STOR 590 9/4/20

$f(y;\theta)$ density of y given (scalar) parameter θ

e.g. Bernoulli (binary) $\theta = \log \frac{p}{1-p}$

continuous or discrete

p.d.f. p.m.f.

Let $\ell(\theta; y) = \log f(y;\theta)$ log likelihood based on 1 obsn. $Y = y$

Let $\ell' = \frac{d\ell}{d\theta}$ $\ell'' = \frac{d^2\ell}{d\theta^2}$

assume these exist

Then:
$$E_Y\{\ell'(\theta; Y)\} = 0$$
$$E_Y\left[\{\ell'(\theta;Y)\}^2\right] = \cancel{-E[\ell''(Y;\theta)]}$$
$$-E[\ell''(\theta;Y)]$$

Proof

$$1 = \int f(y;\theta)\,dy \quad \left[\text{or } \sum_y f(y;\theta)\right]$$
for every θ

$$0 = \frac{d}{d\theta}\left\{\int f(y;\theta)\,dy\right\} = \int \frac{d}{d\theta} f(y;\theta)\,dy$$
$$= \int \frac{d}{d\theta}\{\log f(y;\theta)\}\cdot f(y;\theta)\,dy$$

$$\boxed{\frac{d}{d\theta}\{\log g(\theta)\} = \frac{1}{g(\theta)}\cdot g'(\theta)}$$

$$= E\Big\{\underbrace{\frac{d}{d\theta}\log f(Y;\theta)}_{=\ell'(\theta;Y)}\Big\}$$

Exponential family

$$f(y;\theta,\phi) = \exp\left\{ \frac{y\theta - b(\theta)}{a(\phi)} + c(y;\phi) \right\}$$

. Y discrete or continuous

. θ main parameter of interest, ϕ is "dispersion"

[classic case: linear regression $\theta \equiv \beta$, $\phi = \sigma^2$ or $\frac{1}{\sigma^2}$]

~~$c(y;\phi$~~

. $a(\phi)$: any function of ϕ

. $c(y;\phi)$ normalizing function <u>independent of θ</u>

↓ to make $\int f(y;\theta,\phi)\,dy = 1$

or $\sum f(y;\theta,\phi) = 1$

$$\ell(y;\theta,\phi) = \frac{y\theta - b(\theta)}{a(\phi)} + c(y;\phi)$$

Assume θ scalar

$$\frac{d\ell}{d\theta} = \frac{y - b'(\theta)}{a(\phi)} \qquad \frac{d''\ell}{d\theta} = -\frac{b''(\theta)}{a(\phi)}$$

$$0 = E_Y\left[\frac{d\ell}{d\theta}(Y;\theta,\phi)\right] = \frac{E(Y) - b'(\theta)}{a(\phi)} \qquad \boxed{E(Y) = b'(\theta)}$$

$$\mathrm{var}\left[\ell'(Y;\theta,\phi)\right] = \frac{1}{a^2(\phi)}\mathrm{var}(Y) = \frac{b''(\theta)}{a(\phi)} \qquad \boxed{\mathrm{var}(Y) = a(\phi)\, b''(\theta)}$$

Normal $f(y;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right\}$

$$= \exp\left\{\frac{y\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2} \underbrace{- \frac{y^2}{2\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2)}_{+\,c(y;\phi)}\right\}$$

$\theta = \mu$ $\qquad \phi = \sigma^2 \quad a(\phi) = \phi$

$$\exp\left[\frac{y\theta - \frac{1}{2}\theta^2}{\sigma^2} + c(y;\phi)\right]$$

$b(\theta) = \frac{1}{2}\theta^2 \qquad b'(\theta) = \theta \qquad E(Y) = \mu$

Poisson $f(y;\mu) = \frac{\mu^y e^{-\mu}}{y!} \quad y = 0, 1, 2, \ldots$

$\ell(y;\mu) = y\log\mu - \mu - \log(y!)$

$\theta = \log\mu \qquad y\theta - e^\theta \underbrace{- \log(y!)}_{c(y;\phi)}$

$b(\theta) = e^\theta$

$b'(\theta) = e^\theta = \mu$ Read Binomial Dist^n