-4} + s^2_{1+2-4}) \\
&= atan2(0, 1) = 0 \\
\phi_n &= atan2((-a_y c\phi_a + a_x s\phi_a), (o_y c\phi_a - o_x s\phi_a)) \\
&= atan2((0c\phi_a + 0s\phi_a), (-c_{1+2-4}c\phi_a - s_{1+2-4}s\phi_a)) = atan2(0, -c^2_{1+2-4} - s^2_{1+2-4}) \\
&= atan2(0, -1) = \pi
\end{aligned}
$$

Chapter V: Jacobians

5.1 Analytical Jacobian Matrix

Example

SCARA Robot

$$^{R}T_{H} = \begin{bmatrix} c_{1+2-4} & s_{1+2-4} & 0 & a_1c_1 + a_2c_{1+2} \\ s_{1+2-4} & -c_{1+2-4} & 0 & a_1s_1 + a_2s_{1+2} \\ 0 & 0 & -1 & d_1 - q_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$J(Q) = \begin{bmatrix} \frac{\partial f_x}{\partial q_1} & \cdots & \frac{\partial f_x}{\partial q_n} \\ \vdots & & \vdots \\ \frac{\partial f_{\phi n}}{\partial q_1} & \cdots & \frac{\partial f_{\phi n}}{\partial q_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial a_1c_1+a_2c_{1+2}}{\partial q_1} & \frac{\partial a_1c_1+a_2c_{1+2}}{\partial q_2} & \frac{\partial a_1c_1+a_2c_{1+2}}{\partial q_3} & \frac{\partial a_1c_1+a_2c_{1+2}}{\partial q_4} \\ \frac{\partial a_1s_1+a_2s_{1+2}}{\partial q_1} & \frac{\partial a_1s_1+a_2s_{1+2}}{\partial q_2} & \frac{\partial a_1s_1+a_2s_{1+2}}{\partial q_3} & \frac{\partial a_1s_1+a_2s_{1+2}}{\partial q_4} \\ \frac{\partial d_1-q_3-d_4}{\partial q_1} & \frac{\partial d_1-q_3-d_4}{\partial q_2} & \frac{\partial d_1-q_3-d_4}{\partial q_3} & \frac{\partial d_1-q_3-d_4}{\partial q_4} \\ \frac{\partial q_1+q_2-q_4}{\partial q_1} & \frac{\partial q_1+q_2-q_4}{\partial q_2} & \frac{\partial q_1+q_2-q_4}{\partial q_3} & \frac{\partial q_1+q_2-q_4}{\partial q_4} \\ \frac{\partial 0}{\partial q_1} & \frac{\partial 0}{\partial q_2} & \frac{\partial 0}{\partial q_3} & \frac{\partial 0}{\partial q_4} \\ \frac{\partial \pi}{\partial q_1} & \frac{\partial \pi}{\partial q_2} & \frac{\partial \pi}{\partial q_3} & \frac{\partial \pi}{\partial q_4} \end{bmatrix}$$

$$= \begin{bmatrix} -a_1s_1 - a_2s_{1+2} & -a_2s_{1+2} & 0 & 0 \\ a_1c_1 + a_2c_{1+2} & a_2c_{1+2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Chapter V: Jacobians

5.1 Analytical Jacobian Matrix

Example
SCARA Robot

Ordenant cada terme:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\phi}_a \\ \dot{\phi}_o \\ \dot{\phi}_n \end{bmatrix} = \begin{bmatrix} -a_1 s_1 - a_2 s_{1+2} & -a_2 s_{1+2} & 0 & 0 \\ a_1 c_1 + a_2 c_{1+2} & a_2 c_{1+2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}$$

The z component cannot be controlled (it is independent) by q_1 or q_2 (looking at the previous drawing, we can see that the speed in z only depends on the first joint).

Chapter V: Jacobians

5.1 Analytical Jacobian matrix

If the velocities of the joints are:

$$\dot{q}_1 = \frac{\pi}{2} rad/s \qquad \dot{q}_2 = \frac{\pi}{2} rad/s \qquad \dot{q}_3 = 1m/s$$

Determine the linear velocity of the EE at the following positions:

$$q_1 = 0rad \quad q_2 = {}^{\pi}\!/_{2}\, rad$$

$$a_1 = 1m \qquad a_2 = {}^{1}\!/_{2}\, m$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -{}^{1}\!/_{2} & -{}^{1}\!/_{2} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} {}^{\pi}\!/_{2} \\ {}^{\pi}\!/_{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -{}^{\pi}\!/_{2} \\ {}^{\pi}\!/_{2} \\ -1 \end{bmatrix}$$

$$q_1 = {}^{\pi}\!/_{2}\, rad \qquad q_2 = {}^{\pi}\!/_{2}\, rad$$

$$a_1 = 1m \qquad a_2 = {}^{1}\!/_{2}\, m$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -{}^{1}\!/_{2} & -{}^{1}\!/_{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} {}^{\pi}\!/_{2} \\ {}^{\pi}\!/_{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -{}^{\pi}\!/_{2} \\ -{}^{\pi}\!/_{2} \\ -1 \end{bmatrix}$$

Notice that although the speed of the joints remains constant, the speed of the EE changes according to the position!

5.2 Geometric Jacobian matrix

The Geometric Jacobian matrix, or Jacobian matrix, allows us to find the linear and angular velocity of the end effecto, depending on the velocities of the joints:

$$\boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{J_v}(Q) \\ \boldsymbol{J_\omega}(Q) \end{bmatrix}}_{\boldsymbol{J_g}(Q)} \cdot \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

Chapter V: Jacobians

5.2 Geometric Jacobian matrix

The Jacobian of the linear velocity, ***J_v*** , is equivalent to the part of the analytical Jacobian:

$$\boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \boldsymbol{J_v(Q)} \cdot \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

Chapter V: Jacobians

5.2 Geometric Jacobian matrix

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$$\boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \boldsymbol{J_v}(\boldsymbol{Q}) \cdot \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

Apply the chain rule:

$$v_x = \dot{p}_x = \frac{\partial f_x}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial f_x}{\partial q_2}\dot{q}_n$$

$$v_y = \dot{p}_y = \frac{\partial f_y}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial f_y}{\partial q_2}\dot{q}_n$$

$$v_z = \dot{p}_z = \frac{\partial f_z}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial f_z}{\partial q_2}\dot{q}_n$$

Chapter V: Jacobians

5.2 Geometric Jacobian matrix

The Jacobian of the linear velocity, ***J_v***, is equivalent to the part of the analytical Jacobian:

$$\boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \boldsymbol{J_v(Q)} \cdot \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

Apply the chain rule:

$$v_x = \dot{p}_x = \frac{\partial f_x}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial f_x}{\partial q_n}\dot{q}_n$$

$$v_y = \dot{p}_y = \frac{\partial f_y}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial f_y}{\partial q_n}\dot{q}_n$$

$$v_z = \dot{p}_z = \frac{\partial f_z}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial f_z}{\partial q_n}\dot{q}_n$$

Written as a matrix equation as:

$$\boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f_x}{\partial q_1} & \cdots & \frac{\partial f_x}{\partial q_n} \\ \frac{\partial f_y}{\partial q_1} & \cdots & \frac{\partial f_x}{\partial q_n} \\ \frac{\partial f_z}{\partial q_1} & \cdots & \frac{\partial f_x}{\partial q_n} \end{bmatrix}}_{\boldsymbol{J_v(Q)}} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

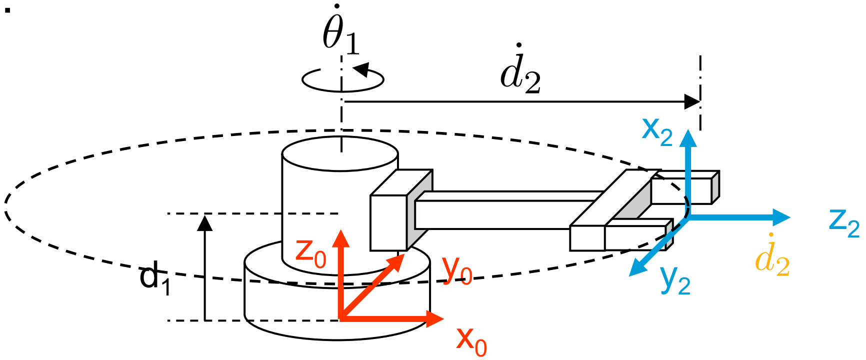
Chapter V: Jacobians

5.2 Geometric Jacobian matrix

- Let us compute ***J_g* *(Q)*** for the ϴ-d:

- Starting from the forward kinematics:

$$ {}^{R}\boldsymbol{T}_{\boldsymbol{H}} = \begin{bmatrix} 0 & -c_1 & s_1 & d_2 s_1 \\ 0 & -s_1 & -c_1 & -d_2 c_1 \\ 1 & 0 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



- The Jacobian ***J_g* *(Q)*** *is given by*:

$$ \boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{J_v}(Q) \\ \boldsymbol{J_\omega}(Q) \end{bmatrix} \cdot \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} $$

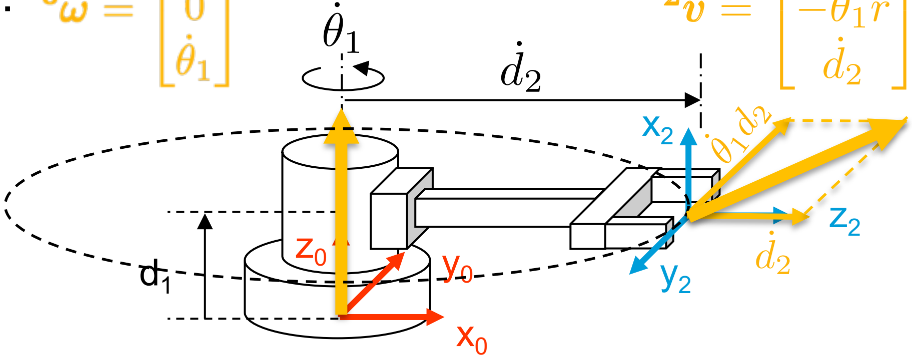
Chapter V: Jacobians

5.2 Geometric Jacobian matrix

- Let us compute $\boldsymbol{J_g}$ ***(Q)*** for the ϴ-d:

- Starting from the forward kinematics: $^{0}\boldsymbol{\omega} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$ $\quad ^{2}\boldsymbol{v} = \begin{bmatrix} 0 \\ -\dot{\theta}_1 r \\ \dot{d}_2 \end{bmatrix}$

$$^{R}\boldsymbol{T_H} = \begin{bmatrix} 0 & -c_1 & s_1 & d_2 s_1 \\ 0 & -s_1 & -c_1 & -d_2 c_1 \\ 1 & 0 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



- The Jacobian $\boldsymbol{J_g}$ ***(Q)*** *is given by*:

$$\boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{J_v}(Q) \\ \boldsymbol{J_\omega}(Q) \end{bmatrix} \cdot \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} \left\{ \begin{matrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} d_2 c_1 & s_1 \\ d_2 s_1 & -c_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix} \\ \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix} \end{matrix} \right\} \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} d_2 c_1 & s_1 \\ d_2 s_1 & -c_1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix}$$

5.2 Geometric Jacobian matrix

- The Jacobian of the angular velocity, ***J*$_\omega$** , we can obtain it from the rotation matrix ***R:***

$$ {}^{R}\boldsymbol{T}_{\boldsymbol{H}} = \begin{bmatrix} \boldsymbol{R}_{\boldsymbol{3\times 3}} & \boldsymbol{p}_{\boldsymbol{3\times 1}} \\ \boldsymbol{0}_{\boldsymbol{1\times 3}} & 1 \end{bmatrix} $$

- We know that: $[\omega]_\times = \dot{\boldsymbol{R}}\boldsymbol{R}^{\boldsymbol{T}}$ where: $[\boldsymbol{\omega}]_\times = \boldsymbol{\Omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$

  velocity tensor

- We can express $\boldsymbol{\Omega}$ as the product of a matrix that depends on ***Q***, $\boldsymbol{J_\omega(Q)}$, by $\dot{\boldsymbol{Q}}$

$$ \boldsymbol{\Omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = \boldsymbol{J_\omega(Q)} \cdot \dot{\boldsymbol{Q}} $$

5.2 Geometric Jacobian matrix

Example
SCARA Robot

$$\boldsymbol{\Omega} = \frac{d\boldsymbol{R}}{dt}\boldsymbol{R}^{T}$$

$${}^{\boldsymbol{R}}\boldsymbol{T}_{\boldsymbol{H}} = \left[\begin{array}{c|c} \overset{\boldsymbol{R}}{\boxed{\begin{matrix} c_{1+2-4} & s_{1+2-4} & 0 \\ s_{1+2-4} & -c_{1+2-4} & 0 \\ 0 & 0 & -1 \end{matrix}}} & \begin{matrix} a_1 c_1 + a_2 c_{1+2} \\ a_1 s_1 + a_2 s_{1+2} \\ d_1 - q_3 - d_4 \end{matrix} \\ \begin{matrix} 0 & \quad 0 & \quad 0 \end{matrix} & 1 \end{array}\right]$$

