# Cryptography: asymmetric encryption

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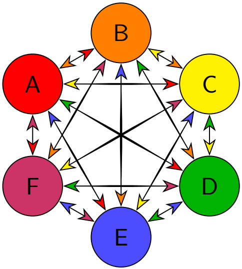
# Introduction

So far: how two users can protect data using a shared secret key

- One shared secret key per pair of users that want to communicate



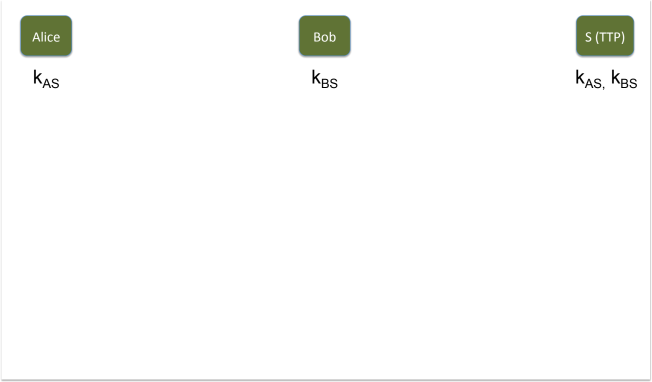

Our goal now: how to establish a shared secret key to begin with?

- Trusted Third Party (TTP)
- Diffie-Hellman (DH) protocol
- *RSA*
- ElGamal (EG)

# Online Trusted Third Party (TTP)

- Users $U_1, U_2, U_3, \ldots, U_n, \ldots$
- Each user $U_i$ has a shared secret key $K_i$ with the TTP
- $U_i$ and $U_j$ can establish a key $K_{i,j}$ with the help of the TTP
- $\{m\}_k$ denotes the symmetric encryption of $m$ under the key $k$

  ex: using Paulson's variant of the Yahalom protocol



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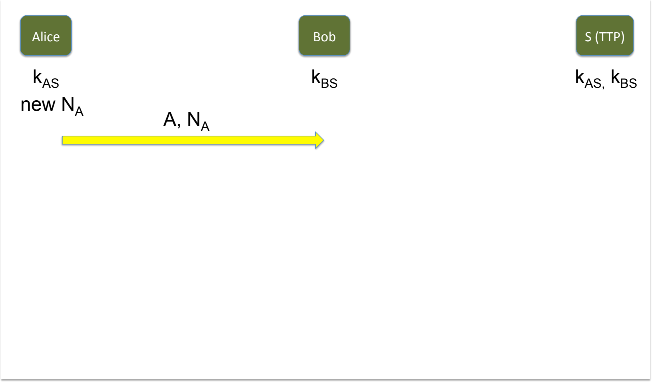

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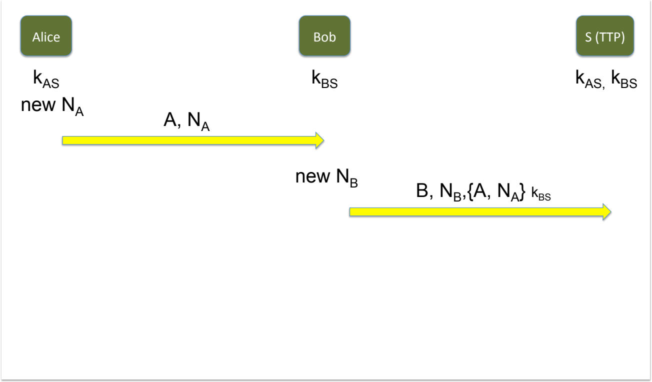

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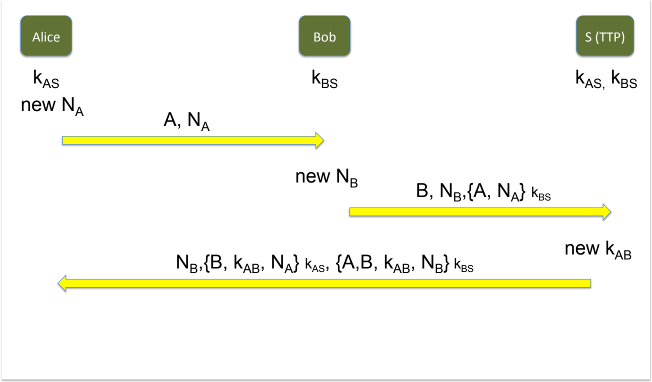

