Parametric Surface

A parametric surface S is a set

$$S = \{(x,y,z) \mid x = x(u,v),\ y = y(u,v),\ z = z(u,v),\ (u,v) \in D\}$$

where D is the region in uv-plane

Parametric equations for S:

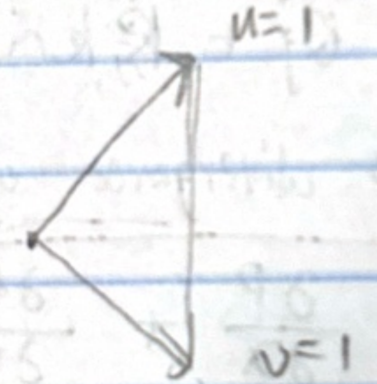
$$r(u,v) = x(u,v)\, i + y(u,v)\, j + z(u,v)\, k$$

describe surface parametrized

eg. by $r(u,v) = \langle u+v,\ u-v,\ 2u+3v \rangle$

over set $D = \{(u,v) \mid 0 \le u, v \le 1,\ u+v \le 1\}$

solve: $r(u,v) = \langle 0,0,0 \rangle + u\langle 1,1,2 \rangle + v\langle 1,-1,3$



is a triangle with

Vertexs are $(0,0,0)$, $(1,1,2)$, $(1,-1,3)$

eg. surface paramatrized by

$$r(u,v) = \langle u, v, v^2 - u^2 \rangle$$

over $D = \{(u,v) \mid u^2 + v^2 \le 1\}$

eg. paramatrize the upper unit hemisphere

Cartesian form

$$\vec{r}(u,v) = \langle u, v, \sqrt{1-u^2-v^2} \rangle$$

over (optional) $D = \{(u,v) \mid 0 \le u^2+v^2 \le 1\}$

cylindrical form

$$\vec{r}(\overset{r}{u},\overset{\theta}{v}) = \langle u\cos v, u\sin v, \sqrt{1-u^2} \rangle$$

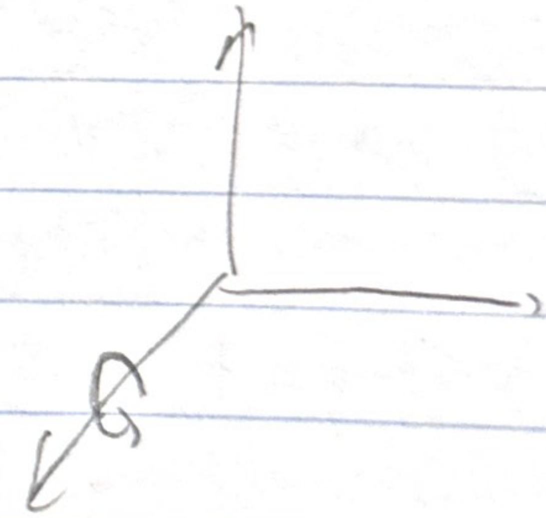
over $D = \{(u,v) \mid 0 \le u \le 1, 0 \le v \le 2\pi\}$

spherical form $\overset{=1 \text{ in this case}}{\rho}, \theta, \varphi$

$$\vec{r}(\overset{\theta}{u},\overset{\varphi}{v}) = \langle 1 \cdot \sin v\cos u, 1 \cdot \sin v \sin u, 1 \cdot \cos v \rangle$$

over $D = \{(u,v) \mid 0 \le v \le \frac{\pi}{2}, 0 \le u \le 2\pi\}$

eg. The graph of $y = x - x^2$ over $0 \le x \le 1$ is revolved around x-axis, paramatrize the surface of revolution



