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Introduction to Optimization

From Linear Programming to Nonlinear Programming

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Chapter 5

Duality Theory and Sensitivity Analysis

5.1 The primal and dual pair

Each linear program is naturally paired with another linear program. One in this pair is called the **primal LP**, and the other is called the **dual LP**. These two LPs are defined by the same data: the coefficients and constants in constraints and the objective function of one LP are used in the other LP, but in a transposed manner. One LP in each primal-dual pair is a minimization problem, and the other is a maximization problem.

Given an LP, we will construct its dual LP by identifying the following attributes:

- The number of dual variables, and the number of dual constraints.
- The direction of optimization: to minimize or to maximize.
- The coefficients in the dual objective function.
- The coefficient matrix of the constraints and the right-hand side constants of the constraints.
- The types of constraints: LE ($\leq$), GE ($\geq$), or equality constraints.
- The types of sign restrictions on the variables: nonnegative, nonpositive, or free.

In this procedure, it will be convenient to classify constraints (not the sign restrictions) into three types: *natural*, *reversed*, or *equality*, as shown in Table 5.1. The class of a constraint depends on its own type ($\leq$, $\geq$, or $=$) as well as the direction of the optimization problem it belongs to. For example, a constraint of the type $\leq$ in a maximization LP is a natural constraint, and a constraint of the type $\leq$ in a minimization LP is a reversed constraint. Also note that Table 5.1 should not be

Types of constraints	in a max LP	in a min LP
Natural constraints	$\leq$ (LE)	$\geq$ (GE)
Reversed constraints	$\geq$ (GE)	$\leq$ (LE)
Equality constraints	$=$ (E)	$=$ (E)

Table 5.1 Natural, reversed, and equality constraints

applied to sign restrictions of variables, which are classified into the three types, nonnegative, nonpositive and free, as before.

The following steps give us a generic procedure to construct the dual LP from a given primal LP. Here, the term "constraint" is used in a restrictive way and does not include sign restrictions on variables.

Step 1: For each constraint i of the primal LP, define a dual variable, e.g., y_i.

- Again, **sign restrictions** will be treated differently. Do not define dual variables for them.

Step 2: Determine the dual objective function.

- If the primal is a max problem, then the dual is a min problem. Conversely, if the primal is a min problem, then the dual is a max problem.
- Use the right-hand side constants of the primal constraints as coefficients in the dual objective function.

Step 3: Determine data in the dual constraints.

- For each primal variable, e.g., x_j, define a corresponding dual constraint j.
- The right-hand side constant of this dual constraint is given by the coefficient of this primal variable in the primal objective function.
- The left-hand side coefficients of this dual constraint are given by the coefficients of the primal variable in primal constraints. In other words, the j-th column of the coefficient matrix for the primal constraints becomes the j-th row of the coefficient matrix in the dual constraints.

Step 4: Determine types of the dual constraints ($\leq$, $\geq$ or $=$).

- By construction, each dual constraint j corresponds to a primal variable x_j. The type of this dual constraint depends on the sign restriction of the primal variable x_j.
  - If its corresponding primal variable is **nonnegative**, then the corresponding dual constraint is **natural**. *Hence, based on Table 5.1, the dual constraint is of type $\leq$ if the dual LP is a max problem, and it is of type $\geq$ if the dual LP is a min problem.*
  - If its corresponding primal variable is **nonpositive**, then the dual constraint is **reversed**. *Again, based on Table 5.1, the dual constraint is of type $\geq$ if the dual LP is a max problem, and it is of type $\leq$ if the dual LP is a min problem.*
  - If its corresponding primal variable is **free**, then the dual constraint is an **equality constraint**.

Step 5: Determine sign restrictions of the dual variables.

- By construction, each dual variable, e.g., y_i, corresponds to a primal constraint i, the one used to define the dual variable.
  - If its corresponding primal constraint is **natural**, then the dual variable is **nonnegative**. *Again, whether the constraint is natural depends on both its type and the optimization direction of the primal LP. Based on Table 5.1, the primal constraint is natural, if it is of type $\leq$ when the primal LP is a max problem, or if it is of type $\geq$ when the primal LP is a min problem.*
  - If its corresponding primal constraint is **reversed**, then the dual variable is **nonpositive**. *Based on Table 5.1, the primal constraint is reversed, if it is of type $\geq$ when the primal LP is a max problem, or if it is of type $\leq$ when the primal LP is a min problem.*
  - If its corresponding primal constraint is an equality, then the dual variable is **free**.

Let us provide some examples to illustrate this procedure.

Example 5.1. Given a primal LP:

$$\begin{cases} \quad\quad\quad\underline{\text{Primal}} \\ \min\limits_{x} \quad z = 2x_1 + 3x_2 \\ \text{s.t.} \quad\quad\quad x_1 + x_2 \geq 5 \\ \text{and } \; x_1 \geq 0, \; x_2 \geq 0. \end{cases} \tag{5.1}$$

By **Step 3**, the dual problem is as follows:

$$\begin{cases} \quad\quad\underline{\text{Dual}} \\ \max\limits_{y} \; v = 5y \\ \text{s.t.} \quad\quad y \,\square\, 2 \\ \quad\quad\quad\quad y \,\square\, 3 \\ \text{and} \quad y \;\square \end{cases}$$

Here, we have yet to decide constraint types and sign restrictions of the dual variable.

Dual constraint types:

- The primal variable $x_1 \geq 0$: the 1^{st} dual constraint should be natural. A natural constraint in a max problem is of the type $\leq$. So the 1^{st} dual constraint is $y \leq 2$.
- The primal variable $x_2 \geq 0$: the 2^{nd} dual constraint should be natural: $y \leq 3$.

