

1. Vectors

Lecture Notes

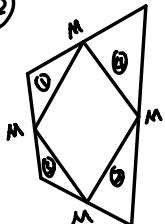
- if $\vec{v} = r\vec{w}$ for some $r \in \mathbb{R}$, then $\vec{v} \parallel \vec{w}$
- $\text{SP}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k) = \{r_1\vec{v}_1 + r_2\vec{v}_2 + \dots + r_k\vec{v}_k \mid r_i \in \mathbb{R}, 1 \leq i \leq k\}$
- the span of two non-parallel vectors in \mathbb{R}^2 is \mathbb{R}^2
- magnitude or norm of \vec{u} is $\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$
- dot product: $\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + \dots + v_nw_n$
- angle between two nonzero vectors in \mathbb{R}^2 is $\arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right)$ where $\theta \in [0, \pi]$ ($0^\circ \sim 90^\circ$)
- perpendicular or orthogonal: $\vec{v} \cdot \vec{u} = 0$
- distance: $\|\vec{v} - \vec{u}\|$
- C-S inequality: $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$
- Triangle Inequality: $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$
- orthogonal projection of \vec{b} on $\vec{a} = \vec{p} = \text{Proj}_{\vec{a}} \vec{b} = \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|^2} \cdot \vec{a}$
- vector component of \vec{b} orthogonal to $\vec{a} = \vec{v} = \vec{b} - \vec{p}$

Assignments

(1) Prove that $\mathbf{u} \cdot \mathbf{v} = \frac{1}{4} \|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4} \|\mathbf{u} - \mathbf{v}\|^2$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 \quad (1) \\ \|\vec{u} - \vec{v}\|^2 &= \|\vec{u}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 \quad (2) \end{aligned}$$
$$\begin{aligned} (1) - (2) \quad &\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 = 4\vec{u} \cdot \vec{v} \\ \therefore \frac{1}{4} \|\vec{u} + \vec{v}\|^2 - \frac{1}{4} \|\vec{u} - \vec{v}\|^2 &= \vec{u} \cdot \vec{v} \end{aligned}$$

(2) show (1) = (2), (3) = (4)



③

\vec{AC}, \vec{BD} bisect each other
 $\star AB = rAB - BC$

$$AB = AO + OB = rAc + kDB = r(AB + BC) + k(AB - BC) = rAB + rBC + kAB - kBC$$

$$\therefore AB = rAB + rBC + kAB - kBC$$

$$0 = rAB + rBC + kAB - kBC - AB$$

$$= AB(r+k-1) + BC(r-k)$$

$$\therefore \begin{cases} r+k-1=0 \\ r-k=0 \end{cases} \quad \begin{cases} r=\frac{1}{2} \\ k=\frac{1}{2} \end{cases}$$

④ three altitudes of \triangle meet.

Given: $AO \cdot BC = 0$
 $BO \cdot AC = 0$

Show: $CO \cdot AB = 0$

$$\begin{aligned} CO \cdot AB &= CO \cdot (AC - BC) \\ &= CO \cdot AC - CO \cdot BC \\ &= (CB + BO) \cdot AC - (CA + AO) \cdot BC \quad \text{make } AO \cdot BC \text{ appear} \\ &= CB \cdot AC + CA \cdot BC - CA \cdot BC - BO \cdot AC \\ &= (-BC) \cdot (CA) - CA \cdot BC \\ &= CA \cdot BC - CA \cdot BC = 0 \end{aligned}$$

⑤ $a_1, \dots, a_m \in \mathbb{R}$

$$\frac{|a_1 + a_2 + \dots + a_m|}{\sqrt{n}} \leq \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$$

let $\vec{u} = [a_1, a_2, \dots, a_m]$ and $\vec{v} = [\underbrace{1, 1, \dots, 1}_n]$

by C-S $|u \cdot v| \leq \|u\| \|v\|$

$$|a_1 + a_2 + \dots + a_m| \leq \sqrt{a_1^2 + a_2^2 + \dots + a_n^2} \sqrt{n}$$

$$\therefore \frac{|a_1 + a_2 + \dots + a_m|}{\sqrt{n}} \leq \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$$

