Random Averaging Operators and Random Tensors

Rui Liang

University of Massachusetts Amherst

August 13-15, 2024

Institute of Mathematical Sciences

ShanghaiTech University, Shanghai, China

Part 1:

Bourgain's works on NLS with random data

- 1-d quintic NLS on $\mathbb{T}$
- 2-d cubic NLS on $\mathbb{T}^2$

Part 2:

The theory of random averaging operators and random tensors

$$\mathrm{i}\partial_t u - \mathrm{D}_x^{\alpha} u = \pm|u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}$$

Part 1:

Bourgain's works on NLS with random data

- 1-d quintic NLS on $\mathbb{T}$
  - $X^{s,b}$ space
  - L^6-Strichartz estimate
  - Globalisation argument
- 2-d cubic NLS on $\mathbb{T}^2$
  - Bourgain trick
  - 2-d quintic NLS on $\mathbb{T}^2$ & random averaging operators

Zakharov's question

Question: Zakharov '83[1].

"Numerical experiments demonstrated dispersive PDEs possess the "returning" property, i.e. solutions appear to be very close to the initial state $\cdots$, after some time of rather chaotic evolution. "

Question by Zakharov

How to explain this phenomenon?

- Poincaré's recurrence theorem
  - Volume preserving measure (invariant measures);
  - Dynamical system (flow property);
  - Global dynamics (Global well-posedness).
- Friedlander '85, Lebowitz-Rose-Speer '88, **Bourgain '94, '96**, Burq-Tzvetkov '05 -'24, Gubinelli-Koch-Oh '18 - '23, Bringmann '20 - '22, **Deng-Nahmod-Yue '22 - '24**.

1. V. Zakharov asked this question during the Sixth I. G. Petrovskii memorial meeting of the Moscow Mathematical Society in 1983.

Hamiltonian flow on $\mathbb{R}^{2n}$:

$$\partial_t q_i = \frac{\partial H}{\partial p_i}, \quad \partial_t p_i = -\frac{\partial H}{\partial q_i}$$

with $H(q,p) = H(q_1, \cdots, q_n, p_1, \cdots, p_n)$.

- Vector field $\left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$ is divergence-free:

  By *Liouville's theorem*, Lebesgue measure $\Pi_{j=1}^{n} \mathrm{d}q_j \mathrm{d}p_j$ is invariant.
- Hamiltonian $H(q(t), p(t))$ is conversed under the flow $\Phi(t)$.

$\Longrightarrow$ Gibbs measure: $\mathrm{d}\mu_{\mathrm{Gibbs}} = \mathrm{e}^{-\beta H(q,p)} \mathrm{d}q \mathrm{d}p$ is invariant, i.e.

$$\mu_{\mathrm{Gibbs}}(\Phi(t, A)) = \mu_{\mathrm{Gibbs}}(A).$$

$\Longrightarrow$ Poincaré recurrence theorem: "returning" property.

Q: Why do we care about *invariant measures*?

Given an invariant measure μ, we can view the system as a dynamical system with *measure-preserving* transformation Φ:

$$\Phi = \text{solution map} : (q(0), p(0)) \mapsto (q(t), p(t)).$$

We have the following theorem on recurrence properties of the dynamics:

Poincaré recurrence theorem

For any measurable set A with $\mu(A) > 0$, there exists n such that

$$\mu(A \cap \Phi^{-n} A) > 0.$$

To: Answer Zakharov's question.

Remark:

Dispersive PDEs viewed as an infinite-dimensional Hamiltonian system.

To answer Zakharov's question

To address Zakharov's question, we need to ask

- Is a dispersive PDE a Hamiltonian system?
- Does there exist a measure which is invariant under the flow of the dispersive PDE?
- Can we extend the solution globally-in-time?
- Does the flow form a dynamical system?

Nonlinear Schrödinger equation (NLS) on the torus $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$:

$$i\partial_t u = -\Delta u + G'(u) \quad \Longleftrightarrow \quad \partial_t u = -i\frac{\delta H}{\delta \bar{u}},$$

with

$$H(u) = \frac{1}{2}\int_{\mathbb{T}^d} |\nabla_x u|^2 \mathrm{d}x + \int_{\mathbb{T}^d} G(u)\mathrm{d}x.$$

If we set $q_k = \mathrm{Re}\, \hat{u}_k$ and $p_k = \mathrm{Im}\, \hat{u}_k$, then

$$H = H(q,p) = \frac{1}{4}\sum_k |k|^\alpha (q_k^2 + p_k^2) + \frac{1}{2}\int_{\mathbb{T}^d} G\Big(\sum_k (q_k + ip_k)e^{ik\cdot x}\Big)\mathrm{d}x,$$

which leads to

$$\partial_t q_k = \frac{\partial H}{\partial p_k}, \quad \partial_t p_k = -\frac{\partial H}{\partial q_k}.$$

Gibbs measure:

$$\mathrm{d}\rho_{\mathrm{Gibbs}}(u) = \mathcal{Z}^{-1} e^{-H(u)} \mathrm{d}u.$$

Consider quintic NLS on $\mathbb{T}$:

$$i\partial_t u + \Delta u = \pm |u|^4 u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}. \tag{NLS}$$

- (NLS) is an infinite-dimensional Hamiltonian system with Hamiltonian

$$H(u) = \frac{1}{2}\int_{\mathbb{T}} |\nabla_x u|^2 \mathrm{d}x \pm \frac{1}{6}\int_{\mathbb{T}} |u|^6 \mathrm{d}x.$$

- (NLS) is L^2-critical (mass-critical).
- + sign: defocusing; - sign: focusing.

LWP above 1/2

Let us prove LWP of (NLS) in $H^{1/2+}$. Note that H^s is an algebra for all $s>1/2$; i.e.

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}. \tag{1}$$

Then from the Duhamel formulation of (NLS), we define a mapping Γ:

$$\Gamma(u)(t) = e^{it\Delta}u(0) \mp i\int_0^t e^{i(t-s)\Delta}|u(s)|^4 u(s)\,ds.$$

Then we have from (1) that

$$\|\Gamma(u)\|_{C_T H_x^s} \lesssim \|u(0)\|_{H_x^s} + T\|u\|_{C_T H_x^s}^5$$

$$\|\Gamma(u)-\Gamma(v)\|_{C_T H_x^s} \lesssim T\sum_{0\le j\le 4} \|u\|_{C_T H_x^s}^{j}\|v\|_{C_T H_x^s}^{4-j}\|u-v\|_{C_T H_x^s}.$$

Then Γ is a contractive mapping on the ball $\{\|u\|_{C_T H_x^s} \le R\}$ with $R = 8C\|u(0)\|_{H_x^s}$, and $T = R^{-4}/8$.

When $s \leq 1/2$, H^s is not an algebra. However, the support of the Gibbs measure is a subset of $H^{\frac{1}{2}-}$. In 1990s, low regularity even just below 1/2, is a major open problem.

Lebowitz-Rose-Speer '88 studied statistical mechanics of nonlinear Schrödinger (NLS) equations and attempted to answer a question posed by Zakharov. In the 1990s, they faced difficulties constructing solutions due to the reliance on energy methods that involved algebra. When Speer attended a talk by Kenig, he inquired if Kenig's method could handle NLS with initial data below 1/2. Kenig replied that it was a formidable challenge and suggested that Bourgain might make headway.

Lebowitz-Rose-Speer's Question: invariance of Gibbs measure for NLS on $\mathbb{T}$?

One of the motivation of low regularity problem.

To answer LRS's question, we need local-in-time dynamics with data in $H^{\frac{1}{2}-}$.

Theorem 1 (Bourgain '93, GAFA)

LWP of (NLS) in $C_T H_x^{0+}$.

The proof of Theorem 1 involves

- introduction of $X^{s,b}$;
- L^6-Strichart estimate.

$X^{s,b}$ space

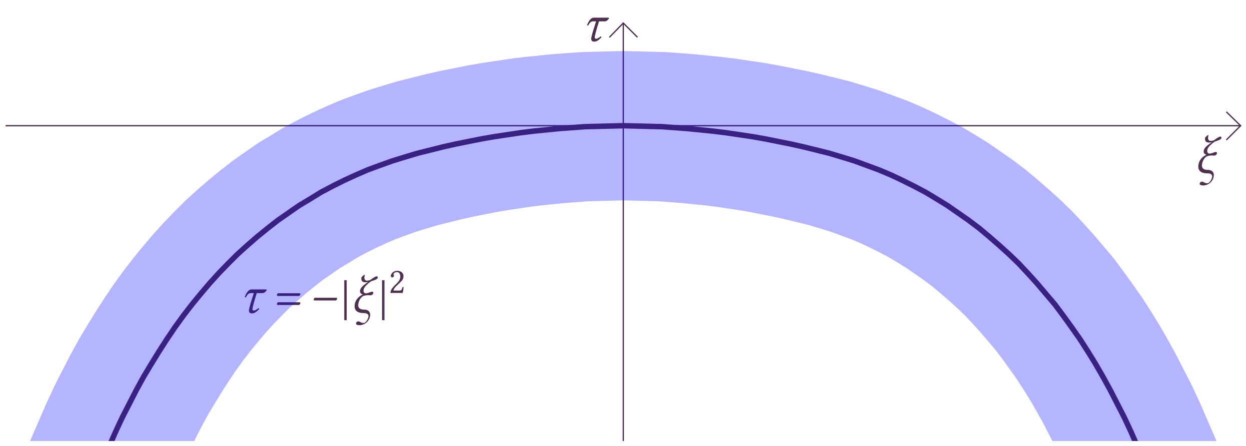
On the Fourier side, the linear part of (NLS) is

$$(\tau + |k|^2)\hat{u}(\tau, k) = 0.$$

Then $\hat{u}(\tau, k)$ is a measure supported in hypersurface $\{\tau = -|k|^2\}$. Consider the norm

$$\|u\|_{X^{s,b}} = \|\langle k\rangle^s \langle \tau + |k|^2\rangle^b \hat{u}(\tau, k)\|_{\ell^2_n L^2_\tau} = \|\mathrm{e}^{-\mathrm{i}t\Delta} u\|_{H^b_t H^s_x}.$$

When $b > 0$, $\langle \tau + |k|^2\rangle^b$ penalises the function if it lies far from the hypersurface.

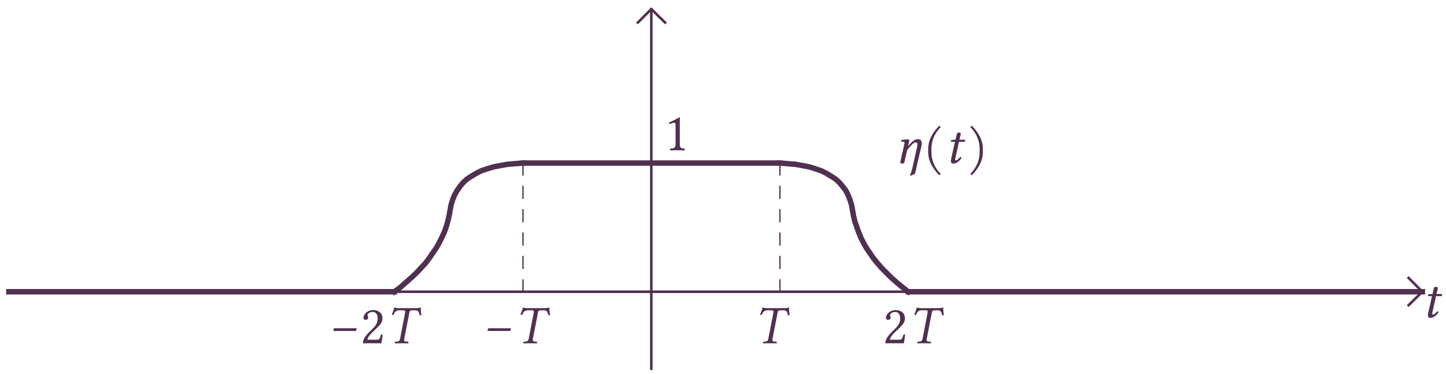


$$\widehat{e^{-it|k|^2}f}(\tau,k) = \delta(\tau+|k|^2)\hat{f}(k) \notin X^{s,b},$$

but

$$\widehat{\eta(t)e^{-it|k|^2}f}(\tau,k) = \hat{\eta}(\tau+|k|^2)\hat{f}(k),$$

with



Homogeneous linear estimate:

$$\|\eta(t)e^{it\Delta}f\|_{X^{s,b}} = \|\eta\|_{H_t^b}\|f\|_{H_x^s}.$$

Properties of $X^{s,b}$

Inhomogeneous linear estimate (Bourgain '93 GAFA):

$$\left\|\eta(t)\int_0^t \mathrm{e}^{\mathrm{i}(t-s)\Delta}F(s)\,\mathrm{d}s\right\|_{X^{s,b}} \lesssim \|F\|_{X^{s,b-1}}, \quad b>\frac{1}{2}.$$

Proof. WLOG, assume $s=0$; then take spacetime Fourier transform. □

Also,

$$\left\|\eta\left(\frac{t}{T}\right)\int_0^t \mathrm{e}^{-\mathrm{i}(t-s)\Delta}F(s)\,\mathrm{d}s\right\|_{X^{s,b}} \lesssim T^{\theta}\|F\|_{X^{s,b-1+\theta}}, \quad b>\frac{1}{2},\ 0<T\ll 1.$$

See Ginibre-Tsutsumi-Velo '97 JFA for proof.

Embedding:

$$X^{s,b}\hookrightarrow C_T H_x^s, \quad \text{for } b>\frac{1}{2}.$$

Proof. $\|u\|_{L_t^\infty H_x^s}=\|\mathrm{e}^{-\mathrm{i}\Delta t}u\|_{L_t^\infty H_x^s}\lesssim\|\mathrm{e}^{-\mathrm{i}\Delta t}u\|_{H_t^b H_x^s}=\|u\|_{X^{s,b}}$. The proof of the continuity in time of u is omitted here. □

L^6-Strichartz estimate:

$$\|\eta(t)\mathrm{e}^{\mathrm{i}t\Delta}f\|_{L^6_{t,x}([0,1]\times\mathbb{T})} \lesssim \|f\|_{H^\varepsilon}, \quad \varepsilon > 0.$$

Ideas of the proof:

L^6-norm ~ L^2-norm of $(\mathrm{e}^{\mathrm{i}t\Delta}f)^3$

use Plancherel $\longrightarrow$ work on Fourier side

$\Longrightarrow$ reduces to counting estimate.

Proof. Let $f = \Delta_N f$. We write

$$\begin{aligned}
\|e^{it\Delta}f\|^3_{L^6_{t,x}} &= \|(e^{it\Delta}f)^3\|_{L^2_{t,x}} \\
&= \Big[\sum_k \Big\| \sum_{k_1+k_2+k_3=k} e^{-it(|k_1|^2+|k_2|^2+|k_3|^2)}\hat{f}(k_1)\hat{f}(k_2)\hat{f}(k_3)\Big\|^2_{L^2_t}\Big]^{1/2}.
\end{aligned}$$

Decompose the summation of k_1, k_2 and k_3,

$$\begin{aligned}
F_k(t) &= \sum_{k_1+k_2+k_3=k} e^{-it(|k_1|^2+|k_2|^2+|k_3|^2)}\hat{f}(k_1)\hat{f}(k_2)\hat{f}(k_3) \\
&= \sum_m e^{itm} \sum_{k_1,k_2,k_3} \mathbf{1}_{k_1+k_2+k_3=k,|k_1|^2+|k_2|^2+|k_3|^2=-m}\hat{f}(k_1)\hat{f}(k_2)\hat{f}(k_3).
\end{aligned}$$

Denoting by $c_k = |\hat{f}(k)|$, by Plancherel identity we have

$$\|F_k\|^2_{L^2_t} \le \Big(\sum_m \Big| \sum_{k_1,k_2,k_3} \mathbf{1}_{k_1+k_2+k_3=k,|k_1|^2+|k_2|^2+|k_3|^2=-m} c_{k_1}c_{k_2}c_{k_3}\Big|^2\Big)^{1/2}.$$

Then by Hölder inequality, the above is bounded by

$$\begin{aligned}
&\sup_m \Big(\sum_{k_1,k_2,k_3} \mathbf{1}_{k_1+k_2+k_3=k,|k_1|^2+|k_2|^2+|k_3|^2=-m}\Big)^{1/2}\Big(\sum_m \sum_{k_1,k_2,k_3} \mathbf{1}_{k_1+k_2+k_3=k,|k_1|^2+|k_2|^2+|k_3|^2=-m} c^2_{k_1}c^2_{k_2}c^2_{k_3}\Big)^{1/2} \\
&\le \sup_m (\#S_{k,m})^{1/2}\Big(\sum_{k_1,k_2} c^2_{k_1}c^2_{k_2}c^2_{k-k_1-k_2}\Big)^{1/2}.
\end{aligned}$$

Taking the l^2_k norm reduce the proof to estimating $\#S_{k,m} \lesssim N^{\varepsilon}$. □

Counting estimate

$$S_{k,m} := \{(k_1, k_2, k_3) \in \mathbb{Z}^3; k_1 + k_2 + k_3 = k, |k_1|^2 + |k_2|^2 + |k_3|^2 = -m\}.$$

When $m < 0$,

$$\Gamma = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 + (k - x - y)^2 = -m\}$$

is an analytic image of a circle.

Counting Theorem (Bombieri-Pila '89 Duke)

If Γ is a real analytic image of the circle S^1, then for $t \longrightarrow \infty$,

$$|\mathbb{Z}^2 \cap t\Gamma| \ll t^{\varepsilon}.$$

Transference principle:

$$\|\eta(t)e^{it\Delta}\Pi_I f\|_{L^6_{t,x}([0.1]\times\mathbb{T})} \lesssim |I|^{\varepsilon}\|f\|_{L^2_x}$$

$$\Longrightarrow \quad \|\eta(t)\Pi_I u\|_{L^6_{t,x}} \lesssim |I|^{0+}\|\Pi_I u\|_{X^{0,b}}, \quad b>1/2.$$

Idea: $u\in X^{s,b}$, $b>1/2 \Longrightarrow u$ is a superposition of linear solutions.

