Ass2 Solutions

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1 Solutions

1.1 Prob 2.12

Prove convexity of halfspaces

1.1.1 Prob a

This set is convex. Let $B = \{x \in R^n | a^T x \geq \alpha\}$, $C = \{x \in R^n | a^T x \leq \beta\}$.
Then the slab $S = B \cap C = \{x \in R^n | \alpha \leq a^T x \leq \beta\}$. Because $B \,\text{and}\, C$ are both halfspaces hence convex and the intersection operation preserves convexity. Therefore slab S is convex.

1.1.2 Prob b

This set is convex. Let $S = \{x \in R^n | \alpha_i \leq x \leq \beta_i, i = 1, \ldots, n\}$ can be written as the intersection of a finite number of slabs

$$S = \cap_{i=1,\ldots,n} \{x \in R^n | \alpha_i \leq x \leq \beta_i\}$$

From Prob a we know that slab is convex. Since the intersection operation preserves convexity, the set S is also convex.

1.1.3 Prob c

This set is convex. Let $S = \{x \in R^n | a_1^T x \leq b_1, a_2^T x \leq b_2\}$, $A = \{x \in R^n | a_1^T x \leq b_1\}$ and $A = \{x \in R^n | a_2^T x \leq b_2\}$. Therefore, $S = A \cap B$. Since A and B are halfspaces hence convex, and the intersection operation preserves convexity, the set S is also convex.

1.1.4 Prob d

This set is convex. Let $A = \{\|x - x_0\|_2 \leq \|x - y\|_2 \,\text{for all}\, y \in S\}$, then A can be written as the intersection of a finite number of sets:

$$A = \cap_{y \in S} \{\|x - x_0\|_2 \leq \|x - y\|_2\}$$

When y is fixed, let $B = \{\|x - x_0\|_2 \leq \|x - y\|_2\}$. We have,

$$\begin{aligned}
\|x - x_0\|_2 \leq \|x - y\|_2 &\Longleftrightarrow (x - x_0)^T(x - x_0) \leq (x - y)^T(x - y) \\
&\Longleftrightarrow x^Tx - 2x_0^Tx + x_0^Tx_0 \leq x^Tx - 2y^Tx + y^Ty \\
&\Longleftrightarrow 2(y - x_0)^Tx \leq y^Ty - x_0^Tx_0 \\
&\Longleftrightarrow Ax \preceq b
\end{aligned}$$

where,

$$A = 2\begin{bmatrix} y_1 - x_0 \\ y_2 - x_0 \\ \cdots \\ y_n - x_0 \end{bmatrix}$$

$$b = \begin{bmatrix} y_1^Ty_1 - x_0^Tx_0 \\ y_2^Ty_2 - x_0^Tx_0 \\ \cdots \\ y_n^Ty_n - x_0^Tx_0 \end{bmatrix}$$

Therefore, for every $y \in S$, B is a halfspace which is convex. Since the intersection operation preserves convexity and A can be written as the intersection of a finite number of halfspaces, A is also convex.

1.1.5 Prob e

In general this set is not convex. As an example, suppose $x \in R$, $S = \{-a, a\}$, $T = \{0\}$ and $a \in R^+$. Then we have,

$$A = \{x|\text{dist}(x, S) \leq \text{dist}(x, T)\} = \{x \in R|x \leq -\frac{a}{2} \text{ or } x \geq \frac{a}{2}\}$$

where $c = -\frac{a}{2} \in A$ and $b = \frac{a}{2} \in a$. However, $0 = \frac{1}{2}c + (1 - \frac{1}{2})b \notin A$. Therefore, A is not convex. So in general the set $\{x|\text{dist}(x, S) \leq \text{dist}(x, T)\}$ is not convex.

1.1.6 Prob f

This set is convex. Let $S = \{x|x + S_2 \subseteq S_1\}$. We have,

$$S = \{x|x + S_2 \subseteq S_1\} = \cap_{\text{for all } y \in S_2}\{x|x + y \subseteq S_1\} = \cap_{\text{for all } y \in S_2}(S_1 - y)$$

Since S_1 is convex, $S_1 - y$ is also convex. This is because for all $z \in S_1 - y$, let $m \in S_1$. Because y is fixed, if $z_1 = x_1 - y \in S_1 - y$ and $z_2 = x_2 - y \in S_1 - y$, then $\theta z_1 + (1 - \theta)z_2 = (\theta x_1 + (1 - \theta)x_2) - y$. Because S_1 is convex, let $x_3 = (\theta x_1 + (1 - \theta)x_2) \in S_1$. Then $\theta z_1 + (1 - \theta)z_2 = x_3 - y \in S_1 - y$.

