Practical no ~ 1

limits & Continuity

1) $\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$

$\Rightarrow \lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \times \frac{\sqrt{a+2x} + \sqrt{3x}}{\sqrt{a+2x} + \sqrt{3x}} \times \frac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{3a+x} + 2\sqrt{x}} \right]$

$\lim_{x \to a} \frac{(a+2x-3x)}{(3a+x-4x)} \frac{(\sqrt{3a+x} + 2\sqrt{x})}{(\sqrt{a+2x} + \sqrt{3x})}$

$\lim_{x \to a} \frac{(a-x)(\sqrt{3a-x} + 2\sqrt{x})}{(3a-3x)(\sqrt{a+2x} + \sqrt{3x})}$

$\frac{1}{3} \lim_{x \to a} \frac{(a-x)}{(a-x)} \frac{(\sqrt{3a+x} + 2\sqrt{x})}{(\sqrt{a+2x} + 3\sqrt{x})}$

$\frac{1}{3} \frac{\sqrt{3a+a} + 2\sqrt{a}}{\sqrt{a+2a} \quad 3\sqrt{a}}$

$\frac{1}{3} \frac{\sqrt{4a} + 2\sqrt{a}}{\sqrt{3a} + \sqrt{3a}}$

$\frac{1}{3} \frac{2\sqrt{a} + 2\sqrt{a}}{2\sqrt{3a}} \Rightarrow \frac{2}{3\sqrt{3}}$

② $\lim_{y\to 0}\left[\frac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\right]$

$\lim_{y\to 0}\left[\frac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}} \times \frac{\sqrt{a+y}+\sqrt{a}}{\sqrt{a+y}+\sqrt{a}}\right]$

$\lim_{y\to 0}\left[\frac{(\sqrt{a+y})^2-(\sqrt{a})^2}{y\sqrt{a+y}\,(\sqrt{a+y}+\sqrt{a})}\right]$

$\lim_{y\to 0} \frac{a+y-a}{y\sqrt{a+y}\,(\sqrt{a+y}+\sqrt{a})}$

$\lim_{y\to 0} \frac{y}{y\sqrt{a+y}\,(\sqrt{a+y}+\sqrt{a})}$

$\frac{1}{\sqrt{a+0}\,(\sqrt{a+0}+\sqrt{a})}$

$\frac{1}{\sqrt{a}(\sqrt{a}+\sqrt{a})}$

$\frac{1}{\sqrt{a}(2\sqrt{a})} = \frac{1}{2a}$

③ $\lim_{x\to\frac{\pi}{6}} \frac{\cos x-\sqrt{3}\sin x}{\pi-6x}$

$\Rightarrow$ Substituting $x-\pi/6 = h$ $\quad\therefore x=\pi/6+h$

$\pi-6x = -6h$ $\quad$ where $h\to 0$

$\therefore \lim_{h\to 0} \frac{\cos\left(\frac{\pi}{6}+h\right) - \sqrt{3}\sin\left(\frac{\pi}{6}+h\right)}{-6h}$

$\therefore \lim_{h\to 0} \frac{\cos\frac{\pi}{6}\cdot\cos h - \sin\frac{\pi}{6}\cdot\sin h - \sqrt{3}\left(\sin\frac{\pi}{6}\cdot\cos h + \cos\frac{\pi}{6}\cdot\sin h\right)}{-6h}$

[$\because$ $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$]

$\therefore \lim_{h\to 0} \frac{\frac{\sqrt{3}}{2}\cos - \frac{1}{2}\sin h - \sqrt{3}\left(\frac{1}{2}\cos h + \frac{\sqrt{3}}{2}\sin h\right)}{-6h}$

$\therefore \lim_{h\to 0} \frac{\frac{\sqrt{3}}{2}\cos h - \frac{1}{2}\sin h - \frac{\sqrt{3}}{2}\cos h - \frac{3}{2}\sin h}{-6h}$

$\therefore \lim \frac{-\frac{4}{2}\sin h}{-6h} \Rightarrow \lim_{h\to 0} \frac{-2\sin h}{-6h}$

$\frac{1}{3}\lim_{h\to 0}\frac{\sin h}{h}$

$1/3$

$$\lim_{x\to\pi/2} f(x) = \lim_{x\to\pi/2} \frac{\cos x}{\pi - 2x}$$

Put $x - \frac{\pi}{2} = h$ $\quad x = \frac{\pi}{2} + h$

as $x \to \frac{\pi}{2}$, $h \to 0$ $\quad \pi - 2x = -2h$

$$\lim_{x\to\pi/2^+} f(x) = \lim_{h\to 0} \frac{\cos(\pi/2 + h)}{-2h}$$

$$= \lim_{h\to 0} \frac{-\sin h}{-2h}$$

$$\frac{1}{2} \lim_{h\to 0} \frac{\sin h}{h}$$

$$\frac{1}{2}(1) = \frac{1}{2}$$

$\therefore \lim_{x\to\pi/2} f(x) \ne \lim_{x\to\pi/2^+} \ne f\left(\frac{\pi}{2}\right)$, f is discontinuous at $x = \frac{\pi}{2}$

6] $\lim_{x\to\pi/2} f(x) = \lim_{x\to\pi/2} \frac{\sin 2x}{\sqrt{1-\cos 2x}}$ Using $\sin 2x = 2\sin x \cdot \cos x$

$$\lim_{x\to\pi/2} \frac{2\sin x\cdot\cos x}{\sqrt{2\sin^2 x}} \Rightarrow \lim_{x\to\pi/2} \frac{2\cancel{\sin x}\cdot\cos x}{\sqrt{2}\,\cancel{\sin x}}$$

$$\lim_{x\to\pi/2} \frac{2\cos x}{\sqrt{2}}$$

$$\frac{2}{\sqrt{2}} \lim_{x\to\pi/2} \cos x$$

$\therefore$ f is not continuos at $x = \pi/4$

5] ii $f(x) = \frac{x^2-9}{x-3}$ $\quad 0 < x < 3$

$= x+3$ $\quad 3 \le x \le 6$ $\quad$ at $x = 3$ & $x = 6$

$= \frac{x^2-9}{x+3}$ $\quad 6 < x < 9$

at $x = 3$

i) $f(3) = \frac{x^2-9}{x-3} = 6$

f at $x = 3$ define

ii) $\lim_{x\to 3^+} f(x)$ $\quad \lim_{x\to 3^+} x+3$

$f(3) = x+3 = 3+3 = 6$

f is define at $x=3$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x+3) = 6$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^2-9}{x-3}$ $\quad \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$

f is continuous at $x = 3$

for $x = 6$

$f(6) = \frac{x^2-9}{x+3} = \frac{36-9}{6+3} = \frac{\cancel{27}^{3}}{\cancel{9}} = 3$

5) $\lim_{x \to 6^+} \frac{x^2-9}{x+3}$

$\lim_{x \to 6^+} \frac{(x-3)(x+3)}{(x+3)}$

$\lim_{x \to 6^+} (x-3) \Rightarrow 6-3 = 3$

$\lim_{x \to 6^-} x+3 = 6+3 = 9$

$\therefore L.H.L \neq R.H.L$

function is not continuous

6 i) $f(x) = \frac{1-\cos 4x}{x^2}$ $\left.\begin{matrix} x \neq 0 \\ x = 0 \end{matrix}\right\}$ at $x = 0$

Soln f is continuous at $x=0$

$\lim_{x \to 0} f(x) = f(0)$

$\lim_{x \to 0} \frac{1-\cos 4x}{x^2} = K$

$2 \lim_{x \to 0} \frac{\sin^2 2x}{x^2} = K$

$2 \lim \left(\frac{\sin 2x}{x}\right)^2 = K$

$2(2)^2 = K$

$\therefore K = 8$

ii) $f(x) = (\sec^2 x)^{\cot^2 x}$ $\left.\begin{matrix} x \neq 0 \\ x = 0 \end{matrix}\right\}$ at $x=0$

$= K$

Soln $f(x) = (\sec^2 x)^{\cot^2 x}$

Using

$\sec^2 x - \tan^2 x = \sec^2 x = 1$

$\therefore \sec^2 x = 1 + \tan^2 x$

& $\cot^2 x = \frac{1}{\tan^2 x}$

$\therefore \lim_{x \to 0} (\sec^2 x)^{\cot^2 x}$

$\lim_{x \to 0} (1 + \tan^2 x)^{\frac{1}{\tan^2 x}}$

We, know that

$\lim_{x \to 0} (1+px)^{\frac{1}{px}} = e$

$\therefore = e$

$\therefore K = e$

iii) $f(x) = \frac{\sqrt{3} - \tan x}{\pi - 3x}$ $\left.\begin{matrix} x \neq \pi/3 \\ x = \pi/3 \end{matrix}\right\}$ at $x = \pi/3$

$= K$

$x - \frac{\pi}{3} = h$

$x = h + \frac{\pi}{3}$

When $h \to 0$

$f\left(\frac{\pi}{3} + h\right) = \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$

$\lim_{h \to 0} \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$

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Using $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

$$\frac{\dfrac{\sqrt{3} - \tan\frac{\pi}{3} + \tan h}{1 - \tan\frac{\pi}{3} \cdot \tan h}}{\pi - \pi - 3h}$$

$$\lim_{h\to 0} \frac{\dfrac{\sqrt{3}\left(1 - \tan\frac{\pi}{3} \cdot \tan h\right) - \left(\tan\frac{\pi}{3} + \tan h\right)}{1 - \tan\frac{\pi}{3} \cdot \tan h}}{-3h}$$