$$\vec{r}(u,v) = \langle u, (u-u^2)\cos v, (u-u^2)\sin v \rangle$$

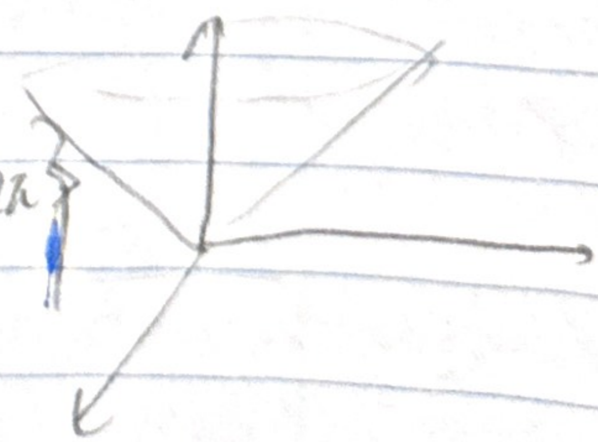
over u: $0 \le u \le 1$

$0 \le v \le 2\pi$

eg. $z = 3x$ on $0 \le x \le 1$ is rotated around z-axis

$$\vec{r}(u,v) = \langle u\cos v, u\sin v, 3u \rangle$$

over $D = \{(u,v) \mid 0 \le u \le 1, 0 \le v \le 2\pi\}$



1. fix the rotating axis which I circle
2. adding after rotating

Tangent Planes

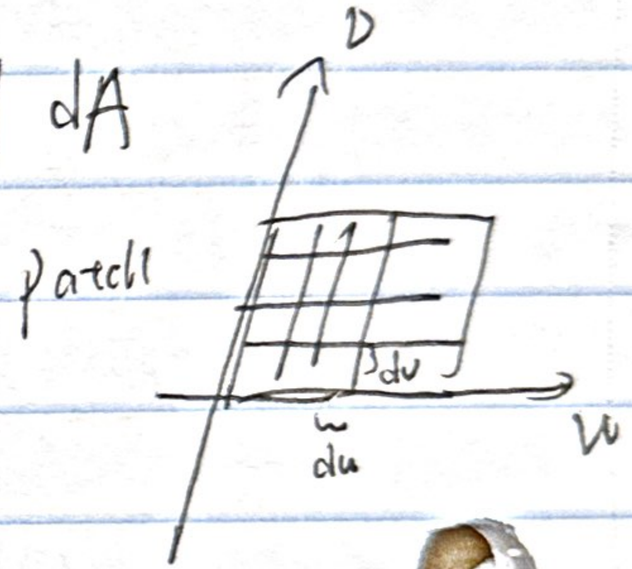
$$r_u = \frac{dx}{du}(u,v)\,i + \frac{dy}{du}\,j + \frac{dz}{du}\,k$$

$$r_v = \frac{dx}{dv}(u,v)\,i + \frac{dy}{dv}\,j + \frac{dz}{dv}\,k$$

are tangent to S

at point $r(u_0, v_0)$ in S, $r_u(u_0, v_0)$ and $r_v(u_0, v_0)$ generate the tangent plane to S

surface area of S = $A(S) = \iint_D |r_u \times r_v| \, dA$



eg. find surface area of the upper unit hemisphere

cylindrical form

step 1: paramatrization of surface, write domain

$$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ \sqrt{1-u^2} \rangle \quad U = \left\{ (u,v) \,\middle|\, \begin{matrix} 0 \le u \le 1 \\ 0 \le v \le 2\pi \end{matrix} \right.$$

step 2: $\|\vec{r_u} \times \vec{r_v}\|$ One patch area on the surface

$$\vec{r_u} = \left\langle \cos v,\ \sin v,\ \frac{-u}{\sqrt{1-u^2}} \right\rangle$$

$$\vec{r_v} = \langle -u\sin v,\ u\cos v,\ 0 \rangle$$

$$\vec{r_u} \times \vec{r_v} = \left\langle \frac{u^2 \cos v}{\sqrt{1-u^2}},\ \frac{u^2 \sin v}{\sqrt{1-u^2}},\ u \right\rangle$$

