STOR 415, Fall 2019
Solutions to Homework Assignment No. 5

1. (a)

$$\begin{array}{llll} \max & z = & 2x_1 + 3x_2 - x_4 & \\ \text{s.t} & & 2x_1 + x_2 + 2x_3 & \leq & 16, \\ & & x_1 + x_2 - x_3 & \geq & 15, \\ & & x_1 \quad\ + x_3 - x_4 & = & -10, \\ & & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

Solution: First, add slack and surplus variables to convert the LP into standard form, and negate the third constraint to make its rhs to become nonnegative:

$$\begin{array}{lllll} \max & z = & 2x_1 + 3x_2 - x_4 & & \\ \text{s.t} & & 2x_1 + x_2 + 2x_3 + s_1 & = & 16, \\ & & x_1 + x_2 - x_3 - s_2 & = & 15, \\ & & -x_1 \quad\ - x_3 + x_4 & = & 10, \\ & & x_1, x_2, x_3, x_4, s_1, s_2 & \geq & 0. \end{array}$$

Since the above LP is not in canonical form, we start the two-phase simplex algorithm by adding an artificial variable to the second constraint. It is fine to add artificial variables to the other two equality constraints, but it is not necessary as they already "isolate" variables. (Strictly speaking, x_4 is not isolated by the third equation as it appears in the objective function. The purpose of adding artificial variables is to construct a canonical form Phase-I LP.)

$$\begin{array}{lllll} \max & t & = -a_1 = x_1 + x_2 - x_3 - s_2 - 15 & & \\ \text{s.t} & & 2x_1 + x_2 + 2x_3 + s_1 & = & 16, \\ & & x_1 + x_2 - x_3 - s_2 + a_1 & = & 15, \\ & & -x_1 \quad\ - x_3 + x_4 & = & 10, \\ & & x_1, x_2, x_3, x_4, s_1, s_2, a_1 & \geq & 0. \end{array}$$

Initial tableau for the Phase-I LP:

t	x_1	x_2	x_3	x_4	s_1	s_2	a_1	rhs	Basic var
1	-1	-1	1	0	0	1	0	-15	$t = -15$
0	2	1	2	0	1	0	0	16	$s_1 = 16$
0	1	1	-1	0	0	-1	1	15	$a_1 = 15$
0	-1	0	-1	1	0	0	0	10	$x_4 = 10$

Choose x_2 as the entering variable and a_1 as the leaving variable, we get the following tableau:

t	x_1	x_2	x_3	x_4	s_1	s_2	a_1	rhs	Basic var
1	0	0	0	0	0	0	1	0	$t = 0$
0	1	0	3	0	1	1	-1	1	$s_1 = 1$
0	1	1	-1	0	0	-1	1	15	$x_2 = 15$
0	-1	0	-1	1	0	0	0	10	$x_4 = 10$

The above tableaus shows an optimal solution for the Phase-I LP. Since its optimal value is 0, the original LP is feasible, with a feasible solution given by $(0, 15, 0, 10)$ which is obtained from the optimal BFS for the Phase-I LP after deleting the artificial and slack variables.

(b)

$$\begin{array}{llll}
\max & x_8 & & \\
\text{s.t.} & -x_1 - x_3 + x_4 + x_5 & = -2, \\
& x_1 - 2x_3 - 3x_4 + x_6 - x_8 & = 4, \\
& x_1 + x_3 - 0.5x_4 + x_7 & = 1, \\
& 2x_1 + x_2 - 5x_4 - x_8 & = 6, \\
& x_i \geq 0 \text{ for all } i &
\end{array}$$

Solution: We negate the first constraint to make its rhs to become nonnegative and add an artificial variable to it:

$$\begin{array}{llll}
\max & t = -a_1 & = x_1 + x_3 - x_4 - x_5 - 2 & \\
\text{s.t} & & x_1 + x_3 - x_4 - x_5 + a_1 & = 2, \\
& & x_1 - 2x_3 - 3x_4 + x_6 - x_8 & = 4, \\
& & x_1 + x_3 - 0.5x_4 + x_7 & = 1, \\
& & 2x_1 + x_2 - 5x_4 - x_8 & = 6, \\
& & x, a & \geq 0.
\end{array}$$

Initial tableau for the Phase-I LP:

t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	a_1	rhs	Basic var
1	-1	0	-1	1	1	0	0	0	0	-2	$t = -2$
0	1	0	1	-1	-1	0	0	0	1	2	$a_1 = 2$
0	1	0	-2	-3	0	1	0	-1	0	4	$x_6 = 4$
0	1	0	1	-1/2	0	0	1	0	0	1	$x_7 = 1$
0	2	1	0	-5	0	0	0	-1	0	6	$x_2 = 6$

Choose x_1 as the entering variable and x_7 as the leaving variable, we get the following tableau:

t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	a_1	rhs	Basic var
1	0	0	2	1/2	1	0	1	0	0	-1	$t = -1$
0	0	0	0	-1/2	-1	0	-1	0	1	1	$a_1 = 1$
0	0	0	-3	-5/2	0	1	-1	-1	0	3	$x_6 = 3$
0	1	0	1	-1/2	0	0	1	0	0	1	$x_1 = 1$
0	0	1	-2	-4	0	0	-2	-1	0	4	$x_2 = 4$

The above tableaus shows an optimal solution for the Phase-I LP. Since its optimal value is -1, the original LP is infeasible.

2. (a) The LP is always feasible (which means that it always has a feasible solution). The reason is the vector $(x, y) = (0, b)$ is always feasible no matter what A and b are.

   (b) This LP cannot be unbounded, because the objective value achieved by any feasible solution is always greater than or equal to 0.

3. (a)

$$\begin{array}{lrrrcr}
\max & 3y_1 & -5y_2 & +2y_3 & & \\
\text{s.t} & y_1 & & +2y_3 & \leq & 1, \\
 & -2y_1 & +y_2 & -3y_3 & = & 2, \\
 & 3y_1 & +2y_2 & -7y_3 & = & -3, \\
 & y_1 & +2y_2 & -4y_3 & \geq & 1, \\
 & \multicolumn{5}{l}{y_1 \leq 0, y_2 \geq 0.}
\end{array}$$

   (b)

$$\begin{array}{lrrcr}
\min & y_1 & +2y_2 & & \\
\text{s.t} & y_1 & -y_2 & \leq & -1, \\
 & y_1 & & \geq & 0, \\
 & & y_2 & = & 2, \\
 & y_1 & & \geq & 0.
\end{array}$$

   (The last constraint is obviously redundant and can be removed.)

   (c)

$$\begin{array}{ll}
\min & b^T y + d^T z \\
 & A^T y + B^T z \geq c, \\
 & y \in \mathbb{R}^m, \\
 & z \leq 0.
\end{array}$$