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Introduction to Optimization

From Linear Programming to Nonlinear Programming

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Chapter 4

Simplex Methods for LPs

This chapter introduces the simplex method for solving linear programs. The simplex method was invented by G. Dantzig in 1947 [1] when he was working at the US Army Air Force. This method is still one of the most efficient methods for solving LPs, and is available in most software packages for LPs.

Before applying the simplex method, we need to convert a general LP into standard form using techniques discussed in the preceding chapter. The complete simplex method consists of two phases. The goal of the first phase is to decide if the LP is feasible, eliminate any redundant equality constraints, and find an initial basic feasible solution when the LP is feasible. Following that, at the second phase, we will start with that initial basic feasible solution, and carry out iterations until we find an optimal basic solution, or find the LP to be unbounded.

For reasons that will become clear, we will start by introducing the *second* phase of the complete simplex method. For brevity, people sometimes just use **the simplex method** to refer to the second phase of the complete simplex method, and use **the two-phase simplex algorithm** to refer to the complete simplex method that consists of the two phases. We will adopt this convention in the sequel. Below, Section 4.1 shows how to perform the simplex method in the matrix form, assuming that an initial basic feasible solution is known. Following that, Section 4.2 shows how to perform the same method in the format of simplex tableaux, assuming that the LP is given in canonical form (which implies the existence of a natural basic feasible solution). Finally, Section 4.3 presents the complete two-phase simplex algorithm, adding the first phase to the second phase to handle any LP.

4.1 The simplex method in matrix form

Let us consider the following standard LP

$$\begin{cases} \max_{x\in\mathbb{R}^n} z = c^T x \\ \text{s.t.} \quad Ax = b \\ \qquad\ x \geq 0, \end{cases} \tag{4.1}$$

where $A \in \mathbb{R}^{m\times n}$. We assume that rows of A are linearly independent, and that we already have a basic feasible solution $\hat{x} = (\hat{x}_B, \hat{x}_N)$, where $\hat{x}_B$ is the subvector consisting of values of the basic variables $x_{B(1)}, \cdots, x_{B(m)}$, and $\hat{x}_N = 0$ is the subvector consisting of values (all zeros) of the nonbasic variables $x_{N(1)}, \cdots, x_{N(n-m)}$. The corresponding decomposition of A is $A = [A_B, A_N]$, where A_B is an $m \times n$ invertible matrix consisting of the basic columns, and A_N is an $m \times (n-m)$ matrix consisting of the nonbasic columns. With $A\hat{x} = A_B\hat{x}_B + A_N\hat{x}_N = b$ and $\hat{x}_N = 0$, we have $A_B\hat{x}_B = b$, which implies $\hat{x}_B = A_B^{-1}b$. Using this decomposition, we can rewrite (4.1) as

$$\begin{cases} \max_{x} z = c_B^T x_B + c_N^T x_N \\ \text{s.t.} \quad A_B x_B + A_N x_N = b, \\ \qquad\ x_B \geq 0,\ x_N \geq 0, \end{cases}$$

where $x_B = (x_{B(1)}, \cdots, x_{B(m)})$ consists of all basic variables and $x_N = (x_{N(1)}, \cdots, x_{N(n-m)})$ consists of all nonbasic variables.

4.1.1 The idea of the simplex method

The idea behind the simplex method is as follows:

- Solve x_B from $A_B x_B + A_N x_N = b$ as $x_B = A_B^{-1}(b - A_N x_N) = A_B^{-1}b - A_B^{-1}A_N x_N$.
- Write the above LP equivalently as

$$\begin{cases} \max_{x} z = c_B^T A_B^{-1} b + (c_N^T - c_B^T A_B^{-1} A_N) x_N \\ \text{s.t.} \quad x_B + A_B^{-1} A_N x_N = A_B^{-1} b, \\ \qquad\ x_B \geq 0,\ x_N \geq 0. \end{cases} \tag{4.2}$$

- Denote $\bar{c}_N^T = c_B^T A_B^{-1} A_N - c_N^T$, the vector of **reduced costs** for the nonbasic variables. We can then write the objective function as $z = c_B^T A_B^{-1} b - \bar{c}_N^T x_N$. If $\bar{c}_N \geq 0$, then for any feasible solution x we have $z \leq c_B^T A_B^{-1} b$, while the z value achieved by the current basic feasible solution $\hat{x} = (\hat{x}_B, \hat{x}_N) = (A_B^{-1}b, 0)$ is exactly $c_B^T A_B^{-1} b$. Hence, the current basic feasible solution $\hat{x}$ is already optimal. We therefore claim that an optimal solution is found and terminate the method.

- **Determine an entering variable:** Otherwise, if the jth element of $\bar{c}_N$ is < 0 for some j, let $j_p = N(j)$. Any increase in the value of x_{j_p} will cause an increase in the objective value z, when the values of other nonbasic variables are fixed at their current value of zero. If there are more than one choice for such an index j_p, we can choose any of them. The variable x_{j_p} is called the **entering variable**, and the index j_p is called the index of the **pivot column**.
- Now, the idea is to try to increase the value of x_{j_p}, whose value in the current basic feasible solution $\hat{x}$ is zero. In the mean time, we keep values of other nonbasic variables at their current value of zero. This will affect values of the basic variables x_B through the equation $x_B + A_B^{-1} A_N x_N = A_B^{-1} b$, which can now be written as

$$x_B + x_{j_p} A_B^{-1} A_{j_p} = A_B^{-1} b \quad \Leftrightarrow \quad x_B = A_B^{-1} b - x_{j_p} A_B^{-1} A_{j_p}.$$

Let us denote by $\bar{b} = A_B^{-1} b$ and the column vector $d_{j_p} = A_B^{-1} A_{j_p}$. Then, the above constraint becomes $x_B = \bar{b} - x_{j_p} d_{j_p}$. Recall the nonnegativity constraint $x_B \geq 0$. Clearly, if $d_{j_p} = A_B^{-1} A_{j_p} \leq 0$, then $x_B \geq 0$ will always hold no matter how much we increase the value of x_{i_p}. In this case, we can keep increasing the value of x_{j_p} with values of other nonbasic variables fixed at zero, without ever violating any nonnegativity sign restrictions on x_B. Hence, the value of z can be made arbitrarily large. We therefore conclude that the LP is unbounded, and terminate the method.

- **Determine a leaving variable:** Otherwise, some entries of d_{j_p} are positive. We choose a row $i_p \in \{1, \cdots, m\}$ from

$$\operatorname*{arg\,min}_{i \in \{1,\cdots,m\}:\ d_{ij_p} > 0} \left\{ \frac{\bar{b}_i}{d_{ij_p}} \right\}.$$

That is: we choose the index that has the minimum ratio $\frac{\bar{b}_i}{d_{ij_p}}$ provided that $d_{ij_p} > 0$. If there are many indices that achieve the minimum ratio, we arbitrarily choose one of them. The idea is then to increase the value of x_{j_p} to

$$\min_{i \in \{1,\cdots,m\}:\ d_{ij_p} > 0} \left\{ \frac{\bar{b}_i}{d_{ij_p}} \right\}$$

with values of other nonbasic variables fixed at zero, which will drive the value of $x_{B(i_p)}$ to 0 and keep $x_{B(i)} \geq 0$ for all other indices $i = 1, \cdots, m$. This will give a feasible solution with a better (at least not worse) objective value, which is in fact also a basic feasible solution because replacing the column $A_{B(i_p)}$ by A_{j_p} in the basic columns preserves the required linear independency [1]. By replacing the variable $x_{B(i_p)}$ by x_{j_p} in the basis, we can obtain this basic feasible solution. We name $x_{B(i_p)}$ as the **leaving variable**, as it is the variable to leave the basis. The row i_p is called the **pivot row**.

We repeat this procedure until one of the two termination conditions is met.

4.1.2 The procedure of the simplex method

The simplex method is an iterative method, each iteration of which computes a new basic feasible solution. We algorithmically present the simplex method here. We have the following remarks on the implementation of this method.