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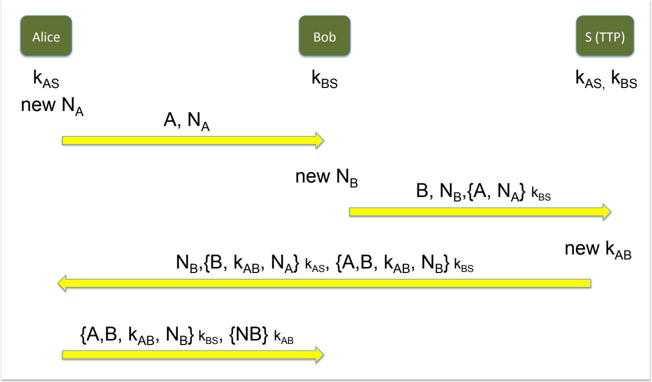

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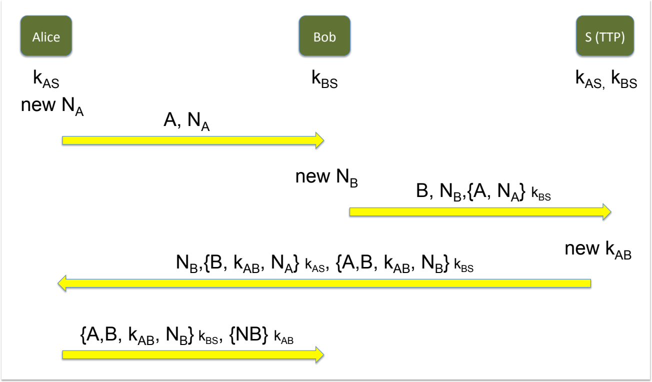
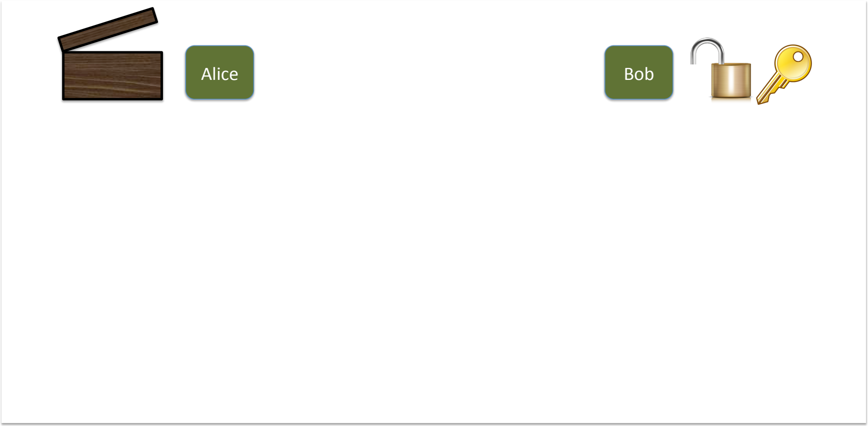



Question: can we establish a shared secret key without a TTP?
Answer: Yes! Using public key cryptography.

# Goal of public-key encryption



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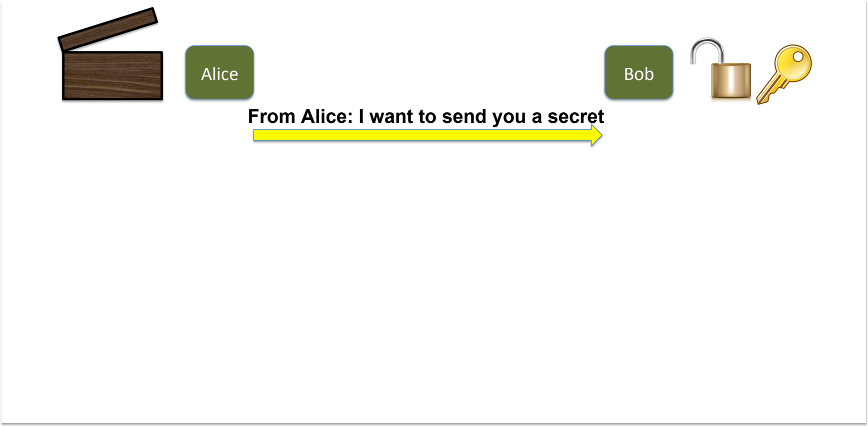

