Market Microstructure Tutorial R/Finance 2009

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24 April 2009

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Who Am I

Assistant Professor of Finance at UIC

– Teach Investments; Market Microstructure; Commodities.

B.S., Electrical Engineering, Cornell U., 1995.
Ph.D., Statistics, U. Chicago, 2008.

Programmer Intern, Listed Equities, Goldman Sachs, 1993–4.
Strategist, Equity Derivatives, LTCM, 1995–2000.
Trader/Researcher, Equity Trading Lab, Morgan Stanley, 2000-3.
Self-employed Trader/Researcher/Coder, Summer 2004.

What Microstructure Is — and Is Not

Market microstructure studies the details of how markets work.

Market microstructure is not "neoclassical" finance[2].

- often directly opposes Efficient Market Hypothesis.

If you believe markets are efficient:

- the details of how markets work are irrelevant since. . .
- you always get efficient price (less universally-known fee).

Thus microstructure embraces:

- the possibility of short-term alpha; and
- behavioral effects.

[2] "So you short one stock..." omits a lot.

What I Cover in a Microstructure Course

When I teach microstructure, I cover many topics:

1. Market Types
2. Orders and Quotes
3. Trades and Traders
4. Market Structures
5. Roll and Sequential Trade Models ← Play with some
6. Strategic Trader and Inventory Models ← of these today.
7. Prices, Sizes, and Times
8. Liquidity and Transactions Costs
9. Market Metrics Across Time
10. Electronic Markets
11. Electronic Trading Tools and Strategies

Introduction

- Today we'll consider models for microstructure phenomena.
- For these models, need to adopt a different perspective.
- Models are simple; lets us conduct controlled experiments.
  - Eliminate all but one or two major factors.
  - Question: Does this model reproduce real-life features?
  - If so: factors we are considering probably matter.
  - We may even have an idea about how those factors matter.
- Less confusion about what matters helps build better models.
- Newest research combining such models with time series.

All models are wrong; some models are useful.
— *George Box*

Asymmetric Information Models

- We will examine two *asymmetric information* models.
  - Asymmetry: Trades may contain private information.
- *Sequential trade*: independent sequence of traders.
  - Traders: informed (know asset value) or not; trade once.
  - Single market maker; learns information by trading.
- *Strategic trader*: one informed trader can trade many times.
  - Informed trader considers own impact on later trades.
  - Uninformed (noise) traders also submit orders.
  - Single MM sees combined order, sets price, fills order.
- Some slides are for you to study later; I'll skip those.

Glosten-Milgrom (1985) Model

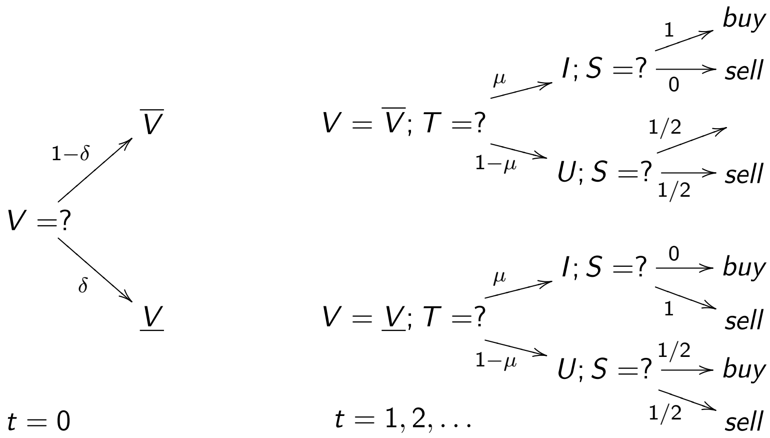
Glosten and Milgrom (1985) Model

- Most famous sequential trade model.
- MM quotes a bid B and ask A.
- Security has value $V = \underline{V}$ or $\overline{V}$, $\underline{V} < \overline{V}$.
- At time $t = 0$, informed traders (only) learn V.
- Time is discretized.
- Trades are for one unit, occur at each time step.
- MM has infinite capital: no inventory/bankruptcy concerns.

