# Lecture 4: Processes with independent increments

## 1. A Wienner process

### 1.1 Definition of a Wienner process

Let $X(t), t \geq 0$ be a real-valued process defind on some probability space $< \Omega, \mathcal{F}, \mathcal{P} >$.

**Definition 4.1**. Stochastic process $X(t), t \geq 0$ is a Markov process if for any $0 \leq t_0 < \ldots < t_n < t < t+s, n \geq 1$,

$$\begin{aligned} &\mathsf{P}\{X(t+s) \leq y/X(t) = x, X(t_k) = x_k, k = 1, \ldots, n\} \qquad (1) \\ &= \mathsf{P}\{X(t+s) \leq y/X(t) = x\} = P(t, x, t+s, y). \end{aligned}$$

**Definition 4.2**. Stochastic process $X(t), t \geq 0$ is a process with independent increments if for any $0 \leq t_0 < \ldots < t_n, n \geq 1$, increments $X(t+s) - X(t), X(t) - X(t_n), X(t_n) - X(t_{n-1}), \ldots, X(t_1) - X(t_0)$ and $X(t_0)$ are independent random variables.

**Lemma 4.1**. Any stochastic process with independent increments is a Markov process.

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$$\begin{aligned} &\mathsf{P}\{X(t+s) \leq y/X(t) = x, X(t_k) = x_k, k = 1, \ldots, n\} \qquad (2) \\ &= \mathsf{P}\{X(t+s) - X(t) \leq y - x/X(t) - X(t_n) = x - x_n, \\ &\quad X(t_n) - X(t_{n-1}) = x_n - x_{n-1}, \ldots, \\ &\quad X(t_1) - X(t_0) = x_1 - x_0, X(t_0) = x_0\} \\ &= \mathsf{P}\{X(t+s) - X(t) \leq y - x\} = P(t, t+s, y-x). \end{aligned}$$

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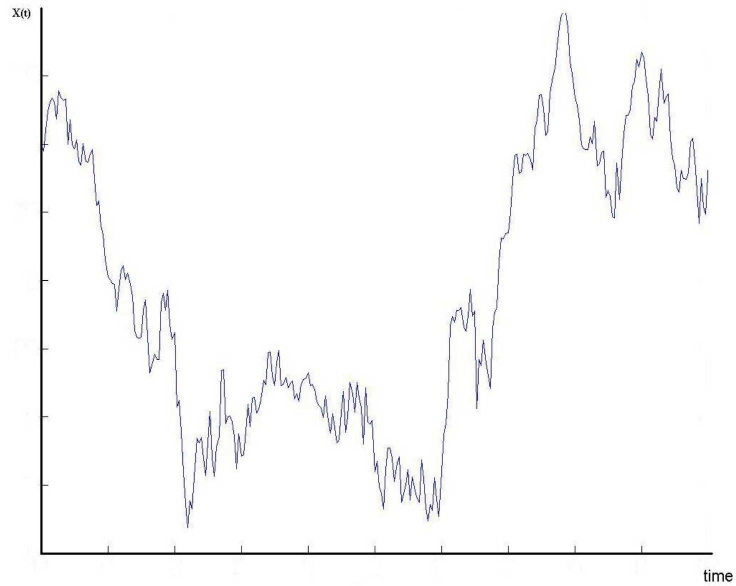
The process with independent increments $X(t), t \geq 0$ is said to be homogeneous in time if for any $t \leq s$,

$$\mathsf{P}\{X(t+s) - X(t) \leq y\} = P(t, t+s, y) = P(s, y).$$




Figur 1: A trajectory of a Wienner process

**Definition 4.3**. A real-valued stochastic process $W(t), t \geq 0$ defined on a probability space $< \Omega, \mathcal{F}, \mathcal{P} >$ is called a standard Wienner process (Brownian motion) if:

**A**: $W(t), t \geq 0$ is a homogenous process with independent increments with the initial value $W(0) \equiv 0$.

**B**: Increment $W(t + s) - W(t)$ has a normal distribution with the mean 0 and the variance $s$, for $0 \leq t \leq t + s < \infty$.

**C**: Process $W(t), t \geq 0$ is continuous, i.e., a trajectory $W(t, \omega), t \geq 0$ is continuous function for any $\omega \in \Omega$.

**Lemma 4.2**. A process $W(t), t \geq 0$ is a Wienner process if and only if it is a real-valued, continuous, Gaussian process with the initial value $W(0) \equiv 0$, the expected values $\mathsf{E}W(t) = 0, t \geq 0$ and the correlation function $\mathsf{E}W(t)W(s) = \min(t, s), t, s \geq 0$.

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(a) A linear transformation of a Gaussian random vector is also a Gaussian random vector;

(b) Let us take arbitrary $n \geq 1$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$. If $(W(t_1), \ldots, W(t_n))$ is a Gaussian random vector then $(W(t_1) - W(t_0), \ldots, W(t_n) - W(t_{n-1}))$ is also a Gaussian random vector and vise versa, since these vectors are linear transformations of each other.

(c) If random variables $W(t_i) - W(t_{i-1}), i = 1, \ldots, n$ are independent and $Var(W(t_i) - W(t_{i-1})) = t_i - t_{i-1}, i = 1, \ldots, n$ then we get $\mathsf{E}W(t_i)W(t_j) = \mathsf{E}(W(t_i) - W(t_0))^2 + \mathsf{E}(W(t_i) - W(t_0))(W(t_j) - W(t_i)) = t_i$, for $i \leq j$.