$\tan \pi/3 = \tan 60 = \sqrt{3}$

$$\lim_{h\to 0} \frac{\dfrac{(\sqrt{3} - \sqrt{3} \times \sqrt{3} \cdot \tan h) - (\sqrt{3} + \tan h)}{1 - \tan \pi/3 \cdot \tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{\dfrac{(\sqrt{3} - 3 \cdot \tan h) - (\sqrt{3} + \tan h)}{1 - \sqrt{3} \cdot \tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{\dfrac{(\sqrt{3} - 3\tan h - \sqrt{3} - \tan h)}{1 - \sqrt{3}\tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{-4\tan h}{-3h(1 - \sqrt{3}\tan h)}$$

$$\lim_{h\to 0} \frac{4\tan h}{3h(1 - \sqrt{3}\tan h)}$$

$$\frac{4}{3} \lim_{h\to 0} \frac{\tan h}{h} \quad \lim_{h\to 0} \frac{1}{(1 - \sqrt{3}\tan h)} \quad \frac{\tan h}{h} = 1$$

$$= \frac{4}{3} \cdot \frac{1}{(1 - \sqrt{3}(0))}$$

$$= \frac{4}{3}(1) = \frac{4}{3}$$

Practical - 02

Derivative

1) Show that the following function defined from IR to IR are differentiable

1) Cot x

$f(x) = \cot x$

$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$= \lim_{x \to a} \frac{\cot x - \cot a}{x - a}$

$= \lim_{x \to a} \frac{\frac{1}{\tan x} - \frac{1}{\tan a}}{x - a}$

$= \lim_{x \to 0} \frac{\tan a - \tan x}{(x-a) \tan x \tan a}$

put $x - a = h$

$x = a + h$

as $x \to a$, $h \to 0$

$Df(h) = \lim_{h \to 0} \frac{\tan a - \tan (a+h)}{(a+h-a) \tan (a+h) \tan a}$

$= \lim_{h \to 0} \frac{\tan a - \tan (a+h)}{h \times \tan (a+h) \tan a}$

Formula: $\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$$\lim_{h\to 0} \frac{-\tan h}{h} \times \frac{1+\tan\tan(a+h)}{\tan(a+h)\tan a}$$

$$= -1 \times \frac{1+\tan^2 a}{\tan^2 a}$$

$$= \frac{-\sec^2 a}{\tan^2 a}$$

$$= \frac{-1}{\cancel{\cos^2 a}} \times \frac{\cancel{\cos^2 a}}{\sin^2 a}$$

$$= -\operatorname{cosec}^2 a$$

$\therefore Df(a) = -\cos^2 a$

$\therefore f$ is differentiable $\forall a \in R$

ii) Cosec x

$f(x) = \operatorname{cosec} x$

$$Df(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{\operatorname{cosec} x - \operatorname{cosec} a}{x-a}$$

$$= \lim_{x\to a} \frac{\frac{1}{\sin x} - \frac{1}{\sin a}}{x-a}$$

$$= \lim_{x\to a} \frac{\sin a - \sin x}{(x-a)\sin a \cdot \sin x}$$

put : $x - a = h$

$x = a+h$

as $x \Rightarrow a,\ h \to 0$

$$Df(n) = \lim_{h\to 0} \frac{\sin a - \sin(a+h)}{(a+h-a)\sin a \sin(a+h)}$$

formula :

$$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{a+a+h}{2}\right)\sin\left(\frac{a-a-h}{2}\right)}{h \times \sin a \sin(a+h)}$$

$$= \lim - \frac{\sin\frac{h}{2}}{h/2}\ \frac{1}{2} \times \frac{2\cos\left(\frac{2a+h}{2}\right)}{\sin a \sin(a+h)}$$

$$= -\frac{1}{2} \times \frac{2\cos\left(\frac{2a+0}{2}\right)}{\sin(a+0)} \Rightarrow \frac{-\cos a}{\sin^2 a}$$

$$= -\cot a \operatorname{cosec} a$$

iii) Sec x

$f(x) = \sec x$

$$Df(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{\sec x - \sec a}{x-a}$$

$$= \lim_{x\to a} \frac{\frac{1}{\cos x} - \frac{1}{\cos a}}{x-a} \Rightarrow \lim_{x\to a} \frac{\cos a - \cos x}{(x-a)\cos a \cos x}$$

Put $x - a = h$

$x = a+h$

as $x \to a,\ h \to 0$

$$Df(n) = \lim_{h\to 0} \frac{\cos a - \cos(a+h)}{h \times \cos a \cos(a+h)}$$

Formula : $-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$$= \lim_{h\to 0} \frac{-2\sin\left(\frac{a+a+h}{2}\right)\sin\left(\frac{a-a-h}{2}\right)}{h \times \cos a \cos(a+h)}$$

$$= \lim_{h\to 0} \frac{-2\sin\left(\frac{2a+h}{2}\right)\sin -\frac{h}{2}}{\cos a \cos(a+h) \times -\frac{h}{2}} \times \frac{-1}{2}$$

$$= \frac{\frac{-1}{2} \times -2\sin\left(\frac{2a+0}{2}\right)}{\cos a \cos(a+0)}$$

$$= -\frac{1}{2} \times 2 \frac{\sin a}{\cos a \times \cos a} \Rightarrow \tan a \sec a$$

Q2] If $f(x) = 4x+1 \quad , x \le 2$

$= x^2+5 \quad x > 0$, at $x=2$, then

find function is diffentiable at $x=2$, then

Soln

L.H.D :

$$Df(2^-) = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2^-} \frac{4x+1-(4\times 2+1)}{x-2} \Rightarrow \lim_{x\to 2^-} \frac{4x+1-9}{x-2}$$

$$= \lim_{x\to 2^-} \frac{4\cancel{(x-2)}}{\cancel{(x-2)}} \Rightarrow 4$$

$Df(2^-) = 4$

RHD :

$$Df(2^+) = \lim_{x\to 2^+} \frac{x^2+5-9}{x-2}$$

$$= \lim_{x\to 2^+} \frac{x^2-4}{x-2}$$

$$= \lim_{x\to 2^+} \frac{(x+2)(x-2)}{x-2}$$

$= 2+2 \Rightarrow 4$

$Df(2^+) = 4$

RHD = LHD

f is differentiable at $x=2$

Q3) If $f(x) = 4x+7 \quad , x < 3$

$= x^2+3x+1 \quad , x \ge 3$ at $x=3$, then

find f is differentiable or not ?

Soln -

RHD :

$$Df(3^+) = \lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3}$$

$$= \lim_{x\to 3^+} \frac{x^2+3x+1-(3^2+3+3+1)}{x-3}$$

$$= \lim_{x\to 3^+} \frac{x^2+3x-18}{x-3}$$

$$= \lim_{x\to 3^+} x^2+6x-3x-18$$

$$= \lim_{x\to 3^+} \frac{x(x+6)-3(x+6)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{(x+6)(x-3)}{(x-3)} = 3+6 \Rightarrow 9$$

$Df(3^+) = 9$

LHD $= Df(3^-)$

$$= \lim_{x \to 3^-} \frac{f(x) - f(3)}{x-3}$$

$$= \lim_{x \to 3^-} \frac{4x + 7 - 19}{x-3}$$

$$= \lim_{x \to 3^+} \frac{4x - 12}{x-3}$$

$$= \lim_{x \to 3} \frac{4(x-3)}{(x-3)}$$

$Df(3^+) = 4$

RHD = LHD

f is not differentiable at $x = 3$

Q2) Is $f(x) = 8x - 5$, $x \le 2$

$= 3x^2 - 4x + 7$, $x > 2$ at $x = 2$, then find f is differentiable or not

Soln

$f(2) = 8 \times 2 - 5 = 16 - 5 = 11$

RHD :

$$Df(2^+) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 4x + 7 - 11}{x-2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 6x + 2x - 4}{x-2}$$

$$= \lim_{x \to 2^+} \frac{3x(x-2) + 2(x-2)}{x-2} \Rightarrow \lim_{x \to 2^+} \frac{(3x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$= 3 \times 2 + 2 = 8$

$Df(2^+) = 8$

L.H.D :

$$Df(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{8x - 5 - 11}{x-2}$$

$$= \lim \frac{8x - 16}{x-2} \Rightarrow \lim_{x \to 2^-} \frac{8\cancel{(x-2)}}{\cancel{(x-2)}}$$

$= 8$

$Df(2^-) = 8$

LHD = RHD

f is differentiable at $x = 3$

AK
06/02/2020

Practical-3

Topic : Application of Derivative

I Find the intervals in which function is increasing or decreasing

(i) $f(x) = x^2 - 5x - 11$
(ii) $f(x) = x^2 - 5x$
(iii) $f(x) = 2x^2 + x^2 - 20x + 5$
(iv) $f(x) = x^3 - 27x + 5$
(v) $f(x) = 69 - 24x - 9x^2 + 2x^3$

II Find the intervals in which function is concave upwards

i) $Y = 3x^2 - 2x^3$
ii) $Y = x^4 - 6x^3 + 12x^2 + 5x + 7$
iii) $Y = x^3 - 27x + 5$
iv) $Y = 69 - 24x - 9x^2 + 2x^3$
v) $Y = 2x^3 + x^2 - 20x + 5$

Soln :

