Multi-Objective Tail Assignment Problem: Formal Definition and QUBO Formulation

Quantum Optimization Framework

1 Mathematical Problem Formulation

We define the multi-objective tail assignment problem (TAP) as follows.

1.1 Sets and Variables

Let $T = \{1, \ldots, N_T\}$ be the set of aircraft (tails) and $F = \{1, \ldots, N_F\}$ be the set of flight legs.

Our primary decision variable is a binary variable $x_{i,j}$:

$$x_{i,j} = \begin{cases} 1 & \text{if tail } i \in T \text{ is assigned to flight } j \in F \\ 0 & \text{otherwise} \end{cases}$$

1.2 Objective Functions

We seek to minimize three competing objectives simultaneously.

1.2.1 Objective 1: Minimize Total Operational Cost (H_{cost})

Given a cost matrix $C_{i,j}$ representing the cost of tail i flying flight j:

$$\text{Minimize} \quad H_{\text{cost}} = \sum_{i \in T} \sum_{j \in F} C_{i,j} \cdot x_{i,j}$$

1.2.2 Objective 2: Maximize Aircraft Utilization (H_{util})

We model this by minimizing the variance in total flight duration across the fleet. Let D_j be the duration of flight j. The mean duration per tail is $\bar{D} = \frac{1}{N_T} \sum_{j \in F} D_j$.

$$\text{Minimize} \quad H_{\text{util}} = \sum_{i \in T} \left(\left(\sum_{j \in F} D_j \cdot x_{i,j} \right) - \bar{D} \right)^2$$

1.2.3 Objective 3: Minimize Propagated Delays (H_{delay})

We penalize assignments that create "tight connections." Let P_i be the set of "at-risk" sequential flight pairs (j, k) for tail i. Let $S_{i,j,k}$ be the pre-calculated penalty for tail i flying flight j immediately followed by flight k.

$$\text{Minimize} \quad H_{\text{delay}} = \sum_{i \in T} \sum_{(j,k) \in P_i} S_{i,j,k} \cdot x_{i,j} \cdot x_{i,k}$$

1.3 Constraints

The assignment must satisfy two hard operational constraints.

1.3.1 Constraint 1: Flight Coverage (C1)

Each flight j must be assigned to exactly one tail i.

$$\sum_{i \in T} x_{i,j} = 1 \quad \forall j \in F$$

1.3.2 Constraint 2: Aircraft Compatibility (C2)

A single tail i cannot be assigned to two incompatible flights. Let I_i be the set of all incompatible flight pairs (j, k) for tail i (e.g., flights that overlap in time or violate minimum turnaround).

$$x_{i,j} \cdot x_{i,k} = 0 \quad \forall i \in T, \forall (j, k) \in I_i$$

2 QUBO Formulation

We formulate the problem as a Quadratic Unconstrained Binary Optimization (QUBO) model by combining objectives with weights (λ) and constraints with penalties (γ).

2.1 Total Hamiltonian

The final Hamiltonian to be minimized is a weighted sum of all components:

$$H_{\text{QUBO}} = H_{\text{Constraints}} + H_{\text{Objectives}}$$

Where:

$$H_{\text{Constraints}} = H_{\text{C1}} + H_{\text{C2}}$$

$$H_{\text{Objectives}} = \lambda_{\text{cost}} H_{\text{cost}} + \lambda_{\text{util}} H_{\text{util}} + \lambda_{\text{delay}} H_{\text{delay}}$$

And γ_1, γ_2 are large penalty coefficients.

2.2 Constraint Hamiltonians (QUBO Form)

We convert the equality and inequality constraints into quadratic penalties.

2.2.1 Flight Coverage Penalty ($H_{\mathbf{C1}}$)

The constraint $\sum_i x_{i,j} = 1$ is quadratized as $\gamma_1 \sum_j (\sum_i x_{i,j} - 1)^2$. We ignore the constant offset term and use $x_{i,j}^2 = x_{i,j}$:

$$H_{\text{C1}} = \gamma_1 \sum_{j \in F} \left(2 \sum_{i \in T} \sum_{k \in T, k > i} x_{i,j} x_{k,j} - \sum_{i \in T} x_{i,j} \right)$$

2.2.2 Aircraft Compatibility Penalty ($H_{\mathbf{C2}}$)

This constraint is already in QUBO form. We simply sum the forbidden interactions:

$$H_{\text{C2}} = \gamma_2 \sum_{i \in T} \sum_{(j,k) \in I_i} x_{i,j} \cdot x_{i,k}$$

2.3 Objective Hamiltonians (QUBO Form)

We expand all objectives into their final linear (biases) and quadratic (couplings) terms.

2.3.1 Cost Objective (H_{cost})

This objective is already linear and serves as the linear bias for λ_{cost}.

$$H_{\text{cost}} = \sum_{i \in T} \sum_{j \in F} C_{i,j} \cdot x_{i,j}$$

2.3.2 Delay Objective (H_{delay})

This objective is already quadratic and serves as the quadratic coupling for λ_{delay}.

$$H_{\text{delay}} = \sum_{i \in T} \sum_{(j,k) \in P_i} S_{i,j,k} \cdot x_{i,j} \cdot x_{i,k}$$

2.3.3 Utilization Objective (H_{util})

We expand the squared term, ignoring the constant offset $\bar{D}^2$.

$$H_{\text{util}} = \sum_{i \in T} \left[\left(\sum_{j \in F} D_j x_{i,j} \right)^2 - 2\bar{D} \sum_{j \in F} D_j x_{i,j} \right]$$

Using $x_{i,j}^2 = x_{i,j}$, the squared sum $\left(\sum_j D_j x_{ij}\right)^2$ expands to $\sum_j D_j^2 x_{ij} + \sum_{j \neq k} D_j D_k x_{ij} x_{ik}$. We group the final expression by its linear and quadratic components:

$$H_{\text{util}} = \underbrace{\sum_{i \in T} \sum_{j \in F} (D_j^2 - 2\bar{D} D_j) x_{i,j}}_{\text{Linear (Biases)}} + \underbrace{\sum_{i \in T} \sum_{j \in F} \sum_{k \in F, k \neq j} (D_j D_k) x_{i,j} x_{i,k}}_{\text{Quadratic (Couplings)}}$$