Glosten-Milgrom Model: Setup

- $V \sim \begin{cases} \overline{V} & w.p.\ 1-\delta \\ \underline{V} & w.p.\ \delta \end{cases}$
- Trader type $T \sim \begin{cases} \text{Informed} & w.p.\ \mu \\ \text{Uninformed} & w.p.\ 1-\mu \end{cases}.$
- Traders take action S (buy/sell from MM), one at a time.
- Informed traders: $S \sim \begin{cases} \text{buy} & \text{if } V = \overline{V} \\ \text{sell} & \text{if } V = \underline{V} \end{cases}.$
- Uninformed traders: $S \sim \begin{cases} \text{buy} & w.p.\ 1/2 \\ \text{sell} & w.p.\ 1/2 \end{cases}.$

Glosten-Milgrom Model: Event Trees



- V = security value ($\overline{V}, \underline{V}$)
- T = trader type (Informed/Uninformed)
- S = trader's side (buy/sell)

Glosten-Milgrom Model: Likelihood of Buys and Sells

- What is P(buy), P(sell) after one trade?

$$
\begin{aligned}
P(\text{buy}) &= P(\text{buy}|\underline{V})P(\underline{V}) + P(\text{buy}|\overline{V})P(\overline{V}) && (1)\\
&= \frac{(1-\mu)\delta + (1+\mu)(1-\delta)}{2} = \frac{1+\mu(1-2\delta)}{2} && (2)\\
P(\text{sell}) &= P(\text{sell}|\underline{V})P(\underline{V}) + P(\text{sell}|\overline{V})P(\overline{V}) && (3)\\
&= \frac{(1+\mu)\delta + (1-\mu)(1-\delta)}{2} = \frac{1-\mu(1-2\delta)}{2} && (4)
\end{aligned}
$$

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Glosten-Milgrom Model: Likelihood of $\overline{V}, \underline{V}$

After one trade, we have information via Bayes' Theorem:

$$P(\overline{V}|\text{buy}) = \frac{P(\text{buy}|\overline{V})P(\overline{V})}{P(\text{buy})} = \frac{(1-\delta)(1+\mu)}{1+\mu(1-2\delta)} \tag{5}$$

$$P(\underline{V}|\text{buy}) = \frac{P(\text{buy}|\underline{V})P(\underline{V})}{P(\text{buy})} = \frac{\delta(1-\mu)}{1+\mu(1-2\delta)} \tag{6}$$

$$P(\overline{V}|\text{sell}) = \frac{P(\text{sell}|\overline{V})P(\overline{V})}{P(\text{sell})} = \frac{(1-\delta)(1-\mu)}{1-\mu(1-2\delta)} \tag{7}$$

$$P(\underline{V}|\text{sell}) = \frac{P(\text{sell}|\underline{V})P(\underline{V})}{P(\text{sell})} = \frac{\delta(1+\mu)}{1-\mu(1-2\delta)} \tag{8}$$

Glosten-Milgrom Model: Bid, Ask, Spread

- If competition narrows profit to 0, before trading...
- $A = E(V|\text{buy})$ and $B = E(V|\text{sell})$.

$$A = \underline{V}P(\underline{V}|\text{buy}) + \overline{V}P(\overline{V}|\text{buy}) \tag{9}$$

$$= \underline{V}\frac{\delta(1-\mu)}{1+\mu(1-2\delta)} + \overline{V}\frac{(1-\delta)(1+\mu)}{1+\mu(1-2\delta)} \tag{10}$$

$$B = \underline{V}P(\underline{V}|\text{sell}) + \overline{V}P(\overline{V}|\text{sell}) \tag{11}$$

$$= \underline{V}\frac{\delta(1+\mu)}{1-\mu(1-2\delta)} + \overline{V}\frac{(1-\delta)(1-\mu)}{1-\mu(1-2\delta)} \tag{12}$$

$$A - B = \frac{4(1-\delta)\delta\mu(\overline{V}-\underline{V})}{1-\mu^2(1-2\delta)^2} \tag{13}$$

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Glosten-Milgrom Model: Updating Bids and Asks

