PMATH 450/650: Introduction to Lebesgue Measure and Fourier Analysis

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Contents

1 Measure

1.1 Borel Set

Definition 1.1

X is a set. We call $a \subseteq \mathcal{P}(x)$ a $\underline{\sigma\text{-algebra}}$ of subsets of X if:

1. $\emptyset \in a$
2. $A \in a \implies X \setminus A \in a$
3. $A_1, A_2, A_3, \ldots, \in a \implies \bigcup_{i=1}^{\infty} A_i \in a$

Remark. $a \subseteq \mathcal{P}(X)$ is a σ-algebra

1. $X \in a,\ X \setminus \emptyset = X \in a$
2. $A, B \in a \implies A \bigcup B \in a$ by $A \bigcup = A \bigcup B \bigcup \underbrace{\emptyset \ldots \bigcup \emptyset \ldots}_{\text{countably many}} \in a$
3. $A_1, A_2, \ldots \in a \implies \bigcap_{i=1}^{\infty} A_i \in a$, by $\bigcap_{i=1}^{\infty} A_i = X \setminus \left(\bigcup_{i=1}^{\infty} (X \setminus A_i) \right) \in a$
4. $A, B \in a \implies A \bigcap B \in a$

Example 1.2: σ-algebra

- $\{\emptyset, X\}$
- $a = \mathcal{P}(x)$
- *$a = \{A \subseteq \mathbb{R} : A$ is open$\}$ is not a σ-algebra. $A = (0,1) \in a$, but $\mathbb{R} \setminus A = (-\infty, 0] \cup [1, \infty) \notin a$ because it's not open*
- *$a = \{A \subseteq \mathbb{R} : A$ is open or closed$\}$ is not a σ-algebra, because $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\} \notin a$ (Q is countable)*

Proposition 1.3

X is a set, $C \subseteq \mathcal{P}(x)$, then

$$a := \bigcap \{\mathbb{B} : \mathbb{B}\ \sigma\text{-algebra},\ C \subseteq \mathbb{B}\} \text{ is a } \sigma\text{-algebra}$$

It's the smallest σ-algebra containing C.

Definition 1.4

$C = \{A \subseteq \mathbb{R} : A \text{ open}\}$, then

$$a = \cap\{\mathbb{B} : C \subseteq \mathbb{B}, \mathbb{B}\ \sigma - algebra\}$$

is a Borel σ-algebra. The elements of a are called the Borel Sets.

Remark.

1. open $\implies$ Borel
2. closed $\implies$ Borel
3. $\{X_1, X_2, \ldots\} = \bigcup_{i=1}^{\infty}\{X_i\}$, so countable $\implies$ Borel. (Note $\mathbb{Q}$ is not open or closed but Borel)
4. $[a, b) = [a, b] \setminus \{b\} = [a, b] \cap (\mathbb{R} \setminus \{b\})$, so a half open interval is also Borel

1.2 Outer Measure

Goal: Define a function

$$m : \mathcal{P}(\mathbb{R}) \mapsto [0, \infty) \cup \{\infty\} \text{ (called a measure)}$$

1. $m((a, b)) = m([a, b]) = m((a, b]) = b - a$
2. $m(A \cup B) \leqslant m(A) + m(B)$
3. $A \cap B = \emptyset$, $m(A \cup B) = m(A) + m(B)$

Definition 1.5

We define a (Lebesgue) <u>*outer measure*</u> *by*

$$m^* : \mathcal{P}(\mathbb{R}) \mapsto [0, \infty) \cup \{\infty\}$$
$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} l(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i,\ I_i \textit{ open, bounded interval} \right\}$$

Example 1.6

$\emptyset \implies m^*(\emptyset) = 0$. *Since* $\forall \varepsilon > 0,\ \emptyset \subseteq (0, \varepsilon) \implies m^*(\emptyset) \leqslant l((0, \varepsilon))$. *Since* $m^*(\emptyset) \geqslant 0$, *we know* $m^*(\emptyset) = 0$

Example 1.7

$A = \{x_1, x_2, \ldots\}$ *is countable, then*

$$A \subseteq \bigcup_{i=1}^{\infty} \left(x_i - \frac{\varepsilon}{2^{i+1}}, x_i + \frac{\varepsilon}{2^{i+1}} \right),\ \varepsilon > 0$$

then

$$\begin{aligned} m^*(A) &\leqslant \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} \\ &= \frac{\varepsilon}{2} \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} \\ &= \frac{\varepsilon}{2} \left(\frac{1}{1 - 1/2} \right) = \varepsilon \end{aligned}$$

Since ε *is arbitrary,*

$$m^*(A) = 0$$

It's also clear that finite set also have measure 0. That is, both countable and finite sets have measure 0

1.3 Outer Measure 2

Proposition 1.8

If $A \subseteq B$, then $m^*(A) \leqslant m^*(B)$

Proof.

$$\begin{aligned} X &:= \left\{ \sum l(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i \right\} \\ Y &:= \left\{ \sum l(I_i) : B \subseteq \bigcup_{i=1}^{\infty} I_i \right\} \\ Y &\subseteq X \\ \inf X &\leqslant \inf Y \end{aligned}$$

□

Lemma 1.9

If $a, b \in \mathbb{R}$ with $a \leqslant b$, then $m^*([a,b]) = b - a$

Proof. Let $\varepsilon > 0$ be given. Since $[a,b] \subseteq \left(a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right)$. We see that $m^*([a,b]) \leqslant b - a + \varepsilon$. Let I_i be bounded, open intervals such that $[a,b] \subseteq \bigcup_{i=1}^{\infty} I_i$. Since $[a,b]$ is compact, then there exists $n \in \mathbb{N}$, such that

$$[a,b] \subseteq \bigcup_{i=1}^{n} I_i$$

so

$$b - a \leqslant \sum_{i=1}^{n} l(I_i) \leqslant \sum_{i=1}^{\infty} l(I_i)$$

and so $m^*([a,b]) \geqslant b - a \implies m^*([a,b]) = b - a$. Note $m^*([a,b]) > 0$ because of the definition of inf. □

Proposition 1.10

If I is an interval, then $m^*(I) = l(I)$

Proof.

1. If I is bounded with endpoints $a \leqslant b$, then

$$\begin{aligned} \varepsilon > 0, I \subseteq [a,b] &\implies m^*(I) \leqslant m^*([a,b]) = b - a \\ \left[a + \frac{\varepsilon}{2}, b - \frac{\varepsilon}{2}\right] \subseteq I &\implies b - a + \varepsilon \leqslant m^*(I) \\ &\implies b - a \leqslant m^*(I) \end{aligned}$$

   then $m^*(I) = b - a$

2. If I is unbounded

$$\begin{aligned} &\forall n \in \mathbb{N}, \exists I_n, l(I_n) = n \\ \Longrightarrow\ &m^*(I) \geqslant m^*(I_n) = n \\ \Longrightarrow\ &m^*(I) = \infty = l(I) \end{aligned}$$

$\square$

1.4 Basic Properties of Outer Measure

Outer measure is

1. Translation Invariant
2. Countably Subadditive

Notation: $x \in \mathbb{R}, A \subseteq \mathbb{R}, x + A := \{x + a : a \in A\}$

Proposition 1.11: Translation Invariant

$m^*(x + A) = m^*(A)$

Proof.

$$\begin{aligned} m^*(x+A) &= \inf\left\{\sum_{i=1}^{\infty} l(I_i) : x + A \subseteq \bigcup_{i=1}^{\infty} I_i,\ \text{bounded, open}\right\} \\ &= \inf\left\{\sum_{i=1}^{\infty} l(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i - x,\ \text{bounded, open}\right\} \\ &= \inf\left\{\sum_{i=1}^{\infty} l(\underbrace{I_i - x}_{J_i}) : A \subseteq \bigcup_{i=1}^{\infty} \underbrace{I_i - x}_{J_i},\ \text{bounded, open}\right\} \\ &= \inf\left\{\sum_{i=1}^{\infty} l(J_i) : A \subseteq \bigcup_{i=1}^{\infty} J_i\right\} \\ &= m^*(A) \end{aligned}$$

□

Proposition 1.12: Countably Subadditivity

If $A_i \subseteq \mathbb{R} (i \in \mathbb{N})$, *then*

$$m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leqslant \bigcup_{i=1}^{\infty} m^*(A_i)$$

Proof. We may assume each $m^*(A_i) < \infty$(otherwise it's trivial). Let $\varepsilon > 0$ be given and let's fix $i \in \mathbb{N}$. There exists open and bounded interval $I_{i,j}$ such that $A_i \subseteq \bigcup\limits_{i=1}^{\infty} I_{i,j}$ and

$$\sum_{i=1}^{\infty} l(I_{i,j}) \leqslant m^*(A_i) + \frac{\varepsilon}{2^i}$$

We see that

$$\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i,j} I_{i,j}$$

and so

$$\begin{aligned} m^*\left(\bigcup_{i=1}^{\infty}\right) &\leqslant \sum_{i,j} l(I_{i,j}) \\ &\leqslant \sum_{i=1}^{\infty}\left(m^*(A_i) + \frac{\varepsilon}{2^i}\right) \\ &= \sum_{i=1}^{\infty} m^*(A_i) + \sum_{i=1}^{\infty}\frac{\varepsilon}{2^i} \\ &= \sum_{i=1}^{\infty} m^*(A_i) + \varepsilon \end{aligned}$$

□

Corollary 1.13: finite subadditivity

If $A_1, \ldots, A_n \in \mathcal{P}(\mathbb{R})$*, then*

$$m^*(A_1 \cup A_2 \ldots \cup A_n) \leqslant m^*(A_1) + m^*(A_2) + \ldots + m^*(A_n)$$

Later we will see that there exists $A, B \subseteq \mathbb{R}$, $A \cap B = \emptyset$ but $m^*(A \cup B) \leqslant m^*(A) + m^*(B)$, we will solve this by restricting the domain of m^* to only include the sets which measure ”nicely”.

1.5 Measurable Sets

Definition 1.14

We say $A \subseteq \mathbb{R}$ is measurable if $\forall X \subseteq \mathbb{R}$,

$$m^*(X) = m^*(X \cap A) + m^*(X \setminus A)$$

Remark. Always have

$$m^*(X) \leqslant m^*(X \cap A) + m^*(X \setminus A)$$

by $X = (X \setminus A) \cup (X \cap A)$

Remark. If $A \subseteq \mathbb{R}$ is measurable and $B \subseteq \mathbb{R}$ with $A \cap B = \emptyset$, then

$$m^*(\underbrace{A \cup B}_{X}) = m^*(X \cap A) + m^*(X \setminus A) = m^*(A) + m^*(B)$$

Proposition 1.15

If $m^*(A) = 0$, then A is measurable

Proof. Let $X \subseteq \mathbb{R}$, since $X \cap A \subseteq A$, we have

$$0 \leqslant m^*(X \cap A) \leqslant m^*(A) = 0$$

so $m^*(X \cap A) = 0$, then

$$\begin{aligned} & m^*(X \cap A) + m^*(X \setminus A) \\ =& m^*(X \setminus A) \\ \leqslant& m^*(X) \end{aligned}$$

the other direction is always true, so

$$m^*(X) = m^*(X \cap A) + m^*(X \setminus A)$$

□

Proposition 1.16

$A_1, \ldots, A_n$ measurable, then $\bigcup_{i=1}^{n} A_i$ is measurable.

Proof. It suffices to prove the result when $n = 2$.
Let $A, B \subseteq \mathbb{R}$ be measurable. Let $X \subseteq \mathbb{R}$, then

$$\begin{aligned} m^*(X) &= m^*(X \cap A) + m^*(\underbrace{X \setminus A}_{Y}) \\ &= m^*(X \cap A) + m^*(Y \cap B) + m^*(Y \setminus B) \\ &= m^*(X \cap A) + m^*((X \setminus A) \cap B) + m^*(X \setminus (A \cup B)) \\ &\geqslant m^*((X \cap A) \cup ((X \setminus A) \cap B)) + m^*(X \setminus (A \cup B)) \\ &= m^*(X \cap (A \cup B)) + m^*(X \setminus (A \cup B)) \end{aligned}$$

□

Proposition 1.17

$A_1, A_2, \ldots, A_n$ measurable, $A_i \cap A_j = \emptyset, i \neq j$. Let $A = A_1 \cup \ldots \cup A_n$. If $X \subseteq \mathbb{R}$, then

$$m^*(X \cap A) = \sum_{i=1}^{n} m^*(X \cap A_i)$$

Proof. For $n = 2$, let $A, B \subseteq \mathbb{R}$ measurable, $A \cap B = \emptyset$. Let $X \subseteq \mathbb{R}$, then

$$\begin{aligned} & m^*(X \cap (A \cup B)) \\ =& m^*((X \cap (A \cup B)) \cap A) + m^*((X \cap (A \cup B)) \setminus A) \\ =& m^*(X \cap A) + m^*(X \cap B) \end{aligned}$$

Note: we only need $n - 1$ sets to be measurable, it's ok if one set is not. □

Corollary 1.18: Finite Additive

$A_1, \ldots, A_n$ measurable, $A_i \cap A_j = \emptyset$, then $m^*(A_1 \cup \ldots \cup A_n) = \sum_{i=1}^{n} m^*(A_i)$

Proof. Take $X = \mathbb{R}$, use the proposition above. □

1.6 Countably Additivity

Lemma 1.19

$A_i \subseteq \mathbb{R}$ measurable ($i \in \mathbb{N}$). If $A_i \cap A_j = \emptyset$ for $i \neq j$, then $A := \bigcup_{i=1}^{\infty} A_i$ is measurable.

Proof. Let $B_n = A_1 \cup \ldots A_n$ and $X \subseteq \mathbb{R}$ arbitrary.

$$\begin{aligned} m^*(X) &= m^*(X \cap B_n) + m^*(X \setminus B_n) \\ &\geqslant m^*(X \cap B_n) + m^*(X \setminus A) \\ &= \sum_{i=1}^{m} m^*(X \cap A_i) + m^*(X \setminus A) \end{aligned}$$

Taking $n \to \infty$,

$$\begin{aligned} m^*(X) &\geqslant \sum_{i=1}^{\infty} m^*(X \cap A_i) + m^*(X \setminus A) \\ &= m^*\left(\bigcup_{i=1}^{\infty}(X \cap A_i)\right) + m^*(X \setminus A) \\ &= m^*(X \cap A) + m^*(X \setminus A) \end{aligned}$$

□

Proposition 1.20

$A \subseteq \mathbb{R}$ measurbale, then $\mathbb{R} \setminus A$ is measurable.

Proof. $X \subseteq \mathbb{R}$,

$$\begin{aligned} &m^*(X \cap (\mathbb{R} \setminus A)) + m^*(X \setminus (\mathbb{R} \setminus A)) \\ =&m^*(X \setminus A) + m^*(X \cap A) \\ =&m^*(X) \text{ by } A \text{ measurable} \end{aligned}$$

□

Proposition 1.21

$A_i \subseteq \mathbb{R}$ measurable $(i \in \mathbb{N})$, then $A = \bigcup_{i=1}^{\infty} A_i$ is measurable.

Proof. $B_n = A_n \setminus (A_1 \cup \ldots \cup A_{n-1}) = A_n \cap (\mathbb{R} \setminus (A_1 \cup \ldots \cup A_{n-1}))$, $(B_1 = A_1)$, $n \geqslant 2$, we can see that B_n is an intersection of measurable sets, hence measurable. And, for $i \neq j$, $B_i \cap B_j = \emptyset$. Also,

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$$

so A is measurable by lemma above. □

Corollary 1.22

The collection $\mathcal{L}$ of (Lebesgue) measurable sets is a σ-algebra of sets in $\mathbb{R}$

Proposition 1.23: Countably Additivity

$A_i \subseteq \mathbb{R}$ measurable ($i \in \mathbb{N}$), if $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$m^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m^*(A_i)$$

Proof.

$$m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \geqslant m^*\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{=1}^{\infty} m^*(A_i)$$

Take $n \to \infty$, then

$$m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \geqslant \sum_{i=1}^{\infty} m^*(A_i)$$

The other direction follows by the subadditivity. $\square$

1.7 Measurable Sets Continued

Proposition 1.24: I

$a \in \mathbb{R}$, then (a, ∞) is measurable

Proof. Let $X \subseteq \mathbb{R}$. We want to show that

$$m^*(X \cap (a, \infty)) + m^*(X \setminus (a, \infty)) \leqslant m^*(X)$$

1. $a \notin X$,
   We show
   $$m^*(\underbrace{X \cap (a, \infty)}_{X_1}) + m^*(\underbrace{X \cap (-\infty, a)}_{X_2}) \leqslant m^*(X)$$
   Let (I_i) be a sequence of bounded, open intervals such that $X \subseteq \bigcup I_i$. Define
   $$I_i' = I_i \cap (a, \infty) \text{ and } I_i'' = I_i \cap (-\infty, a)$$
   Note that
   $$X_1 \subseteq \bigcup I_i', X_2 \subseteq \bigcup I_i''$$
   and so
   $$m^*(X_1) \leqslant \sum l(I_i')$$
   $$m^*(X_2) \leqslant \sum l(I_i'')$$
   We then see that
   $$\begin{aligned} & m^*(X_1) + m^*(X_2) \\ \leqslant & \sum l(I_i') + \sum l(I_i'') \\ = & \sum (l(I_i') + l(I_i'')) \\ = & \sum l(I_i) \end{aligned}$$
   By the definition of $\inf$, we have
   $$m^*(X_1) + m^*(X_2) \leqslant m^*(X)$$
2. $a \in X$, let $X' = X \setminus \{a\}$, then
   $$\begin{aligned} m^*(X \cap (a, \infty)) + m^*(X \setminus (a, \infty)) &= m^*((X' \cup \{a\}) \cap (a, \infty)) + m^*((X' \cup \{a\}) \setminus (a, \infty)) \\ &= m^*(X' \cap (a, \infty)) + m^*((X' \setminus (a, \infty)) \cup \{a\}) \\ &\leqslant m^*(X' \cap (a, \infty)) + m^*(X' \setminus (a, \infty)) + m^*(\{a\}) \\ &= m^*(X') + 0 \leqslant m^*(X) \end{aligned}$$

The other direction is trivial by subadditivity. $\square$

Theorem 1.25

Borel set is measurable

Proof. (a, ∞) is measurable, so $\bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, \infty\right) = [a, \infty)$ is measurable. So $\mathbb{R} \setminus [a, \infty) = (-\infty, a)$ is measurable, then $(a, b) = (a, \infty) \cap (-\infty, b)$ is measurable. Hence, every open set in $\mathbb{R}$ is measurable (open sets can be expressed as countable union of open intervals), so

$$\mathbb{B} \subseteq \mathcal{L}$$

because $\mathbb{B}$ is the smallest σ-algebra containing all open sets and $\mathcal{L}$ is a σ-algebra containing all open sets. □

Definition 1.26

We call $m : \mathcal{L} \mapsto [0, \infty) \cup \{\infty\}$ *given by* $m(A) = m^*(A)$, *the* *Lebesgue Measure*

Remark. $A \subseteq \mathbb{R}$ measurable, then $x + A$ is measurable $\forall x \in \mathbb{R}$

Proof. $\forall K \subseteq \mathbb{R},\ K - x \subseteq \mathbb{R}$,

$$\begin{aligned} m^*(K - x) &= m^*(A \cap (K - x)) + m^*(A \setminus (K - x)) \\ &= m^*((A + x) \cap K) + m^*((A + x) \setminus K) \\ &= m^*(K) \end{aligned}$$

□

1.8 Basic Properties of Lebesgue Measure

Proposition 1.27: Excision Properties

$A \subseteq B$, A measurable, $m(A) < \infty$, then $m^*(B \setminus A) = m^*(B) - m(A)$

Proof.

$$\begin{aligned} m^*(B) &= m^*(B \cap A) + m^*(B \setminus A) \\ &= m^*(A) + m^*(B \setminus A) \\ &= \underbrace{m(A)}_{<\infty} + m^*(B \setminus A) \end{aligned}$$

□

Theorem 1.28: Continuity of Measure

1. *$A_1 \subseteq A_2 \subseteq A_3 \ldots$, measurable, then*

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} m(A_n)$$

2. *$B_1 \supseteq B_2 \supseteq B_3 \ldots$, measurable, and $m(B_1) < \infty$, then*

$$m\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} m(B_n)$$

Proof.

