Ass5 Solutions

Chang Li

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1 Solutions

1.1 Linear Programming

The problem reaches its optimal value when

$$x^* = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}$$

The optimal value p^* is 1.5.

The following is the source code:

```
%% Q1: LINEAR PROGRAMMING
clearvars;
% Input data
n = 4;
% Parameters of objective function
q = [1,2,3,4];
% Parameters of equality constraints
C = [1,1,1,1;1,-1,1,-1];
d = [1;0];
% Parameters of inequal constraints
l = [0,0,0,0]';

% cvx opt
cvx_begin
  variable x(n)
  minimize( q*x )
  subject to
  C*x==d
  l<=x
cvx_end
```

1.2 Regularized Maximum Likelihood Estimation

1.2.1 a

Because dataset D is drawn from a Gaussian distribution with mean μ and covariance Σ. Suppose $x \in R^n$, so the density is:

$$p(x) = (2\pi)^{-n/2} det(\Sigma)^{-1/2} exp(-(x-\mu)^T \Sigma^{-1}(x-\mu)/2)$$

The average log-likelihood function has the form (constant parameters are left out):

$$\begin{aligned} l(D) &= E(\log p(x_1, x_2, \ldots, x_m)) \\ &= E(-\log(2\pi) - \log det\Sigma - \sum_{i=1}^{M}(-(x_i-\mu)^T\Sigma^{-1}(x_i-\mu))) \end{aligned}$$

Since $(x-\mu)^T\Sigma^{-1}(x-\mu)$ is a number and $\text{trace}(AB) = \text{trace}(BA)$ therefore,

$$(x-\mu)^T\Sigma^{-1}(x-\mu) = \text{tr}((x-\mu)^T\Sigma^{-1}(x-\mu)) = \text{tr}(\Sigma^{-1}(x-\mu)^T(x-\mu))$$

Plugging into expectation we get:

$$\begin{aligned} E[(x-\mu)^T\Sigma^{-1}(x-\mu)] &= E[\text{tr}((x-\mu)^T(x-\mu)\Sigma^{-1})] \\ &= \text{tr}(E[(x-\mu)^T(x-\mu)]\Sigma^{-1}) \\ &= \text{tr}(\hat{\Sigma}\Sigma^{-1}) \end{aligned}$$

Plugging into the third term of $l(D)$ we can reformulate it as:

$$\begin{aligned} l(D) &= -\log det\Sigma - tr(\hat{\Sigma}\Sigma^{-1}) - \log(2\pi) \\ &= \log det\Sigma^{-1} - tr(\hat{\Sigma}\Sigma^{-1}) + const \end{aligned}$$

1.2.2 b

Because $K \succ 0$, we use the fact in section 3.1.5 [1] that the log of determinant is concave so its negation is convex and $\text{tr}(\hat{\Sigma}K)$ is also convex. And the absolute value function and summation operation are also convex operations. Therefore the regularized maximum likelihood optimization problem is a summation of convex functions and hence the Total Variation Denoising problem is convex.

1.2.3 c

The following figure 1 uses $\lambda \in [1e-5, 10]$ as x axis. The blue line is number of non-zero elements in the matrix. The red line is the optimal value of the problem. The yellow line is the corresponding value of log-likelihood without regularization term (and constant).

The following is the source code:

```
%% Q2: Regularized ML Estimation
clearvars;
load asgn5q2.mat;
% Input data
[m, n] = size(X);
x_corr = cov(X);
% off diagonal matrix
idx = 1-eye(n,n);
sum_idx = ones(1,n);
% range of lambda
iters_num = 20;
lambda_arr = logspace( -5, 1, iters_num );
% Results history arrays
S_nonzeros_arr = zeros(1,iters_num);
```