The sign restrictions of the dual variable:

- The primal constraint is of type $\geq$ in a min problem, so it is natural. The dual variable corresponding to a natural constraint is always nonnegative, so the sign restriction is $y \geq 0$.

The dual problem of (5.1) becomes

$$\begin{cases} \max\limits_{y} \; v = 5y \\ \text{s.t.} \quad\quad y \;\leq\; 2 \\ \quad\quad\quad\quad y \;\leq\; 3 \\ \quad\quad\quad\quad y \geq 0. \end{cases}$$

Example 5.2. Consider the following primal LP:

$$
\textbf{(Primal)} \begin{cases} \min\limits_{x} z = & 2x_1 & +5x_2 & -4x_3 & \\ \text{s.t.} & 2x_1 & +x_2 & -x_3 & \geq 5 \\ & 4x_1 & -2x_2 & +3x_3 & \leq 7 \\ & x_1 & +3x_2 & -2x_3 & = 12 \\ & x_1 & \leq 0, & x_3 & \geq 0. \end{cases}
$$

This problem has three different types of constraints, and three different types of variables. Following the above procedure, we can determine:

- The number of dual variables $m = 3$, and the number of dual constraints $n = 3$.
- The dual LP is a maximization problem.
- If we denote the dual variables by y_1, y_2, and y_3, then $y_1 \geq 0$ (since the first constraint is natural), $y_2 \leq 0$ (since the second constraint is reversed), and y_3 is free.
- The first dual constraint is GE (reversed), the second is E (equality), and the last is LE (natural).

In summary, we can write the entire dual problem as

$$
\textbf{(Dual)} \begin{cases} \max\ v = & 5y_1 & +7y_2 & +12y_3 & & \\ \text{s.t.} & 2y_1 & +4y_2 & +y_3 & \geq & 2 \\ & y_1 & -2y_2 & +3y_3 & = & 5 \\ & -y_1 & +3y_2 & -2y_3 & \leq & -4 \\ & y_1 & \geq 0, & y_2 & \leq 0. & \end{cases}
$$

We note that if we form the dual problem of the dual LP, then we obtain again the primal LP. That is $\textbf{Dual}\big(\textbf{Dual}\big) = \textbf{Primal}$.

5.2 Weak and strong duality, and complementary slackness

Next, we investigate the relation between the primal and dual linear programs. First, if x is a feasible solution of the primal problem, and y is feasible for the dual problem, is there a relation between them? Second, if x^* is an optimal solution of the primal problem, can we say anything about optimal solutions of the dual problem? Third, given a primal optimal solution x^*, is there a way to compute optimal solutions of the dual problem, or check the optimality of a given dual feasible solution?

5.2.1 Weak and strong duality

We answer the first question by proving the following theorem. This theorem is called the **weak duality theorem** for linear programs.

Theorem 5.1 (Weak duality). *Suppose for a pair of primal-dual LPs that the objective of the primal LP is to maximize $c^T x$, and the objective of the dual LP is to minimize $b^T y$. If x is feasible for the primal LP, and y is feasible for the dual LP, then*

$$c^T x \le y^T b.$$

Proof. Let us suppose the primal LP is of the following form:

$$\begin{cases} \max\limits_{x} & z = c^T x \\ \text{s.t.} & Ax \le b \\ & x \ge 0, \end{cases}$$

where A is an $m \times n$ matrix. The corresponding dual problem becomes:

$$\begin{cases} \min\limits_{y} & v = b^T y \\ \text{s.t.} & A^T y \ge c \\ & y \ge 0. \end{cases}$$

Suppose that x is feasible to the primal problem, and y is feasible to the dual problem. We define

$$u_i = y_i(b_i - \sum_{j=1}^{n} A_{ij}x_j),\ i = 1, \cdots, m, \ \text{ and } \ v_j = (\sum_{i=1}^{m} A_{ij}y_i - c_j)x_j,\ j = 1, \cdots, n.$$

Then, it follows from the relationship between the primal and dual LP's that

- for each $i = 1, \cdots, m$, $y_i \ge 0$ and $b_i - \sum_{j=1}^{n} A_{ij}x_j \ge 0$ and
- for each $j = 1, \cdots, n$, $x_j \ge 0$ and $\sum_{i=1}^{m} A_{ij}y_i - c_j \ge 0$.

Hence, u_i and v_j are all nonnegative. Moreover, we can easily show that

$$\sum_{i=1}^{m} u_i = y^T b - y^T Ax \quad \text{ and } \quad \sum_{j=1}^{n} v_j = y^T Ax - c^T x.$$

These expressions imply

$$y^T b - c^T x = \sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j \ge 0.$$

Hence, $b^T y \geq c^T x$, and we complete the proof. $\square$

We note that the proof of Theorem 5.1 is based on the assumption that the constraints of the primal problem are natural, and all the variables are nonnegative. However, this proof can be extended to general primal-dual pairs, because y_i and $b_i - \sum_{j=1}^n A_{ij}x_j$ always have the same sign in any general primal-dual pair. This means that we always have $u_i = y_i(b_i - \sum_{j=1}^n A_{ij}x_j) \geq 0$. Similarly, x_j and $\sum_{i=1}^m A_{ij}y_i - c_j$ also have the same sign, which implies $v_j = (\sum_{i=1}^m A_{ij}y_i - c_j)x_j \geq 0$ (see Exercise below). The proof of this theorem is therefore valid in general.