Proof. Write u as

$$u(t,x)=\int_{\mathbb{R}}\sum_k \hat{u}(\tau,k)e^{ikx+it\tau}d\tau=\int_{\mathbb{R}}e^{it\tau}e^{it\Delta}f_\tau(x)d\tau,$$

where

$$f_\tau(x):=\sum_k \hat{u}(\tau-k^2,k)e^{ikx+itk^2}.$$

Therefore,

$$\begin{aligned}
\|\eta(t)\Pi_I u\|_{L^6_{t,x}} &\le \int_{\mathbb{R}}|e^{it\tau}|\|\eta(t)\Pi_I f_\tau(x)\|_{L^6_{t,x}}d\tau \le |I|^{\varepsilon}\int_{\mathbb{R}}\|\Pi_I f_\tau(x)\|_{L^2_x}d\tau\\
&=|I|^{\varepsilon}\int_{\mathbb{R}}\langle\tau\rangle^{-b}\langle\tau\rangle^{b}\|\Pi_I f_\tau(x)\|_{L^2_x}d\tau \le |I|^{\varepsilon}\Big(\int_{\mathbb{R}}\langle\tau\rangle^{-2b}d\tau\Big)^{1/2}\Big(\int_{\mathbb{R}}\langle\tau\rangle^{2b}\|\Pi_I f_\tau(x)\|^2_{L^2_x}d\tau\Big)^{1/2}\\
&\lesssim_b |I|^{\varepsilon}\|\langle\tau\rangle^{b}\mathbf{1}_{k\in I}\hat{u}(\tau-k^2,k)\|_{L^2_\tau\ell^2_k}=|I|^{\varepsilon}\|\langle\tau+k^2\rangle^{b}\mathbf{1}_{k\in I}\hat{u}(\tau,k)\|_{L^2_\tau\ell^2_k}=|I|^{\varepsilon}\|\Pi_I u\|_{X^{0,b}}.
\end{aligned}$$

The proof is completed. $\square$

Nonlinear estimate

Nonlinear estimate:

$$\|u_1\overline{u_2}u_3\overline{u_4}u_5\|_{X^{s,-\frac{1}{2}+2\varepsilon}} \lesssim \prod_{j=1}^{5} \|u_j\|_{X^{s,\frac{1}{2}+\varepsilon}}.$$

Proof. Recall that $(X^{s,b})' = X^{-s,-b}$. We have by duality that

$$\begin{aligned}
(\mathrm{LHS}) &= \sup_{\|u_6\|_{X^{0,\frac{1}{2}-2\varepsilon}}=1} \left|\iint \langle\partial_x\rangle^s (u_1\overline{u_2}u_3\overline{u_4}u_5)\, u_6 \mathrm{d}x\mathrm{d}t\right| \\
&\lesssim \sup \sum_{N_1,\cdots,N_6\geq 1} \left|\iint \langle\partial_x\rangle^s (\Pi_{N_1}u_1\Pi_{N_2}\overline{u_2}\Pi_{N_3}u_3\Pi_{N_4}\overline{u_4}\Pi_{N_5}u_5)\Pi_{N_6}u_6 \mathrm{d}x\mathrm{d}t\right|.
\end{aligned}$$

WLOG, assume $N_1 \geq N_2 \geq \cdots \geq N_5$. Note also that $k_1 - k_2 + k_3 - k_4 + k_5 - k_6 = 0$. Only two possibilities: $N_1 \sim N_2 \gtrsim N_6$ or $N_1 \sim N_6 \gg N_2$.
Case 1: $N_1 \sim N_2 \gtrsim N_6$.

$$\left|\iint \langle\partial_x\rangle^s (\Pi_{N_1}u_1\Pi_{N_2}\overline{u_2}\Pi_{N_3}u_3\Pi_{N_4}\overline{u_4}\Pi_{N_5}u_5)\Pi_{N_6}u_6 \mathrm{d}x\mathrm{d}t\right|$$

$$\lesssim N_1^s\left|\iint \Pi_{N_1}u_1\Pi_{N_2}\overline{u_2}\Pi_{N_3}u_3\Pi_{N_4}\overline{u_4}\Pi_{N_5}u_5\Pi_{N_6}u_6\mathrm{d}x\mathrm{d}t\right|.$$

By using $N_1^s \lesssim N_1^{s/2}N_2^{s/2}$ and Hölder inequality, the above can be bounded by

$$N_1^{0-}\left\|N_1^{\frac{s}{2}+}\Pi_{N_1}u_1\right\|_{L^6_{t,x}}\left\|N_2^{\frac{s}{2}}\Pi_{N_2}u_2\right\|_{L^6_{t,x}}\prod_{j=1}^{6}\|\Pi_{N_j}u_j\|_{L^6_{t,x}}.$$

$$\xrightarrow{\substack{L^6\text{-Strichartz}\\ \&\ \text{transference}}}\ \lesssim\ N_1^{0-}\prod_{j=1}^{5}\|u_j\|_{X^{s,\frac{1}{2}+}}\|\Pi_{N_6}u_6\|_{L^6_{t,x}}.$$

Issue: $u_6\in X^{0,\frac{1}{2}-2\varepsilon}$, but transference principle requires $b>1/2$. By transference principle, we have

$$\|\Pi_{N_6}u_6\|_{L^6_tL^6_x}\lesssim\|\Pi_{N_6}u_6\|_{X^{\varepsilon,\frac{1}{2}+\delta}}. \tag{2}$$

By Sobolev embedding, we have

$$\|\Pi_{N_6}u_6\|_{L^6_tL^6_x}\lesssim\|\Pi_{N_6}u_6\|_{X^{\frac{1}{3},\frac{1}{3}}}. \tag{3}$$

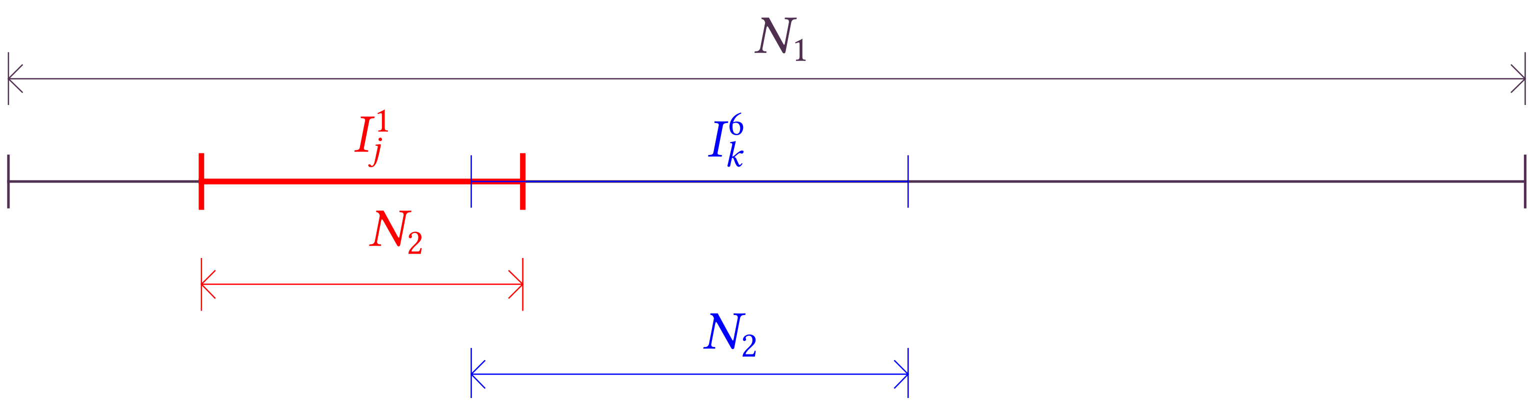
Interpolating (2) and (3) yields

$$\|\Pi_{N_6}u_6\|_{L^6_tL^6_x}\lesssim\|\Pi_{N_6}u_6\|_{X^{\varepsilon,\frac{1}{2}-}}.$$

Case 2: $N_1 \sim N_6 \gg N_2$.
This time, we cannot move the derivative to u_2. Note that $k_1 - k_6 = O(N_2)$. We divide intervals $I_1 = \{|k_1| \sim N_1\}$ and $I_2 = \{|k_6| \sim N_6\}$ into intervals of length $\sim N_2$.

$$I_1 = \bigcup_j I_j^1, \quad I_6 = \bigcup_j I_j^6, \quad |I_j^1| = |I_j^6| = N_2.$$

Note that for fixed $I_j^1 \ni k_1$, there are $O(1)$ many $I_k^6 \ni k_6$ such that $k_1 - k_6 = O(N_2)$.



$$\begin{aligned}
&\sum_{N_1,\cdots,N_6} \left| \iint N_1^s \Pi_{N_1} u_1 \Pi_{N_2} \overline{u_2} \Pi_{N_3} u_3 \Pi_{N_4} \overline{u_4} \Pi_{N_5} u_5 \Pi_{N_6} u_6 \mathrm{d}x\mathrm{d}t \right| \\
&\quad \leq \sum_{N_1,\cdots,N_6} \sum_{j,k} \left| \iint N_1^s \Pi_{I_j^1} u_1 \Pi_{N_2} \overline{u_2} \Pi_{N_3} u_3 \Pi_{N_4} \overline{u_4} \Pi_{N_5} u_5 \Pi_{I_k^6} u_6 \mathrm{d}x\mathrm{d}t \right|
\end{aligned}$$

$$\lesssim \sum_{N_1 \sim N_6} \sum_{N_2} N_2^{\delta} \sum_{j,k} N_1^s \|\Pi_{I_j^1} u_1\|_{L^6_{t,x}} \|\Pi_{N_2} u_2\|_{L^6_{t,x}} \cdots \|\Pi_{I_k^6} u_6\|_{L^6_{t,x}},$$

where we use

$$\sum_{N_3, N_4, N_5 : N_2 \geq N_3 \geq N_4 \geq N_5} 1 \leq (\log N_2)^3 \lesssim N_2^{\delta}.$$

Then by L^6-Strichartz estimate and transference principle, we continue with

$$\lesssim \sum_{N_1 \sim N_6} \sum_{N_2} N_2^{2\delta} \sum_{j,k} \|\Pi_{I_j^1} u_1\|_{X^{s,\frac{1}{2}+}} \|\Pi_{N_2} u_2\|_{X^{0,\frac{1}{2}+}} \cdots \|\Pi_{N_5} u_5\|_{X^{0,\frac{1}{2}+}} \|\Pi_{I_k^6} u_6\|_{X^{0,\frac{1}{2}-2\varepsilon}},$$

where we just lose N_2^{δ} because $|I_j^1| = |I_j^6| = N_2$.

$\underline{\ell = 3, 4, 5}$:

$$\|\Pi_{N_\ell} u_\ell\|_{X^{0,\frac{1}{2}+}} \lesssim \|u_\ell\|_{X^{s,\frac{1}{2}+}}, \quad s \geq 0.$$

$\underline{\ell = 2}$:

$$\underbrace{\sum_{N_2} N^{2\delta - s}}_{\lesssim 1} \underbrace{\|\Pi_{N_2} u_2\|_{X^{s,\frac{1}{2}+}}}_{\lesssim \|u_2\|_{X^{s,\frac{1}{2}+}}}.$$

$\underline{\ell = 1, 6}$:

$$\sum_{N_1 \sim N_6} \sum_j \cancel{\sum_k} \sum_{i=-10}^{10} \|\Pi_{I_j^1} u_1\|_{X^{s,\frac{1}{2}+}} \|\Pi_{I_{j+i}^6} u_6\|_{X^{0,\frac{1}{2}-2\varepsilon}}$$

$$\overset{\text{C-S}}{\lesssim} \sum_{N_1 \sim N_6} \Big(\sum_j \|\Pi_{I_j^1} u_1\|^2_{X^{s,\frac{1}{2}+}} \Big)^{1/2} \Big(\sum_j \sum_{i=-10}^{10} \|\Pi_{I_{j+i}^6} u_6\|^2_{X^{0,\frac{1}{2}-2\varepsilon}} \Big)^{1/2}$$

$$\sim \sum_{N_1 \sim N_6} \|\Pi_{N_1} u_1\|_{X^{s,\frac{1}{2}+}} \|\Pi_{N_6} u_6\|_{X^{0,\frac{1}{2}-2\varepsilon}}$$

$$\overset{\text{C-S}}{\lesssim} \underbrace{\Big(\sum_{N_1} \|\Pi_{N_1} u_1\|^2_{X^{s,\frac{1}{2}+}} \Big)^{1/2}}_{\sim \|u_1\|_{X^{s,\frac{1}{2}+}}} \underbrace{\Big(\sum_{N_6} \|\Pi_{N_6} u_6\|^2_{X^{0,\frac{1}{2}-2\varepsilon}} \Big)^{1/2}}_{\sim \|u_6\|_{X^{0,\frac{1}{2}-2\varepsilon}} = 1}.$$

□

Contraction mapping argument

From the Duhamel formulation of (NLS), we define a mapping Γ:

$$\Gamma(u)(t) = \eta(t)e^{it\Delta}u(0) \mp i\eta\left(\frac{t}{T}\right)\int_0^t e^{i(t-s)\Delta}|u(s)|^4 u(s)\,ds.$$

Then we have from $\|u_1\overline{u_2}u_3\overline{u_4}u_5\|_{X^{s,-\frac{1}{2}+2\varepsilon}} \lesssim \prod_{j=1}^5 \|u_j\|_{X^{s,\frac{1}{2}+\varepsilon}}$ that ($b = \frac{1}{2}+\varepsilon$, $\theta = \varepsilon$)

$$\begin{aligned} \|\Gamma(u)\|_{X^{s,b}} &\lesssim \|u(0)\|_{H_x^s} + T^\theta \||u|^4 u\|_{X^{s,b-1+\theta}} \\ &\lesssim \|u(0)\|_{H_x^s} + T^\theta \|u\|_{X^{s,b}}^5 \end{aligned}$$

$$\|\Gamma(u) - \Gamma(v)\|_{X^{s,b}} \lesssim T^\theta \sum_{0\le j\le 4} \|u\|_{X^{s,b}}^j \|v\|_{X^{s,b}}^{4-j} \|u-v\|_{X^{s,b}}.$$

Then Γ is a contractive mapping on the ball $\{\|u\|_{X^{s,b}} \le R\}$ with $R = 8C\|u(0)\|_{H_x^s}$, and $T = (R^{-4}/8)^{1/\theta}$.

Globalisation argument

If we have LWP in H^1 in a subcritical sense, then we have GWP in H^1.
Issue: $\nexists$ conservation law at the level of the Gibbs measure μ.
Idea: use invariance of μ (in place of a conservation law) to construct global-in-time dynamics (on $\operatorname{supp}\mu$).
Bourgain '94 CMP: use invariance of the “finite dimensional” Gibbs measure μ_N associated to the truncated dynamics $\Longrightarrow$ a.s. GWP $\Longrightarrow$ invariance.

$$\begin{tikzcd}
\text{invariance} \arrow[rr, bend left] & & \text{a.s. GWP} \arrow[ll, bend left]
\end{tikzcd}$$

Consider the finite-dimensional NLS

$$i\partial_t u_N + \partial_x^2 u_N = \Pi_N(|\Pi_N u_N|^4 \Pi_N u_N), \quad (t,x) \in \mathbb{R} \times \mathbb{T}, \tag{FNLS}$$

where

$$(\Pi_N f)(x) := \sum_{|n|\le N} \hat{f}(n) e^{inx}.$$

- (FNLS) is a Hamiltonian PDE with

$$H_N(u_N) = \frac{1}{2}\int_{\mathbb{T}} |\partial_x^2 u_N|^2 dx + \frac{1}{6}\int_{\mathbb{T}} |u_N|^6 dx.$$

- $u_0 = u_0^{\omega} = \sum_{k\in\mathbb{Z}} \frac{g_k(\omega)}{\langle k\rangle} e^{ikx} \Longrightarrow u_0 \in H_x^{\frac{1}{2}-}$ a.s.
- Denote by $u_{\text{low}} := \Pi_N u_N$, $u_{\text{high}} := \Pi_N^{\perp} u_N$, then

$$u_N = u_{\text{low}} + u_{\text{high}},$$

$$\widehat{u_{\text{high}}}(t,k) = e^{-itk^2}\widehat{u_0}(k), \quad |k| > N \quad \Longrightarrow \text{GWP for } u_{\text{high}},$$

  $\widehat{u_{\text{low}}}(t,k)$ satisfies ODEs. Also, $\|u_{\text{low}}\|_{L^2}$ conserved $\Longrightarrow$ GWP for u_{low}.
- $d\rho_N$: low frequency Gaussian measure on $\mathbb{C}^{2N+1} \simeq \mathbb{R}^{2(2N+1)}$.

$$d\rho_N := Z_N^{-1} e^{-\frac{1}{2}\|\Pi_N u\|_{H^1}^2} d(\Pi_N u) = Z_N^{-1} \prod_{|k|\le N} e^{-\frac{1}{2}\langle k\rangle^2 |\hat{u}(k)|^2} d\hat{u}(k).$$

- $\mathrm{d}\rho_N^\perp$: high frequency Gaussian measure.

$$\mathrm{d}\rho_N^\perp := \tilde{Z}_N^{-1} \mathrm{e}^{-\frac{1}{2}\|\Pi_N^\perp u\|_{H^1}^2} \mathrm{d}(\Pi_N^\perp u).$$

- $\mathrm{d}\rho_N^\perp$ is invariant under (FNLS).

$$\Pi_N^\perp u_0(x) = \sum_{|k|>N} \frac{g_k(\omega)}{\langle k \rangle} \mathrm{e}^{\mathrm{i}kx} \Longrightarrow u_{\mathrm{high}}(t,x) = \sum_{|k|>N} \frac{\mathrm{e}^{-\mathrm{i}tk^2} g_n(\omega)}{\langle k \rangle} \mathrm{e}^{\mathrm{i}kx}.$$

  Since g_n is invariant under a rotation, $\mathrm{d}\rho_N^\perp$ is invariant.
- $\mathrm{d}\rho = \mathrm{d}\rho_N \otimes \mathrm{d}\rho_N^\perp$.
- $\mathrm{d}\mu_{N,\mathrm{low}} = Z_N^{-1} \mathrm{e}^{-H_N(u_{\mathrm{low}})} \mathrm{d}u_{\mathrm{low}}$ is invariant.
- $\mathrm{d}\mu_N := \mathrm{d}\mu_{N,\mathrm{low}} \otimes \mathrm{d}\rho_N^\perp$ is invariant under (FNLS).

