Ass3 Solutions

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1 Solutions

1.1 Prob 4.3

Because $\nabla f(x) = \frac{1}{2}(P + P^T)x + q$,

$$\begin{aligned}
\nabla f(x^*) &= \frac{1}{2}(\begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix} + \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}) \begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix} + \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix} \\
&= \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix} + \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}
\end{aligned}$$

According to equation (4.21)[1], because for all y satisfying $y_i \in [-1, 1]$, $i = 1, 2, 3$, the optimality condition holds:

$$\begin{aligned}
\nabla f(x^*)^T (y - x^*) &= \begin{bmatrix} -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 - 1 \\ y_2 - 1/2 \\ y_3 + 1 \end{bmatrix} \\
&= -1(y_1 - 1) + 2(y_3 + 1) \geq 0
\end{aligned}$$

therefore, x^* is optimal for $f(x)$.

1.2 Prob 4.8

1.2.1 a

The Lagrangian of the primal problem is:

$$\begin{aligned}
L(x, v) &= c^T x + v^T (Ax - b) \\
&= -b^T v + (c + A^T v)x
\end{aligned}$$

L is bounded below if and only if $c + A^T v = 0$. Therefore, the primal problem's optimal value $p^* = -b^T v$ for some v satisfying $c + A^T v = 0$. The primal problem is unbounded otherwise.

1.2.2 b

The problem is always feasible. The Lagrangian of the primal problem is:

$$\begin{aligned} L(x, v) &= c^T x + \lambda(a^T x - b) \\ &= -b\lambda + (c + a^T \lambda)x \end{aligned}$$

L is bounded below if and only if $c + a^T\lambda = 0$ and $\lambda \geq 0$. Therefore, the primal problem has optimal value $p^* = -b\lambda$ for some λ satisfying $c + a^T\lambda = 0$ and $\lambda \geq 0$. Otherwise the primal problem is unbounded below.

1.2.3 c

The objective function can be written as $\sum_{i=1}^{n} c_i x_i$. Therefore, the objective function reaches minimal when each of component $c_i x_i$ is minimized subject to $l_i \leq x_i \leq u_i$.

Therefore, for all indices such that $c_i > 0$ we have $x_i = l_i$ and for all indices such that $c_i < 0$ we have $x_i = u_i$. When $c_i = 0$, x_i can be any value in the domain of the problem namely $l_i \leq x_i \leq u_i$.

1.2.4 d

The Lagrangian of the primal problem is:

$$\begin{aligned} L(x, v, \lambda) &= c^T x + v(1^T x - 1) - \lambda^T x \\ &= -v + (c^T + v - \lambda^T)x \end{aligned}$$

L is bounded below if and only if $c^T + v - \lambda^T = 0$ and $\lambda \succeq 0$, namely $-v < c_i \ \forall i$. Therefore, the primal problem has optimal value $p^* \geq -v$ for some λ satisfying $\lambda \succeq 0$ and for some v satisfying $-v < c_i \ \forall i$. Suppose c has minimal values c_{min} at indexes $i, j, k \ldots$, we can always get the optimal solution by setting $x_i, x_j, x_k, \ldots$ to $\sum_{m=i,j,k\ldots} x_m = 1$ at those indexes and 0 at anywhere else. Therefore, we have $p^* = c_{min}$. Otherwise the primal problem is unbounded below.

When we use inequality constraint to replace the equality constraint, the optimal value is equal to $p^* = min\{0, c_{min}\}$. This is because we can assign x as above if $c_{min} < 0$ and $x = 0$ otherwise.

1.2.5 e

Suppose the components of c are sorted in increasing order as:

$$c_1 \leq c_2 \leq \cdots \leq c_\alpha \leq \cdots \leq c_n$$

When α is an integer, the primal problem has optimal value $p^* = \sum_{i=1}^{\alpha} c_i$, namely the smallest α elements of c if and only if $x_i = 1$ for $i \leq \alpha$ and $x_i = 0$ for $i > \alpha$.

