The Mankiw-Romer-Weil model with decreasing population growth rate *†

Juan Gabriel Brida[‡]and Gastón Cayssials[§]

Abstract

This paper studies an extension of the Mankiw-Romer-Weil growth model by departing from the standard assumption of constant population growth rate. More concretely, this rate is assumed to be decreasing over time and a general population growth law verifying this characteristic is introduced. In this setup, the model can be represented by a three dimensional dynamical system which admits a unique solution for any initial condition. It is shown that there is a unique nontrivial equilibrium which is a global attractor. In addition, the speed of convergence to the steady state is characterized, showing that in this model this velocity is lower than in the original Mankiw-Romer-Weil model.

Keywords: Mankiw-Romer-Weil economic growth model; decreasing population growth rate; speed of convergence.

JEL classification: C62; O41

1 Introduction

In the model developed by Mankiw, Romer and Weil [30]-also known as the Solow model extended with human capital, labour force (associated with the size of the population) grows at a constant rate $n > 0$. This assumption, normally used in the classic growth models (Solow [34], Ramsey[33] - Cass[12] -Koopmans [29] among others) implies that the population grows exponentially, i.e. if the initial population is P_0, the population at time

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‡Departamento de Métodos Cuantitativos - Facultad de Ciencias Económicas y de Administración - Universidad de la República (Uruguay). Email: gbrida@ccee.edu.uy

§Departamento de Métodos Cuantitativos - Facultad de Ciencias Económicas y de Administración - Universidad de la República (Uruguay). Email: gacayssials@gmail.com

t is $P(t) = P_0 e^{nt}$. Assuming that population grows exponentially implies that there is no limit to the size of the population (the population tends to infinity as t tends to infinity). This assumption is clearly not sustainable, nor fits in with recent empirical data a hundred years [35].

The exponential model conforms the dynamics of the population in initial periods, but it unable to reflect the fall in the rate of growth due to-for example-lower fertility rate. Verhulst[36] shows that there should be an upper bound for the size of a population, called load-carrying capacity of the environment, a maximum level of population that an environment can support until it is unable to sustain and feed human activity.

Several experts (Daily[15], Brown[10]) reference that humanity is close to that limit. According to data of the United Nations [35] the rate of population growth has decresed in the last one hundred years and is actually close to 1%. Moreover the projections for the coming years is that this trend will continue, due to lower rates of fertility. In summary, empirical data reveals two stylized facts: i)the population does not grow at a constant rate, and ii) this rate decreases to zero.

Maynard [31] proposes the following properties that characterize a law of population verifying these stylized facts: :

1. The population grows but is constrained by a maximum size, carrying capacity P_∞:

$$\dot{P}(t) \geq 0$$

$$\lim_{t \to +\infty} P(t) = P_\infty$$

2. the growth rate of the population decreases to zero, i.e. $n(t) = \frac{\dot{P}(t)}{P(t)}$ decreases to zero: $\dot{n}(t) < 0, \forall t$ and $\lim\limits_{t \to +\infty} n(t) = 0$

The logistic equation, the Verhulst equation, the Richards equation and Von Bertalanffy equation are examples of laws of population used normally by demographers and social scientists that verify these properties [13].

In this paper the model of Mankiw-Romer-Weil is analyzed modifying the hypothesis of exponential growth by introducing a law of general population that safisties the properties listed above. The reformulation of classical growth models has already been studied for the Solow model (using logistic law [14], [38], using a Richards equation [6], von Bertalanffy law [8], bounded population [11], general population laws that verifies the properties mentioned above [9]) and the model of Ramsey (using logistic law [4], [16], [18], [21], [26], [27], [1] or von Bertalanffy law [3], [22], [23], [24], [25], or Richards law [2], [17], or general population law [19], [20],[7]). This paper generalizes [28], where the author modifies the M-R-W model by introducing the logistic law of population.

The model proposed by Mankiw, Romer and Weil in 1992 marks a milestone in the resurgence of neoclassical growth models in the 90s and his work is the most influential and widely cited pieces in the empirical growth literature. When considering a broader capital definition, the model predicts a lower rate of convergence[1] to equilibrium the rate of the Solow model. This implies that the speed of convergence is lower, and that the M-R-W model adjust better to empirical data that the original Solow model. This result, coupled with the emergence of endogenous growth models, promoted the development of an empirical line of research focused on convergence and the dispersal ($\sigma-$convergence) of per capita product between countries, group of countries or regions within the same country. In all these works the Mankiw-Romer-Weil model is a fundamental pillar. Given that the introduction of an alternative law of population growth implies changes in the speed of convergence to the equilibrium, the present study can be seen as a contribution to this empirical line of research.