Suppose that a primal LP and its dual both have optimal solutions. Then, based on the weak duality theorem, the optimal value of the maximization LP in this primal-dual pair is no more than the optimal value of the minimization LP in this pair (exercise). Suppose, on the other hand, that one LP in this pair is unbounded; then the weak duality theorem tells us that the other LP must be infeasible. In other words:

(a) If a minimization LP is unbounded, then its optimal value is $-\infty$, and its dual is infeasible.

(b) If a maximization LP is unbounded, then its optimal value is $+\infty$, and its dual is infeasible.

The following theorem gives a sharper result on the relation between the optimal values of the primal and dual LPs.

Theorem 5.2 (Strong duality). *In a pair of primal-dual LPs, if one problem has an optimal solution, then the other also has an optimal solution, with the same optimal value.*

Proof. Without loss of generality, we can prove this theorem for a canonical LP. Otherwise, we can always transform it into a canonical form that is equivalent to the original problem as long as the problem has an optimal solution. Our proof is based on the simplex method.

Write the primal LP as

$$\begin{cases} \max\limits_{x} & z = c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{cases}$$

The corresponding dual problem becomes:

$$\begin{cases} \min\limits_{y} \ v = b^T y \\ \text{s.t.} \ \ A^T y \geq c. \end{cases}$$

Suppose that the primal LP is in canonical form and has an optimal solution. Then, if we apply the simplex method to solve it, we will end up with an optimal tableau that shows a basic optimal solution (see the previous chapter). Let us write the initial simplex tableau as

z	x	RHS
1	$-c^T$	0
0	A	b

and the final optimal tableau that displays a basic optimal solution x^* as

z	x	RHS	Basic var
1	$\bar{c}^T$	z^*	$z = z^*$
0	$\bar{A}$	$\bar{b}$	$x_B = \bar{b}$

The elementary row operations performed at iterations of the simplex method have led us to the final optimal tableau, starting from the initial simplex tableau. At each such iteration, we have added a multiple of the pivot row to the top row. The pivot row of a tableau is the linear combination of rows (excluding the top row) of its previous tableau. Rows (excluding the top row) of the previous tableau are in turn linear combinations of rows (excluding the top row) of the tableau before it. Hence, each row (excluding the top row) of each tableau is a linear combination of rows (excluding the top row) in the initial tableau. As a result, the reduced cost vector $\bar{c}$ that we obtain in the optimal tableau can be written as $\bar{c}^T = (y^*)^T A - c^T$, where y^* collects all the multiples that we have added up in this procedure (more precisely, $y^* = (A_B^{-1})^T c_B$). Clearly, since x^* is optimal, $\bar{c}^T \geq 0$. This implies $A^T y^* \geq c$.

Since the same elementary row operations have been conducted for the RHS column, the RHS entry in the top row of the optimal tableau is given by $z^* = b^T y^*$. But this is also optimal value of the primal problem, so we have $z^* = b^T y^* = c^T x^*$. The fact $A^T y^* \geq c$ shows that y^* is a feasible solution to the dual problem, and the fact $b^T y^* = c^T x^* \leq b^T y$ shows that $b^T y^* \leq b^T y$ for any feasible solution y due to

Theorem 5.1. Hence, we conclude that y^* is an optimal solution of the dual problem, and $b^T y^* = c^T x^*$. □

5.2.2 The complementary slackness theorem

As in Theorem 5.1, we will use the following primal-dual pair

$$\text{Primal problem:} \begin{cases} \max_x \ z = c^T x \\ \text{s.t.} \quad Ax \leq b \\ \qquad\ x \geq 0, \end{cases} \tag{P}$$

$$\text{Dual problem:} \begin{cases} \min_x \ t = b^T y \\ \text{s.t.} \quad y \geq 0 \\ \qquad A^T y \geq c \end{cases} \tag{D}$$

to illustrate the proof of the complementary slackness theorem below.

Theorem 5.3. *Vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are optimal solutions to the primal LP* (P) *and the dual LP* (D) *respectively, if and only if they simultaneously satisfy the following three conditions:*

1. *x is a primal feasible solution.*
2. *y is a dual feasible solution.*
3. *x and y satisfy the following $m+n$ equalities:*

$$\begin{cases} y_i \left(b_i - \sum_{j=1}^{n} A_{ij} x_j \right) = 0, \quad i = 1, \cdots, m \\ \left(\sum_{i=1}^{m} A_{ij} y_i - c_j \right) x_j = 0, \quad j = 1, \cdots, n. \end{cases} \tag{5.2}$$

Remark 5.1. Equations in (5.2) are called the **complementary slackness condition**. These are not linear equations due to the cross product terms between x_j and y_i.

Proof. Suppose that x and y are feasible solutions to the primal LP (P) and the dual LP (D) respectively. We will show that they are optimal solutions to the primal LP (P) and the dual LP (D), if and only if the complementary slackness condition hold.

Indeed, we define

$$u_i = y_i (b_i - \sum_{j=1}^{n} A_{ij} x_j), \ i = 1, \cdots, m \quad \text{and} \quad v_j = (\sum_{i=1}^{m} A_{ij} y_i - c_j) x_j, \ j = 1, \cdots, n.$$

It was shown in the proof of Theorem 5.1 that u_i and v_j are all nonnegative, with

$$\sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j = y^T b - c^T x. \tag{5.3}$$

Now, suppose that x and y are optimal solutions to the primal LP (P) and the dual LP (D), respectively. By the strong duality theorem (Theorem 5.2), we have $c^T x = y^T b$. This and (5.3) imply that all of the u_i and v_j are zeros. This means that the conditions in (5.2) hold.

