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Introduction to Optimization

From Linear Programming to Nonlinear Programming

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Chapter 3

Foundations of Linear Programming

3.1 Graphical methods for solving LPs with two variables

In practice, we use computer programs to implement some optimization algorithms to solve linear programs to find their optimal solutions, or to find them to be infeasible or unbounded. We will discuss some common algorithms in detail later. Any linear program with only two decision variables can be graphically solved.

Let us illustrate this method with the following example:

$$\begin{cases} \max\limits_{x} & z = 2.25x_1 + 2.60x_2 & \\ \text{subject to} & 2x_1 + x_2 & \leq 4000, \\ & x_1 + 2x_2 & \leq 5000, \\ & x_1 & \geq \quad 0, \\ & x_2 & \geq \quad 0. \end{cases}$$

With only two decision variables, it can be graphically solved by the following procedure.

1. **Plot the feasible set:** We first plot the line $2x_1 + x_2 = 4000$ by connecting the two intercepts $(2000, 0)$ and $(0, 4000)$. This line divides the whole plane into two sides, and we need to decide which side satisfies $2x_1 + x_2 \leq 4000$. It is often convenient to use the origin to check: since $(0, 0)$ satisfies the constraint, points on the same side of the line as the origin all satisfy the constraint. After handling other constraints in a similar manner, we take the intersection to obtain the feasible set, see Figure 3.1. In this example, the quadrilateral with four vertices $(0, 0)$, $(2000, 0)$, $(1000, 2000)$, and $(0, 2500)$ is the feasible set.

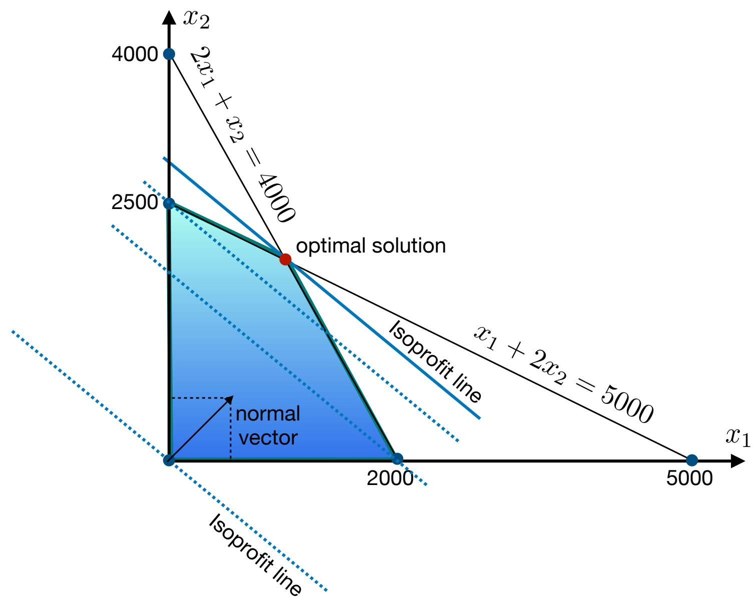

Fig. 3.1 Solving an LP graphically.

2. **Plot the objective line:** Plot lines of constant objective values (these lines are called isoprofit lines for maximization problems as the present example, and isocost lines in minimization problems). For this example, the isoprofit lines are all of the form
$$2.25x_1 + 2.60x_2 = P$$
for a given constant P. Note that isoprofit lines are all parallel. The vector of coefficients of the objective function is called the normal vector. In this example, $\vec{n} = (2.25, 2.6)$ is the normal vector. We can therefore choose an arbitrary value for P to plot the first isoprofit line, and then shift it along the normal vector $\vec{n} = (2.25, 2.6)$ to generate other isoprofit lines.
3. **Shift the isoprofit line:** Push the isoprofit line in the direction of the normal vector $\vec{n}$ to increase the objective value if we are solving a max problem. For a min problem, we need to push the isocost line in the opposite direction of $\vec{n}$ to decrease the objective value. We stop pushing when we would leave the feasible set. The point(s) that lie(s) on the intersection of the last isoprofit line and the feasible set are the optimal solution(s). In the example, the last isoprofit line meets the feasible set at the point where the two lines $2x_1 + x_2 = 4000$ and $x_1 + 2x_2 = 5000$ intersect.

4. **Determine optimal solutions:** Compute the optimal solution and the optimal value. For the example, we can solve the two equations $2x_1 + x_2 = 4000$ and $x_1 + 2x_2 = 5000$, to find the optimal solution to be $(1000, 2000)$. By plugging these values into the objective function, we find the optimal value to be 7450.

3.2 Types of linear programs

Depending on properties of their feasible sets and optimal solutions, linear programs can be classified into the following four types. We will discuss how to identify the type of any linear program in the subsequent sections.

1. **Infeasible linear programs.** These are LPs with no feasible solutions, or equivalently, LPs with empty feasible sets. For instance, consider the following problem:
$$\begin{cases} \min\limits_{x} & x_1 + x_2 \\ \text{subject to} & x_1 + x_2 \leq -1, \\ & x_1 \geq 0,\ x_2 \geq 0. \end{cases}$$
This LP is infeasible because its feasible set $\mathscr{X} = \left\{x \in \mathbb{R}^2 \mid x_1 + x_2 \leq -1,\ x_1 \geq 0,\ x_2 \geq 0\right\}$ is empty.
2. **Linear programs with unique optimal solutions.** These are linear programs that have unique optimal solutions. The example in Figure 3.1 is of this type, because the last isoprofit line meets its feasible set at a single point, which is its only optimal solution.
3. **Linear programs with multiple optimal solutions.** These are linear programs that have more than one optimal solution. Whenever an LP has two different optimal solutions, it has infinitely many optimal solutions. Why? (Exercise.)
4. **Unbounded linear programs.** An LP is said to be unbounded, if its objective value can be made arbitrarily *good* by selecting appropriate feasible solutions. In other words, a maximization LP is unbounded if for any given number M we can find a feasible solution with its objective value bigger than M. When this is the case, we say that the optimal value of this problem is $+\infty$. Conversely, a minimization LP is unbounded if for any given number M we can find a feasible solution with its objective value less than M. In this case, we say that the optimal value of this problem is $-\infty$.

We should distinguish the difference between the unboundedness of an LP and the unboundedness of its feasible set. In general, a set in $\mathbb{R}^n$ is said to be bounded, if there exists a positive number M such that the norm of all elements in this set is no more than M. Hence, the feasible set of an LP is unbounded, if for any positive number M there exists a feasible solution whose norm is bigger than M.

Note that, if an LP is unbounded, then its feasible set must be unbounded. On the other hand, it is possible for an LP with an unbounded feasible set to have either unique or multiple optimal solutions. This is illustrated in the following example.

