Motion of the Simple Pendulum

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Abstract: The aim was to investigate the behaviour of the simple pendulum using numerical techniques such as the Fourth-Order Runge-Kutta method and Trapezoid Rule to integrate the equations of motion. The qualitative differences between the linear and non-linear cases were analysed.

1 Introduction

1.1 The Simple Pendulum

The analysis of simple pendulum is important in numerous area of maths and physics, it is an important example because its equation of motion is non-linear and does not have an analytic solution except for the special case where the angular displacement is small when it can be linearised and the problem reduces to a harmonic oscillator. The equation of motion is given as follows:

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{1}$$

where g is the acceleration due to gravity and l is the length of the pendulum. In the linearised case, the small angle approximation, $\sin\theta \approx \theta$, we have an analytic solution $\theta = a\sin(\omega_0 t + \phi)$ where $\omega_0 = \sqrt{g/l}$ and a and ϕ are arbitrary constants. It is also of physical interest to model the damped driven pendulum, where a frictional damping term proportional to the angular velocity is added and a harmonic driving force. The equation of motion becomes thus:

$$\ddot{\theta} + \gamma\dot{\theta} + \frac{g}{l}\sin\theta = A\cos\omega_D t \tag{2}$$

where γ is the damping coefficient, A is the amplitude of the driving force and ω_D is the driving frequency. For convenience the equation may be reduced to a system of two first order equations by introducing a new variable ω as follows:

$$\begin{aligned} \omega &= \dot{\theta} \\ \dot{\omega} &= -\gamma\omega - g/l \sin\theta + A\cos\omega_D t \end{aligned} \tag{3}$$

1.2 Numerical Integration

The non-linear system does not have an explicit analytic solution, thus it requires numerical methods to determine a solution so that its behaviour may be analysed. There are several different methods which can be used and in this section we shall see how a number of simple ones can be implemented.

1.2.1 The Simple Euler Method

Using a Taylor expansion we have to the second order of smallness in t:

$$\begin{aligned} \theta(t+\Delta t) &= \theta(t) + \omega\Delta t + O(\Delta t)^2 \\ \omega(t+\Delta t) &= \omega(t) + \dot{\omega}\Delta t + O(\Delta t)^2 \end{aligned} \tag{4}$$

Thus we have an iterative method for integrating the equations of motion:

$$\begin{aligned} \theta_{n+1} &= \theta_n + \omega_n\Delta t \\ \omega_{n+1} &= \omega_n + \dot{\omega}_n\Delta t \end{aligned} \tag{5}$$

1.2.2 The Trapezoid Rule

The trapezoid rule is an improved method which is accurate to the third order and averages the rate of change of a curve at the endpoints over an interval, and multiplies this value by Δt, the length of the interval. Thus the iterative equations become

$$\theta_{n+1} \approx \theta_n + \Delta t\frac{(\dot{\theta}(t+\Delta t) + \dot{\theta})}{2} \tag{6}$$

and Taylor expanding $\dot{\theta}(t+\Delta t) \approx \omega_n + \dot{\omega}_n\Delta t$ we have

$$\theta_{n+1} \approx \theta_n + \Delta t\frac{((\omega_n + \dot{\omega}_n\Delta t) + \omega_n)}{2} \tag{7}$$

and we also have

$$\omega_{n+1} \approx \omega_n + \Delta t\frac{(\dot{\omega}(t+\Delta t) + \dot{\omega}(t))}{2} \tag{8}$$

By renaming the variables we can arrange the above equations into a suitable format so they can be easily solved using an iterative method in a simple C program.