If α is not an integer, let $[\alpha]$ denotes the nearest integer less than or equal to α, the optimal value is

$$p^* = \sum_{i=1}^{[\alpha]} c_i + c_{1+[\alpha]}(\alpha - [\alpha])$$

by setting $x_i = 1$ for $i \le [\alpha]$, $x_{1+[\alpha]} = \alpha - [\alpha]$ and $x_i = 0$ for $i > 1 + [\alpha]$.

If change the equality constraint to inequality constraint $1^T x \le \alpha$, then the optimal value is

$$p^* = \begin{cases} \sum_{i=1}^{[\alpha]} c_i + c_{1+[\alpha]}(\alpha - [\alpha]) & ifc_{1+[\alpha]} < 0 \\ \sum_{i=1}^{j} c_i & ifc_j + 1 >= 0 \text{ and } j < 1 + [\alpha] \\ 0 & ifc_1 >= 0 \end{cases}$$

We can always get this by setting:

$$x_i = \begin{cases} 0 & c_i \ge 0 \\ 1 & c_i < 0 \end{cases} \text{ for } i \le [\alpha]$$

$$x_{1+[\alpha]} = \begin{cases} \alpha - [\alpha] & ifc_{1+[\alpha]} < 0 \\ 0 & \text{otherwise} \end{cases}$$

and $x_i = 0$ for $i > [\alpha] + 1$.

1.2.6 f

Let $y_i = d_i x_i$ for $i = 1, \dots, n$ and use y to substitute x in the original problem. Then we have:

$$\text{minimize} \sum_{i=1}^{n} \frac{c_i}{d_i} y_i$$

$$\text{s.t. } 1^T y = \alpha,\ 0 \preceq y \preceq d.$$

This formulation is similar to section 1.2.5. Suppose the components of $\frac{c_i}{d_i}$ are sorted in increasing order as:

$$\frac{c_1}{d_1} \le \dots \le \frac{c_\alpha}{d_\alpha} \le \dots \le \frac{c_n}{d_n}$$

Let $k^* = \text{argmax}_k d_1 + \dots + d_k \le \alpha$, the primal problem reaches optimal value $p^* = \sum_{i=1}^{k^*} c_i + \frac{c_{k^*+1}}{d_{k^*+1}}(\alpha - (d_1 + \dots + d_{k^*}))$ by setting $y_i = d_i$ for $i \le k^*$, $y_{k^*+1} = \alpha - (d_1 + \dots + d_{k^*})$ and $y_i = 0$ for $i > k^* + 1$.

Namely by setting $x_i = 1$ for $i \le k^*$, $x_{k^*+1} = \frac{\alpha - (d_1 + \dots + d_{k^*})}{d_{k+1}}$ and $x_i = 0$ for $i > k^* + 1$.

1.3 Prob 4.9

Because A is nonsingular and square, we have A^{-1}. Substituting $y = Ax$ into the primal problem we have:

$$\text{minimize } c^T A^{-1} y$$

$$\text{s.t. } y \preceq b$$

The primal problem is unbounded below if $c^T A^{-1} \succ 0$. When $c^T A^{-1} \preceq 0$ we have optimal value $p^* = c^T A^{-1} b$ when $y = b$ namely $x = A^{-1} b$.

1.4 Prob 4.47

1.4.1 a

This is because for any unspecified diagonal entry, we can choose an arbitrary large values. By doing this we only need to consider submatrix composed by all specified diagonla entries and their corresponding rows and columns. For example, if one diagonal entry a_{ii} in A is unspecified, we can assign it to inifinity. Then to satisfy the condition $A \succeq 0$ we only need to consider the submatrix of A with ith row and ith column removed $\hat{A} \succeq 0$. By doing this we can only consider the submatrix made by only specified diagonal entries and their corresponding rows and columns.

1.4.2 b

Suppose A_j, b_j where $j = 1, ...m$ are all of those specified entries, we can formulate the problem as:

$$\text{minimize } s$$

$$-A \preceq sI$$

$$A_j A = b_j \text{ , where } j = 1, \ldots m$$

The problem is feasible when $s \leq 0$.

