Recent Patents on Computer Science 2012, 5, 211-219 211

Hurst Exponent and its Applications in Time-series Analysis

Marina Resta[1]*

[1]School of Economics, University of Genova, P.O. Box 16126, Via Vivaldi 5, Genova, Italy

Received: October 28, 2012; Revised: November 11, 2012; Accepted: November 17, 2012

Abstract: The Hurst exponent is an index of fundamental importance in the analysis of the long range dependence features of observable time-series. As such, it has been estimated and analyzed in an astonishing number of physical systems. Over the time, various estimation methods as well as generalizations have been suggested and discussed: we therein judge straightforward to review the most important ones. In addition, we offer some insights on recent literature evolution and on patents that address practical implementation of the Hurst exponent.

Keywords: Hurst Exponent, long-range dependence, time-series analysis.

1. INTRODUCTION

According to the classical statistical approach, any observable system is assumed to be essentially random, with an underlying driving process of white noise-type. White noises are therefore important in time series analysis, and more complicated stochastic processes can be generally defined in terms of them.

In order to capture the spirit of what the white noise is, we can say that it is a discrete time stochastic process whose terms are either independent and identically distributed, with zero mean (*strong* white noise) or with constant second moments, zero autocorrelation and zero mean (*broader sense* white noise): events must not influence one another, and they must all be equally likely to occur.

More formally, let us denote by $X=\{X(t), t\in \mathrm{N}\}$ a one-dimensional random variable. We then say that X is a white noise process if the following conditions hold:

$E(X) = 0$;

$E(XX) = \sigma$;

$E\big(X(t), X(t+k)\big) = 0$ for any integer k.

An independent white noise process whose terms are all normally distributed is called *Gaussian* white noise. Fig. (**1**) shows an example of a Gaussian white noise process: it immediately sticks out that the autocorrelation function exhibits close to zero (and hence negligible) values at least every lag in the range: [1, 50].

Generalizing to multivariate processes, assume $\mathbf{X}=\{\boldsymbol{X}(t), t\in \mathrm{N}\}$ is a p-dimensional process; we say it being a white noise process if:

$$E\big(\boldsymbol{X}(t)\big) = 0$$

$$E\big(\boldsymbol{X}(t), \boldsymbol{X}(t+k)\big) = \begin{cases}\boldsymbol{\Sigma}, k = 0\\ 0, k \neq 0\end{cases}$$

*Address correspondence to this author at the School of Economics, University of Genova, P.O. Box 16126, Via Vivaldi 5, Genova, Italy;
Tel: +390102095469; Fax: +390102095243; E-mail: 55246@unige.it

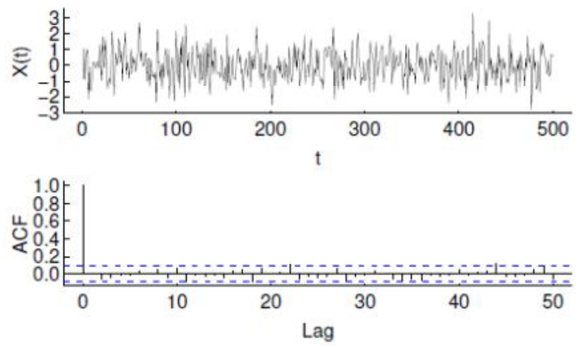

Fig. (1). Example of a Gaussian white noise process (top) with the related autocorrelation function -ACF (down).

where $\boldsymbol{\Sigma}$ is the (constant) covariance matrix. This latter condition does not require $\boldsymbol{X(t)}$s to be independent. If we make this stronger assumption, the process is called *independent* multivariate white noise. If we further assume that $\boldsymbol{X(t)}$s are following a joint normal distribution, the process will be called *Gaussian* multivariate white noise.

However, although quite suggestive and easy to manage, the white noise assumption rarely works well to explain the behavior of observed physical systems: this is particularly true in the case of financial data, precipitations records and heart beat rate observations (to cite some cases), which are said to exhibit a memory structure, i.e. either short-term or long-term memory.

Intuitively, a time series displays long memory when the correlation between current and lagged observations decays slowly.

More formally, let $X=\{X(t), t\in \mathrm{N}\}$ be a (strictly) stationary process with autocorrelation function $\rho(h)$, where h denotes the time lag. If $\sum_{h\in\mathbb{N}}|\rho(h)| = +\infty$, then X is called a long memory process; if $\sum_{h\in\mathbb{N}}|\rho(h)| < \infty$ then X is a short memory process; finally, if $\rho(h) = 0$, for $h\neq 0$, then X has no memory structure and follows a random walk. In a more specific way, one can define stationary processes with long memory (as to say: with long-range dependence, with slowly decaying or long-range correlations), if there exists a real number $\alpha \in (0, 1)$ and a constant $c_\rho > 0$ such that:

$$\lim_{h \to +\infty} \frac{\rho(h)}{c_\rho h^{-\alpha}} = 1$$

Moving from the time domain to the frequency domain, an equivalent definition of long term memory can be given in terms of the spectral density $f_X(\lambda)$ of a stationary process. In this case, X is said to have long-term memory if:

$$\lim_{\lambda \to 0} \frac{f_X(\lambda)}{c_f |\lambda|^{-\beta}} = 1$$

where β is a proper value in (0,1), and c_f is a positive constant.

Several indicators have been proposed over the last years to describe different memory features. Besides the classic way to measure process memory by way of the autocovariance function, the R/S statistic (analysis) that computes the Hurst exponent [2, 3] has revealed to be a very effective tool to determine long-range or short-range dependence.

Moreover, the statistic has several desirable properties. Mandelbrot and Wallis [4], for instance, show using Monte Carlo simulations that the R/S statistic is able to diagnose long-range dependence in highly non-Gaussian time series. Mandelbrot [5] reported the almost sure convergence of the R/S statistic for stochastic processes with infinite variances, and links the index to the concepts of self-similarity and fractal dimension. Mandelbrot (again) with Taqqu [6] derived a robustness property of the R/S statistic. Lo and MacKinlay [7] argued that this classical rescaled range statistic may be sensitive to short-range dependence, and then suggested a modification by introducing a (maximum) time lag that is chosen in dependence of the given data set for the resulting short-term and long-term memory asymptotic. Finally, Palma [8] provided an overview of the theory and methods developed to deal with long-memory structured data.

