

1. How many ways are there to arrange the letters in the word "probabilistic"?

$$\frac{13!}{3! \times 2!}$$

2. With reference to the previous problem, how many ways if all the b's have to precede all the i's?

$$\binom{8+5}{5} * 8! = \binom{13}{5} * 8!$$

3. How many ways are there to put 100 (unlabelled) balls into 50 labelled boxes?

$$\binom{50-1+100}{100} = \binom{149}{100}$$

4. We computed the "mean time to failure" to be  $1/p$  if the probability of failure is  $p$ . What is the mean time to the second failure? In a sense, the answer is obvious, but prove it from first principles. That is, write down an expression for  $q_k$ , the probability that the second failure occurs on the  $k$ -th trial, and then compute and simplify the weighted sum  $\sum kq_k$ .

*HINT: Recall that in class we computed the mean time to failure by determining  $p_k$ , the probability that the first failure occurs on the  $k$ -th trial, to be the quantity in Definition 19.4.6 of the MIT notes. We then simplified  $\sum kp_k$ , the expected number of trials, using the formula  $(1-x)^{-2} = 1 + 2x + 3x^2 + \dots$ , which we obtained by differentiating  $(1-x)^{-1} = 1 + x + x^2 + \dots$*

$$q_k = (1-p)^{k-1}p, \text{ then } \sum_{k=1}^{\infty} (p)_k = \sum_{k=1}^{\infty} (1-p)^{k-1}p = p \sum_{k=0}^{\infty} (1-p)^k$$

$$\sum_{k=1}^{\infty} k(1-p)^{k-1}p = p \sum_{k=1}^{\infty} k(1-p)^{k-1} = p \cdot \frac{1}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}$$

5. What number do you get if you subtract the binomial coefficients  $\binom{n}{k}$  with an even  $k$  from those with an odd  $k$ , where  $n$  is fixed?

If  $n$  is even, then  $\binom{n}{k}$  equal to  $\binom{n-1}{k-1} + \binom{n-1}{k}$ , and  $\binom{n}{k+1}$  equal to  $\binom{n-1}{k} + \binom{n-1}{k+1}$ , thus the sum is 0.

And for  $n$  is odd, then  $k$  is odd, and  $n-k$  is even, thus  $\binom{n}{n-k}$  equal to  $\binom{n}{k}$ , and  $\binom{n}{n-k-1} = \binom{n}{k+1}$ , thus the sum is 0.

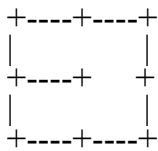
In conclusion, the sum is 0

**6. What difference of binomial coefficients is equal to the sum**

$$\binom{12}{5} + \binom{11}{5} + \binom{10}{5} + \binom{9}{5} + \binom{8}{5} ?$$

$$\binom{12}{5} + \binom{11}{5} + \binom{10}{5} + \binom{9}{5} + \binom{8}{5} = \binom{13}{6} - \binom{8}{6}$$

**7. Imagine a maze created in a  $m$  by  $n$  grid. Assume that there is a unique path from any cell to any other cell. What is the total length of the walls in the maze as a function of  $m$  and  $n$ ? For example, below is a 2 by 2 grid with the required path property and the total length of walls is 9. Explain your answer.**



The formula is  $(m+1)*(n+1)$ . To be precise, assume it is a square, then if it is a 3 by 3 grid, then the total length of walls is 16, which is  $(3+1)*(3+1)$ ; if it is a 4 by 4 grid, then the total length of walls is 25, which is  $(4+1)*(4+1)$ . Thus, the formula should be  $(m+1)*(n+1)$ .

Assume it is a rectangle, then if it is a 3 by 2 grid, then the total length of walls is 12, which is  $(3+1)*(2+1)$ ; if it is a 4 by 3 grid, then the total length of walls is 20, which is  $(4+1)*(3+1)$ . Thus, the formula should be  $m+1)*(n+1)$ .