Lektion 12

Gewöhnliche Differentialgleichungen

$Dgl := diff(y(x), x) = 2 \cdot y(x)$

$$\frac{d}{dx} y(x) = 2\, y(x) \tag{1.1}$$

$dsolve(Dgl, y(x))$

$$y(x) = \_C1\, e^{2x} \tag{1.2}$$

$Ab := y(2) = 4$

$$y(2) = 4 \tag{1.3}$$

$dsolve(\{Dgl, Ab\})$

$$y(x) = \frac{4\, e^{2x}}{e^4} \tag{1.4}$$

$Dgl := y'(x) = y(x) - x^3 + 3 \cdot x - 2$

$$\frac{d}{dx} y(x) = y(x) - x^3 + 3\,x - 2 \tag{1.5}$$

$Lsg1 := dsolve(\{Dgl, y(0) = 1\})$

$$y(x) = x^3 + 3\,x^2 + 3\,x + 5 - 4\, e^x \tag{1.6}$$

$f1 := rhs(Lsg1)$

$$x^3 + 3\,x^2 + 3\,x + 5 - 4\, e^x \tag{1.7}$$

$Lsg2 := dsolve\left(\left\{Dgl, y(0) = \frac{1}{2}\right\}\right)$

$$y(x) = x^3 + 3\,x^2 + 3\,x + 5 - \frac{9}{2}\, e^x \tag{1.8}$$

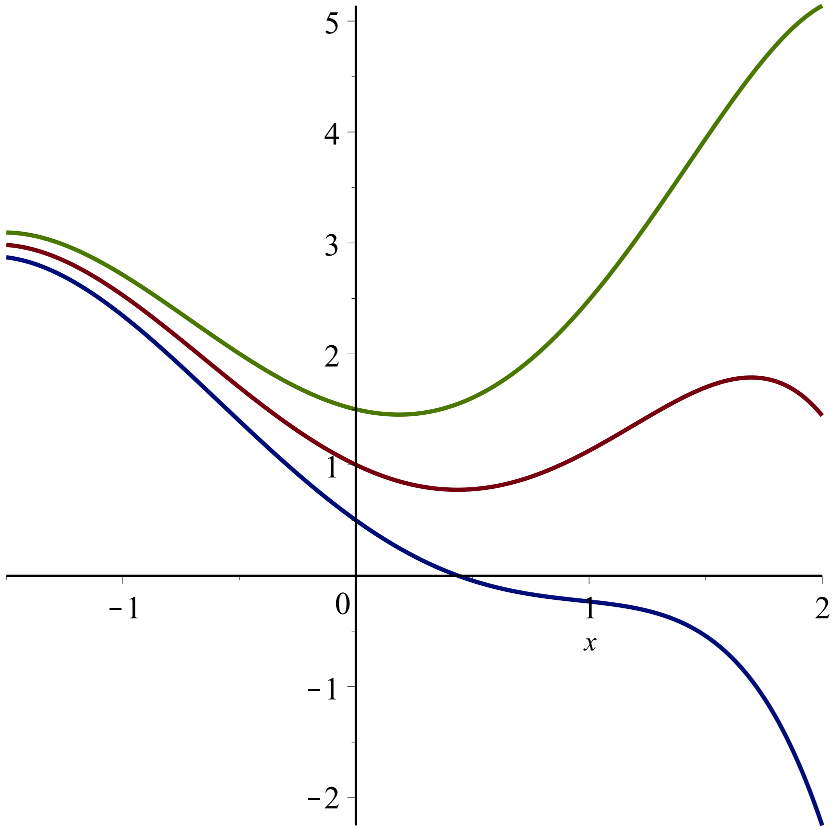
$f2 := eval(y(x), Lsg2)$

$$x^3 + 3\,x^2 + 3\,x + 5 - \frac{9}{2}\, e^x \tag{1.9}$$

$Lsg3 := dsolve\left(\left\{Dgl, y(0) = \frac{3}{2}\right\}\right) :$
$f3 := rhs(Lsg3)$