Example 3.1. In this example, x_1 and x_2 are variables, and c_1 and c_2 are constants.

$$\begin{cases} \min\limits_{x} \ c_1x_1 + c_2x_2 \\ \text{subject to } -x_1 + x_2 \le 1, \\ x_1 \ge 0, \\ x_2 \ge 0. \end{cases} \tag{3.1}$$

Different values of c_1 and c_2 lead to different linear programs with a common feasible set shown in Figure 3.2. We consider three cases, with different values of

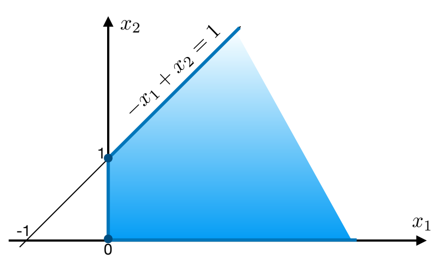

Fig. 3.2 The feasible set in the problem (3.1).

c_1 and c_2.

Case 1: Let us choose $c_1 = 1$ and $c_2 = 1$. Then $c = (1,1)$ is the normal vector of the isocost line. The isocost lines in this case are dashed lines in Figure 3.3. The last isocost line meets the feasible set at the unique optimal solution $x = (0,0)$. The optimal value is 0. The LP has a unique optimal solution.

Case 2: Let us choose $c_1 = 1, c_2 = 0$. The isocost lines in this case are dashed lines in Figure 3.4. The last isocost line that meets the feasible set intersects with it at

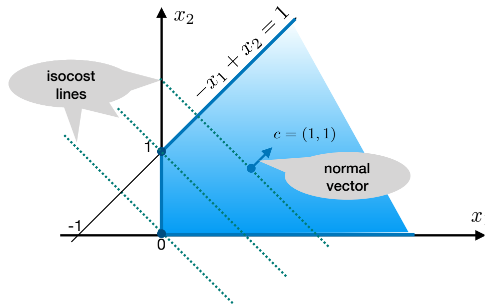

Fig. 3.3 **Case 1** of the LP (3.1) with $c = (1,1)$.

its left edge. Each point on that edge is an optimal solution. For example, $(0,0)$

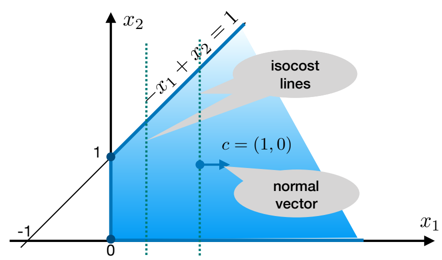

Fig. 3.4 **Case 2** of the LP (3.1) with $c = (1,0)$.

and $(0,1)$ are both optimal solutions. Hence, in this case the LP has multiple optimal solutions. Indeed, the set of optimal solutions can be written as

$$\{(0,x_2) \mid 0 \leq x_2 \leq 1\},$$

which is a bounded set.

Case 3: Let us choose $c_1 = -1$ and $c_2 = -1$. The isocost lines in this case are dashed lines in Figure 3.5. Note that the isocost lines in this case are the same as those in **Case 1**, but we need to push in the opposite direction to decrease the objective value. We can keep pushing the isocost line further and further, without ever leaving the feasible set. Thus, the LP is unbounded, which means that the objective value can be made arbitrarily negative for a minimization problem.

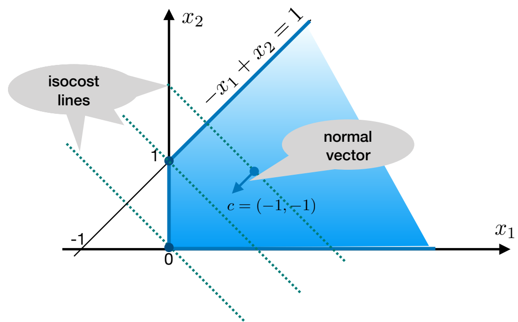

Fig. 3.5 **Case 3** in the LP (3.1) with $c = (-1, -1)$.

3.3 Forms of linear programs

3.3.1 Forms of LPs

An LP can be written in one of the three forms: general form, standard form, or canonical form.

3.3.1.1 The general form

By definition, any linear program can be written into the following **general form**:

$$
\begin{cases}
\displaystyle\min_{x\in\mathbb{R}^n} \text{ (or } \max_{x\in\mathbb{R}^n}\text{) } z = \sum_{j=1}^n c_j x_j & \\
\text{subject to} \quad \displaystyle\sum_{j=1}^n a_{ij}x_j \le b_i, & i = 1, \cdots, m \\
\qquad\qquad\quad \displaystyle\sum_{j=1}^n a_{ij}x_j \ge b_i, & i = m+1, \cdots, m+p \\
\qquad\qquad\quad \displaystyle\sum_{j=1}^n a_{ij}x_j = b_i, & i = m+p+1, \cdots, m+p+q \\
\qquad\qquad\quad x_j \ge 0, & j = 1, \cdots, k \\
\qquad\qquad\quad x_j \le 0, & j = k+1, \cdots, k+l.
\end{cases}
$$

For a general LP:

- its linear objective function $z = c^T x$ can be either **maximized** or **minimized**.
- its constraints can be a mixture of the three types:
  - "$\le$": less than or equal to (LE),
  - "$\ge$": great than or equal to (GE), and
  - "$=$": equal to (E).
- its variables can be any of the three types:

- "≥ 0": nonnegative
- "≤ 0": nonpositive, and
- "free": no sign constraint.

- We note that linear programs may have bound constraints as $l_i \leq x_i \leq u_i$ for some $i \in \{1, \cdots, n\}$, where $l_i, u_i \in \mathbb{R}$ and $l_i \leq u_i$. If $l_i = u_i$, then x_i is called a fixed variable. If $l_i = -\infty$, then x_i does not have a lower bound, and if $u_i = +\infty$, then x_i does not have an upper bound. Bound constraints can be treated as LE and GE constraints, and sometimes it helps to handle them in special ways to improve algorithm efficiency.

Example 3.2. Here is an example of a general LP:

$$\begin{cases} \min\limits_{x} \ z = x_1 - 2x_2 + x_3 - 3x_4 & \\ \text{s.t.} \ \ 2x_1 + x_2 - x_3 + x_4 = 3 & (E) \\ \qquad x_1 - 5x_2 + x_3 \leq 4 & (LE) \\ \qquad 3x_2 + 2x_2 - 3x_4 \geq -2 & (GE) \\ \qquad x_1 \geq 0, \ x_2 \leq 0, \ -2 \leq x_3 \leq 4. & \end{cases}$$

Here, x_4 is a free variable without any sign constraint, and x_3 has a bound constraint with $l_3 = -2$ and $u_3 = 4$.

3.3.1.2 The standard form

An LP is said to be in the **standard form** if it satisfies the following conditions simultaneously:

1. Each variable x_i in the LP is subject to the nonnegativity sign restriction (i.e., $x_i \geq 0$ for $i = 1, \cdots, n$).
2. Each other constraint besides the nonnegativity constraints is an equality constraint of the form $\sum_{j=1}^n a_{ij}x_j = b_i$ for some $i = 1, \cdots, m$.

Note: Some books restrict standard form LPs to either minimization or maximization problems. Our definition for standard form allows both minimization and maximization LPs, and the requirement is only on the format of constraints.