- If the standard LP (4.1) is in canonical form, then we can let $B(i)$ be the index of the variable isolated by the ith equation. In this way, $A_{B(i)}$ is the ith unit vector, and $A_B = \mathbb{I}_m$, the identity matrix. Hence, at the beginning of the first iteration, we have $A_B^{-1} = \mathbb{I}_m$, which implies $\bar{c}_N^T = c_B^T A_N - c_N^T$, $\bar{b} = b$, and $d_{j_p} = A_{j_p}$.
- At the intermediate iterations, instead of directly inverting the newly obtained A_B to obtain A_B^{-1}, we perform elementary row operations on the previous A_B^{-1} to obtain the new A_B^{-1}. More specifically, let $A_B = \left[A_{B(1)} \cdots A_{B(m)}\right]$ be the old matrix, and $A_{B^*} = \left[A_{B(1)} \cdots A_{B(i_p-1)}\ A_{j_p}\ A_{B(i_p+1)} \cdots A_{B(m)}\right]$ be the new matrix. Because $A_B^{-1} A_{B(i)} = e_i$ for each $i = 1, \cdots, m$, we have

$$A_B^{-1} A_{B^*} = \left[e_1 \cdots e_{l-1}\ A_B^{-1} A_{j_p}\ e_{l+1} \cdots e_m\right].$$

  Now put the matrix $A_B^{-1} A_{B^*}$ side by side with A_B^{-1}, and conduct elementary row operations to change $A_B^{-1} A_{B^*}$ to the identity matrix $\mathbb{I}_m$. By conducting the same elementary row operations to the matrix A_B^{-1}, we obtain $A_{B^*}^{-1}$.

[1] Note that $A_B^{-1} A_B = \mathbb{I}_m$, and $A_B^{-1} A_{B(i_p)} = e_{i_p}$ is the i_p'th unit vector. On the other hand, the i_p'th element of $d_{j_p} = A_B^{-1} A_{j_p}$ is > 0, by the definition of i_p. Hence, replacing the column e_{i_p} in the identity matrix $\mathbb{I}_m$ with d_{j_p} preserves the nonsingularity of the matrix. This implies that replacing the column $A_{B(i_p)}$ in the matrix A_B by A_{j_p} preserves the nonsingularity of A_B.

Algorithm 1 (*Simplex method in matrix form*)

1: **Input:**

2: Given a basic feasible solution $\hat{x} = [\hat{x}_B, \hat{x}_N] = [\hat{x}_B, 0]$, where $B = \{B(1), \cdots, B(m)\}$ is the (ordered) index set of basic variables, and $N = \{1, \cdots, n\} \setminus B$

3: **Main loop:**

4: Compute the reduced costs $\bar{c}_N^T = c_B^T A_B^{-1} A_N - c_N^T$, and write $\bar{b} = A_B^{-1} b$ (the values of $\hat{x}_B$).

5: If $\bar{c}_N \geq 0$, then terminate, and return $\hat{x} = [\hat{x}_B, \hat{x}_N]$ as an optimal solution.

6: Select j such that the jth element of $\bar{c}_N$ is < 0, let $j_p = N(j)$, and compute the vector $d_{j_p} = A_B^{-1} A_{j_p}$.

7: If $d_{j_p} \leq 0$, then terminate, and claim that the LP is unbounded.

8: Choose which index to leave the index set B:

$$i_p = \mathop{\arg\min}_{i \in \{1, \cdots, m\}:\ d_{ij_p} > 0} \left\{ \frac{\bar{b}_i}{d_{ij_p}} \right\}.$$

9: Update the index set B by replacing $B(i_p)$ by j_p, and update N accordingly:

$$B(i_p) = j_p \quad \text{and} \quad N \leftarrow \{1, \cdots, n\} \setminus B.$$

10: Update A_B^{-1} and compute a new basic feasible solution $\hat{x} = [\hat{x}_B, \hat{x}_N] = [A_B^{-1} b, 0]$.

11: **Repeat the Main loop**

Example 4.1. Let us apply Algorithm 1 to solve the following canonical LP:

$$\begin{cases} \max\limits_x \ z = & x_1 & +x_2 & & & \\ \text{s.t.} \quad x_1 & +2x_2 & +x_3 & & = 20 \\ \quad\quad 2x_1 & +x_2 & & +x_4 & = 20 \\ \quad\quad x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0. \end{cases} \tag{4.3}$$

We can write

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}, \quad b = \begin{pmatrix} 20 \\ 20 \end{pmatrix}, \quad \text{and} \quad c = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

We have $x^0 = (0,0,20,20)^T$ as a basic feasible solution. In this case, $B = \{3,4\}$ and $N = \{1,2\}$. We can write $x^0 = [x^0_B, x^0_B]$ with $x^0_B = (20,20)^T$ and $x^0_N = (0,0)^T$. We also have $A_B = \mathbb{I}_2$ and $A_N = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Iteration 0: We start with $B = \{3,4\}$ and $N = \{1,2\}$, $x^0_B = (20,20)^T$ and $x^0_N = (0,0)^T$, and $A_B = \mathbb{I}_2$.

1. We compute $\bar{c}_N^T = c_B^T A_B^{-1} A_N - c_N^T = c_B^T A_N - c_N^T = -c_N^T = (-1,-1)$, and $\bar{b} = A_B^{-1} b = b = \begin{pmatrix} 20 \\ 20 \end{pmatrix}$.
2. The first element of $\bar{c}_N$ is $-1 < 0$, we can let $j_p = N(1)$. The variable x_1 is an **entering variable**, and the pivot column is $j_p = 1$.
3. We compute $d_1 = A_B^{-1} A_1 = A_1 = (1,2)^T$.
4. Since there exists $i \in B = \{3,4\}$ such that $d_{i1} > 0$, we compare the two ratios $\left\{\frac{20}{1}, \frac{20}{2}\right\}$ and find the second to be smaller. This means $i_p = 2$. Because $B(2) = 4$, the **leaving variable** is x_4, and the pivot row is the second row.
5. Finally, we form a new B and new N as $B = \{3,1\}$ and $N = \{4,2\}$. As an **optional** step, we can also reorder indices in B and N so that they are in the increasing order, to obtain $B = \{1,3\}$ and $N = \{2,4\}$.

Iteration 1: We have $A_B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, $A_N = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$. (Note: if we choose not to reorder indices in B and N at the end of last iteration, then with $B = \{3,1\}$ and $N = \{4,2\}$ we will have $A_B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $A_N = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$. This apparent difference will not affect the basic feasible solution.) We find $A_B^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$, and obtain a new basic feasible solution $x^1 = [x^1_B, x^1_N]$ such that $x^1_B = A^{-1} b = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{pmatrix} 20 \\ 20 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$. More precisely, we have $x^1 = (0,10,10,0)^T$.

1. We compute $\bar{c}_N^T = c_B^T A_B^{-1} A_N - c_N^T = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$, and $\bar{b} = A_B^{-1} b = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{pmatrix} 20 \\ 20 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$.
2. With the first element of $\bar{c}_N$ being $-\frac{1}{2} < 0$, we let $j_p = N(1) = 2$, and choose the variable x_2 as the entering **entering variable**.
3. We compute $d_2 = A_B^{-1} A_2 = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}$.
4. Since there exists $i \in B$ such that $d_{i2} > 0$, we compare the two $\left\{ \frac{10}{1/2}, \frac{10}{3/2} \right\}$ and find the second to be smaller, which gives $i_p = 2$ and $B(i_p) = 3$. The **leaving variable** is x_3.
5. Finally, we form a new B and a new N as $B = \{1,2\}$ and $N = \{3,4\}$, respectively.

Iteration 2: We have $A_B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $A_N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We can compute $A_B^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$. Hence, we can find a new basic feasible soluion $x^2 = [x_B^2, x_N^2]$ with $x_N^2 = 0$ and $x_B^2 = A^{-1} b = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{pmatrix} 20 \\ 20 \end{pmatrix} = \begin{pmatrix} \frac{20}{3} \\ \frac{20}{3} \end{pmatrix}$.

1. We compute $\bar{c}_N^T = c_B^T A_B^{-1} A_N - c_N^T = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}^T = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \end{pmatrix}$.
2. Since $\bar{c}_N \geq 0$, we conclude that x^2 is optimal. We can write it as $x^* = x^2 = [x_B^2, x_N^2] = \left(\frac{20}{3}, \frac{20}{3}, 0, 0 \right)^T$. The corresponding optimal value is $z^* = x_1^* + x_2^* = \frac{40}{3}$.