The paper is organized as follows. In section 2, the model is presented. The existence and uniqueness of nonzero equilibrium are showed and the stability of the equilibrium is analysed, in section 3 the modified model is presented. Section 4 studies the speed of convergence and transitional dynamics of the model. Finally, section 5 presents some concluding remarks.

2 The model

2.1 The original Mankiw-Romer-Weil model

Let begin by introducing the original Mankiw-Romer-Weil model with exponential population growth law and analysing the main dynamical properties of the model. (see [30] for a detailed description)

Let consider a closed economy, with a single productive sector, which uses physical capital ($K(t)$), labor force ($L(t)$) and human capital ($H(t)$) as factors of production ($Y(t)$). The economy is endowed with a technology defined by a Cobb-Douglas production function with constant returns to scale:

$$Y(t) = K^{\alpha}(t)H^{\beta}(t)L^{1-\alpha-\beta}(t), \ \alpha, \beta, \alpha+\beta \in (0,1)$$

The physical capital stock changes $\dot{K}$ equal the gross investment $I_k = s_k Y(t)$ minus the capital depreciation δK :

$$\dot{K} = s_k Y(t) - \delta K(t) \tag{1}$$

The human capital stock changes $\dot{H}$ equal the gross investment $I_h = s_h Y(t)$ minus the capital depreciation δH :

$$\dot{H} = s_h Y(t) - \delta H(t) \tag{2}$$

[1]or $\beta-$convergence, defined as the time it takes to reach equilibrium economy

The model assumes that the population growths at a constant rate $n > 0$:

$$\begin{cases} \dot{L}(t) = nL(t) \\ L(0) > 0 \end{cases} \tag{3}$$

The production function can be expresed in per capita as:

$$\frac{Y(t)}{L(t)} = \frac{K^{\alpha}(t)H^{\beta}(t)L^{1-\alpha-\beta}(t)}{L(t)} = \left(\frac{K(t)}{L(t)}\right)^{\alpha}\left(\frac{H(t)}{L(t)}\right)^{\beta} = y(t) \tag{4}$$

If we define $K/L = k$ as the physical capital per worker and $H/L = h$ as the human capital per worker. Then the product per capita is:

$$y(t) = k^{\alpha}(t)h^{\beta}(t) \tag{5}$$

Note that

$$\dot{k} = \frac{d(\frac{K(t)}{L(t)})}{dt} = \frac{\dot{K}L - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - \frac{K}{L}\frac{\dot{L}}{L} = \frac{\dot{K}}{L} - kn \tag{6}$$

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{L}\frac{1}{(K/L)} - \frac{kn}{k} = \frac{\dot{K}}{K} - n \tag{7}$$

$$\frac{\dot{k}}{k} = \frac{s_k Y(t) - \delta K(t)}{K(t)} - n = \frac{s_k k^{\alpha}(t)h^{\beta}(t)L(t)}{K(t)} - \delta - n \tag{8}$$

Then the growth rate of physical capital per worker is:

$$\frac{\dot{k}}{k} = \frac{s_k k^{\alpha}(t)h^{\beta}(t)}{(K(t)/L(t))} - \delta - n \tag{9}$$

$$\frac{\dot{k}}{k} = \frac{s_k k^{\alpha}(t)h^{\beta}(t)}{k(t)} - (\delta + n) \tag{10}$$

Thus the accumulation of physical capital by worker is given by:

$$\dot{k} = s_k k^{\alpha}(t)h^{\beta}(t) - (\delta + n)k(t) \tag{11}$$

By a similar reasoning, we arrive to the equation describing the accumulation of per capita human capital:

$$\dot{h} = s_h k^{\alpha}(t)h^{\beta}(t) - (\delta + n)h(t) \tag{12}$$

Then the two dimensional dynamical system:

$$\begin{cases} \dot{k} = s_k k^{\alpha}(t) h^{\beta}(t) - (\delta + n)k(t) \\ \dot{h} = s_h k^{\alpha}(t) h^{\beta}(t) - (\delta + n)h(t) \end{cases} \tag{13}$$

describes the dynamics of the model.

