Stochastic geometry analysis of a narrow-beam LEO uplink with mixed Gaussian shadowing

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Glossary of principal symbols	
Symbol	Explanation
h	Altitude of the SBSs.
ϵ	Elevation angle of the SBSs.
$G[\cdot]$	The SBS antenna gain.
φ_{RX}	Width of the SBSs 3 dB gain.
$\Theta \subset E$	Poisson p.p. on the earth's surface $E \subset \mathbb{R}^3$.
$\Phi \subset \mathbb{R}^2$	Poisson p.p. on the plane.
$\mathcal{G} \subset (0,1)$	A nonhomogeneous Poisson p.p.; the gain process of the approximate signal gains at the typical SBS.
x_0	Nearest point to the origin in Φ.
λ	Density parameter of Φ and Θ.
κ	Parameter that reflects the approximate mean number of UEs inside a SBS's 3 dB footprint; $\kappa = h^2\pi\lambda\varphi_{\mathrm{RX}}^2/\sin^4(\epsilon)$.
$\tilde{\kappa}$	$\kappa/\log(2)$.
g_x	Gamma-distributed random fading gain of a transmitter x.
θ	SIR or SINR threshold for a successful transmission.
I	Interference at the typical SBS in the plane model.
S	The signal power of the served UE at the typical SBS in the plane model.
$\mathring{I}$	Interference at the typical SBS in the spherical model.
$\mathring{S}$	The signal power of the served UE at the typical SBS in the spherical model.
$\hat{d}_{h,\epsilon}$	The distance between the SBS and the center of the footprint in the plane model.
d_0	Normalizing distance.

***Abstract*—**

***Index Terms*—LEO, SIR meta distribution, Nakagami fading**

I. Introduction

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II. Analysis

A. Gain process

We denote the derivative of the function $\|x\| \mapsto \varphi_x$ at $\|x\| = 0$ with $D_{h,\epsilon} \triangleq \sin^2(\epsilon)/h$. Consequently, $\varphi_x \approx D_{h,\epsilon}\|x\|$ for small $\|x\|$. Define the (shadowed) gain process (GP)

$$\mathcal{G} = \{x \in \Phi : H_x G(\|x\|)\}, \tag{1}$$

where $\{H_x\}$ are i.i.d. shadowing variables, possibly degenerate, and

$$G(r) = 2^{-(D_{h,\epsilon} r)^2/\varphi_{\mathrm{RX}}^2} \tag{2}$$

is the Gaussian antenna pattern characterized by the halfwidth of the 3 dB gain φ_{RX}. We use the value $\varphi_{\mathrm{RX}} = 1.6°$ according to the International Telecommunication Union Recommendations (ITU-R) [**?**]. The GP is a *projection process* and, as such, a nonhomogeneous PPP [CITE].

The density of the GP has the following connection to the fading distribution:

Proposition 1 (Density of the GP)**.** *Let $f_H(\cdot)$ and $F_H(\cdot)$ be the pdf and the complementary cdf (ccdf) of H, respectively. The density of $\mathcal{G}$ is given by*

$$\lambda_{\mathcal{G}}(t) = \tilde{\kappa} F_H(t)/t, \text{ for } t > 0, \tag{3}$$

where $\tilde{\kappa} = \kappa/\log(2)$ and

$$\kappa \triangleq \pi\lambda\left(\frac{h\varphi_{\mathrm{RX}}}{\sin^2(\epsilon)}\right)^2 \tag{4}$$

is approximately the mean number of UEs inside the 3 dB footprint of a SBS.

