CO 471/671: Semidefinite Programming

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1 Introduction

For $X \in \mathbb{R}^{n\times n}$, $Tr(X)$ denotes the trace of X,

$$Tr(x) := \sum_{i=1}^{n} X_{ii}$$

For $X, S \in \mathbb{R}^{n\times n}$, a commonly used inner-product is :

$$\begin{aligned}\langle X, S\rangle &:= Tr(X^T S)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} X_{ij}S_{ij}\\ &= Tr(SX^T)\end{aligned}$$

Using the above, we deduce: for every nonsingular matrix $P \in \mathbb{R}^{n\times n}$,

$$Tr(PXP^{-1}) = Tr(XP^{-1}P) = Tr(X), \forall X \in \mathbb{R}^{n\times n}$$

Given $X \in \mathbb{R}^{n\times n}$, the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ (exactly n) of the polynomial equation

$$\det(X - \lambda I) = 0$$

are the eigenvalues of X
We denotes by $\mathbb{S}^n$, the space of n-by-n symmetric matrices:

$$\mathbb{S}^n := \left\{X \in \mathbb{R}^{n\times n} : X = X^T\right\}$$

For every $X \in \mathbb{S}^n$, every eigenvalue of X is real. We usually order them:

$$\lambda_1(X) \geqslant \lambda_2(X) \geqslant \ldots \geqslant \lambda_n(X)$$

Sometimes, we consider $\lambda : \mathbb{S}^n \to \mathbb{R}^n$.
Diag : $\mathbb{R}^n \to \mathbb{S}^n$,

$$\text{Diag}(x) := \begin{bmatrix} x_1 & & & \\ & x_2 & & \text{\Huge 0} \\ & & \ddots & \\ \text{\Huge 0} & & & x_n \end{bmatrix}$$

diag : $\mathbb{S}^n \to \mathbb{R}^n$,

$$\text{diag}(X) := \begin{bmatrix} X_{11} \\ X_{22} \\ \vdots \\ X_{nn} \end{bmatrix}$$

Theorem 1.1: Spectral Decomposition Theorem

For every $X \in \mathbb{S}^n$, $\exists Q \in \mathbb{R}^{n\times n}$, orthogonal ($Q^TQ = I$) such that

$$X = Q\,\mathrm{Diag}(\lambda(X))Q^T$$

In the above, columns of Q are eigenvectors of X. E.g.,

$$j \in \{1, 2, \dots, n\}, X = Q\,\mathrm{Diag}(\lambda(X))Q^T,$$

then,

$$X(Qe_j) = Q\,\mathrm{Diag}(\lambda(X))\underbrace{Q^TQ}_{=I}e_j = \lambda_j(X)(Qe_j)$$

Definition 1.2

$X \in \mathbb{S}^n$ is called <u>positive semidefinite</u> if $h^TXh \geqslant 0$, $\forall h \in \mathbb{R}^n$. $X \in \mathbb{S}^n$ is called <u>positive definite</u> if $h^TXh > 0$, $\forall h \in \mathbb{R}^n \setminus \{0\}$.
We denote the set of p.s.d. matrices in $\mathbb{S}^n$, by $\mathbb{S}^n_+$

Example

$$X := \begin{bmatrix} 4 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \ S := \begin{bmatrix} 2 & 8 & 0 \\ 8 & 4 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

Let $h \in \mathbb{R}^3$ be arbitrary. Then

$$\begin{aligned} h^TXh &= h^T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)h \\ &= \left(\sum_{j=1}^{3} h_j\right)^2 + 3h_1^2 + 4h_2^2 + h_3^2 \geqslant 0, \forall h \in \mathbb{R}^3 \end{aligned}$$

Therefore, X is positive semidefinite. In fact, it is positive definite.

What about S? Consider $h := \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$,

$$h^TSh = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} -6 \\ 4 \\ -1 \end{bmatrix} = -10 > 0$$

Therefore, S is not positive semidefinite.

Theorem 1.3: Cholesky Decomposition Theorem

Let $X \in \mathbb{S}^n$. Then,

1. X is p.s.d. iff $\exists B \in \mathbb{R}^{n \times n}$, lower triangular such that $X = BB^T$
2. X is p.d. iff $\exists B \in \mathbb{R}^{n \times n}$, lower triangular and nonsingular such that $X = BB^T$

Proposition 1.4

Let $X \in \mathbb{S}^n$. Then TFAE:

1. X is p.s.d.
2. $\lambda(X) \geqslant 0$, which is the same as saying every eigenvalue of X is non-negative.
3. $\exists \mu \in \mathbb{R}^n_+$ (non-negative vector) and $h^{(1)}, h^{(2)}, \ldots, h^{(n)} \in \mathbb{R}^n$ such that
$$X = \sum_{i=1}^{n} \mu_i h^{(i)} \left(h^{(i)}\right)^T$$
4. $\exists B \in \mathbb{R}^{n \times n}$ such that $X = BB^T$
5. For every nonempty $J \subseteq \{1, 2, \ldots, n\}$, $\det(X_J) \geqslant 0$, where
$$X_J := \{[X_{ij}] : i, j \in J\}$$
6. For every $S \in \mathbb{S}^n_+$, $Tr(XS) \geqslant 0$

What are Semidefinite Programming Problems?

Let's recall Linear Programming problems first. $a \in \mathbb{R}^n, \alpha \in \mathbb{R}$ are given; $x \in \mathbb{R}^n$ is our variable vector. Then, a linear constraint is

$$\sum_{j=1}^{n} a_j x_j \begin{Bmatrix} \geqslant \\ = \\ \leqslant \end{Bmatrix} \alpha$$

LP is the problem of optimizing (minimizing or maximizing) an affine function of finitely many real valued variables subject to finitely many linear constraints.

SDP is the problem of optimizing (minimizing or maximizing) an affine function of fintiely many matrix variables with real entries, subsject to finitely many linear constraints and some symmetry and positive semidefiniteness constraints on these matrix variables.

To appreciate the power of the generalization from LP to SDP, it is useful to note

$$\begin{array}{lcl}
\text{LP} & \longleftrightarrow & \text{SDP} \\
x_1, x_2, \ldots, x_k \in \mathbb{R} & & X^{(i)} \in \mathbb{R}^{m_i \times n_i}, i \in \{1, \ldots, k\} \\
x_2 \geqslant 0 & & m_2 = n_2, X^{(2)} \in \mathbb{S}^{n_2}, \lambda(X^{(2)}) \geqslant 0 \\
2x_1 - x_2 + x_3 \geqslant 10 & & (A_1x_1 + A_2x_2 + A_3x_3 - 10I) \\
& & \text{ is p.s.d. where} A_1, A_2, A_3 \in \mathbb{S}^n \text{ are given (part of data)}
\end{array}$$

$$\begin{array}{ccc}
\text{LP} & \longleftrightarrow & \text{SDP}, C \in \mathbb{S}^n \\
\left\{\begin{array}{l} \min\ c^T x \\ \qquad Ax = b \\ \qquad x \geqslant 0 \end{array}\right\} & & \left\{\begin{array}{l} \inf\ Tr(CX) \\ \qquad Tr(A_iX) = b_i, \forall i \in \{1, 2, \ldots, m\} \\ \qquad X \in \mathbb{S}_+^{n_1} \oplus \mathbb{S}_+^{n_2} \oplus \ldots \oplus \mathbb{S}_+^{n_k} \end{array}\right\}
\end{array}$$

where $\mathbb{S}_+^{n_1} \oplus \mathbb{S}_+^{n_2} \oplus \ldots \oplus \mathbb{S}_+^{n_k}$ is the set of all $(n_1 + \ldots + n_k) \times (n_1 + \ldots + n_k)$ matrices where the digonal is made of matrices in $\mathbb{S}_+^{n_1}, \ldots, \mathbb{S}_+^{n_k}$

$$\begin{bmatrix} n_1 \times n_1 & & & \text{\huge 0} \\ & n_2 \times n_2 & & \\ & & \ddots & \\ \text{\huge 0} & & & n_k \times n_k \end{bmatrix}$$

More on the power of SDP in mathematical modeling:

Suppose in your application, the variable are $v^{(1)}, v^{(2)}, \ldots, v^{(n)} \in \mathbb{R}^n$, and your objective function and the constraints involve only affine functions of $\langle v^{(i)}, v^{(j)} \rangle, i, j \in \{1, 2, \ldots, n\}$. Then, we can express such a nonlinear and nonconvex optimization problem as an SDP.
Define a new matrix variable $X := VV^T \in \mathbb{S}_+^n$ where $V^T := [v^{(1)}, v^{(2)}, \ldots, v^{(n)}] \in \mathbb{R}^{n \times n}$
Then $X_{ij} = \langle v^{(i)}, v^{(j)} \rangle, \forall i, j \in \{1, 2, \ldots, n\}$ and we can rewrite the original optimization problem using only X variable as an SDP.
E.g.:

$$2\langle v^{(1)}, v^{(3)} \rangle - \langle v^{(1)}, v^{(2)} \rangle \leqslant 64$$
$$\Updownarrow$$
$$2X_{13} - X_{12} \leqslant 64,\ X \in \mathbb{S}_+^n$$

We denote the set of n-by-n symmetric positive definite matrices by $\mathbb{S}_{++}^n$

Lemma 1.5: Schur Complement Lemma

Let $X \in \mathbb{S}^n, T \in \mathbb{S}^m_{++}$. Then,

$$M := \begin{bmatrix} T & U^T \\ U & X \end{bmatrix} \in \mathbb{S}^{m+n}_{+} \iff (X - UT^{-1}U^T) \in \mathbb{S}^n_+$$

Moreover, $M \in \mathbb{S}^{m+n}_{++} \iff (X - UT^{-1}U^T) \in \mathbb{S}^n_{++}$

Proof. Suppose $X \in \mathbb{S}^n$ and $T \in \mathbb{S}^m_{++}$. Then,

$$M = \begin{bmatrix} T & U^T \\ U & X \end{bmatrix} = \underbrace{\begin{bmatrix} I & 0 \\ UT^{-1} & I \end{bmatrix}}_{:=L} \begin{bmatrix} T & 0 \\ 0 & X - UT^{-1}U^T \end{bmatrix} \underbrace{\begin{bmatrix} I & T^{-1}U^T \\ 0 & I \end{bmatrix}}_{=L^T}$$

Then, $\forall h \in \mathbb{R}^{m+n}$,

$$h^T M h = (L^T h)^T \begin{bmatrix} T & 0 \\ 0 & X - UT^{-1}U^T \end{bmatrix} (L^T h)$$

Since L is nonsingular ($\det(L) = 1$), $M \in \mathbb{S}^{m+n}_+ \iff T \in \mathbb{S}^m_+$ and $(X - UT^{-1}U^T) \in \mathbb{S}^n_+$. Similarly $M \in \mathbb{S}^{m+n}_{++} \iff T \in \mathbb{S}^m_+$ and $(X - UT^{-1}U^T) \in \mathbb{S}^n_{++}$ $\square$

For $A, B \in \mathbb{S}^n$, we will also use the notation $A \succcurlyeq B$ to mean $(A - B) \in \mathbb{S}^n_+$ and the notation $A \succ B$ to mean $(A - B) \in \mathbb{S}^n_{++}$. A very special case of the above lemma ($m = 1, T = 1$):

$$\begin{bmatrix} 1 & x^T \\ x^T & X \end{bmatrix} \succcurlyeq 0 \iff X - xx^T \succcurlyeq 0$$
$$\begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succ 0 \iff X - xx^T \succ 0$$

Semidefinite Programming Problems in Standard Equality Form and their Duals:
Given $C \in \mathbb{S}^n$, $b \in \mathbb{R}^m$ and a linear transformation $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$, we define

$$\begin{aligned} (P) \ \ & \inf \ \langle C, X \rangle \\ & s.t. : \mathcal{A}(X) = b \\ & X \succcurlyeq 0 \end{aligned}$$

and its dual

$$\begin{aligned} (D) \ \ & \sup \ b^T y \\ & s.t. \ \mathcal{A}^*(y) + S = C \\ & S \succcurlyeq 0 \end{aligned}$$

where $\mathcal{A}^* : \mathbb{R}^m \to \mathbb{S}^n$ is the adjoint of $\mathcal{A}$:

$$\langle \mathcal{A}^*(y), X \rangle := y^T \mathcal{A}(X), \forall X \in \mathbb{S}^n, \forall y \in \mathbb{R}^m$$

In more explicit form: for every linear transformation $\mathcal{A}: \mathbb{S}^n \to \mathbb{R}^m$, $\exists A_1, A_2, \ldots, A_m \in \mathbb{S}^n$ such that

$$[\mathcal{A}(X)]_i = \langle A_i, X\rangle = Tr(A_i X), \forall i \in \{1, 2, \ldots, m\}$$

and thus, $\mathcal{A}^*(y) = \sum_{i=1}^m y_i A_i$. Therefore,

$$\begin{aligned}(P)\ &\inf\ Tr(CX)\\ &s.t.\ Tr(A_i X) = b_i, i \in \{1, 2, \ldots, m\}\\ &\qquad X \succcurlyeq 0\end{aligned}$$

$$\begin{aligned}(D)\ &\sup\ b^T y\\ &s.t.\ \sum_{i=1}^m y_i A_i + S = C\\ &\qquad S \succcurlyeq 0\end{aligned}$$

Example

$$C := \begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}, A_1 := \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, A_2 := \begin{bmatrix}1 & -1\\ -1 & 5\end{bmatrix}, b := \begin{bmatrix}1\\ 2\end{bmatrix}$$

Then,

$$\begin{aligned}(P)\ &\inf\ Tr(CX) = 3X_{11} + 2X_{21} + 3X_{22}\\ &s.t.\ Tr(A_1 X) = X_{11} + X_{22} = 1\\ &\qquad Tr(A_2 X) = X_{11} - 2X_{21} + 5X_{22} = 2\\ &\qquad X \in \mathbb{S}^2_+\end{aligned}$$

and

$$\begin{aligned}(D)\ &\sup\ y_1 + 2y_2\\ &s.t.\ \begin{bmatrix}y_1 + y_2 & -y_2\\ -y_2 & y_1 + 5y_2\end{bmatrix} + S = \begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}\\ &\qquad S \in \mathbb{S}^2_+\end{aligned}$$

Theorem 1.6: Weak Duality Relation – SDP

Let $\overline{X}$ be feasible in (P) and $(\overline{y}, \overline{S})$ be feasible in (D). Then $\langle C, \overline{X}\rangle - b^T\overline{y} = \langle \overline{X}, \overline{S}\rangle \geqslant 0$.

Proof. Suppose $\overline{X}$, $(\overline{y}, \overline{S})$ are feasible in (P) and (D) respectively. Then,

$$\begin{aligned}\langle C, \overline{X}\rangle - b^T\overline{y} &= \langle \mathcal{A}^*(\overline{y}) + \overline{S}, \overline{X}\rangle - b^T\overline{y} \text{ by } \mathcal{A}^*(\overline{y}) + \overline{S} = C\\ &= \langle \overline{S}, \overline{X}\rangle + \langle \mathcal{A}^*(\overline{y}), \overline{X}\rangle - b^T\overline{y}\\ &= \langle \overline{X}, \overline{S}\rangle + \overline{y}^T\mathcal{A}(\overline{X}) - b^T\overline{y}\\ &= \langle \overline{X}, \overline{S}\rangle \geqslant 0 \text{ by } \mathcal{A}(\overline{X}) = b, \overline{X} \succcurlyeq 0, \overline{S} \succcurlyeq 0, \text{Proposition 4(f)}\end{aligned}$$

□

Corollary 1.7

1. If (P) is unbounded, then (D) is infeasible.
2. If (D) is unbounded, then (P) is infeasible.
3. If for feasible solutions $\overline{X}$ of (P) and $(\overline{y}, \overline{S})$ of (D), we have $\langle \overline{X}, \overline{S} \rangle = 0$ then $\overline{X}$ is optimal in (P) and $(\overline{y}, \overline{S})$ is optimal in (D),

Note: Dual of (D) is "equivalent" to (P). So, SDP duality is an involution. To prove this (and rigorously define "equivalent"), we can put (D) into the form of (P), apply the definition of dual, and then simplify to obtain (P).
Alternatively, we may assume, $\exists \hat{X} \in \mathbb{S}^n$ and $\hat{y} \in \mathbb{R}^m, \hat{S} \in \mathbb{S}^n$ such that

$$\mathcal{A}(\hat{X}) = b,\ \mathcal{A}^*(\hat{y}) + \hat{S} = C, Null(\mathcal{A}) = \{X \in \mathbb{S}^n : \mathcal{A}(X) = 0\} =: L$$

Then, the feasible region of (P) is $\left(L + \{\hat{X}\}\right) \cap \mathbb{S}^n_+$.

$$L^\perp := \{S \in \mathbb{S}^n : \langle X, S \rangle = 0, \forall X \in L\}$$

Note that $Range(\mathcal{A}^*) = L^\perp$. Therfore, $y \in \mathbb{R}^m, S \in \mathbb{S}^n$ satisfy $\mathcal{A}^*(y) + S = C$ if and only if $(S - \hat{S}) \in L^\perp$. Thus, $S \in \mathbb{S}^n$ is a part of a feasible solution of (D) if and only if

$$S \in \left(L^\perp + \{\hat{S}\}\right) \cap \mathbb{S}^n_+$$

Objective function of (P) for X satisfying $\mathcal{A}(X) = b$:

$$\langle C, X \rangle = \langle \mathcal{A}^*(\hat{y}) + \hat{S}, X \rangle = \underbrace{b^T \hat{y}}_{\text{constnat}} + \langle \hat{S}, X \rangle$$

So,

$$(P)\ \inf \left\{ \langle \hat{S}, X \rangle : X \in \left(L + \{\hat{X}\}\right) \cap \mathbb{S}^n_+ \right\}$$

Objective function value of (D) for $(y, S) : \mathcal{A}^*(y) + S = C$ is

$$b^T y = \mathcal{A}(\hat{X})^T y = \langle \hat{X}, \mathcal{A}^*(y) \rangle = \langle \hat{X}, C - S \rangle = \underbrace{\langle C, \hat{X} \rangle}_{\text{constant}} - \langle \hat{X}, S \rangle$$

Therefore, (D) is "equivalent" to

$$\inf \left\{ \langle \hat{X}, S \rangle : S \in \left(L^\perp + \{\hat{S}\}\right) \cap \mathbb{S}^n_+ \right\}$$

Another attractive standard form for SDPs:
Let linear transformation $\mathcal{A} : \mathbb{S}^n \to \mathbb{S}^k$, $C \in \mathbb{S}^n$, $B \in \mathbb{S}^k$ be given. If

$$\begin{aligned} (P)\ \inf\ & \langle C, X \rangle \\ & \mathcal{A}(x) \succcurlyeq B \\ & X \succcurlyeq 0 \end{aligned}$$

, then its dual is equivalent to

$$\begin{aligned} \sup\ & \langle B, Y\rangle \\ & \mathcal{A}^*(Y) \preccurlyeq C \\ & Y \succcurlyeq 0 \end{aligned}$$

Let $X \in \mathbb{S}^n_+$. Define its symmetric positive semidefinite square root by applying the spectral Decomposition theorem to X, $X = Q\,\mathrm{Diag}(\lambda(X))Q^T$ and then

$$X^{\frac{1}{2}} := Q\,[\mathrm{Diag}(\lambda(X))]^{\frac{1}{2}}\,Q^T$$

Proposition 1.8

Let $X, S \in \mathbb{S}^n_+$. Then $\langle X, S\rangle = 0$ if and only if $XS = 0$.

Proof.

- ($\Longleftarrow$) is straight forward, since $Tr(0) = 0$
- ($\Longrightarrow$) Suppose $X, S \in \mathbb{S}^n_+$, $\langle X, S\rangle = 0$. Then

$$0 = Tr(XS) = Tr(\underbrace{X^{\frac{1}{2}}SX^{\frac{1}{2}}}_{\succcurlyeq 0}) \text{ since } S \succcurlyeq 0, X^{\frac{1}{2}} \in \mathbb{S}^n$$

  Thus, by Prop.4(b), $\lambda(X^{\frac{1}{2}}SX^{\frac{1}{2}}) \geqslant 0$, by the above, these eigenvalues add up to zero (trace is the sum of eigenvalues). Therefore, $\lambda(X^{\frac{1}{2}}SX^{\frac{1}{2}}) = 0$ and $0 = X^{\frac{1}{2}}SX^{\frac{1}{2}} = \left(X^{\frac{1}{2}}S^{\frac{1}{2}}\right)\left(X^{\frac{1}{2}}S^{\frac{1}{2}}\right)^T$. Hence, $X^{\frac{1}{2}}S^{\frac{1}{2}} = 0$. Finally, $XS = X^{\frac{1}{2}}\left(X^{\frac{1}{2}}S^{\frac{1}{2}}\right)S^{\frac{1}{2}} = 0$

□

Proposition 1.9

1. $\mathbb{S}^n_{++} = int(\mathbb{S}^n_+)$
2. Let $X \in \mathbb{S}^n$. Then TFAE
   (a) X is positive definite
   (b) $\lambda(X) > 0$
   (c) $\exists \mu \in \mathbb{R}^n_{++}$ and $h^{(1)}, h^{(2)}, \ldots, h^{(n)} \in \mathbb{R}^n$ linearly independent s.t.
   $$X = \sum_{i=1}^n \mu_i h^{(i)}(h^{(i)})^T$$
   (d) $\exists B \in \mathbb{R}^{n\times n}$ nonsingular s.t. $X = BB^T$
   (e) $\forall J_k := \{1, 2, \ldots, k\}, k \in \{1, 2, \ldots, n\}, \det(X_{J_k}) > 0$
   (f) $\forall S \in \mathbb{S}^n_+ \setminus \{0\}, \langle X, S\rangle > 0$
   (g) $X \succcurlyeq 0$ and $rank(X) = n$

Example

Let $X \in \mathbb{R}^6$, $X := \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}$.

Then by Prop.6 part(e), $X \in \mathbb{S}^3_{++}$ if and only if

$$\begin{aligned} x_1 &> 0 \\ x_1x_4 - x_2^2 &> 0 \\ x_1x_4x_6 + 2x_2x_3x_5 - x_3^2x_4 - x_1x_5^2 - x_6x_2^2 &> 0 \end{aligned}$$

2 Duality Theory

Duality theory is very useful in many aspects of optimization including applications, effective utilization of algorithms, the design and analysis of efficient and robust algorithms, and The development and powerful utilization of the theory. For example,

- We can generate (using the Weak Duality Relation) concise and robust evidence/proof that our feasible solutions are optimal or near-optimal.
- We can derive optimality conditions which help design efficient, robust algorithms, including stopping criteria for such algorithms
- We can perform sensitivity and what-if analysis
- Depending on the application, optimal and near-optimal solutions provide information and insights based on our primal optimal solutions (e.g. shadow prices, fair distribution or pricing of resources, outlier defection, infeasibility detection)
- etc

Some notions of duality:

Dual Cone: Given $K \subseteq \mathbb{R}^d$,

$$K^* := \left\{ s \in \mathbb{R}^d, \langle x, s \rangle \geqslant 0, \forall x \in K \right\}$$

Example

1. $K := \mathbb{R}^d_+$, under the usual Euclidean inner-product,

$$K^* = \left\{ s \in \mathbb{R}^d, x^T s \geqslant 0, \forall x \in \mathbb{R}^d_+ \right\} = \mathbb{R}^d_+$$

2. $K := \mathbb{S}^n_+$, under the trace inner-product,

$$K^* = \left\{ S \in \mathbb{S}^n : Tr(XS) \geqslant 0, \forall X \in \mathbb{S}^n_+ \right\} = \mathbb{S}^n_+$$

3. Exercise: Let

$$K := \left\{ \begin{pmatrix} t \\ x \end{pmatrix} \in \mathbb{R} \oplus \mathbb{R}^n : \sum_{j=1}^{n} |x_j| \leqslant t \right\}$$

What is the dual cone of K, under Euclidean inner product?

Polar Set: Given $K \subseteq \mathbb{R}^d$,

$$K^o := \left\{ s \in \mathbb{R}^d : \langle x, s \rangle \leqslant 1, \forall x \in K \right\}$$

Note: if K is a cone, then $K^o = -K^*$

Legendre-Fenchel conjugate of a function

Given $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$,

$$f_*(S) := \sup\{-\langle s, x\rangle - f(x) : x \in \mathbb{R}^d\}$$

Example

$f : \mathbb{S}^n \to \mathbb{R} \cup \{+\infty\}$,

$$f(X) := \begin{cases} -\ln\det(X), & \text{if } X \in \mathbb{S}^n_{++} \\ +\infty, & \text{otherwise} \end{cases}$$

$$\begin{aligned} f_*(S) &= \sup\{-Tr(SX) - f(X) : X \in \mathbb{S}^n\} \\ &= \begin{cases} -\ln\det(S) - n, & \text{if } S \in \mathbb{S}^n_{++} \\ +\infty, & \text{otherwise} \end{cases} \end{aligned}$$

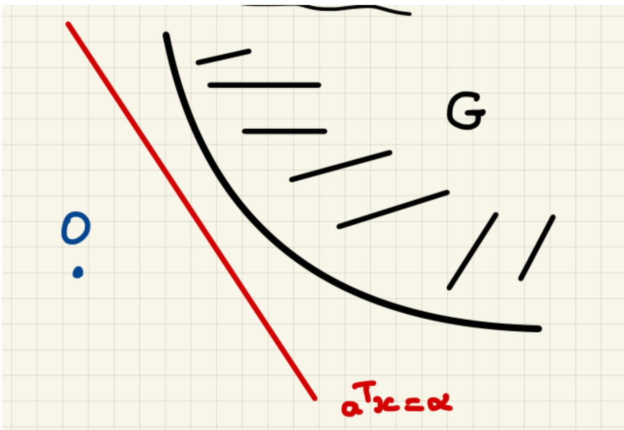
If $S \notin \mathbb{S}^n_{++}$ we can find $\{X^{(k)}\} \subset \mathbb{S}^n_{++}$ such that $Tr(SX^{(k)}) \to$constant and $f(X^{(k)}) \to -\infty$.

If $S \in \mathbb{S}^n_{++}$, then $-f'(X) = S \iff X^{-1} = S \iff X = S^{-1}$.

Theorem 2.1: Hyperplane Separation Theorem for Closed Convex Sets

Let $G \subset \mathbb{R}^d$ be a nonempty, closed convex set. Suppose $0 \notin G$. Then $\exists a \in \mathbb{R}^d \setminus \{0\}$ and $\alpha \in \mathbb{R}_+$ such that

$$G \subseteq \{x \in \mathbb{R}^d : a^T x \geqslant \alpha\}$$

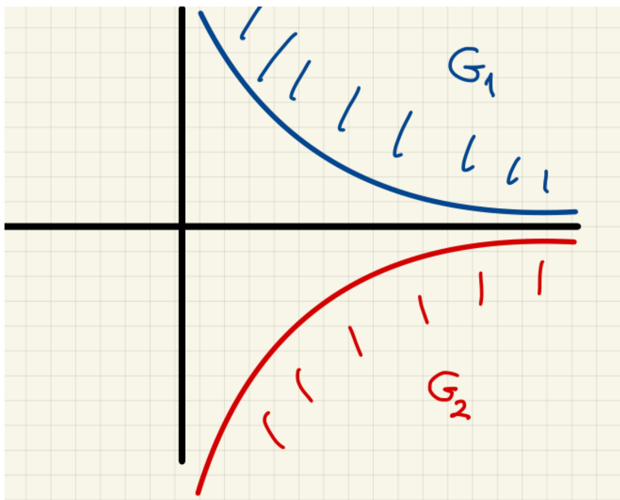


Corollary 2.2

Let $G_1, G_2 \subset \mathbb{R}^d$ be disjoint, nonempty closed convex sets. If G_1 or G_2 is bounded then $\exists a \in \mathbb{R}^d \setminus \{0\}$ such that

$$\inf\{a^T x : x \in G_1\} > \sup\{a^T x : x \in G_2\}$$

If both sets G_1, G_2 are allowed to be unbounded, we cannot guarantee the strict inequality above.



Theorem 2.3

Let $G \subset \mathbb{R}^d$ be a nonempty convex set. Suppose $0 \notin G$. Then, $\exists a \in \mathbb{R}^d \setminus \{0\}$ such that

$$G \subseteq \{x \in \mathbb{R}^d : a^T x \geqslant 0\}$$

Corollary 2.4

Let $G_1, G_2 \subset \mathbb{R}^d$ be nonempty, disjoint convex sets. Then $\exists a \in \mathbb{R}^d \setminus \{0\}$ such that

$$\inf\{a^T x : x \in G_1\} \geqslant \sup\{a^T x : x \in G_2\}$$

Recall,

$$\begin{aligned}
(P)\, &\inf\ Tr(CX) \\
&\quad s.t.\ \mathcal{A}(X) = b \\
&\qquad\quad X \succcurlyeq 0 \\
(D)\, &\sup\ b^T y \\
&\quad s.t.\ \mathcal{A}^*(y) + S = C \\
&\qquad\quad S \succcurlyeq 0
\end{aligned}$$

Definition 2.5

We say that (P) satisfies the slater condition , or (P) has a slater point, if $\exists \overline{X} \in \mathbb{S}^n_{++}$ such that $\mathcal{A}(\overline{X}) = b$. (D) satisfies the slater condition, or (D) has a Slater point if $\exists \overline{y} \in \mathbb{R}^m$ and $\overline{S} \in \mathbb{S}^n_{++}$ such that

$$\mathcal{A}^*(\overline{y}) + \overline{S} = C$$

Theorem 2.6: A Strong Duality Theorem

Suppose (D) has a Slater point and the objective value of (D) is bounded from above. Then (P) attains its optimal value and the optimum values of (P) and (D) are the same.

