# Line Integrals

$$\int_{C} f(x,y)\, ds = \lim_{\max \Delta S_i \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta S_i$$

line C

line integrals

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

长度 |tangent vector| × t

$$\int_C f(x,y)\, ds = \int_{t_1}^{t_2} f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_{t_1}^{t_2} f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

# Partial Integral

$$\int_C f(x,y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$

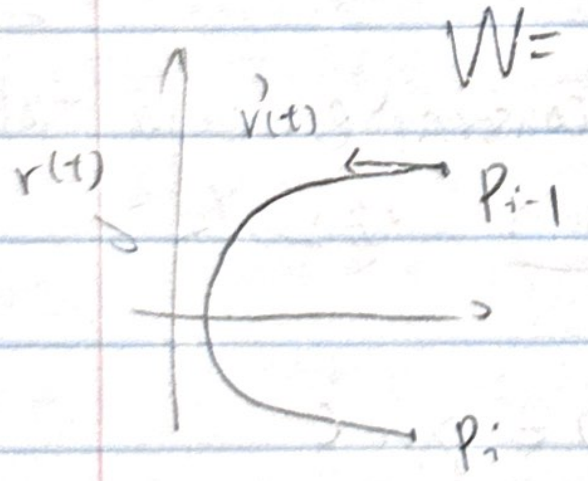
$$\int_C f(x,y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

Application of Line Integral : work done by force F

the work done by force F from $P_{i-1}$ to $P_i$

tangent vector

$$W = \int_a^b F(r(t)) \cdot r'(t)\, dt$$




unit tangent vector

ds

$$= \int_a^b F(r(t)) \cdot \frac{r'(t)}{|r'(t)|} \cdot |r'(t)|\, dt$$

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

$$= \int_C F(r(t)) \cdot T(t)\, ds$$

$\int \nabla f \cdot d\vec{r}$

$= \int_a^b \langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle \, dt$

$= \int_a^b \frac{d}{dt} f(x(t), y(t), z(t)) \, dt$

$= f(x(b), y(b), z(b)) - f(x(a), y(a), z(a))$

## Fundamental Theorem for Line Integrals (FTLI)

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(r(b)) - f(r(a))$$

eg. $F(x,y,z) = -\frac{(xi + yj + zk)}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \langle \frac{-x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \rangle$

move from $A = \langle 0,4,3 \rangle$ to $B = \langle 1,2,0 \rangle$

compute work:

$\because$ $F_x$ $\cancel{f(x)} = \frac{-x}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \cancel{(x^2+y^2+z^2)^{-\frac{1}{2}}}$

$\therefore$ $f(x,y,z) = (x^2+y^2+z^2)^{-\frac{1}{2}} \Big|_{(0,4,3)}^{(1,2,0)}$

$W = \int_C \vec{F} \cdot d\vec{r} = (x^2+y^2+z^2)^{-\frac{1}{2}} \Big|_{(0,4,3)}^{(1,2,0)}$

① 根据 $F(x,y,z) = \langle \_, \_, \_ \rangle$ 中的 $F_x$ 找出,
$f(x,y,z) = \int F_x \, dx + g(y,z)$
② 以此推表, 得到 $f(x,y,z)$
③ $W = f(x_1,y_1,z_1) - f(x_2,y_2,z_2)$

### Independent of path

if $\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$

whenever $C_1$ and $C_2$ are two curves connecting the same two points
any curve — any points

$\int_C F \, dr$ is independent of ~~the~~ path

If $\int_C F \, dr$ is independent of path in D, then $F$ is conservative vector field on D

$\int_C F \, dr$ is independent of path in D if and only if $\int_C F \cdot dr = 0$ for every closed path

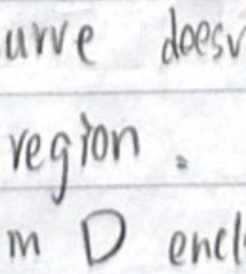
For region D:

open: every point P in D, there is a disk with center P lies entirely in D → doesn't contain any of its boundary points

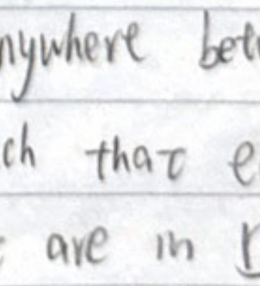
connected: any 2 points can be joined by a path in D

simple curve: a curve doesn't intersect itself anywhere between its endpoints

simply-connected region: connected region D such that every simple closed curve in D encloses only points that are in D



