Lektion 11

Lokale Extrema in 3D

$restart$
$with(VectorCalculus):$
$BasisFormat(false):$
$SetCoordinates(cartesian[x, y, z])$

$$cartesian_{x, y, z} \tag{1.1}$$

$with(plots):$

$k3 := \frac{1}{\mathrm{sqrt}((x-1)^2+y^2+z^2+1)} - \frac{1}{\mathrm{sqrt}((x+1)^2+y^2+z^2+1)}$

$$\frac{1}{\sqrt{(x-1)^2+y^2+z^2+1}} - \frac{1}{\sqrt{(x+1)^2+y^2+z^2+1}} \tag{1.2}$$

$g := Gradient(k3)$

$$\begin{bmatrix} -\frac{1}{2}\frac{2x-2}{((x-1)^2+y^2+z^2+1)^{3/2}} + \frac{1}{2}\frac{2x+2}{((x+1)^2+y^2+z^2+1)^{3/2}} \\ -\frac{y}{((x-1)^2+y^2+z^2+1)^{3/2}} + \frac{y}{((x+1)^2+y^2+z^2+1)^{3/2}} \\ -\frac{z}{((x-1)^2+y^2+z^2+1)^{3/2}} + \frac{z}{((x+1)^2+y^2+z^2+1)^{3/2}} \end{bmatrix} \tag{1.3}$$

$Lsg := solve(convert(g, set))$

$$\Big\{x = -\frac{529}{5792}\, RootOf(4\_Z^6 - 44\_Z^5 + 177\_Z^4 - 552\_Z^3 + 1696\_Z^2 - 2304\_Z + 1024, index = 1)^5 + \frac{5299}{5792}\, RootOf(4\_Z^6 - 44\_Z^5 + 177\_Z^4 - 552\_Z^3 + 1696\_Z^2 - 2304\_Z + 1024, index = 1)^4 - \frac{72929}{23168}\, RootOf(4\_Z^6 - 44\_Z^5 + 177\_Z^4 - 552\_Z^3 + 1696\_Z^2 - 2304\_Z + 1024, index = 1)^3 + \frac{6937}{724}\, RootOf(4\_Z^6 - 44\_Z^5 + 177\_Z^4 - 552\_Z^3 + 1696\_Z^2 - 2304\_Z + 1024, index = 1)^2 - \frac{10809}{362}\, RootOf(4\_Z^6 - 44\_Z^5 + 177\_Z^4 - 552\_Z^3 + 1696\_Z^2 - 2304\_Z + 1024, index = 1) + \frac{4386}{181}, y = 0, z = 0\Big\}, \tag{1.4}$$

$$\left\{x = -\frac{529}{5792}\,\mathit{RootOf}\big(4\_Z^6 - 44\_Z^5 + 177\_Z^4 - 552\_Z^3 + 1696\_Z^2\right.$$

$$- 2304\_Z + 1024,\ \mathit{index} = 4\big)^5 + \frac{5299}{5792}\,\mathit{RootOf}\big(4\_Z^6 - 44\_Z^5$$

$$+ 177\_Z^4 - 552\_Z^3 + 1696\_Z^2 - 2304\_Z + 1024,\ \mathit{index} = 4\big)^4$$

$$- \frac{72929}{23168}\,\mathit{RootOf}\big(4\_Z^6 - 44\_Z^5 + 177\_Z^4 - 552\_Z^3 + 1696\_Z^2$$

$$- 2304\_Z + 1024,\ \mathit{index} = 4\big)^3 + \frac{6937}{724}\,\mathit{RootOf}\big(4\_Z^6 - 44\_Z^5$$

$$+ 177\_Z^4 - 552\_Z^3 + 1696\_Z^2 - 2304\_Z + 1024,\ \mathit{index} = 4\big)^2$$

$$- \frac{10809}{362}\,\mathit{RootOf}\big(4\_Z^6 - 44\_Z^5 + 177\_Z^4 - 552\_Z^3 + 1696\_Z^2 - 2304\_Z$$

$$\left. + 1024,\ \mathit{index} = 4\big) + \frac{4386}{181},\ y = 0,\ z = 0\right\},\ \left\{x = -\frac{529}{5792}\,\mathit{RootOf}\big(4\_Z^6\right.$$

$$- 44\_Z^5 + 177\_Z^4 - 552\_Z^3 + 1696\_Z^2 - 2304\_Z + 1024,\ \mathit{index} = 5\big)^5$$

$$+ \frac{5299}{5792}\,\mathit{RootOf}\big(4\_Z^6 - 44\_Z^5 + 177\_Z^4 - 552\_Z^3 + 1696\_Z^2$$

$$- 2304\_Z + 1024,\ \mathit{index} = 5\big)^4 - \frac{72929}{23168}\,\mathit{RootOf}\big(4\_Z^6 - 44\_Z^5$$

$$+ 177\_Z^4 - 552\_Z^3 + 1696\_Z^2 - 2304\_Z + 1024,\ \mathit{index} = 5\big)^3$$