$$x^3 + 3\,x^2 + 3\,x + 5 - \frac{7}{2}\, e^x \tag{1.10}$$

$pl1 := plot([f1, f2, f3], x = -1.5 ..2, thickness = 2) :$
$pl1$



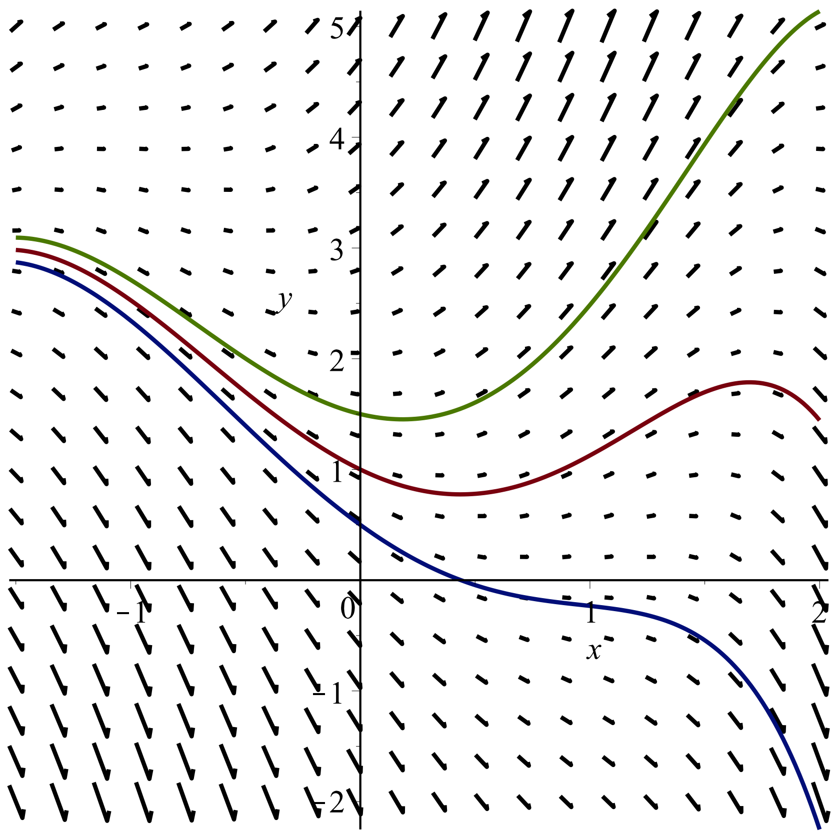
with(*plots*) :
rhs(*Dgl*)

$$y(x) - x^3 + 3\,x - 2 \tag{1.11}$$

$v := [1, eval(rhs(Dgl), y(x) = y)]$

$$[1, -x^3 + 3\,x + y - 2] \tag{1.12}$$

pl2 := *fieldplot*(*v*, *x* = -1.5 ..2, *y* = -2 ..5, *thickness* = 2) :
display({*pl1*, *pl2*})



