STOR 415, Fall 2019
Solutions to HW 11

1. (a) Yes.
   (b) Yes.
   (c) No.
   (d) No.

2. (a) No. The local solution is also a global solution, so 25 is the minimum value $f(x)$ can take over the entire feasible region.
   (b) Yes, it is possible that the local solution is not a global solution.

3. *Proof.* Let $x_1, x_2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$. Then

$$
\begin{aligned}
& F\big(\lambda x_1 + (1-\lambda)x_2\big) \\
=& f\big(g(\lambda x_1 + (1-\lambda)x_2)\big) \text{ by definition of } F \\
=& f\big(\lambda g(x_1) + (1-\lambda)g(x_2)\big) \text{ because } g \text{ is affine} \\
\le& \lambda f(g(x_1)) + (1-\lambda)f(g(x_2)) \text{ by convexity of } f \\
=& \lambda F(x_1) + (1-\lambda)F(x_2) \text{ by definition of } F.
\end{aligned}
$$

   Since the above holds for any $x_1, x_2 \in \mathbb{R}^n$ and any $\lambda \in [0,1]$, the function F is convex. □

4. Consider an affine function $g : \mathbb{R}^4 \to \mathbb{R}^2$ defined as

$$
g(x_1, x_2, x_3, x_4) = \begin{bmatrix} -x_1 + 3x_2 + 5x_3 - 6x_4 \\ x_1 - x_3 \end{bmatrix}
$$

   and a function $F : \mathbb{R}^2 \to \mathbb{R}$ as

$$
F(y_1, y_2) = y_1^2 - 2y_2.
$$

   The function F is convex, because its Hessian matrix is

$$
\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}
$$

   and is positive semidefinite. The function f is the composite of g and F: $f(x) = F(g(x))$. Since g is affine and F is convex, f is convex.