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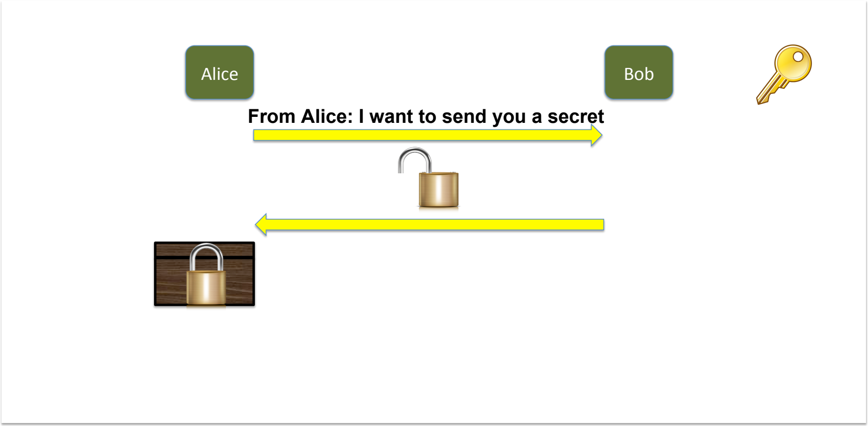

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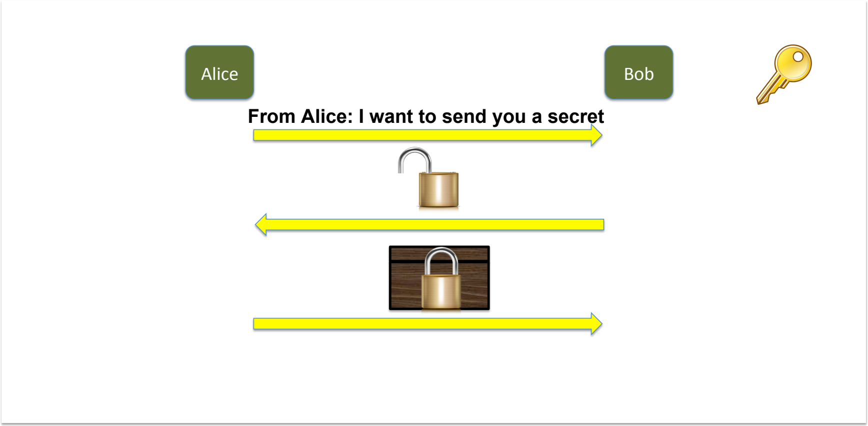

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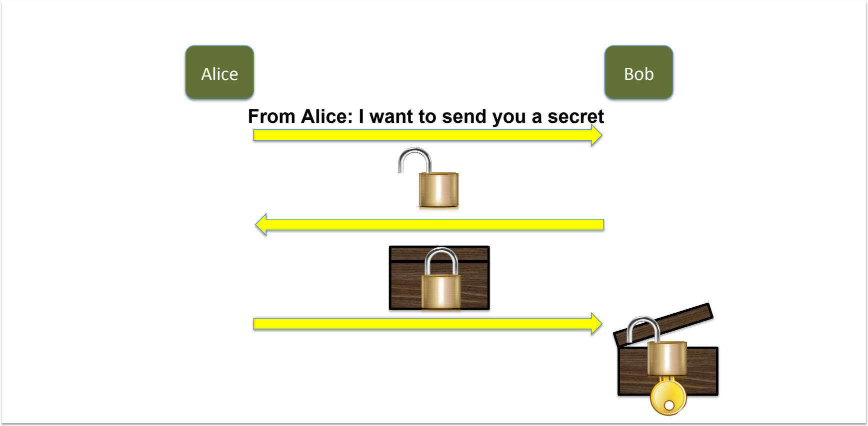
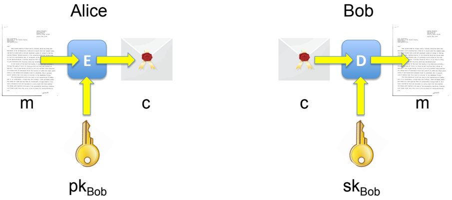