To show how does the simplex method geometrically work, we eliminate variables x_3 and x_4 and rewrite problem (4.3) as

$$\begin{cases} \max\limits_x \ z = & x_1 & +x_2 & & \\ \text{s.t.} \ \ x_1 & & +2x_2 & \leq & 20 \\ \quad\ \ 2x_1 & & +x_2 & \leq & 20 \\ \quad\ \ x_1 \geq 0, \ x_2 \geq 0. & & & & \end{cases}$$

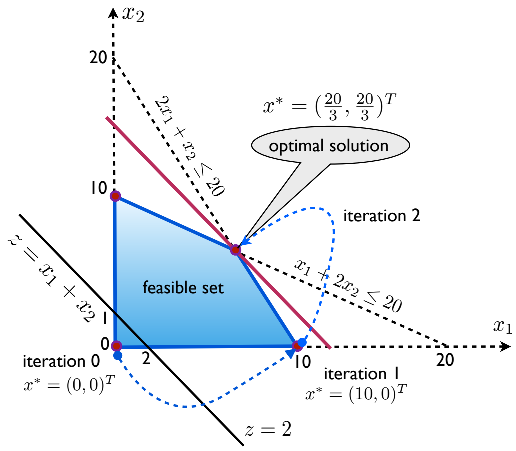

Fig. 4.1 An illustration of the simplex method for solving a simple LP

Figure 4.1 shows the feasible set and the objective line of this problem.

It also shows that we start from the vertex $x^0 = (0,0)^T$ at **iteration 0** with the objective value $z = c^T x^0 = 0$. Then, at **iteration 1**, we move to the next vertex $x^1 = (0,10)^T$ with the objective value $z = c^T x^1 = -10$. At **iteration 2**, we move to the vertex $x^* = x^2 = (20/3, 20/3)^T$, which is optimal, and the optimal value is $z^* = -40/3$. □

4.2 The simplex method with simplex tableaux

We consider the following canonical linear program:

$$
\begin{cases}
\max\limits_{x} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{s.t.} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ = b_1, \\
& \qquad \vdots \qquad\qquad\qquad\quad \vdots \ \ \vdots \\
& a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \ \ = b_i, \\
& \qquad \vdots \qquad\qquad\qquad\quad \vdots \ \ \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \\
& x_j \geq 0, \qquad j = 1, 2, \cdots, n.
\end{cases}
$$

In order to implement the simplex method, we can use simplex tableaux to store data and operate the iterations. At each iteration, we move from one simplex tableau to another by carrying out elementary row operations. We now describe this method in details.

4.2.1 Simplex tabeaux

The initial simplex tableau is given as in Table 4.1. This table has $m+1$ rows

Table 4.1 The initial simplex tableau for the simplex method

z	x_1	$\cdots$	x_n	RHS	Basic var	Ratio test
1	$-c_1$	$\cdots$	$-c_n$	0	$z=0$	
0	a_{11}	$\cdots$	a_{1n}	b_1	$x_{B(1)}=b_1$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
0	a_{i1}	$\cdots$	a_{in}	b_i	$x_{B(i)}=b_i$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
0	a_{m1}	$\cdots$	a_{mn}	b_m	$x_{B(m)}=b_m$	

max z; $x \geq 0$

excluding the title row. The first column consists of coefficients for the objective z, the next n columns include coefficients for variables x_1 to x_n, and the RHS column shows the right-hand side vector b. Moreover:

- The first row in the data section of Table 4.1 represents the equation $z - c_1x_1 - \cdots - c_nx_n = 0$. By convention this row is called “row 0”.
- The remaining m rows represent the m equality constraints, and are called “row 1” $\cdots$, up to “row m”.
- The “Basic var” column gives information on the natural basic feasible solution associated with the canonical LP, where $x_{B(i)}$ is the variable isolated by the ith equation.
- The last column will be used for the ratio test.
- Entries in row 0 of Table 4.1 are called the **reduced costs**. For example the reduced costs of z, x_1 are x_n are 1, $-c_1$ and $-c_n$ respectively in this tableau. (This definition of reduced costs only applies to simplex tableaus for a maximization LP. If the simplex tableau shows a minimization problem, then the reduced costs are the negative entries of row 0. In this course, we only work with simplex tableaus for maximization problems. The name “reduced cost” was chosen for historical reasons and is not particularly useful in understanding the general method.)

4.2.2 Simplex iterations

Starting from the initial simplex tableau, Table 4.1, one can apply the simplex method. Each iteration of this simplex method for solving a canonical LP contains the following steps:

Step 1. (**Choose the pivot column.**) Choose **one** nonbasic variable with a negative (< 0) **reduced cost** to be the **entering variable**. The column under the entering variable is the **pivot column**. If there are more than one negative reduced costs, then we can pick any one of them.

If no entering variable can be selected, we **termninate** the algorithm. The current BFS is an **optimal solution**, and the current simplex tableau is called an **optimal tableau**.

Step 2. (**Choose the pivot row.**) Excluding row 0, compare the ratios

$$\frac{\text{RHS}}{\text{entry in the pivot column}}$$

for each row with a positive (> 0) entry in the pivot column. The row with the minimal ratio is the **pivot row**. If more than one row gives the minimal ratio, then we can pick any of such rows to be the pivot row. The current basic variable in this row is the **leaving variable**. The intersection of the pivot column and the pivot row is the **pivot element**.

If no leaving variable can be selected, we **terminate** the algorithm. The LP is **unbounded**.

Step 3. (**Elementary row operations.**) Multiply the pivot row by the inverse of the pivot element, so that the pivot element becomes one; add a multiple of the pivot row to each other row, so that all other entries of the pivot column become zeros.

Step 4. (**Update the column about basic variables.**) In the pivot row, the basic variable is now the entering variable (it was the leaving variable before this iteration). The basic variables of all other rows remain the same. The values of the basic variables are given by the RHS constants of corresponding rows (which change in different tableaux).

4.2.3 Examples

Let us apply the simplex method in simplex tableaux to solve an LP.

Example 4.2. Consider the following linear program:

$$\begin{cases} \min\limits_{x} & -x_1 - x_2 - x_3 \\ \text{s.t} & -x_1 + x_2 - x_3 \leq 2, \\ & x_1 - x_2 - x_3 \quad \leq 3, \\ & -x_1 - x_2 + x_3 \leq 1, \\ & x_1 + x_2 + x_3 \quad \leq 4, \\ & x_1, x_2, x_3 \qquad \geq 0. \end{cases}$$

We first convert this problem into the maximization problem as

$$\begin{cases} \max\limits_{x} & z = x_1 + x_2 + x_3 \\ \text{s.t} & \quad -x_1 + x_2 - x_3 \leq 2, \\ & \quad x_1 - x_2 - x_3 \quad \leq 3, \\ & \quad -x_1 - x_2 + x_3 \leq 1, \\ & \quad x_1 + x_2 + x_3 \quad \leq 4, \\ & \quad x_1, x_2, x_3 \qquad \geq 0. \end{cases}$$

We then add slack variables x_4, x_5, x_6, x_7 to convert the latter problem into standard form:

$$\begin{cases} \max\limits_{x} & z = x_1 + x_2 + x_3 \\ \text{s.t} & \quad -x_1 + x_2 - x_3 + x_4 = 2, \\ & \quad x_1 - x_2 - x_3 + x_5 = 3, \\ & \quad -x_1 - x_2 + x_3 + x_6 = 1, \\ & \quad x_1 + x_2 + x_3 + x_7 = 4, \\ & \quad x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0. \end{cases}$$

This problem has $m = 3$ constraints and $n = 7$ variables. Moreover, it is easy to check that the last LP is canonical.

Let us write down the initial simplex tableau as follows:

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	Basic var	Ratio test
1	-1	-1	-1	0	0	0	0	0	$z = 0$	
0	-1	1	-1	1	0	0	0	2	$x_4 = 2$	
0	1	-1	-1	0	1	0	0	3	$x_5 = 3$	
0	-1	-1	1	0	0	1	0	1	$x_6 = 1$	
0	1	1	1	0	0	0	1	4	$x_7 = 4$	

max z; $x \geq 0$

Iteration 0: Now, let us start the simplex method from the initial tableau. Since $c_3 = -1 < 0$, we can choose $j_p = 3$ to be a pivot column. We can of course choose $j_p = 1$ or $j_p = 2$ as well.

z	x_1	x_2	x_3 ▼	x_4	x_5	x_6	x_7	RHS	Basic var	Ratio test
1	-1	-1	-1	0	0	0	0	0	$z = 0$	
0	-1	1	-1	1	0	0	0	2	$x_4 = 2$	N/A
0	1	-1	-1	0	1	0	0	3	$x_5 = 3$	N/A
0	-1	-1	$\boxed{1}$	0	0	1	0	1	$x_6 = 1$	$1/1 = 1$
0	1	1	1	0	0	0	1	4	$x_7 = 4$	$4/1 = 4$

max z; $x \geq 0$

1. Entering variable: x_3 (again, it is fine to choose either x_1 or x_2).
2. Ratio test: There are two rows $i = 3$ and $i = 4$ in the tableau with positive entries in the pivot column. For these two rows, we compute the ratio between the RHS and the entry in the pivot column. Row 3 wins, and x_6 is the leaving variable.
3. The pivot element is at the location $(i_p, j_p) = (3, 3)$, with a value of 1.
4. Conduct the following elementary row operations:
   - Add row 3 to row 0, row 3 to row 1, and row 3 to row 2.
   - Subtract row 3 from row 4.