Chapter V: Jacobians

5.2 Geometric Jacobian matrix

Example
SCARA Robot

$$\boldsymbol{\Omega} = \frac{d\boldsymbol{R}}{dt}\boldsymbol{R}^T$$

$${}^{\boldsymbol{R}}\boldsymbol{T}_{\boldsymbol{H}} = \left[\begin{array}{cc} \overset{\boldsymbol{R}}{\boxed{\begin{matrix} c_{1+2-4} & s_{1+2-4} & 0 \\ s_{1+2-4} & -c_{1+2-4} & 0 \\ 0 & 0 & -1 \end{matrix}}} & \begin{matrix} a_1 c_1 + a_2 c_{1+2} \\ a_1 s_1 + a_2 s_{1+2} \\ d_1 - q_3 - d_4 \end{matrix} \\ \begin{matrix} 0 & \quad 0 & \quad 0 \end{matrix} & 1 \end{array}\right]$$

$$\frac{d\boldsymbol{R}}{dt} = \begin{bmatrix} -s_{1+2-4}\dot{q}_1 - s_{1+2-4}\dot{q}_2 + s_{1+2-4}\dot{q}_4 & c_{1+2-4}\dot{q}_1 + c_{1+2-4}\dot{q}_2 - c_{1+2-4}\dot{q}_4 & 0 \\ c_{1+2-4}\dot{q}_1 + c_{1+2-4}\dot{q}_2 - c_{1+2-4}\dot{q}_4 & s_{1+2-4}\dot{q}_1 + s_{1+2-4}\dot{q}_2 - s_{1+2-4}\dot{q}_4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -s_{1+2-4} & c_{1+2-4} & 0 \\ c_{1+2-4} & s_{1+2-4} & 0 \\ 0 & 0 & 0 \end{bmatrix} (\dot{q}_1 + \dot{q}_2 - \dot{q}_4)$$

$$\boldsymbol{\Omega} = \begin{bmatrix} -s_{1+2-4} & c_{1+2-4} & 0 \\ c_{1+2-4} & s_{1+2-4} & 0 \\ 0 & 0 & 0 \end{bmatrix} (\dot{q}_1 + \dot{q}_2 - \dot{q}_4) \begin{bmatrix} c_{1+2-4} & s_{1+2-4} & 0 \\ s_{1+2-4} & -c_{1+2-4} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -s_{1+2-4}c_{1+2-4} + s_{1+2-4}c_{1+2-4} & -s_{1+2-4}^2 - c_{1+2-4}^2 & 0 \\ c_{1+2-4}^2 + s_{1+2-4}^2 & c_{1+2-4}s_{1+2-4} - s_{1+2-4}c_{1+2-4} & 0 \\ 0 & 0 & 0 \end{bmatrix} (\dot{q}_1 + \dot{q}_2 - \dot{q}_4)$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (\dot{q}_1 + \dot{q}_2 - \dot{q}_4)$$

Chapter V: Jacobians

5.2 Geometric Jacobian matrix

Example
SCARA Robot

$$\Omega = \frac{d\boldsymbol{R}}{dt}\boldsymbol{R}^T \quad \Bigg| \quad {}^{\boldsymbol{R}}\boldsymbol{T}_{\boldsymbol{H}} = \begin{bmatrix} \overset{\boldsymbol{R}}{\boxed{\begin{matrix} c_{1+2-4} & s_{1+2-4} & 0 \\ s_{1+2-4} & -c_{1+2-4} & 0 \\ 0 & 0 & -1 \end{matrix}}} & \begin{matrix} a_1c_1 + a_2c_{1+2} \\ a_1s_1 + a_2s_{1+2} \\ d_1 - q_3 - d_4 \end{matrix} \\ \begin{matrix} 0 & \quad 0 & \quad 0 \end{matrix} & 1 \end{bmatrix}$$

Chapter V: Jacobians

5.2 Geometric Jacobian matrix

Example
SCARA Robot

$$\boldsymbol{\Omega} = \frac{d\boldsymbol{R}}{dt}\boldsymbol{R}^T$$

$$^{\boldsymbol{R}}\boldsymbol{T_H} = \begin{bmatrix} \overset{\boldsymbol{R}}{\boxed{\begin{matrix} c_{1+2-4} & s_{1+2-4} & 0 \\ s_{1+2-4} & -c_{1+2-4} & 0 \\ 0 & 0 & -1 \end{matrix}}} & \begin{matrix} a_1c_1 + a_2c_{1+2} \\ a_1s_1 + a_2s_{1+2} \\ d_1 - q_3 - d_4 \end{matrix} \\ \begin{matrix} 0 & \quad 0 & \quad 0 \end{matrix} & 1 \end{bmatrix}$$

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)$$

$$\begin{aligned} \omega_x &= 0 \\ \omega_y &= 0 \\ \omega_z &= \dot{q}_1 + \dot{q}_2 - \dot{q}_4 \end{aligned} \quad \Longrightarrow \quad \boldsymbol{J_\omega(Q)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

Chapter V: Jacobians

5.2 Geometric Jacobian matrix

Example
SCARA Robot

$$\boldsymbol{\Omega} = \frac{d\boldsymbol{R}}{dt}\boldsymbol{R}^T \quad \Bigg| \quad {}^{\boldsymbol{R}}\boldsymbol{T}_{\boldsymbol{H}} = \begin{bmatrix} \overset{\boldsymbol{R}}{\boxed{\begin{matrix} c_{1+2-4} & s_{1+2-4} & 0 \\ s_{1+2-4} & -c_{1+2-4} & 0 \\ 0 & 0 & -1 \end{matrix}}} & \begin{matrix} a_1c_1 + a_2c_{1+2} \\ a_1s_1 + a_2s_{1+2} \\ d_1 - q_3 - d_4 \end{matrix} \\ \begin{matrix} 0 & \qquad 0 & \qquad 0 \end{matrix} & 1 \end{bmatrix}$$

$$J_v(Q) = \begin{bmatrix} -a_1s_1 - a_2s_{1+2} & -a_2s_{1+2} & 0 & 0 \\ a_1c_1 + a_2c_{1+2} & a_2c_{1+2} & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$J_\omega(Q) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

$$J_g(Q) = \begin{bmatrix} J_v(Q) \\ J_\omega(Q) \end{bmatrix} = \begin{bmatrix} -a_1s_1 - a_2s_{1+2} & -a_2s_{1+2} & 0 & 0 \\ a_1c_1 + a_2c_{1+2} & a_2c_{1+2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

5.3 Reverse Jacobian Matrix

- It is used to determine the joint velocities from the velocities of the end effector of the robot.

$$\begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} = \boldsymbol{J_a}(\boldsymbol{Q})^{-1} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \phi_a \\ \phi_o \\ \phi_n \end{bmatrix} \qquad \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} = \boldsymbol{J_g}(\boldsymbol{Q})^{-1} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Calculation of the inverse Jacobian, options:

1. If **n=6** (J_{6x6})
   - **Invert the symbolic Jacobian**.
     - It's not always easy: J depends on trigonometric functions!
   - **Invert the numerical Jacobian.**
     - Evaluate J(Q) for the current Q and invert it ⇒ it is necessary to recompute J(Q) each time Q changes.
   - **Calculate the Symbolic Inverse Jacobian from the Inverse Kinematics.**

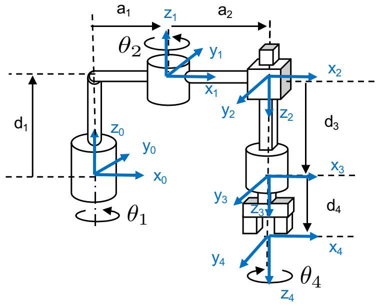
2. If **n<6** it is possible to do the pseudoinverse. $J_e(Q)^+ = \left(J(Q)^T \cdot J(Q)\right)^{-1} J(Q)^T$
   - Might happen that the desired EE Velocity could not be achieved due to missing DOFs.
   - The left pseudo-inverse will provide the EE velocity having the least possible error.

3. If **n>6** there are infinitely many solutions: $J_d(Q)^+ = J(Q)^T \left(J(Q) \cdot J(Q)^T\right)^{-1}$
   - The right pseudo-inverse must be used. It minimizes the norm of the joint velocities providing the desired EE velocity

Chapter V: Jacobians

5.3 Inverse Jacobian Matrix, singular configurations

Singularities of the SCARA robot



$$\begin{bmatrix}\dot{p}_x\\ \dot{p}_y\\ \dot{p}_z\end{bmatrix} = \underbrace{\begin{bmatrix}-a_1s_1-a_2s_{1+2} & -a_2s_{1+2} & 0 & 0\\ a_1c_1+a_2c_{1+2} & a_2c_{1+2} & 0 & 0\\ 0 & 0 & -1 & 0\end{bmatrix}}_{\boldsymbol{J_v}(\boldsymbol{Q})}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\\ \dot{q}_4\end{bmatrix}$$

They arise at the Q where det(Jv(Q))=0 where the inverse does not exist:

$$det(\boldsymbol{J_v}(\boldsymbol{Q})) = a_2c_{1+2}(a_1s_1+a_2s_{1+2}) - a_2s_{1+2}(a_1c_1+a_2c_{1+2})$$

It becomes zero when:

$$a_2c_{1+2}(a_1s_1+a_2s_{1+2}) = a_2s_{1+2}(a_1c_1+a_2c_{1+2})$$

which happens when q_2 = 0 or pi,

$$a_2c_1(a_1s_1+a_2s_1) = a_2s_1(a_1c_1+a_2c_1)$$
$$a_2c_1s_1(a_1+a_2) = a_2s_1c_1(a_1+a_2)$$

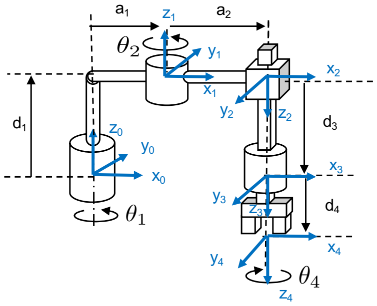
5.3 Inverse Jacobian Matrix, singular configurations

- **The singular configurations** of a robot are those joint positions where the Jacobian determinant is therefore annulled:
  - **the inverse Jacobian does not exist** (it is necessary to divide by the determinant to calculate the inverse).
  - **The robot loses some of its degree of freedom** (it loses mobility in some Cartesian direction).

Chapter V: Jacobians

5.3 Inverse Jacobian Matrix, singular configurations

Singularities of the SCARA robot

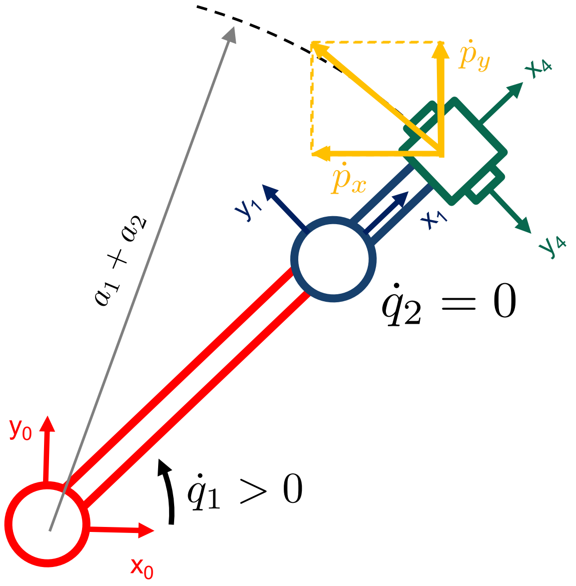


$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \underbrace{\begin{bmatrix} -a_1 s_1 - a_2 s_{1+2} & -a_2 s_{1+2} & 0 & 0 \\ a_1 c_1 + a_2 c_{1+2} & a_2 c_{1+2} & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}}_{\boldsymbol{J_v(Q)}} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}$$