- After each trade, Bayesian update of beliefs about V.
- Idea: How often would we see these trades if $V = \overline{V}$ vs. $\underline{V}$?
- With these ideas, we can update the bid and ask prices.
- Expected bid and ask after k buys and ℓ sells:

$$A_{k+\ell} =$$

$$= \underline{V} P(\underline{V} | k+1 \text{ buys}, \ell \text{ sells}) + \overline{V} P(\overline{V} | k+1 \text{ buys}, \ell \text{ sells}) \tag{14}$$

$$= \frac{\underline{V}\delta(1-\mu)^{k+1}(1+\mu)^{\ell} + \overline{V}(1-\delta)(1+\mu)^{k+1}(1-\mu)^{\ell}}{(1-\delta)(1+\mu)^{k+1}(1-\mu)^{\ell} + \delta(1-\mu)^{k+1}(1+\mu)^{\ell}} \tag{15}$$

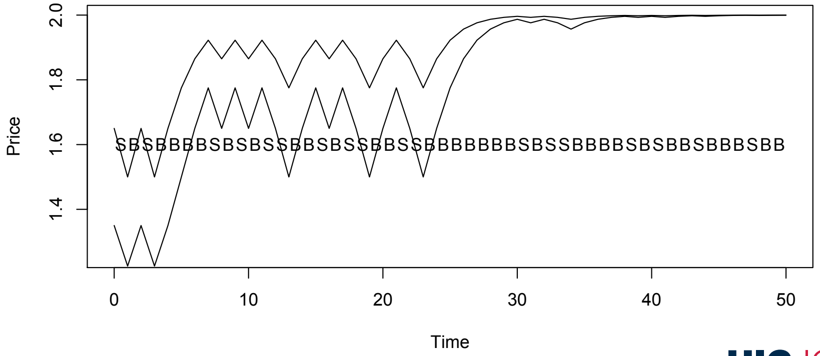
$$B_{k+\ell} =$$

$$= \underline{V} P(\underline{V} | k \text{ buys}, \ell+1 \text{ sells}) + \overline{V} P(\overline{V} | k \text{ buys}, \ell+1 \text{ sells}) \tag{16}$$

$$= \frac{\underline{V}\delta(1-\mu)^{k}(1+\mu)^{\ell+1} + \overline{V}(1-\delta)(1-\mu)^{k}(1+\mu)^{\ell+1}}{(1-\delta)(1+\mu)^{k}(1-\mu)^{\ell+1} + \delta(1-\mu)^{k}(1+\mu)^{\ell+1}} \tag{17}$$

Glosten-Milgrom Model: Simulation Example

- One simulation for $\mu = 0.3$, $\delta = 0.5$; example bids and asks.
- Simple case: $V = \overline{V} = 2$ (versus $\underline{V} = 1$).
- Can see price impact — especially for sequence of orders.

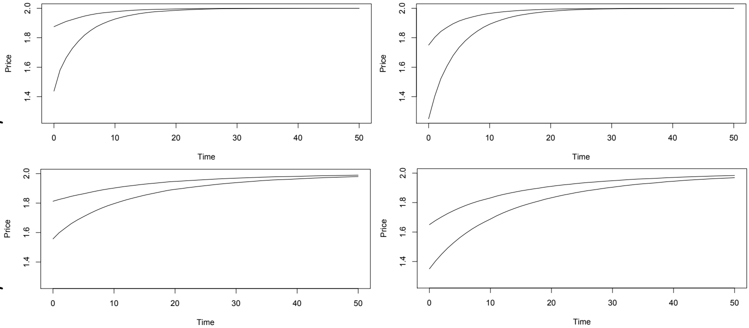


Glosten-Milgrom Model: Simulation Averages

- Simulate to find the average bid and ask.
- 10,000 simulations; tried $\mu = 0.3, 0.5$, $\delta = 0.3, 0.5$.
- Simple case: $V = \overline{V} = 2$ (versus $\underline{V} = 1$).

