Shu Lu *and* Quoc Tran-Dinh

Introduction to Optimization

From Linear Programming to Nonlinear Programming

August 23, 2019

STOR-UNC-Chapel Hill

Chapter 1

Introduction

1.1 Introduction

The process of solving a real-world problem is often complicated and consists of several stages. One often needs to collect and analyze data before proceeding to develop a mathematical model. Figure 1.1 illustrates the three main stages of such a process.

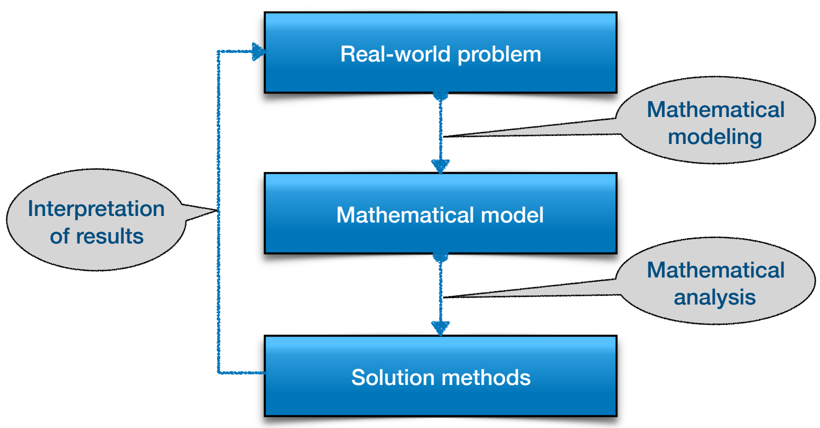


Fig. 1.1 Three main stages of a process of solving real-world problems using mathematical tools

In the mathematical modeling stage, a real-world problem is modeled into a mathematical problem using mathematical concepts such as variables, parameters, equations, constraints and objective functions. Once such a model is in place, we can use mathematical tools to develop algorithms or apply computer software to solve it. This stage is seen as the mathematical analysis stage. The results/solutions of the mathematical model are then analyzed and interpreted in the original real-world setting. If we find a solution that achieves the original goal, then we can recommend implementing it in practice. Otherwise, we should go back to identify issues within the process and fix them, and repeat the process.

In this course, we will cover basic concepts in optimization, fundamental theory in linear programming, brief introductions to nonlinear programming and integer programming, as well as representative solution methods for linear and integer programming. We will also use examples to show how to build optimization models for practical problems in production planning, inventory management, network flow optimization, machine learning, and so on.

1.2 What is mathematical optimization?

A mathematical optimization problem is also called a mathematical programming problem (or a mathematical program). It is the problem of finding the best solutions among all feasible solutions.

Example 1.1. As a very simple example, the problem of finding a rectangle with the maximum area among all rectangles of a fixed perimeter can be modeled as a mathematical optimization problem.

1.2.1 Definitions

An optimization problem must consist of three components:

- **Decision variables** (shortly, variables): Decision variables are the unknown variables which describe the decisions to be made.
- **The objective function:** The objective function is a function of the decision variables that reflects the objective of the decision making. Each optimization problem considered in this course has a single objective function. (Optimization problems with more than one objective functions are called multi-objective optimization problems and are beyond the scope of this course.)
- **Constraints:** These are requirements on values of the decision variables.

Formally, an *optimization problem* **seeks** *values* of the *decision variables* that **optimize** (maximize or minimize) an *objective function* while **satisfying** all the given *constraints*.