1.4.3 c

The optimization problem can be formulated as:

$$\text{maximize } |A|$$

$$-A \preceq 0$$

$$A_j A = b_j \text{ , where } j = 1, \ldots m$$

Because $A \succeq 0$, $-\text{log}det(A)$ is a strictly convex function over positive definite matrix A. The problem can be reformulated as:

$$\text{minimize } - logdet(A)$$

$$-A \preceq 0$$

$$A_j A = b_j \text{ , where } j = 1, \ldots m$$

Suppose S is the set of all positive semidefinite completions of A, S is closed, convex and bounded. This is because all diagonal entries are specified. Therefore, entries which are not on the diagonal cannot exceed the maximum diagonal entry otherwise the matrix will not be positive definite. Therefore, the problem is a convex optimization problem defined on a convex set and hence has a unique optimum point.

When A^* is the optimal solution, suppose the element a_{ij} is unspecified, then the gradient at a_{ij} should be 0. Therefore we have:

$$\frac{\partial f_0(A^*)}{\partial a_{ij}} = 2\text{tr}(A^*)^{-1} E_{ij} = 0$$

Because A is symmetric therefore the ijth element as well as jith element are all 0s. Therefore, A^* has zeros in all unspecified entries of the origin matrix.

1.4.4 d

If there exists a positive definite completion of A, then as section c proved, there exists an unique positive definite completion A^* for A which maximizes the determinant. Because as section c proved A^* has zeros in all unspecified entries of the origin matrix, so $(A^*)^{-1}$ is tridiagonal.

1.5 Prob 5.12

Let A be a matrix whose ith row is a_i^T, by introduce equality constraints $y_i = b_i - a_i^T x$ we have:

$$\begin{aligned} \text{minimize} \ \ & -\sum_i^m \log y_i \\ \text{s.t.} \ \ & y = b - Ax \end{aligned}$$

The Lagrangian of the primal problem is:

$$\begin{aligned} L(x, y, v) &= -\sum_i^m \log y_i + v^T(y - b + Ax) \\ &= v^T Ax - \sum_i^m \log y_i + v^T y - v^T b \end{aligned}$$

Therefore, the dual function is:

$$g(v) = \inf_{x,y}(v^T Ax - \sum_i^m \log y_i + v^T y - v^T b)$$

When we consider in terms of y, this function is unbounded below if $v \preceq 0$. Therefore, the function is bounded only if $v \succ 0$. Because $y_i \geq 0$, if we take derivative in terms of y and set it equals 0 we have the function reaches minimum when:

$$y_i = \frac{1}{v_i}$$

In terms of x the function is unbounded below if $v^T A \neq 0$. Therefore we have the dual function:

$$g(v) = \begin{cases} \sum_i^m \log v_i + m - v^T b & v^T A = 0, v \succ 0 \\ -\infty & otherwise \end{cases}$$

Therefore we have the dual problem:

$$\begin{aligned} \text{maximize} \ \ & \sum_i^m \log v_i + m - v^T b \\ \text{s.t.} \ \ & v^T A = 0, v \succ 0 \end{aligned}$$

1.6 Prob 5.31

Because the problem is convex, every inequality consitraints are hence convex. $x^* \in R^n$ and $\lambda^* \in R^m$. For any x that is feasible we have:

$$0 \geq f_i(x) \geq f_i(x^*) + \nabla f_i(x^*)^T(x - x^*) \text{ for all } i = 1, \ldots, m$$

Therefore, for all inequality constraints we have:

$$0 \geq \sum_{i=1}^{m} \lambda_i^* (f_i(x^*) + \nabla f_i(x^*)^T (x - x^*)) \tag{1}$$

$$= \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*)^T (x - x^*) \tag{2}$$

According to KKT conditions, we have that $\sum_{i=1}^{m} \lambda_i^* f_i(x^*) = 0$ and $\sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*)^T = -\nabla f_0(x^*)$. Introducing those two terms into equation 2 we have:

$$0 \geq -\nabla f_0(x^*)^T (x - x^*)$$

Bibliography

[1] S. Boyd and L. Vandenberghe, *Convex optimization.* Cambridge university press, 2004.