⑥ shortest distance from a to the line $b \vee c$.

$$\text{proj}_{bc} ba = \frac{ba \cdot bc}{\|bc\|^2} bc$$

$$v = ba - \text{proj}_{bc} ba$$

$$\therefore \text{ans} = \|v\|$$

⑦ through $(1, 3)$ and \parallel to $[2, -1]$

$V = [1, 3] \quad a = [2, -1]$
then $x = V + ta \quad t \in \mathbb{R}$
 $x = [x_1, x_2] = [1, 3] + t[2, -1]$

⑧ Find equation of the plane with normal $[2, -1, 5]$ and through $(7, 3, -1)$

Let $\mathbf{a} = [a_1, a_2, a_3]$ on the plane then $[a_1 - 7, a_2 - 3, a_3 + 1] \cdot [2, -1, 5] = 0$

$$\therefore 2a_1 - 14 - a_2 + 3 + 5a_3 + 5 = 0$$

$$2a_1 - a_2 + 5a_3 = 6$$

then the equation of the plane are given by the point a as on
the plane if $a_1 = \frac{6 + s - t}{2}, a_2 = s, a_3 = t$.

⑨ find equation with three points $P(1, 3, 2)$ $Q(0, 6, 2)$ $R(3, 4, -3)$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\begin{aligned}\vec{v}_1 &= \vec{PQ} = [-1, 3, 0] & \begin{vmatrix} i & j & k \\ -1 & 3 & 0 \\ 2 & 1 & -5 \end{vmatrix} &= i \begin{vmatrix} 3 & 0 \\ 1 & -5 \end{vmatrix} - j \begin{vmatrix} -1 & 0 \\ 2 & -5 \end{vmatrix} + k \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} \\ \vec{v}_2 &= \vec{PR} = [2, 1, -5] & &= -16i - 3j - 7k \\ & & \text{a} \quad \text{b} \quad \text{c} &\end{aligned}$$

$$\therefore -16(x - 1) - 3(y - 3) - 7(z - 2) = 0$$

$$-16x + 16 - 3y + 9 - 7z + 14 = 0$$

⑩ P(6, -5, 9) Q(4, 1, 5) R(12, 4, -4)

a) find all $y \in \mathbb{R}$ s.t. PQR is a right angle triangle

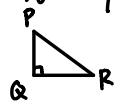
$$\vec{PQ} = [-2, 4, -4], \vec{PR} = [6, 9, y - 9], \vec{PQ} \cdot \vec{PR} = 0 \Rightarrow -12 + 36 + 36 - 4y = 0 \Rightarrow y = 15$$

$$\vec{QP} = [2, -4, 4], \vec{QR} = [8, 5, y - 5], \vec{QP} \cdot \vec{QR} = 6 - 20 + 4y - 20 = 0 \Rightarrow y = 6 \quad q - y = 5 - y$$

$$\vec{RP} = [-6, -9, 9 - y], \vec{RQ} = [-8, -5, 5 - y], \vec{RP} \cdot \vec{RQ} = 48 + 45 + 45 - 9y - 5y + y^2 = y^2 - 14y + 138$$

$$y^2 - 14y + 138 = 0 \quad b^2 - 4ac = 196 - 4 \times 1 \times 138 < 0 \quad \therefore \text{no root} \quad \therefore y = 15, y = 6.$$

b) find equation of the plane that contain PQR and Q is at the right angle



$$\therefore \vec{QP} = [2, -4, 4], \vec{PR} = [6, 9, -5], \vec{QR} = [8, 5, 1]. \text{ let } \mathbf{n} = [a, b, c] \text{ s.t. } \vec{QP} \cdot \mathbf{n} = 0, \vec{QR} \cdot \mathbf{n} = 0, \vec{PR} \cdot \mathbf{n} = 0$$