Much has been said and (probably) much is still waiting to be said on Hurst exponent. However, we think that an essay discussing main techniques to evaluate it, as well as the modifications that the original index has been incurred into since its introduction may help the reader to make an idea about future research directions as well as on practical application towards which research efforts can be addressed.

With this in mind, what remains of the paper is organized as follows. Section 2 reviews the basic idea inside the Hurst exponent H, also discussing the steps leading to its effective computation. Section 3 examines main methodologies that have been suggested in order to provide better H estimates, working either in the time or the frequency domain. The section ends giving emphasis to the wavelet approach that works both in the time and frequency domain. Section 4 provides a review focusing on main application fields of the Hurst statistics and on patents dealing with its practical application. Section 5 concludes.

2. THE HURST EXPONENT: BASIC GUIDELINES.

Harold Edwin Hurst was a hydrologist who spent almost his entire working career in Egypt, struggling with the problem of reservoir control. Hurst studied how the range of the reservoir level fluctuated around its average level; if successive influxes were random (i.e. statistically independent), this range would increase over time in line with the square root of time. In search of a confirmation coming from the study of the Nile river data, Hurst derived a dimensionless statistical exponent by dividing the adjusted range by the standard deviation of the observations. This approach is generally referred to as *rescaled range analysis* (R/S analysis).

In order to recall Hurst's original definition, we may consider the observed time-series $W=\{w(t), t\in \mathrm{N}\}$ and turn it into the series of returns $R=\{r(t), t\in \mathrm{N}\}$, being:

$$r(t) = \frac{w(t) - w(t-1)}{w(t-1)} \approx log \frac{w(t)}{w(t-1)}$$

The procedure leading to the Hurst exponent H may be then summarized in a number of steps [9].

Step 1. Divide R into d sub-series of length n.

Step 2. For each sub-series $m=1,\dots,d$, evaluate the corresponding mean value (E_m) and standard deviation (S_m).

Step 3. Normalize the data by subtracting the sample mean:

$$Z_{i,m} = R_{i,m} - E_m \tag{1}$$

for $i=1,\dots,n$.

Step 4. Create the cumulative time-series $Y_{i,m}$:

$$Y_{i,m} = \sum_{j=1}^{i} Z_{i,m} \tag{2}$$

for $i=1,\dots,n$.

Step 5. Find the range:

$$R_m = max\{Y_{1,m}, Y_{2,m}, \dots, Y_{i,m}\} - min\{Y_{1,m}, Y_{2,m}, \dots, Y_{i,m}\}$$

Step 6. Rescale the range dividing R_m by S_m: R_m/S_m. Averaging over the whole set of sub-samples d, the mean value of the rescaled range for sub-series of length n is then:

$$(R/S)_n = \frac{1}{d} \sum_{m=1}^{d} \frac{R_m}{S_m}$$

After the analysis is conducted for all possible divisors of N, one can plot the $(R/S)_n$ statistics against n on a double-logarithmic scale. If the returns process is white noise, then the plot will be roughly a straight line with slope 0.5. If the process is persistent (i.e. it exhibits long memory), the slope will be greater than 0.5; if it is anti-persistent (and hence it has short-term memory), then the slope will be lower than 0.5.

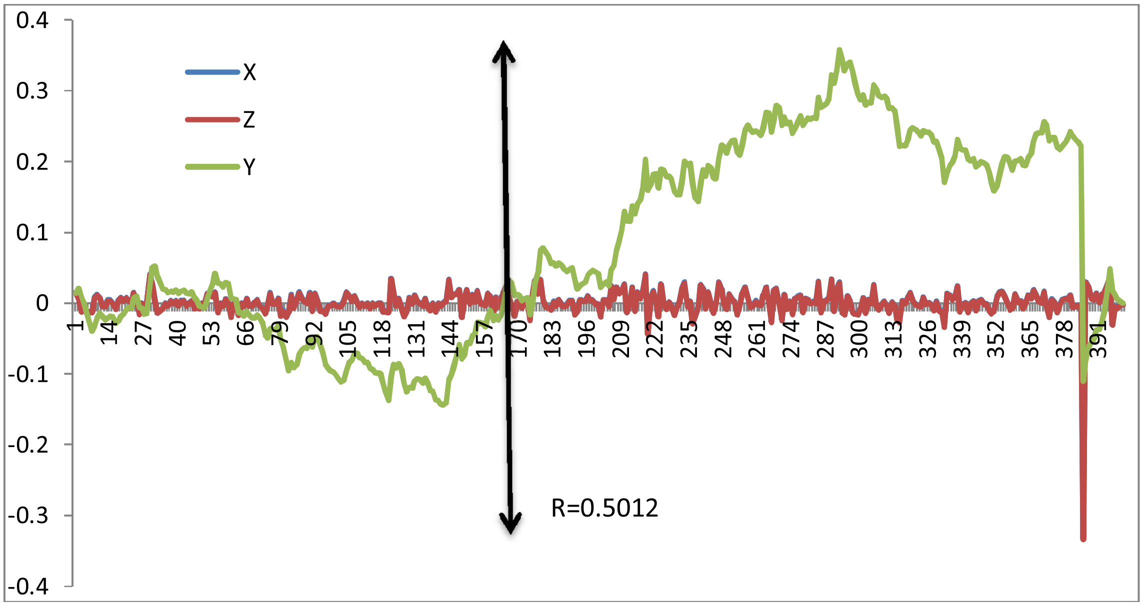

Fig. (2). Behavior of log-returns (*X*), standardized log-returns (*Z*), and cumulative standardized log-returns (*Y*) for the MSFT stock, observed on a time frame of 401 days.

