Math189R SU17
Homework 5
Monday, June 5, 2017

Feel free to work with other students, but make sure you write up the homework and code on your own (no copying homework *or* code; no pair programming). Feel free to ask students or instructors for help debugging code or whatever else, though.

1 (Murphy 12.5 - Deriving the Residual Error for PCA) It may be helpful to reference section 12.2.2 of Murphy.

(a) Prove that

$$\left\|\mathbf{x}_i - \sum_{j=1}^{k} z_{ij}\mathbf{v}_j\right\|^2 = \mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{v}_j.$$

Hint: first consider the case when $k = 2$. Use the fact that $\mathbf{v}_i^\top \mathbf{v}_j$ is 1 if $i = j$ and 0 otherwise. Recall that $z_{ij} = \mathbf{x}_i^\top \mathbf{v}_j$.

(b) Now show that

$$J_k = \frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{v}_j\right) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \lambda_j.$$

Hint: recall that $\mathbf{v}_j^\top \boldsymbol{\Sigma} \mathbf{v}_j = \lambda_j \mathbf{v}_j^\top \mathbf{v}_j = \lambda_j$.

(c) If $k = d$ there is no truncation, so $J_d = 0$. Use this to show that the error from only using $k < d$ terms is given by

$$J_k = \sum_{j=k+1}^{d} \lambda_j.$$

Hint: partition the sum $\sum_{j=1}^{d} \lambda_j$ into $\sum_{j=1}^{k} \lambda_j$ and $\sum_{j=k+1}^{d} \lambda_j$.

(a) $\left\| x_i - \sum_{j=1}^{k} z_{ij} v_j \right\|_2^2 = \left(x_i - \sum_{j=1}^{k} z_{ij} v_j\right)^T \left(x_i - \sum_{j=1}^{k} z_{ij} v_j\right) = x_i^T x_i - \sum_{j=1}^{k} z_{ij} v_j^T x_i - x_i^T \sum_{j=1}^{k} z_{ij} v_j + \left(\sum_{j=1}^{k} z_{ij} v_j\right)^T \left(\sum_{j=1}^{k} z_{ij} v_j\right)$

$= x_i^T x_i - 2\sum_{j=1}^{k} z_{ij} v_j^T x_i + \left(\sum_{j=1}^{k} z_{ij} v_j\right)^T \left(\sum_{j=1}^{k} z_{ij} v_j\right) = x_i^T x_i - 2\sum_{j=1}^{k} z_{ij} v_j^T x_i + \sum_{j=1}^{k} v_j^T z_{ij}^T z_{ij} v_j$

$= x_i^T x_i - 2\sum_{j=1}^{k} z_{ij} v_j^T x_i + \sum_{j=1}^{k} v_j^T x_i x_i^T v_j = x_i^T x_i - 2\sum_{j=1}^{k} v_j^T x_i x_i^T v_j + \sum_{j=1}^{k} v_j^T x_i x_i^T v_j = x_i^T x_i - \sum_{j=1}^{k} v_j^T x_i x_i^T v_j.$

(b) $J_k = \frac{1}{n}\sum_{i=1}^{n}\left(x_i^T x_i - \sum_{j=1}^{k} v_j^T x_i x_i^T v_j\right) = \frac{1}{n}\sum_{i=1}^{n} x_i^T x_i - \sum_{j=1}^{k} v_j^T \frac{1}{n}\left(\sum_{i=1}^{n} x_i x_i^T\right) v_j = \frac{1}{n}\sum_{i=1}^{n} x_i^T x_i - \sum_{j=1}^{k} v_j^T \Sigma v_j$

$= \frac{1}{n}\sum_{i=1}^{n} x_i^T x_i - \sum_{j=1}^{k} \lambda_j.$

(c) Since $J_d = 0$ and $\sum_{j=1}^{d} \lambda_j = \frac{1}{n}\sum_{i=1}^{n} x_i^T x_i$, $J_k = \frac{1}{n}\sum_{i=1}^{n} x_i^T x_i - \sum_{j=1}^{d} \lambda_j + \sum_{j=k+1}^{d} \lambda_j = \sum_{j=k+1}^{d} \lambda_j.$

2 **(ℓ_1-Regularization)** Consider the ℓ_1 norm of a vector $\mathbf{x} \in \mathbb{R}^n$:

$$\|\mathbf{x}\|_1 = \sum_i |\mathbf{x}_i|.$$

Draw the norm-ball $B_k = \{\mathbf{x} : \|\mathbf{x}\|_1 \leq k\}$ for $k = 1$. On the same graph, draw the Euclidean norm-ball $A_k = \{\mathbf{x} : \|\mathbf{x}\|_2 \leq k\}$ for $k = 1$ behind the first plot. (Do not need to write any code, draw the graph by hand).