Q.1) $f(x) = x^3 - 5x - 11$

$f'(x) = 3x^2 - 5$

$\because f$ is increasing iff $f(x) > 0$

$3x^2 - 5 > 0$

$3(x^2 - 5/3) > 0$

$(x - \sqrt{5/3})(x + \sqrt{5/3}) > 0$

$+ \quad - \quad +$

$-\sqrt{5}/3 \qquad \sqrt{5}/3$

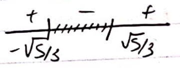
$x \in (-\infty, -\sqrt{5/3}) \cup (\sqrt{5}/3, \infty)$

and f is decreasing iff $f'(x) < 0$

$\therefore 3x^2 - 5 < 0$

$\therefore 3(x^2 - 5/3) < 0$

$\therefore (x - \sqrt{5/3})(x + \sqrt{5/3}) < 0$



$x \in (\sqrt{5/3}, \sqrt{5/3})$

2) $f(x) = x^2 - 5x$

$f'(x) = 2x - 5$

$\therefore f(x)$ is increasing iff $f'(x) > 0$

$\therefore 2x - 5 > 0$

$\therefore 2(x - 2) > 0$

$\therefore x - 2 > 0$

$x \in (2, \infty)$

and f is decreasing iff $f'(x) < 0$

$\therefore 2x - 5 < 0$

$\therefore 2(x-2) < 0$

$x - 2 < 0$

$x \in (-\infty, 2)$

3) $f(x) = 2x^3 + x^2 - 20x + 5$

$f'(x) = 6x^2 + 2x - 20$

$\therefore f$ is increasing iff $f'(x) > 0$

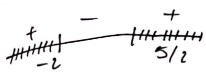
$\therefore 6x^2 + 2x - 20 > 0$

$\therefore 2(3x^2 + x - 10) > 0$

$\therefore 3x^2 + x - 10 > 0$

$\therefore 3x^2 + 6x - 5x - 10 > 0$

$\therefore (x+2)(3x-5) > 0$



$x \in (-\infty, -2) \cup (5/3, \infty)$

and f is decreasing iff $f'(x) < 0$

$\therefore 6x^2 + 2x - 20 < 0$

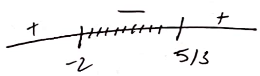
$\therefore 2(3x^2 + x - 10) < 0$

$\therefore 3x^2 + x - 10 < 0$

$\therefore 3x^2 + 6x - 5x - 10 < 0$

$\therefore x(x+2) - 5(x+2) < 0$

$\therefore (x+2)(3x-5) < 0$



$x \in (-2, 5/3)$

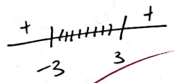
$\therefore x \in (-\infty, -3) \cup (3, \infty)$

and f is decreasing iff $f'(x) < 0$

$\therefore 3x^2 - 27 < 0$

$\therefore 3(x^2 - 9) < 0$

$\therefore (x-3)(x+3) < 0$



$\therefore x \in (-3, 3)$

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5) $f(x) = 2x^3 - 9x^2 - 24x + 64$

$f'(x) = 6x^2 - 18x - 24$

$\because$ f is increased iff $f'(x) > 0$

$\therefore 6x^2 - 18x - 24 > 0$

$\therefore 6(x^2 - 3x - 4) > 0$

$\therefore x^2 - 4x + x - 4 > 0$

$\therefore x(x-4) + 1(x-4) > 0$

$(x-4)(x+1) > 0$

$+ \quad - \quad +$

$-1 \quad 4$

$\therefore x \in (-\infty, -1) \cup (4, \infty)$

and f is decreasing iff $f'(x) < 0$

$\therefore 6x^2 - 18x - 24 < 0$

$\therefore 6(x^2 - 3x - 4) < 0$

$\therefore x^2 - 4x + x - 4 < 0$

$\therefore x(x-4) + 1(x-4) < 0$

$\therefore (x-4)(x+1) < 0$

$+ \quad - \quad +$

$-1 \quad 4$

$\therefore x \in (-1, 4)$

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Q2) 1) $Y = 3x^2 - 2x^3$

$\therefore f(x) = 3x^2 - 2x^3$

$\therefore f'(x) = 6x - 6x^2$

$\therefore f''(x) = 6 - 12x$

f is concave upward if $f''(x) > 0$

$\therefore (6 - 12x) > 0$

$\therefore 12(6/12 - x) > 0$

$x - 1/2 > 0$

$x > 1/2$

$\therefore f''(x) > 0$

$\therefore x \in (1/2, \infty)$

2) $Y = x^4 - 6x^3 + 12x^2 + 5x + 7$

$f'(x) = 4x^3 - 18x^2 + 24x + 5$

$f''(x) = 12x^2 - 36x + 24$

f is concave upward if $f''(x) > 0$

$\therefore 12x^2 - 36x + 24 > 0$

$\therefore 12(x^2 - 3x + 2) > 0$

$\therefore x^2 - 2x - x + 2 > 0$

$\therefore x(x-2) - 1(x-2) > 0$

$\therefore (x-2)(x-1) > 0$

$+ \quad - \quad +$

$1 \quad 2$

$x \in (-\infty, 1) \cup (2, \infty)$

3) $y = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

$f''(x) = 6x$

f is concave upword iff $f''(x) > 0$

$\therefore 6x > 0$

$\therefore x > 0$

$\therefore x \in (0, \infty)$

4) $y = 64 - 24x - 9x^2 + 2x^3$

$f(x) = 2x^3 - 9x^2 - 24x + 64$

$f'(x) = 6x^2 - 18x - 24$

$f''(x) = 12x - 18$

f is concave upword iff $f(x) > 0$

$\therefore 12x - 18 > 0$

$\therefore 12(x - 18/12) > 0$

$\therefore x - 3/2 \quad -x > 3/2 \quad \therefore x \in (3/2), \infty)$

5) $y = 2x^3 + x^2 - 20x + 4$

$f(x) = 2x^3 + x^2 - 20x + 4$

$f'(x) = 6x^2 + 2x - 20$

$f''(x) = 12x + 2$

f is concave upword iff $f(x) > 0$

$\therefore f''(x) > 0$

$\therefore 12x + 2 > 0$

$\therefore 12(x + 2/12) > 0$

$\therefore x + 1/6 > 0$

$\therefore x < -1/6$

$\because f''(x) \not> 0$

$\therefore$ There exist no interval

Practical ~ 4

Q.1]

(i) $f(x) = x^2 + \frac{16}{x^2}$

$f'(x) = 2x - \frac{32}{x^3}$

Now consider,

$f'(x) = 0$

$\therefore \; 2x - \frac{32}{x^3} = 0$

$2x - \frac{32}{x^3} = 0$

$2x = \frac{32}{x^3}$

$x^4 = \frac{32}{2} \Rightarrow x^4 = 16 \Rightarrow x = \pm 2$

$f''(x) = 2 + \frac{96}{x^4} \Rightarrow f''(2) = 2 + \frac{96}{2^4}$

$= 2 + \frac{96}{16} \Rightarrow 2 + 6 \Rightarrow 8 > 0$

$\therefore$ f has minimom value at $x = 2$

$\therefore \; f(2) = 2^2 + \frac{16}{2^2} \Rightarrow 4 + \frac{16}{4} \Rightarrow 8$

$f''(-2) = 2 + \frac{96}{-2^4} \Rightarrow 2 + \frac{96}{16}$

$= 2 + 6 \Rightarrow 8 > 0$

$\therefore$ f has minimom value at $x = -2$

$\therefore$ Function reaches minimom value at $x = 2$ and $x = -2$

ii) $f(x) = 3 - 5x^3 + 3x^5$

$\therefore f'(x) = -15x^2 + 15x^4$

Consider,

$f'(x) = 0$

$\therefore -15x^2 + 15x^4 = 0$

$15x^4 = 15x^2$

$x^2 = 1$

$x = \pm 1$

$\therefore f''(x) = -30x + 60x^3$

$f''(1) = -30 + 60$

$= 30 > 0$

$\therefore$ f has minimum value at $x = 1$

$\therefore f(1) = 3 - 5(1)^3 + 3(1)^5 \Rightarrow 6 - 5 \Rightarrow 1$

$\therefore f''(-1) = -30(-1) + 60(-1)^3$

$= 30 - (60)$

$= -30 < 0$

$\therefore$ f has maximum value at $x = -1$

$\therefore f(-1) = 3 - 5(-1)^3 + 3(-1)^5$

$= 3 + 5 - 3 \Rightarrow 5$

$\therefore$ f has the maximum value 5 at $x = -1$ and has the minimum value 1 at $x = 1$

iii) $f(x) = x^3 - 3x^2 + 1$

$\therefore f'(x) = 3x^2 - 6x$

consider,

$f'(x) = 0$

$\therefore 3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0$

$\therefore 3x = 0$ or $x - 2 = 0$

$x = 0$ or $x = 2$

$f''(x) = 6x - 6$

$f''(0) = 6(0) - 6$

$= -6 < 0$

$\therefore$ f has maximum value at $x = 0$

$\therefore f(0) = (0)^3 - 3(0)^2 + 1$

$= 1$

$\therefore f''(2) = 6(2) - 6$

$= 12 - 6$

$= 6 > 0$

$\therefore$ f has minimum value at $x = 2$

$f(2) = (2)^3 - 3(2)^2 + 1$

$= 8 - 3(4) + 1 \Rightarrow 9 - 12$

$= -3$

$\therefore f$ has maximum value 1 at $x = 0$ and f has minimum value -3 at $x = 2$.

iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$

$f'(x) = 6x^2 - 6x - 12$

consider,

$f'(x) = 0$

$\therefore 6x^2 - 6x - 12 = 0$

$6(x^2 - x - 2) = 0 \quad x^2 - x - 2 = 0$

$x^2 - x - 2 = 0 \Rightarrow x^2 + x - 2x - 2 = 0$

$x(x+1) - 2(x+1) = 0$

$(x-2)(x+1) = 0$

$\therefore x = 2$ or $x = -1$

$\therefore f''(x) = 12x - 6$

$\therefore f''(2) = 12(2) - 6$

$= 24 - 6 \Rightarrow 18 > 0$

$\therefore f$ has minimum value at $x = 2$

$\therefore f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 1$

$= 2(8) - 3(4) - 24 + 1$

$= 16 - 12 - 24 + 1$

$= -19$

$f''(-1) = 12(-1) - 6$

$= -12 - 6$

$= -18 < 0$

$\therefore f$ has maximum value at $x = -1$

$\therefore f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1$

$= -2 - 3 + 12 + 1 \Rightarrow 8$

$\therefore f$ has maximum value 8 at $x = -1$ and

f has minimum value -19 at $x = 2$

Q2 1) $f(x) = x^3 - 3x^2 - 55x + 9.5 \qquad x_0 = 0 \rightarrow$ Given

$f'(x) = 3x^2 - 6x - 55$

By Newton's Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \Rightarrow x_1 = 0 + \frac{9.5}{55}$

$\therefore x_1 = 0.1727$

$\therefore f(x_1) = (0.1727)^3 - 3(0.1727)^2 - 55(0.1727) + 9.5$

$= 0.0051 - 0.0895 - 9.4985 + 9.5$

$= -0.0829$

$$f'(x_1) = 3(0.1727)^2 - 6(0.1727) - 55$$
$$= 0.0895 - 1.0362 - 55$$
$$= -55.9467$$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 0.1727 - \frac{0.0829}{55.9467}$$
$$= 0.1712$$

$$f(x_2) = (0.1712)^3 - 3(0.1712)^2 - 55(0.1712) + 9.5$$
$$= 0.0050 - 0.0879 - 9.416 + 9.5$$
$$= 0.0011$$

$$f'(x_2) = 3(0.1712)^2 - 6(0.1712) - 55$$
$$= 0.0879 - 1.0272 - 55$$
$$= -55.9393$$

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 0.1712 + \frac{0.0011}{55.9393} \Rightarrow 0.1712$$

$\therefore$ The root of the equation is 0.1712.

ii) $f(x) = x^3 - 4x - 9$

$f(x) = 3x^2 - 4$

$$f(2) = 2^3 - 4(2) - 9$$
$$= 8 - 8 - 9 \Rightarrow -9$$

$$f(3) = 3^3 - 4(3) - 9 \Rightarrow 27 - 12 - 9$$
$$= 6$$

Let $x_0 = 3$ be the initial approximation,

$\therefore$ By Newton's Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \Rightarrow 3 - \frac{6}{23} \Rightarrow 2.7392$$

$$f(x_1) = (2.7392)^3 - 4(2.7392) - 9$$
$$= 20.5528 - 10.9568 - 9$$
$$= 0.596$$

$$f'(x_1) = 3(2.7392)^2 - 4$$
$$= 22.5096 - 4$$
$$= 18.5096$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \Rightarrow 2.7392 - \frac{0.596}{18.5096}$$
$$= 2.7071$$

$$f(x_2) = (2.7071)^3 - 4(2.7071) - 9$$
$$= 19.8386 - 10.8284 - 9$$
$$= 0.0102$$

$$f'(x_2) = 3(2.7071)^2 - 4$$
$$= 21.9851 - 4 \Rightarrow 17.9851$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$= 2.7091 - \frac{0.0102}{17.9851}$

$= 2.7071 - 0.0056$

$= 2.7015$

$f(x_3) = (2.7015)^3 - 4(2.7015) - 9$

$= 19.7158 - 10.806 - 9$

$= -0.0901$

$f'(x_3) = 3(2.7015)^2 - 4$

$= 21.8943 - 4$

$= 17.8943$

$x_4 = 2.7015 + \frac{0.0901}{17.8943}$

$= 2.7015 + 0.0050$

$= 2.7065$

iii) $f(x) = x^3 - 1.8x^2 - 10x + 17$

$f'(x) = 3x^2 - 3.6x - 10$

$f(1) = (1)^3 - 1.8(1)^2 - 10(1) + 17$

$= 1.8 - 10 + 17$

$= 6.2$

$f(2) = (2)^3 - 1.8(2)^2 - 10(2) + 17$

$= 8 - 7.2 - 20 + 17$

$= -2.2$

Let $\cancel{x_0 = 2}$ be critical approximation By Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x_1 = x_0 - \frac{f(x_2)}{f'(x_2)}$

$= 2 - \frac{2.2}{5.2}$

$= 2 - \frac{2.2}{5.2} \Rightarrow 2 - 0.0230$

$= 1.577$

$f(x_1) = (1.577)^3 - 1.8(1.577)^2 - 10(1.577) + 17$

$= 3.9219 - 0.0764 - 15.77 + 17$

$= 0.6755$

$f'(x_1) = 3(1.577)^2 - 3.6(1.577) - 10$

$= 7.4608 - 5.6772 - 10$

$= -8.2164$

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1.577 + \frac{0.6755}{8.2164}$

$= 1.577 + 0.0822$

$= \cancel{1.6592}$

$f(x_2) = (1.6592)^3 - 1.8(1.6592) - 10(1.6592) + 1.7$

$= 4.5677 - 4.9553 - 16.592 + 17$

$= 0.0204$

$f'(x_2) = 3(1.6592)^2 - 3.6(1.6592) - 10$

$= 8.2588 - 5.97312 - 10 \Rightarrow -7.7143$

$$x_3 = x_2 - \frac{f(x_1)}{f(x_2)}$$

$$x_3 = x_2 \quad \frac{f(x_2)}{f'(x_2)}$$

$$= 1.6592 + \frac{0.0204}{7.7143}$$

$$= 1.6592 + \frac{0.0204}{7.7143}$$

$$= 1.6592 + 0.0026 \Rightarrow 1.6618$$

$$f(x_3) = (1.6618)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17$$
$$= 4.5892 - 4.9708 - 16.618 + 17$$
$$= 0.0004$$

$$f(3_3) = 3(1.6618)^2 - 3.6(1.6618) - 10$$
$$= 8.2847 - 5.9824 - 10$$
$$= -7.6977$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 1.6618 + \frac{0.0004}{7.6977}$$

$$= 1.6618$$

$\therefore$ The root of equation is 1.6618

Practical ~ 5

(i) $\int \frac{dx}{\sqrt{x^2+2x-3}}$

$I = \int \frac{dx}{\sqrt{x^2+2x-3}}$

$= \int \frac{dx}{\sqrt{x^2+2x+1-4}} \Rightarrow \int \frac{1}{\sqrt{x+2^2}} \cdot dx$

ii) $I = (x) \quad x+1+\sqrt{x^2+2x-3} \quad +c$

$I = \int 4e^{3x} + \int 1\, dx$

$= \frac{4e^2}{3} + x + c$

iii) $I = \int (2x^2 - 3\sin x + 5\sqrt{x})dx$

$= 2\int x^2\, dx - \int \sin x\, dx + 5\int x^{1/2}\, dx$

$= \frac{2x^3}{3} + 3\cos x + \frac{5x^2}{3} x^{3/2} + c$

$I = \frac{2x^3}{3} + 3\cos x + \frac{10}{3} x\sqrt{x} + c$

iv) $I = \int \frac{x^3+3x+4}{\sqrt{x}} \cdot dx$

Subs $\sqrt{x^2}\ t$

$\therefore \frac{1}{2\sqrt{x}} = \frac{\delta t}{\delta x}$, $\frac{\delta x}{\sqrt{x}} = 2\delta t$

$I = \int \frac{(x)^6 + 3(\sqrt{x})^2 + 4\delta x}{\sqrt{x}}$

$= 2\int \frac{t^6 + 3t^2 - 4\delta t}{\sqrt{x}}$

$= 2\left[\frac{x^{7/2}}{7} + t^{7/2} + 4x^{1/2}\right] + C$

v) $I = \int t^2 \sin(2t)^2 \cdot \delta t$

$\int t^4 \sin 2t^4 \delta t$

Substitute $t^4 = x$

$4t^3 = \frac{\delta x}{\delta t}$

$t^3 \cdot \delta t = \frac{1}{4}\delta x$

$I = \frac{1}{4}\int \sin(2x)\,\delta x$

$\therefore I = \frac{1}{4}\left[\sin 2x\, \delta x - \int \frac{\delta x}{d}(\sin 2x \delta x)\delta x\right]$

$= \frac{1}{4}\left[-\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x\, dx\right]$

