Assignment - II
Mathematics for Machine Learning

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1. Vector Spaces

(a) Vectors : $\{(1,2,-3), (1,-3,2), (2,-1,5)\}$.
Check for basis of R^3.
Find coordinates of $(2,1,-2)$.

Consider the vectors : $(1,2,-3)$, $(1,-3,2)$ & $(2,-1,5)$.

Then, $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ -3 & 2 & 5 \end{bmatrix}$; $\cancel{\det(A) = 1(-3\cdot5 - (-1)2)}$

$\det(A) = 1(-3\cdot5 - (-1)(2)) - 1(2\cdot5 - (-1)(-3)) + 2(2\cdot2 - (-3)(-3))$

$= -13 - 7 - 10$

$= -30 \neq 0$

$\therefore$ Given vectors are linearly independent.
Hence, they span R^3 and also form the basis.

Consider, $(2,1,-2)$.

Then, $a(1,2,-3) + b(1,-3,2) + c(2,-1,5) = (2,1,-2)$

$$\rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ -3 & 2 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

This is of the form $AX = b$.

$$\rightarrow A:b = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & -3 & -1 & 1 \\ -3 & 2 & 5 & -2 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 + 3R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -5 & -5 & -3 \\ 0 & 5 & 11 & 4 \end{array}\right] \quad R_3 + R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -5 & -5 & -3 \\ 0 & 0 & 6 & 1 \end{array}\right]$$

Rewriting,

$$a + b + 2c = 2$$
$$-5b - 5c = -3$$
$$6c = 1 \longrightarrow c = 1/6$$

$$\longrightarrow -5b - \frac{5}{6} = -3 \longrightarrow b = \frac{13}{30}$$

$$\longrightarrow a + \frac{13}{30} + 2 \cdot \frac{1}{6} = 2 \longrightarrow a = \frac{37}{30}$$

$\therefore$ $a = \frac{37}{30}$, $b = \frac{13}{30}$, $c = 1/6$ are the coordinates.

(b) Vectors : $\{(1,1,1), (0,1,1), (0,1,-1)\}$

Check for basis of R^3.

Find coordinates of $(1,-1,2)$.

Given vectors : $(1,1,1), (0,1,1), (0,1,-1)$.

Then, $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$$\det(A) = 1\,(1 \cdot (-1) - 1 \cdot 1) - 0 + 0$$
$$= -2 \neq 0$$

$\therefore$ Given vectors are linearly independent

$\therefore$ They form a basis of R^3.

(2) Consider, $(1, -1, 2)$.

Then, $A : b = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 2 \end{array}\right]$ $R_2 - R_1$, $R_3 - R_1$

$\longrightarrow A : b = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right]$ $R_3 - R_2$

$\longrightarrow A : b = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & -2 & 3 \end{array}\right]$

Rewriting, $a = 1$

$b + c = -2$

$-2c = 3 \longrightarrow c = -3/2$

$\longrightarrow b - \frac{3}{2} = -2 \longrightarrow b = -1/2$

$\therefore$ $a = 1$, $b = -1/2$, $c = -3/2$ are the coordinates.

(c) Show that $W = \{(x, y, z) : ax + by - cz = 0\}$ is a subspace of $\mathbb{R}^3$.

Consider $u = (x_1, y_1, z_1)$ & $v = (x_2, y_2, z_2)$ belong to W.

$\longrightarrow ax_1 + by_1 - cz_1 = 0$; $ax_2 + by_2 - cz_2 = 0$

- <u>Zero Vector</u>: Consider $(x_1, y_1, z_1) = (0, 0, 0)$

$\longrightarrow a \cdot 0 + b \cdot 0 - c \cdot 0 = 0$

$\therefore$ zero vector $\in W$

- <u>Addition</u>: Consider $u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

Now, Consider $a(x_1 + x_2) + b(y_1 + y_2) - c(z_1 + z_2)$

$= (ax_1 + by_1 - cz_1) + (ax_2 + by_2 - cz_2)$

$= 0 + 0 = 0$

$\therefore$ Closed under addition.

- Scalar Multiplication:

Consider a scalar $k \in \mathbb{R}$.

Now, $kU = k(x_1, y_1, z_1)$

$= (kx_1, ky_1, kz_1)$

Consider $a.kx_1 + b\,ky_1 - c.kz_1$

$= k(ax_1 + by_1 - cz_1)$

$= k.0$

$= 0$

$\therefore$ Closed under scalar multiplication.

$\therefore$ ω is a subspace of $\mathbb{R}^3$.

(d) Show that $\omega = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \right\}$ is a subspace of $x^2 : x \in \mathbb{R}\}$

$M^2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$.

Consider $\omega = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} ; a, b, d \in \mathbb{R} \right\}$

- Zero Vector:

Consider $U = \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}$; $V = \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix}$ belong to ω.

- Zero Vector:

Consider $\phi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \omega$

$\therefore$ Zero vector belongs to ω.

- Addition:

Consider $U + V = \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix}$

$= \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 + 0 & d_1 + d_2 \end{bmatrix} \in \omega$

$\therefore$ Closed under addition

(3) • <u>Scalar Multiplication</u>:

Consider $k \in \mathbb{R}$ be a scalar.

Consider $kU = k \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}$

$= \begin{bmatrix} ka_1 & kb_1 \\ 0 & kd_1 \end{bmatrix} \in \omega$

$\therefore$ Closed under scalar multiplication.