Therefore, S is convex because it can be written as the intersection of convex sets.

1.1.7 Prob g

This set is convex. Let $S = \{x|\|x - a\|_2 \leq \theta\|x - b\|_2\}$, where $0 \leq \theta \leq 1$.

$$\begin{aligned}
\|x - a\|_2 \leq \theta\|x - b\|_2 &\Longleftrightarrow \|(x - a)^T(x - a) \leq \theta^2(x - b)^T(x - b) \\
&\Longleftrightarrow x^Tx - 2a^Tx + a^Ta \leq \theta^2(x^Tx - 2b^Tx + b^Tb) \\
&\Longleftrightarrow (1 - \theta^2)x^Tx - 2(a - \theta^2b)^Tx + (a^Ta - \theta^2b^Tb) \leq 0
\end{aligned}$$

Because $0 \leq \theta \leq 1$, when $\theta = 1$, $S = \{x| -2(a-b)^T x \leq -(a^T a - b^T b)\}$. Therefore, S is a halfspace hence convex.

When $\theta < 1$

$$S = \{x|(x - \frac{a-\theta^2 b}{1-\theta^2})^T(x - \frac{a-\theta^2 b}{1-\theta^2}) \leq (\frac{\theta^2 b^T b - a^T a}{1-\theta^2} - (\frac{a-\theta^2 b}{1-\theta^2})^T(\frac{a-\theta^2 b}{1-\theta^2}))\}$$

Therefore, S is a ball hence convex.

1.2 Prob 2.15

Let $P = \{p|\mathbf{1}^T p = 1, p \succeq 0\}$. Because $\mathbf{1}^T p = 1$ defines a hyperplane and $p \succeq 0$ define halfspaces, P is an intersection of a hyperplane and halfspaces. Therefore, P is a polyhedron hence convex.

1.2.1 a

This is convex constraint. The constraint can be written as:

$$Ef(x) \leq \beta \Longleftrightarrow \sum_{i=1}^{n} p_i f(a_i) \leq \beta \tag{1}$$

$$Ef(x) \geq \alpha \Longleftrightarrow \sum_{i=1}^{n} p_i f(a_i) \geq \alpha \tag{2}$$

Therefore, both constraints 1 and 2 are halfspaces hence are convex. Since P is convex and the intersection operation preserves convexity, the intersection of P and two halfspaces: $P^* = \{p|\mathbf{1}^T p = 1, p \succeq 0, \alpha \leq \mathbf{E}f(x) \leq \beta\}$ is also convex.

1.2.2 b

This is convex constraint. Because a is in ascent order, let i^* equals the minimum index satisfies $a_i \geq \alpha$. The constraint can be written as:

$$\text{prob}(x \geq \alpha) = \sum_{i^*}^{n} p_i = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \\ \cdots \\ p_i^* \\ p_{i^*+1} \\ \cdots \\ p_n \end{bmatrix} = v^T p \leq \beta$$

Where the first $i^* - 1$ elements of v are zeros and all ones after i^*. Therefore the constraint defines a halfspace hence convex. Since P is convex and the intersection operation preserves convexity, the intersection of P and halfspace: $P^* = \{p|\mathbf{1}^T p = 1, p \succeq 0, \text{prob}(x \geq \alpha)\}$ is also convex.

1.2.3 e

This is convex constraint. The constraint can be written as:

$$\mathbf{E}x^2 = \sum_{i=1}^{n} p_i a_i^2 \geq \alpha$$

which defines a halfspace hence convex. Since P is convex and the intersection operation preserves convexity, the intersection of P and halfspace: $P^* = \{p|\mathbf{1}^T p = 1, p \succeq 0, \mathbf{E}x^2\}$ is also convex.

1.2.4 f

In general this is not a convex constraint. The constraint can be written as:

$$\mathrm{var}(x) = \mathbf{E}x^2 - (\mathbf{E}x)^2 = \sum_{i=1}^{n} p_i a_i^2 - (\sum_{i=1}^{n} p_i a_i)^2 \leq \alpha$$

Let $n = 2$, $\alpha = 0.1$, $a_1 = 1$ and $a_2 = 2$. Let $p_1 = (1, 0)$ and $p_2 = (0, 1)$. We have $var(p_1) = 0 \leq \alpha$ and $var(p_2) = 0 \leq \alpha$. However, the convex combination $p_3 = \frac{1}{2}p_1 + \frac{1}{2}p_2 = (\frac{1}{2}, \frac{1}{2})$ doesn't satisfy the constraint $var(p_3) = 0.25 > \alpha$. Therefore, in general it is not a convex constraint.

