# Stochastic geometry analysis of a narrow-beam LEO uplink with mixed Gaussian shadowing

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| Glossary of principal symbols | |
|---|---|
| Symbol | Explanation |
| $h$ | Altitude of the SBSs. |
| $\epsilon$ | Elevation angle of the SBSs. |
| $G[\cdot]$ | The SBS antenna gain. |
| $\varphi_{\mathrm{RX}}$ | Width of the SBSs 3 dB gain. |
| $\Theta \subset E$ | Poisson p.p. on the earth's surface $E \subset \mathbb{R}^3$. |
| $\Phi \subset \mathbb{R}^2$ | Poisson p.p. on the plane. |
| $x_0$ | Nearest point to the origin in $\Phi$. |
| $\lambda$ | Density parameter of $\Phi$ and $\Theta$. |
| $\kappa$ | Parameter that reflects the approximate mean number of UEs inside a SBS's 3 dB footprint; $\kappa = h^2\pi\lambda\varphi_{\mathrm{RX}}^2/\sin^4(\epsilon)$. |
| $\tilde{\kappa}$ | $\kappa/\log(2)$. |
| $g_x$ | Gamma-distributed random fading gain of a transmitter $x$. |
| $\theta$ | SIR or SINR threshold for a successful transmission. |
| $I$ | Interference at the typical SBS in the plane model. |
| $S$ | The signal power of the served UE at the typical SBS in the plane model. |
| $\mathring{I}$ | Interference at the typical SBS in the spherical model. |
| $\mathring{S}$ | The signal power of the served UE at the typical SBS in the spherical model. |
| $\hat{d}_{h,\epsilon}$ | The distance between the SBS and the center of the footprint in the plane model. |
| $d_0$ | Normalizing distance. |

***Abstract*—**

***Index Terms*—LEO, SIR meta distribution, Nakagami fading**

## I. Introduction

## II. Analysis

### *A. Gain process*

We denote the derivative of the function $\|x\| \mapsto \varphi_x$ at $\|x\| = 0$ with $D_{h,\epsilon} \triangleq \sin^2(\epsilon)/h$. Consequently, $\varphi_x \approx D_{h,\epsilon}\|x\|$ for small $\|x\|$. Given a PPP $\Phi \subset \mathbb{R}^2$ of density $\lambda$, define the shadowed gain process (GP)

$$\mathcal{G} \triangleq \{x \in \Phi : H_x G(D_{h,\epsilon}\|x\|)\}, \tag{1}$$

where $\{H_x\}$ are i.i.d. shadowing variables, possibly degenerate, and

$$G(\varphi) = 2^{-\varphi^2/\varphi_{\mathrm{RX}}^2} \tag{2}$$

is the Gaussian antenna pattern $G : [0,\infty) \to (0,1]$ characterized by the halfwidth of the 3 dB gain $\varphi_{\mathrm{RX}}$. We use the value $\varphi_{\mathrm{RX}} = 1.6°$ according to the International Telecommunication Union Recommendations (ITU-R) [**?**]. The GP is a *projection process* and, as such, a nonhomogeneous PPP [CITE].

The density of the GP has the following connection to the fading distribution:

**Proposition 1** (Density of the GP)**.** *Let $F_H(\cdot)$ be the complementary cdf (ccdf) of $H$. The density of $\mathcal{G}$ is given by*

$$\lambda_{\mathcal{G}}(t) = \tilde{\kappa}F_H(t)/t, \text{ for } t > 0, \tag{3}$$

*where $\tilde{\kappa} = \kappa/\log(2)$ and*

$$\kappa \triangleq \pi\lambda\left(\frac{h\varphi_{\mathrm{RX}}}{\sin^2(\epsilon)}\right)^2 \tag{4}$$