5.3 Inverse Jacobian Matrix, singular configurations

Singularities of the SCARA robot

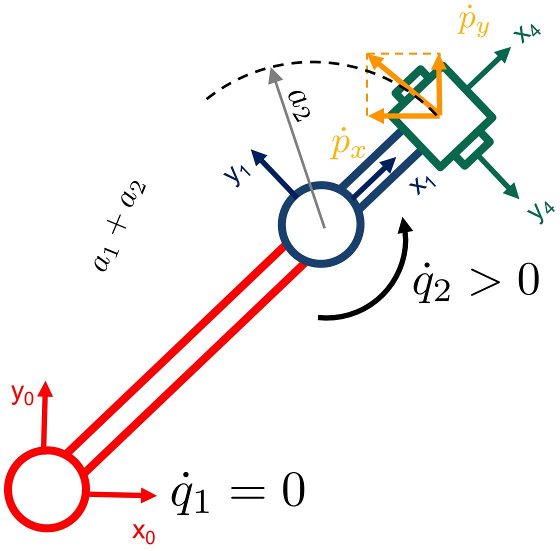
$$q_2 = 0 \Rightarrow \qquad \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \begin{bmatrix} -(a_1+a_2)s_1 & -a_2 s_1 & 0 & 0 \\ (a_1+a_2)c_1 & a_2 c_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}$$



$$\dot{p}_x = -(a_1+a_2)s_1\dot{q}_1$$
$$\dot{p}_y = (a_1+a_2)c_1\dot{q}_1$$

Singularities of the SCARA robot

$$q_2 = 0 \Rightarrow \quad \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \begin{bmatrix} -(a_1+a_2)s_1 & -a_2 s_1 & 0 & 0 \\ (a_1+a_2)c_1 & a_2 c_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}$$



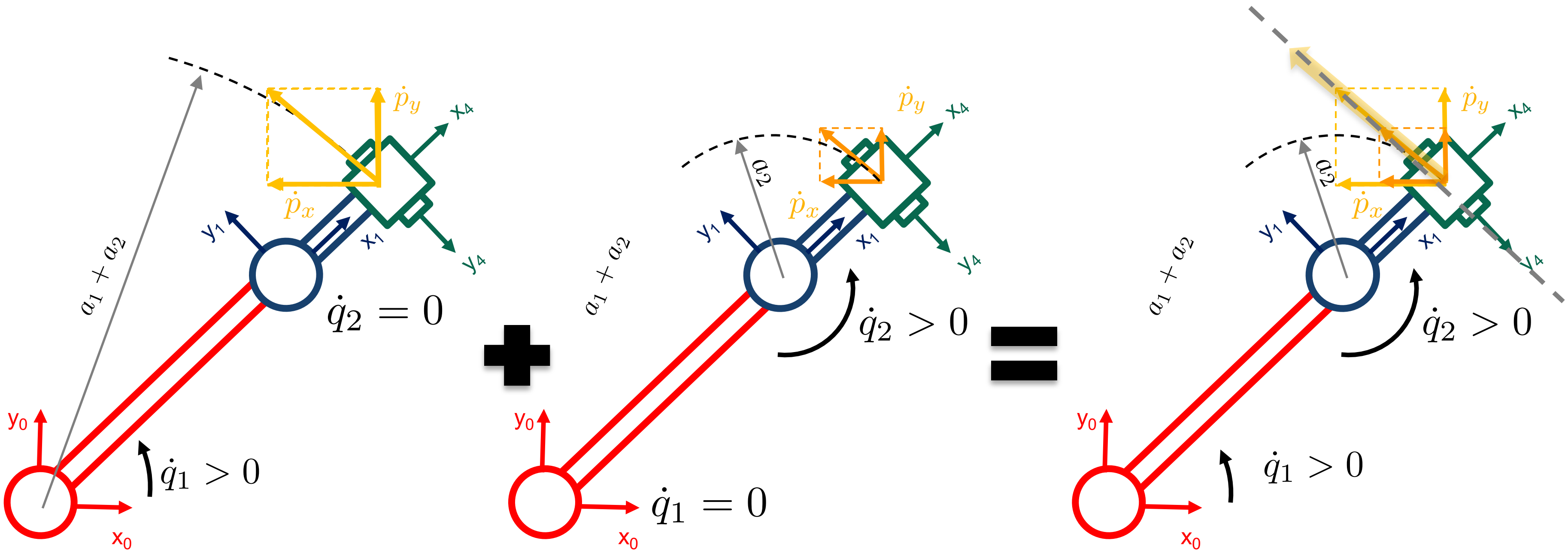
$$\dot{p}_x = -a_2 s_1 \dot{q}_1$$

$$\dot{p}_y = a_2 c_1 \dot{q}_1$$

Singularities of the SCARA robot

$$q_2 = 0 \Rightarrow \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \begin{bmatrix} -(a_1+a_2)s_1 & -a_2 s_1 & 0 & 0 \\ (a_1+a_2)c_1 & a_2 c_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}$$

We can only generate speeds in this direction



$$\dot{p}_x = -(a_1+a_2)s_1\dot{q}_1$$
$$\dot{p}_y = (a_1+a_2)c_1\dot{q}_1$$

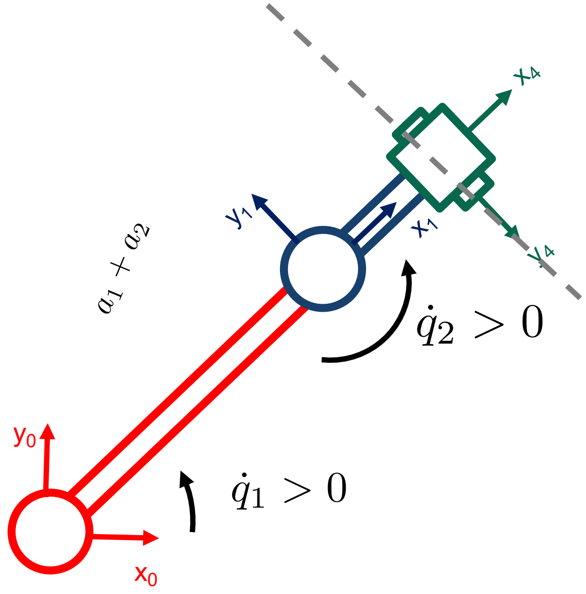
$$\dot{p}_x = -a_2 s_1 \dot{q}_2$$
$$\dot{p}_y = a_2 c_1 \dot{q}_2$$

$$\dot{p}_x = -(a_1+a_2)s_1\dot{q}_1 - a_2 s_1 \dot{q}_2$$
$$\dot{p}_y = (a_1+a_2)c_1\dot{q}_1 + a_2 c_1 \dot{q}_2$$

Singularities of the SCARA robot

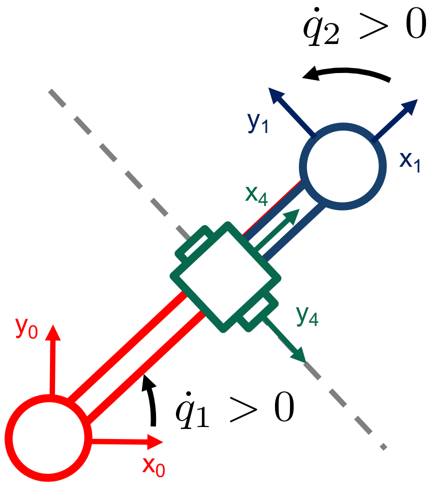
$q_2 = 0$

$$\begin{bmatrix}\dot p_x\\ \dot p_y\\ \dot p_z\end{bmatrix} = \begin{bmatrix}-(a_1+a_2)s_1 & -a_2 s_1 & 0 & 0\\ (a_1+a_2)c_1 & a_2 c_1 & 0 & 0\\ 0 & 0 & -1 & 0\end{bmatrix}\cdot\begin{bmatrix}\dot q_1\\ \dot q_2\\ \dot q_3\\ \dot q_4\end{bmatrix}$$



$q_2 = \pi$

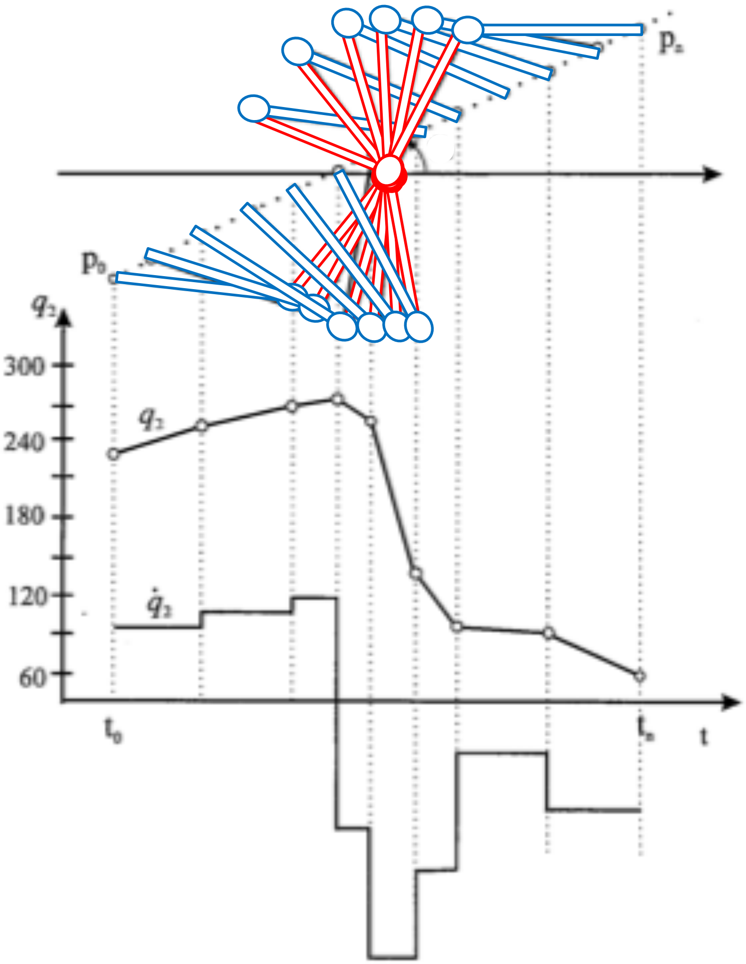
$$\begin{bmatrix}\dot p_x\\ \dot p_y\\ \dot p_z\end{bmatrix} = \begin{bmatrix}(a_1+a_2)s_1 & a_2 s_1 & 0 & 0\\ -(a_1+a_2)c_1 & -a_2 c_1 & 0 & 0\\ 0 & 0 & -1 & 0\end{bmatrix}\cdot\begin{bmatrix}\dot q_1\\ \dot q_2\\ \dot q_3\\ \dot q_4\end{bmatrix}$$