1.3 Prob 3.14

1.3.1 a

Since f(x,z) is second-order differentiable. Because for each fixed x, $f(x, z)$ is a concave function of z, we have $\nabla^2_{zz} f(x, z) \preceq 0$. Because for each fixed x, $f(x, z)$ is a convex function of x, we have $\nabla^2_{xx} f(x, z) \succeq 0$. Therefore, in the Hessian matrix

$$\nabla^2 f(x, z) = \begin{bmatrix} \nabla^2_{xx} f(x, z) & \nabla^2_{xz} f(x, z) \\ \nabla^2_{xz} f(x, z) & \nabla^2_{zz} f(x, z) \end{bmatrix}$$

The element $\nabla^2_{xx} f(x, z) \succeq 0$ and the element $\nabla^2_{zz} f(x, z) \preceq 0$.

1.3.2 b

Because for each fixed $\tilde{x}$, $f(\tilde{x}, z)$ is a concave function of z, we also know that $\nabla f(\tilde{x}, \tilde{z}) = 0$, therefore $\tilde{z}$ maxmizes $f(\tilde{x}, z)$ over z Therefore, we have $f(\tilde{x}, z) \leq f(\tilde{x}, \tilde{z})$, namely $inf_x sup_z f(x, z) \leq sup_z f(\tilde{x}, z) \leq f(\tilde{x}, \tilde{z})$.

For similar reason, because for each fixed $\tilde{z}$, $f(x, \tilde{z})$ is a convex function of x, we also know that $\nabla f(\tilde{x}, \tilde{z}) = 0$, therefore $\tilde{x}$ minimizes $f(x, \tilde{z})$ over x. Therefore, we have $f(\tilde{x}, \tilde{z}) \leq f(x, \tilde{z})$, namely $f(\tilde{x}, \tilde{z}) \leq inf_x f(x, \tilde{z}) \leq sup_z inf_x f(x, z)$.

So we have $f(\tilde{x}, z) \leq f(\tilde{x}, \tilde{z}) \leq f(x, \tilde{z})$.

We also have:

$$inf_x sup_z f(x, z) \leq sup_z f(\tilde{x}, z) \leq f(\tilde{x}, \tilde{z}) \leq inf_x f(x, \tilde{z}) \leq sup_z inf_x f(x, z)$$

However, according to equation 5.46 in Boyd's book[1]. The following condition always hold:

$$sup_z inf_x f(x, z) \leq inf_x sup_z f(x, z)$$

Therefore, we have $sup_z inf_x f(x, z) = inf_x sup_z f(x, z) = f(\tilde{x}, \tilde{z})$.

1.3.3 c

For all x, because the following holds $f(\tilde{x}, \tilde{z}) \leq f(x, \tilde{z})$. Therefore $\tilde{x}$ minimizes $f(x, \tilde{z})$ over all x. Therefore $\nabla f_x(\tilde{x}, \tilde{z}) = 0$.

For all z, because the following holds $f(\tilde{x}, z) \leq f(\tilde{x}, \tilde{z})$. Therefore $\tilde{z}$ maximizes $f(\tilde{x}, z)$ over all z. Therefore $\nabla f_z(\tilde{x}, \tilde{z}) = 0$.

Therefore, $\nabla f(\tilde{x}, \tilde{z}) = 0$.

1.4 Prob 3.16

1.4.1 a

This is both convex and quasiconvex and quasiconcave but not concave. This is because:

$$\nabla^2 f(x) = e^x > 0, \text{ for all } x \in R$$

Therefore $f(x)$ is both convex and quasiconvex.

For its superlevel sets:

$$S_\alpha = \{x | e^x - 1 \geq \alpha\}$$

is also convex. Let $x_1 \leq x_2$, $e^{x_1} - 1 \geq \alpha$ and $e^{x_2} - 1 \geq \alpha$, then $x_1 \leq \theta x_1 + (1-\theta)x_2$ for $\theta \geq 0$. Therefore, $e^{\theta x_1 + (1-\theta)x_2} - 1 \geq e^{x_1} - 1 \geq \alpha$ because $\nabla(e^x) = e^x > 0$ which is increasing.