Show that the optimization problem

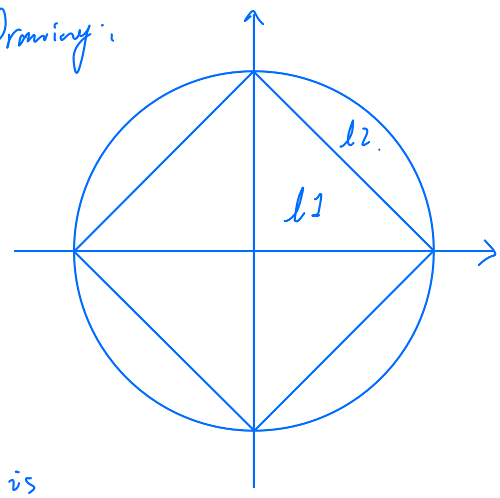
$$\begin{aligned} \text{minimize: } & f(\mathbf{x}) \\ \text{subj. to: } & \|\mathbf{x}\|_p \leq k \end{aligned}$$

is equivalent to

$$\text{minimize: } f(\mathbf{x}) + \lambda\|\mathbf{x}\|_p$$

(hint: create the Lagrangian). With this knowledge, and the plots given above, argue why using ℓ_1 regularization (adding a $\lambda\|\mathbf{x}\|_1$ term to the objective) will give sparser solutions than using ℓ_2 regularization for suitably large λ.

Drawing:



We know the optimization problem:

minimize: $f(x)$ subj. to $\|x\|_p \leq k$

is equivalent to

$$\inf_x \sup_{\lambda \geq 0} \mathcal{L}(x, \lambda) = \inf_x \sup_{\lambda \geq 0} f(x) + \lambda(\|x\|_p - k).$$

In its dual we can flip the inf and sup:

$$\sup_{\lambda \geq 0} \inf_x f(x) + \lambda(\|x\|_p - k) = \sup_{\lambda \geq 0} g(\lambda).$$

Since the minimizing value of $f(x) + \lambda(\|x\|_p - k)$ over x is equivalent to the minimizing value of $f(x) + \lambda\|x\|_p$, we know that optimizing x will solve minimize: $f(x) + \lambda\|x\|_p$ for some suitable value of $\lambda \geq 0$. Looking at the drawing, we can consider l_1 regularization as project the actual optimal solution of the problem onto some suitably sized l_1 norm ball. Since the l_1 ball has sharper edges, the probability of landing on an edge and not on the face is infinitely larger than the l_2 ball. This is due to the rotation invariance of the l_2 that clearly doesn't hold for the l_1 ball! Generalizing to higher dimensions, we can see that the l_1 penalty will encourage more weights to be zero compared to the l_2 ball, which is what we want.

Extra Credit (Lasso) Show that placing an equal zero-mean Laplace prior on each element of the weights $\boldsymbol{\theta}$ of a model is equivelent to ℓ_1 regularization in the Maximum-a-Posteriori estimate

$$\text{maximize: } \mathbb{P}(\boldsymbol{\theta}|\mathcal{D}) = \frac{\mathbb{P}(\mathcal{D}|\boldsymbol{\theta})\mathbb{P}(\boldsymbol{\theta})}{\mathbb{P}(\mathcal{D})}.$$

Note the form of the Laplace distribution is

$$\text{Lap}(x|\mu, b) = \frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right)$$

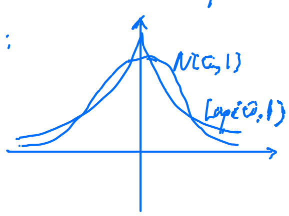
where μ is the location parameter and $b > 0$ controls the variance. Draw (by hand) and compare the density $\text{Lap}(x|0,1)$ and the standard normal $\mathcal{N}(x|0,1)$ and suggest why this would lead to sparser solutions than a Gaussian prior on each elements of the weights (which correspond to ℓ_2 regularization).

We know the Maximum-a-Posteriori problem maximize: $P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)}$ is equivalent to maximizing $\log P(\theta|D)$ given the monotonicity of $\log(x)$. This gives maximize: $\log P(\theta|D) = \log P(D|\theta) + \log P(\theta) - \log P(D)$.

Since $P(D)$ is a constant not dependent on θ, we can drop that term from the problem and flip into a minimization problem, giving minimize: $-\log P(D|\theta) - \log P(\theta)$. Given a prior $\theta_i \sim Lap(0,b)$, $-\log P(\theta) = -\log \prod_i \exp(-\frac{|\theta_i|}{b}) + Z = \frac{1}{b}\sum_i |\theta_i| + Z = \lambda \|\theta\|_1 + Z$.

It following that our original problem is equivalent to minimize: $-\log P(D|\theta) + \lambda\|\theta\|_1$, or a regularized maximum likelihood estimate, as desired. Note the plots of the Standard Normal and Laplace Densities.

Drawing:



We can see that Lap(0,1) will place much more mass at $x=0$. It follows that when we use a Laplace prior instead of a Gaussian prior on our weights, our weights will be more 'encouraged' to be exactly zero, forcing sparsity.