*is approximately the mean number of UEs inside the 3 dB footprint of a SBS.*

*Proof.* Let $f_H(\cdot)$ be the pdf of $H$. We denote $G^{-1}(\cdot)$ as the generalized inverse $G^{-1}(y) = \inf\{x : G(x) < y\}$. By [CITE],

$$\begin{aligned}
\int_t^\infty \lambda_{\mathcal{G}}(y)dy &= \pi\lambda\mathbb{E}\left[\left(\frac{G^{-1}(t/H)}{D_{h,\epsilon}}\right)^2\right] \\
&= \pi\lambda\int_t^\infty\left(-\frac{h\varphi_{\mathrm{RX}}\sqrt{-\log(t/h)}}{\sin^2(\epsilon)\sqrt{\log(2)}}\right)^2 f_H(h)dh \\
&= -\tilde{\kappa}\int_t^\infty \log(t/h)f_H(h)dh \\
&\overset{(a)}{=} -\tilde{\kappa}\Big/_{t}^{\infty}\log(t/h)F_H(h) - \tilde{\kappa}\int_t^\infty\frac{F_H(h)}{h}dh.
\end{aligned}$$

In (a), we use the integration by parts. The result follows by derivating w.r.t. $t$ and taking the minus sign—as long as $\int_t^\infty \log(t/h)f_H(h)dh$ converges for all $t > 0$. $\square$

The mean and the variance of the total received power are given by

$$\begin{aligned}
&\mathbb{E}\left(\sum_{x\in\mathcal{G}} x\right) = \int_0^\infty t\lambda_{\mathcal{G}}(t)dt = \tilde{\kappa}\int_0^\infty F_H(t)dt \\
&= \tilde{\kappa}\mathbb{E}(H),
\end{aligned} \tag{5}$$

$$\begin{aligned}
&\mathrm{var}\left(\sum_{x\in\mathcal{G}} x\right) = \int_0^\infty t^2\lambda_{\mathcal{G}}(t)dt = \tilde{\kappa}\int_0^\infty tF_H(t)dt \\
&\tilde{\kappa}\frac{\mathrm{var}(H) + \mathbb{E}(H)^2}{2} = \tilde{\kappa}\mathbb{E}[H^2]/2,
\end{aligned} \tag{6}$$

respectively.

Unfortunately, unlike in terrestrial networks with a singular path loss, where the density of the projection process is dependent only on a single moment of $H$, $\lambda_{\mathcal{G}}(t)$ has an explicit pointwise dependence on the ccdf of the fading distribution. However, the density functions are similar for certain fading distributions with matched moments. For example,

## B. Multitier network

We consider $J$ different tiers identically and independently randomly assigned for each UE. In this work, the tiers represent different types of shadowing. We denote by $T_x \in \mathcal{T} \triangleq \{1, \dots, J\}$ the type (of the fading) of the UE $x \in \Phi$. Each tier is characterized by its unique fading variable and the probability $\mathbb{P}(T_x = i)$ that the transmitter belongs to tier $i$. It holds that $\sum_{i\in\mathcal{T}} \mathbb{P}(T_x = i) = 1$. In this work, the probability is identical for all UEs, and we can denote $\mathbb{P}(T = i) \triangleq \mathbb{P}(T_x = i)$. Also, the fading variables are i.d.d.; hence, we denote $(H|T = i) \triangleq (H_x|T_x = i)$. We denote by $\Phi_i$ the point process of tier $i$ transmitters and $\mathcal{G}_i$ the corresponding GP. By the thinning theorem [CITE], both are PPPs, and if $\lambda$ is the density of $\Phi = \cup_{i\in\mathcal{T}}\Phi_i$, we can sum the densities of the separate tiers $\lambda = \sum_{i\in\mathcal{T}} \lambda_i$ with $\lambda_i \triangleq \lambda\mathbb{P}(T = i)$. Accordingly, $\kappa_i \triangleq \mathbb{P}(T = i)\kappa$, where $\kappa$ is the mean number of UEs inside the 3 dB footprint in $\Phi$ as in (4), and $\kappa_i$ and $(H|T = i)$ determine the density $\lambda_{\mathcal{G}_i}(t)$ as in (3). Also the GP tiers can be superpositioned: if $\mathcal{G}$ is the GP of $\Phi = \cup_{i\in\mathcal{T}}\Phi_i$, $\lambda_{\mathcal{G}}(t) = \sum_{i\in\mathcal{T}} \lambda_{\mathcal{G}_i}(t)$.