Mathematical formulation: Mathematically, an optimization problem can be written as follows:

$$\begin{cases} \min\limits_{x_1,\cdots,x_n} \ (\text{or} \ \max\limits_{x_1,\cdots,x_n}) & f(x_1,\cdots,x_n) \\ \text{subject to} & (x_1,\cdots,x_n) \in \mathscr{X}. \end{cases} \tag{1.1}$$

In the above formulation, $x_1, x_2, \cdots, x_n$ are n decision variables, f is the objective function that returns a numerical value for any input $(x_1,\cdots,x_n)$, and $\mathscr{X}$ is a subset in $\mathbb{R}^n$ and it contains all values of the vector $(x_1,\cdots,x_n)$ that satisfy the constraints. The notation $\min_{x_1,\cdots,x_n}$ ($\max_{x_1,\cdots,x_n}$) means that the goal is to minimize (maximize) the value of $f(x_1,\cdots,x_n)$ among all vectors $(x_1,\cdots,x_n)$ in the set $\mathscr{X}$.

The following basic concepts in optimization will be used frequently in this course.

- **Feasible solutions:** A feasible solution is an assignment of values to the decision variables such that all the constraints are simultaneously satisfied. If there are n decision variables, then each feasible solution is an n-dimensional vector.

  Example 1.2. Consider the following example:

$$\min_{x\in\mathbb{R}^2} \left\{x_1^2 + (x_2-1)^2 \mid x_1+x_2 = 1,\ 0 \le x_1 \le 1,\ 0 \le x_2 \le 1\right\}. \tag{1.2}$$

  Any point $(x_1,x_2) \in \mathbb{R}^2$ that satisfies $x_1+x_2 = 1$, $0 \le x_1 \le 1$ and $0 \le x_2 \le 1$ simultaneously is a feasible solution. For instance, $x = (0,1)$ is feasible, while $x = (\frac{1}{3},\frac{1}{3})$ is not feasible as it does not satisfy $x_1+x_2 = 1$.

- **The feasible set:** The feasible set (or the feasible region) is the set of all feasible solutions. When there are n decision variables, the feasible set is a subset of $\mathbb{R}^n$. In the formulation (1.1), $\mathscr{X}$ is the feasible set. In the above example, $\mathscr{X} := \left\{x \in \mathbb{R}^2 \mid x_1+x_2 = 1,\ 0 \le x_1 \le 1, 0 \le x_2 \le 1\right\}$ is the feasible set of (1.2).
- **Optimal solutions:** An optimal solution, if any, is a feasible solution that achieves the best objective value among all feasible solutions. For example, the point $x^* = (0,1)$ is an optimal solution of (1.2) since $f(x^*) = 0 \le f(x) = x_1^2 + (x_2-1)^2$ for all $x = (x_1,x_2) \in \mathscr{X}$. In fact, x^* is the unique optimal solution of (1.2).
- **The optimal value:** The optimal value is the objective value achieved at an optimal solution. For example, the optimal value of (1.2) is 0.

We note that an optimization problem can have more than one optimal solution or no optimal solution at all. For instance, consider the following problem with three decision variables:

$$\begin{cases} \min\limits_{x\in\mathbb{R}^3} & x_1^2 + 2x_2 + x_3 \\ \text{s.t.} & x_1+x_2+x_3 \le 3, \\ & 4x_2 + 2x_3 \ge 1, \\ & x_1,x_2,x_3 \ge 0, \end{cases}$$

The vector $x^* = (0,\frac{1}{4},\frac{1}{2})$ is an optimal solution with the optimal value $f(x^*) = \frac{1}{2}$. The vector $x^* = (0,0,\frac{1}{2})$ is another optimal solution with the same optimal value $f(x^*) = \frac{1}{2}$. In fact, the set of optimal solutions is $\mathscr{X}^* = \left\{x \in \mathbb{R}^3 \mid x_1 = 0,\ \ 2x_2 + x_3 = \frac{1}{2},\ \ x_2 \ge 0,\ x_3 \ge 0\right\}$. Therefore, this problem has an infinite number of optimal solutions.

The following example shows that an optimization problem may not have any optimal solution:

$$\min_{x\in\mathbb{R}^2} \{x_1 + x_2 \mid x_1 + x_2 \le 0,\ \ 2x_1 + x_2 \le 3\}.$$

This problem has infinitely many feasible solutions, but it does not have an optimal solution. For any feasible solution x, we can find another feasible solution $\bar{x}$ such that $f(\bar{x}) < f(x)$.