$\therefore$ ω is a subspace of M^2.

(e) $\omega = \{b_0 + b_1 x : x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

Given ω is a subspt of $P_2(x) = \{a_0 + a_1 x + a_2 x^2 : x \in \mathbb{R}\}$

Given $\omega = \{b_0 + b_1 x : x \in \mathbb{R}\}$

Consider $p(x) = b_0 + b_1 x$; $q(x) = c_0 + c_1 x$

• <u>Zero Vector</u>:

Consider $b_0 = 0$; $b_1 = 0$

$\rightarrow b_0 + b_1 x = 0 \in \omega$

$\therefore$ zero vector belongs to ω.

• <u>Addition</u>:

Consider $p(x) + q(x) = b_0 + b_1 x + c_0 + c_1 x$

$= (b_0 + c_0) + (b_1 + c_1) x \in \omega$

$\therefore$ Closed under addition.

• <u>Scalar Multiplication</u>:

Consider $k \in \mathbb{R}$ be a scalar

Consider, $k \cdot p(x) = k(b_0 + b_1 x)$

$= (kb_0 + kb_1 x) \in \omega$

$\therefore$ Closed under scalar multiplication.

$\therefore$ ω is a subspace of $\mathbb{R}^3$.

• <u>Subspace</u>: A non-empty ω of a vector space V is called a subspace if:

(a) $U \in \omega, V \in \omega \rightarrow U + V \in \omega$

(b) $U \in \omega, k \in \mathbb{R} \rightarrow kU \in \omega$

2. Inner Product : $\langle p, q \rangle = \int_0^1 p(x) \cdot q(x)\, dx$.

~~(i) Nor~~ (i) $\| p \|$

(ii) $\| q \|$

(iii) $\langle p, q \rangle$

(a) $p(x) = 1 - x$; $q(x) = x$

(i) $\langle p, p \rangle = \int_0^1 (1-x) \cdot (1-x)\, dx$

$= \int_0^1 x^2 - 2x + 1\, dx$

$= \frac{x^3}{3}\Big]_0^1 - 2 \cdot \frac{x^2}{2}\Big]_0^1 + x\Big]_0^1$

$= \frac{1}{3} - 1 + 1 = 1/3$

$\therefore \| p \| = \sqrt{\langle p, p \rangle} = \sqrt{1/3} = 1/\sqrt{3} = \sqrt{3}/3$

(ii) $\langle q, q \rangle = \int_0^1 x \cdot x\, dx$

$= \int_0^1 x^2\, dx = \frac{x^3}{3}\Big]_0^1$

$= 1/3$

$\therefore \| q \| = \sqrt{\langle q, q \rangle} = \sqrt{1/3} = 1/\sqrt{3} = \sqrt{3}/3$

(iii) $\langle p, q \rangle = \int_0^1 (1-x) \cdot x\, dx$

$= \int_0^1 x - x^2\, dx$

$= \frac{x^2}{2}\Big]_0^1 - \frac{x^3}{3}\Big]_0^1$

$= \frac{1}{2} - \frac{1}{3} = 1/6$

(b) $p(x) = 1 + x + x^2$; $q(x) = 3 - x^2$

(i) $\langle p, p \rangle = \int_0^1 (1 + x + x^2)(1 + x + x^2)\, dx$

$= \int_0^1 x^4 + x^2 + 1 + 2x + 2x^2 + 2x^3\, dx$

$= \int_0^1 x^4 + 2x^3 + 3x^2 + 2x + 1\, dx$

$$= \frac{x^5}{5}\Big]_0^1 + 2\frac{x^4}{4}\Big]_0^1 + 3\frac{x^3}{3}\Big]_0^1 + 2\frac{x^2}{2}\Big]_0^1 + x\Big]_0^1$$

$$= \frac{1}{5} + \frac{1}{2} + 1 + 1 + 1$$

$$= \frac{37}{10}$$

$$\therefore \|p\| = \sqrt{\langle p, p\rangle} = \sqrt{37/10} = \frac{\sqrt{370}}{10}$$

(ii) $$\langle q, q\rangle = \int_0^1 (3-x^2)^2\, dx$$

$$= \int_0^1 x^4 - 6x^2 + 9\, dx$$

$$= \frac{x^5}{5}\Big]_0^1 - 6\frac{x^3}{3}\Big]_0^1 + 9x\Big]_0^1$$

$$= \frac{1}{5} - 2 + 9 = \frac{36}{5}$$

$$\therefore \|q\| = \sqrt{\langle q, q\rangle} = \sqrt{36/5} = \frac{6}{\sqrt{5}} = \frac{6\sqrt{5}}{5}$$