$= \frac{1}{10}\sin 2x - \frac{x\cos 2x}{8} + C$

$I = \frac{1}{10}\sin 2t^4 - t^4\frac{\cos 2t^4}{8} + C$

vi) $I = \int \sqrt{x}\,(x^2 - 1)\cdot\delta x$

$I = \int x^2\sqrt{x}\,dx - \int\sqrt{x}\,\delta x$

$= \int x^{1/2}\delta x - \int x^{1/2}\delta x$

$= \frac{2}{7}x^3\sqrt{x} - 2/3\,x\sqrt{x} + C$

vii) $I = \int \frac{1}{x^3}\sin(1/x^2)\,\delta x$

$1/x^2 = t$

$\frac{-2}{x^3} = \frac{dx}{dx^2}$

$I = -\frac{1}{2}\int \sin t\,\delta t$

$= \frac{\cos t}{t} + C$

viii) $I = \int \frac{\cos x}{3\sqrt{\sin^2 x}}\,\delta x$

Substitute $\sin x = t$

$I = \int \frac{1}{t^{2/3}}\,\delta t$

$= \frac{t^{-2/3} + 1}{-2/3 + 1} + C$

$= 3t^{1/3} + c$

$= 3\sqrt[3]{\sin x} + c$

ix) $I = \int e^{\cos^2 x} \sin 2x \, dx$

Substitute $\cos^2 x = t$

$\therefore -2\cos x \cdot \sin x = \frac{dt}{dx}$

$\therefore \sin 2x \, dx = -dt$

$\therefore I = -\int e^t dt$

$= -e^t + c$

$= -e^{\cos^2 x} + c$

(x) $I = \int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1} \right) \cdot dx$

$x^3 - 3x^2 + 1 = t$

$x^3 - 3x^2 + 1 = t$

$3x^2 - 6x = \frac{dt}{dx}$

$(x^2 - 2x) \cdot dx = \frac{dt}{3}$

$I = 1/3 \int \frac{1}{t} dt$

$= \frac{1}{3} \log |t| + c$

$= \frac{1}{3} \log |x^3 - 3x^2 + 1| + c$

Practical no-6

Topic: Application of Integration & numerical Integration

Q.1) find the length of the following:

① $x = t \sin t$; $y = 1-\cos t \in [0, 2\pi]$

Solution:-

arc length $= \int_0^{4} \sqrt{\left(\frac{\delta x}{dt}\right)^2 + \left(\frac{\delta y}{dt}\right)^3}$

$\delta t = \int_0^{2\pi} \sqrt{(1-\cos t)^2 (\sin t)^2} \cdot \delta t$

$= \int_0^{2\pi} \sqrt{1-2\cos t + \cos^2 t + \sin^2 t} \; + \delta t = \sqrt{1-\cos t + 1} \delta t$

$= \int_0^{2\pi} \sqrt{2-2\cos t} \; + \delta t$

$= \int_0^{2\pi} 2\sin 1/2 \; \delta t$

$= \left[-9\cos 1/2\right]_0^{2\pi}$

$= (-4\cos\pi) + 4\cos 0$

$= 8$ units

2) $y = \sqrt{4-x^2}$ $\quad x \in [-2, 2]$

$\frac{\delta y}{\delta x} = \frac{1}{2\sqrt{4-x^2}}$

$= \frac{-x}{\sqrt{4-x^2}}$

$L = \int_{-2}^{2} \sqrt{1 + \left(\frac{dy}{\delta x}\right)^2} \, \delta x$

$= \int_{-2}^{2} \sqrt{\frac{4-x^2+x^2}{4-x^2}} \cdot \delta x$

$= \int_{-2}^{2} \frac{1}{\sqrt{2^2-x^2}} \cdot \delta x$

$= 2\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$

$= 2\left[\pi/2 + \pi/2\right] = 2\pi$

Scanned with CamScanner

3] $y = x^{3/2}$ in $[0, 4)$

$$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$$

$$L = \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$= \int_0^4 \sqrt{1 + \frac{9}{4}} \cdot dx$$

$$= \frac{1}{2} \int_0^4 \sqrt{4 + 9x} \cdot dx$$

$$= \frac{1}{2} \left[\frac{(4+9x)^{3/2}}{3/2} \times 1/9 \right]_0^4$$

$$= \frac{1}{2} \left[(4+9x)^{3/2} \right]_0^4 \Rightarrow \frac{-1}{27} \left[(4+0)^{3/2} - (4+36)^{3/2} \right]$$

$$= \frac{1}{27} \left(40^{3/2} - 8 \right) \text{units}$$

4) $x = 3\sin t$, $y = 3\cos t$ $t \in (0, 2\pi)$

$$\frac{dx}{dt} = 3\cos t, \quad \frac{dy}{dt} = -3\sin t$$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} \cdot dt$$

$$= \int_0^{2\pi} \sqrt{(3\cos t)^2 + (-3\sin t)^2} \cdot dt$$

$$= \int_0^{2\pi} \sqrt{9\sin^2 + 9\cos^2 t} \cdot dt$$

$$= \int_0^{2\pi} 3\sqrt{x}\, dt \Rightarrow 3\int_0^{2\pi} 3 \cdot dt$$

$$= 6\pi \text{ units}$$

5) $x = \frac{1}{6} y^3 + \frac{1}{2} y$ on $y \in (1, 2)$

$$\frac{dx}{dy} = \frac{y^2}{2} \quad \frac{-1}{2y^2}$$

$$\frac{dx}{dy} = \frac{y^2 - 1}{2y^2}$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy$$

55

$$= \int_1^2 \sqrt{\frac{(y^4-1)^2+4 \times y^4}{4y^4}} \cdot dy$$

$$= \int_1^2 \frac{y^4+1}{2y^2} \, dy$$

$$= \frac{1}{2}\int_1^2 y^2 \cdot dy + \frac{1}{2}\int_1^2 y^{-2}\, dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3} - \frac{y^{-1}}{1}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1\right]$$

$$= \frac{1}{2}\left[\frac{7}{3} + \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{7}{6}\right] \Rightarrow \frac{17}{12} \text{ units}$$

Q II i) $\int_0^4 x^2 dx \qquad x = 4$

$$h = \frac{4-0}{4} = 1$$

x	0	1	2	3	4
y	0	1	4	9	16

$$\int_0^4 x^2 dx = \frac{1}{3}\left[(y_0+y_4) + 4(y_1+y_3) + 2(y_2)\right]$$

$$= \frac{1}{3}\left[16 + 4(10) + 8\right]$$

$$= \frac{64}{3}$$

$$\int_0^4 x^2 dx = 21.533$$

iii) $\int_0^{\pi/3} \sqrt{\sin x} \cdot dx$ with $n = 6$

$$h = \frac{\pi/3 - 0}{6} = \pi/18$$

x	0	$\pi/18$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$	$\frac{5\pi}{18}$	$8\pi/18$
y	0	0.4166	0.58	0.70	0.80	0.87	0.93

$$\int_0^{\pi/3} \sqrt{\sin x} = \frac{h}{3}\left[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{\pi}{54} \times 12.1163$$

