Lektion 9

Norm eines Vektors

$with(LinearAlgebra)$:
$v := \langle 1, 0, -1, 0, 1 \rangle$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad (1.1)$$

Die euklidische Norm von v ist gleich
(1) 1
(2) sqrt(3)
(3) 3
(4) sqrt(5)

$Norm(v, 2)$

$$\sqrt{3} \quad (1.2)$$

$Norm(v, 1)$

$$3 \quad (1.3)$$

$Norm(v, infinity)$

$$1 \quad (1.4)$$

$Norm(v)$

$$1 \quad (1.5)$$

Welche Norm entspricht der anschaulichen Länge?
(1) Die 1-Norm
(2) Die 2-Norm
(3) Die unendlich-Norm

Das Kreuzprodukt

$a := \langle 1, 2, 3 \rangle$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad (2.1)$$

$b := \langle 1, 0, 1 \rangle$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad (2.2)$$

$c := CrossProduct(a, b)$

$$\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} \qquad (2.3)$$

```
with(plots) :
pl1 := spacecurve([[0, 0, 0], convert(a, list)], color = red) :
pl2 := spacecurve([[0, 0, 0], convert(b, list)], color = green) :
pl3 := spacecurve([[0, 0, 0], convert(c, list)], color = blue) :
display({pl1, pl2, pl3}, scaling = constrained)
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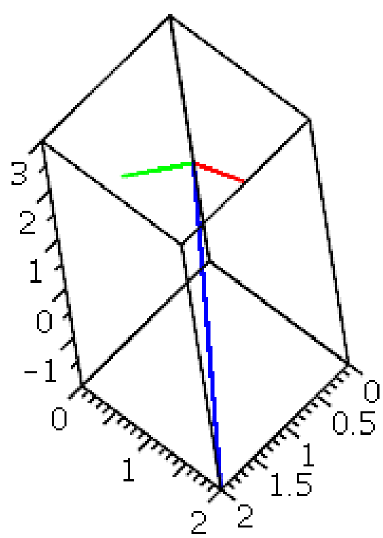