$$\begin{cases} 2a - 4b + 4c = 0 \\ 6a - 9b - 5c = 0 \\ 8a + 5b + c = 0 \end{cases} \Rightarrow \begin{cases} a - 2b + 2c = 0 \\ 21b - 15c = 0 \\ 21b + 5c = 0 \end{cases} \Rightarrow \begin{cases} a = -\frac{4}{3}c \\ b = \frac{5}{3}c \\ c = c \end{cases} \text{ let } \mathbf{n} = [-4, 5, 3]$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) \text{ using } \mathbf{n} = [-4, 5, 3] \text{ and } Q = [4, -1, 5]$$

$$-4(x - 4) + 5(y + 1) + 7(z - 5) = 0$$

$$\therefore -4x + 5y + 7z = 14.$$

2. Matrices & linear systems

Lecture Notes

- diagonal matrix: square $a_{ij}=0$ if $i \neq j$ $\begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{bmatrix}$
- identity matrix: square $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- upper triangular matrix $\begin{bmatrix} a_{11} & a_{12} & \dots \\ 0 & a_{22} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$
- lower triangular matrix $\begin{bmatrix} a_{11} & 0 & \dots \\ a_{21} & a_{22} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$
- $AB = \begin{bmatrix} -a_1 & - \\ -a_2 & - \\ \vdots & \vdots \\ -a_n & - \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \dots & b_k \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & \dots & b_k \end{bmatrix}$
- symmetric matrix $A^T = A$
- skew symmetric matrix $A^T = -A$
- $Ax = b$ has a solution if b is in the span of $\text{col}(A)$
- Gauss' method with back substitution: get REF and back substitute
- Gauss-Jordan method: get RREF

$$\begin{array}{ccc} I_m & & \\ \downarrow & \rightarrow & \\ A_{m \times n} & \rightarrow & B = EA \\ \downarrow & & \\ E & & \end{array}$$

- $A \sim I \Leftrightarrow Ax = b$ has a solution for $\forall b \in \mathbb{R}^n$

Assignments

- ① If $A_{n \times n}, B_{n \times n}$ prove $\text{tr}(AB) = \text{tr}(BA)$

$$\text{tr}(AB) = \sum_{i=1}^n (AB)_{ii} = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ji} = \sum_{j=1}^n \sum_{i=1}^n b_{ji} a_{ij} = \sum_{j=1}^n (BA)_{jj} = \text{tr}(BA)$$

- ② Suppose that the matrices A and H below are row equivalent.

$$A = \begin{bmatrix} 2 & 1 & x_3 & 1 & y_1 \\ 1 & -1 & 5 & 1 & y_2 \\ x_1 & 2 & 5 & -1 & y_3 \\ 0 & -1 & 2 & -1 & y_4 \\ 1 & x_2 & 3 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = H$$

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad \quad h_1 \quad h_2 \quad h_3 \quad h_4 \quad h_5$

Find the values of $x_1, x_2, x_3, y_1, y_2, y_3, y_4, y_5$. Fully explain your reasoning.

$$3h_1 - 2h_2 = h_3 \Leftrightarrow 3a_1 - 2a_2 = a_3 \Rightarrow \begin{bmatrix} x_3 \\ 5 \\ 5 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 5x_1 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ \frac{-2}{4} \\ -2 \\ 2x_2 \end{bmatrix}$$

$$\therefore \begin{cases} 3x_1 - 4 = 5 \\ 6 - 2 = x_3 \Rightarrow \\ 3 - 2x_2 = 3 \end{cases} \quad \begin{cases} x_1 = 3 \\ x_2 = 0 \\ x_3 = 4 \end{cases}$$

$$h_1 + 2h_2 - 3h_4 = h_5 \Rightarrow a_1 + 2a_2 - 3a_4 = a_5$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ -3 \\ -3 \\ 3 \\ 3 \end{bmatrix} \quad \therefore \begin{cases} y_1 = 1 \\ y_2 = -4 \\ y_3 = 10 \\ y_4 = 1 \end{cases}$$