$$+ \frac{6937}{724}\,\mathit{RootOf}\big(4\_Z^6 - 44\_Z^5 + 177\_Z^4 - 552\_Z^3 + 1696\_Z^2$$

$$- 2304\_Z + 1024,\ \mathit{index} = 5\big)^2 - \frac{10809}{362}\,\mathit{RootOf}\big(4\_Z^6 - 44\_Z^5 + 177\_Z^4$$

$$\left. - 552\_Z^3 + 1696\_Z^2 - 2304\_Z + 1024,\ \mathit{index} = 5\big) + \frac{4386}{181},\ y = 0,\ z = 0\right\},$$

$$\left\{x = -\frac{529}{5792}\,\mathit{RootOf}\big(4\_Z^6 - 44\_Z^5 + 177\_Z^4 - 552\_Z^3 + 1696\_Z^2\right.$$

$$- 2304\_Z + 1024,\ \mathit{index} = 6\big)^5 + \frac{5299}{5792}\,\mathit{RootOf}\big(4\_Z^6 - 44\_Z^5$$

$$+ 177\_Z^4 - 552\_Z^3 + 1696\_Z^2 - 2304\_Z + 1024,\ \mathit{index} = 6\big)^4$$

$$- \frac{72929}{23168}\,\mathit{RootOf}\big(4\_Z^6 - 44\_Z^5 + 177\_Z^4 - 552\_Z^3 + 1696\_Z^2$$

$$- 2304\_Z + 1024,\ \mathit{index} = 6\big)^3 + \frac{6937}{724}\,\mathit{RootOf}\big(4\_Z^6 - 44\_Z^5$$

$$+ 177\_Z^4 - 552\_Z^3 + 1696\_Z^2 - 2304\_Z + 1024, index = 6)^2$$

$$- \frac{10809}{362} RootOf(4\_Z^6 - 44\_Z^5 + 177\_Z^4 - 552\_Z^3 + 1696\_Z^2 - 2304\_Z + 1024, index = 6) + \frac{4386}{181}, y = 0, z = 0\}$$

Numerisch kann er das System nicht lösen

$fsolve(convert(g, set), \{x, y, z\})$

$$fsolve\left(\left\{ -\frac{y}{((x-1)^2+y^2+z^2+1)^{3/2}} + \frac{y}{((x+1)^2+y^2+z^2+1)^{3/2}}, -\frac{z}{((x-1)^2+y^2+z^2+1)^{3/2}} + \frac{z}{((x+1)^2+y^2+z^2+1)^{3/2}}, -\frac{1}{2}\frac{2x-2}{((x-1)^2+y^2+z^2+1)^{3/2}} + \frac{1}{2}\frac{2x+2}{((x+1)^2+y^2+z^2+1)^{3/2}} \right\}, \{x, y, z\}\right) \quad (1.5)$$

$numLsg := evalf(Lsg)$

$$\{x = 1.16615264, y = 0., z = 0.\}, \{x = -1.16615220, y = 0., z = 0.\}, \{x = 0. - 1.56786189\,I, y = 0., z = 0.\}, \{x = 0. + 1.56786189\,I, y = 0., z = 0.\} \quad (1.6)$$

$p1 := numLsg[1]$

$$\{x = 1.16615264, y = 0., z = 0.\} \quad (1.7)$$

$p2 := numLsg[2]$

$$\{x = -1.16615220, y = 0., z = 0.\} \quad (1.8)$$

$optionen := axes = boxed, style = patchnogrid, transparency = 0.5, scaling = constrained, numpoints = 18000$

$$axes = boxed, style = patchnogrid, transparency = 0.5, scaling = constrained, numpoints = 18000 \quad (1.9)$$

$pl1 := implicitplot3d\left(k3 = -\frac{1}{3}, x = -2.3..2.3, y = -2.3..2.3, z = -1..1, color = blue, optionen\right):$

