

Stochastic Process 1

Rui Gong

March 9, 2024

Contents

1	Poisson Process	3
1.1	Poisson Approximation to Binomial	3
1.2	Total Variance Distance	3
1.3	Probability Axioms	5
1.4	Cumulative Distribution Function (c.d.f.)	7
1.4.1	Geometric Distribution to Exponential, the Memoryless variables	8
1.5	Point Process	9
1.6	Bernoulli and Poisson	10
2	Markov-Chain	17
2.1	Transition Matrix	19
2.2	Communication, Recurrence and Transience	21
2.3	Stationary Distribution and Positive Recurrence	28
2.4	Period	30
2.4.1	Fundamental Theorem of Markov Chain	30
2.5	Reversibility	32

1 Poisson Process

1.1 Poisson Approximation to Binomial

Given a Poisson random variable $Y \sim \text{Poisson}(\lambda)$ with pdf

$$\Pr[Y = k] = \frac{e^{-\lambda} \lambda^k}{k!}, \forall k \in N_0 = \{0, 1, \dots\}.$$

The probability of a binomial random variable being k is

$$\binom{n}{k} p^k (1-p)^{n-k}$$

Theorem 1.1. Given $p \rightarrow 0$, $np \rightarrow \lambda$, we have that

$$\binom{n}{k} p^k (1-p)^{n-k} \rightarrow \frac{e^{-\lambda} \lambda^k}{k!}$$

Proof.

$$\begin{aligned} \binom{n}{k} p^k (1-p)^{n-k} &= \binom{n}{k} \left(\frac{np}{n}\right)^k \left(1 - \frac{np}{n}\right)^{n-k} \\ &= \frac{n!}{k! (n-k)!} \left(\frac{np}{n}\right)^k \left(1 - \frac{np}{n}\right)^n \left(1 - \frac{np}{n}\right)^{-k} \\ &= \frac{1}{k!} \underbrace{\frac{n(n-1)\dots(n-k+1)}{n^k}}_{\rightarrow 1} \underbrace{(np)^k}_{\rightarrow \lambda^k} \underbrace{\left(1 - \frac{np}{n}\right)^n}_{\rightarrow e^{-\lambda}} \underbrace{\left(1 - \frac{np}{n}\right)^{-k}}_{\rightarrow 1} \\ &\rightarrow \frac{1}{k!} \lambda^k e^{-\lambda} \end{aligned}$$

□

With that, we consider three different binomial random variables:

$$X_n \sim \text{Binomial}(n, p_n), p_n \rightarrow 0, np_n \rightarrow \lambda > 0, \text{ as } n \rightarrow \infty.$$

$$Z_p \sim \text{Binomial}(n(p), p), p \rightarrow 0, n(p)p \rightarrow \lambda > 0, \text{ as } p \rightarrow 0.$$

$$N_x \sim \text{Binomial}(n(x), p(x)), p(x) \rightarrow 0, n(x) \rightarrow \lambda > 0, \text{ as } x.$$

For example, if $X_n \sim \text{Binomial}(n, 2/n)$, then we expect

$$\binom{n}{k} p^k (1-p)^{n-k} \rightarrow \frac{e^{-2} 2^k}{k!}$$

1.2 Total Variance Distance

Let X_1, \dots, X_n be n independent Bernoulli random variables, where $\mathbb{E}[X_i] = p_i$.

Given $S = \sum X_i$ and $T \sim \text{Poisson}(\lambda = \sum p_i)$, how close are these two distributions? Or, how to measure the closeness?

Definition 1.2. Given two random variables X, Y , (which shares the sample space), we have the *total variance distance* defined as

$$d_{TV}(X, Y) = \sup_A |\Pr[X \in A] - \Pr[Y \in A]|$$

where A is a Borel set defined with respect to the sample space σ -algebra.

Example 1.3. Given two distributions

	0	1	2	3
$\Pr[X = k]$	5/10	3/10	1/10	1/10
$\Pr[Y = k]$	2/10	1/10	1/10	6/10

Table 1: Discrete Distribution Distance

If $A = \{0, 1\}$, then

$$|\Pr[X \in A] - \Pr[Y \in A]| = 3/10 + 2/10 = 1/2.$$

If $A = \{3\}$,

$$|\Pr[X \in A] - \Pr[Y \in A]| = |1/10 - 6/10| = 1/2.$$

Lemma 1.4. If X, Y take values in a countable set E ,

$$\begin{aligned} d_{TV}(X, Y) &= \sum_{i \in E} (\Pr[X = i] - \Pr[Y = i])^+ \\ &= \sum_{i \in E} (\Pr[Y = i] - \Pr[X = i])^+ \\ &= \frac{1}{2} \sum_{i \in E} |\Pr[Y = i] - \Pr[X = i]| \end{aligned}$$

Proposition 1.5. Given two random variables, we have

$$d_{TV}(X, Y) \leq \Pr[X \neq Y]$$

Proof. For any A ,

$$\begin{aligned} &|\Pr[X \in A] - \Pr[Y \in A]| \\ &= |\Pr[X \in A, Y \in A] + \Pr[X \in A, Y \notin A] - \Pr[Y \in A, X \in A] - \Pr[Y \in A, X \notin A]| \\ &= |\Pr[X \in A, Y \notin A] - \Pr[Y \in A, X \notin A]| \\ &\leq \max \{\Pr[X \in A, Y \notin A], \Pr[Y \in A, X \notin A]\} \leq \Pr[X \neq Y]. \end{aligned}$$

□

Let X_1, \dots, X_n be independent Bernoulli random variables with $\mathbb{E}[X_i] = p_i$. Let $S := \sum X_i$, and $T \sim \text{Poisson}(\lambda := p_1 + \dots + p_n)$. Then $\mathbb{E}[S] = \mathbb{E}[\sum X_i] = \sum \mathbb{E}[X_i] = \sum p_i = \lambda$.

Let Y_1, \dots, Y_n be independent Poisson random variables with $\mathbb{E}[Y_i] = p_i$. Then $T = \sum Y_i \sim \text{Poisson}(\lambda)$. We also have

$$[S \neq T] \subseteq \underbrace{[X_1 \neq Y_1]}_{B_1} \cup [X_2 \neq Y_2] \cup \dots \cup [X_n \neq Y_n]$$

And hence

$$\begin{aligned} \Pr[S \neq T] &\leq \Pr[B_1 \cup \dots \cup B_n] \\ &\leq \Pr[B_1] + \dots + \Pr[B_n] \\ &\leq p_1^2 + \dots + p_n^2 \end{aligned}$$

where $\Pr[X_i = Y_i] = 1 - p + pe^{-p}$, $\Pr[X_i \neq Y_i] = p - pe^{-p} \leq p(1 - (1 - p + p^2/2! + \dots)) = p(p - p^2/2! + \dots) \leq p^2$.

Hence,

$$d_{TV}(S, T) \leq \Pr[S \neq T] \leq \sum_{i=1}^n p_i^2.$$

Consider $X_1 \sim \text{Bernoulli}(p_1 = 1/5)$, $X_2 \sim \text{Bernoulli}(p_2 = 1/6)$, $X_3 \sim \text{Bernoulli}(p_3 = 1/10)$, $S = X_1 + X_2 + X_3$ and $T \sim \text{Poisson}(\lambda = \frac{7}{15})$. Then if estimate T by S , for example,

$$\Pr[S \text{ is an odd number}] \approx \Pr[T \text{ is an odd number}]$$

the probability of getting an error is at most

$$(1/5)^2 + (1/6)^2 + (1/10)^2$$

by letting A be the set of odd numbers.

1.3 Probablity Axioms

Consider the sample space Ω , the set of events \mathcal{F} and the probability P , where

Ω : sample spaces - set of all outcomes

\mathcal{F} : all events

$P : \mathcal{F} \rightarrow [0, 1]$

. Then we can write a random variable X_1 as:

$$X_1 : \Omega \rightarrow \mathbb{R}$$

and an event as

$$B_1 = [X_1 \neq Y_1] = [w \in \Omega | X_1(w) \neq Y_1(w)].$$

Definition 1.6. Event Axioms:

E.1 $\Omega \in \mathcal{F}$

$$\text{E.2 } A \in \mathcal{F} \implies A^C \in \mathcal{F}$$

$$\text{E.3 } A_1, A_2, \dots \in \mathcal{F} \implies A_1 \cup A_2 \cup \dots \in \mathcal{F}$$

Definition 1.7. Probability Axioms:

$$\text{P.1 } A \in \mathcal{F} \implies P(A) \geq 0$$

$$\text{P.2 Countable additivity. } A_1, A_2, \dots \text{ being disjoint events, then } P(A_1 \cup A_2 \cup \dots) = \sum_{i=1}^{\infty} P(A_i).$$

$$\text{P.3 } P(\Omega) = 1.$$

Example 1.8. $X = (\Omega, \mathcal{F}) \rightarrow \mathbb{R}$, and let B be a Borel set. Then we can write $\{X = 3\} = \{w \in \Omega : X(w) = 3\} \in \mathcal{F}$. Similarly, $P(X \in B) \in \mathcal{F}$.

$\Omega = \{a, b, c\}$, $\mathcal{F} = \{\emptyset, \Omega, \{a\}, \{b, c\}\}$. Given $X(a) = 1, X(b) = 2, X(c) = 3$, we have

$$[X = 3] = [w \in \Omega : X(w) = 3] = [c]$$

which is not in the event, so X is not a random variable. If $X(b) = 3$, then X is a random variable.

Given $X \sim \text{Exp}(\lambda), Y \sim \text{Exp}(\mu)$ and $X \perp Y$, then

$$\begin{aligned} &P(X > s, Y - X > t | X < Y) \\ &= P(X > s | X < Y) P(Y - X > t | X < Y) \end{aligned}$$

$$\lambda_n \rightarrow \lambda \implies (1 + \frac{\lambda_n}{n})^n \rightarrow e^\lambda. f(h) = o(h) \implies f(h)/h \rightarrow 0 \text{ as } h \rightarrow 0.$$

Fix x , a function f is differentiable at x iff there exists a number $f'(x)$ such that

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + o(h) \\ \frac{f(x+h) - f(x)}{h} &= f'(x) + o(h)/h, h \rightarrow 0 \end{aligned}$$

For example, if we want to show $n \log(1 + \frac{\lambda_n}{n}) \rightarrow \lambda$. Take $h_n = \lambda_n/n, x = 1$.

$$n \log(1 + h_n) = nh_n + nO(h_n) = nh_n + \frac{\lambda_n}{h_n} O(h_n)$$

where $\log(1 + h) = \log(1) + h + o(h)$. Then as $n \rightarrow \infty$, we have $h_n \rightarrow 0, nh_n \rightarrow \lambda, n \log(1 + \lambda_n/n) \rightarrow \lambda$.