Lemma (Tail estimate)

Let $\sigma < 1/2$. Then,

$$\mathrm{d}\rho(\|u\|_{H^\sigma} > R) \leq C\mathrm{e}^{-cR^2} \quad \forall R > 0.$$

Remark 1. This follows from Fernique's integrability theorem

$$\int_B \mathrm{e}^{c\|u\|_B^2}\mathrm{d}\rho(u) < \infty \text{ for some } c > 0.$$

Proof. By Chebyshev's inequality ($\rho(|f| > R) \leq R^{-2}\mathbb{E}[|f|^2]$),

$$\begin{aligned} \mathrm{e}^{cR^2}\mathrm{d}\rho(\|u\|_{H^\sigma} > R) &\leq \int_{H^\sigma} \mathrm{e}^{c\|u\|_{H_x^\sigma}^2}\mathrm{d}\rho(u) \\ &= \prod_{k\in\mathbb{Z}} \int_{\mathbb{C}} \mathrm{e}^{c\langle k\rangle^{2\sigma-2}|g_k|^2}\mathrm{e}^{-\frac{1}{2}|g_k|^2}\frac{\mathrm{d}g_k}{2\pi} \end{aligned}$$

$$
\begin{aligned}
&= \prod_{k\in\mathbb{Z}} \frac{1}{1-2c\langle k\rangle^{2\sigma-2}} \\
&= \prod_{k\in\mathbb{Z}} \left(1+\frac{2c\langle k\rangle^{2\sigma-2}}{1-2c\langle k\rangle^{2\sigma-2}}\right) < \infty.
\end{aligned}
$$

This proof is completed. □

Key Lemma (Bourgain '94 CMP)

For any $T>0$, $\varepsilon>0$, there exists $\Omega_N=\Omega_N(T,\varepsilon)$ s.t.

i. $\mu_N(\Omega_N^c)<\varepsilon$;

ii. For $u_0\in\Omega_N$, the solution u_N to (FNLS) with $u_N|_{t=0}=u_0$ satisfies

$$
\|u_N(t)\|_{H^s} \lesssim \left(\log\frac{T}{\varepsilon}\right)^{1/2}, \quad |t|\le T.
$$

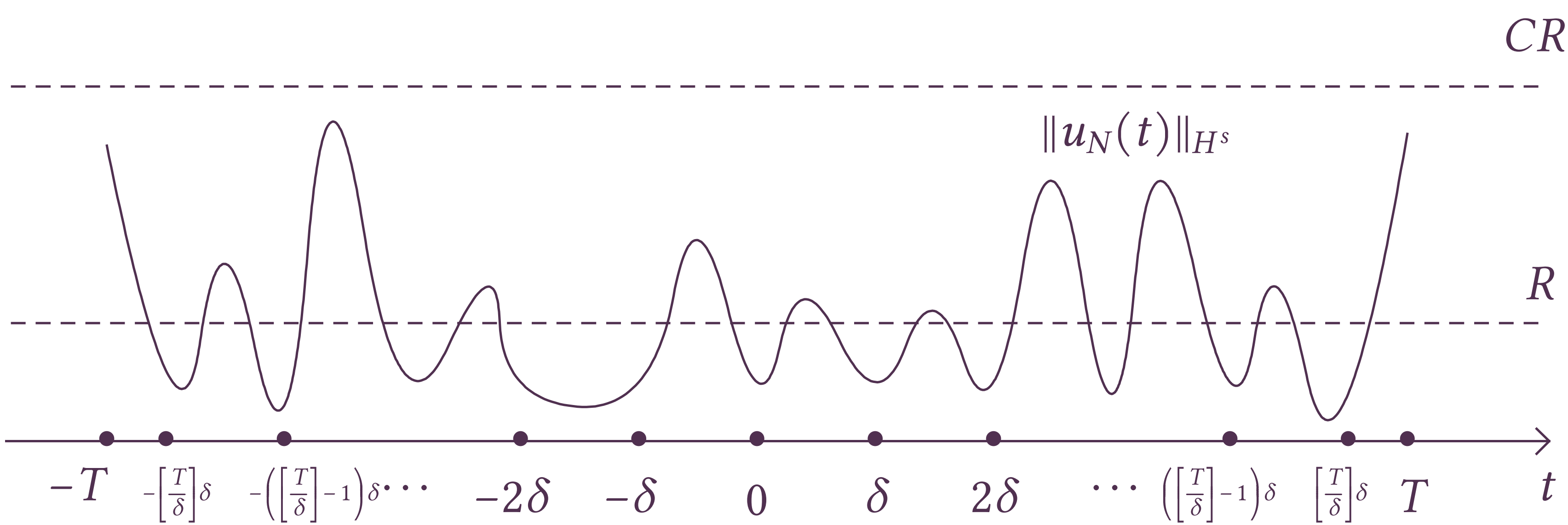
(implicit constant is independent of N).

Proof. By local theory,

$$\|u_0\|_{H^s} \le R \implies \|u_N(t)\|_{H^s} \le CR \text{ for } |t| \le \delta \sim R^{-4} \text{ (independent of } N).$$

Let $\Phi_N(t)\colon u_0 \mapsto u(t)$ be the solution map for (FNLS) and let

$$\Omega_N = \bigcap_{j=-\left[\frac{T}{\delta}\right]}^{\left[\frac{T}{\delta}\right]} \Phi_N(j\delta)\Big(\underbrace{\{\|u_0\|_{H^s} \le R\}}_{B_R}\Big).$$



Then

$$
\begin{aligned}
\mu_N(\Omega_N^c) &\le \sum_{j=-\left[\frac{T}{\delta}\right]}^{\left[\frac{T}{\delta}\right]} \mu_N\Big(\Phi_N\cancel{(}j\delta)\big(\underset{\substack{\uparrow\\ \text{use uniqueness}}}{B_R^c}\big)\Big) \\
&\sim \frac{T}{\delta}\,\mu_N(B_R^c) \\
&\lesssim T R^4 \mathrm{e}^{-cR^2} < \varepsilon.
\end{aligned}
$$

$$
\boxed{\begin{aligned}\mu_N(B_R^c) &\overset{\text{C-S}}{\le} \overbrace{\Big\| \mathrm{e}^{-\frac{1}{p}\|u_N\|_{L_x^p}^p} \Big\|_{L^2(\mathrm{d}\rho)}}^{\le C} \\ &\quad \times \left(\mathrm{d}\rho(B_R^c)\right)^{\frac{1}{2}} \le \mathrm{e}^{-cR^2}\end{aligned}} \longrightarrow
$$

By choosing $R \sim \left(\log \frac{T}{\varepsilon}\right)^{1/2}$ and by local theory,

$$\|u_N(t)\|_{H^s} \leq CR \sim \left(\log \frac{T}{\varepsilon}\right)^{1/2}, \quad \forall\, |t| \leq T.$$

□

Approximation lemma

Let $s < 1/2$, $u_0 \in H^s$ *with* $\|u_0\|_{H^s} \leq R$. *Suppose solution* u_N *to (FNLS) with* $u_N|_{t=0} = u_0$ *satisfies*

$$\|u_N(t)\|_{H^s} \leq R, \quad |t| \leq T.$$

Then, $\exists!$ *solution* u *to (NLS) on* $[-T, T]$ *with* $u|_{t=0} = u_0$. *Moreover, we have*

$$\|u(t) - \Pi_N u_N(t)\|_{H^{s_1}} \leq C_0 \mathrm{e}^{C_1(1+R)^{C_2} T} R \underbrace{N^{s_1 - s}}_{\to 0}$$

for $s_1 < s$ *(for sufficient large* $N \in \mathbb{N}$ *).*

Proof. (FNLS) & (NLS) with $u_N|_{t=0} = u|_{t=0} = u_0$ are locally well-posed on $[-\delta, \delta]$,

$\delta = ((R+1)^{-4}/8)^{1/\theta} \sim (R+1)^{-\gamma}$, independent of N. Let $v_N = \Pi_{\leq N} u_N$.

$$\|u - v_N\|_{X^{s_1,b}([0,\delta])} \lesssim \underbrace{\|u_0 - \Pi_N u_0\|_{H^{s_1}}}_{\leq N^{s_1-s}\|u_0\|_{H^s} \leq N^{s_1-s}R} + \underbrace{\left\|\eta\left(\frac{t}{\delta}\right)\int_0^t e^{i\Delta(t-t')}(|u|^4 u - |v_N|^4 v_N)(t')\,dt'\right\|_{X^{s_1,b}([0,\delta])}}_{(*)},$$

where

$$\begin{aligned}
(*) \ &\leq \left\|\eta\left(\frac{t}{\delta}\right)\int_0^t e^{i\Delta(t-t')}\Pi_N^{\perp}(|u|^4 u - |v_N|^4 v_N)(t')\,dt'\right\|_{X^{s_1,b}([0,\delta])} \\
&\quad + \left\|\eta\left(\frac{t}{\delta}\right)\int_0^t e^{i\Delta(t-t')}\Pi_N(|u|^4 u - |v_N|^4 v_N)(t')\,dt'\right\|_{X^{s_1,b}([0,\delta])} \\
&\lesssim \delta^{\theta}\|\Pi_N^{\perp}(|u|^4 u - |v_N|^4 v_N)\|_{X^{s_1,b-1+\theta}([0,\delta])} + \delta^{\theta}\|\Pi_N(|u|^4 u - |v_N|^4 v_N)\|_{X^{s_1,b-1+\theta}([0,\delta])} \\
&\lesssim \delta^{\theta} N^{s_1-s} R^5 + \delta^{\theta} R^4 \|u - v_N\|_{X^{s_1,b-1+\theta}([0,\delta])}.
\end{aligned}$$

Therefore,

$$\|u - v_N\|_{X^{s_1,b}([0,\delta])} \lesssim N^{s_1-s} R$$

$$\Longrightarrow \|u(\delta)\|_{H^{s_1}} \leq \|v_N(\delta)\|_{H^s} + C N^{s_1-s} R \leq R + o(1).$$

Then u **exists** on $[\delta, \delta + ((R + o(1))^{-4}/8)^{1/\theta}] \supseteq [\delta, 2\delta]$ in $X^{s_1,b}$. Now, iterate the argument for $\sim T/\delta$ many times.

$$\|u - v_N\|_{X^{s_1,b}([\delta,2\delta])} \lesssim \underbrace{\|u(\delta) - v_N(\delta)\|_{H^{s_1}}}_{\lesssim N^{s_1-s}R} + \delta^\theta N^{s_1-s}R^5 + \delta^\theta R^4 \|u - v_N\|_{X^{s_1,b-1+\theta}([\delta,2\delta])}.$$

$$\|u - v_N\|_{X^{s_1,b}([0,T])} \lesssim \mathrm{e}^{c\frac{T}{\delta}} N^{s_1-s}R \sim \mathrm{e}^{cR^{-\gamma}T} N^{s_1-s}R.$$

The proof is completed. □

Lemma 1

For $s < \frac{1}{2}$. Given $T, \varepsilon > 0$, there exists $\Omega_{T,\varepsilon} \subset H^s(\mathbb{T})$ s.t. the following holds.

i. $\mu(\Omega_{T,\varepsilon}^c) < \varepsilon$;

ii. For $u_0 \in \Omega_{T,\varepsilon}$, $\exists!$ solution u to (NLS) on $[-T, T]$ s.t.

$$\|u(t)\|_{H^{s_1}} \lesssim \left(\log \frac{T}{\varepsilon}\right)^{1/2}, \quad |t| \le T$$

for $s_1 < s$.

Proof. Let $\Omega_N(T,\varepsilon)$ be as in Key Lemma. Then $\|\Phi_N(t)(u_0)\|_{H^s} \leq CR$ for $|t| \leq T$ and $u_0 \in \Omega_N$. By Approximation lemma, $\exists!$ solution u to (NLS) on $[-T,T]$ in $X^{s_1,b}$, and $\exists\, N_1 \gg 1$ such that

$$\|u(t) - u_N(t)\|_{H^{s_1}} \ll 1, \quad |t| \leq T$$

for $N \geq N_1$. Therefore,

$$\|u(t)\|_{H^{s_1}} \lesssim R \sim \left(\log \frac{T}{\varepsilon}\right)^{1/2}, \quad |t| \leq T.$$

Also,

$$\mu(\Omega_N^c) \overset{\text{C-S}}{\leq} \underbrace{\left\|\mathrm{e}^{-\frac{1}{6}\|u\|_{L_x^6}^6}\right\|_{L^2(\mathrm{d}\rho)}}_{\leq C} (\mathrm{d}\rho(B_R^c))^{\frac{1}{2}} \lesssim \mathrm{e}^{-cR^2} \sim \varepsilon.$$

□

Theorem (Bourgain '94 CMP)

a.s. (NLS) globally well-posed almost surely with respect to the Gibbs measure μ.

Proof. Let $T_j = 2^j$, $\varepsilon_{i,j} = \frac{1}{2^{i+j}}$. Let

$$\Omega^{(i)} := \bigcap_j \Omega_{T_j, \varepsilon_{i,j}}.$$

Then by Lemma ii, we have

i. $\mu((\Omega^{(i)})^c) \leq \sum_j \frac{1}{2^{i+j}} = \frac{1}{2^i}$;

ii. if $u_0 \in \Omega^{(i)}$, then the solution u to (NLS) with $u|_{t=0} = u_0$ exists on time interval $[-2^j, 2^j]$ for all j (= on $\mathbb{R}$).

$$\Sigma := \bigcup_i \Omega^{(i)}.$$

Then

$$\mu(\Sigma^c) = \inf_i \mu((\Omega^{(i)})^c) \leq \inf_i \frac{1}{2^i} = 0.$$ □

Theorem (Bourgain '94 CMP)

The Gibbs measure μ is invariant under the flow of (NLS).

Proof. By time reversibility of $\Phi(t)$, it suffices to show

$$\mu(A) \leq \mu(\Phi(t)A) \tag{4}$$

for all measurable set $A \subset H^s$ and $t \in \mathbb{R}$ (if u solves (NLS), so does $\bar{u}(-t)$).

By inner regularity

$$\mu(A) = \sup_{F \subset A \text{ closed in } H^s} \mu(F),$$

i.e. $\exists$ closed sets $\{F_n\}$ in H^s s.t.

$$F_n \subset A \text{ and } \mu(A) = \lim_{n \to \infty} \mu(F_n).$$

Let us claim that it suffices to prove (4) for closed sets. It is because

$$\begin{aligned} \mu(A) &= \lim_{n \to \infty} \mu(F_n) \\ &\leq \overline{\lim_{n \to \infty}} \mu(\Phi(t) F_n) \\ \text{uniqueness} \longrightarrow \quad &\leq \mu(\Phi(t) A). \end{aligned}$$

Given a closed set $F \subset H^s$, let

$$K_n = \{u \in F; \|u\|_{H^\sigma} \leq n\}, \quad s < \sigma < \frac{1}{2}.$$

Then by Rellich's lemma, K_n is compact in H^s. Then it suffices to prove (4) for compact sets. In fact, by tail estimate, we have

$$\mu(F) = \lim_{n \to \infty} \mu(K_n).$$

Now, let K be a compact set in H^s. Unsing $\mu_N \rightharpoonup \mu$ and Portmanteau theorem, we have

$$\mu(\Phi(t)K + \overline{B_\varepsilon}) \geq \overline{\lim}\, \mu_N(\Phi(t)K + \overline{B_\varepsilon}). \tag{5}$$

Fix $0 < t \ll 1$. Then,

$$\begin{aligned} \Phi_N(t)(K + B_\delta) &\overset{\text{LWP}}{\subset} \Phi_N(t)K + B_{\varepsilon/2} \\ \text{Approximation lemma} \longrightarrow &\subset \Phi(t)K + B_\varepsilon \end{aligned}$$

By invariance of μ_N,

$$\mu_N(K + B_\delta) \le \mu_N(\Phi_N(t)K + B_\varepsilon). \tag{6}$$

Therefore,

$$\begin{array}{rcl} \mu(K) & \le & \mu(K + B_\delta) \\ & \le & \varliminf \mu_N(K + B_\delta) \\ & \overset{(6)}{\le} & \varliminf \mu_N(\Phi(t)K + B_\varepsilon) \\ & \le & \varlimsup \mu_N(\Phi(t)K + \overline{B_\varepsilon}) \\ & \overset{(5)}{\le} & \mu(\Phi(t)K + \overline{B_\varepsilon}). \end{array}$$

Let $\varepsilon \longrightarrow 0$, we get $\mu(K) \le \mu(\Phi(t)K)$ for $0 < t \ll 1$. $\square$

Cubic NLS on $\mathbb{T}^2$

Consider

$$\mathrm{i}\partial_t u + \Delta u = |u|^2 u, \quad (t,x) \in \mathbb{T}^2. \tag{CNLS}$$

Hamiltonian:

$$H(u) = \frac{1}{2}\int_{\mathbb{T}^2} |\nabla_x u|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{T}^2} |u|^4 \mathrm{d}x.$$

Gibbs measure:

$$\mathrm{d}\mu := \mathrm{e}^{-H(u)}\mathrm{d}u = \mathrm{e}^{-\frac{1}{4}\int_{\mathbb{T}^2}|u|^4\mathrm{d}x}\mathrm{d}\rho,$$

where $\mathrm{d}\rho$ is the Gaussian measure given by

$$\mathrm{d}\rho = \mathrm{e}^{-\frac{1}{2}\|\nabla_x u\|^2_{\dot{H}^1}}\mathrm{d}u = \prod_{k \in \mathbb{Z}^2 \setminus \{0\}} \mathrm{e}^{-\frac{1}{2}|k|^2|g_k|^2}\mathrm{d}g_k.$$

The variance for k-th mode is $|k|^{-2}$.