Proof. Suppose $\exists \overline{y} \in \mathbb{R}^m, \overline{S} \in \mathbb{S}^n_{++}$ such that $\mathcal{A}^*(\overline{y}) + \overline{S} = C$. Further assume $\exists \gamma \in \mathbb{R}$ such that $b^T y \leqslant \gamma$ for all feasible solutions (y, S) of (D).
Let $z^* := \sup\{b^T y : \mathcal{A}^*(y) + S = C, S \succcurlyeq 0\}$. We may assume $b \neq 0$. (If $b = 0$, then $\overline{X} = 0$ is a feasible solution of (P); thus, $\overline{X}, (\overline{y}, \overline{S}))$ are optimal in (P) and (D) respectively, by Corollary 7 par(C), and we are done).
Next, we utilize Corollary 13. Let

$$G_1 := \mathbb{S}^n_{++},\ G_2 := \left\{S \in \mathbb{S}^n : S = C - \mathcal{A}^*(y), b^T y \geqslant z^* \text{ for some } y \in \mathbb{R}^m\right\}$$

G_1 and G_2 are convex, $G_1 \neq \emptyset$. We can check $G_2 \neq \emptyset$ (for example, using the Fundamental Theorem of LP). Also, $G_1 \cap G_2 = \emptyset$ (if not, $\exists \tilde{y} \in \mathbb{R}^m$ s.t. $\mathcal{A}^*(\tilde{y}) \prec C$ and $b^T\tilde{y} \geqslant z^*$; letting $\hat{y} := \tilde{y} + \varepsilon b$ for some $\varepsilon > 0$ small enough yields $\mathcal{A}^*(\hat{y}) \prec C$ and $b^T\hat{y} = \underbrace{b^T\tilde{y}}_{\geqslant z^*} + \underbrace{\varepsilon\|b\|_2^2}_{>0} > z^*$, a contradiction).
Therefore, Corollary 13 applies to G_1, G_2. Thus, $\exists \tilde{X} \in \mathbb{S}^n \setminus \{0\}$ such that

$$\inf\left\{\langle \tilde{X}, S\rangle : S \in \mathbb{S}^n_{++}\right\} \geqslant \sup\left\{\langle \tilde{X}, S\rangle : S \in G_2\right\}$$

Since $G_2 \neq \emptyset$, the LHS is bounded from below. Since $\mathbb{S}^n_{++}$ is a cone, $\langle \tilde{X}, S\rangle \geqslant 0, \forall S \in \mathbb{S}^n_{++}$ (otherwise, the LHS would be $-\infty$). Furthermore, $\langle \tilde{X}, S\rangle \geqslant 0, \forall S \in cl(\mathbb{S}^n_{++}) = \mathbb{S}^n_+$. Hence, by proposition 4, part (f), $\tilde{X} \in \mathbb{S}^n_+$. We already proved LHS $\geqslant 0$. Since we can take a sequence $\{S^{(k)}\} \subset \mathbb{S}^n_{++}$ such that $S^{(k)} \to 0$, $LHS = 0$. Therefore,

$$\begin{aligned} &0 \geqslant \langle \tilde{X}, C\rangle - \langle \tilde{X}, \mathcal{A}^*(y)\rangle \text{ for every } y \in \mathbb{R}^m \text{ s.t. } b^T y \geqslant z^* \\ \iff &\mathcal{A}(\tilde{X})^T y \geqslant \langle C, \tilde{X}\rangle \text{ for every } y \in \mathbb{R}^m \text{ s.t. } b^T y \geqslant z^* \end{aligned}$$

Thus, $\mathcal{A}(\tilde{X})^T y$ is bounded from below on the set $\left\{y \in \mathbb{R}^m : b^T y \geqslant z^*\right\}$. LP duality theorem implies $\mathcal{A}(\tilde{X}) = \alpha b$ for some $\alpha \in \mathbb{R}_+$. If $\alpha = 0$, we get $\mathcal{A}(\tilde{X}) = 0$ and $0 \geqslant \langle C, \tilde{X}\rangle = \langle \mathcal{A}^*(\overline{y}) + \overline{S}, \tilde{X}\rangle = \underbrace{\underbrace{\mathcal{A}(\tilde{X})}_{=0} y}_{=0} + \underbrace{\langle \overline{S}, \tilde{X}\rangle}_{>0,\ prop6(f)} > 0$, a contradiction. Hence, $\alpha > 0$. Define $\hat{X} := \frac{1}{\alpha}\tilde{X} \in \mathbb{S}^n_+$, we have $\mathcal{A}(\hat{X}) = b$ and $\mathcal{A}(\hat{X})^T y \geqslant \langle C, \hat{X} \rangle$ for all $y \in \mathbb{R}^m$ s.t. $b^T y \geqslant z^*$. Therefore, $\langle C, \hat{X} \rangle \leqslant z^*$ and by the Weak Duality Relation (theorem 6), $\hat{X}$ is optimal in (P) and optimal objective values of (P) and (D) are the same. □

Since we estabilished that the dual of (D) is equivalent to (P), we have

Corollary 2.7

If (D) has a feasible solution and (P) has a Slater point, then (D) attains its optimal objective value, and the optimal objective values of (P) and (D) coincide.

Corollary 2.8

If (P) and (D) both have Slater points, then both (P) and (D) attain their optimal objective values and these objective ' values are the same

Remark. The above theorem and its proof generalize to conic convex optimization setting where we may pick our standard form as

$$\begin{aligned} (P)\ \ \inf\ & \langle c, x \rangle \\ & \mathcal{A}(x) = b \\ & \mathcal{B}(x) \geqslant d \\ & x \in K \\ (D)\ \ \sup\ & \langle b, y \rangle + \langle d, u \rangle \\ & \mathcal{A}^*(y) + \mathcal{B}^*(u) + s = c \\ & u \geqslant 0 \\ & s \in K^* \end{aligned}$$

where $c \in \mathbb{R}^n, b \in \mathbb{R}^{m_1}, d \in \mathbb{R}^{m_2}, \mathcal{A} : \mathbb{R}^n \to \mathbb{R}^{m_1}, \mathcal{B} : \mathbb{R}^n \to \mathbb{R}^{m_2}$, $\mathcal{A}, \mathcal{B}$ are linear transformations and $\mathcal{A}, \mathcal{B}, b, c, d$ are all given; $K \subseteq \mathbb{R}^n$ is a closed convex cone.
For this more general set up we can use the following "restricted" notion of Slater point: $\overline{x} \in \mathbb{R}^n$ is a Slater point for (P) if $\overline{x} \in int(K)$ and $\mathcal{A}(\overline{x}) = b, \mathcal{B}(\overline{x}) \geqslant d$. (We do not require that $\overline{x} \in int\{x \in \mathbb{R}^n : \mathcal{B}(\overline{x}) \geqslant d\} = \{x \in \mathbb{R}^n : \mathcal{B}(\overline{x} > d)\}$)
"Restricted" Slater Point (interiosity condition restricted to the nonpolyhedral constraints) for (D) is $(\overline{y}, \overline{u}, \overline{s}) \in \mathbb{R}^{m_1} \oplus \mathbb{R}^{m_2} \oplus \mathbb{R}^n$ such that

$$\mathcal{A}^*(\overline{y}) + \mathcal{B}^*(\overline{u}) + \overline{s} = c, \overline{u} \geqslant 0, \overline{s} \in int(K^*)$$

How useful are these duality theorems of SDP (Weak Duality and Strong Duality Theorem)?
Very useful!
In many applications (engineering, big data, machine learning, statistics, computer science, other areas of mathematics, ...) we construct a mathematical model that might be twoo hard or impossible to solve exactly:

$$(P)\ \ \inf\ \ f(x)\ s.t.\ x \in \Phi$$

We can construct and SDP relaxation (the "=" is by the assumption of Strong Duality Theorem)

$$(P)\ \inf\ f(x),\ x \in \Phi \geqslant \underbrace{\inf\ \langle C, X\rangle,\ \mathcal{A}(X) = b,\ X \succcurlyeq 0}_{SDP} = \underbrace{\sup b^T y,\ \mathcal{A}^*(y) + S = C,\ S \succcurlyeq 0}_{SDD}$$

Suppose we have a fast algorithm (heuristic) which provides feasible solutions to (P) that usually have good objective function values. We run the heuristic and obtain $\hat{x} \in \Phi$, we also solve (SDD) approximately and obtain a feasible solution $(\overline{y}, \overline{s})$ of (SDD). Then,

$$f(\hat{x}) \geqslant v \geqslant b^T \overline{y}$$

where v is the unknown optimal value of (P), then $b^T \overline{y}$ is closed to the optimal value of (SDP) and (SDD).

Pitfalls in SDP Duality:

In the statement of our Strong Duality Theorem (Theorem 15) the assumptions cannot be removed (without changing the rest of the statement). Consider, for example, the statement of Corollary 16 and the following SDP instance.

Example 2.9

$m := 1, n := 2, C := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, b := 2$

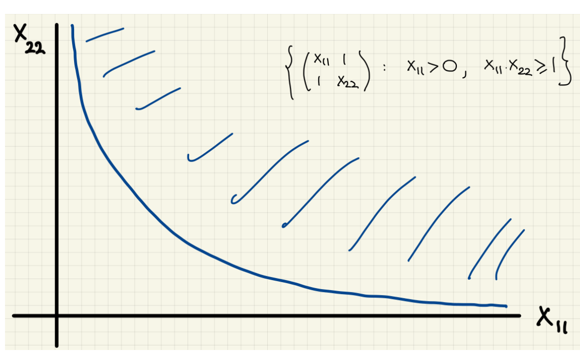
$$\begin{aligned} (P) \ & \inf Tr(CX) = X_{11} \\ & s.t. \ Tr(A_1 X) = b \iff 2X_{21} = 2 \\ & \quad X \succcurlyeq 0 \\ (D) \ & \sup 2y \\ & s.t. \begin{bmatrix} 1 & -y \\ -y & 0 \end{bmatrix} \succcurlyeq 0 \end{aligned}$$

which is

$$\begin{aligned} (P) \ & \inf X_{11} \\ & s.t. \begin{pmatrix} X_{11} & 1 \\ 1 & X_{22} \end{pmatrix} \succcurlyeq 0 \\ (D) \ & \sup 2y \\ & s.t. \begin{pmatrix} 1 & -y \\ -y & 0 \end{pmatrix} \succcurlyeq 0 \end{aligned}$$

(P) has a Slater point $\overline{X} := \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, and (D) has a feasible solution $\overline{y} := 0$. Thus, Corollary 16 applies.
(D) attains its optimal value and the optimal objective values of (P) and (D) are the same.
Feasible region of (D) is a singleton $\{0\}$. Therefore, $\overline{y} = 0$ is the unique optimal solution of (D). Feasible region of (P):

$$\left\{ \begin{pmatrix} X_{11} & 1 \\ 1 & X_{22} \end{pmatrix} : X_{11} > 0, X_{11}X_{22} \geqslant 1 \right\}$$

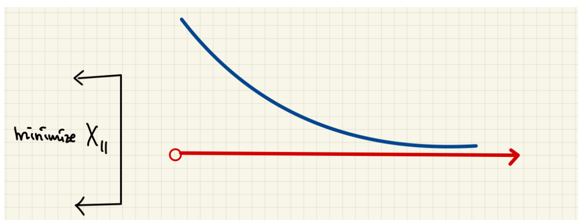


Example: 18 continued

Hence the optimal obj. value of (P) is nonnegative.
Consider the following family of feasible solutions of (P).

$$X(\varepsilon) := \begin{pmatrix} \varepsilon & 1 \\ 1 & \frac{1}{\varepsilon} \end{pmatrix}, \varepsilon > 0$$

As $\varepsilon \searrow 0$, $Tr(CX(\varepsilon)) \searrow 0$; however, there is no feasible solution of (P) with objective value zero. Therefore, optimal obj. value of (P) is zero but it is not attained.
What went wrong with (P)?
Even thought the feasible region of (P) is a closed set (this is always true: intersection of closed sets), its projection onto $\begin{pmatrix} \mathbb{R} \\ 0 \end{pmatrix}$ is not:



Note that the statement of Theorem 15 does note directly apply to this (P) and (D). indeed, (D) does not have a Slater point and (P) does not have an optimal solution.

Things can get worse:

Example 2.10

Consider $m := 2, n := 3, C := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$$A_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 := \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, b := \begin{bmatrix} 0 \\ 2\gamma \end{bmatrix}$$

where $\gamma \in \mathbb{R}_{++}$.

$$\begin{aligned} (P) \ & \inf 0 \\ & s.t. \begin{pmatrix} X_{11} & 0 & X_{31} \\ 0 & 0 & 0 \\ X_{31} & 0 & \gamma \end{pmatrix} \succcurlyeq 0 \end{aligned}$$

Optimal value of (P) is zero, $\forall \gamma \in \mathbb{R}_{++}$ attained by $X(\gamma) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix}$

$$\begin{aligned} (D) \ & \sup 2\gamma y_2 \\ & s.t. \begin{pmatrix} 0 & 1 + y_2 & 0 \\ 1 + y_2 & -y_1 & 0 \\ 0 & 0 & -2y_2 \end{pmatrix} \succcurlyeq 0 \end{aligned}$$

$y_2 = -1, \overline{y} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an optimal solution of (D) with objective value -2γ.

Both optimal values zero for (P), and -2γ for (D) are attained. However, there is a duality gap of -2γ.

Pitfalls in SDP Duality (continued):

In the special case of Linear Programming problems, if (P) and (D) have feasible solutions then they both have optimal solutions and for every pair of optimal solutions $\overline{x}$ of (P) , $(\overline{y}, \overline{s})$ of (D) complementarity holds: $\overline{x}_j \overline{s}_j = 0, \forall j$. Moreover, $\exists$ an optimal pair $\hat{x}$ of (P) and $(\hat{y}, \hat{s})$ of (D) for which strict complementarity holds: $\hat{x} + \hat{s} > 0$.
In the general set up of (SDP), we proved that if (P) and (D) have Slater points, then they both have optimal solutions and for every pair of optimal solutions $\overline{X}$ of (P) and $(\overline{y}, \overline{S})$ of (D) complementarity holds: $\overline{XS} = \overline{SX} = 0$. However, as you are showing in Assignment 1 4(c), strict complementarity may fail to hold for some SDPs, even if both (P) and (D) have Slater points.

Why do we care about strict complementarity?
Very useful and/or necessary for:

1. identifying the set of optimal solutions for (P) and (D)
2. detecting infeasibility, unboundedness
3. establishing optimality conditions
4. robustness and fast local convergence of various algorithms for SDP
5. estabilishing sensitivity analysis, what-if analysis, stability results, error bounds
6. etc

Sort of good news:

Fix positive integers $n > m \geqslant 1$. Consider the set of data $(\mathcal{A}, b, c) \in \mathcal{L}(\mathbb{S}^n, \mathbb{R}^m) \oplus \mathbb{R}^m \oplus \mathbb{R}^n$ for which (P) has a feasible solution. Among the elements of this set of data, "almost" every instance has a Slater point. Similarly, if we focus on the set of data $(\mathcal{A}, b, c)$ for which both (P) and (D) have feasible solutions, then for "almost" every such instance (P) and (D) both have Slater points and they have a strictly complementary pair of optimal solutions. For a more rigorous statement, see Theorem 2.20 (of the textbook).

Certifying Infeasibility and Unboundedness in SDPs

Example 2.11

$$(P)\ \inf\ Tr\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}X\right] : Tr\left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}X\right] = 1, X \succcurlyeq 0$$

$$(D)\ \sup\ y : \begin{pmatrix}y & 0\\ 0 & 0\end{pmatrix} \preccurlyeq \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \text{ Equivalently}$$

$$(P)\ \inf\ 2X_{21}\ s.t.\ \begin{pmatrix}1 & X_{21}\\ X_{21} & X_{22}\end{pmatrix} \succcurlyeq 0$$

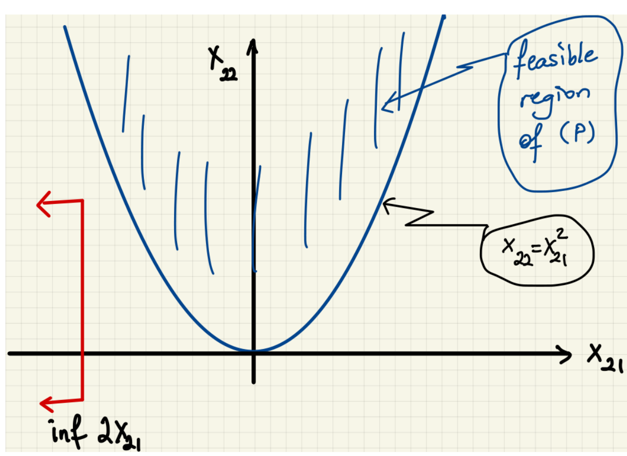
$$(D)\ \sup\ y\ s.t.\ \begin{pmatrix}-y & 1\\ 1 & 0\end{pmatrix} \succcurlyeq 0 \implies (D) \text{ is infeasible.}$$

Consider $X(\alpha) := \begin{pmatrix}1 & \alpha\\ \alpha & \alpha^2\end{pmatrix}, \forall \alpha \in \mathbb{R}$. $X(\alpha)$ is feasible in (P), $\forall \alpha \in \mathbb{R}$ and the objective function value $2[X(\alpha)]_{21} = 2\alpha \to -\infty$ as $\alpha \to -\infty$. Therefore, (P) is unbounded. All seems well here, (P) is unbounded $\implies$ (D) is infeasible.

Do we have certificate of unboundedness for (P) (equivalently, a certificate of infeasibility for (D)) similar to those for LPs?(Generalization of Farkas' Lemma?)
Recall, for every $A \in \mathbb{R}^{m\times n}, c \in \mathbb{R}^n$, exactly one of the following systems has a solutions:

1. $A^T y \leqslant c$
2. $Ad = 0, d \geqslant 0, c^T d < 0$

Since in Example 20, (D) is infeasible, can we find $D \in \mathbb{S}^2_+$ such that $\mathcal{A}(D) = 0$, and $Tr(CD) < 0$?



(D) is infeasible since $\begin{pmatrix}-y & 1\\ 1 & 0\end{pmatrix} \succcurlyeq 0$ cannot hold no matter what $y \in \mathbb{R}$ is. However, (D) is "almost feasible".

Definition 2.12

Let $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ be a linear transformation and $C \in \mathbb{S}^n$. Then, we say that $\mathcal{A}^*(y) \preccurlyeq C$ is almost feasible if for every $\varepsilon > 0$, $\exists C' \in \mathbb{S}^n$ such that $\|C - C'\| < \varepsilon$ and $\mathcal{A}^*(y) \preccurlyeq C'$ is feasible.

Note: $\mathcal{A}^*(y) \preccurlyeq C$ is feasible $\implies$ $\mathcal{A}^*(y) \preccurlyeq C$ is almost feasible.

Theorem 2.13

Let $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ be linear and $C \in \mathbb{S}^n$. Then

1. If $\exists D \in \mathbb{S}^n_+$ such that $\mathcal{A}(D) = 0, Tr(CD) < 0$ then $\nexists y \in \mathbb{R}^m$ such that $\mathcal{A}^*(y) \preccurlyeq C$;
2. If $\nexists D \in \mathbb{S}^n_+$ such that $\mathcal{A}(D) = 0, Tr(CD) < 0$ then $\mathcal{A}^*(y) \preccurlyeq C$ is almost feasible.

Theorem 2.14

$\exists D \in \mathbb{S}^n_+$ such that $\mathcal{A}(D) = 0, Tr(CD) < 0$ if and only if $\mathcal{A}^*(y) \preccurlyeq C$ is not almost feasible.

Proof.

- $\Longleftarrow$ Theorem 22
- $\Longrightarrow$ Suppose there exists such D. We may assume $Tr(CD) = -1$ (replace D by $\frac{1}{|Tr(CD)|}D$). Then for every $C' \in \mathbb{S}^n$ such that $\|C - C'\|_F < \frac{1}{\|D\|_F}$, $\mathcal{A}^*(y) \preccurlyeq C'$ is infeasible (if $\mathcal{A}^*(y) \preccurlyeq C'$, then $\langle D, \mathcal{A}^*(y)\rangle \leqslant Tr(C'D) \implies 0 = y^T\mathcal{A}(D) \leqslant Tr(CD) - Tr[(C - C')D] \leqslant -1 + \|C - C'\|_F \|D\|_F < 0$ a contradiction). Therefore, $\{\mathcal{A}^*(y) \preccurlyeq C\}$ is not almost feasible.

$\square$

Linear Programming	SDP
$A^T y \leqslant c$ is infeasible iff $\exists d \in \mathbb{R}^n$ such that $Ad = 0$ $d \geqslant 0$ $c^T d < 0$	$A^*(y) \preccurlyeq C$ is not almost feasible iff $\exists D \in \mathbb{S}^n$ such that $\mathcal{A}(D) = 0$ $D \succcurlyeq 0$ $Tr(CD) < 0$

Another way to deal with possible dualitty gaps, dual attainment issues, infeasibility and unboundedness certificates, is to keep the statements of theorems analogous to the LP special case but change the definition of the dual.

Slater Condition, Facial Reduction, Extended Lagrange-Slater Dual

Definition 2.15

Let $K \subseteq \mathbb{R}^d$ be a closed convex cone. A convex conve $G \subseteq K$ is a face of K if for every pair $u, v \in K$ such that $(u+v) \in G$, we have $u \in G, v \in G$.

For example, let $K := \mathbb{R}^3_+$. Then, $G := \left\{x \in \mathbb{R}^3_+ : x_3 = 0\right\}$ is a face of K.

A face G of K is exposed, if $\exists a \in \mathbb{R}^d \setminus \{0\}$, such that

$$G = \{x \in K : \langle a, x\rangle = 0\} \text{ and } K \subseteq \left\{x \in \mathbb{R}^d : \langle a, x\rangle \leqslant 0\right\}.$$

Note:$a \in -K^*$.
A face G of K is a proper face of K if

$$\{0\} \subset G \subset K$$

Theorem 2.16

(a) Every nonempty face of $\mathbb{S}^n_+$ is characterized by a unique subspace $L \subseteq \mathbb{R}^n$ such that

$$G = \left\{X \in \mathbb{S}^n_+ : Null(X) \supseteq L\right\},\ \operatorname{relint}(G) = \left\{X \in \mathbb{S}^n_+ : Null(X) = L\right\}$$

(b) Every proper face of $\mathbb{S}^n_+$ is exposed.

(c) $\mathbb{S}^n_+$ is projectionally exposed. That is, every nonempty face G of $\mathbb{S}^n_+$ can be expressed as

$$G = (I-Q)\mathbb{S}^n_+(I-Q)$$

where $Q \in \mathbb{S}^n$ is the projecction onto the unique subspace L defining G.

The above theorem implies that every proper face of $\mathbb{S}^n_+$ is linearly isomorphic to $\mathbb{S}^k_+$ for some $k \in \{1, 2, \ldots, n-1\}$.
So, G is a proper face of $\mathbb{S}^n_+$ iff $\exists k \in \{1, 2, \ldots, n-1\}$ and $Q \in \mathbb{R}^{n\times n}$ orthogonal such that

$$G = \left\{Q \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} Q^T : X \in \mathbb{S}^k_+\right\}$$

Given $(\mathcal{A}, b, c)$ suppose we find the minimal face (with respect to set inclusion) $\overline{G}$ of $\mathbb{S}^n_+$ which contains the feasible region of (P).
Then our problem (P) is equivalent to

$$\begin{aligned} (\tilde{P})\ \inf\ & Tr(CX) \\ & \mathcal{A}(X) = b \\ & X \in \overline{G} \end{aligned}$$

Let $Q \in \mathbb{R}^{n\times n}$ be orthogonal such that $\overline{G} = \left\{ Q \begin{bmatrix} \overline{X} & 0 \\ 0 & 0 \end{bmatrix} Q^T : \overline{X} \in \mathbb{S}^k_+ \right\}$

Define $\overline{C} \in \mathbb{S}^k$, $\overline{\mathcal{A}} : \mathbb{S}^k \to \mathbb{R}^m$ using Q. Then $(\tilde{P})$ is equivalent to

$$
\begin{aligned}
(\overline{P}) \ \inf \ & Tr(\overline{CX}) \\
& \overline{\mathcal{A}}(\overline{X}) = b \\
& \overline{X} \in \mathbb{S}^k_+
\end{aligned}
$$

$(\overline{P})$ satisfies the Slater condition!

To find the minimal face of $\mathbb{S}^n_+$ containing the feasible region of (P) is no easier than finding a solution to (P) in the worst case. However, the following result is useful:

Lemma 2.17

Let $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ be linear and $b \in \mathbb{R}^m$. Then exactly one of the following two systems has a solution

(I) $\mathcal{A}(X) = b, \ X \in \mathbb{S}^n_{++}$

(II) $\mathcal{A}^*(y) \in \mathbb{S}^n_+ \setminus \{0\}, \ b^T y = 0$

Lemma 26 says: either (P) has a Slater point or, we ccan find a supporting Hyperplane $\{X \in \mathbb{S}^n : Tr(\overline{S}X) = 0\}$ of $\mathbb{S}^n_+$ whichi contains the feasible region of (P). Here, if $\overline{u} \in \mathbb{R}^m$ is a solution of system (II), we can choose $\overline{S} := \mathcal{A}^*(\overline{u})$.

Recursive application of the above idea and Lemma 26 eventually results in an SDP like $(\overline{P})$ above which does satisfy the Slater condition.

Note that with each application of Lemma 26 k in $\mathbb{S}^k_+$ goes down by at least 1, so n applications of Lemma 26 suffices.

This process is sometimes called <u>facial reduction</u>.

A related, alternative approach is to write down a modified dual problem directly (instead of doing the computations required to find $\overline{u}$ above, possibly n times).

Suppose the problem we want to solve is

$$
(D) \ \sup \ b^T y \ \mathcal{A}^*(y) \preccurlyeq C
$$

Define its <u>Extended Lagrange-Slater Dual</u> as

$$
\begin{aligned}
(ELSD) \ \inf \ & Tr(C(U+W)) \\
s.t. \ & \mathcal{A}(U+W) = b \\
& W \in \mathcal{W}_n \\
& U \succcurlyeq 0
\end{aligned}
$$

where $\mathcal{W}_n \subseteq \mathbb{S}^n$ a linear subspace defined using $\mathcal{A}$ and C, $n(m+1)$ linear equations and n p.s.d matrix inequalities (" $\succcurlyeq$ ") on 2n-by-2n matrices.

Detailed description of $\mathcal{W}_n$ is Given in the textbook on page 42 (before Theorem 2.28).

Theorem 2.18

If (D) has a finite optimal value, then the optimal values of (D) and $(ELSD)$ are the same and $(ELSD)$ attains its optimal value.

Theorem 2.19

Let $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ linear, $C \in \mathbb{S}^n$. Then, exactly one of the following systems has a solution.

1. $\mathcal{A}^*(y) \preccurlyeq C$
2. $\mathcal{A}(U + W) = 0,\ W \in \mathcal{W}_n,\ U \succcurlyeq 0,\ Tr(C(U + W)) = -1$

Theorem 2.20

In the real number computation model, the problem of deciding SDP feasibility is in $\mathcal{NP} \cap\ co\text{−}\mathcal{NP}$

Open Problems:

1. Does there exist a more efficient representation of the subspace $\mathcal{W}_n$ in (ELSD)?
2. Is SDP feasibility in $\mathcal{NP}$ in the Turing machine model?
3. ...

In many application of SDP, our SDP is not an exact formulation but a relaxation of a much harder problem. So,
When does the Slater Condition hold in SDP relaxations?
Given $c \in \mathbb{R}^n$, our hard optimization problem is

$$\inf\ c^T x,\ s.t.\ x \in F$$

, where $F \subset \mathbb{R}^n$ a difficult nonconvex set.
We can also consider

$$\inf\ c^T x + x^T C x,\ s.t.\ x \in F$$

, where $C \in \mathbb{S}^n$ is also given.

Homogeneous Equality Form:
Suppose $\exists \mathcal{A} : \mathbb{S}^{n+1} \to \mathbb{R}^m$ linear, such that

$$F = \left\{ x \in \mathbb{R}^n : \mathcal{A} \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} = 0 \right\}$$

Which sets F can be represented in Homogeneous Equality Form?

Proposition 2.21

Every system of finitely many multivariate polynomial equations and inequalities can be put into Homogeneous Equality Form.

