Elements of a GLM

Density of i'th obs. is $\exp\left\{ w_i \cdot \frac{y_i\theta_i - b(\theta_i)}{\phi} + c(y_i, \phi) \right\}$

$a_i(\phi) = \frac{\phi}{w_i}$

weight

Mean of y_i is $b'(\theta_i) = \mu_i$

Variance $b''(\theta_i) \cdot a_i(\phi) = V(\mu_i) \cdot \frac{\phi}{w_i}$

Definition $V(\mu_i) = b''(\theta_i)$

scaled variance

Link Function Assume $g(\mu_i) = \sum_{j=0}^{p} x_{ij}\beta_j = \eta_i$

link function

g is arbitrary (continuous, differentiable)

Canonical link function when $g(\mu_i) = \theta_i$

g is inverse of b'

$\left(\cancel{\theta_i = b'^{-1}} \quad \mu_i = b'(\theta_i) \right)$

e.g. **Normal** $\theta_i = \mu_i$

$V(\mu) = 1$ $\quad g(\mu) = \mu$: identity link function

Poisson $\mu_i = e^{\theta_i}$ $\quad \theta_i = \log \mu_i$ $\quad g(\mu) = \log \mu$ Log link

$V(\mu) = \mu$

Binomial $\mu_i = p_i = \frac{e^{\theta_i}}{1+e^{\theta_i}}$ $\quad \theta_i = \log \frac{p_i}{1-p_i}$

logistic regression: $\log \frac{p_i}{1-p_i} = \sum x_{ij}\beta_j$

$g(\mu) = \log \frac{\mu}{1-\mu}$ $\quad V(\mu) = \mu(1-\mu)$

Fitting a GLM

Density of i'th observation $\exp\left\{ w_i \cdot \frac{y_i \theta_i - b(\theta_i)}{\phi} + c(y_i, \phi) \right\}$

~~mean~~

$$\ell(\underset{\sim}{\beta}; y_1, \ldots, y_n) = \sum_i \left\{ w_i \cdot \frac{y_i \theta_i - b(\theta_i)}{\phi} + c(y_i, \phi) \right\}$$

$$g(\mu_i) = \sum x_{ij} \beta_j = \theta_i \quad \text{in canonical case}$$

$$\frac{\partial \ell}{\partial \beta_j} = \sum_i \frac{w_i}{\phi} \left\{ y_i \frac{\partial \theta_i}{\partial \beta_j} - b'(\theta_i) \frac{\partial \theta_i}{\partial \beta_j} \right\}$$

$$\cancel{= \sum_i \frac{w_i}{\phi}}$$

Chainrule

$$\frac{\partial \theta_i}{\partial \beta_j} = \frac{\partial \theta_i}{\partial \mu_i} \cdot \frac{\partial \mu_i}{\partial \beta_j} = \frac{1}{b''(\theta_i)} \cdot \frac{\partial \mu_i}{\partial \beta_j}$$

$$\cancel{\frac{\partial \mu}{\partial \theta}} \qquad \mu = b'(\theta) \qquad \frac{\partial \mu}{\partial \theta} = b''(\theta) = V(\mu)$$

$$\frac{\partial \theta_i}{\partial \beta_j} = \frac{1}{V(\mu_i)} \cdot \frac{\partial \mu_i}{\partial \beta_j}$$

Likelihood equations

$$0 = \frac{\partial \ell}{\partial \beta_j} = \sum \frac{w_i}{\phi} \cdot \frac{1}{V(\mu_i)} \frac{\partial \mu_i}{\partial \beta_j} (y_i - \mu_i)$$

Equivalent to minimizing

$$\sum_i w_i \cdot \left(\frac{y_i - \mu_i}{V(\mu_i)} \right)^2$$