Consider

$$\begin{aligned}\gamma\colon\ \Omega &\longrightarrow \mathcal{S}'(\mathbb{T}^2),\\ \omega &\longmapsto u_0^\omega := \sum_{k\in\mathbb{Z}^2\setminus\{0\}} \frac{g_k(\omega)}{|k|} \mathrm{e}^{\mathrm{i}k\cdot x},\end{aligned}$$

where $\{g_k\}_{k\in\mathbb{Z}^2\setminus\{0\}}$ is a sequence of standard complex-valued Gaussian r.v. i.i.d. on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. The k-th mode $\frac{g_k(\omega)}{|k|}$ is a Gaussian random variable with variance:

$$\mathbb{E}\left[\left|\frac{g_k(\omega)}{|k|}\right|^2\right] = |k|^{-2}.$$

In reality,

$$\mathrm{d}\rho = \mathrm{Law}(\gamma) := \mathbb{P}\circ\gamma^{-1}.$$

Also $\mathrm{d}\mu \ll \mathrm{d}\rho$ & $\mathrm{d}\rho \ll \mathrm{d}\mu$, which implies

$$\text{Data in } \operatorname{supp}\mu \iff u(0) = u_0^\omega,\ \omega\in\Omega.$$

Let us find "s" s.t. $u_0^\omega \in H^s(\mathbb{T}^2)$ $\mathbb{P}$-a.e. (= a.s.).

$$\begin{aligned}
\mathbb{E}\big[\||\nabla|^s u_0^\omega\|_{L_x^2}^2\big] &= \sum_{k\in\mathbb{Z}^2} \mathbb{E}\left[\frac{|g_k|^2}{|k|^{2-2s}}\right] \\
&= \sum_{k\in\mathbb{Z}^2} \frac{1}{|k|^{2-2s}} < \infty,
\end{aligned}$$

if

$$2-2s > 2 \iff s < 0.$$

Therefore, $u_0^\omega \in L^2(\Omega; H^{0-}(\mathbb{T}^2))$, implying

$$\operatorname{supp} d\mu \subseteq H^{0-}(\mathbb{T}^2),$$

which is below L^2, the scaling of (CNLS). Therefore, Gibbs dynamics for (CNLS) is **deterministically super-critical**.

Let us consider

$$i\partial_t u + \Delta u = |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^d$$

in the $H^s(\mathbb{T}^d)$ spaces.

Question: For what s do we have Local wellposedness (LWP)?

Scaling argument for the threshold of s:

Suppose we have initial data.

$$u(0) = f = \sum_{|k|\sim N} N^{-\alpha} e^{ik\cdot x}, \quad \alpha = s + \frac{d}{2},$$

then $\|f\|_{H^s} \sim 1$.

By Duhamel's formulation

$$u(t) = e^{it\Delta} f - i\int_0^t e^{i(t-s)\Delta}(|u(s)|^{p-1}|u(s)|)\,ds.$$

$$\|e^{it\Delta} f\|_{H^s} = \|f\|_{H^s} \sim 1.$$

The Picard 2nd iterate

$$u^1(t) = -\mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}(t-s)\Delta}(|\mathrm{e}^{\mathrm{i}s\Delta}f|^{p-1}\mathrm{e}^{\mathrm{i}s\Delta}f)\mathrm{d}s$$

$$\hat{u}^1(t,k) = \mathrm{e}^{-\mathrm{i}t|k|^2}\sum_{k=k_1-k_2+\cdots+k_p}\int_0^t \mathrm{e}^{\mathrm{i}s\Omega}\mathrm{d}s\cdot N^{-p\alpha},$$

where $\Omega = |k|^2 - |k_1|^2 + \cdots - |k_p|^2$ is the “resonance factor”.

$$\begin{aligned}\Longrightarrow \hat{u}^1(t,k) \ &\sim \ N^{-p\alpha}\sum_{k_1-k_2+\cdots+k_p, |k_j|\sim N}\frac{1}{\langle\Omega\rangle}\\ &\sim \ N^{-p\alpha}\sum_{k_1,\cdots,k_p} h^b_{kk_1\cdots k_p} \sim N^{-p\alpha}\cdot N^{-pd-d-2},\end{aligned}$$

where the base tensor h^b is defined by

$$h^b_{kk_1\cdots k_p} := \mathbf{1}_{k_1-k_2+k_3-\cdots+k_p=k, \Omega=|k|^2-|k_1|^2+\cdots+|k_p|^2=\text{const.}}$$

We want

$$\|u^1(t)\|_{H^s} \lesssim 1.$$

$$\Longleftrightarrow \Bigg[\sum_{|k|\sim N} (N^s N^{-p\alpha+pd-d-2})^2\Bigg]^{1/2} \lesssim 1$$

$$\Longleftrightarrow -p\alpha + pd - d - 2 + s + \frac{d}{2} \leq 0$$

$$\Longleftrightarrow s \geq \underbrace{\frac{d}{2} - \frac{2}{p-1}}_{=:s_{\mathrm{cr}}}.$$

s_{cr} stands for the (deterministic) scaling critical exponent for NLS.

This "s_{cr}" matches the threshold derived by

$$\left\|\lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)\right\|_{\dot{H}^{s_{\mathrm{cr}}}(\mathbb{R}^d)} = \|u\|_{\dot{H}^{s_{\mathrm{cr}}}(\mathbb{R}^d)}.$$

Theorem (Bourgain '93, Bourgain-Demeter '15)

Assume $s_{\mathrm{cr}} \geq 0$. Then NLS is LWP in H^s if $s > s_{\mathrm{cr}}$, and is ill-posed if $s < s_{\mathrm{cr}}$.

When $s = s_{\mathrm{cr}} = 1$, LWP (Herr-Tataru-Tzvetkov '10).

Renormalisation

Apart from the deterministic supercriticality of (CNLS), the nonlinearity is ill-defined as it is, because u is not a function, necessiating the need of renomalisation:

$$\mathrm{i}\partial_t u + \Delta u = :|u|^2 u: \quad (t,x) \in \mathbb{R} \times \mathbb{T}^2,$$

with

$$:|u|^2 u := |u|^2 u - 2u \underbrace{\mathbb{E}\Big[\int_{\mathbb{T}^2} |u|^2 \mathrm{d}x\Big]}_{=\sigma_N},$$

understood as the limit of

$$:|u_N|^2 u_N := |u_N|^2 u_N - 2u_N \mathbb{E}\Big[\int_{\mathbb{T}^2} |u_N|^2 \mathrm{d}x\Big].$$

Redefine the Gibbs measure

$$\mathrm{d}\mu := \mathrm{e}^{-\frac{1}{4}\int_{\mathbb{T}^2} :|u|^4: \mathrm{d}x} \mathrm{d}\rho,$$

with

$$:|u|^4: = |u|^4 - 4\sigma_N|u|^2 + 2\sigma_N^2.$$

It is essential that

$$\widehat{:|u_N|^2 u_N:}(k) = \sum_{\substack{k_1-k_2+k_3=k \\ k_2 \notin \{k_1,k_3\} \\ |k_j| \leq N}} \hat{u}(k_1)\overline{\hat{u}(k_2)}\hat{u}(k_3).$$

Finite-dimensional approximation

Consider the frequency truncated (CNLS):

$$\begin{cases} \mathrm{i}\partial_t u_N + \Delta u_N = \Pi_N(:|u_N|^2 u_N:), \\ u_N(0) = \Pi_N u_0^\omega. \end{cases} \quad (t,x) \in \mathbb{R} \times \mathbb{T}. \tag{FNLS1}$$

Hamiltonian:

$$H_N(u_N) = \frac{1}{2}\int_{\mathbb{T}^2} |\nabla_x u_N|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{T}^2} :|u_N|^4: \mathrm{d}x.$$

Gibbs measure:

$$\mathrm{d}\mu_N := \mathrm{e}^{-\frac{1}{4}\int_{\mathbb{T}^2} :|u_N|^4: \mathrm{d}x} \mathrm{d}\rho.$$

Duhamel formulation:

$$u_N(t) = \mathrm{e}^{\mathrm{i}t\Delta}\Pi_N u_0^\omega - \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}(t-t')\Delta}\Pi_N(:|u_N|^2 u_N:)(t')\mathrm{d}t'.$$

Interaction picture

Denote by

$$v_N(t) = \mathrm{e}^{-\mathrm{i}t\Delta} u_N(t).$$

Then

$$v_N(t) = \Pi_N u_0^\omega - \mathrm{i}\int_0^t \mathrm{e}^{-\mathrm{i}t'\Delta}\Pi_N(:|\mathrm{e}^{\mathrm{i}t'\Delta} v_N|^2 \mathrm{e}^{\mathrm{i}t'\Delta} v_N:)(t')\mathrm{d}t'.$$

Let us save the notation v_N, and rewrite v_N as u_N:

$$u_N(t) = \Pi_N u_0^\omega - \mathrm{i}\int_0^t \mathrm{e}^{-\mathrm{i}t'\Delta}\Pi_N(:|\mathrm{e}^{\mathrm{i}t'\Delta} u_N|^2 \mathrm{e}^{\mathrm{i}t'\Delta} u_N:)(t')\mathrm{d}t'.$$

Or on the Fourier side ($|k| \leq N$).

$$(u_N)_k(t) = (u_0^\omega)_k - \mathrm{i}\sum_{\substack{k-k_1+k_2-k_3=0 \\ k_2 \notin \{k_1,k_3\}}} \int_0^t \mathrm{e}^{\mathrm{i}t'\Omega}(u_N)_{k_1}\overline{(u_N)_{k_2}}(u_N)_{k_3}(t')\mathrm{d}t',$$

where

$$\Omega = |k|^2 - |k_1|^2 + |k_2|^2 - |k_3|^2.$$

Theorem (Bourgain '96 CMP)

Invariance of Gibbs measure for (CNLS).

Decompose y_N of as

$$u_N(t) = \underbrace{\Pi_N u_0^\omega}_{=:F_N} + z_N(t).$$

$$F_N = \sum_{\langle k\rangle \le N} \frac{g_k(\omega)}{|k|} e^{ik\cdot x}.$$

Hoping the regularity of z_N is better than u_N (nonlinear smoothing). We want to put z_N into H^γ.

Then

$$\begin{aligned} z_N(t) &= -i\eta(t)\int_0^t \eta(t')e^{it'\Omega}\Pi_N(: u_N\overline{u_N}u_N :)(t')dt' \\ &= -i\sum_{N_1,N_2,N_3\le N} \eta(t)\int_0^t \eta(t')e^{it'\Omega}\Pi_N(:\Delta_{N_1}u_N\Delta_{N_2}\overline{u_N}\Delta_{N_3}u_N:)(t')dt' \end{aligned}$$

On the Fourier side,

$$(z_N)_k(t) = -\mathrm{i}\sum_{N_1,N_2,N_3\le N}\ \sum_{\substack{k-k_1+k_2-k_3=0\\ k_2\notin\{k_1,k_3\}}} \eta(t)\int_0^t \eta(t')\mathrm{e}^{\mathrm{i}t'\Omega}(\Delta_{N_1}u_N)_{k_1}\overline{(\Delta_{N_2}u_N)_{k_2}}(\Delta_{N_3}u_N)_{k_3}(t')\mathrm{d}t'.$$

- Type (G): $\Delta_{N_j}F_N$;
- Type (D): $\Delta_{N_j}z_N$.

Then we have following terms in the RHS of (7): $(\Delta_{N_1}u_N, \Delta_{N_2}\overline{u_N}, \Delta_{N_3}u_N)$ are of types

- (G, G, G);
- (G, G, D);
- (G, D, G);
- (G, D, D);
- ~~(D, G, G)~~;
- (D, G, D);
- ~~(D, D, G)~~;
- (D, D, D).

Lemma 2 (Deng-Nahmod-Yue CMP)

$$\mathcal{I}_\eta v(t) = \eta(t)\int_0^t \eta(t')v(t')\mathrm{d}t'.$$

$$\widehat{\mathcal{I}_\eta v}(\tau) = \int_{\mathbb{R}} \mathcal{K}(\tau,\tau')\hat{v}(\tau')\mathrm{d}\tau'.$$

Then the kernel $\mathcal{K}$ *satisfies*

$$|\mathcal{K}| + |\partial_{\tau,\tau'}\mathcal{K}| \lesssim \left(\frac{1}{\langle\tau\rangle^3} + \frac{1}{\langle\tau-\tau'\rangle^3}\right)\frac{1}{\langle\tau'\rangle} \lesssim \frac{1}{\langle\tau\rangle\langle\tau-\tau'\rangle}.$$

$$\begin{aligned}(\widehat{z_N})_k(\tau) &= -\mathrm{i}\sum_{N_1,N_2,N_3\le N}\ \sum_{\substack{k-k_1+k_2-k_3=0\\ k_2\notin\{k_1,k_3\}}} \int_{\mathbb{R}^3}\mathcal{K}(\tau,\Omega+\tau_1-\tau_2+\tau_3)\\ &\qquad \times(\Delta_{N_1}u_N)_{k_1}(\tau_1)\overline{(\Delta_{N_2}u_N)_{k_2}}(\tau_2)(\Delta_{N_3}u_N)_{k_3}(\tau_3)\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\tau_3\\ &= -\mathrm{i}\sum_{N_1,N_2,N_3\le N}\ \sum_{k_1,k_2,k_3}\ \sum_{m}\int_{\mathbb{R}^3}\mathcal{K}(\tau,m+\tau_1-\tau_2+\tau_3)\times\mathrm{T}^m_{kk_1k_2k_3}\\ &\qquad \times(\Delta_{N_1}u_N)_{k_1}(\tau_1)\overline{(\Delta_{N_2}u_N)_{k_2}}(\tau_2)(\Delta_{N_3}u_N)_{k_3}(\tau_3)\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\tau_3,\end{aligned} \quad ($$

where

$$\mathrm{T}^m_{kk_1k_2k_3} := \mathbf{1}_{k-k_1+k_2-k_3=0,|k|^2-|k_1|^2+|k_2|^2-|k_3|^2=m,k_2\notin\{k_1,k_3\}}.$$

Let us estimate $\|(z_N)_k\|_{L^2_t\ell^2_k}$ case-by-case.

<u>Case (G, G, G)</u>

$$\begin{aligned}
\|(z_N)_k\|_{H^{1-b}_t\ell^2_k} &= \Big\|\langle\tau\rangle^{1-b}\sum_{N_1,N_2,N_3\le N}\sum_{k_1,k_2,k_3}\sum_m\int_{\mathbb{R}^3}\mathcal{K}(\tau,m+\tau_1-\tau_2+\tau_3)\times\mathrm{T}^m_{kk_1k_2k_3}\\
&\qquad\times(\Delta_{N_1}F_N)_{k_1}(\tau_1)\overline{(\Delta_{N_2}F_N)_{k_2}}(\tau_2)(\Delta_{N_3}F_N)_{k_3}(\tau_3)\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\tau_3\Big\|_{L^2_\tau\ell^2_k}\\
&\lesssim \Big\|\langle\tau\rangle^{-b}\sum_{N_1,N_2,N_3\le N}\sum_{k_1,k_2,k_3}\int_{\mathbb{R}^3}\underbrace{\sum_m\langle\tau-m-\tau_1+\tau_2-\tau_3\rangle^{-1}}_{\lesssim 1+\log N}\times\mathrm{T}^m_{kk_1k_2k_3}\\
&\qquad\times\langle\tau_1\rangle^{-b}\langle\tau_2\rangle^{-b}\langle\tau_3\rangle^{-b}\prod_{i=1}^3\langle\tau_i\rangle^b(\Delta_{N_j}F_N)^{\zeta_i}_{k_i}(\tau_i)\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\tau_3\Big\|_{L^2_\tau\ell^2_k}.
\end{aligned}$$

Using Cauchy-Schwarz inequality in τ_1, τ_2, and τ_3, we continue with

$$\lesssim \sum_{N_1,N_2,N_3\le N} N^{\varepsilon}\Big\|\sum_{k_1,k_2,k_3} \mathrm{T}^m_{kk_1k_2k_3}\prod_{i=1}^{3}\langle\tau_i\rangle^b(\Delta_{N_j}F_N)^{\zeta_i}_{k_i}(\tau_i)\Big\|_{L^2_{\tau_1\tau_2\tau_3}\ell^2_k}.$$

By using meshing argument (Deng-Nahmod Yue '24 Annals), we can reduce to fixing N^κ many choices of τ_i, throwing N^κ many expectional sets when applying Lemma 3.

$$\begin{aligned}
&\sum_{N_1,N_2,N_3\le N} N^{\varepsilon}N_1^{-1}N_2^{-1}N_3^{-1}\Big\|\sum_{k_1,k_2,k_3}\mathrm{T}^m_{kk_1k_2k_3}g_{k_1}\overline{g_{k_2}}g_{k_3}\Big\|_{\ell^2_k}\\
\lesssim &\sum_{N_1,N_2,N_3\le N} N^{\varepsilon+\theta}N_1^{-1}N_2^{-1}N_3^{-1}\|\mathrm{T}^m_{kk_1k_2k_3}\|_{\ell^2_{kk_1k_2k_3}},
\end{aligned}$$

where

$$\begin{aligned}
&\|\mathrm{T}^m_{kk_1k_2k_3}\|^2_{\ell^2_{kk_1k_2k_3}}\\
&\quad\lesssim \#\{(k,k_1,k_2,k_3);\, k-k_1+k_2-k_3=0,\\
&\qquad\qquad |k|^2-|k_1|^2+|k_2|^2-|k_3|^2=m,\, k_2\notin\{k_1,k_3\},\\
&\qquad\qquad |k|,|k_j|\le N\}\\
&\quad\lesssim \min\left(N_1^2N_3^2N_2^\theta,\, N^2N_2^2(N_1\wedge N_3)^\theta,\, N_2^2(N_1N_3)^{1+\theta},\, N_1^2(NN_2)^{1+\theta}\right).
\end{aligned}$$

Proof of counting: The choices to fix k_1 and k_3 is $N_1^2 N_3^2$. After fixing k_1 and k_3, it remains to count (k, k_2). It is equivalent to count k_2 satisfying the restriction $|k_2|^2 + |k_2 + c_1|^2 = c_2^2$. By using Counting Theorem (Bombieri-Pila '89 Duke), we know that there are at most N_2^θ choices of k_2.

