Stochastic Process 1

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1 Poisson Process

1.1 Poisson Approximation to Binomial

Given a Poisson random random variable $Y \sim \text{Poisson}(\lambda)$ with pdf

$$\Pr[Y = k] = \frac{e^{-\lambda}\lambda^k}{k!}, \forall k \in N_0 = \{0, 1 \ldots\}.$$

The probablity of a binomial random variable being k is

$$\binom{n}{k} p^k (1-p)^{n-k}$$

Theorem 1.1. Given $p \to 0$, $np \to \lambda$, we have that

$$\binom{n}{k} p^k (1-p)^{n-k} \to \frac{e^{-\lambda}\lambda^k}{k!}$$

Proof.

$$\begin{aligned}
\binom{n}{k} p^k (1-p)^{n-k} &= \binom{n}{k} (\frac{np}{n})^k (1 - \frac{np}{n})^{n-k} \\
&= \frac{n!}{k!\,(n-k)!} \left(\frac{np}{n}\right)^k \left(1 - \frac{np}{n}\right)^n \left(1 - \frac{np}{n}\right)^{-k} \\
&= \frac{1}{k!} \underbrace{\frac{n(n-1)\ldots(n-k+1)}{n^k}}_{\to 1} \underbrace{(np)^k}_{\to \lambda^k} \underbrace{\left(1 - \frac{np}{n}\right)^n}_{\to e^{-\lambda}} \underbrace{\left(1 - \frac{np}{n}\right)^{-k}}_{\to 1} \\
&\to \frac{1}{k!}\lambda^k e^{-\lambda}
\end{aligned}$$

□

With that, we consider three different binomial random variables:

$$\begin{aligned}
X_n &\sim \text{Binomial}(n, p_n), p_n \to 0, np_n \to \lambda > 0, \text{ as } n \to \infty. \\
Z_p &\sim \text{Binomial}(n(p), p), p \to 0, n(p)p \to \lambda > 0, \text{ as } p \to 0. \\
N_x &\sim \text{Binomial}(n(x), p(x)), p(x) \to 0, n(x) \to \lambda > 0, \text{ as } x.
\end{aligned}$$

For example, if $X_n \sim \text{Binomial}(n, 2/n)$, then we expect

$$\binom{n}{k} p^k (1-p)^{n-k} \to \frac{e^{-2}2^k}{k!}$$

1.2 Total Variance Distance

Let $X_1, \ldots, X_n$ be n independent Bernoulli random variables, where $\mathbb{E}[X_i] = p_i$.

Given $S = \sum X_i$ and $T \sim \text{Poisson}(\lambda = \sum p_i)$, how close are these two distributions? Or, how to measure the closedness?

Definition 1.2. Given two random variables X, Y, (which shares the sample space), we have the *total variance distance* defined as

$$d_{TV}(X,Y) = \sup_A |\Pr[X \in A] - \Pr[Y \in A]|$$

where A is a Borel set defined with respect to the sample space σ-algebra.

Example 1.3. Given two distributions

	0	1	2	3
$\Pr[X = k]$	$5/10$	$3/10$	$1/10$	$1/10$
$\Pr[Y = k]$	$2/10$	$1/10$	$1/10$	$6/10$

Table 1: Discrete Distribution Distance

If $A = \{0, 1\}$, then

$$|\Pr[X \in A] - \Pr[Y \in A]| = 3/10 + 2/10 = 1/2.$$

If $A = \{3\}$,

$$|\Pr[X \in A] - \Pr[Y \in A]| = |1/10 - 6/10| = 1/2.$$

Lemma 1.4. If X, Y take values in a countable set E,

$$\begin{aligned} d_{TV}(X,Y) &= \sum_{i \in E} (\Pr[X = i] - \Pr[Y = i])^+ \\ &= \sum_{i \in E} (\Pr[Y = i] - \Pr[X = i])^+ \\ &= \frac{1}{2} \sum_{i \in E} |\Pr[Y = i] - \Pr[X = i]| \end{aligned}$$

Proposition 1.5. Given two random variables, we have

$$d_{TV}(X,Y) \leq \Pr[X \neq Y]$$

Proof. For any A,

$$\begin{aligned} &|\Pr[X \in A] - \Pr[Y \in A]| \\ =&|\Pr[X \in A, Y \in A] + \Pr[X \in A, Y \notin A] - \Pr[Y \in A, X \in A] - \Pr[Y \in A, X \notin A]| \\ =&|\Pr[X \in A, Y \notin A] - \Pr[Y \in A, X \notin A]| \\ \leq& \max\{\Pr[X \in A, Y \notin A], \Pr[Y \in A, X \notin A]\} \leq \Pr[X \neq Y]. \end{aligned}$$

□

Let $X_1, \ldots, X_n$ be independent Bernoulli random variables with $\mathbb{E}[X_i] = p_i$. Let $S := \sum X_i$, and $T \sim \text{Poisson}(\lambda := p_1 + \ldots + p_n)$. Then $\mathbb{E}[S] = \mathbb{E}[\sum X_i] = \sum \mathbb{E}[X_i] = \sum p_i = \lambda$.

Let $Y_1, \ldots, Y_n$ be independent Poissson random variables with $\mathbb{E}[Y_i] = p_i$. Then $T = \sum Y_i \sim \text{Poisson}(\lambda)$. We also have

$$[S \neq T] \subseteq \underbrace{[X_1 \neq Y_1]}_{B_1} \cup [X_2 \neq Y_2] \cup \ldots [X_n \neq Y_n]$$

And hence

$$\begin{aligned} \Pr[S \neq T] &\leq \Pr[B_1 \cup \ldots \cup B_n] \\ &\leq \Pr[B_1] + \ldots + \Pr[B_n] \\ &\leq p_1^2 + \ldots + p_n^2 \end{aligned}$$

where $\Pr[X_i = Y_i] = 1 - p + pe^{-p}, \Pr[X_i \neq Y_i] = p - pe^{-p} \leq p(1 - (1 - p + p^2/2! + \ldots) = p(p - p^2/2! + \ldots) \leq p^2$.

Hence,

$$d_{TV}(S, T) \leq \Pr[S \neq T] \leq \sum_{i=1}^{n} p_i^2.$$

Consider $X_1 \sim \text{Bernoulli}(p_1 = 1/5), X_2 \sim \text{Bernoulli}(p_2 = 1/6), X_3 \sim \text{Bernoulli}(p_3 = 1/10)$, $S = X_1 + X_2 + X_3$ and $T \sim \text{Poisson}(\lambda = \frac{7}{15})$. Then if estimate T by S, for example,

$$\Pr[S \text{ is an odd number}] \approx \Pr[T \text{ is an odd number}]$$

the probability of getting an error is at most

$$(1/5)^2 + (1/6)^2 + (1/10)^2$$

by letting A be the set of odd numbers.

1.3 Probablity Axioms

Consider the sample space Ω, the set of events $\mathcal{F}$ and the probablity P, where

$$\begin{aligned} &\Omega : \text{sample spaces - set of all outcomes} \\ &\mathcal{F} : \text{all events} \\ &P : \mathcal{F} \rightarrow [0, 1] \end{aligned}$$

. Then we can write a random variable X_1 as:

$$X_1 : \Omega \rightarrow \mathbb{R}$$

and an event as

$$B_1 = [X_1 \neq Y_1] = [w \in \Omega | X_1(w) \neq Y_1(w)].$$

Definition 1.6. Event Axioms:

E.1 $\Omega \in \mathcal{F}$

E.2 $A \in \mathcal{F} \implies A^C \in \mathcal{F}$

E.3 $A_1, A_2, \ldots \in \mathcal{F} \implies A_1 \cup A_2 \cup \ldots \in \mathcal{F}$

Definition 1.7. Probability Axioms:

P.1 $A \in \mathcal{F} \implies P(A) \geq 0$

P.2 Countable additivity. $A_1, A_2, \ldots$ being disjoint events, then $P(A_1 \cup A_2 \cup \ldots) = \sum_{i=1} P(A_i)$.

P.3 $P(\Omega) = 1$.

Example 1.8. $X = (\Omega, \mathcal{F}) \to \mathbb{R}$, and let B be a Borel set. Then we can write $\{X = 3\} = \{w \in \Omega : X(w) = 3\} \in \mathcal{F}$. Similarly, $P(X \in B) \in \mathcal{F}$.

$\Omega = \{a, b, c\}$, $\mathcal{F} = \{\emptyset, \Omega, \{a\}, \{b, c\}\}$. Given $X(a) = 1, X(b) = 2, X(c) = 3$, we have

$$[X = 3] = [w \in \Omega : X(w) = 3] = [c]$$

which is not in the event, so X is not a random variable. If $X(b) = 3$, then X is a random variable.

Given $X \sim \mathrm{Exp}(\lambda), Y \sim \mathrm{Exp}(\mu)$ and $X \perp Y$, then

$$\begin{aligned} &P(X > s, Y - X > t | X < Y) \\ =&P(X > s | X < Y) P(Y - X > t | X < Y) \end{aligned}$$

$\lambda_n \to \lambda \implies (1 + \frac{\lambda_n}{n})^n \to e^\lambda$. $f(h) = o(h) \implies f(h)/h \to 0$ as $h \to 0$.

Fix x, a function f is differentiable at x iff there exists a number $f'(x)$ such that

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + o(h) \\ \frac{f(x+h) - f(x)}{h} &= f'(x) + o(h)/h, h \to 0 \end{aligned}$$

For example, if we want to show $n \log(1 + \frac{\lambda_n}{n}) \to \lambda$. Take $h_n = \lambda_n/n$, $x = 1$.

$$n \log(1 + h_n) = nh_n + nO(h_n) = nh_n + \frac{\lambda_n}{h_n} O(h_n)$$

where $\log(1 + h) = \log(1) + h + o(h)$. Then as $n \to \infty$, we have $h_n \to 0$, $nh_n \to \lambda$, $n \log(1 + \lambda_n/n) \to \lambda$.

Definition 1.9. Suppose X is nonnegative, integer-valued random variable $P(X = k) = p_k$ for $k = 0, 1, 2, \ldots$, then the *probability-generating function* is defined as:

$$G(s) := \mathbb{E}[s^X] = \sum_{k=0}^{\infty} p_k s^k = p_0 + p_1 s + p_2 s^2 + \ldots$$

and $G(s) < \infty$ for $|s| < R$.

Then we have

$$\begin{aligned}
G'(s) &= \sum_{k=0}^{\infty} k p_k s^{k-1} = \mathbb{E}[X s^{X-1}] \\
G'(1) &= \mathbb{E}[X] \\
G''(s) &= \sum_{k=0}^{\infty} k(k-1) p_k s^{k-2} = \mathbb{E}[X(X-1) s^{X-2}] \\
G''(1) &= \mathbb{E}[X(X-1)] = \mathbb{E}[X^2 - X] = \mathbb{E}[X^2] - \mathbb{E}[X] \\
\mathbb{E}[X^2] &= G''(1) + G'(1) \\
var(X) &= G''(1) + G'(1) - [G'(1)]^2 = E[X^2] - E[X]^2
\end{aligned}$$

$$G(0) = p_0, G'(0) = p_1, \frac{G''(0)}{2} = p_2.$$

Let X, Y be independent nonnegative, integer-value random variables.

$$\begin{aligned}
T =& X + Y \\
\mathbb{E}[s^T] =& \mathbb{E}[s^{X+Y}] = \mathbb{E}[s^X s^Y] = \mathbb{E}[s^X]\mathbb{E}[s^Y]
\end{aligned}$$

Example 1.10. Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli random variable.

$$\begin{aligned}
T =& X_1 + \ldots + X_n \\
\mathbb{E}[s^T] =& \mathbb{E}[s^{X_1 + \ldots + X_n}] = (\mathbb{E}[s^X])^n = (1 - p + ps)^n \\
\mathbb{E}[s^{X_1}] =& s^0(1-p) + sp
\end{aligned}$$

Let $X_n \sim \text{Binomial}(n, p_n)$, $p_n \to 0$, $np_n \to \lambda$, $n \to \infty$.

$$\begin{aligned}
G_n(s) &= \mathbb{E}[s^{X_n}] \\
&= (1 - p_n + p_n s)^n \\
&= \left(1 - \frac{np_n}{n} + \frac{np_n s}{n}\right)^n \\
&= \left(1 - \frac{np_n(1-s)}{n}\right)^n \to e^{-\lambda(1-s)}
\end{aligned}$$

as $n \to \infty$.

$X \sim \text{Poisson}(\lambda)$,

$$G(s) = \mathbb{E}(s^X) = \sum_{k=0}^{\infty} s^k P(x = k) = \sum_{k=0}^{\infty} s^k \frac{e^{-\lambda}\lambda^k}{\lambda!} = e^{-\lambda} e^{\lambda s} = e^{-\lambda(1-s)}.$$

1.4 Cumulative Distribution Function (c.d.f.)

Definition 1.11. Given a random variable X, its *cumulative distribution function (c.d.f.)* is defined as

$$F(t) := P(X \leq t), -\inf < t < \inf.$$

Given a Borel set A, we have
$$F(A) = P(X \in A)$$
For example, $P(X \in (a, b]) = F(b) - F(a)$.

Definition 1.12 (Convergence in distribution)**.** Let X_n be a sequence of random variables, X be a random variable. Let F_n be the cdf of X_n and F be the cdf of X. We can X_n converges to X in distribution (written as $X_n \overset{D}{\Longrightarrow} X$, or $X_n \to X$), if

$$F_n(t) \to F(t), \forall t \in \mathcal{C}(F)(\text{ the continuous domain of } F)] \text{ or}$$
$$\mathbb{E}[h(X_n)] \to \mathbb{E}[h(X)], \forall \text{ bounded continuous function of } h$$

Definition 1.13. We say X_n converges to X in (total) variation if $d_{TV}(X_n, X) \to 0$ as $n \to \infty$.

Example 1.14. Let X_n be constant random variable $1/n$ and $X = 0$. For every n, we have

$$d_{TV}(X_n, X) = \sup_A |F_n(A) - F_X(A)|$$

and $P(X_n = 0) = 0$, $P(X = 0) = 1$, so X_n does not converge to X in variation.

Given that $\mathcal{C}(F_X) = (-\infty, 0) \cup (0, \infty)$, we have for every $t \in \mathcal{C}(F_X)$, and for all large n,

$$\begin{cases} F_n(t) = 1, \text{ if } t \in (0, \infty) \\ F_n(t) = 0, \text{ if } t \in (-\infty, 0) \end{cases}$$

Hence, $F_n(t)$ converges to $F_X(t)$ for every $t \in \mathcal{C}(F)$, so $X_n \overset{D}{\Longrightarrow} X$.