Conversely, suppose all conditions in (5.2) hold. Then all components of the u_i and v_j are zeros. By (5.3) we have $c^T x = y^T b$. The weak duality theorem now tells us that x and y are optimal solutions to the primal LP (P) and the dual LP (D). Indeed, if x is not an optimal solution of (P), there exists a feasible solution $\hat{x}$ such that $c^T x < c^T \hat{x}$. By Theorem 5.1, we have $c^T \hat{x} \le b^T y$ since y is a feasible solution to (D). Hence, $c^T x < c^T \hat{x} \le b^T y$, which contradicts $c^T x = b^T y$. Hence, x must be an optimal solution to (P). Similarly, y must be an optimal solution to (D). □

To see the meaning of the equalities in (5.3), pair the primal constraints $Ax \le b$ one by one with the corresponding sign restrictions $y \ge 0$, and pair the sign restrictions $x \ge 0$ with corresponding dual constraints $A^T y \ge c$. Every pair consists of two inequalities, and the equalities in (5.3) say that at least one of those two inequalities is **satisfied as an equality**. An inequality is said to be satisfied as an equality if both sides of it are equal. For example, the inequality $2x_1 + x_2 \ge 8$ is satisfied as an equality at $x = (3, 2)$, and $y_1 \le 0$ is satisfied as an equality at $y_1 = 0$.

When $\sum_{j=1}^{n} A_{ij} x_j \le b_i$ is satisfied with an equality at a given point, we say that the i-th primal constraint is **active** (also called **binding**) at this point. Otherwise, it is said to be **inactive**. Similarly, if $\sum_{i=1}^{m} A_{ij} y_i = c_j$ at a given point, then we say that the j-th dual constraint is **active** there. Otherwise, it is **inactive**.

Although the statement of Theorem 5.3 only covers primal and dual LPs in the above special formats, it can be readily extended to general primal and dual LPs. For a general primal LP and its dual LP, pair the primal constraints one by one with the corresponding sign restrictions for the dual variables, and pair the primal sign restrictions one by one with corresponding dual constraints. Then, the general complementary slackness theorem states:

> Let x and y be a pair of feasible solutions to the primal LP and the dual LP respectively. Then, they are optimal solutions to the primal LP and the dual LP respectively, if and only if at least one in each pair of primal-dual constraints is active at these solutions.

Again, the above condition that "at least one in each pair of primal-dual constraints is active at these solutions" is called the complementary slackness condition. If, in a pair of primal-dual constraints, the primal constraint is an equality, then the corresponding dual sign restriction will be free. The complementary slackness condition for such a pair is automatically satisfied whenever the considered primal solution is feasible, because the equality constraint is satisfied at any primal feasible solution.

Example 5.3. Consider the following primal-dual LP pair:

$$
\text{(P)}\ \left\{\begin{array}{lllll} & \underline{\text{Primal}} & & & \\ \min\limits_{x} & 13x_1 & +10x_2 & +6x_3 & \\ \text{s.t.} & 5x_1 & +x_2 & +3x_3 & = 8 \\ & 3x_1 & +x_2 & & = 3 \\ & x_1, x_2, x_3 \geq 0 & & & \end{array}\right.
\qquad
\text{(D)}\ \left\{\begin{array}{llll} & \underline{\text{Dual}} & & \\ \max\limits_{y} & 8y_1 & +3y_2 & \\ \text{s.t.} & 5y_1 & +3y_2 & \leq 13 \\ & y_1 & +y_2 & \leq 10 \\ & 3y_1 & & \leq 6 \end{array}\right.
$$

- Is $x^{\star} = (1,0,1)$ an optimal solution to the primal?

  Solution: First, note that $x^{\star}$ is a primal feasible solution by directly checking all the constraints. To check its optimality, we need to check if there exists a dual feasible solution $y^{\star}$ such that $(x^{\star}, y^{\star})$ satisfies the complementary slackness condition. Since $x_1^{\star} = 1 > 0$ and $x_3^{\star} = 1 > 0$, we need the first and third dual constraints to be satisfied as equalities at $y^{\star}$. That is, both these constraints need to be **active** at $y^{\star}$. Hence, the conditions on $y^{\star}$ are:

$$
\left\{\begin{array}{llll} 5y_1 & +3y_2 & = 13 \\ y_1 & +y_2 & \leq 10 \\ 3y_1 & & = 6. \end{array}\right. \tag{5.4}
$$

  Solving the system of two equations $5y_1 + 3y_2 = 13$ and $3y_1 = 6$ in y_1 and y_2 we obtain $y^{*} = (2,1)$, which satisfies $y_1 + y_2 \leq 10$. Clearly, y^{*} is feasible to the dual problem (D), and $b^T y^* = c^T x^\star = 19$. Hence, $x^\star$ is a primal optimal solution.

- What is the set of optimal solutions to the dual problem?

  Solution: Since $y^\star = (2,1)$ is dual feasible, and satisfies the complementary slackness condition with $x^\star$, it is a dual optimal solution. Also note that no other vector in $\mathbb{R}^2$ satisfies the three conditions in (5.4) simultaneously. Hence $y^\star = (2,1)$ is the unique dual optimal solution.