Definition 1.9. Suppose X is nonnegative, integer-valued random variable $P(X = k) = p_k$ for $k = 0, 1, 2, \dots$, then the *probability-generating function* is defined as:

$$G(s) := \mathbb{E}[s^X] = \sum_{k=0}^{\infty} p_k s^k = p_0 + p_1 s + p_2 s^2 + \dots$$

and $G(s) < \infty$ for $|s| < R$.

Then we have

$$\begin{aligned}
G'(s) &= \sum_{k=0}^{\infty} k p_k s^{k-1} = \mathbb{E}[X s^{X-1}] \\
G'(1) &= \mathbb{E}[X] \\
G''(s) &= \sum_{k=0}^{\infty} k(k-1) p_k s^{k-2} = \mathbb{E}[X(X-1) s^{X-2}] \\
G''(1) &= \mathbb{E}[X(X-1)] = \mathbb{E}[X^2 - X] = \mathbb{E}[X^2] - \mathbb{E}[X] \\
\mathbb{E}[X^2] &= G''(1) + G'(1) \\
\text{var}(X) &= G''(1) + G'(1) - [G'(1)]^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2
\end{aligned}$$

$$G(0) = p_0, G'(0) = p_1, \frac{G''(0)}{2} = p_2.$$

Let X, Y be independent nonnegative, integer-value random variables.

$$\begin{aligned}
T &= X + Y \\
\mathbb{E}[s^T] &= \mathbb{E}[s^{X+Y}] = \mathbb{E}[s^X s^Y] = \mathbb{E}[s^X] \mathbb{E}[s^Y]
\end{aligned}$$

Example 1.10. Let X_1, \dots, X_n be i.i.d. Bernoulli random variable.

$$\begin{aligned}
T &= X_1 + \dots + X_n \\
\mathbb{E}[s^T] &= \mathbb{E}[s^{X_1 + \dots + X_n}] = (\mathbb{E}[s^{X_1}])^n = (1 - p + ps)^n \\
\mathbb{E}[s^{X_1}] &= s^0(1 - p) + sp
\end{aligned}$$

Let $X_n \sim \text{Binomial}(n, p_n)$, $p_n \rightarrow 0$, $np_n \rightarrow \lambda$, $n \rightarrow \infty$.

$$\begin{aligned}
G_n(s) &= \mathbb{E}[s^{X_n}] \\
&= (1 - p_n + p_n s)^n \\
&= \left(1 - \frac{np_n}{n} + \frac{np_n s}{n}\right)^n \\
&= \left(1 - \frac{np_n(1-s)}{n}\right)^n \rightarrow e^{-\lambda(1-s)}
\end{aligned}$$

as $n \rightarrow \infty$.

$X \sim \text{Poisson}(\lambda)$,

$$G(s) = \mathbb{E}(s^X) = \sum_{k=0}^{\infty} s^k P(X = k) = \sum_{k=0}^{\infty} s^k \frac{e^{-\lambda} \lambda^k}{k!} = e^{-\lambda} e^{\lambda s} = e^{-\lambda(1-s)}.$$

1.4 Cumulative Distribution Function (c.d.f.)

Definition 1.11. Given a random variable X , its *cumulative distribution function (c.d.f.)* is defined as

$$F(t) := P(X \leq t), -\infty < t < \infty.$$

Given a Borel set A , we have

$$F(A) = P(X \in A)$$

For example, $P(X \in (a, b]) = F(b) - F(a)$.

Definition 1.12 (Convergence in distribution). Let X_n be a sequence of random variables, X be a random variable. Let F_n be the cdf of X_n and F be the cdf of X . We can X_n converges to X in distribution (written as $X_n \xrightarrow{D} X$, or $X_n \rightarrow X$), if

$$\begin{aligned} F_n(t) &\rightarrow F(t), \forall t \in \mathcal{C}(F) \text{ (the continuous domain of } F) \text{] or} \\ \mathbb{E}[h(X_n)] &\rightarrow \mathbb{E}[h(X)], \forall \text{ bounded continuous function of } h \end{aligned}$$

Definition 1.13. We say X_n converges to X in (total) variation if $d_{TV}(X_n, X) \rightarrow 0$ as $n \rightarrow \infty$.

Example 1.14. Let X_n be constant random variable $1/n$ and $X = 0$. For every n , we have

$$d_{TV}(X_n, X) = \sup_A |F_n(A) - F_X(A)|$$

and $P(X_n = 0) = 0$, $P(X = 0) = 1$, so X_n does not converge to X in variation.

Given that $\mathcal{C}(F_X) = (-\infty, 0) \cup (0, \infty)$, we have for every $t \in \mathcal{C}(F_X)$, and for all large n ,

$$\begin{cases} F_n(t) = 1, & \text{if } t \in (0, \infty) \\ F_n(t) = 0, & \text{if } t \in (-\infty, 0) \end{cases}$$

Hence, $F_n(t)$ converges to $F_X(t)$ for every $t \in \mathcal{C}(F)$, so $X_n \xrightarrow{D} X$.

Example 1.15. If you have an n -sided die labelled $1/n, 2/n, \dots, n/n$. Then notice that

$$X_n \xrightarrow{D} U \sim \text{Uniform}(0, 1)$$

because if we consider any $t \in (0, 1)$, $F_U(t) = t$, and $F_n(t) = \frac{k}{n}$ where $(k-1)/n < t \leq k/n$. As $n \rightarrow \infty$, k/n converges to t .

Again X_n does not converge to X in variation. Let Q be the set of rational numbers. $P(X \in Q) = 0$ because Q has measure zero, but $P(X_n \in Q) = 1$. Hence $d_{TV}(X_n, X) = 1$ for every n .

1.4.1 Geometric Distribution to Exponential, the Memoryless variables

Let $T_n \sim \text{Geo}(p_n)$, then $\Pr[T_n = k] = (1 - p_n)^{k-1} p_n$, $k = 1, 2, \dots$, $\Pr[T_n > k] = (1 - p_n)^k$, and $\Pr[T_n > k + j | T_n > k] = \Pr[T_n > j]$. Also, $\mathbb{E}[T_n] = 1/p_n$.

And let $X \sim \text{Exp}(\lambda)$, $f_x(t) = \lambda e^{-\lambda t}$, $t \geq 0$, $\Pr[X > t] = e^{-\lambda t}$, $\Pr[X > t + s | X > t] = \Pr[X > s]$, $\mathbb{E}[X] = 1/\lambda$.

We will show

$$\frac{T_n}{n} \xrightarrow{D} X \sim \text{Exp}(\lambda).$$

First, let F_n be the c.d.f. of T_n and F_X be the c.d.f. of X . We need to show that $F_n(t) \rightarrow F_X(t)$ for all $t \in \mathcal{C}(X)$.

Proof.

$$\begin{aligned} 1 - F_n(t) &= \Pr\left[\frac{T_n}{n} > t\right] = \Pr[T_n > nt] = \Pr[T_n > \lfloor nt \rfloor] \\ &= \left(1 - \frac{np_n}{n}\right)^{\lfloor nt \rfloor} = \left(1 - \frac{\lambda_n}{n}\right)^{\lfloor nt \rfloor} \rightarrow e^{-\lambda t} \end{aligned}$$

where $\lambda_n := np_n \rightarrow \lambda$ as $n \rightarrow \infty$, and the convergence to $e^{-\lambda t}$ is by squeeze theorem. \square

1.5 Point Process

Consider $N \sim \text{Poisson}(\lambda)$ and let X_1, \dots, X_N be i.i.d. Bernoulli(p). Define $Y = \sum_{i=1}^N X_i$. If for each of its count of N , it has p chances to be 1 and $(1 - p)$ to be 0, then we can split N into two Poisson distributions

$$\begin{aligned} Y &\sim \text{Poisson}(\lambda p) \\ Z &\sim \text{Poisson}(\lambda(1 - p)) \end{aligned}$$

where $Z := N - Y$ and we have $Z \perp Y$ (seen that in homework 1).

Definition 1.16. A *point process* on $[0, \infty)$ is a mapping, assigning each Borel set $J \subseteq [0, \infty)$, a nonnegative extended integer valued r.v. $N(J) = N_J$, so that if J_1, J_2, \dots , are disjoint, then

$$N(\cup_i J_i) = \sum_i N(J_i)$$

A counting process associated with N (family of random variables), $N(t) = N_t$ for $t \geq 0$ where $N(t) = N((0, t])$ for $t > 0$. By convention, the sample paths are right continuous.

Definition 1.17. A *Poisson point process* with intensity $\lambda > 0$ is a point process with:

- a) If J_1, J_2, \dots , are nonoverlapping intervals, then $N(J_1), N(J_2), \dots$, are independent.
- b) $N(J) \sim \text{Poisson}(\lambda|J|)$ where J is the length of the interval J .

Given a Poisson Point Process above, let $0 = T_0 < T_1 \leq T_2 \leq T_3 \leq \dots$ be the time i^{th} customer arrives and $\tau_n = T_n - T_{n-1}$. Then τ_1, τ_2, \dots , are i.i.d. $\exp(\lambda)$.

Example 1.18. Let $N(t)$ be the number of customers arriving during $(0, t]$ and $N \sim \text{Poisson}(5)$. The probability of 0 arrivals up to time 2 is

$$\Pr[N(2) = 0] = e^{-5(2)} = e^{-10}$$

While the probability of k arrivals up up time 2 is

$$\Pr[N(2) = k] = \frac{e^{-10} 10^k}{k!}.$$

Consider

$$\begin{aligned}
& \{N(5) = 7 | N(2) = 1\} \\
& \{N((2, 5]) = 6 | N(2) = 1\} \\
& \Pr[N(5) - N(2) = 6 | N(2) = 1] \\
& = \Pr[N(5) - N(2) = 6] \\
& = \Pr[N((2, 5]) = 6] \\
& = \Pr[N(3) = 6]
\end{aligned}$$

We can also consider

$$\Pr[T_2 > 5.8 | T_1 = 3.7] = \Pr[\tau_2 > 2.1 | \tau_1 = 3.7] = e^{-\lambda(2.1)}$$

If you look at the store a 100 min, when will the next customer arrive?

We expect $\frac{1}{\lambda} = \frac{1}{5}\text{hr} = 12\text{min}$.

$$\begin{aligned}
\Pr[X_1 > t] &= \Pr[N(t) = 0] = e^{-\lambda t}, t \geq 0 \\
\Pr[X_2 > t | X_1 = s] &= \Pr[N((s, s + t]) = 0 | X_1 = s] \\
&= \Pr[N((s, s + t]) = 0] \\
&= e^{-\lambda t}
\end{aligned}$$

1.6 Bernoulli and Poisson

Let X_1, X_2, \dots , be Bernoulli Process with $p \in (0, 1)$.