Therefore, ($N_{\max} \sim N$, because of $k_1 - k_2 + k_3 = k$).

$$\begin{aligned}
\|(z_N)_k\|_{H_t^{1-b}\ell_k^2} &\lesssim \sum_{N_1, N_2, N_3 \leq N} N^{\varepsilon+\theta} N_1^{-1} N_2^{-1} N_3^{-1} N_{\min} (N_{\text{med}} N)^{\frac{1}{2}+\frac{\theta}{2}} \\
&\lesssim \sum_{N_1, N_2, N_3 \leq N} N^{\varepsilon+\theta+\frac{1}{2}+\frac{\theta}{2}} N_{\max}^{-1} N_{\text{med}}^{-\frac{1}{2}+\varepsilon+2\theta} \\
&\lesssim N^{-\frac{1}{2}+}.
\end{aligned}$$

This case can be put into $H^{\frac{1}{2}-}$.

Case (G, G, D):

For this case, we need to bound

$$N^{\varepsilon} N_1^{-1} N_2^{-1} \Big\| \sum_{k_1, k_2, k_3} \mathrm{T}^m_{k k_1 k_2 k_3} g_{k_1} \overline{g_{k_2}} (\Delta_{N_3} z_N)_{k_3} \Big\|_k$$

$$
\begin{aligned}
&\lesssim N^{\varepsilon} N_1^{-1} N_2^{-1} \|(\Delta_{N_3} z_N)_{k_3}\|_{k_3} \Big\| \sum_{k_1,k_2} \mathrm{T}^m_{kk_1k_2k_3} g_{k_1} \overline{g_{k_2}} \Big\|_{k \to k_3} \\
&\lesssim N^{\varepsilon} N_1^{-1} N_2^{-1} N_3^{-\gamma} \max(\|\mathrm{T}\|_{kk_1 \to k_2k_3}, \|\mathrm{T}\|_{kk_2 \to k_1k_3}, \|\mathrm{T}\|_{kk_1k_2 \to k_3}, \|\mathrm{T}\|_{k \to k_1k_2k_3}) \\
&\lesssim N^{\varepsilon} (N_1N_2)^{-\frac{1}{2}+\frac{\varepsilon}{2}} N_3^{-\gamma},
\end{aligned}
$$

where $\mathrm{T} := \mathrm{T}^m_{kk_1k_2k_3}$. Here we use Schur's test

$$
\|\mathrm{T}^m_{kk_1k_2k_3}\|^2_{kk_1 \to k_2k_3} \lesssim \sup_{k,k_1} \Big(\sum_{k_2,k_3} \mathrm{T}^m_{kk_1k_2k_3} \Big) \sup_{k_2,k_3} \Big(\sum_{k,k_1} \mathrm{T}^m_{kk_1k_2k_3} \Big).
$$

We list the counting estimates.

$$
\begin{aligned}
\|\mathrm{T}\|^2_{kk_1k_2k_3} &\lesssim \min\left(N_1^2 N_3^2 N_2^{\theta}, N^2 N_2^2 (N_1 \wedge N_3)^{\theta}, N_2^2 (N_1N_3)^{1+\theta}, N_1^2 (NN_2)^{1+\theta}\right); \\
\|\mathrm{T}\|^2_{k \to k_1k_2k_3} &\lesssim \min\left(N_2^2 (N_1^2 \wedge N_3^{\theta}), (N_{\mathrm{med}} N_{\mathrm{min}})^{1+\theta}\right); \\
\|\mathrm{T}\|^2_{k_3 \to kk_1k_2} &\lesssim \min\left(N_1^2 N_2^{\theta}, (N_2N)^{1+\theta}, (N_1N_2)^{1+\theta}, (N_1N)^{1+\theta}\right); \\
\|\mathrm{T}\|^2_{k_2 \to kk_1k_3} &\lesssim \min\left(N^2 (N_1 \wedge N_2)^{\theta}, (N_1N_3)^{1+\theta}, (N_1N)^{1+\theta}, (N_3N)^{1+\theta}\right); \\
\|\mathrm{T}\|^2_{k_1 \to kk_2k_3} &\lesssim \min\left(N_3^2 N_2^{\theta}, (N_2N)^{1+\theta}, (N_2N_3)^{1+\theta}, (N_3N)^{1+\theta}\right); \\
\|\mathrm{T}\|^2_{kk_1 \to k_2k_3} &\lesssim (N_2 \wedge N_3)^{1+\theta} N^{1+\theta}; \\
\|\mathrm{T}\|^2_{kk_2 \to k_1k_3} &\lesssim (N_1 \wedge N_3)^{\theta} N_2^{\theta};
\end{aligned}
$$

$$\|\mathrm{T}\|^2_{kk_3\to k_1k_2} \lesssim (N_1\wedge N_2)^{1+\theta}N_3^{1+\theta}.$$

Case (G, D, G):

For this case, we need to bound

$$\begin{aligned}
&N^{\varepsilon}N_1^{-1}N_3^{-1}\Big\|\sum_{k_1,k_2,k_3}\mathrm{T}^m_{kk_1k_2k_3}g_{k_1}g_{k_3}\overline{(\Delta_{N_2}z_N)_{k_2}}\Big\|_k\\
&\lesssim N^{\varepsilon}N_1^{-1}N_3^{-1}\|(\Delta_{N_2}z_N)_{k_2}\|_{k_2}\Big\|\sum_{k_1,k_3}\mathrm{T}^m_{kk_1k_2k_3}g_{k_1}g_{k_3}\Big\|_{k\to k_2}\\
&\lesssim N^{\varepsilon}N_1^{-1}N_3^{-1}N_2^{-\gamma}\max(\|\mathrm{T}\|_{kk_1\to k_2k_3},\|\mathrm{T}\|_{kk_3\to k_1k_2},\|\mathrm{T}\|_{kk_1k_3\to k_2},\|\mathrm{T}\|_{k\to k_1k_2k_3})\\
&\lesssim N^{\varepsilon}(N_1N_3)^{-\frac{1}{2}+\frac{\varepsilon}{2}}N_2^{-\gamma}.
\end{aligned}$$

Case (G, D, D):

For this case, we need to bound

$$\begin{aligned}
&N^{\varepsilon}N_1^{-1}\Big\|\sum_{k_1,k_2,k_3}\mathrm{T}^m_{kk_1k_2k_3}g_{k_1}\overline{(\Delta_{N_2}z_N)_{k_2}}(\Delta_{N_3}z_N)_{k_3}\Big\|_k\\
&\lesssim N^{\varepsilon}N_1^{-1}\|(\Delta_{N_2}z_N)_{k_2}\|_{k_2}\|(\Delta_{N_3}z_N)_{k_3}\|_{k_3}\Big\|\sum_{k_1}\mathrm{T}^m_{kk_1k_2k_3}g_{k_1}\Big\|_{(kk_3\to k_2)\cup(kk_2\to k_3)\cup(k\to k_2k_3)}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim N^{\varepsilon}N_1^{-1}N_2^{-\gamma}N_3^{-\gamma}\min(\|\mathrm{T}\|_{(kk_3\to k_1k_2)\cap(kk_1k_3\to k_2)},\|\mathrm{T}\|_{(kk_2\to k_1k_3)\cap(kk_1k_2\to k_3)},\\
&\quad \|\mathrm{T}\|_{(k\to k_1k_2k_3)\cap(kk_1\to k_2k_3)})\\
&\lesssim N^{\varepsilon}N_1^{-1}N_2^{-\gamma}N_3^{-\gamma}(N_{\mathrm{med}}N_{\min})^{\frac{1}{2}+\theta}.
\end{aligned}
$$

Case (D, G, D):

For this case, we need to bound

$$
\begin{aligned}
&N^{\varepsilon}N_2^{-1}\Big\|\sum_{k_1,k_2,k_3}\mathrm{T}^m_{kk_1k_2k_3}(\Delta_{N_1}z_N)_{k_1}\overline{g_{k_2}}(\Delta_{N_3}z_N)_{k_3}\Big\|_k\\
&\lesssim N^{\varepsilon}N_2^{-1}\|(\Delta_{N_1}z_N)_{k_1}\|_{k_1}\|(\Delta_{N_3}z_N)_{k_3}\|_{k_3}\Big\|\sum_{k_2}\mathrm{T}^m_{kk_1k_2k_3}\overline{g_{k_2}}\Big\|_{(kk_3\to k_1)\cup(kk_1\to k_3)\cup(k\to k_1k_3)}\\
&\lesssim N^{\varepsilon}N_2^{-1}N_1^{-\gamma}N_3^{-\gamma}\min(\|\mathrm{T}\|_{(kk_3\to k_1k_2)\cap(kk_2k_3\to k_1)},\|\mathrm{T}\|_{(kk_1\to k_2k_3)\cap(kk_1k_2\to k_3)},\\
&\quad \|\mathrm{T}\|_{(k\to k_1k_2k_3)\cap(kk_2\to k_1k_3)})\\
&\lesssim N^{\varepsilon}N_2^{-1}N_1^{-\gamma}N_3^{-\gamma}(N_{\mathrm{med}}N_{\min})^{\frac{1}{2}+\theta}.
\end{aligned}
$$

Case (D, D, D):

For this case, we need to bound

$$\begin{aligned}
&N^{\varepsilon}N_1^{-\gamma}N_2^{-\gamma}N_3^{-\gamma}\|\mathrm{T}\|_{kk_2\to k_1k_3}\\
&\quad\lesssim N^{\varepsilon}N_1^{-\gamma}N_2^{-\gamma}N_3^{-\gamma}(N_1\wedge N_3)^{\theta}N_2^{\theta}.
\end{aligned}$$

Futher discussions on $p > 3$ odd cases.

$$\begin{cases} (\mathrm{i}\partial_t + \Delta) u = |u|^{p-1} u, \quad (t, x) \in \mathbb{R} \times \mathbb{T}^2, \\ u(0) = \varphi^\omega = \sum_{k \in \mathbb{Z}^2} \frac{g_k}{\langle k \rangle} \mathrm{e}^{\mathrm{i}k \cdot x}. \end{cases}$$

Question: Why Bourgain's ansatz cannot solve “invariant Gibbs measure for $2D$ NLS with $p > 3$?

Recall Bourgain's ansatz:

$$u_N = \underbrace{\mathrm{e}^{\mathrm{i}t\Delta} \Pi_N \varphi^\omega}_{\text{in } H^{0-} \text{ a.s.}} + \underbrace{z_N}_{\text{in } H^s\text{, we treat it as a deterministic term.}}$$

- We need to make $z_N \in H^s$, $s > s_{\mathrm{cr}} = \frac{d}{2} - \frac{2}{p-1} = 1 - \frac{2}{p-1}$.

- Consider a term $N_1 \gg N_2 \sim N_3 \sim \cdots \sim N_p$

$$
\begin{aligned}
&\|\mathcal{M}_{\mathrm{np}}(\Delta_{N_1}F_N, \Delta_{N_2}F_N, \Delta_{N_3}F_N, \cdots, \Delta_{N_p}F_N)\|_{H^s}^2 \\
&= \sum_k \langle k\rangle^{2s} \left| \sum_{|k_j|\sim N_j} h^b_{kk_1k_2k_3} \frac{g_{k_1}}{\langle k_1\rangle} \frac{\overline{g_{k_2}}}{\langle k_2\rangle} \frac{g_{k_3}}{\langle k_3\rangle} \cdots \frac{g_{k_p}}{\langle k_p\rangle} \right|^2 \\
&\leq \sum_k \langle k\rangle^{2s} (N_1N_2N_3\cdots N_p)^{-2} \Big(\sum_{|k_j|\sim N_j} h^b_{kk_1k_2k_3\cdots k_p} \Big)^2 \\
&\leq N_1^{2s} (N_1N_2N_3\cdots N_p)^{-2} \Big(\sum_{|k_j|\sim N_j} h^b_{kk_1k_2k_3\cdots k_p} \Big)^2, \\
&\leq N_1^{2\left(s-\frac{1}{2}\right)+\theta} N_2^{-1+\theta},
\end{aligned}
$$

where we used

$$
\begin{aligned}
&\sum_{|k_j|\sim N_j} h^b_{kk_1k_2k_3\cdots k_p} \\
=\ &\#\{(k, k_1, \cdots, k_p); |k_j| \sim N_j, k = k_1 - k_2 + \cdots + k_p, |k|^2 = |k_1|^2 - \cdots + |k_p|^2\} \\
\leq\ &(N_3N_4\cdots N_p)^2 \cdot (N_1N_2)^{1+\theta}
\end{aligned}
$$

where $s < \frac{1}{2}$ is needed.

- However,

$$s < \frac{1}{2} \leq s_{\mathrm{cr}} = 1 - \frac{2}{p-1}, \text{ for } p \geq 5.$$

In fact, all terms as

$$\mathcal{M}_{\mathrm{np}}(\Delta_{N_1} F_N, u_L, \cdots, u_L)$$

with $N_1 \gg L$ are problematic.

How to solve this problem?

- Put these high-low-low terms also in the center of the ansatz.

$$u_N = \underbrace{\mathrm{e}^{it\Delta} \Pi_N \varphi^\omega}_{H^{0-}} + \underbrace{(\text{high} - \text{low} - \text{low})_N}_{H^{\frac{1}{2}-} \text{ with Random Averaging Operator structure}} + z_N.$$

Part 2:

The theory of random averaging operators and random tensors

Consider

$$\begin{cases} (\mathrm{i}\partial_t + \Delta)u_N = \Pi_N(:|u_N|^{p-1}u_N:), \quad (t,x) \in \mathbb{R} \times \mathbb{T}^d, \\ u_N(0) = \Pi_N \varphi^\omega. \end{cases} \tag{pNLS}$$

Theorem (Deng-Nahmod-Yue '24 Annals)

i. a.s. LWP of (pNLS). $\{u_N\}$ τ^{-1}-certainly converges in $C_t^0 H_x^{0-}([0,\tau])$ for (pNLS) on $\mathbb{T}^2$ ($p \geq 3$, odd).

ii. Invariant Gibbs measure under the flow & a.s. GWP.

Ansatz

- Decompose the $\{u_N\}$, $y_N := u_N - u_{N/2}$. Recalling $\Pi_N - \Pi_{N/2} = \Delta_N$

$$
\begin{aligned}
(\mathrm{i}\partial_t + \Delta) y_N &= \Pi_N(\mathcal{N}(u_N)) - \Pi_{N/2}(\mathcal{N}(u_{N/2})) \\
&= \Pi_N(\mathcal{N}(y_N + u_{N/2}) - \Pi_{N/2}\mathcal{N}(u_{N/2})) + \underbrace{\Delta_N(\mathcal{N}(u_{N/2}))}_{\text{commutator term}}.
\end{aligned}
$$

- Capture High-low-low terms ($L \ll N$)

$$
\begin{cases}
(\mathrm{i}\partial_t + \Delta)\psi_{N,L} = \Pi_N \mathcal{N}(\psi_{N,L}, u_L, \cdots, u_L), \\
\psi_{N,L}(0) = \Delta_N \varphi^{\omega}.
\end{cases}
$$

k-th Fourier mode

$$
(\psi_{N,L})_k = \sum_{\frac{N}{2} < |k^*| \leq N} H^{N,L}_{kk^*} \frac{g_{k^*}}{\langle k^* \rangle}.
$$

For simplicity,

$$\psi_{N,1/2} = e^{it\Delta}\Delta_N \varphi^{\omega}, \quad H^{N,\frac{1}{2}}_{kk^*} = e^{-i|k|^2 t}\mathbf{1}_{k=k^*}.$$

- $H^{N,L}_{kk^*}$ is a random matrix totally depending on u_L which is a r.v. generated by evolving $\Pi_L \varphi^{\omega}$ $(\{g_k\}, |k| \le L)$ under (pNLS).

  We say $H^{N,L}_{kk^*} \in \mathcal{B}_{\le L}$.
- Hence $H^{N,L}_{kk^*}$ is independent with g_{k^*} $(|k^*| \sim N)$.
- $\zeta_{N,L} = \psi_{N,L} - \psi_{N,L/2}$, $h^{N,L} = H^{N,L} - H^{N,L} \in \mathcal{B}_{\le L}$.
- Full ansatz

$$\begin{aligned} y_N &= \psi_{N,N^{1-\delta}} + z_N \\ &= \psi_{N,\frac{1}{2}} + \sum_{L \le N^{1-\delta}} \zeta_{N,L} + z_N. \end{aligned}$$

$$(y_N)_k = \underbrace{e^{it\Delta}\Delta_N \varphi^{\omega}}_{H^{0-}} + \underbrace{\sum_{L \le N^{1-\delta}} \left(\sum_{|k^*| \sim N} h^{N,L}_{kk^*} \frac{g_{k^*}}{\langle k^* \rangle} \right)}_{H^{\frac{1}{2}-}} + \underbrace{z_N}_{H^-}.$$

Then the equation for z_N: plug the ansatz into

$$(\mathrm{i}\partial_t + \Delta) y_N = \Pi_N\Big(\mathcal{N}\big(y_N + u_{\frac{N}{2}}\big) - \mathcal{N}\big(u_{\frac{N}{2}}\big)\Big) + \underbrace{\Delta_N\Big(\mathcal{N}\big(u_{\frac{N}{2}}\big)\Big)}_{\text{commutator term}}.$$

We have

$$\begin{aligned}(\mathrm{i}\partial_t + \Delta) z_N \ = \ & \Delta_N\left(\mathcal{N}\left(\sum_{M\le N/2} \psi_{M,M^{1-\delta}} + z_M\right)\right)\\ & + \Pi_N(\mathcal{N}(z_N + \psi_{N,N^{1-\delta}} + u_{N/2}) - \mathcal{N}(u_{N/2}))\\ & - \mathcal{N}(\psi_{N,N^{1-\delta}}, u_{N^{1-\delta}}, \cdots, u_{N^{1-\delta}}).\end{aligned}$$

- Bounds with the ansatz.

$$\|h^{N,L}_{kk^*}\|_{k\to k^*} \le L^{-\delta}.$$

$$\|h^{N,L}_{kk^*}\|_{kk^*} \le N^{\frac{1}{2}+\delta} L^{-\frac{1}{2}}.$$

$$\left\|\left(1 + \frac{|k-k^*|}{L}\right)^{\kappa} h^{N,L}_{kk^*}\right\|_{kk^*} \le N.$$

$$\|(z_N)_k\|_k \lesssim N^{-1+\delta}.$$

Fractional nonlinear Schrödinger equation (FNLS) on the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$:

$$\begin{cases} i\partial_t u - D_x^\alpha u = \pm |u|^2 u, \\ u(0,x) = u_0(x), \end{cases} \quad (t,x) \in \mathbb{R} \times \mathbb{T}$$

where $D_x^\alpha = (-\partial_{xx})^{\frac{\alpha}{2}}$. The order of the derivative α measures the strength of the dispersion.