Proof ideas:
First, show that we can handle systems of multivariate quadratic equations and quadratic inequalities.
Multivariate Quadratic Equations
Given $Q \in \mathbb{S}^n, q \in \mathbb{R}^n, \gamma \in \mathbb{R}$,

$$\begin{aligned} 0 =& x^T Q x + 2q^T x + \gamma \\ \iff Tr & \left[\begin{pmatrix} \gamma & q^T \\ q & Q \end{pmatrix} \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \right] = 0 \\ =& \gamma + q^T x + Tr(qx^T) + Tr(Qxx^T) \\ =& \gamma + q^T x + Tr(x^T q) + Tr(x^T Q x) \\ =& \gamma + 2q^T x + x^T Q x \end{aligned}$$

Multivariate Quadratic Inequalities
Given $Q \in \mathbb{S}^n, q \in \mathbb{R}^n, \gamma \in \mathbb{R}$,

$$x^T Q x + 2q^T x + \gamma \leqslant 0$$
$$\text{iff}$$
$$x^T Q x + 2q^T x + \gamma + \tilde{s}^2 = 0$$
$$\text{iff}$$
$$Tr \left[\begin{pmatrix} \gamma & q^T & 0 \\ q & Q & 0 \\ 0 & 0^T & 1 \end{pmatrix} \begin{pmatrix} 1 & x^T & \tilde{s} \\ x & xx^T & \tilde{s}x \\ \tilde{s} & \tilde{s}x^T & \tilde{s}^2 \end{pmatrix} \right] = 0$$

Note the right matrix in the $Tr[\cdot]$ is $[1\ x^T\ \tilde{s}]^T [1\ x^T\ \tilde{s}]$.
Higher degree multivariate polynomials
Ex: $2x_1^4 + x_1 x_2^3 + x_3^2 - 5 = 0$

$$\iff \begin{cases} 2x_4^2 + x_5 x_6 + x_3^2 - 5 & = 0 \\ x_1^2 - x_4 & = 0 \\ x_1 x_2 - x_5 & = 0 \\ x_2^2 - x_6 & = 0 \end{cases}$$

and

$$F = \left\{ x \in \mathbb{R}^n : \mathcal{A} \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} = 0 \right\}$$

SDP relaxation:

$$\hat{P} := \left\{ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathbb{S}^{n+1} : \mathcal{A} \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = 0, \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succcurlyeq 0 \right\}$$

$$\inf\ Tr \left[\begin{pmatrix} 0 & \frac{1}{2}c^T \\ \frac{1}{2}c & 0 \end{pmatrix} \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \right]\ s.t.\ \mathcal{A} \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = 0, \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succcurlyeq 0$$

Theorem 2.22

If $conv(F)$ is full dimensional, then the Slater condition holds for the SDP relaxation above.

Proof. Suppose $conv(F)$ is full-dimensional. Then $\exists v^{(1)}, v^{(2)}, \ldots, v^{(n+1)} \in F$ that are affinely independent . $\iff$ $\begin{pmatrix} 1 \\ v^{(1)} \end{pmatrix}, \begin{pmatrix} 1 \\ v^{(2)} \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ v^{(n+1)} \end{pmatrix}$ are linearly independent. Consider $\overline{X} := \frac{1}{n+1} \sum_{i=1}^{n+1} \begin{pmatrix} 1 \\ v^{(i)} \end{pmatrix} \begin{pmatrix} 1 & v^{(i)T} \end{pmatrix}$. By Proposition 9, part (c), $\overline{X} \in \mathbb{S}^{n+1}_{++}$. Moreover,

$$X \in conv \left\{ \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \in \mathbb{S}^{n+1} : x \in F \right\} \subseteq \hat{P}$$

Therefore, $\overline{X}$ is a Slater point for the SDP relaxation. □

Given $c \in \mathbb{R}^n$, our hard optimization problem is

$$\inf \ c^T x \ s.t. \ x \in F$$

, where $F \subset \mathbb{R}^n$ a difficult nonconvex set.
If $dim(conv(F)) = n$, then a very large class of SDP relaxations for the above-mentioned difficult nonconvex optimiztion problem will have Slater points.
What if $dim(conv(F)) \leqslant n - 1$?
We first determine the affine hull of F (the smallest affine space containing F, equivalently, the intersection of all affine spaces containing F).
Suppose $dim(conv(F)) = d \leqslant n-1$. Then, we find $L \in \mathbb{R}^{d \times n}, l \in \mathbb{R}^n$ such that $rank(L) = d$ and $x \in F \implies x = l + L^T y$ for some $y \in \mathbb{R}^d$.
Define a linear transformation $\mathcal{L} : \mathbb{S}^{n+1} \to \mathbb{S}^{d+1}$,

$$\mathcal{L}(Z) := \begin{pmatrix} 1 & l^T \\ 0 & L \end{pmatrix} Z \begin{pmatrix} 1 & 0^T \\ l & L^T \end{pmatrix}.$$

Its adjoint $\mathcal{L}^* : \mathbb{S}^{d+1} \to \mathbb{S}^{n+1}$ is given by

$$\mathcal{L}^*(W) = \begin{pmatrix} 1 & 0^T \\ l & L^T \end{pmatrix} W \begin{pmatrix} 1 & l^T \\ 0 & L^T \end{pmatrix}.$$

Define $\overline{\mathcal{A}} : \mathbb{S}^{d+1} \to \mathbb{R}^m$,

$$\overline{\mathcal{A}}(W) := \mathcal{A}(\mathcal{L}^*(W)).$$

Then,

$$F = \left\{ l + L^T y : \overline{\mathcal{A}} \begin{pmatrix} 1 & y^T \\ y & yy^T \end{pmatrix} = 0, y \in \mathbb{R}^d \right\}$$

which leads to the SDP relaxation

$$\hat{P}_L := \left\{ \begin{pmatrix} 1 & y^T \\ y & Y \end{pmatrix} \in \mathbb{S}^{d+1} : \overline{\mathcal{A}} \begin{pmatrix} 1 & y^T \\ y^T & Y \end{pmatrix} = 0, \begin{pmatrix} 1 & y^T \\ y & Y \end{pmatrix} \succcurlyeq 0 \right\}$$

Theorem 2.23

$$\hat{P}_L \cap \mathbb{S}^{d+1}_{++} \neq \emptyset$$

That is, we can always guarantee the Slater condition holds in a wide class of SDP relaxations, provided we can identify the affine hull of F. Moreover, in many cases we decrease the size of the SDP.

3 Solving SDP Problems

3.1 Ellipsoid& Ellipsoid Method

$E \subset \mathbb{R}^d$ is an ellipsoid if $\exists c \in \mathbb{R}^d$ (centre) and $A \in \mathbb{S}^d_{++}$ (shape & size) such that

$$E = \left\{x \in \mathbb{R}^d : (x-c)^T A^{-1}(x-c) \leqslant 1\right\} =: E(A,c)$$

Note:

$$\begin{aligned} E(A,c) &= \left\{x \in \mathbb{R}^d : \|A^{-\frac{1}{2}}(x-c)\|_2^2 \leqslant 1\right\} \\ &= \left\{A^{\frac{1}{2}}z + c : \|z\|_2^2 \leqslant 1, z \in \mathbb{R}^d\right\} \\ &= c + A^{\frac{1}{2}}B_d(0,1) \text{ the } B \text{ is unit ball in } \mathbb{R}^d \text{ centred at the origin} \end{aligned}$$

So, ellipsoids are simple convex sets in that they are affine images (under symmetric positive defintie maps plus a shift) of Euclidean unit balls.
Many attributes of ellipsoids are easy to handle

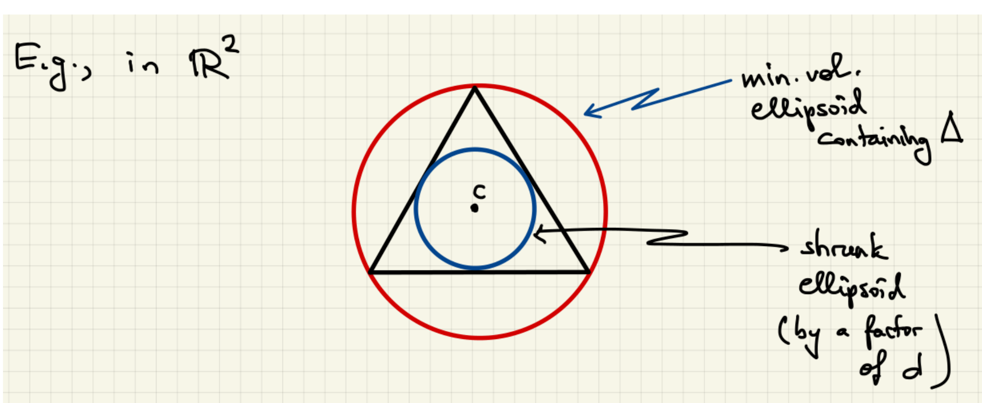
$$vol(E(A,c)) = \sqrt{\det(A)}vol(B_d(0,1)),$$

longest axis of $E(A,c)$ corresponds to an eigenvector of A determining $\lambda_1(A)$; shortest axis of $E(A,c)$ corresponds to an eigenvector of A determining $\lambda_d(A)$.
However, ellipsoids are versatile enough to approximate any given convex set well:

Theorem 3.1

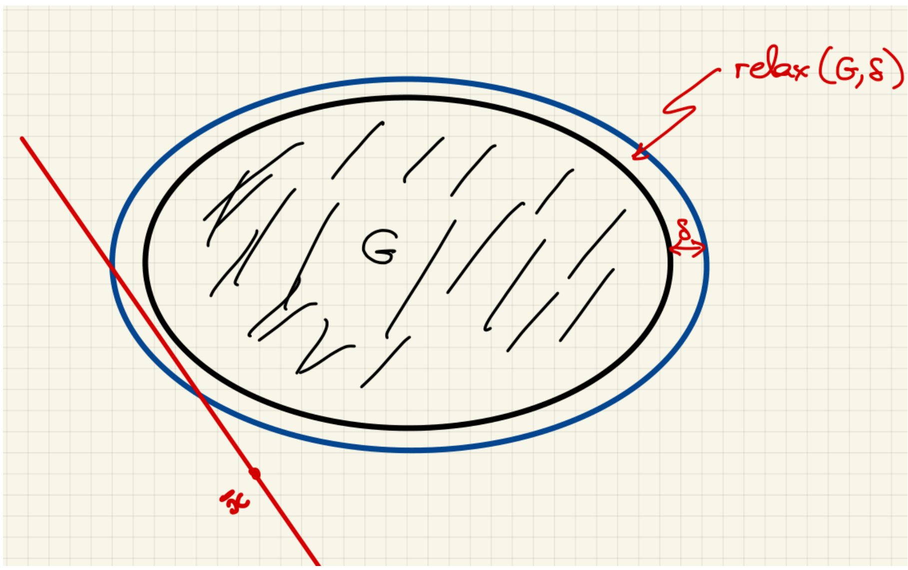
For every compact convex set in $\mathbb{R}^d$ with nonempty interior, there exists a unique minimal volume ellipsoid containing that set. Moreover, shrinking that min. volume ellipsoid (around its centre) by a factor of at most d gives an ellipsoid contained in the convex set. (Löwner-John Theorem, John Theorem, Löwner-John ellipsoid).

The factor d in the statement above is tight: use as the convex set, d-dimensional simplex.



Ellipsoid Method does not require an explicit description of the feasible region, it suffices to have a weak separation oracle.
Let $G \subset \mathbb{R}^d$ be a convex set. Given $\delta > 0$, δ-relaxation of G is

$$\operatorname{relax}(G, \delta) := \left\{ u \in \mathbb{R}^d : \|u - x\|_2 \leqslant \delta \text{ for some } x \in G \right\} \text{ which is convex by definition}$$



A weak separation oracle for G takes as input $\overline{x} \in \mathbb{Q}^d$, $\delta \in \mathbb{Q}_{++}$ and it outputs:

- "$\overline{x} \in \operatorname{relax}(G, \delta)$" OR
- $a \in \mathbb{Q}^d$ such that $\|a\|_\infty = 1$ and

$$\langle a, \overline{x} \rangle \geqslant \langle a, x \rangle - \delta, \forall x \in \operatorname{relax}(G, \delta)$$

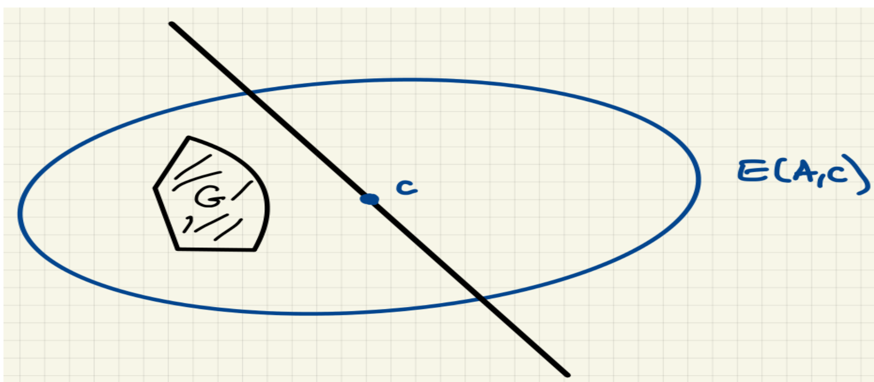
3.1.1 Ellipsoid Method for finding a feasible solution

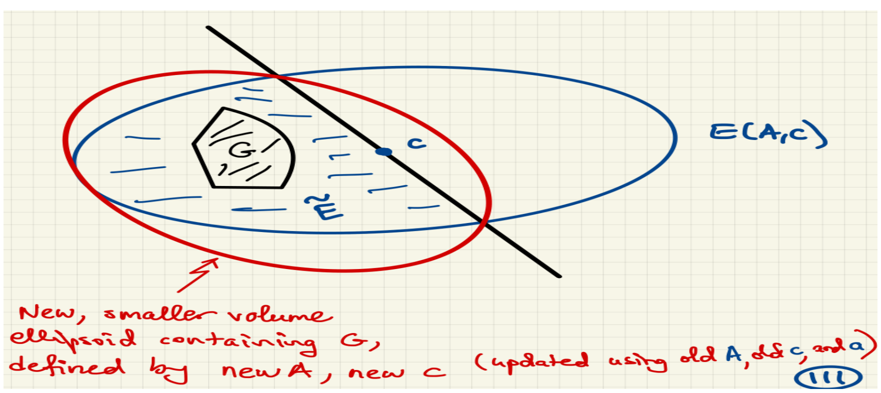
Input: $A \in \mathbb{S}^d_{++}, c \in \mathbb{R}^d$ such that $E(A, c) \supseteq G$ (where $G \subset \mathbb{R}^d$ is a convex set, we want to compute $\overline{x} \in G$), $\varepsilon \in \mathbb{Q}_{++}$.

Proposition 3.2

Algorithm:

1. Call the separation oracle: "is $c \in G$?"
   - if $c \in G$, STOP
   - else retrieve $a \in \mathbb{Q}^d$, $\|a\|_\infty$ separating c from G.
2. - If $vol(E(A,c)) < \varepsilon$, STOP $(vol(G) < \varepsilon)$
   - else $\tilde{E} := \{x \in E(A,c) : \langle a, x\rangle \leqslant \langle a, c\rangle\}$.
3. Compute the minimum volume ellipsoid $E(A,c)$ (A, c are updated) containing $\tilde{E}$; go to step1.





Lemma 3.3

Let $A_+ \in \mathbb{S}^d_{++}$, $c_+ \in \mathbb{R}^d$ such that $E(A_+, c_+)$ is the minimum volume ellipsoid containing

$$\tilde{E} = \{x \in E(A,c) : \langle a, x\rangle \leqslant \langle a, c\rangle\}.$$

Then

$$c_+ = c - \frac{1}{(d+1)\sqrt{a^TAa}}Aa,\ \ A_+ = \frac{d^2}{d^2-1}\left[A - \frac{2}{(d+1)a^TAa}Aaa^TA\right]$$

Moreover,

$$\ln\left(\frac{vol(E(A_+,c_+))}{vol(E(A,c))}\right) \leqslant -\frac{1}{2d}$$

The last lemma tells us that

- every iteration of Algorithm 34 can be implemented efficiently, and
- the volume of the current ellipsoid decreases ”significantly enough”.

Theorem 3.4

Let $G \subset \mathbb{R}^d$ be a convex set such that

(i) We have access to a weak separation oracle for G,

(ii) $G \subseteq B_d(0,R), R \in \mathbb{Q}_{++}$ is given.

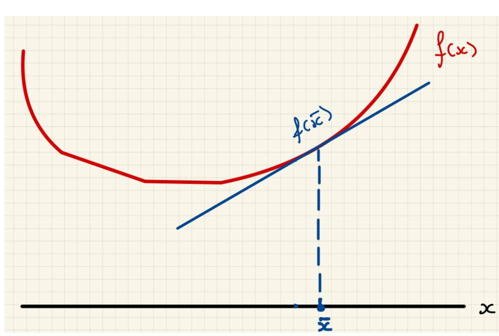
Then, for every given $\varepsilon \in \mathbb{Q}_{++}$, after $O(d^2\ln(R/\varepsilon))$ iterations of Algorithm 34, we either compute $\overline{x} \in \text{relax}(G,\varepsilon)$ or prove that $vol(G) \leqslant \varepsilon$.

We can extend Algorithm 34 to handle convex optimization problems of form:

$$\inf\ \ f(x)\ s.t.\ x \in G$$

where $f : \mathbb{R}^d \to \mathbb{R}$ is a convex function. Just like set G, accessing f via an oracle will suffice. A subgradient oracle for f takes as input $\overline{x} \in \mathbb{R}^d$, returns $f(\overline{x})$ and $h \in \mathbb{R}^d$ such that

$$f(x) \geqslant f(\overline{x}) + h^T(x-\overline{x}), \forall x \in \mathbb{R}^d$$



Theorem 3.5

Let $G \subset \mathbb{R}^n$ be a convex set and $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function such that

(i) $\exists$ a weak separataion oracle for G,

(ii) $\exists$ a subgradient oracle for f,

(iii) r and $R \in \mathbb{Q}_{++}$ are given such that $B_d(\tilde{x}, r) \subseteq G \subseteq B_d(0, R)$ for some not given $\tilde{x} \in \mathbb{R}^d$.

Then after $O\left(d^2\left[\ln(R/r) + \ln(\mu_0/\varepsilon)\right]\right)$ iterations of an ellipsoid method, we obtain $\overline{x} \in G$ such that

$$f(\overline{x}) \leqslant \inf\{f(x) : x \in G\} + \varepsilon$$

In the above,

$$\mu_0 := \varepsilon + \sup_{x \in B_d(0,R)} \{f(x)\} - \inf_{x \in B_d(0,R)} \{f(x)\}$$

Wow! This is nice and applies to all convex optimization problems, including SDPs. Suppose (P) and (D) have Slater points $\overline{X}, (\overline{y}, \overline{S})$ respectively. Then, we can replace (P) by $(\tilde{P})$.

$$\begin{aligned} (\tilde{P}) \ \inf \ & \langle C, X\rangle \\ s.t. \ & \mathcal{A}(X) = b \\ & \langle \overline{S}, X\rangle \leqslant 2\langle \overline{X}, \overline{S}\rangle \\ & X \succcurlyeq 0 \end{aligned}$$

Theorem 3.6

(a) The SDPs (P) and $(\tilde{P})$ have optimal solutions.

(b) The optimal solution sets of (P) and $(\tilde{P})$ are the same.

(c) Let $G \subset \mathbb{S}^n_+$ denote the feasible solution set for $(\tilde{P})$. Then G is compact and convex. Moreover,

$$B_G(\overline{X}, \lambda_n(\overline{X})) \subseteq G \subseteq B_G\left(0, \frac{2\langle \overline{X}, \overline{S}\rangle}{\lambda_n(\overline{S})}\right)$$

where B_G denotes the Euclidean ball in $aff[G]$.

(d) $\max\{\langle C, X\rangle : X \in G\} - \min\{\langle C, X\rangle : X \in G\} \leqslant \frac{4n\|C\|_2\langle \overline{X}, \overline{S}\rangle}{\lambda_n(\overline{S})}$

Theorem 38 and the reformulation of (P) as $(\tilde{P})$ (significantly aided by the given Slater points $\overline{X}$, $(\overline{y}, \overline{S})$), show that we can apply Theorem 37 to SDPs.

Thare are many other approaches for utilizing algorithms like Ellipsoid Method and resulsts like Theorem 37 for SDPs.

3.2 Primal-Dual Interior-Point Methods

Consider algorithms which start with $X^{(0)} \succ 0$, $y^{(0)} \in \mathbb{R}^m, S^{(0)} \succ 0$, and generate

$$\left\{(X^{(k)}, y^{(k)}, S^{(k)}) : k \in \mathbb{Z}_+\right\}$$

such that $X^{(k)} \succ 0, S^{(k)} \succ 0, \forall k \in Z_{++}$.
Assuming (P) and (D) have optimal solutions and their optimal values are the same, we wnat

- $\|\mathcal{A}(X^{(k)}) - b\| \to 0$ with $(X^{(k)} \succ 0)$ Primal Feasibility.
- $\|\mathcal{A}^*(y^{(k)}) + S^{(k)}\| \to 0$ with $(S^{(k)} \succ 0)$ Dual Feasibility.
- $\langle X^{(k)}, S^{(k)} \rangle \to 0$ with $(X^{(k)} \succ 0, S^{(k)} \succ 0)$ Complementarity Slackness

For simplicity of presentation, we will assume $\mathcal{A}(X^{(0)}) = b, \mathcal{A}^*(y^0) + S^{(0)} = C$ and that $\mathcal{A}$ is surjective. Define $f : \mathbb{S}^n \to \mathbb{R} \cup \{\infty\}$ by

$$f(X) := \begin{cases} -\ln(\det(X)), & \text{if } X \in \mathbb{S}^n_{++} \\ +\infty, & \text{otherwise} \end{cases}$$

Note: For every sequence $\left\{X^{(k)}\right\} \subset \mathbb{S}^n_{++}$ such that $X^{(k)} \to \overline{X} \in bd(\mathbb{S}^n_+)$,

$$f(X^{(k)}) \to +\infty$$

So, we can use f to reformulate our problem (P).

Proposition 3.7

The above defined function f is strictly convex on $\mathbb{S}^n_{++}$. Moreover, for every $X \in PD^n$ and $H \in \mathbb{S}^n$, we have

$$\langle f'(X), H \rangle = -Tr(X^{-1}H)$$
$$\langle f''(X)H, H \rangle = Tr(X^{-1}HX^{-1}H) = Tr\left[\left(X^{-\frac{1}{2}}HX^{-\frac{1}{2}}\right)^2\right]$$
$$f'''(X)[H, H, H] = -2Tr\left[\left(X^{-\frac{1}{2}}HX^{-\frac{1}{2}}\right)^3\right]$$

3.2.1 Central Path

For each $\mu > 0$, we define

$$(P_\mu)\ \inf\ \frac{1}{\mu}\langle C, X\rangle + f(X)\ s.t\ \mathcal{A}(X) = b$$

Necessary and sufficient optimality conditions for (P_μ) (under the Slater point assumptions for $(P)\&(D)$):

$$\begin{gathered}\mathcal{A}(X) = b, X \succ 0 \\ -\mathcal{A}^*(y) - X^{-1} + \tfrac{1}{\mu}C = 0\end{gathered}$$

$y \leftarrow \mu y$, $S := \mu X^{-1}$. Then, necessary and sufficient conditions for optimality (for (P_μ)) become

$$\begin{gathered}\mathcal{A}(X) = b, X \succ 0 \\ \mathcal{A}^*(y) + S = C \\ S = \mu X^{-1}\end{gathered}$$

For each $\mu > 0$, the unique solution $(X(\mu), y(\mu), S(\mu))$ defines the primal-dual central path:

$$\{(X(\mu), y(\mu), S(\mu)) \in \mathbb{S}^n \oplus \mathbb{R}^m \oplus \mathbb{S}^n : \mu > 0\}$$

Theorem 3.8

Suppose (P) and (D) have Slater points and $\mathcal{A}$ is surjective. Then, for every $\mu > 0$, (P_μ) has a unique optimal solution $X(\mu)$. Moreover, the following system

$$\begin{gathered}\mathcal{A}(X) = b, X \succ 0 \\ \mathcal{A}^*(y) + S = C \\ S = \mu X^{-1}\end{gathered}$$

has a unique solution $(X(\mu), y(\mu), S(\mu))$.

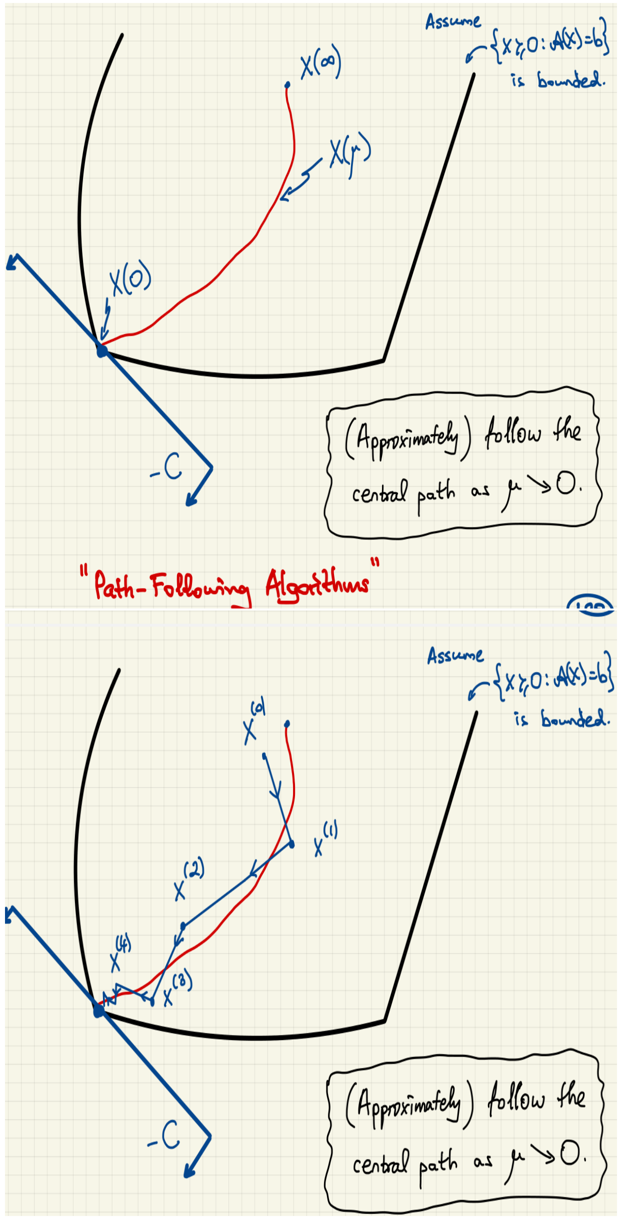
The above system also characterizes the unique otpimal solution of

$$(D_\mu)\ \sup\left\{\frac{1}{\mu}b^T y + f(S) : \mathcal{A}^*(y) + S = C\right\}.$$

Consider the solutions $(X(\mu), y(\mu), S(\mu))$ for $\mu > 0$, and focus on

$$S(\mu) = \mu[X(\mu)]^{-1} \implies \langle X(\mu), S(\mu)\rangle = \langle X(\mu), \mu[X(\mu)]^{-1}\rangle = \mu Tr(I) = n\mu$$

So, as $\mu \searrow 0$, $\langle X(\mu), S(\mu)\rangle \searrow 0$.



To derive path-following algorithms, one can use Newton's Method and its variants locally on the system of (nonlinear) equations:

$$\begin{aligned} \mathcal{A}(X) = b, \\ \mathcal{A}^*(y) + S = C, \\ S = \mu X^{-1}, \end{aligned} \quad or, \quad \begin{aligned} \mathcal{A}(X) = b, \\ \mathcal{A}^*(y) + S = C, \\ X = \mu S^{-1} \end{aligned}$$

or on some equivalent system with the conditions $X^{(k)} \succ 0,\ S^{(k)} \succ 0$ enforced by the step size selection rules.
Given a pair of Slater points X, S for (P) and (D) respectively, we can easily measure how close (X, S) is to the central path.

One way would be $\mu(X,S) := Tr(XS)/n$ and consider

$$\|S - \mu(X,S)X^{-1}\|.$$

Another way, more directly relating to $f(\cdot)$ is:

$$\psi(X,S) := n \ln\left(\frac{Tr(XS)}{n}\right) + f(X) + f(S)$$

and if $X, S \in \mathbb{S}^n_{++}$, then $f(X) + f(S) = -\ln(\det(X)) - \ln(\det(S))$.

Theorem 3.9

For every $(X,S) \in \mathbb{S}^n_{++} \oplus \mathbb{S}^n_{++}$, $\psi(X,S) \geqslant 0$. Moreover, the equality holds iff $S = \mu X^{-1}$ with $\mu := \frac{\langle X,S\rangle}{n}$.

Note:

$$\begin{aligned}
\psi(X,S) =& n \ln\left[\frac{Tr(S^{\frac{1}{2}}XS^{\frac{1}{2}})}{n}\right] - \ln\det(S^{\frac{1}{2}}XS^{\frac{1}{2}}) \\
=& n \ln\left[\left(\sum_{j=1}^{n}\lambda_j\right)/n\right] - \ln\left(\prod_{j=1}^{n}\lambda_j\right) \\
=& n \ln\left(\frac{\text{arithmetic mean}(\lambda_1,\lambda_2,\ldots,\lambda_n)}{\text{geometrix mean}(\lambda_1,\lambda_2,\ldots,\lambda_n)}\right) \geqslant 0
\end{aligned}$$

where $\lambda := \lambda\left(S^{\frac{1}{2}}XS^{\frac{1}{2}}\right)$.