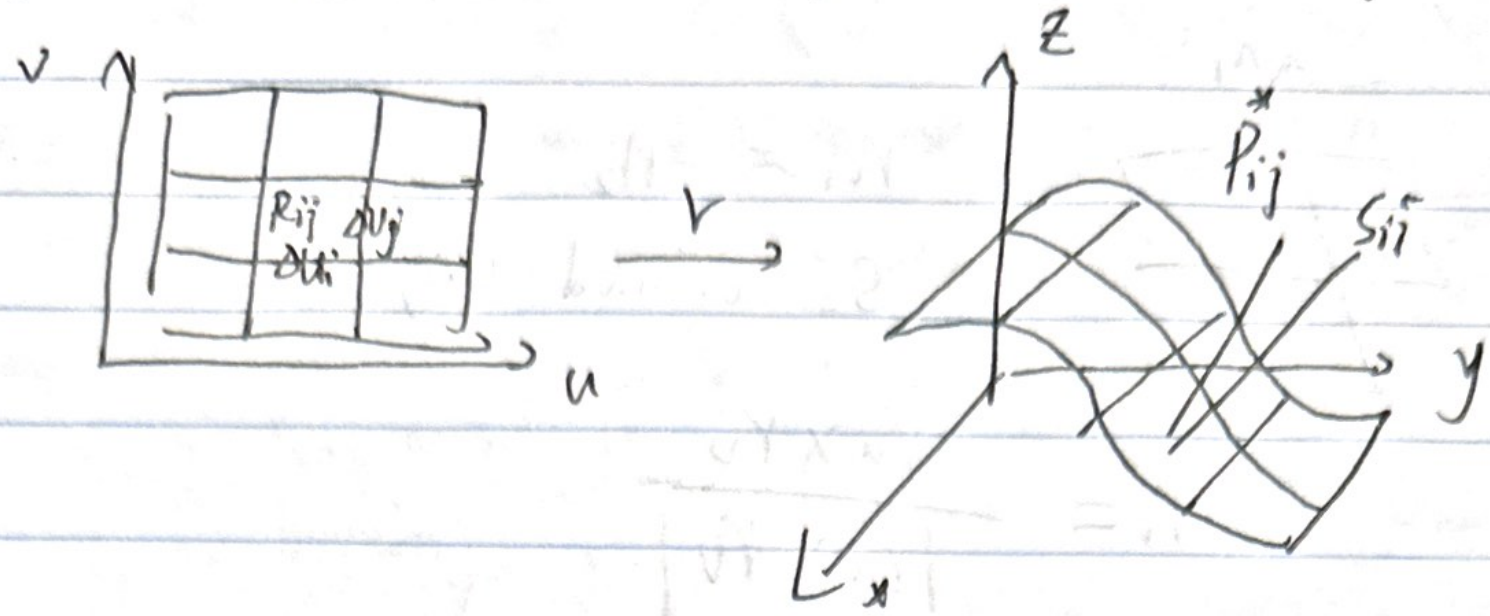
$$\|\vec{r_u} \times \vec{r_v}\| = \frac{u}{\sqrt{1-u^2}}$$

step 3: Integration $A(S) = \iint_D \|r_u \times r_v\| \, dA$

$$A_S = \int_0^{2\pi} \int_0^1 \frac{u}{\sqrt{1-u^2}} \, du \, dv$$

Surface Integral

divergence : flux : amount of F that is caught by S



Surface integral of f over the surface S

$$\iint_S f(x,y,z)\,dS = \lim_{\max \Delta u_i, \Delta v_j \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\,\Delta S_{ij}$$

$$\Delta S_{ij} \approx |r_u \times r_v|\,\Delta u_i\,\Delta v_j$$

面积

$$\iint_S f(x,y,z)\,dS = \iint_D f(r(u,v))\,|r_u \times r_v|\,dA$$

$$= \iint_D f(r(u,v))\,|r_u \times r_v|\,du\,dv$$

eg. Let S be the upper ~~the~~ unit hemisphere. Find $\cancel{\iint z}$ $\iint_S z\,dS$

(Spherical)