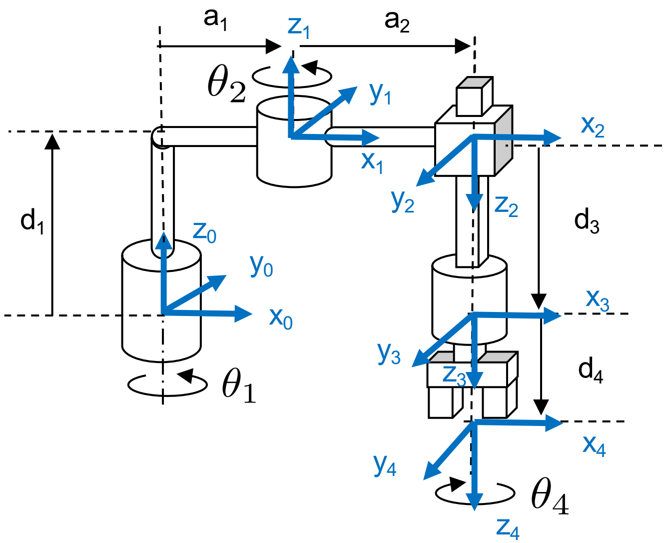


- **Singularities at the limit of the work area**

Chapter V: Jacobians

5.3 Inverse Jacobian Matrix, singular configurations

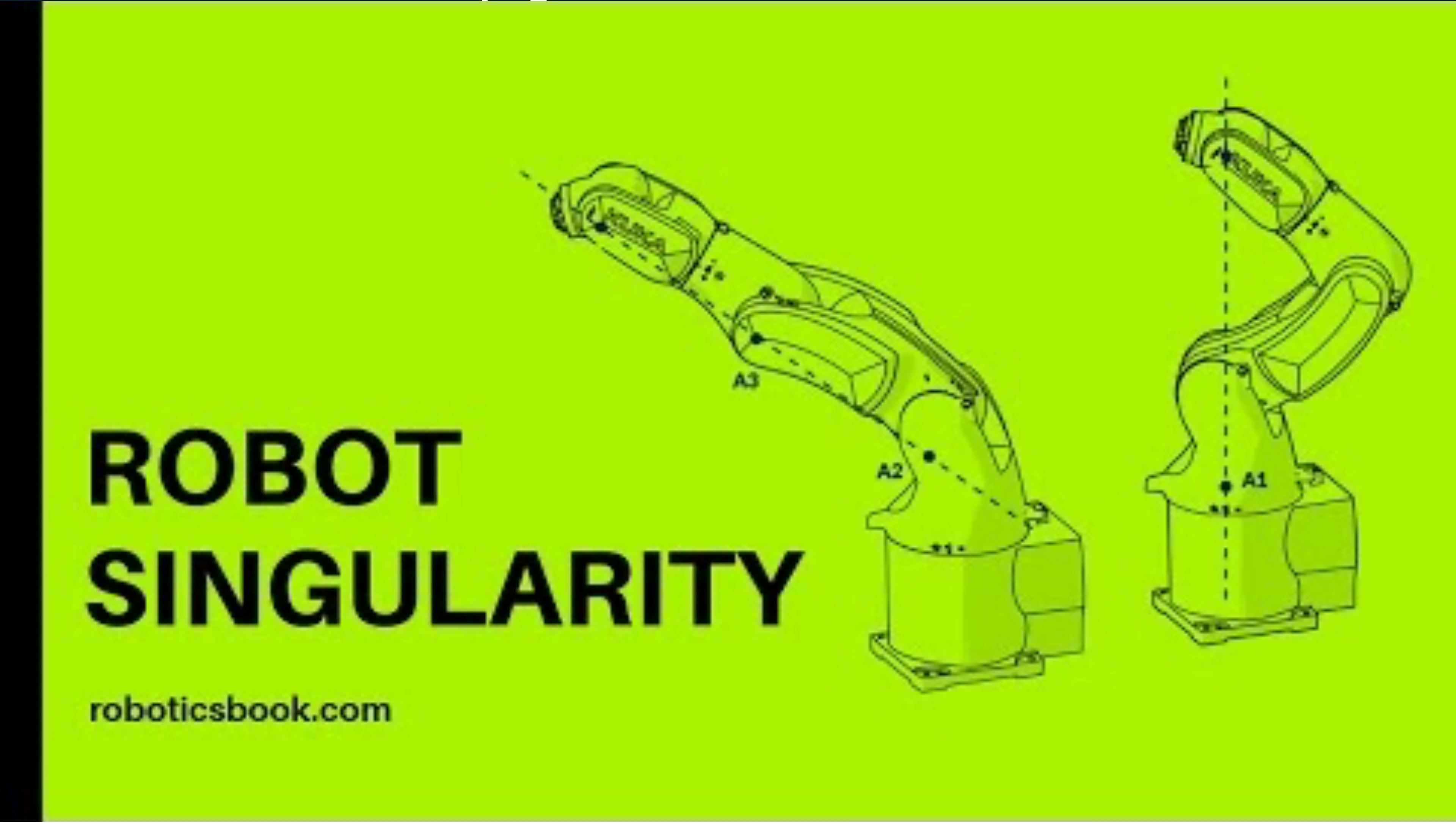




- The robot follows a straight line to get from p_0 to p_n .
- However, within the trajectory there is a singular configuration.
- Note the abrupt change in the speed of q_2 , making real movement unfeasible.

Chapter V: Jacobians

5.4 Jacobian Matrix: Propagation of Velocities

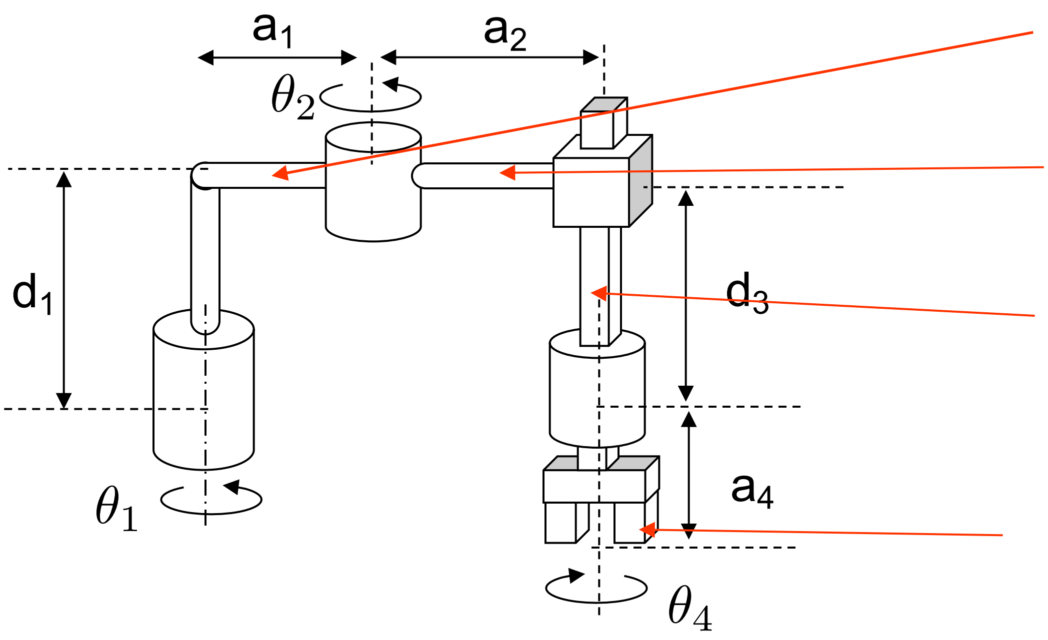


https://www.youtube.com/watch?v=BJnZvwAE0PY

Chapter V: Jacobians

5.4 Jacobian Matrix: Propagation of Velocities

- Calculation of the direct Jacobian from the propagation of the linear and angular velocities of each joint.
- It allows to calculate the matrix:
  - **Symbolic**
  - **Numerical:** In this case it is a generic solution, depending exclusively on the DH parameters.



The speed depends on the speed of the joint 1

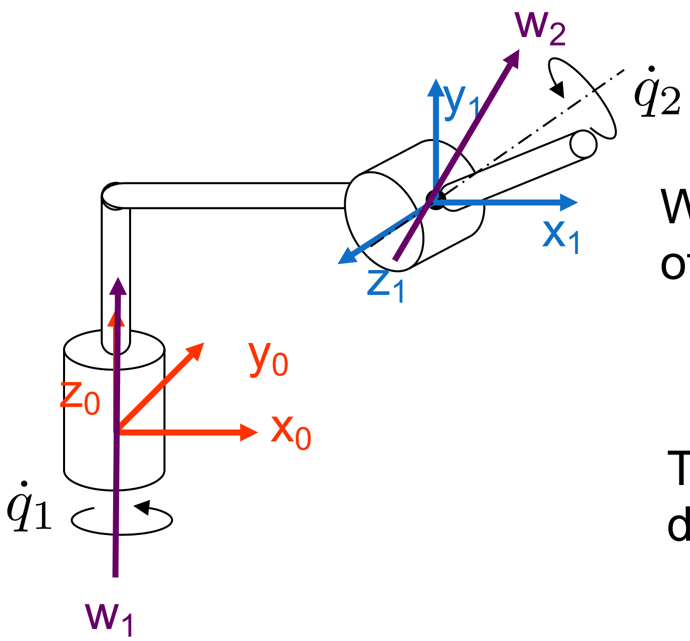
The speed depends on the joint 1 and 2

The speed depends on the joint 1, 2 and 3

The speed depends on the joint 1, 2, 3 and 4

5.4 Jacobian Matrix: Propagation of Velocities

Angular joints: angular velocity



Where do the angular velocities of each joint point to?

$$^0w_1 = {}^0z_0 \cdot \dot{q}_1$$

The angular velocity w_1 goes in the direction z_0 .

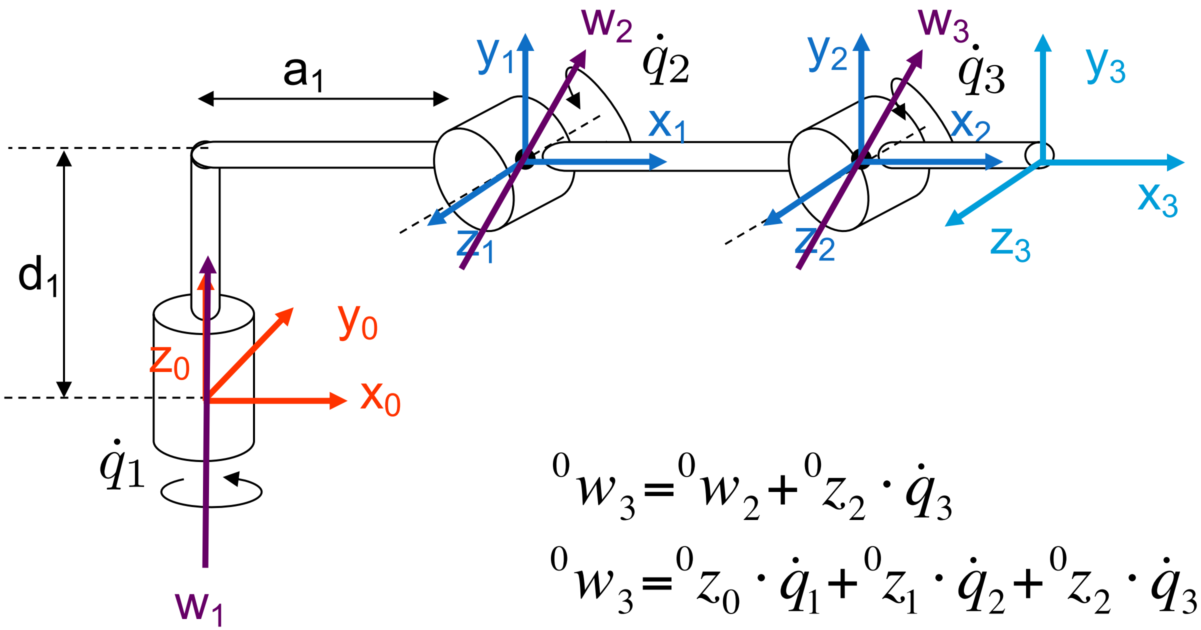
- All the points of the same link have the same angular velocity, therefore the vector w_1 can be represented at any point.
- The angular velocity w_2 is a combination of w_1 and the variation of q_2 over time => it will not go in a privileged direction!

$$^0w_2 = {}^0w_1 + {}^0z_1 \cdot \dot{q}_2$$

Chapter V: Jacobians

5.4 Jacobian Matrix: Velocity Propagation

Angular joints: angular velocity



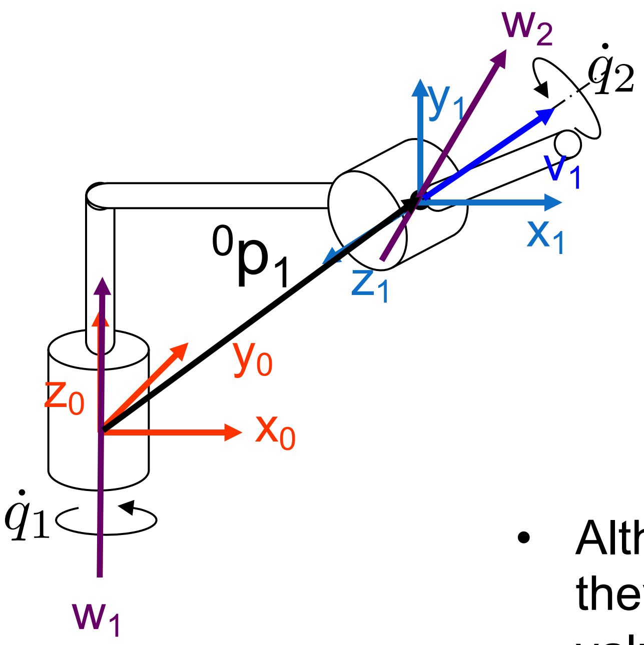
$${}^0w_3 = {}^0w_2 + {}^0z_2 \cdot \dot{q}_3$$

$${}^0w_3 = {}^0z_0 \cdot \dot{q}_1 + {}^0z_1 \cdot \dot{q}_2 + {}^0z_2 \cdot \dot{q}_3$$

So in general: $${}^0w_n = {}^0z_0 \cdot \dot{q}_1 + {}^0z_1 \cdot \dot{q}_2 + \cdots + {}^0z_{n-1} \cdot \dot{q}_n$$