We will use two attributes for judging how good a pair of Slater points (X,S) is:

(i) want small duality gap $\langle X,S\rangle$.

(ii) want to be close to the central path (small $\psi(X,S)$).

3.3 Primal-Dual Potential function

$$\phi_q(X,S) := q\ln\langle X,S\rangle + \psi(X,S), \text{ where } q > 0$$

Theorem 3.10

Suppose $X^{(0)}, S^{(0)} \in \mathbb{S}^n_{++}$ are Slater points for (P), (D) respectively and they satisfy $\psi(X^{(0)}, S^{(0)}) \leqslant \sqrt{n}\ln(1/\varepsilon)$, for some $\varepsilon \in (0,1)$. If we generate a sequence $\left\{\left(X^{(k)}, S^{(k)}\right)\right\}$ of feasible solutions for $(P),(D)$ respectively such that $\phi_{\sqrt{n}}\left(X^{(k)}, S^{(k)}\right) \leqslant \phi_{\sqrt{n}}\left(X^{(k-1)}, S^{(k-1)}\right) - \delta$, for every $k \in \mathbb{Z}_{++}$, for some absolute constant $\delta > 0$, then for some $\overline{k} = O(\sqrt{n}\ln(1/\varepsilon))$, we have

$$\langle X^{(k)}, S^{(k)}\rangle \leqslant \varepsilon\langle X^{(0)}, S^{(0)}\rangle, \ \forall k \geqslant \overline{k}$$

We will design an algorithm that will have the

- property described in the assumptions of Theorem 42,
- primal-dual symmetry property,
- scale-invariance property.

Given the current iterate $(X^{(k)}, S^{(k)})$, we will find a pair of search directions D_X, D_S such that for all $\alpha \in \mathbb{R}_+$ (step size), $\left(X^{(k)} + \alpha D_X\right), \left(S^{(k)} + \alpha D_S\right)$ satisfy $\mathcal{A}(X) = b$ and $\mathcal{A}^*(y) + S = C$ (for some $y \in \mathbb{R}^m$). $\iff$ $\mathcal{A}(D_X) = 0, \mathcal{A}^*(d_y) + D_S = 0$ (for some $d_y \in \mathbb{R}^m$).
To achieve primal-dual symmetry and scale-invariance in an elegant way, for every pair $X, S \in \mathbb{S}^n_{++}$, we will find $T : \mathbb{S}^n \to \mathbb{S}^n$ linear such that

(i) $T \in Aut(\mathbb{S}^n_+)$,

(ii) $T(S) = T^{-1}(X) =: V$,

(iii) $T(X^{-1}) = T^{-1}(S^{-1}) = V^{-1}$.

Then, we transform the X-space via T^{-1}, S-space via T.

$$\begin{gathered}(X, S) \text{ gets mapped to } (V, V).\\ \overline{\mathcal{A}}(\cdot) := \mathcal{A}(T(\cdot)),\\ \overline{C} := T(C),\\ \overline{D}_X := T^{-1}(D_X),\\ \overline{D}_S := T(D_S).\end{gathered}$$

Then $(P), (D)$ become

$$\begin{aligned}&(\overline{P})\ \inf\langle \overline{C}, X\rangle \ s.t.\ \overline{\mathcal{A}}(X) = b,\ X \in T^{-1}(\mathbb{S}^n_+) = \mathbb{S}^n_+\\ &(\overline{D})\ \sup b^T y \ s.t.\ \overline{\mathcal{A}}^*(y) + S = \overline{C}, S \in T(\mathbb{S}^n_+) = \mathbb{S}^n_+\end{aligned}$$

Theorem 3.11

For every pair of $X, S \in \mathbb{S}^n_{++}$, $\exists T \in Aut(\mathbb{S}^n_+)$ such that

(i) $T(S) = T^{-1}(X) =: V$,

(ii) $T(X^{-1}) = T^{-1}(S^{-1}) = V^{-1}$.

Proof. Let $X, S \in \mathbb{S}^n_{++}$. We will find $W \in \mathbb{S}^n_{++}$ such that $T : \mathbb{S}^n \to \mathbb{S}^n$, $T(Z) := WZW$ satisfies the desired condition. Note that for such a W, $\forall z \in \mathbb{S}^n, T(Z) \in \mathbb{S}^n$ and $Z \in \mathbb{S}^n_+ \iff T(Z) \in \mathbb{S}^n_+$.

Therefore, $T \in Aut(\mathbb{S}^n_+)$.
Let us try to solve the equation $T(S) = T^{-1}(X)$. For our choice of T, this equation is:

$$\begin{aligned}
& WSW = W^{-1}XW^{-1} \\
\iff & W^2SW^2 = X \\
\iff & S^{\frac{1}{2}}W^2SW^2S^{\frac{1}{2}} = S^{\frac{1}{2}}XS^{\frac{1}{2}} \\
\iff & \left(S^{\frac{1}{2}}W^2S^{\frac{1}{2}}\right)^2 = S^{\frac{1}{2}}XS^{\frac{1}{2}} \\
\iff & S^{\frac{1}{2}}W^2S^{\frac{1}{2}} = \left(S^{\frac{1}{2}}XS^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
\iff & W^2 = S^{-\frac{1}{2}}\left(S^{\frac{1}{2}}XS^{\frac{1}{2}}\right)^{\frac{1}{2}}S^{-\frac{1}{2}}
\end{aligned}$$

We have $W \in \mathbb{S}^n_{++}$ such that

$$WSW = W^{-1}XW^{-1} =: V.$$

and

$$WSW = W^{-1}XW^{-1} = V \iff \underbrace{W^{-1}S^{-1}W^{-1}}_{T^{-1}(S^{-1})} = \underbrace{WX^{-1}W}_{T(X^{-1})} = V^{-1}$$

□

3.4 Finding a Good Search Direction

Recall, we

(i) want small duality gap $\langle X, S\rangle$.

(ii) want to be close to the central path (small $\psi(X,S) = n\ln(Tr(XS)/n) + f(X) + f(S)$).

Let $D_X, D_S \in \mathbb{S}^n$ denote the search directions.

$$\left.\begin{aligned} X(\alpha) &:= X + \alpha D_X \\ S(\alpha) &:= S + \alpha D_S \end{aligned}\right\} \alpha \in \mathbb{R}_{++}.$$

$$\begin{aligned}
\langle X(\alpha), S(\alpha)\rangle &= \langle X, S\rangle + \alpha[\langle X, D_S\rangle + \langle D_X, S\rangle] + \alpha^2 \underbrace{\langle D_X, D_S\rangle}_{=0} \\
&= \langle X, S\rangle + \alpha[\langle \underbrace{T^{-1}(X)}_{=V}, \underbrace{T(D_S)}_{=\overline{D}_S}\rangle + \langle \underbrace{T^{-1}(D_X)}_{=\overline{D}_X}, \underbrace{T(S)}_{=V}\rangle] \\
&= \langle X, S\rangle + \alpha\langle V, \overline{D}_X + \overline{D}_S\rangle
\end{aligned}$$

Idea: For the largest rate of decrease in the duality gap, take $\overline{D}_X + \overline{D}_S = -V$ (and $\overline{D}_X$ as the orthogonal projection of $-V$ onto $Null(\overline{\mathcal{A}})$, $\overline{D}_S$ as the orthogonal projection of $-V$ onto $Range(\overline{\mathcal{A}}^*)$).
It remains to control the distance to the central path.

Lemma 3.12

Let $X \in \mathbb{S}^n_{++}$, $D \in \mathbb{S}^n$ such that

$$\|D\|_X := \langle D, X^{-1}DX^{-1}\rangle^{\frac{1}{2}} \leqslant 1.$$

Then,

$$f(X) + \langle f'(X), D\rangle \leqslant f(X+D) \leqslant f(X) + \langle f'(X), D\rangle + \frac{\|D\|_X^2}{2(1-\|D\|_X)^2}$$

Note that the assumptions of the lemma, $X \in \mathbb{S}^n_{++}$, $D \in \mathbb{S}^n$, $\|D\|_X \leqslant 1$ imply

$$1 \geqslant \langle D, X^{-1}DX^{-1}\rangle^{\frac{1}{2}} = \left[Tr(X^{-\frac{1}{2}}DX^{-\frac{1}{2}})^2\right]^{1/2} = \left\|X^{-\frac{1}{2}}DX^{-\frac{1}{2}}\right\|_F.$$

This further implies $1 \geqslant \left\|X^{-\frac{1}{2}}DX^{-\frac{1}{2}}\right\|_2$. This is equivalent to

$$-I \preccurlyeq X^{-\frac{1}{2}}DX^{-\frac{1}{2}} \preccurlyeq I \iff X \mp D \succcurlyeq 0$$

(We applied $X^{\frac{1}{2}} \cdot X^{\frac{1}{2}} \in Aut(\mathbb{S}^n_+)$ to all sides).
Focusing on the first-order part of the estimate from Lemma 44 for $[f(X(\alpha)) + f(S(\alpha))]$, we compute

$$\begin{aligned}
&\langle f'(X), D_X\rangle + \langle f'(S), D_S\rangle = \langle -X^{-1}, D_X\rangle + \langle -S^{-1}, D_S\rangle \\
=& -\langle \underbrace{T(X^{-1})}_{=V^{-1}}, \underbrace{T^{-1}(D_X)}_{=\overline{D}_X} - \langle \underbrace{T^{-1}(S^{-1})}_{=V^{-1}}, \underbrace{T(D_S)}_{=\overline{D}_S} \\
=& -\langle V^{-1}, \overline{D}_X + \overline{D}_S\rangle.
\end{aligned}$$

So, we should set

$$\overline{D}_X + \overline{D}_S = \mathcal{K}_1 V^{-1} - \mathcal{K}_2 V$$

, for some $\mathcal{K}_1, \mathcal{K}_2 > 0$.
Setting $\mathcal{K}_1 := 1, \mathcal{K}_2 := \frac{n+\sqrt{n}}{\langle X,S\rangle}$ with a suitable choice for step size like $\alpha := \frac{\lambda_n(V)}{8}$ yields

$$\begin{aligned}
&\phi_{\sqrt{n}}(X(\alpha), S(\alpha)) - \phi_{\sqrt{n}}(X, S) < -\frac{1}{12}, \text{ an absolute constant, recall Theorem 42} \\
&\tilde{\mathcal{U}} := V^{-1} - \frac{n+\sqrt{n}}{\langle X, S\rangle}V, \\
&\mathcal{U} := \frac{\tilde{\mathcal{U}}}{\|\tilde{\mathcal{U}}\|_F}
\end{aligned}$$

where $\tilde{\mathcal{U}} = 0$ iff $\frac{n+\sqrt{n}}{\langle X,S\rangle}V = V^{-1}$ iff $\frac{n+\sqrt{n}}{\langle V,V\rangle}\langle V, V\rangle = \langle V^{-1}, V\rangle = Tr(I) = n$. Therefore, $\|\tilde{\mathcal{U}}\|_F > 0$.

3.5 A Primal-Dual Interio-Point Algorithm

Input: $X^{(0)}, S^{(0)} \in \mathbb{S}^n_{++}, \varepsilon \in (0,1)$ such that $X^{(0)}, S^{(0)}$ are feasible in $(P)\&(D)$ respectively, $\psi(X^{(0)}, S^{(0)}) \leqslant \sqrt{n}\ln(1/\varepsilon)$.

$$
\begin{aligned}
& k := 0 \\
& \quad \text{While} \langle X^{(k)}, S^{(k)}\rangle > \varepsilon \langle X^{(0)}, S^{(0)}\rangle, \\
& \quad\quad W^2 := (S^{(k)})^{-\frac{1}{2}}[(S^{(k)})^{\frac{1}{2}} X^{(k)} (S^{(k)})^{\frac{1}{2}}]^{\frac{1}{2}} (S^{(k)})^{-\frac{1}{2}} \\
& \quad\quad \overline{\mathcal{A}} := \mathcal{A}(W \cdot W), \qquad [\overline{A}_i := W A_i W, \forall i] \\
& \quad\quad V := W S^{(k)} W \\
& \quad\quad \tilde{\mathcal{U}} := V^{-1} - \frac{n + \sqrt{n}}{\langle X^{(k)}, S^{(k)}\rangle} V \\
& \quad\quad \mathcal{U} := \tilde{\mathcal{U}} / \|\tilde{\mathcal{U}}\|_F \\
& \quad\quad \text{Solve the linear system of equations} \\
& \quad\quad\quad \overline{\mathcal{A}}(\overline{D}_X) = 0 \\
& \quad\quad\quad \overline{\mathcal{A}}^*(d_y) + \overline{D}_S = 0 \\
& \quad\quad\quad \overline{D}_X + \overline{D}_S = \mathcal{U} \\
& \quad\quad \text{Compute} \\
& \quad\quad \overline{\alpha} := \arg\min \left\{ \phi_{\sqrt{n}}(X(\alpha), S(\alpha)) : \alpha > 0 \right\} \\
& \quad\quad X^{(k+1)} := X^{(k)} + \overline{\alpha} W \overline{D}_X W, \;\; S^{(k+1)} := S^{(k)} + \overline{\alpha} W^{-1} \overline{D}_S W^{-1} \\
& \quad\quad k := k+1 \\
& \quad \text{end\{while\}}
\end{aligned}
$$

Theorem 3.13

The above algorithm terminates in at most $24\sqrt{n}\ln(1/\varepsilon)$ $(O(\sqrt{n}\ln(1/\varepsilon)))$ with $X^{(k)}, S^{(k)}$ feasible in (P), (D) respectively and satisfying

$$\langle X^{(k)}, S^{(k)}\rangle \leqslant \varepsilon \langle X^{(0)}, S^{(0)}\rangle$$

How about the assumption that Slate points $X^{(0)}$, $S^{(0)}$ for $(P), (D)$ are given?
Introduce an artificial variable $\xi \geqslant 0$ and construct an auxiliary SDP.

$$
\begin{aligned}
(P_{aux}) \;\; & \inf \; \xi \\
& s.t. \; \mathcal{A}(X) + \xi(b - \mathcal{A}(I)) = b \\
& \quad\;\; \langle I, X\rangle \leqslant M \\
& \quad\;\; \xi \geqslant 0 \\
& \quad\;\; X \succcurlyeq 0
\end{aligned}
$$

where M is a large constant we pick, say at least $M > n$.
Then, $(X^{(0)}, \xi_0) := (I, 1)$ is a Slater point for (P_{aux}).

The dual of (P_{aux}) is

$$\begin{aligned}(D_{aux})\ \ &\sup\ b^T y + M\eta \\ &s.t.\ \mathcal{A}^*(y) + \eta I + S = 0 \\ &\qquad b^T y - Tr(A^*(y)) \leqslant 1 \\ &\qquad \eta \leqslant 0 \\ &\qquad S \succcurlyeq 0.\end{aligned}$$

Here, $(y^{(0)}, S^{(0)}, \eta_0) := (0, I, -1)$ is a Slater point for (D_{aux}).
For this pair,

$$\psi(\ldots,\ldots) = (n+1)\ln\left(\frac{M+1}{n+2}\right) - \ln(M-n)$$

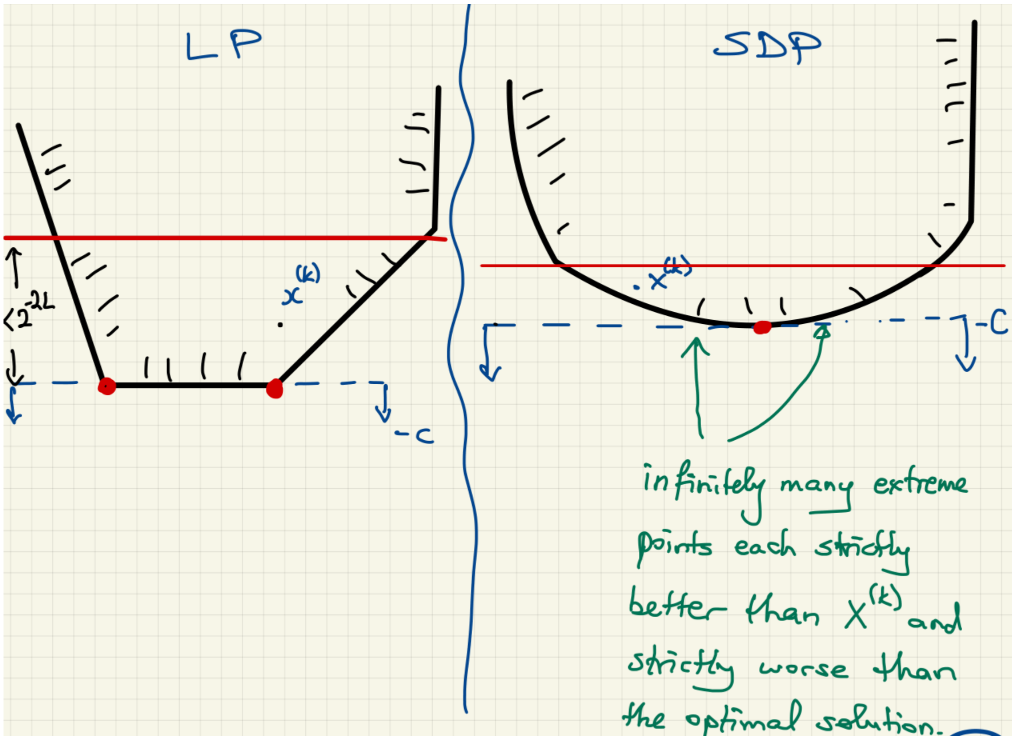
To find a Slater point for (P) via $(P_{aux}), (D_{aux})$, we need to further modify (P_{aux}) by picking anotehr constant γ (this time, tiny) and replace the constraint "$X \succcurlyeq 0$" by "$X \succcurlyeq \gamma I$".
SO, if we find a solution of this modified (P_{aux}) with $\xi = 0$, we have a Slater point for (P).
If the optimal value of ξ is positive, all we can say is

$$\{X \in \mathbb{S}^n : X \succcurlyeq \gamma I \text{ and } Tr(X) \leqslant M\}$$

does not contain any feasible solution of (P).
In the case of linear programming problems with rational data ($A \in \mathbb{Q}^{m\times n}, b \in \mathbb{Q}^m, c \in \mathbb{Q}^n$), we can pick $\gamma \approx 2^{-L}, M \approx 2^L$ where L is the number of bits required to express the data (A, b, c). Write each rational number as p/q where $p \in \mathbb{Z}, q \in \mathbb{Z}\setminus\{0\}$, p and q relatively prime and express p, q in binary.
However, we can construct instances of SDP with data only containing $0, 1$ and 2 where

$$\gamma \approx 2^{-2^n}\ (2^{-2^L}) \text{ and/or } M \approx 2^{2^n}\ (2^{2^L})$$

Linear Programming	SDP
If we have a feasible solution of (P) whose objective value is within 2^{-2L} of the optimum, then every extreme point of (P) with at least as good objective value is optimal, and we can compute an (exact) optimal solution very efficiently.	Given a feasibnle solution of (P) we can compute "in practice" an extreme point solution whose objective value is at least as good, but there may be infinitely many extreme point solutions of (P) that are strictly better.



Furthermore, SDP may have a unique optimal solution that is irrational.

Example

$$n := 2, m := 2, A_1 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, b := \begin{bmatrix} 2 \\ 1 \end{bmatrix}, C := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$(P)\ \inf\ 2X_{21}\ s.t.\ \begin{pmatrix} 2 & X_{21} \\ X_{21} & 1 \end{pmatrix} \succcurlyeq 0 \quad (D)\ \sup\ 2y_1 + y_2\ s.t.\ \begin{pmatrix} -y_1 & 1 \\ 1 & -y_2 \end{pmatrix} \succcurlyeq 0$$

$$\text{Opt.Soln:} \quad \begin{aligned} \overline{X} &:= \begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix} \text{ obj. value} = -2\sqrt{2}, \\ \overline{y} &:= -\begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2} \end{pmatrix}, \text{ obj. value} = -2\sqrt{2} \end{aligned}$$

SDP-Feasibility: Given $A_1, A_2, \ldots, A_m \in \mathbb{S}^n \cap \mathbb{Z}^{n\times n}$ and $b \in \mathbb{Z}^m$, does there exist $\overline{X} \in \mathbb{S}^n_+$ such that $\langle A_i, \overline{X}\rangle = b_i, \forall i \in \{1, 2, \ldots, m\}$?
Open Problem 3: Is SDP-Feasibility is $\mathcal{P}$?
In theoretical applications, Ellipsoid Method is very powerful.
Interior-point algorithms have better complexity bounds and in applications requiring high accuracy, if we can perform one iteration in a reasonable time, they are hard to beat.
When we can't even perform a single iteration of and interior-point algorithm (instance is huge and does not have easily exploitable structure), we resort to first-order algorithms (but not ellipsoid method). We will see some of these other first-order algorithms for SDP,

after we discuss some fundamental approximation algorithms based on SDP for some hard problems.

4 Approximation Algorithms Based on SDP

Recall, in many applications (engineering, big data, machine learning, statistics, computer science, other areas of mathematics, ...) we construct a mathematical model that might be too hard or impossible to solve exactly:

$$(P)\ \sup\ f(x)\ s.t.\ x \in \Phi$$

We can construct an SDP relaxation

$$\{(P)\ \sup\ f(X)\ X \in \Phi\} \leqslant \{\sup\langle C, X\rangle,\ \mathcal{A}(X) = b,\ X \succcurlyeq 0\} = \{\inf\ b^T y,\ \mathcal{A}^*(y) - S = C\ S \succcurlyeq 0\}$$

where the equality is under the assumptions of Strong Duality Theorem.
Suppose we have a fast algorithm (heuristic) which provides feasible solutions to (P) that usually have good objective function values. We run the heuristic and obtain $\hat{x} \in \Phi$, we also solve (SDD) approximately and obtain a feasible solution $(\overline{y}, \overline{S})$ of (SDD). Then,

$$f(\hat{x}) \leqslant v \leqslant b^T\overline{y}$$

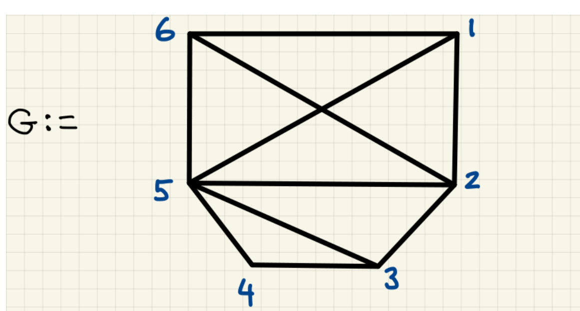
where v is the unknown optimal value of (P), and $b^T\overline{y}$ close to the optimal value of $(SDP)\&(SDD)$.
Next, we focus on a hard combinatorial optimization problem MaxCut (Maximum Cut): given an undirected graph $G = (V, E)$ and $w \in \mathbb{R}^E_+$, find $U \subseteq V$ such that

$$\sum_{\{i,j\} \in \delta(U)} w_{ij} \text{ is maximized,}$$

where $\delta(U) := \{\{i, j\} \in E : i \in U, j \in V \setminus U\}$.

Example

Suppose all weights are one.



1. $U_1 := \{1, 2, 3\} \to$ cut of weight 6
2. $U_2 := \{5\} \to$ cut of weight 5
3. $U_3 := \{2, 5\} \to$ cut of weight 7

4.1 A formulation of MaxCut as a nonconvex optimization problem:

With $n := |V|$, let's represent each cut $(U, V \setminus U)$ by a $u \in \{-1, 1\}^n$.

$$u_i := \begin{cases} 1, & \text{if } i \in U, \\ -1, & \text{if } i \notin U. \end{cases}$$

Also, set $w_{ij} = 0, \forall \{i, j\} \notin E$. Then MaxCut Problem:

$$(P) \;\max \frac{1}{4} \sum_{i \in V} \sum_{j \in V} w_{ij}(1 - u_i u_j) \; s.t. \; u \in \{-1, 1\}^n$$

4.1.1 Another, equivalent nonconvex formultaion:

Define $W \in \mathbb{S}^n$ by $W_{ij} := w_{ij}, \forall i, j$.

$$\begin{aligned} \max & \frac{1}{4}\langle W, \overline{e}\overline{e}^T \rangle - \frac{1}{4}\langle W, X \rangle \longleftarrow \text{ linear} \\ s.t.\, & \operatorname{diag}(X) = \overline{e} \\ & X \succcurlyeq 0 \\ & rank(X) = 1 \end{aligned}$$

Note except the SDP constraints, we have $rank(X) = 1$ which is not convex.
To go between this nonconvex optimization problem and (P), use

$$X \leftrightarrow uu^T.$$

4.1.2 SDP relaxation and its dual:

$$\begin{aligned} (SDP) \;\max \; & -\frac{1}{4} Tr(WX) \; \left(+\frac{1}{4}\overline{e}^T W \overline{e}\right) \\ s.t. \; & \operatorname{diag}(X) = \overline{e}, \\ & X \succcurlyeq 0 \\ (SDD) \;\min \; & \overline{e}^T y \; \left(+\frac{1}{4}\overline{e}^T W \overline{e}\right) \\ s.t. \; & \operatorname{Diag}(y) - S = -\frac{1}{4} W, \\ & S \succcurlyeq 0 \end{aligned}$$

$\overline{X} := I, \overline{y} := \eta \overline{e}$, where $\eta := \frac{1}{4}\overline{e}^T W \overline{e} + 1$ yield Slater points for (SDP) and (SDD).

4.1.3 Goemans-Williamson Approximation Algorithms and Analysis:

If we find an exact optimal solutions of (SDP), $\hat{X}$ such that $rank(\hat{X}) = 1$, then we are done! (We have an optimal solution $\hat{u}$ of (P) and thus, of MaxCut).

$$\hat{X} =: BB^T, \text{ where } B^T =: [v^{(1)}, v^{(2)}, \ldots, v^{(n)}], v^{(i)} \in \mathbb{R}^d, d \leqslant n$$

so we have

$$\begin{cases} \hat{X}_{ij} = \langle v^{(i)}, v^{(j)} \rangle, & \forall i,j \\ \text{and } 1 = \hat{X}_{ii} = \langle v^{(i)}, v^{(i)} \rangle, & \forall i \end{cases}$$

4.1.4 Random Hyperplane Technique

Generate $r \in \mathbb{R}^d$ on the unit hyperplane randomly. Then setting $U := \{i \in V : r^T v^{(i)} \geqslant 0\}$ defines a cut in G.

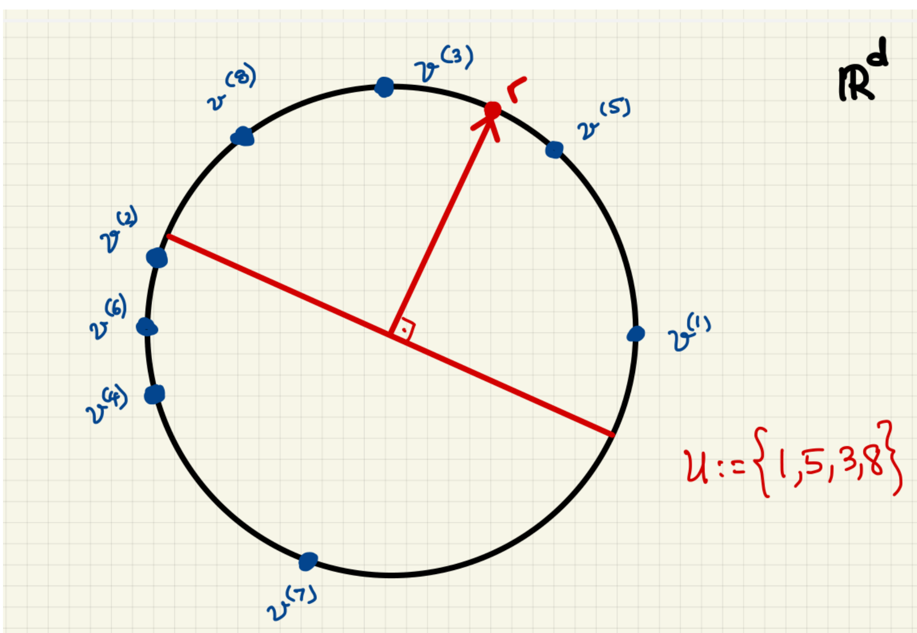
Lemma 4.1

Let $v^{(i)}$ and r be as above. Then,

$$Prob\left\{sign(r^T v^{(i)}) \neq sign(r^T v^{(j)})\right\} = \frac{\theta}{\pi},$$

where $\theta := \arccos\langle v^{(i)}, v^{(j)} \rangle$.

For $v \in \mathbb{R}^d, sign(v) \in \{-1,1\}^d : [sign(v)]_j := \begin{cases} 1, & \text{if } v_j \geqslant 0, \\ -1, & \text{if } v_j < 0. \end{cases}$



Lemma 4.2

For every $u \in [-1, 1]$, we have

$$\frac{1}{\pi} \arccos(u) \geqslant \frac{\rho}{2}(1-u),$$

and

$$1 - \frac{1}{\pi} \arccos(u) \geqslant \frac{\rho}{2}(1+u)$$

where $\rho \approx 0.87856$.