Question:

- a) Is $\Pr[X_n = k | T = n]$ equal $\Pr[X_T = k | T = n]$? **Yes.**
Let $A = \{w \in \Omega : X_n(w) = k\}$, $B = \{w \in \Omega : T(w) = n\}$, $C = \{w \in \Omega : X_{T(w)}(w) = k\}$
and $A \cap B = \{w \in \Omega : X_n(w) = k, T(w) = n\}$, $C \cap B = \{w \in \Omega : X_{T(w)}(w) = k, T(w) = n\}$, which implies $\Pr[A \cap B] / \Pr[B] = \Pr[C \cap B] / \Pr[B]$
- b) Is $\Pr[X_n = k | T = n]$ equal to $\Pr[X_n = k]$? **No.** e.g. $T := \min\{n : X_n = 1\}$, and $\Pr[X_n = 1 | T = n] = 1$, $\Pr[X_n = 1] = p$.
e.g. $X_i \sim \text{Exp}(\lambda)$ where X_1, X_2, \dots , are event times.

$$\begin{aligned}
\Pr[X_2 > t | X_1 = s] &= \Pr[N((s, s + t]) = 0 | X_1 = s] \\
&= \Pr[N(s, s + t] = 0] \text{ by independent increment} \\
&= \Pr[N(X_1, X_1 + t] = 0 | X_1 = s]
\end{aligned}$$

But then let $T := \min\{r : N(r, r + t] = 10\}$. We have

$$\Pr[N(T, T + t) = 0 | T = 3.87] = 0, \Pr[N(3.87, 3.87 + t] = 0] = e^{-\lambda t}$$

Definition 1.19. Let $0 = T_0 < T_1 = \tau_1 \leq T_2 = \tau_1 + \tau_2 \leq \dots$ be the *occurrence times* of a Poisson process which are the successive times $N(t)$ jumps. Let τ_1, τ_2, \dots be the *interoccurrence time*, where $\tau_i := T_i - T_{i-1}$.

Theorem 1.20 (Interoccurrence Time Theorem).

- (A) Interoccurrence times τ_1, τ_2, \dots , of a Poisson process with rate λ are i.i.d. $\text{Exp}(\lambda)$
- (B) Let Y_1, Y_2, \dots , be i.i.d. $\text{Exp}(\lambda)$.

$$N(t) := \max\{n : \sum_{i=1}^n Y_i \leq t\} \implies \{N(t)\}_{t \geq 0} \text{ is a Poisson counting process with rate } \lambda > 0$$

Example 1.21. Consider Bernoulli processes $\{X_k^m\}_{k \in \frac{\mathbb{N}}{m}}$ with parameter $p_m \in (0, 1)$. Then $\tau_1^m = T_1^m = \min\{n \in \frac{\mathbb{N}}{m} : X_n^m = 1\}$. Then $m\tau_1^m \sim \text{Geo}(p_m)$. Let $T_2^m = \min\{n > T_1^m : X_n^m = 1\}$ and $\tau_2^m = T_2^m - T_1^m$, then $m\tau_2^m \sim \text{Geo}(p_m)$ as well. Then with the occurrence time T_i , we have a counting process

$$N^m(t_1) \sim \text{Binomial}(\cdot, p_m)$$

Useful later: $\{T_1 \geq t_1, T_2 \geq t_2\} \iff \{N(t_1) \geq 1, N(t_2) \geq 2\}$.

Theorem 1.22 (The law of small numbers for Bernoulli Process). Let $\{X_r^m\}_{r \in \mathbb{N}/m}$ be a Bernoulli Process with parameter p_m indexed by multipliers of \mathbb{N}/m . Let $N^m(t)$ be the corresponding counting process. If $mp_m \rightarrow \lambda > 0$, then the counting process N^m converges in distribution to the counting process of a Poisson process with rate $\lambda > 0$ in the following sense:

$$\forall n, 0 = t_0 < t_1 < \dots < t_n, (N^m(t_1), \dots, N^m(t_n)) \xrightarrow{D} (N(t_1), \dots, N(t_n))$$

Proof of Interoccurrence Time Theorem.

- a) We showed in the previous section that for a geometric r.v. with p_n with $np_n \rightarrow \lambda$. $T_n/n \xrightarrow{D} \text{Exp}(\lambda)$. And we have seen that the interoccurrence times of Bernoulli $\{X_k^m\}_{k \in \mathbb{N}/n}$ are geometric, $\Delta_k^m = N^m(t_k) - N^m(t_{k-1}) \sim \text{Binomial}(m(t_k - t_{k-1}) \pm 1, p_m)$ where \pm considers the rounding of $m(t_k - t_{k-1})$. And this converges in distribution to $\Delta_k \sim \text{Poisson}(\lambda(t_k - t_{k-1}))$. Thus the occurrence time of $N^m(t)$ converges to $N(t)$ in distribution. Thus, the interoccurrence time of X_k^m , which is the interoccurrence time of $N^m(t)$, converging to $\text{Exp}(\lambda)$ implies that the interoccurrence time of $N(t)$ converges to $\text{Exp}(\lambda)$.
- b) With a Poisson process with rate λ , and let τ_i be its interoccurrence times, and we know $\tau_i \stackrel{iid}{\sim} \text{Exp}(\lambda)$. Let Y_i be another sequence of i.i.d. exponentials with λ . Then since τ_i and Y_i have the same joint distribution, we also have

$$\left(\tau_1, \tau_1 + \tau_2, \dots, \sum_{i=1}^n \tau_i \right) \stackrel{D}{=} \left(Y_1, Y_1 + Y_2, \dots, \sum_{i=1}^n Y_i \right)$$

But $(Y_1, Y_1 + Y_2, \dots, \sum_{i=1}^n Y_i)$ determines the joint distribution of the occurrence time of $N(t)$. That is, the occurrence times of $N(t)$ are the occurrence times of a Poisson distribution. So $N(t)$ is Poisson.

□

Given B), now we can simulate Poisson with $U_i \stackrel{D}{\sim} \text{Uniform}([0, 1])$ and have $\tau_i = -\frac{1}{\lambda} \log(1 - U_i)$. However, if the actual $\lambda > \mu$ and we simulate with μ , then we have

$$\tilde{\tau}_i = -\frac{1}{\mu} \log(1 - U_i) \stackrel{D}{=} \frac{\lambda}{\mu} \tau_k$$

Theorem 1.23 (Generalized Thinning Theorem). Let $N \sim \text{Poisson}(\lambda)$, X_i be iid r.v. with $\Pr[X_i = k] = p_k, k = 1, \dots, m$ and $\sum_{k=1}^m p_k = 1$. And N is independent from X_i for all i . Let $N_k = \sum_{j=1}^N \mathbb{1}_{\{X_j=k\}}$.
e.g:

$$\begin{array}{cccccc} m = 3 & x_1 & x_2 & x_3 & x_4 & x_5 \\ N = 5 & 2 & 3 & 3 & 1 & 2 \end{array}$$

then $N_1 = 1, N_2 = 2, N_3 = 2, N_1 + N_2 + N_3 = N$.

We have that N_1, \dots, N_m are independent Poisson r.v. with $\mathbb{E}[N_k] = \lambda p_k$. (You can consider this as splitting a Poisson process into m different ones with probability p_k .)

And we have

$$\begin{aligned} \Pr[N_1 = j_1, N_2 = j_2, \dots, N_m = j_m] &= \Pr[N = j_1 + \dots + j_m, N_1 = j_1, \dots, N_m = j_m] \\ &= \underbrace{\Pr[N = j_1 + \dots + j_m]}_{\text{Poisson}} \underbrace{\Pr[N_1 = j_1, \dots, N_m = j_m | N = \sum_{i=1}^m j_i]}_{\text{multinomial}} \\ &= \frac{e^{-\lambda} \lambda^{j_1 + \dots + j_m}}{(j_1 + \dots + j_m)!} \binom{j_1 + \dots + j_m}{j_1, \dots, j_m} p_1^{j_1} \dots p_m^{j_m} \\ &= \prod_{i=1}^m \frac{e^{-\lambda p_i} (p_i \lambda)^{j_i}}{j_i!} \end{aligned}$$

Second Construction Let m_1, m_2, \dots be iid $\text{Poisson}(\lambda)$. Let U_1, U_2, \dots be iid $\text{Uniform}(0, 1)$ such that (m_1, m_2, \dots) is independent of (U_1, U_2, \dots) . Put points at U_1, \dots, U_{m_1} if $m_1 > 0$. Put points at $1 + U_{m_1+1}, \dots, 1 + U_{m_1+m_2}$ if $m_2 > 0$ and so on.

Claim 1.23.1. Above points form a Poisson point process (THM 7 of UChicago Notes).

Proof. $0 = t_1 < t_1 < \dots < t_n = 1, J_k = (t_{k-1}, t_k] \implies p_k = t_k - t_{k-1}$. $N(J_1), \dots, N(J_n)$ independent Poisson $\mathbb{E}[N(J_k)] = \lambda p_k = \lambda |J_k|$. \square

Definition 1.24. Poisson point process on \mathbb{R}^k with mean measure Λ is a point process on \mathbb{R}^k with

1. J_1, J_2, \dots disjoint Borel sets in \mathbb{R}^k ; $N(J_1), N(J_2), \dots$ are independent.
2. $N(J_k) \sim \text{Poisson}(\Lambda(J_k))$

Proposition 1.25. To show a point process is a Poisson point process, it suffices to verify the conditions above for rectangles J, J_i with sides parallel to the coordinate axes.

Example 1.26. Let T_i be the occurrence times of a Poisson process on $[0, \infty)$ with rate λ . Let S_j be the iid rv with CDF F . S_j, T_i are indep. Then we have $J = [t_1, t_2] \times [s_1, s_2]$. So $N(J) = \lambda(t_2 - t_1)(s_2 - s_1)$, where $J' \cap J = \emptyset$ implies $N(J)$ independent $N(J')$.

For a Poisson Point Process on \mathbb{R} with rate $\lambda > 0$, then given $t > 0$, we have

$$\begin{aligned}\Pr[N(0, t] = 0] &= e^{-\lambda t} \\ \Pr[N(-t, 0] = 0] &= e^{-\lambda t} \\ \Pr[N(-t, t] = 0] &= e^{-2\lambda t}\end{aligned}$$

Given 2 Poisson Processes on $[0, \infty)$ with $N \sim \text{Poisson}(\lambda)$, $M \sim \text{Poisson}(\mu)$, $\lambda > \mu$, how can we comply them so $N(J) \geq M(J)$ for every Borel set J ?

1. Superposition: Consider M as above and $L \sim \text{Poisson}(\lambda - \mu)$, which are independent, then take the superposition (a process made of all success of M, L) so we get another $\text{Poisson}(\lambda)$.
2. Decomposition: With the N above, for each success of N , split it to M with probability μ/λ , and L with $(1 - \mu/\lambda)$, then M and L are independent Poisson Processes and M is what's required.