Die Länge von a x b ist gleich
(1) 1

(2) der Länge von a plus der Länge von b
(3) der Länge von a mal der Länge von b
(4) dem Flächeninhalt des von a und b aufgespannten Parallelogramms
(5) Antworten (3) und (4) sind beide richtig

$CrossProduct(a, a)$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.4}$$

$e\_a := \dfrac{a}{Norm(a, 2)}$

$$\begin{bmatrix} \frac{1}{14}\sqrt{14} \\ \frac{1}{7}\sqrt{14} \\ \frac{3}{14}\sqrt{14} \end{bmatrix} \tag{2.5}$$

$h := b - b.e\_a \cdot e\_a$

$$\begin{bmatrix} \frac{5}{7} \\ -\frac{4}{7} \\ \frac{1}{7} \end{bmatrix} \tag{2.6}$$

$Norm(a, 2) \cdot Norm(h, 2)$

$$\frac{1}{7}\sqrt{14}\sqrt{42} \tag{2.7}$$

$simplify(\mathbf{(2.7)})$

$$2\sqrt{3} \tag{2.8}$$

$Norm(c, 2)$

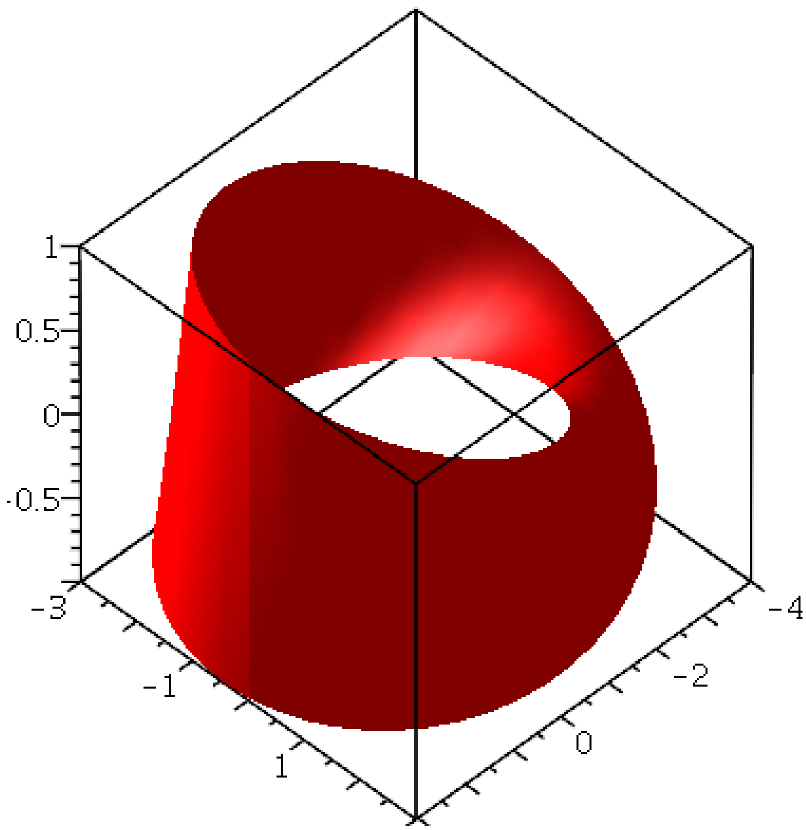
$$2\sqrt{3} \tag{2.9}$$

Das Möbiusband

$M := \left\langle 3 \cdot \cos(t) + s \cdot \sin\left(\frac{t}{2}\right), 3 \cdot \sin(t), s \cdot \cos\left(\frac{t}{2}\right) \right\rangle$

$$\left[\begin{array}{c} 3\cos(t)+s\sin\left(\frac{1}{2}t\right) \\ 3\sin(t) \\ s\cos\left(\frac{1}{2}t\right) \end{array}\right] \quad \textbf{(3.1)}$$