1. Since $m(A_k) \leqslant m(\cup A_i)$, $\forall k \in \mathbb{N}$, we have

$$\lim_{n\to\infty} m(A_n) \leqslant m(\cup A_i)$$

if $\exists k \in \mathbb{N}$ such that $m(A_k) = \infty$, then $\lim_{n\to\infty} m(A_n) = \infty$ and we are done, so assume $m(A_k) < \infty$, $\forall k \in \mathbb{N}$.
For each $k \in \mathbb{N}$, let $D_k = A_k \setminus A_{k-1}$, $A_0 \neq \emptyset$. Note

- D_k's are measurable
- D_k's are parwise disjoint
- $\cup D_i = \cup A_i$

so

$$\begin{aligned} m^*(\cup A_i) &= m^*(\cup D_i) \\ &= \sum_{i=1}^{\infty} m(D_i) \\ &= \sum_{i=1}^{\infty} m(A_i) - m(A_{i-1}) \\ &= \lim_{n\to\infty} \sum_{i=1}^{\infty} m(A_i) - m(A_{i-1}) \\ &= \lim_{n\to\infty} m(A_n) - m(A_0) \\ &= \lim_{n\to\infty} m(A_n) \end{aligned}$$

2. For $k \in \mathbb{N}$, define

$$D_k = B_1 \setminus B_k$$

Note:

- D_k's measurable
- $D_1 \subseteq D_2 \subseteq D_3 \subseteq \dots$

By 1), we know $m(\cup D_i) = \lim_{n\to\infty} m(D_n)$, we see that

$$\cup D_i = \bigcup_{i=1}^{\infty}(B_1 \setminus B_i) = B_1 \setminus \left(\bigcap_{i=1}^{\infty} B_i\right)$$

and so,

$$\lim_{n\to\infty} m(D_n) = m(\cup D_i) = m(B_1 \setminus (\cap B_i)) = m(B_1) - m(\cap B_i)$$

because $\cap B_i$ is measurable and has finite measure.
However,

$$\begin{aligned} \lim_{n\to\infty} m(D_n) &= \lim_{n\to\infty} m(B_1 \setminus B_n) \\ &= \lim_{n\to\infty} m(B_1) - m(B_n) \\ &= m(B_1) - \lim_{n\to\infty} m(B_n) \\ &= m(B_1) - m(\cap B_i) \end{aligned}$$

Hence,

$$\lim_{n\to\infty} m(B_n) = m(\cap B_i)$$

□

Example 1.29

$B_i = (i, \infty)$, *and* $m(\cap B_i) = m(\emptyset) = 0$, *but* $\lim_{n\to\infty} m(B_n) = \infty$

1.9 Non-Measurable Sets

Lemma 1.30

$A \subseteq \mathbb{R}$ bounded, measurable $\Lambda \subseteq \mathbb{R}$ bounded, countably infinite. If $\lambda + A$, $\lambda \in \Lambda$ are pairwise disjoint, then $m(A) = 0$

Proof. $\bigcup_{\lambda \in \Lambda} (\lambda + A)$ is a bounded set, which is measurable, then

$$
\begin{aligned}
m\left(\bigcup_{\lambda}(\lambda + A)\right) &< \infty \\
m\left(\bigcup_{\lambda}(\lambda + A)\right) &= \sum_{\lambda} m(\lambda + A) = \sum_{\lambda} m(A) < \infty
\end{aligned}
$$

and $m(A) \geqslant 0$, so $m(A) = 0$ (Λ is countably infinite) □

Construction: Start with $\emptyset \neq A \subseteq \mathbb{R}$, consider $a \sim b \iff a - b \in \mathbb{R}$. Then $\sim$ is an equivalence relation.
Let C_A denotes a single choice of equivalence class representatives for A relative to $\sim$.

Remark. The sets $\lambda + C_A$, $\lambda \in \mathbb{Q}$ are pairwise disjoint

Proof. say $x \in (\lambda_1 + C_A) \cap (\lambda_2 \cap C_A)$

$$
\begin{aligned}
& x = \lambda_1 + a = \lambda_2 + b \\
\implies & a, b \in C_A \\
\implies & a - b = \lambda_1 - \lambda_2 \in \mathbb{Q} \\
\implies & a \sim b \implies a = b \text{ by each equiv. class has one repre.} \\
\implies & \lambda_1 = \lambda_2
\end{aligned}
$$

□

Theorem 1.31: Vitali

Every set $A \subseteq \mathbb{R}$ with $m^*(A) > 0$ contains a non-measurable subset.

Proof. By Quiz1, we may assume A is bounded, say $A \subseteq [-N, N]$, for some $N \in \mathbb{N}$.

Claim: C_A is non-measurable.
Assume C_A is measurable. Let $\Lambda \subseteq \mathbb{Q}$ be bounded, infinite. By the lemma and remark,

$$m(C_A) = 0$$

Let $a \in A$, then $a \sim b$ for some $b \in C_A$. In particular, $a - b = \lambda \in \mathbb{Q}$. Moreover,

$$\lambda \in [-2N, 2N]$$

Taking $\Lambda_0 = \mathbb{Q} \cap [-2N, 2N]$, have

$$A \subseteq \bigcup_{\lambda \in \Lambda_0} (\lambda + C_A)$$

so $m^*(A) = 0$, contradiction $\square$

Corollary 1.32

$\exists A, B \subseteq \mathbb{R}$, such that

1. *$A \cap B = \emptyset$, and*
2. *$m^*(A \cup B) < m^*(A) + m^*(B)$*

Proof. Let C be a non-measurable set, $\exists X \subseteq \mathbb{R}$ such that

$$m^*(X) < m^*(\underbrace{X \cap C}_{A}) + m^*(\underbrace{X \setminus C}_{B})$$

$\square$

1.10 Cantor-Lebesgue Function

Recall: Cantor Set

$$\begin{aligned} I &= [0,1] \\ C_1 &= [0,1/3] \cup [2/3,1] \\ C_2 &= [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1] \\ &\vdots \\ C &= \bigcap_{k=1}^{\infty} C_k \end{aligned}$$

Note C is <u>countable</u> and <u>closed</u>.

Proposition 1.33

The Cantor Set is Borel and has measure zero.

Proof. Closed $\implies$ Borel. And $C = \bigcap_{k=1}^{\infty} C_k$, where C_k's measurable and

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$$

By continuity of measure,

$$\begin{aligned} m(C) &= \lim_{k\to\infty} m(C_k) \\ &= \lim_{k\to\infty} \frac{2^k}{3^k} = 0 \end{aligned}$$

□

Construction: Cantor-Lebesgue Function (C-L fcn)

1. For $k \in \mathbb{N}$, U_k= Union of open intervals deleted in the process of constructing $C_1, C_2, \ldots, C_k$ i.e. $U_k = [0,1] \setminus C_k$.
2. $U = \bigcup_{k=1}^{\infty} U_k$, i.e. $U = [0,1] \setminus C$
3. Say $U_k = I_{k,1} \cup I_{k,2} \cup \ldots \cup I_{k,2^k-1}$ (In order: from left to right). Define

$$\varphi : U_k \to [0,1] \text{ by } \varphi|_{I_{k,i}} = \frac{i}{2^k}$$

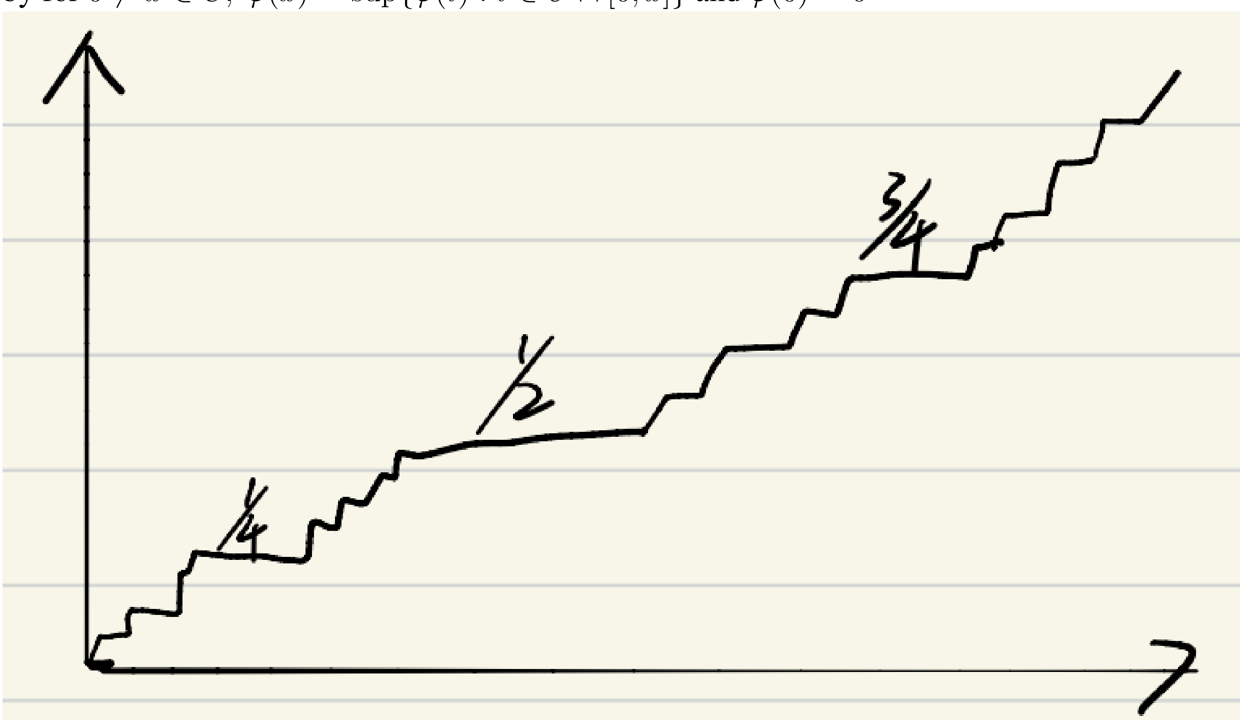
e.g. $U_1 = (1/3, 2/3) \to \frac{1}{2^1} = \frac{1}{2}$ and

$$\begin{aligned} U_2 = &(1/9, 2/9) && \cup(1/3, 2/3) && \cup(7/9, 8/9) \\ &\to \frac{1}{4} && \to \frac{2}{4} && \to \frac{3}{4} \end{aligned}$$

4. Define

$$\varphi : [0,1] \to [0,1]$$

by for $0 \neq x \in C,\ \varphi(x) = \sup\{\varphi(t) : t \in U \cap [0,x]\}$ and $\varphi(0) = 0$



Things to know about φ

1. φ is increasing. Take two points in U, for large enough k, both points in U_k. If they are in the Cantor Set, then it's increasing by definition

2. φ is continuous

   - φ is continuous on U. (It's constant on a small interval)
   - $x \in C,\ x \neq 0, 1$. For large k, $\exists a_k \in I_{k,i}$, $\exists b_k \in I_{k,i+1}$ such that

   $$a_k < x < b_k$$

     but, $\varphi(b_k) - \varphi(a_k) = \frac{i+1}{2^k} - \frac{i}{2^k} = \frac{1}{2^k} \to 0$
   - $x \in \{0, 1\}$

3. $\varphi : u \to [0, 1]$ is differentiable and $\varphi' = 0$

4. φ is onto,

   $$\varphi(0) = 0,\ \varphi(1) = 1$$

   by Intermediate Value Theorem.

1.11 A Non-Borel Set

Let φ be the Cantor-Lebesgue Function. Consider $\psi : [0, 1] \to [0, 2]$ defined by $\psi(x) = x + \varphi(x)$.

1. ψ is strictly increasing
2. ψ is continuous
3. ψ is onto

By 1),3), we know ψ is bijective, hence invertible.

Properties:

1. $\psi(C)$ is measurable and has positive measure.
2. ψ maps a particular (measurable) subset of C to a non-measurable set.

Proof.

1. By A1, ψ^{-1} is continuous, so $\psi(C) = (\psi^{-1})^{-1}(C)$ is closed, so $\psi(C)$ is Borel implies that it's measurable.
   Note that
   $$\begin{aligned} &[0,1] = C\dot{\cup}U \\ \implies &[0,2] = \psi(C\dot{\cup}U) = \psi(C)\dot{\cup}\psi(U) \text{ by bijectivity} \\ \implies &2 = m(\psi(C)) + m(\psi(U)) \end{aligned}$$
   It suffices to show that
   $$m(\psi(U)) = 1$$
   Say $U = \dot{\bigcup}_{i=1}^{\infty} I_i$, where I_i are disjoint open intervals. Then
   $$\psi(U) = \dot{\bigcup_{i=1}^{\infty}} \psi(I_i) \implies m(\psi(U)) = \sum m(\psi(I_i))$$
   Note that $\forall i \in \mathbb{N},\ \exists r \in \mathbb{R}$, such that $\varphi(x) = r, \forall x \in I_i$
   In particular, $\psi(x) = x + r, \forall x \in I_i$ and so
   $$\psi(I_i) = r + I_i$$
   so
   $$m(\psi(U)) = \sum m(\psi(I_i)) = \sum m(I_i) = m(\dot{\cup} I_i) = m(U)$$
   Since $[0, 1] = U\dot{\cup}C$, we have that $1 = m(U) + m(C) = m(U)$, so $m(\psi(U)) = m(U) = 1 > 0 \implies m(\psi(C)) = 1$
2. By Vitali, $\psi(C)$ contains a subset $A \subseteq \psi(C)$ which is non-measurable. Let $B = \psi^{-1}(A) \subseteq C$, B is measurable because $0 = m(C) \geqslant m(B) = 0$. Then $\psi(B) = \psi(\psi^{-1}(A)) = A$

$\square$

Theorem 1.34

Cantor Set contains an element $\mathcal{L} \setminus \mathbb{B}$

Proof. $B \subseteq C \implies B$ measurable. $\psi(B)$ is non-measurable. By A1, if B is Borel, then $\psi(B)$ is Borel, so B cannot be Borel. $\square$

1.12 Measurable Function

Definition 1.35

$A \subseteq \mathbb{R}$ measurable, we say $f : A \to \mathbb{R}$ is measurable iff for all open $U \subseteq \mathbb{R}$, $f^{-1}(U)$ measurable.

Proposition 1.36

If $A \subseteq \mathbb{R}$ is measurable and $f : A \to \mathbb{R}$ is continuous then f is measurable.

Proof. f is continuous $\implies$ $f^{-1}(U)$ open if U open $\implies$ $f^{-1}(U)$ Borel, measurable □

Proposition 1.37

$A \subseteq \mathbb{R}$ measurable, $\chi_A : \mathbb{R} \to \mathbb{R}$, $\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$, then χ_A is measurable.

Proof.

$$\begin{aligned} &U \subseteq \mathbb{R}, \text{ open} \\ &\chi_A^{-1}(U) = \mathbb{R}, \text{ if } 0, 1 \in U \\ &\chi_A^{-1}(U) = A, \text{ if } 1 \in U, 0 \notin U \\ &\chi_A^{-1}(U) = A^C, \text{ if } 0 \in U, 1 \notin U \\ &\chi_A^{-1}(U) = \emptyset, \text{ if } 0, 1 \notin U \end{aligned}$$

In any case, $\chi_A^{-1}(U)$ is measurable. □

Proposition 1.38

$A \subseteq \mathbb{R}$ measurable, $f : A \to \mathbb{R}$, the following are equivalent,

1. *f is measurable*
2. *$\forall a \in \mathbb{R}$, $f^{-1}(a, \infty)$ is measurable*
3. *$\forall a < b$, $f^{-1}(a, b)$ measurable*

Proof.