Definitionsbereiche

restart

$Dgl := diff(y(x), x) = \exp(y(x)) \cdot \sin(x)$

$$\frac{d}{dx} y(x) = e^{y(x)} \sin(x) \tag{2.1}$$

$Lsg := dsolve\left(\left\{Dgl, y(0) = -\frac{1}{2}\right\}\right)$

$$y(x) = -\ln\left(\cos(x) - 1 + e^{\frac{1}{2}}\right) \tag{2.2}$$

$f := rhs(Lsg)$

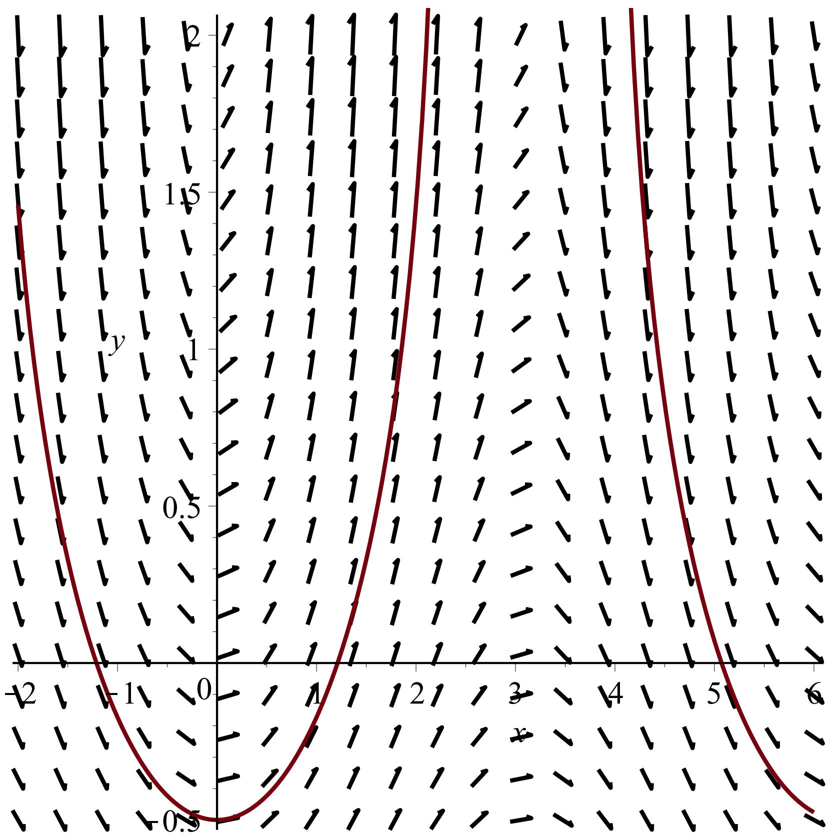
$$-\ln\left(\cos(x) - 1 + e^{\frac{1}{2}}\right) \tag{2.3}$$

$v := [1, \ eval(rhs(Dgl), y(x) = y)]$

$$[1, e^{y} \sin(x)] \tag{2.4}$$

with(plots) :

$pl1 := plot(f, x = -2..6, thickness = 2):$
$pl2 := fieldplot(v, x = -2..6, y = -0.5..2, thickness = 2, fieldstrength = \log):$
$display(\{pl1, pl2\})$



$solve(\mathrm{Im}(f) = 0)$

$$RealRange\left(-3\pi + \arccos\left(-1 + e^{\frac{1}{2}}\right), -\pi - \arccos\left(-1 + e^{\frac{1}{2}}\right)\right), RealRange\left(-\pi + \arccos\left(-1 + e^{\frac{1}{2}}\right), \pi - \arccos\left(-1 + e^{\frac{1}{2}}\right)\right), RealRange\left(\pi + \arccos\left(-1 + e^{\frac{1}{2}}\right), 3\pi - \arccos\left(-1 + e^{\frac{1}{2}}\right)\right) \quad \textbf{(2.5)}$$

Höhere Ordnung

$restart$
$Dgl := diff(y(x), x\$2) + a \cdot diff(y(x), x) + y(x) = 0$

$$\frac{d^2}{dx^2} y(x) + a \left(\frac{d}{dx} y(x) \right) + y(x) = 0 \quad \textbf{(3.1)}$$

$dsolve(Dgl)$

$$y(x) = \_C1\, e^{\left(-\frac{1}{2} a + \frac{1}{2} \sqrt{a^2-4}\right) x} + \_C2\, e^{\left(-\frac{1}{2} a - \frac{1}{2} \sqrt{a^2-4}\right) x} \quad \textbf{(3.2)}$$

$\mathrm{chi} := eval\left(Dgl, \{diff(y(x), x\$2) = t^2, diff(y(x), x) = t, y(x) = 1\}\right)$

$$a\,t + t^2 + 1 = 0 \quad \textbf{(3.3)}$$

$solve(\mathrm{chi}, t)$

$$-\frac{1}{2} a + \frac{1}{2} \sqrt{a^2-4},\ -\frac{1}{2} a - \frac{1}{2} \sqrt{a^2-4} \quad \textbf{(3.4)}$$