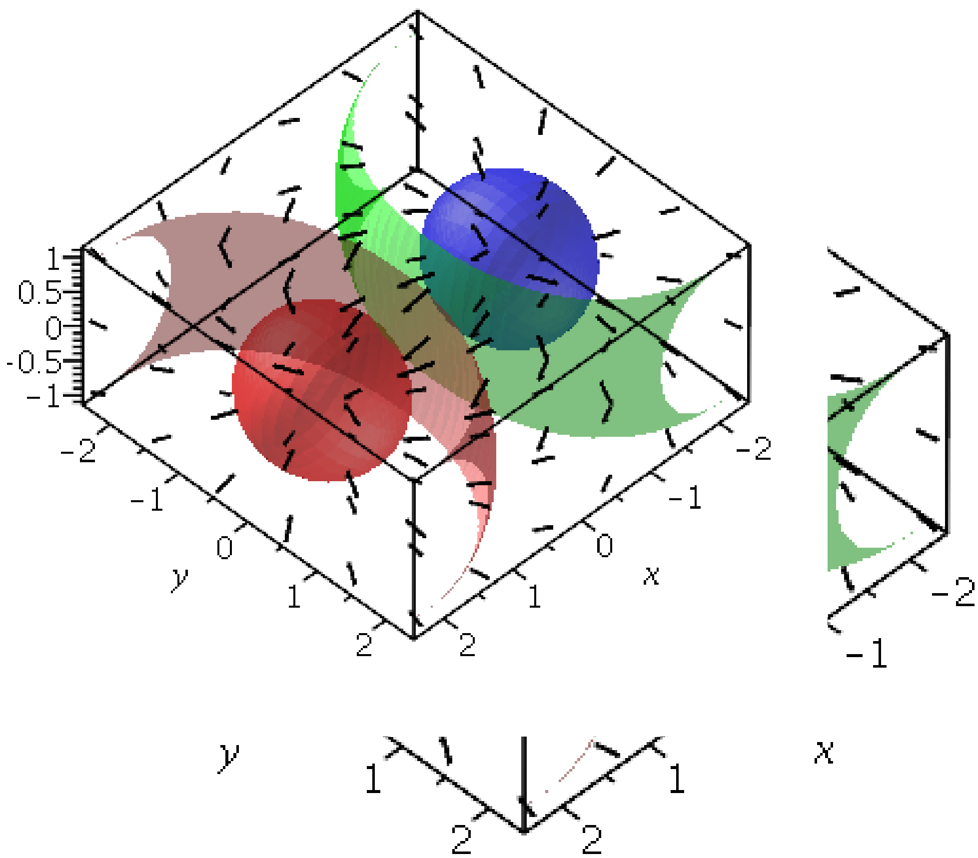
$pl2 := implicitplot3d\left(k3 = -\frac{1}{10}, x = -2.3..2.3, y = -2.3..2.3, z = -1..1, color = green, optionen\right):$

$pl3 := implicitplot3d\left(k3 = \frac{1}{10}, x = -2.3..2.3, y = -2.3..2.3, z = -1..1, color = orange, optionen\right):$

$pl4 := implicitplot3d\left(k3 = \frac{1}{3}, x = -2.3..2.3, y = -2.3..2.3, z = -1..1, color = red, optionen\right):$

$pl5 := fieldplot3d(g, x = -2.3..2.3, y = -2.3..2.3, z = -1..1, grid = [6, 6, 3], thickness = 2, color = black, fieldstrength = \log, scaling = constrained):$

$display(\{pl1, pl2, pl3, pl4, pl5\})$



Extrema unter Nebenbedingungen

$restart$

$with(plots):$

$with(algcurves):$

$with(VectorCalculus):$

$BasisFormat(false):$

$SetCoordinates(cartesian[x, y])$

$$cartesian_{x,\, y} \qquad \textbf{(1.1.1)}$$

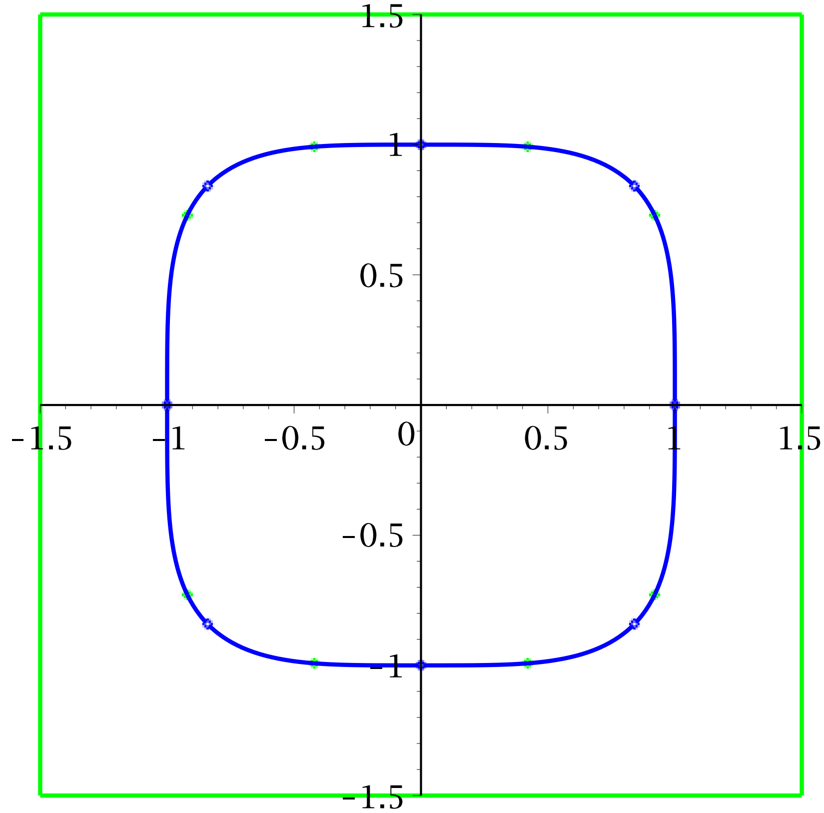
$g := x^4 + y^4 - 1$

$$x^4 + y^4 - 1 \qquad (1.1.2)$$

$optionen := view = [-1.5..1.5, -1.5..1.5], thickness = 2$

$$view = [-1.5..1.5, -1.5..1.5], thickness = 2 \qquad (1.1.3)$$

$NBplot := plot\_real\_curve(g, x, y, optionen):$
$NBplot$



$f := x^2 + y^2$

$$x^2 + y^2 \qquad (1.1.4)$$

Gesucht: Maximum von f unter der Nebenbedingung g=0

$Gls := Gradient(f) - \text{lambda} \cdot Gradient(g)$

$$\begin{bmatrix} -4\lambda x^3 + 2x \\ -4\lambda y^3 + 2y \end{bmatrix} \qquad (1.1.5)$$