Of course, the index set $\mathcal{T}$ does not have to be a set of the first $J$ integers; for example, we could you use $\mathcal{T} = \{\text{NLoS, LoS}\}$.

## C. Signal-to-interference ratio

Let $W > 0$ be a normalized noise power constant. As we consider a NB that decays much faster than the spatial path loss w.r.t. $\|x\|$, $x \in \Phi$ as in the definition of the GP (1), we approximate that **the spatial path loss is equal to all transmitters** and depends on the distance between the SBS and its antenna boresight location on the earth's surface $\hat{d}_{h,\epsilon}$. The path loss law is defined as $\ell(d_x) \triangleq (d_x/d_0)^\gamma$, where $d_x$ is the Euclidean distance between the UE $x \in \mathcal{G}$ and the typical SBS, and $\gamma$ is the power path loss exponent. We define the SINR of the strongest signal $x_0 \triangleq \max\mathcal{G}$ at the typical SBS as

$$\text{SINR} \triangleq \frac{x_0/\ell(\hat{d}_{h,\epsilon})}{\sum\limits_{x\in\mathcal{G}\setminus\{x_0\}} x/\ell(\hat{d}_{h,\epsilon}) + W}. \tag{7}$$

## D. Laplace transform of the total received power

The Laplace function of the total of the points in $\mathcal{G}$ is defined as $S_{\mathcal{G}} \triangleq \sum_{x\in\mathcal{G}} x$, i.e., the *total received power* scaled by the spatial path loss $\ell(\hat{d}_{h,\epsilon})$ from all UEs in $\mathcal{G}$ at the typical SBS, is given by

$$\mathcal{L}_{S_{\mathcal{G}}}(s) \triangleq \mathbb{E}\left(e^{-s\sum\limits_{x\in\mathcal{G}} x}\right) = \exp\left\{-\int_0^\infty (1 - e^{-sr})\lambda_{\mathcal{G}}(r)dr\right\}. \tag{8}$$

For independent $\{\mathcal{G}_i\}_{i\in\mathcal{T}}$, $\mathcal{L}_{\sum_{i\in\mathcal{T}}\mathcal{G}_i}(s) = \prod_{i\in\mathcal{T}} \mathcal{L}_{\mathcal{G}_i}(s)$

*1) Rayleigh fading case:* We are ready to derive a approximation for the probability of coverage of the UE with the srongest signal.

## E. Gaussian mixture model

Let us consider a two-tier $\mathcal{T} = \{\text{LoS}, \text{NLoS}\}$ logarithmic Gaussian mixture fading model with the mean and standard deviation $\mu_i, \sigma_i, i \in \{\text{LoS}, \text{NLoS}\}$, for the line-of-sight and the non-line-of-sight tiers, respectively. Consequently, The power fading RV of a transmitter $x \in \Phi$ is given by

$$H_x \sim \begin{cases} \text{Lognormal}(\mu_{\text{LoS}}, \sigma^2_{\text{LoS}}), \text{ if } U < p_{\text{LoS}} \\ \text{Lognormal}(\mu_{\text{NLoS}}, \sigma^2_{\text{NLoS}}) \text{ if } U \geq p_{\text{NLoS}}, \end{cases} \tag{9}$$

where $U \sim U(0, 1)$ follows the uniform distribution.

The mean of the lognormal RV is given by $m_i = \mathbb{E}(H|T = i) = \exp\{\mu_i + \sigma_i^2/2\}$.