$\mu = 0.5$

$\mu = 0.3$



$\delta = 0.3$ $\delta = 0.5$

Glosten-Milgrom Model: Other Results

- Basic idea: Spreads exist due to adverse selection.
- Buys/sells are unbalanced; but, price series is a martingale.
- Orders are serially correlated: buys tend to follow buys.
  - This and the preceding line seem contradictory.
  - Difference is akin to Pearson ρ versus Kendall τ.
- Trades have price impact: a buy increases B and A.
- Spreads tend to decline over time as MMs figure out V.
- Bid-ask may be such that market effectively shuts down[3]
- If uninformed were price sensitive, spreads would be wider.
- Code in THE SECRET DIRECTORY (glosten-milgrom.r).
- Fun: Add very rare third trader, govt, who always buys at $\underline{V}$.
  - Stunning: still converging after 50,000 trades.

[3] We find ourselves in the game-theoretic Paradox of Trade.

Kyle (1985) Model

One-Period Kyle (1985) Model

- Kyle (1985) proposed a model with a single informed trader.
- The informed trader:
  - Considers price impact in setting trade size;
  - Learns security's terminal value v; and,
  - Submits order for quantity x.
- Liquidity ("noise") traders submit net order u.
- The single market maker (MM):
  - Observes total order $y = x + u$;
  - Makes up the difference; and,
  - Sets the market clearing price p.
- All trades happen at one price; no bid-ask spread.
- All trading occurs in one period.

One-Period Kyle Model: Informed Trader vs. MM

- The informed trader would like to trade aggressively.
- Q1: What sort of function maps v to order size?
- A: Linear function? (Yes.)
- MM knows larger net orders are more likely to be informed.
- Thus MM sets price increasing in net order size.
- Q2: What sort of function maps order size to MM's price?
- A: Linear function? (Yes.)

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One-Period Kyle Model: Setup

$$\begin{aligned}
\text{Security value} \quad & v \sim N(p_0, \Sigma_0) & (18)\\
\text{Noise order} \quad & u \sim N(0, \sigma_u^2) \qquad u \perp\!\!\!\perp v & (19)\\
\text{MM assumes: informed order} \quad & x = \beta v + \alpha & (20)\\
\text{Net order} \quad & y = x + u & (21)\\
\text{Informed assumes: trade price} \quad & p = \underbrace{\lambda}_{\text{illiquidity}} y + \mu & (22)\\
\text{Informed trader profit} \quad & \pi = (v - p)x & (23)\\
& \quad = (v - \lambda(x + u) - \mu)x & (24)
\end{aligned}$$

One-Period Kyle Model: Optimize, Compute Statistics

If we combine the previous formulæ, we get a little further:

$$E(\pi) = \underbrace{E((v - \lambda(x+u) - \mu)x)}_{E(u)=0;\text{x non-random}} = (v - \lambda x - \mu)x \tag{25}$$

$$x^* = \underset{x\in\mathbb{R}}{\operatorname{argmax}}\, E(\pi) = \frac{v-\mu}{2\lambda} \quad \text{if } \lambda > 0 \tag{26}$$

$$\Rightarrow \beta = \frac{1}{2\lambda}, \alpha = \frac{-\mu}{2\lambda} \tag{27}$$

$$E(y) = \alpha + \beta E(v) = \frac{-\mu}{2\lambda} + \frac{p_0}{2\lambda} \tag{28}$$

$$\operatorname{Var}(y) = \operatorname{Var}(x) + \operatorname{Var}(u) = \beta^2 \operatorname{Var}(v) + \operatorname{Var}(u) \tag{29}$$

$$= \beta^2\Sigma_0 + \sigma_u^2 \tag{30}$$

$$\operatorname{Cov}(y, v) = \operatorname{Cov}(\alpha + \beta v, v) = \beta\Sigma_0 \tag{31}$$

One-Period Kyle Model: Find Linear Parameters

Now solve for the linear MM pricing and trader order parameters.