(iii) $$\langle p, q\rangle = \int_0^1 (1+x+x^2)\cdot(3-x^2)\, dx$$

$$= \int_0^1 3 - x^2 + 3x - x^3 + 3x^2 - x^4\, dx$$

$$= \int_0^1 -x^4 - x^3 + 2x^2 + 3x + 3\, dx$$

$$= -\frac{x^5}{5}\Big]_0^1 - \frac{x^4}{4}\Big]_0^1 + 2\frac{x^3}{3}\Big]_0^1 + 3\frac{x^2}{2}\Big]_0^1 + 3x\Big]_0^1$$

$$= \frac{-1}{5} - \frac{1}{4} + \frac{2}{3} + \frac{3}{2} + 3$$

$$= \frac{283}{60}$$

(c) $p(x) = 1-x$; $q(x) = x+1$

(i) $\langle p, p\rangle = \int_0^1 (1-x)^2\, dx$

$= \int_0^1 x^2 - 2x + 1\, dx$

$= \frac{x^3}{3}\Big]_0^1 - 2\frac{x^2}{2}\Big]_0^1 + x\Big]_0^1$

$= \frac{1}{3} - 1 + 1 = 1/3$

$\therefore \|p\| = \sqrt{\langle p, p\rangle} = 1/\sqrt{3} = \frac{\sqrt{3}}{3}$

(ii) $\langle q, q\rangle = \int_0^1 (x+1)^2\, dx$

$= \int_0^1 x^2 + 2x + 1\, dx$

$= \frac{x^3}{3}\Big]_0^1 + 2\frac{x^2}{2}\Big]_0^1 + x\Big]_0^1$

$= \frac{1}{3} + 1 + 1 = 7/3$

$\therefore \|q\| = \sqrt{\langle q, q\rangle} = \sqrt{\frac{7}{3}} = \frac{\sqrt{21}}{3}$

(iii) $\langle p, q\rangle = \int_0^1 (1-x)(x+1)\, dx$

$= \int_0^1 1 - x^2\, dx$

$= x\Big]_0^1 - \frac{x^3}{3}\Big]_0^1$

$= 1 - 1/3 = 2/3$

(d) $p(x) = 1-x$; $q(x) = e^{-2x}$

(i) $\langle p, p\rangle = \int_0^1 (1-x)^2\, dx$

$= \int_0^1 x^2 - 2x + 1\, dx$

$= \frac{x^3}{3}\Big]_0^1 - 2\frac{x^2}{2}\Big]_0^1 + x\Big]_0^1$

$= \frac{1}{3} - 1 + 1 = 1/3$

$\therefore \|p\| = \sqrt{1/3} = \frac{\sqrt{3}}{3}$

(5)

(ii) $\langle q, q \rangle = \int_0^1 e^{-2x} \cdot e^{-2x}\, dx$

$= \int_0^1 e^{-4x}\, dx$

$= \frac{-1}{4} e^{-4x} \Big]_0^1$

$= \frac{-1}{4} \left(e^{-4} - e^{0} \right)$

$= \frac{-1}{4} \left(\frac{1}{e^4} - 1 \right)$

$= \frac{1}{4} \left(1 - \frac{1}{e^4} \right)$

$= \frac{e^4 - 1}{4e^4}$

$\therefore \|q\| = \sqrt{\frac{e^4 - 1}{4e^4}} = \frac{\sqrt{e^4 - 1}}{2e^2}$

(iii) $\langle p, q \rangle = \int_0^1 (1 - x) \cdot e^{-2x}\, dx$

$= \int_0^1 e^{-2x} - x e^{-2x}\, dx$

$= \frac{-1}{2} e^{-2x} \Big]_0^1 - \left(\frac{-1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} \right) \Big]_0^1$

$= \frac{-1}{2} \left(e^{-2} - e^{0} \right) - \left(\frac{-1}{2} \left(e^{-2} - 0 \right) \right) + \frac{1}{4} \left(e^{-2} - e^{0} \right)$

$= \frac{-1}{4} \left(e^{-2} - 1 \right) + \frac{e^{-2}}{2}$

$= \frac{1}{4} e^{-2} + \frac{1}{4}$

$= \frac{e^{-2} + 1}{4}$

(e) $p(x) = \sin 2\pi x$; $q = \cos 2\pi x$

(i) $\langle p, p \rangle = \int_0^1 \sin^2 2\pi x\, dx$

$= \int_0^1 \frac{1 - \cos 4\pi x}{2}\, dx$

$\therefore x - x \cdot a(x) = x$

$$= \frac{1}{2} \int_0^1 1 - \cos 4\pi x \, dx$$

$$= \frac{1}{2} \left\{ x \Big]_0^1 - \frac{1}{4\pi} \cdot \sin 4\pi x \Big]_0^1 \right\}$$

$$= \frac{1}{2} \left\{ 1 - \frac{1}{4\pi} (0 - 0) \right\}$$

$$= 1/2$$

$$\therefore \| p \| = \sqrt{1/2} = \frac{\sqrt{2}}{2}$$

(ii) $\langle q, q \rangle = \int_0^1 \cos^2 2\pi x \, dx$

$$= \int_0^1 1 - \sin^2 2\pi x \, dx$$

$$= \int_0^1 1 - \int_0^1 \sin^2 2\pi x \, dx$$

$$= x \Big]_0^1 - 1/2$$

$$= 1 - 1/2 = 1/2 \rightarrow \| q \| = \frac{\sqrt{2}}{2}$$

(iii) $\langle p, q \rangle = \int_0^1 \sin 2\pi x \cdot \cos 2\pi x \, dx$

$$= \int_0^1 \frac{1}{2} \left[\sin (2\pi x + 2\pi x) + \sin (2\pi x - 2\pi x) \right] dx$$

$$= \frac{1}{2} \int_0^1 \sin 4\pi x \, dx$$

$$= \frac{1}{2} \cdot \frac{1}{4\pi} (-\cos 4\pi x) \Big]_0^1$$

$$= \frac{-1}{8\pi} [1 - 1]$$

$$= 0$$

3. Inner product : $\langle p, q \rangle = \int_{-1}^{1} p(x) \cdot q(x) \, dx$

(i) $\| p \|$

(ii) $\| q \|$

(iii) $\langle p, q \rangle$

(a) $p(x) = k - x$; $q(x) = x$

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(i) $\langle p, p \rangle = \int_{-1}^{1} (k-x)^2 \, dx$

