Ass3 Solutions

Chang Li

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1 Solutions

1.1 Prob 4.3

Because $\nabla f(x) = \frac{1}{2}(P + P^T)x + q$,

$$\begin{aligned}
\nabla f(x^*) &= \frac{1}{2}(\begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix} + \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}) \begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix} + \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix} \\
&= \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix} + \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}
\end{aligned}$$

According to equation (4.21)[?], because for all y satisfying $y_i \in [-1, 1]$, $i = 1, 2, 3$, the optimality condition holds:

$$\begin{aligned}
\nabla f(x^*)^T(y - x^*) &= \begin{bmatrix} -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 - 1 \\ y_2 - 1/2 \\ y_3 + 1 \end{bmatrix} \\
&= -1(y_1 - 1) + 2(y_3 + 1) \geq 0
\end{aligned}$$

therefore, x^* is optimal for $f(x)$.

1.2 Prob 4.8

1.2.1 a

The Lagrangian of the primal problem is:

$$\begin{aligned}
L(x, v) &= c^T x + v^T(Ax - b) \\
&= -b^T v + (c + A^T v)x
\end{aligned}$$

L is bounded below if and only if $c + A^T v = 0$. Therefore, the primal problem's optimal value $p^* = -b^T v$ for some v satisfying $c + A^T v = 0$. The primal problem is unbounded otherwise.

1.2.2 b

The problem is always feasible. The Lagrangian of the primal problem is:

$$\begin{aligned} L(x, v) &= c^T x + \lambda(a^T x - b) \\ &= -b\lambda + (c + a^T \lambda)x \end{aligned}$$

L is bounded below if and only if $c + a^T\lambda = 0$ and $\lambda \geq 0$. Therefore, the primal problem has optimal value $p^* = -b\lambda$ for some λ satisfying $c + a^T\lambda = 0$ and $\lambda \geq 0$. Otherwise the primal problem is unbounded below.

1.2.3 c

The objective function can be written as $\sum_{i=1}^{n} c_i x_i$. Therefore, the objective function reaches minimal when each of component $c_i x_i$ is minimized subject to $l_i \leq x_i \leq u_i$.

Therefore, for all indices such that $c_i > 0$ we have $x_i = l_i$ and for all indices such that $c_i < 0$ we have $x_i = u_i$. When $c_i = 0$, x_i can be any value in the domain of the problem namely $l_i \leq x_i \leq u_i$.

1.2.4 d

The Lagrangian of the primal problem is:

$$\begin{aligned} L(x, v, \lambda) &= c^T x + v(1^T x - 1) - \lambda^T x \\ &= -v + (c^T + v - \lambda^T)x \end{aligned}$$

L is bounded below if and only if $c^T + v - \lambda^T = 0$ and $\lambda \succeq 0$, namely $-v < c_i \ \forall i$. Therefore, the primal problem has optimal value $p^* \geq -v$ for some λ satisfying $\lambda \succeq 0$ and for some v satisfying $-v < c_i \ \forall i$. Suppose c has minimal values c_{min} at indexes $i, j, k \ldots$, we can always get the optimal solution by setting $x_i, x_j, x_k, \ldots$ to $\sum_{m=i,j,k\ldots} x_m = 1$ at those indexes and 0 at anywhere else. Therefore, we have $p^* = c_{min}$. Otherwise the primal problem is unbounded below.

When we use inequality constraint to replace the equality constraint, the optimal value is equal to $p^* = min\{0, c_{min}\}$. This is because we can assign x as above if $c_{min} < 0$ and $x = 0$ otherwise.

1.2.5 e

Suppose the components of c are sorted in increasing order as:

$$c_1 \leq c_2 \leq \cdots \leq c_\alpha \leq \cdots \leq c_n$$

When α is an integer, the primal problem has optimal value $p^* = \sum_{i=1}^{\alpha} c_i$, namely the smallest α elements of c if and only if $x_i = 1$ for $i \leq \alpha$ and $x_i = 0$ for $i > \alpha$.

If α is not an integer, let $[\alpha]$ denotes the nearest integer less than or equal to α, the optimal value is

$$p^* = \sum_{i=1}^{[\alpha]} c_i + c_{1+[\alpha]}(\alpha - [\alpha])$$

by setting $x_i = 1$ for $i \leq [\alpha]$, $x_{1+[\alpha]} = \alpha - [\alpha]$ and $x_i = 0$ for $i > 1 + [\alpha]$.