Consider N, M with the distributions above, let T_1 be the time of first success of N , then what's the probability that $M(T_1) = k$? If we directly compute it, it will be

$$\Pr[M(T_1) = k] = \int_0^\infty \Pr[M(T_1) = k | T_1 = s] \underbrace{\lambda e^{-\lambda s}}_{\Pr[T_1=s]} ds$$

which is not that easy to compute. But we can consider $N + M \sim \text{Poisson}(\lambda + \mu)$. And split its success to N, M with probability $\frac{\lambda}{\lambda + \mu}$ and $\frac{\mu}{\lambda + \mu}$ respectively. Then T_1 is the time when a success is splitted to N the first time. That is, $M(T_1 = k)$ can be considered as a geometric process with k failure and one success, so

$$\Pr[M(T_1) = k] = \left(\frac{\mu}{\lambda + \mu}\right)^k \left(\frac{\lambda}{\lambda + \mu}\right)$$

Let $\{N(t)\}_{t \geq 0}$ be a counting process on $[0, \infty)$. Prove or disprove: If $N(t) \sim \text{Poisson}(\lambda t)$ for all $t > 0$, then N is a Poisson Process.

Let T_i be the occurrence times and τ_i be the interoccurrence times as before. Then $T_n = \tau_1 + \dots + \tau_n$. If τ_i are independent $\text{Exp}(\lambda)$, we know $T_n \sim \text{Erlang}(n, \lambda)$, so $\mathbb{E}[T_n] = n/\lambda$ and

$$F_n(t) = \Pr[T_n \leq t] = \Pr[N(t) \geq n] = \sum_{k=n}^{\infty} \frac{e^{-\lambda t} (\lambda t)^k}{k!} = 1 - \sum_{k=0}^{n-1} \frac{e^{-\lambda t} (\lambda t)^k}{k!}$$

so if T_1, T_2, \dots , have the "right" distribution, then $N(t)$ will be $\text{Poisson}(\lambda t)$. What if we don't have the independence? Consider $T_i := F_i^{-1}(U)$ where F_i is the cdf of $\text{Erlang}(i, \lambda)$ and $U \sim \text{Uniform}(0, 1)$. Then it's not hard to see that each $T_i \sim \text{Erlang}(i, \lambda)$, however, once T_1 is given, we can compute U_1 and hence all T_2, T_3, \dots are know, so the process with T_i being the occurrence time is not a Poisson.

limits of expectation and expectation of limits

Theorem 1.27 (Monotone Convergence Theorem). Let $\{X_n\}_{n \geq 1}$ be a sequence of random variables such that for all $n \geq 1$,

$$0 \leq X_n \leq X_{n+1}, \text{ Probably a.s.,}$$

then

$$\mathbb{E}[\lim_{n \rightarrow \infty} X_n] = \lim_{n \rightarrow \infty} \mathbb{E}[X_n].$$

Theorem 1.28 (Dominated Convergence Theorem). Let $\{X_n\}_{n \geq 1}$ be a sequence of random variable such that for all ω outside a set \mathcal{N} of null probability there exists $\lim_{n \rightarrow \infty} X_n(\omega)$ and such that for all $n \geq 1$

$$|X_n| \leq Y, \text{ Probably a.s.,}$$

where Y is some integrable random variable. Then

$$\mathbb{E}[\lim_{n \rightarrow \infty} X_n] = \lim_{n \rightarrow \infty} \mathbb{E}[X_n].$$

Example 1.29 ("Counter Example"). Suppose we are rolling a fair dice independently. Every time we get 6, we lose all the money, otherwise, we double the current amount. Starting with $X_0 = 100$, we have

$$X_n = \begin{cases} 100 * 2^n, & \text{with prob } (5/6)^n \\ 0, & \text{with prob } 1 - (5/6)^n \end{cases}$$

$$\begin{aligned} \mathbb{E}[X_n] &= 100 * (5/3)^n \\ \lim_{n \rightarrow \infty} \mathbb{E}[X_n] &= \infty \\ \mathbb{E}[\lim_{n \rightarrow \infty} X_n] &= 0 \end{aligned}$$

where the last inequality is by $\lim_{n \rightarrow \infty} \Pr[X_n > 0] = 0$ and $\lim_{n \rightarrow \infty} \Pr[X_n = 0] = 1$, so $X_n \rightarrow 0$ almost surely.

Let N be a Poisson on $[0, \infty)$ with rate λ . Let $T \geq 0$ be a r.v. such that N, T are independent. If we know the distribution of $N(T)$, can we determine the distribution of T ? First consider the *probability generating function* (p.g.f.) of a Poisson $X \sim \text{Poisson}(\lambda)$, we have

$$G(s) = \mathbb{E}[s^X] = \sum_{k=0}^{\infty} s^k \Pr[X = k] = \sum_{k=0}^{\infty} s^k \frac{e^{-\lambda} \lambda^k}{k!} = e^{-\lambda} e^{\lambda s} = e^{-\lambda(1-s)}$$

Or let x be nonnegative, integer-valued r.v. the *Laplace-Stieltjes Transformation* of X is

$$L(s) = \mathbb{E}[e^{-sX}] = \int_0^{\infty} e^{-st} dF(t) = \int_0^{\infty} e^{-st} F(dt)$$

note this formula prevent us from worrying about the continuity of X by $F(t)$.

Recall the moment generating function (m.g.f.) $m_X(\theta) = \mathbb{E}[e^{\theta X}]$. We give some examples,

Example 1.30.

1. When $\Pr[T = t] = 1$, we have $\mathbb{E}[e^{-sT}] = e^{-st}$.

2. When $T \sim \text{Bernoulli}(p)$,

$$L(s) = \mathbb{E}[e^{-sT}] = (1-p) * 1 + p * e^{-s} = \int_{[0,\infty)} e^{-st} dF(t)$$

3. $T \sim \text{Binomial}(n, p)$. $T = X_1 + \dots + X_n$, where X_i are i.i.d. Bernoulli.

$$\begin{aligned} L(S) &= \mathbb{E}[e^{-sT}] \\ &= \int_{[0,\infty)} e^{-st} dF(t) \\ &= \mathbb{E}[e^{-s(X_1 + \dots + X_n)}] \\ &= \mathbb{E}[e^{-sX_1} \dots e^{-sX_n}] \\ &= \mathbb{E}[e^{-sX_1}] \dots \mathbb{E}[e^{-sX_n}] \\ &= (1-p + pe^{-s})^n \end{aligned}$$

4. Let $X \sim \text{Exp}(\lambda)$, we have

$$\mathbb{E}[e^{-sX}] = \int_0^\infty e^{-st} \lambda e^{-\lambda t} dt = \frac{\lambda}{s + \lambda}. \quad (\text{L.S. of Exp})$$

Lemma 1.31. Given a $N(T) \sim \text{Poisson}(\lambda)$, and N being independent from T , we have $L_T(s) = G(1 - s/\lambda)$.

Proof.

$$\begin{aligned} G(z) &= \mathbb{E}[z^{N(T)}] \\ &= \mathbb{E}[\mathbb{E}[z^{N(T)} | T]] \\ &= \mathbb{E}[e^{-\lambda T(1-z)}] \\ &= L(\lambda(1-z)) \end{aligned}$$

where the second last inequality is by

$$G(z) = \mathbb{E}[z^{N(T)}] = \sum_{k=0}^{\infty} z^k \frac{e^{-\lambda T} (\lambda T)^k}{k!} = e^{-\lambda T(1-z)}.$$

And then let $s = \lambda(1 - z)$, we are done. □

Thus, when $N(T) \sim \text{Poisson}(\lambda T)$,

$$L(s) = G(1 - s/\lambda) = e^{-\lambda T(1 - (1 - s/\lambda))} = e^{-st}$$

so $\Pr[T = t] = 1$.

Theorem 1.32 (Not gonna prove). Like p.g.f. and m.g.f., $L(s)$ uniquely corresponds to a random distribution.

Example 1.33. Let $\Pr[N(T) = k] = \rho^k(1 - \rho)$, $k = 0, 1, \dots$. Then

$$\begin{aligned} G(z) &= \mathbb{E}[z^{N(T)}] = \sum_{k=0}^{\infty} z^k \rho^k (1 - \rho) = \frac{1 - \rho}{1 - \rho z}. \\ L(s) &= \mathbb{E}[e^{-sT}] = G(1 - s/\lambda) = \frac{1 - \rho}{1 - \rho(1 - s/\lambda)} \\ &= \frac{1 - \rho}{1 - \rho + \rho s/\lambda} = \frac{\frac{\lambda}{\rho}(1 - \rho)}{\frac{\lambda}{\rho}(1 - \rho) + s} \end{aligned}$$

which shows that $T \sim \text{Exp}(\frac{\lambda}{\rho}(1 - \rho))$ by **(L.S. of Exp)**.

2 Markov-Chain

Let X_0, X_1, \dots be discrete-time stochastic processes and let the state space be countable.

$$\Pr[X_0 = i_0, \dots, X_n = i_n], \forall n, i_0, \dots, i_n \in \text{state space}.$$

1. Markov Property:

$$\Pr[\underbrace{X_{n+1} = j}_{\text{future}} | \underbrace{X_n = i_n}_{\text{present}}, \underbrace{\dots, X_0 = i_0}_{\text{past}}] = \Pr[X_{n+1} = j | X_n = i_n]$$

2. Time Homogeneity:

$$\Pr[X_{n+1} = j | X_n = i] = \Pr[X_1 = j | X_0 = i] = \Pr(i, j)$$

Definition 2.1. X_0, X_1, \dots is a *discrete-time Markov chain (DTMC)* if X_0, X_1, \dots has the two properties above.

Example 2.2. Let X_0, X_1, \dots be an independent Bernoulli process with parameter p . Then the state space is $\{0, 1\}$.

$$\begin{aligned} \Pr[X_{n+1} = j | X_n = i_n, \dots, X_0 = i_0] &= \Pr[X_{n+1} = j] \\ \Pr[X_{n+1} = j | X_n = i_n] &= \Pr[X_{n+1} = j] \\ \Pr[X_{n+1} = j | X_n = j] &= \Pr(i, j). \end{aligned}$$

This forms a really special DTMC, basically every r.v. are i.i.d.. Its transition matrix looks like

$$P = \begin{bmatrix} 1-p & p & \dots \\ 1-p & p & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

where the rows represent the "from" and columns represent the "to". That is, $[P]_{ij} = \Pr(i, j)$.