We obtain the second tableau below.

▼

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	Basic var	Ratio
1	-2	-2	0	0	0	1	0	1	$z = 1$	
0	-2	0	0	1	0	1	0	3	$x_4 = 3$	N/A
0	0	-2	0	0	1	1	0	4	$x_5 = 4$	N/A
0	-1	-1	1	0	0	1	0	1	$x_3 = 1$	N/A
0	$\boxed{2}$	2	0	0	0	-1	1	3	$x_7 = 3$	$3/2 = 1.5$

max z; $x \geq 0$

Iteration 1: We repeat the simplex iteration as follows.

1. Entering variable: Since $c_1 = -2 < 0$, we choose x_1 as an entering variable (it is fine to choose x_2).
2. Ratio test: row 4 is the only row with positive entry in the pivot column, so we choose it as the pivot row and x_7 as the leaving variable.
3. Conduct elementary row operations:
   - Divide row 4 by 2.
   - Add twice row 4 to row 0, twice row 4 to row 1, and row 4 to row 3.
   - Keep row 2 unchanged.

We obtain the third tableau below.

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	Basic var
1	0	0	0	0	0	0	1	4	$z = 4$
0	0	2	0	1	0	0	1	6	$x_4 = 6$
0	0	-2	0	0	1	1	0	4	$x_5 = 4$
0	0	0	1	0	0	0.5	0.5	2.5	$x_3 = 2.5$
0	1	1	0	0	0	-0.5	0.5	1.5	$x_1 = 1.5$

max z; $x \geq 0$

We see that the first $n+1$ columns of row 0 do not include any negative values. Hence, there is no entering variable to select, and we can conclude that the current BFS

$$x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)^T = (1.5, 0, 2.5, 6, 4, 0, 0)^T$$

is optimal, and the objective value attained by this BFS, $z = 4$, is the optimal value. The algorithm is terminated with the above tableau being an optimal tableau. By removing the slack variables, we obtain $\hat{x}^* = (1.5, 0, 2.5)^T$ as an optimal solution of the original problem, with the optimal value $\hat{z} = -4$.

4.2.4 Features of simplex tableaux

With a closer look at the simplex tableaux, we observe the following features:

- The data section of any simplex tableau contains columns of the form

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

  where the first column is the column under z, and the others are columns associated with the current basic variables. The basic variable of each row is the variable isolated by this row (having a coefficient of 1 in this row and a coefficient of 0 in any other row), and its value is exactly the current RHS of this row.
- All the entries (except the one in row 0) of the RHS column of any simplex tableau are nonnegative numbers, which are values of the basic variables.
- Each simplex tableau represents a transformation of the original LP. All these LPs are equivalent to each other, and they are all in canonical form, with different isolated variables.

4.2.5 Uniqueness of the optimal solution

An LP may have more than one optimal solution. To illustrate this, we consider LPs of two variables. Figure 4.2 shows two different LPs. The goal in both cases are to maximize the objective functions. By shifting the iso-profit line along the normal vector c until it would leave the feasible set, we obtain the last iso-profit line. The intersection between the last iso-proft line and the feasible set gives the optimal solutions. In both cases, that intersection consists of an entire edge of the feasible set, so any point on that edge is an optimal solution of the LP. The two cases are different in the boundedness of the set of the optimal solutions:

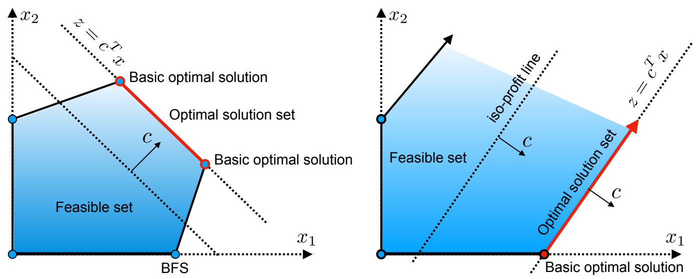

Fig. 4.2 Multiple optimal solutions in LPs. Left: **Case 1:** Bounded solution set, Right: **Case 2:** Unbounded solution set

- **Case 1:** The optimal edge is of finite length. In other words, the set of optimal solutions is bounded.
- **Case 2:** The optimal edge is of infinite length. In other words, the set of optimal solutions is unbounded.

Next, we consider how to treat these two situations in the simplex method with simplex tableaux.

Consider the optimal tableau obtained from applying the simplex method to an LP. All reduced costs in such a tableau are nonnegative.

- If all reduced costs associated with the nonbasic variables are strictly positive, then the LP has a unique optimal solution.
- If there exists a nonbasic variable x_j with zero reduced cost, then the LP can have multiple solutions. In this case, we can increase this variable from zero to a positive number without decreasing the objective value, to obtain a new optimal solution. This can be done by performing an extra simplex iteration . In this simplex iteration, x_j is selected as the **entering variable**. In trying to determine the **leaving variable**, there are two situations:
  - **Case 1:** If there exists a positive entry in the j-th column, then we can choose the leaving variable by using the ratio test rule. Hence, by performing an extra simplex iteration on the optimal simplex tableau, we can move from one basic optimal solution to another one.
  - **Cases 2:** If all the entries of the j-th column are nonpositive, then we can increase the value of x_j from zero to any positive number without making other basic variables to be negative. Hence, we obtain an unbounded

halfline starting from the current basic optimal solution, with each point on this halfline being an optimal solution.

Note: if x^* and $\hat{x}^*$ are two optimal solutions of an LP, then for any $\alpha \in (0,1)$, the point $x = \alpha x^* + (1-\alpha)\hat{x}^*$ is also an optimal solution of this problem.

Let us consider three examples that illustrate three situations described above.

Example 4.3. Suppose that we obtain the following optimal simplex tableau when solving a maximization LP of 4 variables and 2 equality constraints:

z	x_1	x_2	x_3	x_4	RHS	Basic var
1	0	0	2	3	10	$z = 10$
0	1	0	3	2	4	$x_1 = 4$
0	0	1	1	1	3	$x_2 = 3$

The BFS $x^* = (4,3,0,0)^T$ is the unique optimal solution, since the reduced costs of both nonbasic variables x_3 and x_4 are strictly positive.

Example 4.4. Suppose that we obtain the following optimal simplex tableau when solving a maximization LP of 4 variables and 2 equality constraints:

z	x_1	x_2	x_3	x_4	RHS	Basic var
1	2	0	0	0	2	$z = 2$
0	1	0	1	1	2	$x_4 = 2$
0	3	1	-2	0	3	$x_2 = 3$

The BFS $x^* = (0,3,0,2)^T$ is optimal, but there can be other optimal solutions, because the reduced cost of the nonbasic variable x_3 is zero. To find other optimal solutions, we select x_3 as the entering variable, and conduct an extra simplex iteration to find a different basic optimal solution $\hat{x}^* = (0,7,2,0)^T$ (more details about how to conduct such an extra iteration are provided in Example 4.6 below). Then, any vector of the format $x = \alpha x^* + (1-\alpha)\hat{x}^* = \alpha(0,3,0,2)^T + (1-\alpha)(0,7,2,0)^T$ with some $0 \le \alpha \le 1$ is an optimal solution.

Alternatively, one can let $x_3 = t \ge 0$, fix $x_1 = 0$, and let x_2 and x_4 be expressed as functions of t as $x_2 = 3 + 2t$ and $x_4 = 2 - t$ respectively. The non-negativity requirements on x_2, x_3 and x_4 restrict t to the range $[0,2]$. Any vector $x^* = (0, 3 + 2t, t, 2 - t)^T$ is an optimal solution of the LP for any $t \in [0,2]$.

Example 4.5. Suppose that we obtain the following optimal simplex tableau when solving a maximization LP of 4 variables and 2 equality constraints:

z	x_1	x_2	x_3	x_4	RHS	Basic var
1	2	0	0	0	2	$z = 2$
0	1	0	-1	1	2	$x_4 = 2$
0	3	1	-2	0	3	$x_2 = 3$

The BFS $x^* = (0,3,0,2)^T$ is optimal, but there can be other optimal solutions because the reduced cost of the nonbasic variable x_3 is zero. To find other optimal solutions, let $x_3 = t$, and fix $x_1 = 0$, and let x_2 and x_4 be expressed as functions of t as $3+2t$ and $2+t$ respectively. Here, the nonnegativity requirements on x_2 and x_4 are automatically satisfied for any $t \geq 0$, so $x = (0, 3+2t, t, 2+t)$ is an optimal solution for any $t \geq 0$. In this example, if one would select x_3 as the entering variable in the extra simplex iteration, then the ratio test would be void and one would not be able to find a different basic optimal solution.