# Public-key encryption - Definition

- key generation algorithm: $G : \to \mathcal{K} \times \mathcal{K}$
  encryption algorithm $E : \mathcal{K} \times \mathcal{M} \to \mathcal{C}$
  decryption algorithm $D : \mathcal{K} \times \mathcal{C} \to \mathcal{M}$
  st. $\forall (sk, pk) \in G$, and $\forall m \in \mathcal{M}$, $D(sk, E(pk, m)) = m$




- the decryption key $sk_{Bob}$ is secret (only known to Bob). The encryption key $pk_{Bob}$ is known to everyone. And $sk_{Bob} \neq pk_{Bob}$

We need a bit of number theory now

# Primes

**Definition**

$p \in \mathbb{N}$ is a **prime** if its only divisors are 1 and $p$

Ex: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29

**Theorem**

*Every $n \in \mathbb{N}$ has a* **unique factorization** *as a product of prime numbers (which are called its factors)*

Ex: $23244 = 2 \times 2 \times 3 \times 13 \times 149$

# Relative primes

**Definition**

$a$ and $b$ in $\mathbb{Z}$ are **relative primes** if they have no common factors

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The Euler function $\phi(n)$ is the number of elements that are relative primes with $n$:

$$\phi(n) = |\{m \mid 0 < m < n \text{ and } \gcd(m, n) = 1\}|$$

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- For $p$ and $q$ primes: $\phi(p \cdot q) = (p\text{-}1)(q\text{-}1)$

# $\mathbb{Z}_n$

- Let $n \in \mathbb{N}$. We define $\mathbb{Z}_n = \{0, \ldots n\text{-}1\}$

$$\forall a \in \mathbb{Z},\ \forall b \in \mathbb{Z}_n,\ a \equiv b \pmod{n} \iff \exists k \in \mathbb{N}.\ a = b + k \cdot n$$

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**Theorem**

*Let $n \in \mathbb{N}$. Let $x \in \mathbb{Z}_n$. $x$ has a inverse in $\mathbb{Z}_n$ iff* $\gcd(x, n) = 1$

# $\mathbb{Z}_N^*$

- Let $n \in \mathbb{N}$. We define $\mathbb{Z}_n^* = \{x \in \mathbb{Z}_n \mid \gcd(x, n) = 1\}$
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$\forall p$ *prime*, $\mathbb{Z}_p^*$ *is a cyclic group, i.e.*

$$\exists g \in \mathbb{Z}_p^*,\ \{1, g, g^2, g^3, \ldots, g^{p-2}\} = \mathbb{Z}_p^*$$

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- DHP:
  input: prime $p$, generator $g$ of $\mathbb{Z}_p^*$, $g^a \pmod{p}$, $g^b \pmod{p}$
  output: $g^{ab} \pmod{p}$

We can now go back and see how to establish a key without a TTP

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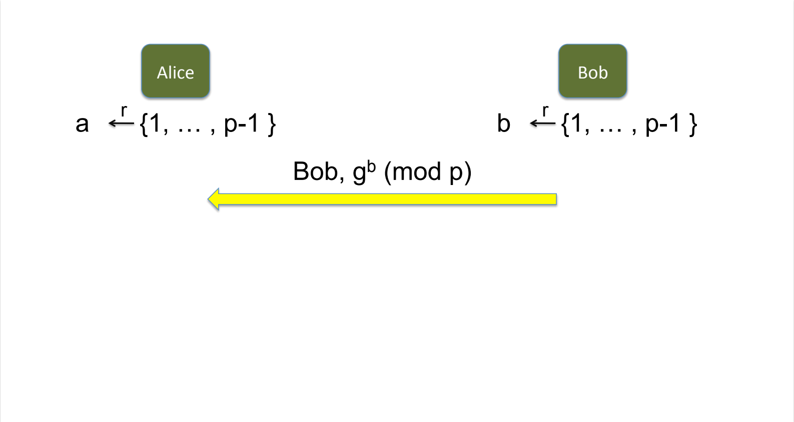