Example 1.15. If you have an n-sided die labelled $1/n, 2/n, \ldots, n/n$. Then notice that

$$X_n \overset{D}{\Longrightarrow} U \sim \text{Uniform}(0, 1)$$

because if we consider any $t \in (0, 1)$, $F_U(t) = t$, and $F_n(t) = \frac{k}{n}$ where $(k-1)/n < t \leq k/n$. As $n \to \infty$, k/n converges to t.

Again X_n does not converge to X in variation. Let Q be the set of rational numbers. $P(X \in Q) = 0$ because Q has measure zero, but $P(X_n \in Q) = 1$. Hence $d_{TV}(X_n, X) = 1$ for every n.

1.4.1 Geometric Distribution to Exponential, the Memoryless variables

Let $T_n \sim \text{Geo}(p_n)$, then $\Pr[T_n = k] = (1 - p_n)^{k-1} p_n, k = 1, 2, \ldots$, $\Pr[T_n > k] = (1 - p_n)^k$, and $\Pr[T_n > k + j | T_n > k] = \Pr[T_n > j]$. Also, $\mathbb{E}[T_n] = 1/p_n$.

And let $X \sim \text{Exp}(\lambda)$, $f_x(t) = \lambda e^{-\lambda t}, t \geq 0$, $\Pr[X > t] = e^{-\lambda t}, \Pr[X > t + s | X > t] = \Pr[X > s], \mathbb{E}[X] = 1/\lambda$.

We will show
$$\frac{T_n}{n} \overset{D}{\to} X \sim \text{Exp}(\lambda).$$
First, let F_n be the c.d.f. of T_n and F_X be the c.d.f. of X. We need to show that $F_n(t) \to F_X(T)$ for all $t \in C(X)$.

Proof.

$$\begin{aligned}1 - F_n(t) = \Pr\left[\frac{T_n}{n} > t\right] &= \Pr[T_n > nt] = \Pr[T_n > \lfloor nt \rfloor] \\ &= (1 - \frac{np_n}{n})^{\lfloor nt \rfloor} = (1 - \frac{\lambda_n}{n})^{\lfloor nt \rfloor} \to e^{-\lambda t}\end{aligned}$$

where $\lambda_n := np_n \to \lambda$ as $n \to \infty$, and the convergence to $e^{-\lambda t}$ is by squeeze theorem. $\square$

1.5 Point Process

Consider $N \sim \text{Poisson}(\lambda)$ and let $X_1, \ldots, X_N$ be i.i.d. $\text{Bernoulli}(p)$. Define $Y = \sum_{i=1}^{N} X_i$. If for each of its count of N, it has p chances to be 1 and $(1-p)$ to be 0, then we can split N into two Poisson distributions

$$\begin{aligned}Y &\sim \text{Poisson}(\lambda p) \\ Z &\sim \text{Poisson}(\lambda(1-p))\end{aligned}$$

where $Z := N - Y$ and we have $Z \perp Y$ (seen that in homework 1).

Definition 1.16. A *point process* on $[0, \infty)$ is a mapping, assigning each Borel set $J \subseteq [0, \infty)$, a nonnegative extended integer valuesd r.v. $N(J) = N_J$, so that if $J_1, J_2, \ldots,$ are disjoint, then

$$N(\cup_i J_i) = \sum_i N(J_i)$$

A counting process associated with N (family of random variables), $N(t) = N_t$ for $t \geq 0$ where $N(t) = N((0, t])$ for $t > 0$. By convention, the sample paths are right continuous.

Definition 1.17. A *Poisson point process* with intensity $\lambda > 0$ is a point process with:

a) If $J_1, J_2, \ldots,$ are nonoverlapping intervals, then $N(J_1), N(J_2), \ldots,$ are independent.

b) $N(J) \sim \text{Poisson}(\lambda |J|)$ where J is the length of the interval J.

Given a Poissson Point Process above, let $0 = T_0 < T_1 \leq T_2 \leq T_3 \leq \ldots$ be the time i^{th} customer arrives and $\tau_n = T_n - T_{n-1}$. Then $\tau_1, \tau_2, \ldots,$ are i.i.d. $\exp(\lambda)$.

Example 1.18. Let $N(t)$ be the number of customers arriving during $(0, t]$ and $N \sim \text{Poisson}(5)$. The probability of 0 arrivals up to time 2 is

$$\Pr[N(2) = 0] = e^{-5(2)} = e^{-10}$$

While the probability of k arrivals up up time 2 is

$$\Pr[N(2) = k] = \frac{e^{-10} 10^k}{k!}.$$

Consider

$$\begin{aligned}
&\{N(5) = 7|N(2) = 1\} \\
&\{N((2,5]) = 6|N(2) = 1\} \\
&\Pr[N(5) - N(2) = 6|N(2) = 1] \\
= &\Pr[N(5) - N(2) = 6] \\
= &\Pr[N((2,5]) = 6] \\
= &\Pr[N(3) = 6]
\end{aligned}$$

We can also consider

$$\Pr[T_2 > 5.8|T_1 = 3.7] = \Pr[\tau_2 > 2.1|\tau_1 = 3.7] = e^{-\lambda(2.1)}$$

If you look at the store a 100 min, when will the next customer arrive?
We expect $\frac{1}{\lambda} = \frac{1}{5}\text{hr} = 12\text{min}$.

$$\begin{aligned}
\Pr[X_1 > t] = \Pr[N(t) = 0] &= e^{-\lambda t}, t \geq 0 \\
\Pr[X_2 > t|X_1 = s] &= \Pr[N((s, s+t]) = 0|X_1 = s] \\
&= \Pr[N((s, s+t]) = 0] \\
&= e^{-\lambda t}
\end{aligned}$$

1.6 Bernoulli and Poisson

Let $X_1, X_2, \ldots$, be Bernoulli Process with $p \in (0,1)$.
Question:

a) Is $\Pr[X_n = k|T = n]$ equal $\Pr[X_T = k|T = n]$? Yes.
Let $A = \{w \in \Omega : X_n(w) = k\}$, $B = \{w \in \Omega : T(w) = n\}$, $C = \{w \in \Omega : X_{T(w)}(w) = k\}$ and $A \cap B = \{w \in \Omega : X_n(w) = k, T(w) = n\}$, $C \cap B = \{w \in \Omega : X_{T(w)}(w) = k, T(w) = n\}$, which implies $\Pr[A \cap B]/\Pr[B] = \Pr[C \cap B]/\Pr[B]$

b) Is $\Pr[X_n = k|T = n]$ equal to $\Pr[X_n = k]$? No. e.g. $T := \min\{n : X_n = 1\}$, and $\Pr[X_n = 1|T = n] = 1$, $\Pr[X_n = 1] = p$.
e.g. $X_i \sim \text{Exp}(\lambda)$ where $X_1, X_2, \ldots,$ are event times.

$$\begin{aligned}
\Pr[X_2 > t|X_1 = s] &= \Pr[N((s, s+t]) = 0|X_1 = s] \\
&= \Pr[N(s, s+t] = 0] \text{ by independent increment} \\
&= \Pr[N(X_1, X_1 + t] = 0|X_1 = s]
\end{aligned}$$

But then let $T := \min\{r : N(r, r+t] = 10\}$. We have

$$\Pr[N(T, T+t) = 0|T = 3.87] = 0, \Pr[N(3.87, 3.87+t] = 0] = e^{-\lambda t}$$

Definition 1.19. Let $0 = T_0 < T_1 = \tau_1 \leq T_2 = \tau_1 + \tau_2 \leq \ldots$ be the *occurence times* of a Poisson process which are the successive times $N(t)$ jumps. Let $\tau_1, \tau_2, \ldots$ be the *interoccurence time*, where $\tau_i := T_i - T_{i-1}$.

Theorem 1.20 (Interoccurence Time Theorem)**.**

(A) Interoccurence times $\tau_1, \tau_2, \dots,$ of a Poisson process with rate λ are i.i.d. $\mathrm{Exp}(\lambda)$

(B) Let $Y_1, Y_2, \dots,$ be i.i.d. $\mathrm{Exp}(\lambda)$.

$$N(t) := \max\{n : \sum_{i=1}^{n} Y_i \le t\} \implies \{N(t)\}_{t\ge 0} \text{ is a Poisson counting process with rate } \lambda > 0$$

Example 1.21. Consider Bernoulli processes $\{X_k^m\}_{k\in\frac{\mathbb{N}}{m}}$ with parameter $p_m \in (0,1)$. Then $\tau_1^m = T_1^m = \min\{n \in \frac{\mathbb{N}}{m} = X_n^m = 1\}$ Then $m\tau_1^m \sim \mathrm{Geo}(p_m)$. Let $T_2^m = \min\{n > T_1^m : X_n^m = 1\}$ and $\tau_2^m = T_2^m - T_1^m$, then $m\tau_2^m \sim \mathrm{Geo}(p_m)$ as well. Then with the occurrence time T_i, we have a counting process

$$N^m(t_1) \sim \mathrm{Binomial}(?\,, p_m)$$

Useful later: $\{T_1 \ge t_1, T_2 \ge t_2\} \iff \{N(t_1) \ge 1, N(t_2) \ge 2\}$.

Theorem 1.22 (The law of small numbers for Bernoulli Process)**.** Let $\{X_r^m\}_{r\in\mathbb{N}/m}$ be a Bernoulli Process with parameter p_m indexed by multipliers of $\mathbb{N}/m$. Let $N^m(t)$ be the corresponding counting process. If $mp_m \to \lambda > 0$, then the counting process N^m converges in distribution to the counting process of a Poisson process with rate $\lambda > 0$ in the following sense:

$$\forall n, 0 = t_0 < t_1 < \ldots < t_n, (N^m(t_1), \ldots, N^m(t_n)) \xrightarrow{D} (N(t_1), \ldots, N(t_n))$$

Proof of Interoccurence Time Theorem.

a) We showed in the previous section that for a geometric r.v. with p_n with $np_n \to \lambda$. $T_n/n \xrightarrow{D} \mathrm{Exp}(\lambda)$. And we have seen that the interoccurrence times of Bernoulli $\{X_k^m\}_{k\in\mathbb{N}/n}$ are geometric, $\Delta_k^m = N^m(t_k) - N^m(t_{k-1}) \sim \mathrm{Binomial}(m(t_k - t_{k-1}) \pm 1, p_m)$ where $\pm$ considers the rounding of $m(t_k - t_{k-1})$. And this converges in distribution to $\Delta_k \sim \mathrm{Poisson}(\lambda(t_k - t_{k-1}))$. Thus the occurrence time of $N^m(t)$ converges to $N(t)$ in distribution. Thus, the interoccurence time of X_k^m, which is the interoccurence time of $N^m(t)$, converging to $\mathrm{Exp}(\lambda)$ implies that the interoccurence time of $N(t)$ converges to $\mathrm{Exp}(\lambda)$.

b) With a Poisson process with rate λ, and let τ_i be its interoccurence times, and we know $\tau_i \overset{iid}{\sim} \mathrm{Exp}(\lambda)$. Let Y_i be another sequence of i.i.d. expenentials with λ. Then since τ_i and Y_i have the same joint distribution, we also have

$$\left(\tau_1, \tau_1 + \tau_2, \ldots, \sum_{i=1}^{n} \tau_i\right) \overset{D}{=} \left(Y_1, Y_1 + Y_2, \ldots, \sum_{i=1}^{n} Y_i\right)$$

But $(Y_1, Y_1 + Y_2, \ldots, \sum_{i=1}^{n} Y_i)$ determines the joint distribution of the occurrence time of $N(t)$. That is, the occurence times of $N(t)$ are the occurence times of a Poisson distribution. So $N(t)$ is Poisson.

□

Given B), now we can simulate Poisson with $U_i \overset{D}{\sim} \text{Uniform}([0,1])$ and have $\tau_i = -\frac{1}{\lambda}\log(1-U_i)$. However, if the actual $\lambda > \mu$ and we simulate with μ, then we have

$$\tilde{\tau}_i = -\frac{1}{\mu}\log(1-U_i) \overset{D}{=} \frac{\lambda}{\mu}\tau_k$$

Theorem 1.23 (Generalized Thinning Theorem)**.** Let $N \sim \text{Poisson}(\lambda)$, X_i be iid r.v. with $\Pr[X_i = k] = p_k, k = 1, \ldots, m$ and $\sum_{i=1}^m p_k = 1$. And N is independent from X_i for all i. Let $N_k = \sum_{j=1}^N \mathbb{1}_{\{X_j = k\}}$.
e.g:

$$\begin{array}{cccccc} m=3 & x_1 & x_2 & x_3 & x_4 & x_5 \\ N=5 & 2 & 3 & 3 & 1 & 2 \end{array}$$

then $N_1 = 1$, $N_2 = 2$, $N_3 = 2$, $N_1 + N_2 + N_3 = N$.
We have that $N_1, \ldots, N_m$ are independent Poisson r.v. with $\mathbb{E}[N_k] = \lambda p_k$. (You can consider this as splitting a Poisson process into m different ones with probability p_k.)
And we have

$$\begin{aligned}
\Pr[N_1 = j_1, N_2 = j_2, \ldots, N_m = j_m] &= \Pr[N = j_1 + \ldots + j_m, N_1 = j_1, \ldots, N_m = j_m] \\
&= \underbrace{\Pr[N = j_1 + \ldots + j_m]}_{\text{Poisson}} \underbrace{\Pr[N_1 = j_1, \ldots, N_m = j_m | N = \sum_{i=1}^m j_i]}_{\text{multinomial}} \\
&= \frac{e^{-\lambda}\lambda^{j_1 + \ldots + j_m}}{(j_1 + \ldots + j_m)!}\binom{j_1 + \ldots + j_m}{j_1, \ldots, j_m} p_1^{j_1} \ldots p_m^{j_m} \\
&= \prod_{i=1}^m \frac{e^{-\lambda p_i}(p_i\lambda)^{j_i}}{j_i!}
\end{aligned}$$

Scecond Construction Let $m_1, m_2, \ldots$ be iid $\text{Poisson}(\lambda)$. Let $U_1, U_2, \ldots$ be iid $\text{Uniform}(0,1)$ such that $(m_1, m_2, \ldots)$ independs $(U_1, U_2, \ldots)$. Put points at $U_1, \ldots, U_{m_1}$ if $m_1 > 0$. Put points at $1 + U_{m_1+1}, \ldots, 1 + U_{m_2}$ if $M_2 > 0$ and so on.

Claim 1.23.1. Above points form a Poisson point process (THM 7 of UChichago Notes).

Proof. $0 = t_1 < t_1 < \ldots < t_n = 1, J_k = (t_{k-1}, t_k] \implies p_k = t_k - t_{k-1}$. $N(J_1), \ldots, N(J_n)$ independent Poisson $\mathbb{E}[N(J_k)] = \lambda p_k = \lambda |J_k|$. □

Definition 1.24. Poisson point process on $\mathbb{R}^k$ with mean measure Λ is a point process on $\mathbb{R}^k$ with

1. $J_1, J_2, \ldots$ disjoint Borel sets in $\mathbb{R}^k$; $N(J_1), N(J_2), \ldots$ are independent.
2. $N(J_k) \sim \text{Poisson}(\Lambda(J_k))$

Proposition 1.25. To show a point process is a Poisson point process, it suffices to verify the conditions above for rectangles J, J_i with sides parallel to the coordinate axes.