Theorem 4.3

The expected weight of the cut generated by the Random Hyperplane Technique based on $\hat{X}$ is at least

$$\frac{\rho}{4} \sum_{i \in V} \sum_{j \in V} w_{ij} \left(1 - \langle v^{(i)}, v^{(j)} \rangle\right) = \rho \cdot \mathit{opt}(SDP)$$

Proof. Let $\hat{X} \in \mathbb{S}^n_+$ be an optimal solution of (SDP).

$$\begin{aligned}
\mathit{opt}(SDP) &= \frac{1}{4} \sum_i \sum_j W_{ij}(1 - \hat{X}_{ij}) = \frac{1}{4} \sum_i \sum_j w_{ij}(1 - \langle v^{(i)}, v^{(j)} \rangle). \\
E(\text{RHT-cut}) &= \sum_i \sum_j w_{ij} \frac{\arccos(\langle v^{(i)}, v^{(j)} \rangle)}{2\pi} \\
&\geqslant \frac{\rho}{4} \sum_i \sum_j w_{ij}(1 - \langle v^{(i)}, v^{(j)} \rangle) \\
&= \rho \cdot \mathit{opt}(SDP)
\end{aligned}$$

$\square$

Theorem 4.4

Let $G = (V, E)$ with $w \in \mathbb{Q}^E_+$ be given. Then a cut of value at least $\rho \cdot$ (optimal value of the MaxCut) can be obtained in polynomial time.

Note that we do <u>not</u> need an exact optimization solution of (SDP), an approximate solution $\tilde{X}$ in place of $\hat{X}$ would work.
The algorithm can be "derandomized."
This approximation ratio is the best possible (polynomial time approximation algorithms for MaxCut) unless the "Unique Games Conjecture" is false.

4.2 Maximum Satisfiability (MaxSat) Problem

- <u>Boolean variables:</u>$x_1, x_2, \ldots, x_n \in \{0, 1\}$

- Literals:$x_i, \overline{x}_i$ (complement of x_i)
- Clauses:conjunction of a subset of literals e.g. $(x_3 \vee \overline{x}_4 \vee x_1)$.
- (satisfiability) Formula: disjunction of the clauses e.g. $(x_3 \vee \overline{x}_4 \vee x_1) \wedge (x_2) \wedge (x_1 \vee \overline{x}_2)$

4.2.1 Satisfiability Problem (SAT)

Given a formula as above, decide whether $\exists$ an assignment of values to the varaibles so that the formula evalutates to "True".
An Integer Programming (feasibility) formulation: Suppose the given formula is $C_1 \wedge C_2 \wedge \ldots \wedge C_m$.

$$\sum_{j: x_j \in C_i} x_j + \sum_{j: \overline{x}_j \in C_i} (1 - x_j) \geqslant 1, \forall i \in \{1, 2, \ldots, n\}, x \in \{0, 1\}^n$$

MaxSat: Given a Boolean formula $C_1 \wedge C_2 \wedge \ldots \wedge C_m$ and weights on the clauses $w_i \in \mathbb{R}_+, i \in \{1, 2, \ldots, m\}$, find an assignment of values to the variables which maximizes the total weight of satisfied clauses.
Note: A given formula is satisfiable iff $\forall w \in \mathbb{R}^m_+$ the corresponding MaxSat instance has the optimal value $\overline{e}^T w$.

- K-SAT: Satisfiability problem where every clause has at most k literals.
- Max k-Sat: MaxSat problem where every clause has at most k literals.

Theorem 4.5

For every $k \geqslant 3$, k-Sat is NP-complete.
For every $k \geqslant 2$, Max k-Sat is NP-hard.

Max 2-Sat is closely related to MaxCut:
Let $G = (V, E)$ be a given instance of MaxCut, assume every edge has weight one.
Make a variable $x_v, \forall v \in V$,
make a clause $(x_u \vee x_v), \forall \{u, v\} \in E, w_{uv} := 2$,
make a clause $(\overline{x}_v), \forall v \in V, w_v := |\delta(v)|$.
Then $opt(Max2 - Sat) = opt(MaxCut) + 2|E|$.
Approximation results for MaxCut extend to Max 2-Sat. Can we extend them to more general nonconvex optimization problems? Yes!

4.3 Quadratic Optimization over Sign Vectors

Let $W \in \mathbb{S}^n$.

$$\overline{f}(W) := \max_{x\in\{-1,1\}^n} x^T W x = \max Tr(WX) \ s.t. \begin{cases} \text{diag}(X) = \overline{e}, \\ X \succcurlyeq 0, \\ rank(X) = 1. \end{cases}$$

$$\underline{f}(W) := \min_{x\in\{-1,1\}^n} x^T W x = \min Tr(WX) \ s.t. \begin{cases} \text{diag}(X) = \overline{e}, \\ X \succcurlyeq 0, \\ rank(X) = 1. \end{cases}$$

SDP relaxations:

$$\overline{F}(W) := \max Tr(WX) \ s.t. \begin{cases} \text{diag}(X) = \overline{e}, \\ X \succcurlyeq 0. \end{cases} = \min \overline{e}^T y \ \ \text{Diag}(y) \succcurlyeq W$$

$$\underline{F}(W) := \min Tr(WX) \ s.t. \begin{cases} \text{diag}(X) = \overline{e}, \\ X \succcurlyeq 0. \end{cases} = \max \overline{e}^T y \ \ \text{Diag}(y) \preccurlyeq W$$

where the two equalities are from the Corollary of the Strong Duality Theorem (both (P) and (D) have Slater points).

Proposition 4.6

For every $W \in \mathbb{S}^n$, we have

$$\begin{aligned} &\underline{f}(W) = -\overline{f}(-W), \\ &\underline{F}(W) = -\overline{F}(-W), \text{ and} \\ &\underline{F}(W) \leqslant \underline{f}(W) \leqslant \overline{f}(W) \leqslant \overline{F}(W) \end{aligned}$$

We can apply the Random Hyperplane Technique here!

Lemma 4.7

Let $W \in \mathbb{S}^n$. Then

$$\overline{f}(W) = \max \xi^T W \xi \ s.t. \begin{cases} \xi = sign(Br) \\ \|B^T e_i\|_2 = 1, \forall i \\ \|r\|_2 = 1 \\ B \in \mathbb{R}^{n\times n}, r \in \mathbb{R}^n \end{cases}$$

$$= \max E_r(\xi^T W \xi) \ s.t. \begin{cases} \xi = sign(Br) \\ \|B^T e_i\|_2 = 1, \forall i \\ \|r\|_2 = 1 \\ B \in \mathbb{R}^{n\times n}, r \in \mathbb{R}^n \end{cases}$$

Proof. First equations:" $\geqslant$ " is clear since $\xi \in \{-1,1\}^n$ for every feasible solution.
" $\leqslant$ ": Let $\hat{x} \in \{-1,1\}^n$ such that $\overline{f}(W) = \hat{x}^T W \hat{x}$. Pick any $r \in \mathbb{R}^n$ with $\|r\|_2 = 1$. Define $B \in \mathbb{R}^{n\times n}$ by

$$B^T e_i := \begin{cases} r, & \text{if } \hat{x}_i = 1 \\ -r, & \text{if } \hat{x}_i = -1. \end{cases}$$

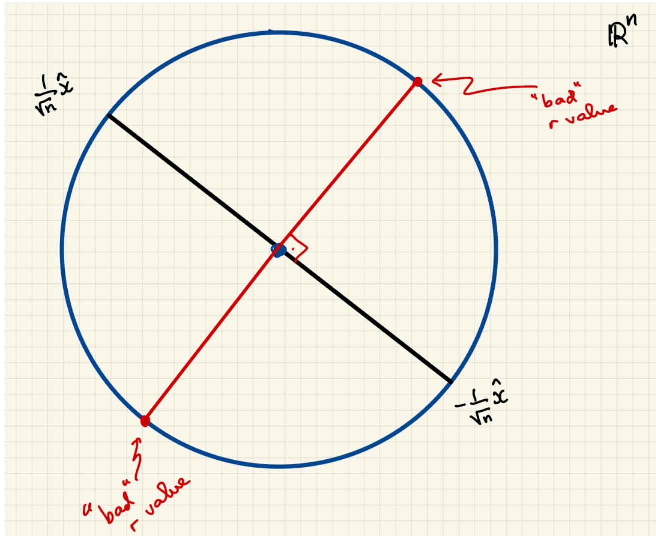
Then $\xi = \hat{x}$.
Second equation: " $\geqslant$ " is clear, since we are taking an expectation of the objective value over all possible choices for r and B.
" $\leqslant$ " Let $\hat{x} \in \{-1,1\}^n$ such that $\overline{f}(W) = \hat{x}^T W \hat{x}$.
Define $B \in \mathbb{R}^{n\times n}$ by

$$B^T e_i := \begin{cases} \frac{1}{\sqrt{n}}\hat{x}, & \text{if } \hat{x}_i = 1 \\ -\frac{1}{\sqrt{n}}\hat{x}, & \text{if } \hat{x}_i = -1. \end{cases}$$

Then,

$$E_r\left[sign(r^T B^T e_i) \cdot sign(r^T B^T e_j)\right] = \begin{cases} 1, & \text{if } \hat{x}_i = \hat{x}_j \\ -1, & \text{if } \hat{x}_i \neq \hat{x}_j \end{cases} = \hat{x}_i \hat{x}_j, \forall i, j$$

for the first equality, we used the fact that $\{r \in B_n(0,1) : r \text{ is orthogonal to both } B^T e_i \text{ and } B^T e_j\}$ has zero $(n-1)$-dimensional measure.
Therefore, $E_r(\xi^T W \xi) = \sum_i \sum_j W_{ij} \hat{x}_i \hat{x}_j = \hat{x}^T W \hat{x} = \overline{f}(W)$. $\square$



Lemma 4.8

For every $W \in \mathbb{S}^n$,

$$\overline{f}(W) = \max \frac{2}{\pi}\langle W, \arcsin(X)\rangle$$
$$s.t.\ \operatorname{diag}(X) = \overline{e}$$
$$X \succcurlyeq 0$$

Note that

$$\begin{aligned} &\max x^T W x,\ x \in \{-1,1\}^n \\ =&\max Tr(WX),\ s.t.\ \operatorname{diag}(X) = \overline{e},\ X \succcurlyeq 0,\ rank(X) = 1 \\ =&\max \frac{2}{\pi} Tr(W \arcsin(X)),\ \operatorname{diag}(X) = \overline{e},\ X \succcurlyeq 0 \end{aligned}$$

Proof. Since $X \succcurlyeq 0$, $\operatorname{diag}(X) = \overline{e}$ imply $|X_{ij}| \leqslant 1$, the optimization problem is well-defined. Feasible region is nonempty and compact, the objective function is continuous over the feasible region. Therefore, the maximum is finite and is attained.
Given W, apply Lemma 52. Then

$$\overline{f}(W) = \max\left\{E_r(\xi^T W \xi) : \xi = sign(Br), \|B^T e_i\|_2 = 1, \forall i, \|r\|_2 = 1, B \in \mathbb{R}^{n\times n}, r \in \mathbb{R}^n\right\}$$

Let $\hat{B} \in \mathbb{R}^{n\times n}$ be an optimal solution of this last problem.

$$\hat{B}^T =: [v^{(1)}\ v^{(2)}\ \ldots\ v^{(n)}]$$

Then,

$$\begin{aligned}
&E_r[sign(r^T\hat{B}^Te_i)\cdot sign(r^T\hat{B}^Te_j)]\\
=&Prob\left\{sign\langle r, v^{(i)}\rangle = sign\langle r, v^{(j)}\rangle\right\} - Prob\left\{sign\langle r, v^{(i)}\rangle \neq sign\langle r, v^{(j)}\rangle\right\}\\
=&1-2Prob\left\{sign\langle r, v^{(i)} \neq sign\langle r, v^{(j)}\rangle\right\}\\
=&1-\frac{2}{\pi}\arccos\langle v^{(i)}, v^{(j)}\rangle = \frac{2}{\pi}\arcsin\langle v^{(i)}, v^{(j)}\rangle, \forall v, j. \text{ by Lemma 46}
\end{aligned}$$

Thus, the optimal objective value is

$$E_r\left[[sign(\hat{B}r)]^TW[sign(\hat{B}r)]\right] = \frac{2}{\pi}\langle W, \arcsin(\hat{B}\hat{B}^T)\rangle.$$

Since $\tilde{X} := \hat{B}\hat{B}^T$ satisfies $\operatorname{diag}(\tilde{X}) = \bar{e}, \tilde{X} \succcurlyeq 0$, we proved " $\leqslant$ ".
" $\geqslant$ " Let $\overline{X} \in \mathbb{S}^n_+$ be an optimal solution of $\max\left\{Tr(W\arcsin(X)) : \operatorname{diag}(X) = \bar{e}, X \succcurlyeq 0\right\}$. Let $\overline{B} \in \mathbb{R}^{n\times n}$, $\overline{X} =: \overline{B}\overline{B}^T$. We have

$$\frac{2}{\pi}Tr(W\arcsin(\overline{X})) = E_r\left[[sign(\overline{B}r)]^TW[sign(\overline{B}r)]\right]$$

Using Lemma 52, we have the desired inequality. □

Lemma 4.9

For every $X \in \mathbb{S}^n_+$ such that $|X_{ij}| \leqslant 1, \forall i, j$, we have

$$\arcsin(X) \succcurlyeq X.$$

Proof.

$$\arcsin(X) = X + \frac{1}{2\cdot 3}X\odot X\odot X + \frac{1\cdot 3}{2\cdot 4\cdot 5}X\odot X\odot X\odot X\odot X + \dots$$

and recall Assignment 1. □

Theorem 4.10

For every $W \in \mathbb{S}^n_+$, we have

$$\begin{aligned}
\overline{f}(W) \geqslant \max\ &\frac{2}{\pi}\langle W, X\rangle\\
s.t.\ &\begin{cases}\operatorname{diag}(X) = \bar{e}\\ X \succcurlyeq 0\end{cases}
\end{aligned}$$

$\forall W \in \mathbb{S}^n_+$,

$$\frac{2}{\pi}\overline{F}(W) \leqslant \overline{f}(W) \leqslant \overline{F}(W)$$

The last theorem assume $W \in \mathbb{S}^n_+$, which includes MaxCut instances as a special case: Given graph $G = (V, E)$ and $W \in \mathbb{R}^E_+$, define $W \in \mathbb{S}^V$ as the Laplacian of G with respect to

weight w.
(See page 105 of the textbook). Note that W is <u>diagonally dominant</u>, hence by Gershgorin Disk Theorem (Theorem 1.12 in the textbook), $W \succcurlyeq 0$.

What if W is <u>not</u> PSD?

4.3.1 Arbitrary $W \in \mathbb{S}^n$ (not necessarily p.s.d)

In SDP relaxation defining $\underline{F}(W)$ amd $\overline{F}(W)$ we have the dual constraints $[W - \text{Diag}(y)] \succcurlyeq 0$ and $[\text{Diag}(y) - W] \succcurlyeq 0$ respectively. Moreover, we can make <u>any</u> $W \in \mathbb{S}^n$ positive semidefinite by adding to it a diagonal matrix.
This motivates investigating changes to $\underline{f}, \overline{f}, \underline{F}, \overline{F}$ under diagonal perturbations.
Let $y \in \mathbb{R}^n$ be given. Then,

$$\begin{aligned}
\underline{f}(W + \text{Diag}(y)) &= \min_{x \in \{-1,1\}^n} \left\{ x^T W x + x^T \text{Diag}(y) x \right\} \\
&= \min_{x \in \{-1,1\}^n} \left\{ x^T W x + \underbrace{\sum_{i=1}^{n} y_i \underbrace{x_i^2}_{=1}}_{\text{constant}} \right\} \\
&= \min_{x \in \{-1,1\}^n} \{x^T W x\} + \bar{e}^T y \\
&= \underline{f}(W) + \bar{e}^T y
\end{aligned}$$

Similarly, $\overline{f}(W + \text{Diag}(y)) = \overline{f}(W) + \bar{e}^T y$.

$$\begin{aligned}
\underline{F}(W + \text{Diag}(y)) &= \min_{\substack{X \succcurlyeq 0 \\ \text{diag}(X) = \bar{e}}} \left\{ Tr(WX) + \underbrace{\langle \text{Diag}(y), X \rangle}_{y^T \text{diag}(X)} \right\} \\
&= \underline{F}(W) + \bar{e}^T y
\end{aligned}$$

Similarly, $\overline{F}(W + \text{Diag}(y)) = \overline{F}(W) + \bar{e}^T y$.

Theorem 4.11

For every $W \in \mathbb{S}^n$, we have

$$\begin{aligned}
\underline{F}(W) \leqslant \underline{f}(W) &\leqslant \frac{2}{\pi} \underline{F}(W) + \left(1 - \frac{2}{\pi}\right) \overline{F}(W) \\
&\leqslant \left(1 - \frac{2}{\pi}\right) \underline{F}(W) + \frac{2}{\pi} \overline{F}(W) \leqslant \overline{f}(W) \leqslant \overline{F}(W)
\end{aligned}$$

Corollary 4.12

For every $W \in \mathbb{S}^n$, the value $v := \left(1 - \frac{2}{\pi}\right) \underline{F}(W) + \frac{2}{\pi}\overline{F}(W)$ satisfies

$$\frac{\overline{f}(W) - v}{\overline{f}(W) - \underline{f}(W)} < \frac{4}{7}$$

Proof. (for Theorem 56)
Let $\overline{y} \in \mathbb{R}^n$ be an optimal solution to the dual of the SDP relaxation determining $\overline{F}(W)$:

$$\begin{aligned}\overline{F}(W) &= \max\left\{Tr(WX) : \operatorname{diag}(X) = \overline{e}, X \in \mathbb{S}^n_+\right\} \\ &= \min\left\{\overline{e}^T y : \operatorname{Diag}(y) \succcurlyeq W\right\} = \overline{e}^T\overline{y},\end{aligned}$$

and $\operatorname{Diag}(\overline{y}) - W \succcurlyeq 0$.

$$\begin{aligned}\overline{F}(W) - \underline{f}(W) &= \overline{e}^T\overline{y} + \overline{f}(-W) \text{ Defn of } \overline{y}\text{, strong duality, Prop51} \\ &= \overline{f}(\operatorname{Diag}(\overline{y}) - W) \text{ Diagonal perturbation property} \\ &\geqslant \frac{2}{\pi}\overline{F}(\operatorname{Diag}(\overline{y}) - W) \text{ } \operatorname{Diag}(\overline{y}) - W \succcurlyeq 0\text{, Theorem 55} \\ &= \frac{2}{\pi}[\overline{e}^T\overline{y} - \underline{F}(W)] \text{ Diagonal perturbation, Prop51} \\ &= \frac{2}{\pi}[\overline{F}(W) - \underline{F}(W)] \text{ Definition of } \overline{y}\text{, strong duality}\end{aligned}$$

Therefore,

$$\underline{f}(W) \leqslant \frac{2}{\pi}\underline{F}(W) + \left(1 - \frac{2}{\pi}\right)\overline{F}(W).$$

Similarly, defining $\hat{y} \in \mathbb{R}^m$ to be an optimal solution of the dual of the SDP describing $\underline{F}(W)$, we can prove

$$\overline{f}(W) \geqslant \left(1 - \frac{2}{\pi}\right)\underline{F}(W) + \frac{2}{\pi}\overline{F}(W).$$

The remaining inequalities are elementary. □

Now that we can handle any $W \in \mathbb{S}^n$, this also allows us to handle linear terms in the objective function. Suppose $W \in \mathbb{S}^n$, $w \in \mathbb{R}^n$ are given. We have

$$\begin{aligned}\max_{x \in \{-1,1\}^n}\left\{2w^T x + x^T W x\right\} &= \max_{\substack{x \in \{-1,1\}^n \\ x_0 \in \{-1,1\}}}\left\{2x_0 w^T x + x^T W x\right\} \\ &= \max_{[x_0 \ \ x]^T \in \{-1,1\}^{n+1}}\left\{\begin{bmatrix} x_0 & x^T\end{bmatrix}\begin{bmatrix} 0 & w^T \\ w & W\end{bmatrix}\begin{bmatrix} x_0 \\ x\end{bmatrix}\right\}\end{aligned}$$

4.4 Burer-Monteiro Approach for Solving MaxCut SDPs (and generalization)

This is a first-order algorithm which has good practical performance in some very large-scale instances. It involves a avery simple nonconvex reformulation.

$$\max Tr(WX),\ \text{diag}(X) = \bar{e},\ X \succcurlyeq 0 = (P_\triangle)\ \max Tr(WLL^T),\ \text{diag}(LL^T) = \bar{e},\ (L \in \mathbb{T}^n)$$

where $\mathbb{T}^n$: The space of n-by-n lower triangular matrices.
Note that:

- any $L \in \mathbb{T}^n$ with no zero rows can be made feasible for $(P_\triangle)$ by simply scaling each row by its 2-norm.
- we can restrict L to $\mathbb{T}^{n,r}$ (lower triangular matrices that n-by-r, $r < n$. E.g. $\begin{bmatrix} l_1 & 0 \\ l_2 & l_{n+1} \\ \vdots & \vdots \\ l_n & l_{2n-1} \end{bmatrix} \in \mathbb{T}^{n,2}$). This way we are automatically restricting for $X := LL^T, rank(X) = rank(L) \leqslant r$.

Once we choose r, we can easily construct $L^{(0)} \in \mathbb{T}^{n,r}$ such that $\text{diag}(L^{(0)}(L^{(0)})^T) = \bar{e}$.
Then, in each iteration k, we compute the gradient of the objective function at $L^{(k-1)}$ project this gradient so that a linearization of the constraints is satisfied:

$$\text{diag}\left((L^{(k-1)} + d_L)(L^{(k-1)} + d_L)^T\right) = \bar{e}$$

Ignore the quadratic term in d_L.

$$\text{diag}(L^{(k-1)} d_L^T + d_L {L^{(k-1)}}^T) = 0$$

So, this projected gradient determines the search direction d_L. Then choose a step size $\alpha > 0$ (satisfying Armijo-Goldstein-Wolfe conditions, or similar) for the objectice function.

$$L^{(k)} := L^{(k-1)} + \alpha d_L.$$

Scale the rows of $L^{(k)}$ so that every row has 2-norm equal to one (i.e., $\text{diag}(L^{(k)} {L^{(k)}}^T) = \bar{e}$). There are very many first-order algorithms for solving the SDP relaxation of MaxCut problem, as well as general SDPs. Among others, consider bundle methods, mulplicative weights based methods, proximal point algorithms. Also consider the software: SDPNAL+

5 Geometric Representations of Graphs via SDPs

Let $G = (V, E)$ be an undirected graph.

Definition 5.1

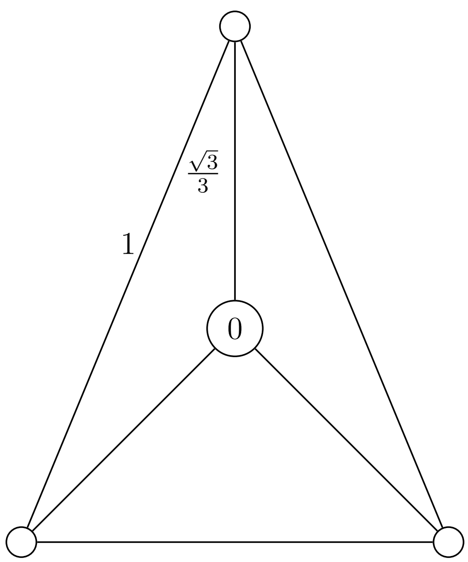
A map $u : V \mapsto \mathbb{R}^d$ for some nonnegative integer d, is called a geometric representation of G.

Definition 5.2

A geometric representation u of G is called a unit-distance representation of G, if

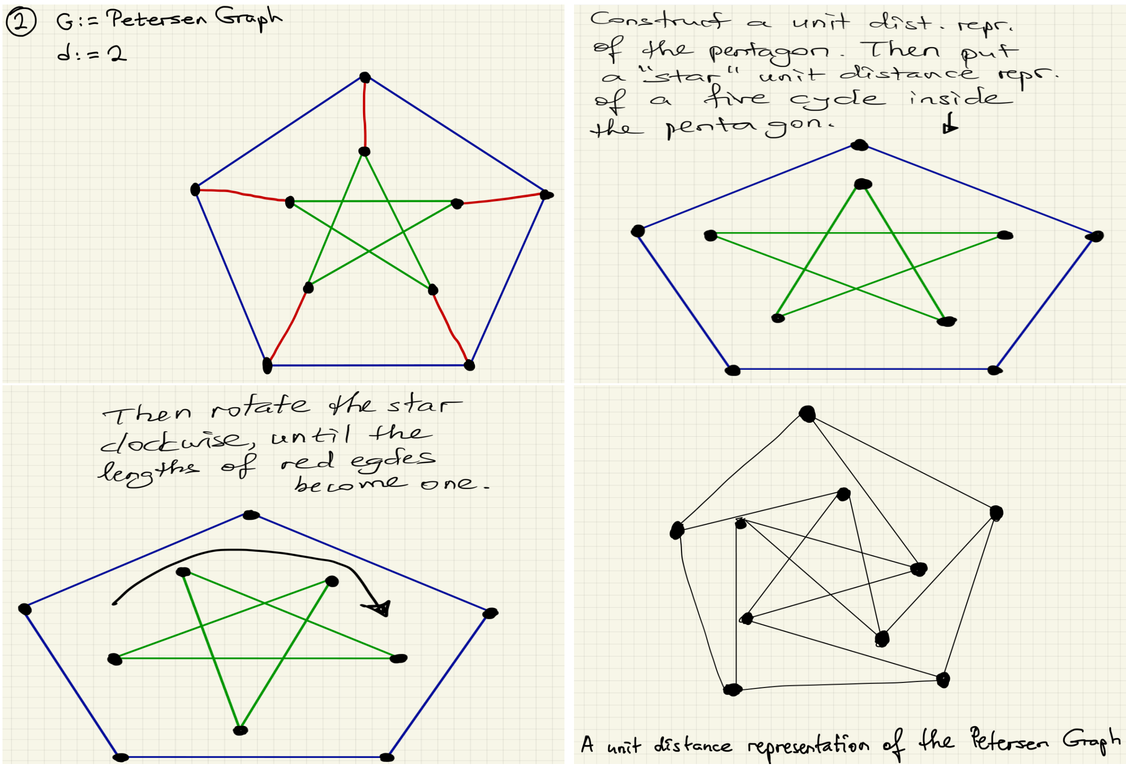
$$\|u(i) - u(j)\|_2 = 1, \forall \{i, j\} \in E$$

Example

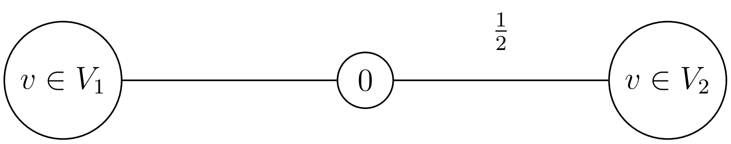


So here $G :=$ 3-cluque, $d := 2$. This unit distance representation is contained in a ball of radiu $\frac{\sqrt{3}}{3}$.

We can also have



or we consider bi partite graphs, $d := 1$



where $[V_1, V_2]$ is a bipartition of V.
Do unit distance representations exist for every graph G?

Theorem 5.3

Every graph $G = (V, E)$ admits a unit distance representation in $\mathbb{R}^{n-1}$, where $n := |V|$.

Proof. It suffices to prove that for every positive integer n, the clique on n vertices admits a unit distance representation in $\mathbb{R}^{n-1}$.
Embed the n-clique as the vertices of a simplex in $\mathbb{R}^{n-1}$ where every edge of the simnplex is of unit length $\square$

Our first example above has $n = 3$ and the 3-clique.

Geometric representations of graphs have an amazing range of applications:
Graph embeddability, graph realization, matrix completion, molecular confirmation, structural engineering (tensegrity theory), dimensionality reduction, data sparsification, computer

vision, data clustering, multi-class learning, signal processing, coding theory, communication complexity, recommender systems, outlier detection, some combinatorial packing and covering problems, ...
We can $\dim(G) := \min d \in \mathbb{Z}_+$, for which G admits a unit distance representation in $\mathbb{R}^d$.

Theorem 5.4

Deciding whether $\dim(G) \leqslant 2$ is $\mathcal{NP}$-hard.

Consider instead, computing a unit distance representation of G which is contained in an Euclidean Ball with smallest possible radius. (Recall the first example in this subsection again). Let $t_b(G)$ dentoe the square of this minimum radiu.

Theorem 5.5

For every graph $G = (V, E)$,

$$t_b(G) = \min\ t,\ s.t. \begin{cases} X_{ii} \leqslant t, & \forall i \in V; \\ X_{ii} + X_{jj} - 2X_{ij} = 1, & \forall \{i,j\} \in E; \\ X \in \mathbb{S}^V_+. \end{cases}$$

Next, consider computing a unit distance representation of G contained in a hypersphere of minimum radius. Let $t_h(G)$ denote the square of this minimum radius.
Let discuss some of the main ingredients for a proof.

