

Ass1 Solutions

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1 Solutions

1.1 Prob1.

If $\det(A) = 0$, then $\det(A^T) = 0$, therefore $\det(A) = \det(A^T)$.

If $\det(A) \neq 0$, then matrix has inverse. So A is a full rank matrix. Since every full rank matrix can be written as a product of elementary matrices:

$$R_1 \dots R_t I_n C_1 \dots C_s$$

where R_i are row elementary matrices and C_i are column elementary matrices. We know that for all elementary matrices, $\det(E) = \det(E^T)$ (because column expansion or row expansion of \det doesn't change the value) and $(A_1 \dots A_n)^T = A_n^T \dots A_1^T$. $\det(I_n) = \det(I_n^T) = 1$ Therefore,

$$\begin{aligned} \det(A^T) &= \det((R_1 \dots R_t I_n C_1 \dots C_s)^T) \\ &= \det(C_s^T \dots C_1^T I_n^T R_t^T \dots R_1^T) \\ &= \det(C_s^T) \dots \det(C_1^T) \det(I_n^T) \det(R_t^T) \dots \det(R_1^T) \\ &= \det(C_s) \dots \det(C_1) \det(I_n) \det(R_t) \dots \det(R_1) \\ &= \det(R_1) \dots \det(R_t) \det(I_n) \det(C_1) \dots \det(C_s) \\ &= \det(R_1 \dots R_t I_n C_1 \dots C_s) \\ &= \det(A) \end{aligned}$$

1.2 Prob 2.

For any n , an identity matrix $A \in R^{n \times n}$ is a matrix only have element equal to 1 on diagonal and all other places are 0s

$$A = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Assume that when A is an indentity matrix of $A \in R^{n \times n}$, $\det(A) = \prod_{i=1}^n a_{ii} = 1$. We prove this using induction.

When $n = 1$, A is a scalar and $\det(A) = a_{11} = 1$.

When the size is $n - 1$, $\det(A^{(n-1) \times (n-1)}) = \prod_{i=1}^{n-1} a_{ii} = 1$. When consider the identity matrix of size n , expanding the matrix along the first row. Since only the first element is 1 and all the other are 0s, we have:

$$\begin{aligned} & 1 \times \det(A^{(n-1) \times (n-1)}) + 0 + \dots + 0 \\ &= a_{11} \times \prod_{i=2}^n a_{ii} \\ &= \prod_{i=1}^n a_{ii} = 1 \end{aligned}$$

Therefore, for any n we have $\det(A) = 1$, when A is an identity matrix and $A \in R^{n \times n}$.

1.3 Prob 3.

For any triangular matrix A of size $n \times n$, $\det(A) = \prod_{i=1}^n a_{ii}$. Therefore, if A has eigen values then,

$$\det(\lambda I - A) = \prod_{i=1}^n (\lambda - a_{ii}) \quad (1)$$

$$= 0 \quad (2)$$

The equation 1 holds if and only if $\lambda_i = a_{ii}$ for some i . Therefore, the diagonal elements are equal to eigen values.

1.4 Prob 4.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} \\ &= \lambda^2 - 1 = 0 \end{aligned}$$

Therefore, matrix A has two eigen values. $\lambda_1 = 1$ and $\lambda_2 = -1$. Matrix A is non-negative but $\lambda_2 = -1 < 0$.

1.5 Prob 5.

(a)

$$\begin{aligned} \frac{\partial(a^T x)}{\partial x} &= \frac{\partial(x^T a)}{\partial x} = a \\ \nabla f(x) &= a \end{aligned}$$

(b) Suppose $y = Px$,

$$\begin{aligned}\frac{\partial(x^T Px)}{\partial x} &= \frac{\partial(x^T y)}{\partial x} + \frac{\partial y^T}{\partial x} \frac{\partial(x^T y)}{\partial y} \\ &= y + \frac{\partial(x^T P^T)}{\partial x} \frac{\partial(x^T y)}{\partial y} \\ &= y + P^T x = (P + P^T)x\end{aligned}$$

