Lektion 6

Grenzwerte

$L := Limit\left(\left(1-\frac{1}{n}\right)^{n}, n = infinity\right):$
$L = value(L)$

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n} = e^{-1} \tag{1.1}$$

$L := Limit(x\cdot\log(x), x = 0):$
$L = value(L)$

$$\lim_{x\to 0} x\ln(x) = 0 \tag{1.2}$$

$L := Limit((-1)^{2\cdot n}, n = infinity):$
$L = value(L)$

$$\lim_{n\to\infty}(-1)^{2n} = -1-I..1+I \tag{1.3}$$

$value(L)$assuming $n :: integer$

$$-1-I..1+I \tag{1.4}$$

$L := Limit((-1)^{2\cdot\ n}, n = infinity):$
$L = value(L)$

$$\lim_{n\to\infty}(-1)^{2n} = -1-I..1+I \tag{1.5}$$

$L = value(L)$ assuming $n :: integer$

$$\lim_{n\to\infty}(-1)^{2n} = -1-I..1+I \tag{1.6}$$

$simplify(L)$ assuming $n :: integer$

$$\lim_{n\to\infty}(-1)^{2n} \tag{1.7}$$

$a := (-1)^{2\cdot n}$

$$(-1)^{2n} \tag{1.8}$$

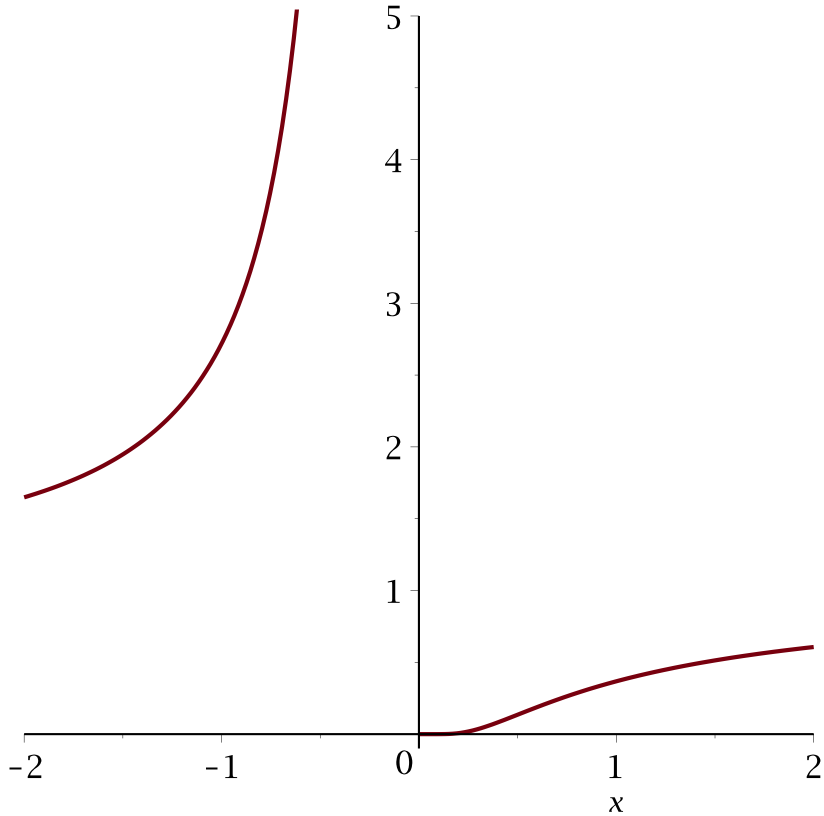
$simplify(a)$ assuming $integer$

$$1 \tag{1.9}$$

$f := \exp\left(-\frac{1}{x}\right)$

$$e^{-\frac{1}{x}} \tag{1.10}$$

$plot(f, x = -2..2, -0.1..5, thickness = 2)$



$L := Limit(f, x = 0) :$
$L = value(L)$

$$\lim_{x \to 0} \mathrm{e}^{-\frac{1}{x}} = undefined \tag{1.11}$$

$L := Limit(f, x = 0, right) :$
$L = value(L)$

$$\lim_{x \to 0+} \mathrm{e}^{-\frac{1}{x}} = 0 \tag{1.12}$$

$L := Limit(f, x = 0, left) :$
$L = value(L)$

$$\lim_{x \to 0-} \mathrm{e}^{-\frac{1}{x}} = \infty \tag{1.13}$$

Reihen

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for s from 2 to 6 do
  C := Sum(1/n^s, n = 1 ..infinity);
  print(C = value(C));
end do:
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$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6}\pi^2$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{90}\pi^4$$

$$\sum_{n=1}^{\infty} \frac{1}{n^5} = \zeta(5)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{1}{945}\pi^6 \tag{2.1}$$