In the two-tier network $\mathcal{T} = \{\text{LoS}, \text{NLoS}\}$ with the lognormal fading variables (**??**) and (9), in the simple coverage region, the approximate probability of coverage is given by evaluating (16) with $m_{LoS} = \exp\{\mu_{\text{LoS}} + \sigma^2_{\text{LoS}}/2\}$, $m_{NLoS} = \exp\{\mu_{\text{NLoS}} + \sigma^2_{\text{NLoS}}/2\}$, $\kappa_{\text{LoS}} = p_{\text{LoS}}\kappa$ and $\kappa_{\text{NLoS}} = (1 - p_{\text{LoS}})\kappa$, where $\kappa$ is given in given in (4), for a density $\lambda$ of the entire netwrok.

## F. Approximation of the lognormal distribution

For a tradeoff between accuracy and analytical tractability, we approximate the mixed lognormal shadowing distribution with a mixture distribution consisting of an exponential distribution and an atomic probability measure at 0, where the parameters are chosen by matching the mean, and the variance of the total received power. Accordingly, the exponential shadowing approximation is defined by

$$\hat{H}_x \sim \begin{cases} 0, \text{ if } U < 1 - b, \\ \text{Exp}(a), \text{ if } U \geq 1 - b, \end{cases} \tag{10}$$

with $a > 0$ and $0 < b \leq 1$.

The complementary CDF (CCDF) is given by the defective distribution function

$$F_{\hat{H}_{\text{exp}}}(t) = e^{-at}b, \tag{11}$$

where

$$a = \frac{2(p_{\text{LoS}} - 1)e^{\mu_{NLoS} + \frac{\sigma_{NLoS}{}^2}{2}} - 2p_{\text{LoS}}e^{\mu_{\text{LoS}} + \frac{\sigma_{\text{LoS}}{}^2}{2}}}{(p_{\text{LoS}} - 1)e^{2(\mu_{\text{NLoS}} + \sigma_{NLoS}{}^2)} - p_{\text{LoS}}e^{2(\mu_{\text{LoS}} + \sigma_{\text{LoS}}{}^2)}}, \tag{12}$$

$$b = \frac{2\left((p_{\text{LoS}} - 1)e^{\mu_{\text{NLoS}} + \frac{\sigma_{\text{NLoS}}{}^2}{2}} - p_{\text{LoS}}e^{\mu_{\text{LoS}} + \frac{\sigma_{\text{LoS}}{}^2}{2}}\right)^2}{(p_{\text{LoS}} - 1)e^{2(\mu_{\text{NLoS}} + \sigma_{\text{NLoS}}{}^2)} - p_{\text{LoS}}e^{2(\mu_{\text{LoS}} + \sigma_{\text{LoS}}{}^2)}}. \tag{13}$$

At the probability $1 - b$, the signal power maps to 0: this reflects the severe shadowing occurrences when the signal strength in the lognormal shadowing model gets mapped very close to 0. Notice that the exponential approximation is applicable only if $0 < b \leq 1$; hence, it does not apply under general shadowing.

*G. Distribution of the total received power*

Under the shadowing approximation (11), the total received power can be simply characterized by the gamma distribution.

$$\mathcal{L}_{S_{\mathcal{G}}}(s) \approx \exp\left\{-\kappa \int_0^\infty (1-e^{-sr})\frac{F_{\hat{H}}(t)}{r}dr\right\}, \tag{14}$$

$$= \exp\left\{-\kappa \int_0^\infty (1-e^{-sr})\frac{e^{-ra}b}{r}dr\right\} = (1+s/a)^{-b\tilde{\kappa}}, \tag{15}$$

which is the Laplace transform of the gamma distribution with shape parameter $b\tilde{\kappa}$ and scale parameter $1/a$.