Example 5.4. Consider the following primal and dual LP pair:

$$
(\mathrm{P}) \left\{ \begin{array}{lllllll}
\max\limits_{x} z = & x_1 & +2x_2 & +2x_3 & +3x_4 & & \\
\text{s.t.} & 2x_1 & +2x_2 & & +2x_4 & \leq 64 \\
& & x_2 & +x_3 & & \leq 32 \\
& & & 2x_3 & +2x_4 & \leq 64 \\
& x_1, x_2, x_3, x_4 \geq 0, & & & & &
\end{array} \right.
$$

and

$$
(\mathrm{D}) \left\{ \begin{array}{llllll}
\min\limits_{y} v = & 64y_1 & +32y_2 & +64y_3 & \\
\text{s.t.} & 2y_1 & & & \geq 1 \\
& 2y_1 & +y_2 & & \geq 2 \\
& & y_2 & +2y_3 & \geq 2 \\
& 2y_1 & & +2y_3 & \geq 3 \\
& & y_1, y_2, y_3 \geq 0. & &
\end{array} \right.
$$

Verify that $y^* = (\frac{3}{4}, \frac{1}{2}, \frac{3}{4})^T$ is an optimal solution of (D). Compute an optimal solution x^* of (P).

Solution: Let us assume that y^* is optimal to (D). We denote x^* an optimal solution to (P). By substituting y^* into the constraints of (D), we can see that the last three constraints are active (i.e., satisfied as equalities). The first dual constraint is not active at y^*, so $x_1^* = 0$ must hold. Moreover, since y_1^*, y_2^* and y_3^* are all strictly positive, their corresponding constraints in (P) need to be active at x^*. This means that x^* is a solution of the following three equations:

$$
\begin{cases}
2x_2 + 2x_4 = 64 \\
x_2 + x_3 = 32 \\
2x_3 + 2x_4 = 64.
\end{cases}
$$

The only solution to the above equations is $x_2 = x_3 = x_4 = 16$, so we find $x^* = (0,16,16,16)^T$. We can check that x^* is feasible to (P). Hence, y^* is optimal to the dual problem (D), and x^* is the only optimal solution to the primal problem (P).

5.3 Sensitivity Analysis

Sensitivity analysis in linear programming analyzes the effects of perturbation of the input data on solutions of LPs. This section introduces sensitivity analysis techniques for LPs. More details can be found in linear programming textbooks, e.g., [1, 2, 3].

5.3.1 Introduction

We consider the following LP in standard form:

$$\begin{cases} \max\limits_{x} & z = c^T x \\ \text{s.t.} & Ax = b, \\ & x \geq 0. \end{cases} \tag{5.5}$$

Here, the matrix $A \in \mathbb{R}^{m \times n}$, vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ are input data for this LP. We can put them into a triple (A,b,c). In many situations the input data (A,b,c) may not be accurate and may change over time.

Suppose that this LP has an optimal solution x^* with the optimal value z^*. The following questions can be asked:

1. How do the optimal solution x^* and/or the optimal value z^* change if
   - the objective coefficient vector c changes?
   - the right-hand side vector b changes?
   - the constraint coefficient matrix A changes?
2. How much can we change the input data (A,b,c) so that x^* remains an optimal solution?

When the right hand side constant of the ith constraint of (5.5) changes from b_i to $b_i + \delta$, the optimal value of (5.5) changes accordingly. Under some conditions, the change in the optimal value of (5.5) can be represented as a linear function of δ, for δ belonging to a certain interval that includes 0. The slope of such a linear function is called the **shadow price** on this constraint, which represents the rate of change in the optimal value as the right-hand side of this constraint changes from its current value b_i.

The following theorem makes precise the above claim. A proof can be found in [1]. Again, this theorem can be extended to general linear programs.

Theorem 5.4. *Suppose that the LP* (5.5) *has a nondegenerate optimal basic solution x^*. Then its dual has a unique optimal solution y^*. Moreover, for each $i = 1, \cdots, m$, there exists an interval $[l_i, u_i]$ with $l_i < 0 < u_i$ such that for each $\delta \in [l_i, u_i]$ the optimal value of the LP* (5.5) *with b_i replaced by $b_i + \delta$ is given by $z^* + y_i^* \delta$.*

Below, we consider the following example for a simple illustration.

$$\begin{cases} \max\limits_{x} \ z = 500x_1 + 450x_2 \\ \text{s.t.} \quad 6x_1 + 5x_2 \leq 60 \\ \qquad\ 10x_1 + 20x_2 \leq 150 \\ \qquad\ 0 \leq x_1 \leq 8, \ x_2 \geq 0. \end{cases} \tag{5.6}$$

This LP has a unique optimal solution $x^* = (\frac{45}{7}, \frac{30}{7})^T$ with the optimal value $z^* = 5142\frac{6}{7}$.

If we increase the RHS of the first constraint by 1 unit, so that the first constraint becomes $6x_1 + 5x_2 \leq 61$, how much does the optimal value change? If we solve this LP graphically, we can find that the optimal solution after the change is still determined by the following equations

$$\begin{cases} 6x_1 + 5x_2 = 61 \\ 10x_1 + 20x_2 = 150. \end{cases}$$

The new optimal solution is $\hat{x}^* = (6\frac{5}{7}, 4\frac{1}{7})$ and the new optimal value is $\hat{z}^* = 5221\frac{3}{7}$. The change in the optimal value is $\hat{z}^* - z^* = 5221\frac{3}{7} - 5142\frac{6}{7} = 78\frac{4}{7}$.