Note that the non trivial equilibrium is the point (k^*, h^*) such that:

$$\begin{cases} k^* = \left[\frac{s_k^{1-\beta} s_h^{\beta}}{\delta+n}\right]^{\frac{1}{1-\alpha-\beta}} \\ h^* = \left[\frac{s_h^{1-\alpha} s_k^{\alpha}}{\delta+n}\right]^{\frac{1}{1-\alpha-\beta}} \end{cases} \tag{14}$$

and the equilibrium of the product is:

$$y^* = (k^*)^{\alpha}(h^*)^{\beta} = \left[\frac{s_k}{\delta + n}\right]^{\frac{\alpha}{1-\alpha-\beta}} \left[\frac{s_h}{\delta + n}\right]^{\frac{\beta}{1-\alpha-\beta}}$$

Then the equilibrium values of long-term capital (physical and human) and product, depend positively on the savings rates (s_k, s_h) and on the degree of efficiency of scale of reproducible factors (α, β) and negatively on the rate of depreciation (δ) and on population growth (n).

In order to analyze the stability of the stationary state we consider the linear approximation of the function $G : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$G(k, h) = \big(s_k k^{\alpha} h^{\beta} - (\delta + n)k, s_h k^{\alpha} h^{\beta} - (\delta + n)h\big)$$

Which gives the first order approximation of the model as:

$$\begin{pmatrix} \dot{k} \\ \dot{h} \end{pmatrix} = G(k, h)$$

The transitional dynamic around the equilibrium (k^*, h^*) can be quantified from the linearization of the system:

$$\begin{pmatrix} \dot{k} \\ \dot{h} \end{pmatrix} = G(k^*, h^*) + J_G \begin{pmatrix} k - k^* \\ h - h^* \end{pmatrix} = J_G \begin{pmatrix} k - k^* \\ h - h^* \end{pmatrix}$$

where J_G is the Jacobian matrix of G evaluated in the equilibrium.

$$J_G = \begin{pmatrix} s_k \alpha \frac{(\delta+n)}{s_k} - (\delta + n) & \frac{s_k}{s_h} \beta(\delta + n) \\ \frac{s_h}{s_k} \alpha(\delta + n) & s_h \beta \frac{(\delta+n)}{s_h} - (\delta + n) \end{pmatrix}$$

$$J_G = \begin{pmatrix} (\delta + n)(\alpha - 1) & \frac{s_k}{s_h} \beta(\delta + n) \\ \frac{s_h}{s_k} \alpha(\delta + n) & (\delta + n)(\beta - 1) \end{pmatrix}$$

The characteristic polynomial of the Jacobian matrix is:

$$P(X) = (\delta + n)^2(1 - \alpha - \beta) - (\alpha + \beta - 2)(\delta + n)X + X^2$$

that presents two negative eigenvalues: $\lambda_1 = (\delta + n)(\alpha + \beta - 1)$ y $\lambda_2 = -(\delta + n)$. This implies that the equilibrium is a global attractor.

Remark 1. *The convergence rate is determined by the smallest eigenvalue in absolute value, ie by $\lambda_1 = (\delta+n)(\alpha+\beta-1)$. One of the characteristics of the model is that the convergence rate is lower than in the Solow model*[2]*. This implies that the M-R-W model fits better to empirical data than the Solow model (see [5] chap.* 1*).*

An alternative approach to analyse the dynamical properties of the model (as presented in the seminal paper [30]) is to introduce log-linear approximation of system 13:

$$\begin{cases} \dot{\bar{k}} = (\delta + n)\left[(\alpha - 1)(\bar{k} - \bar{k}^*) + \beta(\bar{h} - \bar{h}^*)\right] \\ \dot{\bar{h}} = (\delta + n)\left[\alpha(\bar{k} - \bar{k}^*) + (\beta - 1)(\bar{h} - \bar{h}^*)\right] \end{cases} \tag{15}$$

and substituting in $\frac{d(log(y)}{dt} = \dot{\bar{y}} = \alpha\dot{\bar{k}} + \beta\dot{\bar{h}}$

$$\dot{\bar{y}} = (\delta + n)(\alpha + \beta - 1)\left[\bar{y} - \bar{y}^*\right] = \lambda_1\left[\bar{y} - \bar{y}^*\right]$$