Example 1.26. Let T_i be the occurence times of a Poisson process on $[0, \infty)$ with rate λ. Let S_j be the iid rv with CDF F. S_j, T_i are indep. Then we have $J = [t_1, t_2] \times [s_1, s_2]$. So $N(J) = \lambda(t_2 - t_1)(s_2 - s_1)$, where $J' \cap J = \emptyset$ implies $N(J)$ independent $N(J')$.

For a Poisson Point Process on $\mathbb{R}$ with rate $\lambda > 0$, then given $t > 0$, we have

$$\begin{aligned} \Pr[N(0,t] = 0] &= e^{-\lambda t} \\ \Pr[N(-t,0] = 0] &= e^{-\lambda t} \\ \Pr[N(-t,t] = 0] &= e^{-2\lambda t} \end{aligned}$$

Given 2 Poisson Processes on $[0, \infty)$ with $N \sim \mathrm{Poisson}(\lambda)$, $M \sim \mathrm{Poisson}(\mu)$, $\lambda > \mu$, how can we comply them so $N(J) \geq M(J)$ for every Borel set J?

1. Superposition: Consider M as above and $L \sim \mathrm{Poisson}(\lambda - \mu)$, which are independent, then take the superposition (a process made of all success of M, L) so we get another $\mathrm{Poisson}(\lambda)$.

2. Decomposition: With the N above, for each success of N, split it to M with probability μ/λ, and L with $(1 - \mu/\lambda)$, then M and L are independent Poisson Processes and M is what's required.

Consider N, M with the distributions above, let T_1 be the time of first success of N, then what's the probability that $M(T_1) = k$? If we directly compute it, it will be

$$\Pr[M(T_1) = k] = \int_0^\infty \Pr[M(T_1) = k | T_1 = s] \underbrace{\lambda e^{-\lambda s}}_{\Pr[T_1 = s]} \, ds$$

which is not that easy to compute. But we can consider $N + M \sim \mathrm{Poisson}(\lambda + \mu)$. And split its success to N, M with probability $\frac{\lambda}{\mu+\lambda}$ and $\frac{\mu}{\mu+\lambda}$ respectively. Then T_1 is the time when a success is splitted to N the first time. That is, $M(T_1 = k)$ can be considered as a geometric process with k failure and one success, so

$$\Pr[M(T_1) = k] = \left(\frac{\mu}{\lambda + \mu}\right)^k \left(\frac{\lambda}{\lambda + \mu}\right)$$

Let $\{N(t)\}_{t \geq 0}$ be a counting process on $[0, \infty)$. Prove or disprove: If $N(t) \sim \mathrm{Poisson}(\lambda t)$ for all $t > 0$, then N is a Poisson Process.
Let T_i be the occurence times and τ_i be the interoccurence times as before. Then $T_n = \tau_1 + \ldots + \tau_n$. If τ_i are independent $\mathrm{Exp}(\lambda)$, we know $T_n \sim \mathrm{Erlang}(n, \lambda)$, so $\mathbb{E}[T_n] = n/\lambda$ and

$$F_n(t) = \Pr[T_n \leq t] = \Pr[N(t) \geq n] = \sum_{k=n}^\infty \frac{e^{-\lambda t (\lambda t)^k}}{k!} = 1 - \sum_{k=0}^{n-1} \frac{e^{-\lambda t (\lambda t)^k}}{k!}$$

so if $T_1, T_2, \ldots,$ have the "right" distribution, then $N(t)$ will be $\mathrm{Poisson}(\lambda t)$. What if we don't have the independence? Consider $T_i := F_i^{-1}(U)$ where F_i is the cdf of $\mathrm{Erlang}(i, \lambda)$ and $U \sim \mathrm{Uniform}(0, 1)$. Then it's not hard to see that each $T_i \sim \mathrm{Erlang}(i, \lambda)$, however, once T_1 is given, we can compute U_1 and hence all $T_2, T_3, \ldots$ are know, so the process with T_i being the occurence time is not a Poisson.

limits of expectation and expectation of limits

Theorem 1.27 (Monotone Convergence Theorem)**.** Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables such that for all $n \geq 1$,

$$0 \leq X_n \leq X_{n+1}, \text{ Probably a.s.,}$$

then

$$\mathbb{E}[\lim_{n\to\infty} X_n] = \lim_{n\to\infty} \mathbb{E}[X_n].$$

Theorem 1.28 (Deominated Convergence Theorem)**.** Let $\{X_n\}_{n\geq 1}$ be a sequence of random variable such that for all ω outside a set $\mathcal{N}$ of null probability there exists $\lim_{n\to\infty} X_n(\omega)$ and such that for all $n \geq 1$

$$|X_n| \leq Y, \text{ Probably a.s.,}$$

where Y is some integrable random variable. Then

$$\mathbb{E}[\lim_{n\to\infty} X_n] = \lim_{n\to\infty} \mathbb{E}[X_n].$$

Example 1.29 ("Counter Example")**.** Suppose we are rolling a fair dice independently. Every time we get 6, we lose all the money, otherwise, we double the current amount. Starting with $X_0 = 100$, we have

$$X_n = \begin{cases} 100 * 2^n, & \text{with prob } (5/6)^n \\ 0, & \text{with prob } 1 - (5/6)^n \end{cases}$$

$$\begin{aligned} \mathbb{E}[X_n] &= 100 * (5/3)^n \\ \lim_{n\to\infty} \mathbb{E}[X_n] &= \infty \\ \mathbb{E}[\lim_{n\to\infty} X_n] &= 0 \end{aligned}$$

where the last inequality is by $\lim_{n\to\infty} \Pr[X_n > 0] = 0$ and $\lim_{n\to\infty} \Pr[X_n = 0] = 1$, so $X_n \to 0$ almost surely.

Let N be a Poisson on $[0, \infty)$ with rate λ. Let $T \geq 0$ be a r.v. such that N, T are independent. If we know the distribution of $N(T)$, can we determine the distribution of T? First consider the *probability generating function* (p.g.f.) of a Poisson $X \sim \text{Poisson}(\lambda)$, we have

$$G(s) = \mathbb{E}[s^X] = \sum_{k=0}^{\infty} s^k \Pr[X = k] = \sum_{k=0}^{\infty} s^k \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} e^{\lambda s} = e^{-\lambda(1-s)}$$

Or let x be nonnegative, integer-valued r.v.m the *Laplace-Stieltjes Transformation* of X is

$$L(s) = \mathbb{E}[e^{-sX}] = \int_0^\infty e^{-st} dF(t) = \int_0^\infty e^{-st} F(dt)$$

note this formula prevent us from worrying about the continuity of X by $F(t)$.
Recall the moment generating function (m.g.f.) $m_X(\theta) = \mathbb{E}[e^{\theta X}]$. We give some examples,

Example 1.30.

1. When $\Pr[T = t] = 1$, we have $\mathbb{E}[e^{-sT}] = e^{-st}$.

2. When $T \sim \text{Bernoulli}(p)$,

$$L(s) = \mathbb{E}[e^{-sT}] = (1-p) * 1 + p * e^{-s} = \int_{[0,\infty)} e^{-st} dF(t)$$

3. $T \sim \text{Binomial}(n, p)$. $T = X_1 + \ldots + X_n$, where X_i are i.i.d. Bernoulli.

$$\begin{aligned} L(S) &= \mathbb{E}[e^{-sT}] \\ &= \int_{[0,\infty)} e^{-st} dF(t) \\ &= \mathbb{E}[e^{-s(X_1+\ldots+X_n})] \\ &= \mathbb{E}[e^{-sX_1} \ldots e^{-sX_n}] \\ &= \mathbb{E}[e^{-sX_1}] \ldots \mathbb{E}[e^{-sX_n}] \\ &= (1 - p + pe^{-s})^n \end{aligned}$$

4. Let $X \sim \text{Exp}(\lambda)$, we have

$$\mathbb{E}[e^{-sX}] = \int_0^\infty e^{-st} \lambda e^{-\lambda t} dt = \frac{\lambda}{s + \lambda}. \tag{L.S. of Exp}$$

Lemma 1.31. Given a $N(T) \sim \text{Poisson}(\lambda)$, and N being independent from T, we have $L_T(s) = G(1 - s/\lambda)$.

Proof.

$$\begin{aligned} G(z) &= \mathbb{E}[z^{N(T)}] \\ &= \mathbb{E}[\mathbb{E}[z^{N(T)} | T]] \\ &= \mathbb{E}[e^{-\lambda T(1-z)}] \\ &= L(\lambda(1 - z)) \end{aligned}$$

where the second last inequality is by

$$G(z) = \mathbb{E}[z^{N(T)}] = \sum_{k=0}^{\infty} z^k \frac{e^{-\lambda T} (\lambda T)^k}{k!} = e^{-\lambda T(1-z)}.$$

And then let $s = \lambda(1 - z)$, we are done. $\square$

Thus, when $N(T) \sim \text{Poisson}(\lambda t)$,

$$L(s) = G(1 - s/\lambda) = e^{-\lambda t(1-(1-s/\lambda))} = e^{-st}$$

so $\Pr[T = t] = 1$.

Theorem 1.32 (Not gonna prove)**.** Like p.g.f. and m.g.f., $L(s)$ uniquely corresponds to a random distribution.

Example 1.33. Let $\Pr[N(T) = k] = \rho^k(1-\rho), k = 0, 1, \ldots$. Then

$$
\begin{aligned}
G(z) = \mathbb{E}[z^{N(T)}] &= \sum_{k=0}^{\infty} z^k \rho^k (1-\rho) = \frac{1-\rho}{1-\rho z}. \\
L(s) = \mathbb{E}[e^{-sT}] &= G(1 - s/\lambda) = \frac{1-\rho}{1-\rho(1-s/\lambda)} \\
&= \frac{1-\rho}{1-\rho+\rho s/\lambda} = \frac{\frac{\lambda}{\rho}(1-\rho)}{\frac{\lambda}{\rho}(1-\rho)+s}
\end{aligned}
$$

which shows that $T \sim \mathrm{Exp}(\frac{\lambda}{\rho}(1-\rho))$ by (L.S. of Exp).

2 Markov-Chain

Let $X_0, X_1, \ldots$ be discrete-time stochastic processes and let the state space be countable.

$$\Pr[X_0 = i_0, \ldots, X_n = i_n], \forall n,\ i_0, \ldots, i_n \in \text{ state space.}$$

1. Markov Property:

$$\Pr[\underbrace{X_{n+1} = j}_{\text{future}} | \underbrace{X_n = i_n}_{\text{present}}, \underbrace{\ldots, X_0 = i_0}_{\text{past}}] = \Pr[X_{n+1} = j | X_n = i_n]$$

2. Time Homogeneity:

$$\Pr[X_{n+1} = j | X_n = i] = \Pr[X_1 = j | X_0 = i] = \Pr(i, j)$$

Definition 2.1. $X_0, X_1, \ldots$ is a *discrete-time Markov chain (DTMC)* if $X_0, X_1, \ldots$ has the two properties above.

Example 2.2. Let $X_0, X_1, \ldots$ be an independent Bernoulli process with parameter p. Then the state space is $\{0, 1\}$.

$$\begin{array}{lcl} \Pr[X_{n+1} = j | X_n = i_n, \ldots, X_0 = i_0] & = & \Pr[X_{n+1} = j] \\ \Pr[X_{n+1} = j | X_n = i_n] & = & \Pr[X_{n+1} = j] \\ \Pr[X_{n+1} = j | X_n = j] & = & \Pr(j, j). \end{array}$$

This forms a really special DTMC, basically every r.v. are i.i.d.. Its transition matrix looks like

$$P = \begin{bmatrix} 1-p & p \\ 1-p & p \end{bmatrix}$$

where the rows represent the "from" and columns represent the "to". That is, $[P]_{ij} = \Pr(i, j)$.

Example 2.3. Let $X_0, X_1, \ldots \sim \text{Bernoulli}(p), p \in (0, 1)$. $Y_n = X_n + X_{n+1} \in \{0, 1, 2\}$. Is $Y_0, Y_1, \ldots$ a Markov Chain? No.

$$\Pr[Y_2 = 0 | Y_1 = 1, Y_0 = 0] = 0$$
$$\Pr[Y_2 = 0 | Y_1 = 1, Y_0 = 2] = 1 - p$$

because $Y_0 = 0, Y_1 = 1$ implies that $X_2 = 1, X_0 = X_1 = 0$, first probability is the probability that $X_3 = -1$ and the second one is the probability that $X_3 = 0$.
What can we add to make it a DTMC?
Acquire more information. Let $Z_n = (X_n, Y_n)$, then we consider

$$\Pr[Z_{n+1} = (j_1, j_2) | Z_n = (i_1, i_2), Z_{n-1} = (k_{n-1}, \ell_{n-1}), \ldots, Z_0 = (k_0, \ell_0)]$$

And the transition matrix is

	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	1-p	p	0	0
(0,1)	0	0	1-p	p
(1,0)	1-p	p	0	0
(1,1)	0	0	1-p	p

$M/M/1$ **Queue** Consider an $M/M/1$ queue, which is the queue with customers arriving according to $\mathrm{Poisson}(\lambda)$, service time following i.i.d. $\exp(\mu)$ with 1 server. The model records the number of customers whenever a process (arrival or service) is done. Note that this process or a point from the Poisson process does not have to "happen". You can treat all events as a $\mathrm{Poisson}(\lambda+\mu)$. For each point, there is a chance we have a service done, and another chance the we have an arrival. However, since this is an event, when there is 0 customer in the system, next point can still be a departure point, but the number of customers will stay at 0 instead of going to -1. When there are at least one customer in the system, the server actually serves the customer and make the number of customers minus 1.

For example, if we have $X_0 = 0$ and the next event is finishing a service, $X_1 = 0$, if it's a customer arrival, $X_1 = 1$. This model is also called the birth and death model, basically we add one when we have a birth and minus one when we have a death. Since the moment starts, we can only have "deaths" (or departures) until the first arrival. That is, given $X_n = 0$, the probability that $X_{n+1} = 0$ is the probability that

$$\Pr[D < A] = \frac{\mu}{\lambda+\mu}$$

where $D \sim \exp(\mu)$ is the service time and $A \sim \exp(\lambda)$ is the interoccurence time of $\mathrm{Poisson}(\lambda)$ (i.e. the arrival time). Similarly, given $X_n = 0$, the probability that $X_{n+1} = 1$ is the probability that the customer arrives before the service time. So the transition matrix looks like

$$\begin{bmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} & 0 & \dots & \dots & \\ \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & \dots & \\ 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & \dots \end{bmatrix}$$

where rows and columns are from 0 to infinity.