$a := \dfrac{1}{q \cdot k^2 - k + 3}$

$$\frac{1}{k^2 q - k + 3} \tag{2.2}$$

$S := Sum(a, k = 1 ..infinity) :$

$S = value(S)$

$$\sum_{k=1}^{\infty} \frac{1}{k^2 q - k + 3} = -\frac{\Psi\left(-\frac{1}{2} \frac{-2q + 1 + \sqrt{1 - 12q}}{q}\right) - \Psi\left(\frac{1}{2} \frac{2q - 1 + \sqrt{1 - 12q}}{q}\right)}{\sqrt{1 - 12q}} \tag{2.3}$$

$Diff(\log(\mathrm{GAMMA}(x)), x)$

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(\Gamma(x)) \tag{2.4}$$

$\% = value(\%)$

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(\Gamma(x)) = \Psi(x) \tag{2.5}$$

$series(value(S), q = infinity, 3)$

$$\frac{1}{6} \frac{\pi^2}{q} + \frac{\frac{1}{144}\pi^2 + \frac{1}{6} \mathrm{I}\left(\frac{1}{72} \mathrm{I}\pi^2\sqrt{3} - 2\mathrm{I}\zeta(3)\sqrt{3} + \frac{1}{15}\mathrm{I}\pi^4\sqrt{3}\right)\sqrt{3}}{q^2} + \mathrm{O}\left(\frac{1}{q^3}\right) \tag{2.6}$$

$map(simplify, \textbf{(2.6)})$

$$\frac{1}{6}\,\frac{\pi^2}{q} - \frac{1}{30}\,\frac{\pi^4 - 30\,\zeta(3)}{q^2} + \mathrm{O}\left(\frac{1}{q^3}\right) \tag{2.7}$$

simplify(**(2.6)**)

$$\frac{1}{30}\,\frac{-\pi^4 + 30\,\mathrm{O}\left(\frac{1}{q^3}\right)\,q^2 + 5\,\pi^2\,q + 30\,\zeta(3)}{q^2} \tag{2.8}$$

$b := (-1)^k \cdot k^2 - (-1)^k \cdot \left(k - \frac{1}{k^3}\right)^2$

$$(-1)^k\,k^2 - (-1)^k\left(k - \frac{1}{k^3}\right)^2 \tag{2.9}$$

$S := Sum(b, k = 1\,..infinity)$

$$\sum_{k=1}^{\infty}\left((-1)^k\,k^2 - (-1)^k\left(k - \frac{1}{k^3}\right)^2\right) \tag{2.10}$$

 # S=value(S) # Nicht auslösen!

Produkte

$a := \frac{4 \cdot k^2}{4 \cdot k^2 - 1}$

$$\frac{4\,k^2}{4\,k^2 - 1} \tag{3.1}$$

$A := Product(a,\ k = 1\,..infinity)$:

$A = value(A)$

$$\prod_{k=1}^{\infty}\left(\frac{4\,k^2}{4\,k^2 - 1}\right) = \frac{1}{2}\,\pi \tag{3.2}$$

Das Taylorpolynom

$f := \frac{\mathrm{sqrt}(1 + x)}{\mathrm{sqrt}(1 - x^2)}$

$$\frac{\sqrt{1 + x}}{\sqrt{-x^2 + 1}} \tag{4.1}$$

$t := series(f,\ x = 0,\ 8)$

$$1 + \frac{1}{2}\,x + \frac{3}{8}\,x^2 + \frac{5}{16}\,x^3 + \frac{35}{128}\,x^4 + \frac{63}{256}\,x^5 + \frac{231}{1024}\,x^6 + \frac{429}{2048}\,x^7 + \mathrm{O}(x^8) \tag{4.2}$$