Example 4.6. In Example 4.2, we note that the objective function of the LP in the optimal simplex tableau is

$$z = x_7 - 4,$$

which holds for each feasible solution. If we increase the nonbasic variable x_6 from zero to a positive number while keeping the other nonbasic variables x_2 and x_7 at zero, z will remain to be 4. Hence, for the extra pivot, we can choose

1. Entering variable: x_6
2. Ratio test: Row 2 wins, and x_5 is the leaving variable.

By elementary row operations we obtain the following tableau:

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	Basic var	Ratio test
1	0	0	0	0	0	0	1	4	$z = 4$	
0	0	2	0	1	0	0	1	6	$x_4 = 6$	
0	0	-2	0	0	1	1	0	4	$x_6 = 4$	
0	0	1	1	0	-0.5	0	0.5	0.5	$x_3 = 0.5$	
0	1	0	0	0	0.5	0	0.5	3.5	$x_1 = 3.5$	

max z; $x \geq 0$

Again no entering variable to select: the above tableau is another optimal tableau. The current BFS is

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)^T = (3.5, 0, 0.5, 6, 0, 4, 0)^T,$$

which is an alternative optimal solution. The optimal value at this optimal solution is still $z = 4$.

We have found 2 **basic optimal solutions** so far. In fact, there are 4 more **basic optimal solutions**. We can find them by continuing extra pivots from the present optimal simplex tableau. We may also find them by selecting different entering variables in the process of the algorithm. In total, there are 6 **basic optimal solutions**, and infinitely many optimal solutions!

4.2.6 Unbounded linear programs

In the previous chapter, we have illustrated the situation in which an LP is unbounded. A maximization (minimization) LP is unbounded, when its feasible set is unbounded and there exists a halfline $\{x + td \mid t \geq 0\}$ entirely included in the feasible set, with x being the starting point and d being the direction, such that the objective value of $x + td$ goes to ∞ ($-\infty$) as t increases to ∞. Such a direction d is called the **direction of unboundedness** of the LP. While an unbounded LP must have an unbounded feasible set, an LP with an unbounded feasible set may have unique or multiple optimal solutions, and the set of its optimal solutions can be bounded or unbounded.

Below, we give details on how to identify unbounded LPs from simplex tableaux. Consider a simplex tableau obtained by solving a maximization LP. If the tableau contains a column that has a negative entry in row 0 and nonpositive entries in the all remaining rows, then the LP is unbounded. In other words, if we can select an entering variable but not a leaving variable, then the LP is unbounded. With such a tableau, one can find feasible solutions with arbitrarily large objective values, by letting the entering variable take any nonnegative value, fixing other nonbasic variables at zero, and letting basic variables be expressed as functions of the entering variable.

Example 4.7. Suppose that we are solving a maximization LP of 4 variables and 2 equality constraints, and obtain the following simplex tableau:

z	x_1	x_2	x_3	x_4	RHS	Basic var
1	-3	-2	0	0	0	$z = 0$
0	1	-5	1	0	3	$x_3 = 3$
0	2	0	0	1	4	$x_4 = 4$

The LP is unbounded, because the nonbasic variable x_2 has a negative reduced cost, and all entries in the column under x_2 are nonpositive.

To find the direction of unboundedness, let $x_2 = t \geq 0$, and fix the other nonbasic variable $x_1 = 0$, and let x_3 and x_4 be expressed as functions of t as $x_3 = 3 + 5t$ and $x_4 = 4$, respectively. In summary, $x = (0, t, 3+5t, 4)^T$ is a feasible solution for any $t \geq 0$, with the objective value $z = 2t$. For example, with $t = 1000$, we have a feasible solution $(0, 1000, 5003, 4)^T$ that gives $z = 2000$. Furthermore, by a decomposition, we can write any feasible solution x as

$$\begin{pmatrix} 0 \\ t \\ 3+5t \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 4 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 5 \\ 0 \end{pmatrix}.$$

Hence, we have found a starting point $x^0 = (0,0,3,4)^T$ and a direction $d = (0,1,5,0)^T$, so that the half-line $x = x^0 + td$ from the starting point x^0 along the direction d is entirely contained in the feasible set and the objective value $z = 2t$ increases along the direction d as $t \to +\infty$.

4.3 The two-phase simplex algorithm

In real-world problems, we often obtain a general linear program instead of a canonical LP. To solve such a general LP, we need to find an initial basic feasible solution to apply the simplex method as described in the preceding sections to solve the LP by starting from this basic feasible solution. However, for a general LP, finding a basic feasible solution is almost as hard as finding an optimal solution given a initial basic feasible solution. In this section, we describe a two-phase simplex method to solve a general LP. First, we convert a general LP into a standard LP, which can always be done. If the resulting LP is canonical, we can easily obtain a natural BFS and directly apply the simplex method described in Section 4.2. Otherwise, we apply the two-phase simplex method as described below, to solve the standard LP. The two phases of this method are discussed in the following two subsections respectively.

4.3.1 Phase 1: Solving the auxiliary LP

Let us consider the following standard LP:

$$\begin{cases} \max\limits_{x} & z = c^T x \\ \text{s.t.} & Ax = b, \\ & x \geq 0. \end{cases} \tag{4.4}$$

Here, without loss of generality, we assume that the right-hand side vector b is nonnegative, i.e., $b \geq 0$. (If $b_i < 0$ for some index i, then we can simply multiply the i-th constraint by -1 on both sides to obtain a positive right-hand side.) We also assume that rows of A are linearly independent.

Step 1: Our next step is to form an auxiliary LP for (4.4) as

$$\begin{cases} \max\limits_{x,y} & t = -y_1 - y_2 - \cdots - y_m \\ \text{s.t.} & Ax + y = b, \\ & x \geq 0, \\ & y \geq 0. \end{cases} \tag{4.5}$$

Here, we add an **artificial variable** y_i to the i-th equality constraint of the original LP (4.4). We ignore the objective function of (4.4), and define a new objective function to be maximizing the sum of additive inverses of all artificial variables y, i.e., $t = -\sum_{i=1}^n y_i$. The resulting LP has $n+m$ variables and m constraints, and is called the **Phase I LP** or the **auxiliary LP**.

Fact 4.3.1 *The auxiliary LP* (4.5) *always has an optimal solution.*

Indeed, the feasible set of this problem is nonempty since $(x,y) = (0,b)$ is a feasible solution. Moreover, the objective function $t = -\sum_{i=1}^m y_i \leq 0$ for all feasible solutions $y \geq 0$, which shows that t is bounded from above. It follows that (4.5) must have an optimal solution.

Step 2: To transform (4.5) into canonical form, we note from the constraints of (4.5) that $y = b - Ax$. Hence, we can rewrite the objective of (4.5) as

$$t = -\sum_{i=1}^m y_i = -\sum_{i=1}^m b_i + \sum_{i=1}^m \sum_{j=1}^n A_{ij} x_j.$$

If we denote by $1 = (1, 1, \cdots, 1)^T$ the vector of all ones in $\mathbb{R}^m$, then $t = (A^T 1)^T x - b^T 1 = \bar{c}^T x - b^T 1$, where $\bar{c} = A^T 1$. Hence, the auxiliary LP (4.5) is equivalent to the following canonical LP:

$$\begin{cases} \max\limits_{x,y} & t = \bar{c}^T x - b^T 1 \\ \text{s.t.} & \quad Ax + y \quad = b, \\ & \quad x \quad \geq 0, \\ & \quad y \quad \geq 0, \end{cases} \tag{4.6}$$

in which y_i is the variable isolated by the ith equation. Accordingly, $(x, y) = (0, b)$ is the basic feasible solution naturally associated with (4.6).

Fact 4.3.2 *The standard LP* (4.4) *has a feasible solution if and only if the optimal value t^* of the auxiliary problem* (4.5) *is zero.*

Indeed, if (4.4) has a feasible solution $\bar{x}$, then $A\bar{x} = b$ and $\bar{x} \geq 0$. Then $(x, y) = (\bar{x}, 0)$ is a feasible solution for (4.5) with $t = 0$. On the other hand, any feasible solution (x, y) of (4.5) must have $y \geq 0$ and therefore $t \leq 0$. Hence, the optimal value of (4.5) is $t = 0$.