The significance level is usually chosen to be $\sqrt{1/N}$, as to say: the standard deviation of a Gaussian white noise. Hall *et al*. [10] discuss the asymptotic distribution of $(R/S)_n$: for 3/4 < H < 1, the asymptotic distribution of $(R/S)_n$ is the Rosenblatt distribution [11], while for 0 < H ≤ 3/4, one obtains the normal distribution.

Note that, in general, since for small *n* values there is a significant deviation from the 0.5 slope, the theoretical values of the R/S statistics are approximated according to the following formula [11]:

$$E(R/S)_n = \begin{cases} \dfrac{n-\frac{1}{2}}{n} \dfrac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)} \displaystyle\sum_{i=1}^{n-1} \sqrt{\frac{n-i}{i}}, n \leq 340 \\ \dfrac{n-\frac{1}{2}}{n} \dfrac{1}{\sqrt{n\frac{\pi}{2}}} \displaystyle\sum_{i=1}^{n-1} \sqrt{\frac{n-i}{i}}, n > 340 \end{cases}$$

As a matter of fact, so far we provided evidence that (*a*) the *H* value is relatively easy to be evaluated and (*b*) it offers a powerful tool to analyze observable time-series. In order to put more light on this concept, we are going to discuss an application on close prices of the Microsoft (MSFT) stock, observed in the time range: 13 March 1986- 31 October 2012, for an overall number of 6718 records.

Let us consider latest 401 close prices for Microsoft (MSFT) stock (source: Yahoo Finance). We will denote them by: w_1, w_2, …, w_{401}. Now, we turn them into the corresponding log-returns: $X=\{ r_1, r_2, \ldots, r_{400}\}$, and we compute their mean value *E*: E=(1/400) [r_1+ r_2+, …, +r_{400}]. We move to calculate the deviations from the mean z_i, and the cumulative time-series y_i, (*i*=1,400), as shown in Eqs. (1)-(2). We can plot them together, as it has been done in Fig. (**2**).

The behavior of *X* is shown in gray, that of *Z* in dark gray, and finally the dynamic of *Y* is given in soft gray. Since *X* and *Z* values are quite similar, as a result we have two time-series practically overlapping in Fig. (**2**).

Now we find the maximum and the minimum *Y* values: by subtracting one to each other we get the range *R* =0.501234 that we have indicated in Fig. (**2**) by the double-oriented black arrow. Finally, we calculate the standard deviation over the log-returns' time-series.

In order to summarize what we have done, we can say that for *n*=400 we get: R/S_n=0.501234/0.020719=24.1918. That will give us one point in the chart opposing log(R/S_n) *vs* log(*n*), namely: log(400) = 2.6021, and log(24.1918) = 1.3836.

Now we can repeat the above scheme for 410 points, then for 420 points, and so on: each time we will generate a point on our chart, as we have shown in Fig. (**3**).

The slope of the interpolating line joining together the points (log(*n*),log(R/S_n)) provides an estimation of the *H* value. In particular, in the examined case we have got H=0.4975, that means that the returns of MSFT stock are very close to be truly independent and uncorrelated, and there is no trend effect in the stock price.

What is true for the MSFT stock, however, does not necessarily hold for other market stocks. Figure **4** provides the chart opposing log(R/S_n) vs log(*n*) for the quoted stocks: Intel Corporation (INTC), General Electric Company (GE), Wal Mart Stores (WMT), and Exxon (XOM).

The estimated value of the Hurst exponent corresponds to the slope of the interpolating line. While in the case of INTC, GE and XOM, the H value is sensitively lower that 0.5, in the case of WMT it maintains closer to 1.

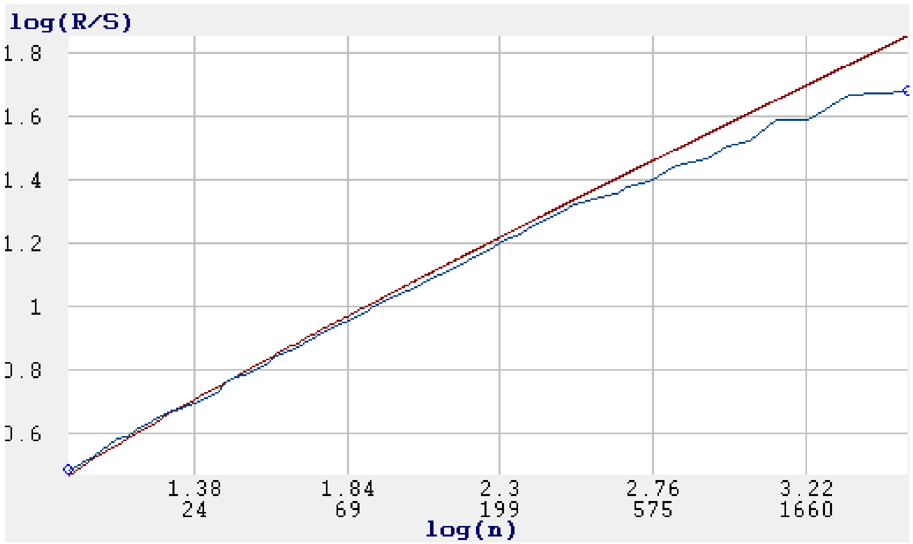

Fig. (3). Chart of log(R/Sn) vs log(*n*) for the MSFT stock. The upper straight line has slope equal to 0.5, while the other line has been obtained by joining together the points (log(*n*),log(R/Sn)) for n=10 to 1660**,** and has slope equal to (approximately) 0.4975.