$P := convert(t,\ polynom)$

$$1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4+\frac{63}{256}x^5+\frac{231}{1024}x^6+\frac{429}{2048}x^7 \quad \textbf{(4.3)}$$

for n **from** 1 **to** 3 **do**
  $t := series(f,\ x=0,\ n+1);$
  $P[n] := convert(t,\ polynom);$
  $print(P[n]);$
end do:

$$1+\frac{1}{2}x$$

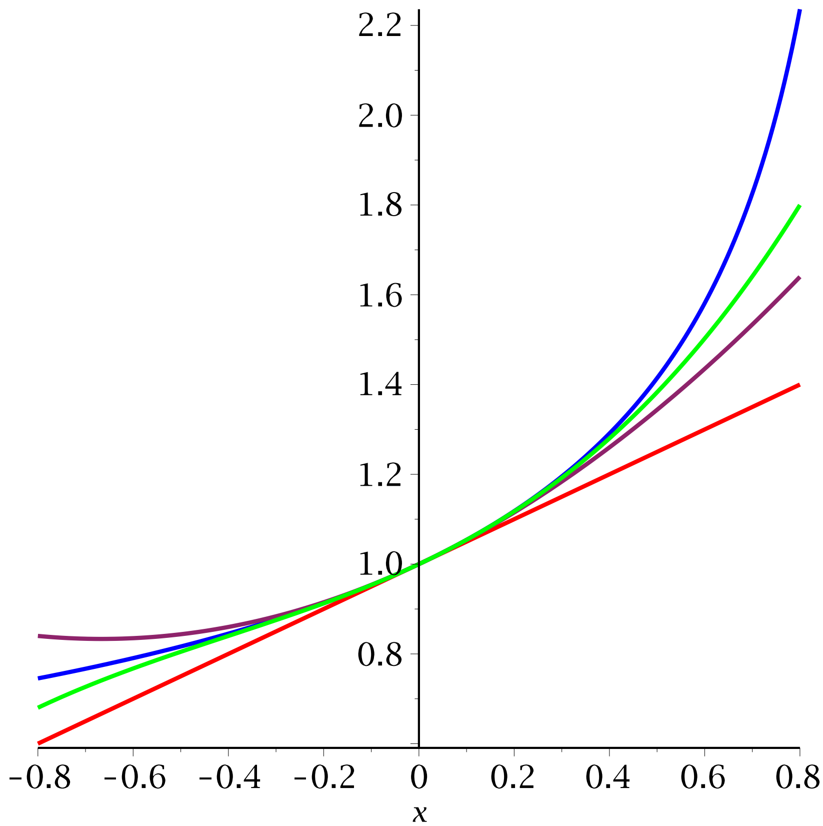
$$1+\frac{1}{2}x+\frac{3}{8}x^2$$

$$1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3 \quad \textbf{(4.4)}$$

$P[0] := f:$
$farbe := [blue,\ red,\ maroon,\ green]$

$$[blue,\ red,\ maroon,\ green] \quad \textbf{(4.5)}$$

$plot([seq(P[n],\ n=0..3)],\ x=-0.8..0.8,\ color=farbe,\ thickness=2)$

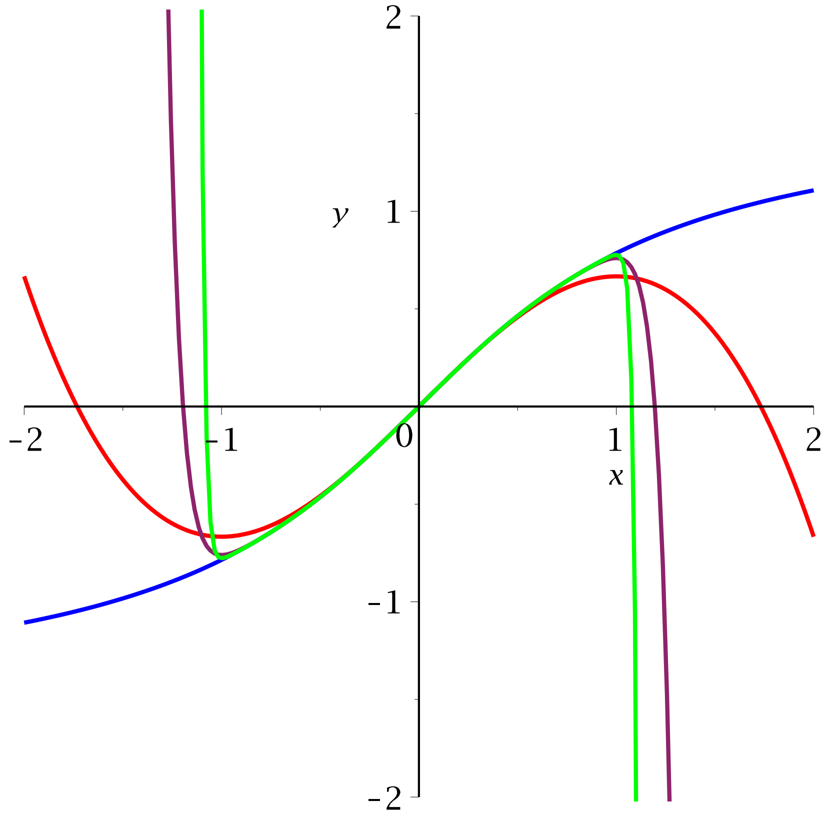