To use this example to verify the above theorem, we can write the dual LP of (5.6) as

$$\begin{cases} \min\limits_{y} \ t = 60y_1 + 150y_2 + 8y_3 \\ \text{s.t.} \quad 6y_1 + 10y_2 + y_3 \geq 500 \\ \qquad\ 5y_1 + 20y_2 \geq 450 \\ \qquad\ y \geq 0. \end{cases}$$

Solving the dual LP we obtain $y_1^* = 78\frac{4}{7}$, exactly the same value as $\hat{z}^* - z^*$.

In the subsequent sections, we use the following example to describe a method that uses information in the optimal tableau to conduct sensitivity analysis. This method works whenever an optimal tableau for the original LP is available.

Example 5.5. Consider the following LP:

$$\begin{cases} \max\limits_{x} \ z = 41x_1 + 25x_2 \\ \quad \text{s.t.} \quad 3x_1 + 2x_2 + x_3 = 10 \\ \qquad\quad 5x_1 + 3x_2 + x_4 = 16 \\ \qquad x_1, \cdots, x_4 \geq 0. \end{cases} \tag{5.7}$$

Since (5.7) is canonical, we can apply the simplex method to solve it. The initial tableau (called Tableau 1) in solving (5.7) is:

z	x_1	x_2	x_3	x_4	RHS	Basic var
1	-41	-25	0	0	0	$z = 0$
0	3	2	1	0	10	$x_3 = 10$
0	5	3	0	1	16	$x_4 = 16$

After several simplex iterations, we arrive at the optimal tableau below (called Tableau 2), which shows the optimal basic solution in which x_1 and x_2 are basic variables. We say that $\{x_1, x_2\}$ is an **optimal basis**.

z	x_1	x_2	x_3	x_4	RHS	Basic var
1	0	0	2	7	132	$z = 132$
0	1	0	-3	2	2	$x_1 = 2$
0	0	1	5	-3	2	$x_2 = 2$

5.3.2 Changes in the objective coefficients

We consider two cases:

- **Case 1:** The coefficient is associated with a nonbasic variable.
- **Case 2:** The coefficient is associated with a basic variable.

Case 1: Consider changing the coefficient for x_3 in the objective function from 0 to δ. Note that x_3 is a nonbasic variable in Tableau 2. Tableau 1 becomes Tableau 1a below:

z	x_1	x_2	x_3	x_4	RHS	Basic var
1	-41	-25	$-\delta$	0	0	$z = 0$
0	3	2	1	0	10	$x_3 = 10$
0	5	3	0	1	16	$x_4 = 16$

Starting from Tableau 1a, we conduct the same EROs (elementary row operations) that bring Tableau 1 to Tableau 2, to arrive at Tableau 2a below:

z	x_1	x_2	x_3	x_4	RHS	Basic var
1	0	0	$2-\delta$	7	132	$z = 132$
0	1	0	-3	2	2	$x_1 = 2$
0	0	1	5	-3	2	$x_2 = 2$

Clearly, if $\delta \leq 2$, the current BFS (basic feasible solution) $x = (2,2,0,0)^T$ remains optimal, and the optimal value is $z = 132$. If $\delta > 2$, then extra simplex iterations are needed to find the new optimal solution.

<u>Case 2:</u> Now, consider changing the coefficient in the objective function for x_1 from 41 to $41 + \delta$. Note that x_1 is a basic variable in Tableau 2. Tableau 1 becomes Tableau 1b below:

z	x_1	x_2	x_3	x_4	RHS	Basic var
1	$-41-\delta$	-25	0	0	0	$z = 0$
0	3	2	1	0	10	$x_3 = 10$
0	5	3	0	1	16	$x_4 = 16$

Starting from Tableau 1b, we conduct the same EROs that bring Tableau 1 to Tableau 2, to arrive at Tableau 2b below:

z	x_1	x_2	x_3	x_4	RHS	Basic var
1	$-\delta$	0	2	7	132	$z = 132$
0	1	0	-3	2	2	$x_1 = 2$
0	0	1	5	-3	2	$x_2 = 2$

Tableau 2b is not a valid simplex tableau, because the coefficient for x_1 in row 0 is nonzero. It needs to be zero since x_1 is a basic variable in this tableau. We add δ times row 1 to row 0 to obtain Tableau 3b below:

z	x_1	x_2	x_3	x_4	RHS	Basic var
1	0	0	$2-3\delta$	$7+2\delta$	$132+2\delta$	$z=132+2\delta$
0	1	0	-3	2	2	$x_1=2$
0	0	1	5	-3	2	$x_2=2$

In order for the current BFS to remain optimal, we need $2-3\delta$ and $7+2\delta$ to be nonnegative. Hence, when $-7/2 \le \delta \le 2/3$, the current BFS $x=(2,2,0,0)$ remains optimal, and the optimal value is $z=132+2\delta$. If δ does not belong to the above range, then extra iterations need to be conducted.

5.3.3 Changes in the right hand side coefficients

Now, consider changing the right hand side constant for the first constraint from 10 to $10+\delta$. Tableau 1 becomes Tableau 1c below:

z	x_1	x_2	x_3	x_4	RHS	Basic var
1	-41	-25	0	0	0	$z=0$
0	3	2	1	0	$10+\delta$	$x_3=10+\delta$
0	5	3	0	1	16	$x_4=16$

Starting from Tableau 1c, we conduct the same EROs that bring Tableau 1 to Tableau 2, to arrive at Tableau 2c below:

z	x_1	x_2	x_3	x_4	RHS	Basic var
1	0	0	2	7	132+2δ	$z=132+2\delta$
0	1	0	-3	2	$2-3\delta$	$x_1=2-3\delta$
0	0	1	5	-3	2+5δ	$x_2=2+5\delta$

The right hand side column in Tableau 2c is obtained by noting that the column under x_3 changes from

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

after the elementary row operations. So a column that starts with $(0,\delta,0)^T$ will become $(2\delta,-3\delta,5\delta)^T$ after the same EROs.