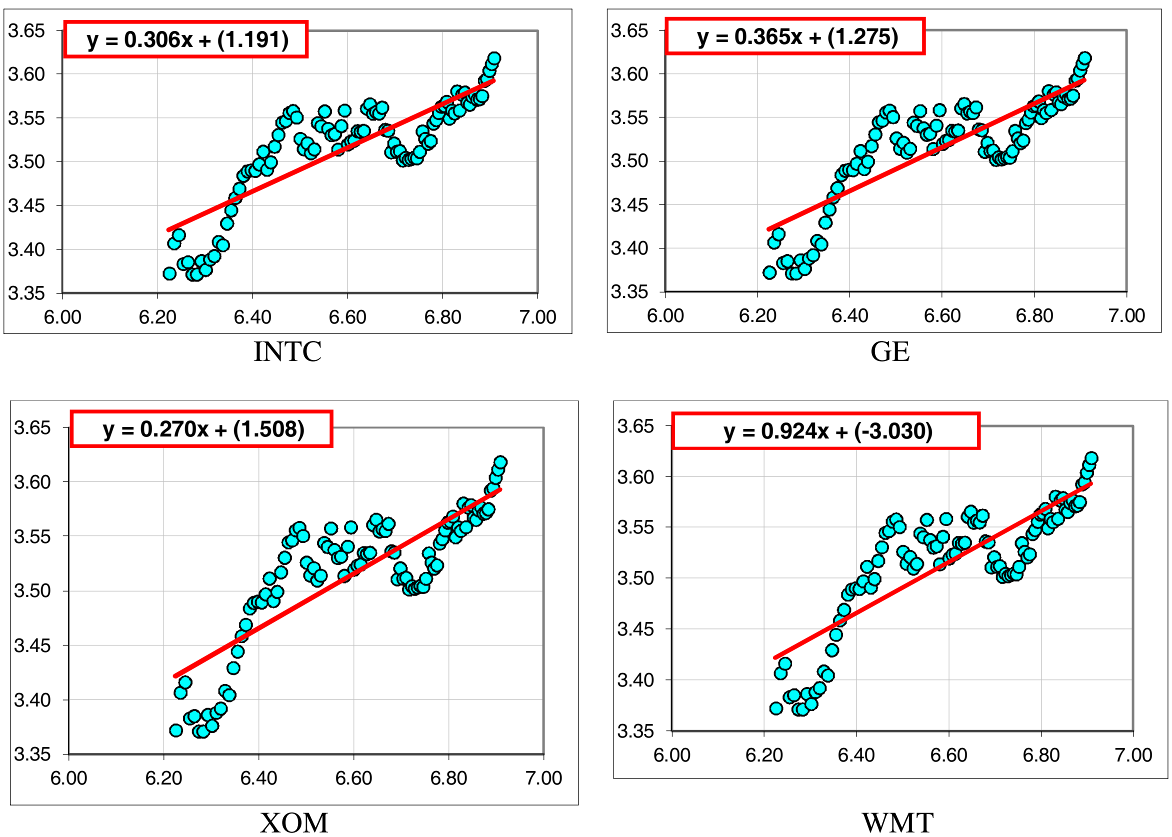

Fig. (4). From top to bottom in clock-wise sense: chart of log(R/S_n) vs log(*n*) for the stocks: Intel Corporation (INTC), General Electric Company (GE), Wal Mart Stores (WMT), and Exxon (XOM).

Those value provide a key to analyze the behavior of the related stocks. WMT returns, in fact, are told to exhibit persistence: this means that positive returns tend to follow positive returns and negative returns tend to follow negative returns. That is the so-called *Joseph Effect* (see [4] again), as in the Old Testament, where the prophet foretold of seven years of prosperity followed by seven years of famine. INTC, GE and XOM, on the other hand, seem to follow a Mean Reverting Process [13], that is, if a return is far from the mean, the subsequent returns tend to move toward the mean rather than continuing to be far from the mean.

3. METHODS OF COMPUTATION: A SURVEY

Starting from the original technique suggested by Hurst, several methods have been developed to estimate the parameter *H*, which operates both in time and frequency domain.

The aggregate variance (see [1] again) is a time domain method useful for non-stationary time series. Using the same notational conventions adopted in previous section, assume that *R* is the series of returns derived from the original time-series *W*. Then:

Step 1. Aggregate *R* into *d* sub-series of length *n*, with *n*=2,...,[*N*]/2, where *N* is the original length of *R*, and [.] indicates the integer part.

Step 2. For each sub-series consider the related sample variance.

Step 3. Graph the variances of such aggregate time-series in a log-log plot versus the different levels of aggregation and provide a least square line to fit the data. The slope of such line gives the estimation for *H*.

The modulus of the aggregate series [1] works in a similar fashion, but uses the modulus of the aggregate time series variance instead of its variance.

The periodogram method, on the other hand, is a technique working in the frequencies domain. The periodogram [14] for the returns series *R* is defined as:

$$I(\lambda) = \frac{1}{2\pi N}\left|\sum_{j=1}^{N} R_j e^{ij\lambda}\right|^2$$

where λ is the frequency. For a series with finite variance, I(λ) is an estimate of the time-series spectral density. Fitting a log-log plot of I(λ) with a least square line, one should obtain a slope of 1-2*H* close to the origin.

Another approach is discussed by the Whittle's estimator [15] which is a Maximum Likelihood Estimator. It can be defined as the value η that minimizes:

$$Q^*(\eta) = \sum_{j=1}^{\frac{N-1}{2}} \frac{I(\lambda_j)}{f^*(\lambda_j \eta)}$$

where λ is the frequency and I(λ) is an estimate of the time-series spectral density. The basic intuition is that when the underlying process is either a Gaussian white noise (GWN) or a FARIMA(0,*d*,0) process, then the optimal value η should approximate *H* or *d*, respectively. A slight issue with this technique, however, is that the user must specify the expected functional form, (either GWN or FARIMA, this latter with the order specified). In case of wrong choice of the underlying model, errors might occur. A partial softening of this issue is provided by the local Whittle's estimator [15] which is a semi-parametric version, and assumes a functional form for the spectral density at frequencies near zero.

Turning back to time-domain models, the Higuchi estimator [16] is based on the fractal dimension *D* of a time series. Starting from the original time-series *R*, the procedure works as follows.

Step 1. Create sample sequences:

$$Z_k^m = \left\{Y(m), Y(m+k), Y(m+2k), \ldots, Y\left(m + \left[\frac{(N-m)}{k}\right]k\right)\right\}$$

where $Y(m) = r_1 + r_2 + \cdots + r_m$ are partial sums, *k*= 1, 2,…,*N*; *m* = 1, 2,…, *k*, and the operator [.] stands, as already seen in previous rows, for the integer part.

