A Novel Framework for Policy Mirror Descent with General parameterization and Linear Convergence

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Introduction

Modern policy optimization algorithm, such as TRPO and PPO, owe their success to the use of parameterized policies such as

$$\pi(a|s) \propto \exp(f^\theta(s,a)),$$

where f^θ is a neural network. However, the use general parameterization schemes still lacks theoretical justification.

Contribution: A novel framework for policy optimization based on mirror descent that naturally accommodates general parameterizations and enjoys theoretical guarantees.

Preliminaries

Consider a discounted MDP $(\mathcal{S}, \mathcal{A}, P, r, \gamma, \mu)$. Given a policy π, define the value function

$$V^\pi(s) := \mathbb{E}_{a_t \sim \pi(\cdot|s_t), s_{t+1} \sim P(\cdot|s_t,a_t)} \left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \Big| \pi, s_0 = s \right]$$

and the Q-function

$$Q^\pi(s,a) := \mathbb{E}_{a_t \sim \pi(\cdot|s_t), s_{t+1} \sim P(\cdot|s_t,a_t)} \left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \mid \pi, s_0 = s, a_0 = a \right].$$

Letting $V^\pi(\mu) := \mathbb{E}_{s\sim\mu}[V^\pi(s)]$, our objective is for the agent to find an optimal policy

$$\pi^\star \in \operatorname{argmax}_{\pi \in (\Delta(\mathcal{A}))^{\mathcal{S}}} V^\pi(\mu).$$

Define the discounted state visitation distribution by

$$d_\mu^\pi(s) := (1-\gamma)\mathbb{E}_{s_0\sim\mu}\left[\sum_{t=0}^{\infty} \gamma^t P(s_t = s \mid \pi, s_0) \right].$$

Mirror Descent. Let $\mathcal{Y} \subseteq \mathbb{R}^{|\mathcal{A}|}$ be a convex set. A *mirror map* $h : \mathcal{Y} \to \mathbb{R}$ is a strictly convex, continuously differentiable and essentially smooth function[a] such that $\nabla h(\mathcal{Y}) = \mathbb{R}^{|\mathcal{A}|}$. The convex conjugate of h, denoted by h^*, is given by

$$h^*(x^*) := \sup_{x\in\mathcal{Y}} \langle x^*, x \rangle - h(x), \quad x^* \in \mathbb{R}^{|\mathcal{A}|}.$$

The mirror map h induces a *Bregman divergence*, defined as

$$\mathcal{D}_h(x,y) := h(x) - h(y) - \langle \nabla h(y), x - y \rangle,$$

where $\mathcal{D}_h(x,y) \geqslant 0$ for all $x, y \in \mathcal{Y}$. Let $\mathcal{X} \subseteq \mathcal{Y}$ be a convex set and $V : \mathcal{X} \to \mathbb{R}$ be a differentiable function. To solve $\min_{x\in\mathcal{X}} V(x)$, MD consists in the updates: for all $t \geqslant 0$,

$$y^{t+1} = \nabla h(x^t) - \eta_t \nabla V(x)|_{x=x^t},$$
$$x^{t+1} = \mathrm{Proj}^h_{\mathcal{X}}(\nabla h^*(y^{t+1})) = \operatorname{argmin}_{x\in\mathcal{X}} \mathcal{D}_h(x, \nabla h^*(y^{t+1})).$$

Notation. At each time t, let $\pi^t := \pi^{\theta_t}$, $f^t := f^{\theta_t}$, $V^t := V^{\pi^t}$, $Q^t := Q^{\pi^t}$, and $d^t_\mu := d^{\pi^t}_\mu$. Further, for any function $y : \mathcal{S}\times\mathcal{A} \to \mathbb{R}$ and distribution v over $\mathcal{S}\times\mathcal{A}$, let $y_s := y(s,\cdot) \in \mathbb{R}^{|\mathcal{A}|}$ and $\|y\|^2_{L_2(v)} = \mathbb{E}_v[(y(s,a))^2]$. Let $\mathcal{D}_0^\star = \mathbb{E}_{s\sim d^\star_\mu}[\mathcal{D}_h(\pi^\star_s, \pi^0_s)]$.

[a] h is essentially smooth if $\lim_{x\to\partial\mathcal{Y}} \|\nabla h(x)\|_2 = +\infty$, where $\partial\mathcal{Y}$ denotes the boundary of $\mathcal{Y}$.