For the maximum area rectangle problem above, we can define x as the width and y as the length of a rectangle. Hence, x and y are referred to as decision variables. We have two constraints as $x, y \ge 0$, and $2(x+y) = d$, where d is the given perimeter of the rectangle. Hence, we can write $\mathscr{X} = \left\{(x,y) \in \mathbb{R}^2 \mid x, y \ge 0,\ x+y = \frac{d}{2}\right\}$ as the feasible set. The objective function is $f(x,y) = xy$, which we need to maximize. By putting the objective function and the constraints together, we obtain an optimization problem of the form (1.1).

Continuous vs. discrete: In a continuous optimization problem, values of the decision variables are allowed to be continuous numbers (real numbers with no gaps) in a given subset of $\mathbb{R}^n$. In contrast, the values of decision variables are discrete (finite, or countably infinite) in a discrete optimization problem. The main focus of this course is on continuous optimization problems, with a brief introduction to discrete problems.

1.2.2 More examples

Let us provide more motivating examples of optimization problems used in different fields.

*Example 1.3 (**Optimizing resources**).* We are making an open-topped box from some material so that it can contain 15 cubic inches. We want to make the bottom thicker than the sides of the box. The material for the bottom costs \$0.5 per square inch, and the material for the sides costs \$0.35 per square inch. Our goal is to find the dimensions (width, length, and height) of the box to minimize the total material cost.

Mathematical model: To model this problem, we denote by x, y and z the width, length and height of the box, respectively. Here, $x \ge 0$, $y \ge 0$ and $z \ge 0$. The volume of the box is $V = xyz = 15$, while the total material cost is $C = 0.5xy + 0.7(xz + yz)$. Hence, the problem becomes

$$\min_{x,y,z} \{C = 0.5xy + 0.7(xz + yz)\} \quad \text{subject to} \quad xyz = 15,\ \ x \ge 0,\ y \ge 0,\ z \ge 0.$$

This problem can be solved analytically to obtain a unique optimal solution.

*Example 1.4 (**The transportation problem**).* There are three warehouses WH_1, WH_2 and WH_3 for some commodity located in nearby cities. The minimum demand at each warehouse is given in the last row of Table 1.1. Two factories F_1 and F_2 can supply this commodity to the warehouses, and the capacities of their supply are given in the last column of Table 1.1. The cost to transport each unit of commodity from each factory to each warehouse is also given in Table 1.1. The goal is to find a transportation plan

	WH_1	WH_2	WH_3	Supply
F_1	4	3	7	15
F_2	5	4	6	25
Demand	8	12	20	40

Table 1.1 Demands, supply capacities, and transportation costs.

to minimize the total shipping cost, while fulfilling all the demand and supply requirements.

Mathematical model: To formulate this into an optimization problem, we denote by x_{ij} the units of commodity to transport from F_i to WH_j for $i = 1,2$ and $j = 1,2,3$. We must have $x_{ij} \geq 0$. The total cost of shipping is

$$c(x) = 4x_{11} + 3x_{12} + 7x_{13} + 5x_{21} + 4x_{22} + 6x_{23}.$$

To obey the capacities of supply, we have two constraints $x_{11} + x_{12} + x_{13} \leq 15$ and $x_{21} + x_{22} + x_{23} \leq 40$. To meet the demands, we write three constraints $x_{11} + x_{21} \geq 8$, $x_{12} + x_{22} \geq 12$ and $x_{13} + x_{23} \geq 20$.