Step 2. For each sequence Z_k^m the normalized curve length is computed as:

$$L_m(k) = \frac{N-1}{k^2\left[\frac{N-m}{k}\right]} \sum_{i=1}^{\left[\frac{N-m}{k}\right]} |Y(m+ik) - Y(m+(i-1)k)|$$

and the curve length *L*(*k*) for each lag *k* is:

$$L(k) = \frac{1}{k}\sum_{m=1}^{k} L_m(k)$$

Hence, $E(L(k)) \approx C_{2k}^{-D}$ for $k \to \infty$, where $D = 2 - H$. The *H* parameter is then estimated by classical regression techniques with *log L*(*k*) opposed to *log*(*k*).

So far we have described approaches that assume the stationarity of the time-series to work with. This is not the case of the Detrended Fluctuation Analysis (DFA) method [17] that makes possible to estimate *H* also in case of non-stationary time-series.

At first, one should divide the sequence *R* of length *N* into [*N*]/*s* non-overlapping boxes, each containing *s* points. The linear local trend z(*t*) = *a t*+*b* in each box is the standard linear least-square fit of the data points belonging to that box.

The detrended fluctuation function *F* is then defined by:

$$F_k^2(s) = \sum_{t=ks+1}^{(k+1)s} |r(t) - z(t)|^2$$

Averaging $F_k^2(s)$ over the *[N]/s* intervals gives the fluctuation $E(F^2(s))$ as a function of s , being:

$$E(F^2(s)) = \frac{s}{N}\sum_{k=0}^{\frac{[N]}{s}-1} F_k^2(s) \tag{3}$$

If the observable returns are realizations of random uncorrelated variables or short-range correlated variables, the behavior of (3) is expected to be a power law:

$$\sqrt{E(F^2(s))} \sim s^H \tag{4}$$

The value *H* can be then easily derived from (4).

To conclude, an interesting estimation method has been introduced at the eve of the new century [18, 19], based on the wavelet approach [20].

The wavelet transform is a mathematical tool for representing signals as sum of "small waves". It is a better substitute of the Fourier transform: while this latter is used to transform a signal from the time domain to the frequency domain, the wavelet transform, on the other hand, is capable of providing the time and frequency information of a signal simultaneously.

The input signal is represented in terms of dilated versions of a prototype of high-pass wavelet function $\psi_{ij}$and shifted version of a low-pass scaling function (ϕ_{ij}), based on the scaling function ϕ_0 and the mother wavelet basis function (ψ_0), being:

$$\phi_{ij}(t) = 2^{-i/2}\phi_0(2^{-i}t - j),\ i\in \mathrm{Z}$$

$$\psi_{ij}(t) = 2^{-j/2}\psi_0(2^{-j}t - j),\ j\in \mathrm{Z}$$

The approximation information of sequence R is then given by:

$$approx_i(t) = \sum_j a_r(i,j)\phi_{ij}(t)$$

where the coefficient $a_r(i,j)$ is given by calculating the inner product:

$$a_r(i,j) = \langle R, \phi_{ij}\rangle$$

The detail information ($detail_i$) of sequence R is given by:

$$detail_i(t) = \sum_j d_r(i,j)\psi_{ij}(t)$$

where the coefficient $d_r(i,j)$ is given by calculating the inner product of R:

$$d_r(i,j) = \langle R, \psi_{ij}\rangle$$

Multi Resolution Analysis (MRA) represents the information about the sequence X as a collection of details and a low resolution approximation:

$$r(t) = approx_N(t) + \textstyle\sum_{i=1}^{N} detail_i(t) =$$

$$= \textstyle\sum_j a_r(N,j)\phi_{Nj}(t) + \sum_{i=1}^{N}\sum_j d_r(i,j)\psi_{ij}(t)$$

The function ϕ_0 produces an approximation of signal R and it must be a low-pass filter. The mother wavelet function ψ_0 must be high-pass filter, and it performs a differential operation on the input signal to produce the detail version.

The wavelet-based Hurst parameter estimator is based on a spectral estimator obtained by performing a time average of the wavelet *detail* coefficients $|d_r(i,j)|^2$ at a given scale:

$$S_r = \frac{1}{N_i}\sum_j |d_r(i,j)|^2$$

where N_i is the number of wavelet coefficients at scale i, i.e., $N_i=2^{-i}N$, and N is number of data points. The estimator first performs Discrete Wavelet Transform (DWT) on the input signal, employing wavelets from the Daubechies family. After computing the DWT, the estimator calculates the estimates of $log_2E[d(i,j)]^2$ and variance of these estimates and performs a linear regression hence finding the slope b. H is then calculated as: $H=0.5(1+b)$, $0<b<1$.

4. REVIEW OF RECENT LITERATURE AND APPLICATIONS IN FINANCIAL MARKETS

The potential of the Hurst exponent has been explored in a very variety of applications: a recent web query in search for Hurst exponent applications retrieved approximately 904,000 results. There is then a high probability to omit some notable results. For this reason, we are going to discuss the relevance of studies about H on financial markets, since the debate is still maintaining at top levels in this research strand.

Practical applications began to flourish after the publication of [21] which is worldwide considered a pioneering contribution, because it traced the way to find a connection between self-similarity and Hurst exponent.

In the mid seventies, [22] studied 200 daily stock returns of securities listed on the New York Stock Exchange and found significant long range dependence.

After a period of apparent low activity in the field, a new refreshing age started in the last decade of past century. In this period, in fact, [23] popularized the notion of Hurst exponent among the practitioners' community, and provided H estimations for monthly returns on the S&P 500 from January 1950 to July 1988. Focusing on a bunch of S&P, Hurst exponents were observed to vary in the range [0.54, 0.75] thus accounting for the presence of persistence.

Another research strand was debated on [24], which discussed how to use H in order to avoid misspecification of long memory in seasonal time series, while [25] examined long memory features in inflation rates, and [26] gave some tips to practitioners approaching to the evaluation and interpretation of H values.

