STOR 415, Fall 2019
Solutions to Homework Assignment No. 6

1. (a) Solution: The statement is false; in the following example, the primal is feasible (indeed it is unbounded), and the dual is infeasible.

   Primal: max y, s.t. $y \geq 2$.

   Dual: min $2x$, s.t. $x = 1, \quad x \leq 0$.

   A general counter example can be constructed by using an unbounded LP as the primal. The dual of such a primal is infeasible.

   (b) Solution: The statement is false; in the following example, both the primal and the dual are infeasible.

   Primal: max $x_1 + 2x_2$, s.t. $x_1 + x_2 = 1, \quad x_1 + x_2 = 2$.

   Dual: min $y_1 + 2y_2$, s.t. $y_1 + y_2 = 1, \quad y_1 + y_2 = 2$.

   One way to construct such a counter example is to choose a matrix $A \in \mathbb{R}^{m \times n}$ and vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ such that neither $Ax = b$ nor $A^T y = c$ has a solution, and let the primal LP be maximizing $c^T x$ subject to $Ax = b$.

2. (a) Solution: The dual LP is

$$\begin{aligned} \min \quad & 6y_1 + 7y_2 \\ & 2y_1 + y_2 \geq 5 \\ & y_1 + 2y_2 \geq 3 \\ & y_1 + y_2 \geq 1 \\ & y_1 \geq 0, \quad y_2 \geq 0 \end{aligned}$$

Graphically solve the dual LP to find its optimal solution to be $y^* = (7/3, 1/3)$. Below, we pair each primal constraint with its corresponding dual constraint, and write down the complementary slackness condition as follows:

$$\begin{array}{lcl} (2x_1 + x_2 + x_3 - 6)y_1 & = & 0 \\ (x_1 + 2x_2 + x_3 - 7)y_2 & = & 0 \\ x_1(2y_1 + y_2 - 5) & = & 0 \\ x_2(y_1 + 2y_2 - 3) & = & 0 \\ x_3(y_1 + y_2 - 1) & = & 0. \end{array}$$

By plugging $y^* = (7/3, 1/3)$ in the above conditions, we find that the following equalities and inequalities have to be satisfied by x^* for it to be a primal optimal solution:

$$\begin{aligned} & 2x_1 + x_2 + x_3 = 6 \\ & x_1 + 2x_2 + x_3 = 7 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_3 = 0 \end{aligned}$$

The three equations above have a unique solution $x^* = (5/3, 8/3, 0)$, which also satisfies the two inequalities $x_1 \geq 0$ and $x_2 \geq 0$. Hence, $x^* = (5/3, 8/3, 0)$ is the unique primal optimal solution.

(b) Solution: The dual is

$$
\begin{aligned}
\min \quad & 5y_1 + 3y_2 \\
& 2y_1 + y_2 \geq 5 \\
& y_1 - y_2 \geq -7 \\
& y_1 + y_2 \geq 5 \\
& y_1 \geq 0, \quad y_2 \text{ free}
\end{aligned}
$$

Graphically solve the dual LP to find its optimal solution to be $y^* = (0, 5)$. Below, each primal constraint is paired with its corresponding dual constraint in the complementary slackness conditions:

$$
\begin{array}{lcl}
(2x_1 + x_2 + x_3 - 5)y_1 & = & 0 \\
(x_1 - x_2 + x_3 - 3)y_2 & = & 0 \\
x_1(2y_1 + y_2 - 5) & = & 0 \\
x_2(y_1 - y_2 + 7) & = & 0 \\
x_3(y_1 + y_2 - 5) & = & 0.
\end{array}
$$

By plugging $y^* = (0, 5)$ in the above conditions, we find that the following equalities and inequalities have to be satisfied by x^* for it to be a primal optimal solution:

$$
\begin{aligned}
& 2x_1 + x_2 + x_3 \leq 5 \\
& x_1 + 2x_2 + x_3 = 3 \\
& x_1 \geq 0 \\
& x_2 = 0 \\
& x_3 \geq 0
\end{aligned}
$$

The two equations above give $x_2 = 0$ and $x_1 = 3 - x_3$. Replacing x_1 by $3 - x_3$ and x_2 by 0 in the first inequality, we find $2(3 - x_3) + x_3 \leq 5$, so that $x_3 \geq 1$. The third inequality implies $x_3 \leq 3$. We conclude that $x = (3 - x_3, 0, x_3)$ is an optimal solution for each $x_3 \in [1, 3]$, i.e., the set of optimal solutions is $\{x = (3 - x_3, 0, x_3) \mid x_3 \in [1, 3]\}$.

3. Solution:

(a) x^* satisfies the constraints $x_5 \geq 0$ and $x_6 \geq 0$ as strict inequalities, and all other constraints as equalities. Accordingly, any dual optimal solution needs to satisfy $\pi_1 + 3\pi_2 = 27$ and $2\pi_1 + 3\pi_2 = 22$, in addition to other dual constraints. The only 2-dimensional vector that satisfies all those conditions is $\pi^* = (3, 8)$, which is the unique dual optimal solution.

To determine whether x^* is the unique or one of multiple optimal solutions to the primal LP, we use π^* to write down conditions for a 6-dimensional vector x to be a primal optimal solution. Note that π^* satisfies $\pi_1 + 3\pi_2 \leq 27$ and $2\pi_1 + 3\pi_2 \leq 22$ as equalities, and all other dual constraints as strictly inequalities. Thus, for a primal feasible solution x to be optimal, it needs to satisfy all the primal constraints (except the two constraints $x_5 \geq 0$ and $x_6 \geq 0$) as equalities. The only 6-dimensional vector that satisfies those conditions is x^*, which is therefore the unique primal optimal solution.

(b) After the change, $\pi^* = (3, 8)$ remains feasible to the dual LP. To check if it is a dual optimal solution, we need to check if there is a 6-dimensional vector x that satisfies the complementary slackness conditions with π^*. Again, the requirement on x is that it is primal feasible and satisfies all the primal constraints (except the two constraints $x_5 \geq 0$ and $x_6 \geq 0$) as equalities. There is a unique vector $x' = (0, 0, 0, 0, 6, 2.5)$ that satisfies those conditions. Thus, x' is the unique primal optimal solution, and π^* remains as a dual optimal solution.