A standard LP can be represented as follows:

$$\begin{cases} \min\limits_{x\in\mathbb{R}^n} \ (\text{or } \max\limits_{x\in\mathbb{R}^n}) \ z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{s.t.} \qquad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ \qquad\quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \qquad\quad \vdots \qquad\qquad\qquad\qquad \vdots \quad \vdots \\ \qquad\quad a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \\ \qquad\quad x_i \geq 0, \qquad i = 1, 2, \cdots, n. \end{cases}$$

Matrix representation of the standard form: If we define

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

then we can write the above standard LP as

$$\begin{cases} \min\limits_{x} \ (\text{or } \max\limits_{x}) \ z = c^T x \\ \text{s.t.} \qquad Ax = b, \\ \qquad\quad x \geq 0. \end{cases} \tag{3.2}$$

Example 3.3. The following LP is in the standard form:

$$\begin{cases} \min\limits_{x} z = 2x_1 + x_2 - 3x_3 \\ \text{s.t.} \ \ 2x_1 + x_2 + x_3 = 6 \\ \qquad x_1 - 2x_2 + 3x_3 = 12 \\ \qquad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{cases}$$

Let us define

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & 3 \end{bmatrix}, \quad b = \begin{pmatrix} 6 \\ 12 \end{pmatrix}, \quad c = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \text{ and } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Then, we can write this LP into the matrix form (3.2).

Example 3.4. Consider the following LP:

$$\begin{cases} \max\limits_{x} & z = x_1 + x_2 \\ \text{s.t} & 2x_1 + x_2 \leq 60, \\ & x_1 + x_2 \quad \leq 20, \\ & x_1 \qquad \geq \ \ 0. \end{cases}$$

This LP is in general form but not in the standard form, because (i) the first two constraints are not equalities; and (ii) there is no nonnegativity requirement on the variable x_2.

3.3.1.3 The canonical form

We have seen that the standard form of an LP is a special case of the general form. There is a special case of the standard form, called the **canonical form**. That means

$$\text{canonical form} \quad \subset \quad \text{standard form} \quad \subset \quad \text{general form}.$$

Here, $A \subset B$ means that A is a subset of B.

An LP is said to be in the canonical form if it simultaneously satisfies the following conditions:

1. It is in the standard form.
2. The entries of the right-hand-side vector of the constraints are all nonnegative.
3. Each equality constraint **isolates** a variable. That is
   - This variable has a coefficient of 1 in this constraint;
   - This variable has a zero coefficient in all other equality constraints;
   - This variable has a zero coefficient in the objective function.

Example 3.5. The following LP is in a canonical form as it satisfies all above conditions.

$$\begin{cases} \max\limits_{x} z = x_1 + x_2 \\ \text{s.t} \qquad 2x_1 + x_2 + x_3 \qquad = 60, \\ \qquad\quad x_1 + x_2 \qquad + x_4 = 20, \\ \qquad\quad x_1, x_2, x_3, x_4 \qquad \geq \ \ 0 \end{cases}$$

Why?

1. It is in the standard form.
2. The right-hand side vector of its equality constraints is $(60,20)^T$, which is nonnegative.
3. The coefficient of x_3 is 1 in the first equality constraint, and is 0 in the other equality constraint and the objective function, so x_3 is isolated by the first equality constraint. Similarly, x_4 is isolated by the second equality constraint.

If we define

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \quad b = \begin{pmatrix} 60 \\ 20 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

then we can rewrite this problem into the matrix form (3.2).

For a canonical LP with m equality constraints, we use the following notation.

- $x_{B(1)}$: the variable isolated by the 1^{st} equality constraint,
- $x_{B(2)}$: the variable isolated by the 2^{nd} equality constraint,
- $\cdots$
- $x_{B(m)}$: the variable isolated by the m^{th} equality constraint.

Then, from the definition, it follows that

$$c_{B(1)} = c_{B(2)} = \cdots = c_{B(m)} = 0,$$

and

$$A_{B(1)} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad A_{B(2)} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots \quad A_{B(m)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Later, we will denote $c_B = (c_{B(1)}, \cdots, c_{B(m)})^T$, a column vector whose entries are $c_{B(1)}, \cdots, c_{B(m)}$, and $A_B = (A_{B(1)}, \cdots, A_{B(m)})$, an $m \times m$ submatrix of A whose columns are formed from m columns $A_{B(1)}, \cdots, A_{B(m)}$.

Note: If an LP is given by

$$
\begin{cases}
\min\limits_{x\in\mathbb{R}^n} \ (\text{or} \ \max\limits_{x\in\mathbb{R}^n}) & z = c^T x \\
\text{s.t.} & Ax \le b, \\
 & x \ge 0,
\end{cases}
$$

where b is a vector of nonnegative entries, then we can always convert it into the canonical form by introducing m slack variables $x_{n+1},\ldots,x_{n+m}$ as we will see in the next subsection.

3.3.2 Converting from the general form to the standard form

Two linear programs are said to be *equivalent* if for every feasible solution of one problem there exists a feasible solution of the other problem with the same objective value. (In fact, as long as the difference between the two objective values is a constant for any such pair of feasible solutions, we can say the two LPs are equivalent.) Any LP in the general form can be converted into an *equivalent* LP in the standard form, by the following procedure. It is a step of the preprocessing stage in linear programming.

- **The objective function:** We note that $\max z = -\min(-z)$. Therefore, to convert from the maximization form to the minimization form, we simply need to change the values c_i in the objective to $-c_i$, and vice versa.
- **Nonpositive variables:** For a nonpositive variable $x_j \le 0$, we conduct a change of variable as $\hat{x}_j = -x_j$, where $\hat{x}_j$ is a new variable on which we impose the sign restriction $\hat{x}_j \ge 0$. We then substitute x_j by $-\hat{x}_j$ in the objective functions and all constraints.
- **Free variables:** For a free variable x_j, we introduce two new variables x_j^+ and x_j^- and write $x_j = x_j^+ - x_j^-$. In this case, x_j^+ and x_j^- are both required to be nonnegative, i.e., $x_j^+ \ge 0$ and $x_j^- \ge 0$. We then substitute x_j by $x_j^+ - x_j^-$ in the objective function and all constraints.
- **LE constraints:** For an inequality constraint

  $$\sum_{j=1}^n a_{ij}x_j \le b_i,$$

  we introduce a new variable s_i and transform this constraint into

  $$
  \begin{aligned}
  \textstyle\sum_{j=1}^n a_{ij}x_j + s_i &= b_i, \\
  s_i &\ge 0.
  \end{aligned}
  $$

Such a variable s_i is called a **slack** variable.

- **GE constraints:** For an inequality constraint

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i,$$

we introduce a new variable s_i and transform it into

$$\begin{aligned} \textstyle\sum_{j=1}^{n} a_{ij}x_j - s_i &= b_i, \\ s_i &\geq 0. \end{aligned}$$

Such a variable s_i is called a **surplus** variable.