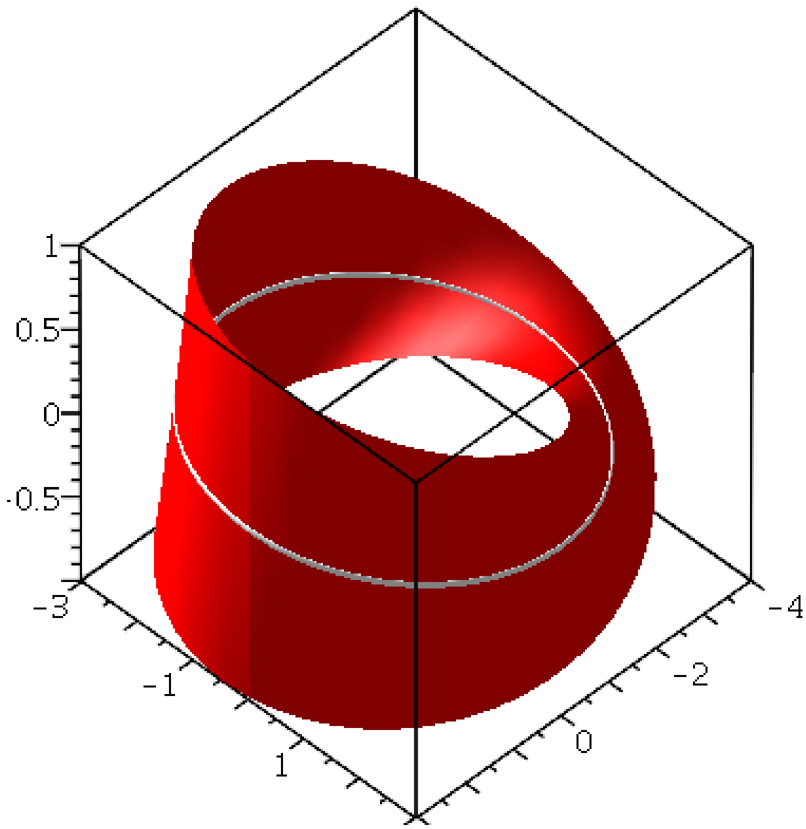
$pl1 := plot3d(M, t = 0..2\cdot \mathrm{Pi}, s = -1..1, color = red, style = patchnogrid) :$
$pl1$



$Seele := eval(M, s = 0)$

$$\left[\begin{array}{c} 3\cos(t) \\ 3\sin(t) \\ 0 \end{array}\right] \quad \textbf{(3.2)}$$

$pl2 := spacecurve(Seele, t = 0..2\cdot \mathrm{Pi}, thickness = 2, color = white) :$
$pl2 := tubeplot(convert(Seele, list), t = 0..2\cdot \mathrm{Pi}, radius = 0.02, color = white, style = patchnogrid) :$
$display(\{pl1, pl2\})$



$with(VectorCalculus):$
$BasisFormat(false)$

$$true \tag{3.3}$$

$Mt := diff(Seele, t)$

$$\begin{bmatrix} -3\sin(t) \\ 3\cos(t) \\ 0 \end{bmatrix} \tag{3.4}$$

$Ms := diff(M, s)$

$$\begin{bmatrix} \sin\left(\frac{1}{2}t\right) \\ 0 \\ \cos\left(\frac{1}{2}t\right) \end{bmatrix} \tag{3.5}$$

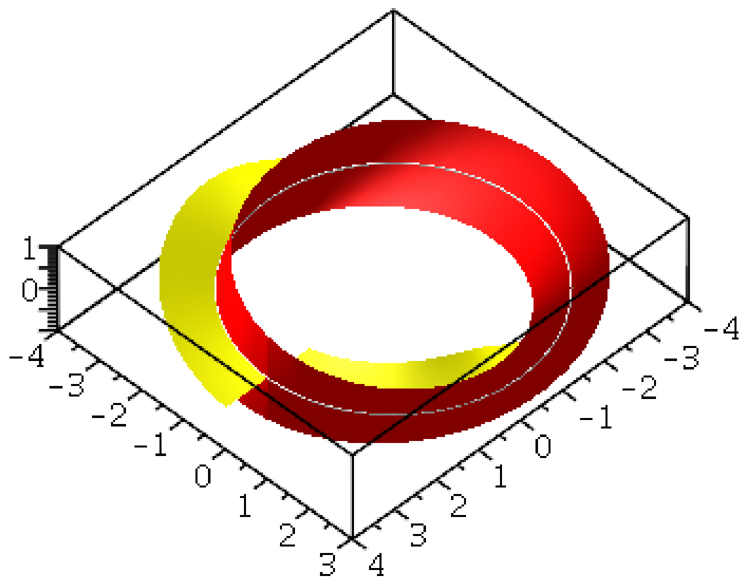
$Normale := CrossProduct(Ms, Mt)$

$$\begin{bmatrix} -3\cos\left(\frac{1}{2}t\right)\cos(t) \\ -3\cos\left(\frac{1}{2}t\right)\sin(t) \\ 3\sin\left(\frac{1}{2}t\right)\cos(t) \end{bmatrix} \tag{3.6}$$

$e\_Normale := simplify\left(\frac{Normale}{Norm(Normale, 2)}\right)$