- 1) $\implies$ 2), trivial
- 2) $\implies$ 3), let $b \in \mathbb{R}$ such that $f^{-1}(b, \infty)$ is measurable, then $\mathbb{R} \setminus f^{-1}(b, \infty) = f^{-1}(\mathbb{R} \setminus (b, \infty) = f^{-1}((-\infty, b])$ is measurable as well.
  We see that $(-\infty, b) = \bigcup_{n=1}^{\infty}(-\infty, b - \frac{1}{n}]$ and so

$$f^{-1}(-\infty, b) = \bigcup_{n=1}^{\infty} f^{-1}((-\infty, b - \frac{1}{n}])$$

so it's measurable.
Finally, for $a < b$,

$$(a, b) = (a, \infty) \cap (-\infty, b)$$

so

$$f^{-1}((a, b)) = f^{-1}((a, \infty) \cap (-\infty, b)) = f^{-1}((a, \infty)) \cap f^{-1}((-\infty, b))$$

so it's measurable.

- $3) \implies 1)$ Trivial. Any open set is a countable union of intervals.

$\square$

1.13 Properties of Measurable Function

Proposition 1.39

$A \subseteq \mathbb{R}$ measurable, $f, g : A \to \mathbb{R}$ measurable.

1. *$\forall a, b \in \mathbb{R},\ af + bg$ is measurable*
2. *The function fg is measurable.*

Proof.

1. Let $a \in \mathbb{R}$. For $\alpha \in \mathbb{R}$, $(af)^{-1}(\alpha, \infty) = \{x \in A : af(x) > \alpha\}$
   (a) if $a > 0$,
   $$(af)^{-1}(\alpha, \infty) = \{x \in A : f(x) > \alpha/a\} = f^{-1}(\alpha/a, \infty) \implies \text{ measurable}$$
   (b) $a < 0$,
   $$(af)^{-1}(\alpha, \infty) = f^{-1}(-\infty, \alpha/a) \implies \text{ measurable}$$
   (c) $a = 0$,
   $$af \text{ constant} \implies \text{ continuous} \implies \text{ measurable}$$

   We now show that $f + g$ measurable. For $\alpha \in \mathbb{R}$,
   $$\begin{aligned}(f+g)^{-1}(\alpha, \infty) &= \{x \in A : f(x) + g(x) > \alpha\} \\ &= \{x \in A : f(x) > \alpha - g(x)\} \\ &= \{x \in A : \exists q \in \mathbb{Q}, f(x) > q > \alpha - g(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} (\{x \in A : f(X) > q\} \cap \{x \in A : g(x) > \alpha - q\}) \\ &= \bigcup_{q \in \mathbb{Q}} f^{-1}(q, \infty) \cap g^{-1}(\alpha - q, \infty) \implies \text{ measurable}\end{aligned}$$

   so $f + g$ is measurable.

2. By the quiz, $|f|$ is measurable. For $\alpha \in \mathbb{R}$,
   $$\begin{aligned}&(f^2)^{-1}(\alpha, \infty) \\ =&\{x \in A : f(x)^2 > \alpha\} \\ =&\begin{cases} A, & \alpha < 0 \\ \{x \in A : |f(x)| > \sqrt{\alpha}\}, & \alpha \geqslant 0 \end{cases} \\ =&\begin{cases} A, & \alpha < 0 \\ |f|^{-1}(\sqrt{\alpha}, \infty), & \alpha \geqslant 0 \end{cases}\end{aligned}$$

is measurable, so f^2 is measurable.
Since $(f+g)^2$ is also measurable, and

$$2fg = (f+g)^2 - f^2 - g^2$$

so $2fg$ is measurable. By 1),

□

Example 1.40

$\psi : [0,1] \to \mathbb{R}$, $\psi(x) = x + \varphi(x)$. There exists $A \subseteq [0,1]$ such that A is measurable but $\psi(A)$ is not measurable. Extend $\psi : \mathbb{R} \to \mathbb{R}$ continuously to a strictly increasing surjective function such that ψ^{-1} is continuous. Consider $\chi_A \circ \psi^{-1}$ where both χ_A and ψ^{-1} are measurable. Then,

$$\begin{aligned}&(\chi_A \circ \psi^{-1})^{-1}\left(\frac{1}{2}, \frac{3}{2}\right)\\ =&\psi(\chi_A^{-1}(1/2, 3/2))\\ =&\psi(A) \textit{ NOT measurable}\end{aligned}$$

Proposition 1.41

$A \subseteq \mathbb{R}$ measurable. If $g : A \to \mathbb{R}$ is measurable and $f : \mathbb{R} \to \mathbb{R}$ is continuous then $f \circ g$ is measurable.

Proof. Let $U \subseteq \mathbb{R}$ open, then

$$(f \circ g)^{-1}(U) = g^{-1}(\underbrace{f^{-1}(U)}_{\text{open}})$$

which is always measurable by g being measurable. □

1.14 More Properties for Measurable Functions

Definition 1.42

$A \subseteq \mathbb{R}$, we say a property $P(x)$ ($x \in A$) is true almost everywhere if

$$m(\{x \in A : P(x) \text{ false}\}) = 0$$

Proposition 1.43

$f : A \to \mathbb{R}$ measurable. If $g : A \to \mathbb{R}$ is a function and $f = g$ a.e., then g is measurable.

Proof. $B := \{x \in A : f(x) \neq g(x)\}$, and $m(B) = 0$. Let $\alpha \in \mathbb{R}$, then

$$\begin{aligned} &g^{-1}(\alpha,\infty) = \{x \in A : g(x) > \alpha\} \\ =&\{x \in A \setminus B : g(x) > \alpha\} \cup \{x \in B : g(x) > \alpha\} \\ =&\{x \in A \setminus B : f(x) > \alpha\} \cup \{x \in B : g(x) > \alpha\} \\ =&(\underbrace{f^{-1}(\alpha,\infty)}_{\text{measurable}} \cap \underbrace{A \setminus B}_{A,B\text{measurable, so it's measurable}}) \cup \underbrace{\{x \in B : g(x) > \alpha\}}_{\subseteq B\text{,so measure zero, measurable}} \end{aligned}$$

Hence, $g^{-1}(a,\infty)$ is measurable, so g is measurable. □

Proposition 1.44

A is measurable, and $B \subseteq A$ is measurable. A function $f : A \to \mathbb{R}$ is measurable if and only if $f|_B$ and $f|_{A \setminus B}$ are measurable.

Proof.

- $\implies$ Suppose $f : A \to \mathbb{R}$ is measurable. Let $\alpha \in \mathbb{R}$, then,

  $$(f|_B)^{-1}(\alpha,\infty) = \{x \in B : f(x) > \alpha\} = f^{-1}(\alpha,\infty) \cap B \implies \text{ measurable}$$

  so $f|_B$ is measurable, the proof for $f|_{A \setminus B}$ is identical.

- $\impliedby$ Suppose $f|_B$ and $f|_{A \setminus B}$ are measurable. For $\alpha \in \mathbb{R}$,

  $$\begin{aligned} f^{-1}(\alpha,\infty) &= \{x \in A : f(x) > \alpha\} \\ &= \{x \in B : f(x) > \alpha\} \cup \{x \in A \setminus B : f(x) > \alpha\} \\ &= (f|_B)^{-1}(\alpha,\infty) \cup (f|_{A \setminus B})^{-1}(\alpha,\infty) \end{aligned}$$

  is measurable, so f is measurable.

□

Proposition 1.45

(f_n) measurable, $A \to \mathbb{R}$. If $f_n \to f$ pointwise a.e. then f is measurable.

Proof. Let $B = \{x \in A : f_n(x) \not\to f(x)\}$ so that $m(B) = 0$.
For $\alpha \in \mathbb{R}$,

$$(f|_B)^{-1}(\alpha, \infty) = \underbrace{f^{-1}(\alpha, \infty) \cap B}_{\text{measure zero}} \text{ is measurable}$$

It suffices to show that $f|_{A\setminus B}$ is measurable. By replacing f by $f|_{A\setminus B}$, we may assume $f_n \to f$ pointwise. Let $\alpha \in \mathbb{R}$, since $f_n \to f$ pointwise, we set that for $x \in A$,

$$f(x) > \alpha \iff \exists n, N \in \mathbb{N}, \forall i \in \mathbb{N}, f_i(x) > \alpha + \frac{1}{n}(\text{ to avoid } f_n \to \alpha)$$

We then see that

$$\begin{aligned} &f^{-1}(\alpha, \infty) \\ =&\bigcup_{n\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \underbrace{f_i^{-1}(\alpha + \frac{1}{n}, \infty)}_{\text{measurable}} \end{aligned}$$

is measurable, which implies that f is measurable. $\square$

1.15 Simple Approximation

Definition 1.46

A function $\varphi : A \to \mathbb{R}$ is called simple if

1. *φ is measurable*
2. *$\varphi(A)$ is finite*

Remark. [Conical Representation]

$$\varphi : A \to \mathbb{R} \text{ is simple}$$

and

$$\varphi(A) = \{\underbrace{c_1, c_2, \ldots, c_k}_{\text{distinct}}\}$$

then

$$A_i = \varphi^{-1}(\{c_i\}) \text{ measurable}$$
$$A = \dot{\bigcup_{i=1}^{k}} A_i$$
$$\varphi = \sum_{i=1}^{k} c_i \chi_{A_i}$$

Lemma 1.47

$f : A \to \mathbb{R}$ measurable and bounded. $\forall \varepsilon > 0$, there exists simple function, $\varphi_\varepsilon, \psi_\varepsilon : A \to \mathbb{R}$ such that $\forall x \in A$,

1. *$\varphi_\varepsilon \leqslant f \leqslant \psi_\varepsilon$ and*
2. *$0 \leqslant \psi_\varepsilon - \varphi_\varepsilon < \varepsilon$*

Proof.

$$f(A) \subseteq [a, b]$$

Given $\varepsilon > 0$,

$$a = y_0 < y_1 < y_2 \ldots < y_n = b$$
$$y_{i+1} - y_i < \varepsilon$$
$$\underbrace{I_k}_{\text{Borel}} = [y_{k-1}, y_k),\ A_k = f^{-1}(I_k) \implies \text{measurable}$$
$$\varphi_\varepsilon : A \to \mathbb{R}, \psi_\varepsilon : A \to \mathbb{R}$$
$$\varphi_\varepsilon = \sum_{k=1}^{n} y_{k-1}\chi_{A_k}$$
$$\psi_\varepsilon = \sum_{k=1}^{n} y_k \chi_{A_k}$$

Let $x \in A$. Since $f(x) \in [a, b]$, $\exists k \in \{1, \ldots, n\}$ such that $f(x) \in I_k$ i.e. $y_{k-1} \leqslant f(x) \leqslant y_k$, $x \in A_k$. Moreover,

$$\varphi_\varepsilon(x) = y_{k-1} \leqslant f(x) \leqslant y_k = \psi_\varepsilon(x)$$

and so

$$\varphi_\varepsilon \leqslant f \leqslant \psi_\varepsilon$$

For the same x,

$$0 \leqslant \psi_\varepsilon(x) - \varphi_\varepsilon(x) = y_k - y_{k-1} < \varepsilon$$

□

Theorem 1.48: Simple Approximation

$A \subseteq \mathbb{R}$ is measurable. A function $f : A \to \mathbb{R}$ is measurable if and only if there is a sequence (φ_n) of simple functions on A such that

1. *$\varphi_n \to f$ pointwise*
2. *$\forall n,\ |\varphi_n| \leqslant |f|$*

Proof.

- $\impliedby$ Simple functions are measurable and pointwise limit of measurable functions is also measurable
- $\implies$ Suppose $f : A \to \mathbb{R}$ is measurable,
    1. $f \geqslant 0$
    For $n \in \mathbb{N}$, define
    $$A_n = \{x \in A : f(x) \leqslant n\}$$
    such that A_n is measurable and $f|_{A_n}$ is measurable and bounded.
    By the lemma, there exists simple functions φ_n and ψ_n such that
    $$0 \leqslant \varphi_n \leqslant f \leqslant \psi_n \text{ on } A_n \text{ and } 0 \leqslant \psi_n - \varphi_n < \frac{1}{n}$$
    Fix $n \in \mathbb{N}$, extend $\varphi_n : A \to \mathbb{R}$ by setting $\varphi_n(x) = n$ if $x \notin A_n$, so $0 \leqslant \varphi_n \leqslant f$
    For each $n \in \mathbb{N}$, $\varphi_n : A \to \mathbb{R}$ is simple (it's just a simple function with one more value on a disjoint set).
    <u>Claim:</u> $\varphi_n \to f$ pointwise
    Let $x \in A$ and let $N \in \mathbb{N}$ such that $f(x) \leqslant N$ (i.e. $x \in A_N$). For $n \geqslant N,\ x \in A_n$ and so
    $$0 \leqslant f(x) - \varphi_n(x) \leqslant \psi_n(x) - \varphi_n(x) < \frac{1}{n}$$
    2. $f : A \to \mathbb{R}$ is measurable. And $B = \{x \in A : f(x) \geqslant 0\}$ and $C = \{x \in A : f(x) < 0\}$ are both measurable.
    Define $g, h : A \to \mathbb{R}$,
    $$g = \chi_B f,\ h = -\chi_B f$$

so that g, h measurable and non-negative.
By Case 1, there exists a sequence (φ_n), (ψ_n) of simple functions such that $\varphi_n \to g$ pointwise, $\psi_n \to h$ pointwsie, $0 \leqslant \varphi_n \leqslant g,\ 0 \leqslant \psi_n \leqslant h$. Then

$$\underbrace{\varphi_n - \psi_n}_{\text{simple}} \to g - h = f \text{ pointwise}$$

and

$$|\varphi_n - \psi_n| \leqslant |\psi_n| + |\varphi_n| = \varphi_n + \psi_n \leqslant g + h = |f|$$

$\square$

1.16 Littlewood's Principle

Up to certain finiteness conditions

1. Measurable sets are "almost" finite, disjoint unions of bounded open intervals.
2. Measurable functions are "almost" continuous.
3. Pointwise limits of measurable functions are "almost" uniform limits

Theorem 1.49: [Littlewood 1]

A be measurable set, $m(A) < \infty$. $\forall \varepsilon > 0$, there exists finitely many open, bounded, disjoint intervals $I_1, I_2, \ldots, I_n$ such that $m(A \triangle U) < \varepsilon$, where $U = I_1 \cup I_2 \cup \ldots \cup I_n$. Note: $m(A \triangle U) = m(A \setminus U) + m(U \setminus A)$.

Proof. Let $\varepsilon > 0$ be given. We may find an open set U and $A \subseteq U$ and

$$m(U \setminus A) < \frac{\varepsilon}{2}$$

By PMATH351, there exists open, bounded, disjoint intervals $I_i (i \in \mathbb{N})$ such that

$$U = \dot{\bigcup}_{i=1}^{\infty} I_i$$

Note that,

$$\sum_{i=1}^{\infty} l(I_i) = m(U) = m(U \setminus A) + m(A) < \infty$$

In particular, there exists $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} l(I_i) = \frac{\varepsilon}{2}$$

Take $V = I_1 \cup \ldots \cup I_N$, we see that

$$\begin{aligned} m(A \setminus V) &\leqslant m(U \setminus V) \\ &= m\left(\bigcup_{N+1}^{\infty} I_i\right) \\ &= \sum_{N+1}^{\infty} l(I_i) < \frac{\varepsilon}{2} \end{aligned}$$

and

$$m(V \setminus A) \leqslant m(U \setminus A) < \frac{\varepsilon}{2}$$

□

Lemma 1.50

Let A be measurable and $m(A) < \infty$, (f_n) be measurable, $A \to \mathbb{R}$. Assume $f : A \to \mathbb{R}$ such that $f_n \to f$ pointwise. $\forall \alpha, \beta > 0$, there exists a measurable subset $B \subseteq A$ and $N \in \mathbb{N}$ such that

1. $|f_n(x) - f(x)| < \alpha, \forall\, x \in B, n \geqslant N$
2. $m(A \setminus B) < \beta$

Proof. Let $\alpha, \beta > 0$ be given. For $n \in \mathbb{N}$, define

$$\begin{aligned} A_n &= \{x \in A : |f_k(x) - f(x)| < \alpha, \forall k \geqslant n\} \\ &= \bigcap_{k=n}^{\infty} \underbrace{|f_k - f|^{-1}(-\infty, \alpha)}_{\text{measurable}} \end{aligned}$$

So every A_n is measurable. Since $f_n \to f$ pointwise,

$$A = \bigcup_{n=1}^{\infty} A_n$$

Since (A_n) is ascending, by continuity of measure,

$$m(A) = \lim_{n \to \infty} m(A_n) < \infty$$

we may find $N \in \mathbb{N}$ such that $\forall n \geqslant N$,

$$m(A) - m(A_n) < \beta$$

Pick $B = A_N$ we get what's required. $\square$

Theorem 1.51: Littlewood 3, Egoroff's Theorem

A is measurable, $m(A) < \infty$, (f_n) is measurable, $A \to \mathbb{R}$, $f_n \to f$ pointwise. $\forall \varepsilon > 0$, there exists a closed set $C \subseteq A$ such that

1. $f_n \to f$ *uniformly on* C
2. $m(A \setminus C) < \varepsilon$

Proof. Let $\varepsilon > 0$ be given. By the lemma, for every $n \in \mathbb{N}$, there exists a measurable set $A_n \subseteq A$ and $N(n) \in \mathbb{N}$ such that