$Lsg := [solve(convert(Gls, set)\ \mathbf{union}\ \{g\}, Explicit = true)]$

$$\left[\left\{\lambda = \frac{1}{2}, x = 0, y = 1\right\}, \left\{\lambda = \frac{1}{2}, x = 0, y = -1\right\}, \left\{\lambda = -\frac{1}{2}, x = 0, y = \mathrm{I}\right\}, \left\{\lambda = \right.\right. \qquad (1.1.6)$$

$-\frac{1}{2}, x=0, y=-\mathrm{I}\Big\}, \Big\{\lambda=\frac{1}{2}, x=1, y=0\Big\}, \Big\{\lambda=\frac{1}{2}, x=-1, y=0\Big\}, \Big\{\lambda=$
$-\frac{1}{2}, x=\mathrm{I}, y=0\Big\}, \Big\{\lambda=-\frac{1}{2}, x=-\mathrm{I}, y=0\Big\}, \Big\{\lambda=\frac{1}{2}\sqrt{2}, x=\frac{1}{2}2^{3/4}, y$
$=\frac{1}{2}2^{3/4}\Big\}, \Big\{\lambda=\frac{1}{2}\sqrt{2}, x=-\frac{1}{2}2^{3/4}, y=-\frac{1}{2}2^{3/4}\Big\}, \Big\{\lambda=-\frac{1}{2}\sqrt{2}, x$
$=\frac{1}{2}\mathrm{I}\,2^{3/4}, y=\frac{1}{2}\mathrm{I}\,2^{3/4}\Big\}, \Big\{\lambda=-\frac{1}{2}\sqrt{2}, x=-\frac{1}{2}\mathrm{I}\,2^{3/4}, y=-\frac{1}{2}\mathrm{I}\,2^{3/4}\Big\}, \Big\{\lambda$
$=\frac{1}{2}\sqrt{2}, x=\frac{1}{2}2^{3/4}, y=-\frac{1}{2}2^{3/4}\Big\}, \Big\{\lambda=\frac{1}{2}\sqrt{2}, x=-\frac{1}{2}2^{3/4}, y$
$=\frac{1}{2}2^{3/4}\Big\}, \Big\{\lambda=-\frac{1}{2}\sqrt{2}, x=\frac{1}{2}\mathrm{I}\,2^{3/4}, y=-\frac{1}{2}\mathrm{I}\,2^{3/4}\Big\}, \Big\{\lambda=-\frac{1}{2}\sqrt{2}, x=$
$-\frac{1}{2}\mathrm{I}\,2^{3/4}, y=\frac{1}{2}\mathrm{I}\,2^{3/4}\Big\}\Big]$

$isprime(21)$

$$false \qquad \textbf{(1.1.7)}$$

$L := [2, 3, 4, 5]$

$$[2, 3, 4, 5] \qquad \textbf{(1.1.8)}$$

$select(isprime, L)$

$$[2, 3, 5] \qquad \textbf{(1.1.9)}$$

$has(2+3\cdot I, I)$

$$true \qquad \textbf{(1.1.10)}$$

$has(25!, I)$

$$false \qquad \textbf{(1.1.11)}$$

$ist\_reell := t \to \textbf{not}\ has(t, I)$

$$t \to \textbf{not}\ has(t, \mathrm{I}) \qquad \textbf{(1.1.12)}$$

$select(ist\_reell, \{I, 1+2\cdot I, 20!\})$

$$\{2432902008176640000\} \qquad \textbf{(1.1.13)}$$

$ist\_reell(\log(-\varepsilon^2))$

$$true \qquad \textbf{(1.1.14)}$$

Der Test hat Potential nach oben.

$reelleLsg := select(ist\_reell, Lsg)$

$\Big[\Big\{\lambda=\frac{1}{2}, x=0, y=1\Big\}, \Big\{\lambda=\frac{1}{2}, x=0, y=-1\Big\}, \Big\{\lambda=\frac{1}{2}, x=1, y=0\Big\}, \Big\{\lambda$ **(1.1.15)**
$=\frac{1}{2}, x=-1, y=0\Big\}, \Big\{\lambda=\frac{1}{2}\sqrt{2}, x=\frac{1}{2}2^{3/4}, y=\frac{1}{2}2^{3/4}\Big\}, \Big\{\lambda=\frac{1}{2}\sqrt{2},$
$x=-\frac{1}{2}2^{3/4}, y=-\frac{1}{2}2^{3/4}\Big\}, \Big\{\lambda=\frac{1}{2}\sqrt{2}, x=\frac{1}{2}2^{3/4}, y=-\frac{1}{2}2^{3/4}\Big\}, \Big\{\lambda$

