double Riemann sum : $V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$

region $R = [a,b] \times [c,d] = \{(x,y) \in R^2 \mid a \le x \le b,\ c \le y \le d\}$

sample point (x_{ij}^*, y_{ij}^*)

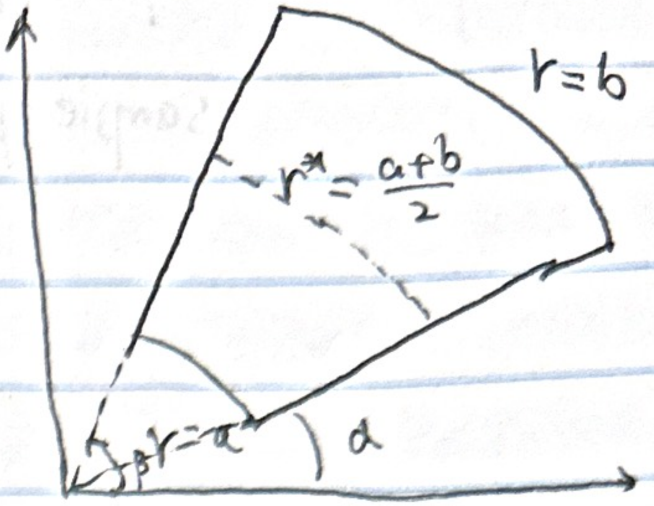
$$V = \iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Type I $V = \iint_D f(x,y)\,dA = \int_a^b \int_{g_2(x)}^{g_1(x)} f(x,y)\,dy\,dx$ x, y not interchangeable

Type II $V = \iint_D f(x,y)\,dA = \int_c^d \int_{h_2(y)}^{h_1(x)} f(x,y)\,dx\,dy$

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

Polar rectangle



$$R = \{(r,\theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$$

$$A(R) = \cancel{\pi r^2 (\beta - \alpha)}\ \pi r^2 \cdot \frac{\beta-\alpha}{2\pi}$$

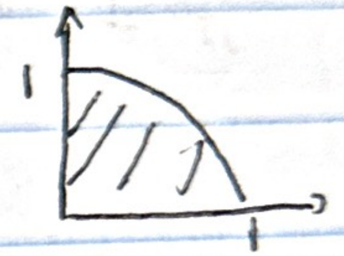
$$= \frac{(b-a)^2}{2}(\beta - \alpha)$$

$$= \frac{b+a}{2} \Delta r \Delta \theta$$

$$dA = \Delta A = r^* \cdot \Delta r \Delta \theta = r^* \, dr \, d\theta$$

$$\iint_R f(x,y)\, dA = \lim_{\max \Delta r_i,\ \Delta\theta_j \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^* \cos\theta_j^*,\ r_i^* \sin\theta_j^*)\, \Delta A_{ij}$$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta,\ r\sin\theta)\, r \, dr\, d\theta$$



eg. $\iint_D \frac{1}{1+x^2+y^2}\, dA \Rightarrow \int_0^{\frac{\pi}{2}} \int_0^1 \frac{1}{1+r^2}\, r\, dr\, d\theta$

hint $[\ln(1+r^2)]' = \frac{2r}{1+r^2}$ $\quad = \int_0^{\frac{\pi}{2}} \frac{1}{2} \ln(1+r^2)\Big|_0^1 \, d\theta$

Triple Intergral

$$\cancel{\iint_R g(x)} \quad \iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