Consequently, whenever $2-3\delta \ge 0$ and $2+5\delta \ge 0$ hold simultaneously, Tableau 2c shows an optimal solution. This happens when $-2/5 \le \delta \le 2/3$. For each δ belonging to this range, $\{x_1, x_2\}$ is an optimal basis, and an optimal solution is given by $x=(2-3\delta, 2+5\delta, 0, 0)$, with the optimal value $132+2\delta$.

5.4 Exercises

Exercise 5.1. Consider a primal-dual pair of LPs, where the primal problem is a maximization problem. Using primal-dual rules to show that the vectors u and v computed by $u = y^T(A^T x - c)$ and $v = y^T(b - Ax)$ are always nonnegative, where x is the vector of primal variables and y is the vector of dual variables.

Exercise 5.2. For each LP problem below, write down the corresponding dual LP problem. Check if the dual problem is in a standard or in a canonical form (Explain why?). Explain how do you conduct the sign of the variables and the constraints in the dual problem?

1.
$$\begin{cases} \min\limits_{x} & z = x_1 + 2x_2 - 3x_3 + x_4 \\ \text{s.t.} & x_1 - 2x_2 + 3x_3 + x_4 \leq 3, \\ & x_2 + 2x_3 + 2x_4 \geq -5, \\ & 2x_1 - 3x_2 - 7x_3 - 4x_4 = 2, \\ & x_1 \geq 0, \ \ x_4 \leq 0. \end{cases}$$

2.
$$\begin{cases} \max\limits_{x} & z = -x_1 + 2x_3 \\ \text{s.t.} & x_1 + x_2 \leq 1, \\ & -x_1 + x_3 = 2, \\ & x_1 \leq 0, \ \ x_2 \geq 0 \end{cases}$$

3.
$$\begin{cases} \max\limits_{x} & z = c^T x \\ \text{s.t.} & Ax = b, \\ & Bx \geq d, \\ & x \geq 0, \end{cases}$$

where $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $d \in \mathbb{R}^p$.

4.
$$\begin{cases} \min\limits_{x} & z = c^T x \\ \text{s.t.} & a \leq Ax \leq b, \\ & 0 \leq x \leq u, \end{cases}$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $a, b \in \mathbb{R}^m$ $(a \leq b)$ and $u \in \mathbb{R}^n$ $(u \geq 0)$.

Exercise 5.3. Using the duality theorem, verify if the following LP problems are unbounded:

$$
(a)\ \begin{cases} \max\limits_{x}\ z = 2x_1 + x_2 \\ \text{s.t.}\quad x_1 - x_2 \leq 4, \\ \qquad\ \ x_1 - x_2 \leq 2 \\ \qquad\ \ x_1 \geq 0, x_2 \geq 0. \end{cases}
\qquad
(b)\ \begin{cases} \min\limits_{x}\ z = -4x_1 + 2x_2 \\ \text{s.t.}\quad -x_1 + x_2 \geq 2, \\ \qquad\ \ -x_1 + x_2 \geq 1 \\ \qquad\ \ x_1 \geq 0, x_2 \geq 0. \end{cases}
$$

Explain your answer using the duality theory we studied in class, do not use simplex methods or graphs.

Exercise 5.4. When modeling a portfolio selection problem, we obtain the following LP problem:

$$
\begin{array}{ll}
\max\limits_{x} & z = 0.043x_1 + 0.027x_2 + 0.025x_3 + 0.022x_4 + 0.045x_5 \\
\text{s.t.} & \begin{cases} \text{Cash:} & x_1 + x_2 + x_3 + x_4 + x_5 & \leq 10, \\ \text{Government:} & x_2 + x_3 + x_4 & \geq 4, \\ \text{Quality:} & 0.6x_1 + 0.6x_2 - 0.4x_3 - 0.4x_4 + 3.6x_5 & \leq 0, \\ \text{Maturity:} & 4x_1 + 10x_2 - x_3 - 2x_4 - 3x_5 & \leq 0, \\ & x \geq 0. & \end{cases}
\end{array}
$$

(a) Write down the dual problem of this LP.

(b) Verify that $x^* = (2.1818, 0, 7.3636, 0, 0.4545)^T$ is a feasible solution of the primal problem rounding up to 4 digits after the decimal point.

(c) Using the complementarity slackness theory to check if x^* is an optimal solution of the primal LP. If x^* is optimal, compute an optimal solution for the dual problem.

Note that all the computation must be done approximately up to 4 digits after the decimal point.

Exercise 5.5. Consider the following pair of primal and dual LPs:

$$
(\text{Primal})\quad \begin{cases} \min\limits_{x} & z = 35x_1 + 30x_2 + 60x_3 + 50x_4 + 27x_5 + 22x_6 \\ \text{s.t} & x_1 \qquad\quad + 2x_3 + 2x_4 + x_5 + 2x_6 \geq 9, \\ & \qquad x_2 + 3x_3 + x_4 + 3x_5 + 2x_6 \geq 19, \\ & x \geq 0. \end{cases}
$$

$$
\text{(Dual)} \quad \begin{cases} \max\limits_{y} & v = 9y_1 + 19y_2 \\ \text{s.t} & y_1 \leq 35, \\ & y_2 \leq 30, \\ & 2y_1 + 3y_2 \leq 60, \\ & 2y_1 + y_2 \leq 50, \\ & y_1 + 3y_2 \leq 27, \\ & 2y_1 + 2y_2 \leq 22, \\ & y \geq 0. \end{cases}
$$

(a) Suppose that $x^* = (0,0,0,0,5,2)^T$ is feasible solution of the primal problem, verify that this vector x^* is an optimal solution to the primal LP using the complementary slackness condition.