Conversely, let (x^*, y^*) be an optimal solution of (4.5) with $t^* = 0$. Then, since $y^* \geq 0$, the fact $t^* = -\sum_{i=1}^m y_i^* = 0$ implies $y^* = 0$. This means $Ax^* = b$, which together with $x^* \geq 0$ implies that x^* is a feasible solution to (4.4).

Remark 4.1. If the ith equality constraint of the standard LP (4.4) already isolates a variable, then we do not have to add an artificial variable y_i into this constraint. The objective function of the auxiliary problem (4.5) is always to maximize the sum of the additive inverses of all artificial variables. (See the example below for such a situation.)

Step 3: Our final step is to apply the simplex method to solve the canonical auxiliary LP (4.6), to find a basic optimal solution (x^*, y^*) with the optimal value $t^* \leq 0$. Depending on the value of t^*, there are the following two cases to consider:

Case 1: The optimal value t^* of (4.6) is negative ($t^* < 0$). In this case, we conclude that the standard LP (4.4) is infeasible. Hence, the original general LP is also infeasible.

Case 2: The optimal value $t^* = 0$. In this case, $y^* = 0$ and the standard LP (4.4) is feasible. Since $(x^*, 0)$ is a BFS of (4.6), x^* is a BFS of (4.4). (This can be seen, e.g., from the fact that basic feasible solutions are exactly extreme points.) We will then apply the simplex method to solve (4.4) using this BFS as the initial point. This phase is called Phase 2, as presented below.

4.3.2 Phase 2: Performing the simplex method on the standard LP

If we directly apply the simplex method to solve the standard LP (4.4) using the BFS x^* obtained at the end of Phase 1, then we will need to initialize a simplex tableau so that all the basic variables defining x^* are isolated variables in the tableau. In other words, we will need to identity the matrix A_B that consists of columns for the basic variables defining x^*, and then multiply both sides of the equation $Ax = b$ by A_B^{-1} to transform the LP from (4.4) to the canonical form (4.2). However, by exploiting the optimal simplex tableau from Phase 1, we can construct the initial simplex tableau for the original LP (4.4) without directly computing A_B^{-1}. Below we explain how to proceed.

Examining the optimal simplex tableau at Phase 1, we can encounter the following two cases:

Case 1: None of the artificial variables is a basic variable in the optimal tableau of Phase 1. In this case, we eliminate the artificial variables and the columns under these variables from the tableau, and express the objective function of the original LP (4.4) using the remaining nonbasic variables. This yields a canonical form of the original LP. From there, one can apply the simplex method to solve (4.4).

More precisely, after removing all artificial variables from the optimal tableau of Phase 1, we can write all basic variables x_B as $x_B = \bar{b} - \bar{A}_N x_N$, where $\bar{A}_N$ is the matrix formed from the nonbasic columns in that tableau, and $\bar{b}$ is the right-hand side vector in that tableau. Substituting x_B into the objective function

$$z = c^T x = c_B^T(\bar{b} - \bar{A}_N x_N) + c_N^T x_N = c_B^T \bar{b} + (c_N - \bar{A}_N^T c_B)^T x_N,$$

we obtain the reduced costs for nonbasic variables in the initial simplex tableau for (4.4) as $\bar{c}_N = \bar{A}_N^T c_B - c_N$.

Case 2: If there exists an artificial variable y_i that is a basic variable in the optimal tableau of Phase 1, then we can change this basic variable to an non-artificial variable by elementary row operations. More specifically, suppose that y_i is the basic variable in row l. Because the optimal value is $t^* = 0$, all artificial variables are zero in this BFS, which implies that the right hand side of row l is zero. By the linear independency of rows of A, there is at least one nonzero entry of row l under some non-artificial variable. Suppose the entry in row l under a non-artificial variable x_j is nonzero. Then, x_j must be a nonbasic variable in this tableau. Choose this entry as the pivot element, and conduct elementary row operations so that x_j becomes the basic variable in row l (i.e., use ERO's to make the entry in row l under x_j to become 1, and other entries in the x_j column to become zeros). During those ERO's, the right hand side coefficients in the tableau do not change because the right hand side of row l is zero. We repeat this procedure until we remove all artificial variables from the basis.

4.3.3 Examples

Let us consider some examples to illustrate how the two-phase simplex method works.

Example 4.8. Consider the following standard LP:

$$\begin{cases} \max\limits_{x} & z = -2x_1 - 4x_2 + 2x_3 \\ \text{s.t} & \quad x_1 - 2x_2 + x_3 \qquad\quad = 27, \\ & \quad 2x_1 + x_2 + 2x_3 \qquad\quad = 50, \\ & \quad x_1 - x_2 - x_3 + x_4 = 18, \\ & \quad x_1, \cdots, x_4 \geq 0. \end{cases}$$

Since the last constraint isolates the variable x_4, we do not need an artificial variable for this constraint. We can write the auxiliary LP for Phase 1 by adding two artificial variables y_1 and y_2 as:

$$\begin{cases} \max\limits_{x,y} & t = 3x_1 - x_2 + 3x_3 - 77 \\ \text{s.t} & \quad x_1 - 2x_2 + x_3 \qquad\quad + y_1 \qquad = 27, \\ & \quad 2x_1 + x_2 + 2x_3 \qquad\qquad\quad + y_2 = 50, \\ & \quad x_1 - x_2 - x_3 + x_4 \qquad\qquad\quad = 18, \\ & \quad x_1, \cdots, x_4, y_1, y_2 \geq 0. \end{cases}$$

Here, the objective function $t = -y_1 - y_2$ is equivalently written as $t = -77 + 3x_1 - x_2 + 3x_3$, because summing up the two first constraints gives $-y_1 - y_2 = -77 + 3x_1 - x_2 + 3x_3$.

Phase 1: Now, we apply the simplex method to solve the auxiliary problem. The initial tableau is as follows.

t	x_1	x_2	x_3 ▼	x_4	y_1	y_2	RHS	Basic var	Ratio test
1	-3	1	-3	0	0	0	-77	$t = -77$	
0	1	-2	1	0	1	0	27	$y_1 = 27$	$\frac{27}{1} = 27$
0	2	1	$\boxed{2}$	0	0	1	50	$y_2 = 50$	$\frac{50}{2} = 25$
0	1	-1	-1	1	0	0	18	$x_4 = 18$	N/A

Iteration 0: x_3 is the entering variable, and y_2 is the leaving variable.

t	x_1 ▼	x_2	x_3	x_4	y_1	y_2	RHS	Basic var	Ratio test
1	0	5/2	0	3	0	3/2	-2	$t = -2$	
0	0	-5/2	0	0	1	-1/2	2	$y_1 = 2$	
0	1	1/2	1	0	0	1/2	25	$x_3 = 25$	
0	2	-1/2	0	1	0	1/2	43	$x_4 = 43$	

This simplex tableau is already optimal. Here, we have $t^* = -2 < 0$. We conclude that the original problem is infeasible.

Example 4.9. We consider the following standard LP:

$$\begin{cases} \max_x \ z = -x_1 - x_2 - x_3 \\ \text{s.t} \quad x_1 + 2x_2 + 3x_3 = 3, \\ \quad -x_1 + 2x_2 + 6x_3 = 2, \\ \quad 3x_3 + x_4 = 1, \\ \quad x_1, \cdots, x_4 \geq 0. \end{cases} \tag{4.7}$$

Because x_4 has coefficient 1 in the third equation and zero coefficients in the other two equations, we do not need to add an artificial variable in the third equation. We only add two artificial variables x_5 and x_6 to obtain the following auxiliary LP:

$$\begin{cases} \max_x \ t = & -x_5 - x_6 \\ \text{s.t} & x_1 + 2x_2 + 3x_3 + x_5 = 3, \\ & -x_1 + 2x_2 + 6x_3 + x_6 = 2, \\ & 3x_3 + x_4 = 1, \\ & x_1, \cdots, x_6 \geq 0. \end{cases}$$

Using the first two constraints one can express x_5 and x_6 as functions of x_1, x_2 and x_3 as

$$x_5 = 3 - x_1 + 2x_2 + 3x_3 \quad \text{and} \quad x_6 = 2 + x_1 - 2x_2 - 6x_3.$$

Substitute these two expressions into t to transform the auxiliary LP into:

$$\begin{cases} \max_x \ t = & 4x_2 + 9x_3 - 5 \\ \text{s.t} & x_1 + 2x_2 + 3x_3 + x_5 = 3, \\ & -x_1 + 2x_2 + 6x_3 + x_6 = 2, \\ & 3x_3 + x_4 = 1, \\ & x_1, \cdots, x_6 \geq 0. \end{cases}$$

This problem is canonical. The initial tableau is as follows.

t	x_1	x_2	x_3	x_4	x_5	x_6	RHS	Basic var	Ratio test
1	0	-4	-9	0	0	0	-5	$t = -5$	
0	1	2	3	0	1	0	3	$x_5 = 3$	$\frac{3}{2}$
0	-1	2	6	0	0	1	2	$x_6 = 2$	$\frac{2}{2}$
0	0	0	3	1	0	0	1	$x_4 = 1$	

We apply the simplex iterations to operate on this simplex tableau.