1. $\forall x \in A_n$ and $k \geqslant N(n)$,
$$|f_k(x) - f(x)| < \frac{1}{n}$$
2. $m(A \setminus A_n) < \frac{\varepsilon}{2^{n+1}}$

Take $B = \bigcap_{n=1}^{\infty} A_n$ (measurable). For $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon,\ k \geqslant N(n)$, and $x \in B$,

$$|f_k(x) - f(x)| < \frac{1}{n} < \varepsilon$$

so $f_n \to f$ uniformly on B. Moreover,

$$m(A \setminus B) = m(A \setminus \cap A_n) = m(\cup(A \setminus A_n)) \leqslant \sum m(A \setminus A_n) < \sum \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2}$$

By A1, there exists a closed set C such that $C \subseteq B$ and $m(B \setminus C) < \frac{\varepsilon}{2}$, so

1. Since $C \subseteq B$, $f_k \to f$ uniformly on C
2. $m(A \setminus C) = m(A \setminus B) + m(B \setminus C) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

□

Warning:
$f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \frac{x}{n}$ and $f_n \to 0$ pointwise. But $f_n \not\to 0$ uniformly on any measurable set $B \subseteq \mathbb{R}$ such that $m(\mathbb{R} \setminus B) < 1$

Proof. Suppose such B exists. Since B measurable, $B \subseteq \mathbb{R}$, we know

$$m(\mathbb{R} \setminus B) = m(\mathbb{R}) - m(B) < 1 \implies m(B) = \infty$$

That is, B has to be unbounded.
Since $f_n \to 0$ uniformly on B, $\forall \varepsilon > 0, \exists N \in \mathbb{N}, s/t\ \forall k \geqslant N, \forall x \in B$,

$$|0 - f_k(x)| < \varepsilon \implies \left|\frac{x}{k}\right| < \varepsilon$$

However, since B is unbounede, we can always find $x \in B$ such that $|x| = (\varepsilon + 1)|k|$, so $|x/k| = \varepsilon + 1 > \varepsilon$.
That is, no matter how big the N is, I can always find points where the uniformly convergence condition fails. Contradiction! So no such B exists. □

Lemma 1.52

$f : A \to \mathbb{R}$ simple. $\forall \varepsilon > 0$, there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ and a closed $C \subseteq A$ such that

1. *$f = g$ on C*
2. *$m(A \setminus C) < \varepsilon$*

Proof. $f = \sum_{i=1}^{n} a_i \chi_{A_i}$, conical representation. $A_i = \{x \in A : f(x) = a_i\}$ is measurable. By A1, $C_i \subseteq A_i$ closed,

$$m(A_i \setminus C_i) < \frac{\varepsilon}{n}$$

AND

$$A = \dot{\bigcup}_{i=1}^{n} A_i,\ C := \dot{\bigcup}_{i=1}^{n} C_i \text{ closed}$$

1. $\forall x \in C_i, f(x) = a_i$. By A1, f is continuous on $C \implies$ we then extend $f|_C$ to a continuous function $g : \mathbb{R} \to \mathbb{R}$
2. $m(A \setminus C) = m(\cup_{i=1}^n A_i \setminus C_i) = \sum_{i=1}^n m(A_i \setminus C_i) < \varepsilon$

$\square$

Theorem 1.53: Littlewood 2, Lusin Theorem

$f : A \to \mathbb{R}$ is measurable. $\forall \varepsilon > 0$, there exists a continuous $g : \mathbb{R} \to \mathbb{R}$ and a closed set $C \subseteq A$ such that

1. *$f = g$ on C and*
2. *$m(A \setminus C) < \varepsilon$*

Proof. Let $\varepsilon > 0$ given.

1. $m(A) < \infty$
   Let $f : A \to \mathbb{R}$ be measurable. By the Simple Approximation Theorem, there exists (f_n) simple such that $f_n \to f$ pointwise. By the lemma, there exists continuous $g_n : \mathbb{R} \to \mathbb{R}$ and closed $C_n \subseteq A$ such that

   (a) $f_n = g_n$ on C_n
   (b) $m(A \setminus C_n) < \frac{\varepsilon}{2^{n+1}}$

   By Egoroff, there exists a closed set $C_0 \subseteq A$ such that $f_n \to f$ uniformly on C_0 and $m(A \setminus C_0) < \frac{\varepsilon}{2}$.

   Let $C = \bigcap_{i=0}^{\infty} C_i$

   (a) $g_n = f_n \to f$ uniformly on $C \subseteq C_0$, so f is continuous on C. By A1, extend $f|_C$ to a continuouse function $g : \mathbb{R} \to \mathbb{R}$.
   (b)
   $$\begin{aligned} m(A \setminus C) = m(A \setminus \cap_{i=0}^{\infty} C_i) &= m(\cup_{i=0}^{\infty}(A \setminus C_i)) \\ &\leqslant \sum_{i=0}^{\infty} m(A \setminus C_i) = m(A \setminus C_0) + \sum_{i=1}^{\infty} m(A \setminus C_i) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

2. $m(A) = \infty$
   For $n \in \mathbb{N}$,
   $$A_n = \{a \in A : |a| \in [n-1, n)\}$$
   such that
   $$A = \dot{\bigcup}_{n=1}^{\infty} A_n$$
   By case 1, there exists continuous functions $g_n : \mathbb{R} \to \mathbb{R}$ and closed $C_n \subseteq A_n$ such that

(a) $f = g_n$ on C_n
(b) $m(A_n \setminus C_n) < \frac{\varepsilon}{2^n}$

Consider $C = \dot{\bigcup}_{n=1}^{\infty} C_n$, and C is closed.

(a) $m(A \setminus C) = m(\dot{\cup}(A_n \setminus C_n)) = \sum m(A_n \setminus C_n) < \varepsilon$
(b) $g : C \to \mathbb{R}$. Let $x \in C$ such that $x \in C_n$ for one $n \in \mathbb{N}$. Define $g(x) = g_n(x) = f(x)$. By A1, extend g on $\mathbb{R}$.

$\square$

2 Integration

2.1 Integration

1. Simple functions
$$\varphi : A \to \mathbb{R},\ m(A) < \infty$$
2. $f : A \to \mathbb{R}$, bounded measure, $m(A) < \infty$,
$$\varphi_\varepsilon \leqslant f \leqslant \psi_\varepsilon$$
3. $f : A \to \mathbb{R}$ measurable, $f \geqslant 0$,
$$\sup \left\{ \int_A h : h \in (2), 0 \leqslant h \leqslant f \right\}$$
4. $f : A \to \mathbb{R}$ measurable,
$$f^+ = \max\{f, 0\}$$
$$f^- = \max\{-f, 0\}$$

Step 1: $\varphi : A \to \mathbb{R}$ simple, $m(A) < \infty$

Definition 2.1

$m(A) < \infty$, $\varphi : A \to \mathbb{R}$ simple. Conical Rep.:$\varphi = \sum_{i=1}^n a_i \chi_{A_i}$. The (Lebesgue) Integral of φ over A is

$$\int_A \varphi = \sum_{i=1}^n a_i m(A_i)$$

Lemma 2.2

$m(A) < \infty$ (A measurable). If $B_1, B_2, \ldots, B_n \subseteq A$ are measurable and disjoint and $\varphi : A \to \mathbb{R}$ defined by

$$\varphi = \sum_{i=1}^n b_i \chi_{B_i}$$

then

$$\int_A \varphi = \sum_{i=1}^n b_i m(B_i)$$

Proof. For $n = 2$,
If $b_1 \neq b_2$, then $\varphi = b_1\chi_{B_1} + b_2\chi_{B_2}$ is the conical representation.
If $b_1 = b_2$, then
$$b_1\chi_{B_1} + b_1\chi_{B_2} = b_1(\chi_{B_1} + \chi_{B_2}) = \underbrace{b_1\chi_{B_1\cup B_2}}_{\text{conical rep.}}$$

so

$$\begin{aligned}\int_A \varphi &= b_1 m(B_1 \dot\cup B_2)\\ &= b_1(m(B_1)+m(B_2))\\ &= b_1 m(B_1) + b_2 m(B_2)\end{aligned}$$

Then simple dicuss other cases. □

Proposition 2.3

$\varphi, \psi : A \to \mathbb{R}$ *simple,* $m(A) < \infty$*. For all* $\alpha, \beta \in \mathbb{R}$*,*

$$\int_A (\alpha\varphi + \beta\psi) = \alpha \int_A \varphi + \beta \int_A \psi$$

Proof.

$$\varphi(A) = \{a_1, a_2, \dots, a_n\}$$
$$\psi(A) = \{b_1, b_2, \dots, b_m\}$$

where the elements are distinct for each set.

Define

$$C_{ij} = \{x \in A : \varphi(x) = a_i, \psi(x) = b_j\} = \varphi^{-1}(\{a_i\}) \cap \psi^{-1}(\{b_j\})$$

which is measurable.

$$\alpha\varphi + \beta\psi = \sum_{i,j} (\alpha a_i + \beta b_j)\chi_{C_{ij}}$$

By the lemma,

$$\begin{aligned}\int_A \alpha\varphi + \beta\psi &= \sum_{i,j}(\alpha a_i + \beta b_j) m(C_{ij})\\ &= \sum_{i,j} \alpha a_i m(C_{ij}) + \sum_{i,j} \beta b_j m(C_{ij})\\ &= \sum_i \alpha a_i \sum_j m(C_{ij}) + \sum_j \beta b_j \sum_i m(C_{ij})\\ &= \sum_i \alpha a_i m(\{x \in A : \varphi(x) = a_i\}) + \sum_j \beta b_j m(\{x \in A : \varphi(x) = a_i\})\\ &= \alpha \int_A \varphi + \beta \int_A \psi\end{aligned}$$

□

Proposition 2.4

$\varphi, \psi : A \to \mathbb{R}$ *simple,* $m(A) < \infty$*. If* $\varphi \leqslant \psi$*, then*

$$\int_A \varphi \leqslant \int_A \psi$$

Proof.

$$\int_A \psi - \int_A \varphi = \int_A \underbrace{(\psi - \varphi)}_{\geqslant 0} \geqslant 0$$

$\square$

Step2: $f : A \to \mathbb{R}$ bounded, measurable $m(A) < \infty$

Definition 2.5

$f : A \to \mathbb{R}$ be bounded, measurable and $m(A) < \infty$. Then

- *Lower Lebesgue Integral:*

$$\underline{\int_A} f = \sup \left\{ \int_A \varphi : \varphi \leqslant f \text{ simple} \right\}$$

- *Lower Lebesgue Integral:*

$$\overline{\int_A} f = \inf \left\{ \int_A \psi : f \leqslant \psi \text{ simple} \right\}$$

Proposition 2.6

$m(A) < \infty$, $f : A \to \mathbb{R}$ bounded, measurable. Then

$$\underline{\int_A} f = \overline{\int_A} f$$

Proof. $\forall n \in \mathbb{N}$, there exists simple functions, $\varphi_n, \psi_n : A \to \mathbb{R}$ such that

1. $\varphi_n \leqslant f \leqslant \psi_n$
2. $\psi_n - \varphi_n \leqslant \frac{1}{n}$

We see that

$$\begin{aligned} 0 &\leqslant \overline{\int_A} f - \underline{\int_A} f \\ &\leqslant \int_A \psi_n - \int_A \varphi_n \\ &= \int_A (\psi_n - \varphi_n) \\ &\leqslant \int_A \frac{1}{n} \\ &= \frac{1}{n} m(A) < \infty \\ &\to 0 \end{aligned}$$

$\square$

Definition 2.7

$m(A) < \infty$, $f : A \to \mathbb{R}$ bounded, measurable, we define the (Lebesgue) integral of f over A by

$$\int_A f := \underline{\int_A} f = \overline{\int_A} f$$

Proposition 2.8

$f, g : A \to \mathbb{R}$ bounded, measurable, $m(A) < \infty$. For $\alpha, \beta \in \mathbb{R}$,

$$\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$$

Proof. Scalar multiplication is easy.
Now, have $\varphi_1, \varphi_2, \psi_1, \psi_2$ all simple,

$$\varphi_1 \leqslant f \leqslant \psi_1,\ \varphi_2 \leqslant g \leqslant \psi_2$$

1.
$$\begin{aligned}\int_A f + g &= \overline{\int_A} f + g \\ &\leqslant \int_A \psi_1 + \psi_2 \\ &= \int_A \psi_1 + \int_A \psi_2\end{aligned}$$

so

$$\begin{aligned}\int_A f + g &\leqslant \inf\left\{\int_A \psi_1 + \int_A \psi_2 : f \leqslant \psi_1, g \leqslant \psi_2, \psi_1, \psi_2 \text{ simple}\right\} \\ &= \inf\left\{\int_A \psi_1 : f \leqslant \psi_1 \text{ simple}\right\} + \inf\left\{\int_A \psi_2 : g \leqslant \psi_2 \text{ simple}\right\} \\ &= \int_A f + \int_A g\end{aligned}$$

2.
$$\int_A f + g = \underline{\int_A} f + g \geqslant \int_A \varphi_1 + \int_A \varphi_2$$

so

$$\begin{aligned}\int_A f + g &\geqslant \sup\left\{\int_A \varphi_1 + \int_A \varphi_2 : f \geqslant \varphi_1, g \geqslant \varphi_2, \varphi_1, \varphi_2 \text{ simple}\right\} \\ &= \sup\left\{\int_A \varphi_1 : f \geqslant \varphi_1, \varphi_1 \text{ simple}\right\} + \sup\left\{\int_A \varphi_2 : f \geqslant \varphi_2, \varphi_2 \text{ simple}\right\} \\ &= \int_A f + \int_A g\end{aligned}$$

so

$$\int_A f+g=\int_A f+\int_A g$$

□

Proposition 2.9

$f,g:A\to\mathbb{R}$ bounded, measurable and $m(A)\leqslant\infty$. If $f\leqslant g$, then $\int_A f\leqslant\int_A g$.

Proof. Since $g-f\geqslant 0$, where 0 is also a simple function, we have

$$\int_A(g-f)=\underline{\int_A}(g-f)\geqslant\int_A 0=0\implies\int_A g\geqslant\int_A f$$

□

2.2 Bounded Convergence Theorem

Proposition 2.10

$f : A \to \mathbb{R}$ bounded, measurable, $B \subseteq A$ measurable, $m(A) < \infty$, then

$$\int_B f = \int_A f\chi_B$$

Proof.

1. $f = \chi_C$, $C \subseteq A$ measurable.

$$\begin{aligned}\int_A \chi_C\chi_B &= \int_A \chi_{B\cap C}\\ &= 1 * m(B\cap C)\\ &= \int_B \chi_{C|_B}\end{aligned}$$

2. f is simple, $f = \sum_{i=1}^n a_i\chi_{A_i}$,

$$\int_A f\chi_B = \sum a_i \int_A \chi_{A_i}\chi_B = \sum a_i \int_B \chi_{A_i} = \int_B (\sum a_i\chi_{A_i|_B}) = \int_B f$$

3. $f : A \to \mathbb{R}$ be bounded and measurable.
First we take $f \leqslant \psi$, simple, then

$$\int_A f\chi_B \leqslant \int_A \psi\chi_B = \int_B \psi$$

By taking the $\inf$ over all such ψ, we have that

$$\int_A f\chi_B \leqslant \overline{\int_A} f = \int_B f$$

Similarly, taking $\varphi \leqslant f$, φ simple, we obtain,

$$\underline{\int_B} f = \int_B f \leqslant \int_A f\chi_B$$

so we have

$$\int_A f\chi_B = \int_B f$$

$\square$

Proposition 2.11

$f : A \to \mathbb{R}$ be bounded, measurable, $m(A) < \infty$. If $B, C \subseteq A$ are measurable and disjoint, then

$$\int_{B\cup C} f = \int_B f + \int_C f$$

Proof.

$$\int_{B\cup C} f = \int_A f\chi_{B\cup C}$$
$$= \int_A f(\chi_B + \chi_C)$$
$$= \int_A f\chi_B + \int_A f\chi_C$$
$$= \int_B f + \int_C f$$

□

Proposition 2.12

$f : A \to \mathbb{R}$ be bounded, measurable, $m(A) < \infty$, then $\left|\int_A f\right| \leqslant \int_A |f|$.