We can also consider $X_n :=$ number of customers in the system just before n-th arrival. For example, given $X_n = 0$, the probability $X_{n+1} = 0$ is $\frac{\mu}{\lambda+\mu}$, because $X_n = 0$, so between n-th and $n+1$th arrival, there is at most one customer in the system, and we have the probability $\frac{\mu}{\lambda+\mu}$ to finish the service before $n+1$-th arrival, otherwise, with probability $\frac{\lambda}{\mu+\lambda}$, we still have a customer in the system when $n+1$-th customer arrives.

Another way of considering this is treating the arrivals as a geometric distribution with $\frac{\lambda}{\lambda+\mu}$ success rate. For example, if $X_n = 1$. That means between n and $n+1$ arrivals, there are 2 customers in the system, and we do the geometric experiment. The probability that there is no customer in the system when $n+1$th customer arrives is the probability we "fail" at least twice before the "success". Similarly, the probability that there is one customer in the system when $n+1$th customer arrives is the prob that we "fail" exactly once before the first success, and so on. So the transition matrix looks like:

$$\begin{bmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} & 0 & \dots & \dots \\ \left(\frac{\mu}{\lambda+\mu}\right)^2 & \frac{\mu\lambda}{(\lambda+\mu)^2} & \frac{\lambda}{\lambda+\mu} & 0 & \dots \\ \left(\frac{\mu}{\lambda+\mu}\right)^3 & \left(\frac{\mu}{\lambda+\mu}\right)^2 \frac{\lambda}{(\lambda+\mu)} & \frac{\mu\lambda}{(\lambda+\mu)^2} & \frac{\lambda}{\mu+\lambda} & \dots \end{bmatrix}$$

$M/M/1/3$ **Queue** Consider the $M/M/1/3$ queue where the 3 means the capacity of the system. Let $Y_n :=$ number of customers in the system just after the n-th departure, so now the state space is $\{0, 1, 2\}$. Then let's say $Y_n = 0$, then the probability $Y_{n+1} = 0$ is the probability that there is an arrival between n-th and $n+1$-th departures. In other words, for $n+1$-th departure to happen, there has to be an arrival, so the probability is actually the probability that the $(n+1)$-th departure happen before any arrivals except for the necessary one, which is $\frac{\mu}{\lambda+\mu}$, similar to other cases. So the transition matrix looks like:

$$\begin{bmatrix} \frac{\mu}{\mu+\lambda} & \frac{\lambda\mu}{(\mu+\lambda)^2} & \left(\frac{\lambda}{\lambda+\mu}\right)^2 \\ \frac{\mu}{\mu+\lambda} & \frac{\lambda\mu}{(\mu+\lambda)^2} & \left(\frac{\lambda}{\lambda+\mu}\right)^2 \\ 0 & \frac{\mu}{\mu+\lambda} & \frac{\lambda}{\mu+\lambda} \end{bmatrix}$$

2.1 Transition Matrix

Definition 2.4. A matrix P is a *stochastic matrix* if $P(i,j) \geq 0$, and $\sum_{j \in S} P(i,j) = 1$. It is called a *doubly stochastic matrix* if it is a stochastic matrix and $\sum_{i \in S} P(i,j) = 1$. It is called a *substochastic matrix* if $P(i,j) \geq 0$ and $\sum_{j \in S} P(i,j) \leq 1$.

Given $S = \{0, 1, 2\}$, and a transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}. \tag{2.1}$$

We have the transition plot of the above matrix,

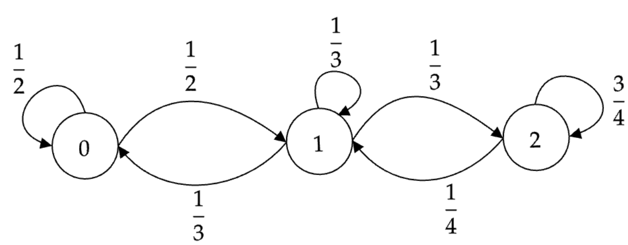


Figure 1: Transition Plot of P

Lemma 2.5. $\Pr[A, B, C, D] = \Pr[A]\Pr[B|A]\Pr[C|AB]\Pr[D|ABC]$

Example 2.6. Given $X_0, X_1, \ldots$, we have

$$\begin{aligned} &\Pr[X_0 = i_0, \ldots, X_n = i_n] \\ =&\Pr[X_0 = i_0]\Pr[X_1 = i_1 | X_0 = i_0] \ldots \Pr[X_n = i_n | X_{n-1} = i_{n-1}, \ldots, X_0 = i_0] \\ =&\Pr[X_0 = i_0]P(i_0, i_1)P(i_1, i_2) \ldots P(i_{n-1}, i_n) \end{aligned}$$

Definition 2.7. We use *measure distributions* on S that are functions from S to $\mathbb{R}$ to describe a distribution of a random variable. We use α, β, μ, π to describe row vectors, and use f, g, h to describe column vectors. For example,

$$X_0 \sim \alpha = (1/3, 1/2, 1/6)$$

and a function

$$f = \begin{pmatrix} f(0) \\ f(1) \\ f(2) \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix},$$

then $\alpha f = \mathbb{E}[f(X_0)] \in \mathbb{R}$.

Example 2.8.

$$\begin{aligned} \Pr[X_2 = j | X_0 = i] &= \sum_{k \in S} \Pr[X_2 = j, X_1 = k | X_0 = i] \\ &= \sum_{k \in S} P(i,k) P(k,j) \\ &= P^2(i,j). \end{aligned}$$

For our P, we have $P^2(1,1) = \frac{1}{6} + \frac{1}{9} + \frac{1}{12}$.

Lemma 2.9 (Chapman-Kolmogorov)**.**

$$P^{m+n}(i,j) = \sum_{k \in S} P^m(i,k) P^n(k,j)$$

where $P^{m+n} = P^m P^n$.

Example 2.10. $\Pr[X_4 = 1, X_2 = 0, X_7 = 1 | X_1 = 2] = P(2,0)P^2(0,1)P^3(1,1)$.

Lemma 2.11.

$$X_0 \sim \alpha \implies X_1 \sim \alpha P, \ldots, X_n \sim \alpha P^n$$

And

$$\begin{aligned} \Pr[X_1 = j] &= \sum_i \Pr[X_1 = j | X_0 = i] \Pr[X_0 = i] \\ &= \sum_i \alpha(i) P(i,j) \end{aligned}$$

Example 2.12.

$$\Pr[X_4 = 1 | X_5 = 1] = \frac{\Pr[X_4 = 1, X_5 = 1]}{\Pr[X_5 = 1]} = \frac{\Pr[X_5 = 1 | X_4 = 1] \Pr[X_4 = 1]}{\Pr[X_5 = 1]} = \frac{\alpha P^4(1) P(1,1)}{\alpha P^5(1)}$$

With the properties above, we can let f be a vector and have

$$\begin{aligned} [Pf]_i &= \mathbb{E}[f(X_1) | X_0 = i] \\ [P^n f]_i &= \mathbb{E}[f(X_n) | X_0 = i] \\ \alpha P^n f &= \mathbb{E}[f(X_n)] \end{aligned}$$

Definition 2.13. An *invariant measure* μ is a measure that $\mu = \mu P$. For our matrix P in (2.1), $\mu = (1, 3/2, 2)$ is an invariant measure.
A *stationary distribution* is an invariant measure that sums to 1. For our P in (2.1), $(2/9, 3/9, 4/9)$ is one.

2.2 Communication, Recurence and Transience

Definition 2.14. We say j is *accessible* from i if $\exists n \geq 0$ such that $P^n(i,j) > 0$.
We say i and j *communicate* ($i \sim j$) if i is accessible from j and vice versa.
We say i is *absorbing* j if $P(i,j) = 1$.

Proposition 2.15. Communication is an equivalent relation being:

- reflective: $i \sim i$, which is always true by letting $n = 0$ and hence $P = I$.
- symmetric: $i \sim j \implies j \sim i$.
- transitive: $i \sim j, j \sim k \implies i \sim k$. (If there exists n with $P^n(i,j) > 0$ and m with $P^m(j,k) > 0$ then $m + n$ takes us from i to j.

Example 2.16. For the following plot, we see that for each state, they only communicate with themselves.

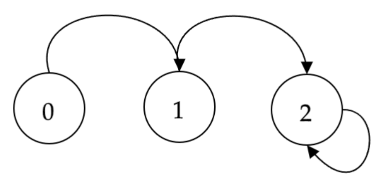


Figure 2: Self Commu States

Definition 2.17. If every two states communicate, then we say this Markov Chain is *irreducible*.

Definition 2.18. The *period* of state i is $d(i)$ defined as the greatest common divider of $\{n > 0 | P^n(i,i) > 0\}$. If $d(i) = 1$ for every state i, then the Markov Chain is *aperiodic*.

Example 2.19. Given the following graph:

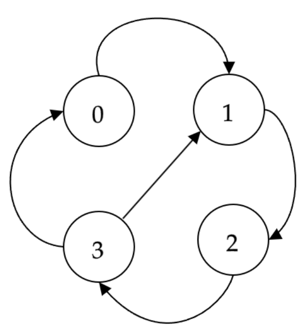


Figure 3: Period 1

Consider $i = 0$, then

$$\{n > 0 | P^n(0,0) > 0\} = \{4, 7, 10, 13, \ldots\} \implies d(0) = 1$$

For the following graph:

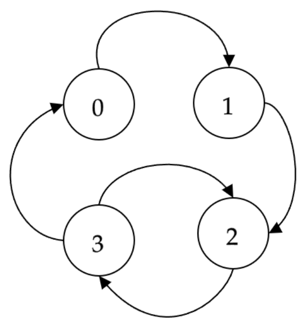


Figure 4: Period 2

Consider $i = 0$, then

$$\{n > 0 | P^n(0,0) > 0\} = \{4, 6, 8, \ldots\} \implies d(0) = 2$$

Proposition 2.20. If i and j communicate, $d(i) = d(j)$.

Proof. We know there exist m and n such that $P^m(i,j) > 0$ and $P^n(j,i) > 0$, so $P^{m+n}(i,i) > 0$, and $m + n$ is a multiplier of $d(i)$. Let ℓ be an integer such that $P^\ell(j,j) > 0$. Then

$$P^{m+n+\ell}(i,i) \geq P^m(i,j)P^\ell(j,j)P^n(j,i) > 0$$

so $m + n + \ell$ is a multiplier of $d(i)$. Hence, we know $m + n + \ell$ is a multiplier of $d(i)$, so ℓ is a multiplier of $d(i)$ which implies $d(j) \geq d(i)$. The argument for $d(i) \geq d(j)$ is similar, so $d(i) = d(j)$. $\square$

Definition 2.21. T is called a stopping time if $\{T = n\}$ can be determined from $X_0, \ldots, X_n$, i.e.

$$\mathbb{1}_{T=n} = g_n(X_0, \ldots, X_n).$$

for some function g_n.

Example 2.22. $T_x = \inf\{n \geq 0 | X_n = x\}$ is a stopping time. $T_x^k =$ time of k^{th} visit of x is also a stopping time.

Let T be a stopping time, then

$$\Pr[X_{T+1} = i_{m+1}, X_{T+2} = i_{m+2}, \ldots, X_{T+n} = i_{m+n} | T = m, X_m = i_m, X_{m-1} = i_{m-1}, \ldots, X_0 = i_0]$$
$$= P(i_m, i_{m+1}) \ldots P(i_{m+n-1}, i_{m+n})$$

and since T is a stopping time, $T = m$ is redundant by knowing $X_m, \ldots, X_0$. This is called *Strong Markov Property*. That is, the Strong Markov Property says that if we know a stopping time $T = m$, then we can treat the Markov chain after T as one Markov chain Y with the same transition matrix P but starting with $Y_0 = X_m$.

Definition 2.23. Let $T_x^1 = T_x = \inf\{n \geq 1 | X_n = x\}$, $T_x^k = \inf\{n > T_x^{k-1} | X_n = x\}$, $k = 2, 3, \ldots$, and $\Pr[X_0 = x] = 1$.

- State x is *recurrent* if $\Pr_x[T_x < \infty] = 1$.
- State x is *transient* if $\Pr_x[T_x < \infty] < 1$.
- State x is *positive recurrent* if $\mathbb{E}_x[T_x] < \infty$.
- State x is *null recurrent* if x is recurrent and $\mathbb{E}_x[T_x] = \infty$.

Example 2.24. Let $\Pr[X = k] = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ for $k = 1, 2, \ldots$. Then

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \frac{1}{k+1} = \infty$$

$$\Pr[X \leq n] = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1}$$

Suppose x is recurrent. How many times will x be revisited is represented as

$$N_x = \sum_{k=0}^{\infty} [X_k = x].$$

Suppose state x is transient, by Strong Markov property,

$$\Pr[T_x^k < \infty] = \Pr_x[T_x < \infty]^k.$$

Assuming $X_0 = x$, $N_x \sim \mathrm{Geo}(\Pr[T_x = \infty])$. That is, N_x stops (the number will not increase) once we fall into the case X_n never comes to x.

Proposition 2.25. State x is recurrent if and only if $\mathbb{E}_X[N_X] = \infty$.

Proof.

$$\begin{aligned}
\mathbb{E}_X[N_X] &= \mathbb{E}_X \sum_{k=0}^{\infty} \mathbb{1}[X_k = x] \\
&= \sum_{k=0}^{\infty} \mathbb{E}_x[\mathbb{1}[X_k = x]] \\
&= \sum_{k=0}^{\infty} \Pr_x[X_k = x] = \sum_{k=0}^{\infty} P^k(x, x) \\
N_X &= 1 + \sum_{k=1}^{\infty} \mathbb{1}[T_x^k < \infty]
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}_X[N_X] &= 1 + \sum_{k=1}^{\infty} \mathbb{E}[\mathbb{1}[T_x^k < \infty]] \\
&= 1 + \sum_{k=1}^{\infty} \Pr[T_x^k < \infty] \\
&= 1 + \sum_{k=1}^{\infty} \Pr[T_x < \infty]^k \\
&= \begin{cases} \infty, & \text{if recurrent.} \\ \frac{1}{1-\Pr[T_x<\infty]}, & \text{transient.} \end{cases}
\end{aligned}
$$

□

Proposition 2.26. If x is recurrent and x, y communicate, then y is recurrent.

