Stochastic geometry analysis of a narrow-beam LEO uplink with mixed Gaussian shadowing

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Glossary of principal symbols	
Symbol	Explanation
h	Altitude of the SBSs.
ϵ	Elevation angle of the SBSs.
$G[\cdot]$	The SBS antenna gain.
φ_{RX}	Width of the SBSs 3 dB gain.
$\Theta \subset E$	Poisson p.p. on the earth's surface $E \subset \mathbb{R}^3$.
$\Phi \subset \mathbb{R}^2$	Poisson p.p. on the plane.
$\mathcal{G} \subset (0,1)$	A nonhomogeneous Poisson p.p.; the gain process of the approximate signal gains at the typical SBS.
x_0	Nearest point to the origin in Φ.
λ	Density parameter of Φ and Θ.
κ	Parameter that reflects the approximate mean number of UEs inside a SBS's 3 dB footprint; $\kappa = h^2\pi\lambda\varphi_{\mathrm{RX}}^2/\sin^4(\epsilon)$.
$\tilde{\kappa}$	$\kappa/\log(2)$.
g_x	Gamma distributed random fading gain of a transmitter x.
θ	SIR or SINR threshold for a successful transmission.
I	Interference at the typical SBS in the plane model.
S	The signal power of the served UE at the typical SBS in the plane model.
$\mathring{I}$	Interference at the typical SBS in the spherical model.
$\mathring{S}$	The signal power of the served UE at the typical SBS in the spherical model.
$\hat{d}_{h,\epsilon}$	The distance between the SBS and the center of the footprint in the plane model.
d_0	Normalizing distance.

***Abstract*—**

***Index Terms*—LEO, SIR meta distribution, Nakagami fading**

I. Introduction

hhihhi

II. Analysis

A. Gain process

Let the constant $D_{h,\epsilon} \triangleq \sin^2(\epsilon)/h$ be the derivative of the function $\|x\| \mapsto \varphi_x$ at $\|x\| = 0$. Consequently, $\varphi_x \approx D_{h,\epsilon}\|x\|$ for small $\|x\|$. Define the gain process (GP)

$$\mathcal{G} = \{x \in \Phi : H_x G[D_{h,\epsilon}\|x\|]\}, \tag{1}$$

where $\{H_x\}$ are i.i.d. shadowing variables (possibly degenerate), and

$$G[\cdot] = 2^{-(\cdot)^2/\varphi_{\mathrm{RX}}^2} \tag{2}$$

is the Gaussian antenna pattern $G : [0,\infty) \to [0,1]$ characterized by the halfwidth of the 3 dB gain φ_{RX}. We use the value $\varphi_{\mathrm{RX}} = 1.6^\circ$ according to the International Telecommunication Union Recommendations (ITU-R) [**?**]. The GP is a *projection process* and, as such, a nonhomogeneous PPP [CITE].

The density of the GP has the following connection to the fading distribution:

Proposition 1 (Density of the GP)**.** *Let $f_H(\cdot)$ and $F_H(\cdot)$ be the pdf and the complementary cdf (ccdf) of H, respectively. The density of $\mathcal{G}$ is given by*

$$\lambda_{\mathcal{G}}(t) = \tilde{\kappa} F_H(t)/t \tag{3}$$

for $t > 1$.

Proof. By [CITE],

$$\lambda_{\mathcal{G}}(t) = \pi\lambda\mathbb{E}\left[\left(G^{-1}[t/H]\right)^2\right] = \tilde{\kappa}\frac{d}{dt}\int_t^\infty G^{-1}[t/y] f_H(y)dy$$

$$= \tilde{\kappa}\frac{d}{dt}\int_t^\infty \log(t/y) f_H(y)dy = \tilde{\kappa}\int_t^\infty \log(t/y)F_H(y)$$

$$+ \tilde{\kappa}\frac{d}{dt}\int_0^t F_H(y)/y dy = \tilde{\kappa}F_H(t)/t,$$

as long as $\log(t/y)F_H(y) = 0$ as $y \to 0$ for all $t > 0$. $G^{-1}[\cdot]$ is considered to be the generalized inverse $G^{-1}[\cdot] = \inf\{x : G[x] < y\}$. □