- $\alpha = 1$: no dispersion (half-wave).
- $\alpha \in (1,2)$: weak dispersion.
- $\alpha = 2$: Schrödinger equation.
- $\alpha > 2$: strong dispersion.

FNLS

$$i\partial_t u - D_x^\alpha u = \pm|u|^2 u.$$

- When $\alpha = 2$, i.e. $D_x^2 = -\partial_{xx}$, it is the famous NLS, Bourgain, Kenig-Ponce-Vega, Tao, ...
- When $\alpha \neq 2$,
  - Laskin '00 fractional quantum mechanics;
  - Ionescu-Pusateri '14 water wave problems;
  - Kirkpatrick-Lenzmann-Staffilani '13 continuum limit of lattice interactions:
    - long-range interaction $\alpha \in (1,2)$,
    - short-range interaction $\alpha = 2$;
  - Half-wave equations $\alpha = 1$, Lenzmann, Gerard, Raphael, ...

Gibbs measure

The Gibbs measure ρ_{Gibbs} on distributions $\mathcal{S}'(\mathbb{R}^d)$ formally given by

$$\begin{aligned} d\rho_{\text{Gibbs}}(\phi) &= \mathcal{Z}^{-1} e^{\pm\int_{\mathbb{T}}|\phi|^4 dx} e^{-\frac{1}{2}\int_{\mathbb{T}}|\langle\partial_x\rangle^{\alpha/2}\phi|^2 dx} d\phi \\ &= \mathcal{Z}^{-1} \underbrace{e^{\pm\int_{\mathbb{T}}|\phi|^4 dx}}_{weight} \overbrace{\prod_k e^{-\frac{1}{2}\langle k\rangle^{\alpha}(p_k^2+q_k^2)} dp_k dq_k}^{Gaussian\ measure}, \end{aligned}$$

where $\mathcal{Z}$ is a normalisation constant.

Constructive quantum field theory:

- Defocusing when $-$ (conjectured for $\alpha > \frac{1}{2}$):
  - Sun-Tzvetkov '20 $\alpha > \frac{7}{8}$;
  - Tanaka-Wang '23 $\alpha > \frac{2}{3}$.
- Focusing when $+$:
  - Normalisability for $\alpha > 1$, Lebowitz-Rose-Speer '88, Bourgain '94, Oh-Sosoe-Tolomeo '20, L.-Wang '22.
  - Non-normalisability for $\alpha \leq 1$, Oh-Seong-Tolomeo '20.

Deterministic well-posedness:

- When $\alpha = 2$:
  - LWP in $L^2(\mathbb{T})$: Bourgain '94.
  - results below L^2,
    - norm-inflation below $H^{-\frac{1}{2}}$: Christ-Colliander-Tao '03;
    - non-uniform continuity below L^2: Burq-Gérard-Tzvetkov '09;
    - non-uniqueness below L^2: Guo-Oh '18.
- When $\alpha \in (1, 2)$: Well(ill)-posedness in H^s for $s \geq \frac{2-\alpha}{4}$ ($s < \frac{2-\alpha}{4}$), Cho-Hwang-Kwon-Lee '15.
- When $\alpha > 2$: Well-posedness in H^s for $s \geq \frac{2-\alpha}{4}$. Miyaji-Tsutsumi '17, '18, Oh-Wang '18, Brun-Li-Liu-Zine '23.

Probabilistic well-posedness:

- When $\alpha = 2$:
  - a.s. GWP in $H^{\frac{1}{2}-}(\mathbb{T})$: Bourgain '94;
  - a.s. LWP in $H^{-\frac{1}{2}+}(\mathbb{T})$: Deng-Nahmod-Yue '22.
- When $\alpha \in (1,2)$: Gibbs dynamics in $H^{\frac{\alpha-1}{2}-}(\mathbb{T})$:
  - Mild solution with flow property:
    - for $\alpha > \frac{4}{3}$, Demirbas '15;
    - for $\alpha > \frac{6}{5}$, Sun-Tzvetkov '20;
    - for $\alpha > \frac{31-\sqrt{233}}{14}$, Sun-Tzvetkov '21;
  - Weak solution *without* proving flow property: for $\alpha > 1$, Sun-Tzvetkov '20.

$\Longrightarrow$ answer to Zakharov's question for $\alpha > \frac{31-\sqrt{233}}{14}$.

Main Problem

Zakharov's question for full dispersive range $\alpha > 1$.

Key: Flow property of the solution.

Stability & flow property

We consider the frequency truncated FNLS:

$$\begin{cases} \mathrm{i}\partial_t u_N - \mathrm{D}_x^\alpha u_N - u_N = \Pi_{\le N}(|u_N|^2 u_N), \\ u_N(0,x) = \sum_{|k|\le N} \frac{g_k(\omega)}{\langle k\rangle^{\alpha/2}} \mathrm{e}^{\mathrm{i}k\cdot x}, \end{cases} \qquad (t,x)\in\mathbb{R}\times\mathbb{T}.$$

To show the flow property

$$u(t+s,u(0)) = u(t,u(s)),$$

we see that

$$\begin{aligned} u(t+s,u(0)) &= \lim_{N\to\infty} u_N(t+s,u_N(0)) \\ &= \lim_{N\to\infty} u_N(t,u_N(s)) \\ &\overset{?}{=} \lim_{N\to\infty} u_N\Big(t, \lim_{M\to\infty} u_M(s)\Big) \\ &= u(t,u(s)), \end{aligned}$$

where we need stability.

Main results

Now we are ready to state our main results.

Theorem (L.-Wang '23)

For all $\alpha > 1$:

- GWP a.s. with flow property.
- Gibbs measure is invariant under the dynamics.
- Returning property, i.e. Zakharov's question.

The Gibbs measure supports on rough function spaces:

$$\omega \mapsto u_0^\omega(x) = \sum_{k\in\mathbb{Z}} \frac{g_k(\omega)}{\langle k\rangle^{\alpha/2}} e^{ikx} \in \bigcap_{s<\frac{\alpha-1}{2}} H^s(\mathbb{T}) \setminus H^{\frac{\alpha-1}{2}}(\mathbb{T})$$

- As $\alpha \longrightarrow 1$, we see that $s \longrightarrow 0$.
- FNLS is (deterministically) well-posed in $H^s(\mathbb{T})$ for $s \geq \frac{2-\alpha}{4} > \frac{1}{4}$.

Probabilistic argument.

- Sun-Tzvetkov '20: $s > \frac{1}{10}$.
- Sun-Tzvetkov '21: $s > \frac{17-\sqrt{233}}{28} \approx 0.062$.

$$y_N = u_N - u_{N/2}. \quad y_N = \underbrace{f_N}_{\widehat{F_N}(k) = \mathbf{1}_{\langle k\rangle \sim N}\langle k\rangle^{-\alpha/2} g_k} + \underbrace{z_N}_{\text{remainder}}.$$

Say, we want to put remainder $z_N \in H^\gamma(\mathbb{T})$.

$$\|\widehat{z_N}(k)\|_{\ell^2_k} \lesssim \Big\| \sum_{k_1-k_2+k_3=k} \mathbf{1}_{|k|^\alpha - |k_1|^\alpha + |k_2|^\alpha - |k_3|^\alpha \approx m} \widehat{z_{N_{\min}}}(k_1)\overline{\widehat{z_{N_{\text{med}}}}}(k_2)\hat{f}_N(k_2) \Big\|_{\ell^2_k} + \cdots$$

$$\lesssim \underbrace{N^{1-\frac{\alpha}{2}} N_{\min}^{1-\frac{\alpha}{2}}}_{\text{counting estimate}} \times N_{\min}^{-\gamma} N_{\text{med}}^{-\gamma} N^{-\frac{\alpha}{2}} \le N_{\min}^{1-\frac{\alpha}{2}-2\gamma} N^{1-\alpha} \underset{?????}{\le} N^{-\gamma}$$

if

$$\begin{cases} 1 - \frac{\alpha}{2} - 2\gamma \le 0 \\ 1 - \alpha \le -\gamma \end{cases} \Longrightarrow 1 - \frac{\alpha}{2} \le 2\gamma \le 2\alpha - 2 \Longrightarrow \alpha > \frac{6}{5} \text{ (Sun-Tzvetkov '20).}$$

How to improve? Observation: this argument wastes $N_{\text{med}}^{-\gamma}$ when $N_{\text{med}} \gg N_{\min}$.
Improvement: if $N_{\text{med}} > N^{1-\delta}$, then

$$N^{1-\frac{\alpha}{2}} N_{\min}^{1-\frac{\alpha}{2}} \times N_{\min}^{-\gamma} N_{\text{med}}^{-\gamma} N^{-\frac{\alpha}{2}} \le N^{1-\alpha-\gamma(1-\delta)} \le N^{-\gamma} \Longrightarrow \alpha > 1.$$

We deal with the high-low interaction ($L < N^{1-\delta}$) by random averaging operator (Deng-Nahmod–Yue '24, to appear in Ann. of Math.):

$$\widehat{\mathcal{P}^{N,L}(w)}(k) \approx \mathbf{1}_{\langle k\rangle \leq N} \sum_{k_1-k_2+k_3=k} \mathbf{1}_{|k|^\alpha-|k_1|^\alpha+|k_2|^\alpha-|k_3|^\alpha \approx m} \widehat{u_L}(k_1)\overline{\widehat{u_L}}(k_2)\hat{w}(k_2).$$

Ideas:

- use operator norm $\|\mathcal{P}^{N,L}\|_{k\to k'}$ which is finer than Hilbert-Schmidt norm (Gubinelli '15).
- use induction w.r.t N to prove estimates.
- F_N's independence to u_L allows usages of random tensor estimate.

By removing the high-low interaction from the remainder, we get the ansatz

$$y_N(t) = f_N + \underbrace{\sum_{1\leq L<N^{1-\delta}} \mathfrak{h}^{N,L}(f_N)(t)}_{\text{paralinear term}} + z_N(t),$$

with $f_N, \mathfrak{h}^{N,L}(f_N)(t) \in H^{(\alpha-1)/2-}$ and $z_N \in H^{1/2+}$.

Weak dispersion $\alpha < 2 \implies$ worse counting estimates

$$\phi(k_1, k_2, k_3) = |k_1|^\alpha - |k_2|^\alpha + |k_3|^\alpha - |k_1 - k_2 + k_3|^\alpha$$

may be dense in the uniform interval $[c, c+1]$ for some $c \in \mathbb{R}$.

Improvements compared to Sun-Tzvetkov '21:

- Better counting estimates (frequency dependent counting), L.-Wang '23.
  - Worst-case scenario of counting do not happen for low-high-low.
- Better estimates from random tensor estimates, Deng-Nahmod-Yue '22.
- Structures of the equation:
  - Unitary property, Bourgain '96, Deng-Nahmod-Yue '21.
  - Γ-condition, Deng-Nahmod-Yue '24.

Let $\alpha \in (1,2)$ and $b \in \mathbb{R}$. Define

$$\phi_{b,\pm}(x) = |x|^\alpha \pm |x-b|^\alpha,$$

Lemma (L.-Wang '23)

Let $\phi_{b,\pm}$ be as above.
(i) *Let $\alpha \in (1,2)$. Then, we have the following sharp estimates*

$$|\phi'_{b,-}(x)| \gtrsim_\alpha \min\left(|b||x|^{\alpha-2}, |b|^{\alpha-1}\right)$$

provided $x \neq 0$.
(ii) *Let $\alpha \in (1,2)$ and $|b| \geq 1$. Then, we have*

$$|\phi'_{b,+}(x)| \gtrsim_\alpha |b|^{\alpha-1}$$

for $|2x-b| \gtrsim |b|$. For $|2x-b| \ll |b|$, if we further assume that $|2x-b| \gtrsim |b|^{1-\frac{\alpha}{2}}$, we have

$$|\phi'_{b,+}(x)| \gtrsim_\alpha |b|^{\frac{\alpha}{2}-1}.$$

Let

$$S_{kk_1} = \{k_3 \in \mathbb{Z}; k_3 \neq k, |k_3| \leq N_3, |k_1 + k_3 - k| \leq N_2,$$
$$|k_3|^\alpha - |k_1 + k_3 - k|^\alpha = |k|^\alpha - |k_1|^\alpha - m + O(1)\}.$$

Then we have

$$|S_{kk_1}| \lesssim (N_2 \wedge N_3)^{2-\alpha} \langle k_1 - k \rangle^{-1} + 1$$

Remark. In Sun-Tzvetkov '21, they obtained

$$|S_{kk_1}| \lesssim N^{2-\alpha+\varepsilon} \langle k_1 - k \rangle^{-1} + 1$$

provided $N_1 \sim N_2 \sim N_3 \sim N_4 \sim N$.

For the countings of $|S_{k_1k_3}|$ and $|S_{kk_2}|$, we need the following further decomposition.

$$\begin{cases} S_{k_1k_3}^{\mathrm{bad}} = \{(k, k_2) \in S_{k_1k_3}; |2k - (k_1 + k_3)| \ll |k_1 + k_3|\}; \\ S_{k_1k_3}^{\mathrm{good}} = \{(k, k_2) \in S_{k_1k_3}; |2k - (k_1 + k_3)| \gtrsim |k_1 + k_3|\}. \end{cases}$$

Then we have the following counting estimates

$$\begin{aligned} &|S_{k_1k_3}^{\mathrm{good}}| \lesssim 1; \\ &|S_{k_1k_3}^{\mathrm{bad}}| \lesssim |k_1 + k_3|^{1-\frac{\alpha}{2}}. \end{aligned}$$

Define the set

$$S_k = \{(k_1, k_2, k_3) \in \mathbb{Z}^3; k_3 \notin \{k, k_2\}, |k_j| \le N, k_1 - k_2 + k_3 = k,$$
$$|k_1| - |k_2| + |k_3| - |k| = m + O(1)\}.$$

We have the following counting estimates

$$|S_k| \sim N^2.$$

From the above we have

(L.-Wang-Yue '24+)

The probabilistic scaling critical space for the half-wave equation is $L^2(\mathbb{T})$; while the support of its Gibbs measure is $H^{0-}(\mathbb{T})$.

Remark. When $\alpha > 1$, $|S_k| \lesssim (N_2 \wedge N_3)^{2-\alpha} \log N_1 + N_1$. There is a sharp contrast.

Remark. No dimension reduction for $\alpha = 1$.

Probabilistic scaling for half-wave

$$(\widehat{\mathcal{N}(u)})_k := \sum_{k_1 - k_2 + k_3} u_{k_1} \overline{u_{k_2}} u_{k_3}.$$

Let $u(0) \in H^s$. To prove LWP in H^s, we want Picard 1^{st} iterate

$$u^{(1)}(t) := \int_0^1 e^{-i(t-s)|\partial_x|} \mathcal{N}(e^{-is|\partial_x|} u(0)) ds \quad \in H^s$$

Consider

$$u(0) = N^{-\alpha/2} \sum_{|k| \sim N} g_k(\omega) e^{ikx}.$$

We have

$$\left(\widehat{u^{(1)}(t)}\right)_k \sim N^{-3\alpha/2} \sum_{\substack{k_j \in \mathbb{Z}, |k_j| \sim N \\ k_1 - k_2 + k_3 = k}} g_{k_1} \overline{g_{k_2}} g_{k_3}.$$

Due to square root cancellation, we have with high probability that

$$\|u^{(1)}(t)\|_{H^s} \sim N^{s - 3\alpha/2} N^{\frac{3}{2}} \stackrel{\alpha = 2s+1}{=\!=\!=} N^{-2s}; \quad \|u^{(1)}(t)\|_{H^s} \lesssim 1 \iff s \geq 0 := s_{pr}.$$

Given $N \in \mathbb{N}$, define the following Picard 2nd iterate

$$Z_N^{(2)}(t) = \int_0^t e^{i(t-t')D_x^\alpha} \Pi_N \left[\left(|F_N(t')|^2 - \frac{1}{\pi} \int_{\mathbb{T}} |F_N(t')|^2 dx \right) F_N(t') \right] dt',$$

where

$$F_N(t) = \sum_{N^{1-\delta} < k < N} \frac{g_k(\omega)}{\langle k \rangle^{\alpha/2}} e^{it|k|^\alpha + ikx}$$

is the truncation of the random linear solution. We have the following result.

Theorem (L.-Wang '23)

With the above notation and $|t| \sim 1$, we have

(i) *When $\alpha > 1$, we have* $\left(\mathbb{E}\left[\|Z_N^{(2)}(t)\|_{L^2(\mathbb{T})}^2\right]\right)^{1/2} \lesssim N^{-\frac{1}{2}-}$;

(ii) *When $\alpha = 1$, we have* $\left(\mathbb{E}\left[\|Z_N^{(2)}(t)\|_{L^2(\mathbb{T})}^2\right]\right)^{1/2} \gtrsim (\log N)^3$.

High-low $N_{\mathrm{med}} < N^{1-\delta}$ is not very bad. Still waste the high-intermediate interaction.