Note that in this case the convergence rate can be interpreted as the speed with which an economy approaches equilibrium at time t. By solving this differential equation the product can be written as:

$$\bar{y}(t) = \bar{y}(0)e^{\lambda_1 t} + (1 - e^{\lambda_1 t})\bar{y}^*$$

and replacing $\bar{y}^*$ with the equilibrium value, $\bar{y}^* = \alpha\bar{k}^* + \beta\bar{h}^*$, the following equation is obtained:

$$log(y(t)) - log(y(0)) = -(1 - e^{\lambda_1 t})log(y(0)) + (1 - e^{\lambda_1 t})\left[\frac{\alpha}{1 - \alpha - \beta}log(\frac{s_k}{\delta + n}) + \frac{\beta}{1 - \alpha - \beta}log(\frac{s_h}{\delta + n})\right]$$

This equation is used to estimate the model in empirical studies of growth. In particular to contrast what in the literature on growth is known as *convergence hypothesis*, that is, there is a negative relationship between the distance to the equilibrium and the speed of convergence.

[2]The speed of convergence that Solow model predicts is: $(\delta + n)(\alpha - 1)$

2.2 The modified Mankiw-Romer-Weil model with a decreasing population growth rate

In the previous model, we replace the growth population law $\dot{L}(t) = nL(t)$ by a law $\dot{L}(t) = p(t)L(t)$ that verifies the following properties:

1. $L(0) = L_0 > 0, \dot{L}(t) \geq 0, \forall t \geq 0$ y $\lim_{t \longrightarrow +\infty} L(t) = L_\infty$.
   Growing and bounded population.

2. If $p(t) = \frac{\dot{L}(t)}{L(t)}$ then: $\dot{p}(t) < 0, \forall t \geq 0$ and $\lim_{t \longrightarrow +\infty} p(t) = 0$
   The population growth rate is decreasing and tends to 0.

Some well known examples of population laws verifying these properties are described in the following table 1:

Table 1: Example of population laws

Population Laws		$L(t)$	$p(t)$
Logistic [13]	$\dot{L} = aL - bL^2$ $L(0) = L_0 > 0$	$\frac{aL_0e^{at}}{a+bL_0(e^{at}-1)}$	$\frac{a(a-bL_0)}{a-bL_0(e^{at}-1)}$
Verhulst [36]	$\dot{L} = rL(1 - \frac{L}{L_\infty})$ $L(0) = L_0 > 0$	$\frac{L_0L_\infty e^{rt}}{L_0e^{rt}+L_\infty-L_0}$	$\frac{r(L_\infty-L_0)}{e^{rt}+L_\infty-L_0}$
Von Bertalanffy [37]	$\dot{L} = r(L_\infty - L)$ $L(0) = L_0 > 0$	$\frac{e^{rt}L_\infty+L_0-L_\infty}{e^{rt}}$	$\frac{r(L_\infty-L_0)}{L_0-L_\infty+L_\infty e^{rt}}$

After replacing the exponential law by the equation $\dot{L}(t) = p(t)L(t)$ and redoing the step of the previus subsection, the dynamical system describing the modified model can be represented by the following system of differential equation of order 3:

$$\begin{cases} \dot{k} = s_k k^\alpha(t)h^\beta(t) - (\delta + p(L(t)))k(t) \\ \dot{h} = s_h k^\alpha(t)h^\beta(t) - (\delta + p(L(t)))h(t) \\ \dot{L}(t) = L(t)p(L(t)) \end{cases} \tag{16}$$

3 Equilibria and stability: qualitative analysis

3.1 The positive steady state

This section investigates the dynamic behaviour of the model's solution $(k(t), h(t), L(t))$.

Lemma 2. *If excluded the trivial solutions obtained by considering $k = 0$, $h = 0$ and $L = 0$, then the model has a unique positive equilibrium (k^*, h^*, L^*) verifying:*

$$
\begin{cases}
k^* = \left[\frac{s_k^{1-\beta}s_h^{\beta}}{\delta}\right]^{\frac{1}{1-\alpha-\beta}} \\
h^* = \left[\frac{s_h^{1-\alpha}s_k^{\alpha}}{\delta}\right]^{\frac{1}{1-\alpha-\beta}} \\
L^* = L_\infty
\end{cases}
\tag{17}
$$

Proof. The proof is immediate from solving the system (15) looking for a constant solution. ∎