Step 1 : paramatrization of surface, write domain

$$\vec{r}(u,v) = \langle 1\cdot \sin v \cos u,\ 1\cdot \sin v \sin u,\ 1\cdot \cos v \rangle$$

$$= \langle \sin v \cos u,\ \sin v \sin u,\ \cos v \rangle$$

$$D = \{(u,v) \mid 0 \le u \le 2\pi,\ 0 \le v \le \tfrac{\pi}{2}\}$$

step 2 : $\|\vec{r_u} \times \vec{r_v}\|$

$$\vec{r_u} = \langle -\sin v \sin u,\ \sin v \cos u,\ 0 \rangle$$

$$\vec{r_v} = \langle \cos v \cos u,\ \cos v \sin u,\ -\sin v \rangle$$

$$|\vec{r_u} \times \vec{r_v}| = \sin v$$

plugin parametrization

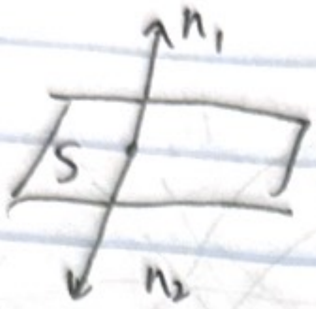
$\|r_u \times r_v\|\,du\,dv$

Step 3 : Integral $\iint_S z\,dS$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \cos v \cdot \sin v\,du\,dv$$

$$= \pi$$

Orientation = a choice of unit normal vector n that varies continuously over surface S



$n_1 = -n_2$

S = oriented surface

$$n = \frac{r_u \times r_v}{|r_u \times r_v|}$$

for $z = g(x)$

$$\text{unit } n = \frac{-\frac{\partial g}{\partial x} i - \frac{\partial g}{\partial y} j + k}{\sqrt{1 + (\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2}}$$

k component is positive ⇓ outward

$$\underset{\text{(not unit)}}{\vec{n}} = \langle -g_x, -g_y, 1 \rangle$$

$$\text{flux} = (\underbrace{F}_{\text{vector field}} \cdot \underbrace{n}_{\text{unit normal vector}}) A(S) = |\vec{F}| \cdot |\vec{n}| \cancel{A(S)} \cos\theta \, A(S)$$

the flux of F, the surface Integral of F over S

$$\iint_S F \cdot dS = \iint_S F \cdot \underbrace{n}_{\text{unit normal vector}} \, dS$$

$$= \iint_D F(r(\cancel{t} u, v)) \cdot \underbrace{(r_u \times r_v)}_{\text{just normal vector}} \, dA$$

	Curve (line Integral)	Surface
function	$\int_C f \, ds = \int_a^b f(r(t)) \underbrace{\|r'(t)\| dt}_{\text{length of a small piece}}$	$\iint_S f \, dS = \iint_D f(r(u,v)) \underbrace{\|r_u \times r_v\| dA_{u,v}}_{\text{area of small piece}}$
Vector Field	$\int_C F \, dr = \int_a^b F(r(t)) \cdot \underbrace{r'(t) \, dt}_{\text{(length with direction) of a small piece}}$	$\iint_S F \cdot dS = \iint_D F(r(u,v)) \cdot \underbrace{(r_u \times r_v) \, dA_{u,v}}_{\text{Area (with direction) of small piece}}$