Approximate Mirror Policy Optimization

Given a parameterized function class $\mathcal{F}^\Theta = \{f^\theta : \mathcal{S}\times\mathcal{A} \to \mathbb{R}, \theta \in \Theta\}$, ideally, we would like to execute the exact MD-based algorithm: for all $t \geqslant 0$ and for all $s \in \mathcal{S}$,

$$f_s^{t+1} = \nabla h(\pi^t_s) + \eta_t(1-\gamma)\nabla_s V^t(\mu)/d^t_\mu(s) = \nabla h(\pi^t_s) + \eta_t Q^t_s, \tag{1}$$
$$\pi^{t+1}_s = \mathrm{Proj}^h_{\Delta(\mathcal{A})}(\nabla h^*(\eta_t f^{t+1}_s)).$$

However, there may not be any $\theta^{t+1} \in \Theta$ such that (1) is satisfied for all $s \in \mathcal{S}$. To remedy this issue, we propose Approximate Mirror Policy Optimization (AMPO).

Algorithm 1: Approximate Mirror Policy Optimization

Input: Initial policy π^0, mirror map h, parameterization class $\mathcal{F}^\Theta$, iteration number T, step-size schedule $(\eta_t)_{t\geqslant 0}$, state-action distribution sequence $(v_t)_{t\geqslant0}$.

1: For $t = 0, \dots, T-1$ do:

2: Obtain $\theta^{t+1} \in \Theta$ such that

$$\theta^{t+1} \in \operatorname{argmin}_{\theta\in\Theta} \left\| f^\theta - Q^t - \eta_t^{-1}\nabla h(\pi^t) \right\|^2_{L_2(v_t)}.$$

3: Update

$$\pi^{t+1}_s = \operatorname*{argmin}_{\pi' \in \Delta(\mathcal{A})} \mathcal{D}_h(\pi', \nabla h^*(\eta_t f^{\theta^{t+1}}_s)), \ \forall s \in \mathcal{S}.$$

Output: $(\pi^1, \dots, \pi^T)$

Comparison with previous frameworks

Similarly to AMPO, previous approximations of PMD [1, 2] provide an expression to be optimized. For instance, [1] aim to maximize an expression equivalent to

$$\pi^{t+1} = \operatorname*{argmax}_{\pi^\theta \in \Pi(\Theta)} \mathbb{E}_{s\sim d^t_\mu}[\eta_t \langle Q^t_s, \pi^\theta_s \rangle - \mathcal{D}_h(\pi^\theta_s, \pi^t_s)], \tag{2}$$

where $\Pi(\Theta)$ is a given parameterized policy class. The improvement of AMPO over this type of update is twofold.

▹ The parameterized policy class $\Pi(\Theta)$ is often non-convex with respect to θ in practice, which prevents the application of existing proof techniques that rely on the convexity of the tabular parameterization [3]. On the contrary, AMPO avoids this problem thanks to the Bregman projection and the update in Line 2 of Algorithm 1.

▹ AMPO involves a subroutine optimization procedure that is structurally different from the update in (2). Our approach employs a standard regression procedure, which has been extensively researched and benefits from established solving methods.

A practical class of mirror maps

For $a \in (-\infty, +\infty]$, $\omega \leqslant 0$, let an ω-*potential* be an increasing C^1-diffeomorphism $\phi : (-\infty, a) \to (\omega, +\infty)$ such that

$$\lim_{u\to-\infty}\phi(u) = \omega, \qquad \lim_{u\to a}\phi(u) = +\infty, \qquad \int_0^1 \phi^{-1}(u)du \leqslant \infty.$$

For any ω-potential ϕ, the associated mirror map h_ϕ is defined as

$$h_\phi(\pi_s) = \sum_{a\in\mathcal{A}} \int_1^{\pi(a|s)} \phi^{-1}(u)du.$$

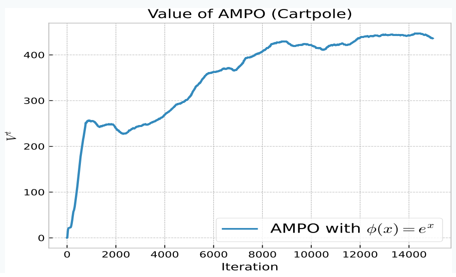
Thanks to [4, Proposition 2], the policy π^{t+1} in Line 3 induced by the ω-potential mirror map can be obtained with $\widetilde{\mathcal{O}}(|\mathcal{A}|)$ computations and can be written as

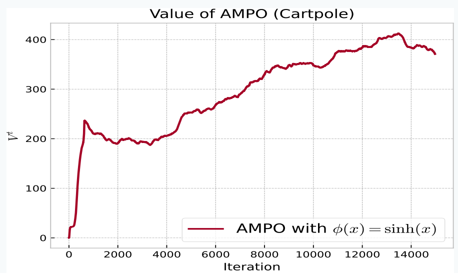
$$\pi^{t+1}(a|s) = \sigma(\phi(\eta_t f^{t+1}(s,a) + \lambda^{t+1}_s)) \quad \forall s \in \mathcal{S}, a \in \mathcal{A},$$

where $\lambda_s \in \mathbb{R}$ is a normalization factor to ensure $\sum_{a\in\mathcal{A}} \pi^{t+1}(a|s) = 1$ for all $s \in \mathcal{S}$, and $\sigma(z) = \max(z, 0)$ for $z \in \mathbb{R}$. The minimization problem in Line 2 is simplified to be

$$\theta^{t+1} \in \operatorname*{argmin}_{\theta\in\Theta} \left\| f^\theta - Q^t - \eta_t^{-1}\max(\eta_{t-1}f^t, \phi^{-1}(0) - \lambda^t_s) \right\|^2_{L_2(v_t)}.$$

When $\phi(x) = e^x$, we recover an approximation of NPG.