(d) If $\mathsf{E}W(t_i)W(t_j) = t_i$, for $i \leq j$, then $\mathsf{E}(W(t_i) - W(t_{i-1}))^2 = \mathsf{E}W(t_i)^2 - 2\mathsf{E}W(t_i)W(t_{i-1}) + \mathsf{E}W(t_{i-1})^2 = t_i - 2t_{i-1} + t_{i-1} = t_i - t_{i-1}$, for $i = 1, \ldots, n$, and $\mathsf{E}(W(t_i) - W(t_{i-1}))(W(t_j) - W(t_{j-1})) = t_i - t_i - t_{i-1} + t_{i-1} = 0$, for $i < j$.

(e) Thus increments $W(t_i) - W(t_{i-1}), i = 1, \ldots, n$ are independent since Gaussian random vector has independent components if and only if they are not correlated.

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**(1)** Wienner process posesses the following symmetric property,

$$W(t), t \geq 0 \stackrel{d}{=} -W(t), t \geq 0.$$

**(2)** Trajectories of a Wienner process are continuous but non-differentiable functions.

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(a) Indeed, for any $\Delta > 0$, the random variable $\frac{W(t+\Delta)-W(t)}{\Delta} \stackrel{d}{=} \frac{1}{\sqrt{\Delta}}N(0,1)$, and, therefore, taking $\Delta_n = \frac{1}{n^4}$, we get for any $K > 0$,

$$\sum_{n=1}^{\infty} \mathsf{P}\{|\frac{W(t + \Delta_n) - W(t)}{\Delta_n}| < K\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-K\sqrt{\Delta_n}}^{K\sqrt{\Delta_n}} e^{-\frac{u^2}{2}} du \leq \sum_{n=1}^{\infty} \frac{2K}{\sqrt{2\pi}} \cdot \frac{1}{n^2} < \infty.$$

(b) This relation implies, by Borel-Kantelli lemma, that for any $K > 0$ the only a finite number of events from the sequence $A_{n,K} = \{|\frac{W(t+\Delta_n)-W(t)}{\Delta_n}| < K\}, n = 1, 2, \ldots$ occur and, thus,

$$|\frac{W(t+\Delta_n) - W(t)}{\Delta_n}| \xrightarrow{a.s.} \infty \text{ as } n \to \infty.$$

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**(3)** Trajectories of a Wienner process have unbounded variation, that means that for any $t_{k,n} = a + k\Delta_n$, where $\Delta_n = \frac{(b-a)}{2^n}, k = 1, \ldots 2^n, n \geq 1$

$$L_n = \sum_{k=1}^{2^n} \sqrt{\Delta_n^2 + |W(t_{k,n}) - W(t_{k-1,n}|^2} \xrightarrow{a.s.} \infty \text{ as } n \to \infty.$$

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(a) Indeed, $L_n \geq M_n = \sum_{k=1}^{2^n} |W(t_{k,n}) - W(t_{k-1,n}|$.

(b) $\mathsf{E}M_n = 2^n\sqrt{\Delta_n}\mathsf{E}|N(0,1)| = c2^{n/2}$, where $c = \sqrt{b-a} \cdot \mathsf{E}|N(0,1)|$;

(c) $VarM_n = 2^n Var|W(t_{1,n}) - W(t_{0,n}| \leq 2^n\mathsf{E}|W(t_{1,n}) - W(t_{0,n}|^2 = b - a$.

(d) Let $d_n = o(2^{n/2}) \to \infty$. Obviously $\mathsf{P}\{|M_n - \mathsf{E}M_n| \geq d_n\} \leq \frac{b-a}{d_n^2} \to 0$.

(e) $M_n \xrightarrow{P} \infty$ as $n \to \infty$.

(f) $M_n, n = 1, 2, \ldots$ is a monotonic sequence, and, thus, $M_n \leq L_n \xrightarrow{a.s.} \infty$ as $n \to \infty$.

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**(4)** Wienner process has a fractal self-similarity property,

$$\frac{1}{\sqrt{c}} W(ct), t \geq 0 \overset{d}{=} W(t), t \geq 0.$$

A process $Z(t) = x + \mu t + \sigma W(t), t \geq 0$, where $x, \mu \in R_1, \sigma > 0$, is a Wienner process with the initial state $x$, the drift $\mu$, and the diffusion coefficient $\sigma$.

**(5)** Transition probabilities for Wienner process have the following form,

$$P(x, y, t) = \mathsf{P}\{Z(s+t) \leq y / Z(s) = x\}$$

$$= \mathsf{P}\{x + \mu t + \sigma W(t) \leq y\} = \Phi(\frac{y - x - \mu t}{\sigma\sqrt{t}}),$$

where

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{v^2}{2}} dv, \ u \in R_1.$$

**(6)** They satisfy the Kolmogorov backward equation,

$$\frac{\partial P(x, y, t)}{\partial t} = \mu \frac{\partial}{\partial x} P(x, y, t) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} P(x, y, t).$$