$= \int_{-1}^{1} x^2 - 2kx + k^2 \, dx$

$= \frac{x^3}{3}\Big]_{-1}^{1} - 2k\frac{x^2}{2}\Big]_{-1}^{1} + k^2 x\Big]_{-1}^{1}$

$= \frac{2}{3} - 0 + 2k^2$

$\therefore \|p\| = \sqrt{\frac{2}{3} + 2k^2}$

(ii) $\langle q, q \rangle = \int_{-1}^{1} x^2 \, dx$

$= \frac{x^3}{3}\Big]_{-1}^{1} = 2/3$

$\therefore \|q\| = \sqrt{2/3} = \frac{\sqrt{6}}{3}$

(iii) $\langle p, q \rangle = \int_{-1}^{1} (k-x) . x \, dx$

$= \int_{-1}^{1} kx - x^2 \, dx$

$= k\frac{x^2}{2}\Big]_{-1}^{1} - \frac{x^3}{3}\Big]_{-1}^{1}$

$= \frac{-2}{3}$

(b) $p(x) = 2 - x$; $q(x) = 2 + x$

i) $\langle p, p \rangle = \int_{-1}^{1} (2-x)^2 \, dx$

$= \int_{-1}^{1} x^2 - 4x + 4 \, dx$

$= \frac{x^3}{3}\Big]_{-1}^{1} - 4\frac{x^2}{2}\Big]_{-1}^{1} + 4x\Big]_{-1}^{1}$

$= \frac{2}{3} + 8 = \frac{26}{3}$

$\therefore \|p\| = \sqrt{26/3} = \frac{\sqrt{78}}{3}$

(ii) $\langle q, q \rangle = \int_{-1}^{1} (2+x)^2 \, dx$

$= \int_{-1}^{1} x^2 + 4x + 4 \, dx$

$= \frac{x^3}{3}\Big]_{-1}^{1} + 4\frac{x^2}{2}\Big]_{-1}^{1} + 4x\Big]_{-1}^{1}$

$= \frac{2}{3} + 8 = 26/3$

$\therefore \|q\| = \sqrt{26/3} = \frac{\sqrt{78}}{3}$

(iii) $\langle p, q \rangle = \int_{-1}^{1} (2-x)(2+x) \, dx$

$= \int_{-1}^{1} 4 - x^2 \, dx$

$= 4x\Big]_{-1}^{1} - \frac{x^3}{3}\Big]_{-1}^{1}$

$= 8 - 2/3 = 22/3$

(c) $p(x) = 1 - x$; $q(x) = x^2$

(i) $\langle p, p \rangle = \int_{-1}^{1} (1-x)^2 \, dx$

$= \int_{-1}^{1} x^2 - 2x + 1 \, dx$

$= \frac{x^3}{3}\Big]_{-1}^{1} - 2\frac{x^2}{2}\Big]_{-1}^{1} + x\Big]_{-1}^{1}$

$= \frac{2}{3} + 2 = \cancel{\frac{26}{3}} \; 8/3$

$\therefore \|p\| = \sqrt{8/3} = \frac{2\sqrt{6}}{3}$

(ii) $\langle q, q \rangle = \int_{-1}^{1} x^4 \, dx$

$= \frac{x^5}{5}\Big]_{-1}^{1} = 2/5$

$\therefore \|q\| = \sqrt{2/5} = \frac{\sqrt{10}}{5}$

(iii) $\langle p, q \rangle = \int_{-1}^{1} (1-x) \cdot x^2 \, dx$

$= \int_{-1}^{1} x^2 - x^3 \, dx$

$$= \frac{x^3}{3}\Big]_{-1}^{1} - \frac{x^4}{4}\Big]_{-1}^{1}$$

$$= 2/3$$

(d) $p(x) = 1-x$; $q(x) = x^2$

(i) $\|p\| = 2\sqrt{6}/3$

(ii) $\|q\| = \sqrt{10}/5$

(iii) $\langle p, q\rangle = 2/3$

(e) ~~e^{-2x}~~ $p(x) = \sin 2\pi x$; $q(x) = \cos 2\pi x$

(i) $$\langle p, p\rangle = \int_{-1}^{1} \sin^2 2\pi x \, dx$$

$$= \int_{-1}^{1} \frac{1-\cos 4\pi x}{2} \, dx$$

$$= \frac{1}{2}\left\{ x\Big]_{-1}^{1} - \frac{1}{4}\sin 4\pi x\Big]_{-1}^{1} \right\}$$