Finally, we can write the overall problem as

$$\min_x \; 4x_{11} + 3x_{12} + 7x_{13} + 5x_{21} + 4x_{22} + 6x_{23}$$

subject to

$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} &\leq 15 \\ x_{21} + x_{22} + x_{23} &\leq 25 \end{aligned}\right\} \text{(supply constraints)}$$

$$\left.\begin{aligned} x_{11} + x_{21} &\geq 8 \\ x_{12} + x_{22} &\geq 12 \\ x_{13} + x_{23} &\geq 20 \end{aligned}\right\} \text{(demand constraints)}$$

$$x_{ij} \geq 0, \;\; i = 1,2; \;\; j = 1,2,3.$$

*Example 1.5 (**The knapsack problem**).* There are n objects that have different weights $w_1, w_2, \ldots, w_n$ in certain weight units (e.g., in pounds), and different values $v_1, v_2, \ldots, v_n$ (e.g., in dollars). One needs to select a number of objects to pack into a box that can hold a maximum of W weight units. The goal is to select the right objects such that total value is maximized.

Mathematical model: Let x_i be a binary variable defined as follows:

- $x_i = 1$ if the i-th object is selected, and
- $x_i = 0$ if the i-th object is not selected.

Then, the total value is computed as $V = \sum_{i=1}^n x_i v_i$. We also have one constraint on the maximum weight capacity of the box as $\sum_{i=1}^n x_i w_i \leq W$.

To this end, the knapsack problem can be formulated into the following optimization problem:

$$\begin{cases} \max_x & \sum_{i=1}^n v_i x_i \\ \text{s.t.} & \sum_{i=1}^n w_i x_i \le W, \\ & x_i \in \{0,1\}. \end{cases}$$

Since the values of x_i are binary, i.e., $x_i \in \{0,1\}$, this problem is a discrete optimization problem.

1.2.3 Optimization in operations research

Operations research is a discipline dealing with the application of advanced analytical methods, such as optimization, simulation and stochastic models, to make better decisions. Operations research techniques are widely used in areas such as

- Computing and information technologies
- Environment, energy, and natural resources
- Financial engineering
- Manufacturing, service science, and supply chain management
- Marketing science
- Policy modeling and public sector work
- Revenue management
- Transportation.

1.3 Linear programming and nonlinear programming

Continuous optimization problems can be further classified into linear optimization (linear programming) and nonlinear optimization (nonlinear programming) problems. This classification is somewhat unbalanced, because nonlinear programming covers many more problems than linear programming. Nonetheless, understanding of the mathematical properties and solution algorithms for linear programming problems is very important for understanding the more complicated nonlinear programming problems.

1.3.1 Linear programming

Linear programming (LP) is a special class of optimization problems, that has one linear objective function and possibly many linear constraints. More specifically, suppose that there are n decision variables, $x_1, \cdots, x_n$. A linear objective function is of the form

$$f(x_1, x_2, \cdots, x_n) \overset{\text{def}}{=} c_1 x_1 + c_2 x_2 + \cdots + c_n x_n.$$

A linear constraint is of the form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i,$$

or

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i,$$

or

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i,$$

where $a_{i1}, \ldots, a_{in}$, $c_1, \cdots, c_n$ and b_i are constants for all i.

- We can have three different types of constraints: equality (E) constraint ($=$), "less than or equal to" (LE) constraint ($\leq$), or "great than or equal to" (GE) constraint ($\geq$).
- We can also have three different types of variables: nonnegative variables ($x_i \geq 0$), non-positive variables ($x_i \leq 0$), or free variables (i.e., variables not subject to sign constraints).

We can write an LP problem as follows:

$$\begin{cases} \min\limits_{x_1,\cdots,x_n} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} & \\ & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq (\geq \text{ or } =) b_i, \quad \text{for } i = 1, \cdots, m, \\ & x_i \geq 0 \ \ (x_i \leq 0, \text{ or } x_i \text{ is free}). \end{cases} \tag{LP}$$

Here, the notation $\min\limits_{x_1,\cdots,x_n}$ means that the goal is to minimize the objective z over the variables $x_1, \cdots, x_n$ subject to the constraints. By noting that $\min z = -\max(-z)$, we can convert any min problem into a max problem, and vice versa. The objective of a minimization problem is often called the cost/risk function, while the objective of the maximization problem is often referred to as the profit or utility function.