**(7)** The characteristic function of a Wienner process $Z(t)$ has the following form, for $t \geq 0$,

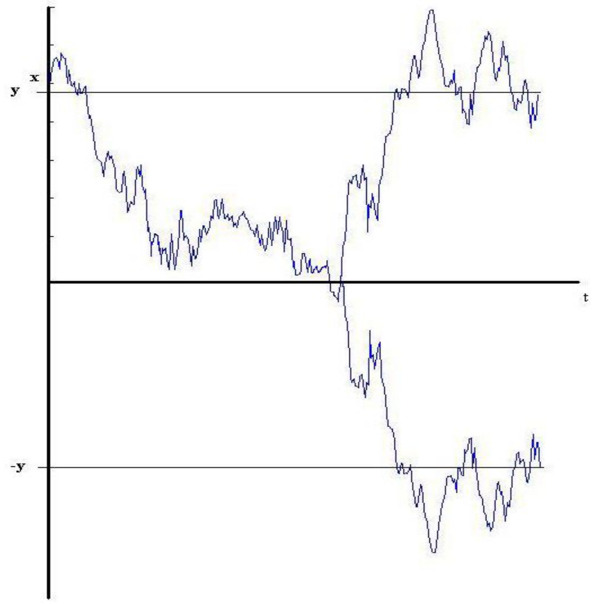
$$\mathsf{E} e^{\mathbf{i} z Z(t)} = e^{\mathbf{i} z (x + \mu t) - \frac{z^2 \sigma^2}{2} t}, \ z \in R_1.$$

**1.2 Reflection principle**

Let consider the minimum functionals of a standard Wienner process $W(t), t \geq 0$,

$$M(t) = \min_{0 \leq s \leq t} W(s), \ t \geq 0.$$

**(8)** Process $Z_x(t) = x + W(t), t \geq 0$ possesses a strong Markov property at hitting moments $\tau_y = \inf(t : Z_x(t) = y)$ that means that $\sigma$-algebras of random events $\sigma[Z_x(s \wedge \tau_y), s \geq 0]$ and $\sigma[Z_x(\tau_y + t), t \geq 0)]$ are independent and, moreover, $Z_x(\tau_y + t), t \geq 0 \stackrel{d}{=} Z_y(t), t \geq 0$.




Figur 2: Reflection transformation

**(9)** The strong Markov property **(8)** implies the following reflection principle, which is expressed by the following equality, for $x, y \geq 0$,

$$\mathsf{P}\{x + W(t) > y, x + M(t) \leq 0\} = \mathsf{P}\{x + W(t) < -y\} \tag{3}$$

$$= \Phi(\frac{-x-y}{\sqrt{t}}) = 1 - \Phi(\frac{x+y}{\sqrt{t}}),\ x, y \geq 0.$$

**Theorem 4.1**. The following formula take place for the standard Wiener process,

$$\mathsf{P}\{W(t) > u, M(t) > -x\}$$

$$= \begin{cases} \Phi(\frac{2x+u}{\sqrt{t}}) - \Phi(\frac{u}{\sqrt{t}}) & \text{for } u \geq -x, x \geq 0 \\ \Phi(\frac{x}{\sqrt{t}}) - \Phi(\frac{-x}{\sqrt{t}}) & \text{for } u < -x, x \geq 0 \end{cases} \tag{4}$$

**(10)** $-M(t) \stackrel{d}{=} \sup_{0 \leq s \leq t} W(s) \stackrel{d}{=} |N(0, t)|$.

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(a) Using reflection equality (5), we get for $x, y \geq 0$,

$$\mathsf{P}\{x + W(t) > y,\ x + M(t) > 0\} \tag{5}$$

$$= \mathsf{P}\{x + W(t) > y\} - \mathsf{P}\{x + W(t) > y,\ x + M(t) \leq 0\}$$

$$= 1 - \Phi(\frac{y - x}{\sqrt{t}}) - (1 - \Phi(\frac{x + y}{\sqrt{t}})) = \Phi(\frac{x + y}{\sqrt{t}}) - \Phi(\frac{y - x}{\sqrt{t}}).$$

(b) Using change of variables $y - x = u \Leftrightarrow y + x = 2x + u$, we get for $x, y \geq 0$,

$$\mathsf{P}\{W(t) > y - x,\ M(t) > -x\} = \mathsf{P}\{W(t) > u,\ M(t) > -x\}$$

$$= \Phi(\frac{2x + u}{\sqrt{t}}) - \Phi(\frac{u}{\sqrt{t}}), u \geq -x, x \geq 0,$$

and

$$\mathsf{P}\{W(t) > u,\ M(t) > -x\} = \mathsf{P}\{M(t) > -x\}$$

$$= \mathsf{P}\{W(t) > -x,\ M(t) > -x\} = \Phi(\frac{x}{\sqrt{t}}) - \Phi(\frac{-x}{\sqrt{t}}), u < -x, x \geq 0.$$

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**1.3 Exponential Brownian motion**

**Definition 4.3**. An exponential Brownian motion is the process given by the following relation,

$$S(t) = Se^{\mu t + \sigma W(t)},\ t \geq 0.$$

where $S, \mu \in R_1, \sigma > 0$.

*Example*

**A European option contract**. This is a contract between two parties, a seller and a buyer, in which the buyer pays to the seller the price $C > 0$ at moment 0 for the right to get from the seller the revenue $e^{-rT}[S(T) - K]_+ =$

$e^{-rT}[S(T) - K]I(S(T) \geq K)$. Here, $r > 0$ is the free interest rate, $K$ is a so-calledstrike price, and $T$ is a maturity of the option contract.