Proof. There exists k such that $P^k(x,y) > 0$, and there exists ℓ such that $P^\ell(y,x) > 0$.

$$
\sum_{n=1}^{\infty} P^{k+\ell+n}(y,y) \geq \sum_{n=1}^{\infty} P^\ell(y,x)P^n(x,x)P^k(x,y) = \infty.
$$

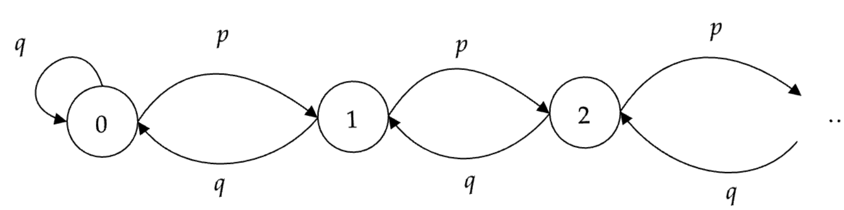
which implies that y is recurrent. □

Example 2.27.

$$
P = \begin{bmatrix} 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{6} & 0 & \frac{5}{6} \\ \frac{1}{6} & \frac{5}{6} & 0 \end{bmatrix}
$$

and all states are recurrent.

Example 2.28. Consider the below Markov chain with $0 < p < 1$.



Consider the probability of starting at 1 and first time visit 0 at k,

$$
P_1[T_0 = k] = p_k,
$$

and we have

$$
\begin{aligned}
\Phi(s) &= \sum_{k=0}^{\infty} p_k s^k \\
\Phi(s) &= qs + ps\Phi^2(s)
\end{aligned}
$$

where the second equality is by the fact that, $T_0 = 1$ when we go from 1 to 0 directly with probability q, otherwise, we go to 2 in the first step and then consider the steps required for us to go from 2 to 0, which is 2 to 1 then 1 to 0. In other words, we write

$$\begin{aligned}\Phi(s) &= \sum_{k=0}^{\infty} p_k s^k \\ &= 0 * 1 + qs + \sum_{k=2} p_k s^k \\ &= qs + s\sum_{k=0} p_{k+1} s^k \\ &= qs + ps\sum_{k=0} P_2[T_0 = k] s^k \\ &= qs + ps\mathbb{E}[s^{X+Y}]\end{aligned}$$

where $p_{k+1} = p * P_2[T_0 = k]$, and X is the random variable of number of steps from 0 to 1 and Y is from 2 to 1 which follow the same distribution as T_0 starting at 1 and are independent, so $\mathbb{E}[s^{X+Y}] = \mathbb{E}[s^X]\mathbb{E}[s^Y] = \Phi^2(s)$.
Then we can have that

$$\begin{aligned}\Phi(s) &= \frac{1-\sqrt{1-4pqs^2}}{2ps} \\ \Phi(1) &= \frac{1-\sqrt{1-4pq}}{2p} = \begin{cases} 1, \text{ if } p \le 1/2 \\ \frac{q}{p}, \text{ if } p > 1/2 \end{cases}\end{aligned}$$

That is, when $p > 1/2$, there is a chance we never go to 0. Or we can find the expectation by

$$E_1[T_0] = \lim_{s\to 1} \Phi'(s).$$

Definition 2.29. We call π a *stationary distribution* for a Markov chain with transition matrix P, if

$$\pi = \pi P, \sum \pi(i) = 1.$$

Example 2.30.

$$(\pi(0), \pi(1), \pi(2)) \begin{bmatrix} 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{6} & 0 & \frac{5}{6} \\ \frac{1}{6} & \frac{5}{6} & 0 \end{bmatrix} = (\pi(0), \pi(1), \pi(2))$$

solve to get

$$(\pi(0), \pi(1), \pi(2)) = (11/77, 31/77, 35/77)$$

and then

$$\mathbb{E}_0[T_0] = \frac{77}{11} = \frac{1}{\pi(0)}$$

because now we can consider it as a geometric distribution with parameter $\pi(0)$, starting from $X_0 = 0$, you have $11/77$ chance to get 0 at X_1, similarly, if you get $X_1 \neq 0$, then you still have

$11/77$ for $X_2 = 0$ by π being stationary, and so on.
We can also consider the central limit theorem which gives:

$$\frac{f(x_0) + \ldots + f(x_n)}{n+1} \to \pi f$$

for a function f valued on the states of the Markov chain X_i.

Example 2.31 (x-excursion chain)**.** Let $X_0, X_1, \ldots$ be an irreducible Markov chain with stationary distribution π, transition matrix P and state space S. Let's consider words (or strings if you prefer) that are finite, starting with x and containing only one x, call the set of all such words, S_y. Consider random variables Y_i with state space S_y, defined as

$$\begin{aligned}
Y_0 &= x \\
Y_1 &= xX_1 \\
Y_2 &= xX_1X_2 \\
Y_3 &= xX_1X_2X_3 \\
&\vdots
\end{aligned}$$

where we keep $X_0 = x$. So

$$\Pr[Y_3 = xy_1y_2y_3] = P(x, y_1)P(y_1, y_2)P(y_2, y_3).$$

and we can build the transition matrix Q for Y_i as

$$\begin{aligned}
&Q(xy_1 \ldots y_k, xy_1 \ldots y_ky_{k+1}) = P(y_k, y_{k+1}) \\
&Q(xy_1 \ldots y_k, x) = P(y_k, x) \\
&Q(x, xy) = P(x, y) \\
&Q(x, x) = P(x, x).
\end{aligned}$$

And we define $F : S_y \to S$ where $F(w)$ is the last letter of w.

Fact 2.32. If $Y_0, Y_1, \ldots$ is a Markov chain with transition matrix Q and state space S_y, then $F(Y_0), F(Y_1), \ldots$ is a Markov chain with state space S and transition matrix P.

Now let's consider the stationary distribution for Y. Let ν be a stationary distribution of Y_i, then

$$\begin{aligned}
\nu &= \nu Q \\
\nu(w) &= \sum_{w' \in S_y} \nu(w')Q(w', w), \sum_{w \in S_y} \nu(w) = 1
\end{aligned}$$

Let $w = xy_1 \ldots y_{k-1}y_ky_{k+1}$, we have

$$\begin{aligned}
\nu(xy_1 \ldots y_{k+1}) &= \nu(xy_1 \ldots y_k)Q(y_k, y_{k+1}) \\
\nu(xy_1 \ldots y_k) &= \nu(x)P(x, y_1)P(y_1, y_2) \ldots P(y_{k-1}, y_k)
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{w\in S_y}\nu(w) &= \nu(x)+\sum_{k=1}^{\infty}\sum_{y_1\ldots y_k}\nu(x)P(x,y_1)P(y_1,y_2)\ldots P(y_{k-1},y_k)\\
&= \nu(x)+\nu(x)\sum_{k=1}^{\infty}\sum_{y_1\ldots y_k}P(x,y_1)P(y_1,y_2)\ldots P(y_{k-1},y_k)\\
&= \nu(x)P_x(T_x>0)+\nu(x)\sum_{k=1}^{\infty}P_x(T_x>k)\\
&= \nu(x)\sum_{k=0}^{\infty}P_x(T_x>k)\\
&= \nu(x)\mathbb{E}_x[T_x]=1
\end{aligned}$$

If state x is recurrent, then we have $\nu(x)=\frac{1}{\mathbb{E}_x[T_x]}$, otherwise, Q does not have a stationary distribution. Thus if $X_0, X_1, \ldots$ has a positive recurrent state x, then there exists at least one stationary distribution ν by the fact $\nu(w)$ can be defined by $\nu(x)$ and $P(x,y_1),\ldots,P(y_{k-1},y_k)$.
If $Y_0\sim\nu$, and $Y_1,\ldots\sim\nu$, let $\pi(z)=\sum_{w,F(w)=z}\nu(w)$, we have $\pi=\pi P$ and $\sum_{x\in S}\pi(x)=1$.

Example 2.33. We consider a Markov chain $X_0, X_1, \ldots$. For the case we start with $X_0=x$, denote P_x, if we start with $X_0=y$, denote P_y. Let $\tau(i)$ be the time we have the i-th x excluding X_0, that is, $\tau(i)=T_x$, $\tau(2)=T_x^2$ and $\tau(0)=0$. Define

$$\begin{aligned}
W_1 &= (X_0, X_1, \ldots, X_{\tau(1)-1})\\
W_2 &= (X_{\tau(1)}, \ldots, X_{\tau(2)-1})\\
&\vdots
\end{aligned}$$

Under P_x, the words $W_1, W_2, \ldots$ are i.i.d. Under P_y, $y\neq x$, the workds $W_1, W_2, \ldots$ are independent, and $W_2, W_3, \ldots$ are identically distributed. Let $W_j=(X_{j,1},\ldots,X_{j,m(j)})$, then

$$\begin{aligned}
&P_x(W_1=w_1, W_2=w_2,\ldots,W_k=w_k)\\
=&\prod_{j=1}^{k}\left(\prod_{\ell=1}^{m(j)-1}P(x_{j,\ell},x_{j,\ell+1})\right)P(x_{j,m(j)},x)\\
=&\prod_{j=1}^{k}P(W_j=w_j)
\end{aligned}$$

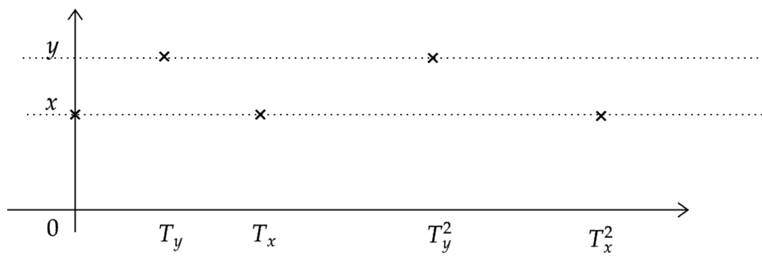
For P_y, $X_{1,1}=y$, all other $X_{j,1}$ remains at x, so $w_2, w_3, \ldots$ are identically distributed.

Proposition 2.34. WLOG, assume $x\neq y$, if x and y communicate, and x is positive recurrent, then y is positive recurrent.

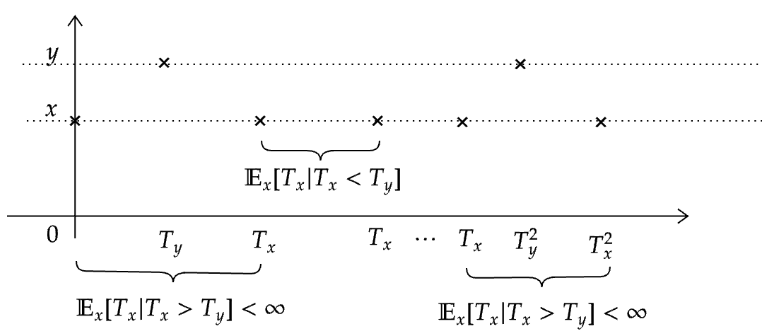
Proof.

$$\infty>\mathbb{E}_x[T_x]=\mathbb{E}_x[T_x|T_x>T_y]P_x[T_x>T_y]+\mathbb{E}_x[T_x|T_x<T_y]P_x[T_x<T_y]$$

If $P_x[T_x < T_y] = 0$, then $\mathbb{E}_y[T_y] \leq 2\mathbb{E}_x[T_x] < \infty$. The reason is that, we have $T_y \leq T_x$, then $\mathbb{E}_y[T_y]$ can be considered as $\mathbb{E}_x[T_y^2] - \mathbb{E}_x[T_y]$, but by $P_x[T_x < T_y] = 0$, we know for if we start at $X_0 = x$, then $T_y^2 \leq T_x^2$, see the plot below



If $P_x[T_x < T_y] > 0$, consider the plot



Similar, we have $\mathbb{E}_y[T_y] < \infty$. $\square$

2.3 Stationary Distribution and Positive Reucurrence

Consider a random variable X, we can write it as $X = X^+ + X^-$, where $X^+ := \max(X, 0)$ and $X^- := \max(-X, 0)$. If both $\mathbb{E}[X^+], \mathbb{E}[X^-]$ are well-defined with value in $[0, \infty]$. Then

$$\mu := \mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-]$$

unless it is $\infty - \infty$. To avoid this, we can assume either X is nonnegative, or $|X|$ integrable ($\mathbb{E}[|X|] < \infty$), or $\mathbb{E}[X^-] < \infty$, then we have $\mu < \infty$ or μ is well-defined as ∞.

Theorem 2.35 (Strong Law of Large Number)**.** Consider $S_n = X_1 + \ldots + X_n$

1. If $X_1, X_2, \ldots$ are pairwise i.i.d. integrable with mean μ, then

2. Or if $X_1, X_2, \ldots$ are i.i.d. with $\mathbb{E}[X^+] = \infty$, $\mathbb{E}[X^-] < \infty$ with mean $\mu = \infty$, then

$$\frac{S_n}{n} \to \mu \text{ a.s. w.p. } 1$$

almost surely with probability 1.

When we say with almost surely with probability 1, we mean that the set

$$A = \left\{ \omega \in \Omega : \frac{S_n(\omega)}{n} \to \mu \right\}$$

has a probability 1 when $n \to \infty$.

Example 2.36. Recall our "string" example, where $W_1 = (X_0, \ldots, X_{\tau(1)-1}, W_2 = (X_{\tau(0)}, \ldots, X_{\tau(2)-1}), \ldots$. Under P_x (start with $X_0 = x$), $W_1, W_2, \ldots$ are i.i.d., while under P_y, for $y \neq x$, $W_2, W_3, \ldots$ are i.i.d. and $W_1, W_2, \ldots$ are independent. Write $W_j = (X_{j,1}, \ldots, X_{j,m(j)})$, then

$$\Pr_x[W_1 = w_1, \ldots, W_k = w_k] = \prod_{j=1}^{k} \left(\prod_{\ell=1}^{m(j)-1} P(x_{j,\ell}, x_{j,\ell+1}) \right) P(x_{j,m(j)}, x).$$

Definition 2.37. Let $f : S \to \mathbb{R}_+$. The additive extension to the set of finite "words" with letters in S is the function f_+ where for $w = (x_1, \ldots, x_m)$,

$$f_+(w) = \sum_{i=1}^{m} f(x_i).$$

For any initial state $y \in S$ by the Strong Law of Large Number,

$$\lim_{k \to \infty} \frac{\sum_{i=1}^{k} f_+(w_i)}{k} = \mathbb{E}_x[f_+(w_1)] = \mathbb{E}_x[\sum_{j=0}^{\tau(1)-1} f(x_j)]$$

with P_y almost surely, because if $y \neq x$, then

$$\frac{f_+(w_1) + \ldots + f_+(w_k)}{k} = \frac{f_+(w_1)}{k} + \frac{f_+(w_2) + \ldots + f_+(w_k)}{k-1} \frac{k-1}{k} \to 0 + \mathbb{E}_x[f_+(w_2)] * 1.$$

In particular, if we set $f \equiv 1$, then

$$\lim_{k \to \infty} \tau(k)/k = \mathbb{E}_x[\tau(1)]$$

with P_y almost surely.

Let $N_n^x =$ the number of visits to state x up to time $n = \sum_{k=1}^{n} \mathbb{1}\{X_k = x\}$.