$$\begin{bmatrix} -\frac{\cos\left(\frac{1}{2}t\right)\left(2\cos\left(\frac{1}{2}t\right)^2-1\right)}{\sqrt{-4\cos\left(\frac{1}{2}t\right)^6+8\cos\left(\frac{1}{2}t\right)^4-4\cos\left(\frac{1}{2}t\right)^2+1}} \\ -\frac{2\cos\left(\frac{1}{2}t\right)^2\sin\left(\frac{1}{2}t\right)}{\sqrt{-4\cos\left(\frac{1}{2}t\right)^6+8\cos\left(\frac{1}{2}t\right)^4-4\cos\left(\frac{1}{2}t\right)^2+1}} \\ \frac{\sin\left(\frac{1}{2}t\right)\left(2\cos\left(\frac{1}{2}t\right)^2-1\right)}{\sqrt{-4\cos\left(\frac{1}{2}t\right)^6+8\cos\left(\frac{1}{2}t\right)^4-4\cos\left(\frac{1}{2}t\right)^2+1}} \end{bmatrix} \tag{3.7}$$

$Normalenflaeche := Seele + s \cdot e\_Normale :$
$pl3 := plot3d(Normalenflaeche, t = 0..2 \cdot \mathrm{Pi}, s = 0..1, color = yellow, style = patchnogrid) :$
$display(\{pl1, pl2, pl3\}, scaling = constrained)$



$eval(Normale, \{t = 0, s = 0\})$

$$\begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} \quad \textbf{(3.8)}$$

$eval(Normale, \{t = 2 \cdot \mathrm{Pi}, s = 0\})$

$$\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \quad \textbf{(3.9)}$$

Vektorfelder

restart