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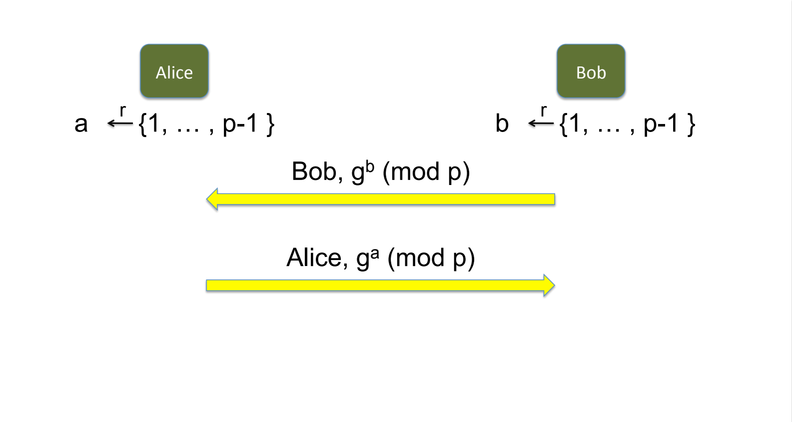

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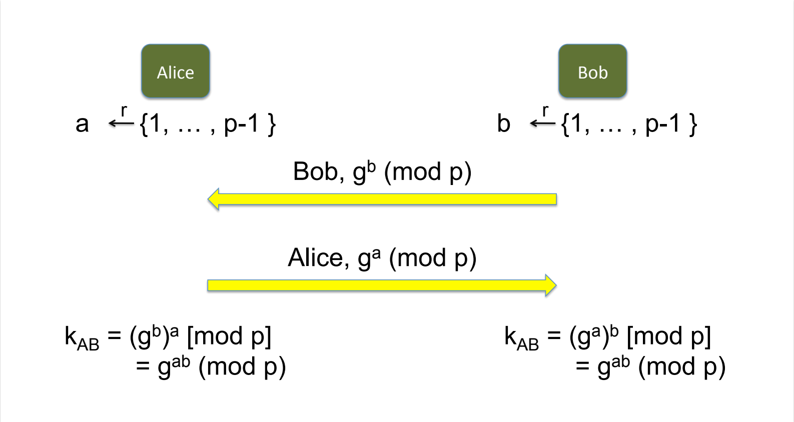

# Man in the middle attack on DH

| Alice | Attacker | Bob |
| --- | --- | --- |
| $a \xleftarrow{r} \{1, \dots, p-1\}$ | $a' \xleftarrow{r} \{1, \dots, p-1\}$ | $b \xleftarrow{r} \{1, \dots, p-1\}$ |
| | $b' \xleftarrow{r} \{1, \dots, p-1\}$ | |

# Man in the middle attack on DH

Alice

$a \xleftarrow{r} \{1, \ldots, p-1\}$

Attacker

$a' \xleftarrow{r} \{1, \ldots, p-1\}$

$b' \xleftarrow{r} \{1, \ldots, p-1\}$

Bob

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Bob, $g^b \pmod p$

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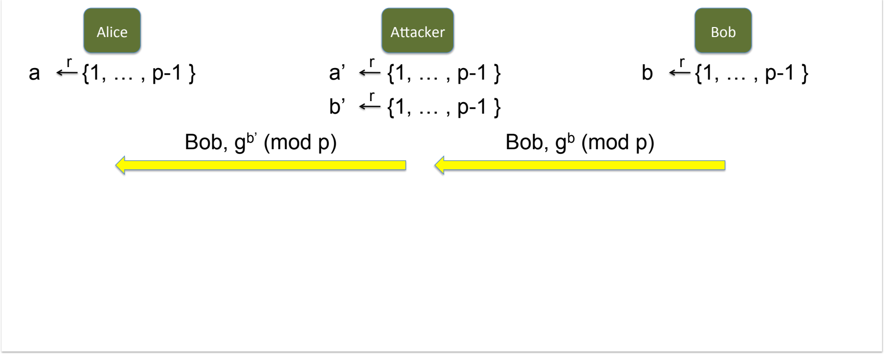