← simply connected



← connected but not simply connected

## Green's Theorem

| | C is closed | C is not closed |
|---|---|---|
| F is conservative | $\oint F \cdot dr = 0$ | $\int_C F \cdot dr = f(B) - f(A)$ |
| F is not conservative | $\int_C \vec{F} \cdot d\vec{r} = \oint (P\,dx + Q\,dy) = \iint_D \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dA$ | $\int_C F \cdot dr = \int_C F \cdot T \cdot ds = \int_a^b F(r(t)) \cdot r'(t)\,dt$ |

$ds = |r'(t)|\,dt$

$T = \frac{r'(t)}{|r'(t)|}$

counter clockwise

Green's: for $\vec{F} = \langle P, Q \rangle$, let $C$ be a <u>positive oriented</u>, simple closed curve in the plane and let $D$ be the region bounded by $C$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + \int_C Q\,dy = \iint_D \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dA$$

eg. $C$ be the circle of radius 2 in the plane

$$\oint_C \left(2y + \sqrt{9 + x^3}\right) dx + \left(5x + e^{\tan^{-1}(y)}\right) dy$$

$$= \int_0^{2\pi} \int_0^2 (5 - 2)\, r\, dr\, d\theta$$

## Curl and Divergence (旋度和散度)

$\nabla$ Gradient：梯度

Curl: for $F = Pi + Qj + Rk$ which is a vector field on $\mathbb{R}^3$ and the partial derivative of $P$、$Q$、$R$ exist, the curl of $F$ is the vector field on $\mathbb{R}^3$ =

$$\boxed{\text{curl } F = \underbrace{\left(\frac{dR}{dy} - \frac{dQ}{dz}\right)}_{yz \text{ plane}} i + \underbrace{\left(\frac{dP}{dz} - \frac{dR}{dx}\right)}_{xz \text{ plane}} j + \underbrace{\left(\frac{dQ}{dx} - \frac{dP}{dy}\right)}_{xy \text{ plane}} k}$$

$$= \underset{\text{gradient operator}}{\nabla} \times F$$

$$\nabla = \left\langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right\rangle$$

$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ P & Q & R \end{vmatrix}$$

If $\mathrm{curl}(\nabla f) = \vec{0}$

curl (conservative vector field)

※ If $F$ is conservative, $\mathrm{curl}\, F = 0$

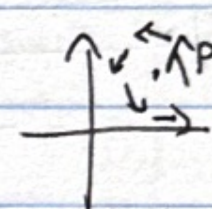
proof:

$$\mathrm{curl}(\nabla f) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left\langle \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, -\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right), \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right\rangle = \langle 0, 0, 0 \rangle$$

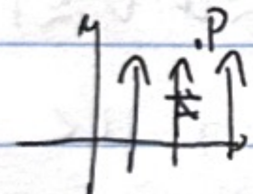
※ If $\mathrm{curl}\, F = 0$, $F$ is conservative vector field

proof:

curl 表明旋度，假设一风扇放向量场。



顺时针：$\mathrm{curl}\, F < 0$



不旋转：$\mathrm{curl}\, F = 0$

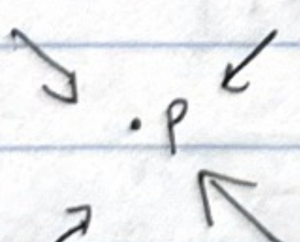
逆时针旋转：$\mathrm{curl}\, F > 0$

**Divergence**: for $F = Pi + Qj + Rk$ which is a vector field on $R^3$ and $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z}$ exist, then the divergence of $F$ is the function of three variables:

$$\boxed{\mathrm{div}\, F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

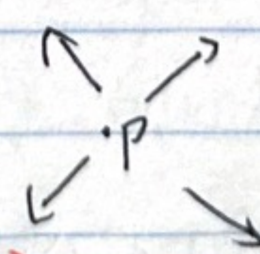
$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$\mathrm{div}\, F = \nabla \cdot F$ describe rotation phenomena



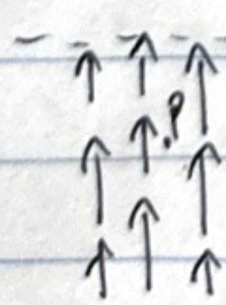
P sink point (汇点) (getting into P)

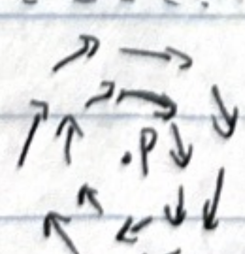
$\mathrm{div}\, \vec{F} < 0$ 流出<流入



P source point (源点) (getting out of P) emanates from the origin

$\mathrm{div}\, \vec{F} > 0$ 流出>流入





$\mathrm{div}\, \vec{F} = 0$ 流出=流入

F (vector field) is incompressible (or solenoidal)