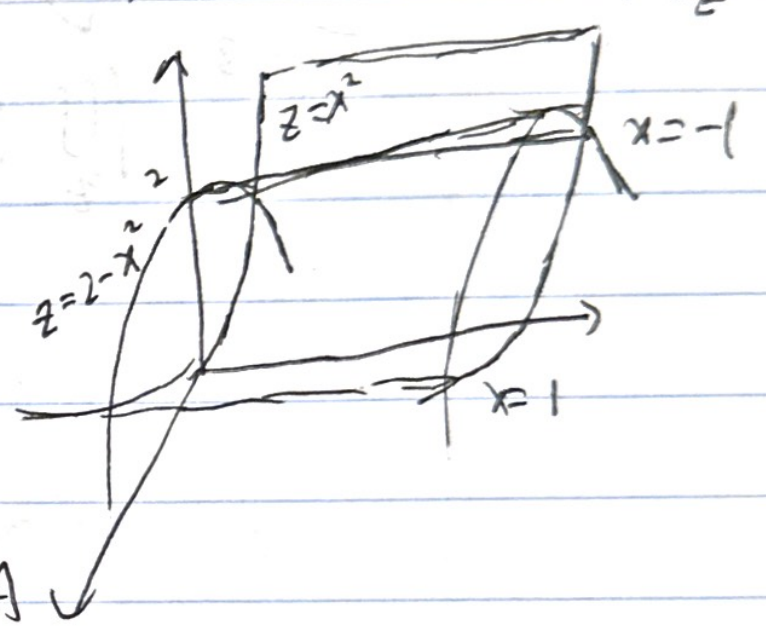
if $f(x,y,z) = g(x)h(y)k(z)$

$$\int_a^b \int_c^d \int_r^s f(x,y,z)\,dz\,dy\,dx = \int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy \cdot \int_r^s k(z)\,dz$$

eg. Let E be solid betwen surface $z = 2-x^2$ and $z = x^2$ for $0 \le y \le 3$, find $\iiint_E (x+y)\,dV$

$$\begin{cases} z = 2-x^2 \\ z = x^2 \end{cases}$$

$x = \pm 1$ on $z = 1$



$$\int_0^3 \int_{-1}^{1} \int_{x^2}^{2-x^2} (x+y)\,dz\,dx\,dy$$

get volume of E: $\iiint_E 1\,dV = \iiint_{x^2}^{2-x^2} 1\,dz\,dA$

or $\iint (2-x^2) - x^2\,dA$

$V(E) = \iiint_E dV = \iiint 1\,dV$

if $f(x,y,z)$ is given as the density function for E, then $\iiint_E f(x,y,z)\,dV$ is the total mass

Center of the mass:

$m = \iiint_E \rho(x,y,z)\,dV$ (only in exam)

$M_{yz} = \iiint_E x\rho(x,y,z)\,dV \quad \overset{\text{moment}}{M_{xz}} = \iiint_E y\,\rho(x,y,z)\,dV$

$M_{xy} = \iiint_E z\,\rho(x,y,z)\,dV$

$\bar{x} = \dfrac{M_{yz}}{m}, \quad \bar{y} = \dfrac{M_{xz}}{m}, \quad \bar{z} = \dfrac{M_{xy}}{m}$

Type 2 :

