IE 5531 – Practice Midterm #1

Prof. John Gunnar Carlsson

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Problem 1: Some simple LPs Give an explicit solution for each of the following linear programs:

1. A linear function over an affine set

$$\begin{array}{rcl} \text{minimize } \mathbf{c}^T\mathbf{x} & & s.t. \\ A\mathbf{x} & = & \mathbf{b} \end{array}$$

Solution The dual of this problem is

$$\begin{array}{rcl} \text{maximize } \mathbf{b}^T\mathbf{y} & & s.t. \\ A^T\mathbf{y} & = & \mathbf{c} \end{array}$$

Suppose that the primal problem is feasible. The dual is feasible only if the system $A^T\mathbf{y} = \mathbf{c}$ has a solution, in which case the optimal value is $\mathbf{c}^T\mathbf{x}^* = \mathbf{y}^{*T}A\mathbf{x}$. The primal problem is unbounded otherwise.

2. A linear function over a half space

$$\begin{array}{rcl} \text{minimize } \mathbf{c}^T\mathbf{x} & & s.t. \\ \mathbf{a}^T\mathbf{x} & \leq & b \end{array}$$

(with $\mathbf{a} \neq \mathbf{0}$)

Solution The primal problem is clearly feasible. The dual is

$$\begin{array}{rcl} \text{maximize } by & & s.t. \\ \mathbf{a}y & = & \mathbf{c} \\ y & \leq & 0 \end{array}$$

which is feasible only if $\mathbf{a}y = \mathbf{c}$ for some $y \leq 0$ and infeasible otherwise. Therefore the primal problem has objective function value ba_1/c_1 if $\mathbf{a}y = \mathbf{c}$ for some y, and unbounded otherwise.

3. A linear function over a rectangle

$$\text{minimize } \mathbf{c}^T\mathbf{x} \quad s.t.$$

$$\ell \leq \mathbf{x} \leq \mathbf{u}$$

where $\ell, \mathbf{u} \geq 0$ and $\ell \leq \mathbf{u}$.

Solution Without using duality it is clear that the optimal solution has $x_i = \ell_i$ for all indices i such that $c_i > 0$ and $x_i = u_i$ for all indices i such that $c_i < 0$.

4. A linear function over the probability simplex

$$\begin{array}{rcl} \text{minimize } \mathbf{c}^T\mathbf{x} & & s.t. \\ \mathbf{1}^T\mathbf{x} & = & 1 \\ \mathbf{x} & \geq & \mathbf{0} \end{array}$$

where $\mathbf{1}$ is a vector consisting of all 1's.

Solution The dual is

$$\begin{aligned} \text{maximize } y \quad & s.t. \\ y \leq \; & c_i \, \forall i \end{aligned}$$

which clearly has an optimal solution $\min_i \{c_i\}$. Therefore the optimal solution to the primal is to set $x_{\bar{i}} = 1$ where $\bar{i}$ is the index of the smallest element of $\mathbf{c}$.

5. A linear function over a unit box with a budget constraint

$$\begin{aligned} \text{minimize } \mathbf{c}^T\mathbf{x} & \quad s.t. \\ \mathbf{1}^T\mathbf{x} &= \alpha \end{aligned}$$

$$\mathbf{0} \leq \mathbf{x} \leq \mathbf{1}$$

where α is an integer between 0 and n.

Solution Suppose without loss of generality that $c_1 \leq c_2 \leq \ldots \leq c_\alpha \leq \cdots \leq c_n$. Clearly the optimal solution is to set $x_1 = \cdots = x_\alpha = 1$ and $x_{\alpha+1} = \cdots = x_n = 0$ at which point the optimal objective value is $c_1 + \cdots + c_\alpha$. We can verify this by complementary slackness. The dual problem is

$$\begin{aligned} \text{maximize } (\alpha, \mathbf{1}^T) \begin{pmatrix} y \\ \mathbf{z} \end{pmatrix} & \quad s.t. \\ y + z_i & \leq c_i \; \forall i \\ y & \quad \text{free} \\ \mathbf{z} & \leq \mathbf{0} \end{aligned}$$

and therefore the proposed solution is optimal if and only if the complementary slackness conditions hold, i.e.

$$\begin{aligned} y + z_i &= c_i \; i \in \{1, \ldots, \alpha\} \\ z_i &= 0 \; i \in \{\alpha + 1, \ldots, n\} \end{aligned}$$

By setting $y = c_\alpha$, we can find feasible z_i by setting $z_i = c_i - y \leq 0$ for $i \in \{1, \ldots, \alpha\}$. The dual objective value is

$$\begin{aligned} \alpha c_\alpha + z_1 + \cdots + z_\alpha &= \alpha c_\alpha + (c_1 - y) + \cdots + (c_\alpha - y) \\ &= c_1 + \cdots + c_\alpha \end{aligned}$$

as desired.