Theorem 2.38. Fix $x \in S$. If the Markov Chain is irreducible and positive recurrent, then $\exists!$ (there exists a unique) stationary distribution π and for all states x, y,

$$\lim_{n \to \infty} N_n^x/n = \pi(x), \ P_y\text{-a.s.}$$

If the chain is null recurrent, then there does not exist a stationary distribution and for all x, y,

$$\lim_{n \to \infty} N_n^x/n = 0, \ P_y\text{-a.s.}$$

Proof. First, we show $N_n^x/n \to 1/\mathbb{E}_x[T_x]$, P_y-a.s. Note, $N_n^x \le n$, and $N_n^x \to \infty$ P_y a.s.,

$$\frac{\tau(N_n^x)}{N_n^x} \le \frac{n}{N_n^x} < \frac{\tau(1+N_n^x)}{1+N_n^x}\frac{1+N_n^x}{N_n^x}.$$

where $n < \tau(1+N_n^x)$. And $\frac{\tau(N_n^x)}{N_n^x} \to \mathbb{E}_x[\tau(1)]$, $\frac{\tau(1+N_n^x)}{1+N_n^x} \to \mathbb{E}_x[\tau(1)]$, so $n/N_n^x \to \mathbb{E}_x[\tau(1)]$ with P_y-a.s..

Second, assume the Markov Chain has a stationary distribution π, then define $P_\pi(\cdot) = \sum_y \pi(y)P_y(\cdot)$,

$$N_n^x/n \to 1/\mathbb{E}_x[T_x],\ P_\pi\text{-a.s.}$$

by P_y-a.s and

$$\lim_{n\to\infty} \mathbb{E}_\pi[N_n^x/n] = \mathbb{E}_\pi \lim_{n\to\infty} N_n^x/n = \mathbb{E}_\pi[1/\mathbb{E}_x[T_x]] = 1/\mathbb{E}_x[T_x]$$

where the first equality is by $|N_n^x/n| \le 1$, $\mathbb{E}_\pi(1) = 1 < \infty$ by Dominant Convergence Theorem. The above equation is equivalent to

$$\lim_{n\to\infty} \mathbb{E}_\pi[N_n^x/n] = \lim_{n\to\infty} \mathbb{E}_\pi \frac{\sum_{j=1}^n \mathbb{1}[X_j = x]}{n} = \lim_{n\to\infty} \frac{n\pi(x)}{n} = \pi(x)$$

by π being stationary, $\mathbb{E}_x[\mathbb{1}[X_j = x]] = 1 * P_\pi(x) = \sum_y \pi(y)P_y(x) = \pi(x)$. Hence, for all state x,

$$\pi(x) = \frac{1}{\mathbb{E}_x[T_x]}.$$

For the positive recurrent case, π is uniquely defined as above. If it's the null recurrent case, then $\mathbb{E}_x[T_x] = \infty$, $\pi(x) = 0$, which is not even a distribution. □

Lemma 2.39. If $X_0, X_1, \ldots$ is recurrent, then the invariant measure is unique up to multiplication by constants.

Proof. See Bremaul's book. □

Combining the Lemma and Theorem, we know a recurrent Markov Chain's invariant measure sometimes does not give a stationary distribution because the sum of measure goes to infinity.

2.4 Period

2.4.1 Fundamental Theorem of Markov Chain

Let $a_1, a_2, \ldots$ be a sequence of integers. $d_k = g.c.d.(a_1, \ldots, a_k)$, if $1 \le d_k$ is nondecreasing and $d_k \to d$, then there exists k_0 such that $d_k = d$ for $k \ge k_0$.

Lemma 2.40. Let $S \subseteq \mathbb{Z}$ contain at least one non-zero element and be closed under addition and subtraction. Then S contains a smallest, positive integer a and $S = \{ka : k \in \mathbb{Z}\}$.

Proof. Let $c \in S$ with $c \neq 0$, then $0 = c - c \in S$ and $-c = 0 - c \in S$. Hence S contains at least one positive, one negative value. Then S contains a smallest positive element a. So

$$a, 2a, 3a, \ldots \in S$$
$$-a, -2a, -3a, \ldots \in S$$

so $\{ka : k \in \mathbb{Z}\} \subseteq S$. Let $c \in S$, $c = ka + r$, $0 \leq r \leq a - 1, r \in \mathbb{Z}$. And $0 \leq r = c - ka \in S$ by subtraction, but $r < a$ and a is the smallest positive integer in S, so $r = 0$. $\square$

Lemma 2.41. Let $a_1, a_2, \ldots, a_k$ be positive integer with g.c.d. d, there exist $n_1, n_2, \ldots, n_k \in \mathbb{Z}$ such that $d = \sum_{i=1}^{k} n_i a_i$.

Proof. The set $S = \{\sum_{i=1}^{k} n_i a_i : n_1, \ldots, n_k \in \mathbb{Z}\}$ is closed under additions and subtractions. So $S = \{ka : k \in \mathbb{Z}\}$ with $a = \sum_{i=1}^{k} n_i a_i$ being the smallest positive integer in S. Hence, d is a divisor of a by $a = \sum_{i=1}^{k} n_i a_i$. Then by $a_i = ka$, we know a is a divisor of a_i, so $a \leq g.c.d.(a_1, \ldots, a_k) = d$, so $a = d$. $\square$

Theorem 2.42. $A = \{a_1, a_2, \ldots\}$ which is a set of positive integers. Let $d = g.c.d.(A)$, and A is closed under addition. Then A contains, all but a finite number of multiples of d.

Proof. WLOG, $d = 1$. For some k, we have $d = g.c.d.(a_1, \ldots, a_k)$. By Lemma (2.41).

$$1 = \sum_{i=1}^{k} n_i a_i, \text{ for some } n_1, \ldots, n_k \in \mathbb{Z}, 1 \quad = M - P, \text{ where } M \geq 0, P < 0, M, P \in A.$$

Let $n \in \mathbb{N}$, $n \geq P(P-1)$, $n = aP + r$, $0 \leq r \leq P - 1$, so $a \geq P - 1$ (If $a \leq P - 2, aP + r < P(P-1)$). By $1 = M - P$, we have

$$n = aP + r(M - P) = (a - r)P + rM$$

and $a - r \geq 0$ by $a \geq P - 1 \geq r$, which implies $n \in A$. Hence, $n \in A$ except for $n < P(P-1)$, $n \in \mathbb{N}$. $\square$

Theorem 2.43 (Fundamental Theorem of Markov Chain)**.** For an irreducible positive recurrent aperiodic Markov X_i chain with the stationary distribution π and transition matrix P, we have

$$\lim_{n \to \infty} \Pr_i[X_n = j] = \lim_{n \to \infty} P^n(i, j) = \pi(j)$$

Proof. Consider two sequence of variables. Let $x = X_0, X_1, \ldots$ be the Markov chain starting with x, and $X_0^*, X_1^*, \ldots$ be a Markov chain where each $X_i^* \sim \pi$.

We have

$$\begin{aligned}
|\Pr_x[X_n = y] - \Pr_\pi[X_n = y]| =& |\Pr_x[X_n = y] - \pi(y)| \\
=& |\Pr[X_n = y, X_n^* = y] + \Pr[X_n = y, X_n^* \neq y] \\
& - \Pr[X_n = y, X_n^* = y] - \Pr[X_n^* = y, X_n \neq y]| \\
\leq& \Pr[X_n \neq X_n^*]
\end{aligned}$$

and we want to show $\Pr[X_n \neq X_n^*]$ goes to 0. Let $\tau := \min\{n \geq 0 : X_n = X_n^*\}$. And consider another independent Markov chain $X_0', X_1', \ldots$ which use the same transition matrix P. Consider a Markov chain V_n and its transition matrix Q:

$$V_n = (X_n, X_n'), \Pr[V_{n+1} = (y, y') | V_n = (x, x'), V_{n-1}, \ldots, V_0] = Q((x, x'), (y, y')) = P(x, y)P(x', y').$$

V_n has a stationary distribution where $\pi(x, x') = \pi(x)\pi(x')$ and

$$\begin{aligned}
\pi(y, y') &= \sum_x \sum_{x'} \pi(x, x') Q((x, x'), (y, y')) \\
&= \sum_x \sum_{x'} \pi(x)\pi(x')P(x, y)P(x', y') \\
&= \sum_x \pi(x)P(x, y) \sum_{x'} \pi(x')P(x', y') \\
&= \pi(y)\pi(y')
\end{aligned}$$

Consider

$$A_x = \{n \geq 1 : P^n(x, x) > 0\}$$

Then Theorem 2.42, there exists n_x such that $\forall n \geq n_x$, $P^n(x, x) > 0$ and there exists $k_{x,y}$ such that $P^{k_{x,y}}(x, y) > 0$, so $P^{n+k_{x,y}}(x, y) \geq P^{k_{x,y}}(x, y)P^n(x, x)$. Hence

$$P^n(x, y) > 0, \forall n \geq k_{x,y} + n_x,$$

similarly, we also have

$$P^n(x', y') > 0, \forall n \geq k_{x',y'} + n_{x'}.$$

Then for all $n \geq \max\{k_{x,y} + n_x, k_{x',y'} + n_{x'}\}$, we have

$$Q^n((x, x'), (y, y'))) > 0,$$

so V_n is irreducible and aperiodic (by letting $y, y' = x, x'$) and positive recurrent by having a stationary distribution.

Hence, all states are expected to be visited in finite time. $\tau' = \min\{n \geq 0 : X_n = X_n'\}, \tau < \infty$ almost surely by considering arbitrary (x, x). Consider

$$\bar{X}_n = \begin{cases} X_n', & n \leq \pi \\ X_n, & n > \pi \end{cases}.$$

By the Strong Markov Property, the part of X_n' and X_n for $n \geq \tau'$ are i.i.d. Markov chain, so the $\bar{X}_n$ we constrauct follow the same distribution as X_n^* follows. That is,

$$\Pr[X_n \neq X_n^*] = \Pr[X_n \neq \bar{X}_n] = \Pr[\tau' > n] \to 0$$

by $\tau < \infty$ almost surely. $\square$

2.5 Reversibility

Let P be a transition matrix for an irreducible Markov chain. Take a guess for stationary distribution π and a reverse transition matrix $\overleftarrow{P}$ with the same state space. If $\pi(j)\overleftarrow{P}(j,i) = \pi(i)P(i,j)$, then both guesses are right, we know this Markov chain is reversible with $\overleftarrow{P}$ and positive recurrent with π.

Consider $X_0, X_1, \ldots$ being stationary with the stationary distribution π, then

$$\begin{aligned}\Pr[X_n = i, X_{n+1} = j] &= \Pr[X_n = i] \Pr[X_{n+1} = j | X_n = i] = \pi(i)P(i,j) \\ &= \Pr[X_{n+1} = j] \Pr[X_n = i | X_{n+1} = j] \\ &= \pi(j)\overleftarrow{P}(j,i)\end{aligned}$$

Example 2.44. Consider a simple graph $G = (V, E)$ with vertices $0, \ldots, n$, then consider a Markov chain with states being the vertices with the transition matrix:

$$P(i,j) = \begin{cases} \frac{1}{d(i)}, & \text{if } ij \in E \\ 0, & \text{otherwise.} \end{cases}$$

Then $v = (d(0), d(1), \ldots)$ is an invariant measure. Consider

$$\begin{aligned} v(j)\overleftarrow{P}(j,i) &= v(i)P(i,j) \\ \iff d(j)\overleftarrow{P}(j,i) &= d(i)\frac{1}{d(i)} \end{aligned}$$

so $\overleftarrow{P}(j,i) = \frac{1}{d(j)}$, that is, $\overleftarrow{P} = P$.

2.6 Wald's (First) Lemma

Theorem 2.45. Wald's (First) Lemma Let $X_1, X_2, \ldots,$ be i.i.d. integrable random variables with $\mathbb{E}[|X_1|] < \infty$, and let T be a stopping time with $\mathbb{E}[T] < \infty$, then

$$\mathbb{E}[\underbrace{X_1 + X_2 + \ldots + X_T}_{=:S_T}] = \mathbb{E}[X_1]\mathbb{E}[T]$$

where $S_0 := 0$.

Proof. We have

$$S_T = \sum_{n=1}^{T} X_n = \sum_{n=1}^{\infty} X_n \mathbb{1}\{n \le T\} = \sum_{n=1}^{\infty} X_n \mathbb{1}\{T \le n-1\}^c,$$

where the supscript c means the complement of the set. Also consider

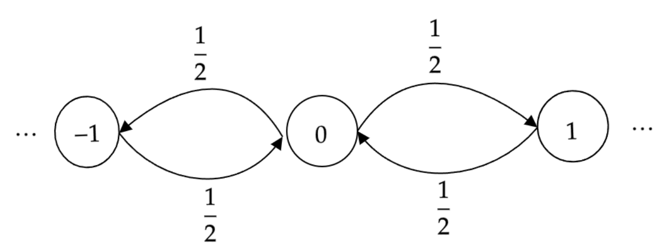
$$\begin{aligned}\mathbb{E}[S_T] &\overset{?}{=} \sum_{n=1}^{\infty} \mathbb{E}[X_n]\mathbb{E}[\mathbb{1}\{T \le n-1\}^c] \\ &= \sum_{n=1}^{\infty} \mathbb{E}[X_n] \Pr[T \ge n] \\ &= \mathbb{E}[X_1] \sum_{n=1}^{\infty} \Pr[T \ge n] \\ &= \mathbb{E}[X_1]\mathbb{E}[T].\end{aligned}$$

For the "?" equation, note

$$\begin{aligned}&\mathbb{E}[\sum_{n=1}^{\infty} X_n \mathbb{1}\{T \ge n\}] \\ \le&\mathbb{E}\left[\sum_{n=1}^{\infty} \underbrace{|X_n|}_{\ge 0} \underbrace{\mathbb{1}\{T \ge n\}}_{\ge 0}\right] \\ =&\sum_{n=1}^{\infty} \mathbb{E}\left[|X_n| \mathbb{1}\{T \le n-1\}^c\right] \\ =&\mathbb{E}[|X_1|] \sum_{n=1}^{\infty} \mathbb{E}\left[\mathbb{1}\{T \le n-1\}^c\right] \\ =&\mathbb{E}[|X_1|] \sum_{n=1}^{\infty} \Pr[T \ge n] \\ =&\mathbb{E}[|X_1|]\mathbb{E}[T] < \infty.\end{aligned}$$

then we can apply dominant expectation theorem and get "?". $\square$

Example 2.46. We consider an example fails the assumptions. Consider

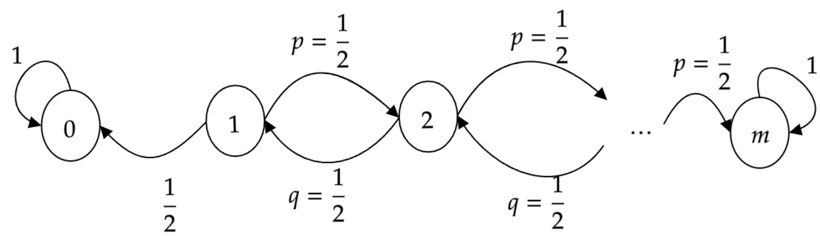


This Markov Chain is null recurrent. Let T be the first time reach 1 starting from 0.

$$X_i = \begin{cases} 1, \text{ with probability } \frac{1}{2}, \\ -1, \text{ with probability } \frac{1}{2}. \end{cases}$$

Then $\mathbb{E}[X_i] = 0$, $T = \inf\{n \ge 0 : S_n = 1 = X_1 + \ldots + X_n\}$ and $\mathbb{E}[T] = \infty$, so we cannot consider $\mathbb{E}[X_1]\mathbb{E}[T] = \mathbb{E}[S_T]$.