There are other procedures for converting an LP in the general form into the standard form, see, e.g., [**?**]. We omit details of those procedures, but mention two common cases:

- **Fixed variables:** If x_j is a fixed variable, i.e., $x_j = u_j$ for a given number u_j, then we eliminate this variable by moving it to the right-hand side of the constraint, and remove it from the objective function.
- **Bound constraints:** For a bound constraint $l_j \leq x_j \leq u_j$, we can process as follows:
  - Introduce two new variables s_j^+ and s_j^- and split the constraint into two constraints as

$$x_j - s_j^- = l_i, \quad x_j + s_j^+ = u_j, \quad \text{where } s_j^+ \geq 0,\ s_j^- \geq 0.$$

  - Or we can do a change of variable $\hat{x}_j = x_j - l_j$ and introduce one slack variable s_j to write it as

$$\hat{x}_j + s_j = u_j - l_j, \ \text{ where } \hat{x}_j \geq 0,\ s_j \geq 0.$$

    Then, substitute x_j by $\hat{x}_j + l_j$ into the objective function and all the constraints. Finally, rearrange the resulting problem to remove constant terms.

- **Redundant constraints:** The row vectors of the constraint matrix A may be linearly dependent. In this case, we say that the LP has redundant constraints. One can remove such a redundant row by applying elementary row operations as discussed in the previous chapter. We skip the details.

Example 3.6. The following LP is in the general form, but not in the standard form. Convert it into the standard form.

$$\begin{cases} \max\limits_{x} & z = x_2 - 2x_3 \\ \text{s.t.} & x_1 + x_2 - 2x_3 \leq 1, \\ & 2x_1 - x_2 = 0, \\ & x_1 + x_3 \geq 0, \\ & 2 \leq x_2 \leq 5. \end{cases}$$

Solution: Applying the above procedure, we process as follows:

- Since x_1 is free, we introduce x_1^+ and x_1^- to write $x_1 = x_1^+ - x_1^-$ with $x_1^+ \geq 0$ and $x_1^- \geq 0$.
- Since x_2 has a bound constraint, we define $\hat{x}_2 = x_2 - 2$, and write $2 \leq x_2 \leq 5$ as $0 \leq \hat{x}_2 \leq 3$.
- Since x_3 is free, we introduce x_3^+ and x_3^- to write $x_3 = x_3^+ - x_3^-$ with $x_3^+ \geq 0$ and $x_3^- \geq 0$.

To process the objective, we note that $x_2 = \hat{x}_2 + 2$. Hence, we have $z = x_2 - 2x_3 = \hat{x}_2 + 2 - 2(x_3^+ - x_3^-) = \hat{x}_2 - 2x_3^+ + 2x_3^- + 2$. We define $w = \hat{x}_2 - 2x_3^+ + 2x_3^-$ by ignoring the constant 2.

Now, we process the constraints:

- For the first constraint $x_1 + x_2 - 2x_3 \leq 1$, we need a slack variable $s_1 \geq 0$, and write it as $x_1^+ - x_1^- + \hat{x}_2 + 2 - 2x_3^+ + 2x_3^- + s_1 = 1$, or equivalent to $x_1^+ - x_1^- + \hat{x}_2 - 2x_3^+ + 2x_3^- + s_1 = -1$.
- For the second constraint $2x_1 - x_2 = 0$, we write it as $2x_1^+ - 2x_1^- - \hat{x}_2 - 2 = 0$, or equivalent to $2x_1^+ - 2x_1^- - \hat{x}_2 = 2$.
- For the third constraint $x_1 + x_3 \geq 0$, we introduce a surplus variable $s_2 \geq 0$, and write it as $x_1^+ - x_1^- + x_3^+ - x_3^- - s_2 = 0$.
- For the last constraint $2 \leq x_2 \leq 5$, we already write it as $\hat{x}_2 \leq 3$ and $\hat{x}_2 \geq 0$. Hence, we need a slack variable $s_3 \geq 0$ to write it as $\hat{x}_2 + s_3 = 3$.

Finally, we write the entire standard LP as

$$\begin{cases} \max\limits_{x,s} & w = \hat{x}_2 - 2x_3^+ + 2x_3^- \\ \text{s.t.} & x_1^+ - x_1^- + \hat{x}_2 - 2x_3^+ + 2x_3^- + s_1 = -1, \\ & 2x_1^+ - 2x_1^- - \hat{x}_2 = 2, \\ & x_1^+ - x_1^- + x_3^+ - x_3^- - s_2 = 0, \\ & \hat{x}_2 + s_3 = 3, \\ & x_1^+, x_1^-, \hat{x}_2, x_3^+, x_3^-, s_1, s_2, s_3 \geq 0. \end{cases} \tag{3.3}$$

For notational simplicity, we can redefine $x = (x_1, x_2, \cdots, x_8)^T$ for $(x_1^+, x_1^-, \hat{x}_2, x_3^+, x_3^-, s_1, s_2, s_3)^T$.

3.3.3 Equivalence between the two LPs

To prove that the original LP in general form and the transformed LP in standard form (shortly, the standard LP) are equivalent, we show they satisfy the condition given in the definition of equivalent LPs at the beginning of the preceding subsection.

Given a feasible solution x of the original LP, we can perform the following steps to construct a corresponding feasible solution of the standard LP:

- For a free variable x_j, since $x_j = x_j^+ - x_j^-$, with $x_j^+ \geq 0$ and $x_j^- \geq 0$, we take $x_j^+ = \max\{0, x_j\} \geq 0$ and $x_j^- = \max\{0, -x_j\} = -\min\{0, x_j\} \geq 0$.
- For a slack variable s_i in the constraint $a_i^T x + s_i = b_i$, we define $s_i = b_i - a_i^T x \geq 0$.
- For a surplus variable s_i in the constraint $a_i^T x - s_i = b_i$, we define $s_i = a_i^T x - b_i \geq 0$.

Conversely, given a feasible solution of the standard LP, we can also construct the corresponding feasible solution of the original LP using similar rules.

Example 3.7. In Example 3.6, $x = (1, 2, 1)^T$ is a feasible solution to the first LP with objective value $z = 0$. We construct a feasible solution of the standard LP (3.3) as follows:

- given any feasible solution $(x_1, x_2, x_3) = (1, 2, 1)$ to the first LP, we can construct a feasible solution to the second LP as

$$\big(\max(x_1, 0), -\min(0, x_1), x_2 - 2, \max(x_3, 0), -\min(0, x_3), 1 - x_1 - x_2 + 2x_3, x_1 + x_3, 5 - x_2\big)^T$$
$$= (1, 0, 0, 1, 0, 0, 2, 3)^T.$$

In this case, the objective value is $w = \hat{x}_2 - 2x_3^+ + 2x_3^- = -2$. Hence, $z = w + 2 = 0$, which shows the difference between two objective functions is a fixed constant 2.

- Conversely, given any feasible solution

$$(x_1^+, x_1^-, \hat{x}_2, x_3^+, x_3^-, s_1, s_2, s_3)^T$$

to the second LP (3.3), we can construct a feasible solution to the first LP as $(x_1^+ - x_1^-, \hat{x}_2 + 2, x_3^+ - x_3^-)$, which has the same objective value for the first LP as the given feasible solution for the second LP. For example, if $(x_1^+, x_1^-, \hat{x}_2, x_3^+, x_3^-, s_1, s_2, s_3)^T = (1,0,0,1,0,0,2,3)^T$, then $x = (1,2,1)^T$.