Remark 3. *The values for k^*, h^* and y^* match with the ones of the original model of Mankiw-Romer-Weil when $n = 0$. This implies that the equilibrium values of the modified model are higher than those of the original model.(i.e., $n = 0$; see equations (14)) and then the parameters of the population law do not enter in the determinants of k^*, h^* and y^*. The steady state values of physical capital and human capital depend only on the parameters of technology α, β y δ and the exogenous savings rates s_k and s_h. This is an importan difference with the original model, where an increase in the intrinsec rate of population growth leads to lower leves of these variables in the long run. In adition, given that in the modified model the population size is bounded by the carring capacity $L^* = L_\infty$, then the agregate quantity of fisical and human capital in the long run are finite and equals $K^* = L^*k^*$ and $H^* = L^*h^*$ respectively (while in the original Mankiw-Romer-Weil model are infinite).*

Proposition 4. *The equilibrium point (k^*, h^*, L^*) is a global attractor.*

Proof. From the system (16) :

$$
\begin{cases}
\dot{k} = s_k k^\alpha(t)h^\beta(t) - (\delta + p(L(t)))k(t) \\
\dot{h} = s_h k^\alpha(t)h^\beta(t) - (\delta + p(L(t)))h(t) \\
\dot{L}(t) = L(t)p(L(t))
\end{cases}
\tag{18}
$$

To analyse the stability of the steady state solution, let consider the linear approximation of the function $M : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$
M(k, h, L) = \left(s_k k^\alpha h^\beta - (\delta + p(L)k, s_h k^\alpha h^\beta - (\delta + p(L))h, Lp(L)\right)
$$

arround the equilibrium point (k^*, h^*, L^*). The Jacobian matrix of the linear approximation is given by:

$$
J_M = \begin{pmatrix}
\delta(\alpha - 1) & \frac{s_k}{s_h}\beta\delta & k^*p\prime(L_\infty) \\
\frac{s_h}{s_k}\alpha\delta & \delta(\beta - 1) & h^*p\prime(L_\infty) \\
0 & 0 & L_\infty p\prime(L_\infty)
\end{pmatrix}
$$

Then the characteristic polynomial of this matrix is given by

$$
R(X) = (L_\infty p\prime(L_\infty) - X)((\delta(\alpha - 1) - X)(\delta(\beta - 1) - X) - \alpha\beta\delta^2)
\tag{19}
$$

This polynomial presents three real negative eigenvalues: $\lambda_1 = \delta(\alpha + \beta - 1) < 0$, $\lambda_2 = -\delta < 0$ and $\lambda_3 = L_\infty p\prime(L_\infty) < 0$. Then the steady-state is a global attractor. ∎

4 Transitional dynamics and speed of convergence

The Mankiw-Romer-Weil model is a good approximation to the real world, empirically proved to be more robust, better suited to the empirical data, that the Solow model, nevertheless it describes the economic reality incompletely. The modified model has richer dynamics. In particular, in the modified model one can have positive growth at the equilibrium levels of physical and human capital.

The transitional dynamics around the steady state (k^*, h^*, L^*) can be quantified by using the linearization of system (16):

$$\begin{pmatrix} \dot{k} \\ \dot{h} \\ \dot{L} \end{pmatrix} = \begin{pmatrix} \delta(\alpha-1) & \frac{s_k}{s_h}\beta\delta & k^* p'(L_\infty) \\ \frac{s_h}{s_k}\alpha\delta & \delta(\beta-1) & h^* p'(L_\infty) \\ 0 & 0 & L_\infty p'(L_\infty) \end{pmatrix} \begin{pmatrix} k-k^* \\ h-h^* \\ L-L_\infty \end{pmatrix}$$

We know that the matrix representing this linear system has three negative eigenvalues: $\lambda_1 = \delta(\alpha+\beta-1) < 0$, $\lambda_2 = -\delta < 0$ and $\lambda_3 = L_\infty p^{'}(L_\infty) < 0$.

In this section we provide a quantitative assessment of the speed of convergence of transitional dynamics. The speed depends on the parameters of technology and preferences and can be computed from the matrix $J_M(k^*, h^*, L_\infty)$. The eigenvalues λ_1 and λ_1 are analogous with the convergence coefficients in the standard model when the rate of population growth is zero. The eigenvalue $\lambda_3 = L_\infty p'(L_\infty)$ corresponds to the speed of convergence of population to the carrying capacity L_∞. Each eigenvalue corresponds to one source of convergence and each stable transition path to the steady state of the system takes the form

$$\begin{cases} k(t) = k^* + C_1 v_{11} e^{\lambda_1 t} + C_2 v_{21} e^{\lambda_2 t} + C_3 v_{31} e^{L_\infty (p'(L_\infty) t} \\ h(t) = h^* + C_1 v_{12} e^{\lambda_1 t} + C_2 v_{22} e^{\lambda_2 t} + C_3 v_{32} e^{L_\infty (p'(L_\infty) t} \\ L(t) = L_\infty + (L_0 - L_\infty)\, e^{L_\infty (p'(L_\infty))t} \end{cases} \tag{20}$$