2.6.1 Gambler's Ruin



Suppose we start with i, then $S_0 = i$. Let $T = \inf\{n \geq 0 : X_n \in \{0, m\}\}$, $h_i = \Pr_i[S_T = m]$, $h_0 = 0, h_m = 1$. First, we show that $\mathbb{E}[T]$ is finite. Let $X_1, X_2, \ldots$ be i.i.d. r.v.s where $\Pr[X_i = 1] = p$ and $\Pr[X_i = -1] = q = 1 - p$ where $p \in (0, 1)$. Let $S_0 = i$, and $S_n = S_0 + X_1 + \ldots + X_n$. Now define

$$N = \inf\{k | X_{(k-1)m+1} = X_{X(k-1)m+2} = ... = X_{km} = 1\}.$$

That is, we are looking at blocks of size m. First, we look at $X_1, \ldots, X_m$, then we look at $X_{m+1}, ..., X_{2m}$, and so on until we find a block that are all the X's are $+1$. Now, $T <= mN$ since if we have m successes in a row, then the gambler must have either m or 0 dollars. But, N is a geometric r.v. with parameter $p^m > 0$. Since $\mathbb{E}[N] = 1/p^m$, we have that $\mathbb{E}[T] <= \mathbb{E}[mN] = m/p^m < \infty$.

A way to interpret the setup is that we start with i dollars, for each play, we have p chances to win and q to lose. And we have to stop playing when we reach m or 0 dollar.

$$\mathbb{E}[S_T | S_0 = i] = mh_i, \mathbb{E}[X_1 + \ldots + X_T] = (m-i)h_i + (-i)(1-h_i) = mh_i - i = 0 \text{ by } \mathbb{E}[X_1] = 0$$

where $E[X_1 + \ldots + X_T] = (m - i)h_i + (-i)(1 - h_i)$ is by the fact that $X_1 + \ldots + X_T$ is either $-i$ or $(m - i)$, and it is $m - i$ with probability h_i.

Consider $S'_n := S_n - i$, when $p = 1/2 = q$, as we have seen, $h_i = \frac{i}{m}$, then $\mathbb{E}[S_T] = mh_i + 0 = i$ or we can get it by $\mathbb{E}[S_T] = \mathbb{E}[S'_T] + i = 0 + i$, where $S'_T = X_1 + \ldots + X_T$ by the fact that $S_0 = i$ and $\mathbb{E}[S'_T] = \mathbb{E}[X_1 + \ldots + X_T] = \mathbb{E}[X_1]\mathbb{E}[T] = 0$.

If $p \neq q$, then note that $h_i = ph_{i+1} + qh_{i-1}$. Consider the probability matrix P of this chain, we know $h = Ph$. Consider

$$\begin{aligned}
(p+q)h_i &= ph_{i+1} + qh_{i-1} \\
qh_i - qh_{i-1} &= ph_{i+1} - ph_i \\
\frac{q}{p}(h_i - h_{i-1}) &= h_{i+1} - h_i
\end{aligned}$$

Notice we can write

$$0 = ah_{i+1} + bh_i + ch_{i-1}$$

which is in the form of a second order difference equation. The characterization equation of it is

$$0 = ar^2 + br + c$$

and once it is solved, we have two roots r_1, r_2 such that

$$h_i = c_1 r_1^i + c_2 r_2^i, \text{ when } r_1 \neq r_2$$
$$h_i = c_1 r_1^i + c_2 i r_2^i, \text{ when } r_1 = r_2$$

for some constants c_1, c_2.

For our case, we need to consider

$$0 = pr^2 - r + q \iff 0 = (r-1)(pr-q) \implies r_1 = 1, r_2 = q/p.$$

When $p = q$, $h_i = c_1 1^i + c_2 i 1^i = c_1 + c_2 i$. Note $h_0 = 0 = c_1 + c_2 * 0 = c_1 = 0$ and $h_m = 0 + c_2 m = 1$ so $c_2 = \frac{1}{m}$ and $h_i = \frac{i}{m}$ which agrees with what we had before. When $p \neq q$, $h_i = c_1 + c_2(\frac{q}{p})^i$ and $h_0 = 0 \implies c_1 + c_2 = 0, c_2 = -c_1$. And $h_m = c_1 - c_1(\frac{q}{p})^m = 1 \implies c_1 = \frac{1}{1-(\frac{q}{p})^m}$, so $h_i = \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^m}$.

Let $g(i) := \mathbb{E}_i[T]$, note $g(0) = 0 = g(m)$ and $g(i) = 1 + pg(i+1) + qg(i-1)$. For the same S_T defined before, consider $S'_T = S_T - i$ where $S'_n = S_n - i$. We know

$$\begin{aligned}\mathbb{E}_i[S'_T] &= \mathbb{E}_i[X_1] \underbrace{\mathbb{E}_i[T]}_{\leq \frac{m}{p^m} < \infty} \\ &= (p-q)\mathbb{E}_i[T].\end{aligned}$$

When $p \neq q$, $\mathbb{E}_i[S'_T] = (m-i)h_i + (-i)(1-h_i) = mh_i - i$, so

$$\mathbb{E}_i[T] = \frac{mp_i - i}{p-q}.$$

When $p = q$, use second order difference equation, we have

$$\mathbb{E}_i[T] = g(i) = i(m-i)$$

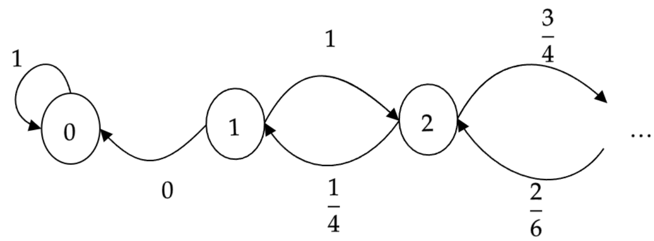
which can be verified that

$$i(m-i) = 1 + \frac{1}{2}(i+1)(m-i+1) + \frac{1}{2}(i-1)(m-i+1).$$

Now we consider the reverse probability, by supposing $h > 0$, then

$$\tilde{P}(i,j) = \frac{P(i,j)h_j}{h_i}$$
$$\sum_j \tilde{P}(i,j) = \frac{\sum_j P(i,j)h_j}{h_i} = \frac{h_i}{h_i} = 1.$$

When $p = q$, $\tilde{P}(i, i+1) = \frac{P(i,i+1)\frac{i+1}{m}}{\frac{i}{m}} = p\frac{i+1}{i} = \frac{i+1}{2i}$,


1
0
1
1
2
1
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0
1/4
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...

3 Renewal Theory

Definition 3.1. Let nonnegative random variables $S_0, S_1, \ldots$ be a *renewal sequence* where $0 \leq S_0 < S_1 < \ldots$. That is, consider S_i the time of i-th renewal. Let X_i be the time between $(i-1)$-th and i-th renewal, which is i.i.d. and nonnegative.

We call a renewal sequence an ordinary renewal process if

$$S_n = X_1 + \ldots + X_n.$$

We call it a delayed renewal process if

$$S_n = S_0 + X_1 + \ldots + X_n,$$

where S_0 is the delay.

We let $N(t) := \max\{n : S_n \leq t\}$ which is the last renewal time before t.

If you wish, you can think about a renewal process as considering returning to a specific state of a Markov chain. Then it is recurrent if $\Pr[X_i < \infty] = 1$ and transient if $\Pr[X_i < \infty] < 1$.

Some examples of renewal processes are:

- **Arithmetic Bernoulli Process**: $\Pr[X_i \in h\mathbb{Z}] = 1$ with the largest possible h, assume, WLOG, $h = 1$.
- **Nonarithmetic Poisson Continuous Process**: For $X_i \sim \exp(\pi)$,

3.1 Erdös Feller Pollard

The Erdos Feller Pollard (EFP) theorem we are discussing in this subsection is for arithmetic processes. For the non-arithmetic ones, one can check the Blackwell's theorem.

Let $f_k = \Pr[X_i = k]$, $\sum_{k=1}^{\infty} f_k = 1$, and $\mu = \sum_{k=1}^{\infty} k f_k < \infty$.

Definition 3.2. The *renewal measure* of a renewal process is defined as:

$$u(m) = \Pr[S_n = m \text{ for some } n \geq 0] = \sum_{n=0}^{\infty} \Pr[S_n = m],$$

which is the probablity that the renewal ever happens at time m.

Theorem 3.3 (Elementary Renewal Theorem)**.** For an ordinary, recurrent and arithmetic renewal process,

$$\frac{N(m)}{m} \to \frac{1}{\mu} \text{ a.s.}$$
$$\mathbb{E}\left[\frac{N(m)}{m}\right] \to \frac{1}{\mu}$$

Proof. Notice that

$$S_{N(m)} \leq m \leq S_{N(m)+1} \iff \frac{S_{N(m)}}{N(m)} \leq \frac{m}{N(m)} \leq \frac{S_{N(m)+1}}{N(m)}$$

where $\frac{S_{N(m)}}{N(m)} \to \mu$ a.s. by $\frac{S_n}{n} = \frac{X_1+\ldots+X_n}{n} \to \mu$ a.s., and we also have $\frac{S_{N(m)+1}}{N(m)} = \frac{S_{N(m)+1}}{N(m)+1}\frac{N(m)+1}{N(m)} \to \mu * 1$ a.s..

Also, since $N(m)/m \leq 1$, by Dominent Convergence Theorem or Bounded Convergence Theorem, we have

$$\lim_{n\to\infty} \mathbb{E}[N(m)/n] = \mathbb{E} \lim_{n\to\infty} N(m)/m = 1/\mu.$$

□

Theorem 3.4 (Erdös Feller Pollard (EFP) Theorem)**.** Let $\{S_n, n \geq 0\}$ be an ordinary, recurrent arithmetic renewal process whose interoccurence times have distribution $f_k = \Pr[X_i = k]$, $\mu = \mathbb{E}[X_i] < \infty$ that is not supported by any proper subgroup of the integers (i.e., there does not exists an $m \geq 2$ such that $\Pr[X_i \in m\mathbb{Z}] = 1$). Then

$$\lim_{m\to\infty} u(m) = \frac{1}{\mu} = \frac{1}{\mathbb{E}[X_i]}.$$

Corollary 3.5. With the same conditions as EFP, except having a delayed renewal process

$$\lim_{m\to\infty} \Pr[S_n = m \text{ for some } n \geq 0] = \frac{1}{\mu}.$$

Proof. We can condition on S_0, then

$$\begin{aligned}
&\lim_{m\to\infty} \Pr[S_n = m, \text{ for some } n \geq 0] \\
=&\lim_{m\to\infty} \sum_{k=0}^{\infty} \Pr[S_0 = k] \Pr[S_n - S_0 = m - k, \text{ for some } n \geq 0] \\
=&\sum_{k=0}^{\infty} \Pr[S_0 = k] \frac{1}{\mu} \\
=&\frac{1}{\mu},
\end{aligned}$$

where the second equality is by DCT, so we can put $\lim$ inside. so we have □

3.2 Life Time, Age, Remaining Time and Renewal Equation

Consider a point m, and the *age* $A(m)$ at m is the time from the last renewal to m, that is, $m-S_{N(m)}$ and the *remaining time* $R(m)$ at m is the time from m to the next renewal, that is, $S_{N(m)+1} - m$, and we also have the life time $L(m) = A(m) + R(m)$.

Example 3.6. Suppose we have a renewal process with $X_i \in \{1, \ldots, k\}$. Is $R_0, R_1, \ldots$ a Markov Chain?

Consider its probability matrix, $P(1,i) = f_i$, because when $R_t = 1$, at $t+1$, a renewal happens, so the remaining time is actually, X_{t+1}, which follows the distribution f_i. For $j > 1$, $P(j, j-1) = 1$ and 0 otherwise. We can consider the stationary distribution, where

$$\begin{aligned}
\pi_1 &= \pi_1 f_1 + \pi_2 \\
\pi_2 &= \pi_1(1 - f_1) \\
&= f_2\pi_1 + \pi_3 \\
\pi_3 &= \pi_2 - f_2\pi_1 = \pi_1(1 - f_1 - f_2),
\end{aligned}$$

and

$$\sum_{i=1}^{\infty} \pi_i = \sum_{i=1}^{\infty} \pi_1 \Pr[X_1 \geq i] = \pi_1 \sum_{i=1}^{\infty} \Pr[X_i \geq i] = \pi_1 \mathbb{E}[X_1] = 1.$$

so $\pi_1 = 1/\mathbb{E}[X_1]$.

Notice that $u(m) = \Pr[R(m-1) = 1]$, where $R(m-1)$ is an aperiodic and irreducible Markov Chain, so

$$\lim_{m\to\infty} \Pr[R(m-1) = 1] \to 1/\pi_1 = 1/\mu = 1/\mathbb{E}[X_1],$$

which agrees EFP.

3.2.1 Renewal Equation

Consider bounded sequences $\{z(m), m \geq 0\}$ and $\{b(m), m \geq 0\}$, where the renewal equation is

$$z(m) = b(m) + \sum_{k=1}^{\infty} f_k z(m-k), m \geq 0.$$

Let $z(m) = b(m) = 0$ for $m < 0$,

$$\begin{aligned}
z(m) &= b(m) + \sum_{k=1}^{\infty} f_k z(m-k) \\
&= b(m) + \mathbb{E}[z(m-k)]
\end{aligned}$$

1. Set up renewal equation.
2. Solution is $z(m) = \sum_{k=0}^{\infty} b(m-k)u(k) = \sum_{k=0}^{m} b(m-k)u(k) + 0$
3. Key Renewal Theorem: if $\sum_{k=0}^{\infty}|b(k)| < \infty$, then

$$\lim_{m\to\infty} z(m) = \sum_{k=1}^{\infty} b(k)/m.$$

Now we consider one renewal equation:

$$\begin{aligned}
z(m) &= \Pr[A(m) = r] \\
z(m) &= \sum_{k=m+1}^{\infty} f_k \Pr[A(m) = r | X_1 = k] + \sum_{k=1}^{m} f_k \Pr[A(m) = r | X_1 = k]. \\
&= \mathbb{1}\{m = r\} \sum_{k=m+1}^{\infty} f_k + \sum_{k=1}^{m} f_k z(m-k).
\end{aligned}$$

notice when $k > m$, $A(m) = m - 0 = m$, so $A(m) = r$ if and only if $m = r$. Let $\delta_{m,r} = \mathbb{1}\{m = r\}$. We then have

$$b(m) = \mathbb{1}\{m = r\} \sum_{k=m+1}^{\infty} f_k = \sum_{k=m+1}^{\infty} f_k \delta_{m,r}$$

from renewal equation.