3.4 Basic solutions and basic feasible solutions

Let us consider a standard form LP written into the following matrix form:

$$\begin{cases} \max\limits_{x \in \mathbb{R}^n} & z = c^T x \\ \text{s.t.} & Ax = b, \\ & x \geq 0. \end{cases} \tag{3.4}$$

Throughout this section, we assume that A is an $m \times n$ matrix whose rows are linearly independent (which implies $m \leq n$).

3.4.1 Basic solutions

By linear algebra theory (see the preceding chapter), the above assumption implies that there exist m columns of A that are linearly independent. One can find a **basic solution** by the following procedure:

1. Choose m linearly independent columns $A_{B(1)}, \cdots, A_{B(m)}$. These columns are called **basic columns**. Let variables $x_{B(1)}, \cdots, x_{B(m)}$ be called **basic variables**, and the remaining variables be **nonbasic variables**. The collection of all basic variables, $\{x_{B(1)}, \cdots, x_{B(m)}\}$, is called a **basis**.
2. Set the nonbasic variables to zero. That is, let $x_i = 0$ for $i \notin \{B(1), \cdots, B(m)\}$. Then solve the system of m equations $Ax = b$ for values of the basic variables. When nonbasic variables are fixed to be zero, the system $Ax = b$ reduces to a system of m equations with m variables:

$$A_{B(1)}x_{B(1)} + \cdots + A_{B(m)}x_{B(m)} = b, \tag{3.5}$$

which has a unique solution because its coefficient matrix is invertible.

3. Put values of the nonbasic variables and basic variables together to obtain a **basic solution**.

Example 3.8. Consider the system of linear constraints of a standard LP:

$$\begin{cases} x_1 + 2x_2 + x_3 - x_4 &= 4 \\ 2x_1 + x_2 + 2x_3 + x_4 &= 5. \end{cases} \quad \text{or equivalent} \quad \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 1 & 2 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

Here $m = 2$ and $n = 4$. It is easy to check that the row vectors of A are linearly independent. Note that the two first columns of A as $A_1 = (1,2)^T$ and $A_2 = (2,1)^T$ are linearly independent. By letting $B(1) = 1$ and $B(2) = 2$, we can form a basic solution $x = (x_1, x_2, 0, 0)^T$ where x_1 and x_2 are the solutions of the following square system:

$$\begin{cases} x_1 + 2x_2 = 4 \\ 2x_1 + x_2 = 5. \end{cases}$$

Solving this system, we have $x_1 = 2$ and $x_2 = 1$. This gives a basic solution $x = (2,1,0,0)^T$. $\square$

From the above construction, one can see that a vector $x^* \in \mathbb{R}^n$ is a basic solution for the LP (3.4) if and only if it satisfies the following conditions:

1. $Ax^* = b$.
2. Columns of A corresponding to nonzero entries of x^* are linearly independent.

3.4.2 Basic feasible solutions

If a **basic solution** is also a feasible solution of the LP (3.4), then it is called a **basic feasible solution**. From its construction, the basic solution must satisfy the constraint $Ax = b$, so the additional requirement for it to be a basic feasible solution is the nonnegativity of all its components. Equivalently, we say that:

A vector $x^* \in \mathbb{R}^n$ is a basic feasible solution of (3.4) if and only if it satisfies the following conditions:

1. $Ax^* = b$ and $x^* \geq 0$.

2. Columns of A corresponding to positive entries of x^* are linearly independent.

For convenience, we define $B = \{B(1), \cdots, B(m)\}$ the index set of the basic columns of A; A_B is an $m \times m$ matrix formed from $A_{B(i)}$ for $i = 1, \ldots, m$. The nonbasic index set is denoted by $N = \{1, 2, \cdots, n\} \setminus B$. Hence, we can also define the nonbasic submatrix as A_N. In this case, the matrix A can be decomposed as $A = [A_B, A_N]$, and vector c can also be decomposed as $c = [c_B, c_N]$. [1]

Example 3.9. Consider the following LP:

$$\begin{cases} \min\limits_{x} & z = x_1 - x_2 + x_3 \\ \text{s.t.} & x_1 - 2x_2 - x_3 = 2, \\ & x_1 + x_2 - x_3 = 2, \\ & x \geq 0. \end{cases} \tag{3.6}$$

There are 3 possible ways to choose basic columns:

- **Choice 1:** $B(1) = 1, B(2) = 2$: We have $A_B = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$. Hence, a basic solution $x = (x_1, x_2, x_3) = (2, 0, 0)$. This is also a basic feasible solution since $x \geq 0$.
- **Choice 2:** $B(1) = 2, B(2) = 3$: We have $A_B = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix}$. A basic solution $x = (x_1, x_2, x_3) = (0, 0, -2)$. This is not a basic feasible solution since $x_3 < 0$.
- **Choice 3:** $B(1) = 1, B(2) = 3$: We have $A_B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$. These two columns are linearly dependent. Accordingly, this choice does not provide a basic solution.

A basic feasible solution of (3.4) is said to be **nondegenerate**, if exactly $n - m$ of its components are zero. It is **degenerate** if more than $n - m$ of its components are zero.

Since every basic feasible solution is a basic solution, the nonbasic variables in it are all zero. The nondegeneracy requires all the basic variables in a basic feasible solution to be strictly positive.

[1] Here, we need to re-order the variables so that we can concatenate both submatrices to form A.

Example 3.10. In the above example, we have $n = 3$ and $m = 2$. The basic solution $(2,0,0)$ is degenerate since x_2, one of its basic variables, is 0.

Basic optimal solutions:

If an optimal solution of the LP (3.4) is also a **basic feasible solution**, then we say that it is a basic optimal solution. The following is an important fact for linear programming, whose proof is out of scope of this course.

Fact 3.4.1 *If a linear program of the form* (3.4) *has an optimal solution, then it has a* ***basic optimal solution****.*

3.4.3 Geometric interpretation of basic feasible solutions

The feasible set $\mathscr{X} := \{x \mid Ax = b, x \geq 0\}$ of the standard LP (3.4) is a **polyhedron**. In general, a polyhedron in $\mathbb{R}^n$ is a set defined by finitely many linear constraints. A point x in a polyhedron is called an **extreme point** of this set, if there do not exist two distinct points y and z in this set, such that $x = \frac{1}{2}y + \frac{1}{2}z$ (i.e., x is not the middle point of any two points y and z in this set). Geometrically, extreme points of a polyhedron are its vertices (or corner points). Below, we show that basic feasible solutions of (3.4) are exactly the extreme points of the feasible set $\mathscr{X}$.