Example 2.3. Let $X_0, X_1, \dots \sim \text{Bernoulli}(p), p \in (0, 1)$. $Y_n = X_n + X_{n+1} \in \{0, 1, 2\}$. Is Y_0, Y_1, \dots a Markov Chain? No.

$$\begin{aligned} \Pr[Y_2 = 0 | Y_1 = 1, Y_0 = 0] &= 0 \\ \Pr[Y_2 = 0 | Y_1 = 1, Y_0 = 2] &= 1 - p \end{aligned}$$

because $Y_0 = 0, Y_1 = 1$ implies that $X_2 = 1, X_0 = X_1 = 0$, first probability is the probability that $X_3 = -1$ and the second one is the probability that $X_3 = 0$.

What can we add to make it a DTMC?

Acquire more information. Let $Z_n = (X_n, Y_n)$, then we consider

$$\Pr[Z_{n+1} = (j_1, j_2) | Z_n = (i_1, i_2), Z_{n-1} = (k_{n-1}, \ell_{n-1}), \dots, Z_0 = (k_0, \ell_0)]$$

And the transition matrix is

	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	1-p	p	0	0
(0,1)	0	0	1-p	p
(1,0)	1-p	p	0	0
(1,1)	0	0	1-p	p

$M/M/1$ Queue Consider an $M/M/1$ queue, which is the queue with customers arriving according to $\text{Poisson}(\lambda)$, service time following i.i.d. $\exp(\mu)$ with 1 server. The model records the number of customers whenever a process (arrival or service) is done. Note that this process or a point from the Poisson process does not have to "happen". You can treat all events as a $\text{Poisson}(\lambda + \mu)$. For each point, there is a chance we have a service done, and another chance the we have an arrival. However, since this is an event, when there is 0 customer in the system, next point can still be a departure point, but the number of customers will stay at 0 instead of going to -1 . When there are at least one customer in the system, the server actually serves the customer and make the number of customers minus 1.

For example, if we have $X_0 = 0$ and the next event is finishing a service, $X_1 = 0$, if it's a customer arrival, $X_1 = 1$. This model is also called the birth and death model, basically we add one when we have a birth and minus one when we have a death. Since the moment starts, we can only have "deaths" (or departures) until the first arrival. That is, given $X_n = 0$, the probability that $X_{n+1} = 0$ is the probability that

$$\Pr[D < A] = \frac{\mu}{\lambda + \mu}$$

where $D \sim \exp(\mu)$ is the service time and $A \sim \exp(\lambda)$ is the interoccurrence time of $\text{Poisson}(\lambda)$ (i.e. the arrival time). Similarly, given $X_n = 0$, the probability that $X_{n+1} = 1$ is the probability that the customer arrives before the service time. So the transition matrix looks like

$$\begin{bmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} & 0 & \dots & \dots \\ \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & \dots \\ 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & \dots \end{bmatrix}$$

where rows and columns are from 0 to infinity.

We can also consider $X_n :=$ number of customers in the system just before n -th arrival. For example, given $X_n = 0$, the probability $X_{n+1} = 0$ is $\frac{\mu}{\lambda+\mu}$, because $X_n = 0$, so between n -th and $n + 1$ th arrival, there is at most one customer in the system, and we have the probability $\frac{\mu}{\lambda+\mu}$ to finish the service before $n + 1$ -th arrival, otherwise, with probability $\frac{\lambda}{\mu+\lambda}$, we still have a customer in the system when $n + 1$ -th customer arrives.

Another way of considering this is treating the arrivals as a geometric distribution with $\frac{\lambda}{\lambda+\mu}$ success rate. For example, if $X_n = 1$. That means between n and $n + 1$ arrivals, there are 2 customers in the system, and we do the geometric experiment. The probability that there is no customer in the system when $n + 1$ th customer arrives is the probability we "fail" at least twice before the "success". Similarly, the probability that there is one customer in the system when $n + 1$ th customer arrives is the prob that we "fail" exactly once before the first success, and so on. So the transition matrix looks like:

$$\begin{bmatrix} \left(\frac{\mu}{\lambda+\mu}\right)^2 & \frac{\lambda}{\lambda+\mu} & 0 & \dots & \dots \\ \left(\frac{\mu}{\lambda+\mu}\right)^3 & \left(\frac{\mu}{\lambda+\mu}\right)^2 \frac{\lambda}{(\lambda+\mu)^2} & \frac{\lambda}{\lambda+\mu} & 0 & \dots \\ \left(\frac{\mu}{\lambda+\mu}\right)^4 & \left(\frac{\mu}{\lambda+\mu}\right)^3 \frac{\lambda}{(\lambda+\mu)^3} & \frac{\mu\lambda}{(\lambda+\mu)^2} & \frac{\lambda}{\mu+\lambda} & \dots \end{bmatrix}$$

$M/M/1/3$ Queue Consider the $M/M/1/3$ queue where the 3 means the capacity of the system. Let $Y_n :=$ number of customers in the system just after the n -th departure, so now the state space

is $\{0, 1, 2\}$. Then let's say $Y_n = 0$, then the probability $Y_{n+1} = 0$ is the probability that there is an arrival between n -th and $n + 1$ -th departures. In other words, for $n + 1$ -th departure to happen, there has to be an arrival, so the probability is actually the probability that the $(n + 1)$ -th departure happen before any arrivals except for the necessary one, which is $\frac{\mu}{\lambda + \mu}$, similar to other cases. So the transition matrix looks like:

$$\begin{bmatrix} \frac{\mu}{\mu + \lambda} & \frac{\lambda\mu}{(\mu + \lambda)^2} & \left(\frac{\lambda}{\lambda + \mu}\right)^2 \\ \frac{\mu}{\mu + \lambda} & \frac{\lambda\mu}{(\mu + \lambda)^2} & \left(\frac{\lambda}{\lambda + \mu}\right)^2 \\ 0 & \frac{\mu}{\mu + \lambda} & \frac{\lambda}{\mu + \lambda} \end{bmatrix}$$

2.1 Transition Matrix

Definition 2.4. A matrix P is a *stochastic matrix* if $P(i, j) \geq 0$, and $\sum_{j \in S} P(i, j) = 1$. It is called a *doubly stochastic matrix* if it is a stochastic matrix and $\sum_{i \in S} P(i, j) = 1$. It is called a *substochastic matrix* if $P(i, j) \geq 0$ and $\sum_{j \in S} P(i, j) \leq 1$.

Given $S = \{0, 1, 2\}$, and a transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}. \quad (2.1)$$

We have the transition plot of the above matrix,

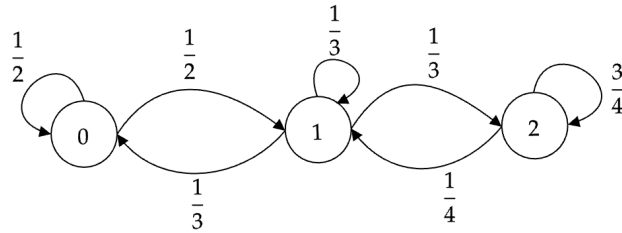


Figure 1: Transition Plot of P

Lemma 2.5. $\Pr[A, B, C, D] = \Pr[A] \Pr[B|A] \Pr[C|AB] \Pr[D|ABC]$

Example 2.6. Given X_0, X_1, \dots , we have

$$\begin{aligned} & \Pr[X_0 = i_0, \dots, X_n = i_n] \\ &= \Pr[X_0 = i_0] \Pr[X_1 = i_1 | X_0 = i_0] \dots \Pr[X_n = i_n | X_{n-1} = i_{n-1}, \dots, X_0 = i_0] \\ &= \Pr[X_0 = i_0] P(i_0, i_1) P(i_1, i_2) \dots P(i_{n-1}, i_n) \end{aligned}$$

Definition 2.7. We use *measure distributions* on S that are functions from S to \mathbb{R} to describe a distribution of a random variable. We use α, β, μ, π to describe row vectors, and use f, g, h to describe column vectors. For example,

$$X_0 \sim \alpha = (1/3, 1/2, 1/6)$$

and a function

$$f = \begin{pmatrix} f(0) \\ f(1) \\ f(2) \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix},$$

then $\alpha f = \mathbb{E}[f(X_0)] \in \mathbb{R}$.

Example 2.8.

$$\begin{aligned} \Pr[X_2 = j | X_0 = i] &= \sum_{k \in S} \Pr[X_2 = j, X_1 = k | X_0 = i] \\ &= \sum_{k \in S} P(i, k) P(k, j) \\ &= P^2(i, j). \end{aligned}$$

For our P , we have $P^2(1, 1) = \frac{1}{6} + \frac{1}{9} + \frac{1}{12}$.

Lemma 2.9 (Chapman-Kolmogorov).

$$P^{m+n}(i, j) = \sum_{k \in S} P^m(i, k) P^n(k, j)$$

where $P^{m+n} = P^m P^n$.

Example 2.10. $\Pr[X_4 = 1, X_2 = 0, X_7 = 1 | X_1 = 2] = P(2, 0) P^2(0, 1) P^3(1, 1)$.

Lemma 2.11.

$$X_0 \sim \alpha \implies X_1 \sim \alpha P, \dots, X_n \sim \alpha P^n$$

And

$$\begin{aligned} \Pr[X_1 = j] &= \sum_i \Pr[X_1 = j | X_0 = i] \Pr[X_0 = i] \\ &= \sum_i \alpha(i) P(i, j) \end{aligned}$$

Example 2.12.

$$\Pr[X_4 = 1 | X_5 = 1] = \frac{\Pr[X_4 = 1, X_5 = 1]}{\Pr[X_5 = 1]} = \frac{\Pr[X_5 = 1 | X_4 = 1] \Pr[X_4 = 1]}{\Pr[X_5 = 1]} = \frac{\alpha P^4(1) P(1, 1)}{\alpha P^5(1)}$$

With the properties above, we can let f be a vector and have

$$\begin{aligned} [Pf]_i &= \mathbb{E}[f(X_1) | X_0 = i] \\ [P^n f]_i &= \mathbb{E}[f(X_n) | X_0 = i] \\ \alpha P^n f &= \mathbb{E}[f(X_n)] \end{aligned}$$

Definition 2.13. An *invariant measure* μ is a measure that $\mu = \mu P$. For our matrix P in (2.1), $\mu = (1, 3/2, 2)$ is an invariant measure.

A *stationary distribution* is an invariant measure that sums to 1. For our P in (2.1), $(2/9, 3/9, 4/9)$ is one.

2.2 Communication, Recurrence and Transience

Definition 2.14. We say j is *accessible* from i if $\exists n \geq 0$ such that $P^n(i, j) > 0$.

We say i and j *communicate* ($i \sim j$) if i is accessible from j and vice versa.

We say i is *absorbing* if $P(i, i) = 1$.