Stokes' Theorem

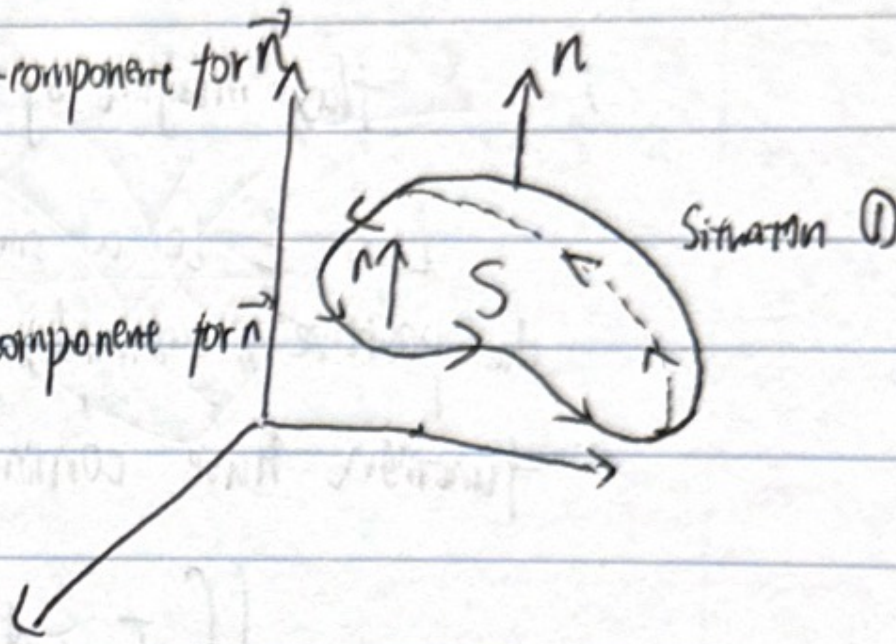
Direction : The following situation, no sign flip when applying Stokes' Theorem

Situation 1 : ① S : ~~attend~~ outward $\vec{n}$ / positive k-componenent for $\vec{n}$

② boundary C : counterclockwise

Situation 2 : ① S : inward $\vec{n}$ / negative k-componenent for $\vec{n}$

② boundary C : clockwise



Stoke's Theorem : Let S be an surface that is bounded by a simple, closed boundary curve C with positive orientation. Let F be a vector field whose componenets have continuos partial derivatives

$$\oint_C F \cdot dr = \iint_S \text{curl } F \cdot dS$$

bounded curve ~~shade~~ surface

eg. Let C be the intersection of $x^2+y^2=1$ with the plane $z=$

$F = \langle 3z, 5x, -2y \rangle$ Find $\oint_C F \cdot dr$

solve: $\text{curl}\vec{F} = \langle -2, 3, 5 \rangle$

By Stoke's Thm, do surface Integral of $\text{curl}\vec{F}$ on the region intersected

$\vec{r}(u,v) = \langle u, v, v+3 \rangle$

$0 \le u^2+v^2 \le 1$

$\vec{n} = \langle 0, -1, 1 \rangle$

$$\iint_S \text{curl } F \cdot dS = \int_{-1}^{1}\int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} \text{curl}\vec{F} \cdot \vec{n} \, du \, dv$$

当题目出现 Find $\iint_S \text{curl } F \cdot dS$，用 Stokes' Thm to find $\int_C F \cdot dr$

使用情况 : ① C is closed curve : ~~sphere, cylinder~~, circle, ellipse, intersection of surface, go back

② 3D : $\vec{F} = \langle P, Q, R \rangle$ can't apply Green Theorem

③ $\text{curl}\vec{F}$ a reasonable vector

Divergence Theorem

$$\operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

flux integral of F over S is the sum of the divergence div F over S

Let E be a simple solid and let S be the boundary surface of E, given the positive (outward) orientation. Let F be a vector field whose component functions have continuous partial derivaties on an open region that contains E

$$\iint_S F \cdot dS = \iiint_E \operatorname{div} F \, dV$$

使用条件： surface integral + multiple surface of objat 一定闭

Compare

Green's Theorem $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$

double integral $\longleftrightarrow$ line integral ($\int_C \vec{F} \cdot d\vec{r}$) $\vec{F} = \langle P, Q \rangle$

Stokes' Theorem $\iint_S \operatorname{curl} F \cdot dS = \int_C F \cdot dr$

surface integral $\longleftrightarrow$ line integral

Divergence Theorem $\iiint_E \operatorname{div} F \, dV = \iint_S F \cdot dS$

triple integral $\longleftrightarrow$ Surface Integral