Convergence Rates

Assumption (A1) (*Approximation error*). There exists $\varepsilon_{\mathrm{approx}} \geqslant 0$ such that, $\forall t \geqslant 0$,

$$\mathbb{E}\left[\left\| f^{t+1} - Q^t - \eta_t^{-1}\nabla h(\pi^t) \right\|^2_{L_2(v_t)} \right] \leqslant \varepsilon_{\mathrm{approx}},$$

where $(v^t)_{t\geqslant0}$ is a sequence of distributions over states and actions and the expectation is taken over the randomness of AMPO.

Assumption (A2) (*Concentrability coefficient*). There exists $C_v \geqslant 0$ such that, $\forall t \geqslant 0$,

$$\mathbb{E}_{(s,a)\sim v^t}\left[\left(\frac{d^\pi_\mu(s)\pi(a|s)}{v^t(s,a)} \right)^2 \right] \leqslant C_v,$$

whenever (d^π_μ, π) is either $(d^\star_\mu, \pi^\star)$, (d^{t+1}_μ, π^{t+1}), $(d^\star_\mu, \pi^t)$, or (d^{t+1}_μ, π^t).

Assumption (A3) (*Distribution mismatch coefficient*). There exists $\nu_\mu \geqslant 0$ such that

$$\max_{s\in\mathcal{S}} \frac{d^\star_\mu(s)}{d^t_\mu(s)} \leqslant \nu_\mu, \quad \text{for all times } t \geqslant 0.$$

Theorem 4.3. Let Assumptions (A1), (A2), and (A3) be true. If the step-size schedule is non-decreasing, i.e., $\eta_t \leqslant \eta_{t+1}$ for all $t \geqslant 0$, the iterates of Algorithm 1 satisfy: $\forall T \geqslant 0$,

$$V^\star(\mu) - \frac{1}{T}\sum_{t<T}\mathbb{E}\left[V^t(\mu)\right] \leqslant \frac{1}{T}\left(\frac{\mathcal{D}^\star_0}{(1-\gamma)\eta_0} + \frac{\nu_\mu}{1-\gamma} \right) + \frac{2(1+\nu_\mu)\sqrt{C_v \varepsilon_{\mathrm{approx}}}}{1-\gamma}.$$

Furthermore, if the step-size schedule is geometrically increasing, i.e., satisfies

$$\eta_{t+1} \geqslant \frac{\nu_\mu}{\nu_\mu - 1}\eta_t \qquad \forall t \geqslant 0,$$

we have: for every $T \geqslant 0$,

$$V^\star(\mu) - \mathbb{E}\left[V^T(\mu)\right] \leqslant \frac{1}{1-\gamma}\left(1 - \frac{1}{\nu_\mu}\right)^T \left(1 + \frac{\mathcal{D}^\star_0}{\eta_0(\nu_\mu - 1)}\right) + \frac{2(1+\nu_\mu)\sqrt{C_v\varepsilon_{\mathrm{approx}}}}{1-\gamma}.$$

▹ First result that establishes linear convergence for a PG-based method involving general policy parameterization and mirror maps.

▹ For the same setting, it is also the first result that establishes $O(1/T)$ convergence without regularization.

▹ First result that provides a convergence rate for a PMD-based algorithm that allows any mirror map and non-tabular policies.

Sample complexity for neural network parameterization

Let $\mathcal{F}^\Theta$ be a class of shallow neural networks. At each iteration t of AMPO, we set $v^t = d^t_\mu$ and solve the regression problem in Line 2 of Algorithm 1 through SGD. Then, thank to Theorem 4.3 and an existing analysis of neural networks [5, Theorem 1], we have the sample complexity of AMPO

Corollary 4.4. In the setting of Theorem 4.3, let the parameterization class $\mathcal{F}^\Theta$ consist of sufficiently wide shallow ReLU neural networks. Using an exponentially increasing step-size and solving the minimization problem in Line 2 with SGD, the number of samples required by AMPO to find an ε-optimal policy with high probability is $\widetilde{\mathcal{O}}(C_v^2\nu_\mu^5/\varepsilon^4(1-\gamma)^6)$, where ε has to be larger than a non-vanishing error floor.

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