5.4 Jacobian Matrix: Propagation of Velocities

Angular joints: linear speed



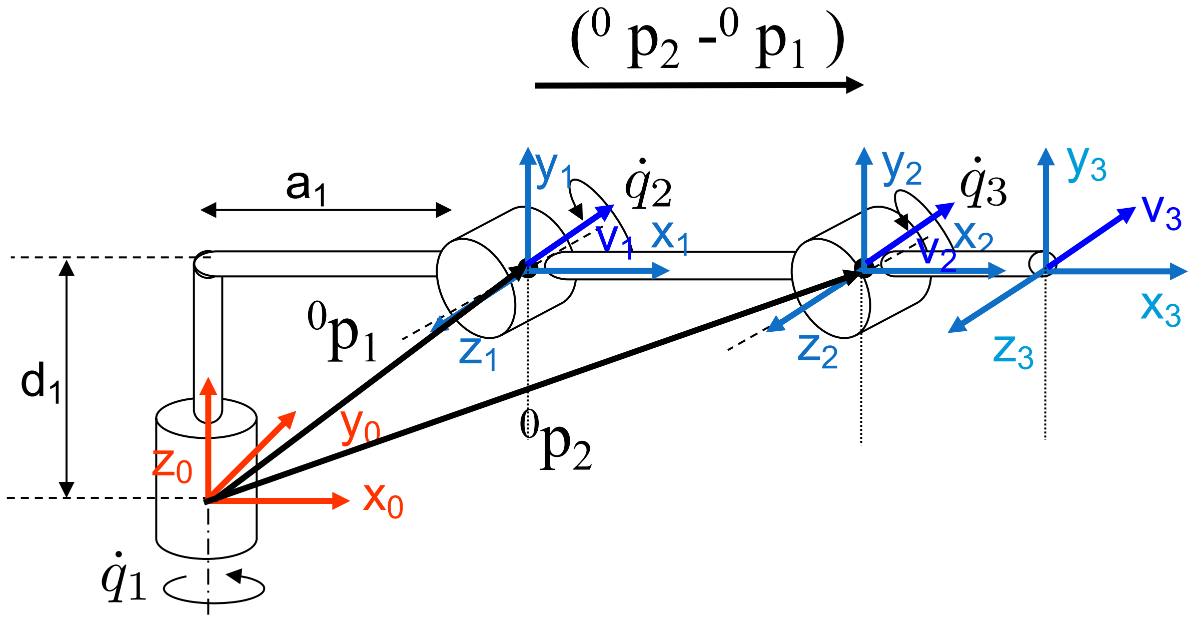
$$\vec{v} = \vec{w} \times \vec{r}$$

$$v_1 = w_1 \times {}^0p_1$$

$$v_1 = \left({}^0z_0 \times {}^0p_1\right)\dot{q}_1$$

- Although the joints are only angular, they also provide a linear velocity value!
- The linear velocity does **depend** on the location of the point in the link!!!

Angular joints: linear speed

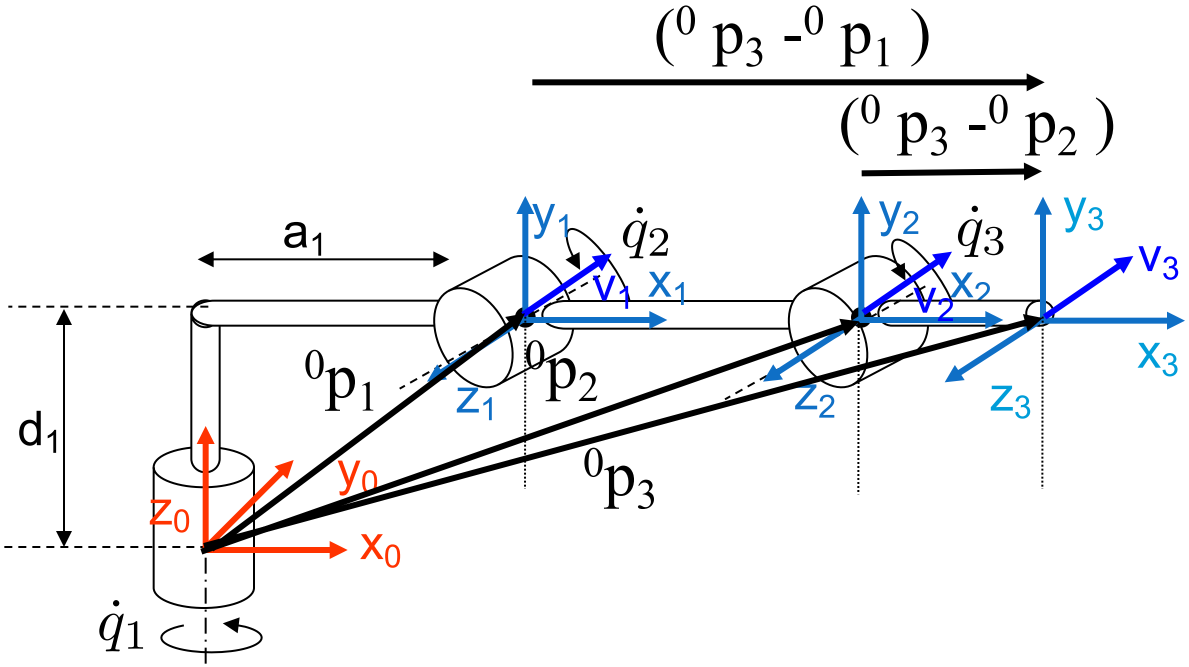


$$v_2 = \left({}^0z_0 \times {}^0p_2\right)\dot{q}_1 + \left({}^0z_1 \times \left({}^0p_2 - {}^0p_1\right)\right)\dot{q}_2$$

Chapter V: Jacobians

5.4 Jacobian Matrix: Propagation of Velocities

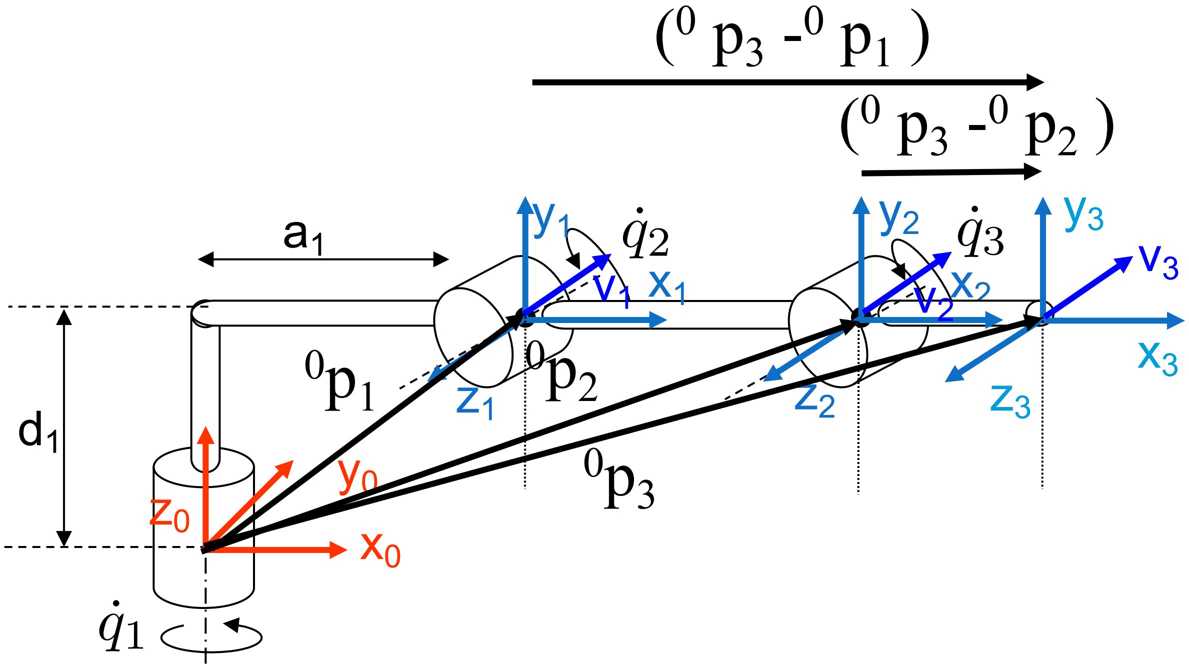
Angular joints: linear speed



$$v_3 = \left({}^0z_0 \times {}^0p_3\right)\dot{q}_1 + \left({}^0z_1 \times \left({}^0p_3 - {}^0p_1\right)\right)\dot{q}_2 + \left({}^0z_2 \times \left({}^0p_3 - {}^0p_2\right)\right)\dot{q}_3$$

5.4 Jacobian Matrix: Propagation of Velocities

Angular joints: linear speed



So in general:

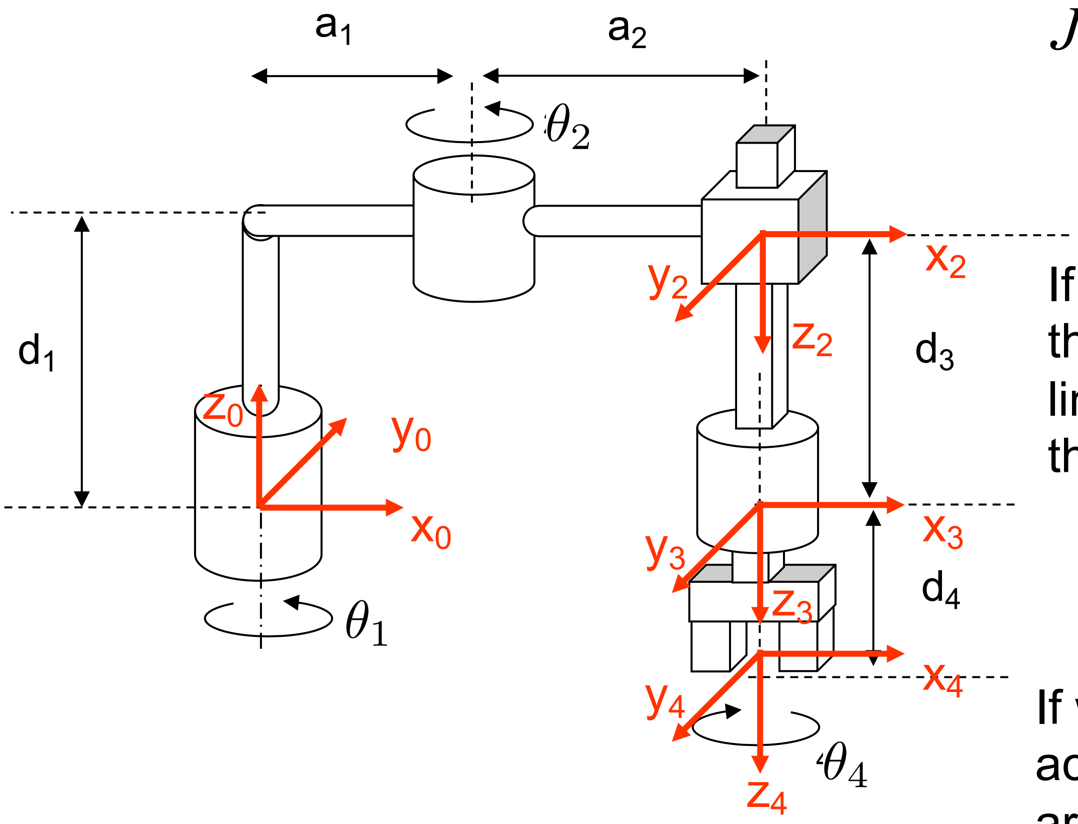
$$v_n = \left({}^0z_0 \times {}^0p_n\right)\dot{q}_1 + \cdots + \left({}^0z_{i-1} \times \left({}^0p_n - {}^0p_{i-1}\right)\right)\dot{q}_i + \cdots + \left({}^0z_{n-1} \times \left({}^0p_n - {}^0p_{n-1}\right)\right)\dot{q}_n$$

5.4 Jacobian Matrix: Propagation of Velocities

Prismatic joints:

If the joint is prismatic, **it does not provide angular velocity**, but linear velocity. Moreover, this **new linear velocity will be at the z-axis** of the joint:

$$J_i = \begin{pmatrix} {}^{0}(z_{i-1}) \\ 0 \end{pmatrix}$$



If this link (3) moves, the speed will only be linear and will go in the direction z_2.

$$v_4 = {}^{0}z_2 \cdot \dot{q}_3$$

If we don't take into account the angular articulations!