The mean and the variance of the total received power are given by

$$\mathbb{E}\left(\sum_{x\in\mathcal{G}} x\right) = \int_0^\infty t\lambda_{\mathcal{G}}(t)dt = \tilde{\kappa}\int_0^\infty F_H(t)dt$$
$$= \tilde{\kappa}\mathbb{E}(H), \tag{4}$$

$$\mathrm{var}\left(\sum_{x\in\mathcal{G}} x\right) = \int_0^\infty t^2\lambda_{\mathcal{G}}(t)dt = \tilde{\kappa}\int_0^\infty tF_H(t)dt$$
$$\tilde{\kappa}\frac{\mathrm{var}(H) + \mathbb{E}(H)^2}{2} = \tilde{\kappa}\mathbb{E}[H^2]/2, \tag{5}$$

respectively.

Unfortunately, unlike in terrestrial networks with a singular path loss, where the density of the projection process is dependent only on a single moment of H, $\lambda_{\mathcal{G}}(t)$ has an explicit pointwise dependence on the ccdf of the fading distribution.

However, the density functions are similar for certain fading distributions with matched moments. For example,

Proposition 2. *Let H_0 the fading variable of the UE x_0 with the strongest signal. In the simple coverage coverage region $\theta \geq 1$,*

$$
\begin{aligned}
&p_c(\theta) \triangleq \mathbb{P}\left(\frac{x_0}{\sum_{x\in\cup\mathcal{G}_i} x + N} > \theta\right) \\
&= \lambda \int_{x\in\mathbb{R}^2} \mathbb{P}\left(H_0 > \theta/\mu 2^{(D_{h,\epsilon}\|x\|)^2/\varphi_{\text{RX}}^2}\left(\sum_{x\in\cup\mathcal{G}_i} x + W\right)\right) dx \\
&\overset{(a)}{\approx} 2\pi \int_0^\infty r \prod_{i=1}^N \mathbb{E}_{\mathcal{G}_i} \exp\left\{-\theta 2^{(D_{h,\epsilon} r)^2/\varphi_{\text{RX}}^2}/\mu \sum_{x\in\mathcal{G}_i} x\right\} \cdot \\
&\qquad\qquad \exp\left\{-\theta/\mu 2^{(D_{h,\epsilon} r)^2/\varphi_{\text{RX}}^2} W\right\} dr \\
&= 2\pi \int_0^\infty r \prod_{i=1}^N \left(\frac{1}{1+2^{(D_{h,\epsilon} r)^2/\varphi_{\text{RX}}^2}\theta}\right)^{\kappa_i} \cdot \\
&\qquad\qquad \exp\left\{\theta/\mu 2^{(D_{h,\epsilon} r)^2/\varphi_{\text{RX}}^2} W\right\} dr. \qquad (6)
\end{aligned}
$$

Proof. In the simple coverage region, the UE is covered only if it have the strongest signal, and utlilizing Slivnyak's theorem leads to the probability of coverage by the integral. In (a), a exponential H_0 allows the expression in terms of product of Laplace transforms of $\{\sum_{x\in\mathcal{G}_1} x, \sum_{x\in\mathcal{G}_2} x, \ldots, \sum_{x\in\mathcal{G}_i} x\}$ and N. In the exponential case, the formers are gamma distributed. □

For $W = 0$, the scale parameter μ is cancelled out in $p_c(\theta)$, meaning that the severity of the fading does not have any effect on the probability of coverage.

In a single-tier interference-only channel, (6) has the analytical expression

$$p_c(\theta) = \theta^{-\kappa} {}_2F_1\left(\kappa, \kappa; \kappa+1; -1/\theta\right). \qquad (7)$$

where ${}_2F_1(\cdot)$ is the hypergeometric function.

B. Explicit fading model

The power fading distribution for the mixed Gaussian model is given by