$$\int_0^{\pi/3} \sqrt{\sin x \, dx} = 0.7049$$

Practical 07

Topic : Differential Equation

Q) Solve the following

1) $x \frac{dy}{dx} + y = e^x$

2) $e^x \frac{dy}{dx} + 2e^x y = 1$

3) $x \frac{dy}{dx} = \frac{\cos x}{x} - 2y$

4) $x \frac{dy}{dx} = \frac{\cos x}{x} - 2y$

5) $x \frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

6) $e^2 x \frac{dy}{dx} + 2e^{2x} y = 2x$

7) $\sec^2 x \tan x y \, dx + \sec^2 y \tan x \cdot dx = 0$

8) $\frac{dy}{dx} = \sin^2 (x - y + 1)$

9) $\frac{dy}{dx} = \frac{2x + 3y - 1}{6x + 9y + 6}$

Q1) $x \frac{dy}{dx} + y = e^x$

$\frac{dy}{dx} + \frac{1}{x} y = \frac{e^x}{x}$

$f(x) = 1/x \qquad Q(x) = e^x/x$

$IF = e^{\int 1/x \, dx}$

$= e$

$= x$

$y(I.F) = \int Q(x)(I.F)dx + C$

$= \int \frac{e^x}{x} \times x \, dx + C$

$= \int e^x dx + C$

$= e^x dx + C$

Q2) $e^x \frac{dy}{dx} + 2e^x y = 1$

$\frac{dy}{dx} + 2y = e^{-x}$

$f(x) = 2 \qquad Q(x) = e^{-x}$

$If = e^{\int f(x) dx}$

$= e^{\int 2 dx}$

$= e^{2x}$

$\therefore y(I.F) = \int Q(x)(I.F)dx + C$

$= \int e^{-x} \times e^{2x} dx + C$

$= \int e^{-x+2x} dx + C$

$= \int e^x dx + C$

$= e^x + C$

3) $\frac{dy}{dx} + \frac{2}{x} y = \frac{\cos x}{x^2}$

$f(x) = \frac{2}{x} \qquad Q(x) = \cos x / x^2$

$I.F = e^{\int f(x) dx}$

$= e^{\int 2/x \, dx}$

$= e^{2/x}$

$= e^{x^2}$

$= x^2$

$y(I.F) = \int Q(x) \cdot (I.F) \cdot dx + C$

$= \int \frac{\cos^2}{x^2} x^2 \cdot dx + C$

$= \int \cos x + C \Rightarrow -\sin x + C$

Q4) $x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

$\frac{dy}{dx} + \frac{3}{x}y = \frac{\sin x}{x^2}$

$f(x) = \frac{3}{x}$ $\quad Q(x) = \frac{\sin x}{x^2}$

$(I.F) = e^{\int f(x)\,dx}$

$= e^{\int 3/x \cdot dx}$

$= e^{\int \ln 3}$

$= x^3$

$y(I.F) = \int Q(x).(I.F)dx + C$

$= \int \frac{\sin x}{x^2} \times x^3 dx + C$

$= \int \sin x\,dx + C$

$= -\cos x + C$

Q5] $e^{2x}\frac{dy}{dx} + 2e^{2x}y = 2x$

$\frac{dy}{dx} + 2y = \frac{2x}{e^{2x}}$

$f(x) = 2$ $\quad Q(x) = 2x/e^{2x}$

$(I.F) = e^{\int f(x)dx}$

$= e^{\int 2\,dx}$

$= e^{2x}$

$y(I.F) = \int Q(x).(I.F)dx + C$

$= \int \frac{2x}{e^{2x}} \times e^{2x} \cdot dx + C$

$= \int 2x \cdot dx + C$

$\Rightarrow x^2 + C$

Q6) $\sec^2 x \cdot \tan y\,dx + \sec^2 y \tan x\,dy = 0$

$\Rightarrow \sec^2 x \cdot \tan y\,dx = \sec^2 y \cdot \tan dy$

$\frac{\sec^2 x\,dx}{\tan x} = -\frac{\sec^2 y\,dy}{\tan y}$

$= \therefore \int \frac{\sec^2 x\,dx}{\tan x} = -\frac{\sec^2 y\,dy}{\tan y}$

$\therefore \log|\tan x| = -\log|\tan y| + 1$

$\therefore \log|\tan x| - \tan| = c$

$\tan x \cdot \tan y = e$

Q7) $\frac{dy}{dx} = \sin^2(x-y-z) - x$

Differentiating both side

$x - y + 1 2 V$

$\therefore 1 - \frac{dy}{dx} = \frac{dv}{dx}$

$1 - \frac{dv}{dx} = \frac{dv}{dx}$

$1 = \frac{dv}{dx} = \sin^2 v$

$\frac{dy}{dx} = \cos^2 x$

$\int \sec^2 v \, dv = \int dx$

$\tan v = x + C$

$\tan(x + y - 1) = x + C$

Q8) $\frac{dy}{dx} = \frac{2x + 3y - 1}{6x + 9y + 6}$

Put $2x + 3y = V$

$2 + 3\frac{dy}{dx} = \frac{dv}{dx}$

$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$

$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{1}{3}\left(\frac{v-1}{v+2}\right)$

$= \frac{dv}{dx} = \frac{v-1}{v+2} + 2$

$= \frac{dv}{dx} = \frac{v - 1 + 2v + 4}{v+2} \Rightarrow \frac{3v+3}{v+2} = 3\left(\frac{v+1}{v+2}\right)$

$= \int \frac{v+1}{v+1} dv + \int \frac{1}{v+1} dv = 3x + C$

$= v + \log|v+1| = 3x + C$

$= 2x + 3y + \log|2x + 3y + 1| = 3x + C$

$\Rightarrow 3y = x - \log|2x + 3y + 1| + C$

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Practical no. 8

* Using Euler's method

1) $\frac{dy}{dx} = y + e^x - 2$, $y(0)$, $h = 0.5$ Find $y(2)$

2) $\frac{dy}{dx} = 1 + y^2$, $y(0) = 0$, $h = 0.2$, Find $y(1)$

3) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$, $y(0) = 1$, $h = 0.2$. Find $y(2)$

4) $\frac{dy}{dx} = 3x^2 + 1$, $y(1) = 2$. Find $y(2)$
For $h = 0.5$ & $h = 0.25$

5) $\frac{dy}{dx} = \sqrt{xy} + 2$, $y(1) = 1$. Find $y(1.2)$ with $h = 0.2$

Solution

$f(x, y) = y + e^x - 2$

$y(0) = 2$ $h = 0.5$

x	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.1487	3.2231
2	1	3.2231	3.9414	5.1938
3	1.5	5.1938	7.6755	9.0313

$y(2) = 9.0315$

2) $\frac{dy}{dx} = 1 + y^2 = f(x, y)$

$y(0) = 0$; $h = 0.2$

x	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	0	1	0.2
1	0.2	0.2	1.04	0.405
2	0.4	0.408	1.1665	0.6413
3	0.6	0.6418	1.4113	0.9236
4	0.8	0.9233	1.8520	1.2442

3) $f(x, y) = \sqrt{\frac{x}{y}}$; $y(0) = 1$; $h = 0.2$

x	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	1	0	1
1	0.2	1	0.4472	1.0894
2	0.4	1.0894	0.6059	1.2106
3	0.6	1.2106	0.704	1.3514
4	0.8	1.3514	0.7694	1.5053

~~y(1)~~ $y(1) = 1.5053$

Q4) $f(x,y) = 3x^2 + 1$; $y(1) = 2$; $h = 0.5$

x	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
	1	2	4	4
1				
2	1.5	4	7.75	7.875

$y(2) = 7.875$

~~Q5~~ $h = 0.25$

x	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
		2	4	4
1	1			
2	1.25	3	5.6875	4.4219
3	1.5	4.4219	7.75	6.3594
4	1.73	6.3594	10.1875	8.9063

$y(2) = 8.9063$

Q5) $f(x,y) = \sqrt{xy} + 2$; $y(1) = 1$, $h = 0.2$

x	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
1	1	1	3	1.6

$y(1.2) = 1.6$

AK
22/01/2020

Practical - 9

Limits & Partial order Derivative

1) Evaluate the following limits

(i) $\lim\limits_{(x,y)\to(-4,-1)} \dfrac{x^3 - 3y + y^2 - 1}{xy + 5}$

Applying limit

$$= \frac{(-4)^3 - 3(-1) + (-1)^2 - 1}{(-4)(-1) + 5}$$

$$= \frac{-64 + 3 + 1 - 1}{4 + 5} = -\frac{61}{9}$$

ii) $\lim\limits_{(x,y)\to(2,0)} \dfrac{(y+1)(x^2+y^2-4x)}{x+3y}$

Applying limit

$$\frac{(0+1)((2)^2 + (0)^2 - 4(2))}{2 + 3(0)}$$

$$= \frac{1(4-8)}{2} = -\frac{4}{2} = -2$$

iii) $\lim\limits_{(x,y,z)\to(1,1,1)} \dfrac{x^2 - y^2 z^2}{x^3 - x^2 yz}$

$\Rightarrow \lim\limits_{(x,y,z)\to(1,1,1)} \dfrac{(x)^2 - (yz)^2}{x^2(x - yz)}$

$\lim\limits_{(x,y,z)\to(1,1,1)} \dfrac{(x+yz)(x-yz)}{x^2(x-yz)}$ $[\because (a)^2 - b^2 = (a+b)(a-b)]$

$\lim\limits_{(x,y,z)\to(1,1,1)} \dfrac{x + yz}{x}$

$$\frac{1 + (1)(1)}{(1)^2} = 2$$

2) Find f_x, f_y, for each of the following

(i) $f(x,y) = xy\,e^{x^2+y^2}$

$f(x) = \dfrac{\partial f}{\partial x}$

$= \partial(xy\,e^{x^2+y^2})$

$= y\,\dfrac{\partial(x\cdot e^{x^2+y^2})}{\partial x}$

$= y\,\dfrac{\partial(x\cdot e^{x^2+y^2})}{\partial x}$

$= y\left[x\cdot\dfrac{d}{dy}(e^{x^2+y^2}) + e^{x^2+y^2}\cdot\dfrac{d}{dx}(x)\right]$

$\left[\because \dfrac{d}{dx}(uv) = uv' + v\cdot u'\right]$

$= y[x\cdot e^{x^2+y^2}\cdot 2x + e^{x^2+y^2}(1)]$

$= y\cdot e^{x^2+y^2}[2x + 1]$

Now, $f(y) = \dfrac{\delta f}{\delta y}$

$= \dfrac{\partial(xy\,e^{x^2+y^2})}{\partial y}$

$= x \cdot \frac{\partial}{\partial y}(y \cdot e^{x^2+y^2})$

$= x\left[y \frac{d}{dy}(e^{x^2+y^2}) + e^{x^2+y^2} \cdot \frac{d}{dy}(y)\right]$

$\because \left[\frac{d}{dx}(uv) = uv' + vu'\right]$

$= x\left[2y^2 \cdot e^{x^2+y^2} + e^{x^2+y^2}\right]$

$= x \cdot e^{x^2+y^2}\left[2y^2+1\right]$

(ii) $f(x,y) = e^x \cos y$

$f(x) = e^x \cos y$

$\therefore f(y) = e^x \frac{\partial}{\partial y}(\cos y)$

$= e^x(-\sin y)$

$= -e^x \sin y$

iii) $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1$

$f(x) = \frac{\partial f}{\partial x} = \frac{\partial(x^3y^2 - 3x^2y + y^3 + 1)}{\partial x}$

$= 3x^2y^2 - 3(2x)y$

$= 3x^2y^2 - 6xy$

$f(y) = \frac{\partial f}{\partial y} = \frac{\partial(x^3y^2 - 3x^2y + y^3 + 1)}{\partial y}$

$= x^3(2y) - 3(1)x^2 + 3y^2$

$= 2x^3y - 3x^2 + 3y^2$

Q3) Using definition find values of f_x, f_y at $(0,0)$ for $f(x,y) = \frac{2x}{1+y^2}$

$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$

where $(a,b) = (0,0)$

$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$

$= \lim_{h\to 0} \frac{2h - 0}{h} = 2$

Similarly

$f_y(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$

$= \lim_{h\to 0} \frac{0-0}{h} = 0$

$\therefore f_x = 2, \; f_y = 0$

Q4) Find all second order partial derivative of f also verify whether $f_{xy} = f_{yx}$