Problem 2: Production problem duality Consider a production problem with m constraints given by

$$\begin{aligned} \text{maximize } \mathbf{c}^T\mathbf{x} & & s.t. \\ A\mathbf{x} &\leq & \mathbf{b} \\ \mathbf{x} &\geq & \mathbf{0} \end{aligned}$$

Consider the standard form representation

$$\begin{aligned} \text{minimize } -\mathbf{c}^T\mathbf{x} & & s.t. \\ A\mathbf{x}+\mathbf{s} &= & \mathbf{b} \\ \mathbf{x},\mathbf{s} &\geq & \mathbf{0} \end{aligned}$$

Show that the optimal dual variables $\mathbf{y}^*$ can be obtained directly from the final simplex tableau.

Hint 1 Write $\tilde{A} = (A, I)$, where $I \in \mathbb{R}^{m\times m}$ is the identity matrix, and $\tilde{\mathbf{c}} = \begin{pmatrix} \mathbf{c} \\ \mathbf{0} \end{pmatrix}$ to accommodate the slack variables. Then, recall that the reduced cost vector (the top row of the simplex tableau) is the transpose of

$$\mathbf{r} = -\left[\tilde{\mathbf{c}} - \tilde{A}^T \left(\tilde{A}_B^{-1}\right)^T \tilde{\mathbf{c}}_B\right].$$

Write a formula for the last m elements of $\mathbf{r}$ at the final simplex tableau.

Hint 2 Also remember that $\mathbf{y}^* = \left(\tilde{A}_B^{-1}\right)^T \tilde{\mathbf{c}}_B$.

Solution Following the hint, we have

$$\mathbf{r} = -\left[\tilde{\mathbf{c}} - \tilde{A}^T \left(\tilde{A}_B^{-1}\right)^T \tilde{\mathbf{c}}_B\right] \geq \mathbf{0}$$

The last m elements of $\mathbf{r}$ are precisely the last m elements of the vector

$$-\begin{pmatrix} \mathbf{c} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} A^T \\ I \end{pmatrix} \left(\tilde{A}_B^{-1}\right)^T \tilde{\mathbf{c}}_B$$

which is simply

$$(\mathbf{0}) + (I)\left(\tilde{A}_B^{-1}\right)^T \tilde{\mathbf{c}}_B = \left(\tilde{A}_B^{-1}\right)^T \tilde{\mathbf{c}}_B$$

Thus, the optimal dual variables (the marginal prices) are precisely the last m elements of $\mathbf{r}$ in the final simplex tableau.

Problem 3: Parametric linear programming Consider a market that involves a stock A, which has a price of \$1 per share today. The price of A tomorrow is a random variable S, which can take two values, \$2 or \$0.50. The market also has a "put option" P, which is a financial product that gives the owner the right to sell one share of A at a strike price K tomorrow. Each share of P has a price of \$0.10 today and has a payoff $\max\{K-S,0\}$ tomorrow. We use the notation $(K-S)^+ := \max\{K-S,0\}$ throughout this problem. We want to determine the optimal amounts of stocks θ_A and put options θ_P to purchase, given by the linear program below[1]:

$$\begin{aligned} \text{minimize } \theta_A + 0.1\theta_P & \quad s.t. \\ 2\theta_A + (K-2)^+\theta_P &\geq 0 \\ 0.5\theta_A + (K-0.5)^+\theta_P &\geq 0 \\ \theta_A, \theta_P & \quad \text{free} \end{aligned}$$

A market has an *arbitrage opportunity* if there is a way to earn a strictly positive payoff by buying and selling assets in the market today, with nonnegative payoffs in the future (in other words, guaranteed free money, regardless of which outcome happens). A market with no arbitrage opportunity is called *arbitrage free*.

1. Suppose that $K = 0.5$ and answer the questions below. You do not need to use the simplex method to solve any of this.