where $C_1, C_2, C_3, v_{11}, v_{21}, v_{31}, v_{12}, v_{22}$ and v_{32} depends on the initial conditions and coefficients of $J_M(k^*, h^*, L_\infty)$. Then the speed of convergence of physical and human capital depends on eigenvalues $\lambda_1 = \delta(\alpha+\beta-1)$ and $L_\infty p'(L_\infty)$. Note that, being population given exogenously, the speed of convergence of population only depends on $L_\infty p'(L_\infty)$. In fact, the transition depends on eigenvalue with higher absolute value. If $|L_\infty p'(L_\infty)| < |\lambda_1|$, then the speed of convergence of L_t is faster than that of $k(t)$ and $h(t)$ and if $|L_\infty p'(L_\infty)| > |\lambda_1|$ then all variables converge at speed $L_\infty p'(L_\infty)$.

Remark 5. *Regardless of whether the speed of convergence is λ_1 (it just depends on the degree of efficiency of scale reproducible factor and on the rate of depreciation) or λ_3 (it just depends on the population law), in both cases it is lower than in the original model.*

An alternative approach to analyze the dynamic properties of the model (as presented in the seminal paper [30]), in particular the speed of convergence in this new framework, is to introduce log-linear approximation of system (16):

$$\begin{cases} \dot{\bar{k}} = (\alpha-1)\delta(\bar{k}-\bar{k}^*) + \beta\delta(\bar{h}-\bar{h}^*) - p'(L_\infty)L_\infty(\bar{L}-\bar{L}^*) \\ \dot{\bar{h}} = \alpha\delta(\bar{k}-\bar{k}^*) + (\beta-1)\delta(\bar{h}-\bar{h}^*) - p'(L_\infty)L_\infty(\bar{L}-\bar{L}^*) \\ \dot{\bar{L}} = p'(L_\infty)L_\infty(\bar{L}-\bar{L}^*) \end{cases} \tag{21}$$

and substituting in $\frac{d(log(y)}{dt} = \dot{\bar{y}} = \alpha\dot{\bar{k}} + \beta\dot{\bar{h}}$

$$\dot{\bar{y}} = \delta(1-\alpha-\beta)(\bar{y}-\bar{y}^*) - (\alpha+\beta)p'(L_\infty)L_\infty(\bar{L}-\bar{L}^*)$$

If this differential equation is solved, equation is obtained empirically to estimate the rate of convergence in this new framework, it is required to specify a population law.

5 Concluding Remarks

In economic growth theory it is usually assumed that population growth follows an exponential law. This is clearly unrealistic because it implies that population goes to infinity when time goes to infinity. In this study an improved version of the Mankiw-Romer-Weil growth model is developed by introducing a general population law.

The model is presented as a dynamic system of dimension three, who supports a unique equilibrium than the trivial, which is as in the original model a global attractor. In the modified model equilibrium values of the product, physical capital and human capital per capita depend on the degree of efficiency of scale of reproducible factors (α, β), depreciation rate (δ) and the savings rate (s_k, s_h), but do not depend on the parameters of the population. Besides their values are greater than the classical model, whatever the constant rate $n > 0$ of population growth

In the equilibrium of the classical Mankiw-Romer-Weil model, aggregate physical and human capital tends unrealistically to infinity as t tends to infinity, because population grows to infinity. This situation is improved in the modified model, where in equilibrium aggregate physical and human capital tends to the finite values $K^* = L_\infty k^*$ and $H^* = L_\infty h^*$.

Finally, the document shows that the model has a finite speed of convergence, it just depends on technology parameters and the rate of depreciation or the law of population, but not both, and in any case it is smaller than in the original model.

Future research can include modelling population by an equation that depends on other variables of the model; i.e., to endogenize population.

A second line of research is to analyze the modified model in discrete time. Empirical studies based on the M-R-W model implicitly assume that the continuous-time dynamic properties are the same than in discrete time, but the logistics equation is a classic example of that is not always so. If the dynamic is different in a model or another, the conclusions and policy recommendations will also be different.

A third line line of research can include the possibility of migration.

Finally, a last line of investigation is the empirical study under a specification that follows the modified model.

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