It is known, that the *fair* price $C^*$ of the European contract is given by the following formula,

$$C^* = \mathsf{E}e^{-rT}[S(T) - K]_+,$$

where the expectation should be computed for the so-called risk-neutral model with parameter $\mu = r - \frac{\sigma^2}{2}$.

Note, that, in this case, the process $e^{-rt}S(t), t \geq 0$ possesses the martingale property,

$$\mathsf{E}\{e^{-r(t+s)}S(t+s)/S(t)\} = \mathsf{E}\{e^{-rt}S(t)e^{(-r+\mu)s+\sigma(W(t+s)-W(t))}/S(t)\}$$

$$e^{-rt}S(t)e^{(-r+\mu+\frac{\sigma^2}{2})s} = e^{-rt}S(t), \ t, s \geq 0.$$

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### 1.4 Exchange of measure (Girsanov theorem)

Let $W(t), t \geq 0$ be a standard Wienner process defined on a probability space $< \Omega, \mathcal{F}, \mathcal{P} >$.

Let us introduce random variables, for $\beta \in R_1, T > 0$,

$$Y_\beta(T) = e^{\beta W(T) - \frac{\beta^2}{2}T}. \tag{6}$$

Note that, by the definition, (a) $Y_\beta(T) > 0$ (for every $\omega \in \Omega$), and (b) $\mathsf{E}Y_\beta(T) = 1$.

Let us also define a new probability measure on the $\sigma$-algebra $\mathcal{F}$ using the following $\beta$-transformation,

$$P^*(A) = \mathsf{E}I(A)Y_\beta(T), \ A \in \mathcal{F}. \tag{7}$$

**Lemma 4.3**. The measures $P(A) = \mathsf{E}I(A)$ and $P^*(A)$ are equivalent, i.e., $P(A) = 0 \Leftrightarrow P^*(A) = 0$.

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(a) This readily follows from the positivity of the random variable $Y_\beta(T)$.

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**Theorem 4.2 (Girsanov)**. The process $\tilde{W}(t) = W(t) - \beta t, t \geq 0$ is a standard Wienner process under measure $P^*(A)$ or, equivalently, $W(t) = \tilde{W}(t) + \beta t, t \geq 0$ is a Wienner process with drift $\beta$ and diffusion 1 under measure $P^*(A)$.

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(a) The moment generating function,

$$
\begin{aligned}
m_t(z) &= \mathsf{E}^* e^{z(W(t)-\beta t)} = \mathsf{E} e^{z(W(t)-\beta t)} e^{\beta W(T) - \frac{\beta^2}{2}T} \qquad (8)\\
&= e^{-z\beta t - \frac{\beta^2}{2}t} \times \mathsf{E} e^{(z+\beta)W(t)} \times \mathsf{E} e^{\beta(W(T)-W(t)) - \frac{\beta^2}{2}(T-t)}\\
&= e^{-z\beta t - \frac{\beta^2}{2}t} \times e^{\frac{(z+\beta)^2}{2}t} \times 1 = e^{\frac{z^2}{2}t} = \mathsf{E}^* e^{z\tilde{W}(t)}, \ z \in R_1.
\end{aligned}
$$

(b) A proof for multivariate distributions is similar.

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*Example*

**Black-Scholes formula**. Let us illustrate Girsanov theorem using it for proving the celebrating Black-Scholes formula for the fair price of an European option.

(a) According Girsanov theorem, the $\beta$-transform transforms the process $\mu t + \sigma W(t)$ to the process $\mu t + \sigma(\tilde{W}(t) + \beta t) = (\mu + \sigma\beta)t + \sigma\tilde{W}(t)$, where $\tilde{W}(t)$ is a standard Wienner process under measure $P^*(A) = \mathsf{E} I(A) Y_\beta(T)$.

(b) If to choose $\beta = \frac{(r - \frac{\sigma^2}{2}) - \mu}{\sigma}$, then the process $\mu t + \sigma W(t)$ will be transformed to the process $(r - \frac{\sigma^2}{2})t + \sigma\tilde{W}(t)$ under measure $P^*(A)$.

(c) In this case, the the process $S(t) = Se^{\mu t + \sigma W(t)}$ will be transformed in the process $\tilde{S}(t) = Se^{(r - \frac{\sigma^2}{2})t + \sigma\tilde{W}(t)}$ under measure $P^*(A)$.

(d) The fair price of an European option,

$$C^* = e^{-rT}\mathsf{E}^*[Se^{(r-\frac{\sigma^2}{2})T+\sigma\tilde{W}(T)} - K]_+ \tag{9}$$
$$= S\mathsf{E}^* e^{-\frac{\sigma^2}{2}T+\sigma\tilde{W}(T)} I\Big(Se^{-\frac{\sigma^2}{2}T+\sigma\tilde{W}(T)} \geq K\Big)$$
$$-e^{-rT}KP^*\{Se^{(r-\frac{\sigma^2}{2})T+\sigma\tilde{W}(T)} \geq K\}.$$

(e) The probability in the second term of the above formula,

$$P^*\{Se^{(r-\frac{\sigma^2}{2})T+\sigma\tilde{W}(T)} \geq K\} = P^*\{\sigma\tilde{W}(T) \geq \ln\frac{K}{S} - (r-\frac{\sigma^2}{2})T\} \tag{10}$$
$$= P^*\{N(0,1) \geq \frac{\ln\frac{K}{S} - (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\} = \Phi(\xi - \sigma\sqrt{T}),$$

where $\xi = \frac{\ln\frac{K}{S} - (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$.