**Proposition 2.** *Assume a network of density $\lambda$ with the tiers $\mathcal{T}$. Let $H_{x_0}$ be the shadowing variable, and $x'_0 = H_{x_0}G(\|x_0\|) \in \mathcal{G} = \cup_{i\in\mathcal{T}}\mathcal{G}_i$ be the strongest signal of the UEs. In the simple coverage region $\theta \geq 1$,*

$$p_c(\theta) \approx \mathbb{P}\left(\frac{\hat{H}_{x_0}G(\|x_0\|)}{\sum\limits_{x'\in\mathcal{G}\setminus\{x'_0\}} x' + W\ell(\hat{d}_{h,\epsilon})} > \theta\right)$$

$$\stackrel{(a)}{=} \lambda \int_{\mathbb{R}^2} \mathbb{P}\left(\hat{H}_{x_0} > \theta 2^{\left(\frac{D_{h,\epsilon}\|y\|}{\varphi_{\text{RX}}}\right)^2} \cdot \left(\sum_{x\in\mathcal{G}_i} x + W\ell(\hat{d}_{h,\epsilon})\right)\right)dy$$

$$\stackrel{(b)}{\approx} 2\pi\lambda b \cdot \int_0^\infty r \prod_{k\in\mathcal{T}} \mathbb{E}\left(e^{-\frac{\theta a \sum_{x\in\mathcal{G}} x}{2^{-(D_{h,\epsilon}r/\varphi_{\text{RX}})^2}}} \cdot e^{-\theta\frac{W\ell(\hat{d}_{h,\epsilon})a}{2^{-(D_{h,\epsilon}r/\varphi_{\text{RX}})^2}}}\right)dr$$

$$\stackrel{(c)}{=} \tilde{\kappa}b \int_1^\infty \mathcal{L}_{S_{\mathcal{G}}}(\theta va)\,\mathcal{L}_W\left(\theta v W\ell(\hat{d}_{h,\epsilon})a\right)/v dv$$

$$= \tilde{\kappa}b \int_1^\infty \frac{\exp\{-\theta v W(\hat{d}_{h,\epsilon}/d_0)^\gamma a\}}{v\,(1+\theta v)^{\tilde{\kappa}b}}dv. \tag{16}$$

*Proof.* The detailed proof is presented in [CITE]. Step (a) heuristically: In the simple coverage region, the UE is covered *only if* it has the strongest signal. Using the translation invariance of the stationary PPP and utilizing Slivnyak's and Gambell's theorems leads to the probability of coverage given by the integral over the plane. In (b), we assume an exponential $(H_{x_0}|T_{x_0} = i)$ to approximate the probability of coverage, which allows the probability to be expressed in terms of the Laplace transforms of $S_{\mathcal{G}_i} = \sum_{x\in\cup\mathcal{G}_i} x$ and $W$. By the independent shadowing, they are independent and can be expressed as the product. In (c), we made the substitutions $D_{h,\epsilon}r/\varphi_{\text{RX}} \mapsto u$, and further, $2^{u^2} \mapsto v$. □

Notice that $p_c(\theta)$ is the probability of coverage of the UE with the *strongest signal*, which is *not* necessarily the spatially *nearest transmitter*.

In an interference-only channel, $W = 0$, the coverage probability (16) has the analytical expression

$$p_c(\theta) = \theta^{-\tilde{\kappa}b}{}_2F_1\left(\tilde{\kappa}b, \tilde{\kappa}b; \tilde{\kappa}b+1; -1/\theta\right), \tag{17}$$

where ${}_2F_1(\cdot)$ is the hypergeometric function. The coverage probability does not depend on the scale parameter $a$. However, the probability $b$ that the signal strength of the UEs is more than $0$ has an effect on the coverage probability in a decreasing sense w.r.t. the $b$: for $b \to 0$, $p_c(\theta) = 1$; hence, small $p_{\text{LoS}}$ causing severe shadowing (which is the case with the low elevation angles) is beneficial for the coverage. Directly seen from (16), the same insight holds in the interference-plus-noise-limited channel $W > 0$.