1.2.3 The Runge-Kutta Fourth Order Method

This final method is the most accurate, to the fifth order in t. This is necessary for analysing the motion of the full damped driven pendulum problem. The 'RK4' method uses weighted averages to compute the slope over the interval. The iterative equations of the method are given by:

$$\begin{aligned} \theta_{n+1} &= \theta_n + \tfrac{1}{6}(k_{1a} + 2k_{2a} + 2k_{3a} + k_{4a}) \\ \omega_{n+1} &= \omega_n + \tfrac{1}{6}(k_{1b} + 2k_{2b} + 2k_{3b} + k_{4b}) \end{aligned} \tag{9}$$

where the new variables are given as follows with $\dot{\omega} = f(\theta_n, \omega_n, t)$ the right hand side of the second equation in the system we wish to solve.

$$\begin{array}{rl} k_{1a} = \omega_n \Delta t & k_{1b} = f(\theta_n, \omega_n, t)\Delta t \\ k_{2a} = (\omega_n + \frac{k_{1b}}{2})\Delta t & k_{2b} = f(\theta_n + \frac{k_{1a}}{2}, \omega_n + \frac{k_{1b}}{2}, t + \frac{\Delta t}{2})\Delta t \\ k_{3a} = (\omega_n + \frac{k_{2b}}{2})\Delta t & k_{3b} = f(\theta_n + \frac{k_{2a}}{2}, \omega_n + \frac{k_{2b}}{2}, t + \frac{\Delta t}{2})\Delta t \\ k_{4a} = (\omega_n + k_{3b})\Delta t & k_{4b} = f(\theta_n + k_{3a}, \omega_n + k_{3b}, t + \Delta t)\Delta t \end{array} \tag{10}$$

2 Implementation

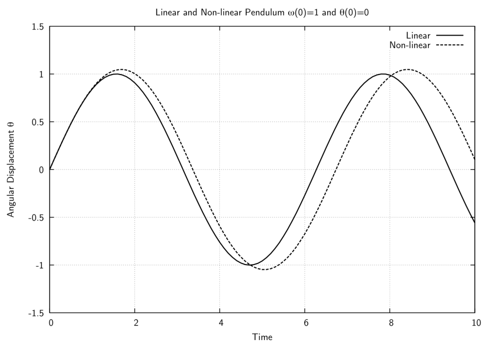
The numerical integration methods were realised by using the `for` loop construction in a simple C program to compute the iterations for a sufficient number of timesteps that allowed the behaviour to be analysed adequately. The `fprintf` statement was used to write the output to a file which was later used in GNUPlot to plot the data and compare the motion of the linear and non-linear cases. For example in the Trapezoid Method

```
for(i=1;i<=n;i++){
            k1a=h*omega;
            k1b=h*f(theta,omega,t);
            k2a=h*(omega+k1b);
            k2b=h*f(theta+k1a,omega+k1b,t+h);

            theta = theta + (k1a + k2a)/2;
            omega = omega + (k1b + k2b)/2;
            t=t+h;

            fprintf(Oscp,"%lf\t%lf\t%lf\n", t, theta, omega);
            }
```

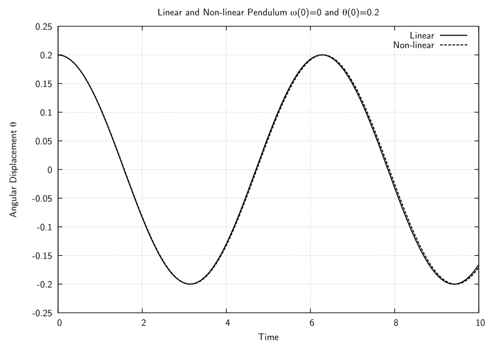
An `if` statement was also included in the later programs to prevent the first 500 or so iterations from being written to the file as to allow time for the motion to settle.