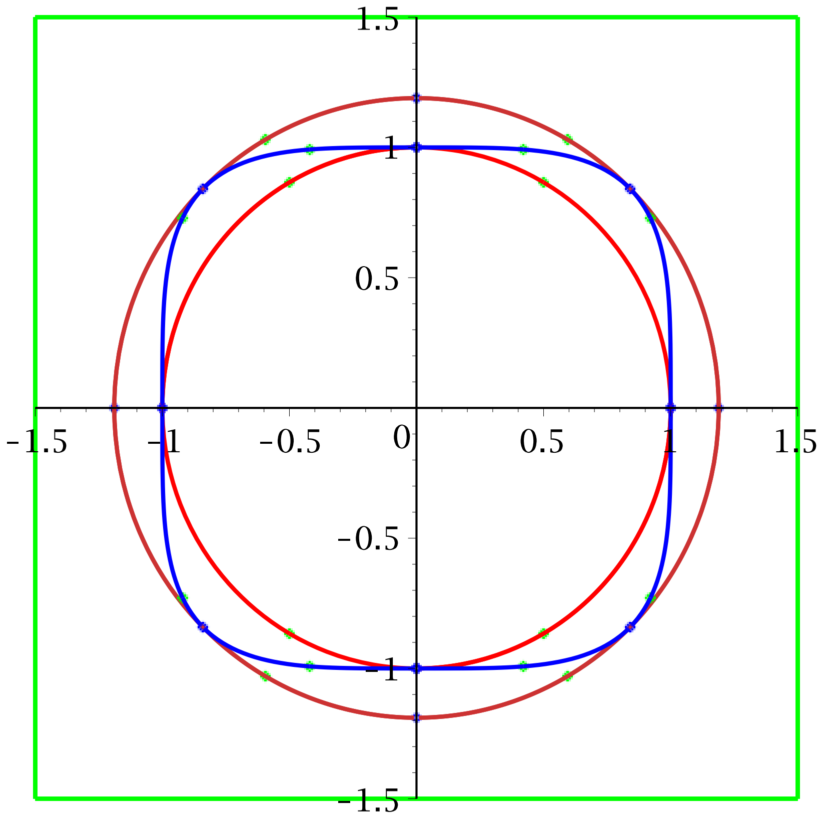
$= \frac{1}{2}\sqrt{2},\ x = -\frac{1}{2}\,2^{3/4},\ y = \frac{1}{2}\,2^{3/4}\}]$

$werte := map(lsg \to eval(f, lsg), reelleLsg)$

$$[1, 1, 1, 1, \sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}] \qquad \textbf{(1.1.16)}$$

Das Minimum ist 1, das Maximum ist sqrt(2)

$Mplot := plot\_real\_curve(f - \mathrm{sqrt}(2), x, y, colorOfCurve = orange, optionen) :$
$mplot := plot\_real\_curve(f - 1, x, y, colorOfCurve = red, optionen) :$
$display(\{NBplot, mplot, Mplot\})$



Man sieht, dass in den Extremalstellen die Niveaulinien von Zielfunktion f und Nebenbedingung g tangetial zueinander liegen. Das ist die Gleichung des Lagrangeschen Multiplikators.

Nebenbedingungen in 3D

$SetCoordinates(cartesian[x, y, z])$

$$\mathit{cartesian}_{x,\, y,\, z} \tag{2.1}$$

$a, b, c := 1, 2, 3$

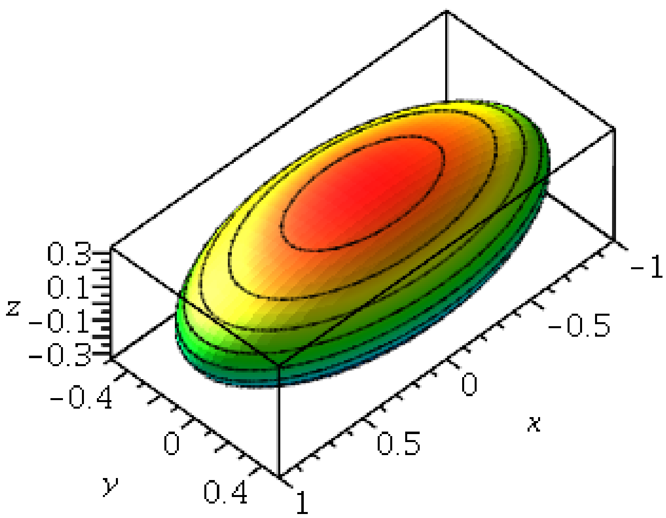
$$1, 2, 3 \tag{2.2}$$

$g := a^2 \cdot x^2 + b^2 \cdot y^2 + c^2 \cdot z^2 - 1$

$$x^2 + 4\,y^2 + 9\,z^2 - 1 \tag{2.3}$$

$NB3plot := implicitplot3d\left(g, x = -\frac{1}{a} .. \frac{1}{a}, y = -\frac{1}{b} .. \frac{1}{b}, z = -\frac{1}{c} .. \frac{1}{c}, style = patchcontour, shading = zhue, scaling = constrained, numpoints = 15000\right) :$