5.4 Jacobian Matrix: Propagation of Velocities

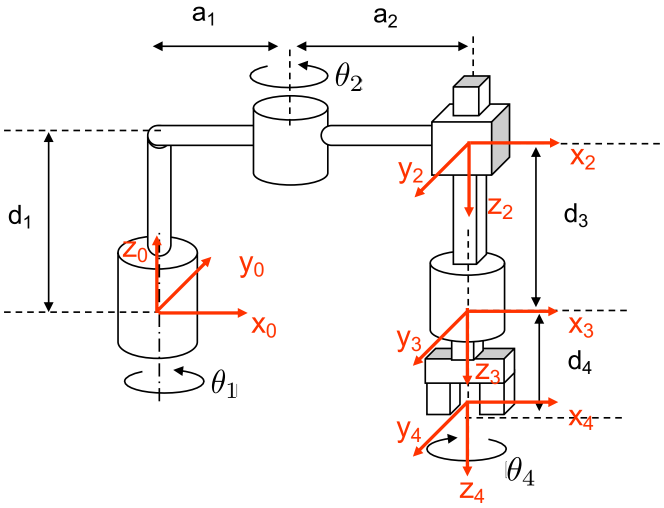
- We have just seen a way to find the Jacobian by going link by link, and without the need to derive. In short, the method consists of:

$$\begin{pmatrix} v_n \\ w_n \end{pmatrix} = J \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = \left(J_1 \middle| J_2 \middle| \cdots \middle| J_n \right) \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$

$$\text{on} \begin{cases} J_i = \begin{pmatrix} {}^0 z_{i-1} \\ 0 \end{pmatrix} & \text{If the joint is prismatic} \\ \\ J_i = \begin{pmatrix} {}^0 z_{i-1} \times \left({}^0 p_n - {}^0 p_{i-1} \right) \\ {}^0 z_{i-1} \end{pmatrix} & \text{If the joint is angular} \end{cases}$$

5.4 Jacobian Matrix: Propagation of Velocities

Example
SCARA Robot



$$\begin{pmatrix} v_n \\ w_n \end{pmatrix} = J \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = \left(J_1 | J_2 | \cdots | J_n\right) \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$

$$\text{on} \begin{cases} J_i = \begin{pmatrix} {}^0z_{i-1} \\ 0 \end{pmatrix} \\ J_i = \begin{pmatrix} {}^0z_{i-1} \times \left({}^0p_n - {}^0p_{i-1}\right) \\ {}^0z_{i-1} \end{pmatrix} \end{cases}$$

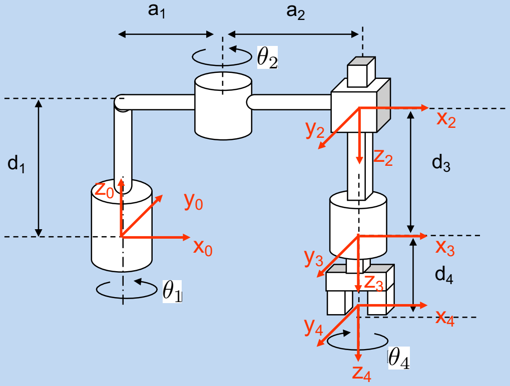
$$J_1 = \begin{pmatrix} {}^0z_0 \times \left({}^0p_4 - {}^0p_0\right) \\ {}^0z_0 \end{pmatrix} \quad J_2 = \begin{pmatrix} {}^0z_1 \times \left({}^0p_4 - {}^0p_1\right) \\ {}^0z_1 \end{pmatrix} \quad J_4 = \begin{pmatrix} {}^0z_3 \times \left({}^0p_4 - {}^0p_3\right) \\ {}^0z_3 \end{pmatrix} \quad J_3 = \begin{pmatrix} {}^0z_2 \\ 0 \end{pmatrix}$$

Chapter V: Jacobians

5.4 Jacobian Matrix: Propagation of Velocities

Example

SCARA Robot



$$\begin{pmatrix} v_n \\ w_n \end{pmatrix} = J \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = \left(J_1 | J_2 | \cdots | J_n\right) \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} \text{ on } \begin{cases} J_i = \begin{pmatrix} {}^0z_{i-1} \\ 0 \end{pmatrix} \\ J_i = \begin{pmatrix} {}^0z_{i-1} \times \left({}^0p_n - {}^0p_{i-1}\right) \\ {}^0z_{i-1} \end{pmatrix} \end{cases}$$

$${}^0A_1 = \begin{pmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^1A_2 = \begin{pmatrix} c_2 & s_2 & 0 & a_1c_1 \\ s_2 & -c_2 & 0 & a_1s_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^2A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^3A_4 = \begin{pmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0A_2 = \begin{pmatrix} c_{1+2} & s_{1+2} & 0 & a_1c_1 + a_2c_{1+2} \\ s_{1+2} & -c_{1+2} & 0 & a_1s_1 + a_2s_{1+2} \\ 0 & 0 & -1 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0A_3 = \begin{pmatrix} c_{1+2} & s_{1+2} & 0 & a_1c_1 + a_2c_{1+2} \\ s_{1+2} & -c_{1+2} & 0 & a_1s_1 + a_2s_{1+2} \\ 0 & 0 & -1 & d_1 - q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

${}^0p_4$

$${}^0A_4 = \begin{pmatrix} c_{1+2-4} & s_{1+2-4} & 0 & \boxed{\begin{matrix} a_1c_1 + a_2c_{1+2} \\ a_1s_1 + a_2s_{1+2} \\ d_1 - q_3 - d_4 \end{matrix}} \\ s_{1+2-4} & -c_{1+2-4} & 0 & \\ 0 & 0 & -1 & \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

q1 angular joint

$$J_1 = \begin{bmatrix} {}^0z_0 \times \left({}^0p_4 - {}^0p_0\right) \\ {}^0z_0 \end{bmatrix} = \begin{bmatrix} -a_1s_1 - a_2s_{1+2} \\ a_1c_1 + a_2c_{1+2} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$${}^0z_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \left({}^0p_4 - {}^0p_0\right) = \begin{pmatrix} a_1c_1 + a_2c_{1+2} \\ a_1s_1 + a_2s_{1+2} \\ d_1 - q_3 - d_4 \end{pmatrix}$$

$${}^0z_0 \times \left({}^0p_4 - {}^0p_0\right) = \begin{pmatrix} i & j & k \\ 0 & 0 & 1 \\ a_1c_1 + a_2c_{1+2} & a_1s_1 + a_2s_{1+2} & d_1 - q_3 - d_4 \end{pmatrix}$$

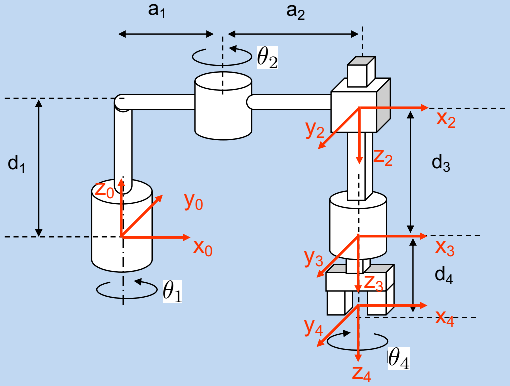
$$= \begin{pmatrix} -a_1s_1 - a_2s_{1+2} \\ a_1c_1 + a_2c_{1+2} \\ 0 \end{pmatrix}$$

Chapter V: Jacobians

5.4 Jacobian Matrix: Propagation of Velocities

Example

SCARA Robot



$$\begin{pmatrix} v_n \\ w_n \end{pmatrix} = J \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = \left(J_1 | J_2 | \cdots | J_n \right) \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} \text{ on } \left\{ \begin{array}{l} J_i = \begin{pmatrix} {}^0z_{i-1} \\ 0 \end{pmatrix} \\ \\ J_i = \begin{pmatrix} {}^0z_{i-1} \times \left({}^0p_n - {}^0p_{i-1} \right) \\ {}^0z_{i-1} \end{pmatrix} \end{array} \right.$$

$$\underbrace{{}^0A_1 = \left(\begin{array}{cccc} & & {}^0z_1 & {}^0p_1 \\ c_1 & -s_1 & \boxed{0} & \boxed{a_1c_1} \\ s_1 & c_1 & \boxed{0} & \boxed{a_1s_1} \\ 0 & 0 & \boxed{1} & \boxed{d_1} \\ 0 & 0 & 0 & 1 \end{array} \right) \quad {}^1A_2 = \begin{pmatrix} c_2 & s_2 & 0 & a_1c_1 \\ s_2 & -c_2 & 0 & a_1s_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \quad {}^2A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^3A_4 = \begin{pmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underbrace{{}^0A_2 = \begin{pmatrix} c_{1+2} & s_{1+2} & 0 & a_1c_1 + a_2c_{1+2} \\ s_{1+2} & -c_{1+2} & 0 & a_1s_1 + a_2s_{1+2} \\ 0 & 0 & -1 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^2A_3}$$

$$\underbrace{{}^0A_3 = \begin{pmatrix} c_{1+2} & s_{1+2} & 0 & a_1c_1 + a_2c_{1+2} \\ s_{1+2} & -c_{1+2} & 0 & a_1s_1 + a_2s_{1+2} \\ 0 & 0 & -1 & d_1 - q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^3A_4}$$

$${}^0A_4 = \left(\begin{array}{cccc} & & & {}^0p_4 \\ c_{1+2-4} & s_{1+2-4} & 0 & \boxed{a_1c_1 + a_2c_{1+2}} \\ s_{1+2-4} & -c_{1+2-4} & 0 & \boxed{a_1s_1 + a_2s_{1+2}} \\ 0 & 0 & -1 & \boxed{d_1 - q_3 - d_4} \\ 0 & 0 & 0 & 1 \end{array} \right)$$

q_2 angular joint

$$J_2 = \begin{bmatrix} {}^0z_1 \times \left({}^0p_4 - {}^0p_1 \right) \\ {}^0z_1 \end{bmatrix} = \begin{bmatrix} -a_2s_{1+2} \\ a_2c_{1+2} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$${}^0z_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \left({}^0p_4 - {}^0p_1 \right) = \begin{pmatrix} a_2c_{1+2} \\ a_2s_{1+2} \\ -q_3 - d_4 \end{pmatrix}$$

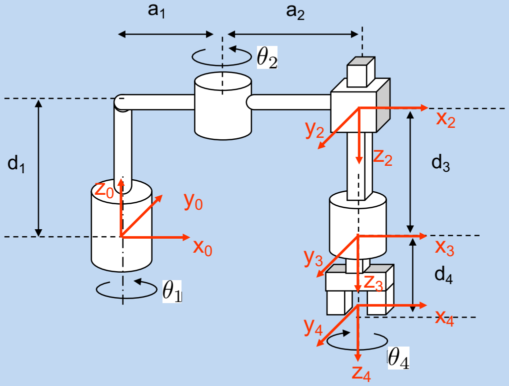
$${}^0z_1 \times \left({}^0p_4 - {}^0p_1 \right) = \begin{pmatrix} i & j & k \\ 0 & 0 & 1 \\ a_2c_{1+2} & a_2s_{1+2} & -q_3 - d_4 \end{pmatrix}$$

$$= \begin{pmatrix} -a_2s_{1+2} \\ a_2c_{1+2} \\ 0 \end{pmatrix}$$

5.4 Jacobian Matrix: Propagation of Velocities

Example

SCARA Robot



$$\begin{pmatrix} v_n \\ w_n \end{pmatrix} = J \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = \left(J_1 \middle| J_2 \middle| \cdots \middle| J_n\right) \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} \text{ on } \begin{cases} J_i = \begin{pmatrix} {}^0z_{i-1} \\ 0 \end{pmatrix} \\ J_i = \begin{pmatrix} {}^0z_{i-1} \times \left({}^0p_n - {}^0p_{i-1}\right) \\ {}^0z_{i-1} \end{pmatrix} \end{cases}$$

$${}^0A_1 = \begin{pmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^1A_2 = \begin{pmatrix} c_2 & s_2 & 0 & a_1c_1 \\ s_2 & -c_2 & 0 & a_1s_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^2A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^3A_4 = \begin{pmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0Z_2$$

$${}^0A_2 = \begin{pmatrix} c_{1+2} & s_{1+2} & 0 & a_1c_1 + a_2c_{1+2} \\ s_{1+2} & -c_{1+2} & 0 & a_1s_1 + a_2s_{1+2} \\ 0 & 0 & -1 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0A_3 = \begin{pmatrix} c_{1+2} & s_{1+2} & 0 & a_1c_1 + a_2c_{1+2} \\ s_{1+2} & -c_{1+2} & 0 & a_1s_1 + a_2s_{1+2} \\ 0 & 0 & -1 & d_1 - q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0A_4 = \begin{pmatrix} c_{1+2-4} & s_{1+2-4} & 0 & a_1c_1 + a_2c_{1+2} \\ s_{1+2-4} & -c_{1+2-4} & 0 & a_1s_1 + a_2s_{1+2} \\ 0 & 0 & -1 & d_1 - q_3 - d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

<u>q_3 prismatic joint</u>

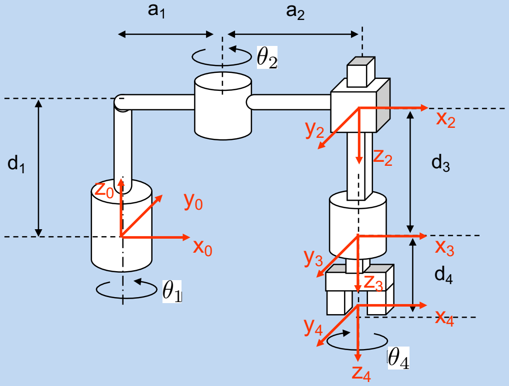
$$\boldsymbol{J_3} = \begin{bmatrix} {}^0\boldsymbol{z_2} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$${}^0z_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