$$= \frac{1}{2}\{2-0\}$$

$$= 1$$

$\therefore \|p\| = \sqrt{1} = 1$

(ii) $$\langle q, q\rangle = \int_{-1}^{1} \cos^2 2\pi x \, dx$$

$$= \int_{-1}^{1} 1 - \sin^2 2\pi x \, dx$$

$$= \int_{-1}^{1} 1 \, dx - \int_{-1}^{1} \sin^2 2\pi x \, dx$$

$$= x\Big]_{-1}^{1} - 1 = 2-1 = 1$$

$\therefore \|q\| = \sqrt{1} = 1$

(iii) $$\langle p, q\rangle = \int_{-1}^{1} \sin 2\pi x \cos 2\pi x \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} \sin 4\pi x \, dx$$

$$= \frac{1}{2} \cdot \frac{1}{4\pi} \; -\cos 4\pi x\Big]_{-1}^{1}$$

$$= \frac{-1}{8\pi} \cdot (1-1) = 0$$

4. Eigen values and Eigen vectors

A(a) $A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$

$S_1 = \text{Trace} = 4 + 3 + 1 = 8$

$S_2 = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ -5 & 3 \end{vmatrix}$

$= -5 + 0 + 22 = 17$

$|A| = 4\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} -5 & 2 \\ -2 & 1 \end{vmatrix} - 2\begin{vmatrix} -5 & 3 \\ -2 & 4 \end{vmatrix}$

$= 4(-5) - 2(-1) - 2(-14) = 10$

Char eq: $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$\longrightarrow \lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0$

1 is a root by inspection

$\longrightarrow$

1	1	-8	17	-10
	0	1	-7	10
	1	-7	10	0

$\therefore \lambda^2 - 7\lambda + 10 = 0 \longrightarrow \lambda^2 - 2\lambda - 5\lambda + 10 = 0$

$\longrightarrow \lambda = 2, 5$

$\therefore \lambda = 5, 2, 1$

When $\lambda = 5$,

$\longrightarrow \begin{bmatrix} 4-5 & 2 & -2 \\ -5 & 3-5 & 2 \\ -2 & 4 & 1-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix}$

Cramer's Rule:

$$\begin{array}{ccc} 2 & -2 & -1 & 2 \\ -2 & 2 & -5 & -2 \end{array}$$

$$\frac{x_1}{0} = \frac{x_2}{12} = \frac{x_3}{12}$$

$$\therefore X_1 = \begin{bmatrix} 0 \\ 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

When $\lambda = 2$,

$$\longrightarrow \begin{bmatrix} 4-2 & 2 & -2 \\ -5 & 3-2 & 2 \\ -2 & 4 & 1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix}$$

Cramer's Rule:

$$\begin{array}{ccc} 2 & -2 & 2 & 2 \\ 1 & 2 & -5 & 1 \end{array}$$

$$\frac{x_1}{6} = \frac{x_2}{6} = \frac{x_3}{12}$$

$$\therefore X_2 = \begin{bmatrix} 6 \\ 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

When $\lambda = 1$,

$$\longrightarrow \begin{bmatrix} 4-1 & 2 & -2 \\ -5 & 3-1 & 2 \\ -2 & 4 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix}$$

Cramer's Rule:

$$\begin{array}{ccc} 2 & -2 & 3 & 2 \\ 4 & 0 & -2 & 4 \end{array}$$

$$\frac{x_1}{8} = \frac{x_2}{4} = \frac{x_3}{16}$$

$$\therefore X_3 = \begin{bmatrix} 8 \\ 4 \\ 16 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$S_1 = \cancel{9}\ 3 + 3 + 3 = 9$

$S_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$

$= 8 + 8 + 8 = 24$

$|A| = 3 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix}$

$= 24 - 4 + (-4) = \cancel{20}\ 16$

Char Eq: $\lambda^3 - S_1 \lambda^2 + S_2 \lambda - |A| = 0$

$\longrightarrow \lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$

1 is a root by inspection.

$\longrightarrow$

1	1	-9	24	-16
	0	1	-8	-16
	1	-8	16	0

$\therefore \lambda^2 - 8\lambda + 16 = 0 \longrightarrow (\lambda - 4)^2 = 0$

$\therefore \lambda = 4, 4, 1$ are the eigen values.

When $\lambda = 4$, the matrix becomes:

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Then, $-x_1 + x_2 + x_3 = 0$ (Two free variables)

Put $x_1 = t_1$; $x_2 = t_2$

$\longrightarrow -t_1 + t_2 + x_3 = 0 \longrightarrow x_3 = t_1 - t_2$

$$\therefore X_1 = \begin{bmatrix} t_1 \\ t_2 \\ t_1 - t_2 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(9) When $\lambda = 1$, the matrix becomes:

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Cramer's Rule:

$$\begin{matrix} 2 & -1 & 1 & 2 \\ -1 & 2 & 1 & -1 \end{matrix}$$

$$\frac{x_1}{3} = \frac{x_2}{-3} = \frac{x_3}{-3}$$

$$\therefore X_3 = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

(c) $$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$$

$S_1 = 4 + 3 - 2 = 5$

$$S_2 = \begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}$$

$$= 4 - 2 + 6 = 8$$

$$|A| = 4\begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} - 1\begin{vmatrix} 6 & 6 \\ -5 & -2 \end{vmatrix} - 1\begin{vmatrix} 6 & 6 \\ 3 & 2 \end{vmatrix}$$

$$= 4 \cdot 4 - 1 \cdot 18 - 1 \cdot (-6)$$

$$= 4$$

Char Eq: $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

1 is a root by inspection

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 8 & -4 \\ & 0 & 1 & -4 & 4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$\therefore \lambda^2 - 4\lambda + 4 = 0 \longrightarrow (\lambda - 2)^2 = 0$

$\therefore \lambda = 2, 2, 1$ are the eigenvalues.