The characterization of financial markets by means of H values is another topic of intense debate: [27], for instance, characterized the efficiency level of the Brazilian market by way of the Hurst exponent; similar arguments, but more generally referring to financial markets, are discussed in [28]. The characterization of African and Middle East financial markets under the H statistics profile, on the other hand, is a more recent phenomenon. In particular, [29] focused on the analysis of MENA markets, to conclude by claiming for their substantial inefficiency, while [30] examined the Indian stock market and found long-term memory for all-time lags, hence inferring that Indian asset log-returns do not follow a random walk. In contrast to this conclusion, [31] discussed the real informative impact of H on a bunch of Asian markets, evidencing that its estimates become unstable when the time-series length falls below proper threshold levels.

An interesting viewpoint is then discussed in [32], where some foreign currency markets were studied: in most samples H revealed to be far from 0.5. Besides, it has also been found that the Hurst exponent is not constant, but it changes dynamically over time, thus introducing the possibility to distinguish between a global H value and local H values, as to say mono-fractal versus multifractal markets. In this spirit, [33] measured the degrees of persistence of the daily returns of several European stock market indexes and found that the FTSE turns out to be an ultra-efficient market, with abnormally fast mean-reversion. Multifractional properties of stock indexes are also discussed in [34] and [35]. In addition, [36] studied Bombay stock exchange (BSE) index financial time series for fractal and multifractal behavior, observing that BSE index time series is mono-fractal and can be characterized by a fractional Brownian motion. In search for diversity in local and global H behavior, [37] analyzed the Hurst exponent for all trading-day periods of the Dow-Jones index from January 1930 to May 2004, inferring that when Hurst exponent has higher values, then the corresponding market behavior is easier to predict. In order to give robustness to those conclusions, financial data series were divided into different periods and examined through back-propagation neural networks, to show that series with large H can be predicted more accurately [38] is a study on the Chinese markets that identified a link between the degree of persistence and the regulation level: by examining the Shanghai (SHI) stock market, the Shenzhen A-shares (SZI) and the Shenzhen B-shares (SZBI), before and after various deregulation and re-regulations processes, the empirical evidence revealed that SHI, SZI, and SZBI are moderately persistent, with Hurst exponents slightly greater than 0.5. In addition, these stock markets were more persistent before the deregulation, i.e. the markets have become more efficient in recent times.

Finally, [39] tried to find synthesis between technical analysis and fractal geometry, arguing that time series with high H resulted in higher profits in case of trending trading rules, and time series with low H resulted in higher profits from trading rules moving in the opposite direction to the market.

Moving to practical implementation, one can observe that despite the great intensity of the debate around H, however, it is really hard to find patents related to the use of Hurst exponent in financial applications. An example is provided by US 8170894 [40] that make use of H to predict the effects of innovation in business activity. Rather some implementations are documented in the medical research strand. This is, for instance, the case of US6144877 [41] that discuss a method using H into a computer environment for determining statistical information for time series data for a measurable activity. US 20060129069 A1 [42] describes a nonlinear technique that applies to pain time-series when they are proved to be scale-free and permits to detect in advance pain status. Similarly, US 7117108 [43] uses H to characterize an ensemble of different physiological time-series, while US 6438419 [44] discusses a method that is able to extract non-linearity features from time-series, even in the case only a limited observation period is available.

5. CURRENT & FUTURE DEVELOPMENTS

This work discussed the relevance of the Hurst exponent H in time-series analysis, with a privileged view on financial markets.

In particular, we started by providing some basic definitions, useful to understand the impact of the Hurst exponent estimation to describe the memory features of the process driving observable physical systems. We have then given some insights about the rough procedure to calculate H, and we have discussed an example focused on the estimation of H for some financial time-series. Most important refinements of the original technique proposed by Hurst have been briefly summarized to move to discuss the practical significance of the Hurst exponent in the analysis f financial time-series. Our declared aim was let the (not necessarily informed and update) reader with some background on the topic, giving him the chance to acquire a basic familiarity in it and hence to decide towards which direction to address his own research efforts.

Our review focused on financial markets applications for the motivation we are going to provide. Related literature is astonishingly wide: more than 1,000,000 records can be returned from an Internet research query, just by typing: "Hurst exponent". Focusing on financial markets studies made us possible to limit the range of analysis, although there is still the probability to omit some very relevant reference. At the same time, this does not exclude the relevance and the intensity of the debate in other fields. As an example, an active debate on the true nature of Hurst phenomenon and its consequences still concerns hydrological and climate studies, in the path of the original Hurst work [45-50].

Something similar applies for medical studies (we have referred to some of them in the section devoted to practical applications).

Going back to our analysis on financial markets, our review highlighted a number of issues that deserve deepest investigations.

As first remark, we observe that from the financial literature point of view, the interest for the Hurst exponent is still mainly theoretical. So far, only a very low number of patents dealing with the use of the Hurst exponent in financial applications has been known. Only in very recent times we can find papers addressing the issue in a more specific way: [51], for instance, examined the fractal features of a bunch of indexes that practitioners generally maintain under strong control to capture the movements of Eurozone stocks, and usED them to develop a trading system on the Eurostoxx 50 index.

A second issue concerns the estimation of H related to the length of observable time-series: in finance (as well as in many other fields) a single empirical time series is available, and it is not possible to restart the history to get another sample. A notable result allowing for both global and local H estimates is contained in [52], where a method to evaluate H from very short samples is described. Another way to get around such difficulty would be to have a very long history and to extract many samples from the whole time series: one could then regard these samples as shorter histories generated by the same physical mechanism. This is the main idea of the Bootstrap method [53]. Moreover, a related challenge is to get H by means of more and more accurate and fast algorithms. In this spirit, the Detrended Moving Average (DMA) has been proposed and discussed [54-56]. This latter method is quite similar to both the original Hurst's R/S procedure (see Section 2), and the DFA technique which we have described in Section 3. However, whereas both R/S, and DFA methods require the division of the series in sub-samples, this is not needed by the DMA algorithm. The scaling property, in fact, is obtained by using the moving average. Under this profile DMA is highly efficient from the computational point of view.