- MM earns no expected profit[4], prices trade at $p = E(v|y)$.
- Since v, y normal, form of $E(v|y)$ is like linear regression.

$$p = E(v|y) = E(v) + \frac{\text{Cov}(v,y)}{\text{Var}(y)}(y - E(y)) \tag{32}$$

$$= p_0 + \frac{\beta\Sigma_0}{\sigma_u^2 + \beta^2\Sigma_0}(y - \alpha - \beta p_0) \tag{33}$$

- Use (33), (27), and (22) to solve for $\alpha, \beta, \mu, \lambda$:

$$\alpha = p_0 \frac{\sigma_u}{\sqrt{\Sigma_0}}; \quad \beta = \frac{\sigma_u}{\sqrt{\Sigma_0}}; \quad \mu = p_0; \quad \lambda = \frac{\sqrt{\Sigma_0}}{2\sigma_u}. \tag{34}$$

[4] Due to competition, again.

One-Period Kyle Model: MM Price; Informed Order, Profit

$$\text{MM trade price} \quad p = E(v|y) = \lambda y + \mu = \frac{\sqrt{\Sigma_0}}{2\sigma_u} \cdot y + p_0 \quad (35)$$

$$\text{Value uncertainty} \quad \mathrm{Var}(v|y) = \frac{\sigma_u^2 \Sigma_0}{\sigma_u^2 + \beta^2 \Sigma_0} = \frac{\Sigma_0}{2} \quad (36)$$

$$\text{Informed order} \quad x^* = \beta v + \alpha = \frac{(v - p_0)\sigma_u}{\sqrt{\Sigma_0}} \quad (37)$$

$$\text{Expected profit} \quad E(\pi) = (v - \lambda x - \mu)x = \frac{(v - p_0)^2 \sigma_u}{2\sqrt{\Sigma_0}} \quad (38)$$

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One-Period Kyle Model: Commentary

What can we learn from the one-period Kyle model?

- Trade price linear in net order size, security volatility.
- Trade price inverse to noise order volatility.
- Informed order linear in security's deviation from mean.
- Expected profit quadratic in security's deviation from mean.
  - Large deviations matter much more than small deviations[5].
- Informed order, expected profit linear in noise order volatility[6].
- $\mathrm{Var}(v|y) = \frac{\Sigma_0}{2} \Rightarrow$ Half of information[7] leaks after one trade.
- Negative net order ($i.e.$ $u < -x$) yields negative price. (!)

[5] Except data errors and adverse selection will hurt you.

[6] Use uninformed orders to hide.

[7] Information in a Fisher sense.

One-Period Kyle Model: Illiquidity Parameter

Finally: Consider the illiquidity parameter.

- Illiquidity parameter $\lambda = \frac{\sqrt{\Sigma_0}}{2\sigma_u}$.
- $\sqrt{\Sigma_o}/\sigma_u$ is ratio of volatilities.
  - Value uncertainty vs. noise order uncertainty.
- $\lambda y = \sqrt{\Sigma_0}\frac{y}{2\sigma_u}$: like liquidity risk:
  - Scaled by volatility of security; and,
  - y/σ_u is similar/proportional to percentage of volume.
- Nice: Demanding liquidity has a cost.
- Full course covers more such ideas.

Multi-Period Kyle (1985) Model

- Kyle also discussed a multi-period model.
- Slice time $t \in \{0, 1\}$ into N bins, $n = 1, \ldots, N$:

$$
\begin{aligned}
\text{Time:} \quad & \Delta t_n = 1/N, t_n = n/N. && (39) \\
\text{Noise order:} \quad & \Delta u_n \sim N(0, \sigma_u^2 \Delta t_n). && (40) \\
\text{Informed order:} \quad & \Delta x_n = \beta_n (v - p_{n-1})/N. && (41) \\
\text{Price change:} \quad & \Delta p_n = \lambda_n (\Delta x_n + \Delta u_n). && (42) \\
\text{E(Later profit):} \quad & E(\pi_n | p_{t<n}) = \alpha_{n-1}(v - p_{n-1})^2 + \delta_{n-1} && (43)
\end{aligned}
$$

Competition $\Rightarrow p_n = E(v | \Delta x_1 + \Delta u_1, \ldots, \Delta x_n + \Delta u_n)$
Informed orders not autocorrelated, by construction.