3 Results & Analysis

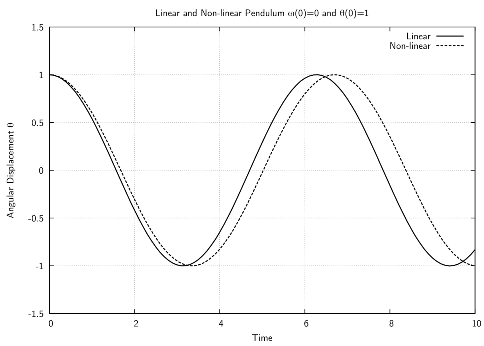
GNUPlot was used to plot the data created from the programs for various initial conditions so that the behaviour of the pendulum could be investigated and also the accuracy of the different computational methods could be observed. Fig. 1 and Fig. 2 show the angular displacement and velocity of the linear and non-linear pendula as a function of time. It is clear that the linear approximation works well for the case of small angular displacement, Fig. 1 (b) as opposed to the case with large initial angular displacement, Fig. 1 (d), as expected. This is also evident in Fig. 2.



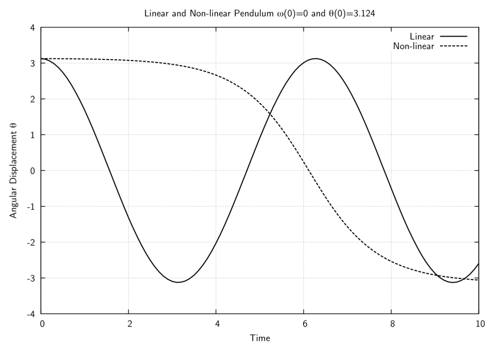
(a)



(b)

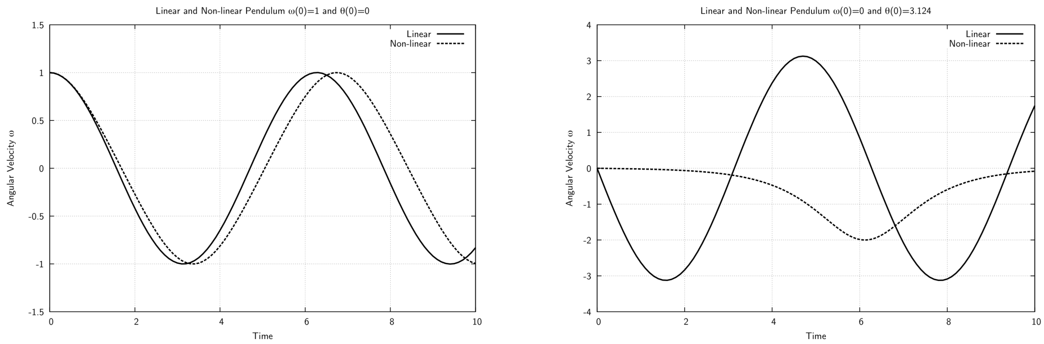


(c)



(d)

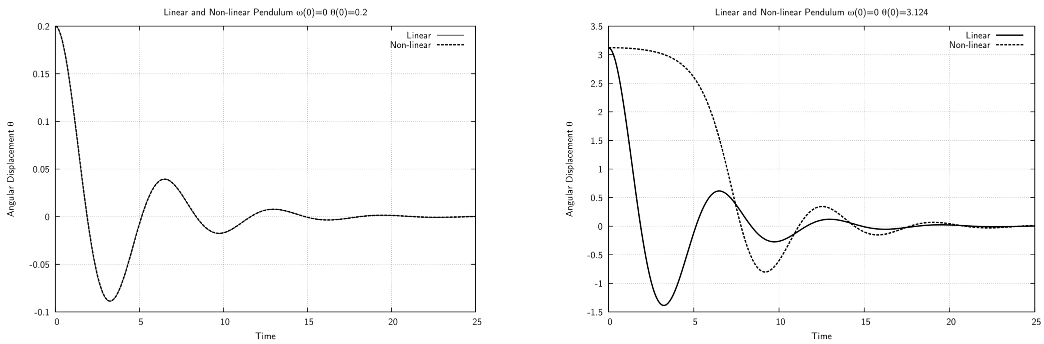
Figure 1: Comparison of Linear and Non-linear Cases Using Trapezoid Method



(a) (b)

Figure 2: Comparison of Linear and Non-linear Cases Using Trapezoid Method

Using the data generated from the Fourth-Order Runge-Kutta method it is clear also in the damped pendulum the linear approximation is a good fit for small initial displacement, however for large initial displacement it is a poor fit. It is also seen that the fit improves as the time increases as the displacement amplitude decreases and approaches the small angle approximation.