1.4.2 b

This is quasiconcave, but not convex, concave or quasiconvex. This is because

$$\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Therefore, the Hessian matrix of f is neither positive semidefinite nor negative semidefinite. Therefore, it is neither convex nor concave.

However, it is quasiconcave, since the superlevel sets

$$\{x \in R^2_{++} | x_1 x_2 \geq \alpha\}$$

are convex sets for all α.

1.4.3 c

This is convex and quasiconvex, but not concave or quasiconcave. This is because,

$$\nabla^2 f(x) = \frac{1}{x_1 x_2} \begin{bmatrix} \frac{2}{x_1^2} & \frac{1}{x_1 x_2} \\ \frac{1}{x_1 x_2} & \frac{2}{x_2^2} \end{bmatrix} \succeq 0$$

Therefore, $f(x)$ is convex and hence quasiconvex.

1.4.4 d

This is both quasiconvex and quasiconcave, but not convex or concave. This is because,

$$\nabla^2 f(x) = \begin{bmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & 2\frac{x_1}{x_2^3} \end{bmatrix}$$

Therefore, the Hessian matrix of f is neither positive semidefinite nor negative semidefinite. Therefore, it is neither convex nor concave.

However, the superlevel is:

$$\{x \in R^2_{++} | \frac{x_1}{x_2} \geq \alpha\} = \{x \in R^2_{++} | x_1 - \alpha x_2 \geq 0\}$$

the sublevel is:

$$\{x \in R^2_{++} | \frac{x_1}{x_2} \leq \alpha\} = \{x \in R^2_{++} | x_1 - \alpha x_2 \leq 0\}$$

Since they are all halfspaces therefore convex. Therefore, $f(x)$ is both quasiconvex and quasiconcave.

1.4.5 e

This is convex and quasiconvex, but not concave or quasiconcave. This is because:

$$\nabla^2 f(x) = \frac{2}{x_2} \begin{bmatrix} 1 & -\frac{x_1}{x_1 x_2} \\ -\frac{x_1}{x_1 x_2} & \frac{x_1^2}{x_2^2} \end{bmatrix} = \frac{2}{x_2} \begin{bmatrix} 1 \\ -\frac{x_1}{x_2} \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{x_1}{x_2} \end{bmatrix}^T \succeq 0$$

Therefore, $f(x)$ is convex and hence quasiconvex.

1.4.6 f

This is concave and quasiconcave, but not convex or quasiconvex. This is because:

$$\begin{aligned} \nabla^2 f(x) &= \alpha(\alpha - 1) x_1^\alpha x_2^{1-\alpha} \begin{bmatrix} \frac{1}{x_1^2} & -\frac{1}{x_1 x_2} \\ -\frac{1}{x_1 x_2} & \frac{1}{x_2^2} \end{bmatrix} \\ &= \alpha(\alpha - 1) x_1^\alpha x_2^{1-\alpha} \begin{bmatrix} \frac{1}{x_1} \\ -\frac{1}{x_2} \end{bmatrix} \begin{bmatrix} \frac{1}{x_1} \\ -\frac{1}{x_2} \end{bmatrix}^T \preceq 0 \end{aligned}$$

Therefore, $f(x)$ is concave and hence quasiconcave.

1.5 Prob 3.16

1.5.1 a

We first determine the domain for y of the conjugate function.

$$f^*(y) = sup_{x \in \mathrm{dom} f}(y^T x - max_{i=1,\dots,n} x_i) \tag{3}$$

First, let $y \prec 0$. Suppose $x_j = -t$, $x_i = 0$ for $i \neq k$ and let t go to infinity, so that:

$$y^T x - max_{i=1,\dots,n} x_i = -ty_j \to \infty$$

Therefore, $y \prec 0$ is not in the domain $\mathrm{dom} f^*$.