(i) $f(x,y) = \frac{y^2 - xy}{x^2}$

$f(x) = \frac{f}{\partial x} = \frac{\partial(y^2 - xy)}{x^2}$

$= \frac{x^2 \frac{\partial}{\partial x}(y^2 - xy) - (y^2 - xy)\cdot\frac{\partial}{\partial x}(x^2)}{(x^2)^2}$

$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}\right]$

$= \frac{x^2(-y) - (y^2 - xy)(2x)}{x^4}$

$$= \frac{-x^2y - 2xy^2 + 2x^2y}{x^3}$$

$$= \frac{-x^2y - 2xy^2 + 2x^2y}{x^3} = \frac{x(xy - 2y^2)}{x^3}$$

$$\delta(x) = \frac{xy - 2y^2}{x^3}$$

$$\delta(y) = \frac{\delta f}{\delta y} = \delta\left(\frac{y^2 - xy}{x^2}\right)$$

$$= \frac{\partial\left(\frac{y^2}{x^2} - \frac{xy}{x^2}\right)}{\partial y} \Rightarrow \partial\left(\frac{y^2}{x^2} - \frac{y}{x}\right) \Big/ \partial y$$

$$= \frac{1}{x^2} 2y - \frac{1}{x}$$

$$\delta(y) = \frac{2y - x}{x^2}$$

$$\delta(xx) = \frac{\partial\left(\frac{xy - 2y^2}{x^3}\right)}{\partial x}$$

$$= \frac{x^3 \cdot \frac{d}{dx}(xy - 2y^2) - (xy - 2y^2)\frac{d}{dx}(x^3)}{(x^3)^2}$$

$$= \frac{x^3(y) - (xy - 2y^2)(3x^2)}{6}$$

$$= \frac{x^3y - 3x^3y + 6x^2y^2}{6}$$

$$= \frac{6x^2y^2 - 2x^3y}{x^6} \Rightarrow \frac{x^2(6y^2 - 2xy)}{x^6}$$

$$= \frac{6y^2 - 2xy}{x^4}$$

$$\delta(yy) = \frac{\partial\left(\frac{2y - x}{x^2}\right)}{\partial y}$$

$$= \frac{\frac{1}{x^2}\,\partial(2y - x)}{\partial y} \Rightarrow \frac{1}{x}(2)$$

$$\delta(xy) = \frac{\partial\left(\frac{xy - 2y^2}{x^3}\right)}{\partial y} = \frac{\partial\left(\frac{xy}{x^3} - \frac{2y^2}{x^3}\right)}{\partial y}$$

$$= \frac{\partial\left(\frac{y}{x^2} - \frac{2y^2}{x^3}\right)}{\partial y}$$

$$= \frac{1}{x^2} - \frac{1}{x^2}\,2(2y)$$

$$= \frac{1}{x^2} - \frac{4y}{x^3} \Rightarrow \frac{x^3 - 4yx^2}{x^6}$$

$$= \frac{\cancel{x^2}(x - 4y)}{\cancel{x^6}}$$

$$= \frac{x - 4y}{x^4}$$

$$\delta(yx) = \frac{\partial\left(\frac{2y - x}{x^2}\right)}{\partial x}$$

$$= \frac{\partial\left(\frac{2y}{x^2} - \frac{x}{x^2}\right)}{\partial x} \Rightarrow \frac{\partial\left(\frac{2y}{x^2} - \frac{1}{x}\right)}{\partial x}$$

$= 2y\left(\frac{-2}{x^3}\right) - \left(\frac{-1}{x^2}\right)$

$= -\frac{4y}{x^3} + \frac{1}{x^2}$

$= \frac{-4yx^2 + x^3}{x^6}$

$= \frac{x^2(x-4y)}{x^6}$

$= \frac{x-4y}{x^4}$

$\therefore f(xy) = f(yx) = \frac{x-4y}{x^4}$

Hence verified

Q) $f(x,y) = x^3 + 3x^2y^2 - \log(x^2+1)$

$\therefore f(x) = \frac{\partial f}{\partial x} = \frac{\partial(x^3+3x^2y^2-\log(x^2+1))}{\partial x}$

$= 3x^2 + 3(2x)y^2 - \frac{1}{x^2+1}(2x)$

$\therefore f(x) = 3x^2 + 6xy^2 - \frac{2x}{x^2+1}$

$f(y) = \frac{\partial f}{\partial y} = \frac{\partial(x^2+3x^2y^2-\log(x^2+1))}{\partial y}$

$= 0 + 3(2y)(x^2) + 0$

$f(y) = 6x^2y$

$f(xx) = \frac{\partial fx}{\partial x} = \frac{\partial}{\partial x}\left(3x^2 + 6xy^2 - \frac{2x}{x^2+1}\right)$

$= 6x + 6y^2(1) - 2\left[\frac{x^2+1(1) - x(2x)}{(x^2+1)^2}\right]$

$\left[\because \frac{d}{dx}\frac{u}{v} = \frac{v.u' - v.u'}{v^2}\right]$

$= 6x + 6y^2 - 2$

$f(yy) = \frac{\partial}{\partial y}(fy)$

$= \frac{\partial}{\partial y}(6x^2y)$

$= 6x^2$

$fxy = \frac{\partial}{\partial y} fx \Rightarrow \frac{\partial}{\partial y}\left(3x^2+6xy^2-\frac{2x}{x^2+2}\right)$

$= 12xy$

$fyx = \frac{\partial}{\partial y} fx$

$= \frac{\partial}{\partial y}\left(3x^2+6xy^2-\frac{2x}{x^2+1}\right)$

$= 12xy$

Q5)

i) $f(x,y) = \sqrt{x^2+y^2}$ $\quad (a,b) = (1,1)$

$f_x = \frac{1}{2\sqrt{x^2+y^2}} \times 2x$ $\qquad f_y = \frac{2y}{2\sqrt{y^2+x^2}}$

$= \frac{x}{\sqrt{x^2+y^2}}$ $\qquad = \frac{y}{\sqrt{x^2+y^2}}$

$f_x(1,1) = \frac{1}{\sqrt{2}}$ $\qquad = \frac{1}{\sqrt{2}}$

$\therefore L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= \sqrt{2} + \frac{\cancel{x-1}\ x-1}{\sqrt{2}} + \frac{y-1}{\sqrt{2}}$

$= \sqrt{2} + \frac{x+y-2}{\sqrt{2}} \Rightarrow \frac{x+y}{\sqrt{2}}$

ii) $f(x,y) = 1 - x + y\sin x$ $\qquad (a,b) = (\pi/2, 0)$

$f(\pi/2, 0) = 1 - \frac{\pi}{2} + 0 \sin\frac{\pi}{2}$

$= \frac{2-\pi}{2}$

$f_x = -1 + y\cos x$ $\qquad f_y = \sin x$

$f_x = (\pi/2, 0) = -1 + 0 \times \cos \pi/2$ $\qquad f_y(\pi/2, 0) = \sin\pi/2$

$= -1$ $\qquad = 1$

$L(x,y) = f(\pi/2, 0) + f_x(\pi/2,0)(x-\pi/2) + f_y(\pi/2,0)(y-0)$

$= \frac{2-\pi}{2} + (-1)(x - \frac{\pi}{2}) + 1(y)$

$= 1 - \frac{\pi}{2} - x + \frac{\pi}{2} + y$

$= 1 - x + y$

iii) $f(x,y) = \log x + \log y$ $\qquad (a,b) = (1,1)$

$f(1,1) = \log(1) + \log(1)$

$= 0$

$f_x = \frac{1}{x}$ $\qquad f_y = \frac{1}{y}$

$f(1,1) = 1$ $\qquad f(1,1) = 1$

$L(x,y) = 0 + 1(x-1) + 1(y-1)$

$= x + y - 2$

Practical-10

Q.1) Find the directional derivative of the following function at given points and in the direction of given vector

i) $g(x,y) = x+2y-3$ $a=(1,-1)$ $g = 3i - j$

Here, $u = 3i - j$ is not a unit vector

$|\bar{u}| = \sqrt{(3)^2+(-1)^2} = \sqrt{9+1} = \sqrt{10}$

Unit vector along $u = \frac{u}{|u|} = \frac{1}{\sqrt{10}}(3,-1)$

$= \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}}\right)$

$f(a) = f(1,-1) = (1)+2(-1)-3 = 1-2-3 = -4$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$= f\left(1+\frac{3}{\sqrt{10}}\right), \left(-1-\frac{h}{\sqrt{10}}\right)$

$f(a+hu) = \left(1+\frac{3}{10}\right)+2\left(-1 \ \frac{-h}{\sqrt{10}}\right)-3$

$= 1+\frac{3}{\sqrt{10}} - 2 - \frac{2h}{\sqrt{10}} - 3$

$f(a+hu) = -4 + \frac{h}{\sqrt{10}}$

$D_u f(a) = \lim_{h\to 0} \frac{f(a+hu)-f(a)}{h}$

$= \lim_{h\to 0} \frac{-4 + {}^{h}/_{\sqrt{10}} + 4}{4h}$

$D_u f(a) = \frac{1}{\sqrt{10}}$

ii) $f(x) = y^2 - 4x + 1$ $a = (3,4)$ $u = i+5j$

Here $u = i+5j$, is not a unit vector

$|\bar{u}| = \sqrt{(1)^2+(5)^2} = \sqrt{26}$

unit vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{26}}(1,5)$

$= \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

$f(a) = f(3,4) = (4)^2 - 4(3)+1 = 5$

$f + \left(3+\frac{h}{\sqrt{30}}\right), 4+\frac{5h}{\sqrt{26}}\right)$

$f_{xy}(a+hu) = \left(4+\frac{5h}{\sqrt{26}}\right)^2 - 4\left(3+\frac{h}{\sqrt{20}}\right)+1$