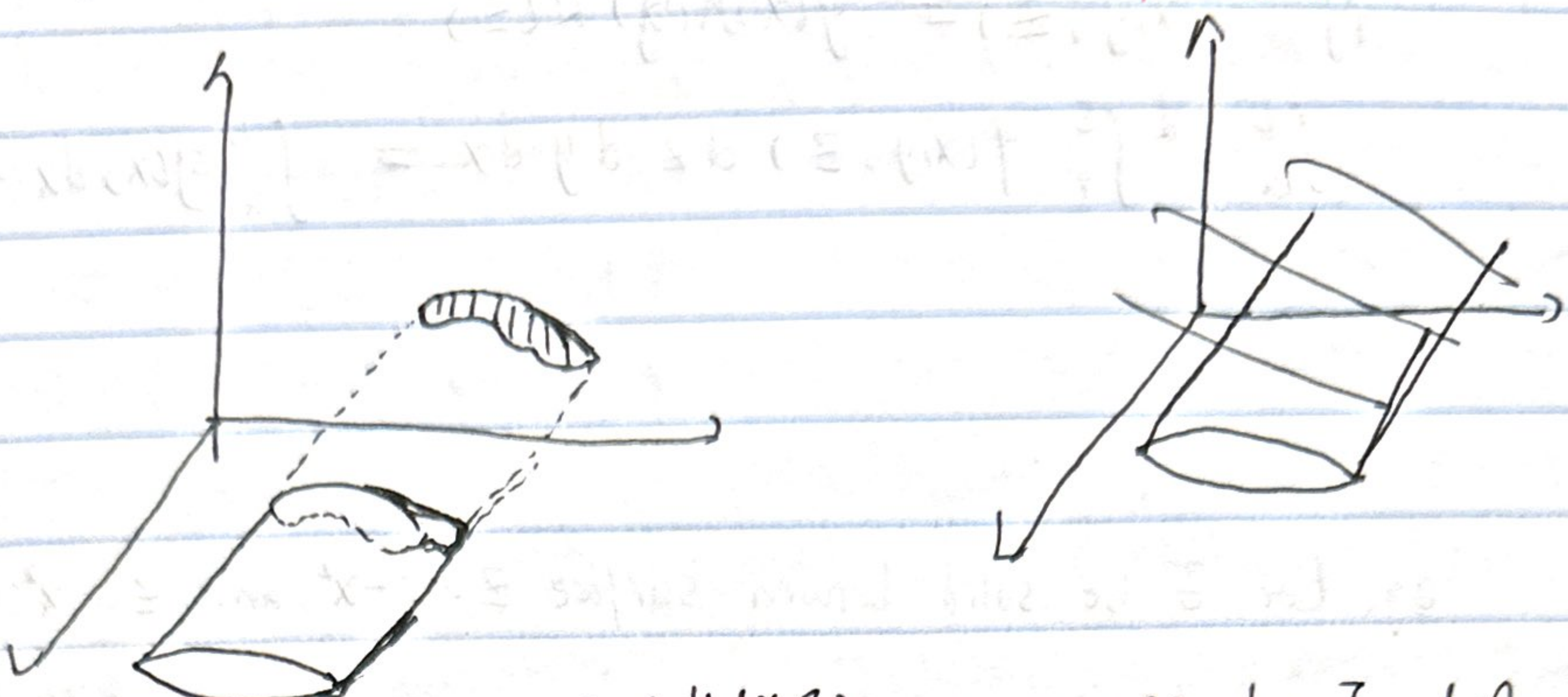
$$\iiint_E f(x,y,z)\,dV = \iint \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx \right] dA$$

$u_2(y,z)$ → x surface closer to me

$u_1(y,z)$ → x surface away from me

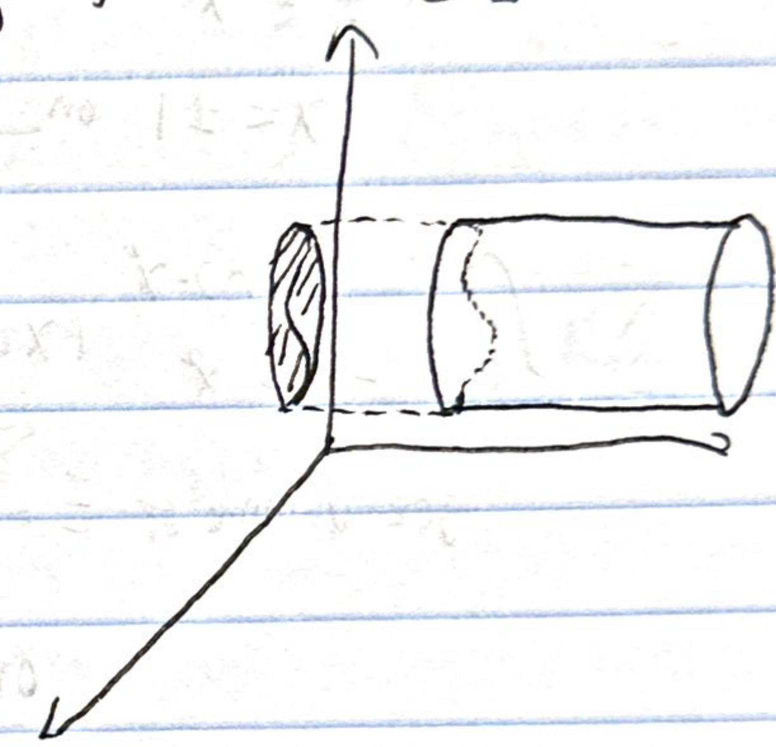
dA → projection on yz plane

{ intersection $x = 0$



Type 3 :

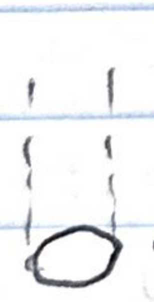
$$\iiint_E f(x,y,z)\,dV = \iint \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy \right] dA$$