5.4 Jacobian Matrix: Propagation of Velocities

Example

SCARA Robot



$$\begin{pmatrix} v_n \\ w_n \end{pmatrix} = J \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = \left(J_1 \middle| J_2 \middle| \cdots \middle| J_n \right) \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} \text{ on } \begin{cases} J_i = \begin{pmatrix} {}^0z_{i-1} \\ 0 \end{pmatrix} \\ J_i = \begin{pmatrix} {}^0z_{i-1} \times \left({}^0p_n - {}^0p_{i-1} \right) \\ {}^0z_{i-1} \end{pmatrix} \end{cases}$$

$$\underbrace{{}^0A_1 = \begin{pmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^1A_2 = \begin{pmatrix} c_2 & s_2 & 0 & a_1c_1 \\ s_2 & -c_2 & 0 & a_1s_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \quad {}^2A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^3A_4 = \begin{pmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0A_2 = \begin{pmatrix} c_{1+2} & s_{1+2} & 0 & a_1c_1 + a_2c_{1+2} \\ s_{1+2} & -c_{1+2} & 0 & a_1s_1 + a_2s_{1+2} \\ 0 & 0 & -1 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0z_3 \qquad {}^0p_3$$

$${}^0A_3 = \begin{pmatrix} c_{1+2} & s_{1+2} & \boxed{0} & \boxed{a_1c_1 + a_2c_{1+2}} \\ s_{1+2} & -c_{1+2} & \boxed{0} & \boxed{a_1s_1 + a_2s_{1+2}} \\ 0 & 0 & \boxed{-1} & \boxed{d_1 - q_3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0p_4$$

$${}^0A_4 = \begin{pmatrix} c_{1+2-4} & s_{1+2-4} & 0 & \boxed{a_1c_1 + a_2c_{1+2}} \\ s_{1+2-4} & -c_{1+2-4} & 0 & \boxed{a_1s_1 + a_2s_{1+2}} \\ 0 & 0 & -1 & \boxed{d_1 - q_3 - d_4} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

q_4 angular joint

$$\mathbf{J_4} = \begin{bmatrix} {}^0\mathbf{z_3} \times \left({}^0\mathbf{p_4} - {}^0\mathbf{p_3} \right) \\ {}^0\mathbf{z_3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$${}^0z_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \qquad \left({}^0p_4 - {}^0p_3 \right) = \begin{pmatrix} 0 \\ 0 \\ -d_4 \end{pmatrix}$$

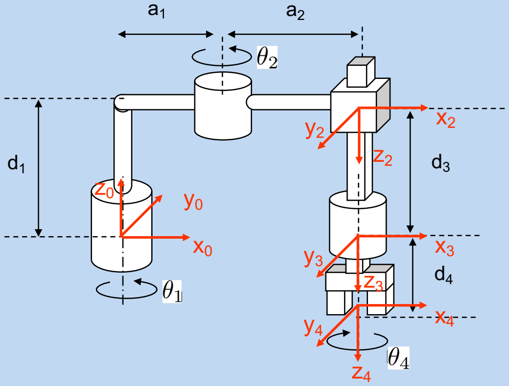
$${}^0z_3 \times \left({}^0p_4 - {}^0p_3 \right) = \begin{pmatrix} i & j & k \\ 0 & 0 & -1 \\ 0 & 0 & -d_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Chapter V: Jacobians

5.4 Jacobian Matrix: Propagation of Velocities

Example

SCARA Robot



$$\begin{pmatrix} v_n \\ w_n \end{pmatrix} = J\begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = \left(J_1 | J_2 | \cdots | J_n\right)\begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} \text{ on } \left\{ \begin{array}{l} J_i = \begin{pmatrix} {}^0z_{i-1} \\ 0 \end{pmatrix} \\ J_i = \begin{pmatrix} {}^0z_{i-1} \times \left({}^0p_n - {}^0p_{i-1}\right) \\ {}^0z_{i-1} \end{pmatrix} \end{array} \right.$$

$${}^0A_1 = \begin{pmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^1A_2 = \begin{pmatrix} c_2 & s_2 & 0 & a_1c_1 \\ s_2 & -c_2 & 0 & a_1s_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^2A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^3A_4 = \begin{pmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0A_2 = \begin{pmatrix} c_{1+2} & s_{1+2} & 0 & a_1c_1 + a_2c_{1+2} \\ s_{1+2} & -c_{1+2} & 0 & a_1s_1 + a_2s_{1+2} \\ 0 & 0 & -1 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0A_3 = \begin{pmatrix} c_{1+2} & s_{1+2} & 0 & a_1c_1 + a_2c_{1+2} \\ s_{1+2} & -c_{1+2} & 0 & a_1s_1 + a_2s_{1+2} \\ 0 & 0 & -1 & d_1 - q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0A_4 = \begin{pmatrix} c_{1+2-4} & s_{1+2-4} & 0 & a_1c_1 + a_2c_{1+2} \\ s_{1+2-4} & -c_{1+2-4} & 0 & a_1s_1 + a_2s_{1+2} \\ 0 & 0 & -1 & d_1 - q_3 - d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

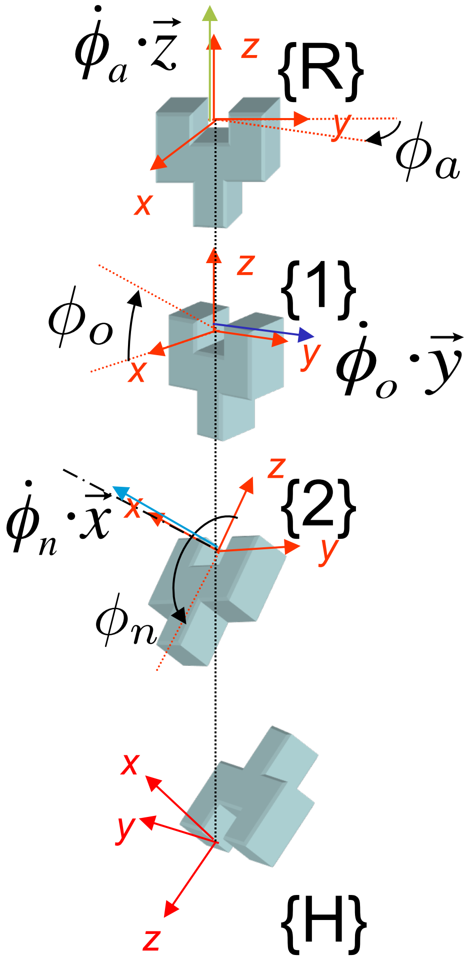
$$\boldsymbol{J_1} = \begin{bmatrix} -a_1s_1 - a_2s_{1+2} \\ a_1c_1 + a_2c_{1+2} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{J_2} = \begin{bmatrix} -a_2s_{1+2} \\ a_2c_{1+2} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{J_3} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{J_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\boldsymbol{J(Q)} = \begin{bmatrix} -a_1s_1 - a_2s_{1+2} & -a_2s_{1+2} & 0 & 0 \\ a_1c_1 + a_2c_{1+2} & a_2c_{1+2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

5.5 Relationship between analytical and geometrical Jacobian

Relationship between angular velocity and the RPY rate of change



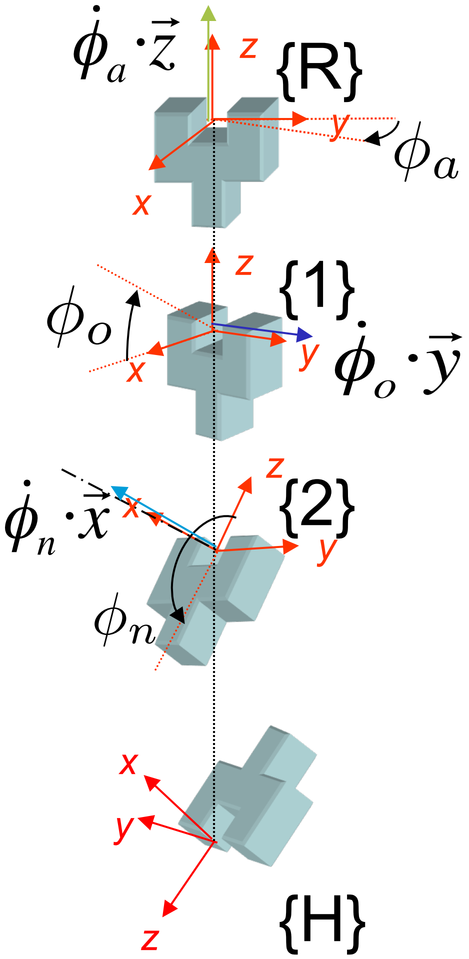
$$^{R}\omega = {}^{R}\begin{pmatrix} 0 \\ 0 \\ \dot{\phi}_a \end{pmatrix} + {}^{R}R_1 \, {}^{1}\begin{pmatrix} 0 \\ \dot{\phi}_o \\ 0 \end{pmatrix} + {}^{R}R_2 \, {}^{2}\begin{pmatrix} \dot{\phi}_n \\ 0 \\ 0 \end{pmatrix} =$$

$$= {}^{R}\begin{pmatrix} 0 \\ 0 \\ \dot{\phi}_a \end{pmatrix} + Rot(\phi_a, z) \, {}^{1}\begin{pmatrix} 0 \\ \dot{\phi}_o \\ 0 \end{pmatrix} + Rot(\phi_a, z) Rot(\phi_o, y) \, {}^{2}\begin{pmatrix} \dot{\phi}_n \\ 0 \\ 0 \end{pmatrix} =$$

$$= {}^{R}\begin{pmatrix} 0 \\ 0 \\ \dot{\phi}_a \end{pmatrix} + \begin{pmatrix} c\phi_a & -s\phi_a & 0 \\ s\phi_a & c\phi_a & 0 \\ 0 & 0 & 1 \end{pmatrix} {}^{1}\begin{pmatrix} 0 \\ \dot{\phi}_o \\ 0 \end{pmatrix} + \underbrace{\begin{pmatrix} c\phi_a & -s\phi_a & 0 \\ s\phi_a & c\phi_a & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\phi_o & 0 & s\phi_o \\ 0 & 1 & 0 \\ -s\phi_o & 0 & c\phi_o \end{pmatrix}}_{\begin{pmatrix} c\phi_a c\phi_o & -s\phi_a & c\phi_a s\phi_o \\ s\phi_a c\phi_o & c\phi_a & s\phi_a s\phi_o \\ -s\phi_o & 0 & c\phi_o \end{pmatrix}} {}^{2}\begin{pmatrix} \dot{\phi}_n \\ 0 \\ 0 \end{pmatrix}$$

5.5 Relationship between analytical and geometrical Jacobian

Relationship between angular velocity and the RPY rate of change



$$^{R}\omega = {}^{R}\begin{pmatrix} 0 \\ 0 \\ \dot{\phi}_a \end{pmatrix} + \begin{pmatrix} c\phi_a & -s\phi_a & 0 \\ s\phi_a & c\phi_a & 0 \\ 0 & 0 & 1 \end{pmatrix} {}^{1}\begin{pmatrix} 0 \\ \dot{\phi}_o \\ 0 \end{pmatrix} + \begin{pmatrix} c\phi_a c\phi_o & -s\phi_a & c\phi_a s\phi_o \\ s\phi_a c\phi_o & c\phi_a & s\phi_a s\phi_o \\ -s\phi_o & 0 & c\phi_a \end{pmatrix} {}^{2}\begin{pmatrix} \dot{\phi}_n \\ 0 \\ 0 \end{pmatrix} =$$

$$= {}^{R}\begin{pmatrix} 0 \\ 0 \\ \dot{\phi}_a \end{pmatrix} + {}^{R}\begin{pmatrix} -s\phi_a \dot{\phi}_o \\ c\phi_a \dot{\phi}_o \\ 0 \end{pmatrix} + {}^{R}\begin{pmatrix} c\phi_a c\phi_o \dot{\phi}_n \\ s\phi_a c\phi_o \dot{\phi}_n \\ -s\phi_o \dot{\phi}_n \end{pmatrix} = {}^{R}\begin{pmatrix} -s\phi_a \dot{\phi}_o + c\phi_a c\phi_o \dot{\phi}_n \\ c\phi_a \dot{\phi}_o + s\phi_a c\phi_o \dot{\phi}_n \\ \dot{\phi}_a - s\phi_o \dot{\phi}_n \end{pmatrix} \Rightarrow$$

$$^{R}\omega = \underbrace{\begin{pmatrix} 0 & -s\phi_a & c\phi_a c\phi_o \\ 0 & c\phi_a & s\phi_a c\phi_o \\ 1 & 0 & -s\phi_o \end{pmatrix}}_{B(\Gamma)} \underbrace{\begin{pmatrix} \dot{\phi}_a \\ \dot{\phi}_o \\ \dot{\phi}_n \end{pmatrix}}_{\dot{\Gamma}} \Rightarrow {}^{R}\begin{pmatrix} v \\ \omega \end{pmatrix} = \underbrace{\begin{pmatrix} I & 0 \\ 0 & B(\Gamma) \end{pmatrix} \underbrace{\begin{pmatrix} J_v \\ J_{\dot{\Gamma}} \end{pmatrix}}_{Jacob.analitic.}}_{Jac.geomètric} \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$