a=2 ist der aperiodische Grenzfall

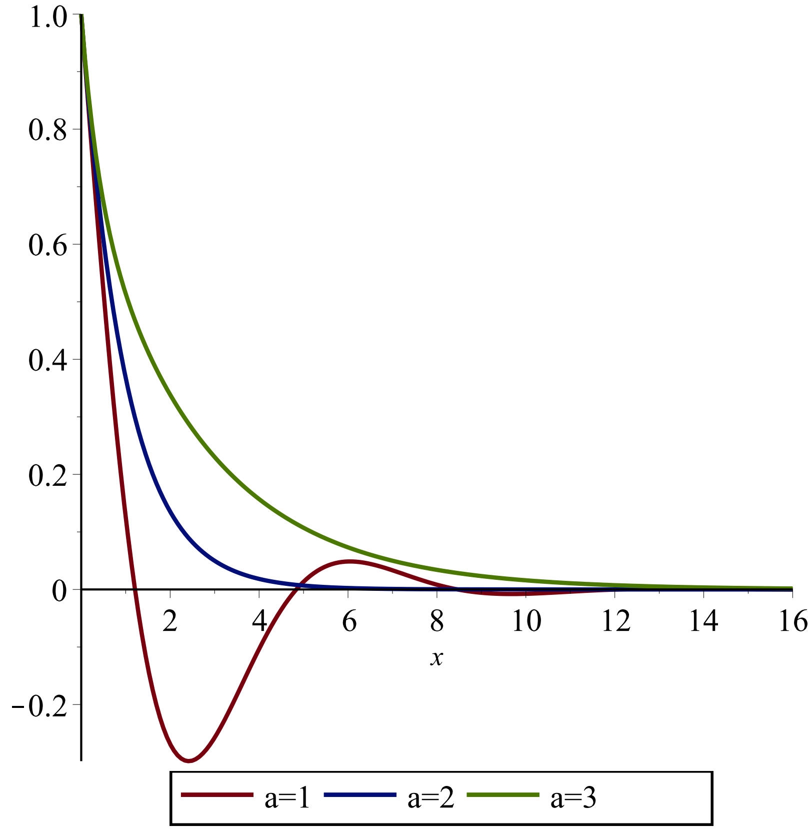
for a **in** $[1, 2, 3]$ **do**
$Lsg := dsolve(\{Dgl, y(0) = 1, \mathrm{D}(y)(0) = -1\});$
$f[a] := rhs(Lsg);$
$print(a, f[a]);$
end do:

$$1, -\frac{1}{3} \sqrt{3}\, e^{-\frac{1}{2} x} \sin\left(\frac{1}{2} \sqrt{3}\, x\right) + e^{-\frac{1}{2} x} \cos\left(\frac{1}{2} \sqrt{3}\, x\right)$$

$$2, e^{-x}$$

$$3, \left(\frac{1}{2} + \frac{1}{10} \sqrt{5}\right) e^{\frac{1}{2} \left(\sqrt{5} - 3\right) x} + \left(\frac{1}{2} - \frac{1}{10} \sqrt{5}\right) e^{-\frac{1}{2} \left(\sqrt{5} + 3\right) x} \quad \textbf{(3.5)}$$

$plot(convert(f, list), x = 0..16, thickness = 2, legend = [\text{"a=1"}, \text{"a=2"}, \text{"a=3"}])$



$Dgl := diff(y(x), x\$2) = y(x) \cdot (1 + x^{-2})$

$$\frac{d^2}{dx^2} y(x) = y(x) \left(1 + \frac{1}{x^2}\right) \quad \textbf{(3.6)}$$

$dsolve(Dgl)$

$$y(x) = \_C1 \sqrt{x}\, \mathrm{BesselI}\left(\frac{1}{2}\sqrt{5}, x\right) + \_C2 \sqrt{x}\, \mathrm{BesselK}\left(\frac{1}{2}\sqrt{5}, x\right) \quad \textbf{(3.7)}$$