Multi-Period Kyle Model: Further Definitions

Use the following definitions:

$$\text{Strategy:} \quad X = (x_1, \ldots, x_N). \tag{44}$$

$$\text{Pricing rule:} \quad P = (p_1, \ldots, p_N). \tag{45}$$

X chosen to always maximize expected future profit:

$$E(\pi_n(X, P) | v, p_1, \ldots, p_{n-1}) \geq E(\pi_n(X', P') | v, p_1, \ldots, p_{n-1}) \quad n = 1, \ldots, N. \tag{46}$$

Multi-Period Kyle Model: Dynamics Equations

Model dynamics are given by difference equations for $n = 1, \ldots, N$:

$$\delta_{n-1} = \delta_n + \alpha_n \lambda_n^2 \sigma_u^2 \Delta t_n; \tag{47}$$

$$\alpha_{n-1} = 1/(4\lambda_n(1 - \alpha_n \lambda_n)); \tag{48}$$

$$\beta_n \Delta t_n = \frac{1 - 2\alpha_n \lambda_n}{2\lambda_n(1 - \alpha_n \lambda_n)}; \tag{49}$$

$$\lambda_n = \beta_n \Sigma_n / \sigma_u^2; \quad \text{and,} \tag{50}$$

$$\Sigma_n = (1 - \beta_n \lambda_n \Delta t_n) \Sigma_{n-1}. \tag{51}$$

λ_n is the middle root of the cubic equation:

$$(1 - \lambda_n^2 \sigma_u^2 \Delta t_n / \Sigma_n)(1 - \alpha_n \lambda_n) = \frac{1}{2}. \tag{52}$$

Multi-Period Kyle Model: Solving for Dynamics

Solving for the model dynamics is a bit crusty:

1. Guess Σ_N (call the guess Σ_N^*).
2. Use $\alpha_N = \delta_N = 0$ to get $\lambda_N = \frac{\sqrt{\Sigma_N^*}}{\sigma_u \sqrt{2\Delta t_N}}$.
3. Set $n = N$.
4. Solve for β_n and Σ_{n-1}^*.
5. Find α_{n-1} for λ_n.
6. Solve (52)[8], using middle root for λ_{n-1}.
7. $n = n - 1$; if $n > 0$, go to step 4.
8. If $|\Sigma_0^* - \Sigma_0| > \epsilon$: try another Σ_N^*, go to step 1.
9. Solve for β_0

[8] Numerically or via Cardano's formula.

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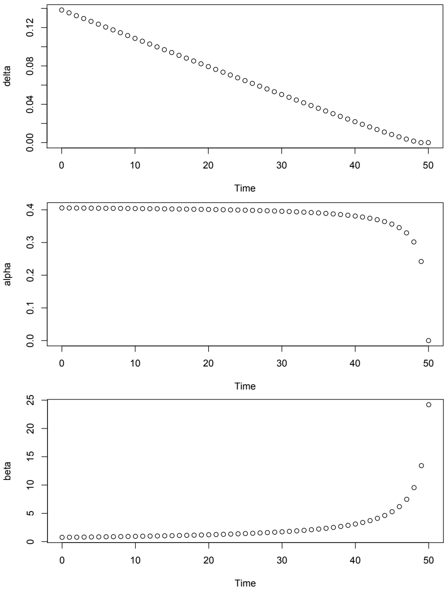
- We can simulate the Kyle model to see its behavior.
- Use $p_0 = 2$, $\Sigma_0 = 0.4$, and $\sigma_u = 0.5$.
- Run one simulation. What do we get?
- The parameter evolution is not so surprising.
- The action evolution is more illuminating.