   (a) Will you buy any positive amount of put options with this strike price?

   Solution No. θ_P has no impact on feasibility and it strictly increases the objective function value.

   (b) Give a feasible solution (θ_A, θ_P) to the above linear program with negative objective function value. Argue that the problem is unbounded, and explain why this corresponds to an arbitrage opportunity in the market.

   Solution Setting $\theta_A = 0$ and $\theta_P = -1$ is a feasible solution with objective function value -0.1. We could take any multiple of θ_P and remain feasible.

2. Suppose $K = 0.7$ and answer the questions below. You do not need to use the simplex method to solve any of this.

   (a) Write the dual program.

   Solution The dual is

$$\begin{aligned} \text{maximize } 0 & \quad s.t. \\ 2y_1 + 0.5y_2 &= 1 \\ 0.2y_2 &= 0.1 \\ y_1, y_2 &\geq 0 \end{aligned}$$

   (b) Is the dual feasible? What is the optimal solution of the dual program?

   Solution The dual is feasible. Set $y_2 = 0.5$ and $y_1 = 0.375$.

   (c) Argue that if the dual program is feasible then the market is arbitrage-free (in other words, there is no portfolio with a negative cost today and nonnegative payoff tomorrow).

3. The case $K = 0.5$ shows that improper pricing of the put option leads to an arbitrage opportunity in the market. Find the range of K for whch there is no arbitrage opportunity (hint: there is an arbitrage opportunity if and only if the dual program is infeasible).

[1] The payoff of an option P with strike price K is $(K-S)^+$; if $K > S$ then we can buy a stock at price S and sell it immediately at price K and earn a profit, and if $K \leq S$ then we lose money by selling the good at price K, so we gain nothing. The modern financial market allows people to sell stocks and options "short", meaning that one can sell items without actually having any to start with; this corresponds to a negative value for θ_A or θ_P. Here the first constraint means that we must have a nonnegative payoff if the first scenario occurs, and the second constraint means that we must have a nonnegative payoff if the second scenario occurs.

Solution The dual program is feasible if and only the system

$$\begin{pmatrix} 2 & 0.5 \\ (K-2)^+ & (K-0.5)^+ \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.1 \end{pmatrix}$$

has a nonnegative solution. For $K \le 0.5$ this system is infeasible. If $0.5 < K \le 2$ then the system takes the form

$$\begin{pmatrix} 2 & 0.5 \\ 0 & s \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.1 \end{pmatrix}$$

where $0 < s \le 1.5$. We can see that $y_1 \ge 0$ if and only if $s \ge 0.05$ and therefore $K \ge 0.55$. Finally, if $K > 2$, the system takes the form

$$\begin{pmatrix} 2 & 0.5 \\ s & s+1.5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.1 \end{pmatrix}$$

with $s > 0$. We can write

$$\begin{pmatrix} 2 & 0.5 \\ s & s+1.5 \end{pmatrix}^{-1} = \frac{1}{2\,(s+1.5) - 0.5s} \begin{pmatrix} s+1.5 & -0.5 \\ -s & 2 \end{pmatrix}$$

Note that since $s > 0$ the denominator is always positive, so we simply require

$$\begin{pmatrix} s+1.5 & -0.5 \\ -s & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0.1 \end{pmatrix} \ge \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which is true for $K \le 2.2$.

Problem 4: Graphical simplex method Consider the problem

$$
\begin{aligned}
\text{minimize } -2x_1 - x_2 & \qquad s.t. \\
x_1 - x_2 &\leq 2 \\
x_1 + x_2 &\leq 6 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

1. Convert the problem into standard form and construct a basic feasible solution at which $x_1 = x_2 = 0$.

   Solution The BFS is $x_3 = 2$ and $x_4 = 6$.

2. Carry out the simplex method, starting with the BFS from the previous part.

   Solution The optimal basic set is $\{1, 2\}$, at which point $x_1 = 4$ and $x_2 = 2$.

3. Draw a graphical representation of the problem interms of the original variables x_1, x_2, and indicate the path taken by the simplex algorithm.

   Solution Depending on the choice of entering and exiting variables, the path should either be $(0,0), (0,6), (4,2)$ or $(0,0), (2,0), (4,2)$.

4. Determine the range of values of $b_1 = 2$ for which the current basic set remains optimal.

   Solution Following the directions in lecture we find that we require that $-6 \leq b_1 \leq 6$.