$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$\nabla F = \langle F_x, F_y, F_z \rangle$

tangent plane : $F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0$

☆ Directional derivative of f at (x_0, y_0) in direction of unit vector $\vec{u} = \langle a, b \rangle$

$$\underline{D_{\vec{u}} f(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

$$= \underline{f_x(x,y)\, a + f_y(x,y)\, b} = \underline{|\nabla f(x,y)| \cdot \underbrace{|\vec{v}|}_{=1} \cdot \cos\theta}$$

rate of change

when $\dfrac{f_x(x,y)}{f_y(x,y)} = \dfrac{a}{b}$

$\Downarrow$

max rate of change

☆ how to find critical point

$f_x(a,b) = 0 \qquad f_y(a,b) = 0$

then $f(a,b)$ is the critical point

[illegible] $\underline{D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2}$ $\quad D = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$

$$\begin{cases} \text{If } D > 0 \text{ 且 } f_{xx}(a,b) > 0 \implies f(a,b) \text{ local minimum} \\ \text{If } D > 0 \text{ and } f_{xx}(a,b) < 0 \implies \text{local maximum} \\ \text{If } D < 0 \quad \text{neither, saddle point} \end{cases}$$

☆ how to find max and min point on closed, bounded set D:

candidate $\begin{cases} \text{1. find critical point} \\ \text{2. find extreme value of } f\text{: point on boundary, with } f'(x) = 0 \text{ or } f'(y) = 0 \end{cases}$

3. compare these point's value of f

Method of Lagrange Multipliers

To find max and min value of $f(x,y,z)$ subject to constrain $g(x,y,z)=k$

1. find
$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = k \end{cases}$$

2. evaluate all points. Largest = max ; Smallest = min
再找一个点 to check max or min

If $g(x,y,z)$ ~~=k~~ is a boundary region. eg. $g(x,y,z) \le k$

eg. $f(x,y) = x^2 - y^2$ in region $x^2+y^2 \le 4$

1. find candidate

① critical point
$$f_x = 0 \Rightarrow x = a$$
$$f_y = 0 \Rightarrow y = b$$
(a, b)

② point on boundary
use lagrange multiplier
$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = k \end{cases}$$

2. evaluate all points

Lagrange Multipliers with 2 constraints

~~$\nabla f(x,y,z)$~~ f has extreme value at $P(x_0, y_0, z_0)$

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

eg. $x+y+z=12$ intersects the paraboloid $z = x^2+y^2$ in an ellipse
find highest and lowest points

$$f(x,y,z) = z$$