$NB3plot$



$f := x + y + z$

$$x + y + z \tag{2.4}$$

$Gls := Gradient(f) - \text{lambda} \cdot Gradient(g)$

$$\begin{bmatrix} -2\lambda x+1 \\ -8\lambda y+1 \\ -18\lambda z+1 \end{bmatrix} \tag{2.5}$$

$Lsg := \{solve(convert(Gls, set)\ \mathbf{union}\ \{g\})\}$

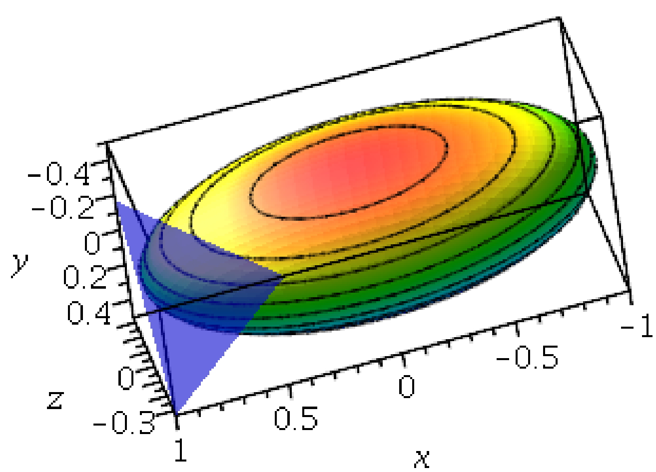
$$\left\{\left\{\lambda = -\frac{7}{12}, x = -\frac{6}{7}, y = -\frac{3}{14}, z = -\frac{2}{21}\right\}, \left\{\lambda = \frac{7}{12}, x = \frac{6}{7}, y = \frac{3}{14}, z = \frac{2}{21}\right\}\right\} \tag{2.6}$$

$werte := map(lsg \rightarrow eval(f, lsg), Lsg)$

$$\left\{-\frac{7}{6}, \frac{7}{6}\right\} \tag{2.7}$$

$fplot := implicitplot3d\left(f - eval(f, Lsg[2]), x = -\frac{1}{a} .. \frac{1}{a}, y = -\frac{1}{b} .. \frac{1}{b}, z = -\frac{1}{c} .. \frac{1}{c}, color = blue, transparency = 0.4, style = patchnogrid\right):$

$display(\{NB3plot, fplot\}, orientation = [75, 30])$



Zeichenkettenverarbeitung und Plotverschönerung

$restart$
for j **from** 1 **to** 3 **do**
  $a[j] := \text{"Zeile Nummer "} \| j$
end do

$$\text{"Zeile Nummer 1"}$$

$$\text{"Zeile Nummer 2"}$$

$$\text{"Zeile Nummer 3"} \tag{3.1}$$

$z := \text{"Zeile Nummer "}$

$$\text{"Zeile Nummer "} \tag{3.2}$$

j

$$4 \tag{3.3}$$

$z \| j$

$$z4 \tag{3.4}$$

$cat(z, j)$

$$\text{"Zeile Nummer 4"} \tag{3.5}$$

for j **from** 2 **to** 5 **do**
  $Lsg := solve(x^j = 1)$:
  $print(cat(j, \text{"-te Einheitswurzel:"}, Lsg[j]))$
end do:

$$\textit{2-te Einheitswurzel:-1}$$

$$\textit{3-te Einheitswurzel:} \| \left(-\frac{1}{2} + \frac{1}{2}\,\mathrm{I}\sqrt{3}\right)$$

$$\textit{4-te Einheitswurzel:} \| (-\mathrm{I})$$

$$\textit{5-te Einheitswurzel:} \| \left(\frac{1}{4}\sqrt{5} - \frac{1}{4} - \frac{1}{4}\,\mathrm{I}\sqrt{2}\,\sqrt{5+\sqrt{5}}\right) \tag{3.6}$$