Das komplexe Bild

$h := \arctan(x)$

$$\arctan(x) \tag{5.1}$$

for n **in** $[4, 20, 60]$ **do**
  $t := series(h, x = 0, n + 1)$;
  $S[n] := convert(t, polynom)$;
 end do:
$S[0] := h$:
$plot([seq(S[n], n \textbf{ in } [0, 4, 20, 60])], x = -2..2, y = -2..2, color = farbe, thickness = 2)$



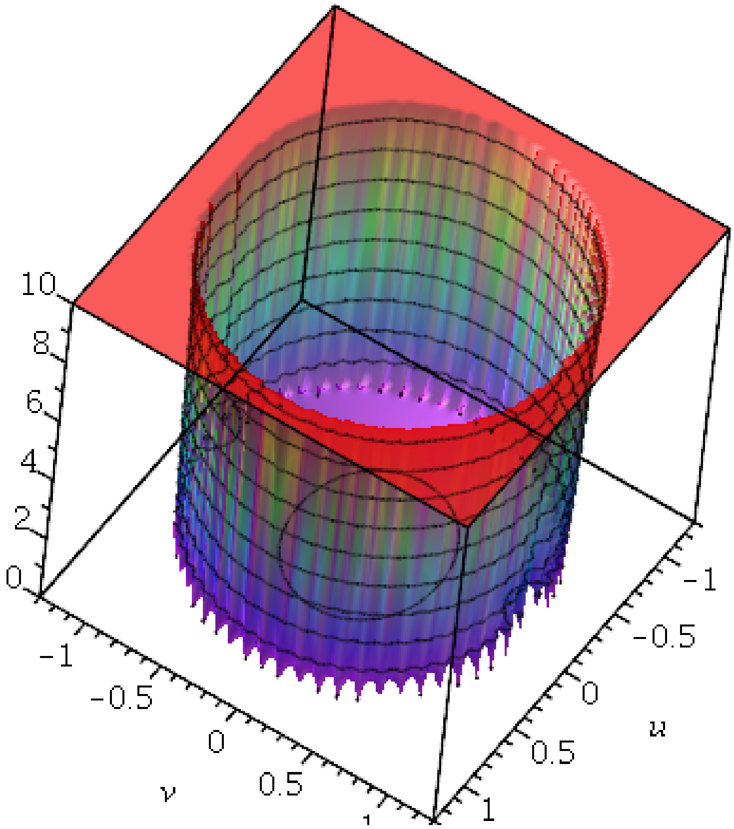
$with(plots)$:
$x := u + I \cdot v$

$$u + \mathrm{I}\, v \tag{5.2}$$

$pl1 := plot3d(\min(10, \mathrm{abs}(S[60])), u = -1.3..1.3, v = -1.3..1.3, style = patchcontour, shading = zhue, transparency = 0.3, numpoints = 10000)$

$$PLOT3D(...) \tag{5.3}$$

$pl1$



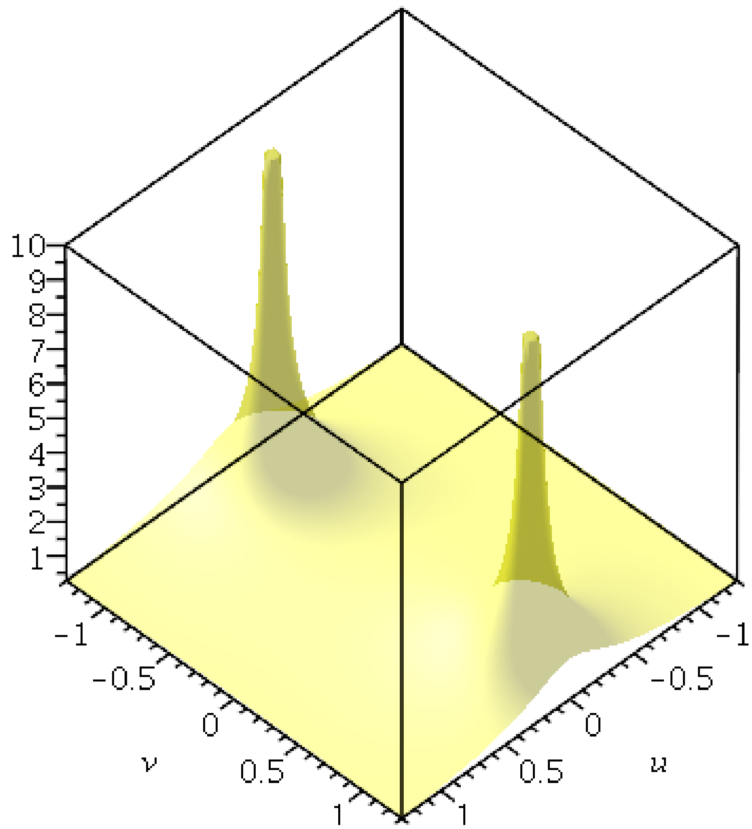
diff(arctan(xi), xi)