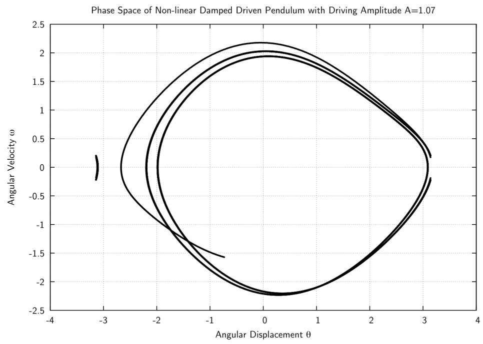


(a) (b)

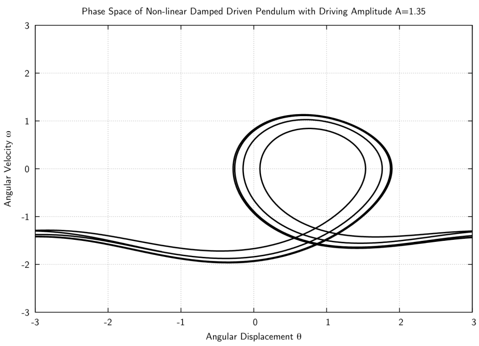
Figure 3: Comparison of Linear and Non-linear Damped Cases Using 'RK4' Method

Phase space plots were then made to analyse the motion of the damped driven pendulum for various values of the driving amplitude. It can be seen from Fig. 4 (a) that the motion is periodic as the in the phase diagram the trajectory approaches a limit cycle. In Fig. 4 (b) period doubling from the existence of two limit cycles. In Fig. 4 (c) we have period three behaviour and for even larger amplitudes the motion becomes chaotic.

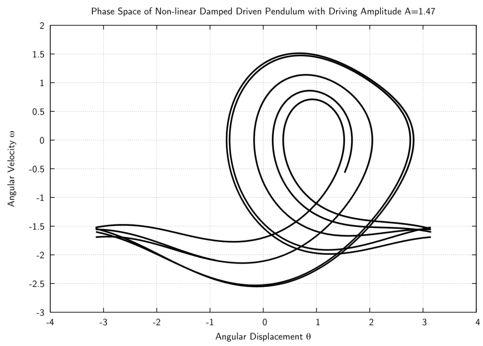
(a)



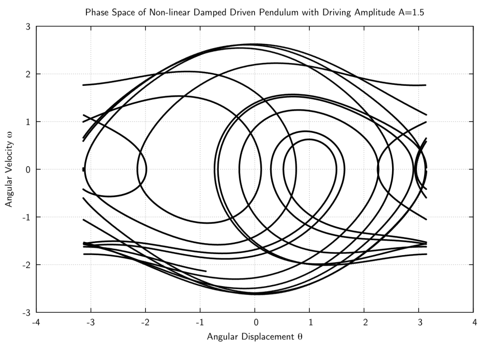
(b)



(c)



(d)



(e)

Figure 4: Phase Space Diagrams for Damped Driven Pendulum Using ‘RK4’ Method

4 Conclusions

It was seen how numerical methods such as the Trapezoid Rule and Runge-Kutta methods could be used in a simple C program to integrate the equations of motion of the simple pendulum for which there is no simple analytic solution. Thus the dynamics of the non-linear pendulum were analysed and compared with the linearised pendulum. Chaotic behaviour was also observed in the damped driven system.

References

[1] Hutzler, S., PY2002 Lecture Notes

[2] Hilborn, R. 2000, Chaos and Nonlinear Dynamics, 2nd edn, Oxford University Press