Proof.

$$-|f| \leqslant f \leqslant |f|$$
$$-\int_A |f| \leqslant \int_A |f| \leqslant \int_A |f|$$

□

Proposition 2.13

(f_n) is bounded, measurable, $A :\to \mathbb{R}$, $m(A) < \infty$. If $f_n \to f$ uniformly, then

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

Proof. Let $\varepsilon > 0$ be given, let $N \in \mathbb{N}$ such that

$$|f_n - f| \leqslant \frac{\varepsilon}{m(A)+1}$$

then, for $n \geqslant N$

$$\left|\int_A f_n - \int_A f\right|$$
$$= \left|\int_A (f_n - f)\right|$$
$$\leqslant \int_A |f_n - f|$$
$$\leqslant m(A) * \frac{\varepsilon}{m(A)+1}$$
$$< \varepsilon$$

□

Example 2.14

$f_n : [0,1] \to \mathbb{R}$,

$$f_n(x) = \begin{cases} 0, & 0 \leqslant x < \frac{1}{n} \\ n, & \frac{1}{n} \leqslant x < \frac{2}{n} \\ 0, & \frac{2}{n} \leqslant x \end{cases}$$

then $f_n \to 0$ pointwisely, but

$$\int_{[0,1]} f_n = 1, \int_{[0,1]} 0 = 0$$

Theorem 2.15: [BCT]

$(f_n) : A \to \mathbb{R}$ measurable, $m(A) < \infty$. If there exists $M > 0$ such that $|f_n| \leqslant M$ for all n and $f_n \to f$ pointwise then

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

Proof. Let $\varepsilon > 0$ be given. By Egoroff's theorem, there exists measurable $B \subseteq A$ and $N \in \mathbb{N}$ such that for $n \geqslant N$,

1. $|f_n - f| < \frac{\varepsilon}{2(m(B)+1)}$ on B
2. $m(A \setminus B) < \frac{\varepsilon}{4M}$

$\forall n \geqslant N$,

$$\begin{aligned} \left| \int_A f_n - \int_A f \right| &\leqslant \int_A |f_n - f| \\ &= \int_B |f_n - f| + \int_{A \setminus B} |f_n - f| \\ &\leqslant \int_B |f_n - f| + \int_{A \setminus B} (|f_n| + |f|) \\ &\leqslant \int_B |f_n - f| + 2M * m(A \setminus B) \\ &= \leqslant m(B) \frac{\varepsilon}{2(M(B)+1)} + 2M \frac{\varepsilon}{4M} \\ &\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

$\square$

Definition 2.16

$f : A \to \mathbb{R}$ measurable

1. *We say f has finite support if*
$$A_0 := \{x \in A : f(x) \neq 0\}$$
has finite measure.
2. *We say f is a BF function. If f is bounded and has finite support.*
3. *If $f : A \to \mathbb{R}$ is BF, then*
$$\int_A f := \int_{A_0} f$$

Definition 2.17

$f : A \to \mathbb{R}$ measurable, $f \geqslant 0$,
$$\int_A f = \sup\left\{\int_A h : 0 \leqslant h \leqslant f,\ \textit{BF}\right\}$$

Proposition 2.18

$f, g : A \to \mathbb{R}$ measurable, $f, g \geqslant 0$

1. *$\forall \alpha, \beta \in \mathbb{R}$,*
$$\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$$
2. *If $f \leqslant g$, then $\int_A f \leqslant \int_A g$*
3. *If $B, C \subseteq A$ are measurable and $B \cap C = \emptyset$ then*
$$\int_{B \cup C} f = \int_B f + \int_C f$$

Theorem 2.19: [Chebychev's Inequality]

$f : A \to \mathbb{R}$ measurable, non-negative; $\forall \varepsilon > 0$,
$$m\left(\{x \in A : f(x) \geqslant \varepsilon\}\right) \leqslant \frac{1}{\varepsilon} \int_A f$$

Proof. Let $\varepsilon > 0$ given and let
$$A_\varepsilon = \{x \in A : f(x) \geqslant \varepsilon\}$$

1. $m(A_\varepsilon) < \infty$

$$\underbrace{\varphi}_{\text{BF}} = \varepsilon\chi_{A_\varepsilon} \leqslant f$$

so

$$\varepsilon m(A_\varepsilon) = \int_A \varphi \leqslant \int_A f$$

2. $m(A_\varepsilon) = \infty$ For $n \in \mathbb{N}$, $A_{\varepsilon,n} := A_\varepsilon \cap [-n, n]$. By the continuity of measure,

$$\infty = m(A_\varepsilon) = \lim_{n\to\infty} m(A_{\varepsilon,n})$$

For $n \in \mathbb{N}$, $\varphi_n := \varepsilon\chi_{\varepsilon,n}$(BF), we see that $\varphi_n \leqslant f$.
Therefore,

$$\begin{aligned}\infty =& m(A_\varepsilon)\\ =& \lim_{n\to\infty} m(A_{\varepsilon,n})\\ =& \lim_{n\to\infty} \frac{1}{\varepsilon}\int_A \varphi_n\\ \leqslant& \frac{1}{\varepsilon}\int_A f\end{aligned}$$

□

Proposition 2.20

$f : A \to \mathbb{R}$ *measurable,* $f \geqslant 0$

$$\int_A f = 0 \iff f = 0 \textit{ a.e.}$$

Proof.

- ($\implies$) Suppose $\int_A (f) = 0$,

$$\begin{aligned}&m\left(\{x \in A : f(x) \neq 0\}\right)\\ \leqslant& \sum m\left(\left\{x \in A : f(x) \geqslant \frac{1}{n}\right\}\right)\\ \underbrace{\leqslant}_{\text{Chebychev}}& \sum n \int_A f = 0\end{aligned}$$

- $\impliedby$ Suppose $B = \{x \in A : f(x) \neq 0\}$ has measure 0.

$$\begin{aligned}\int_A f =& \int_B f + \int_{A\setminus B} \underbrace{f}_{=0}\\ =& \int_B f + 0\\ =& 0\end{aligned}$$

$\int_B f = 0$ because for any h BF and $0 \leqslant h \leqslant f$, there is a $M_h \geqslant 0$ such that $h \leqslant M_h$, then

$$\int_B 0 \leqslant \int_B h \leqslant \int_B M_h = \int_B M_h \chi_B = M_h m(B) = M_h * 0 = 0$$

so $\int_B h$ is always zero, hence

$$\int_B f = \sup \left\{ \int_B h : 0 \leqslant h \leqslant f,\ h \text{ BF} \right\} = 0$$

□

2.3 Fatou's Lemma and MCT

Theorem 2.21: Fatou's Lemma

(f_n) measurable, non-negative, $A \to \mathbb{R}$. If $f_n \to f$ pointwise then

$$\int_A f \leqslant \liminf \int_A f_n$$

Proof. Let $0 \leqslant h \leqslant f$ be a BF function. Say $A_0 = \{x \in A : h(x) \neq 0\}$. It suffices to show

$$\int_A h \leqslant \liminf \int_A f_n$$

Since h is BF, $m(A_0) < \infty$. For each $n \in \mathbb{N}$, let

$$h_n = \min\{h, f_n\} \text{ (meas.)}$$

Note:

1. $0 \leqslant h_n \leqslant h \leqslant M$, for some $M > 0$, $\forall n \in \mathbb{N}$
2. For $x \in A_0$ and $n \in \mathbb{N}$,
   (a) $h_n(x) = h(x)$ or
   (b) $h_n(x) = f_n(x) \leqslant h(x)$ and
   $$0 \leqslant h(x) - h_n(x) = h(x) - f_n(x) \leqslant f(x) - f_n(x) \to 0$$
   so $h_n(x) \to h$ on A_0

By BCT,

$$\lim_{n\to\infty} \int_{A_0} h_n = \int_{A_0} h \implies \lim_{n\to\infty} \int_A h_n = \int_A h$$

Since $h_n \leqslant f_n$ on A,

$$\int_A = \lim_{n\to\infty} \int_A h_n = \liminf_{n\to\infty} \int_A h_n \leqslant \liminf_{n\to\infty} \int_A f_n$$

□

Example 2.22

$$A = (0,1]$$
$$f_n = n\chi(0,1/n)$$
$$f_n \to 0 \textit{ pointwise}$$
$$\int_A 0 = 0$$
$$\int_A f_n = n \cdot m(0,1/n) = 1$$
$$\liminf \int_A f_n = 1$$

Theorem 2.23: [MCT]

(f_n) non-negative, measurable, $A \to \mathbb{R}$. If (f_n) is increasing and $f_n \to f$ pointwise, then

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

Proof.

$$\begin{aligned} \int_A f &\leqslant \liminf \int_A f_n \text{ by Fatou's Lemma} \\ &\leqslant \limsup \int_A f_n \\ &\leqslant \int_A f \text{ by } f_n \nearrow \text{ and converge to } f \end{aligned}$$

so $\lim_{n\to\infty} \int_A f_n = \liminf \int_A f_n = \limsup \int_A f_n$ □

Remark.

1. If $\varphi : A \to \mathbb{R}$ is simple and $m(A) < \infty$, then
$$\int_A \varphi < \infty$$
2. If $f : A \to \mathbb{R}$ is bounded, measurable and $m(A) < \infty$, then
$$\int_A f < \infty$$

Definition 2.24

If $f : A \to \mathbb{R}$ is measurable and $f \geqslant 0$, then we say f is integrable if and only if

$$\int_A f < \infty$$

2.4 The General Integral

Definition 2.25

$f : A \to \mathbb{R}$ measurable,

$$f^+(x) = \max\{f(x), 0\}$$
$$f^-(x) = \max\{-f(x), 0\}$$

Notes:

1. $f^+ + f^- = |f|$
2. $f^+ - f^- = f$
3. f^+, f^- measurable

Proposition 2.26

$f : A \to \mathbb{R}$ measurable. Then f^+, f^- are integrable if and only if $|f|$ is integrable.

Proof.

- $|f| = f^+ + f^-$
$$\int_A |f| = \underbrace{\int_A f^+}_{<\infty} + \underbrace{\int_A f^-}_{<\infty} < \infty$$
- $$\int_A f^+ \leqslant \int_A |f| < \infty; \int_A f^- \leqslant \int_A |f| < \infty$$

□

Definition 2.27

$f : A \to \mathbb{R}$ measurable. We say f is integrable if and only if $|f|$ is integrable if and only if f^+, f^- are integrable, and define

$$\int_A f = \int_A f^+ - \int_A f^-$$

Proposition 2.28: [Comparison Test]

$f : A \to \mathbb{R}$ measurable, $g : A \to \mathbb{R}$ non-negative integrable. If $|f| \leqslant g$ then f is integrable and $\left|\int_A f\right| \leqslant \int_A |f|$

Proof.

1. $\underbrace{\int_A |f|}_{<\infty} \leqslant \int_A g < \infty$

2. $|\int_A f| = \left|\int_A f^+ - \int_A f^-\right| \leqslant |\int_A f^+| + |\int_A f^-| = \int_A f^+ + \int_A f^- = \int_A (f^+ + f^-) = \int_A (f)$

$\square$

Proposition 2.29

$f, g : A \to \mathbb{R}$ integrable.

1. *$\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is integrable, and*
$$\int_A \alpha f + \beta g = \alpha \int_A f + \beta \int_A g$$
2. *If $f \leqslant g$, then $\int_A f \leqslant \int_A g$*
3. *If $B, C \subseteq A$ are measurable with $B \cap C = \emptyset$, then*
$$\int_{B \cup C} f = \int_B f + \int_C f$$

Proof.

- Comparison Test
- Results hold for f^+, f^-, g^+, g^-

$\square$

Theorem 2.30: [Lebesgue Dominated Convergence Theorem]

$f_n : A \to \mathbb{R}$ measurable. $f_n \to f$ pointwise. If there exists a $g : A \to \mathbb{R}$ integrable such that $|f_n| \leqslant g$, $\forall n \in \mathbb{N}$, then f is integrable and $\lim_{n\to\infty} \int_A f_n = \int_A f$

Proof. Since $|f_n| \to |f|$, and so $|f| \leqslant g$.
By comparison test, f is integrable. Next, observe $g - f \geqslant 0$. By Fatou,

1.

$$\begin{aligned}\int_A g - \int_A f = \int_A g - f \\ &\leqslant \liminf \int_A g - f_n \\ &= \int_A g - \limsup \int_A f_n\end{aligned}$$

$$\implies \limsup \int_A f_n \leqslant \int_A f$$

2.

$$\int_A g + \int_A f = \int_A g + f \leqslant \liminf \int_A g + f_n = \int_A g + \liminf \int_A f_n$$

$$\implies \int_A f = \liminf \int_A f_n = \limsup \int_A f_n = \lim \int_A f_n$$

□

2.5 Riemann Integration

Definition 2.31

$f : [a, b] \to \mathbb{R}$ bounded

1. *A partition of $[a, b]$ is a finite set such that*

$$P = \{x_0, x_1, \ldots, x_n\} \subseteq \mathbb{R} \text{ and } a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

2. *Relative to P, we define the lower Darboux sum:*

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$
$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$

3. *Similarly, we define the upper Darboux sum:*

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$
$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

Definition 2.32

$f : [a, b] \to \mathbb{R}$, bounded,

1. *Lower Riemann Integral*

$$R\underline{\int_a^b} f = \sup\{L(f, P) : P \text{ partition}\}$$

2. *Upper Riemann Integral*

$$R\overline{\int_a^b} f = \inf\{U(f, P) : P \text{ partition}\}$$

3. *We say f is Riemann Integrable if and only if*

$$\underbrace{R\underline{\int_a^b} f = R\overline{\int_a^b} f}_{R\int_a^b f}$$

Definition 2.33

Let $I_1, \ldots, I_n$ be pairwise disjoint intervals such that

$$[a, b] = \dot{\cup}_{i=1}^n I_i$$

A step function is a functions of the form

$$f = \sum_{i=1}^n a_i \chi_{I_i}$$

for some $a_i \in \mathbb{R}$

Remark. $f : [a, b] \to \mathbb{R}$ bounded. $a = x_0 < x_1 < \ldots < x_n = b$. $I_i = [x_{i-1}, x_i]$, $i = 1, \ldots, n$. Then

$$L(f, P) = \sum_{i=1}^n m_i \cdot l(I_i) = R\int_a^b \varphi$$

where $\varphi(x) = mi$ on I_i ($\varphi \leqslant f$).

$$U(f, P) = \sum_{i=1}^n M_i \cdot l(I_i) = R\int_a^b \psi$$

where $\psi(x) = Mi$ on I_i ($f \leqslant \psi$).

Remark. $f : [a, b] \to \mathbb{R}$ bounded,

$$R\underline{\int_a^b} f = \sup\{L(f, P) : P\} = \sup\left\{R\int_a^b \varphi : \varphi \leqslant f \text{ step}\right\}$$
$$R\overline{\int_a^b} f = \inf\{U(f, P) : P\} = \inf\left\{R\int_a^b \psi : f \leqslant \psi \text{ step}\right\}$$

2.5.1 Riemann Integral VS Lebesgue Integral

Definition 2.34

Let $f : [a, b] \to \mathbb{R}$ *bounded. Let* $x \in [a, b]$ *and* $\delta > 0$

1. $m_\delta(x) = \inf\{f(x) : x \in (x - \delta, x + \delta) \cap [a, b]\}$
2. $M_\delta(x) = \sup\{f(x) : x \in (x - \delta, x + \delta) \cap [a, b]\}$
3. *Lower Boundary of* f,
$$m(x) = \lim_{\delta \to 0} m_\delta(x)$$
4. *Upper Boundary of* f,
$$M(x) = \lim_{\delta \to 0} M_\delta(x)$$
5. *<u>Oscillation</u> of* f,
$$\omega(x) = M(x) - m(x)$$

Remark. $f : [a, b] \to \mathbb{R}$ bounded, TFAE

1. f is continuous at $x \in [a, b]$
2. $M(x) = m(x)$
3. $\omega(x) = 0$

Lemma 2.35

$f : [a, b] \to \mathbb{R}$ *bounded,*

1. m *is measure*
2. *If* $\varphi : [a, b] \to \mathbb{R}$ *is a step function with* $\varphi \leqslant f$*, then* $\varphi(x) \leqslant m(x)$ *at all points of continuity of* φ
3. $R\underline{\int_a^b} f = \int_{[a,b]} m$

Lemma 2.36

$f : [a,b] \to \mathbb{R}$ bounded,

1. *M is measure*
2. *If $\psi : [a,b] \to \mathbb{R}$ is a step function with $\psi \geqslant f$, then $\psi(x) \geqslant M(x)$ at all points of continuity of ψ*
3. *$R\overline{\int_a^b} f = \int_{[a,b]} M$*

Theorem 2.37: [Lebesgue]

Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f Riemann integrable if and only if f is continuous a.e., in that case,

$$R\int_a^b f = \int_{[a,b]} f$$

Proof.

$$R\underline{\int_a^b} f = \int_{[a,b]} m \leqslant \int_{[a,b]} M = R\overline{\int_a^b} f$$

$$\begin{aligned} & f \text{ Riemann Integrable} \\ \iff & \int_{[a,b]} m = \int_{[a,b]} M \\ \iff & \int_{[a,b]} (\underbrace{M - m}_{\geqslant 0}) = 0 \\ \iff & M = m \text{ a.e.} \\ \iff & \omega = 0 \text{ a.e.} \\ \iff & f \text{ is continuous a.e.} \end{aligned}$$

If f is continuous a.e. $\implies$ f is measurable and

$$R\underline{\int_a^b} f = \int_{[a,b]} m \leqslant \int_{[a,b]} f \leqslant \int_{[a,b]} M = R\overline{\int_a^b} f \implies R\int_a^b f = \int_{[a,b]} f$$

because $M = m$ a.e. $\square$

Example 2.38

$f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \notin \mathbb{Q} \end{cases}$$

f is discontinuous on $[0,1] \implies f$ is NOT Riemann Integrable. But $f = 0$ a.e. ans so

$$\int_{[0,1]} f = \int_{[0,1]} 0 = 0$$

Example 2.39

$\mathbb{Q} \cap [0,1] = \{q_1, q_2, \ldots\}$, $f_n = \chi_{\{q_1,\ldots,q_n\}}$. *$f_n \to f$ pointwise (f in the previous example). f_n is increasing, continuous a.e. on $[0,1]$, and it's bounded by 1, so it's Riemann Integrable.*

$$0 = R\int_{[0,1]} f_n \not\to R\int_{[0,1]} f$$

3 L^p Spaces

3.1 L^P Spaces

Recall

1. For $1 \leqslant p < \infty$, $(C([a,b]), \|\cdot\|_p)$ is a normed-vector space, where $\|f\|_p^p = \int_a^b |f|^p$
2. For $p = \infty$, $(C([a,b]), \|\cdot\|_\infty)$, $\|f\|_\infty = \sup\{|f(x)| : x \in [a,b]\}$ is a Banach Space.

Problem: $A \subseteq \mathbb{R}$ measurable, $1 \leqslant p < \infty$, $\|f\|_p = \left(\int_A |f|^p\right)^{\frac{1}{p}}$ is NOT a norm on the vector space of integrable functions $f : A \to \mathbb{R}$. WHY? $\int_A |f|^p = 0 \iff f = 0$ a.e.