$$\frac{1}{\xi^2+1} \tag{5.4}$$

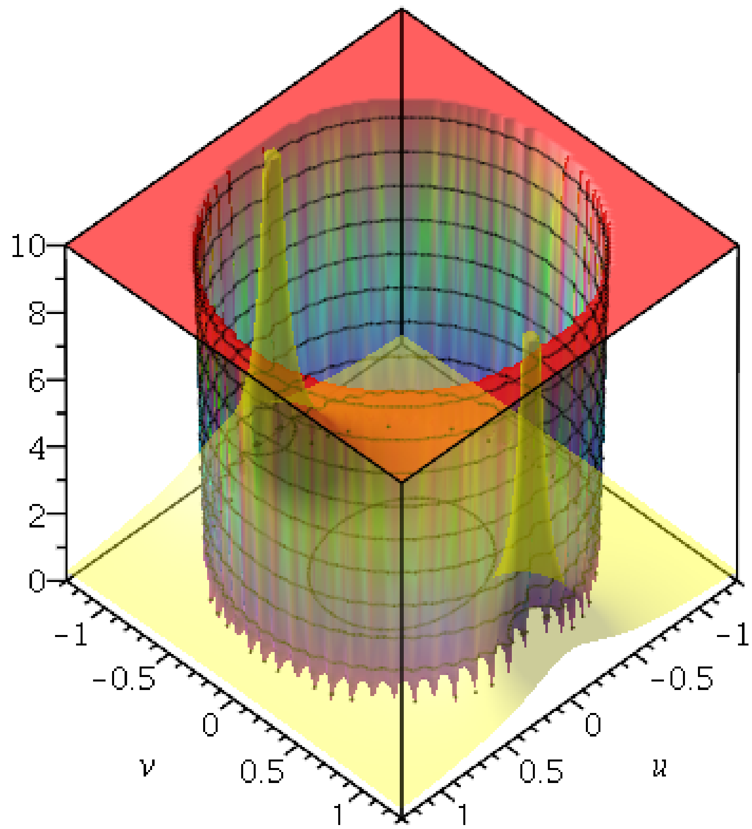
$pl2 := plot3d\left(\min\left(10, \operatorname{abs}\left(\frac{1}{1+x^2}\right)\right), u = -1.3..1.3, v = -1.3..1.3, style = patchnogrid, color = yellow, transparency = 0.6, numpoints = 30000\right)$

$$PLOT3D(\ldots) \tag{5.5}$$

pl2



display([*pl1*, *pl2*])



Allgemeinere Reihenentwicklungen

$x := 'x':$

$t1 := convert(series(h, x = 0, 12), polynom)$

$$x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 + \frac{1}{9} x^9 - \frac{1}{11} x^{11} \tag{6.1}$$