Let us redefine the high-low interaction by $N_{\mathrm{med}} < N^{\delta}$. In

$$y_N(t) = -\mathrm{i} \sum_{N_{\max}=N} \int_0^t \Pi_N \mathcal{M}(y_{N_1}, y_{N_2}, y_{N_3})(t')\mathrm{d}t' + \cdots,$$

we plug the ansatz $y_N(t) = \sum_{1/2 \le L < N^{\delta}} \mathfrak{h}^{N,L}(f_N)(t) + z_N(t)$ in to get a new ansatz

$$y_N(t) = \underbrace{-\mathrm{i} \sum_{N_{\max}=N} \sum_{1/2 \le L_j < N^{\delta}} \int_0^t \Pi_N \mathcal{M}(\mathfrak{h}^{N_1,L_1}(f_{N_1}), \mathfrak{h}^{N_2,L_2}(f_{N_2}), \mathfrak{h}^{N_3,L_3}(f_{N_3}))(t')\mathrm{d}t' + \cdots}_{\text{(para-) trilinear in } F_{N_j}\text{'s}}$$

Iterate some times more to get ansatz (para-)multilinear in f_{N_j}'s:

$$(\widehat{y_N})_k(t) = \sum_{\text{plants } \mathcal{S}} \sum_{k_{\mathcal{U}}, k_{\mathcal{V}}} \int \mathrm{d}\lambda_{\mathcal{V}} \cdot \mathfrak{h}^{\mathcal{S}}_{k k_{\mathcal{U}}}(t, k_{\mathcal{V}}, \lambda_{\mathcal{V}}) \times \prod_{\mathfrak{l} \in \mathcal{U}} (f_{N_{\mathfrak{l}}})^{\zeta_{\mathfrak{l}}}_{k_{\mathfrak{l}}} \prod_{\mathfrak{f} \in \mathcal{V}} (\widehat{z_{N_{\mathfrak{f}}}})^{\zeta_{\mathfrak{l}}}_{k_{\mathfrak{l}}}(\lambda_{\mathfrak{f}}) + (\widehat{z_N})_k(t).$$

High-low $N_{\mathrm{med}} < N^{1-\delta}$ is not very bad. Still waste the high-intermediate interaction.

Let us redefine the high-low interaction by $N_{\mathrm{med}} < N^{\delta}$. In

$$y_N(t) = -\mathrm{i} \sum_{N_{\max}=N} \int_0^t \Pi_N \mathcal{M}(y_{N_1}, y_{N_2}, y_{N_3})(t')\mathrm{d}t' + \cdots,$$

we plug the ansatz $y_N(t) = \downarrow + z_N(t)$ in to get a new ansatz

$$y_N(t) \;=\; \underbrace{-\mathrm{i} \sum_{N_{\max}=N} \sum_{1/2\leq L_j < N_j^{\delta}} \int_0^t \Pi_N \mathcal{M}(\mathfrak{h}^{N_1,L_1}(f_{N_1}), \mathfrak{h}^{N_2,L_2}(f_{N_2}), \mathfrak{h}^{N_3,L_3}(f_{N_3}))(t')\mathrm{d}t'}_{\text{(para-) trilinear in } F_{N_j}\text{'s}} + \cdots.$$

Iterate some times more to get ansatz (para-)multilinear in f_{N_j}'s:

$$(\widehat{y_N})_k(t) = \sum_{\text{plants } \mathcal{S}} \sum_{k_{\mathcal{U}}, k_{\mathcal{V}}} \int \mathrm{d}\lambda_{\mathcal{V}} \cdot \mathfrak{h}^{\mathcal{S}}_{k k_{\mathcal{U}}}(t, k_{\mathcal{V}}, \lambda_{\mathcal{V}}) \times \prod_{\mathfrak{l}\in\mathcal{U}} (f_{N_{\mathfrak{l}}})^{\zeta_{\mathfrak{l}}}_{k_{\mathfrak{l}}} \prod_{\mathfrak{f}\in\mathcal{V}} (\widehat{z_{N_{\mathfrak{f}}}})^{\zeta_{\mathfrak{l}}}_{k_{\mathfrak{l}}}(\lambda_{\mathfrak{f}}) + (\widehat{z_N})_k(t).$$

Random tensor theory & half-wave

High-low $N_{\text{med}} < N^{1-\delta}$ is not very bad. Still waste the high-intermediate interaction.
Let us redefine the high-low interaction by $N_{\text{med}} < N^{\delta}$. In

$$y_N(t) = -\mathrm{i} \sum_{N_{\max}=N} \int_0^t \Pi_N \mathcal{M}(y_{N_1}, y_{N_2}, y_{N_3})(t')\mathrm{d}t' + \cdots,$$

we plug the ansatz $y_N(t) =$  $+ z_N(t)$ in to get a new ansatz

$$y_N(t) \quad = \quad$$  $$+\cdots.$$

Iterate some times more to get ansatz (para-)multilinear in f_{N_j}'s:

$$(\widehat{y_N})_k(t) = \sum_{\text{plants } \mathcal{S}} \sum_{k_{\mathcal{L}}, k_{\mathcal{V}}} \int \mathrm{d}\lambda_{\mathcal{V}} \cdot \mathfrak{h}^{\mathcal{S}}_{k k_{\mathcal{L}}}(t, k_{\mathcal{V}}, \lambda_{\mathcal{V}}) \times \prod_{\mathfrak{l} \in \mathcal{L}} (f_{N_{\mathfrak{l}}})^{\zeta_{\mathfrak{l}}}_{k_{\mathfrak{l}}} \prod_{\mathfrak{f} \in \mathcal{V}} (\widehat{z_{N_{\mathfrak{f}}}})^{\zeta_{\mathfrak{l}}}_{k_{\mathfrak{l}}}(\lambda_{\mathfrak{f}}) + (\widehat{z_N})_k(t).$$

We end the iteration when $|\mathcal{S}| > \delta^{-100}$. Thus, in the remainder we gain from accumulation of high-intermediate interaction $N_{\mathfrak{n}} \geq N^{\delta}$ that

$$\|\widehat{z_N}(k)\|_{\ell^2_k} \lesssim \prod_{\mathfrak{n} \in \mathcal{S}} N_{\mathfrak{n}}^{-\gamma} \leq ((N^{\delta})^{-\gamma})^{|\mathcal{S}|} < N^{-\gamma \delta^{-99}}.$$

Merging estimates & selection algorithm

We define the random tensors inductively

$$h^{\mathcal{S}}_{kk_{\mathcal{L}}} = \sum_{k_1-k_2+k_3=k} \mathbf{1}_{|k|-|k_1|+|k_2|-|k_3|=m} h^{\mathcal{S}_1}_{k_1k_{\mathcal{L}_1}} h^{\mathcal{S}_2}_{k_2k_{\mathcal{L}_2}} h^{\mathcal{S}_3}_{k_3k_{\mathcal{L}_3}},$$

where $\mathcal{S} = (\mathcal{L}, \mathcal{V})$ is a plant "merged" from $\mathcal{S}_j = (\mathcal{L}_j, \mathcal{V}_j)$ $(j = 1, 2, 3)$ via

$$\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3, \quad \mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3.$$

Merging estimates: subject to the location of the highest leaves, we design a selection algorithm to distribute the operator norms.

$$\|h^{\mathcal{S}}_{kk_{\mathcal{L}}}\|_{\ell^2_{kk_B} \to \ell^2_C} \le \|h^{\mathcal{S}_1}_{k_1k_{\mathcal{L}_1}}\|_{\ell^2_{k_1k_{C_1}} \to \ell^2_{k_{B_1}}} \|\mathbf{1}_{k_1-k_2+k_3=k}\|_{\ell^2_{kk_1} \to \ell^2_{k_2k_3}} \|h^{\mathcal{S}_2}_{k_2k_{\mathcal{L}_2}}\|_{\ell^2_{k_2k_{B_2}} \to \ell^2_{k_{C_2}}} \|h^{\mathcal{S}_3}_{k_3k_{\mathcal{L}_3}}\|_{\ell^2_{k_3k_{B_3}} \to \ell^2_{k_{C_3}}},$$

where

$$B_j = B \cap \mathcal{L}_j, \quad C_j = C \cap \mathcal{L}_j.$$

Then we use induction to prove estimates for random tensors.

Remark 2. Elements in $\mathcal{L}$ and $\mathcal{V}$ are placeholds for the tensors. In general relativity community, they are known as the abstact index notations. See Liang-Zhou '23, Wald '84, Penrose '87.

Then we cover the full probabilistically subcritical range by using the theory of random tensors (Deng-Nahmod-Yue '22) described in last few slides.

Theorem A (L.-Wang-Yue '24)

a.s. LWP of half-wave equation in $H^{0+}(\mathbb{T})$.

Remark. Due to the non-negativity of the probabilistic scaling, we view the pairings as perturbations.

$$\Big\| \sum_{\substack{k_1=k_2\\ \langle k_1\rangle\sim N_1}} h_{kk_1k_2}\frac{g_{k_1}}{\langle k_1\rangle^{\alpha/2}}\frac{\overline{g_{k_2}}}{\langle k_2\rangle^{\alpha/2}}\Big\|_{\ell^2_k} \le \sum_{\langle k_1\rangle\sim N_1} N_1^{-\alpha+\theta}\sup_{k_1,k_2}\|h_{kk_1k_2}\|_{\ell^2_k} \le N_1^{-\varepsilon}\sup_{k_1,k_2}\|h_{kk_1k_2}\|_{\ell^2_k},$$

where $\alpha=2s+1>1$ and

$$\|h_{kk_1k_2}\|_{\ell^2_k} \le \min\left(\|h_{kk_1k_2}\|_{\ell^2_k\to\ell^2_{k_1k_2}}, \|h_{kk_1k_2}\|_{\ell^2_{kk_1}\to\ell^2_{k_2}}, \|h_{kk_1k_2}\|_{\ell^2_{kk_2}\to\ell^2_{k_1}}, \|h_{kk_1k_2}\|_{\ell^2_{kk_1k_2}}\right).$$

Details of random tensor theory

Let us expand more details of the proof of Theorem A (Liang-Wang-Yue '24+) in the remaining slides.

Recall the half-wave equation

$$\begin{cases} (\mathrm{i}\partial_t u - |\partial_x|)\, u = |u|^2 u, \\ u(0) = u_0^\omega, \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{T},$$

with

$$u_0^\omega = \sum_{k \in \mathbb{Z}} \frac{g_k(\omega)}{\langle k \rangle^{\alpha/2}} \mathrm{e}^{\mathrm{i}kx}.$$

$$\frac{\alpha}{2} = s + \frac{1}{2}, \qquad s > s_{\mathrm{pr}} := 0.$$

The probabilistic scaling:

Defined by

$$(\mathcal{N}(u))_k := \sum_{k_1-k_2+k_3=k} u_{k_1}\overline{u_{k_2}}u_{k_3}.$$

Consider

$$u(0) = N^{-\alpha/2}\sum_{|k|\sim N} g_k(\omega)\mathrm{e}^{\mathrm{i}kx}.$$

Then $\|u(0)\|_{H^s}\sim 1$. Write down the Picard 2nd iterate

$$u^{(1)}(t) := \int_0^t \mathrm{e}^{-\mathrm{i}(t-t')|\partial_x|}\mathcal{N}(\mathrm{e}^{-\mathrm{i}t'|\partial_x|}u(0))\mathrm{d}t'.$$

Then on the Fourier side, we have

$$u_k^{(1)}(t)\sim N^{-3\alpha/2}\sum_{\substack{k_j\in\mathbb{Z},|k_j|\sim N\\ k_1-k_2+k_3=k}} \frac{1}{\langle\Omega\rangle} g_{k_1}\overline{g_{k_2}}g_{k_3}, \quad \Omega := |k|-|k_1|+|k_2|-|k_3|.$$

Due to square root cancellation, we have with high probability that

$$\|u^{(1)}(t)\|_{H^s}\sim N^{s-3\alpha/2}N^{\frac{3}{2}} \overset{\alpha=2s+1}{=\!=\!=} N^{-2s}; \quad \|u^{(1)}(t)\|_{H^s}\lesssim 1 \Longleftrightarrow s\geq 0 := s_{pr}.$$

Plants and plant ternsors:

- Ansatz

$$(y_N)_k(t) = \sum_{\substack{\mathcal{S}:N(\mathcal{S})=N \\ |\mathcal{S}|\le D}} \sum_{k_{\mathcal{L}},k_{\mathcal{V}}} \int \mathrm{d}\lambda_{\mathcal{V}} \cdot h^{\mathcal{S}}_{kk_{\mathcal{L}}}(t,k_{\mathcal{V}},\lambda_{\mathcal{V}}) \cdot \prod_{\mathfrak{l}\in\mathcal{L}} (f_{N_{\mathfrak{l}}})^{\zeta_{\mathfrak{l}}}_{k_{\mathfrak{l}}} \prod_{\mathfrak{f}\in\mathcal{V}} (\widehat{z_{N_{\mathfrak{f}}}})^{\zeta_{\mathfrak{f}}}_{k_{\mathfrak{f}}}(\lambda_{\mathfrak{f}}) + (z_N)_k(t)$$

- Plant $\mathcal{S} = (\mathcal{L},\mathcal{V},\mathcal{Y})$: assign dyadic $N(S)$
  - Tree $\mathcal{L}$: set of abstract index that acts on Type (C);
  - blossom set $\mathcal{V}$: set of abstract indices that acts on Type (D);
  - memory set $\mathcal{Y}$: record skeletons, faciliate induction in the size of plant, contributing smoothing effects. (not placeholder of tensor);
  - for $\mathfrak{n}\in\mathcal{L}\cup\mathcal{V}\cup\mathcal{Y}$, assign dyadic N_n;
  - for $\mathfrak{n}\in\mathcal{L}\cup\mathcal{V}$, assign a sign $\zeta_{\mathfrak{n}}$;
  - regular plant $N_{\mathfrak{n}} \ge N(\mathcal{S})^{\delta}$.

- Plant tensors $h^{\mathcal{S}}_{kk_{\mathcal{L}}}(k_{\mathcal{V}}, \lambda_{\mathcal{V}})$

$$\Psi_k = \sum_{k_{\mathcal{L}}, k_{\mathcal{V}}} \int \mathrm{d}\lambda_{\mathcal{V}} \cdot h_{kk_{\mathcal{L}}}(k_{\mathcal{V}}, \lambda_{\mathcal{V}}) \cdot \prod_{\mathfrak{l} \in \mathcal{L}} (f_{N_{\mathfrak{l}}})^{\zeta_{\mathfrak{l}}}_{k_{\mathfrak{l}}} \prod_{\mathfrak{f} \in \mathcal{V}} (\widehat{z_{N_{\mathfrak{f}}}})^{\zeta_{\mathfrak{f}}}_{k_{\mathfrak{f}}}(\lambda_{\mathfrak{f}}).$$

Trimming & Merging:

- **Trimming**

  We need to hide some low frequency randomness and remainders into the tensor to (1) ensure independence; (2) to exclude low frequencies that makes the plant larger but does not contribute smoothing.

  Given $R \geq 1$, $\mathcal{S} = (\mathcal{L}, \mathcal{V}, \mathcal{Y})$.

$$\mathcal{L}' = \{\mathfrak{l} \in \mathcal{L}; N_{\mathfrak{l}} \geq R\}, \quad \mathcal{V}' = \{\mathfrak{f} \in \mathcal{V}; N_{\mathfrak{f}} \geq R\}, \quad \mathcal{Y}' = \{\mathfrak{p} \in \mathcal{Y}; N_{\mathfrak{p}} \geq R\}.$$

$$\mathcal{S}' = \mathtt{Trim}(\mathcal{S}, R) = (\mathcal{L}', \mathcal{V}', \mathcal{Y}').$$

$$(h')_{kk_{\mathcal{L}'}}(k_{\mathcal{V}'}, \lambda_{\mathcal{V}'}) = \sum_{k_{\mathcal{L}\setminus\mathcal{L}'}} \sum_{k_{\mathcal{V}\setminus\mathcal{V}'}} \int \mathrm{d}\lambda_{\mathcal{V}\setminus\mathcal{V}'} \cdot h_{kk_{\mathcal{L}}}(k_{\mathcal{V}}, \lambda_{\mathcal{V}}) \cdot \prod_{\mathfrak{l} \in \mathcal{L}\setminus\mathcal{L}'} (f_{N_{\mathfrak{l}}})_{k_{\mathfrak{l}}}^{\zeta_{\mathfrak{l}}} \prod_{\mathfrak{f} \in \mathcal{V}\setminus\mathcal{V}'} (\widehat{z_{N_{\mathfrak{f}}}})_{k_{\mathfrak{f}}}^{\zeta_{\mathfrak{f}}}(\lambda_{\mathfrak{f}}).$$

$$h' = \mathtt{Trim}(h, R).$$

- **Merging**

  A way to illustrate tensor products. Grouping abstract indices to describe tensor products.

