ISyE 7203: Logistics and System Engineering

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Contents

1 Network Flows

Definition 1.1

A network (graph) *is made of the following:*

- (N) *nodes (vertices): a finite set.*
- *For undirected network:* (E) *edges:* $\{i, j\}, i, j \in N$ *drawn as* $i - j$.
- *For directed network:* (A) *arcs* $(i, j), i, j \in N$ *drawn as* $i \to j$, *where we call* i *the tail and* j *the head of the arc.*

For this course, we focus on the directed networks and the following definitions are all on directed networks.

Definition 1.2

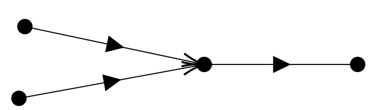
- path*: a sequence of arcs where the head of the first is the tail of the second and so forth, and no nodes repeat.*
- walk*: a sequence of arcs where the head of the first is the tail of the second and so forth, and nodes may repeat.*
- cycle*: a path except that the first and the last nodes are the same. If a cycles has repeated nodes in between, it is a* closed walk *instead.*

Definition 1.3: Connectedness

A directed network is:

- strongly connected*: for any* $i, j \in N$*, there exists an* $i - j$ *(directed) path and a* $j - i$ *directed path.*
- weakly connected*: for any* $i, j \in N$*, there exists an (undirected)* $i - j$ *path if we ignore orientations.*

Example 1.1. Consider the following network:



It is weakly but not strongly connected.

Definition 1.4: network flow

Given the network by (N, A)*,* $b \in \mathbb{R}^N, c \in \mathbb{R}^A, u \in (\mathbb{R} \cup \{\infty\})^A$. b *represents the net supply where* i *is a supplier, consumer, trans-shipment respectively if* $b_i > 0, b_i < 0, b_i = 0$ *respectively;* c *represents the cost, and* u *represent the capacity.*
For modeling, X_{ij} *is used to represent the amount of flow on* $(i, j) \in A$*. Then the* network flow problem *is:*

$$\begin{aligned} \min \ & \sum_{a \in A} c_a x_a \\ \textit{s.t.} \ & \sum_{a \in \delta^+(i)} - \sum_{a \in \delta^-(i)} x_i = b_i, i \in N \\ & 0 \le x_a \le u_a, \ \forall a \in A \end{aligned} \tag{NF}$$

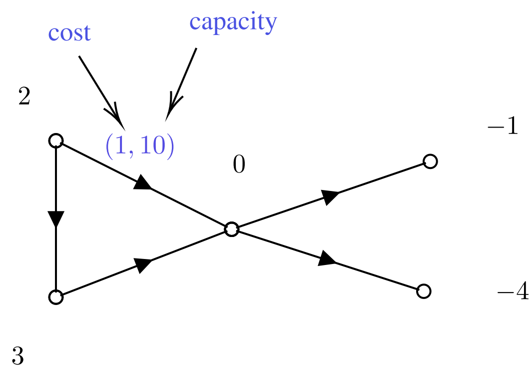
where $\delta^+(i)$ *is the set of arcs having* i *as the tail and* $\delta^-(i)$ *is the set of arcs having* i *as the head. The equality constraints are called the flow balance constraint.*

Remark. Aggregate the flow balance constraints, we get

$$\sum_i \left(\sum_{a\in\delta^+(i)} - \sum_{a\in\delta^-(i)} x_i \right) = \sum_{a\in A} x_a - x_a = 0 = \sum_i b_i.$$

Thus, $\sum_{i=1}^N b_i = 0$ is a needed assumption.

Example 1.2. The following is an instance of a network flow problem:



Theorem 1.5

The constraint matrix of NF is always TU (totally unimodular). Thus, if b, u are integral, all extreme points are integral.

Example 1.3. Assignment Problem Considering assigning n workers to n tasks, one for each. Let c_{ij} be the cost of assigning worker i to task j. In this way, we can consider $1, \ldots, n$ nodes as workers and $n+1, \ldots, 2n$ as tasks. For each node representing a worker, it has arcs from it to all task nodes. The supply of each worker and task are 1 and -1 respetively. Then the problem can be written as:

$$\begin{aligned} \min\ & \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} \\ \text{s.t.}\ & \sum_{j=n+1}^{2n} x_{ij} = 1,\ i = 1, \ldots, n \\ & \sum_{i=1}^n -x_{ij} = -1 \\ & 0 \leq x_{ij}. \end{aligned}$$

Notice that there is no need to have $x_{ij} \leq 1$, it is implied by the balance flow constraints and nonnegative constraints.