Proof. By [CITE],

$$\begin{aligned}
\lambda_{\mathcal{G}}(t) &= \frac{d}{dt}\pi\lambda\mathbb{E}\left[\left(G^{-1}[t/H]\right)^2\right] \\
&= \tilde{\kappa}\frac{d}{dt}\int_t^\infty G^{-1}(t/y) f_H(y)dy = \tilde{\kappa}\frac{d}{dt}\int_t^\infty \log(t/y) f_H(y)dy \\
&= \tilde{\kappa} \Big/_{t}^{\infty} \log(t/y)F_H(y) + \tilde{\kappa}\frac{d}{dt}\int_0^t \frac{F_H(y)}{y}dy = \tilde{\kappa}F_H(t)/t,
\end{aligned}$$

as long as $\log(t/y)F_H(y) = 0$ as $y \to 0$ for all $t > 0$. $G^{-1}(\cdot)$ is considered to be the generalized inverse $G^{-1}(y) = \inf\{x : G(x) < y\}$. □

The mean and the variance of the total received power are given by

$$\begin{aligned}
&\mathbb{E}\left(\sum_{x\in\mathcal{G}} x\right) = \int_0^\infty t\lambda_{\mathcal{G}}(t)dt = \tilde{\kappa}\int_0^\infty F_H(t)dt \\
&= \tilde{\kappa}\mathbb{E}(H),
\end{aligned} \tag{5}$$

$$\begin{aligned}
&\mathrm{var}\left(\sum_{x\in\mathcal{G}} x\right) = \int_0^\infty t^2\lambda_{\mathcal{G}}(t)dt = \tilde{\kappa}\int_0^\infty tF_H(t)dt \\
&\tilde{\kappa}\frac{\mathrm{var}(H) + \mathbb{E}(H)^2}{2} = \tilde{\kappa}\mathbb{E}[H^2]/2,
\end{aligned} \tag{6}$$

respectively.

Unfortunately, unlike in terrestrial networks with a singular path loss, where the density of the projection process is dependent only on a single moment of H, $\lambda_{\mathcal{G}}(t)$ has an explicit pointwise dependence on the ccdf of the fading distribution. However, the density functions are similar for certain fading distributions with matched moments. For example,

B. Multitier network

We consider J different tiers identically and independently randomly assigned for each UE. In this work, the tiers represent different types of shadowing. We denote by $T_x \in \mathcal{T} \triangleq \{1,\ldots,J\}$ the type (of the fading) of the UE $x \in \Phi$. Each tier is characterized by its unique fading variable and the probability $\mathbb{P}(T_x = i)$ that the transmitter belongs to tier i. It holds that $\sum_{i\in\mathcal{T}} \mathbb{P}(T_x = i) = 1$. In this work, the probability is identical for all UEs, and we can denote $\mathbb{P}(T = i) \triangleq \mathbb{P}(T_x = i)$. Also, the fading variables are i.d.d.; hence, we denote $(H|T = i) \triangleq (H_x|T_x = i)$. We denote by Φ_i the point process of tier i transmitters and $\mathcal{G}_i$ the corresponding GP. By the thinning theorem [CITE], both are PPPs, and if λ is the density of $\Phi = \cup_{i\in\mathcal{T}}\Phi_i$, we can sum the densities of the separate tiers $\lambda = \sum_{i\in\mathcal{T}} \lambda_i$ with $\lambda_i \triangleq \lambda\mathbb{P}(T = i)$. Accordingly, $\kappa_i \triangleq \mathbb{P}(T = i)\kappa$, where κ is the mean number of UEs inside the 3 dB footprint in Φ as in (4), and κ_i and $(H|T = i)$ determine the density $\lambda_{\mathcal{G}_i}(t)$ as in (3). Also the GP tiers can be superpositioned: if $\mathcal{G}$ is the GP of $\Phi = \cup_{i\in\mathcal{T}}\Phi_i$, $\lambda_{\mathcal{G}}(t) = \sum_{i\in\mathcal{T}} \lambda_{\mathcal{G}_i}(t)$.

Of course, the index set $\mathcal{T}$ does not have to be a set of the first J integers; for example, we could you use $\mathcal{T} = \{\text{NLoS, LoS}\}$.