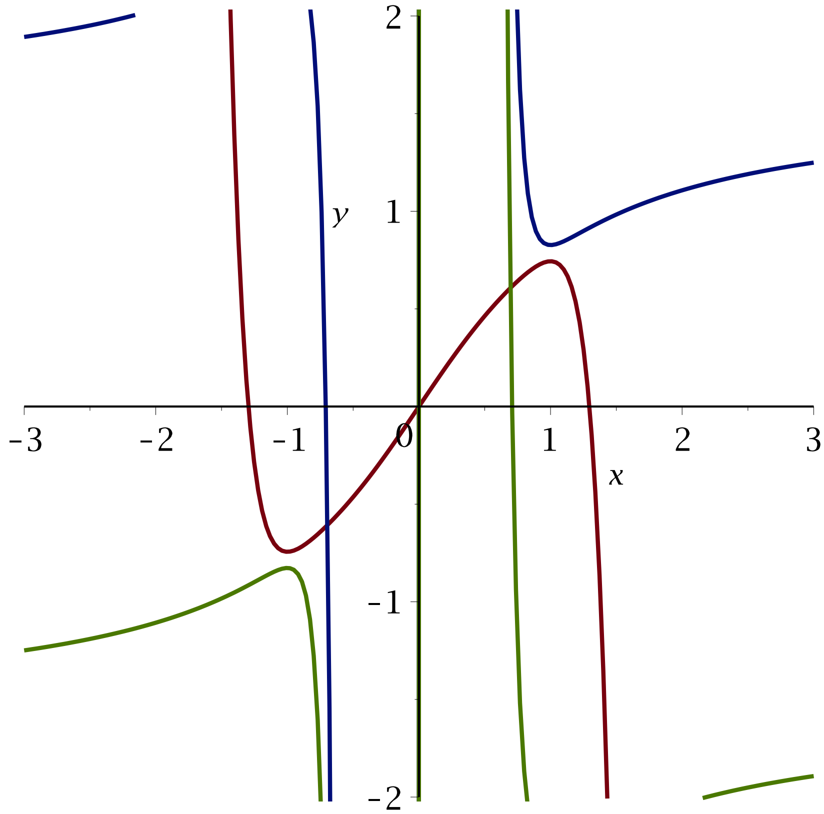
$t2 := convert(series(h, x = infinity, 12), polynom)$

$$\frac{1}{2} \pi - \frac{1}{x} + \frac{1}{3 x^3} - \frac{1}{5 x^5} + \frac{1}{7 x^7} - \frac{1}{9 x^9} + \frac{1}{11 x^{11}} \tag{6.2}$$

$t3 := convert(series(h, x = -infinity, 12), polynom)$

$$-\frac{1}{2} \pi - \frac{1}{x} + \frac{1}{3 x^3} - \frac{1}{5 x^5} + \frac{1}{7 x^7} - \frac{1}{9 x^9} + \frac{1}{11 x^{11}} \tag{6.3}$$

$plot([t1, t2, t3], x = -3..3, y = -2..2, thickness = 2)$



series(Psi(x), x = *infinity*)

$$\ln(x) - \frac{1}{2\,x} - \frac{1}{12\,x^2} + \frac{1}{120\,x^4} + \mathrm{O}\left(\frac{1}{x^6}\right) \tag{6.4}$$

series$\left(x\cdot\left(\log(\sin(x)\right)^2 + \log(x)\right)^2, x = 0\right)$

$$\left(\ln(x)^2 + \ln(x)\right)^2 x - \frac{2}{3}\left(\ln(x)^2 + \ln(x)\right)\ln(x)\,x^3 + \left(2\left(\ln(x)^2 + \ln(x)\right)\left(-\frac{1}{90}\ln(x) + \frac{1}{36}\right) + \frac{1}{9}\ln(x)^2\right)x^5 + \mathrm{O}(x^7) \tag{6.5}$$

Grenzwertbestimmung mittels Reihenentwicklung

$f := 1 - \cos(x^2)$

$$1 - \cos(x^2) \tag{7.1}$$

$g := x\cdot(x - \sin(x))$

$$x\,(x - \sin(x)) \tag{7.2}$$

$L := Limit\left(\frac{f}{g}, x = 0\right):$

$L = value(L)$

$$\lim_{x \to 0} \frac{1 - \cos(x^2)}{x\,(x - \sin(x))} = 3 \tag{7.3}$$

$fr := series(f, x = 0, 12)$

$$\frac{1}{2}\,x^4 - \frac{1}{24}\,x^8 + \mathrm{O}(x^{12}) \tag{7.4}$$

$gr := series(g, x = 0, 8)$

$$\frac{1}{6}\,x^4 - \frac{1}{120}\,x^6 + \mathrm{O}(x^8) \tag{7.5}$$

$b1 := \frac{convert(fr, polynom)}{convert(gr, polynom)}$

$$\frac{\frac{1}{2}\,x^4 - \frac{1}{24}\,x^8}{\frac{1}{6}\,x^4 - \frac{1}{120}\,x^6} \tag{7.6}$$

$b2 := normal(b1)$

$$\frac{5\,(x^4 - 12)}{x^2 - 20} \tag{7.7}$$

$eval(b2, x = 0)$

$$3 \tag{7.8}$$