with(*VectorCalculus*) :
BasisFormat(*false*)

$$false \quad \textbf{(4.1)}$$

$SetCoordinates(cartesian[x, y])$

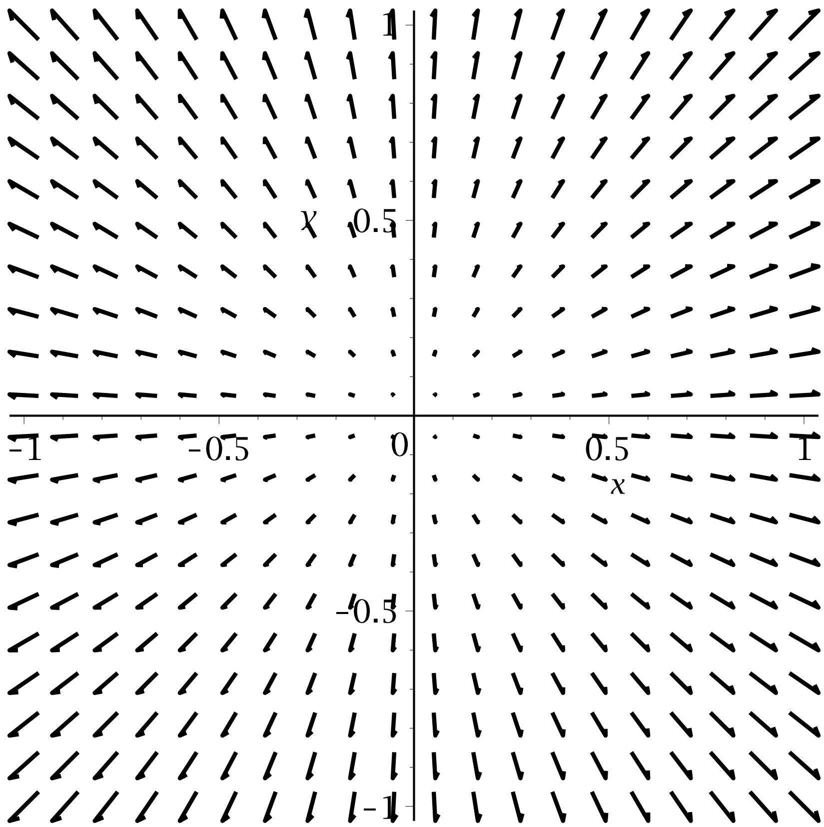
$$cartesian_{x, y} \quad \textbf{(4.2)}$$

$vf := VectorField(\langle x, y \rangle)$

$$(x)\bar{e}_x + (y)\bar{e}_y \quad \textbf{(4.3)}$$

$with(plots):$

$fieldplot(vf, x = -1 ..1, y = -1 ..1, thickness = 2)$



$n := Norm(vf)$

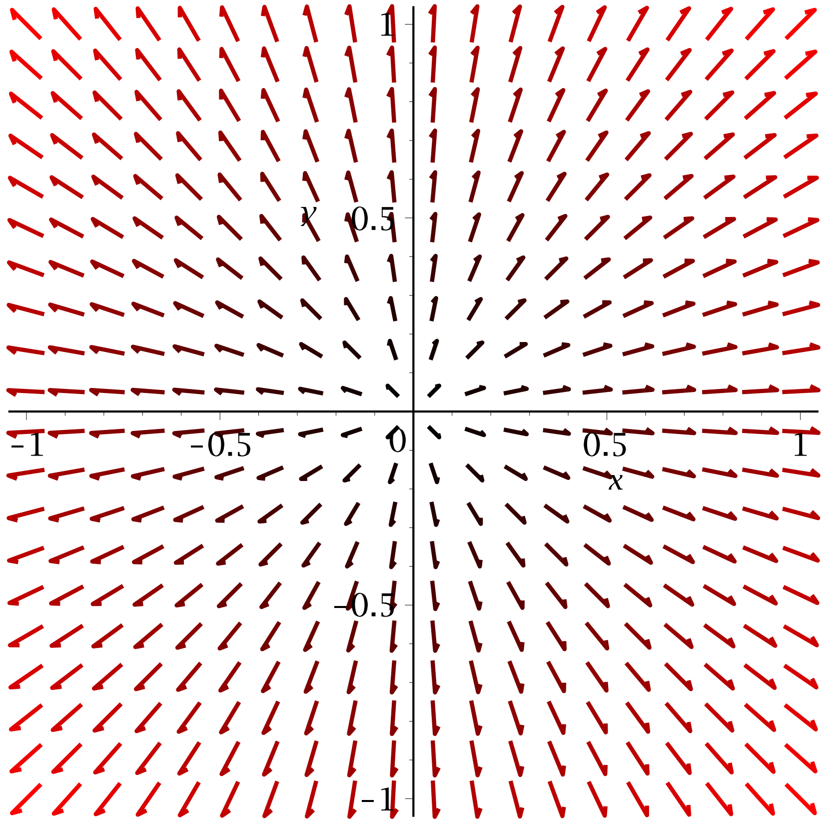
$$w \quad \textbf{(4.4)}$$

$\rightarrow VectorCalculus:-Norm(VectorCalculus:-evalVF(Vector(2, \{(1) = x, (2) = y\}, attributes = [vectorfield, coords = cartesian[x, y]]), w), 2)$

$n(\langle 1, 2 \rangle)$

$$\sqrt{5} \quad \textbf{(4.5)}$$

$fieldplot(vf, x = -1 ..1, y = -1 ..1, thickness = 2, fieldstrength = \log, color = RGB(n(\langle x, y \rangle), 0, 0))$



$e\_r := VectorField(\langle 1, 0\rangle, \mathrm{polar}[r, \mathrm{phi}])$

$$\bar{e}_r \tag{4.6}$$

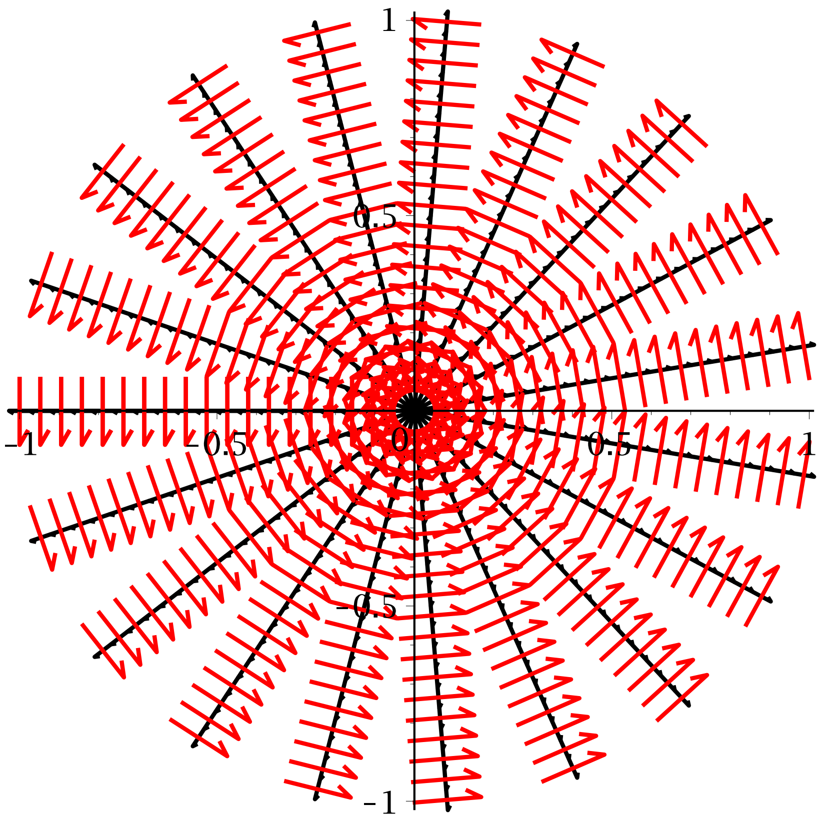
$pl1 := fieldplot(e\_r, r = 0..1, \mathrm{phi} = -\mathrm{Pi}..\mathrm{Pi}, thickness = 2):$

$e\_phi := VectorField(\langle 0, 1\rangle, \mathrm{polar}[r, \mathrm{phi}])$