Therefore,

$$\nabla f(x) = \frac{1}{2}(P + P^T)x + q$$

(c) Because $P = P^T$,

$$\nabla f(x) = \frac{1}{2}(P + P^T)x = Px$$

(d) Suppose $g(x) = \exp(a^T x + b)$, then $f(x) = f(g(x)) = \frac{g(x)}{1 + g(x)}$

$$\frac{\partial g(x)}{\partial x} = \frac{\partial(\exp(a^T x + b))}{\partial x} = \exp(a^T x + b)a$$

Therefore,

$$\nabla f(x) = a \frac{\exp(a^T x + b)}{(1 + \exp(a^T x + b))^2}$$

1.6 Prob 6.

Suppose $X^T = [X_1^T, X_2^T, \dots, X_n^T]^T$, where $X_i \in R^m$ is the i th column of X . Similarly $Y = [Y_1, Y_2, \dots, Y_n]$ where $Y_i \in R^m$ is the i th column of Y . Therefore, let $C = X^T Y$ then,

$$C = \begin{bmatrix} X_1^T Y_1 & X_1^T Y_2 & \dots & X_1^T Y_n \\ X_2^T Y_1 & X_2^T Y_2 & \dots & X_2^T Y_n \\ \vdots & \vdots & \ddots & \vdots \\ X_n^T Y_1 & X_n^T Y_2 & \dots & X_n^T Y_n \end{bmatrix}$$

Therefore,

$$\langle X, Y \rangle = \text{tr}(X^T Y) = \sum_{i=1}^n X_i^T Y_i = \sum_{i=1}^n \sum_{j=1}^m x_{ji} y_{ji} \quad (3)$$

Property1: symmetry

$$\begin{aligned}\langle X, Y \rangle &= \text{tr}(X^T Y) \\ &= \sum_{i=1}^n \sum_{j=1}^m y_{ji} x_{ji} \\ &= \text{tr}(Y^T X) \\ &= \langle Y, X \rangle\end{aligned}$$

Property2: linearity

Since $\text{tr}(X + Y) = \text{tr}(X) + \text{tr}(Y)$ and $\text{tr}(\lambda X) = \lambda \text{tr}(X)$.

$$\begin{aligned}\langle cA + B, Y \rangle &= \text{tr}((cA + B)^T Y) \\ &= \text{tr}(cA^T Y + B^T Y) \\ &= c \text{tr}(A^T Y) + \text{tr}(B^T Y) \\ &= c \langle A, Y \rangle + \langle B, Y \rangle\end{aligned}$$

Property3: definiteness

From equation 3, $\langle A, A \rangle = \sum_{i=1}^n \sum_{j=1}^m a_{ji}^2$. Therefore, $\langle A, A \rangle = 0$ iff $A = 0$, otherwise $\langle A, A \rangle > 0$.

1.7 Prob 7.

Proof:

Suppose $x^T A x = a$. Because $x \in R^n, A \in R^{n \times n}$, then $a \in R$, then $(x^T A x)^T = a^T = a$. Because $(A A^T)^T = A^T A$,

$$\begin{aligned}(x^T A x)^T &= x^T A^T x \\ &= x^T (-A) x \\ &= -(x^T A x) \\ &= -a\end{aligned}$$

Therefore, $a = -a, a = 0$, namely $x^T A x = 0$.

1.8 Prob 8.

Proof:

Because $E(X) = 0, E(Y) = 0$. Therefore, $\text{Var}(X) = E(X^2) - E(X)^2 = E(X^2)$ and similarly, $\text{Var}(Y) = E(Y^2)$. $\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = E(XY)$. According to Cauchy - Schwarz inequality $|\langle X, Y \rangle|^2 \leq \langle X, X \rangle \langle Y, Y \rangle$

$$\begin{aligned}|\text{Cov}(X, Y)|^2 &= |E(XY)|^2 \\ &= |\langle X, Y \rangle|^2 \\ &\leq \langle X, X \rangle \langle Y, Y \rangle \\ &= E(X^2) E(Y^2) \\ &= \text{Var}(X) \text{Var}(Y)\end{aligned}$$