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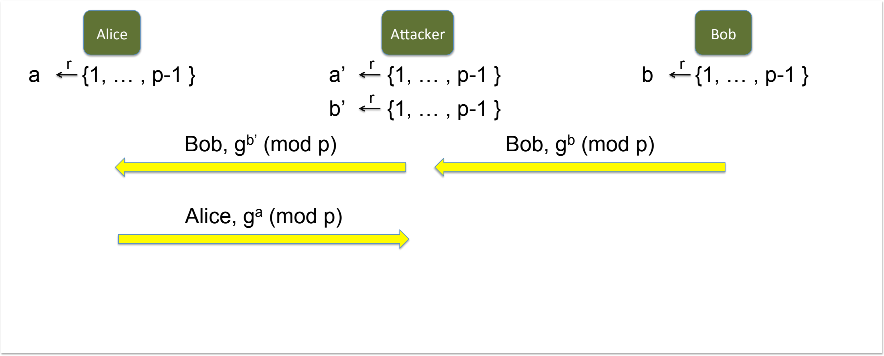

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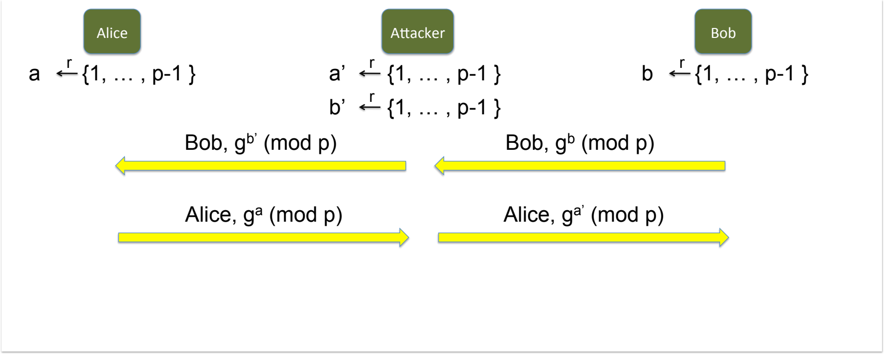

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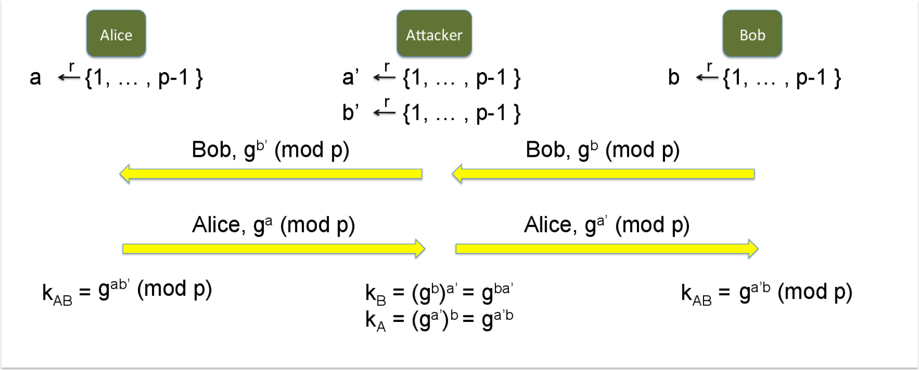

# *RSA* trapdoor permutation

- $G_{RSA}() = (pk, sk)$ where $pk = (N, e)$ and $sk = (N, d)$ and $N = p \cdot q$ with $p, q$ random primes and $e, d \in \mathbb{Z}$ st. $e \cdot d \equiv 1 \pmod{\phi(N)}$

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  Proof: Let $pk = (N, e)$, $sk = (N, d)$. and $x \in \mathbb{Z}_N$. Easy case where $x$ and $N$ are relatively prime