$$\bar{e}_\phi \tag{4.7}$$

$pl2 := fieldplot(e\_phi, r = 0..1, \mathrm{phi} = -\mathrm{Pi}..\mathrm{Pi}, thickness = 2, color = red):$

$display(\{pl1, pl2\})$



Wie wird das kartesische Vektorfeld <x,y> in Polarkoordinaten angegeben?
(1) <r cos(phi), r sin(phi)>
(2) <r, 0>
(3) <1, 0>
(4) <r,phi>

$pf1 := VectorField(\langle r \cdot \cos(\text{phi}), r \cdot \sin(\text{phi}) \rangle, \text{polar}[r, \text{phi}])$

$$(r\cos(\phi))\bar{e}_r + (r\sin(\phi))\bar{e}_\phi \tag{4.8}$$

$pf2 := VectorField(\langle r, 0 \rangle, \text{polar}[r, \text{phi}])$

$$(r)\bar{e}_r \tag{4.9}$$

$pf3 := VectorField(\langle 1, 0 \rangle, \text{polar}[r, \text{phi}])$

$$\bar{e}_r \tag{4.10}$$

$pf4 := VectorField(\langle r, \text{phi} \rangle, \text{polar}[r, \text{phi}])$

$$(r)\bar{e}_r + (\phi)\bar{e}_\phi \tag{4.11}$$

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for pf in [pf1, pf2, pf3, pf4] do
  fieldplot(pf, r = 0..1, phi = -Pi..Pi, thickness = 2)
```
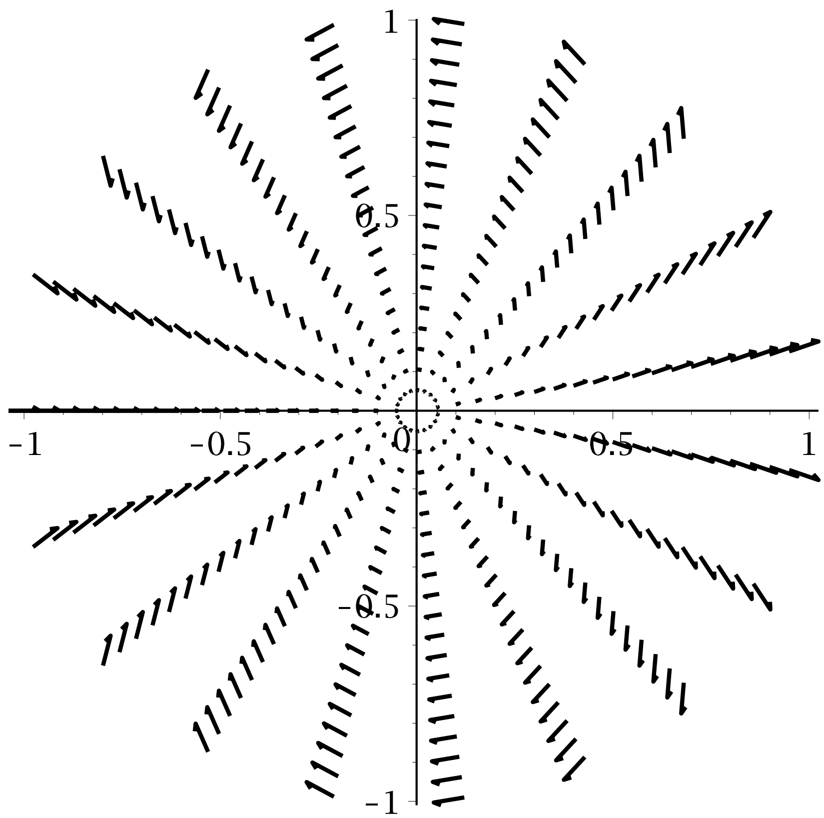
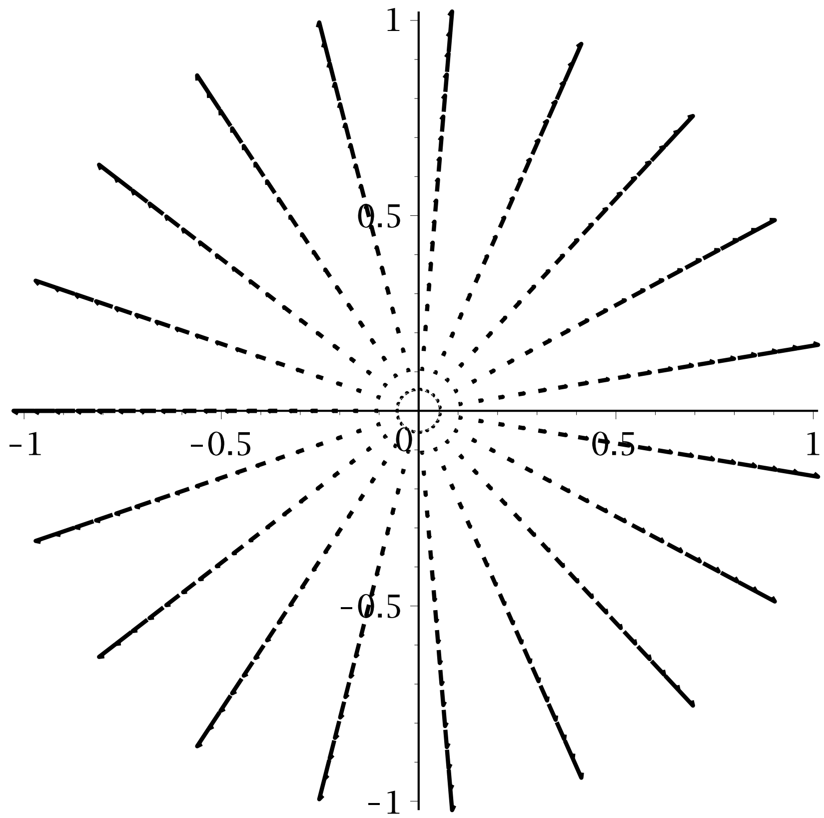
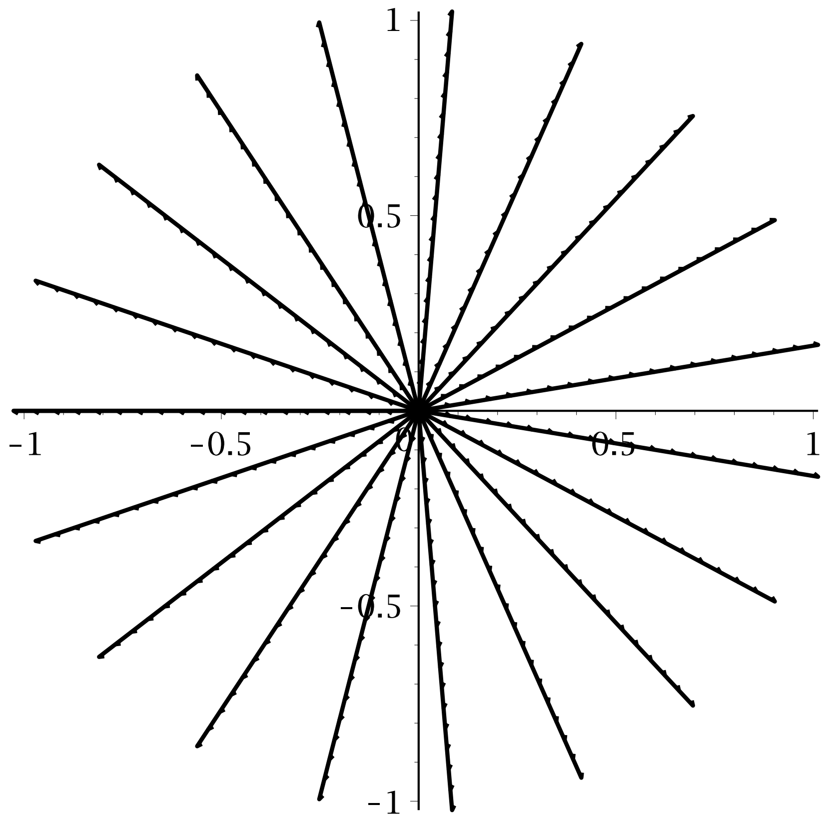
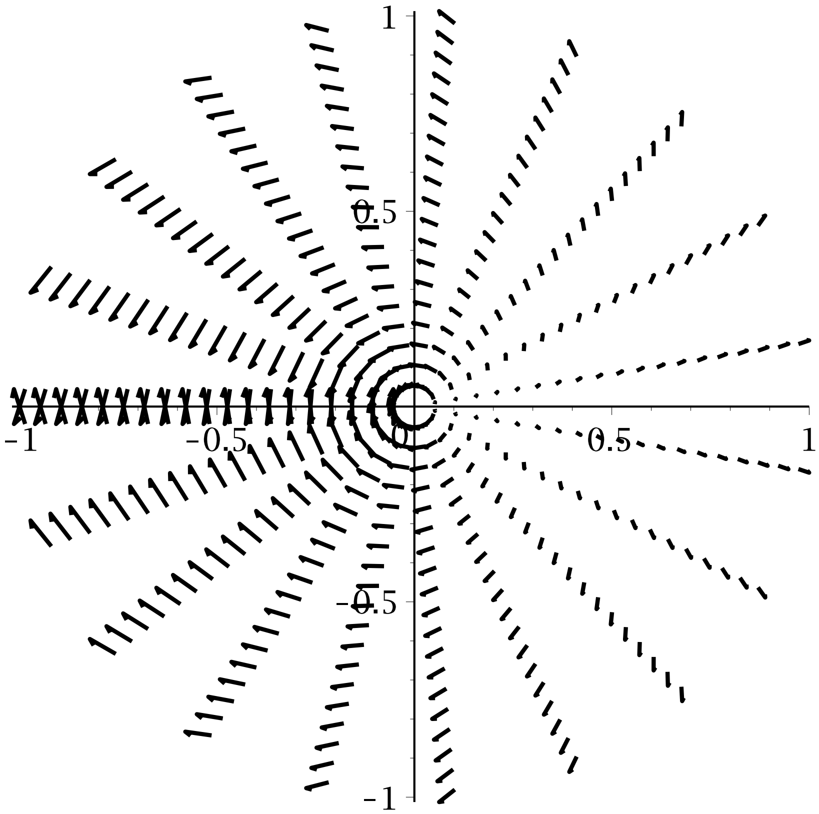