As a simple example, consider the following problem:

$$\begin{cases} \min\limits_{x_1,x_2,x_3} & 2x_1 + x_2 + 3x_3 \\ \text{subject to} & x_1 + x_2 + x_3 = 10 \quad \text{(E)}, \\ & 2x_1 + 3x_2 - x_3 \leq 8 \quad \text{(LE)}, \\ & 3x_1 + 2x_2 + x_3 \geq 5 \quad \text{(GE)}, \\ & x_1 \geq 0, \ x_2 \leq 0. \end{cases}$$

Since the objective function and the constraints are all linear, the problem is an LP. It has three types (E, LE, and GE) of constraints, and three types of variables: x_1 is nonnegative, x_2 is nonpositive, and x_3 is free.

Example 1.6. A dietician wishes to plan a meal using ground beef, potatoes, and spinach to satisfy minimum daily requirements on protein, carbohydrates, and iron. The nutritional makeup of each ounce of foodstuff (in the appropriate nutritional units) is given below.

	Nutrient		
Food	protein	carbohydrates	iron
beef	20	10	5
potatoes	10	30	6
spinach	6	7	20

The minimum daily requirements of protein, carbohydrates, and iron in the diet are 500, 400, and 75, respectively, and the cost of beef, potatoes, and spinach are \$0.35, \$0.20, and \$0.15 per ounce, respectively.

Goal: How should the dietician plan the menu to satisfy the minimum requirements with the lowest total cost?

Mathematical model: To model this problem as an optimization problem, we first choose the decision variables x_1, x_2, and x_3 to denote the amount in ounces of beef, potatoes, and spinach to put in the meal, respectively. The objective function is the total cost to be minimized. The minimum requirements on the nutritional makeup need to be considered as constraints. In addition, all three decision variables should be nonnegative. This leads to the following optimization problem, which is a linear program:

$$
\begin{cases}
\min\limits_{x_1,x_2,x_3} \; z = 0.35x_1 + 0.20x_2 + 0.15x_3 \\
\text{subject to} \\
\qquad 20x_1 + 10x_2 + 6x_3 \geq 500, \\
\qquad 10x_1 + 30x_2 + 7x_3 \geq 400, \\
\qquad 5x_1 + 6x_2 + 20x_3 \geq 75, \\
\qquad x_1 \geq 0, \\
\qquad x_2 \geq 0, \\
\qquad x_3 \geq 0.
\end{cases}
$$

1.3.2 Nonlinear programming

A nonlinear program (NLP) is an optimization problem that has a nonlinear objective function, or some nonlinear constraints. Below is a list of sub-classes of problems that belong to NLP:

- quadratic programs (QPs);
- second order cone programs (SOCPs);
- semidefinite programs (SDPs);
- convex programs, and
- nonconvex programs.

We give a few examples.

Example 1.7. Consider the following optimization problem:

$$\begin{cases} \min\limits_{x_1,x_2,x_3} & \frac{1}{2}x_1^2 + x_2^2 + 2x_3^2 + x_2x_3 - x_1 + x_2 - 2x_3 \\ \text{subject to} & x_1 + x_2 + x_3 = 1 \\ & x_1 - 2x_2 \le 0 \\ & x_1, x_2, x_3 \ge 0. \end{cases}$$

The objective function of this problem is $\frac{1}{2}x_1^2 + x_2^2 + 2x_3^2 + x_2x_3 - x_1 + x_2 - 2x_3$, which is nonlinear (indeed, it is quadratic). Hence, the problem is nonlinear.

Example 1.8. We consider the following problem

$$\begin{cases} \min\limits_{x\in\mathbb{R}^3} & x_1 + x_2 + x_3 \\ \text{subject to} & x_1^2 + x_2^2 + x_3^2 \le 1. \end{cases}$$

While the objective function is linear, the constraint is nonlinear. Hence, this is a nonlinear program (indeed, it is a second order cone program).