When $\lambda = 2$, the matrix becomes:

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}$$

Here, $R_1 = R_2 - R_3$

$\therefore$ Rank $= 1$; $n = 3$

~~Cramer's Rule~~

$\therefore$ No. of independent variables $= 2$.

Consider $2x_1 + 6x_2 + 6x_3 = 0$

Put $x_2 = t_1$; $x_3 = t_2$

$\longrightarrow 2x_1 + 6t_1 + 6t_2 = 0$

$\longrightarrow x_1 = -3(t_1 + t_2)$

$$\therefore X_1 = \begin{bmatrix} -3t_1 - 3t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

When $\lambda = 3$, the matrix becomes:

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix}$$

Cramer's Rule:

$$\begin{matrix} 2 & & 2 & & 1 & & 2 \\ -5 & & -3 & & -1 & & -5 \end{matrix}$$

$$\frac{x_1}{4} = \frac{x_2}{1} = \frac{x_3}{-3}$$

$$\therefore X_3 = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$$

(d) $$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

$S_1 = -9 + 3 + 7 = 1$

$$S_2 = \begin{vmatrix} 3 & 4 \\ 8 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 \\ -16 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 \\ -8 & 3 \end{vmatrix} = -11 + 1 + 5$$

$= -5$

$$|A| = -9\begin{vmatrix} 3 & 4 \\ 8 & 7 \end{vmatrix} - 4\begin{vmatrix} -8 & 4 \\ -16 & 7 \end{vmatrix} + 4\begin{vmatrix} -8 & 3 \\ -16 & 8 \end{vmatrix}$$

$$= -9\cdot(-11) - 4\cdot(8) + 4(-16)$$

$$= 99 - 32 - 64 = 3$$

Char Eq: $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$\longrightarrow \lambda^3 - \lambda^2 - 5\lambda - 3 = 0$

-1 is a root by inspection

$$\longrightarrow \begin{array}{c|cccc} -1 & 1 & -1 & -5 & -3 \\ & 0 & -1 & 2 & 3 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

$\longrightarrow \lambda^2 - 2\lambda - 3 = 0 \longrightarrow \lambda^2 + \lambda - 3\lambda - 3 = 0$

$\longrightarrow \lambda(\lambda+1) - 3(\lambda+1) = 0$

$\longrightarrow \lambda = -1, 3$

$\therefore \lambda = 3, -1, -1$

When $\lambda = 3$, the matrix becomes:

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}$$

Cramer's Rule:

$$\begin{matrix} 0 & 4 & -8 & 0 \\ 8 & 4 & -16 & 8 \end{matrix}$$

$$\frac{x_1}{-32} = \frac{x_2}{-32} = \frac{x_3}{-64}$$

$$\therefore X_1 = \begin{bmatrix} -32 \\ -32 \\ -64 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

When $\lambda = -1$, the matrix becomes:

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}$$

All the rows are identical

$\therefore$ Rank $= 1$

No of independent variables $= 3 - 1 = 2$

Consider, $-8x_1 + 4x_2 + 4x_3 = 0$

Put $x_2 = t_1$; $x_3 = t_2$ $\longrightarrow$ $-8x_1 + 4t_1 + 4t_2 = 0$

$\longrightarrow x_1 = \dfrac{t_1 + t_2}{2}$

$$\therefore X_2 = \begin{bmatrix} t_1 + t_2 \\ 2t_1 \\ 2t_2 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

(e) ~~(d)~~

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$S_1 = 8 + 7 + 3 = 18$

$$S_2 = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$= 5 + 20 + 20 = 45$

$$|A| = 8 \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + 6 \begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix} + 2 \begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}$$

$= 8 \cdot 5 + 6(-10) + 2(10)$

$= 0$

Char Eq: $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$\longrightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\longrightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0$

$\longrightarrow \lambda(\lambda^2 - 3\lambda - 15\lambda + 45) = 0$

$\longrightarrow \lambda = 15, 3, 0$

When $\lambda = 15$, the matrix becomes:

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}$$

Cramer's Rule:

$$\begin{matrix} -8 & -4 & -6 & -8 \\ -4 & -12 & 2 & -4 \end{matrix}$$

$$\frac{x_1}{80} = \frac{x_2}{-80} = \frac{x_3}{40}$$

$$\rightarrow X_1 = \begin{bmatrix} 80 \\ -80 \\ 40 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

When $\lambda = 3$, the matrix becomes:

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}$$

Cramer's Rule:

$$\begin{matrix} 4 & -4 & -6 & 4 \\ -4 & 0 & 2 & -4 \end{matrix}$$

$$\frac{x_1}{-16} = \frac{x_2}{-8} = \frac{x_3}{16}$$

$$\rightarrow X_2 = \begin{bmatrix} -16 \\ -8 \\ 16 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

When $\lambda = 0$, the matrix becomes:

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Cramer's Rule:

$$\begin{matrix} 7 & -4 & -6 & 7 \\ -4 & 3 & 2 & -4 \end{matrix}$$

$$\frac{x_1}{5} = \frac{x_2}{10} = \frac{x_3}{10}$$

$$\rightarrow X_3 = \begin{bmatrix} 5 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