Proposition 2.15. Communication is an equivalent relation being:

- reflective: $i \sim i$, which is always true by letting $n = 0$ and hence $P = I$.
- symmetric: $i \sim j \implies j \sim i$.
- transitive: $i \sim j, j \sim k \implies i \sim k$. (If there exists n with $P^n(i, j) > 0$ and m with $P^m(j, k) > 0$ then $m + n$ takes us from i to k).

Example 2.16. For the following plot, we see that for each state, they only communicate with themselves.

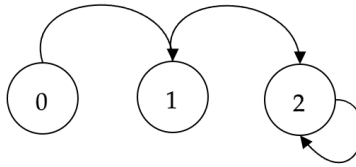


Figure 2: Self Commu States

Definition 2.17. If every two states communicate, then we say this Markov Chain is *irreducible*.

Definition 2.18. The *period* of state i is $d(i)$ defined as the greatest common divider of $\{n > 0 | P^n(i, i) > 0\}$. If $d(i) = 1$ for every state i , then the Markov Chain is *aperiodic*.

Example 2.19. Given the following graph:

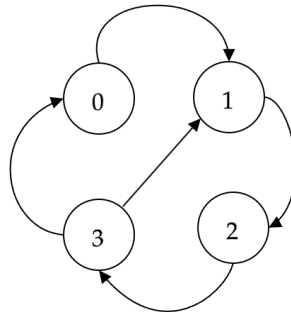


Figure 3: Period 1

Consider $i = 0$, then

$$\{n > 0 | P^n(0, 0) > 0\} = \{4, 7, 10, 13, \dots\} \implies d(0) = 1$$

For the following graph:

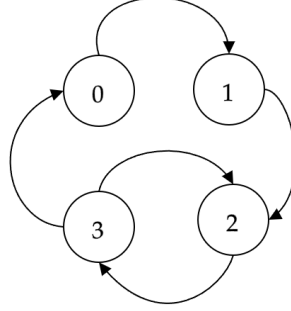


Figure 4: Period 2

Consider $i = 0$, then

$$\{n > 0 | P^n(0, 0) > 0\} = \{4, 6, 8, \dots\} \implies d(0) = 2$$

Proposition 2.20. If i and j communicate, $d(i) = d(j)$.

Proof. We know there exist m and n such that $P^m(i, j) > 0$ and $P^n(j, i) > 0$, so $P^{m+n}(i, i) > 0$, and $m + n$ is a multiplier of $d(i)$. Let ℓ be an integer such that $P^\ell(j, j) > 0$. Then

$$P^{m+n+\ell}(i, i) \geq P^m(i, j)P^\ell(j, j)P^n(j, i) > 0$$

so $m + n + \ell$ is a multiplier of $d(i)$. Hence, we know $m + n + \ell$ is a multiplier of $d(i)$, so ℓ is a multiplier of $d(i)$ which implies $d(j) \geq d(i)$. The argument for $d(i) \geq d(j)$ is similar, so $d(i) = d(j)$. \square

Definition 2.21. T is called a stopping time of $\{T = n\}$ can be determined from X_0, \dots, X_n , i.e.

$$\mathbb{1}_{T=n} = g_n(X_0, \dots, X_n).$$

Example 2.22. $T_x = \inf\{n \geq 0 | X_n = x\}$ is a stopping time. $T_x^k = \text{time of } k^{\text{th}} \text{ visit of } x$ is also a stopping time.

Let T be a stopping time, then

$$\begin{aligned} & \Pr[X_{T+1} = i_{m+1}, X_{T+2} = i_{m+2}, \dots, X_{T+n} = i_{m+n} | T = m, X_m = i_m, X_{m-1} = i_{m-1}, \dots, X_0 = i_0] \\ &= P(i_m, i_{m+1}) \dots P(i_{m+n-1}, i_{m+n}) \end{aligned}$$

and since T is a stopping time, $T = m$ is redundant by knowing X_m, \dots, X_0 . This is called *Strong Markov Property*. That is, Strong Markov Property says that if we know a stopping time $T = m$, then we can treat the Markov chain after T as one Markov chain Y with the same transition matrix P but starting with $Y_0 = X_m$.

Definition 2.23. Let $T_x^1 = T_x = \inf\{n \geq 1 | X_n = x\}$, $T_x^k = \inf\{n > T_x^{k-1} | X_n = x\}$, $k = 2, 3, \dots$, and $\Pr[X_0 = x] = 1$.

- State x is *recurrent* if $\Pr_x[T_x < \infty] = 1$.
- State x is *transient* if $\Pr_x[T_x < \infty] < 1$.
- State x is *positive recurrent* if $\mathbb{E}_x[T_x] < \infty$.
- State x is *null* if x is recurrent and $\mathbb{E}_x[T_x] = \infty$.

Example 2.24. Let $\Pr[X = k] = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ for $k = 1, 2, \dots$. Then

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \frac{1}{k+1} = \infty$$

$$\Pr[X \leq n] = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \dots + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1}$$

Suppose x is recurrent. How many times will x be revisited is represented

$$N_x = \sum_{k=0}^{\infty} [X_k = x].$$

Suppose state x is transient, by Strong Markov property,

$$\Pr[T_x^k < \infty] = \Pr_x[T_x < \infty]^k.$$

Assuming $X_0 = x$, $N_x \sim \text{Geo}(\Pr[T_x = \infty])$. That is, N_x stops (the number will not increase) once we fall into the case X_n never comes to x .

Proposition 2.25. State x is recurrent if and only if $\mathbb{E}_X[N_X] = \infty$.

Proof.

$$\begin{aligned} \mathbb{E}_X[N_X] &= \mathbb{E}_X \sum_{k=0}^{\infty} \mathbb{1}[X_k = x] \\ &= \sum_{k=0}^{\infty} \mathbb{E}_x[\mathbb{1}[X_k = x]] \\ &= \sum_{k=0}^{\infty} \Pr_X[X_k = x] = \sum_{k=0}^{\infty} P^k(x, x) \\ N_X &= 1 + \sum_{k=1}^{\infty} \mathbb{1}[T_x^k < \infty] \end{aligned}$$

$$\begin{aligned}
\mathbb{E}_X[N_X] &= 1 + \sum_{k=1}^{\infty} \mathbb{E}[\mathbb{1}[T_x^k < \infty]] \\
&= 1 + \sum_{k=1}^{\infty} \Pr[T_x^k < \infty] \\
&= 1 + \sum_{k=1}^{\infty} \Pr[T_x < \infty]^k \\
&= \begin{cases} \infty, & \text{if recurrent.} \\ \frac{1}{1 - \Pr[T_x < \infty]}, & \text{transient.} \end{cases}
\end{aligned}$$

□

Proposition 2.26. If x is recurrent and x, y communicate, then y is recurrent.

Proof. There exists k such that $P^k(x, y) > 0$, and there exists ℓ such that $P^\ell(y, x) > 0$.

$$\sum_{n=1}^{\infty} P^{k+\ell+n}(y, y) \geq \sum_{n=1}^{\infty} P^\ell(y, x) P^n(x, x) P^k(x, y) = \infty.$$

which implies that y is recurrent.

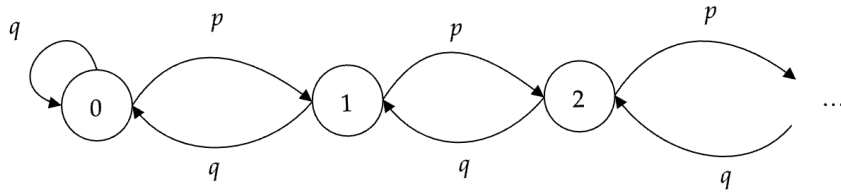
□

Example 2.27.

$$P = \begin{bmatrix} 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{6} & 0 & \frac{5}{6} \\ \frac{1}{6} & \frac{5}{6} & 0 \end{bmatrix}$$

and all states are recurrent.

Example 2.28. Consider the below Markov chain with $0 < p < 1$.



Consider the probability of starting at 1 and first time visit 0 at k ,

$$P_1[T_0 = k] = p_k,$$

and we have

$$\begin{aligned}
\Phi(s) &= \sum_{k=0}^{\infty} p_k s^k \\
\Phi(s) &= qs + ps\Phi(s)
\end{aligned}$$

where the second equality is by the fact that, $T_0 = 1$ when we go from 1 to 0 directly with probability q , otherwise, we go to 2 in the first step and then consider the steps required for us to go from 2 to 0, which is 2 to 1 then 1 to 0. In other words, we write

$$\begin{aligned}
\Phi(s) &= \sum_{k=0}^{\infty} p_k s^k \\
&= 0 * 1 + qs + \sum_{k=2} p_k s^k \\
&= qs + s \sum_{k=0} p_{k+1} s^k \\
&= qs + ps \sum_{k=0} P_2[T_0 = k] s^k \\
&= qs + ps \mathbb{E}[s^{X+Y}]
\end{aligned}$$

where $p_{k+1} = p * P_2[T_0 = k]$, and X is the random variable of number of steps from 0 to 1 and Y is from 2 to 1 which follow the same distribution as T_0 starting at 1 and are independent, so $\mathbb{E}[s^{X+Y}] = \mathbb{E}[s^X] \mathbb{E}[s^Y] = \Phi^2(s)$.

Then we can have that

$$\begin{aligned}
\Phi(s) &= \frac{1 - \sqrt{1 - 4pqs^2}}{2ps} \\
\Phi(1) &= \frac{1 - \sqrt{1 - 4pq}}{2p} = \begin{cases} 1, & \text{if } p \leq 1/2 \\ \frac{q}{p}, & \text{if } p > 1/2 \end{cases}
\end{aligned}$$

That is, when $p > 1/2$, there is a chance we never go to 0. Or we can find the expectation by

$$E_1[T_0] = \lim_{s \rightarrow 1} \Phi'(s).$$

Definition 2.29. We call π a *stationary distribution* for a Markov chain with transition matrix P , if

$$\pi = \pi P, \sum \pi(i) = 1.$$

Example 2.30.

$$(\pi(0), \pi(1), \pi(2)) \begin{bmatrix} 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{6} & 0 & \frac{5}{6} \\ \frac{1}{6} & \frac{5}{6} & 0 \end{bmatrix} = (\pi(0), \pi(1), \pi(2))$$

solve to get

$$(\pi(0), \pi(1), \pi(2)) = (11/77, 31/77, 35/77)$$

and then

$$\mathbb{E}_0[T_0] = \frac{77}{11} = \frac{1}{\pi(0)}$$

because now we can consider it as a geometric distribution with parameter $\pi(0)$, starting from $X_0 = 0$, you have 11/77 chance to get 0 at X_1 , similarly, if you get $X_1 \neq 0$, then you still have

11/77 for $X_2 = 0$ by π being stationary, and so on.

We can also consider the central limit theorem which gives:

$$\frac{f(x_0) + \dots + f(x_n)}{n+1} \rightarrow \pi f$$

for a function f valued on the states of the Markov chain X_i .