(f) Let us apply the $\beta$-transform with parameter $\beta = \sigma$ to the standard Wienner process $\tilde{W}(t)$ (under measure $P^*(A)$), which, in this case, will be transformed to the process $\tilde{W}'(t) + \sigma t$), where $\tilde{W}'(t)$ is a standard Wienner process under the new measure $P'(A) = \mathsf{E}^* I(A)\tilde{Y}_\sigma(T)$, where $\tilde{Y}_\beta(T) = e^{\beta\tilde{W}(T) - \frac{\beta^2}{2}T}$.

(g) In this case, the expectation,

$$\mathsf{E}^* e^{-\frac{\sigma^2}{2}T+\sigma\tilde{W}(T)} I\Big(Se^{-\frac{\sigma^2}{2}T+\sigma\tilde{W}(T)} \geq K\Big) = P'\{Se^{(r-\frac{\sigma^2}{2})T+\sigma(\tilde{W}'(T)+\sigma T)} \geq K\}$$
$$= P'\{N(0,1) \geq \frac{\ln\frac{K}{S} - (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\} = \Phi(\xi). \tag{11}$$

(h) Formulas (9) – (11) yields the desired Black-Scholes formula,

$$C^* = S\Phi(\xi) - e^{-rT}K\Phi(\xi - \sigma\sqrt{T}). \tag{12}$$

## 1.5 A multivariate Wienner processes

**Definition 4.4**. A stochastic process $\overline{W}(t) = (W_1(t), \ldots, W_k(t)), t \geq 0$ with real-valued components, defined on a probability space $< \Omega, \mathcal{F}, \mathcal{P} >$, is called a standard $k$-dimensional Wienner process (Brownian motion) if:

**A'**: $\overline{W}(t), t \geq 0$ is a homogenous process with independent increments with the initial values of components $W_i(0) \equiv 0, i = 1, \ldots, k$.

**B'**: An increment $\overline{W}(t+s) - \overline{W}(t)$ has a multivariate normal distribution with means $\mathsf{E}(W_i(t+s) - W_i(t)) = 0, i = 1, \ldots, k$ and correlation coefficients $\mathsf{E}(W_i(t+s) - W_i(t))(W_j(t+s) - W_j(t)) = sI(i=j), i, j = 1, \ldots, k$, for $0 \leq t \leq t+s < \infty$.

**C'**: Process $\overline{W}(t), t \geq 0$ is continuous, i.e., a trajectory $\overline{W}(t, \omega), t \geq 0$ is continuous function for any $\omega \in \Omega$.

**Definition 4.5**. A stochastic process $\overline{Z}(t) = (Z_1(t), \ldots, Z_k(t)), t \geq 0$ is a $k$-dimensional Wienner process if it is a linear transformation of a standard $k$-dimensional Wienner process $\overline{W}(t), t \geq 0$, i.e, it is given by the following formula,

$$\overline{Z}(t) = \bar{x} + \bar{\mu}t + \Lambda \overline{W}(t), t \geq 0,$$

where $\bar{x} = (x_1, \ldots, x_k), \bar{\mu} = (\mu_1, \ldots, \mu_k)$ are $k$-dimensional vectors with real-valued components, and $\Lambda = \|\lambda_{ij}\|$ is a $k \times k$ matrix with real-valued elements.

**Lemma 4.4**. Process $\overline{Z}(t), t \geq 0$ is a continuous process with independent increments, initial state $\overline{Z}(0) = \bar{x}$, and $k$-dimensional Gaussian distribution of an increment $\overline{Z}(t+s) - \overline{Z}(t)$ with the mean vector $\bar{\mu}s$ and the correlation matrix $\Sigma \cdot s = \|\sigma_{ij} \cdot s\|$, where $\sigma_{ij} = \sum_{r=1}^{k} \lambda_{ir}\lambda_{jr}, i, j = 1, \ldots, k$, for $t, s \geq 0$.

**Lemma 4.5**. A linear transformation $\overline{Z}'(t) = \Gamma \overline{Z}(t) = \Gamma \bar{x} + \Gamma \bar{\mu} t + \Gamma \Lambda \overline{W}(t), t \geq 0$ of a $k$-dimensional Wienner process $\overline{Z}(t) = \bar{z} + \bar{\mu}t + \Lambda \overline{W}(t)$, $t \geq 0$, where $\Gamma = \|\gamma_{ij}\|$ is a $l \times k$ matrix with real-valued elements, is a

$l$-dimensional Wienner process.

## 2. Poisson and related processes

### 2.1 A Poisson process

**Definition 4.6**. A stochastic process $N(t), t \geq 0$ with real-valued components, defined on a probability space $< \Omega, \mathcal{F}, \mathcal{P} >$ is called a Poisson process if:

**D**: $N(t), t \geq 0$ is a homogenous process with independent increments with the initial value $N(0) \equiv 0$.

**E**: Increment $N(t+s) - N(t)$ has a Poisson distribution with mean $\lambda > 0$, i.e., $\mathsf{P}\{N(t+s) - N(t) = n\} = e^{-\lambda s} \frac{(\lambda s)^n}{n!}, n = 0, 1, \ldots$.

**F**: Process $N(t), t \geq 0$ has trajectories continuous from the right.

**(1)** Process $N(t+s) - N(t), t \geq 0$ takes only non-negative integer values and, thus, $N(t)$ is a non-decreasing process.