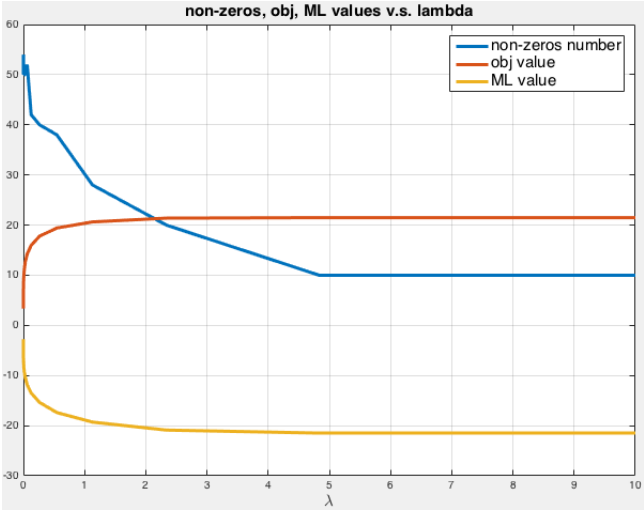

Figure 1: values change versus lambda

```
15    obj_arr = zeros(1,iters_num);
16    ml_arr = zeros(1,iters_num);
17
18    % cvx opt loop
19    for i=1:iters_num
20      lambda = lambda_arr(i);
21      cvx_begin sdp
22        variable S(n,n) symmetric
23        minimize ( -log_det(S) + trace(x_corr*S)...
24        + lambda * sum_idx * abs(idx.*S)*sum_idx' );
25        subject to
26          S==semidefinite(n,n);
27      cvx_end
28
29      % record history
30      ml_arr(i) = log_det(S) - trace(x_corr*S);
31      obj_arr(i) = cvx_optval;
32      S_nonzeros_arr(i) = sum(sum(S>1e-6));
33    end
34
35    % plot graph
36    figure();
37    plot(lambda_arr,[S_nonzeros_arr;obj_arr;ml_arr], 'LineWidth', 3);
38    xlabel('\lambda','FontSize',15);
39    l = legend('non-zeros number', 'obj value', 'ML value');
40    set(l, 'FontSize',15);
41    title('non-zeros, obj, ML values v.s. lambda', 'FontSize',15);
42    grid on;
```

1.3 Total Variation Denoising

To find a "good" value of λ, we print out the trade-off curve between $||D * \hat{x}||_1$ and $||\hat{x} - x_{corr}||_2$ in the following figure 2:

We can find that when $||D * \hat{x}||_1$ is around $[4.8, 5.2]$, the curve has a clear knee. After some experiments we found when $\lambda = 0.2212$ the recovered signal

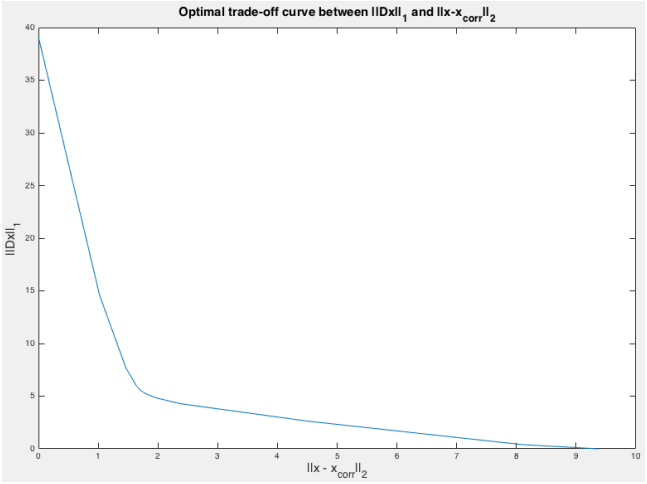

Figure 2: trade-off curve

has relatively good performance. The following is a figure 3 to show our contrast between the original corrupted signal and the signal recovered when $\lambda = 0.2212$:

In the figure 3 we can observe that most of the noisy are smoothed away, which means the regularize term gives a significant noisy reduction. However, at the same time the rapid variations in the original signal are also preserved. So we think this value is much preferable.