$latex(Lsg[5])$

```
1/4\,\sqrt {5}-1/4-i/4\sqrt {2}\sqrt {5+\sqrt {5}}
```

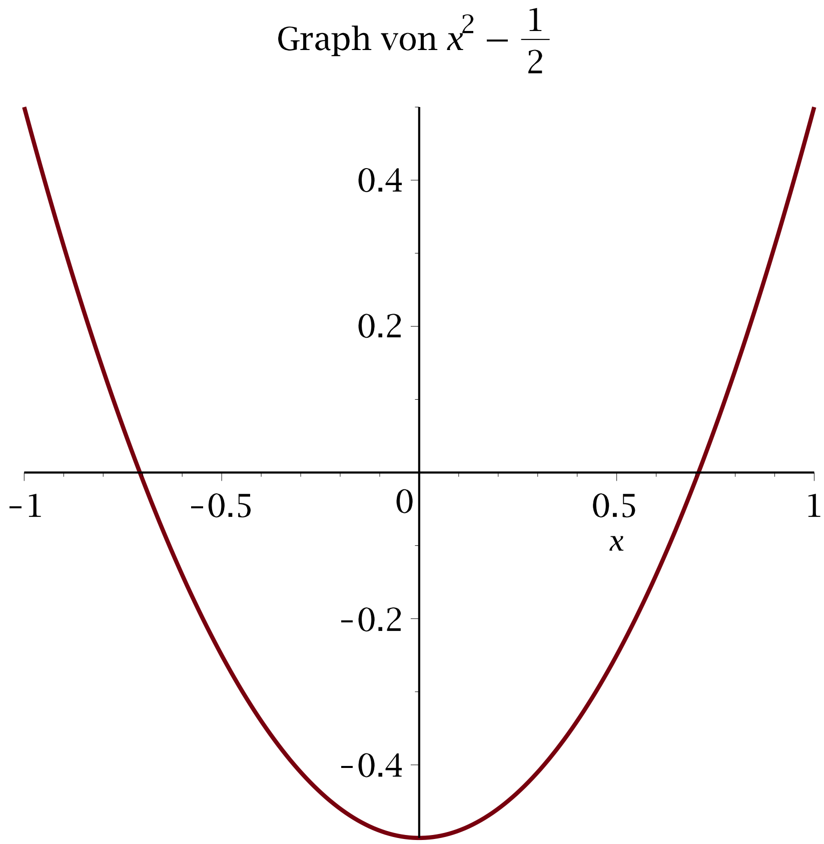
$f := x^2 - \frac{1}{2}$

$$x^2 - \frac{1}{2} \tag{3.7}$$

$Titel := typeset(\text{"Graph von "}, f)$

$$typeset\left(\text{"Graph von "}, x^2 - \frac{1}{2}\right) \tag{3.8}$$

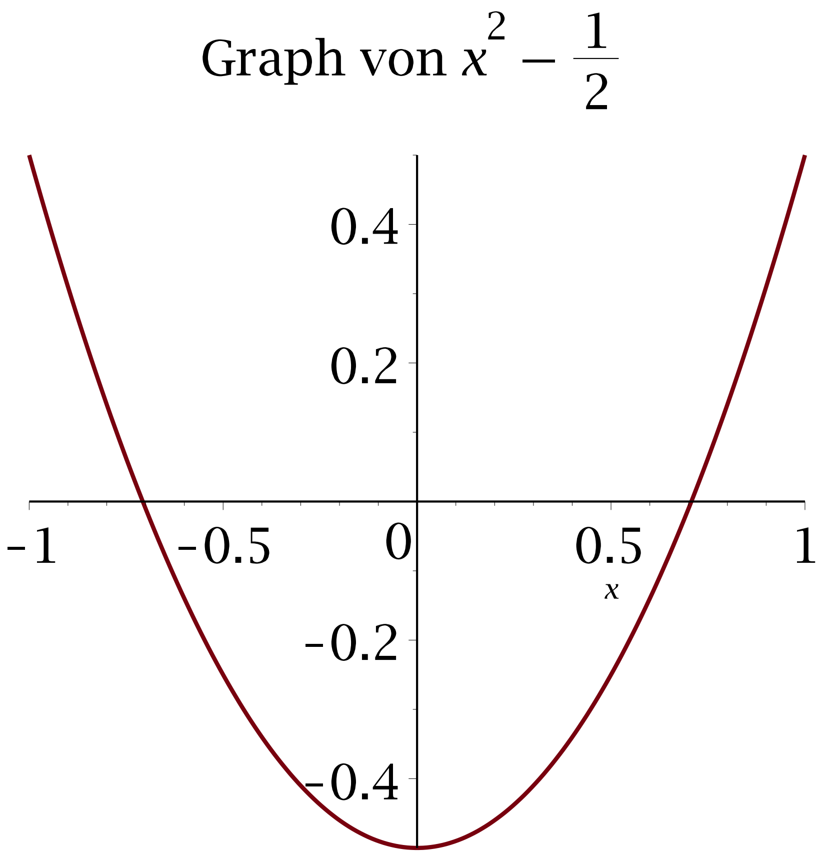
$plot(f, x = -1 ..1, title = Titel, thickness = 2)$