Let $y \succeq 0$. Let $x = t\mathbf{1}$ and t go to infinity, we have,

$$y^T x - max_{i=1,\dots,n} x_i = t\mathbf{1}^T y - t = (\mathbf{1}^T y - 1)t$$

Therefore, when $\mathbf{1}^T y \neq 1$, we always have $y^T x - max_{i=1,\dots,n} x_i \to \infty$ (Let $x = -t\mathbf{1}$ when $\mathbf{1}^T y < 1$). When $\mathbf{1}^T y = 1$, so that:

$$y^T x - max_{i=1,\dots,n} x_i \leq 0 \ \text{ for all } \ x$$

When $x = 0$, $y^T x - max_{i=1,\dots,n} x_i = 0$. Therefore, we have

$$f^*(y) = \begin{cases} 0 & if\ y \succeq 0, \mathbf{1}^T y = 1 \\ \infty & otherwise \end{cases}$$

1.5.2 b

We first determine the domain for y of the conjugate function.

$$f^*(y) = sup_{x \in \mathrm{dom} f} (y^T x - \sum_{i=1}^{r} x_{[i]}) \tag{4}$$

First, let $y \prec 0$. Suppose $x_j = -t$, $x_i = 0$ for $i \neq k$ and let t go to infinity, so that:

$$y^T x - \sum_{i=1}^{r} x_{[i]} = -ty_j \to \infty$$

Therefore, $y \prec 0$ is not in the domain $\mathrm{dom} f^*$.

Let $y \succeq 0$. Let $x = t\mathbf{1}$ and t go to infinity, we have,

$$y^T x - \sum_{i=1}^{r} x_{[i]} = t\mathbf{1}^T y - tr = (\mathbf{1}^T y - r)t$$

Therefore, when $\mathbf{1}^T y \neq r$, we always have $y^T x - \sum_{i=1}^{r} x_{[i]} \to \infty$ (Let $x = -t\mathbf{1}$ when $\mathbf{1}^T y < r$). When $\mathbf{1}^T y = r$, so that:

$$y^T x - \sum_{i=1}^{r} x_{[i]} \leq 0 \ \text{ for all } \ x$$

When $x = 0$, $y^T x - \sum_{i=1}^{r} x_{[i]} = 0$. Therefore, we have

$$f^*(y) = \begin{cases} 0 & if\ y \succeq 0, \mathbf{1}^T y = r \\ \infty & otherwise \end{cases}$$

1.5.3 e

We first determine the domain for y of the conjugate function.

$$f^*(y) = sup_{x\in \mathrm{dom} f}(y^T x - (-(\prod x_i)^{1/n})) \tag{5}$$

First, let $y \succ 0$. Suppose $x_j = t$, $x_i = 1$ for $i \neq k$ and let t go to infinity, so that:

$$y^T x - (-(\prod x_i)^{1/n}) = ty_j + \sum_{i\neq j} y_i + t^{1/n} \to \infty$$

Therefore, $y \succ 0$ is not in the domain $\mathrm{dom} f^*$.

Let $y \preceq 0$. Let $x_i = -\frac{t}{y_i}$ and t go to infinity, we have,

$$y^T x - (-(\prod x_i)^{1/n}) = -tn - t(-(\prod(-\frac{1}{y_i}))^{1/n}) = ((\prod(-\frac{1}{y_i}))^{1/n} - n)t$$

Therefore, when $\prod(-y_i)^{1/n} \leq \frac{1}{n}$, we always have $y^T x - (-(\prod x_i)^{1/n}) \to \infty$. When $\prod(-y_i)^{1/n} \geq \frac{1}{n}$, because $x \in R^n_{++}$, according to arithmetic-geometric mean inequality we have that:

$$-\frac{y^T x}{n} \geq (\prod_i(-y_i x_i))^{1/n} \geq \frac{1}{n}(\prod_i x_i)^{1/n}$$

Namely, $y^T x - f(x) \leq 0$. When $x_i = -\frac{t}{y_i}$, $y^T x - (-(\prod x_i)^{1/n}) = 0$. Therefore, we have

$$f^*(y) = \begin{cases} 0 & if\ y \preceq 0, \prod(-y_i)^{1/n} \geq \frac{1}{n} \\ \infty & otherwise \end{cases}$$

1.6 Polyhedron

Let $S = \{A, B, C, D, E, F\} = \{(-1, 2), (0, 3), (2, 0), (2, -2), (0, 0), (-1, 0)\}$. According to Quickhull algorithm[2], ultimate points are A, B, C, D and F. So the polyhedron can be expressed as:

$$\begin{bmatrix} -1 & 1 \\ 3 & 2 \\ -2 & -3 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \preceq \begin{bmatrix} 3 \\ 6 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

Bibliography

[1] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.

[2] Wikipedia, "Quickhull — wikipedia, the free encyclopedia," 2016. [Online; accessed 14-August-2016].