③ $A_{n \times n}$ is skew symmetric matrix, I_n is identity matrix.

a) show that $A - I$ and $A + I$ are invertible

b) show that if $B = (I - A)(I + A)^{-1}$, then $B^T B = I$

a) since we know that A is invertible $\Leftrightarrow \vec{x} = \vec{0}$ is the only solution to $A\vec{x} = \vec{0}$

This is equivalently proving $(A + I)\vec{x} = \vec{0} \Leftrightarrow \vec{x} = \vec{0}$

$$\begin{aligned} \therefore Ax &= -x \\ (Ax)^T &= -x^T \\ x^T A^T &= -x^T \end{aligned}$$

$$x^T A = -x^T$$

$$x^T A = x^T$$

$$x^T A x = x^T x$$

$$x^T (-x) = x^T x$$

$$-||x||^2 = ||x||^2$$

$$\therefore ||x|| = 0$$

$$x = 0$$

$$\text{b) Since } B = (I - A)(I + A)^{-1}$$

$$\therefore B^T = ((I - A)(I + A)^{-1})^T$$

$$= ((I + A)^{-1})^T (I - A)^T$$

$$= (I^T + A^T)^{-1} (I^T - A^T)$$

$$= (I - A)^{-1} (I + A)$$

$$B^T B = (I - A)^{-1} (I + A) (I - A) (I + A)^{-1}$$

$$= (I - A)^{-1} (I - A) (I + A) (I + A)^{-1}$$

$$= I$$

④ An $n \times n$ $\mathbf{0}_{nn}$ zero matrix

$$A^3 - 2A^2 + 3A - I = 0$$

Show A is invertible and find A^{-1} in terms of A .

$$A^3 - 2A^2 + 3A - I = 0 \quad A^{-1} = B = A^2 - 2A + 3I.$$

$$A^3 - 2A^2 + 3A = I$$

$$\underbrace{A(A^2 - 2A + 3I)}_B = I$$

⑤

Let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5 \in \mathbb{R}^5$ satisfy the following two equations:

$$\begin{array}{rcl} 2\mathbf{c}_1 + \mathbf{c}_2 & - 3\mathbf{c}_4 - \mathbf{c}_5 & = \mathbf{0} \\ -\mathbf{c}_1 + 2\mathbf{c}_2 - \mathbf{c}_3 & & = \mathbf{0} \end{array}$$

and let vectors $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4$ be linearly independent.

If A is the matrix that has as its i^{th} column the vector \mathbf{c}_i for $i = 1, 2, 3, 4, 5$, then find the row reduced echelon form of A . Justify your answer.

Let $A \in \mathbb{R}^{5 \times 5}$ in RREF

$$\therefore A\vec{x} = 0 \Leftrightarrow H\vec{x} = 0$$

$$\therefore 2\mathbf{c}_1 + \mathbf{c}_2 - 3\mathbf{c}_4 - \mathbf{c}_5 = \mathbf{0}$$

$$\therefore A\vec{v} = 0 \text{ where } \vec{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -3 \\ -5 \end{bmatrix}$$

$$\therefore H\vec{v} = 0 \text{ where } \vec{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -3 \\ -5 \end{bmatrix}$$

$$\text{and } h_5 = 2h_1 + h_2 - 3h_4$$

$$h_5 = 2h_2 - h_1$$

$$\therefore \begin{bmatrix} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \quad \text{linearly independent} \Rightarrow \text{pivot in column.}$$