Example 1.9 (Weber problem). Given m points $z^{(1)} = (x_1, y_1), \ldots, z^{(m)} = (x_m, y_m)$ in the xy-plane, find a point $\bar{z} = (\bar{x}, \bar{y})$ such that the total distance from $\bar{z}$ to all the points $z^{(i)}$ for $i = 1, \ldots, m$ is minimized. This problem can be formulated as follows:

$$\min_{\bar{z}\in\mathbb{R}^2} \sum_{i=1}^{m} d(\bar{z}, z^{(i)}),$$

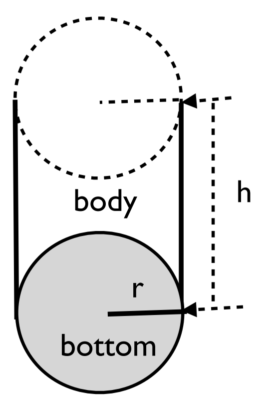
where $d(\bar{z}, z^{(i)}) \overset{\text{def}}{=} \sqrt{(\bar{x}-x_i)^2 + (\bar{y}-y_i)^2}$ is the distance from $\bar{z}$ to $z^{(i)}$ for $i = 1, \cdots, m$. Clearly, this problem is nonlinear since the objective function is nonlinear, even though it does not have any constraint.

1.4 Exercises

Exercise 1.1. Furnco manufactures desks and chairs. Each desk uses 4 units of wood, and each chair uses 3 units. A desk contributes \$40 to profit, and a chair contributes \$25. Marketing restrictions require that the number of chairs produced be at least twice the number of desks produced. There are totally 20 units of wood available.

1. Assume that all decision variables are continuous, formulate an LP to maximize Furnco's profit: define the variables, write down the objective function, and all constraints.
2. Is $(0,0)$ a feasible solution?
3. Is $(-3,2)$ a feasible solution?
4. Is $(2,4)$ a feasible solution?
5. Graphically solve the LP problem obtained from Item 1. Label the vertices (i.e., corners) of the feasible set with their coordinates. Write down the optimal solution(s) and the optimal value.

Exercise 1.2. A factory is manufacturing open-top cylinder containers using new polymeric material. Each square inch of the material costs \$0.4 to make the bottom, and costs \$0.25 to make the sides (body). The factory wants to find suitable radius r and height h of each container.



(a) If the volume of each container is fixed at 15 cubic inches, then find r and h to minimize the total material cost of each container. Model this problem into an optimization problem and solve it.

(b) If the total material cost is fixed at \$7.5 per container, then find r and h to maximize the volume V, and compute the value of V in this case. Model this problem into an optimization problem. Solve the resulting problem.

Exercise 1.3. U.S. Labs manufactures mechanical heart valves from the heart valves of pigs. Different heart operations require valves of different sizes. U.S. Labs purchases pig valves from three different suppliers. The cost and size mix of the valves purchased from each supplier are given in the following table.

Supplier	Cost per Valve ($)	Percent Large	Percent Medium	Percent Small
1	5	40	40	20
2	4	30	35	35
3	3	20	20	60

For example, any K pig valves from supplier 1 will contain $0.4K$ large, $0.4K$ medium and $0.2K$ small ones; customers must purchase the mix.

Each month, U.S. Labs needs at least 500 large, 300 medium, and 300 small valves. Because of limited availability of pig valves, at most 700 valves per month can be purchased from each supplier.

Formulate an LP that can be used to minimize the cost of acquiring the needed valves: define the variables, write down the objective function, and all constraints. Do not try to solve the obtained LP.