Differentialgleichungssysteme

$restart$

$Dgl1 := diff(x(t), t\$2) = -(1 + a) \cdot x(t) - a \cdot y(t)$

$$\frac{d^2}{dt^2} x(t) = -(1 + a)\, x(t) - a\, y(t) \quad \textbf{(4.1)}$$

$Dgl2 := diff(y(t), t\$2) = -(1 + b) \cdot y(t) - b \cdot x(t)$

$$\frac{d^2}{dt^2} y(t) = -(1 + b)\, y(t) - b\, x(t) \quad \textbf{(4.2)}$$

$Dgs := \{Dgl1, Dgl2\}$

$$\left\{\frac{d^2}{dt^2}\,x(t) = -(1+a)\,x(t) - a\,y(t),\ \frac{d^2}{dt^2}\,y(t) = -(1+b)\,y(t) - b\,x(t)\right\} \tag{4.3}$$

$dsolve(Dgs)$

$$\left\{x(t) = \_C1\,\sin(t) + \_C2\,\cos(t) + \_C3\,\sin\left(\sqrt{a+b+1}\;t\right) + \_C4\,\cos\left(\sqrt{a+b+1}\;t\right), y(t) = -\frac{1}{a}\left(\sin(t)\,\_C1\,a + \cos(t)\,\_C2\,a - \_C3\,\sin\left(\sqrt{a+b+1}\;t\right)b - \_C4\,\cos\left(\sqrt{a+b+1}\;t\right)b\right)\right\} \tag{4.4}$$

$Ab := \{x(0) = 0, D(x)(0) = 0, y(0) = 1, D(y)(0) = 0\}$

$$\{x(0) = 0, y(0) = 1, D(x)(0) = 0, D(y)(0) = 0\} \tag{4.5}$$

$Lsg := dsolve(Dgs\ \textbf{union}\ Ab):$

$normal(Lsg)$

$$\left\{x(t) = -\frac{a\left(\cos(t) - \cos\left(\sqrt{a+b+1}\;t\right)\right)}{a+b}, y(t) = \frac{a\cos(t) + \cos\left(\sqrt{a+b+1}\;t\right)b}{a+b}\right\} \tag{4.6}$$