Theorem 3.1. *A vector $x^* \in \mathbb{R}^n$ is a basic feasible solution of* (3.4) *if and only if it is an extreme point of $\mathscr{X}$.*

Proof. First, we prove that any basic feasible solution is an extreme point. Suppose x^* is a BFS, and let $\{x_{B(1)}, \cdots, x_{B(m)}\}$ be a basis that can be used to construct x^*. This means that $x_i^* = 0$ for each $i \notin \{B(1), \cdots, B(m)\}$ and that columns $A_{B(1)}, \cdots, A_{B(m)}$ are linearly independent. Suppose that $y \in \mathscr{X}$ and $z \in \mathscr{X}$ satisfy $x^* = \frac{y+z}{2}$; we will show that $y = z = x^*$.

Consider an index $i \notin \{B(1), \cdots, B(m)\}$. For such an index, we have $x_i^* = 0$, so that

$$y_i + z_i = 2x_i^* = 0.$$

Since y and z both belong to $\mathscr{X}$, they satisfy $y_i \geq 0$ and $z_i \geq 0$. The only possibility for two nonnegative numbers to sum to 0 is that both numbers are exactly 0. We have therefore shown that $y_i = z_i = 0 = x_i^*$ for each index $i \notin \{B(1), \cdots, B(m)\}$.

The two points y and z also satisfy $Ay = Az = b$. Eliminating the zero components gives

$$A_{B(1)}y_{B(1)} + A_{B(2)}y_{B(2)} + \cdots + A_{B(m)}y_{B(m)} = b$$

and

$$A_{B(1)}z_{B(1)} + A_{B(2)}z_{B(2)} + \cdots + A_{B(m)}z_{B(m)} = b.$$

However, the columns $A_{B(1)}, \cdots, A_{B(m)}$ are known to be linearly independent, so the matrix A_B that consists of those columns is an invertible matrix. Thus,

$$\begin{bmatrix} y_{B(1)} \\ y_{B(2)} \\ \vdots \\ y_{B(m)} \end{bmatrix} = \begin{bmatrix} z_{B(1)} \\ z_{B(2)} \\ \vdots \\ z_{B(m)} \end{bmatrix} = A_B^{-1}b = \begin{bmatrix} x^*_{B(1)} \\ x^*_{B(2)} \\ \vdots \\ x^*_{B(m)} \end{bmatrix}.$$

This proves that $y = z = x^*$. We have thereby shown that any BFS is an extreme point.

Next, we prove that any extreme point is a basic feasible solution. Suppose x^* is an extreme point. To prove that x^* is a BFS, it suffices to show that columns of A corresponding to positive entries of x^* are linearly independent. Suppose without loss of generality that $x^*_i > 0$ for $i = 1, \cdots, k$ and $x^*_i = 0$ for $i = k+1, \cdots, n$. We need to show that $A_1, \cdots, A_k$ are linearly independent. Suppose they are not linearly independent. Then there exist scalars $\lambda_1, \cdots, \lambda_k$, not all zero, such that $\lambda_1 A_1 + \cdots \lambda_k A_k = 0$. Let t be a small positive number. Define two points $y \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$ as follows:

$$y_i = \begin{cases} x^*_i + t\lambda_i & i = 1, \cdots, k \\ 0 & i = k+1, \cdots, n \end{cases}$$

and

$$z_i = \begin{cases} x^*_i - t\lambda_i & i = 1, \cdots, k \\ 0 & i = k+1, \cdots, n \end{cases}$$

We have

$$Ay = A_1y_1 + A_2y_2 + \cdots + A_ky_k = A_1(x^*_1 + t\lambda_1) + A_2(x^*_2 + t\lambda_2) + \cdots + A_k(x^*_k + t\lambda_k) = b$$

because $A_1x^*_1 + A_2x^*_2 + \cdots + A_kx^*_k = b$ and $\lambda_1 A_1 + \cdots \lambda_k A_k = 0$. Similarly we have $Az = b$. By choosing t to be sufficiently small, it can be guaranteed that $y_i \geq 0$ and $z_i \geq 0$ for all $i = 1, \cdots, k$, so that $y \in \mathscr{X}$ and $z \in \mathscr{X}$. Moreover, $x^* = \frac{y+z}{2}$. The fact that λ_i's are not all zero implies that y and z are two different points in $\mathscr{X}$. We have thereby written x^* as the midpoint of two distinct points in $\mathscr{X}$, which contradicts with the initial assumption that x^* is an extreme point. This contradiction is caused by assuming the columns $A_1, \cdots, A_k$ to be linearly dependent.

We can also "project" the feasible set $\mathscr{X}$ of the standard LP (3.4) in $\mathbb{R}^n$ onto a set in the smaller space $\mathbb{R}^{n-m}$ in the following way. Let A_B be an $m \times m$ invertible submatrix of A, write $A = [A_B, A_N]$ and decompose $x = (x_B, x_N)$ correspondingly. The constraint $Ax = b$ can be written as $A_B x_B + A_N x_N = b$. In this case, we can write

$$x_B + A_B^{-1} A_N x_N = A_B^{-1} b \quad \text{or equivalent to} \quad x_B = A_B^{-1} b - A_B^{-1} A_N x_N.$$

The constraint $x_B \geq 0$ then translates to the condition that $A_B^{-1} A_N \bar{x}_N \leq A_B^{-1} b$. Hence, if we define $\bar{A}_N = A_B^{-1} A_N$ and $\bar{b} := A_B^{-1} b$, then the feasible set $\mathscr{X}$ of the standard LP (3.4) in $\mathbb{R}^n$ can be projected to the subspace $\mathbb{R}^{n-m}$ as

$$\bar{\mathscr{X}} := \left\{ x_N \in \mathbb{R}^{n-m} \mid \bar{A}_N x_N \leq \bar{b}, \ \ x_N \geq 0 \right\}.$$

This set is again a polyhedron in $\mathbb{R}^{n-m}$. Each point x_N of this polyhedron can be extended into a feasible solution of (3.4) as (x_B, x_N) with $x_B = A_B^{-1} b - A_B^{-1} A_N x_N$, which projects back to x_N. The two functions (the extension and the projection) together define a one-to-one correspondence between points in $\mathscr{X}$ and $\bar{\mathscr{X}}$. In particular, each extreme point of $\bar{\mathscr{X}}$ corresponds to a basic feasible solution of (3.4).