- Construct Slater points for the SDP in the theorem statement and its dual SDP. Then use a corollary of the Strong Duality Theorem.
- Given an optimal solution $\hat{X}$ of the SDP, define $\hat{X} =: BB^T (B \in \mathbb{R}^{n\times k}, k \leqslant n-1)$ and then $B^T =: \begin{bmatrix} u(1) & u(2) & \dots & u(n) \end{bmatrix}$, where $u(i) \in \mathbb{R}^k, \forall i \in V$. Then $\langle u(i), u(i) \rangle = \hat{X}_{ii}, \forall i \in V$,
$$\|u(i) - u(j)\|_2^2 = \hat{X}_{ii} + \hat{X}_{jj} - 2\hat{X}_{ij}, \forall \{i,j\} \in E.$$
- Given a unit distance representation $u : V \mapsto \mathbb{R}^k$ $(k \leqslant n)$, $B^T =: \begin{bmatrix} u(1) & u(2) & \dots & u(n) \end{bmatrix}$, $\tilde{X} := BB^T$, ...

Theorem 5.6

For every graph $G = (V, E)$,

$$t_h(G) = \min\ t,\ s.t. \begin{cases} \operatorname{diag}(X) = t\bar{e}, \\ X_{ii} + X_{jj} - 2X_{ij} = 1, \quad \forall \{i,j\} \in E \\ X \in \mathbb{S}^V_+. \end{cases}$$

Moreover, $t_h(G) = t_b(G)$.

These two SDPs provide exact mathematical models for their respective problems (not relaxations or approximations). Moreover, we may use (for other applications) $X_{ii}+X_{jj}-2X_{ij} = l_{ij}, \{i,j\} \in E$, for any $l \in \mathbb{R}^E_+$ given.
Let us prove the following statement

$$\text{For every graph} G=(V,E),\ t_b(G) \leqslant t_h(G) \leqslant \frac{1}{2}-\frac{1}{2|v|} < \frac{1}{2}.$$

For every graph G, every unit distance representation of G that is contained in a hypersphere of radius r, is also contain in an Euclidean Ball of radius r. Thus,

$$t_b(G) \leqslant t_h(G).$$

Let $G=(V,E), n := |V|$. Then $t_h(G) \leqslant t_h(\text{n-clique})$. For every $\varepsilon > 0$ consider

$$\begin{aligned}
&X(\varepsilon) := \frac{1}{2}I - \varepsilon\overline{e}\overline{e}^T, t(\varepsilon) := \frac{1}{2}-\varepsilon. \text{ Then,}\\
&[X(\varepsilon)]_{ii} = t(\varepsilon), \forall i \in V \text{ and}\\
&[X(\varepsilon)]_{ii} + [X(\varepsilon)]_{jj} - 2[X(\varepsilon)]_{ij} = \frac{1}{2}-\varepsilon+\frac{1}{2}-\varepsilon+2\varepsilon = 1, \forall i \neq j.
\end{aligned}$$

Moreover, $\forall h \in \mathbb{R}^n,\ s.t. \|h\|_2 = 1$,

$$h^T X(\varepsilon) h = \frac{1}{2}\|h\|_2^2 - \varepsilon(\overline{e}^T h)^2 \geqslant \frac{1}{2} - n\varepsilon \geqslant 0, \forall \varepsilon \leqslant \frac{1}{2n}.$$

Therefore, $[X(\frac{1}{2n}), t(\frac{1}{2n})]$ is a feasible solution to the SDP in Theorem 63, for the n-clique...

5.1 Orthonormal Representations of Graph

Definition 5.7

Given a graph $G=(V,E)$, $v: V \mapsto \mathbb{R}^d$ is an orthonormal representation of G if

$$\begin{cases} \|v(i)\|_2 = 1, & \forall i \in V \text{ and} \\ \langle v(i), v(j)\rangle = 0, & \forall \{i,j\} \in \overline{E} \end{cases}$$

I.e., unrelated pairs of vertices of G are represented by orthogonal unit vector. And $\overline{E} :=$ edges in $\overline{G}$, which is the complement of G ($\overline{G} = (V, \overline{E})$).
So we moved from unit distance hypersphere representation of G to the orthonormal representation of $\overline{G}$ by

$$u : V \mapsto \mathbb{R}^d \text{ hypersphere radius} = \sqrt{t}, (t < \frac{1}{2}) \Rightarrow v : V \mapsto \mathbb{R}^{d+1},\ v(i) := \sqrt{2}\begin{bmatrix} \sqrt{\frac{1}{2}-t} \\ u(i) \end{bmatrix}, \forall i \in v$$

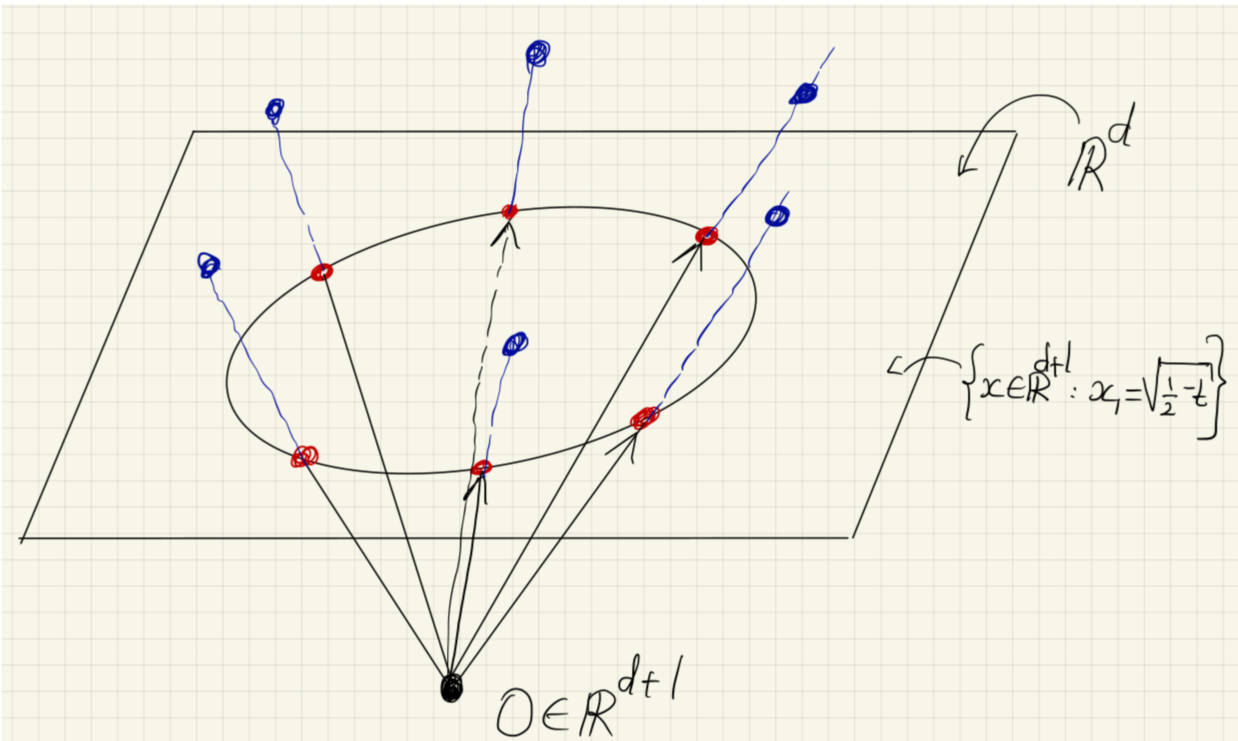
Then $\forall i \in V$,

$$\|v(i)\|_2^2 = 2\left(\frac{1}{2} - t + \underbrace{\langle u(i), u(i)\rangle}_{=t}\right) = 1$$

and, $\forall \{i,j\} \in E$,

$$\langle v(i), v(j)\rangle = 2\left(\frac{1}{2} - t + \underbrace{\langle u(i), u(j)\rangle}_{=\frac{1}{2}(2t-1)}\right) = 0$$

Therefore, $v : V \mapsto \mathbb{R}^{d+1}$ is an orthonormal representation of $\overline{G}$.



We essentially proved

Theorem 5.8

Every graph $G = (V, E)$ admits an orthonormal representation in $\mathbb{R}^n$, where $n := |V|$. Moreover, all orthonormal representations of G can be realized in $\mathbb{R}^n$.

We moved from orthonormal representation of G to the unit distance hypersphere representation of $\overline{G}$ by

$$v : V \mapsto \mathbb{R}^d \Longrightarrow u : V \mapsto \mathbb{R}^d,\ u(i) := \frac{1}{\sqrt{2}} v(i), \forall i \in V$$

Then, $\forall \{i,j\} \in \overline{E}$,

$$\|u(i) - u(j)\|_2^2 = \frac{1}{2} + \frac{1}{2} - 0 = 1$$

$\forall i \in V$,

$$\|u(i)\|_2^2 = \frac{1}{2}\|v(i)\|_2^2 = \frac{1}{2}$$

Therefore, $u : V \mapsto \mathbb{R}^d$ is a unit distance representation of G that lies on a hypersphere of radius $\frac{\sqrt{2}}{2}$.
A very important application of orthonormal representations of graphs is to the Stable Set Problem
$S \subseteq V$ is a stable set in G if for every $\{i, j\} \in E$, at most one of i, j is in S.
Another common term for "stablet set" is "independent set".

$$\underbrace{\alpha(G)}_{\text{stability number of } G} := \max\{|S| \colon S \text{ is a stable set in } G\}$$

$$\underbrace{STAB(G)}_{\text{stable set polytope of } G} := conv\left\{x \in \{0,1\}^V : x \text{ is an incidence vector of a stable set in } G\right\}$$

$$\underbrace{FRAC(G)}_{\substack{\text{fractional stable set}\\ \text{polytope of G}}} := \left\{x \in [0,1]^V : x_i + x_j \leqslant 1, \forall\{i,j\} \in E\right\}$$

Note:

$$STAB(G) = conv\left(FRAC(G) \cap \{0,1\}^V\right)$$

For every clique $\mathcal{C}$ in G, the clique inequality $\sum_{i\in\mathcal{C}} x_i \leqslant 1$ is a valid inequality for $STAB(G)$.
Let $A_{clq}(G)$ denote clique-node incidence matrix of G (it has the number of cliques rows and number of vertices columns) so each row represents if a vertex in the clique or not. Then,

$$\underbrace{CLQ(G)}_{\text{clique polytope of } G} := \left\{x \in \mathbb{R}^V_+ : A_{clq}(G)x \leqslant \bar{e}\right\}$$

5.1.1 Theta Body of G, Lovász Theta Number

$$\underbrace{TH(G)}_{\text{Theta Body of } G} := \left\{ x \in \mathbb{R}^V_+ : \underbrace{\sum_{j=1}^{|V|} [c^T u(j)]^2 x_j \leqslant 1}_{\text{orthonormal repr. constraint}} \;,\; \begin{array}{c} \forall c \in \mathbb{R}^n \; s.t. \; \|c\|_2^2 = 1 \\ \text{and } \forall u : V \mapsto \mathbb{R}^n, \text{ ortho. repr. of } G \end{array} \right\}$$

Theorem 5.9

For every graph G, $TH(G)$ is a nonempty, compact convex set such that

$$STAB(G) \subseteq TH(G) \subseteq CLQ(G) \subseteq FRAC(G)$$

Proof.

- "$CLQ(G) \subseteq FRAC(G)$"
  By definition, every pair $\{i, j\} \in E$ is a clique; thus, $CLQ(G) \subseteq FRAC(G)$ for every graph G.

- "$TH(G) \subseteq CLQ(G)$"
  Let $\mathcal{C} \subseteq V$ be a clique in G. pick any $c \in \mathbb{R}^n$ with $\|c\|_2 = 1$. Define $u(i) := c, \; \forall i \in \mathcal{C}$.

For all other nodes, $i \in V \setminus \mathcal{C}$, choose an orthonormal system of vectors in $(span\{c\})^{\perp}$. Then $u : V \mapsto \mathbb{R}^n$ is an orthonormal representation of G and hence the inequality

$$1 \geqslant \sum_{j=1}^{|V|} [c^T u(j)]^2 x_j = \sum_{j \in \mathcal{G}} (c^T c)^2 x_j = \sum_{j \in \mathcal{C}} x_j$$

is valid for $TH(G)$. Since $TH(G) \subseteq \mathbb{R}^V_+$, all constraints defining $CLQ(G)$ are valid for $TH(G)$. Therefore, $TH(G) \subseteq CLQ(G)$.

- Since $CLQ(G) \subseteq [0,1]^V$, we conclude $TH(G)$ is bounded. Since $TH(G)$ is defined as the intersection of closed convex sets (intersection of the nonnegative orthant with closed half-spaces), $TH(G)$ is closed and convex.

- "$STAB(G) \subseteq TH(G)$"
  Let $S \subseteq V$ be a stable set in G, $\chi_S \in \{0,1\}^V$ denote its incidence vector, $u : V \mapsto \mathbb{R}^n$ be any orthonormal representation of G and let $c \in \mathbb{R}^n$ satisfy $\|c\|_2^2 = 1$. Then

$$\sum_{j=1}^{|V|} [c^T u(j)]^2 (\chi_S)_j = \sum_{j \in S} [c^T u(j)]^2 \leqslant \|Q^T c\|_2^2 = \|c\|_2^2 = 1$$

  where $Q \in \mathbb{R}^{n \times n}$ is an orthogonal matrix defined by

$$Q := \left[\underbrace{u(1)\ u(2)\ \ldots\ u(|S|)}_{\text{orthonormal system}} \quad \underbrace{\ldots}_{\text{complete to an orthonormal basis}} \right]$$

  .
  Since $\chi_S \geqslant 0$, and it satisfies all orthonormal representation constraints for G, $\chi_S \in TH(G)$.
  Since we proved that $TH(G)$ is convex,

$$\underbrace{conv\left(\{\chi_S : S \text{ is a stable set in } G\}\right)}_{=STAB(G)} \subseteq TH(G)$$

  Since $0 \in STAB(G)$, $TH(G)$ is nonempty.

□

Given $w \in \mathbb{R}^V_+$,

$$\underbrace{\vartheta(G, w)}_{\text{Lovász Theta Function}} := \max\left\{\omega^T x : x \in TH(G)\right\}$$

Define $W \in \mathbb{S}^V$ by

$$W_{ij} := \sqrt{w_i \cdot w_j},\ \forall i, j \in V$$

Theorem 5.10

Let $G = (V, E)$, $w \in \mathbb{R}^V_+$. Then, the following are equal:

(i) $\vartheta(G, w)$;

(ii) $\text{minimum}_{u:V\mapsto\mathbb{R}^n \text{ ortho. repr.} c\in\mathbb{R}^n:\|c\|_2=1} \max_{i\in V} \left\{\frac{w_i}{[c^T u(i)]^2}\right\}$;

(iii) $\min\left\{\eta : \text{diag}(S) = 0,\ S_{ij} = 0, \forall\{i,j\} \in \bar{E},\ \eta I + S \succcurlyeq W\right\}$;

(iv) $\max\left\{Tr(WX) : X_{ij} = 0,\ \forall\{i,j\} \in E,\ Tr(X) = 1,\ X \in \mathbb{S}^V_+\right\}$.

In the above, if $w_i = 0$ then $\frac{w_i}{[c^T u(i)]^2} := 0$.
Using the above theorem, we can prove that $\vartheta(G, w)$ can ba approximated to any precision in polynomial time (in $|V|$ and $\ln(1/\varepsilon)$) via approximately solving an SDP.

Definition 5.11

A graph $G = (V, E)$ is <u>perfect</u> if for every node induced subgraph H of G,

$$w(H) = \chi(H)$$

where $w(H)$ is the clique number of H (max. cardinality of a clique in H) and $\chi(H)$ is the chromatic number of H (min. number of colours required to colour all vertices of H).
An <u>odd-hole</u> is a chordless cycle of length at least five. An <u>odd-antihole</u> is the complement of an odd-hole.

Theorem 5.12

Let G be a graph. Then, TFAE

(i) G is perfect;

(ii) $\overline{G}$ is perfect;

(iii) G does not contain an odd-hole or an odd-antihole;

(iv) $STAB(G) = CLQ(G)$;

(v) $STAB(G) = TH(G)$;

(vi) $TH(G) = CLQ(G)$;

(vii) $TH(G)$ is a polytope;

(viii) $\{A_{clq}(G)x \leqslant \overline{e}, x \geqslant 0\}$ is Totally Dual Integral(TDI);

(ix) $T\hat{H}(G) := \left\{Y \in \mathbb{S}_+^{\{0\}\cup V} : Y_{00} = 1, \operatorname{diag}(Y) = Ye_0, Y_{ij} = 0, \forall \{i,j\} \in E\right\}$ is SDP-TDI. (see de Carli Silva and Tuncel (2020) SIAM Journal on Discrete Mathematics)

Theorem 5.13

For every graph $G = (V, E)$, the theta body of the complement of G is equal to the antiblocker of the theta body of G:

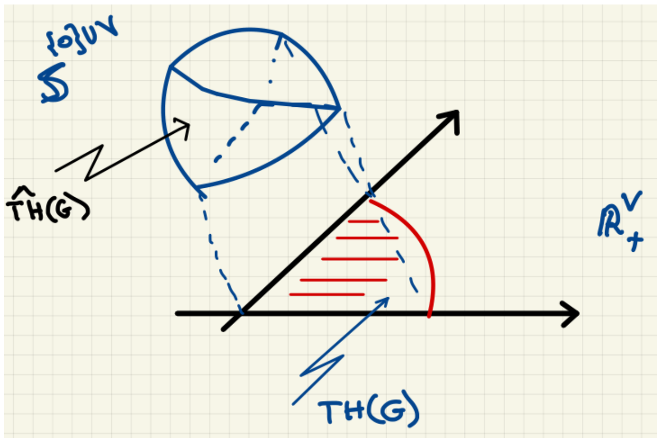
$$[TH(G)]^o \cap \mathbb{R}_+^V = TH(\overline{G})$$

Recall, $[TH(G)]^o = \left\{s \in \mathbb{R}^V : x^T s \leqslant 1, \forall x \in TH(G)\right\}$.

Theorem 5.14

For every graph $G = (V, E)$, we have

$$TH(G) = \left\{ x \in \mathbb{R}^V : \begin{pmatrix} 1 \\ x \end{pmatrix} = Ye_0, \text{ for some } Y \in T\hat{H}(G) \right\}$$



5.1.2 Products of Graphs, Kronecker Products

Given graphs $G = (V, E)$ and $H = (W, F)$,

$$G \otimes H := (V(G \otimes H), E(G \otimes H))$$

which is called the strong product of G and H, and

$$V(G \otimes H) := V \times W$$

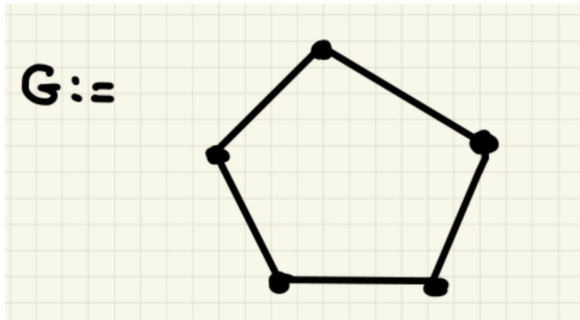
$$E(G \otimes H) := \left\{ \{(i, u), (j, v)\} : \substack{\{i,j\} \in E \text{ and } \{u,v\} \in F \\ \text{or } \{i,j\} \in E \text{ and } u=v \\ \text{or } i=j \text{ and } \{u,v\} \in F} \right\}$$

5.1.3 Stable Set Problem and Shannon Capacity

Suppose we are trying to communicate through a noisy channel. We are using an alphabet and some letters may be confused with each other. Let $G = (V, E)$ model this situation:

$$V \iff \text{the set of letters in the alphabet}$$
$$\{i, j\} \in E \text{ iff letter } i \& j \text{ may be confused with each other.}$$

Example



What's the maximum number of letters that can be safely used? (What is the maximum number of letters such that no pair may be confused with each other?)
The answer is :$\alpha(G)$.
Now, suppose we want to know the maximum number of k-letter words (using the same alphabet given by G) so that no pair of words may be confused with each other. Word1 and word2 may not be confused with each other if there exists a position at which word1 and word2 have different letters AND these different letters do not share an edge. $G^k := \underbrace{G \otimes G \ldots \otimes G}_{k}$. Then the answer is $\alpha(G^k)$.

Definition

Shannon Capacity of G is

$$\mathcal{H}(G) = \lim_{k\to\infty} \sup[\alpha(G^k)]^{\frac{1}{k}}$$

We can show, $\alpha(G^k) \geqslant [\alpha(G)]^k, \forall k \in \mathbb{Z}_{++}, \forall$graphs G which implies $\mathcal{H}(G) \geqslant \alpha(G)$.

Using the fact that Kronecker products of orthonormal representations for G and H give rise to orthonormal representations for $G \otimes H$, we can prove;

Theorem 5.15

For every pair of graphs G and H,

$$\vartheta(G \otimes H) = \vartheta(G)\vartheta(H).$$

Corollary 5.16

For every graph G and every $k \in \mathbb{Z}_{++}$,

$$\vartheta(G^k) = [\vartheta(G)]^k$$

where $\vartheta(G) := \vartheta(G, \bar{e})$.

Theorem 5.17

For every graph $G = (V, E)$,

$$\begin{array}{rl} \vartheta(G) = \max & \bar{e}^T X \bar{e} \\ & X_{ij} = 0, \forall \{i,j\} \in E \\ & Tr(X) = 1 \\ & X \in \mathbb{S}^V_+ \end{array} = \begin{array}{rl} \min & t \\ & \mathrm{diag}(Z) = (t-1)\bar{e} \\ & Z_{ij} = -1, \forall \{i,j\} \in \overline{E} \\ & Z \in \mathbb{S}^V_+ \end{array}$$

Moreover, $\alpha(G) \leqslant \mathcal{H}(G) \leqslant \vartheta(G) \leqslant \chi(\overline{G})$. Finally, we have equality all the way through if G is a perfect graph.

Let's discuss some elements of a proof of Theorem 73. The SDPs in the statement are dual to each other and they are specializations of the SDPs from Theorem 67 with $w := \bar{e}$ (and hence $W := \bar{e}\bar{e}^T$). Both SDPs have Slater points.
Let $S \subseteq V$ be a stable set in G. Then $\overline{X} \in \mathbb{S}^V$ defined by

$$\overline{X}_{ij} := \begin{cases} \frac{1}{|S|}, & \text{if } i \text{ and } j \in S, \\ 0, & \text{otherwise} \end{cases}$$

yield a feasible solution of the first SDP and $\bar{e}^T \overline{X} \bar{e} = |S|$. Thus, $\alpha(G) \leqslant \vartheta(G)$.
We also consider $\mathcal{H} \leqslant \vartheta(G)$, Note, $STAB(G) \subseteq TH(G)$, so any incidence vector for a stable set will be in $TH(G)$, and the inner product of $\bar{e}$ and the incidence vector is just the cardinality of the stable set, and it will be less than or equal to $\vartheta(G)$ (by the definition of it), so $\alpha(G) \leqslant \theta(G)$, because $\alpha(G) = a^T \bar{e}$ for some incidence vector a of a stable set.
Then,

$$\vartheta(G) = \lim(\vartheta(G^k))^{1/k} = \limsup(\vartheta(G^k))^{1/k} \geqslant \limsup(\alpha(G^k))^{1/k} = \mathcal{H}(G)$$

Suppose we have a colouring of $\overline{G}$ with k colours. Then, for the dual SDP, define $\bar{t} := k, \overline{Z} \in \mathbb{S}^V$ such that

$$\overline{Z}_{ij} := \begin{cases} -1, & \text{if } colour(i) \neq colour(j) \\ (k-1), & \text{if } colour(i) = colour(j) \end{cases}$$

$(\overline{Z}, \bar{t})$ is a feasible solution of the dual SDP with objective value k. Therefore, $\vartheta(G) \leqslant \chi(\overline{G})$.
To see that $\overline{Z} \succcurlyeq 0$, note that under a suitable permutation (which groups vertices in the

same colour class together) we have the matrix

$$\begin{bmatrix} (k-1) & \dots & (k-1) & -1 & \dots & -1 \\ \vdots & \dots & \vdots & -1 & \dots & -1 \\ (k-1) & \dots & (k-1) & -1 & \dots & -1 \\ -1 & \dots & -1 & \begin{bmatrix} (k-1) & \dots & (k-1) \\ \vdots & \ddots & \vdots \\ (k-1) & \dots & (k-1) \end{bmatrix} & \dots & \dots \\ \vdots & \ddots & \vdots & \dots & \ddots & \dots \\ -1 & \dots & -1 & \dots & \dots & \begin{bmatrix} (k-1) & \dots & (k-1) \\ \vdots & \ddots & \vdots \\ (k-1) & \dots & (k-1) \end{bmatrix} \end{bmatrix} \succcurlyeq 0$$

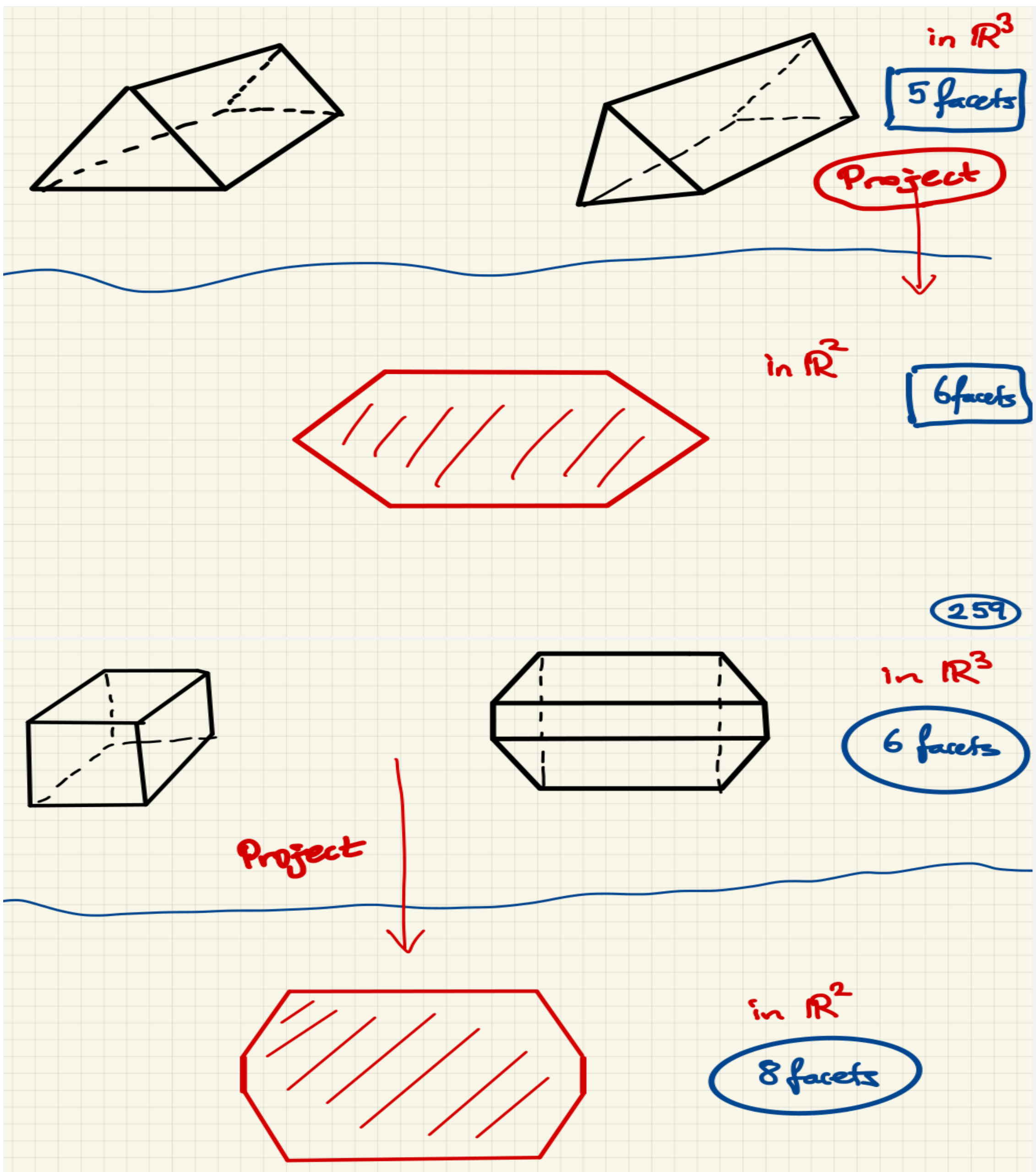
$\iff$

$$\begin{bmatrix} (k-1) & -1 & \dots & \dots & -1 \\ -1 & (k-1) & -1 & \dots & -1 \\ \vdots & -1 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ -1 & -1 & \dots & \dots & (k-1) \end{bmatrix} \succcurlyeq 0, \text{ this is psd since diagonally dominant}$$

By theorem 69, G is perfect iff $\overline{G}$ is. Thus, if G is perfect, we have $\alpha(G) = w(\overline{G}) = \chi(\overline{G})$ which estabilishes the last statement of Theorem 74.

