Maximum Likelihood Estimation

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Distribution Family

Given: Family $\mathcal{P} = \{f_\theta : \theta \in \Theta\}$ of probability mass/density functions on $\mathcal{X}$

- $\Theta \subset \mathbb{R}^d$ called parameter space, $\theta \in \Theta$ called parameters
- parameter θ fully specifies mass/density function f_θ

Examples

- Normal $\mathcal{P} = \{\mathcal{N}(\mu, \sigma^2) : \mu \in \mathbb{R}, \sigma^2 > 0\}$
- Exponential $\mathcal{P} = \{\text{Exp}(\lambda) : \lambda > 0\}$
- Poisson $\mathcal{P} = \{\text{Poiss}(\lambda) : \lambda > 0\}$
- Binomial $\mathcal{P} = \{\text{Bin}(n, p) : p \in [0, 1]\}$

Distribution Family

Given

- Family $\mathcal{P} = \{f_\theta : \theta \in \Theta\}$ of interest
- Data $x_1, \dots, x_n \in \mathcal{X}$
- Assume data drawn independently from unknown $f_{\theta_0} \in \mathcal{P}$

Goal: Estimate θ_0 (and therefore f_{θ_0}) from data $x_1, \dots, x_n$

Idea: Select $\theta \in \Theta$ that makes given $x_1, \dots, x_n$ most likely

Maximum Likelihood Estimation

Definition: The likelihood of $\theta \in \Theta$ is joint density of $x_1, \ldots, x_n$ under f_θ

$$L(\theta) = \prod_{i=1}^{n} f_\theta(x_i)$$

Definition: The maximum likelihood estimator (MLE) of θ_0 is

$$\hat{\theta}_n^{\text{MLE}}(x_1^n) \;=\; \operatorname*{argmax}_{\theta \in \Theta} L(\theta)$$

Note: As $\log(u)$ strictly increasing, MLE can be written in equivalent form

$$\hat{\theta}_n^{\text{MLE}}(x_1^n) \;=\; \operatorname*{argmax}_{\theta \in \Theta} \log L(\theta) \;=\; \operatorname*{argmax}_{\theta \in \Theta} \sum_{i=1}^{n} \log f_\theta(x_i)$$

Maximum Likelihood Estimation

Fact: Under appropriate conditions the MLE is

- *Consistent:* $\hat{\theta}_n^{\text{MLE}}(X_1^n) \to \theta_0$ in probability
- *Asymptotically Normal:* $n^{1/2}\,(\hat{\theta}_n^{\text{MLE}}(X_1^n) - \theta_0) \Rightarrow \mathcal{N}(0, I(\theta_0)^{-1})$

Ex1. $X_1, \ldots, X_n$ iid $\sim f \in \mathcal{P} = \{\mathcal{N}(\mu, \sigma^2) : \mu \in \mathbb{R}\}$ with $\sigma^2 > 0$ known