Definition 3.1

$A \subseteq \mathbb{R}$ measurable,

1. $M(A) = \{f : A \to \mathbb{R}$ *measurable*$\} \to$ *vector space,*

$$f \sim g \iff f = g \text{ a.e.}$$

   let $[f]$ represent the equivalence class.

2. $M(A)/\sim = \{[f] : f \in M(A)\}$. $\alpha[f] + \beta[g] = [\alpha f + \beta g]$ *shows that it's a vector space.*

Remark. If $f \sim g$ and f is integrable, then g is integrable and $\int_A f = \int_A g$

Definition 3.2

$A \subseteq \mathbb{R}$ measurable, $1 \leqslant p < \infty$,

$$L^p(A) = \left\{[f] \in M(A)/\sim : \int_A |f|^p < \infty\right\}$$

Remark. Suppose $[f], [g] \in L^p(A)$. Then $\int_A |f|^p, \int_A |g|^p < \infty$

1. $|f+g|^p \leqslant (|f|+|g|)^p \leqslant (2\max\{|f|,|g|\})^p \leqslant 2^p(|f|^p + |g|^p) \implies |f+g|^p$ integrable by comparison.
2. so $L^p(A)$ is a subspace of $M(A)/\sim$

Definition 3.3

$A \subseteq \mathbb{R}$ measurable,

$$L^\infty(A) = \{[f] \in M(A)/\sim : f \text{ bounded a.e.}\}$$

Remark.

1. $[f], [g] \in L^\infty(A)$

$$|f| \leqslant M \text{ off } B \subseteq A, \ m(B) = 0$$
$$|g| \leqslant N \text{ off } C \subseteq A, \ m(C) = 0$$

off $B \subseteq A$ means on $A \setminus B$.
For $x \notin B \cup C$,

$$|f(x) + g(x)| \leqslant |f(x)| + |g(x)| \leqslant M + N$$

2. $L^\infty(A)$ is a subspace of $M(A)/\sim$

Proposition 3.4

$A \subseteq \mathbb{R}$ measurable, then

$$\|[f]\|_\infty = \inf\{M \geqslant 0 : |f| \leqslant M \text{ a.e.}\}$$

is a norm on $L^\infty(A)$

Remark.

1. $|f| \leqslant \|[f]\|_\infty + \frac{1}{n}$ off $m(A_N) = 0$, and $B = \cup_{n=1}^\infty A_n$ has measure 0
2. $|f| \leqslant \|f\|_\infty$ off B.

Proof.

1. $\|[f]\|_\infty = 0 \implies |f| \leqslant \|[f]\|_\infty$ a.e. $\implies |f| = 0$ a.e. $\implies f = 0$ a.e., then

$$[f] = [0]$$

in $L^\infty(A)$.

2. $|f| \leqslant \|[f]\|_\infty$ off B, $|g| \leqslant \|[g]\|_\infty$ off C. Off $B \cup C \implies$ measure 0:

$$|f + g| \leqslant |f| + |g| \leqslant \|[f]\|_\infty + \|[g]\|_\infty$$

By the definition of inf, we have

$$\|[f + g]\|_\infty = \|[f] + [g]\|_\infty \leqslant \|[f]\|_\infty + \|[g]\|_\infty$$

$\square$

3.2 L^p Norm

Example 3.5

$p = 1$, $A \subseteq \mathbb{R}$ measurable, $[f], [g] \in L^1(A)$,

$$\begin{aligned} &|f+g| \leqslant |f| + |g| \\ \implies &\int_A |f+g| \leqslant \int_A |f| + \int_A |g| \\ \implies &\|f+g\|_1 \leqslant \|[f]\|_1 + \|[g]\|_1 \end{aligned}$$

Abusive Notation:

$$f \equiv [f] \in L^p(A)$$

Remember !

$f = g$ in $L^p(A)$ means $f = g$ a.e.

Definition 3.6

For $p \in (1, \infty)$ we define $q = \frac{p}{p-1}$ to be the Holder Conjugate of p.

Note:

1. $q = \frac{p}{p-1} \iff p = \frac{q}{q-1}$
2. $\frac{1}{p} + \frac{1}{q} = 1$

Definition 3.7

We define 1 and ∞ to be a pair of Holder conjugate.

Proposition 3.8: [Young's Inequality

$p, q \in (1, \infty)$ Holder conjugate. $\forall a, b > 0$,

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$$

Proof.

$$\begin{aligned}
&f(x)=\frac{1}{p}x^p+\frac{1}{q}-x \text{ on } (0,\infty)\\
&f'(x)=x^{p-1}-1\\
&f(1)=\frac{1}{p}+\frac{1}{q}-1=0\\
\implies &f\geqslant 0 \text{ on } (0,\infty)\\
\implies &x\leqslant \frac{1}{p}x^p+\frac{1}{q},\ \forall x>0
\end{aligned}$$

Taking:

$$\begin{aligned}
&x=\frac{q}{b^{q-1}}\\
\implies &\frac{a}{b^{q-1}}\leqslant \frac{1}{p}\frac{a^p}{b^{(q-1)p}}\\
\implies &\frac{a}{b^{q-1}}\leqslant \frac{1}{p}\frac{a^p}{b^p}+\frac{1}{q}\\
\implies &ab\leqslant \frac{1}{p}a^p+\frac{1}{q}b^q
\end{aligned}$$

□

Proposition 3.9: [Holder's Inequality]

$A\subseteq\mathbb{R}$ measurable, $1\leqslant p<\infty$, q is the Holder Conjugate. If $f\in L^p(A)$ and $g\in L^q(A)$ then $fg\in L^1(A)$ and $\int_A|fg|\leqslant\|f\|_p\|g\|_q$

Proof.

1. $p=1,q=\infty$

$$|fg|=|f||g|\leqslant|f|\|g\|_\infty \text{ a.e.}$$

then $fg\in L^1(A)$ and

$$\int_A|fg|\leqslant\int_A|f|\|g\|_\infty=\|g\|_\infty\|f\|_1$$

2. $1<p<\infty$, q HC,

$$|fg|=|f||g|\leqslant\frac{|f|^p}{p}+\frac{|g|^q}{q}\implies fg\in L^1(A)$$

Also,

$$\int_A|fg|\leqslant\frac{1}{p}\int_A|f|^p+\frac{1}{q}\int_A|g|^q=\frac{1}{p}\|f\|_p^p+\frac{1}{q}\|g\|_q^q$$

(a) $\|f\|_p=\|g\|_q=1$,

$$\int_A|fg|\leqslant\frac{1}{p}+\frac{1}{q}=1=\|f\|_p\|g\|_q$$

(b) $\frac{f}{\|f\|_p}, \frac{g}{\|g\|_q}$. By case a),

$$\frac{1}{\|f\|_p\|g\|_q}\int_A |fg| \leqslant 1 \implies \int_A |fg| \leqslant \|f\|_p\|g\|_q$$

□

Lemma 3.10

p, q HC, $f \in L^p(A)$. If $f \neq 0$,

$$f^* = \|f\|_p^{1-p}\mathrm{sgn(f)|f|^{p-1}}$$

is in $L^q(A)$ and

$$\int_A ff^* = \|f\|_p, \text{ and } \|f^*\|_q = 1$$

Proof.

1. $p = 1, q = \infty$

$$f^* = \mathrm{sgn(f) \in L^\infty(A)}$$
$$\int_A ff^* = \int_A |f| = \|f\|_1, \ \|f^*\|_\infty = 1$$

2. $1 < p < \infty$, q HC

$$\int_A ff^* = \|f\|_p^{1-p}\int_A |f|^p = \|f\|_p^{1-p}\|f\|_p^p = \|f\|_p$$
$$\begin{aligned}\|f^*\|_q^q &= \|f\|_p^{(1-p)q}\int_A |f|^{(p-1)q} \\ &= \|f\|_p^{-p}\int_A |f|^p \\ &= \|f\|_p^{-p}\|f\|_p^p = 1\end{aligned}$$

□

Theorem 3.11: [Minkowski's Inequality]

$A \subseteq \mathbb{R}$ measurable and $1 \leqslant p < \infty$. If $f, g \in L^p(A)$ then

$$\|f + g\|_p \leqslant \|f\|_p + \|g\|_p$$

Proof. 1. $p = 1$ Done

2. $1 < p < \infty$

$$\begin{aligned}
\|f+g\|_p &= \int_A (f+g)(f+g)^* \\
&= \int_a f(f+g)^* + \int_A g(f+g)^* \\
&\underbrace{\leqslant}_{Holder} \|f\|_p \|(f+g)^*\|_q + \|g\|_p \|(f+g)^*\|_q \\
&= \|f\|_p + \|g\|_p
\end{aligned}$$

$\square$

3.3 Completeness

Theorem 3.12: [Riesz-Fisher]

For all measurable $A \subseteq \mathbb{R}$ and $1 \leqslant p \leqslant \infty$, $L^P(A)$ is a Banach space.

Proof.

1. $p = \infty$, piazza
2. $1 \leqslant p < \infty$, Let $(f_n) \subseteq L^P(A)$ be strongly Cauchy Sequence. Therefore, there exists $(\varepsilon_n) \subseteq \mathbb{R}$ suCh that
   (a) $\|f_{n+1} - f_n\|_p \leqslant \varepsilon_n^2$
   (b) $\sum \varepsilon_n < \infty$

   Idea: Since $\mathbb{R}$ is complete, if $(f_n(x))$ is strongly-Cauchy then it converges. For each $n \in \mathbb{N}$,

$$\begin{aligned} A_n &= \{x \in A : |f_{n+1}(x) - f_n(x)| \geqslant \varepsilon_n\} \\ &= \{x \in A : |f_{n+1}(x) - f_n(x)|^p \geqslant \varepsilon_n^p\} \end{aligned}$$

   By Chebyshev's Inequality:

$$m(A_n) \leqslant \frac{1}{\varepsilon_n^p} \int_A |f_{n+1} - f_n|^p \leqslant \frac{1}{\varepsilon_n^p} \varepsilon_n^{2P} = \varepsilon_n^p$$
$$\sum m(A_n) \leqslant \sum \varepsilon_n^p \leqslant \left(\sum \varepsilon_n\right)^p < \infty$$

   which implies that $m(\limsup(A_n)) = 0$
   Fix $x \notin \limsup(A_n)$. Let $N = \max\{n : x \in A_n\}$. For $n > N$,

$$\begin{aligned} &|f_{n+1}(x) - f_n(x)| < \varepsilon_n^2, \ \sum \varepsilon_n < \infty \\ \implies &(f_n(x)) \text{ Cauchy} \\ \implies &f_n(x) \to f(x) \in \mathbb{R} \end{aligned}$$

   so $f_n \to f$ pointwise a.e.
   For $k \in \mathbb{N}$,

$$\|f_{n+k} - f_n\|_p \leqslant \|f_{n+k} - f_{n+k-1}\|_p + \ldots + \|f_{n+1} - f_n\|_p \leqslant \varepsilon_{n+k-1}^2 + \ldots + \varepsilon_n^2 \leqslant \sum_{i=n}^{\infty} \varepsilon_i^2$$

so $|f_{n+k} - f_n|^p \to |f_n - f|^p$ pointwise a.e. as $k \to \infty$.

By Fatou's Lemma,

$$\begin{aligned}
&\int_A |f_n - f|^p \\
\leqslant &\liminf_{k\to\infty} \int_A |f_{n+k} - f_n|^p \\
= &\liminf_{k\to\infty} \|f_{n+k} - f_n\|_p^p \\
\leqslant &\left[\sum_{i=n}^{\infty} \varepsilon_i^2\right]^p \to 0
\end{aligned}$$

so f_n converges w.r.t p-norm. $\square$

3.3.1 Separability:

Recall: A metric space X is separable if it has a countable, dense subset.

Example 3.13

$p = \infty$?
Suppose $\{f_n : n \in \mathbb{N}\}$ is dense in $L^\infty[0,1]$. For every $x \in [0,1]$, we may find

$$\|\chi_{[0,x]} - f_{\theta(x)}\|_\infty < \frac{1}{2}$$

For $x \neq y$ in $[0,1]$,

$$\|x_{[0,x]} - \chi_{[0,y]}\|_\infty = 1$$

so $\theta(x) \neq \theta(y)$ and $\theta[0,1] \to \mathbb{N}$ is injective, contradiction ($[0,1]$ not countable).

Notation:

- $Simp(A)$ = Simple functions on measure$A$
- $Step[a,b]$ = Step functions on$[a,b]$
- $Step_{\mathbb{Q}}[a,b]$ =Step functions on $[a,b]$ with rational partition (not including a, b) and functions values.

Proposition 3.14

$A \subseteq \mathbb{R}$ measurable, $1 \leqslant p < \infty$, $Simp(A)$ is dense in $L^P(A)$

Proof.

$$fr \in L^P(A) \to f \text{ measurable}$$

then there exists φ_n simple

1. $\varphi_n \to f$ pointwise
2. $|\varphi_n| \leqslant |f| \implies |\varphi_n|^p \leqslant |f|^p$

By comparison, $(\varphi_n) \subseteq L^P(A)$.
Note,

$$\|\varphi_n - f\|_p^p = \int_A |\varphi_n - f|^p$$
$$|\varphi_n - f|^p \leqslant 2^p(|\varphi_n|^p + |f|^p)$$
$$\leqslant 2^{p+1}|f|^p$$

so by the Lebesgue Dominate Convergence Theorem

$$\lim_{n\to\infty} \|\varphi_n - f\|_p^p = \lim_{n\to\infty} \int_A |\varphi_n - f|^p = \int 0 = 0$$

□

Fact: the above proposition is true for $p = \infty$ (but it's not seperable).

Proposition 3.15

$1 \leqslant p < \infty$. *$Step[a,b]$ is dense in $L^P[a,b]$*

Proof. $A \subseteq [a,b]$ measurable, $\chi_A[a,b] \to \mathbb{R}$.
Littlewood 1:$\exists \dot{\cup}_{i=1}^{n} I_i = U$, where I_is are bounded open intervals. And $m(U \triangle A < \varepsilon$ and $\chi_U : [a,b] \to \mathbb{R}$ is a step function.

$$\begin{aligned}
&\|\chi_U - \chi_A\|_p^p \\
=&\int_A \||\chi_U - \chi_A|^p \\
=&\int_{U\triangle A} 1^p \\
=&m(U \triangle A) \\
\implies &\|\chi_U - \chi_A\|_p < \varepsilon
\end{aligned}$$

so for all characteristic function, we can approach as close as we want by a step function. Simple function is just made of **finitely** many characteristic functions. □

Corollary 3.16

$1 \leqslant p < \infty$. *$Step_{\mathbb{Q}}[a,b]$ is dense in $L^p[a,b]$ (step functions are dense, so for each step function, you can modify the function a little bit by rationals). Therefore, $L^p[a,b]$ is separable.*

Proposition 3.17

$1 \leqslant p < \infty$, *$L^p(\mathbb{R})$ is separable.*

Proof. $1 \leqslant p < \infty$, $L^p(\mathbb{R})$ is separable.

$$F_n = \left\{ f \in L^p(\mathbb{R}) | f|_{[-n,n]} \in Step_{\mathbb{Q}}[-n,n], f|_{\mathbb{R}\setminus[-n,n]} = 0 \right\}$$

$F = \cup_{n=1}^{\infty} F_n$ countable. Take $f \in L^p(\mathbb{R})$. Fix $n \in \mathbb{N}$, we have $f|_{[-n,n]} \in L^p([-n,n])$ We show

$$f\chi_{[-n,n]} \to f \text{ in } L^p(\mathbb{R})$$

Note:

1.

$$\begin{aligned}
&\|f\chi_{[-n,n]} - f\|_p^p \\
=&\int_{\mathbb{R}} |f\chi_{[-n,n]} - f|^p \\
=&\int_{\mathbb{R}\setminus[-n,n]} |f|^p \\
=&\int_{\mathbb{R}} |f|^p \chi_{\mathbb{R}\setminus[-n,n]}
\end{aligned}$$

2. $||f|^p\chi_{\mathbb{R}\setminus[-n,n]}|\leqslant |f|^p$ which is integrable

3. By the Lebesgue Dominated Convergence Theorem

$$\begin{aligned}&\lim_{n\to\infty}\|f\chi_{[-n,n]}-f\|_p^p\\ =&\lim_{n\to\infty}\int_{\mathbb{R}}|f\chi_{[-n,n]}-f|^p=\int_{\mathbb{R}}0=0\end{aligned}$$

   so $\|f\chi_{[-n,n]}-f\|_p\to 0$
   For each $n\in\mathbb{N}$, $\exists\varphi_n\in F$ such that $\|f\chi_{[-n,n]}-\varphi_n\|_p<\frac{1}{n}$, so

$$\|\varphi_n-f\|_p\to 0$$

□

Theorem 3.18

$1\leqslant p<\infty$, $A\subseteq\mathbb{R}$ measurable, $L^p(A)$ is separable.

Proof. F as before, $\{f|_A\colon f\in F\}$ is a countable dense subset of $L^p(A)$ □

4 Fourier Analysis

4.1 Hilbert Space

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Definition 4.1

V is a vector space over $\mathbb{F}$. An inner product on V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ such that

1. *$\forall v \in V$, $\langle v, \rangle \in \mathbb{F}$, $\langle v, v \rangle \geqslant 0$ with $\langle v, v \rangle = 0$ if and only $v = 0$*
2. *$\forall v, w \in V$,*
$$\langle v, w, \rangle = \overline{\langle w, v \rangle}$$
3. *$\forall \alpha \in \mathbb{F}$, $u, v, w \in V$,*
$$\langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle$$

We call $(V, \langle \cdot, \cdot \rangle$ an inner product space.