A third problem concerns the systematic study of the effects due to the finiteness of the sample, to Pareto tails and to the volatility clustering.

Probably, however, the most urgent issue that needs to be assessed concerns the very existence of the Hurst exponent. The calculation of the Hurst exponent, in fact, can be based on the *q*-order moment of the return distribution, but this moment does not necessarily exist at all. It is worth noting that here we are dealing with the theoretical moment, namely with the moment we would expect for the density function of returns we asymptotically extrapolate from our data. Of course the empirical moments, being based on a finite sample of data, are always finite, but when the theoretical limit does not exist the consistency of the Hurst exponent should be carefully checked.

A conclusive remark deserves the readers' attention, at last. The Hurst exponent revealed to be a powerful tool for time-series analysis, but it has to be managed with care: as highlighted in [57], in fact, the *H* index assures robust results only if the right estimation technique (and hence the correct signal identification) is performed, otherwise it can lead to inappropriate conclusions.

CONFLICT OF INTEREST

The authors confirm that this article content has no conflict of interest.

ACKNOWLEDGEMENTS

The author would thank the anonymous referees who contributed with their precious suggestions to improve the quality of the paper.

REFERENCES

[1] J. Beran, Statistics for Long-Memory Processes. New York: Chapman and Hall, 1994.

[2] H. Hurst, "Long-term storage capacity of reservoirs", *Trans. Am. Soc. Civ. Eng*., no. 116, pp. 770-808, 1951.

[3] H. Hurst, "Methods of using long-term storage in reservoirs", *Proc. Inst. Civ. Eng*., Part I, pp. 519-577, 1955.

[4] B. Mandelbrot and J. Wallis, "Computer experiments with fractional Gaussian noises". Parts 1, 2, and 3, *Water Resources Res*., Vol. 5, pp. 228-267, 1969.

[5] B. Mandelbrot, Statistical methodology for non-periodic cycles: From the covariance to R/S analysis, Ann. Ec. *Social Measurement*, Vol. 1, pp. 259-290, 1972.

[6] B. Mandelbrot and M.S. Taqqu, "Robust R/S analysis of long-run serial correlation", *Bulletin of the International Statistical Institute*, Vol. 48, no.2, pp. 59-104, 1979.

[7] W. Lo, and A.C. MacKinlay, A Non-Random Walk Down Wall Street. New Jersey: Princeton University Press, Princeton, 1999.

[8] W. Palma, W., Long-Memory Time Series. New York: John Wiley and Sons, 2007.

[9] Edgar E. Peters, Fractal Market Analysis, Wiley, New York, 1994.

[10] P. Hall, W. Hardle, T. Kleinow and P. Schmidt, "Semiparametric bootstrap approach to hypothesis tests and confidence intervals for the Hurst coefficient", *Stat. Inf. Stoch. Proc*., Vol. 3, pp. 263-76, 2000.

[11] M. Rosenblatt, "Independence and dependance", Proc. 4th Berkeley Symp. Math. Statist. Probab., J. Neyman, ed.), University of California Press, Berkeley, pp. 411-33, 1961.

[12] Murad S. Taqqu, V. Teverovsk and W. Willinger, "Estimators for Long-Range Dependence: An Empirical 0Study", Fractals, Vol. 3, pp.785-98, 1995.

[13] E.U. George, and L.S. Ornstein, "On the theory of Brownian Motion", *Phys. Rev*., Vol. 36, pp. 823-41, 1930.

[14] J. Geweke and S. Porter-Hudak, "The estimation and application of long memory time series models", *J. Time Ser. Anal*., Vol. 4, pp. 221-38, 1983.

[15] R. Fox and Murad S. Taqqu, "Large-sample properties of parameter estimates for strongly dependent stationary Gaussian time series", *Ann. Stats*, Vol. 14, pp. 517-32, 1986.

[16] T. Higuchi, "Approach to an irregular time series on the basis of the fractal theory2, *Phys D*, Vol. 31, pp. 277-83, 1988

[17] Chun-K. Peng, Sergey V. Buldyrev, S. Havlin, M. Simons, Eugene H. Stanley and Ary L. Golderberger, "On the mosaic organization of DNA sequences", *Phys. Rev. E*, Vol. 49, pp. 1685-9, 1994.

[18] P. Abry and D. Veitch, "Wavelet analysis of long range dependent traffic", *IEEE Trans. Inf. Th*., Vol. 44, nr 1, pp. 2-15, 1998.

[19] P. Abry, P. Flandrin, Murad S., Taqqu and D. Veitch, "Self-similarity and long-range dependence through the wavelet lens", in P. Doukhan, G. Oppenheim and M.S. Taqqu, eds, Theory and Applications of Long-range Dependence, *Birkhauser*, pp. 527-56, 2003

[20] Daubechies, Ten Lectures on Wavelets. SIAM. 1992. ISBN 0-89871-274-2.

[21] B. Mandelbrot, "The variation of certain speculative prices", *J. Business*, Vol. 36, pp.394-419, 1963.

[22] Myron T. Greene and Bruce D. Fielitz , "Long-Term Dependence in Common Stock Returns", *J. Fin. Econ*., Vol. 4, pp. 339-349, 1977.

[23] E. Peters, Chaos and Order in the Capital Markets: A new view of cycles, prices, and market volatility, John Wiley & Sons: New York, 1991.

[24] U. Hassler, "(Mis)specification of long memory in seasonal time series", *J. Time Series An*., Vol. 16, pp. 19-30, 1994.

[25] U. Hassler and J. Wolters, "Long memory in inflation rates: International evidence", *J. Bus. Ec. Stat*., Vol. 13, pp. 37-45, 1995.

[26] J. Hampton, "Rescaled range analysis: Approaches for the financial practitioners", Part 3, *NeuroVest J*., pp. 27-30, 1996.

[27] R.L. Costa and G.L. Vasconcelos, "Long-range correlations and nonstationarity in the Brazilian stock market", *Phys. A*, Vol. 329, no. 1-2, pp. 231-248, 2003.