(8)

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$S_1 = 3 + 5 + 3 = 11$

$$S_2 = \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$= 14 + 8 + 14 = 36$

$$|A| = 3\begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} -1 & 5 \\ 1 & -1 \end{vmatrix}$$

$= 3 \cdot 14 + 1 \cdot (-2) + 1 \cdot (-4)$

$= 36$

Char Eq: $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$\longrightarrow \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

3 is a root by inspection.

$$\longrightarrow 3 \quad \begin{array}{|cccc} 1 & -11 & 36 & -36 \\ 0 & 3 & -24 & 36 \\ \hline 1 & -8 & 12 & 0 \end{array}$$

$\longrightarrow \lambda^2 - 8\lambda + 12 = 0$

$\longrightarrow \lambda^2 - 6\lambda - 2\lambda + 12 = 0$

$\longrightarrow \lambda = 2, 6$

$\therefore \lambda = 6, 3, 2$ are the eigenvalues

When $\lambda = 6$, the matrix becomes:

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}$$

Cramer's rule:

$$\begin{matrix} -1 & & -1 & & -1 & & -1 \\ -1 & & -3 & & 1 & & -1 \end{matrix}$$

$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2} \longrightarrow X_1 = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

(12) When $\lambda = 3$, the matrix becomes:

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

Cramer's Rule:

$$\begin{matrix} -1 & 1 & 0 & -1 \\ -1 & 0 & 1 & -1 \end{matrix}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} \longrightarrow X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

When $\lambda = 2$, the matrix becomes:

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Cramer's rule:

$$\begin{matrix} -1 & 1 & 1 & -1 \\ 3 & -1 & -1 & 3 \end{matrix}$$

$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2} \longrightarrow X_3 = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

5. Calculus

(a) $ux = yz$; $vy = zx$; $wz = xy$.

Find $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$

$u = \dfrac{yz}{x}$; $v = \dfrac{zx}{y}$; $w = \dfrac{xy}{z}$

$\dfrac{\partial u}{\partial x} = \dfrac{-yz}{x^2}$; $\dfrac{\partial u}{\partial y} = \dfrac{z}{x}$; $\dfrac{\partial u}{\partial z} = \dfrac{y}{x}$

$\dfrac{\partial v}{\partial x} = \dfrac{z}{y}$; $\dfrac{\partial v}{\partial y} = \dfrac{-zx}{y^2}$; $\dfrac{\partial v}{\partial z} = \dfrac{x}{y}$

$\dfrac{\partial w}{\partial x} = \dfrac{y}{z}$; $\dfrac{\partial w}{\partial y} = \dfrac{x}{z}$; $\dfrac{\partial w}{\partial z} = \dfrac{-xy}{z^2}$

$$\therefore \; J_{mat} = \begin{bmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{bmatrix}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = |J| = \frac{-yz}{x^2}\left(\frac{-xz}{y^2}\cdot\frac{-xy}{z^2} - \frac{x}{y}\cdot\frac{x}{z}\right)$$

$$- \frac{z}{x}\left(\frac{z}{y}\cdot\frac{-xy}{z^2} - \frac{x}{y}\cdot\frac{y}{z}\right) + \frac{y}{x}\left(\frac{z}{y}\cdot\frac{x}{z} - \frac{y}{z}\cdot\frac{-xz}{y^2}\right)$$

$$\rightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{-yz}{x^2}\left(\frac{x^2}{yz} - \frac{x^2}{yz}\right) - \frac{z}{x}\left(\frac{-x}{z} - \frac{x}{z}\right)$$

$$+ \frac{y}{x}\left(\frac{x}{y} + \frac{x}{y}\right)$$

$$= \frac{-yz}{x^2}\cdot(0) - \frac{z}{x}\left(\frac{-2x}{z}\right) + \frac{y}{x}\left(\frac{2x}{y}\right)$$

$$= 4$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$$

(b) $u = x+y+z$; $v = x^2+y^2+z^2$; $w = xy+yz+zx$

$$J_{mat} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{bmatrix}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = |J| = 1\left(2y(x+y) - 2z(x+z)\right) - 1\left(2x(x+y) - 2z(y+z)\right)$$

$$+ 1\left(2x(x+z) - 2y(y+z)\right)$$

$$= 2xy + 2y^2 - 2zx - 2z^2 - 2x^2 - 2xy + 2yz + 2z^2$$

$$+ 2x^2 + 2xz - 2y^2 - 2yz$$

$$= 0$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

(13)

(c) $u = xyz$; $v = x^2+y^2+z^2$; $w = x+y+z$

$$J_{mat} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{bmatrix}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 1\,(2z.xz - 2y.xy) - 1\,(2z.yz - 2x.xy) + 1\,(2y.yz - 2x.xz)$$

$$= 2xz^2 - 2xy^2 - 2yz^2 + 2x^2y + 2y^2z - 2x^2z$$

(d) Verify $JJ' = 1$ for $u = xy$; $v = x+y$.