Example 3.7. Consider a renewal process as a $\mathrm{Bernoulli}(p)$, that is, we have a renewal when the Bernoulli hits a success. Then $u(0) = 1$ by starting with a success and $u(k) = p$, and $X_i \sim \mathrm{Geo}(p)$, we have that, by EFP,

$$u(m) \to 1/p$$

Example 3.8. Consider a renewal process where $X_i \in \{1, 2\}$ and $f_1 = f_2 = 1/2$, from Lalley's notes,

$$\begin{aligned} u(m) &= \frac{2}{3} + \frac{1}{3}(-1/2)^m \to 2/3 \\ u(1) &= 2/3 - 1/3(1/2)^1 = 2/3 - 1/6 = 1/2 \\ u(2) &= f_2 + f_1^2 = 3/4. \end{aligned}$$

We can now start to try to solve the renewal equation.

Example 3.9. Back to the example above with Bernoulli process.

$$\begin{aligned} z(m) = \Pr[A(m) = r] &= \sum_{k=0}^{\infty} b(m-k)u(k) \\ &= \sum_{k=0}^{m} b(m-k)u(k) + 0 \\ &= b(m) + p\sum_{k=1}^{m} b(m-k) \\ &= \sum_{k=m+1}^{\infty} f_k \delta_{m,r} + p\sum_{k=1}^{m} b(m-k) \end{aligned}$$

say $m = 7$, $r = 5$, so $z(m) = \Pr[A(7) = 5]$, and $b(m-k) = \sum_{j=m-k+1}^{\infty} f_j \delta_{m-k,r}$, where $b(5) = \sum_{k=6}^{\infty} f_k \delta_{5,5} = \Pr[X_1 > 5] = (1-p)^5$. We have the above equation becoming:

$$\begin{aligned} &= 0 + p\sum_{k=1}^{7} b(7-k) \\ &= 0 + pb(7-2) \\ &= p(1-p)^5 \end{aligned}$$

which makes sense because $A(7) = 5$ is saying the last renewal is at 2 and there is no renewal at $3, 4, 5, 6, 7$, and the probability for that is $p(1-p)^5$.

Another question we have is that for $z(m) = \Pr[A(m) = r]$, do we have

$$\sum_{k=0}^{\infty} |b(k)| < \infty?$$

The answer is yes.

$$\begin{aligned}
\sum_{m=0}^{\infty} |b(m)| = \sum_{m=0}^{\infty} b(m) &= \sum_{m=0}^{\infty} \sum_{k=m+1}^{\infty} f_k \delta_{m,r} \\
&= \sum_{k=1}^{\infty} f_k \sum_{m=0}^{k-1} \delta_{m,r} \\
&= \sum_{k=r+1}^{\infty} f_k = \Pr[X_1 > r] < \infty
\end{aligned}$$

Then by Key Renewal Theorem, we have

$$z(m) = \Pr[A(m) = r] \to \frac{\Pr[X_1 > r]}{\mu}.$$

For our Bernoulli example again, with $z(m) = \Pr[A(m) = r] = p(1-p)^r$, where

$$\frac{\Pr[X_1 > r]}{\mu} = \frac{(1-p)^r}{1/p} = p(1-p)^r$$

which agrees with what we have.

Now we consider another renewal equation, where $z(m) = \Pr[L(m) = r]$. Then

$$z(m) = \sum_{k=m+1}^{\infty} f_k \mathbb{1}\{k = r\} + \sum_{k=1}^{m} f_k z(m-k)$$

where for the first term, given $X_1 = k$, by $m\{0, 1, \ldots, k\}$, $L(m) = r$ is equivalent to the process renewing at $k = r$, otherwise, $L(m) = k$.

3.3 $\mathbb{E}[N(m) - m/\mu]$

Recall that $\mathbb{E}[N(m)/m] = 1/\mu$ by the elementary renewal theorem, so we can approximate $\mathbb{E}[N(m)]$ by

$$\mathbb{E}[N(m)] \approx m/\mu.$$

Consider $z(m) = \mathbb{E}[N(m)] - m/\mu$, and then

$$z(m) = \sum_{k=m+1}^{\infty} f_k \mathbb{E}[N(m) - m/\mu | X_1 = k] + \sum_{k=1}^{m} f_k \mathbb{E}[N(m) - m/\mu | X_1 = k]$$

Note, when $k > m$, $\mathbb{E}[N(m)|X_1 = k] = 0$, so the first term is $\sum_{k=m+1}^{\infty} f_k(-m/\mu)$. For the second one, since $m \geq k$, $\mathbb{E}[N(m)|X_1 = k] = 1 + \mathbb{E}[N(m-k)]$, and so

$$\mathbb{E}[N(m) - m/\mu | X_1 = k] = 1 + \mathbb{E}[N(m-k)] - \frac{m-k}{\mu} + \frac{m-k}{\mu} - \frac{m}{\mu}$$

So the second term becomes

$$\begin{aligned}
&\sum_{k=1}^{m} f_k(1 + z(m-k) + \frac{m-k}{\mu} - \frac{m}{\mu}) \\
=&\sum_{k=1}^{m} f_k(1 + \frac{m-k}{\mu} - \frac{m}{\mu}) + \sum_{k=1}^{m} f_k z(m-k) \\
=&\sum_{k=1}^{m} f_k(1 - \frac{k}{\mu}) + \sum_{k=1}^{m} f_k z(m-k) \\
=&(1 - \sum_{k=1}^{m} f_k) - \frac{\mu - \sum_{k=m+1}^{\infty} k f_k}{\mu} + \sum_{k=1}^{m} f_k z(m-k) \\
=&\sum_{k=m+1}^{\infty} (k/\mu - 1) f_k + \sum_{k=1}^{m} f_k z(m-k)
\end{aligned}$$

so

$$z(m) = \underbrace{\sum_{k=m+1}^{\infty} \left(\frac{k-m}{\mu} - 1\right) f_k}_{b(m)} + \sum_{k=1}^{m} f_k z(m-k).$$

Then we apply Key Renewal Theorem and assume $\mathbb{E}[X_i^2]$ or $\mathrm{var}(X_i)$ is finite. Then

$$\begin{aligned}
\sum_{m=0}^{\infty} |b(m)| &\leq \sum_{m=0}^{\infty} \sum_{k=m+1}^{\infty} \underbrace{f_k}_{\geq 0} \underbrace{\left(\frac{k-m}{\mu} + 1\right)}_{>0} \\
&= \sum_{k=1}^{\infty} f_k \left(\sum_{m=0}^{k-1} \frac{k-m}{\mu} + \right) \\
&= \sum_{k=1}^{\infty} f_k (\frac{k}{\mu} + \ldots + \frac{1}{\mu} + k) \\
&= \sum_{k=1}^{\infty} f_k \frac{k(k+1)}{2\mu} + \sum_{k=1}^{\infty} k f_k \\
&= \sum_{k=1}^{\infty} \frac{k^2 f_k}{2\mu} + \frac{\sum_{k=1}^{\infty} k f_k}{\mu} + \mu \\
&= \frac{\sigma^2 + \mu^2}{2} + \frac{1}{2} + \mu < \infty,
\end{aligned}$$

then KRT says,

$$
\begin{aligned}
z(m) \to \frac{\sum_{k=0}^{\infty} b(k)}{\mu} = \sum_{m=0}^{\infty}\sum_{k=m+1}^{\infty} f_k\left(\frac{k-m}{\mu}-1\right) &= \sum_{k=1}^{\infty} f_k \sum_{m=0}^{k-1}\left(\frac{k-m}{\mu}-1\right) \\
&= \sum_{k=1}^{\infty} f_k\left(\frac{k(k+1)}{2\mu}-k\right) \\
&= \frac{\sigma^2+\mu^2+\mu}{2\mu}-\mu \\
&= \frac{\sigma^2-\mu^2+\mu}{2\mu}
\end{aligned}
$$

so $\mathbb{E}[N(m)-m/\mu] \to \frac{\sigma^2-\mu^2+\mu}{2\mu^2}$.

Note it "seems" like $\mathbb{E}[N(m)] = 1*u_1 + 1*u_2 + \ldots + 1*u_m = mp = \frac{m}{\mu}$ as $u(m) \to \frac{1}{\mu}$, but is it true? Do we always have $\mathbb{E}[N(m)-m/\mu] = 0$?

Example 3.10. Consider Bernoulli renewal, $\text{Bernoulli}(P)$, where $X_1 \sim \text{Geo}(p)$, $\mu = 1/p$ and $\sigma^2 = q/p^2$. Then

$$\frac{\frac{q}{p^2}-\frac{1}{p^2}+\frac{p}{p^2}}{2/p^2} = 0.$$

Example 3.11. $f_1 = f_2 = \frac{1}{2}$. Then recall we condition on the first renewal time,

$$u(m) = \delta_{0,m} + \sum_{k=1}^{m} f_k u(m-k)$$

by the fact that the first renewal is at $k \geq m$, then having a renewal at m if and only if $m = k$. Hence,

$$
\begin{aligned}
u(m) &= \frac{1}{2}u(m-1) + \frac{1}{2}u(m-2) \\
0 &= u(m) - \frac{1}{2}u(m) - \frac{1}{2}u(m-2) \\
0 &= r^2 - \frac{1}{2}r - \frac{1}{2} \\
0 &= (r-1)(r+\frac{1}{2}) \\
u(m) &= c_1 1^m + c_2(-\frac{1}{2})^m = c_1 + c_2(-1/2)^m \\
u(0) &= 1 = c_1 + c_2 \\
u(1) &= \frac{1}{2} = c_1 - \frac{1}{2}c_2 \\
\frac{1}{2} &= \frac{3}{2}c_2
\end{aligned}
$$

$z(0)$	$z(1)$	$z(2)$	$z(3)$	$z(4)$	$z(5)$
0	12	18	24	20	21

so $c_2 = \frac{1}{3}$, $c_1 = \frac{2}{3}$. Now $\mu = \frac{1}{2} * 1 + \frac{1}{2} * 2 = \frac{3}{2}$ and $\sigma^2 = \frac{1}{4}$ (same as $\text{Bernoulli}(1/2)$ shifted by 1). Then

$$\mathbb{E}[N(m) - m/\mu] = \frac{1/4 - 9/4 + 6/4}{2 * 9/4} = -1/9.$$

and

$$\begin{aligned}
\mathbb{E}[N(m) - m/\mu] &= \mathbb{E}[N(m)] - 2m/3 \\
&= \sum_{k=1}^{m} u(k) - 2m/3 \\
&= \sum_{k=1}^{m} (2/3 + (1/3)(-1/2)^k) - 2m/3 \\
&= 2m/3 + \frac{1}{3}\sum_{k=1}^{m} (-1/2)^k - 2m/3 \\
&= \frac{1}{3}\sum_{k=1}^{m} (-1/2)^k \\
&= \frac{1}{3}\frac{-1}{2}\sum_{k=0}^{m-1} (-1/2)^j = \frac{1}{3}(-1/2)\frac{1}{1+\frac{1}{2}} = -\frac{1}{9}.
\end{aligned}$$

In fact, we can say

$$\mathbb{E}[N(m)] = m/\mu + \frac{\sigma^2 - \mu^2 + \mu}{2\mu^2} + o(1).$$

Example 3.12. Consider the following example: and $f_1 = 3/6$, $f_2 = 2/6$, $f_3 = 1/6$. This $z(m)$ is constructed as, $z(1), z(2), z(3)$ are given, $z(m) = f_1 z(m-1) + f_2 z(m-2) + f_3 z(m-3)$. Is this z a renewal equation? Define $z(0) = b(0) = 0$. We have $z(1) = b(1) + \sum_{k=1}^{1} f_k z(1-k)$.

$$\begin{aligned}
12 = z(1) = b(1) + f_1 z(0) \implies b(1) = 12 z(2) &\qquad = b(2) + f_1 z(1) + f_2 z(0) \\
z(3) = b(3) + f_1 z(2) + f_2 z(1) + f_3 z(0) \\
z(4) = b(4) + f_1 z(3) + f_2 z(2) + f_3 z(1) \\
\implies b(4) = 0, b(5) = 0,
\end{aligned}$$

so $z(2) = b(2) + \frac{3}{6}12 + \frac{2}{6}0 \implies b(2) = 12$. Similarly, $b(3) = 11$.

$$\sum_{m=0}^{\infty} b(m) = 12 + 12 + 11 + 35$$

because by the way $z(m)$ being constructed, $b(m) = 0$ for all $m \geq 4$.

Proof of Key Renewal Theorem. Note $b(m) = z(m) = 0$ for $m \leq -1$. First we have that $\mathbb{E}[z(m - X_1)] = b(m - X_1) + \mathbb{E}[z(m - X_1 - X_2)]$. And hence

$$\begin{aligned} z(m) &= b(m) + \mathbb{E}[z(m - X_1)] \\ &= b(m) + \mathbb{E}[b(m - X_1)] + \mathbb{E}[z(m - X_1 - X_2)] \end{aligned}$$

$$b(m) + \mathbb{E}[b(m - X_1)] + \mathbb{E}[z(m - X_1 - X_2)]$$

$$= b(m) + \mathbb{E}[b(m - X_1)] + \mathbb{E}[b(m - X_1 - X_2)] + \ldots + \mathbb{E}[b(m -$$

because $m - X_1 - X_2 - \ldots - X_m - X_{m+1} < 0$. Now

$$\begin{aligned} z(m) = \sum_{n=0}^{m} \mathbb{E}[b(m - S_n)] &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} [b(m - k)] \Pr[S_n = k] \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} b(m - k) \Pr[S_n = k] \\ z(m) &= \sum_{k=0}^{\infty} b(m - k) u(k) \\ &= \sum_{k=0}^{\infty} b(k) u(m - k). \end{aligned}$$

With this,

$$\begin{aligned} \lim_{m \to \infty} z(m) &= \lim_{m \to \infty} \sum_{k=0}^{\infty} b(k) u(m - k) \\ &= \sum_{k=0} \infty b(k) \underbrace{\lim_{m \to \infty} u(m - k)}_{\to 1/\mu} \\ &= \sum_{k=0}^{\infty} b(k)/\mu \end{aligned}$$

where the second equality is by $\sum_{k=0}^{\infty} b(k) u(m - k) \leq \sum_{k=0}^{\infty} |b(k)|$ since $u(m - k) \in [0, 1]$ and dominant convergence theorem. $\square$