[28] T. Di Matteo , T. Aste , and Michel M. Dacorogna , "Using the Scaling Analysis to Characterize Financial Markets", preprint available at: http://econpapers.repec.org/paper/arxpapers/cond-mat_2f0302434.htm.

[29] Imen Z. Rejichi, C. Aloui, "Hurst exponent behavior and assessment of the MENA stock markets efficiency", *Res. Intl. Bus. Fin*., Vol. 26, no. 3, pp. 353-70, 2012.

[30] Golaka C. Nath, "Long Memory and Indian Stock Market - An Empirical Evidence", UTIICM Conference Paper, 2001.

[31] Sanjit K. Mitra, "Is Hurst Exponent Value Useful in Forecasting Financial Time Series?", *Asian Soc. Sci*., Vol. 8, no. 8, 2012

[32] M.Corazza and Apostolos G. Malliaris, "Multifractality in Foreign Currency Markets", *Mult. Fin. J*., Vol. 6, pp. 387-401, 2002.

[33] J. Lipka, and C. Los, "Persistence characteristics of European stock indexes", Working Paper (Kent State University, Kent, OH), 2002

[34] J. Baruník, T. Aste, T. Di Matteo, and R. Liu, "Understanding the source of multifractality in financial markets", Quantitative Finance Papers, arXiv.org, http://EconPapers.repec.org/RePEc:arx:papers:1201.1535, 2012

[35] S. Bianchi, I. De Bellis and A. Pianese, "Fractal properties of some European electricity markets", *Int. J. Fin. Markets and Derivative*, Vol. 1, no. 4, pp. 395-421, 2010.

[36] A. Razdan, "Scaling in the Bombay stock exchange index", PRAMANA- *J. Physics*, Vol. 58, pp. 537-544, 2002.

[37] B. Qian and K. Rasheed, "Hurst Exponent and Financial Market Predictability", Proc. Fin. Eng. Appl., MIT, Cambridge, USA, 2004.

[38] C.A. Los and Y. Bing, "Persistence characteristics of the Chinese stock markets. International Review of Financial Analysis, Vol. 17, 64-82, 2008.

[39] C. Lento, "A Synthesis of Technical Analysis and Fractal Geometry - Evidence from the Dow Jones Industrial Average Components", Available: http://ssrn.com/abstract=1263615, 2009.

[40] M. Yitts, "*Method of identifying innovations possessing business disrupting properties*", U.S. Patent 8,170,894, May 1, 2012.

[41] P. DePetrillo, "*Determining the Hurst exponent for time series data*", U.S. Patent 6,144,877, August 11, 1998.

[42] A.V. Apkarian and D. Chialvo, "*Apparatus and method for pain measurement*", U.S. Patent 20060129069 A1, June 15, 2006.

[43] P. Rapp, C. Cellucci and T. Schmah, "*System and method for categorical analysis of time dependent dynamic processes*", U.S. Patent 7,117,108, October 3, 2006.

[44] Callaway and L. Sherman, "*Method and apparatus employing a scaling exponent for selectively defibrillating a patient*", U.S. Patent 6,954,700, October 11 2005.

[45] D. Koutsoyiannis, "Climate change, the Hurst phenomenon, and hydrological statistics", *Hydrological Sc. J.*, Vol. 48, nr.1, pp. 3–24, 2003.

[46] D.Koutsoyiannis, "Hurst-Kolmogorov dynamics and uncertainty, Journal of the American Water Resources Association, Vol. 47 (3), pp. 481–495, 2011.

[47] D. Koutsoyiannis, A. Paschalis and N. Theodoratos, "Two-dimensional Hurst-Kolmogorov process and its application to rainfall fields", J. Hydrology, Vol. 398 no.1-2, pp. 91-100, 2011.

[48] D. Koutsoyiannis, "Hurst-Kolmogorov dynamics as a result of extremal entropy production", *Phys. A*, Vol. 390 no. 8, pp. 1424-32, 2011.

[49] S. Papalexiou, D. Koutsoyiannis, and A. Montanari, "Can a simple stochastic model generate rich patterns of rainfall events?", *J. Hydrology*, Vol. 411, no. 3-4, pp. 279-89, 2011.

[50] F. Lombardo, E. Volpi and D. Koutsoyiannis, "Rainfall downscaling in time: Theoretical and empirical comparison between multifractal and Hurst-Kolmogorov discrete random cascades", *Hydrological Sci. J.*, Vol. 57, no. 6, pp. 1052-66, 2012.

[51] M. Resta, "On the fractal characterization of a system for tradings on Eurozone stocks", present to MAF 2012 Conference, Venice, 11-13 April 2012. To appear in M. Faggini: "Chaos Theory: What Perspectives for the Economics", New series, Springer, 2013.

[52] S. Bianchi, "Pathwise Identification of the Memory Function of the Multifractional Brownian Motion with Application to Finance", *Int. J. Th. Appl. Fin.*, Vol. 8, pp. 255-81, 2005.

[53] E. Carlstein, K. Do, P. Hall, T. Hesterberg and H. Kunsch, "Matched-block bootstrap for dependent data", *Bernoulli*, Vol. 4, pp. 305-28, 1998.

[54] E. Alessio, A. Carbone, G. Castelli, V. Frappietro, "Second-order moving average and scaling of stochastic time series ", *Eur. J. Phys. B*, Vol. 27, pp. 197-200, 2002.

[55] Carbone, G. Castelli and Eugene H. Stanley, "Analysis of the clusters formed by the moving average of a long-range correlated stochastic series", *Phys. Rev. E*, Vol. 69, 0161, 2004.

[56] Carbone, G. Castelli and Eugene H. Stanley, "Time-Dependent Hurst Exponent in Financial Time Series", *Phys. A*, Vol. 344, pp. 267-271, 2004.

[57] F. Serinaldi, "Use and misuse of some Hurst parameter estimators applied to stationary and non-stationary financial time series", *Phys. A*, Vol. 389, no. 14, pp. 2770-81, 2010.