$= 16 + \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - 12 - \frac{4h}{\sqrt{26}} + 1$

$= \frac{25h^2}{20} + \frac{40h}{\sqrt{26}} - \frac{4h}{\sqrt{26}} + 5$

$= \frac{25h^2}{20} + \frac{40h - 4h}{\sqrt{26}} + 5$

$= \frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5$

$$D_u f(a) = \lim_{h \to 0} \frac{\frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5 - 5}{h}$$

$$h\left(\frac{25h}{26} + \frac{36}{\sqrt{26}}\right)$$

$$\therefore D_u f(a) = \frac{25h}{26} + \frac{36}{\sqrt{26}}$$

iii) $3x + 3y \quad a = (1,2), \; u\,(3i + 4j)$

Here $u = 3i + 4j$ is not a unit vector

$$|\bar{u}| = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$$

Unit vector along u is $\frac{u}{|u|} = \frac{1}{5}(3,4)$

$$= \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$f(a) = f(1,2) = 2(1) + 3(2) = 8$$

$$f(a+hu) + f(1,2) + h\left(\frac{3}{5}, \frac{4}{5}\right) \Rightarrow f\left(1 + \frac{3h}{5}, 2 + \frac{4h}{5}\right)$$

$$f(a+hu) = 2\left(1 + \frac{3h}{5}\right) + 3\left(2 + \frac{4h}{5}\right)$$

$$= 2 + \frac{6h}{5} + 6 + \frac{12h}{5}$$

$$= \frac{18h}{5} + 8$$

$$D_u f(a) = \lim_{h \to 0} \frac{\frac{18h}{5} + 8 - 8}{h}$$

$$= \frac{18}{5}$$

Q2) Find gradiant vector.

(i) $f(x,y) = x^y + y^x \Rightarrow a = (1,1)$

$$f_x = y \cdot x^{y-1} + y^x \log y$$

$$f_y = x^y \log x + x y^{x-1}$$

$$\nabla f(x,y) = (f_x, f_y)$$

$$= (y x^{y-1} + y^x \log y, \; x^y \log x + x y^{x-1})$$

$$\nabla f(1,1) = (1+0, 1+0)$$

$$= (1,1)$$

ii) $f(x,y) = (\tan^{-1} x) \cdot y^2 \qquad a = (1,-1)$

$$f_x = \frac{1}{1+x^2} \cdot y^2$$

$$f_y = 2y \tan^{-1} x$$

$$\nabla f(x,y) = \nabla f(x,y)$$

$$= \left(\frac{y^2}{1+x^2}, 2y \tan^{-1} x\right)$$

$$\nabla f(1,-1) = \left(\frac{1}{2}, \tan^{-1}(1)(-2)\right)$$

$$= \left(\frac{1}{2}, \frac{-\pi}{2}\right)$$

iii) $f(x,y,z) = xyz - e^{x+y+z}$, $a = (1,-1,0)$

$f_x = yz - e^{x+y+z}$

$f_y = xz - e^{x+y+z}$

$f_z = xy - e^{x+y+z}$

$(x,y,z) = f_x, f_y, f_z$

$= yz - e^{x+y+z}, xz - e^{x+y+z}, xy - e^{x+y+z}$

$f(1,-1,0) = ((-1)(0) - e^{1+(-1)+0}, (1)(0) - e^{1+(-1)+0}, (1)(-1) - e^{1+(-1)+0})$

$= (0 - e^0, 0e^0, -1 - e^0)$
$= (-1, -1, -2)$

Q3) Find the equation of tangent & normal

(i) $x^2 \cos y + e^{xy} = 2$ at $(1, 0)$

$f_x = \cos y \, 2x + e^{xy} y$
$f_y = x^2(-\sin y) + e^{xy} \cdot x$

$(x_0, y_0) = (1, 0)$ $\therefore x_0 = 1$, $y_0 = 0$

equation of tangent

$f_x(x - x_0) + f_y(y - y_0) = 0$

$f_x(x_0, y_0) = \cos 0 \, (2)(1) + e^0 \, 0$
$= 1(2) + 0$
$= 2$

$f_y(x_0, y_0) = (1)^2(-\sin 0) + e^0 \cdot 1$
$= 0 + 1 \cdot 1$
$= 1$

$2(x-1) + 1(y-0) = 0$
$2x - 2 + y = 0$
$2x + y - 2 = 0$ → it is required equation of tangent

equation of normal

$= ax + by + c = 0$
$= bx + ay + d = 0$
$1(1) + 2(y) + d = 0$
$1 + 2y + d = 0$
$= 1 + 2(0) + d = 0$
$= d + 1 = 0$
$\therefore d = -1$

ii) $x^2 + y^2 - 2y + 3y + z = 0$ at $(2, -2)$

$f_x = 2x + 0 - 2 + 0 + 0$
$= 2x - 2$
$f_y = 0 - 2y - 0 + 3 + 0$
$= 2y + 3$
$(x_0, y_0) = (2, -2)$ $\therefore x_0 = 2$, $y_0 = -2$

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$f_x(x_0, y_0) = 2(2) - 2 = 2$
$f_y(x_0, y_0) = 2(-2) + 3 = -1$

Equation of tangent

$f(x)(x - x_0) + f_y(y - y_0) = 0$
$2(x - 2) + (-1)(y + 2)) = 0$
$2x - 2 - y - 2 = 0$
$2x - y - 4 = 0 \Rightarrow$ & is the required equation of tangent

Equation of normal

$= ax + by + c = 0$
$= 6y + 9y + d = 0$

$= -(x) + 2(y) + d = 0$
$-2 + 2(-2) + d = 0$
$-2 - 4 + d = 0$
$-6 + d = 0$
$\therefore d = 6$

Q5) Find the equation of tangent & normal lines to each other of the following surface

i) $x^2 - 2yz + 3y + xz = 7$ at $(2, 1, 0)$

$f_x = 2x - 0 + 0 + 2$
$f_x = 2x + 2$
$f_y = 0 - 2z + 3 + 0$
$= 2z + 3$

$f_z = 0 - 2y + 0 + x$
$-2y + 0$

$(x_0, y_0, z_0) = (2, 1, 0) \therefore x_0 = 2, y_0 = 1, z_0 = 0$

$f_x(x_0, y_0, z_0) = 2(2) + 0 = 4$
$f_y(x_0, y_0, z_0) = 2(0) + 3 = 3$
$f_z(x_0, y_0, z_0) = 2(1) + 2 = 0$

Equation of tangent

$f_x(x - x_0) + f_y(y - y_0) + f_z(-z_0) = 0$
$= 4(x - 2) + 3(y - 1) + 0(z - 0) = 0$
$= 4x - 8 + 3y - 3 = 0$
$4x + 3y - 11 = 0 \Rightarrow$ This is the required equation

Equation of normal at $(4, 3, -11)$

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

$$= \frac{x - 2}{4} = \frac{y - 1}{3} = \frac{z + 11}{0}$$

65) Find the local maxima & minima

i) $f(x,y) = 3x^2 + y^2 - 3xy + 6x - 4y$

$f_x = 6x + 0 - 3y + 6 - 0$
$= 6x - 3y + 6$

$f_y = 0 + 2y - 3x + 0 - 4$
$= 2y - 3x - 4$

$f_x = 0$
$6x - 3y + 6 = 0$
$3(2x - y + 2) = 0$
$2x - y + 2 = 0$
$2x - y = -2$ .... (1)

$f_y = 0$
$2y - 3x - 4 = 0$
$2y - 3x = 4$ .... (2)

Multiply equation (1) & (2)

$4x - \cancel{2y} = -4$
$\underline{\cancel{2y} - 3x = 4}$
$x = 0$

Substitute value of x in equation (1)

$2(0) - y = -2$
$-y = -2$
$\therefore y = 2$

$\therefore$ critical point on $(0, 2)$

$r = f_{xx} = 6$
$t = f_{yy} = 2$
$s = f_{xy} = -3$

Here $r > 0$
$= rt - s^2$
$= 6(2) - (-3)^2$
$= 12 - 9$
$= 3 > 0$

$\therefore$ f has maxima at $(0, 2)$

$3x^2 + y^2 - 3xy + 6x - 4y$ at $(0, 2)$
$3(0)^2 + (2)^2 - 3(0)(2) + 6(0) - 4(2)$
$0 + 4 - 0 + 0 - 8$
$= -3$

ii) $f(x,y) = x^2 - y^2 + 2x + 8y - 70$

$f_x = 2x + 2$
$f_y = -2y + 8$

$f_x = 0$ $\quad \therefore 2x + 2 = 0$
$x = -1$

$f_y = 0$ $\quad \therefore -2y + 8 = 0$
$y = \frac{-8}{-2} = 4$
$\therefore y = 4$

$\therefore$ critical point is $(-1, 4)$

$r = f_{xx} \quad x = 2$

$t = f_{yy} \quad = -2$

$s = f_{xy} \quad = 0$

$r > 0$

$rt - s^2 = 2(-2) - (0)^2$

$= -4 - 0$

$= -4 < 0$

$f(x, y)$ at $(-1, 4)$

$(-1)^2 - (4)^2 + 2(-1) + 8(4) - 70$

$1 + 16 - 2 + 32 - 70$

$\Rightarrow 17 + 30 - 70$

$\Rightarrow 37 - 70 = 33$

AK

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