3. Homogeneous System, Subspaces, Bases and Rank

Lecture Notes

- homogeneous system is consistent and $\Leftrightarrow Ax = \vec{0}$
- if the solution is $\vec{x} = \begin{bmatrix} ? \\ 0 \end{bmatrix} \Rightarrow$ trivial solution
- if A and H has pivots in every column $\Rightarrow Ax = 0$ has the trivial solution \Leftrightarrow invertible
- $\text{null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$
- if $\gamma_1\vec{v}_1 + \gamma_2\vec{v}_2 + \dots + \gamma_n\vec{v}_n = 0$ has one solution $\Rightarrow v_1, v_2, \dots, v_n$ are linearly independent
- if $\gamma_1\vec{v}_1 + \gamma_2\vec{v}_2 + \dots + \gamma_n\vec{v}_n = 0$ has more than one solution $\Rightarrow v_1, v_2, \dots, v_n$ are linearly dependent
- Two vectors in \mathbb{R}^n are linearly indep. if they are nonzero and nonparallel.
- Let $W \subset \mathbb{R}^n$

- ① W is nonempty
- ② if $u, v \in W$, $u + v \in W$ (closure under vector addition)
- ③ if $u \in W$, $r \in \mathbb{R}$, $ru \in W$ (closure under scalar multiplication)

then W is a subspace of \mathbb{R}^n

DEFINITION: Let W be a subspace of \mathbb{R}^n . If $B = \{b_1, b_2, \dots, b_k\}$ is a subset of W , then we say that B is a basis for W if **every** vector in W can be written **uniquely** as a linear combination of the vectors in B . The plural for the word basis is "bases".

THEOREM: Let A be an $n \times n$ matrix. The following are equivalent:

1. The linear system $Ax = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \mathbb{R}^n$.
 2. The matrix A is row equivalent to the $n \times n$ identity, I .
 3. The matrix A is invertible.
 4. The column vectors of A form a basis for \mathbb{R}^n .
5. $\text{rank}(A) = \dim(A)$ 6. $\det(A) \neq 0$
- $\text{rank}(A) + \text{nullity}(A) = n = \dim(A)$

Assignments

① U is a nonzero column vector $\in \mathbb{R}^m$ and V is a column vector \mathbb{R}^n

$$A = UV^T$$

a) show $\text{col}(A) = \text{span}(U)$ and $\text{row}(A) = \text{span}(V)$

b) nullity of A

c) show $\text{null}(A)$ contains all the vector orthogonal to V

$$\begin{aligned} a) U &= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad A = UV^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} [v_1 \ v_2 \ \dots \ v_n] = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \dots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \dots & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_m v_1 & u_m v_2 & \dots & u_m v_n \end{bmatrix} \\ &= \begin{bmatrix} u_1 v \\ u_2 v \\ \vdots \\ u_m v \end{bmatrix} = \begin{bmatrix} U \cdot V \\ U_2 \cdot V \\ \vdots \\ U_m \cdot V \end{bmatrix} \end{aligned}$$

b) nullity of $A = \dim(A) - \text{rank}(A)$

$$= n - 1$$

c) $\text{null}(A) = \{x \in \mathbb{R}^m \mid Ax = 0\} = \{x \in \mathbb{R}^m \mid U \cdot \vec{v} \cdot x = 0\}$ since \vec{v} is non-zero

$$\therefore \{x \in \mathbb{R}^m \mid \vec{v} \cdot x = 0\}$$

② prove $\{U_1, U_2, \dots, U_n\}$ of non-zero pairwise-orthogonal vector in \mathbb{R}^n is a basis for \mathbb{R}^n .

$$\text{Let } A = \begin{bmatrix} U_1 & U_2 & \dots & U_n \end{bmatrix} \text{ and } A^T = \begin{bmatrix} U_1^T \\ U_2^T \\ \vdots \\ U_n^T \end{bmatrix}$$

$$\therefore A^T A = \begin{bmatrix} U_1 U_1^T & U_1 U_2^T \dots U_1 U_n^T \\ U_2 U_1^T & U_2 U_2^T \dots U_2 U_n^T \\ \vdots & \vdots \\ U_n U_1^T & U_n U_2^T \dots U_n U_n^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = I$$

$\therefore A$ has pivot in every column

since $U_i \cdot U_j = 0$ for $i \neq j$

$\therefore \{U_1, U_2, \dots, U_n\}$ form a basis in \mathbb{R}^n .