Iteration 0: We can identify that x_2 is the entering variable and x_6 is the leaving variable.

t	x_1	x_2	x_3	x_4	x_5	x_6	RHS	Basic var	Ratio test
1	-2	0	3	0	0	2	-1	$t = -1$	
0	2	0	-3	0	1	-1	1	$x_5 = 1$	$\frac{1}{2}$
0	-1/2	1	3	0	0	1/2	1	$x_2 = 1$	
0	0	0	3	1	0	0	1	$x_4 = 1$	

Iteration 1: x_1 is the entering variable, and x_5 is the leaving variable.

t	x_1	x_2	x_3	x_4	x_5	x_6	RHS	Basic var	Ratio test
1	0	0	0	0	1	1	0	$t = 0$	
0	1	0	-3/2	0	1/2	-1/2	1/2	$x_1 = 1/2$	
0	0	1	9/4	0	1/4	1/4	5/4	$x_2 = 5/4$	
0	0	0	3	1	0	0	1	$x_4 = 1$	

The above simplex tableau is the optimal simplex tableau we obtain for the auxiliary LP. It shows that the optimal value of the auxiliary LP is $t^* = 0$. Moreover, in this optimal tableau, the basic variables are x_1, x_2, x_4, none of which is an artificial variable. This means that we are in Case 1. By removing columns under the artificial variables, we obtain a tableau for the original LP, where we leave row 0 undetermined for now.

z	x_1	x_2	x_3	x_4	RHS	Basic var
1	*	*	*	*	*	$z = *$
0	1	0	-3/2	0	1/2	$x_1 = 1/2$
0	0	1	9/4	0	5/4	$x_2 = 5/4$
0	0	0	3	1	1	$x_4 = 1$

To complete row 0, we need to express the objective function z as a function of the nonbasic variables (because the reduced costs for basic variables have to be zero in order for the tableau to be a valid simplex tableau). To this end, we use information from rows 1 to 3 in the above tableau, which provides expressions of x_1, x_2 and x_4 as functions of x_3:

$$x_1 = 1/2 + 3/2x_3, \qquad x_2 = 5/4 - 9/4x_3, \qquad \text{and} \quad x_4 = 1 - 3x_3.$$

By substitution, we have

$$z = -x_1 - x_2 - x_3 = -1/4x_3 - 7/4.$$

We now have a complete simplex tableau for the original LP:

z	x_1	x_2	x_3	x_4	RHS	Basic var
1	0	0	1/4	0	-7/4	$z = -7/4$
0	1	0	-3/2	0	1/2	$x_1 = 1/2$
0	0	1	9/4	0	5/4	$x_2 = 5/4$
0	0	0	3	1	1	$x_4 = 1$

Luckily, this simplex tableau is already optimal since all reduced costs are nonnegative. We obtain the optimal solution of this LP as $x^* = (1/2, 5/4, 0, 1)^T$ and the optimal value as $z = -7/4$.

In general, at the end of Phase I, one will obtain an initial simplex tableau for the original LP (when the original LP is feasible) and will then need to continue iterations from there.

Example 4.10. Let us consider the following standard LP:

$$\begin{cases} \max\limits_{x} \ z = -3x_1 \qquad\quad + x_3 \\ \text{s.t} \qquad\quad x_1 + \ \ x_2 + x_3 + x_4 = 4, \\ \qquad\quad -2x_1 + \ \ x_2 - x_3 \qquad\ \ = 1, \\ \qquad\qquad\qquad 3x_2 + x_3 + x_4 = 9, \\ \qquad\quad x_1, \cdots, x_4 \geq 0. \end{cases}$$

With the same procedure as above, we can form the following auxiliary LP:

$$\begin{cases} \max\limits_{x} \ t = \ -x_1 + 5x_2 + x_3 + 2x_4 - 14 \\ \text{s.t} \qquad x_1 + x_2 + x_3 + x_4 + x_5 \qquad\qquad = \ 4, \\ \qquad -2x_1 + x_2 - x_3 \qquad\qquad + x_6 \quad = \ 1, \\ \qquad\qquad 3x_2 + x_3 + x_4 \qquad\qquad + x_7 = 9, \\ \qquad x_1, \cdots, x_7 \geq 0. \end{cases}$$

Here, x_5, x_6 and x_7 are three artificial variables. The objective function is given by $t = -x_5 - x_6 - x_7$, which can be equivalently written as $t = -14 - x_1 + 5x_2 + x_3 + 2x_4$ by summing up the three constraints.

Phase 1: Using the simplex method to solve the auxiliary LP, we have the initial tableau as

t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	Basic var	Ratio test
1	1	-5	-1	-2	0	0	0	-14	$t = -14$	
0	1	1	1	1	1	0	0	4	$x_5 = 4$	$\frac{4}{1} = 4$
0	-2	$\boxed{1}$	-1	0	0	1	0	1	$x_6 = 1$	$\frac{1}{1} = 1$
0	0	3	1	1	0	0	1	9	$x_7 = 9$	$\frac{9}{3} = 3$

Iteration 0: x_2 is the entering variable, and x_6 is the leaving variable.

t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	Basic var	Ratio test
1	-9	0	-6	-2	5	0	0	-9	$t=-9$	
0	$\boxed{3}$	0	2	1	1	0	0	3	$x_5=3$	$\frac{3}{3}=1$
0	-2	1	-1	0	0	1	0	1	$x_2=1$	N/A
0	6	0	4	1	0	-3	1	6	$x_7=6$	$\frac{6}{6}=1$

Iteration 1: x_1 is the entering variable, and x_5 is the leaving variable.

t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	Basic var	Ratio test
1	0	0	0	1	8	0	0	0	$t=0$	
0	1	0	2/3	1/3	1/3	0	0	1	$x_1=1$	
0	0	1	1/3	2/3	2/3	1	0	3	$x_2=3$	
0	0	0	0	-1	-2	-3	1	0	$x_7=0$	

This tableau is optimal for the auxiliary LP, and it shows that the optimal value of the auxiliary LP is 0. This means that the original LP is feasible. However, in the basic optimal solution shown in this tableau, x_7, an artificial variable, is in the basis. We replace x_7 with x_4 as the basic variable in row 3, by performing the following extra step. The elementary row operations in this step do not change the right hand sides because the right hand side of row 3 (which is also the value of x_7) is zero.

t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	Basic var	Ratio test
1	0	0	0	0	6	-3	1	0	$t=0$	
0	1	0	2/3	0	-1/3	-1	1/3	1	$x_1=1$	
0	0	1	1/3	0	-2/3	-1	2/3	3	$x_2=3$	
0	0	0	0	1	2	3	-1	0	$x_4=0$	

Now, we can remove all artificial variables x_5, x_6 and x_7 to obtain a basic feasible solution $x=(1,3,0)^T$ to start Phase II.

Phase 2: The initial simplex tableau of Phase II is as below, where the objective is $z=-3x_1+x_3=-3(1-\frac{2}{3}x_3)+x_3=-3+3x_3$.

z	x_1	x_2	x_3	x_4	RHS	Basic var	Ratio test
1	0	0	-3	0	0	$z=-3$	
0	1	0	$\boxed{2/3}$	0	1	$x_1=1$	$\frac{1}{2/3}=3/2$
0	0	1	1/3	0	3	$x_2=3$	$\frac{3}{1/3}=9$
0	0	0	0	1	0	$x_4=0$	N/A

Iteration 1: The entering variable is x_3 and the leaving variable is x_1.

z	x_1	x_2	x_3	x_4	RHS	Basic var	Ratio test
1	9/2	0	0	0	0	$z = 3/2$	
0	-3/2	0	1	0	3/2	$x_3 = 3/2$	
0	-1/2	1	0	0	5/2	$x_2 = 5/2$	
0	0	0	0	1	0	$x_4 = 0$	

Luckily, this simplex tableau is already optimal. The optimal solution of the original problem is $x^* = (0, 5/2, 3/2, 0)^T$, and the optimal value is $z^* = 3/2$.