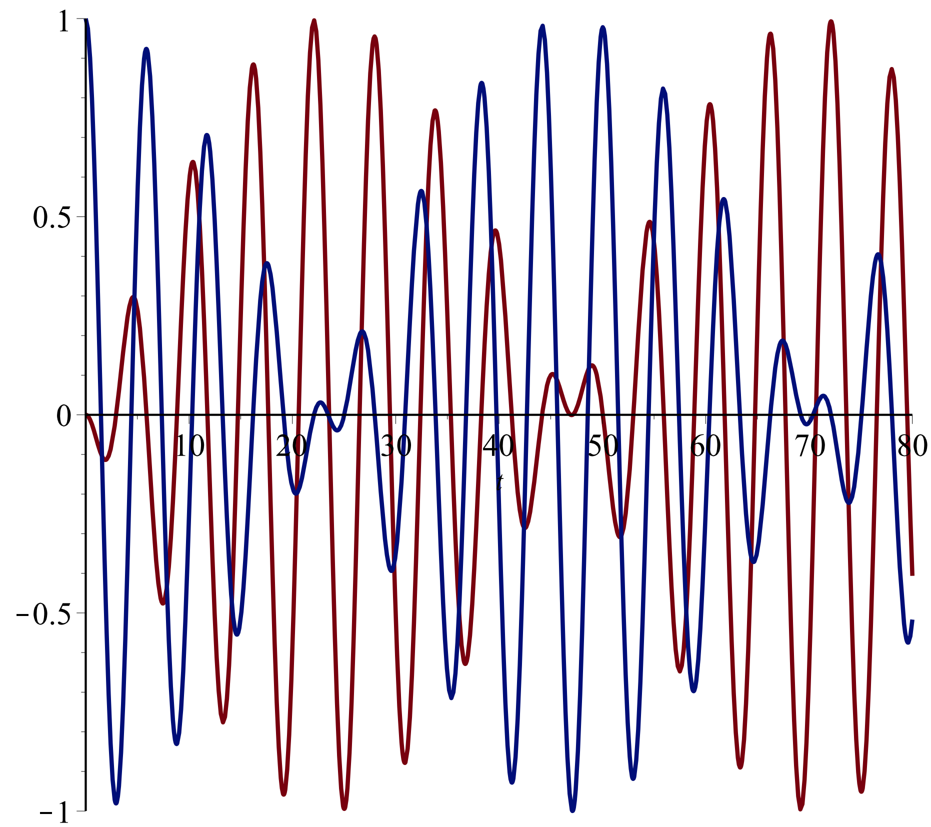
$fx := eval(x(t), Lsg)$

$$-\frac{a\cos(t)}{a+b} + \frac{a\cos\left(\sqrt{a+b+1}\;t\right)}{a+b} \tag{4.7}$$

$fy := eval(y(t), Lsg)$

$$-\frac{-\frac{\cos(t)\,a^2}{a+b} - \frac{a\cos\left(\sqrt{a+b+1}\;t\right)b}{a+b}}{a} \tag{4.8}$$

$plot\left(eval\left([fx, fy], \left\{a = \frac{1}{7}, b = \frac{1}{7}\right\}\right), t = 0..80, thickness = 2\right)$



Numerische Lösung

restart

$Dgl := diff(y(x), x\$2) = -\sin(y(x))$

$$\frac{d^2}{dx^2}\, y(x) = -\sin(y(x)) \tag{5.1}$$

$Ab := y(0) = \frac{\mathrm{Pi}}{2}, \mathrm{D}(y)(0) = 0$

$$y(0) = \frac{1}{2}\pi, \mathrm{D}(y)(0) = 0 \tag{5.2}$$

$numLsg := dsolve(\{Dgl, Ab\}, type = numeric, output = listprocedure)$

$$\left[x = \mathbf{proc}(x)\ \ldots\ \mathbf{end\ proc}, y(x) = \mathbf{proc}(x)\ \ldots\ \mathbf{end\ proc}, \frac{d}{dx}\, y(x) = \mathbf{proc}(x)\ \ldots\ \mathbf{end\ proc}\right] \tag{5.3}$$