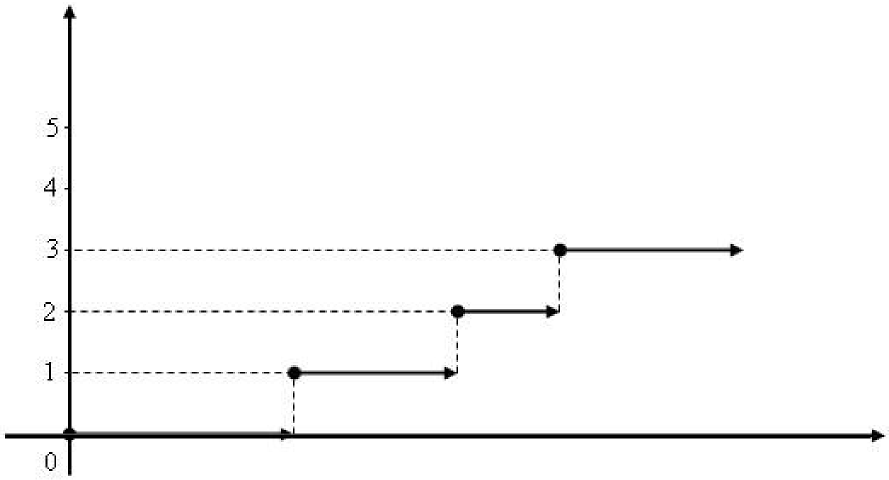
**(2)** Process $N(t), t \geq 0$ has stepwise trajectories.

**(3)** Process $N(t), t \geq 0$ is stochastically continuous.

**(4)** Let $T_n = \min(t : t > T_{n-1}, N(t) > N(T_{n-1})),\ n = 1, 2, \ldots$, where $T_0 = 0$. By the definition, $T_n$ is the moment of n-th jump for the process $N(t)$. The inter-jump times $X_n = T_n - T_{n-1}, n = 1, 2, \ldots$ are mutually independent random variables.

**(5)** Random variables $X_n$ have exponential distribution with parameter $\lambda$, i.e., $\mathsf{P}\{X_n < x\} = 1 - e^{-\lambda x}$, $x \geq 0$, for $n = 1, 2, \ldots$.

**(6)** Random variables $T_n$ have the Erlang distribution with the pdf $p_n(t) = \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}, t \geq 0$, for $n = 1, 2, \ldots$.




Figur 3: A trajectory of a Poisson process

**(7)** Random variables $N(T_n) = n$ with probability 1 for $n = 0, 1, \ldots$.

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(a) $N_\delta(t) = N((n+1)\delta)$ if $n\delta \leq t < (n+1)\delta,\ n = 0, 1, \ldots$, for $\delta > 0$.

(b) $N(t) \leq N_\delta(t) \leq N(t+\delta)$ and, thus, $N_\delta(t) \xrightarrow{a.s.} N(t)$ as $\delta \to 0$.

(c) $T_{\delta,n} = \min(t : t > T_{\delta,n-1}, N_\delta(t) > N_\delta(T_{\delta,n-1})),\ n = 1, 2, \ldots$, where $T_{\delta,0} = 0$.

(d) $T_{\delta,n} \leq T_n \leq T_{\delta,n} + \delta, n = 0, 1, \ldots$.

(e) $\mathsf{P}\{T_{\delta,1} > k\delta\} = e^{-k\delta\lambda} \to e^{-\lambda t}$ as $\delta \to 0, k\delta \to t$, for $t > 0$.

(f) $\mathsf{P}\{N_\delta(T_{\delta,1}) = 1\} = \sum_{k=0}^{\infty} e^{-k\delta\lambda} \cdot \delta\lambda e^{-\delta\lambda} = \frac{\delta\lambda e^{-\delta\lambda}}{1-e^{-\delta\lambda}} \to 1$ as $\delta \to 0$.

(g) etc.

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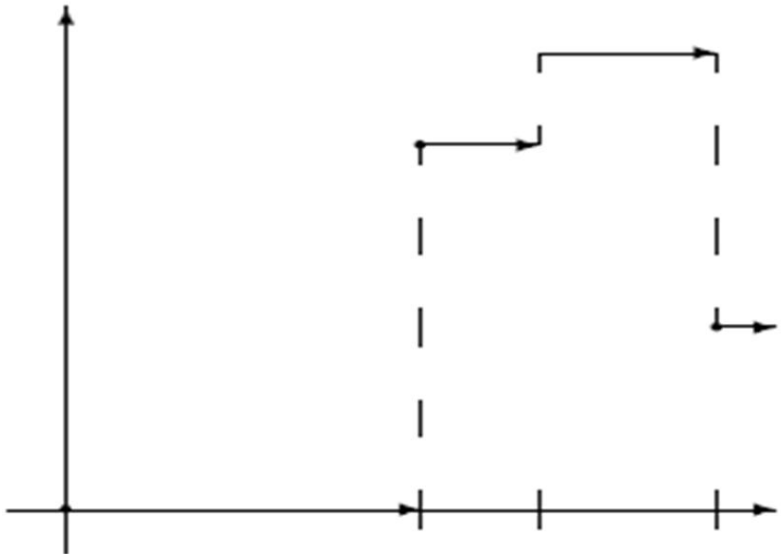
**(8)** Let $N(t), t \geq 0$ is a Poisson process with parameter $\lambda = 1$ and $\lambda(t), t \geq 0$ be a non-negative, continuous function. Define the process $\tilde{N}(t) =$



Figur 4: A trajectory of a compound Poisson process

$N(\Lambda(t))$, $t \geq 0$, where $\Lambda(t) = \int_0^t \lambda(u)du$. Then, $\tilde{N}(t)$ is a process with independent increments such that the increment $\tilde{N}(t+s) - \tilde{N}(t)$ has a Poisson distribution with parameter $\Lambda(t, t+s) = \Lambda(t+s) - \Lambda(t) = \int_t^{t+s} \lambda(u)du$, for $t, s \geq 0$.