4.4 Exercises

Exercise 4.1. Solve the following problems using the simplex method in tableau form. If one is not canonical, transform it into the canonical form first. Describe your steps in detail and give explicit simplex tableau for each iteration.

$$(a)\ \begin{cases} \max\limits_x\ z = x_1 + x_2 \\ \text{s.t.}\quad 2x_1 + x_2 + x_3 = 5 \\ \qquad\ x_1 + 2x_2 + x_4 = 6 \\ \qquad\ x_1, x_2, x_3, x_4 \geq 0. \end{cases} \qquad (b)\ \begin{cases} \min\limits_x\ z = 2x_1 - 3x_2 + x_3 \\ \text{s.t.}\quad x_1 + x_2 + x_3 \leq 10 \\ \qquad\ x_1 + 2x_2 - x_3 \leq 8 \\ \qquad\ x_1, x_2, x_3 \geq 0. \end{cases}$$

Exercise 4.2. The bartender of your local pub asks you to assist him in finding the combination of mixed drinks to maximize his revenue. He has the following bottles available:

- 1 quart (32 oz.) Old Cambridge (a fine whiskey)
- 1 quart Joy Juice (another fine whiskey)
- 1 quart Ma's Wicked Vermouth
- 2 quarts Gil-boy's Gin

Since he is new to the business, his knowledge is limited to the following drinks:

- *Whiskey Sour*. Each serving uses 2 oz. whiskey and sells for \$1.
- *Manhattan*. Each serving uses 2 oz. whiskey and 1 oz. vermouth, and sells for \$2.
- *Martini*. Each serving uses 2 oz. gin and 1 oz. vermouth, and sells for \$2.
- *Pub Special*. Each serving uses 2 oz. gin and 2 oz. whiskey, and sells for \$3.

First, formulate an LP model to maximize the bar's profit. Ignore the fact that the numbers of servings have to be integers. Second, if the problem is not canonical, convert it into a standard form, and you will see the resulting problem is canonical. Next, solve the LP problem using the simplex method. Is there a unique optimal solution, or multiple optimal solutions? Finally, convert the solution to the solution of the original problem and interpret the result.

Exercise 4.3. Suppose that, after applying the simplex method to solve a maximization LP problem, you obtain the tableau below, where $a_{13}, a_{14}, a_{23}, a_{24}, a_{33}, a_{34}, b_1, b_2, b_3, c_3, c_4, d$ are real numbers.

z	x_1	x_2	x_3	x_4	x_5	RHS	Basic var
1	0	0	c_3	c_4	0	d	$z = d$
0	0	1	a_{13}	a_{14}	0	b_1	$x_2 = b_1$
0	0	0	a_{23}	a_{24}	1	b_2	$x_5 = b_2$
0	1	0	a_{33}	a_{34}	0	b_3	$x_1 = b_3$

max z; $\quad x \geq 0$

Suppose $b_1 \geq 0$, $b_2 \geq 0$ and $b_3 \geq 0$. For each of the scenarios below, describe the next action to take in applying the simplex method:

1. $c_3 < 0$, $c_4 > 0$, $a_{13} < 0$, $a_{23} < 0$ and $a_{33} > 0$.
2. $c_3 < 0$, $c_4 > 0$, $a_{13} < 0$, $a_{23} < 0$ and $a_{33} = 0$.
3. $c_3 > 0$, $c_4 > 0$.
4. $c_3 > 0$, $c_4 = 0$.
5. $c_3 > 0$, $c_4 < 0$, $a_{14} > 0$, $a_{24} > 0$ and $a_{34} < 0$.

Note that

- If you conclude that an optimal solution is found, write down an optimal solution.
- If you conclude that the problem has multiple solutions, then can you conclude if the solution set is bounded or unbounded?
- If you conclude that the LP is unbounded, it suffices to state that.
- If the next action is about another iteration, describe the entering variable and the way to choose the leaving variable, but you do not have to conduct the actual row operations to write down the next tableau.

Exercise 4.4. Follow the following steps for each of the problems given below.

- First, convert any minimization problem into a maximization problem by changing the signs of coefficients in the objective function.
- Next, convert the problem into standard form; you will find the resulted LP to be in canonical form.
- Then, apply the simplex method, and decide if the problem is unbounded or has unique or multiple optimal solutions.
- Visualize the result in the x_1x_2-plane (that is, solve these problems graphically) and compare with the result you obtained from the simplex methods.

1.
$$\begin{cases} \max\limits_{x} & z = 2x_1 + 3x_2 \\ \text{s.t.} & \quad x_1 + 2x_2 \ \leq 6, \\ & \quad 2x_1 + x_2 \ \leq 8, \\ & \quad x_1, x_2 \quad \geq 0 \end{cases}$$

2.
$$\begin{cases} \max\limits_{x} & z = -3x_1 + 6x_2 \\ \text{s.t.} & \quad 5x_1 + 7x_2 \ \leq 35, \\ & \quad -x_1 + 2x_2 \ \leq 2, \\ & \quad x_1, x_2 \quad \geq 0 \end{cases}$$

3.
$$\begin{cases} \min\limits_{x} & z = -x_1 - 3x_2 \\ \text{s.t.} & \quad x_1 - 2x_2 \ \leq 4, \\ & \quad -x_1 + x_2 \ \leq 3, \\ & \quad x_1, x_2 \quad \geq 0 \end{cases}$$

Exercise 4.5. Determine the feasibility of each LP problem below by constructing an auxiliary LP problem and perform Phase 1 of the two-phase simplex method. If an LP problem is not in standard form, first convert it into standard form. For each LP problem, determined as feasible, write down its feasible solution obtained from the last tableau of Phase 1, but you do not need to find its optimal solution. To avoid unnecessary calculations, do not define artificial variables for equations that already isolate variables.

1.

$$\begin{cases} \max_x & z = 2x_1 + 3x_2 - x_4 \\ \text{s.t.} & 2x_1 + x_2 + 2x_3 \leq 16, \\ & x_1 + x_2 - x_3 \geq 15, \\ & x_1 + x_3 - x_4 = -10, \\ & x_i \geq 0, \ \ i = 1, \cdots, 4. \end{cases}$$

2.

$$\begin{cases} \max_x z = x_8 \\ \text{s.t.} \ -x_1 - x_3 + x_4 + x_5 = -2, \\ \quad x_1 - 2x_3 - 3x_4 + x_6 - x_8 = 4, \\ \quad x_1 + x_3 - 0.5x_4 + x_7 = 1, \\ \quad 2x_1 + x_2 - 5x_4 - x_8 = 6, \\ \quad x_i \geq 0, \quad i = 1, \cdots, 8. \end{cases}$$

Exercise 4.6. Consider the following LP problem:

$$\begin{cases} \min_{x,y} & z = y_1 + 2y_2 + 3y_3 \cdots + my_m \\ \text{s.t.} & Ax + y = 1 \\ & x \geq 0, \ \ y \geq 0, \end{cases}$$

where A is some $m \times n$ matrix, $1 = (1, 1, \cdots, 1)^T$ is the vector of all ones in $\mathbb{R}^m$, and $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are the variables.

1. Is this LP problem always feasible? Why?
2. Is it possible for this LP problem to be unbounded? Why?

Please explain your answers in detail.

Exercise 4.7. Consider the following linear program:

$$\begin{cases} \min_x z = x_1 - 2x_2 - x_4 + x_5 - x_6 \\ \text{s.t.} \ \ x_1 + x_2 - x_3 + 2x_4 = 10 \\ \quad 2x_2 + 2x_3 + x_5 = 12 \\ \quad -2x_2 + 3x_3 - x_4 + x_6 = 8 \\ \quad x_1, \cdots, x_6 \geq 0. \end{cases}$$

1. Is this LP canonical? Is this LP always feasible? If yes, find a feasible solution. Is it a basic feasible solution. Explain in detail your answers.

2. If you find a basic feasible solution in part 1, using it as a starting point and solve the problem by using simplex method.

Exercise 4.8. Given the following linear program:

$$\begin{cases} \min\limits_{x} & z = 3x_1 + x_2 - 3x_3 \\ \text{s.t.} & x_1 + 2x_2 - x_3 = 2 \\ & 10x_2 - 5x_3 = -5 \\ & -3x_2 + 2x_3 = 4, \\ & x_1, x_2, x_3 \geq 0. \end{cases}$$

Answer the following questions:

1. First, convert this problem into a standard maximization LP problem.
2. Next, form the corresponding auxiliary problem.
3. Using simplex method to solve this auxiliary problem as Phase 1.
4. If the standard LP problem is feasible, and all the artificial variables are non-basic, then move to Phase 2 to solve this standard LP problem.
5. Finally, if you answer part 4, then convert the solution you obtain back to the original problem.

References

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