Exercise 1.4 (Road lighting). [1] A road is divided into n segments that are illuminated by m lamps. Let us denote by p_j the power of the j-th lamp, and I_i the illumination of the i-th segment, which is assumed to be $I_i = \sum_{j=1}^m a_{ij} p_j$, where a_{ij} are known coefficients. Let I_i^* be the desired illumination of the segment i of the road, which is given. The goal is to choose the lamp power p_j so that the illuminations I_i are close to the desired illumination I_i^*.

Model this problem into a reasonable LP problem. Note that the wording of the problem is loose and there may be more than one possible formulation.

Hints: We can measure the difference between I_i and I_i^* as $|I_i - I_i^*| \le t_i$, and try to minimize the sum of upper bound errors $\sum_{i=1}^m t_i$.

Exercise 1.5. A factory is manufacturing and selling two types of commodity, called C1 and C2. Every unit of C1 requires 3 machine hours, and every unit of C2 requires 4 machine hours to manufacture. The material cost of C1 is $3 per unit, but it can be sold at the price of $6 per unit when it is completed. The material cost of C2 is $2 per unit, and it can be sold at the price of $5.4 per unit.

Due to the limitation of resources, the factory has at most $20,000$ machine hours per week, and at most $4,000 of cost per week. Moreover, 45% of the sales revenues from C1 and 30% of the sales revenues from C2 will be made available to financial operations during the current week. The aim of the factory is to maximize the net income subject to the availability of resources.

(a) Model this problem into an optimization problem. Is it a linear program?

(b) Graph the feasible set of this problem and compute all the vertices of the feasible set.

Exercise 1.6. We are going to make two different types of beer: light beer and dark beer using malt, hops and yeast. Currently, we have the following raw material in our inventory:

$$\begin{aligned} &\text{Malt} && 75 \text{ units},\\ &\text{Hops} && 60 \text{ units},\\ &\text{Yeast} && 50 \text{ units}. \end{aligned}$$

Each type of beer requires different amount of malt, hops and yeast as given in the following table:

Requirement per gallon			
	Malt	Hops	Yeast
Light beer	2	3	2
Dark beer	3	1	5/3

If we sell the beers, then the light beer brings \$2.00/gallon profit, and the dark beer brings \$1.00/gallon profit. Given that our customers are willing to buy whatever is made available. Our goal is to make a production plan so that we can maximize our profit while satisfying all the requirements on our available material in our inventory.

(a) Model this problem into an optimization problem. Is it a linear program? Be sure to define all of your variables.

(b) Graph the feasible set of this problem and find all the vertices of this feasible set. Check the objective value at these vertices and show the one that gives the maximum profit.

Exercise 1.7. Given a system of four machines that are configured in a given layout in a two-dimensional plane. These four machines are located at a given coordinate (x_i, y_i) as $(3,0)$, $(0,-3)$, $(-2,1)$ and $(1,4)$ for $i = 1,\cdots,4$, respectively. Assume that we want to add a new machine to the system at a location (x,y). Our goal is to find the location of this new machine so that the total of distances from this new machine to other four machines is minimized. Here, the distance between two points $P(x,y)$ and $Q(\hat{x},\hat{y})$ is measured as $d(P,Q) = |x-\hat{x}| + |y-\hat{y}|$ (which is different from the length between P and Q).

(a) Model this problem as an optimization problem. Define variables, objective functions, and constraints (if any).

(b) Can you geometrically solve this problem? If yes, please solve it.

(c) Transform the resulting problem into a linear program.

Exercise 1.8. A store has hired n staffs to work for a short period during Christmas time. The manager of the store plans to assign each staff one job among n different available jobs. Since each staff can do each job with different cost, he wants to assign each job to the right person to minimize the total cost. Formulate this problem as an optimization problem such that the following constraints are satisfied:

- if the i-th staff is assigned to the j-th job, then it costs c_{ij} dollars for $i, j = 1, \cdot, n$;
- each staff must be only assigned exactly to one job.

References

1. Dimitris Bertsimas and John N Tsitsiklis. *Introduction to linear optimization*, volume 6. Athena Scientific Belmont, MA, 1997.