Given $\mathscr{B} = (N, N_j, \zeta_j, r)$, plants $\mathcal{S}_j = (\mathcal{L}_j, \mathcal{V}_j, \mathcal{Y}_j)$, plant tensors $h^{\mathcal{S}_j}$, $1 \leq j \leq r \leq 3$, and a base tensor h. Define

$$\mathcal{L} := \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_r,$$

$$\mathcal{V} := \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_r \cup \{ \underbrace{\boldsymbol{r+1}, \cdots, \mathbf{3}}_{\text{abstract placeholders to put in remainders}} \},$$

$$\mathcal{Y} := \mathcal{Y}_1 \cup \cdots \cup \mathcal{Y}_r \cup \{ \underset{\substack{\uparrow \\ \text{an abstract recorder of the skeleton of this layer}}}{\mathbf{med}} \},$$

$$\mathcal{S} = \mathtt{Merge}(\mathcal{S}_1, \cdots, \mathcal{S}_r, \mathscr{B}) := (\mathcal{L}, \mathcal{V}, \mathcal{Y}).$$

$$N(\mathcal{S}) := N.$$

$$H_{kk_{\mathcal{L}}}(k_{\mathcal{V}}, \lambda_{\mathcal{V}}) = \sum_{(k_1, \cdots, k_r)} h_{kk_1k_2k_3}(\lambda_{r+1}, \cdots, \lambda_3) \prod_{j=1}^{r} [h^{(j)}_{k_jk_{\mathcal{L}_j}}(k_{\mathcal{V}_j}, \lambda_{\mathcal{V}_j})]^{\zeta_j}.$$

$$H = \mathtt{Merge}(h^{(1)}, \cdots, h^{(r)}, h, \mathscr{B}).$$

Random averaging operators:

Interaction picture:

$$(v_N)_k(t) := \mathrm{e}^{\mathrm{i}t|k|}(u_N)_k(t).$$

Duhamel formulation:

$$(v_N)_k(t) = (F_N)_k - \mathrm{i}\int_0^t \Pi_N \mathcal{M}(v_N, v_N, v_N)_k(t')\,\mathrm{d}t'.$$

$$(F_N)_k := \mathbf{1}_{\langle k\rangle \le N}\langle k\rangle^{-\alpha/2} g_k.$$

$$\mathcal{M}(u, v, w)_k(t) = \sum_{k_1 - k_2 + k_3 = k} e^{\mathrm{i}t\Omega}\, u_{k_1}(t)\overline{v_{k_2}(t)}w_{k_3}(t).$$

$$\Omega := |k| - |k_1| + |k_2| - |k_3|.$$

$$(v_N)_k(t) = \eta(t)(F_N)_k - \mathrm{i}\cdot\eta_\tau(t)\mathcal{I}_\eta \Pi_N \mathcal{M}(v_N, v_N, v_N)_k(t).$$

Dyadic decomposition:

$$y_N := v_N - v_{N/2}, \quad v_N = \sum_{N' \leq N} y_N.$$

$$(y_N)_k(t) = \eta(t)(f_N)_k - \mathrm{i} \sum_{N_1, N_2, N_3 \leq N} \eta_\tau(t) \mathcal{I}_\eta \Pi \mathcal{M}(y_{N_1}, y_{N_2}, y_{N_3})_k(t).$$

$$f_N = F_N - F_{N/2}.$$

$\mathbb{R}$-linear random averaging operators:

$$(\mathscr{L}^M w)_k(t) = -\mathrm{i}\chi_\tau(t) \cdot \mathcal{I}_\chi \Pi_M \sum_{\mathrm{sym}} \mathcal{M}(w, v_{M^{[\delta]}}, v_{M^{[\delta]}})_k(t),$$

where

$$\sum_{\mathrm{sym}} \mathcal{M}(w, v, v) = \mathcal{M}(w, v, v) + \mathcal{M}(v, w, v) + \mathcal{M}(v, v, w).$$

Put out high-low interaction from the Duhamel.

$$(y_N)_k = \eta(t)(f_N)_k + (\mathscr{L}^N \underset{\substack{\uparrow \\ \text{used to} \\ \text{be type (C)} \\ \text{only}}}{y_N} \quad)_k - \mathrm{i} \sum_{\substack{N_1,N_2,N_3 \le N \\ (N^\delta < N_{\text{med}} \le N_{\max} = N \& \Pi = \Pi_N) \\ \text{or} \\ (N_{\max} \le \frac{N}{2} \& \Pi = \Delta_N)}} \eta_\tau(t) \mathcal{I}_\eta \Pi \mathcal{M}(y_{N_1}, y_{N_2}, y_{N_3})_k.$$

$$\mathscr{R}^N := (1 - \mathscr{L}^N)^{-1}.$$

ITERATION TEMPLATE EQUATION

$$\begin{aligned}(y_M)_k(t) = &\sum_{\zeta \in \{\pm\}} \sum_{k'} \int \mathrm{d}t' \cdot (\mathscr{R}^{M,\zeta})_{kk'}(t,t')\, \eta(t') \cdot (f_M)^\zeta_{k'} \\ &+ \sum_{\zeta \in \{\pm\}} \sum_{k'} \int \mathrm{d}t' \cdot (\mathscr{R}^{M,\zeta})_{kk'}(t,t') \sum_{N_1,N_2,N_3} \eta_\tau(t') \left[\mathcal{I}_\eta \Pi \mathcal{M}(y_{N_1}, y_{N_2}, y_{N_3})\right]^\zeta_{k'}(t').\end{aligned}$$

Inductive definition of the random tensors:

In the above few slides, we explain how to GUESS a multilinear ansatz, which was

$$\boxed{(y_N)_k(t) = \sum_{\substack{\mathcal{S}:N(\mathcal{S})=N\\|\mathcal{S}|\leq D}} \sum_{k_\mathcal{L},k_\mathcal{V}} \int \mathrm{d}\lambda_\mathcal{V} \cdot h^\mathcal{S}_{kk_\mathcal{L}}(t,k_\mathcal{V},\lambda_\mathcal{V}) \cdot \prod_{\mathfrak{l}\in\mathcal{L}} (f_{N_\mathfrak{l}})^{\zeta_\mathfrak{l}}_{k_\mathfrak{l}} \prod_{\mathfrak{f}\in\mathcal{V}} (\widehat{z_{N_\mathfrak{f}}})^{\zeta_\mathfrak{f}}_{k_\mathfrak{f}}(\lambda_\mathfrak{f}) + (z_N)_k(t).}$$

Given $M\geq 1$, let us assume we have already defined the $\mathcal{S}$-tensors $h^\mathcal{S}$, all regular plants $\mathcal{S}$ with $N(\mathcal{S})<M$ and $|\mathcal{S}|\leq D$, as well as $z_{N'}=(z_{N'})_k(t)$ for $N'<M$. For $N<M$, assume the existing tensors and remainders are such that the above ansatz holds. Then we construct the tensors $h^\mathcal{S}$ with $N(\mathcal{S})=M$ inductively in $|\mathcal{S}|$, such that when we plug the ansatz into **ITERATION TEMPLATE EQUATION** allowing $N=M$, the terms on the left and right sides cancel to a sufficiently high order so that the remainder would be regular enough and can be put in z_M.

Concretely, let

$$\mathcal{S} = \mathtt{Trim}(\mathtt{Merge}(\mathtt{Trim}(\mathcal{S}_1, M^\delta), \cdots, \mathtt{Trim}(\mathcal{S}_r, M^\delta), \mathscr{B}), M^\delta), \tag{7}$$

$$H = \mathtt{Trim}(\mathtt{Merge}(\mathtt{Trim}(h^{\mathcal{S}_1}, M^\delta), \cdots, \mathtt{Trim}(h^{\mathcal{S}_r}, M^\delta), h, \mathscr{B}), M^\delta),$$

where

$$h_{kk_1k_2k_3}(t',\lambda_{r+1},\cdots,\lambda_3)=1_{k=k_1-k_2+k_3}\cdot 1_{\langle k\rangle\leq M}\prod_{j=1}^{3}1_{\langle k_j\rangle\leq N_j}\\ \times\prod_{j=r+1}^{3}\chi(N^{-\kappa^2}\lambda_j)\,e^{it'(\Phi+\zeta_{r+1}\lambda_{r+1}+\cdots+\zeta_3\lambda_3)}.$$

Then we define the $h^{\mathcal{S}}$ tensors inductively in $|\mathcal{S}|$, by the equations

$$h^{\mathcal{S}}_{kk_{\mathcal{L}}}(t,k_{\mathcal{V}},\lambda_{\mathcal{V}})=\sum_{\zeta\in\{\pm\}}\mathbf{1}_{\mathcal{S}=\mathcal{S}_M^{\zeta}}\int \mathrm{d}t'\cdot\mathbf{1}_{M/2<\langle k_{\mathfrak{l}}\rangle\leq M}\cdot\mathscr{R}^{M,\zeta}_{kk_{\mathfrak{l}}}(t,t')\,\chi(t')\\ +\sum_{\zeta\in\{\pm\}}\sum_{k'}\int \mathrm{d}t'\cdot(\mathscr{R}^{M,\zeta})_{kk'}(t,t')\sum_{\mathrm{sym}}\sum_{(c[\zeta])}\chi_\tau(t')\,[\mathcal{I}_\chi\Pi H_{k'k_{\mathcal{L}}}](t',k_{\mathcal{V}},\lambda_{\mathcal{V}})^{\zeta}.$$

Here $\mathcal{S}_M^{\zeta}=(\{\mathfrak{l}\},\varnothing,\varnothing)$ are the mini plants. The summation $\sum_{(c[\zeta])}$ is taken over $\mathscr{B}$ and regular plants $\mathcal{S}_j$ with frequency $N_j\leq M$ and size $|\mathcal{S}_j|\leq D$ for $1\leq j\leq r$, such that

1. if $N_j=M$ for some $1\leq j\leq 3$ then there is $3\geq j'\neq j$ with $N_{j'}\geq M^{\delta}$;

2. $N_j \le M/2$ for $r+1 \le j \le 3$;

3. if $\zeta = +$ then (7) is true with the given $\mathcal{S}$, and if $\zeta = -$ then (7) is true with the left hand side replaced by $\bar{\mathcal{S}}$.

The term $H_{k' k_{\mathcal{L}}}(t', k_{\mathcal{V}}, \lambda_{\mathcal{V}})$ that appears in the summand is defined in (7) with $h^{\mathcal{S}_j}$ given by the induction hypothesis.

The above is a valid inductive definition, i.e. the tensors $h^{\mathcal{S}_j}$ in (7) are already defined when we use them to define $h^{\mathcal{S}}$, thanks to THE MEMORY $|\mathcal{S}| \ge |\mathcal{S}_j| + 1$. The size of the merged plant do grow (if $\Pi = \Pi_M$).

$$(z_M)_k(t) = \sum_{\zeta \in \{\pm\}} \sum_{k'} \int \mathrm{d}t' \cdot (\mathscr{R}^{M,\zeta})_{kk'}(t,t') \sum_{\text{sym}} \sum_{(d)} \chi_\tau(t')$$
$$\times [\mathcal{I}_\chi \Pi \mathcal{M}(\Psi_{k_1}^{\mathcal{S}_1}, \cdots, \Psi_{k_r}^{\mathcal{S}_r}, z_{N_{r+1}}^*, \cdots, z_{N_3}^*)]_{k'}^{\zeta}(t'),$$

where $z_{N_j}^*$ $(r+1 \le j \le 3)$ is either z_{N_j} or $z_{N_j}^{\text{lo}}$ or the high-modulation cutoff $z_{N_j}^{\text{hi}} := z_{N_j} - z_{N_j}^{\text{lo}}$. Here the sum $\sum_{(d)}$ is taken over $\mathscr{B}$, regular plants $\mathcal{S}_j$ with frequency N_j and size $|\mathcal{S}_j| \le D$ for $1 \le j \le r$, and choices of $z_{N_j}^*$, under the restrictions that (i) if

$N_j = M$ for some $1 \le j \le 3$ then there is $3 \ge j' \ne j$ with $N_{j'} \ge M^{\delta}$, (ii) either $N_j = M$ for at least one $r+1 \le j \le 3$ and $z^*_{N_j} = z_{N_j}$ for all $r+1 \le j \le 3$, or $N_j \le M/2$ for all $r+1 \le j \le 3$ and $z^*_{N_j} = z^{\mathrm{hi}}_{N_j}$ for at least one $r+1 \le j \le 3$, or $(N_j \le M/2) \wedge (z^*_{N_j} = z^{\mathrm{lo}}_{N_j})$ for all $r+1 \le j \le 3$ and the plant

$$\mathcal{S} = \mathtt{Trim}(\mathtt{Merge}(\mathtt{Trim}(\mathcal{S}_1, M^{\delta}), \cdots, \mathtt{Trim}(\mathcal{S}_r, M^{\delta}), \mathscr{B}), M^{\delta})$$

has size $|\mathcal{S}| > D$.

contraction mapping from the set $\{z_M : \|z_M\|_{X^{b_0}} \le M^{-D_1}\}$ to itself, we define z_M to be the unique fixed point of this mapping; otherwise define $z_M = 0$.

Bounds for random tensors:

Define an event called $\mathtt{Local}(M)$.

- Let (B, C) be a subpartition of $\mathcal{L}$, $E = \mathcal{L} \setminus (B \cup C)$. $\alpha/2 > \beta > 1/2$, $\theta \ll \delta \ll \varepsilon \ll \frac{\alpha - 1}{2}$. Then

$$\|h^{\mathcal{S}}_{kk_{\mathcal{L}}}(t, k_{\mathcal{V}}, \lambda_{\mathcal{V}})\|_{X^{1-b,-b_0}_{\mathcal{V}}[kk_B \to k_C]} \leq \prod_{\mathfrak{l} \in B \cup C} N_{\mathfrak{l}}^{\beta} \prod_{\mathfrak{l} \in E} N_{\mathfrak{l}}^{8\varepsilon} \prod_{\mathfrak{p} \in \mathcal{Y}} N_{\mathfrak{p}}^{-\delta^3} \prod_{\mathfrak{f} \in \mathcal{V}} N_{\mathfrak{f}} \cdot (\max_{\mathfrak{l} \in C} N_{\mathfrak{l}})^{-\beta},$$

$$\|h^{\mathcal{S}}_{kk_{\mathcal{L}}}(t, k_{\mathcal{V}}, \lambda_{\mathcal{V}})\|_{X^{\tilde{b},-b_0}_{\mathcal{V}}[kk_B]} \leq \prod_{\mathfrak{l} \in B} N_{\mathfrak{l}}^{\beta} \prod_{\mathfrak{l} \in E} N_{\mathfrak{l}}^{8\varepsilon} \prod_{\mathfrak{p} \in \mathcal{Y}} N_{\mathfrak{p}}^{-\delta^3} \prod_{\mathfrak{f} \in \mathcal{V}} N_{\mathfrak{f}} \cdot N^{-\varepsilon}.$$

- Let $D_1 = \delta^{-100}$. Then

$$\|z_N\|_{X^{b_0}} \leq N^{-D_1}.$$

-

$$\|\mathscr{R}^N\|_{X^{b,-b}[k \to k']} \lesssim 1.$$

$$\mathbb{P}(\mathtt{Local}(M) \wedge \neg\mathtt{Local}(2M)) \leq C_\theta e^{-(\tau^{-1}M)^\theta}.$$

Proof of a.s. LWP. It follows from above that τ^{-1}-certainly, the event $\texttt{Local}(M)$ holds for all M. Recall the ansatz that

$$
\begin{aligned}
(y_N)_k(t) = &\sum_{\substack{\mathcal{S}:N(\mathcal{S})=N \\ |\mathcal{S}|\le D}} \sum_{k_{\mathcal{L}},k_{\mathcal{V}}} \int \mathrm{d}\lambda_{\mathcal{V}} \cdot h^{\mathcal{S}}_{kk_{\mathcal{L}}}(t, k_{\mathcal{V}}, \lambda_{\mathcal{V}}) \cdot \prod_{\mathfrak{l}\in\mathcal{L}} (f_{N_{\mathfrak{l}}})^{\zeta_{\mathfrak{l}}}_{k_{\mathfrak{l}}} \prod_{\mathfrak{f}\in\mathcal{V}} (\widehat{z_{N_{\mathfrak{f}}}})^{\zeta_{\mathfrak{f}}}_{k_{\mathfrak{f}}}(\lambda_{\mathfrak{f}}) + (z_N)_k(t) \\
=: &(\psi_N)_k(t) + (z_N)_k(t).
\end{aligned}
$$

To prove the local well-posedness, it suffices to justify the convergence of the summation $\sum_N y_N$ in some proper sense.

When we fix $k_{\mathcal{L}}$, we will denote by $\mathcal{Q}$ the set of all paired or over-paired leaves in $\mathcal{L}$. Then we see that

$$
\begin{aligned}
\|\psi_N\|_{X^b(\mathcal{J})} \lesssim &\sum_{\mathcal{S},\mathcal{Q}} \|h^{\mathcal{S}}_{kk_{\mathcal{L}}}(k_{\mathcal{V}}, \lambda_{\mathcal{V}})\|_{X^{b,-b_0}_{\mathcal{V}}[kk_{\mathcal{L}\setminus\mathcal{Q}}]} \prod_{\mathfrak{l}\in\mathcal{L}\setminus\mathcal{Q}} N_{\mathfrak{l}}^{-\alpha} \prod_{\mathfrak{f}\in\mathcal{V}\setminus\mathcal{V}'} \|z_{N_{\mathfrak{f}}}\|_{X^{b_0}} \\
\lesssim &(\log N)^{\kappa} N^{\theta} \prod_{\mathfrak{l}\in\mathcal{Q}} N_{\mathfrak{l}}^{\frac{1}{2}-\alpha+\theta} \prod_{\mathfrak{l}\in\mathcal{L}\setminus\mathcal{Q}} N_{\mathfrak{l}}^{\beta-\alpha} \prod_{\mathfrak{p}\in\mathcal{Y}} N_{\mathfrak{p}}^{-\delta^3} \prod_{\mathfrak{f}\in\mathcal{V}} N_{\mathfrak{f}}^{1-D_1} \cdot N^{-\varepsilon} \\
\lesssim &N^{\frac{1}{2}-\alpha+2\varepsilon},
\end{aligned}
$$

where we use interpolation between $1-b$ and D^D in the first step, $\sum_{\mathcal{S},\mathcal{Q}} 1 \lesssim (\log N)^\kappa$ in the second step, and $N \sim \max_{\mathfrak{l}\in\mathcal{L}} N_{\mathfrak{l}}$ and $\beta = 1/2 + 2\varepsilon$ in the final step. Therefore,

$$\sum_N y_N \quad \text{converges in } C_t(\mathcal{J}; H^{s-3\varepsilon}(\mathbb{T})). \tag{8}$$

where $s = \alpha - 1/2$. From Proposition-(2), we also know that

$$\sum_N z_N \quad \text{converges in } C_t(\mathcal{J}; H^{D_1-\varepsilon}(\mathbb{T})). \tag{9}$$

Finally, by collecting (8) and (9), we conclude that the sequence

$$u_N = \sum_{N'=1}^{N} y_{N'}$$

converges in $C_t(\mathcal{J}; H^{s-3\varepsilon}(\mathbb{T}))$, which completes the proof. $\square$

Thank you for your attention!