# STOR 415, Fall 2019
# Solutions to Homework Assignment No. 2

1. $AB = \begin{bmatrix} 16 & 17 \\ 25 & 26 \end{bmatrix}$

2. (a) The equality $x^T y = y^T x$ holds for any two $n$-dimensional column vectors $x$ and $y$.

   Yes, because

$$x^T y = \begin{bmatrix} x_1, & x_2, & \cdots, x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i = \begin{bmatrix} y_1, & y_2, & \cdots, y_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = y^T x.$$

   (b) If $A$, $B$ and $C$ are invertible matrices of the same dimension, then the product of their transposes, $A^T B^T C^T$, is invertible.

   True. If $A$, $B$ and $C$ are invertible matrices of the same dimension, then their transposes are invertible matrices of the same dimension. Accordingly, the product of their transposes is also invertible.

   (c) If $A$, $B$ and $C$ are invertible matrices of the same dimension, then their sum $A + B + C$ is invertible.

   False. Let $A = [1]$, $B = [1]$ and $C = [-2]$.

   (d) Suppose that $A$ is an invertible matrix. Switch its top two rows to obtain a new matrix $B$. The matrix $B$ must be invertible as well.

   True. If $A$ is an invertible matrix, then its rows are linearly independent (by the theorem). After switching the first row with the second row, the new matrix contains exactly the same row vectors. So these rows continue to be linearly independent, and the new matrix is invertible (again by the theorem).

   (e) Let $v_1, v_2, \cdots, v_k$ be column vectors of the same dimension, and let $v = v_1 + v_2 + \cdots + v_k$. The vectors $v_1, v_2, \cdots, v_k, v$ are linearly dependent.

   True, because $v_1 + v_2 + \cdots + v_k - v = 0$.

   (f) The system $Ax = b$ has a solution, if $b$ is a linear combination of columns of $A$.

   True. Let $A_1, \cdots, A_n$ be columns of $A$. If $b$ is a linear combination of $A_1, \cdots, A_n$ then there exist coefficients $x_1, \cdots, x_n$ such that $b = \sum_{i=1}^{n} A_i x_i$. So the column vector $x = (x_1, \cdots, x_n)$ is a solution.

   (g) For a linear system that contains 2 equations and 3 variables (i.e., $A \in \mathbb{R}^{2\times 3}$), there are always infinitely many solutions.

False. Consider the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 1, \\ x_1 + x_2 + x_3 &= 2. \end{aligned}$$

There are no solutions.

3. (a) Final system:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

There is a unique solution $(1,1,1)$.

(b) Final system:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & -0.5 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There are infinitely many solutions and the solution set is

$$\{(-2, 0.5 + 0.5x_3, x_3) \mid x_3 \in \mathbb{R}\}.$$

(c) Final system:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 0.5 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The system has no solution.

4. (a) $\{(4 - 2x_4, -2, 2, x_4) \mid x_4 \in \mathbb{R}\}$.

(b) $\{(3 - x_3 - 9x_4, 2 + x_3 - 3x_4, x_3, x_4) \mid x_3 \in \mathbb{R}, x_4 \in \mathbb{R}\}$.

(c) $\{(0, 0, -x_4, x_4) \mid x_4 \in \mathbb{R}\}$.

(d) No solution.

5. (a) Linearly dependent: because $v_1 + v_2 - v_3 = 0$.

(b) Linearly independent, because the equation $c_1v_1 + c_2v_2 + c_3v_3 = 0$ has a unique solution which is zero.

6. (a) row 3 = -3 (row 1) + (row 2)

(b) Regardless of which three columns of $A$ are contained in $B$, $B$ is not invertible. If $B$ would be invertible, then its three columns would be linearly independent, which means that $A$ has three linearly independent columns. Since $A$ has three rows this would imply all rows of $A$ to be linearly independent, contradicting with the fact in (a).