Proposition 4.2

Let V be an inner product space. Then

$$\|v\| = \sqrt{\langle v, v \rangle}$$

is a norm on V. We call $\|\cdot\|$ the norm induced by $\langle \cdot, \cdot \rangle$

Example 4.3

$A \subseteq \mathbb{R}$ measurable. $V = L^2(A)$, $\langle f, g \rangle = \int_A fg$ is an inner product space.
<u>Note:</u> $\sqrt{\langle f, f \rangle} = \left(\int_A |f|^2 \right)^{\frac{1}{2}} = \|f\|_2$

Example 4.4

$A \subseteq \mathbb{R}$ measurable. $V = L^2(A, \mathbb{C})$, $\langle f, g \rangle = \int_A f\overline{g}$ and $\sqrt{\langle f, f \rangle} = ||f||_2$

Proposition 4.5: [Parallelogram Law]

Let V be an inner product space. $\forall u, v \in V$,

$$\|u + v\|^2 + \|u - v\|^2 = 2 \left(\|u\|^2 + \|v\|^2 \right)$$

Proof.

$$\begin{aligned}
&\|u+v\|^2+\|u-v\|^2 \\
=&\langle u+v,u+v\rangle+\langle u-v,u-v\rangle \\
=&\langle u,u\rangle+2\langle u,v\rangle+\langle v,v\rangle+\langle u,u\rangle-2\langle u,v\rangle+\langle v,v\rangle \\
=&2\|u\|^2+2\|v\|^2 \\
=&2\left(\|u\|^2+\|v\|^2\right)
\end{aligned}$$

□

Example 4.6

$1 \leqslant p < \infty$, $V = L^p[0,2]$ and $f = \chi_{[0,1]}, g = \chi_{[1,2]}$

$$\begin{aligned}
\|f\|_p^2 &= \left(\int_{[0,2]} |f|^p\right)^{\frac{2}{p}} \\
&= 1^{\frac{2}{p}} = 1 \\
\|g\|_p^2 &= 1^{\frac{2}{p}} = 1
\end{aligned}$$

$$\begin{aligned}
\|f+g\|_p^2 &= 2^{\frac{2}{p}} \\
\|f-g\|_p^2 &= 2^{\frac{2}{p}}
\end{aligned}$$

so by Parallelogram Law

$$2^{\frac{2}{p}}+2^{\frac{2}{p}} = 2(1+1) \iff 2^{\frac{2}{2}} = 2 \iff p = 2$$

so $\|\cdot\|_p$ is induced by an inner product if and only if $p = 2$. You can also show that $\|\cdot\|_\infty$ is not induced by an inner product.

Definition 4.7

A Hilbert Space is a complete inner product space (i.e. A Banach Space whose norm is induced by an inner product).

Example 4.8

$L^2(A), L^2(A,\mathbb{C})$ are Hilbert Spaces.

4.2 Orthogonality

Definition 4.9

Let V be an inner product space. We say $v, w \in V$ are orthogonal if $\langle v, w\rangle = 0$.

Example 4.10

$f, g \in L^2([-\pi,\pi],\mathbb{C})\,,\ m \neq n,\ f(x) = e^{inx},\ g(x) = e^{imx}$, *then*

$$\begin{aligned}
\langle f, g\rangle &= \int_{[-\pi,\pi]} f\overline{g} \\
&= \int_{[-\pi,\pi]} e^{inx}e^{-imx}dx \\
&= \int_{[-\pi,\pi]} e^{ix(n-m)}dx \\
&= \int_{[-\pi,\pi]} \cos((n-m)x) + i\int_{[-\pi,\pi]} \sin((n-m)x) \\
&= R\int_{-\pi}^{\pi} \cos((n-m)x) + iR\int_{-\pi}^{\pi} \sin((n-m)x)dx \\
&= 0
\end{aligned}$$

Theorem 4.11: [Pythagorean Theorem]

Let V be an inner product space. If $v_1, \dots, v_n \in V$ are pairwise orthogonal, then,

$$\left\|\sum V_i\right\|^2 = \sum\|V_i\|^2$$

Definition 4.12

Let V be an inner product space. We say $A \subseteq V$ is orthonormal if the elements of A are pairwise orthogonal and $\|v\| = 1, \forall v \in A$.

Corollary 4.13

Let V be an inner product space, $\{v_1, \dots, v_n\}$ orthonormal,

$$\left\|\sum \alpha_i v_i\right\|^2 = \sum|\alpha_i|^2$$

Example 4.14

$L^2([-\pi,\pi],\mathbb{C})$, $A = \left\{\frac{1}{\sqrt{2\pi}}e^{inx} : n \in \mathbb{Z}\right\} \implies$ pairwise orthogonal.

$$\begin{aligned}&\frac{1}{2\pi}\left\|e^{inx}\right\|_2^2\\ &= \frac{1}{2\pi}\int_{[-\pi,\pi]} e^{inx}e^{-inx}dx\\ &= \frac{1}{2\pi}\int_{[-\pi,\pi]} 1 = 1\end{aligned}$$

so A is orthonormal

Definition 4.15

Let V be an inner product space. An <u>orthonormal basis</u> is a maximal (w.r.t $\subseteq$) orthonormal subset of V. (Note it might not ba basis).

<u>Fact:</u> An inner product space always has an orthonormal basis.

<u>Fact:</u> Let H be a Hilbert space. If $W \subseteq H$ is closed subspace then there exists a subspace $W^\perp \subseteq H$ such that

$$H = W \oplus W^\perp$$

and $\langle w, z\rangle = 0$ for all $w \in W$ and $z \in W^\perp$.

Theorem 4.16

Let H be a Hilbert space, then H has a <u>countable</u> ONB (orthonormal basis) if and only if H is separable.

Proof.

- $\implies$ Let be B be a countable orthonormal basis for H.
  <u>Claim:</u> $w = \mathrm{Span(B)}$, $\overline{w} = H$
  Suppose $\overline{w} \neq H$. Since $H = \overline{w} \oplus \overline{w}^\perp$. We may find $0 \neq x \in \overline{w}^\perp$. We may assume $\|x\| = 1$. so $B \cup \{x\}$ is orthonormal. Contradiction! So $\overline{w} = H$.
  We can also show that $\overline{\mathrm{Span}_{\mathbb{Q}}(\mathrm{B})} = H$ where $\mathrm{Span}_{\mathbb{Q}}(\mathrm{B})$ is the span of B only using rational numbers as the coefficients. Hence, H is separable.

- $\impliedby$ Suppose H doesn't have an orthonormal basis which is countable. Let B be ONB for H, so B is uncountable.
  For $u \neq v$ in B,
  $$\|u - v\|^2 = \|u\|^2 + \|v\|^2 = 2 \implies \|u - v\| = \sqrt{2}$$

Suppose $X \subseteq H$ such that $\overline{X} = H$. $\forall u \in B$, there exists $x_n \in X$ such that

$$\|x_n - u\| < \frac{\sqrt{2}}{2}$$

but for $u \neq v$ in B, we have that

$$x_u \neq x_v$$

so

$$\varphi : B \mapsto X,\ \varphi(u) = x_u$$

is an injection. So X is uncountable because B is uncountable, so H is not separable, contradiction.

□

Example 4.17

$\left\{ \frac{1}{\sqrt{2\pi}} e^{inx} : n \in \mathbb{Z} \right\}$ *is a countable orthonormal set in* $L^2([-\pi, \pi], \mathbb{C})$. *We can clearly see that it is countable, orthonormal, but what about maximal?*

4.3 Big Theorems

Remark. Let H be an inner product space, $\{v_1, \dots, v_n\}$ orthonormal.
If $v = \sum \lambda_i v_i$ then $\lambda_i = \langle v, v_i \rangle$. We call $\langle v, v_i \rangle$ the Fourier Coefficients of v w.r.t. $\{v_1, \dots, v_n\}$

Definition 4.18

Let H be a Hilbert space, $\{v_1, v_2, \dots\}$ orthonormal. For $v \in H$, we call

$$\sum_{i=1}^{\infty} \langle v, v_i \rangle v_i$$

the Fourier Series of v relative to $\{v_1, v_2, \dots\}$ and write

$$v \sim \sum_{i=1}^{\infty} \langle v, v_i \rangle v_i$$

- *Does this series converge?*
- *Does it converge to v?*

Theorem 4.19: [Best Approximation]

Let H be a Hilbert Space, $\{v_1, \dots, v_n\}$ orthonormal. For $v \in H$, $\|v - \sum \lambda_i v_i\|$ is minimized when $\lambda_i = \langle v, v_i \rangle$
Moreover,

$$\left\| v - \sum \langle v, v_i \rangle v_i \right\|^2 = \|v\|^2 - \sum |\langle v, v_i \rangle|^2$$

Proof.

1. $W = \mathrm{Span}\{\mathrm{v_1}, \dots, \mathrm{v_n}\}$ closed, $v = W \oplus W^{\perp}$
2. $x \in W,\ v = w + z,\ w \in W, z \in W^{\perp}$,

$$\|v - x\|^2 = \|w + z - x\|^2 = \|w - x + z\|^2 = \|w - x\|^2 + \|z\|^2 \geqslant \|z\|^2 = \|v - x\|^2$$

   so $\|v - x\| \geqslant \|v - w\|$, the closet point in W to v is w, the orthonormal projection.

3. $v = \sum \lambda_i v_i + z,\ z \in W^{\perp}$,

$$\langle v, v_i \rangle = \lambda_i + \underbrace{\langle z, v_i \rangle}_{0} = \lambda_i$$

4. $v = \sum \langle v, v_i \rangle v_i + z,\ z \in W^{\perp}$, then

$$\begin{aligned} \|v\|^2 &= \left\| \sum \langle v, v_i \rangle v_i \right\|^2 + \|z\|^2 \\ &= \sum |\langle v, v_i \rangle|^2 + \|z\|^2 \end{aligned}$$

so,

$$\left\| v - \sum \langle v, v_i \rangle v_i \right\|^2 = \|z\|^2 = \|v\|^2 - \sum |\langle v, v_i \rangle|^2$$

□

Theorem 4.20: [Bessel's Inequality]

Let H be a Hilber Space, $\{v_1, \ldots, v_n\}$ be orthonormal. If $v \in H$,

$$\sum_{i=1}^{n} |\langle v, v_i \rangle|^2 \leqslant \|v\|^2$$

Proof.

$$\|v\|^2 - \sum |\langle v, v_i \rangle|^2 = \left\| v - \sum \langle v, v_i \rangle v_i \right\|^2 \geqslant 0$$

□

Theorem 4.21: [Parseral's Identity]

Let H be a Hilbert space, $\{v_1, v_2, \ldots\}$ orthonormal.
For $v \in H$,

$$\sum_{i=1}^{\infty} |\langle v, v_i \rangle|^2 = \|v\|^2 \iff \lim_{n\to\infty} \left\| v - \sum_{i=1}^{n} \langle v, v_i \rangle v_i \right\|^2 = 0$$

Theorem 4.22: [Orthonormal Basis Test]

Let H be a separable Hilbert Space $\{v_1, v_2, \ldots\}$ orthonormal. TFAE:

1. *$\{v_1, v_2, \ldots\}$ is an orthonormal basis.*
2. *$\overline{\mathrm{Span}\{\mathrm{v}_1, \mathrm{v}_2, \ldots\}} = H$*
3. *$\lim_{n\to\infty} \|v - \sum_{i=1}^{n} \langle v, v_i \rangle v_i\| = 0, \ \forall v \in H$*

Proof.

- (1) $\implies$ (2) Done
- (2) $\implies$ (3)
  If $\{v_1, v_2, \ldots\}$ is not maximal then we may find $u \in H$, $\|u\| = 1$ such that $\langle u, v_i \rangle = 0, \ \forall i \in \mathbb{N}$. Since $C = \{x \in H : \langle x, u \rangle = 0\}$ is closed, $u \notin \overline{\mathrm{Span}\{\mathrm{v}_1, \mathrm{v}_2, \ldots\}}$ ($u \notin C$, $\langle u, u \rangle = 1$, $\overline{\mathrm{Span}\{\mathrm{v}_1, \mathrm{v}_2, \ldots\}} \subseteq C$).
- (2) $\implies$ (3)
  Let $v \in H$ and let $\varepsilon > 0$ be given. Let $\sum_{i=1}^{N} \alpha_i v_i \in \mathrm{Span}\{\mathrm{v}_1, \ldots\}$ such that

$$\left\| v - \sum_{i=1}^{N} \alpha_i v_i \right\| < \varepsilon$$

so $\left\|v - \sum_{i=1}^{N} \langle v, v_i \rangle v_i\right\| < \varepsilon$.
For $n \geqslant \mathbb{N}$,

$$\begin{aligned}
&\left\|v - \sum_{i=1}^{n} \langle v, v_i \rangle v_i\right\| \\
\leqslant &\left\|v - \sum_{i=1}^{N} \langle v, v_i \rangle v_i\right\| + \left\|\sum_{i=N+1}^{n} \langle v, v_i \rangle v_i\right\| \\
< &\varepsilon + \sqrt{\sum_{N+1}^{\infty} |\langle v, v_i \rangle|^2} \longrightarrow 0 \text{ as } N \to \infty
\end{aligned}$$

because by Bessel's Inequality, $\sum_{i=1}^{N} |\langle v, v_o \rangle|^2$ is a bounded increasing sequence, so $\sum_{N+1}^{\infty} |\langle v, v_i \rangle|^2$ will go to 0.

- $(3) \implies (2)$, similar.

$\square$

4.4 Fourier Series

1. Is $\left\{\frac{1}{\sqrt{2\pi}}e^{inx} : n \in \mathbb{Z}\right\}$ an ONB for $L^2([-\pi,\pi],\mathbb{C})$?
2. Is $\mathrm{Span}\left\{\mathrm{e}^{\mathrm{inx}} : \mathrm{n} \in \mathbb{Z}\right\}$ dense in Ł$^2([-\pi,\pi],\mathbb{C})$?
3. Is $\mathrm{Span}\left\{\mathrm{e}^{\mathrm{inx}} : \mathrm{n} \in \mathbb{Z}\right\}$ dense in $L^1([-\pi,\pi],\mathbb{C})$

Definition 4.23

Let $T = [-\pi,\pi)$. We call T the Torus or the circle. We define.

$$L^p(T) = L^p([-\pi,\pi],\mathbb{C})$$

for $1 \leqslant p < \infty$.
Using the norm,

$$\|f\|_p = \left(\frac{1}{2\pi}\int_T |f|^p\right)^{\frac{1}{p}}$$

$L^p(T)$ is a separale Banach Space.

Remark.

1. As a group under addition module 2π,
$$T \cong \mathbb{R}/\mathbb{Z} \cong \{z \in \mathbb{C} : |z| = 1\}$$
2. In this way, T is a locally compact abelion group.
3. There is a one-to-one correspondence between
$$f : T \mapsto \mathbb{C}$$
and 2π-periodic function
$$f : \mathbb{R} \mapsto \mathbb{C}$$

Definition 4.24

$f \in L^1(T)$

1. *We define the* n^{th} $(n \in \mathbb{Z})$ *Fourier Coefficients of* f *by*

$$\langle f, e^{inx} \rangle := \frac{1}{2\pi} \int_T f(x) e^{-inx} dx$$

2. *We define the Fourier Series of* f *by*

$$f \sim \sum_{n \in \mathbb{Z}} a_n e^{inx}$$

where $a_n = \langle f, e^{inx} \rangle$.

3. *We let*

$$S_N(f, x) = \sum_{-N}^{N} a_n e^{inx}$$

denote the N^{th} *partial sum of the above Fourier series.*

Proposition 4.25

Consider the trignometric polynomial $f \in L^1(T)$ *given by*

$$f(x) = \sum_{n=-N}^{N} a_n e^{inx}$$

for some $a_i \in \mathbb{C}$.
For each $-N \leqslant n \leqslant N$,

$$\langle f, e^{inx} \rangle = a_n$$

Why?

$$\frac{1}{2\pi} \int_T e^{imx} e^{-inx} dx = \delta_{m,n} = \begin{cases} 1, m = n \\ 0, m \neq n \end{cases}$$

Remark. Suppose $f \in L^1(T)$ is real-valued, $f \sim \sum_{n \in \mathbb{Z}} a_n e^{inx}$.

For $N \in \mathbb{N}$,

$$
\begin{aligned}
S_N(f,x) &= \sum_{n=-N}^{N} a_n e^{inx} \\
&= a_0 + \sum_{n=1}^{N} (a_n e^{inx} + a_{-n} e^{-inx}) \\
&= a_0 + \sum_{n=1}^{N} (a_n + a_{-n}) \cos(nx) + i(a_n - a_{-n}) \sin(nx) \\
\\
&= a_0 + \sum_{n=1}^{N} b_n \cos(nx) + c_n \sin(nx)
\end{aligned}
$$

Now,

$$a_0 = \frac{1}{2\pi} \int_T f(x) e^{-i0x} dx = \frac{1}{2\pi} \int_T f(x) dx$$

$$
\begin{aligned}
b_n &= a_n + a_{-n} \\
&= \frac{1}{2\pi} \int_T f(x)(e^{-inx} + e^{inx}) dx \\
&= \frac{1}{\pi} \int_T f(x) \cos(nx) dx
\end{aligned}
$$

$$
\begin{aligned}
c_n &= i(a_n - a_{-n}) \\
&= \frac{i}{2\pi} \int_T f(x)(e^{-inx} - e^{inx}) dx \\
&= \frac{1}{\pi} \int_T f(x) \sin(nx) dx
\end{aligned}
$$

are all real-valued.