6 Lift and Project Methods

Recall Theorem71, where we project $T\hat{H}(G) \subset \mathbb{S}^{\{0\}\cup V}$ into $\mathbb{R}^V_+$. It estabilished a representation of the theta body of G (a compact convex set which requires infinitely many linear inequalities to describe in $\mathbb{R}^n, n := |V|$) as a projection of a spectrahedron in $\mathbb{S}^{1+n}$ which is described using $O(n^2)$ linear equations and a single positive semidefiniteness constraint. This kind of efficiency gain in representations of convex sets can be seen in many settings, including polyhedra.



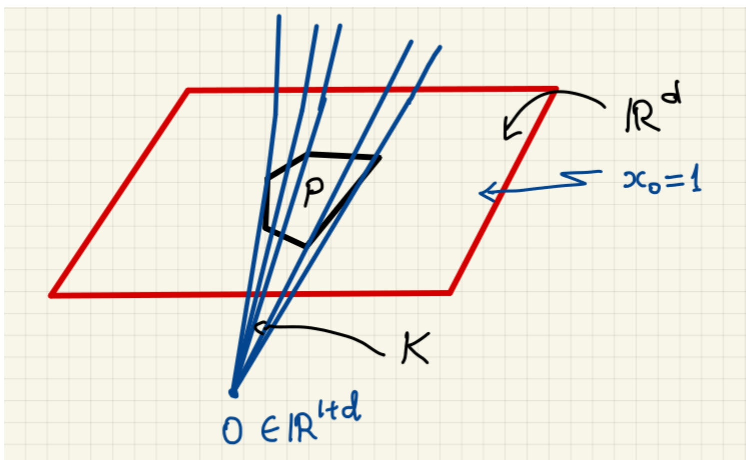
Can we generalize the approach to other combinatorial optimization problem? Yes! Given a polytope $P \subset [0,1]^d$, suppose we are interested in

$$P_I := conv(P \cap \{0,1\}^d)$$

Examples:$P := FRAC(G), P_I = STAB(G)$.
Introduce a new variable x_0 and define

$$K := \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R}^{1+d} : Ax \leqslant x_0 b, 0 \leqslant x \leqslant x_0 \bar{e} \right\}$$

where $P =: \left\{ x \in \mathbb{R}^d : Ax \leqslant b, 0 \leqslant x \leqslant \bar{e} \right\}$.



Consider the set

$$M_+(K) := \left\{ Y \in \mathbb{S}_+^{1+d} : \substack{Ye_0 = \text{diag}(Y) \\ Ye_i \in K, \forall i \in \{1,2,\ldots,d\} \\ Y(e_0 - e_i) \in K, \forall i \in \{1,2,\ldots,d\}} \right\}$$

Suppose $\bar{x} \in P \cap \{0,1\}^d$. Define

$$\bar{Y} := \begin{bmatrix} 1 \\ \bar{x} \end{bmatrix} \begin{bmatrix} 1 & \bar{x}^T \end{bmatrix} = \begin{bmatrix} 1 & \bar{x}^T \\ \bar{x} & \bar{x}\bar{x}^T \end{bmatrix}.$$

Then, $\bar{Y} \in \mathbb{S}_+^{1+d}, \bar{Y}e_0 = \text{diag}(\bar{Y}), \bar{Y}e_i = \bar{x}_i \begin{bmatrix} 1 \\ \bar{x} \end{bmatrix} \in K, \bar{Y}(e_0 - e_i) = (1 - \bar{x}_i) \begin{bmatrix} 1 \\ \bar{x} \end{bmatrix} \in K, \forall i \in \{1, 2, \ldots, d\}$.
Therefore,

$$P \cap \{0,1\}^d \subseteq LS_+(P) := \left\{ x \in \mathbb{R}^d : \begin{pmatrix} 1 \\ x \end{pmatrix} = Ye_0, \text{ for some } Y \in M_+(K) \right\}$$

Since $M_+(K)$ is a spectrahedron, it is convex. Since $LS_+(P)$ is a projection of a convex set, $LS_+(P)$ is also convex. Hence,

$$conv(P \cap \{0,1\}^d) \subseteq LS_+(P).$$

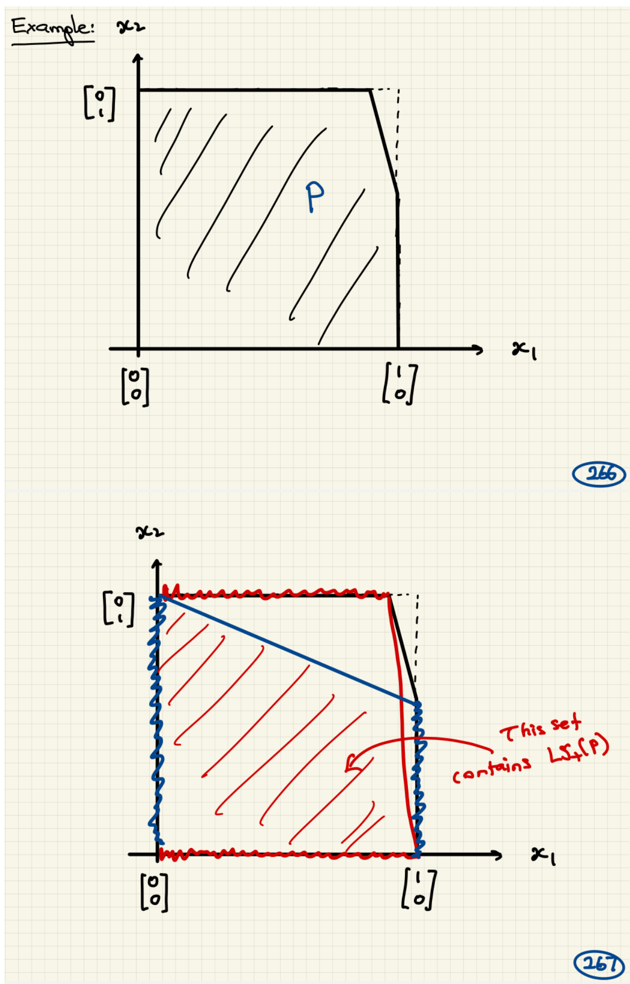
Is $LS_+(P)$ a better approximation to $conv(P \cap \{0,1\}^d)$ than P?

Lemma 6.1

Let $P \subseteq [0,1]^d$ be a convex set. Then,

$$conv(P \cap \{0,1\}^d) \subseteq LS_+(P) \subseteq \bigcap_{j=1}^{d} conv\left[(P \cap H_j^0) \cup (P \cap H_j^1)\right]$$

where $H_j^0 := \left\{x \in \mathbb{R}^d : x_j = 0\right\}, H_j^1 := \left\{x \in \mathbb{R}^d : x_j = 1\right\}$.



Proof. (Lemma 6.1)
We proved the inclusion $conv(P \cap \{0,1\}^d) \subseteq LS_+(P)$ before stating the lemma. To prove the remaining inclusion, let $\overline{x} \in LS_+(P)$. Then $\exists \overline{Y} \in M_+(K)$ such that $\overline{Y} e_0 = \begin{bmatrix} 1 \\ \overline{x} \end{bmatrix}$.

By definition of $M_+(K)$,

$$\begin{bmatrix} 1 \\ \overline{x} \end{bmatrix} = \overline{Y}e_0 = \underbrace{\overline{Y}e_i}_{\in K \cap \{y \in \mathbb{R}^{d+1}: y_i = y_0\}} + \underbrace{\overline{Y}(e_0 - e_i)}_{\in K \cap \{y \in \mathbb{R}^{d+1}: y_i = 0\}}, \forall i \in \{1, 2, \ldots, d\}$$

Thus,

$$\overline{x} \in conv\left[(P \cap H_i^0) \cup (P \cap H_i^1)\right], \forall i \in \{1, 2, \ldots, n\}.$$

Therefore,

$$\overline{x} \in \bigcap_{i=1}^{d} conv\left[(P \cap H_i^0) \cup (P \cap H_i^1)\right]$$

□

Define

$$LS_+^k(P) := \underbrace{LS_+(LS_+(\ldots(P)))}_{k-times}$$

Theorem 6.2

Let $P \subseteq [0,1]^d$ be a convex set. Then

$$P \supseteq LS_+(P) \supseteq LS_+^2(P) \supseteq \ldots \supseteq LS_+^d(P) = conv(P \cap \{0,1\}^d)$$

Moreover, if for some $k \in \{0, 1, \ldots, d-1\}$, $LS_+^k(P) \neq conv(P \cap \{0,1\}^d)$, then

$$LS_+^k(P) \supset LS_+^{k+1}(P).$$

The last theorem indicates that in principle every $0, 1$ integer programming problem can be solved by solving some convex optimization problem based on SDPs.

- Not surprising in the sense that the number of variables and number of constraints can be huge (and we can also derive methods achieving the same goal via LP problems).
- Still interesting, because these strictly improving convex relaxations are generated automatically.

6.1 Lift-and-Project Operator LS_+ applied to $FRAC(G)$

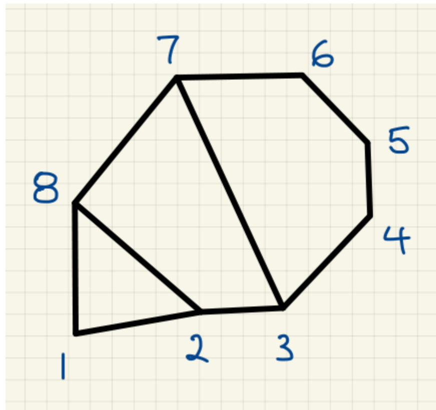
Recall, given a graph $G = (V, E)$,

$$\underbrace{STAB(G)}_{\text{stable set polytope of } G} := conv\left\{x \in \{0,1\}^V : x \text{ is an incidence vector of a stable set in } G\right\}$$

$$\underbrace{FRAC(G)}_{\text{fractional stable set polytope of G}} := \left\{x \in [0,1]^V : x_i + x_j \leqslant 1, \forall \{i,j\} \in E\right\}$$

Note:

$$STAB(G) = conv\left(FRAC(G) \cap \{0,1\}^V\right)$$

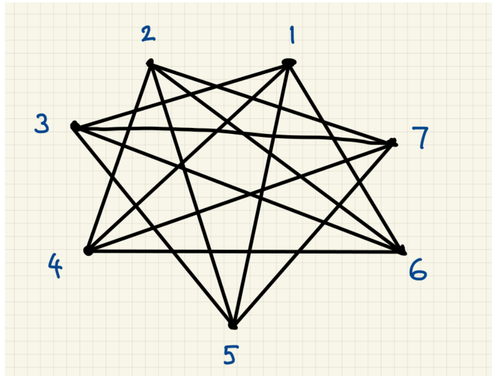


Let $\mathcal{H}$ be the vertex set of an odd-cycle in G. Then the inequality

$$\sum_{i \in \mathcal{H}} x_i \leqslant \frac{|\mathcal{H}|-1}{2}$$

is valid for $STAB(G)$.

$$OC(G) := \left\{x \in FRAC(G) : \substack{x \text{ satisfies all odd-cycle} \\ \text{constraints for } G}\right\}$$

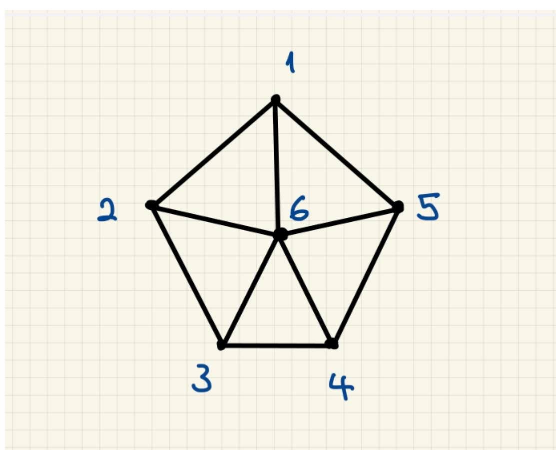


Let $\mathcal{H}$ be the vertex set of an odd-antihole in G. Then the inequality

$$\sum_{i \in \mathcal{H}} x_i \leqslant 2$$

is valid for $STAB(G)$.

$$ANTI-HOLE(G) := \{x \in FRAC(G) : {}^{x \text{ satisfies all odd-antihole}}_{\text{constraints for } G}\}$$



If we have an odd-wheel in G with <u>hub</u> vertex represented by x_{2k+2} and <u>rim</u> vertices represented by $x_1, x_2, \ldots, x_{2k+1}$ then odd-wheel inequality

$$kx_{2k+2} + \sum_{i=1}^{2k+1} x_i \leqslant k$$

is valid for $STAB(G)$.

$$WHEEL(G) := \{x \in FRAC(G) : {}^{x \text{ satisfies all odd-wheel}}_{\text{constraints for } G}\}$$

Theorem 6.3

For every graph G,

$$
\begin{aligned}
&STAB(G) \\
\subseteq &LS_+(FRAC(G)) \\
\subseteq &OC(G) \cap ANTI-HOLE(G) \cap WHEEL(G) \cap CLQ(G) \cap TH(G)
\end{aligned}
$$

Open Problem: Give a full, elegant, combinatorial charaterization for $LS_+(G), \forall G$.
Note: The last inclusion in the statement of Theorem 77 is sometimes strict.
While the above theorem shows the impressive power of LS_+ operator on $FRAC(G)$, on many much easier $0, 1$ integer programming problems LS_+ operator and many many other lift-and-project operators do poorly. See chapter 8 of the textbook and the references therein.

6.2 Successive Nonconvex Relaxation

We can generalize our approach to lift-and-project methods to compute the convex hull of any compact set, hence in principle solve any optimization problem

$$\inf \ f(x), x \in F$$

where f is continuous, F is compact, by solving possibly a very very large scale SDP problem.
Introduce a new variable x_{n+1},

$$
\begin{aligned}
\inf\ & x_{n+1} \\
s.t.\ & f(x) \leqslant x_{n+1} \\
& x \in F \\
& l \leqslant x_{n+1} \leqslant u
\end{aligned}
\quad \longrightarrow \quad
\begin{aligned}
\min\ & c^T x \\
& x \in F
\end{aligned}
$$

where $c := e_{n+1}, F \oplus [l, u] \longrightarrow F$

Theorem 6.4

Every compact set in $\mathbb{R}^d$ can be expressed as the feasible region of a system of quadratice inequalities.

Proof. Let $F \subset \mathbb{R}^d$ be a compact set. Then $\mathbb{R}^d \setminus F$ is open, hence can be expressed as a union of open Euclidean balls. Then,

$$F = \mathbb{R}^d \setminus (\mathbb{R}^d \setminus F) = \cap \left(\text{a collection of quadratic inequalities } \|x - \overline{x}\|_2^2 \geqslant r^2\right)$$

$\square$

Recall, Prop 30 is a similar result.

Lemma 6.5

For every triple $(Q, q, \gamma) \in \mathbb{S}^d \oplus \mathbb{R}^d \oplus \mathbb{R}$,

$$\begin{aligned} &\{x \in \mathbb{R}^d : x^T Q x + 2q^T x + \gamma \leqslant 0\} \\ \subseteq &\left\{ x \in \mathbb{R}^d : Tr \begin{bmatrix} \gamma & q^T \\ q & Q \end{bmatrix} \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \leqslant 0, \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \in \mathbb{S}^{d+1}_+ \right\}. \end{aligned}$$

If $rank \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} = 1$, then equality holds above.

Suppose we are given a set $\mathcal{P} \subset \mathbb{S}^d \oplus \mathbb{R}^d \oplus \mathbb{R}$ so that

$$F = \left\{ x \in \mathbb{R}^d : x^T Q x + 2q^T x + \gamma \leqslant 0, \forall (Q, q, x) \in \mathcal{P} \right\}$$

Note that we may replace $\mathcal{P}$ by $cone(\mathcal{P})$ or by the generators of $cone(\mathcal{P})$ (because mulplication of a positive real number won't change the inequality).
Define $\mathcal{P}_+ := cone(\mathcal{P}) \cap (\mathbb{S}^d_+ \oplus \mathbb{R}^d \oplus \mathbb{R})$ (i,e, collect all the convex quadratic inequalities from this decription, so a convex relaxation of F).

Theorem 6.6

Let $\mathcal{P} \subset \mathbb{S}^d \oplus \mathbb{R}^d \oplus \mathbb{R}$ be a clsoed convex cone containing $(I, 0, \underbrace{R}_{>0})$. Then the convex sets

$$\{x \in \mathbb{R}^d : x^T Q x + 2q^T x + \gamma \leqslant 0, \forall (Q, q, \gamma) \in \mathcal{P}_+\}$$

and

$$\left\{ x \in \mathbb{R}^d : Tr \begin{bmatrix} \gamma & q^T \\ q & Q \end{bmatrix} \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \leqslant 0, \forall (Q, q, \gamma) \in \mathcal{P}; \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \in \mathbb{S}^{d+1}_+ \right\}.$$

are identical. Moreover, in the second description we may replace $\mathcal{P}$ by its generators.

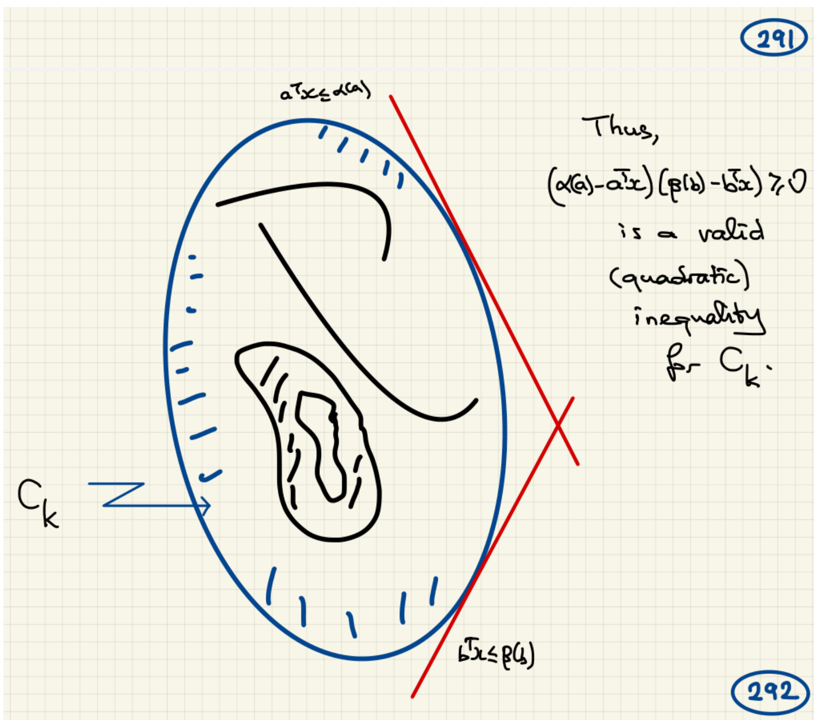
6.3 Successive Convex Relaxation Method

Given $\mathcal{P} \subset \mathbb{S}^d \oplus \mathbb{R}^d \oplus \mathbb{R}$ containing $(I, 0, R)$ for $R > 0$,

$$\begin{aligned} C_0 &:= \left\{ x \in \mathbb{R}^d : x^T Q x + 2q^T x + \gamma \leqslant 0, \forall (Q, q, \gamma) \in \mathcal{P}_+ \right\} \\ D_1 &:= \{d \in \mathbb{R}^d : \|d\|_2 = 1\} \\ D_2 &:= \{e_i, -e_i : i \in \{1, 2, \dots, d\}\} \\ k &:= 0 \end{aligned}$$

<u>At iteration k:</u>

$$
\begin{aligned}
&\forall a \in D_1, \alpha(a) := \max\{a^T x : x \in C_k\} \\
&\forall b \in D_2, \beta(b) := \max\{b^T x : x \in C_k\} \\
&\mathcal{P}_k := \text{ coefficient of } (\alpha - a^T x)(b^T x - \beta) \leqslant 0 \\
&C_{k+1} := \left\{x \in \mathbb{R}^d : x^T Q x + 2q^T x + \gamma \leqslant 0, \forall (Q, q, \gamma) \in (\mathcal{P} \cup \mathcal{P}_k)_+\right\} \\
&k := k + 1
\end{aligned}
$$



Theorem 6.7

With the above definition, the sequence of convex relaxations C_k of F generated by SCRM satisfies

(a) $\forall k \in Z_+, conv(F) \subseteq C_{k+1} \subseteq C_k$, moreoever,

$$C_{k+1} = C_k \iff C_k = conv(F);$$

(b) $\bigcap_{k=1}^{\tau} C_k = \emptyset$ for some finite number τ, if $F = \emptyset$;

(c) $\bigcap_{k=1}^{\infty} C_k = conv(F)$

Theorem 6.8

Let $F \subseteq \{0,1\}^d$ and the set C_0 be defined by quadratic inequalities such that

$$conv(F) \subseteq C_0 \subseteq [0,1]^d.$$

Suppose the quadratic inequalities $x_i^2 - x_i \leqslant 0, -x_i^2 + x_i \leqslant 0, \forall i \in \{1,2,\ldots,d\}$ are included in the quadratic inequality system. Let $\{C_k\}$ denote the sequence of compact convex sets generated by the SCRM. Then,

$$C_k = LS_+^k(C_0), \forall k \in \mathbb{Z}_+$$

Why? Recall the definition of LS_+ via M_+ :

$$M_+(C_k) = \left\{ \ldots \underbrace{Ye_i, Y(e_0 - e_i) \in \underbrace{cone(1 \oplus C_k)}_{K}}_{(1)} \ldots \right\}$$

where

$$(1) \equiv \left.\begin{array}{l} [s_0, s^T]Ye_i \geqslant 0 \\ [s_0, s^T]Y(e_0 - e_i) \geqslant 0 \end{array}\right\}, \forall i \in \{1,2,\ldots,d\}, \forall \begin{bmatrix} s_0 \\ s \end{bmatrix} \in K^*$$

Also,

$$\begin{bmatrix} s_0 \\ s \end{bmatrix} \in K^* \iff \begin{bmatrix} s_0 & s^T \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \geqslant 0, \forall x \in C_k \iff s_0 \geqslant -s^T x, \forall x \in C_k.$$

Our current descriptions of sets C_k are as feasible regions of semi-infinite SDPs (in this case infinitely many constraints on a matrix variable in $\mathbb{S}^{d+1}$).
We can take a finite subset of $D_1 =: \left\{d \in \mathbb{R}^d : \|d\|_2 = 1\right\}$. Then, if the initial system of quadratic inequalities is finite, each C_k will be a projection of a spectrahedron (a typical SDP feasible region).

7 Convex Algebraic Geometry

How can I convince you that

$$\begin{aligned} f(x) :=& 830108x_1 + 216x_2 + x_3^2x_1^2 - 2x_3^3x_1 - 32x_1x_2^3 + 24x_1^2x_2^2 \\ & \dots \\ & + x_1^2x_2^6 + 2x_3^2x_1 - 2x_3^3 \geqslant 2, \forall x \in \mathbb{R}^4 \end{aligned}$$

What if I claim

$$\begin{aligned} f(x) =& (x_1 - 2x_2 - 3)^4 + x_2^2(x_3 - x_1 - 1)^2 \\ & + x_1^2x_2^2(3x_1 - x_2 + 4x_4)^2 + 2 \\ & \geqslant 2, \forall x \in \mathbb{R}^4 \end{aligned}$$

Given a polynomial $f : \mathbb{R}^n \to \mathbb{R}$ of degree of $2d, d \in \mathbb{Z}_{++}$, let

$$h(x) := [1, x_1, x_2, \dots, x_n, x_1^2, x_1x_2, x_1x_3, \dots, x_2^2, \dots, x_n^d]^T \in \mathbb{R}^N,$$

where $N := \binom{n+d}{d}$. We are interested in the set

$$F(f) := \left\{ X \in \mathbb{S}^N : [h(x)]^T X h(x) = f(x) \right\}$$

Theorem 7.1

Let $\overline{z} \in \mathbb{R}$ and f be a multivariate polynomial over the reals. Then,

$$[f(x) - \overline{z}] \text{ if SoS } \iff \underbrace{\left\{ X \in F(f) : X \succcurlyeq \overline{z} e_1 e_1^T \right\}}_{\text{A feasible region of a trivial SDP, even } \overline{z} \text{ not given}} \neq \emptyset$$

Example

$f(x) := x_1^2 + 4x_2^2 - 4x_1x_2 - 6x_1 + 12x_2 + 12, h(x) := [1, x_1, x_2]^T$.

$$\begin{aligned} & \sup \ \overline{z} \\ & s.t. \ \begin{array}{c c} & \begin{array}{ccc} 1 & x_1 & x_2 \end{array} \\ \begin{array}{c} 1 \\ x_1 \\ x_2 \end{array} & \begin{pmatrix} 12 - \overline{z} & -3 & 6 \\ -3 & 1 & -2 \\ 6 & -2 & 4 \end{pmatrix} \end{array} \end{aligned}$$

We can extend the idea of using SoS relaxations of nonnegativity of polynomials to PoPs (Polynomial Optimization Problems):

$$\begin{aligned} \inf\ & P_0(x) \\ s.t.\ & P_1(x) \geqslant 0 \\ & P_2(x) \geqslant 0 \\ & \vdots \\ & P_m(x) \geqslant 0 \end{aligned}$$

We already have seen methods to handle such optimization problems (via Reformulating the problem by quardatic polynomials). We can also treat PoPs directly. We will explain on the feasibility version of (PoP).

$$\text{Is } F := \{x \in \mathbb{R}^n : P_1(x) \geqslant 0, P_2(x) \geqslant 0, \ldots, P_m(x) \geqslant 0\} = \emptyset?$$

Theorem 7.2

Let $P_1, P_2, \ldots, P_m$ be given multivariate polynomials over n real variables. Then,

$$\begin{gathered} F := \{x \in \mathbb{R}^n : P_i(x) \geqslant 0, \forall i \in \{1, 2, \ldots, m\}\} = \emptyset \text{ iff} \\ \exists s_0, \ldots, s_J, \ldots \in SoS(n, *) \text{ such that} \\ g := \sum_{J \subseteq \{1,2,\ldots,m\}} s_J \left(\prod_{i \in J} P_i\right) = -1 \end{gathered}$$

where $SoS(n, *)$ denotes the set of sum of squares polynomials with degree bound $*$.

IN a way, Theorem 84 is a ”common” generalization of Farkas’ Lemma and Hilbert’s Nullstellenstaz.

Theorem: Farkas’ Lemma

Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ be given. Then, exactly one of the following holds:

(i) $\exists x \in \mathbb{R}^n : Ax = b, x \geqslant 0;$

(ii) $\exists y \in \mathbb{R}^m : A^T y \geqslant 0, b^T y < 0;$

For this side only $x \in \mathbb{C}^n$.

Theorem: Hilbert's Nullstellensatz

Given multivariate polynomials $P_1, \dots, P_m : \mathbb{C}^n \mapsto \mathbb{C}$, exactly one of the following systems has a solution (in $\mathbb{C}^n$):

(i) $P_i(x) = 0, \forall i \in \{1, 2, \dots, m\}$;

(ii) $\exists$ polynomials h_i such that

$$\sum_{i=1}^{m} h_i(x) P_i(x) = -1$$

Note that theorem 84 can be implemented computationally (although, in general, none of these methods has been effective on nontrivial, large-scale instances).

Guess an upper bound on the degree of the polynomials s_J's, treat the coefficients of the monomial of S_J's as variables then we have an SDP to solve.

Considering larger degree certificates s_J's lead to very large scale SDP problems.

Recall Theorem 83. Suppose we guessed the maximum degrees of s_J's as two. Suppose $n = 3$.

Represent coefficients of monomials of

$$s_J(x) := a_0 + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_1^2 + a_5x_1x_2 + a_6x_1x_3 + a_7x_2^2 + a_8x_2x_3 + a_9x_3^2 \text{ by}$$

$$X := \begin{array}{c c} & \begin{array}{cccc} 1 & x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} 1 \\ x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} a_0 & a_1/2 & a_2/2 & a_3/2 \\ a_1/2 & a_4 & a_5/2 & a_6/2 \\ a_2/2 & a_5/2 & a_7 & a_8/2 \\ a_3/2 & a_6/2 & a_8/2 & a_9 \end{pmatrix} \end{array} \succcurlyeq 0$$

Note: $s_J(x)$ is a degree 2, SoS if and only if $X \succcurlyeq 0$.

Now if we recall the g in theorem 84, we will see that it actually requires the most of coefficients after the summation becomes zero while we can represent the coefficient of each s_J as a positive semidefinite matrix.