If change the equality constraint to inequality constraint $1^T x \leq \alpha$, then the optimal value is

$$p^* = \begin{cases} \sum_{i=1}^{[\alpha]} c_i + c_{1+[\alpha]}(\alpha - [\alpha]) & if c_{1+[\alpha]} < 0 \\ \sum_{i=1}^{j} c_i & if c_j + 1 >= 0 \text{ and } j < 1 + [\alpha] \\ 0 & if c_1 >= 0 \end{cases}$$

We can always get this by setting:

$$x_i = \begin{cases} 0 & c_i \geq 0 \\ 1 & c_i < 0 \end{cases} \text{ for } i \leq [\alpha]$$

$$x_{1+[\alpha]} = \begin{cases} \alpha - [\alpha] & if c_{1+[\alpha]} < 0 \\ 0 & \text{otherwise} \end{cases}$$

and $x_i = 0$ for $i > [\alpha] + 1$.

1.2.6 f

Let $y_i = d_i x_i$ for $i = 1, \ldots, n$ and use y to substitute x in the original problem. Then we have:

$$\text{minimize} \sum_{i=1}^{n} \frac{c_i}{d_i} y_i$$

$$\text{s.t. } 1^T y = \alpha,\ \ 0 \preceq y \preceq d.$$

This formulation is similar to section 1.2.5. Suppose the components of $\frac{c_i}{d_i}$ are sorted in increasing order as:

$$\frac{c_1}{d_1} \leq \cdots \leq \frac{c_\alpha}{d_\alpha} \leq \cdots \leq \frac{c_n}{d_n}$$

Let $k^* = \text{argmax}_k d_1 + \cdots + d_k \leq \alpha$, the primal problem reaches optimal value $p^* = \sum_{i=1}^{k^*} c_i + \frac{c_{k^*+1}}{d_{k^*+1}}(\alpha - (d_1 + \cdots + d_{k^*}))$ by setting $y_i = d_i$ for $i \leq k^*$, $y_{k^*+1} = \alpha - (d_1 + \cdots + d_{k^*})$ and $y_i = 0$ for $i > k^* + 1$.

Namely by setting $x_i = 1$ for $i \leq k^*$, $x_{k^*+1} = \frac{\alpha - (d_1 + \cdots + d_{k^*})}{d_{k+1}}$ and $x_i = 0$ for $i > k^* + 1$.

1.3 Prob 4.9

Because A is nonsingular, we have A^{-1}. Substituting $y = Ax$ into the primal problem we have:

$$\text{minimize } c^T A^{-1} y$$

$$\text{s.t. } \ y \preceq b$$

The primal problem is unbounded below if $c^T A^{-1} \succ 0$. When $c^T A^{-1} \preceq 0$ we have optimal value $p^* = c^T A^{-1} b$ when $y = b$ namely $x = A^{-1} b$.

1.4 Prob 4.47

1.5 Prob 5.12

Let A be a matrix whose ith row is a_i^T, by introduce equality constraints $y_i = b_i - a_i^T x$ we have:

$$\begin{aligned} &\text{minimize} \;\; -\sum_i^m \log y_i \\ &\text{s.t.} \;\; y = b - Ax \end{aligned}$$

The Lagrangian of the primal problem is:

$$\begin{aligned} L(x, y, v) &= -\sum_i^m \log y_i + v^T(y - b + Ax) \\ &= v^T Ax - \sum_i^m \log y_i + v^T y - v^T b \end{aligned}$$

Therefore, the dual function is:

$$g(v) = \inf_{x,y}(v^T Ax - \sum_i^m \log y_i + v^T y - v^T b)$$

When we consider in terms of y, this function is unbounded below if $v \preceq 0$. Therefore, the function is bounded only if $v \succ 0$. Because $y_i \geq 0$, if we take derivative in terms of y and set it equals 0 we have the function reaches minimum when:

$$y_i = \frac{1}{v_i}$$

In terms of x the function is unbounded below if $v^T A \neq 0$. Therefore we have the dual function:

$$g(v) = \begin{cases} \sum_i^m \log v_i + m - v^T b & v^T A = 0, v \succ 0 \\ -\infty & otherwise \end{cases}$$

Therefore we have the dual problem:

$$\begin{aligned} &\text{maximize} \;\; \sum_i^m \log v_i + m - v^T b \\ &\text{s.t.} \;\; v^T A = 0, v \succ 0 \end{aligned}$$

1.6 Prob 5.31

Because the problem is convex, every inequality consitraints are hence convex. $x^* \in R^n$ and $\lambda^* \in R^m$. For any x that is feasible we have:

$$0 \geq f_i(x) \geq f_i(x^*) + \nabla f_i(x^*)^T (x - x^*) \text{ for all } i = 1, \ldots, m$$

Therefore, for all inequality constraints we have:

$$0 \geq \sum_{i=1}^{m} \lambda_i^* (f_i(x^*) + \nabla f_i(x^*)^T (x - x^*)) \tag{1}$$

$$= \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*)^T (x - x^*) \tag{2}$$

According to KKT conditions, we have that $\sum_{i=1}^{m} \lambda_i^* f_i(x^*) = 0$ and $\sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*)^T = -\nabla f_0(x^*)$. Introducing those two terms into equation 2 we have:

$$0 \geq -\nabla f_0(x^*)^T (x - x^*)$$