4.5 Fourier Coefficients

Proposition 4.26

$f, g \in L^1(T)$

1. $\langle f+g, e^{inx} \rangle = \langle f, e^{inx} \rangle + \langle g, e^{inx} \rangle$
2. *For* $\alpha \in \mathbb{C}$, $\langle \alpha f, e^{inx} \rangle = \alpha \langle f, e^{inx} \rangle$
3. *If* $\overline{f} : T \mapsto \mathbb{C}$ *is defined by* $\overline{f}(x) = \overline{f(x)}$, *then* $\overline{f} \in L^1(T)$ *and* $\langle \overline{f}, e^{inx} \rangle = \overline{\langle f, e^{inx} \rangle}$

Proof.

1. Trivial

2. Trivial

3. $|f| = |\overline{f}| \implies \overline{f} \in L^1(T)$,

$$\begin{aligned}
&\left\langle \overline{f}, e^{inx} \right\rangle \\
=&\frac{1}{2\pi}\int_T \overline{f}(x)e^{-inx}dx \\
=&\frac{1}{2\pi}\int_T \overline{f(x)e^{inx}}dx \\
=&\frac{1}{2\pi}\int_T Re(\overline{f(x)e^{inx}})dx + \frac{i}{2\pi}\int_T Im(\overline{f(x)e^{inx}})dx \\
=&\frac{1}{2\pi}\int_T Re(f(x)e^{inx})dx - \frac{i}{2\pi}\int_T Im(f(x)e^{inx})dx \\
=&\overline{\frac{1}{2\pi}\int_T f(x)e^{inx}dx} \\
=&\overline{\langle f, e^{-inx} \rangle}
\end{aligned}$$

□

Proposition 4.27

$f \in L^1(T)$, $\alpha \in \mathbb{R}$. By a previous remark, we may view $f : \mathbb{R} \mapsto \mathbb{C}$ as a 2π-periodic function which is integrable over T. For $\alpha \in \mathbb{R}$, $f_\alpha : \mathbb{R} \mapsto \mathbb{C}$ given by $f_\alpha(x) = f(x - \alpha)$ is integrable over T and $\langle f_\alpha, e^{inx} \rangle = \langle f, e^{inx} \rangle e^{-in\alpha}$

Proposition 4.28

$f \in L^1(T)$. $\forall n \in \mathbb{Z}$, $|\langle f, e^{inx} \rangle| \leqslant \|f\|_1$

Proof.

$$\begin{aligned}
|\langle f, e^{inx} \rangle| &= \left| \frac{1}{2\pi}\int_T f(x)e^{-inx}dx \right| \\
&\leqslant \frac{1}{2\pi}\int_T \left| f(x)e^{-inx} \right| dx \\
&= \frac{1}{2\pi}\int_T |f(x)|dx
\end{aligned}$$

□

Corollary 4.29

$f_k \mapsto f$ in $L^1(t)$,

$$\forall n \in \mathbb{Z},\ \left\langle f_k, e^{inx} \right\rangle \mapsto \left\langle f, e^{inx} \right\rangle$$

Proof.

$$\begin{aligned}
&\left|\left\langle f_k, e^{inx}\right\rangle - \left\langle f, e^{inx}\right\rangle\right| \\
=&\left|\left\langle f_k - f, e^{inx}\right\rangle\right| \\
\leqslant&\|f_k - f\|_1 \longrightarrow 0
\end{aligned}$$

□

Remark. Let $\mathrm{Trig(T)}$ denote the set of Trigonometric polynomials on T. By A3, $\overline{\mathrm{Trig(T)}} = L^1(T)$

Theorem 4.30: [Riemann-Lebesgue Lemma]

If $f \in L^1(T)$, then

$$\lim_{|n|\to\infty} \left\langle f, e^{inx}\right\rangle = 0$$

Proof. Let $\varepsilon > 0$ be given and let $P \in \mathrm{Trig(T)}$ such that $\|f - P\|_1 \leqslant \varepsilon$. Say $P(x) = \sum_{k=-N}^{N} a_k e^{ikx}$. For $|n| > N$, we have that $\langle P, e^{inx}\rangle = 0$, so

$$\left|\left\langle f, e^{inx}\right\rangle\right| = \left|\left\langle f - P, e^{inx}\right\rangle\right| \leqslant \|f - P\|_1 < \varepsilon$$

□

4.6 Vector-Valued Integration

Definition 4.31

Let B be a Banch space and let $f : [a, b] \to B$ be a function. Consider a partition $P : a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$. We define a Riemann sum of f over P by

$$S(f, P) = \sum_{i=1}^{n} f(t_i^*)(t_i - t_{i-1}) \in B$$

where each $t_i^* \in [t_{i-1}, t_i]$.

Definition 4.32

Let B and f Be as above. We say f is Riemann Integrable if there exists $z \in B$ such that $\forall \varepsilon > 0$, there is a partition P_ε of $[a, b]$ such that whenever P is a refinement of P_ε and $S(f, p)$ is a Riemann sum then

$$\|S(f, P) - z\| < \varepsilon$$

We call z the integral of f over $[a, b]$ and write $z = R\int_a^b f(x)dx$.

A natural question to ask would be: Why are we doing this only for Banach Space?

Theorem 4.33: [Cauchy Criterion]

Let B be a Banach space and let $f : [a, b] \to B$ be a function. Then f is Riemann Integrable if and only if $\forall \varepsilon > 0$, there exists a partition P_ε of $[a, b]$ such that whenever P and Q are refinements of P_ε we have,

$$\|S(f, p) - S(f, Q)\| < \varepsilon$$

for any Riemann sums $S(f, P)$ and $S(f, Q)$.

Proof. Suppose f is Riemann integrable with $z = R\int_a^b f(x)dx$. Let $\varepsilon > 0$ be given. We may find a partition $P_{\varepsilon/2}$ such that whenever P is a refinement partition of $P_{\varepsilon/2}$ then

$$\|S(f, P) - S(f, Q)\| \leqslant \|S(f, P) - z\| + \|z - S(f, Q)\| < \varepsilon$$

Conversely, assume the Cauchy Criterion holds. In particular, for each $n \in \mathbb{N}$, we may find a partition P_n of $[a, b]$ which corresponds to $\varepsilon = \frac{1}{n}$, as per Cauchy Criterion. Without loss of generality, we may assume that each P_{n+1} is a refinement of P_n. For each $n \in \mathbb{N}$, let $S(f, P_n)$ be a Riemann sum. Let $\varepsilon > 0$ be given. Choosing $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$, we see that for $m, n \geqslant \mathbb{N}$,

$$\|S(f, P_m) - S(f, P_n)\| < \frac{1}{N} < \varepsilon$$

Since B is a Banach Space, $S(f, P_n) \to z \in B$

We claim that f is Riemann Integrable with $R\int_a^b dx = z$. Let N and P_N be as above. Moreover, we know $\exists M > N$ such that $\|S(f, P_M) - z\| < \frac{\varepsilon}{2}\|$. Now if P is any refinement partition of P_N, then

$$\|S(f,P) - z\| \leqslant \|S(f,P) - S(f,P_M)\| + \|S(f,P_M) - z\| < \varepsilon$$

$\square$

Theorem 4.34

If B is a Banach Space and $f : [a,b] \to B$ is continuous, then f is Riemann integrable.

4.7 Summability Kernels

Definition 4.35

$f, g \in L^1(T)$. The convolution of f and g is the functions

$$f * g : T \mapsto \mathbb{C}$$

given by

$$(f * g)(x) = \frac{1}{2\pi}\int_T f(t)g(x-t)dt = \frac{1}{2\pi}\int_T f(t)g_t(x)dt$$

Facts:

1. Given $f, g \in L^1(T)$, $f * g \in L^1(T)$ as well.
2. $\|f * g\|_1 \leqslant \|f\|_1 \|g\|_1$
3. This means $L^1(T)$ a Banach Algebra (Banach Space with continuous multiplication, we can think convolution as a "multiplication").

Let $C(T)$ denote the set of continuous functions $T \to \mathbb{C}$

Definition 4.36

A summability kernel is a sequence $(K_n) \subseteq C(T)$ such that

1. $\frac{1}{2\pi}\int_T K_n = 1$
2. $\exists M$, $\forall n$, $\|K_n\|_1 \leqslant M$
3. $\forall 0 < \delta < \pi$,

$$\lim_{n\to\infty}\left(\int_{-\pi}^{-\delta}|K_n| + \int_{\delta}^{\pi}|K_n|\right) = 0$$

This means summability kernels are concentrated at 0.

Proposition 4.37

Let $(B, \|\cdot\|_B)$ be a Banach Space (with scaler $\mathbb{C}$. Let $\varphi : T \mapsto B$ be continuous. Let $(K_n) \subseteq C(t)$ be a summability kernel. Then

$$\lim_{n\to\infty}\frac{1}{2\pi}\underbrace{\int_T K_n(t)\varphi(t)dt}_{\text{Riemann vector-valued integral}} = \varphi(0)$$

in the B-norm.

Proof. Appendix using (2), (3) □

Remark. $\varphi : T \to L^1(T)$, given by

$$\varphi(t) = f_t = f(x - t)$$

is continuous.

Theorem 4.38

$f \in L^1(T)$, K_n is a summability kernel. In $L^1(T)$,

$$f = \lim_{n\to\infty} K_n * f$$

Proof. Let $\varphi(t) = f(x - t)$

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{2\pi} \int_T K_n(t)\varphi(t)dt = \varphi(0) \\
\implies &\lim_{n\to\infty} \frac{1}{2\pi} \int_T K_n(t)f(x-t)dt = \varphi(0) = f(x-0) = f(x) \\
\implies &\lim_{n\to\infty} (K_n * f)(x) = f(x)
\end{aligned}$$

$\square$

4.8 Dirichlet Kernel

We want to find (K_n) such that $K_n * f = S_n(f)$, which is the n^{th} partial sum of Fourier Series of f.

Remark. Let $f \in L^1(T)$. For $n \in \mathbb{Z}$ consider

$$\varphi_n(x) = e^{inx} \in L^1(T)$$

Then

$$\begin{aligned}
&(\varphi_n * f)(x) \\
=&\frac{1}{2\pi}\int_T \varphi_n(t) f_t(x) dt \\
=&\frac{1}{2\pi}\int_T e^{int} f(x-t) dt \\
=&\frac{1}{2\pi}e^{inx}\int_T e^{-in(x-t)} f(x-t) dt \\
=&\frac{1}{2\pi}e^{inx}\int_T e^{-in(-t)} f(-t) dt \\
=&\frac{1}{2\pi}e^{inx}\int_T e^{-int} f(t) dt \\
=&e^{inx}\left\langle f, e^{inx}\right\rangle
\end{aligned}$$

Remark. $f \in L^1(T)$, if $P(x) = \sum_{k=-n}^{n} a_k e^{ikx}$, then

$$\begin{aligned}
&(P * f)(x) \\
=&\frac{1}{2\pi}\int_T P(t) f(x-t) dt \\
=&\sum_{k=-n}^{n} \frac{a_n}{2\pi}\int_T e^{ikt} f(x-t) dt \\
=&\sum_{k=-n}^{n} a_n (\varphi_n * f)(x) \\
=&\sum_{k=-n}^{n} a_n e^{ikx}\left\langle f, e^{ikx}\right\rangle
\end{aligned}$$

Definition 4.39

$D_n(x) = \sum_{k=-n}^{n} e^{ikx}$ is the Dirichlet Kernel of order n. And

$$\begin{aligned}&(D_n * f)(x)\\ =&\sum_{k=-n}^{n} e^{ikx}\left\langle f, e^{ikx}\right\rangle\\ =&S_n(f,x)\end{aligned}$$

which is the n^{th} partial sum we want.However, it's NOT a summability kernel.

4.9 Fejér Kernel

Idea: $(x_n) \subseteq \mathbb{C}$, consider

$$y_n = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

Exer: If $x_n \to x$, then $y_n \to y$.

Definition 4.40

The Fejér Kernel of order n is

$$F_n(x) = \frac{D_0(x) + D_1(x) + \ldots + D_n(x)}{n+1}$$

Remark.

$$\begin{aligned}
F_0(x) &= D_0(x) = 1 \\
F_1(x) &= \frac{e^{-x} + 2e^{i0x} + e^{ix}}{2} \\
F_2(x) &= \frac{e^{-2x} + 2e^{-x} + 3e^{i0x} + 2e^{ix} + e^{i2x}}{3} \\
&\vdots \\
F_n(x) &= \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikx}
\end{aligned}$$

Remark. (F_n) is a summability kernel.

Definition 4.41

$$\begin{aligned}
F_n * f &= \frac{1}{n+1} \sum_{k=0}^{n} D_k * f \\
&= \frac{1}{n+1} \sum_{k=0}^{n} S_k(f) \\
&= \frac{S_0(f) + S_1(f) + \ldots + S_n(f)}{n+1} \\
&=: \sigma_n(f)
\end{aligned}$$

which is the n^{th} Cesaro mean.

Theorem 4.42

$f \in L^1(T)$, (F_n) Fejér.

$$\begin{aligned}&\lim_{n\to\infty} F_n * f\\ =&\lim_{n\to\infty} \sigma_n(f)\\ =&f\end{aligned}$$

in $L^1(T)$.

Remark. If $(S_n(f))$ converges in $L^1(T)$ then $S_n(f) \to f$ in $L^1(T)$.

4.10 Fejér's Theorem

Idea: L^1 convergence is great theoretically, but pointwise convergence is practical.

Theorem 4.43: [Fejér's Theorem]

For $f \in L^1(T)$ and $t \in T$ consider

$$\omega_f(t) = \frac{1}{2} \lim_{x \to 0^+} (f(t+x) + f(t-x))$$

provided the limit exists, then

$$\sigma_n(f,t) \to \omega_f(t)$$

In particular, if f is continuous at t then

$$\sigma_n(f,t) \to f(t)$$

In practice:

1. Fix $x \in T$
2. Prove $(S_n(f,x))$ converged
3. Then
$$S_n(f,x) \to \omega_f(x)$$
4. If f is continuous at x then $S_n(f,x) \to f(x)$, i.e. $S(f,x) = f(x)$.

Example 4.44

$f \in L^1(T),\ f(x) = |x|,$

$$S_n(f,x) = a_0 + \sum_{k=1}^{n}(b_k \cos(kx) + c_k \sin(kx))$$

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| dx = \frac{\pi}{2} \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(kx) dx \\ &= \frac{2(-1)^k - 2}{k^2\pi} \\ c_k &= \frac{1}{\pi}\int_{\pi}^{\pi} |x| \sin(kx) dx = 0 \end{aligned}$$

so

$$\begin{aligned} & S_n(f,x) \\ =& \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^{n}\left(\frac{(-1)^k - 1}{k^2}\cos(kx)\right) \\ =& \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^{(n+1)/2}\left(\frac{-2}{(2k-1)^2}\cos((2k-1)x)\right) \end{aligned}$$

Note: $(S_n(f,x))$ *converges by comparison with* $\sum \frac{1}{(2x-1)^2}$.
Since f *is continuous,*

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos((2k-1)x)}{(2k-1)^2}$$

1. Taking $x = 0$:

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} \implies \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

2.

$$\begin{aligned} \sum_{k=1}^{\infty}\frac{1}{k^2} &= \sum_{k=1}^{\infty}\frac{1}{(2k)^2} + \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} \\ &= \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} + \frac{\pi^2}{8} \\ \implies \quad & \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6} \end{aligned}$$

4.11 Homogeneous Banach Space

Definition 4.45

A homogeneous Banach Space is a Banach Space $(B, \|\cdot\|_b)$ *such that*

1. B *is a subspace of* $L^1(T)$
2. $\|\cdot\|_1 \leqslant \|\cdot\|_b$
3. $\forall f \in B, \forall \alpha \in T$, $\|f_\alpha\|_B = \|f\|_B$ *(assuming* $f_\alpha \in B$*).*
4. $\forall f \in B, \forall t_0 \in T$,
$$\lim_{t \to t_0} \|f_t - f_{t_0}\|_B = 0$$

Example 4.46

$(L^p(T), \|\cdot\|_p)$ *(*$p < \infty$*).*

Theorem 4.47

Let B *be a homogeneous Banach Space* (K_n) *summability kernel.* $\forall f \in B$,
$$\lim_{n \to \infty} \|K_n * f - f\|_B = 0$$

Proof.

1. $\underbrace{\frac{1}{2\pi} \int_T K_n(t) f_t dt}_{\text{B-valued}} = \underbrace{K_n * f}_{L^1-valued}$
2. $\lim_{n\to\infty} \frac{1}{2\pi} \int_T K_n(t)\varphi(t)dt = \varphi(0)$, for all continuous $\varphi : T \to B$
3. $\varphi : T \to B$, $\varphi(t) = f_t$ is continuous $\forall f \in B$
4. $\|K_n * f - f\|_B \to 0$

□

Remark.

1. B norm Banach Space. Taking $K_n = F_n$ we have
$$\|\sigma_n(f) - f\|_B \to 0$$
for all $f \in B$.
2. Taking $B = L^p(T)$
   (a) $\|\sigma_n(f) - f\|_p \to 0$
   (b) $\overline{Trig(T)} = L^p(T)$

Remark. In $L^2(T)$

1. $\overline{Trig(T)} = L^2(T)$
2. $\overline{\mathrm{Span}\{\mathrm{e^{inx} : n \in \mathbb{Z}}\}} = L^2(T)$
3. $\{e^{inx} : n \in \mathbb{Z}\}$ ONB
4. Let the above ONB be written as $\{v_1, v_2, \ldots\}$, for all $f \in L^2(T)$
$$\lim_{n\to\infty} \sum_{i=1}^{n} \langle f, v_i \rangle \, v_i = f$$
5. If $v = e^{ikx}$,
$$\langle f, v \rangle \, v = \left(\frac{1}{2\pi} \int_T f(x) e^{-ikx} dx \right) e^{ikx} = \left\langle f, e^{ikx} \right\rangle e^{ikx}$$
6. $\forall f \in L^2(T)$,
$$\| S_n(f) - f \|_2 \to 0$$