Cylindrical Coordinates

when : projection on xy-plane is circle, half circle, quarter circle

cylindrical



← projection on xy plane

Type 1 (only) : $\iint \int_{\text{bottom } z}^{\text{top } z} f(r\cos\theta, r\sin\theta)\, dz\, dA$

$dA \rightarrow r\,dr\,d\theta$

cylindrical coordinates (r, θ, z) ⟷ Cartesian coordinates / rectangular coordinates (x, y, z)

$r = \sqrt{x^2+y^2}$

$\tan\theta = \dfrac{y}{x}$

$z = k$: horizontal plane

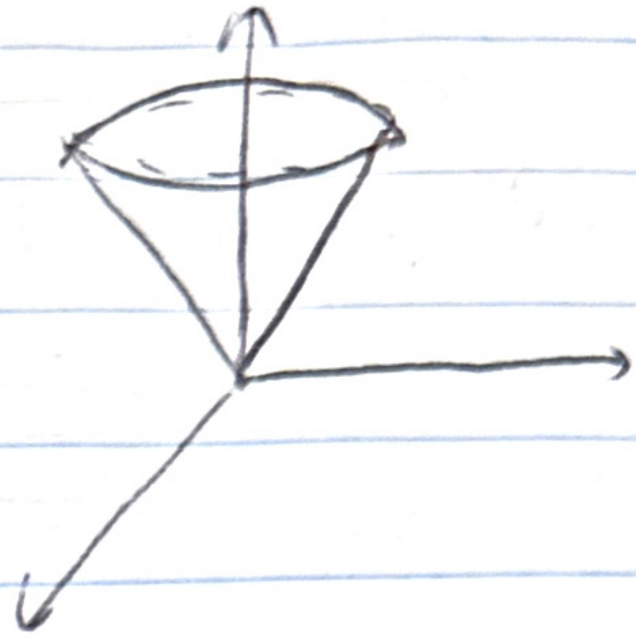
$r = k$: cylinder - infinitely tall, base center on origin, radius k

$\theta = k$: half vertical plane

describe: (a) $z = r$

$z = \sqrt{x^2+y^2}$

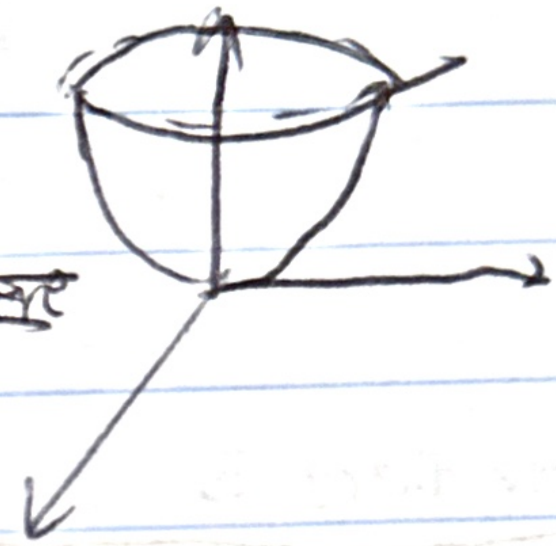
Single top cone



(b) $z = r^2$

$z = x^2+y^2$

circular ~~hemisphere~~ paraboloid



(c) $z = \sqrt{1-r^2}$

$z^2 = 1-x^2-y^2$

eg. $z = x^2+y^2$, $\cancel{z=x}$ $x^2+y^2+z^2 = 2$

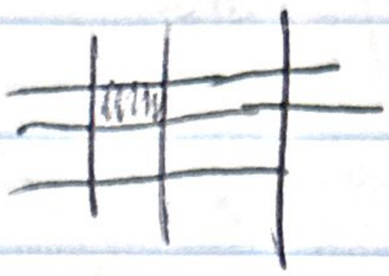
$z = r^2$ $\qquad r^2+z^2 = 2$

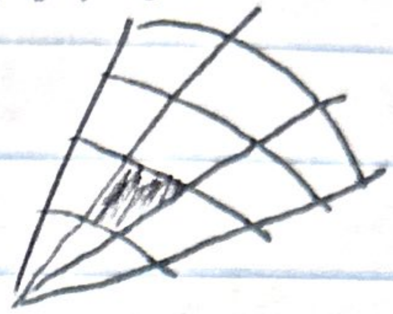
$r^2(r^2+1) = 2 \qquad r = \pm 1$

$$V = \int_0^{2\pi}\int_0^1\int_{\cancel{x^2+y^2}\ r^2}^{\sqrt{2-\cancel{x^2-y^2}\ r^2}} r\, dz\, dr\, d\theta$$