4. linear Transformation

Lecture Notes

- if $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{v}) = rT(\vec{v})$
then a function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation $\forall \vec{v}, \vec{u} \in \mathbb{R}^n$
 \mathbb{R}^n is the domain of T \mathbb{R}^m is the codomain of T
- if $W \subset \mathbb{R}^n$ then $T[W] = \{T(\vec{w}) \mid \vec{w} \in W\}$
- The range of T is $T[\mathbb{R}^n] = \{T(\vec{v}) \mid \vec{v} \in \mathbb{R}^n\}$
- $T^{-1}[w] = \{\vec{v} \in \mathbb{R}^n \mid T(\vec{v}) \in W\}$
- Kernal of T : $T^{-1}\{0\} = \{v \in \mathbb{R}^n \mid T(v) = 0\}$
- $T(r_1 v_1 + r_2 v_2 + \dots + r_n v_n) = r_1 T(v_1) + \dots + r_n T(v_n)$
- $T(0) = 0' \quad 0' \in \mathbb{R}^m$
- if $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $B = \{b_1, b_2, \dots, b_m\}$ is a basis for \mathbb{R}^m , and $v \in \mathbb{R}^n$
then $T(v)$ is determined by $T(b_1), T(b_2), \dots, T(b_m)$
- if $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$, then $T(x) = Ax \quad \forall x \in \mathbb{R}^n$

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix}$$

$$\ker(T) = \text{null}(A) \quad \text{range}(T) = \text{col}(A)$$

A is the standard matrix representation of T

- $\text{rank}(T) = \dim(\text{col}(A)) = \text{rank}(A)$
- $\text{nullity}(T) = \dim(\ker(T)) = \text{nullity}(A)$
- T is invertible $\Leftrightarrow T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ or A is invertible
- $\text{rank}(T) + \text{nullity}(T) = \dim(\text{domain of } T) = \dim(\text{codomain of } T)$
- one to one: $\ker(T) = \text{null}(A)$
H has pivot in every column
- onto: $\text{rank}(A) = m$
H has a pivot in every row
- isomorphism: both $\Leftrightarrow A^{-1}$ exists
 $H \sim I$

Assignments

① Suppose $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $T[1, 1] = [3, 2]$, $T[2, 3] = [-7, 7]$. Find $T[x, y]$

since $[1, 1]$ and $[2, 3]$ are basis for \mathbb{R}^2 , then $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore 3x - 2y [1, 1] + y - x [2, 3] = [x, y]$$

$$\therefore T[x, y] = 3x - 2y T[1, 1] + y - x T[2, 3]$$

$$= 3x - 2y [3, 2] + y - x [1, 1]$$

$$= [2x + y, -x + 3y]$$

② $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ $T[x_1, x_2, x_3, x_4] = [x_2 - 3x_3 - x_4, 6x_1 + 5x_2, x_3 + 2x_4]$ find A

$$T[1, 0, 0, 0] = [0, 6, 2]$$

$$T[0, 1, 0, 0] = [1, 5, 0] \quad A \begin{bmatrix} 0 & 1 \\ 6 & 5 \\ 1 & 0 \end{bmatrix} \dots$$

③ $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $w \in \mathbb{R}^n$, prove $T[w]$ is a subspace for \mathbb{R}^m

i) since $w \in \mathbb{R}^n$ then w is non empty, let $\vec{v} \in w$ then $T(\vec{v}) \in T[w] \therefore T[w]$ is not empty

ii) if $x, y \in w$ then $\exists u, v \in T[w]$ s.t. $T[x] = u$ and $T[y] = v$

since $x + y \in w$ then $T(x+y) \in T[w] \therefore u + v = T(x) + T(y) = T(x+y) \in T[w]$