5.4 Jacobian Matrix: Propagation of Velocities

Example
Spherical Robot

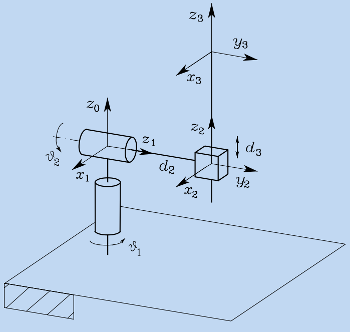


Fig. 2.22. Spherical arm

$$\begin{pmatrix} v_n \\ w_n \end{pmatrix} = J \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = \left(J_1 | J_2 | \cdots | J_n\right) \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} \text{ on } \begin{cases} J_i = \begin{pmatrix} {}^0 z_{i-1} \\ 0 \end{pmatrix} \\ J_i = \begin{pmatrix} {}^0 z_{i-1} \times \left({}^0 p_n - {}^0 p_{i-1}\right) \\ {}^0 z_{i-1} \end{pmatrix} \end{cases}$$

$$\mathbf{{}^0A_1} = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{{}^1A_2} = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{{}^2A_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{{}^1A_2} = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & -s_1d_2 \\ s_1c_2 & c_1 & s_1s_2 & c_1d_2 \\ -s_2 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{{}^1A_3} = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & c_1s_2d_3 - s_1d_2 \\ s_1c_2 & c_1 & s_1s_2 & s_1s_2d_3 + c_1d_2 \\ -s_2 & 0 & c_2 & c_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

<u>q_1 angular joint</u>

$$\mathbf{J_1} = \begin{bmatrix} \mathbf{{}^0z_0} \times (\mathbf{{}^0p_3} - \mathbf{{}^0\,p_1}) \\ \mathbf{{}^0z_0} \end{bmatrix} = \begin{bmatrix} -s_1s_2d_3 - c_1d_2 \\ c_1s_2d_3 - s_1d_2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{{}^0z_0} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{{}^0p_3} - \mathbf{{}^0\,p_0} = \begin{bmatrix} c_1s_2d_3 - s_1d_2 \\ s_1s_2d_3 + c_1d_2 \\ c_2d_3 \end{bmatrix}$$

$$\mathbf{{}^0z_0} = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ c_1s_2d_3 - s_1d_2 & s_1s_2d_3 + c_1d_2 & c_2d_3 \end{vmatrix} = \begin{bmatrix} -s_1s_2d_3 - c_1d_2 \\ c_1s_2d_3 - s_1d_2 \\ 0 \end{bmatrix}$$

Chapter V: Jacobians

5.4 Jacobian Matrix: Propagation of Velocities

Example

Spherical Robot

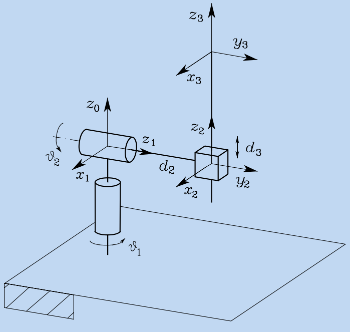


Fig. 2.22. Spherical arm

$$\begin{pmatrix} v_n \\ w_n \end{pmatrix} = J \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = \left(J_1 | J_2 | \cdots | J_n\right) \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} \text{ on } \left\{ \begin{array}{l} J_i = \begin{pmatrix} {}^0z_{i-1} \\ 0 \end{pmatrix} \\ \\ J_i = \begin{pmatrix} {}^0z_{i-1} \times \left({}^0p_n - {}^0p_{i-1}\right) \\ {}^0z_{i-1} \end{pmatrix} \end{array} \right.$$

$$\mathbf{{}^0A_1} = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{{}^1A_2} = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{{}^2A_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{{}^1A_2} = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & -s_1d_2 \\ s_1c_2 & c_1 & s_1s_2 & c_1d_2 \\ -s_2 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{{}^1A_3} = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & c_1s_2d_3 - s_1d_2 \\ s_1c_2 & c_1 & s_1s_2 & s_1s_2d_3 + c_1d_2 \\ -s_2 & 0 & c_2 & c_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

<u>q_2 angular joint</u>

$$\mathbf{J_2} = \begin{bmatrix} \mathbf{{}^0z_1 \times \left({}^0p_3 - {}^0p_1\right)} \\ \mathbf{{}^0z_1} \end{bmatrix} = \begin{bmatrix} c_1c_2d_3 \\ s_1c_2d_3 \\ -s_2d_3 \\ -s_1 \\ c_1 \\ 0 \end{bmatrix}$$

$$\mathbf{{}^0z_1} = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \quad \mathbf{{}^0p_3 - {}^0p_1} = \begin{bmatrix} c_1s_2d_3 - s_1d_2 \\ s_1s_2d_3 + c_1d_2 \\ c_2d_3 \end{bmatrix}$$

$$\mathbf{{}^0z_1 \times \left({}^0p_3 - {}^0p_1\right)} = \begin{vmatrix} i & j & k \\ -s_1 & c_1 & 0 \\ c_1s_2d_3 - s_1d_2 & s_1s_2d_3 + c_1d_2 & c_2d_3 \end{vmatrix} = \begin{bmatrix} c_1c_2d_3 \\ s_1c_2d_3 \\ -s_2d_3 \end{bmatrix}$$

5.4 Jacobian Matrix: Propagation of Velocities

Example

Spherical Robot

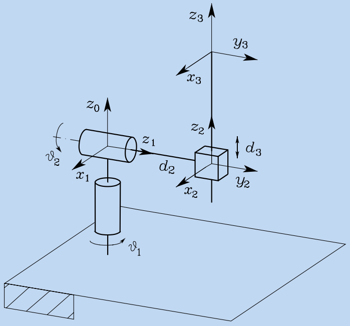


Fig. 2.22. Spherical arm

$$\begin{pmatrix} v_n \\ w_n \end{pmatrix} = J \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = \left(J_1 | J_2 | \cdots | J_n \right) \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} \text{ on } \left\{ \begin{array}{l} J_i = \begin{pmatrix} {}^0z_{i-1} \\ 0 \end{pmatrix} \\ \\ J_i = \begin{pmatrix} {}^0z_{i-1} \times \left({}^0p_n - {}^0p_{i-1} \right) \\ {}^0z_{i-1} \end{pmatrix} \end{array} \right.$$

$$\mathbf{{}^0A_1} = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{{}^1A_2} = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{{}^2A_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{{}^1A_2} = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & -s_1d_2 \\ s_1c_2 & c_1 & s_1s_2 & c_1d_2 \\ -s_2 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{{}^1A_3} = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & c_1s_2d_3 - s_1d_2 \\ s_1c_2 & c_1 & s_1s_2 & s_1s_2d_3 + c_1d_2 \\ -s_2 & 0 & c_2 & c_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

<u>q</u>

q_3 prismatic joint

$$\mathbf{J_3} = \begin{bmatrix} \mathbf{{}^0z_2} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} c_1s_2 \\ s_1s_2 \\ c_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{{}^0z_2} = \begin{bmatrix} c_1s_2 \\ s_1s_2 \\ c_2 \end{bmatrix}$$

5.4 Jacobian Matrix: Propagation of Velocities

Example

Spherical Robot

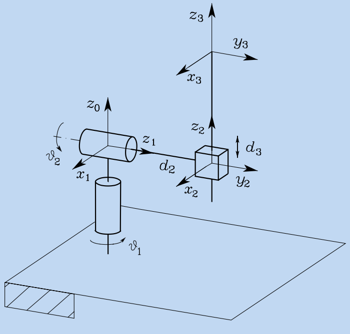


Fig. 2.22. Spherical arm

$$\begin{pmatrix} v_n \\ w_n \end{pmatrix} = J\begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = \left(J_1|J_2|\cdots|J_n\right)\begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} \text{ on } \begin{cases} J_i = \begin{pmatrix} {}^0z_{i-1} \\ 0 \end{pmatrix} \\ J_i = \begin{pmatrix} {}^0z_{i-1} \times \left({}^0p_n - {}^0p_{i-1}\right) \\ {}^0z_{i-1} \end{pmatrix} \end{cases}$$

$$\underbrace{{}^0A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \quad {}^2A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{{}^1A_2 = \begin{bmatrix} c_1c_2 & -s_1 & \boxed{c_1s_2} & -s_1d_2 \\ s_1c_2 & c_1 & \boxed{s_1s_2} & c_1d_2 \\ -s_2 & 0 & \boxed{c_2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad\qquad {}^2A_3}$$

$${}^1A_3 = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & c_1s_2d_3 - s_1d_2 \\ s_1c_2 & c_1 & s_1s_2 & s_1s_2d_3 + c_1d_2 \\ -s_2 & 0 & c_2 & c_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$J_1 = \begin{bmatrix} -s_1s_2d_3 - c_1d_2 \\ c_1s_2d_3 - s_1d_2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad J_2 = \begin{bmatrix} c_1c_2d_3 \\ s_1c_2d_3 \\ -s_2d_3 \\ -s_1 \\ c_1 \\ 0 \end{bmatrix} \quad J_3 = \begin{bmatrix} c_1s_2 \\ s_1s_2 \\ c_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J = \begin{bmatrix} -s_1s_2d_3 - c_1d_2 & c_1c_2d_3 & c_1s_2 \\ c_1s_2d_3 - s_1d_2 & s_1c_2d_3 & s_1s_2 \\ 0 & -s_2d_3 & c_2 \\ 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

5.4 Lineal Velocity Jacobian of the spherical robot RRP

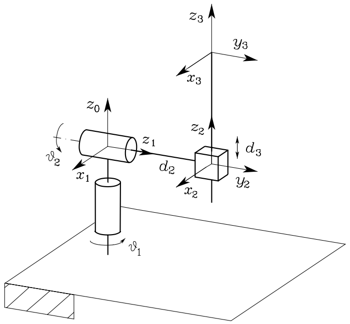


Fig. 2.22. Spherical arm

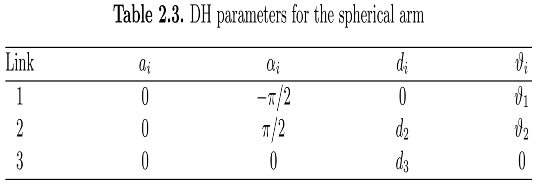


Table 2.3. DH parameters for the spherical arm

Link	a_i	α_i	d_i	ϑ_i
1	0	$-\pi/2$	0	ϑ_1
2	0	$\pi/2$	d_2	ϑ_2
3	0	0	d_3	0

$$^{\boldsymbol{R}}\boldsymbol{T_H} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & c_1s_2d_3 - s_1d_2 \\ s_1c_2 & c_1 & s_1s_2 & s_1s_2d_3 + c_1d_2 \\ -s_2 & 0 & c_2 & c_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{J_v(Q)} = \begin{bmatrix} \frac{\partial p_x}{\partial q_1} & \frac{\partial p_x}{\partial q_2} & \frac{\partial p_x}{\partial q_3} \\ \frac{\partial p_y}{\partial q_1} & \frac{\partial p_y}{\partial q_2} & \frac{\partial p_y}{\partial q_3} \\ \frac{\partial p_z}{\partial q_1} & \frac{\partial p_z}{\partial q_2} & \frac{\partial p_z}{\partial q_3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial c_1s_2d_3 - s_1d_2}{\partial q_1} & \frac{\partial c_1s_2d_3 - s_1d_2}{\partial q_2} & \frac{\partial c_1s_2d_3 - s_1d_2}{\partial q_3} \\ \frac{\partial s_1s_2d_3 + c_1d_2}{\partial q_1} & \frac{\partial s_1s_2d_3 + c_1d_2}{\partial q_2} & \frac{\partial s_1s_2d_3 + c_1d_2}{\partial q_3} \\ \frac{\partial c_2d_3}{\partial q_1} & \frac{\partial c_2d_3}{\partial q_2} & \frac{\partial c_2d_3}{\partial q_3} \end{bmatrix}$$

$$= \begin{bmatrix} -s_1s_2d_3 - c_1d_2 & c_1c_2d_3 & c_1s_2 \\ c_1s_2d_3 - s_1d_2 & s_1c_2d_3 & s_1s_2 \\ 0 & -s_2d_3 & c_2 \end{bmatrix}$$