Example 3.11. To illustrate the geometric interpretation of basic feasible solutions in LPs, let us consider the standard LP (3.4) in $\mathbb{R}^4$ with the following feasible set

$$\mathscr{X} = \left\{ x \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 4, \ \ x_1 + 2x_2 - 2x_3 + 4x_4 = 6, \ \ x \geq 0 \right\}.$$

This problem has 4 variables ($n = 4$) and 2 constraints ($m = 2$). It is easy to check that $\bar{x} = (2, 2, 0, 0)^T$ is a basic feasible solution of this problem. Hence, we can let $B = \{1, 2\}$, $N = \{3, 4\}$, and form submatrices $A_B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $A_N = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. We then express x_B as a function of x_N as

$$x_B = A_B^{-1} b - A_B^{-1} A_N x_N = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{pmatrix} 4 \\ 6 \end{pmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} x_N = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{bmatrix} 4 & -2 \\ -3 & 3 \end{bmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}.$$

Now, if we define $\bar{A} = \begin{bmatrix} 4 & -2 \\ -3 & 3 \end{bmatrix}$ and $\bar{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, then the projected feasible set of this problem becomes

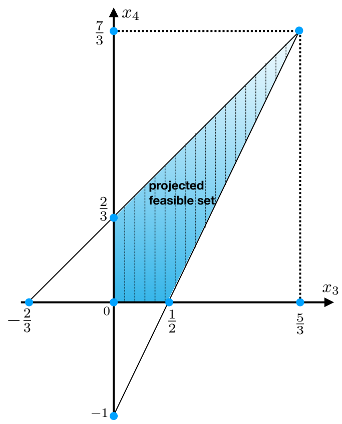

Fig. 3.6 The projected feasible set onto $\mathbb{R}^2$ of the standard LP

$$\bar{\mathscr{X}} = \left\{x_N = (x_3,x_4)^T \in \mathbb{R}^2 \mid 4x_3 - 2x_4 \leq 2, \ \ -3x_3 + 3x_4 \leq 2, \ \ x_3 \geq 0, \ x_4 \geq 0\right\}.$$

This is a subset in $\mathbb{R}^2$. We can plot this projected feasible set as in Figure 3.6, which is a quadrilateral.

This quadrilateral has 4 vertices at $x_N = (0,0)^T$, $x_N = (\frac{1}{2},0)^T$, $x_N = (\frac{5}{3},\frac{7}{3})^T$ and $x_N = (0,\frac{2}{3})^T$. For each x_N, we can compute the corresponding x_B as $x_B = (2,2)^T$, $x_B = (0,\frac{7}{2})^T$, $x_B = (0,0)^T$ and $x_B = (\frac{10}{3},0)^T$, respectively. Hence, we obtain 4 basic feasible solutions of the original problem as

$$\bar{x}^{(1)} = (2,2,0,0)^T, \ \bar{x}^{(2)} = (0,\tfrac{7}{2},\tfrac{1}{2},0)^T, \ \bar{x}^{(3)} = (0,0,\tfrac{5}{3},\tfrac{7}{3})^T, \ \text{and} \ \bar{x}^{(4)} = (\tfrac{10}{3},0,0,\tfrac{2}{3})^T.$$

These basic feasible solutions are all nondegenerate.

3.4.4 Finding a basic feasible solution

To find a basic feasible solution for an LP in the standard form, one would need to try various choices of basic columns, compute the basic solutions and check if they are feasible. We can perform two steps:

- Choose m basic columns A_B of A, and compute $\bar{x}_B = A_B^{-1}b$. If $\bar{x}_B \geq 0$, then let $\bar{x}_N = 0$ and form $\bar{x} = (\bar{x}_B, \bar{x}_N)$ as a basic feasible solution.
- Otherwise, repeat this step by choosing another set of basic columns A_B.

This procedure may require many iterations, but it terminates after at most $\frac{n!}{m!(n-m)!}$ iterations, a very big number when n is large.

Example 3.12. In Example 3.11, we can choose $B = \{1,2\}$ and $A_B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, which is invertible. We compute $\bar{x}_B = A_B^{-1} b = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \geq 0$. Hence, we can form a basic feasible solution $\bar{x} = (2,2,0,0)^T$.

If the LP (3.4) is not only in the standard form but also in the canonical form, then it is much easier to find a basic feasible solution of it. As in Section 3.3, let $x_{B(i)}$ be the variable isolated by the i-th equality constraint for each $i = 1, \cdots, m$. Their corresponding columns satisfy

$$A_{B(1)} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad A_{B(2)} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad A_{B(m)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

which are linearly independent. Let $x_{B(1)}, \cdots, x_{B(m)}$ be the basic variables, and set other variables to zero. Since columns $A_{B(1)}, \cdots, A_{B(m)}$ form the identity matrix $\mathbb{I}_m$ (i.e., $A_B = \mathbb{I}_m$), the system (3.5) has a unique solution given by $x_{B(i)} = b_i$ for each i. Since the LP is in the canonical form, its right-hand-side constants b_i are all nonnegative. The basic solution obtained by this way is therefore a basic feasible solution, and is called *the basic feasible solution naturally associated with the LP in canonical form*, or simply the *natural basic feasible solution*.

Example 3.13. In Example 3.5, the basic feasible solution naturally associated with the LP in the canonical form is $x = (0,0,60,20)^T$, with the isolated variables x_3 and x_4 being basic variables.

3.4.5 Existence of optimal solutions

The following theorem summarizes conditions for existence of an optimal solution (or a basic optimal solution) in linear programs. The phrase "has a" here means "has at least one." The condition "the objective value is bounded from above on its feasible set" means that you can find a fixed constant M such that the objective value achieved by any feasible solution is no more than M. The first statement of this theorem justifies the completeness of the four-type classification of linear programs in Section 3.2. The second statement restates Fact 3.4.1. The proof can be found in [**?**], for example.

Theorem 3.2. *A maximization linear program has an optimal solution if and only if its feasible set is nonempty and the objective value is bounded from above on its feasible set. For an LP in standard form* (3.4)*, if it has an optimal solution, then it has a basic optimal solution.*

3.5 Exercises

Exercise 3.1. Megabuck Hospital Corporation is going to build a subsidized nursing home catering to homeless and high-income patients.

- The overall capacity of the new hospital should be 2100 patients.
- State regulations require such hospitals to house a minimum of 1000 homeless patients and no more than 750 high-income patients in order to receive the subsidy.
- The board of directors, under pressure from a neighborhood group, insists that the number of homeless patients will not exceed twice the number of high-income patients.
- The hospital will make a profit of $\$10,000$ per month for each homeless patient, and $\$8,000$ per month for each high-income patient.

1. Formulate an LP problem to find how many beds should be reserved for each type of patient in order to maximize the hospital's profit.
2. Solve the LP problem graphically. Label the vertices (i.e., corners) of the feasible set with their coordinates, and then find the optimal solution(s) and the optimal value. You can use the method of graphing isoprofit lines as did in class, or you can compare the objective function value at each vertex and choose the best.
3. Interpret your solution.

Exercise 3.2. Consider the following LP problem:

$$\begin{cases} \min\limits_{x_1,x_2} & z = 3x_1 + x_2 \\ \text{subject to} & x_1 \qquad\ \ \geq 3, \\ & x_1 + x_2 \ \geq 4, \\ & x_2 \qquad\ \ \geq 0. \end{cases}$$

1. Graphically show the feasible set. Label the vertices (i.e., corners) of the feasible set with their coordinates.
2. Find the optimal solution(s) and the optimal value. What type of LP is this?
3. Suppose now the objective function changes to min $x_1 + x_2$. What type of LP is this? Find the optimal solution(s) and the optimal value if they exist.
4. Suppose now the objective function changes to min $x_1 - x_2$. What type of LP is this? Find the optimal solution(s) and the optimal value if they exist.
5. Suppose now that we add an additional constraint $x_1 + 2x_2 \leq 3$ to the original LP. What type of LP is this? Find the optimal solution(s) and the optimal value if they exist.