Let $W > 0$ be a normalized noise power constant. As we consider a NB and define the spatial path-loss to be equal to all transmitters, and depending only on the distance between the SBS and its antenna boresights location on the earth's surface $d_{h,\epsilon}$. The path-loss law is defined as $\ell(d) \triangleq (d/d_0)^\gamma$, where γ is the power path-loss exponent. The SINR of the strongest signal $x_0 \triangleq \max\mathcal{G}$ at the typical SBS is

$$\text{SINR} = \frac{x_0/\ell(\hat{d}_{h,\epsilon})}{\sum\limits_{x\in\mathcal{G}\setminus\{x_0\}} x/\ell(\hat{d}_{h,\epsilon}) + W}. \tag{7}$$

C. Laplace transform of the total received power

The Laplace function of the total received power is given as [CITE]

$$\mathcal{L}_{\mathcal{G}}(s) \triangleq \mathbb{E}\left(e^{-s\sum\limits_{x\in\mathcal{G}} x}\right) = \exp\left\{-\int_0^\infty (1-e^{-sr})\lambda_{\mathcal{G}}(r)dr\right\}. \tag{8}$$

Assuming a exponential fading variable H with mean m;

$$\mathcal{L}_{\mathcal{G}}(s) = \exp\left\{-\kappa\int_0^\infty (1-e^{-sr})\frac{e^{-r/m}}{r}dr\right\} = (1+sm)^{-\tilde{\kappa}}, \tag{9}$$

which is the Laplace transform of the **gamma distribution** with shape parameter $\tilde{\kappa}$ and scale parameter m: the total received power is gamma-distributed with exponential fading and approximately gamma-distributed with any fading that produces the GP that can be approximated with the exponential-type density function – like the log-normal fading in suitable regimes.

For independent $\{\mathcal{G}_i\}_{i\in\tau}$, $\mathcal{L}_{\sum_{i\in\mathcal{T}}\mathcal{G}_i}(s) = \prod_{i\in\mathcal{T}}\mathcal{L}_{\mathcal{G}_i}(s)$

We are ready to derive a approximation for the probability of coverage of the UE with the srongest signal.

Proposition 2. *Assume a network of density λ with the tiers $\mathcal{T}$. Let H_{x_0} be the shadowing variable, and $x'_0 = H_{x_0}G(\|x_0\|) \in \mathcal{G} = \cup_{i\in\mathcal{T}}\mathcal{G}_i$ be the strongest signal of the UEs. In the simple coverage region $\theta \geq 1$,*

$$\begin{aligned}
p_c(\theta) &\triangleq \mathbb{P}\left(\frac{H_{x_0}G(\|x_0\|)}{\sum\limits_{x'\in\mathcal{G}\setminus\{x'_0\}} x' + W\ell(\hat{d}_{h,\epsilon})} > \theta\right) \\
&\overset{(a)}{=} \sum_{i\in\mathcal{T}}\mathbb{P}(T_{x_0}=i)\lambda\int_{\mathbb{R}^2}\mathbb{P}\Bigg((H_{x_0}|T_{x_0}=i) > \theta 2^{\left(\frac{D_{h,\epsilon}\|y\|}{\varphi_{\text{RX}}}\right)^2} \\
&\quad\cdot\left(\sum_{x\in\cup\mathcal{G}_i} x + W\ell(\hat{d}_{h,\epsilon})\right)\Bigg)dy \\
&\overset{(b)}{\approx} 2\pi\lambda\sum_{i\in\mathcal{T}}\mathbb{P}(T=i)\cdot \\
&\quad\int_0^\infty r\prod_{k\in\mathcal{T}}\mathbb{E}\left(e^{-\frac{\theta\sum_{x\in\mathcal{G}_k}x}{2^{-(D_{h,\epsilon}r/\varphi_{\text{RX}})^2}m_i}}\cdot e^{-\theta\frac{W\ell(\hat{d}_{h,\epsilon})}{2^{-(D_{h,\epsilon}r/\varphi_{\text{RX}})^2}m_i}}\right)dr \\
&\overset{(c)}{=} 2\sum_{i\in\mathcal{T}}\kappa_i\int_0^\infty\mathcal{L}_{\mathcal{G}_i}\left(\frac{\theta 2^{u^2}m_k}{m_i}\right)\mathcal{L}_W\left(\frac{\theta 2^{u^2}W\ell(\hat{d}_{h,\epsilon})}{m_i}\right)du \\
&= 2\sum_{i\in\mathcal{T}}\kappa_i\int_0^\infty\frac{u\exp\{-\theta 2^{u^2}W\ell(\hat{d}_{h,\epsilon})/m_i\}}{\prod\limits_{k\in\mathcal{T}}\left(1+\theta 2^{u^2}m_k/m_i\right)^{\tilde{\kappa}_k}}du,
\end{aligned} \tag{10}$$