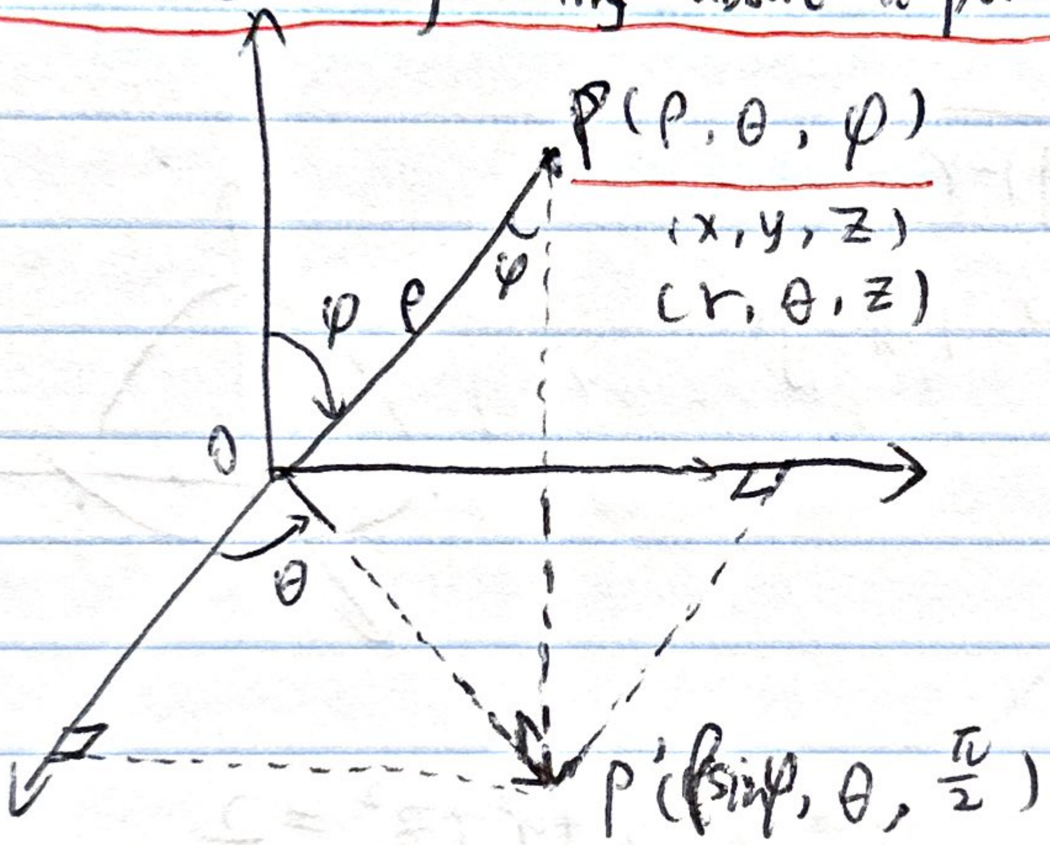
TI in Spherical Coordinates

dA: 2D　　$dx\,dy$　　$r\,dr\,d\theta$





when: where there is symmetry about a point, and the origin is at this point



$\rho = |OP|$

ϕ: 角 between Z-pos and OP

$\rho \geq 0$　　$0 \leq \varphi \leq \pi$

① $z = \rho\cos\varphi$　　$r = \rho\sin\varphi \Rightarrow \begin{cases} ② x = \rho\sin\varphi\cos\theta \\ ③ y = \rho\sin\varphi\sin\varphi \end{cases}$