Example 2.31 (x -excursion chain). Let X_0, X_1, \dots be an irreducible Markov chain with stationary distribution π , transition matrix P and state space S . Let consider words (or strings if you prefer) that are finite, starting with x and containing only one x , call the set of all such words, S_y . Consider random variables Y_i with state space S_y , defined as

$$\begin{aligned} Y_0 &= x \\ Y_1 &= xX_1 \\ Y_2 &= xX_1X_2 \\ Y_3 &= xX_1X_2X_3 \\ &\vdots \end{aligned}$$

where we keep $X_0 = x$. So

$$\Pr[Y_3 = xy_1y_2y_3] = P(x, y_1)P(y_1, y_2)P(y_2, y_3).$$

and we can build the transition matrix Q for Y_i as

$$\begin{aligned} Q(xy_1 \dots y_k, xy_1 \dots y_k y_{k+1}) &= P(y_k, y_{k+1}) \\ Q(xy_1 \dots y_k, xy_1 \dots y_k x) &= P(y_k, x) \\ Q(x, xy) &= P(x, y) \\ Q(x, x) &= P(x, x). \end{aligned}$$

And we let $F : S_y \rightarrow S$ where $F(w)$ is the last letter of w .

Fact 2.32. If Y_0, Y_1, \dots is a Markov chain with transition matrix Q and state space S_y , then $F(Y_0), F(Y_1), \dots$ is a Markov chain with state space S and transition matrix P .

Now let's consider the stationary distribution for Y . Let ν be a stationary distribution of Y_i , then

$$\begin{aligned} \nu &= \nu Q \\ \nu(w) &= \sum_{w' \in S_y} \nu(w')Q(w', w), \quad \sum_{w \in S_y} \nu(w) = 1 \end{aligned}$$

Let $w = xy_1 \dots y_{k-1}y_k y_{k+1}$, we have

$$\begin{aligned} \nu(xy_1 \dots y_{k+1}) &= \nu(xy_1 \dots y_k)Q(y_k, y_{k+1}) \\ \nu(xy_1 \dots y_k) &= \nu(x)P(x, y_1)P(y_1, y_2) \dots P(y_{k-1}, y_k) \end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{w \in S_y} \nu(w) &= \nu(x) + \sum_{k=1}^{\infty} \sum_{y_1 \dots y_k} \nu(x) P(x, y_1) P(y_1, y_2) \dots P(y_{k-1}, y_k) \\
&= \nu(x) + \nu(x) \sum_{k=1}^{\infty} \sum_{y_1 \dots y_k} P(x, y_1) P(y_1, y_2) \dots P(y_{k-1}, y_k) \\
&= \nu(x) P_x(T_x > 0) + \nu(x) \sum_{k=1}^{\infty} P_x(T_x > k) \\
&= \nu(x) \sum_{k=0}^{\infty} P_x(T_x > k) \\
&= \nu(x) \mathbb{E}_x[T_x] = 1
\end{aligned}$$

If state x is recurrent, then we have $\nu(x) = \frac{1}{\mathbb{E}_x[T_x]}$, otherwise, Q does not have a stationary distribution. Thus if X_0, X_1, \dots has a positive recurrent state x , then there exists at least one stationary distribution ν by the fact $\nu(w)$ can be defined by $\nu(x)$ and $P(x, y_1), \dots, P(y_{k-1}, y_k)$.

If $Y_0 \sim \nu$, and $Y_1, \dots \sim \nu$, let $\pi(z) = \sum_{w, F(w)=z} \nu(w)$, we have $\pi = \pi P$ and $\sum_{x \in S} \pi(x) = 1$.

Example 2.33. We consider a Markov chain X_0, X_1, \dots . For the case we start with $X_0 = x$, denote P_x , if we start with $X_0 = y$, denote P_y . Let $\tau(i)$ be the time we have the i -th x excluding X_0 , that is, $\tau(1) = T_x$, $\tau(2) = T_x^2$ and $\tau(0) = 0$. Define

$$\begin{aligned}
W_1 &= (X_0, X_1, \dots, X_{\tau(1)-1}) \\
W_2 &= (X_{\tau(1)}, \dots, X_{\tau(2)-1}) \\
&\vdots
\end{aligned}$$

Under P_x , the words W_1, W_2, \dots are i.i.d. Under P_y , $y \neq x$, the words W_1, W_2, \dots are independent, and W_2, W_3, \dots are identically distributed. Let $W_j = (X_{j,1}, \dots, X_{j,m(j)})$, then

$$\begin{aligned}
&P_x(W_1 = w_1, W_2 = w_2, \dots, W_k = w_k) \\
&= \prod_{j=1}^k \left(\prod_{\ell=1}^{m(j)-1} P(X_{j,\ell}, X_{j,\ell+1}) \right) P(X_{j,m(j)}, x) \\
&= \prod_{j=1}^k P(W_j = w_j)
\end{aligned}$$

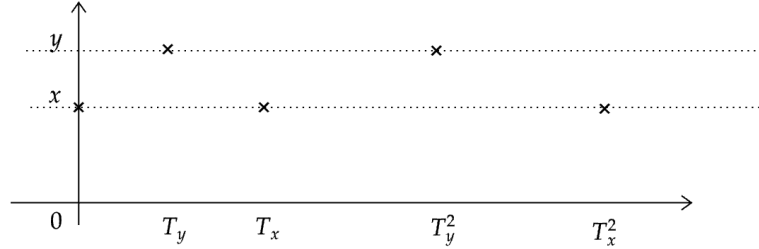
For P_y , $X_{1,1} = y$, all other $X_{j,1}$ remains at x , so w_2, w_3, \dots are identically distributed.

Proposition 2.34. WLOG, assume $x \neq y$, if x and y communicate, and x is positive recurrent, then y is positive recurrent.

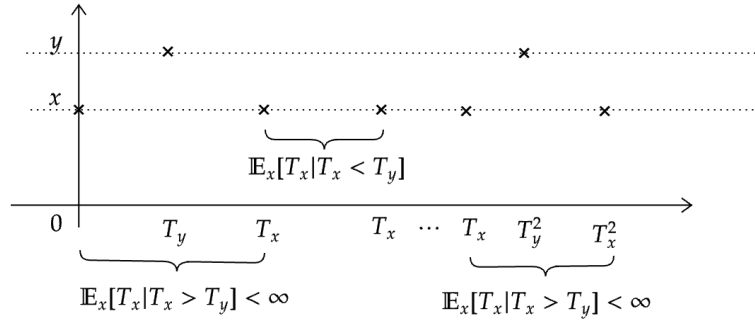
Proof.

$$\infty > \mathbb{E}_x[T_x] = \mathbb{E}_x[T_x | T_x > T_y] P_x[T_x > T_y] + \mathbb{E}_x[T_x | T_x < T_y] P_x[T_x < T_y]$$

If $P_x[T_x < T_y] = 0$, then $\mathbb{E}_y[T_y] \leq 2\mathbb{E}_x[T_x] < \infty$. The reason is that, we have $T_y \leq T_x$, then $\mathbb{E}_y[T_y]$ can be considered as $\mathbb{E}_x[T_y^2] - \mathbb{E}_x[T_y]$, but by $P_x[T_x < T_y] = 0$, we know for if we start at $X_0 = x$, then $T_y^2 \leq T_x^2$, see the plot below



If $P_x[T_x < T_y] > 0$, consider the plot



Similar, we have $\mathbb{E}_y[T_y] < \infty$. □

2.3 Stationary Distribution and Positive Recurrence

Consider a random variable X , we can write it as $X = X^+ + X^-$, where $X^+ := \max(X, 0)$ and $X^- := \max(-X, 0)$. If both $\mathbb{E}[X^+]$, $\mathbb{E}[X^-]$ are well-defined with value in $[0, \infty]$. Then

$$\mu := \mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-]$$

unless it is $\infty - \infty$. To avoid this, we can assume either X is nonnegative, or $|X|$ integrable ($\mathbb{E}[|X|] < \infty$), or $\mathbb{E}[X^-] < \infty$, then we have $\mu < \infty$ or μ is well-defined as ∞ .

Theorem 2.35 (Strong Law of Large Number). Consider $S_n = X_1 + \dots + X_n$

1. If X_1, X_2, \dots are pairwise i.i.d. integrable with mean μ , then

2. Or if X_1, X_2, \dots are i.i.d. with $\mathbb{E}[X^+] = \infty$, $\mathbb{E}[X^-] < \infty$ with mean $\mu < \infty$, then

$$\frac{S_n}{n} \rightarrow \mu \text{ a.s. w.p. } 1$$

almost surely with probability 1.

When we say with almost surely with probability 1, we mean that the set

$$A = \left\{ \omega \in \Omega : \frac{S_n(\omega)}{n} \rightarrow \mu \right\}$$

has a probability 1 when $n \rightarrow \infty$.

Example 2.36. Recall our "string" example, where $W_1 = (X_0, \dots, X_{\tau(1)-1})$, $W_2 = (X_{\tau(0)}, \dots, X_{\tau(2)-1})$, \dots . Under P_x (start with $X_0 = x$), W_1, W_2, \dots are i.i.d., while under P_y , for $y \neq x$, W_2, W_3, \dots are i.i.d. and W_1, W_2, \dots are independent. Write $W_j = (X_{j,1}, \dots, X_{j,m(j)})$, then

$$\Pr_x[W_1 = w_1, \dots, W_k = w_k] = \prod_{j=1}^k \left(\prod_{\ell=1}^{m(j)-1} P(x_{j,\ell}, x_{j,\ell+1}) \right) P(x_{j,m(j)}, x).$$

Definition 2.37. Let $f : S \rightarrow \mathbb{R}_+$. The additive extension to the set of finite "words" with letters in S is the function f_+ where for $w = (x_1, \dots, x_m)$,

$$f_+(w) = \sum_{i=1}^m f(x_i).$$

For any initial state $y \in S$ by the Strong Law of Large Number,

$$\lim_{k \rightarrow \infty} \frac{\sum_{i=1}^k f_+(x_i)}{k} = \mathbb{E}_x[f_+(w_1)] = \mathbb{E}_x \left[\sum_{j=0}^{\tau(1)-1} f(x_j) \right]$$

with P_y almost surely, because if $y \neq x$, then

$$\frac{f_+(w_1) + \dots + f_+(w_k)}{k} = \frac{f_+(w_1)}{k} + \frac{f_+(w_2) + \dots + f_+(w_k)}{k-1} \frac{k-1}{k} \rightarrow 0 + \mathbb{E}_x[f_+(w_2)] * 1.$$

In particular, if we set $f \equiv 1$, then

$$\lim_{k \rightarrow \infty} \tau(k)/k = \mathbb{E}_x[\tau(1)]$$

with P_y almost surely.