④ $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for \mathbb{R}^3 . $T(\vec{v}_1) = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $T(\vec{v}_2) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $T(\vec{v}_3) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

a) Determine if $w = \begin{bmatrix} -6 \\ 0 \\ 0 \end{bmatrix}$ is in range(T)

$$\text{range}(T) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\left[\begin{array}{ccc|c} -2 & 0 & 2 & -6 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} -2 & 0 & 2 & -6 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \text{ inconsistent} \Rightarrow \text{not in range of } T$$

b) basis for range(T)?

$$\left[\begin{array}{ccc|c} -2 & 0 & 2 & -6 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

c) nullity(T)?

$$\text{nullity}(T) = \text{nullity}(A)$$

4. $a = [a_1, a_2] \in \mathbb{R}^2$ and $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x) = \begin{pmatrix} x \cdot a \\ a \cdot a \end{pmatrix}$

a) Find $T[x, y]$

$$T([x, y]) = \left(\frac{[x, y] \cdot [a_1, a_2]}{[a_1, a_2] \cdot [a_1, a_2]} \right) [a_1, a_2] = \frac{a_1 x + a_2 y}{a_1^2 + 2a_1 a_2 + a_2^2} [a_1, a_2] = \begin{pmatrix} a_1(a_1 x + a_2 y) \\ a_1^2 + 2a_1 a_2 + a_2^2 \end{pmatrix}$$

b) Prove T is a linear transformation

(i) Let $\vec{u} = [u_1, u_2]$ and $\vec{v} = [v_1, v_2]$ WTS $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

$$\begin{aligned} T(\vec{u} + \vec{v}) &= T([u_1 + v_1, u_2 + v_2]) = \left(\frac{[u_1 + v_1, u_2 + v_2] \cdot [a_1, a_2]}{a_1^2 + 2a_1 a_2 + a_2^2} \right) [a_1, a_2] \\ &= \left(\frac{(u_1 + v_1)a_1 + (u_2 + v_2)a_2}{a_1^2 + 2a_1 a_2 + a_2^2} \right) [a_1, a_2] \end{aligned}$$

$$\begin{aligned} \text{where } T(\vec{u}) + T(\vec{v}) &= \left(\frac{[u_1, u_2] \cdot [a_1, a_2]}{a_1^2 + 2a_1 a_2 + a_2^2} \right) [a_1, a_2] + \left(\frac{[v_1, v_2] \cdot [a_1, a_2]}{a_1^2 + 2a_1 a_2 + a_2^2} \right) [a_1, a_2] \\ &= \left(\frac{u_1 a_1 + u_2 a_2}{a_1^2 + 2a_1 a_2 + a_2^2} \right) [a_1, a_2] + \left(\frac{v_1 a_1 + v_2 a_2}{a_1^2 + 2a_1 a_2 + a_2^2} \right) [a_1, a_2] \\ &= \end{aligned}$$

c) find A

$$T(e_1) = T[1, 0] = \left(\frac{1}{a_1^2 + a_2^2} \right) [a_1^2, a_1 a_2]$$

$$T(e_2) = T[0, 1] = \left(\frac{1}{a_1^2 + a_2^2} \right) [a_1 a_2, a_2^2]$$

$$\therefore A = \left(\frac{1}{a_1^2 + a_2^2} \right) \begin{bmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix}$$

