# ISyE 7203: Logistics and System Engineering

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# Acknowledgements

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# Contents

# 1 Network Flows

**Definition 1.1**

*A* network (graph) *is made of the following:*

- $(N)$ *nodes (vertices): a finite set.*
- *For undirected network:* $(E)$ *edges:* $\{i, j\}, i, j \in N$ *drawn as* $i - j$.
- *For directed network:* $(A)$ *arcs* $(i, j), i, j \in N$ *drawn as* $i \to j$, *where we call* $i$ *the tail and* $j$ *the head of the arc.*

For this course, we focus on the directed networks and the following definitions are all on directed networks.

**Definition 1.2**

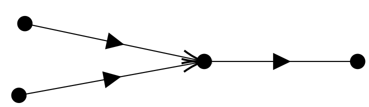
- path*: a sequence of arcs where the head of the first is the tail of the second and so forth, and no nodes repeat.*
- walk*: a sequence of arcs where the head of the first is the tail of the second and so forth, and nodes may repeat.*
- cycle*: a path except that the first and the last nodes are the same. If a cycles has repeated nodes in between, it is a* closed walk *instead.*

**Definition 1.3: Connectedness**

*A directed network is:*

- strongly connected*: for any* $i, j \in N$, *there exists an* $i - j$ *(directed) path and a* $j - i$ *directed path.*
- weakly connected*: for any* $i, j \in N$, *there exists an (undirected)* $i - j$ *path if we ignore orientations.*

*Example* 1.1. Consider the following network:



It is weakly but not strongly connected.

**Definition 1.4: network flow**

*Given the network by* $(N, A)$, $b \in \mathbb{R}^N, c \in \mathbb{R}^A, u \in (\mathbb{R} \cup \{\infty\})^A$. $b$ *represents the net supply where* $i$ *is a supplier, consumer, trans-shipment respectively if* $b_i > 0, b_i < 0, b_i = 0$ *respectively;* $c$ *represents the cost, and* $u$ *represent the capacity.*
*For modeling,* $X_{ij}$ *is used to represent the amount of flow on* $(i, j) \in A$. *Then the* network flow problem *is:*

$$\begin{aligned} \min \; & \sum_{a \in A} c_a x_a \\ \textit{s.t.} \; & \sum_{a \in \delta^+(i)} - \sum_{a \in \delta^-(i)} x_i = b_i, i \in N \\ & 0 \le x_a \le u_a, \; \forall a \in A \end{aligned} \tag{NF}$$

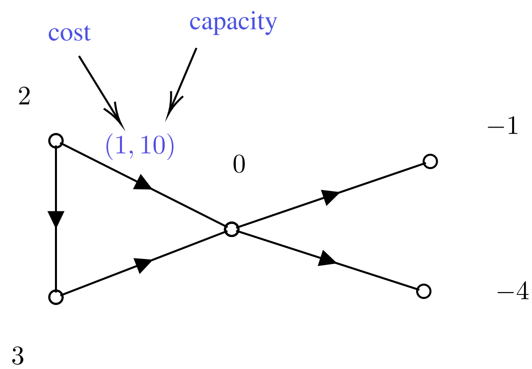
*where* $\delta^+(i)$ *is the set of arcs having* $i$ *as the tail and* $\delta^-(i)$ *is the set of arcs having* $i$ *as the head. The equality constraints are called the flow balance constraint.*

*Remark.* Aggregate the flow balance constraints, we get

$$\sum_i \left( \sum_{a \in \delta^+(i)} - \sum_{a \in \delta^-(i)} x_i \right) = \sum_{a \in A} x_a - x_a = 0 = \sum_i b_i.$$

Thus, $\sum_{i=1}^N b_i = 0$ is a needed assumption.

*Example* 1.2. The following is an instance of a network flow problem:




**Theorem 1.5**

*The constraint matrix of* (NF) *is always TU (totally unimodular). Thus, if $b, u$ are integral, all extreme points are integral.*

*Example* 1.3. Assignment Problem Considering assigning $n$ workers to $n$ tasks, one for each. Let $c_{ij}$ be the cost of assigning worker $i$ to task $j$. In this way, we can consider $1, \ldots, n$ nodes as workers and $n+1, \ldots, 2n$ as tasks. For each node representing a worker, it has arcs from it to all task nodes. The supply of each worker and task are $1$ and $-1$ respetively. Then the problem can be written as:

$$\begin{aligned}
\min \ & \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{s.t. } & \sum_{j=n+1}^{2n} x_{ij} = 1, \ i = 1, \ldots, n \\
& \sum_{i=1}^{n} -x_{ij} = -1 \\
& 0 \leq x_{ij}.
\end{aligned}$$

Notice that there is no need to have $x_{ij} \leq 1$, it is implied by the balance flow constraints and nonnegative constraints.