④ $\rho^2 = x^2 + y^2 + z^2$

for converting $(\rho, \theta, \varphi) \leftrightarrow (x, y, \ldots$

注意 象限 (从 x, y 轴)

$\rho = k$: sphere center at origin, radius $= k$

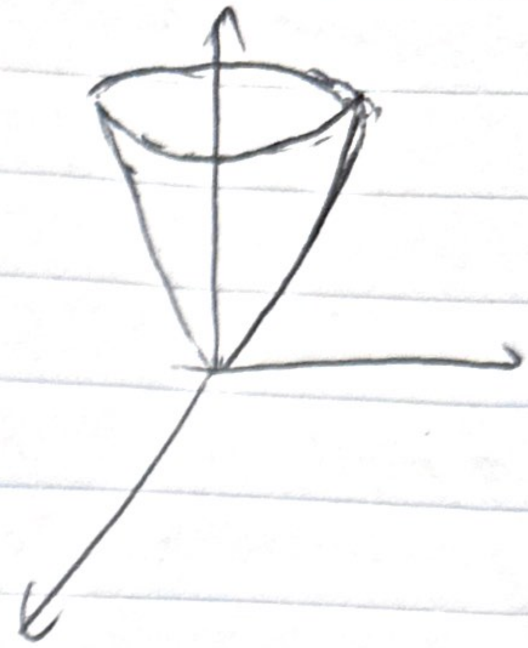
$\theta = k$: verticle half plane (same as cylindrical)

$\varphi = k$: one-piece cone, vertex at origin, extended infinitely

$\varphi = \frac{\pi}{2}$: xy plane

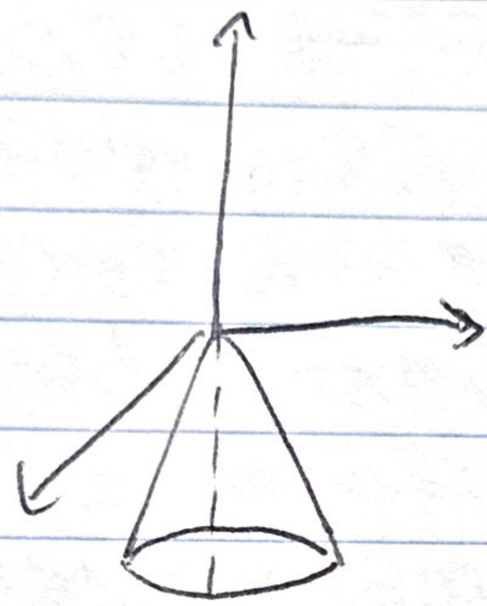
describe (a) 1. $\varphi = \frac{\pi}{4}$: single Standard cone $\Leftrightarrow$ $z = \sqrt{x^2+y^2}$

$$\begin{cases} x = \rho \sin\varphi \cos\theta = \frac{\sqrt{2}}{2} \rho \cos\theta \\ y = \rho \sin\varphi \sin\theta = \frac{\sqrt{2}}{2} \rho \sin\theta \\ z = \rho \cos\varphi = \frac{\sqrt{2}}{2} \rho > 0 \end{cases}$$



$$z^2 = x^2 + y^2 \Rightarrow z = \sqrt{x^2+y^2}$$

2. $\varphi = \frac{3}{4}\pi$: $\Leftrightarrow$ $z = -\sqrt{x^2+y^2}$



(b) $\rho = 5$ $\quad x^2+y^2+z^2 = 25$

sphere center at origin

(c) $\rho = 4\cos\varphi$

$\cos\varphi = \frac{z}{\rho}$ $\quad \rho = 4\frac{z}{\rho}$ $\quad \rho^2 = 4z$

$x^2+y^2+z^2 = 4z \Rightarrow x^2+y^2+(z-2)^2 = 2^2$

sphere center at $(0,0,2)$

ΔV = base Area × height

$= w l h$

$= \Delta\rho_i \cdot (\rho_i \Delta\phi_k) \cdot (\rho_i \sin\phi_k \cdot \Delta\theta_j) = \rho_i^2 \sin\phi_k \Delta\rho_i \Delta\theta_j \Delta\phi_k$

$$\iiint_E f(x,y,z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$