**Proposition 3.** *Let $\mu_M \triangleq \sum_{i=1}^n t'_i/G(D_{h,\epsilon}\|x_i\|)$ and $T'_n \triangleq T_n(t_1,\cdots,t_n) = 1/(1-\sum_{i=1}^n t'_i)$. The $n$-moment measure of the STINR process is given by*

$$M'^{(n)}(t'_1,\ldots,t'_n) = (2\pi b)^n$$
$$\times \int\limits_{(\mathbb{R}_+)^n} \mathcal{L}_W(a\mu_M T_n)\mathcal{L}_I(a\mu_M T_n)\mathcal{L}_D(a\mu_M T_n)\prod_{i=1}^n r_i dr_i$$
$$= (2\pi b)^n \int\limits_{(\mathbb{R}_+)^n} \exp\left\{-aW\ell(\hat{d}_{h,\epsilon})T_n\sum_{i=1}^n t'_i 2^{\left(\frac{D_{h,\epsilon}r_i}{\varphi_{RX}}\right)^2}\right\}$$
$$\times \left(1+T_n\sum_{i=1}^n t'_i 2^{\left(\frac{D_{h,\epsilon}r_i}{\varphi_{RX}}\right)^2}\right)^{-\tilde{\kappa}b} \frac{\prod\limits_{i=1}^n t'_i 2^{\left(\frac{D_{h,\epsilon}r_i}{\varphi_{RX}}\right)^2}}{\sum\limits_{i=1}^n t'_i 2^{\left(\frac{D_{h,\epsilon}r_i}{\varphi_{RX}}\right)^2}}$$
$$\times \left[\sum_{j=1}^n \frac{1}{\prod\limits_{i\neq j}\left(t'_i 2^{\left(\frac{D_{h,\epsilon}r_i}{\varphi_{RX}}\right)^2} + a t'_i T_n \sum\limits_{k=1}^n t'_k 2^{\left(\frac{D_{h,\epsilon}r_k}{\varphi_{RX}}\right)^2}\right)}\right]$$
$$\times r_1 dr_1 \ldots r_n dr_n. \tag{18}$$

*With the successive substitutions $\{u_i\}_{i=1}^n = \{(r_i D_{h,\epsilon}/\varphi_{RX})^2\}_{i=1}^n$ and $\{v_i\}_{i=1}^n = \{2^{u_i^2}\}_{i=1}^n$,*

$$M'^{(n)}(t'_1,\ldots,t'_n) = \left(\frac{\tilde{\kappa}b}{\lambda}\right)^n$$
$$\times \int\limits_{(1,\infty)^n} \exp\left\{-aW\ell(\hat{d}_{h,\epsilon})T_n\sum_{i=1}^n t'_i v_i\right\}\left(1+T_n\sum_{i=1}^n t'_i v_i\right)^{-\tilde{\kappa}b}$$
$$\times \frac{\prod\limits_{i=1}^n t'_i}{\sum\limits_{i=1}^n t'_i v_i}\left[\sum_{j=1}^n \frac{1}{\prod\limits_{i\neq j}\left(t'_i v_i + a t'_i T_n \sum\limits_{k=1}^n t'_k v_k\right)}\right]\prod_{i=1}^n dv_i. \tag{19}$$

*1) Average SIR:* The average SIR conditioning that $\theta \geq 1$ is given by

$$\mathbb{E}(\text{SIR}) = \int_1^\infty \frac{p_c(y)}{p_c(1)}dy = \frac{{}_2F_1(\tilde{\kappa}b-1, \tilde{\kappa}b; \tilde{\kappa}b+1; -1)}{p_c(1)(\tilde{\kappa}b-1)}, \tag{20}$$

which goes to $\infty$ as $\tilde{\kappa}b \to 1+$. For $\tilde{\kappa}b < 1$, the expectation is undefined.

## III. Numerical results