# *RSA* trapdoor permutation

- $G_{RSA}() = (pk, sk)$ where $pk = (N, e)$ and $sk = (N, d)$ and $N = p \cdot q$ with $p, q$ random primes and $e, d \in \mathbb{Z}$ st. $e \cdot d \equiv 1 \pmod{\phi(N)}$
- $\mathcal{M} = \mathcal{C} = \mathbb{Z}_N$
- $RSA(pk, x) = x^e \pmod{N}$ where $pk = (N, e)$
- $RSA^{-1}(sk, x) = x^d \pmod{N}$ where $sk = (N, d)$
- Consistency: $\forall (pk, sk) = G_{RSA}(), \forall x,\ RSA^{-1}(sk, RSA(pk, x)) = x$
  <u>Proof:</u> Let $pk = (N, e)$, $sk = (N, d)$. and $x \in \mathbb{Z}_N$. Easy case where $x$ and $N$ are relatively prime

$$
\begin{aligned}
RSA^{-1}(sk, RSA(pk, x)) &= (x^e)^d \pmod{N} \\
&= x^{e \cdot d} \pmod{N} \\
&= x^{1+k\phi(N)} \pmod{N} \\
&= x \cdot x^{k\phi(N)} \pmod{N} \\
&= x \cdot (x^{\phi(N)})^k \pmod{N} \\
&\overset{\text{Euler}}{=} x \pmod{N}
\end{aligned}
$$

# How **NOT** to use *RSA*

$(G_{RSA}, RSA, RSA^{-1})$ is called raw $RSA$. Do not use raw $RSA$directly as an asymmetric cipher!
$RSA$ is deterministic $\Rightarrow$ not secure against chosen plaintext attacks

# ISO standard

Goal: build a CPA secure asymmetric cipher using $(G_{RSA}, RSA, RSA^{-1})$

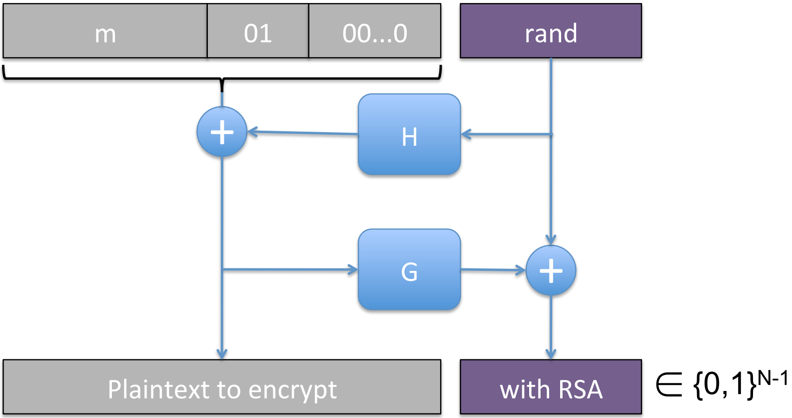
Let $(E_s, D_s)$ be a symmetric encryption scheme over $(\mathcal{M}, \mathcal{C}, \mathcal{K})$
Let $H: \mathbb{Z}_N^* \to \mathcal{K}$

Build $(G_{RSA}, E_{RSA}, D_{RSA})$ as follows

- $G_{RSA}()$ as described above
- $E_{RSA}(pk, m)$:
  - pick random $x \in \mathbb{Z}_N^*$
  - $y \leftarrow RSA(pk, x)(= x^e \mod N)$
  - $k \leftarrow H(x)$
  - $E_{RSA}(pk, m) = y || E_s(k, m)$
- $D_{RSA}(sk, y||c) = D_s(H(RSA^{-1}(sk, y)), c)$

# PKCS1 v2.0: *RSA*-OAEP

Goal: build a CCA secure asymmetric cipher using $(G_{RSA}, RSA, RSA^{-1})$



# ElGamal (EG)

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$$D_{EG}(sk, E_{EG}(pk, x)) \quad = \quad (g^r)^{-d} \cdot m \cdot (g^d)^r \pmod p$$

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$$\begin{aligned} D_{EG}(sk, E_{EG}(pk, x)) &= (g^r)^{\text{-}d} \cdot m \cdot (g^d)^r \pmod{p} \\ &= m \pmod{p} \end{aligned}$$