Graph Coloring and Maximum Clique: A Branch and Bound Approach

EUROHPC Summit 2025

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Objectives

Develop an algorithm to determine the chromatic number $\chi(G)$ of a graph using a branch and bound technique, finding an optimized and parallel version of the algorithm using MPI and OpenMP on VEGA Supercomputer.

VEGA features:
CPU CLUSTER: 960 Nodes, each with two AMD Rome 7H12 CPUs.
GPU CLUSTER: 60 Nodes, each with two AMD EPYC 7H12 CPUs.

Mathematical Background

Graph Coloring involves assigning a unique color to each vertex of a graph G so that no two adjacent vertices share the same color. A graph is considered *k-colorable* if its vertices can be colored using at most k distinct colors while satisfying this constraint. The smallest number of colors required for a proper coloring is called the *chromatic number*, $\chi(G)$. Thus, the graph coloring problem focuses on determining this value.

A *clique* in a graph G is a subset of vertices in which every pair of vertices is directly connected by an edge. The **Maximum Clique** problem seeks to find the *clique number*, $\omega(G)$, which is the size of the largest such subset. Since a proper coloring must assign different colors to all vertices in a clique, it follows that: $\omega(G) \leq \chi(G)$

Methodology

Connected components: The graph is decomposed into connected components in order to have independent subproblems.

Branching: For every pair of non-adjacent vertices (v_1, v_2), two branches are created:

- Merge (v_1, v_2) $\rightarrow$ same color
- Add Edge (v_1, v_2) $\rightarrow$ different colors

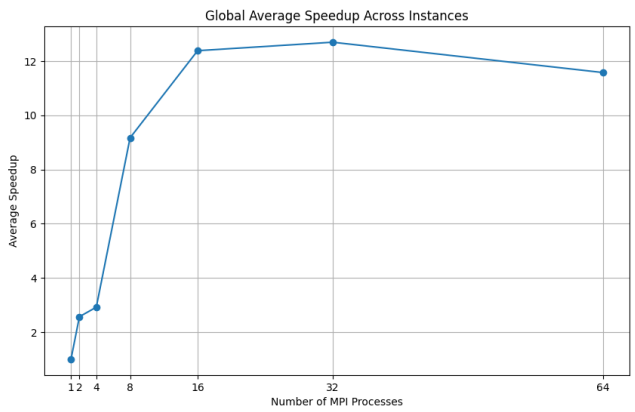
Bounds for $\chi(G)$:

- *Lower Bound:* As lower bound, the Maximum Clique is found thanks to a Bron-Kerbosch based algorithm, which provides $\omega(G) \leq \chi(G)$.
- *Upper Bound:* A DSATUR based heuristic provides a valid coloring, establishing an upper bound for the chromatic number such that $\chi(G) \leq \text{ub}(\chi(G))$.

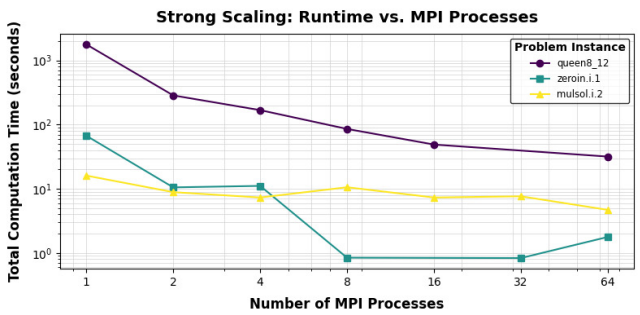
Parallelization:

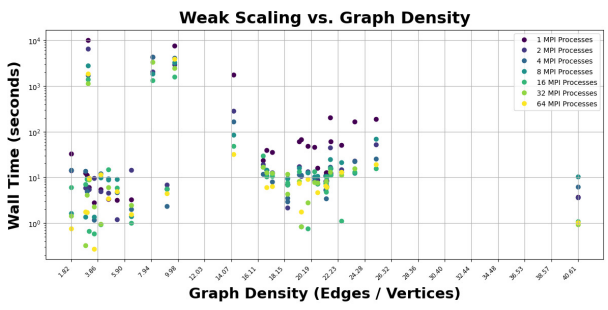
- *MPI:* Subproblems are statically distributed among processes to exploit coarse-grained parallelism.
- *OpenMP:* Within each MPI process, the recursive branch and bound calls are parallelized for larger subproblems.

Results



Up to 16 processors, the speedup rapidly increases as expected. Beyond that, it still improves slightly until 32, but then it moderately decreases. This decline suggests that communication overhead and load imbalance begin to outweigh the benefits of adding more processes.





Acknowledgements

The authors thank Dr. Di Wang for his valuable guidance and the EuroHPC staff for organizing the Challenge.
They also appreciate the VEGA supercomputer team for their support and for providing the computing resources.

References

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This project has received funding from the European High Performance Computing Joint Undertaking under grant agreement No. 101051997. EUROHPC SUMMIT 2025, Krakow — EUMaster4HPC.

See more: https://github.com/Challenge-Team-6

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