SVD

$$\underset{m\times n}{A} = \underset{\substack{m\times m\\ \text{orthogonal}}}{U}\ \underset{\substack{m\times n\\ \text{diagonal}}}{\Sigma}\ \underset{\substack{n\times n\\ \text{orthogonal}}}{V^T}$$

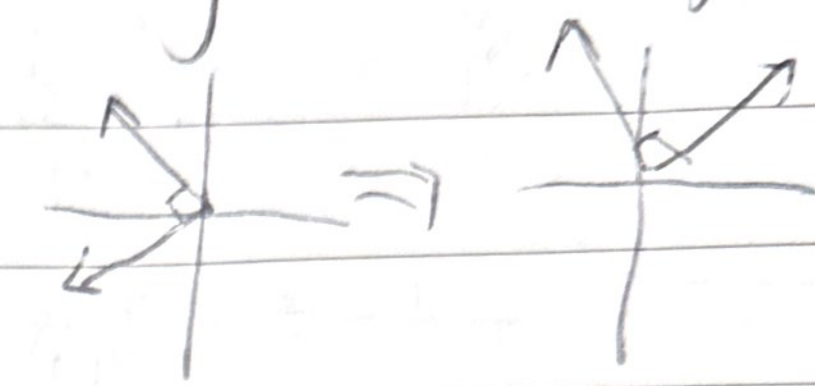
$$A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma^T\Sigma V^T$$

$$AA^T = U\Sigma\Sigma^T U^T$$

what is SVD ? orthogonal sets of vectors transformed by A still orthogonal

$$A\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

$$A \quad V \quad = \quad U \quad \Sigma$$



① $A^TA = V\Sigma^2V^T$

A^TA is symmetric => eigenvector V are orthogonal

Find V by $A^TA = V\Sigma^2V^T$

② $AA^T = U\Sigma^2U^T$ => Find U

③ Find Σ

Σ^2 = eigenvalues matrix of A^TA and AA^T $\begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & & \sigma_n^2 \end{bmatrix}$

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$

note : $\sigma_1 > \sigma_2 \cdots > \sigma_n$

if $m > n$

$$\Sigma = m\left\{\underbrace{\begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ & & \sigma_n \\ 0 & 0\ 0\ 0 & 0 \end{bmatrix}}_{n}\right.$$

if $m < n$

$$\Sigma = m\left\{\underbrace{\begin{bmatrix} \sigma_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & & \vdots \\ 0 & \cdots & \sigma_m & 0 & \cdots & 0 \end{bmatrix}}_{n}\right.$$

$$A = \quad U \quad \Sigma \quad V^T$$

$$\underset{m\times n}{A} = \underset{m\times m}{U}\ \underset{m\times n}{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}}\ \underset{n\times n}{V^T}$$

<=>

$$\underset{m\times n}{A} = \underset{m\times n}{U}\ \underset{n\times n}{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}}\ \underset{n\times n}{V^T}$$

Subspaces of SVD

$$A = U \, \Sigma \, V^T$$

$$\underbrace{\begin{bmatrix} \; \end{bmatrix}}_{m\times n}^{A} = \underbrace{\begin{bmatrix} u_1 & \cdots & u_r & \cdots & u_m \end{bmatrix}}_{m\times m} \underbrace{\left[\begin{array}{ccc|ccc} \sigma_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & \sigma_r & & & \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right]}_{m\times n} \underbrace{\begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \\ \hline \vdots \\ v_n^T \end{bmatrix}}_{n\times n}$$

$$A = U_L \, \Sigma_{TL} \, V_L^T$$

$$\underbrace{\begin{bmatrix} \; \end{bmatrix}}_{m\times n} = \underbrace{\begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_r \end{bmatrix}}_{m\times r} \underbrace{\begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix}}_{r\times r} \underbrace{\begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_r^T \end{bmatrix}}_{r\times n}$$

$$\mathrm{col}(A) = \mathrm{col}(U_L) = \{\vec{u}_1, \vec{u}_2, \cdots \vec{u}_r\}$$

$$\text{left } \mathrm{Nul}(A) = \mathrm{col}(U_R) = \{\vec{u}_{r+1}, \cdots \vec{u}_m\}$$

$$\mathrm{row}(A) = \mathrm{col}(V_L) = \{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_r\}$$

$$\mathrm{Nul}(A) = \mathrm{col}(V_R) = \{\vec{v}_{r+1}, \cdots \vec{v}_n\}$$