where $m_j \triangleq \mathbb{E}(H|T=j)$ is the mean of the fading variable of tier j, $\kappa_i = \mathbb{P}(T=i)\kappa = \mathbb{P}(T=i)\tilde{\kappa}\log(2)$, where $\kappa = \tilde{\kappa}\log(2) = \pi\lambda\left(h\varphi_{\text{RX}}/\sin^2(\epsilon)\right)^2$ reflects the mean number of UEs inside the (spatial) 3 dB footprint of all tiers. Recall that the Laplace transform $\mathcal{L}_{\mathcal{G}}(s)$ of the sum of $\mathcal{G}$ is defined in (8) *and given in the exponential fading case in* (9)*. The Laplace transform of W is defined accordingly as $\mathcal{L}_W(s) \triangleq \mathbb{E}\left(e^{-sW}\right)$, which reduces to e^{-sW} for deterministic W.*

Proof. The detailed proof is presented in [CITE]. Step (a) heuristically: In the simple coverage region, the UE is covered *only if* it has the strongest signal. Using the translation invariance of the stationary PPP and utilizing Slivnyak's and Gambell's theorems leads to the probability of coverage given by the integral over the plane.

In (b), an exponential $(H_{x_0}|T_{x_0} = i)$ allows the expression of the probability in terms of the Laplace transforms of $\sum_{x\in\cup\mathcal{G}_i} x$ and W. By the independent shadowing, they are independent and can be expressed as the product. A change of variable $D_{h,\epsilon}r/\varphi_{\text{RX}} \mapsto u$ was made in (c).

$\square$

Notice that $p_c(\theta)$ is the probability of coverage of the UE with the *strongest signal*, which is *not* necessarily the spatially *nearest transmitter*.

For a single-tier network in an interference-only channel $W = 0$, the coverage probability depends on the mean number $\kappa = \tilde{\kappa}\log(2)$ of UEs inside the 3 dB footprint of the typical SBS and the threshold θ, and (10) has the analytical expression

$$p_c(\theta) = \theta^{-\tilde{\kappa}}{}_2F_1\left(\tilde{\kappa}, \tilde{\kappa}; \tilde{\kappa}+1; -1/\theta\right). \tag{11}$$

where ${}_2F_1(\cdot)$ is the hypergeometric function.

D. Explicit fading model

The power fading distribution for the mixed Gaussian model is given by

E. Gaussian mixture model

The power fading RV of a transmitter $x \in \Phi$ in the two-tier $\mathcal{T} = \{\mathrm{LoS}, \mathrm{NLoS}\}$ Gaussian mixture model is given by

$$\begin{aligned} &H_x \\ &= \mathbb{1}(T_x = \mathrm{LoS})e^{\mu_{\mathrm{LoS}}+\sigma_{\mathrm{LoS}}Z} + \mathbb{1}(T_x = \mathrm{NLoS})e^{\mu_{\mathrm{NLoS}}+\sigma_{\mathrm{NLoS}}Z}, \end{aligned} \tag{12}$$

$\mathbb{1}(\cdot)$ is the indicator function and Z is a standard normal RV. Accordingly, the conditional fading variables follow a log-normal distribution

$$(H_x | T_x = \mathrm{LoS}) \sim \mathrm{Lognormal}(\mu_{\mathrm{LoS}}, \sigma^2_{\mathrm{LoS}}) \tag{13}$$
$$(H_x | T_x = \mathrm{NLoS}) \sim \mathrm{Lognormal}(\mu_{\mathrm{NLoS}}, \sigma^2_{\mathrm{NLoS}}). \tag{14}$$