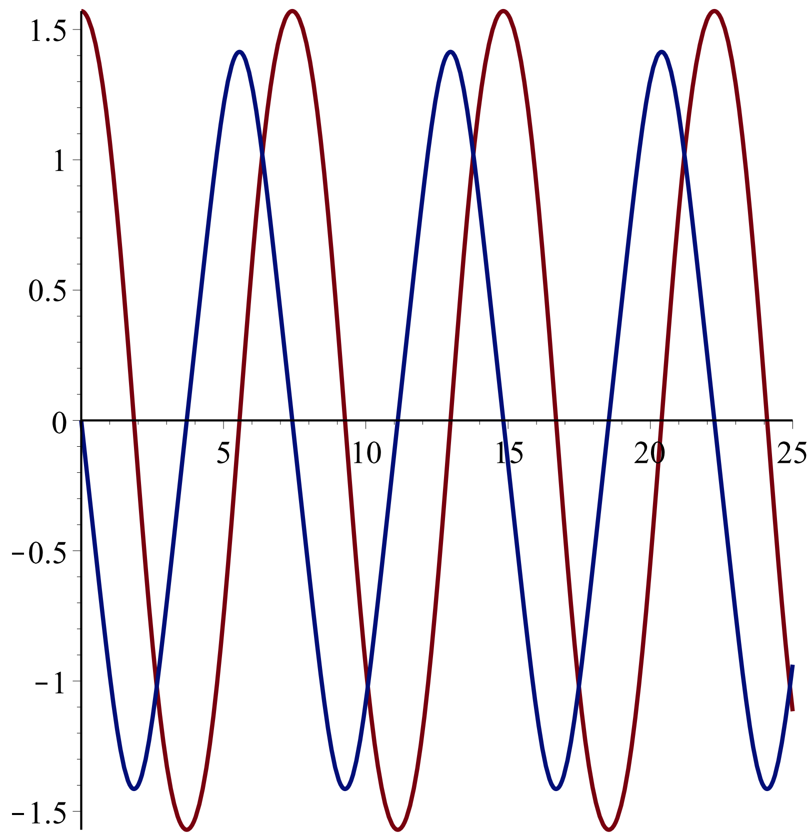
$num\_y := eval(y(x), numLsg)$

$$\mathbf{proc}(x)\ \ldots\ \mathbf{end\ proc} \tag{5.4}$$

$num\_dy := eval(diff(y(x), x), numLsg)$

$$\mathbf{proc}(x) \ \ldots \ \mathbf{end\ proc} \tag{5.5}$$

$plot([num\_y, num\_dy], 0..25., thickness = 2)$



Die Pendelgleichung

$dsolve(Dgl)$

$$\int^{y(x)} \frac{1}{\sqrt{2\cos(\_a) + \_C1}}\, \mathrm{d}\_a - x - \_C2 = 0, \int^{y(x)} \left(-\frac{1}{\sqrt{2\cos(\_a) + \_C1}}\right) \mathrm{d}\_a - x - \_C2 = 0 \tag{6.1}$$

$lprint(\mathbf{(6.1)}[1])$

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Intat(1/(2*cos(_a)+_C1)^(1/2), _a = y(x))-x-_C2 = 0
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$Lsg := dsolve(\{Dgl, Ab\})$

$$y(x) = RootOf\left(\int_{\_Z}^{\frac{1}{2}\pi} \left(-\frac{1}{2}\frac{\sqrt{2}}{\sqrt{\cos(\_a)}}\right) \mathrm{d}\_a + x\right), y(x) = RootOf\left(\int_{\frac{1}{2}\pi}^{\_Z} \left(-\frac{1}{2}\frac{\sqrt{2}}{\sqrt{\cos(\_a)}}\right) \mathrm{d}\_a + x\right) \tag{6.2}$$

Bestimme Nullstelle von y(x)

$Ro := eval(y(x), Lsg[1])$

$$RootOf\left(\int_{\_Z}^{\frac{1}{2}\pi} \left(-\frac{1}{2}\frac{\sqrt{2}}{\sqrt{\cos(\_a)}}\right) \mathrm{d}\_a + x\right) \tag{6.3}$$

$L := convert(Ro, list)$

$$\left[\int_{\_Z}^{\frac{1}{2}\pi} \left(-\frac{1}{2}\frac{\sqrt{2}}{\sqrt{\cos(\_a)}}\right) \mathrm{d}\_a + x\right] \tag{6.4}$$

$Glg := eval(L[1], \_Z = 0) = 0$

$$\int_{0}^{\frac{1}{2}\pi} \left(-\frac{1}{2}\frac{\sqrt{2}}{\sqrt{\cos(\_a)}}\right) \mathrm{d}\_a + x = 0 \tag{6.5}$$

$solve(Glg, x)$

$$-\left(\int_{0}^{\frac{1}{2}\pi} \left(-\frac{1}{2}\frac{\sqrt{2}}{\sqrt{\cos(\_a)}}\right) \mathrm{d}\_a\right) \tag{6.6}$$

$T := value(\mathbf{(6.6)})$

$$\mathrm{EllipticK}\left(\frac{1}{2}\sqrt{2}\right) \tag{6.7}$$

$DglOsz := diff(y(x), x\$2) = -\frac{\pi^2}{4 \cdot T^2} \cdot y(x)$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} y(x) = -\frac{1}{4}\frac{\pi^2 y(x)}{\mathrm{EllipticK}\left(\frac{1}{2}\sqrt{2}\right)^2} \tag{6.8}$$

$LsgOsz := dsolve(\{DglOsz, Ab\})$

$$y(x) = \frac{1}{2}\,\pi\cos\left(\frac{1}{2}\,\frac{\pi\sqrt{\mathrm{EllipticK}\left(\frac{1}{2}\sqrt{2}\right)^2}\,x}{\mathrm{EllipticK}\left(\frac{1}{2}\sqrt{2}\right)^2}\right) \tag{6.9}$$

$y\_Osz := eval(y(x), LsgOsz)$:
$plot([num\_y(x), y\_Osz], x = 0\,..2\cdot\,T, thickness = 2, legend = [$"Pendel", "harm. Oszillator"$])$