end do




1
0.5
−1
−0.5
0
0.5
1
−0.5
−1


1
0.5
-1
-0.5
0
0.5
1
-0.5
-1



$$k := -\frac{1}{\mathrm{sqrt}(x^2+(y-1)^2+1)} + \frac{1}{\mathrm{sqrt}((x-1)^2+(y+1)^2+1)} + \frac{1}{\mathrm{sqrt}((x+1)^2+(y+1)^2+1)}$$

$$-\frac{1}{\sqrt{x^2+(y-1)^2+1}} + \frac{1}{\sqrt{(x-1)^2+(y+1)^2+1}} + \frac{1}{\sqrt{(x+1)^2+(y+1)^2+1}} \quad \textbf{(4.12)}$$

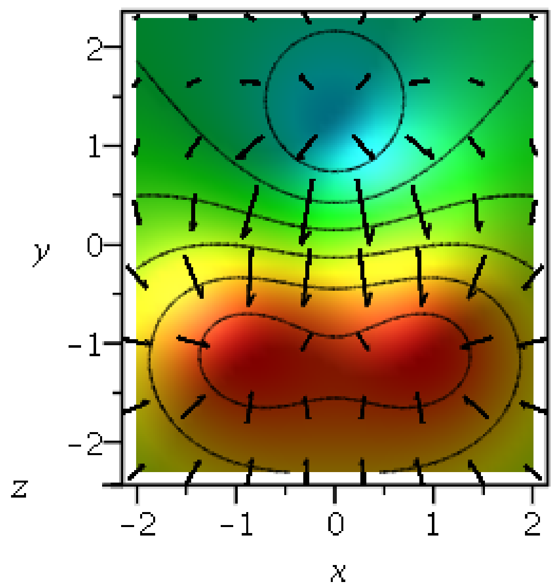
(1) Gradienten stehen senkrecht auf den Niveaulinien
(2) Gradienten verlaufen parallel zu den Niveaulinien
(3) es gibt keinen Zusammenhang zwischen Gradienten und Niveaulinien

$$gr3 := Gradient(k, [x, y, z])$$

$$\left(\frac{x}{(x^2+(y-1)^2+1)^{3/2}} - \frac{1}{2}\frac{2x-2}{((x-1)^2+(y+1)^2+1)^{3/2}} - \frac{1}{2}\frac{2x+2}{((x+1)^2+(y+1)^2+1)^{3/2}}\right)\bar{e}_x + \left(\frac{1}{2}\frac{2y-2}{(x^2+(y-1)^2+1)^{3/2}}\right. \quad \textbf{(4.13)}$$

$$-\frac{1}{2}\frac{2y+2}{((x-1)^2+(y+1)^2+1)^{3/2}}-\frac{1}{2}\frac{2y+2}{((x+1)^2+(y+1)^2+1)^{3/2}}\Big)\bar{e}_y$$

$pl1 := fieldplot3d(gr3, x=-2..2, y=-2.3..2.3, z=-1..1, color=black):$
$pl2 := plot3d(k, x=-2..2, y=-2.3..2.3, shading=zhue, style=patchcontour):$
$display(\{pl1, pl2\}, scaling=constrained, orientation=[-90, 0])$