Let N_n^x = the number of visits to state x up to time $n = \sum_{k=1}^n \mathbb{1}\{X_k = x\}$.

Theorem 2.38. Fix $x \in S$. If the Markov Chain is irreducible and positive recurrent, then $\exists!$ (there exists a unique) stationary distribution π and for all states x, y ,

$$\lim_{n \rightarrow \infty} N_n^x/n = \pi(x), \text{ } P_y\text{-a.s.}$$

If the chain is null recurrent, then there does not exist a stationary distribution and for all x, y ,

$$\lim_{n \rightarrow \infty} N_n^x/n = 0, \text{ } P_y\text{-a.s.}$$

Proof. First, we show $N_n^x/n \rightarrow 1/\mathbb{E}_x[T_x]$, P_y -a.s. Note, $N_n^x \leq n$, and $N_n^x \rightarrow \infty$ P_y a.s.,

$$\frac{\tau(N_n^x)}{N_n^x} \leq \frac{n}{N_n^x} < \frac{\tau(1+N_n^x)}{1+N_n^x} \frac{1+N_n^x}{N_n^x}.$$

where $n < \tau(1+N_n^x)$. And $\frac{\tau(N_n^x)}{N_n^x} \rightarrow \mathbb{E}_x[\tau(1)]$, $\frac{\tau(1+N_n^x)}{1+N_n^x} \rightarrow \mathbb{E}_x[\tau(1)]$, so $n/N_n^x \rightarrow \mathbb{E}_x[\tau(1)]$ with P_y -a.s..

Second, assume the Markov Chain has a stationary distribution π , then define $P_\pi(\cdot) = \sum_y \pi(y)P_y(\cdot)$,

$$N_n^x/n \rightarrow 1/\mathbb{E}_x[T_x], \text{ } P_\pi\text{-a.s.}$$

by P_y -a.s and

$$\lim_{n \rightarrow \infty} \mathbb{E}_\pi[N_n^x/n] = \mathbb{E}_\pi \lim_{n \rightarrow \infty} N_n^x/n = \mathbb{E}_\pi[1/\mathbb{E}_x[T_x]] = 1/\mathbb{E}_x[T_x]$$

where the first equality is by $|N_n^x/n| \leq 1$, $\mathbb{E}_\pi(1) = 1 < \infty$ by Dominant Consequence Theorem. The above equation is equivalent to

$$\lim_{n \rightarrow \infty} \mathbb{E}_\pi[N_n^x/n] = \lim_{n \rightarrow \infty} \mathbb{E}_\pi \frac{\sum_{j=1}^n \mathbb{1}[X_j = x]}{n} = \lim_{n \rightarrow \infty} \frac{\pi(x)}{n} = \pi(x)$$

by π being stationary, $\mathbb{E}_x[\mathbb{1}[X_j = x]] = 1 * P_\pi(x) = \sum_y \pi(y)P_y(x) = \pi(x)$. Hence, for all state x ,

$$\pi(x) = \frac{1}{\mathbb{E}_x[T_x]}.$$

For the positive recurrent case, π is uniquely defined as above. If it's the null recurrent case, then $\mathbb{E}_x[T_x] = \infty$, $\pi(x) = 0$, which is not even a distribution. \square

Lemma 2.39. If X_0, X_1, \dots is recurrent, then the invariant measure is unique up to multiplication by constants.

Proof. See Bremaul's book. \square

Combining the Lemma and Theorem, we know a recurrent Markov Chain's invariant measure sometimes does not give a stationary distribution because the sum of measure goes to infinity.

2.4 Period

2.4.1 Fundamental Theorem of Markov Chain

Let a_1, a_2, \dots be a sequence of integers. $d_k = g.c.d.(a_1, \dots, a_k)$, if $1 \leq d_k$ is nondecreasing and $d_k \rightarrow d$, then there exists k_0 such that $d_k = d$ for $k \geq k_0$.

Lemma 2.40. Let $S \subseteq \mathbb{Z}$ contain at least one non-zero element and be closed under addition and subtraction. Then S contains a smallest, positive integer a and $S = \{ka : k \in \mathbb{Z}\}$.

Proof. Let $c \in S$ with $c \neq 0$, then $0 = c - c \in S$ and $-c = 0 - c \in S$. Hence S contains at least one positive, one negative value. Then S contains a smallest positive element a . So

$$\begin{aligned} a, 2a, 3a, \dots &\in S \\ -a, -2a, -3a, \dots &\in S \end{aligned}$$

so $\{ka : k \in \mathbb{Z}\} \subseteq S$. Let $c \in S$, $c = ka + r$, $0 \leq r \leq a - 1$, $r \in \mathbb{Z}$. And $0 \leq r = c - ka \in S$ by subtraction, but $r < a$ and a is the smallest positive integer in S , so $r = 0$. \square

Lemma 2.41. Let a_1, a_2, \dots, a_k be positive integer with g.c.d. d , there exist $n_1, n_2, \dots, n_k \in \mathbb{Z}$ such that $d = \sum_{i=1}^k n_i a_i$.

Proof. The set $S = \{\sum_{i=1}^k n_i a_i : n_1, \dots, n_k \in \mathbb{Z}\}$ is closed under additions and subtractions. So $S = \{ka : k \in \mathbb{Z}\}$ with $a = \sum_{i=1}^k n_i a_i$ being the smallest positive integer in S . Hence, d is a divisor of a by $a = \sum_{i=1}^k n_i a_i$. Then by $a_i = ka$, we know a is a divisor of a_i , so $a \leq \text{g.c.d.}(a_1, \dots, a_k) = d$, so $a = d$. \square

Theorem 2.42. $A = \{a_1, a_2, \dots\}$ which is a set of positive integers. Let $d = \text{g.c.d.}(A)$, and A is closed under addition. Then A contains, all but a finite number of multiples of d .

Proof. WLOG, $d = 1$. For some k , we have $d = \text{g.c.d.}(a_1, \dots, a_k)$. By Lemma (2.41).

$$1 = \sum_{i=1}^k n_i a_i, \text{ for some } n_1, \dots, n_k \in \mathbb{Z}, 1 = M - P, \text{ where } M \geq 0, P < 0, M, P \in A.$$

Let $n \in \mathbb{N}$, $n \geq P(P - 1)$, $n = aP + r$, $0 \leq r \leq P - 1$, so $a \geq P - 1$ (If $a \leq P - 2$, $aP + r < P(P - 1)$). By $1 = M - P$, we have

$$n = aP + r(M - P) = (a - r)P + rM$$

and $a - r \geq 0$ by $a \geq P - 1 \geq r$, which implies $n \in A$. Hence, $n \in A$ except for $n < P(P - 1)$, $n \in \mathbb{N}$. \square

Theorem 2.43 (Fundamental Theorem of Markov Chain). For an irreducible positive recurrent aperiodic Markov X_i chain with the stationary distribution π and transition matrix P , we have

$$\lim_{n \rightarrow \infty} \Pr[X_n = j] = \lim_{n \rightarrow \infty} P^n(i, j) = \pi(j)$$

Proof. Consider two sequence of variables. Let $x = X_0, X_1, \dots$ be the Markov chain starting with x , and X_0^*, X_1^*, \dots be a Markov chain where each $X_i^* \sim \pi$.

We have

$$\begin{aligned} |\Pr_x[X_n = y] - \Pr_\pi[X_n = y]| &= |\Pr_x[X_n = y] - \pi(y)| \\ &= |\Pr[X_0 = y, X_n^* = y] + \Pr[X_n = y, X_n^* \neq y] \\ &\quad - \Pr[X_n = y, X_n^* = y] - \Pr[X_n^* = y, X_n \neq y]| \\ &\leq \Pr[X_n \neq X_n^*] \end{aligned}$$

and we want to show $\Pr[X_n \neq X_n^*]$ goes to 0. Let $\tau := \min\{n \geq 0 : X_n = X_n^*\}$. And consider another independent Markov chain X'_0, X'_1, \dots which use the same transition matrix P . Consider a Markov chain V_n and its transition matrix Q :

$$V_n = (X_n, X'_n), \Pr[V_{n+1} = (y, y') | V_n = (x, x'), V_{n-1}, \dots, V_0] = Q((x, x'), (y, y')) = P(x, y)P(x', y').$$

V_n has a stationary distribution where $\pi(x, x') = \pi(x)\pi(x')$ and

$$\begin{aligned} \pi(y, y') &= \sum_x \sum_{x'} \pi(x, x') Q((x, x'), (y, y')) \\ &= \sum_x \sum_{x'} \pi(x)\pi(x') P(x, y)P(x', y') \\ &= \sum_x \pi(x)P(x, y) \sum_{x'} \pi(x')P(x', y') \\ &= \pi(y)\pi(y') \end{aligned}$$

Consider

$$A_x = \{n \geq 1 : P^n(x, x) > 0\}$$

Then Theorem 2.42, there exists n_x such that $\forall n \geq n_x, P^n(x, x) > 0$ and there exists $k_{x,y}$ such that $P^{k_{x,y}}(x, y) > 0$, so $P^{n+k_{x,y}}(x, y) \geq P^{k_{x,y}}(x, y)P^n(x, x)$. Hence

$$P^n(x, y) > 0, \forall n \geq k_{x,y} + n_x,$$

similarly, we also have

$$P^n(x', y') > 0, \forall n \geq k_{x',y'} + n_{x'}.$$

Then for all $n \geq \max\{k_{x,y} + n_x, k_{x',y'} + n_{x'}\}$, we have

$$Q^n((x, x'), (y, y')) > 0,$$

so V_n is irreducible and aperiodic (by letting $y, y' = x, x'$) and positive recurrent by having a stationary distribution.

Hence, all states are expected to be visited in finite time. $\tau' = \min\{n \geq 0 : X_n = X'_n\}$, $\tau' < \infty$ almost surely by considering arbitrary (x, x) . Consider

$$\bar{X}_n = \begin{cases} X'_n, & n \leq \tau' \\ X_n, & n > \tau' \end{cases}.$$

By the Strong Markov Property, the part of X'_n and X_n for $n \geq \tau'$ are i.i.d. Markov chain, so the \bar{X}_n we construct follow the same distribution as X_n^* follows. That is,

$$\Pr[X_n \neq X_n^*] = \Pr[X_n \neq \bar{X}_n] = \Pr[\tau' > n] \rightarrow 0$$

by $\tau' < \infty$ almost surely. □

2.5 Reversibility

Let P be a transition matrix for an irreducible Markov chain. Take a guess for stationary distribution π and a reverse transition matrix \tilde{P} . If $\pi(j)\tilde{P}(j, i) = \pi(i)P(i, j)$, then both guesses are right, we know this Markov chain is reversible with \tilde{P} and positive recurrent with π