## 2.2 Compound Poisson processes

**Definition 4.7**. A stochastic process $Y(t), t \geq 0$ with real-valued components, defined on a probability space $< \Omega, \mathcal{F}, \mathcal{P} >$ is called a compound Poisson process if it has the following form:

$$X(t) = \sum_{k=1}^{N(t)} X_k,\ t \geq 0,$$

where (a) $N(t), t \geq 0$ is a Poisson process with parameter $\lambda > 0$; (b) $X_n, n = 1, 2, \ldots$ is a sequence of i.i.d. real-valued random variables with a distribution function $F(x)$; (c) the process $N(t), t \geq 0$ and the random sequence $X_n, n = 1, 2, \ldots$ are independent.

**(1)** Process $X(t), t \geq 0$ is a homogeneous process with independent increments.

**(2)** Process $X(t), t \geq 0$ has continuous from the right stepwise trajectories.

**(3)** Process $X(t), t \geq 0$ is stochastically continuous.

**(4)** The characteristic function of the compound Poisson process $X(t)$ has the following form, for $t \geq 0$,

$$\Psi_t(z) = \mathsf{E}e^{\mathbf{i}zX(t)} = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}(\mathsf{E}e^{\mathbf{i}zX_1})^n = \exp\{\lambda t(\mathsf{E}e^{\mathbf{i}zX_1} - 1)\}$$

$$= \exp\{\lambda t \int_{-\infty}^{\infty}(e^{\mathbf{i}zx} - 1)dF(x)\},\ z \in R_1.$$

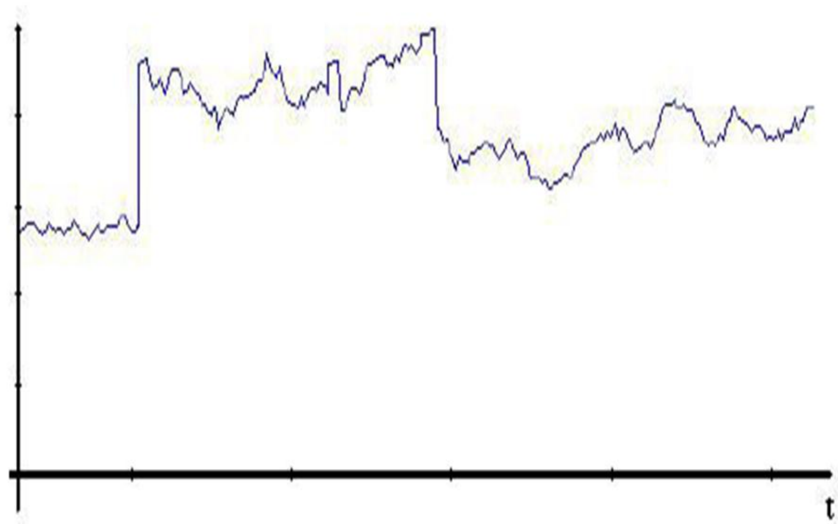
**(5)** $\mathsf{E}X(t) = \lambda t\alpha,\ VarX(t) = \lambda t\beta$, where $\alpha = \mathsf{E}X_1, \beta = \mathsf{E}X_1^2$.

**2.3 Jump-diffusion processes**

**Definition 4.7**. A stochastic process $Y(t), t \geq 0$ with real-valued components, defined on a probability space $< \Omega, \mathcal{F}, \mathcal{P} >$ is called a jump-diffusion process if it has the following form:

$$Y(t) = y + \mu t + \sigma W(t) + \sum_{k=1}^{N(t)} X_k,\ t \geq 0,$$

where (a) $y, \mu \in R_1, \sigma > 0$; (b) $W(t)$ is a standard Brownian motion; (c) $N(t), t \geq 0$ is a Poisson process with parameter $\lambda > 0$; (d) $X_n, n = 1, 2, \ldots$ is a sequence of i.i.d. real-valued random variables with a distribution function $F(x)$; (e) the processes $W(t), t \geq 0$, $N(t), t \geq 0$ and a sequence of random variables $X_n, n = 1, 2, \ldots$ are independent.