Exercise 3.3. Consider the following linear programming problem of two variables:

$$\begin{cases} \max\limits_{x_1,x_2} z = 2x_1 + x_2 \\ \text{s.t.} \quad a_1x_1 + a_2x_2 \leq b \\ \qquad\quad x_1 \geq 0,\ x_2 \geq 0. \end{cases} \tag{3.7}$$

where a_1, a_2 and b are given

1. Find one value of a_1, a_2, and b such that the feasible set is empty (infeasible LP problem).
2. Find one value of a_1, a_2, and b such that the feasible set is unbounded.
3. Find one value of a_1, a_2, and b such that the problem is unbounded.
4. Find one value of a_1, a_2, and b such that the problem has multiple solutions.
5. Is there any value of a_1, a_2, and b such that the feasible set is unbounded, but the problem has an optimal solution?

Show your results graphically on an x_1x_2-plane.

Make sure that you understand the difference between the unbounded feasible set and the unbounded problem. The unbounded problem must have an unbounded feasible set, and its objective function goes to infinity on such a set. But an LP problem has an unbounded feasible set, it may still have optimal solution with finite optimal value.

Exercise 3.4. Given the following linear programming problem with three variables:

$$\begin{cases} \min\limits_{x_1,x_2,x_3} & z = c_1x_1 + c_2x_2 + c_3x_3 \\ \text{s.t.} & -x_1 + x_2 + x_3 = 1 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{cases} \tag{3.8}$$

where the normal vector $c = (c_1, c_2, c_3)$ of the objective function is given.

1. Reformulate this problem into a two variables linear program (we call it a *reduced LP problem*).
2. Plot the feasible domain of the reduced problem in a plane.
3. Is there any triple (c_1, c_2, c_3) such that the LP problem (3.8) is unbounded? If yes, find one triple.
4. Is there any triple (c_1, c_2, c_3) such that the LP problem (3.8) has multiple solutions? If yes, find one triple.

Exercise 3.5. You are given the following linear programming problems:

1.
$$\begin{cases} \min\limits_{x} & z = 2x_1 + 5x_2 \\ \text{s.t.} & x_1 \geq 5, \\ & x_1 + x_2 \leq 9, \\ & -2x_1 + x_2 = -4, \\ & x_1, x_2 \geq 0. \end{cases}$$

2.
$$\begin{cases} \max\limits_{x} & z = x_1 + 2x_2 + 2x_3 + 3x_4 \\ \text{s.t.} & 2x_1 + x_2 + x_3 + x_4 \geq 15, \\ & x_1 + x_2 \leq 6, \\ & x_1 \geq 0,\ \ x_2 \leq 0,\ \ x_4 \geq 0. \end{cases}$$

3.
$$\begin{cases} \max\limits_{x} & z = x_1 + 3x_3 \\ \text{s.t.} & x_1 + x_3 + x_4 = 12, \\ & x_1 + x_2 + x_3 = 8, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{cases}$$

4.

$$
\begin{cases}
\max_x & z = x_1 - 2x_2 - x_3 + x_4 \\
\text{s.t.} & x_1 \quad + x_3 + x_4 + x_5 \quad = \; 14, \\
 & x_1 + x_2 + x_3 \qquad - x_6 = -12, \\
 & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.
\end{cases}
$$

Solve the following questions for each problem:

(a) Determine if each of the above linear programs is in standard form or canonical form. For linear programs in canonical form, specify the isolated variables.

(b) For linear programs not in standard form, convert them into standard form. Check if the converted problem is canonical.

(c) If the problem is feasible, then given a feasible solution $\bar{x}$ of each problem above, how can you form a feasible solution for the corresponding problem that you convert to?

(d) If the problem is feasible, then find a specific feasible solution for each problem above. Then, form a corresponding feasible solution of the converted one from this feasible solution.

Exercise 3.6. Consider the following LP problem:

$$
\begin{cases}
\max_x & z = 2x_1 + 3x_2 \\
\text{s.t.} & x_1 + x_2 + x_3 - x_4 = 2 \\
 & x_1 - x_2 \qquad + x_4 = 0 \\
 & x \geq 0.
\end{cases}
\tag{3.9}
$$

1. List all the basic solutions. Identify all the nondegnerate basic feasible solutions and degenerate basic feasible solutions.
2. Project the feasible set onto the (x_1, x_2) plane: first express x_3 and x_4 as functions of x_1 and x_2, then convert the nonnegativity constraints on x_3 and x_4 into constraints on x_1 and x_2, and finally write down the set of (x_1, x_2) that satisfies the nonnegativity constraints on themselves and the constraints transformed from the nonnegativity of x_3 and x_4.
3. Graph the set you obtain in question 2 in the (x_1, x_2) plane, and label all of its vertices.
4. Does the LP in (3.9) have an optimal solution? If yes, what are the solutions, and the optimal value?

Exercise 3.7. Consider the following LP problem in standard form, with a single equality constraint:

$$\begin{cases} \max_x \ \sum_{i=1}^n c_i x_i \\ \text{s.t.} \ \sum_{i=1}^n a_i x_i = b \\ \qquad x_i \geq 0, \ i = 1, \cdots, n \end{cases} \tag{3.10}$$

Suppose $b > 0$, and $a_i > 0$ for each $i = 1, \cdots, n$.

1. Write down all the basic feasible solutions.
2. For each $i = 1, \cdots, n$, find a number M_i, which may depend on b and a_i, such that any feasible solution x satisfies $x_i \leq M_i$.
3. If $c_i \geq 0$ for all $i = 1, \cdots, n$, then does the LP in (3.10) have an optimal solution? If yes, what is it? and what is the optimal value?
4. If there is no condition on c, then does the LP in (3.10) have an optimal solution? If yes, what is it? and what is the optimal value?
5. Is the feasible set of (3.10) bounded? (This is an extra question without any credit, just for you to understand LPs).

Exercise 3.8. The bartender of your local pub asks you to assist him in finding the combination of mixed drinks to maximize his revenue. He has the following bottles available:

- 1 quart (32 oz.) Old Cambridge (a fine whiskey)
- 1 quart Joy Juice (another fine whiskey)
- 1 quart Ma's Wicked Vermouth
- 2 quarts Gil-boy's Gin

Since he is new to the business, his knowledge is limited to the following drinks:

- *Whiskey Sour*. Each serving uses 2 oz. whiskey and sells for \$1.
- *Manhattan*. Each serving uses 2 oz. whiskey and 1 oz. vermouth, and sells for \$2.
- *Martini*. Each serving uses 2 oz. gin and 1 oz. vermouth, and sells for \$2.
- *Pub Special*. Each serving uses 2 oz. gin and 2 oz. whiskey, and sells for \$3.

First, formulate an LP model to maximize the bar's profit. Ignore the fact that the numbers of servings have to be integers. Second, if the problem is not canonical, convert it into a standard form, and you will see the resulting problem is canonical.

References

1. G. B. Dantzig. *Linear Programming and Extensions*. Princeton University Press, 1963.
2. R.J. Vanderbei. *Linear Programming: Foundations and Extensions*. Springer, 2015.