$u = xy$; $v = x+y$

$$J_{mat} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = J = y - x$$

$u = xy$; $v = x+y \longrightarrow x = v-y$; $y = v-x$

$\longrightarrow u = x(v-x) \longrightarrow u = xv - x^2$

$\longrightarrow x^2 - xv + u = 0$

$$\therefore x = \frac{v \pm \sqrt{v^2 - 4u}}{2};$$

(e) Verify $JJ' = 1$; $x = au + bv - cw$, $y = -au + bv + cw$, $z = au - bv + cw$.

$$J_{mat} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} = \begin{bmatrix} a & b & -c \\ -a & b & c \\ a & -b & c \end{bmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = J = a(bc+bc) - b(-ac-ac) + c(ab-ab)$$

$$= 4abc$$

$x = au + bv - cw$; $y = -au + bv + cw$, $z = au - bv + cw$

$\rightarrow u = \frac{x+z}{2a}$; $v = \frac{x+y}{2b}$; $w = \frac{y+z}{2c}$

$$J'_{mat} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{1}{2a} & 0 & \frac{1}{2a} \\ \frac{1}{2b} & \frac{1}{2b} & 0 \\ 0 & \frac{1}{2c} & \frac{1}{2c} \end{bmatrix}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = J' = \frac{1}{2a}\left(\frac{1}{2b}\cdot\frac{1}{2c} - 0\right) - 0 + \frac{1}{2a}\left(\frac{1}{2b}\cdot\frac{1}{2c} - 0\right)$$

$$= \frac{1}{8abc} + \frac{1}{8abc} = \frac{2}{8abc} = \frac{1}{4abc}$$

$$JJ' = 4abc \cdot \frac{1}{4abc} = 1.$$

Hence, proved.

(f) Hessian at $(1,-3)$ for $f(x,y) = x^2y + xy^2 - 4$

$\frac{\partial f}{\partial x} = 2xy + y^2$; $\frac{\partial f}{\partial y} = 2xy + x^2$

$\frac{\partial^2 f}{\partial x^2} = 2y$; $\frac{\partial^2 f}{\partial x \partial y} = 2x + 2y$; $\frac{\partial^2 f}{\partial y^2} = 2x$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2y & 2x+2y \\ 2x+2y & 2x \end{bmatrix}$$

(14) At the point $(1,-3)$

$$H_{(1,-3)} = \begin{bmatrix} -6 & -4 \\ -4 & 2 \end{bmatrix}$$

(g) Hessian at $(1, -3\pi/4)$ for $u = 2e^{2x}\sin y$

$$\frac{\partial u}{\partial x} = 2e^{2x}\sin y \; ; \; \frac{\partial u}{\partial y} = e^{2x}\cos y$$

$$\frac{\partial^2 u}{\partial x^2} = 4e^{2x}\sin y \; ; \; \frac{\partial^2 u}{\partial x \partial y} = 2e^{2x}\cos y \; ; \; \frac{\partial^2 u}{\partial y^2} = -e^{2x}\sin y$$

$$H = \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 4e^{2x}\sin y & 2e^{2x}\cos y \\ 2e^{2x}\cos y & -e^{2x}\sin y \end{bmatrix}$$

$$H_{(1,-\frac{3\pi}{4})} = \begin{bmatrix} -4e^2 \cdot 1/\sqrt{2} & 2e^2 \cdot -1/\sqrt{2} \\ 2e^2 \cdot -1/\sqrt{2} & e^2 \cdot 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -2\sqrt{2}e^2 & -\sqrt{2}e^2 \\ -\sqrt{2}e^2 & \frac{\sqrt{2}}{2}e^2 \end{bmatrix}$$

(h) First order partial derivatives

(i) $x^4 + y^4 = 5a^2xy$

Here, $f = x^4 + y^4 - 5a^2xy$

$$\frac{\partial f}{\partial x} = 4x^3 - 5a^2y \; ; \; \frac{\partial f}{\partial y} = 4y^3 - 5a^2x$$

(ii) $f(x,y,z) = 4x^3y^2 - e^zy^4 + \frac{z^3}{x^2} + 4y - x^{16}$

$$\frac{\partial f}{\partial x} = 12x^2y^2 - \frac{2z^3}{x^3} - 16x^{15}$$

$$\frac{\partial f}{\partial y} = 8x^3y - 4e^zy^3 + 4$$

$$\frac{\partial f}{\partial z} = -e^zy^4 + \frac{3z^2}{x^2}$$

(i) First & Second Order ~~Partial~~ derivatives

(i) $f(x) = (6x^2 + 7x)^4$

$$\frac{df}{dx} = 4(6x^2+7x)^3(12x+7)$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(4(6x^2+7x)^3(12x+7)\right)$$

$$= \cancel{4 \cdot \{ 3(6x^2+7x)^2 (12x+7) + 6x^2}$$

$$= 4 \left\{ 3(6x^2+7x)^2 \cdot (12x+7)(12x+7) + (6x^2+7x)^3 \cdot 12 \right\}$$

$$= 4 \left\{ 3(6x^2+7x)^2 (12x+7)^2 + (6x^2+7x)^3 \cdot 12 \right\}$$

(ii) $g(z) = 3z^7 - \sin(z^2+6)$

$$\frac{dg}{dz} = 21z^6 - \cos(z^2+6) \cdot 2z$$

$$\frac{d^2 g}{dz^2} = 126z^5 - \left[-\sin(z^2+6) \cdot 2z \cdot 2z + \cos(z^2+6) \cdot 2 \right]$$

$$= 126z^5 + 4z^2 \sin(z^2+6) - 2\cos(z^2+6)$$