Figur 5: A trajectory of a jump-diffusion process

**(1)** Process $Y(t), t \geq 0$ is a homogeneous process with independent increments.

**(2)** Process $Y(t), t \geq 0$ has continuous from the right stepwise trajectories.

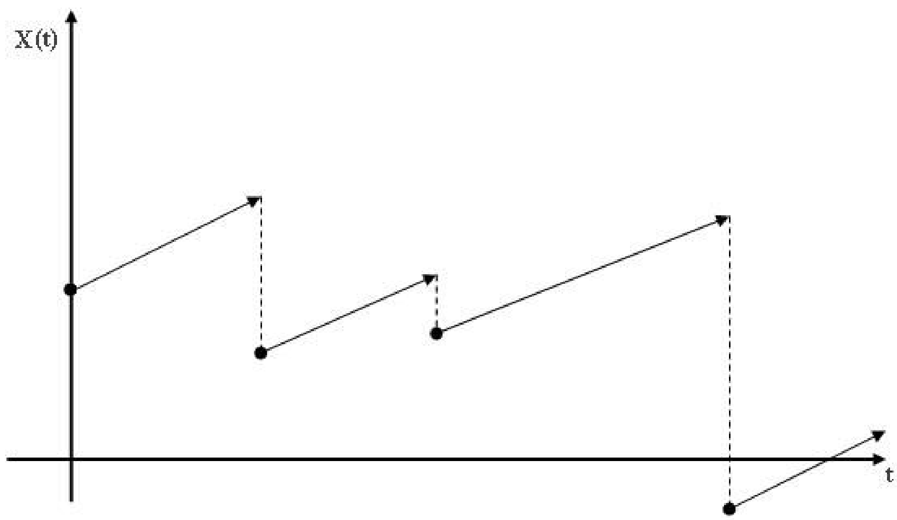
**(3)** Process $Y(t), t \geq 0$ is stochastically continuous.

**(4)** The characteristic function of the jump-diffusion process $Y(t)$ has the following form, for $t \geq 0$,

$$\Psi_t(z) = \mathsf{E}e^{\mathbf{i}zY(t)} = \exp\left\{\mathbf{i}z(y+\mu t) - \frac{z^2\sigma^2}{2}t + \lambda t \int_{-\infty}^{\infty} (e^{\mathbf{i}zx} - 1)dF(x)\right\},\ z \in R_1.$$

**(5)** The process $S(t) = s\exp\{\mu t + \sigma W(t) + \sum_{k=1}^{N(t)} X_k\}, t \geq 0$ is referred as an exponential jump-diffusion process.

**2.4 Risk processes**




Figur 6: A trajectory of a risk process

**Definition 4.8**. A stochastic process $X(t), t \geq 0$ with real-valued components, defined on a probability space $< \Omega, \mathcal{F}, \mathcal{P} >$ is called a jump-diffusion process if it has the following form:

$$X(t) = x + ct - \sum_{k=1}^{N(t)} X_k, \ t \geq 0,$$

where (a) $x, c > 0$; (b) $N(t), t \geq 0$ is a Poisson process with parameter $\lambda > 0$; (c) $X_1, X_2, \ldots$ is a sequence of i.i.d. non-negative random variables with a distribution function $F(x)$; (d) the process $N(t), t \geq 0$ and a sequence of random variables $X_n, n = 1, 2, \ldots$ are independent.

In insurance applications, $c$ is interpreted as a premium rate, $N(t)$ as a process counting the number of claims received by an insurance company in an interval $[0, t]$, $X_n$ as sequential random insurance claims.

A risk process is a particular case of a jump-diffusion process and, thus, has analogous properties.

**(1)** Process $X(t), t \geq 0$ is a homogeneous process with independent increments.

**(2)** Process $X(t), t \geq 0$ has continuous from the right stepwise trajectories.

**(3)** Process $X(t), t \geq 0$ is stochastically continuous.

**(4)** The characteristic function of the risk process $X(t)$ has the following form, for $t \geq 0$,

$$\Psi_t(z) = \mathsf{E}e^{\mathbf{i}zX(t)} = \exp\left\{\mathbf{i}z(x+ct) + \lambda t \int_0^\infty (e^{-\mathbf{i}zx} - 1)dF(x)\right\}, \ z \in R_1.$$

**3. <u>LN Problems</u>**

**1**. Let $U(t,x) = \mathsf{E}f(x+\mu t+\sigma W(t))$, where $f(x)$ be a bounded continuous function. Then $U(t,x)$ satisfies the equation,

$$\frac{\partial U(x,t)}{\partial t} = \mu \frac{\partial}{\partial x} U(x,t) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} U(x,t).$$

**2**. $V(t,x) = \mathsf{E} \int_0^t f(x + \mu s + \sigma W(s))ds$, where $f(x)$ be a bounded continuous function. The following formula takes place,

$$V(t,x) = \int_0^t \frac{1}{\sqrt{2\pi s}\sigma} \int_{-\infty}^\infty f(x+\mu s + y) \exp\{-\frac{y^2}{2\sigma^2 s}\} dyds.$$

**3**. Let $N(t) = \max(n : X_1 + \cdots + X_n \leq t), t \geq 0$, where $X_n, n = 1,2,\ldots$ is a sequence of non-negative i.i.d. random variables. Using formula $\mathsf{P}\{N(t) \geq n\} = \mathsf{P}\{X_1 + \cdots + X_n \leq t\}$ prove that $N(t)$ has a Poisson distribution, if random variables $X_n, n = 1,2,\ldots$ has and exponential distribution with parameter $\lambda$.

**4**. Prove the proposition **(8)** formulated at the Page 13.

**5**. Find conditions under which an exponential jump-diffusion process $S(t), t \geq 0$, introduced in Sub-section 2.3, possesses the martingale property, i.e., $\mathsf{E}\{S(t+s)/S(t)\} = S(t), s,t \geq 0$.

**6**. Find conditions under which a risk process $X(t), t \geq 0$, introduced in Sub-section 2.4, possesses the martingale property, i.e.,
$\mathsf{E}\{X(t+s)/X(t)\} = X(t), s, t \geq 0$.