(b) Find the set of all optimal solutions of the dual LP. Does the primal LP have a unique optimal solution, or multiple optimal solutions?

(c) Now suppose that the right hand side constants of the first two primal constraints are changed to 11 and 23 respectively (from the current values of 9 and 19). Does the dual optimal solution(s) you found in (b) remain optimal after the change? If so, use it to find the set of all optimal solutions to the primal LP.

Exercise 5.6. Consider the following LP problem:

$$
\begin{cases} \max\limits_{x} & z = 3x_1 + 7x_2 + 5x_3 \\ \text{s.t} & x_1 + x_2 + x_3 + s_1 = 50, \\ & 2x_1 + 3x_2 + x_3 + s_2 = 100, \\ & x_1, \ x_2, \ x_3, \ s_1, \ s_2 \geq 0. \end{cases}
$$

After using the simplex method to solve it, we obtain the following tableau, in which x_3 and x_2 are basic variables. For each question below, you should briefly write down the details of your answer to support your claim.

z	x_1	x_2	x_3	s_1	s_2	RHS	Basic var
1	3	0	0	4	1	300	$z = 300$
0	0.5	0	1	1.5	-0.5	25	$x_3 = 25$
0	0.5	1	0	-0.5	0.5	25	$x_2 = 25$

$\max z; x, s \geq 0$

1. Write down the basic optimal solution obtained from this simplex tableau, and the corresponding optimal value of this LP. Explain if this LP has a unique or multiple solutions.
2. Suppose that the objective function changes to $\max z = (3+\Delta)x_1 + 7x_2 + 5x_3$. How does this change affect the simplex tableau in which x_3 and x_2 are basic variables? For what range of Δ does that tableau show an optimal solution? Write down an optimal solution and the optimal value for all Δ belonging to that range. Does $\Delta = 4$ belong to that range? Find an optimal solution and the optimal value for $\Delta = 4$.
3. Suppose that the objective function changes to $\max z = 3x_1 + (7+\Delta)x_2 + 5x_3$. Write down the new simplex tableau in which x_3 and x_2 are basic variables. For what range of Δ does that tableau show an optimal solution? Write down an optimal solution and a formula for the optimal value for Δ in that range. Does $\Delta = 4$ belong to that range? Find an optimal solution and the optimal value for $\Delta = 4$.
4. Suppose that the objective function changes to $\max z = 3x_1 + 7x_2 + (5+\Delta)x_3 + \Theta s_1$. Write down the new simplex tableau in which x_3 and x_2 are basic variables. For which Δ and Θ does that tableau show an optimal solution? Write down an optimal solution and a formula for the optimal value for Δ belonging to that range. Does $(\Delta, \Theta) = (2, 2)$ belong to that range?
5. Let the objective function be the original function. Suppose that the right hand side constant of the first constraint changes from 50 to 50+Δ. For what range of Δ does the basis $\{x_3, x_2\}$ continue to be optimal? What is the optimal value when Δ belongs to this range?
6. Let the objective function be the original function. Suppose that the right hand side constant of the second constraint changes from 100 to 100+Δ. For what range of Δ does the basis $\{x_3, x_2\}$ continue to be optimal? What is the optimal value when Δ belongs to this range?

Exercise 5.7. Consider the following linear program:

$$\begin{cases} \max_x z = 3x_1 + x_2 + 4x_3 + x_4 \\ 6x_1 + 3x_2 + 5x_3 + 4x_4 \le 25 \\ 3x_1 + 2x_2 + 3x_3 + x_4 \le 15 \\ 3x_1 + 4x_2 + 5x_3 + 2x_4 \le 20 \\ x_1 \ge 0\ x_2 \ge 0\ x_3 \ge 0\ x_4 \ge 0\,. \end{cases}$$

By adding three slack variables x_5, x_6 and x_7 to the first, second, and third constraints, we convert the above LP into canonical form. Then we apply the simplex method to it and obtain the following simplex tableau, in which the value in the RHS entry of row 0 is missing. For each question below, you should briefly write down the details of your answer to support your claim.

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	Basic Var
1	0	2	0	1	$1/5$	0	$3/5$	?	$z = ?$
0	1	$-1/3$	0	$2/3$	$1/3$	0	$-1/3$	$5/3$	$x_1 = 5/3$
0	0	0	0	-1	$-2/5$	1	$-1/5$	1	$x_6 = 1$
0	0	1	1	0	$-1/5$	0	$2/5$	3	$x_3 = 3$

1. What is the value of the RHS entry in row 0 of the above tableau? Justify your answer.
2. Suppose that the objective coefficient of x_3 (currently 4) changes to $4+\Delta$. For what range of Δ does the current optimal basis $\{x_1, x_6, x_3\}$ continue to be optimal? What is the optimal value for Δ in this range?
3. Let the objective function be the original function. Suppose that the right hand side constants for the first and second constraints change to $25+\Delta$ and $15+\Delta$ simultaneously (for example if $\Delta = 1$ then the RHS constants in the first two constraints become 26 and 16 respectively). For what range of Δ does the current optimal basis $\{x_1, x_6, x_3\}$ continue to be optimal? What is the optimal value for Δ in this range?

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