8 Extension Complexity

Recall facets plot and theorem 71 and theorem 77.
Given a polyhedra (or a family of polyhedra), what is the smallest number of linear inequalities necessary to represent this polyhedron as a projection of another polyhedra?
Similarly, given a closed convex set which can be expressed as the feasible region of an SDP, what is the smallest size and number of matrix variables and p.s.d. constraints which allow us to represent the given convex set as a projection of a spectrahedron?
Let $A \in \mathbb{R}^{m\times d}, b \in \mathbb{R}^m$.

$$P := \{x \in \mathbb{R}^d : Ax \leqslant b\};$$

suppose $\dim(P) = d$, P is bounded, and P has m facets, n extreme points.
Slack matrix of P is $S \in \mathbb{R}^{m\times n}$,

$$S_{ij} := b_i - \langle a^{(i)}, v^{(j)}\rangle, \forall i, j,$$

where $a^{(i)}$ is the i^{th} row of A, $v^{(j)}$ is an extreme point of P.
Nonnegative rank of S (and P) is the smallest ineger k such that

$$S = FV^T, \text{ where } F \in \mathbb{R}_+^{m\times k}, V \in \mathbb{R}_+^{n\times k}$$

Then $rank_+(P) := rank_+(S) := k$.

Theorem 8.1: Yannakakis[1991]

Let $P \subset \mathbb{R}^d$ be a polytope, $k := rank_+(P)$. Then every lifted representation (extended formulation) of P has at least k constraints. Moreover, there exists a lifted representation of P with at most $(k + d)$ constraints and $(k + d)$ variables.

Note that in the above theorem, by "lifted representation" or " extended formulation" we are only refering to polyhedral representations. So, "number of constraints" refers to the number of linear equations and inequalities.

Proof. Sketch
Note the every valid inequality for P is a linear consequence of facet defining inequalities for P.
Suppose all facets of P are expressed as $\{Ax \leqslant b\}$. Let S be the slack matrix of P and

$$rank_+(S) = k, S = FV^T, F \in \mathbb{R}_+^{m\times k}, V \in \mathbb{R}_+^{n\times k}.$$
$$\hat{P} := \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{R}^d \oplus \mathbb{R}^k : Ax + Fu = b, u \geqslant 0 \right\}$$

Claim: $P = \left\{ x \in \mathbb{R}^d : \begin{pmatrix} x \\ u \end{pmatrix} \in \hat{P} \text{ for some } u \in \mathbb{R}_+^K \right\}$.
So, $\hat{P}$ is a lifted representation with $(k + d)$ variables and $(m + k)$ linear constraints.
We can eliminate $(m - d)$ constraints from the description of $\hat{P}$ which finishes one direction

of the proof.
For the other direction, suppose there exists a polytope $\tilde{P}$ with q facets so that the projection of $\tilde{P}$ is P. Consider the slack matrix $\tilde{S}$ of $\tilde{P}$, but only focus on the submatrix of $\tilde{S}$ whose columns correspond to extreme points of $\tilde{P}$ projecting to extreme points of P. Every facet inducing inequality for P comes from a valid inequality for $\tilde{P}$ which by our first observation is a nononegative linear combination of facet inducing inequalities for $\tilde{P}$. Collecting these facets of $\tilde{P}$ in a matrix $\tilde{F}$, and defining the submatrix of $\tilde{S}$ as $\tilde{V}$, we have $S = \tilde{F}\tilde{V}^T$ which is the slack matrix of P, which shows $rank_+(S) \leqslant q$. □

The smallest number of facets needed in a lifted polyhedral representation is called the extension complexity of P, and is denoted by

$$xc(P).$$

How about lifted SDP representations and SDP extension complexity?
Given a convex set G and a convex cone K do there exist an affine subspace V and linear subspace W such that

$$G = \underbrace{\Pi_W}_{\substack{\text{Projection} \\ \text{onto subspace} \\ W}} \underbrace{(K \cap V)}_{\substack{\underline{\text{Proper}}\text{ if} \\ V\cap int(K)\neq\emptyset}}$$

We say that G admits a lifted representation by K.
Suppose G is a compact convex set with nonempty interior. We may assume $0 \in int(G)$. Recall the polar

$$G^0 := \{s : \langle x, s\rangle \leqslant 1, \forall x \in G\}$$

Slack function of G:$S_G : ext(G) \oplus ext(G^0) \mapsto \mathbb{R}, S_G(x, s) := 1 - \langle x, s\rangle$.
A K-factorization of S_C is a pair of maps $V : ext(G) \mapsto K$, $F : ext(G^0) \mapsto K^*$ such that $S_G(x, y) = \langle V(x), F(s)\rangle, \forall(x, s) \in ext(G) \oplus ext(G^0)$.

Theorem 8.2: Gouveia,Parrilo, Thomas[2013]

If S_G has a K-factorization, then G has a lifted K-representation. If G has a proper lifted K-representation, then S_G has a K-factorization.

Where is the result about lifted-SDP representations?
Set $K := \mathbb{S}^n_+$ or $K := \mathbb{S}^{n_1}_+ \oplus \mathbb{S}^{n_2}_+ \oplus \ldots \oplus \mathbb{S}^{n_r}_+$ then $K^* = K$.
For combinatorial optimization applications, G is a polytope in $[0, 1]^d$, such as $STAB$.

9 Application to Discrepancy Theory

Discrepancy Theory involves studying and quantifying regularities and irregularities in typically discrete mathematical structures. These studies sometimes include approximations to discrete mathematical structures by continuous structures.
Applications include those in other areas such as:
communication complexity, data analysis, design of polynomial time approximation algorithms, computational geometry, computational complexity theory, Monte-Carlo algorithms, computational finance.
Let $\mathcal{F}$ be a family of subsets of $\{0, 1, \ldots, (n-1)\}$. Given $x \in \{-1, 1\}^{\{0,1,\ldots,n-1\}}$ the discrepancy of x is

$$\Delta(x) := \max_{J \in \mathcal{F}} \left| \sum_{j \in J} x_i \right|$$

Example

$n = 5, \mathcal{F} := \{\{0, 2, 4\}, \{1, 3\}, \{2, 3\}\}, \overline{x} := [-1, 1, -1, -1, 1]^T$. Then,

$$\Delta(\overline{x}) = \max\{1, 0, 2\} = 2, \Delta(\mathcal{F}) := \min_{x \in \{-1,1\}^n} \{\Delta(x)\}$$

$\Delta(\mathcal{F}) = 1$, attained by $\hat{x} := [-1, 1, 1, -1, 1]^T$ among others.

We want to find a sign vector x (which partitions the ground set) so that the discrepancy is minimized:

$$\Delta(\mathcal{F}) := \min_{x \in \{-1,1\}^n} \{\Delta(x)\}$$

Consider, as an approximation to $\Delta(\mathcal{F})$, instead of starting with $\Delta(x) = \max_{J \in \mathcal{J}} \left|\sum_{i \in J} x_i\right|$, let's start with $\Delta_2(x) := \sum_{J \in \mathcal{F}} \left(\sum_{i \in J} x_i\right)^2$, and define

$$\Delta_2(\mathcal{F}) := \min_{x \in \{-1,1\}^n} \left\{ \frac{1}{|\mathcal{F}|} \Delta_2(x) \right\}$$

We have $\Delta_2(\mathcal{F}) \leqslant [\Delta(\mathcal{F})]^2$.
An integer programming formulation to compute $\Delta(\mathcal{F})$:

$$\begin{gathered} \min\ t \\ -t \leqslant \textstyle\sum_{i \in J} x_i \leqslant t, \forall J \in \mathcal{F} \\ x \in \{-1, 1\}^n. \end{gathered}$$

Consider

$$\min \frac{1}{|\mathcal{F}|} \sum_{J \in \mathcal{F}} \left(\sum_{i \in J} x_i \right)^2, x \in \{-1, 1\}^n \text{ as a lower bound}$$

Equivalently,

$$\begin{aligned}
\min\ & \frac{1}{|\mathcal{F}|}\sum_{J\in\mathcal{F}}\left(\sum_{i\in J}\sum_{j\in J}X_{ij}\right)\\
& \operatorname{diag}(X)=\bar{e}\\
& X\in\mathbb{S}^n_+\\
& rank(X)=1
\end{aligned}$$

Note in the above, we write $X \leftrightarrow xx^T$ and we know $(\sum_{i\in J}x_i)^2=\sum x_i^2+2\sum_{i<j}x_ix_j$.
If we represent $x_i \to v^{(i)} \in \mathbb{R}^n$ with $\|v^{(i)}\|_2 = 1$, $X \in \mathbb{S}^n_+$ with $\operatorname{diag}(X)=\bar{e}$ and $X = VV^T, V^T := \begin{bmatrix} v^{(0)} & v^{(1)} & \dots & v^{(n-1)} \end{bmatrix}$.
we have the SDP relaxation

- (P) $\inf\left\{\frac{1}{|\mathcal{F}|}Tr(CX) : \operatorname{diag}(X)=\bar{e}, X\succcurlyeq 0\right\}$ and its dual
- (D) $\sup\left\{\bar{e}^Ty : \operatorname{Diag}(y)\preccurlyeq \frac{1}{|\mathcal{F}|}C\right\}$

Optimal objective value of (P) lower bounds $\Delta_2(\mathcal{F})$ whose square-root lower bounds the discrepancy of $\mathcal{F}, \Delta(\mathcal{F})$.
Primal SDP has the Slater point $\overline{X} := I$ and the dual SDP has the Slater point $\bar{y} := -(\|C\|_2+1)\,\bar{e}$. Therefore, by a corollary of the Strong Duality Theorem, both (P) and (D) attain their optimal values and these values are the same.

Theorem 9.1: (Roth[1964])

For every partition $[N_1, N_2]$ of integers $\{0,1,2,\dots,n-1\}$, there exists an arithmetic progression $\mathcal{G} := \{l, l+\alpha, l+2\alpha,\dots,l+k\alpha\}\subseteq\{0,1,\dots,n-1\}$ such that

$$||\mathcal{G}\cap N_1|-|\mathcal{G}\cap N_2|| > \frac{n^{\frac{1}{4}}}{20}.$$

We can prove this theorem, utilizing the SDP relaxation above.
We will prove something slightly stronger, that there exists an arithmetic progression of length $k := \lfloor\sqrt{n/8}\rfloor$ such that $\alpha\in\{1,2,\dots,8k\}$ and the conclusion of the theorem holds.
We consider progressions modulo n (we allow them to wrap around). Note,

$$(k-1)8k \leqslant \lfloor\sqrt{n/8}-1\rfloor\sqrt{8n}\leqslant n-1.$$

Thus, our arithmetic progressions never wrap around more than once.
Let $\mathcal{H}$ denote this family of progressions.
Note, $|\mathcal{H}| = 8kn$. So, our SDP relaxation is:

$$\begin{aligned}
\min & \frac{1}{8kn}\sum_{J\in\mathcal{H}}\left(\sum_{i\in J}\sum_{j\in J}X_{ij}\right)\\
& \operatorname{diag}(X)=\bar{e}\\
& X\in\mathbb{S}^n_+
\end{aligned}$$

Lemma 9.2

The feasible region and the objective function of the above SDP are invariant under cyclic shifts:

$$X_{ij} \mapsto X_{i+1,j+1},$$

where indices are interpreted modulo n.

Using the above lemma, we can deduce $\exists \beta \in \mathbb{R}^{n-1}$ such that an optimal solution of SDP is

$$\hat{X} := \begin{bmatrix} 1 & \beta_1 & \beta_2 & \dots & \beta_{n-1} \\ \beta_{n-1} & 1 & \beta_1 & \dots & \beta_{n-2} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \beta_1 & & & & 1 \end{bmatrix}.$$

and by X being symmetric, we have

$$\hat{X} := \begin{bmatrix} 1 & \beta_1 & \beta_2 & \dots & \beta_{n-1} \\ \beta_1 & 1 & \beta_1 & \dots & \beta_{n-2} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \beta_{n-1} & & & & 1 \end{bmatrix}.$$

which means $\beta_{n-i} = \beta_i$.
I.e., SDP has an optimal solution that is a symmetric, positive semidefinite circulant matrix with every diagonal entry equal to 1.
Using the structure of such a special optimal solution $\hat{X}$ of the SDP relaxation, we can prove that its objective function value is at least $\frac{\sqrt{n/8}-1}{16}$. For all large n, this yields a proof that $\Delta(\mathcal{F}) > \frac{n^{\frac{1}{4}}}{14}$.
We worked on the primal SDP; we could have worked on dual SDP instead:
Let $A_j \in \{0,1\}^{n\times n}$ denote the matrix whose i^{th} row is the characteristic vector of the arithmetic progression with starting point i and stepsize $j \in \{1, 2, \dots, 8k\}$. Then,

$$C := \sum_{j=1}^{8k} A_j A_j^T.$$

Our dual SDP is

$$\max \bar{e}^T y$$
$$s.t.\ \operatorname{Diag}(y) \preccurlyeq \frac{1}{8kn} C$$

It turns out, if suffices to consider a very special family of dual solutions: $y = \eta\bar{e}$ for $\eta \in \mathbb{R}$.
We immediately have that

$$opt(SDP) \geqslant \frac{1}{8kn}\lambda_n(C).$$

Note, C is a symmetric, positive definite circulant matrix.

10 SDP Representability and Some Other Applications

On the spectrum of

1. Theory of SDPs: which convex sets can be expressed as feasible regions of SDPs (perhaps allowing auxiliary variables)?

2. Applications of SDPs: which applications can be efficiently treated via SDPs?

[No bound on $XC_{SDP}(\cdot)$ is required]
Given a convex function $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$, we can formulate convex optimization problems involving f (either in the objective function as " $\inf\ f(x)\ldots$" or in the constraints as "$f(x) \leqslant g(x)$" where g is an affine function) via representing its epigraph,

$$epi(f) := \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R} \oplus \mathbb{R}^d : f(x) \leqslant t \right\}$$

we will focus on the latter.

Affine functions, polyhedra:

Given $A \in \mathbb{R}^{m\times d}, b \in \mathbb{R}^m$,

$$\{x \in \mathbb{R}^d : Ax \leqslant b\} = \{x \in \mathbb{R}^d : \mathrm{Diag}(b - Ax) \succcurlyeq 0\}$$

Euclidean Norm:

$f : \mathbb{R}^d \to \mathbb{R}, f(x) := \|x\|_2$,

$$\begin{aligned} epi(f) &= \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R} \oplus \mathbb{R}^d : \|x\|_2 \leqslant t \right\} \\ &= \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R} \oplus \mathbb{R}^d : \begin{bmatrix} t & x^T \\ x & tI \end{bmatrix} \succcurlyeq 0 \right\}. \end{aligned}$$

Matrix 2-norm(Operator 2-norm)

$f : \mathbb{R}^{m\times n} \to \mathbb{R}, f(x) := \|X\|_2 := \max_{h\in\mathbb{R}^n:\|h\|_2=1} \|Xh\|_2$.

$$\begin{aligned} epi(f) &= \{(t, X) \in \mathbb{R} \oplus \mathbb{R}^{m\times n} : \|X\|_2 \leqslant t\} \\ &= \left\{ (t, X) \in \mathbb{R} \oplus \mathbb{R}^{m\times n} : \begin{bmatrix} tI & X^T \\ X & tI \end{bmatrix} \succcurlyeq 0 \right\} \end{aligned}$$

10.1 A Homogeneous Cone and Nuclear Norm Minimization:

Consider the convex cone

$$cl\left\{(t,U,X)\in\mathbb{R}\oplus\mathbb{R}^{n\times m}\oplus\mathbb{S}^n : tX-UU^T\succcurlyeq 0, t>0\right\}$$
$$=\left\{(t,U,X)\in\mathbb{R}\oplus\mathbb{R}^{n\times m}\oplus\mathbb{S}^n : \begin{bmatrix} tI & U^T \\ U & X\end{bmatrix}\succcurlyeq 0\right\}$$

In many applications of data sparsification (finding sparse representations capturing "important" parts of the data), compressed sensing and machine learning, problems of minimizing the nuclear norm of a matrix over an affine subspace (or its variants) arise.
Given $X\in\mathbb{R}^{m\times n}$, its nuclear norm is

$$\|X\|_*:=Tr\left((XX^T)^{\frac{1}{2}}\right)=\sum_{i=1}^{\min\{m,n\}}\sigma_i(X),$$

where $\sigma_i(X)$ denotes the i^{th} singular value of X.

$$(P)\ \inf\ \|X\|_* \qquad (D)\ \sup\ b^Ty$$
$$s.t.\ \mathcal{A}(X)=b \qquad \|\mathcal{A}^*(y)\|_2\leqslant 1$$

The dual problem can be formulated using the operator 2-norm formulation ($U:=\mathcal{A}^*(y), t:=1$) or the more general formulation ($U:=\mathcal{A}^*(y), t:=I, X:=tI$).

$$\begin{bmatrix} tI & [\mathcal{A}^*(y)]^T \\ \mathcal{A}^*(y) & tI\end{bmatrix}\succcurlyeq 0, t=1 \iff \|\mathcal{A}^*(y)\|_2\leqslant 1$$

10.2 Maximum Eigenvalue of a Symmetric Matrix:

$f:\mathbb{S}^n\to\mathbb{R}, f(X):=\lambda_1(X)$.

$$epi(f)=\{(t,X)\in\mathbb{R}\oplus\mathbb{S}^n:\lambda_1(X)\leqslant t\}$$
$$=\{(t,X)\in\mathbb{R}\oplus\mathbb{S}^n: tI-X\succcurlyeq 0\}$$

10.3 Condition number of a symmetric, positive definite matrix pencil:

Suppoe $A_0,A_1,\ldots,A_m\in\mathbb{S}^n$ are given and we want a matrix of the form $(A_0+\sum_{i=1}^m y_iA_i)$ such that the min. eigenvalue is at least one, and that its condition number is minimized:

$$\inf\ t$$
$$s.t.\ I\preccurlyeq A_0+\sum_{i=1}^m y_iA_i\preccurlyeq tI$$

10.4 Sum of k-largest eigenvalues:

$f : \mathbb{S}^n_+ \to \mathbb{R}, f(X) := \mathcal{K}_k(X) := \sum_{i=1}^k \lambda_i(X)$.

$$\begin{aligned} epi(f) &= \left\{(t, X) \in \mathbb{R} \oplus \mathbb{S}^n_+ : \mathcal{K}_k(X) \leqslant t\right\} \\ &= \left\{ (t,X)\in\mathbb{R}\oplus\mathbb{S}^n_+ : \underbrace{M(t,X,\eta,Y) := \begin{bmatrix} (t-k\eta) - Tr(Y) & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Y - X + \eta I \end{bmatrix}}_{\text{for some } Y\in\mathbb{S}^n, \eta\in\mathbb{R}} \succcurlyeq 0, \right\} \end{aligned}$$

Proposition 10.1

Let f and $M(t, X, \eta, Y)$ be as defined above. Then, $(t, X) \in epi(f)$ if and only if there exists $\eta \in \mathbb{R}$ and $Y \in \mathbb{S}^n$ such that $M(t, X, \eta, Y) \succcurlyeq 0$.

Proof.

1. ($\Longleftarrow$) Suppose $\exists \eta \in \mathbb{R}, Y \in \mathbb{S}^n$ such that $M(t, X, \eta, Y) \succcurlyeq 0$. Then, Theorem 1.20 of the textbook (variational charaterization of eigenvalues of symmetric matrices Courant-Fischer-Weyl Theorem) and $Y - X + \eta I \succcurlyeq 0$ implies

$$\lambda_j(Y) - \lambda_j(X) \geqslant -\eta, \forall j \in \{1, 2, \ldots, n\}$$

   Summing up both sides of the first k inequalities, we obtain $\mathcal{K}_k(Y) + k\eta \geqslant \mathcal{K}_k(X)$. Since $Y \succcurlyeq 0$, we have $Tr(Y) \geqslant \mathcal{K}_k(Y)$. Combining this with $Tr(Y) + k\eta \leqslant t$ (from $(1, 1)$ block of $M(t, X, \eta, y) \succcurlyeq 0$), we conclude

$$\mathcal{K}_k(X) \leqslant \mathcal{K}_k(Y) + k\eta \leqslant Tr(Y) + k\eta \leqslant t$$

   We proved $(t, X) \in epi(f)$.

2. ($\Longrightarrow$) Let $(t, X) \in epi(f)$. Then, $X \succcurlyeq 0$ and $\mathcal{K}_k \leqslant t$. Let $\eta := \lambda_k(X)$. We have

$$\begin{aligned} &\lambda_j(X - \eta I) \geqslant 0, \forall j \in \{1, 2, \ldots, k\} \text{ and} \\ &\lambda_j(X - \eta I) \leqslant 0, \forall j \in \{k + 1, \ldots, n\}. \end{aligned}$$

Let $u^{(1)}, u^{(2)}, \ldots, u^{(n)} \in \mathbb{R}^n$ be the eigenvalues of $(X - \eta I)$ corresponding to the eigenvalues $\lambda_1(X - \eta I), \ldots, \lambda_n(X - \eta I)$ respectively. Let

$$Y := \sum_{j=1}^k \lambda_j(X - \eta I)u^{(j)}{u^{(j)}}^T, Z := Y - (X - \eta I)$$

Note, $Y \succcurlyeq 0$ and $Y - (X - \eta I) = Z \succcurlyeq 0$. Finally,

$$\begin{aligned} t - k\eta - Tr(Y) &= t - k\lambda_k(X) - \mathcal{K}_k(X) + k\lambda_k(X) \\ &= t - \mathcal{K}_k(X) \geqslant 0 \end{aligned}$$

Therefore, $M(t, X, \eta, Y) \succcurlyeq 0$ as desired. $\square$

We can extend the above SDP representation to handle $f : \mathbb{R}^{m\times n} \to \mathbb{R}, f(X) := \sum_{i=1}^{k} \sigma_i(X), epi(f) = \left\{(t, X) \in \mathbb{R} \oplus \mathbb{R}^{m\times n} : \sum_{i=1}^{k} \sigma_i(X) \leqslant t\right\}$, the epigraph of sum of k-largest singular values of X.

Note that the eigenvalues of $\begin{bmatrix} 0 & X^T \\ X & 0 \end{bmatrix}$ are the singular values of X and their negation.

10.5 Geometric Mean

$f : \mathbb{R}^n \to \mathbb{R}, f(x) := -\left(\prod_{j=1}^{n} x_j\right)^{\frac{1}{n}}$.

$$epi(f) = \left\{(t, X) \in \mathbb{R} \oplus \mathbb{R}^n : -\left(\prod_{j=1}^{n} x_j\right)^{\frac{1}{n}} \leqslant t.\right\}$$

10.6 Determinant of $X \in \mathbb{S}_{++}^n$

$f : \mathbb{S}_{++}^n \to \mathbb{R}, f(X) := [\det(X)]^{\frac{1}{n}}$.

$$epi(f) = \left\{(t, X) \in \mathbb{R} \oplus \mathbb{S}_{++}^n : -[\det(X)]^{\frac{1}{n}} \leqslant t\right\}$$

10.7 Determinant of $X \in \mathbb{S}_{+}^n$

$f : \mathbb{S}_{+}^n \to \mathbb{R}, f(X) := [\det(X)]^{\frac{1}{n}}$.

$$epi(f) = \left\{(t, X) \in \mathbb{R} \oplus \mathbb{S}_{+}^n : -[\det(X)]^{\frac{1}{n}} \leqslant t\right\}$$

We will construct an SDP-representation for the above epigraph by utilizing an SDP-representation for the Geometric Mean application. For $\xi \in \mathbb{R}^{n(n+1)/3}$ index its entries by ij, where $i, j \in \{1, 2, \ldots, n\}$ so that ξ gives a v ector representation of the entries of a lower triangular matrix

$$Y(\xi) := \begin{bmatrix} \xi_{11} & & & \\ \xi_{21} & \xi_{22} & & \text{\Large 0} \\ \vdots & & \ddots & \\ \xi_{n1} & \xi_{n2} & & \xi_{nn} \end{bmatrix}$$

Let $Z(t, \xi)$ denote the SDP representation of the set

$$epi(g) := \left\{\begin{pmatrix} y \\ \xi \end{pmatrix} \in \mathbb{R} \oplus \mathbb{R}^{n(n+1)/2} : t \geqslant -\left(\prod_{i=1}^{n} \xi_{ii}\right)^{\frac{2}{n}}, \xi_{ii} \geqslant 0, \forall i\right\}$$

That is, $Z(t, \xi) \succcurlyeq 0$ if and only if $\begin{bmatrix} t \\ xi \end{bmatrix} \in epi(g)$. Define

$$M(t, X, \xi) := \begin{pmatrix} z(t, \xi) & 0 & 0 \\ 0 & I & [Y(\xi)]^T \\ 0 & Y(\xi) & X \end{pmatrix}$$

Theorem 10.2

Let f and $M(t, X, \xi)$ be as above. Then, $\begin{pmatrix} y \\ C \end{pmatrix}$ if and only if $\exists \xi \in \mathbb{R}^{n(n+1)/2}$ with $\xi_{ii} \geqslant 0, \forall i$ such that $M(t, X, \xi) \succcurlyeq 0$.

Proof.

- $\Longleftarrow$ Suppose $\exists \xi \in \mathbb{R}^{n(n+1)/2}$ with $\xi_{ii} \geqslant 0, \forall i$ and $t \in \mathbb{R}, X \in \mathbb{S}^n_+$ such that $M(t, Xi, \xi) \succcurlyeq 0$. Then, by the block structure of $M(t, X, \xi)$ and Lemma 5 (Schur Complement Lemma), we have $Z(t, \xi) \succcurlyeq 0$ and $X \succcurlyeq Y(\xi)[Y(\xi)]^T$.
  The former implies $t \geqslant -\begin{pmatrix} \xi_{11} & \xi_{22} & \dots & \xi_{nn} \end{pmatrix}^{\frac{2}{n}}$. The latter implies $\det(X) \geqslant \det(Y(\xi))^2 = \xi_{11}^2 \xi_{22}^2 \dots \xi_{nn}^2$. Thus, $-(\det(X))^{\frac{1}{n}} \leqslant t, (t, X) \in epi(f)$ as desired.
- $\Longrightarrow$ Suppose $(t, X) \in epi(f)$. Let
$$\xi_{ij} := \begin{cases} \sqrt{\lambda_i(X)} & , \text{ if } i = j \\ 0 & , \text{ otherwise} \end{cases}$$
  Then, $M(t, X, \xi) \succcurlyeq 0$, as desired.

$\square$

10.8 Univariate, nonnegative polynomials

Given $p_0, p_1, \dots, p_n$, we have

$$p(t) := \sum_{k=0}^{n} p_k t^k, \ \ \upsilon_n := [1, t, t^2, \dots, t^n] \in \mathcal{P}^n.$$

Then, $p(t) = \langle p, \upsilon_n \rangle$.

$$K_{2n} := \left\{ \underbrace{p \in \mathcal{P}^{2n}}_{p \in \mathbb{R}^{2n+1}} : p(t) \geqslant 0, \forall t \in \mathbb{R} \right\}.$$

$\tilde{E}_k \in \mathbb{S}^{n+1}$ dentoes the k^{th} cross-diagonal matrix $k \in \{0, 1, \dots, 2n\}$:

$$\tilde{E}_0 = e_1 e_1^T, \tilde{E}_1 = \begin{bmatrix} 0 & 1 & 0 & \dots \\ 1 & 0 & & \\ 0 & & \ddots & \\ \vdots & & & \end{bmatrix}, \tilde{E}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & & \\ 1 & 0 & 0 & & \\ 0 & & & \ddots & \\ \vdots & & & & \end{bmatrix}, \dots, \tilde{E}_{2n} = e_n e_n^T$$

Theorem 10.3

For every positive integer n,

$$K_{2n} = \left\{ p \in \mathbb{R}^{2n+1} : p_K = \langle \tilde{E}_k, X \rangle, k \in \{0, 1, \dots, 2n\}, X \in \mathbb{S}^{n+1}_+ \right\}.$$

Proposition 10.4

$$(K_{2n})^* = \left\{ s \in \mathbb{R}^{2n+1} : \sum_{k=0}^{2n} s_K \tilde{E}_k \succcurlyeq 0 \right\}.$$

Proposition 10.5

For every positive integer n,

$$int(K_{2n}) = \left\{ p \in \mathbb{R}^{2n+1} : p_{2n} > 0 \text{ and } p(t) > 0, \forall t \in \mathbb{R} \right\}$$

Theorem 10.6

For every positive integer n, K_{2n} and K_{2n}^* are pointed closed convex cones with nonempty interiors.

Theorem 10.7

If $p \in int(K_{2n})$ then the set

$$\left\{ X \in \mathbb{S}_+^{n+1} : \langle \overline{E}_k, X \rangle = p_k, \forall k \in \{0, 1, \ldots, 2n\} \right\}$$

is bounded and it contains some $\overline{X} \succ 0$.

Univariate polynomials that are nonnegative on $\mathbb{R}_+$ or on $[0, 1]$ can be treated. Trigonometric polynomials

$$p(t) := \sum_{k=0}^{n} p_k(\cos t + \underbrace{i}_{i:=\sqrt{-1}} \sin t) \text{ can also be treated}$$

More open problems:

- characterize the set of convex cones that are spectrahedral.
- characterize the set of convex sets that are spectrahedral shadows.
- find new applications of SDPs in
  - approximation algorithms
  - quantum computing & information
  - graph theory
  - combinatorics
  - ⋮
- settle the Unique Games Conjecture

- design faster and more robust algorithms for solving SDPs
  - first-order
  - second-order and higher but adaptive
  - exploit sparsity and special structure better
- better understanding of the boundary structure of spectrahedra.
- applications, applications, applications. Data Science, Machine Learning, System&Control Theory,...