$Font := [TIMES, ROMAN, 18]$

$$[TIMES, ROMAN, 18] \tag{3.9}$$

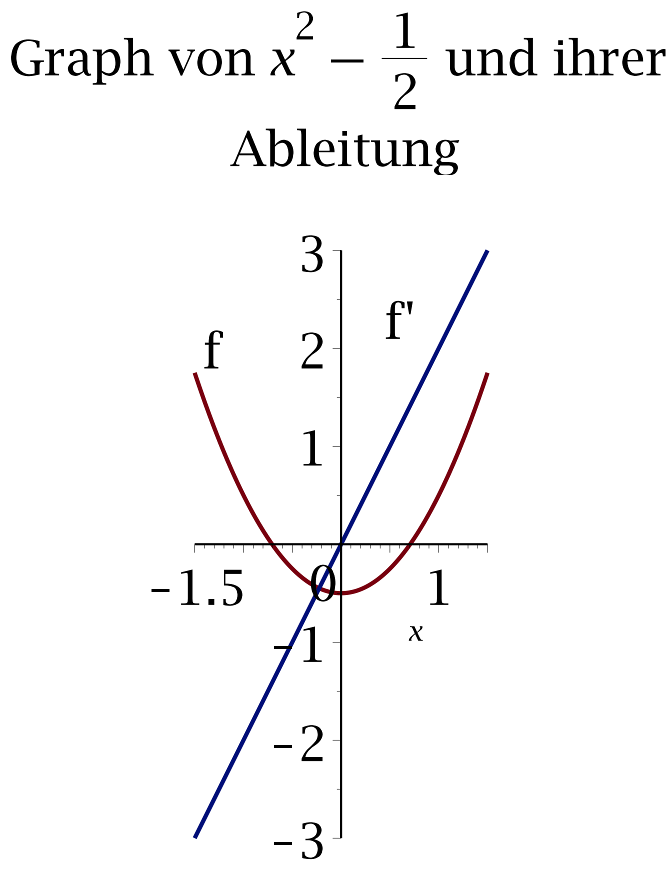
$plot(f, x = -1 ..1, title = Titel, thickness = 2, font = Font)$



$Titel := typeset(\text{"Graph von "}, f, \text{" und ihrer Ableitung"})$

$$typeset\left(\text{"Graph von "}, x^2 - \frac{1}{2}, \text{" und ihrer Ableitung"}\right) \quad \textbf{(3.10)}$$

$with(plots):$
$pl1 := plot([f, diff(f, x)], x = -1.5..1.5, thickness = 2, title = Titel, font = Font, scaling = constrained):$
$pl2 := textplot([-1.5, 1.7, \text{"f"}], align = \{ABOVE, RIGHT\}, font = Font):$
$pl3 := textplot([1, 2, \text{"f' "}], align = \{ABOVE, LEFT\}, font = Font):$
$display(\{seq(pl||j, j = 1..3)\})$



$t3 := textplot3d\left(\left[\frac{1}{2}, 0, 1, \text{"Text schwebt im Raum"}\right], font = Font, color = red\right):$

$x := r \cdot \cos(\text{theta})$

$$r\cos(\theta) \tag{3.11}$$

$y := r \cdot \sin(\text{theta})$

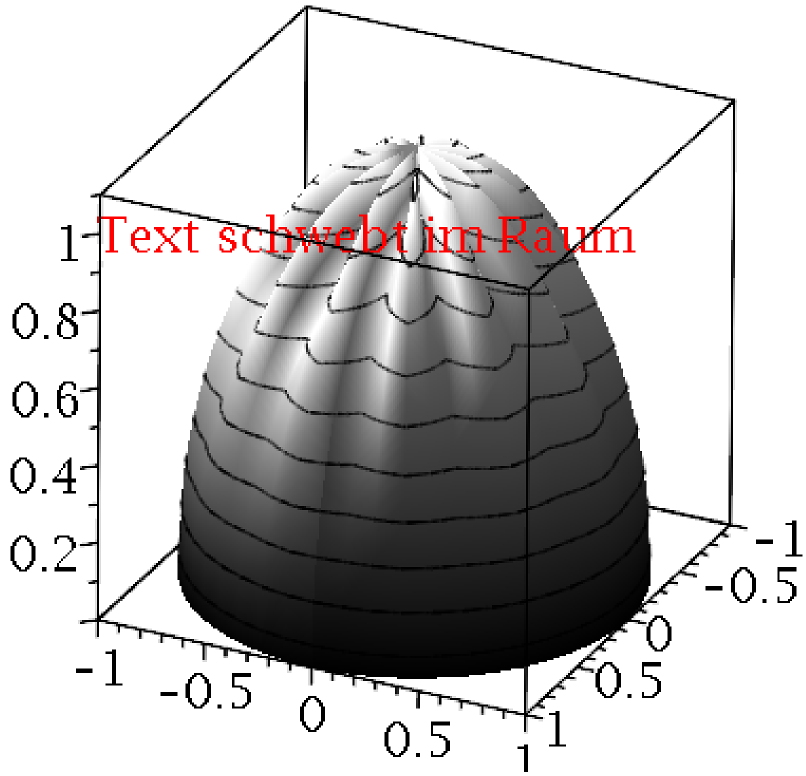
$$r\sin(\theta) \tag{3.12}$$

$f := \text{sqrt}(1 - r^2) \cdot \left(1 - \frac{\sin(12 \cdot \text{theta})}{10}\right)$

$$\sqrt{-r^2 + 1}\left(1 - \frac{1}{10}\sin(12\,\theta)\right) \tag{3.13}$$

$pl := plot3d([x, y, f], r = 0..1, \text{theta} = 0..2 \cdot \text{Pi}, shading = zgrayscale, style = patchcontour, font = Font):$

$display(\{pl, t3\})$



Text schwebt im Raum
1
0.8
0.6
0.4
0.2
-1
-0.5
0
0.5
1
-1
-0.5
0
0.5
1