The following is the source code:

```
%% Q3: Total Variation Denoising
clearvars;
load asgn5q3.mat;
% input data
n = length(x_corr);
% difference matrix
e = ones(n,1);
D = spdiags([-e e], -1:0, n, n);
D(1)=0;
% range of lambda
iters_num = 30;
lambda_arr = logspace( -5,1, iters_num );
% Results history arrays
obj_arr = zeros(1,iters_num);
norm_error = zeros(1,iters_num);
norm_regu = zeros(1,iters_num);

% cvx opt loop
for i = 1:iters_num
  lambda = lambda_arr(i);
  cvx_begin
    variable x(n)
    minimize( norm(x-x_corr) + lambda*norm(D*x,1));
  cvx_end
  % record history
  obj_arr(i) = cvx_optval;
  norm_error(i) = norm(x-x_corr);
  norm_regu(i) = norm(D*x,1);
```

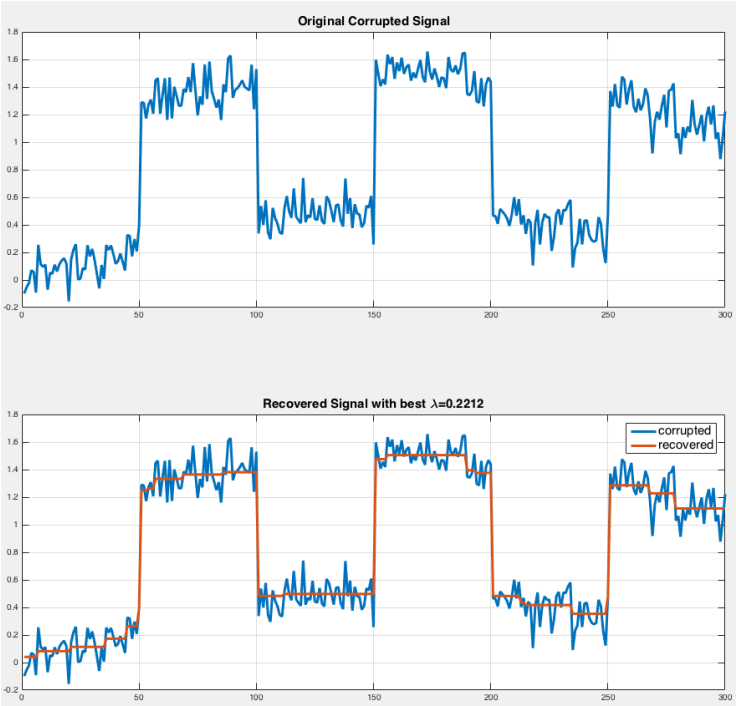

Figure 3: original and recovered signal

```
29    end
30
31    % plot graph
32    figure();
33    plot(1:iters_num,[lambda_arr;obj_arr]);
34    l_lambda = legend('\lambda','obj value');
35    set(l_lambda,'FontSize',15);
36    title('obj value v.s. \lambda', 'FontSize',15);
37    figure();
38    plot(norm_error, norm_regu);
39    xlabel('||x - x_{corr}||_2', 'FontSize',15);
40    ylabel('||Dx||_1', 'FontSize',15);
41    title('Optimal trade-off curve between ||Dx||_1 and ...
          ||x-x_{corr}||_2', 'FontSize',15);
42
43    % Best lambda we choose
44    lambda=0.2212;
45    cvx_begin
46    variable x(n)
47    minimize( norm(x-x_corr) + lambda*norm(D*x,1));
48    cvx_end
49    % Plot Recovered Results
50    figure();
51    subplot(2,1,1);
52    plot(1:n, x_corr, 'LineWidth', 3);
53    grid on;
54    title('Original Corrupted Signal','FontSize', 15);
55    subplot(2,1,2);
56    plot(1:n, [x_corr';x'], 'LineWidth', 3);
57    l = legend('corrupted', 'recovered');
```

```
58    set(l,'FontSize',15);
59    grid on;
60    title('Recovered Signal with best \lambda=0.2212','FontSize',15);
```

Bibliography

[1] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.