Taylorpolynom in mehreren Veränderlichen

$f := \cos(x^2+y^2)$

$$\cos(x^2+y^2) \tag{5.1}$$

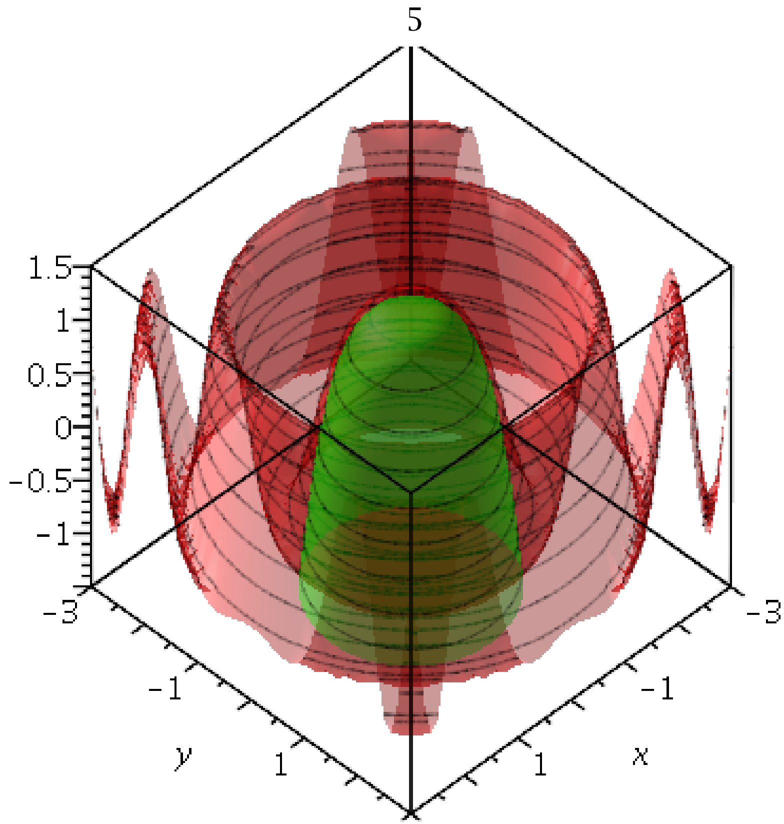
$mtaylor(f, [x=0, y=0], 9)$

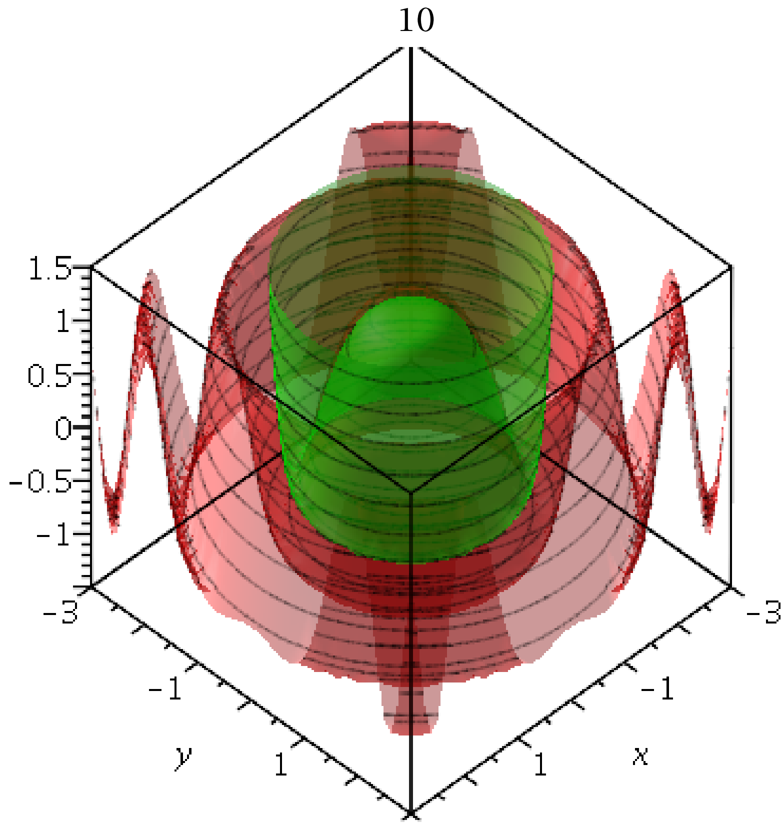
$$1-\frac{1}{2}x^4-y^2x^2-\frac{1}{2}y^4+\frac{1}{24}x^8+\frac{1}{6}y^2x^6+\frac{1}{4}y^4x^4+\frac{1}{6}y^6x^2+\frac{1}{24}y^8 \tag{5.2}$$

```
for n in [5, 10, 30] do
  p[n] := mtaylor(f, [x=0 , y=0], n):
```

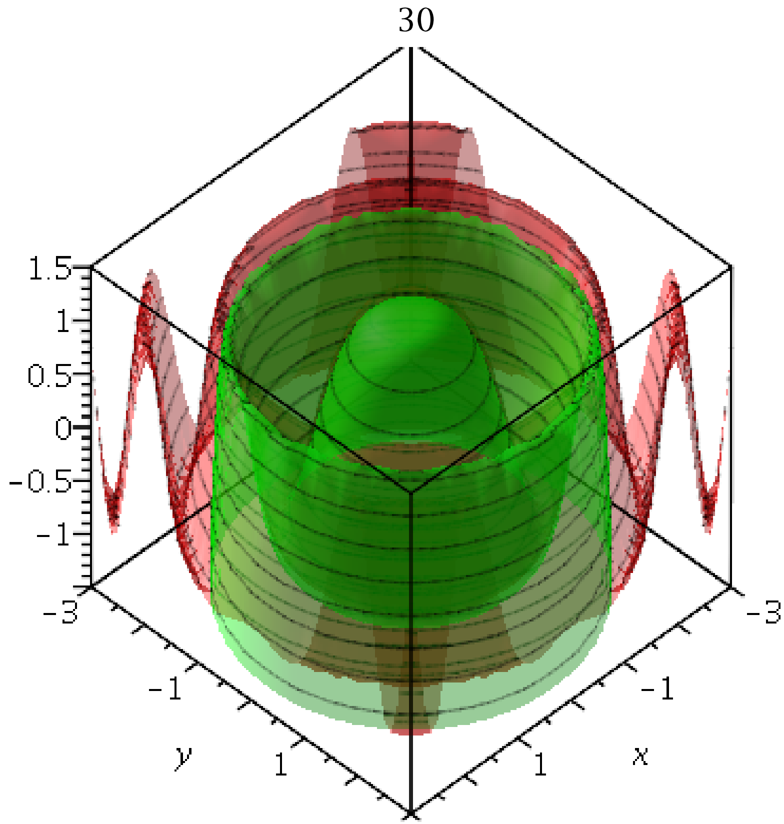
end do:

for *n* **in** [5, 10, 30] **do**
  plot3d([*f* + 0.1, *p*[*n*]], *x* = -3 ..3, *y* = -3 ..3, *view* = -1.5 .. 1.5, *color* = [*red*, *green*], *title* = *n*, *transparency* = 0.6, *style* = *patchcontour*)
end do





10
1.5
1
0.5
0
-0.5
-1
-3
-1
y
1
-3
-1
1
x



30
1.5
1
0.5
0
-0.5
-1
-3
-1
y
1
-3
-1
1
x