STOR 415, Fall 2019
Solutions to Homework Assignment No. 3

1. (a)

$$\begin{array}{llll} \min & z = & 3x_1 + x_2 & \\ \text{s.t} & & x_1 & \geq 3, \\ & & x_1 + x_2 & \leq 4, \\ & & 2x_1 - x_2 & = 3, \\ & & x_1, x_2 & \geq 0 \end{array}$$

Not in standard form or canonical form. Convert it into standard form to obtain the following LP:

$$\begin{array}{llll} \min & z = & 3x_1 + x_2 & \\ \text{s.t} & & x_1 - s_1 & = 3, \\ & & x_1 + x_2 + s_2 & = 4, \\ & & 2x_1 - x_2 & = 3, \\ & & x_1, x_2, s_1, s_2 & \geq 0 \end{array}$$

(b)

$$\begin{array}{llll} \max & z = & 3x_1 + x_2 + x_3 + x_4 & \\ \text{s.t} & & x_1 + 2x_2 + x_3 + x_4 & \geq 10, \\ & & x_1 + x_2 & \leq 4, \\ & & x_1 \geq 0, \quad x_2 \leq 0, \quad x_4 \geq 0 & \end{array}$$

Not in standard form or canonical form. Convert it into standard form to obtain the following LP:

$$\begin{array}{llll} \max & z = & 3x_1 - \hat{x}_2 + x_3^+ - x_3^- + x_4 & \\ \text{s.t} & & x_1 - 2\hat{x}_2 + x_3^+ - x_3^- + x_4 - s_1 & = 10, \\ & & x_1 - \hat{x}_2 \qquad\qquad\qquad\qquad + s_2 & = 4, \\ & & x_1, \hat{x}_2, x_3^+, x_3^-, x_4, s_1, s_2 \geq 0 & \end{array}$$

(c)

$$\begin{array}{llll} \max & z = & x_1 + x_2 - x_3 & \\ \text{s.t} & & x_1 \qquad + x_3 + x_4 & = 10, \\ & & x_1 + x_2 + x_3 & = 6, \\ & & x_1, x_2, x_3, x_4 \geq 0 & \end{array}$$

In standard form, but not in canonical form. (The second equation does not isolate a variable.)

(d)

$$\begin{array}{llll} \max & z = & x_1 - 2x_2 - x_3 + x_4 & \\ \text{s.t} & & x_1 \qquad + x_3 + x_4 + s_1 & = 10, \\ & & x_1 + x_2 + x_3 \qquad\qquad + s_2 & = 6, \\ & & x_1, x_2, x_3, x_4, s_1, s_2 \geq 0 & \end{array}$$

In canonical form. The first equation isolates s_1, and the second equation isolates s_2.

2. (a) Choice 1: B(1)=1,B(2)=2: basic solution $x = (1, 1, 0, 0)$. It is a nondegenerate BFS.

   Choice 2: B(1)=1,B(2)=3: basic solution $x = (0, 0, 2, 0)$. It is a degenerate BFS.

   Choice 3: B(1)=1,B(2)=4: basic solution $x = (1, 0, 0, -1)$. It is not a BFS.

   Choice 4: B(1)=2,B(2)=3: basic solution $x = (0, 0, 2, 0)$. It is a degenerate BFS. (Same as the BFS produced in choice 2)

   Choice 5: B(1)=3,B(2)=4: basic solution $x = (0, 0, 2, 0)$. It is a degenerate BFS. (Same as the BFS produced in choices 2 and 4)

   (b) $x_3 = 2 - 2x_1$ and $x_4 = x_2 - x_1$.

   The projection is

$$\{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0, x_1 \leq 1, x_2 - x_1 \geq 0\}$$

   (c) The set has two vertices at (0,0) and (1,1). It has three edges, including the line segment connecting the two vertices, and two vertical halflines starting from (0,0) and from (1,1) respectively.

   (d) No, it is unbounded because the objective value can be made arbitrarily large by choosing x_2 to be an arbitrarily large number.

3. (a) Each $i = 1, \cdots, n$ is associated with a basic feasible solution:

$$x_i = b/a_i, \qquad x_j = 0 \text{ for each } j \neq i.$$

   All basic feasible solutions can be obtained in such a way.

   (b) $M_i = b/a_i$.

   (c) The optimal value is

$$\max_{i=1,\cdots,n} \frac{bc_i}{a_i}.$$

   Justification (optional). Let

$$\lambda = \max_{i=1,\cdots,n} \frac{c_i}{a_i}.$$

   We have $c_i \leq \lambda a_i$ for each $i = 1, \cdots, n$. Then for any feasible solution x we have

$$\sum_{i=1}^{n} c_i x_i \leq \sum_{i=1}^{n} \lambda a_i x_i = \lambda \sum_{i=1}^{n} a_i x_i = \lambda b,$$

   in which we have used the facts that $x_i \geq 0$ for each $i = 1, \cdots, n$ and that $\sum_{i=1}^{n} a_i x_i = b$. This shows that λb is an upper bound for the objective value of any feasible solution. On the other hand, if we let i_0 to be the index that has the largest value $\frac{c_i}{a_i}$ among all indices i, then the BFS associated with i_0 attains the objective value λb. Thus, λb is the optimal value.