Multi-Period Kyle Model: Evolution of Parameters I



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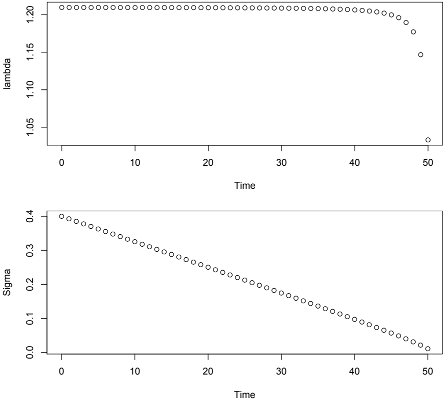
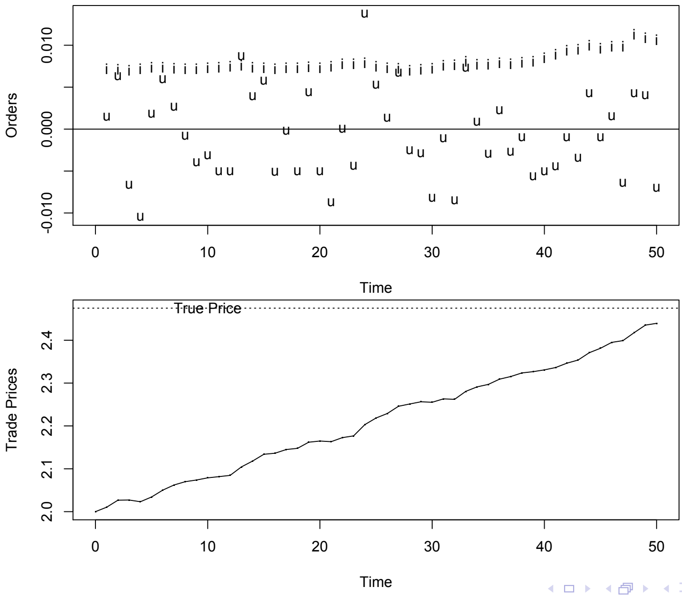
Multi-Period Kyle Model: Evolution of Parameters II[9]



[9] E.B.

Multi-Period Kyle Model: Evolution of Actions



Multi-Period Kyle Model: Commentary

A few details nobody has previously noted[10]:

- Informed orders[11] are larger after negative uninformed trades.
- Informed orders decrease after larger net orders, price moves.
  - Recall: one leads to the other; confounding lives here.
- Informed order size increases slightly with time.
- Trade price moves toward the true value.
- Trade price may not converge to true value by end of trading.

Code in THE SECRET DIRECTORY (kyle.r).

[10] As with the Glosten-Milgrom model, I have yet to see plots from anybody else who has simulated the Kyle model.

[11] Informed trades are shown as 'i's; uninformed trades as 'u's.

If You Want More: to Read

- Journals (and associated societies):
    - *Journal of Financial Markets*
    - *Journal of Business and Economic Statistics*/ASA
    - *Journal of Financial Econometrics*/SoFiE
    - *Journal of Financial Economics*
    - *Journal of Financial and Quantitative Analysis*
    - *Review of Financial Studies*/SFS
- Books:
    - O'Hara, *Market Microstructure Theory*
    - Harris, *Trading and Exchanges*
    - Hasbrouck, *Empirical Market Microstructure*
    - Weisberg, *Applied Regression Analysis*
    - Montgomery, *Design and Analysis of Experiments*
    - McCullagh and Nelder, *Generalized Linear Models*
    - Box, Jenkins, Reinsel, *Time Series Analysis*
    - Osborne and Rubinstein, *A Course in Game Theory*

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If You Want More: to Interact With

- Seminars (times may change next year)

UIC		Northwestern			U. Chicago		
Finance	MSCS	Finance	IEMS	Stat	E&S	Stat	FinMath
Fri	Wed	Wed	Tue	Wed	Thu	Mon	Fri
10:30	4:15	11:00	4:00	11:00	1:20	4:00	4:30

- Center Events: UIC ICFD, NWU Zell, UofC Stevanovich
- Conferences: ASSA, SoFiE, Oxford-Man Institute

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If You Want: to Support Work Like This

- Talk to academics at the conference; some glad to consult.
- Take courses through UIC External Ed:
  - Market Microstructure and Electronic Trading
  - Commodities, Energy, and Related Markets
  - Fixed Income/Structured Products
  - Empirical Methods for Finance
  - Univariate and Mutivariate Time Series Analysis
- Donate to the ICFD.

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