5. Determinants

Lecture Notes

- the determinant of $[3]$ is 3
- the determinant of $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is $a_1b_2 - a_2b_1$
- the determinant of $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ is $a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$
 $= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$
- $a = [a_1, a_2, a_3] \quad b = [b_1, b_2, b_3]$
then $a \times b = \begin{bmatrix} |a_2 a_3| \\ |b_2 b_3| \\ a_1 a_2 a_3 \end{bmatrix}, \begin{bmatrix} |a_1 a_3| \\ |b_1 b_3| \\ a_1 a_2 a_3 \end{bmatrix}, \begin{bmatrix} |a_1 a_2| \\ |b_1 b_2| \\ a_1 a_2 a_3 \end{bmatrix}$
 $b_1 \quad b_2 \quad b_3$
- $a \times b$ is perpendicular to both a and b .
- parallelogram $\vec{a} = [a_1, a_2], \vec{b} = [b_1, b_2]$ in \mathbb{R}^2 Area = $\left| \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \right|$
- parallelogram $\vec{a} = [a_1, a_2, a_3], \vec{b} = [b_1, b_2, b_3]$ in \mathbb{R}^3 Area = $\left| \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \right|$
- parallelepiped $\vec{a} = [a_1, a_2, a_3], \vec{b} = [b_1, b_2, b_3], \vec{c} = [c_1, c_2, c_3]$
Volume = $\left| \begin{bmatrix} -\vec{a}^2 \\ -\vec{b}^2 \\ -\vec{c}^2 \end{bmatrix} \right|$
- $\det(A) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{vmatrix} = a_{11} \underset{\text{minor}}{\underbrace{a'_{11}}} + a_{12} \underset{\text{cofactor}}{\underbrace{a'_{12}}} + \dots + a_{1n} \underset{\text{minor}}{\underbrace{a'_{1n}}}$
- $\det(A) = \det(A^T) \quad \det(A) = r^n \det(A)$
- Elem. Row. Operation in terms of determinant
 - $A \xrightarrow{R_i \leftrightarrow R_j} B \quad \det(B) = -\det(A)$
 - A has two equal rows $\Rightarrow \det(A) = 0$
 - $A \xrightarrow{R_i \leftrightarrow r R_i} B \quad \det(B) = r \det(A)$
 - $A \xrightarrow{R_i \rightarrow R_i + r R_j} B \quad \det(B) = \det(A)$
- $\det(EA) = \det(E)\det(A)$
- if A contains proportional rows or columns, then $\det(A) = 0$
- A is invertible $\Leftrightarrow \det(A) \neq 0$
- $\det(AB) = \det(A)\det(B)$
- if A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$
- if A is non then $(\text{adj}(A))A = A(\text{adj}(A)) = \det(A)I$

where $\text{adj}(A) = (A')^T$ $A' = [a_{ij}']$

• if A is $n \times n$ and $\det(A) \neq 0$ then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

• If $Ax = b$ is $n \times n$ and $\det(A) \neq 0$, then the unique solution $x = [x_1, x_2, \dots, x_n]$

is in the form: $x_k = \frac{\det(B_k)}{\det(A)}$ $k=1, \dots, n$ where B_k is the matrix A with the k th column replaced by b

Assignments

① Compute $\begin{vmatrix} 1 & 2 & 0 & 1 \\ 3 & 3 & 3 & 9 \\ 1 & 4 & 1 & 4 \\ 1 & 1 & 2 & 0 \end{vmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 3 & 3 & 9 \\ 1 & 4 & 1 & 4 \\ 1 & 1 & 2 & 0 \end{bmatrix} \begin{array}{l} R_2 - 3R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -3 & 3 & 6 \\ 0 & 3 & 1 & 3 \\ 0 & -2 & 1 & 1 \end{bmatrix} \begin{array}{l} R_2 \leftrightarrow R_3 \\ R_3 + 3R_2 \\ R_4 + R_2 \end{array} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 3 & 9 \end{bmatrix} \begin{array}{l} 3R_3 \\ SR_3 \end{array} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 15 & 3 \\ 0 & 0 & -15 & 45 \end{bmatrix} \begin{array}{l} R_4 + R_3 \\ R_4 + R_3 \end{array} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 15 & 3 \\ 0 & 0 & 0 & 45 \end{bmatrix}$$

$$|A| = |A_1|$$

$$\begin{array}{c} A_2 \\ A_3 \\ |A_2| = -|A| \\ |A_3| = -|A| \end{array}$$

$$\begin{array}{c} A_4 \\ A_5 \\ |A_4| = -15|A| \\ |A_5| = -15|A| \end{array}$$

$$|A_5| = (-15)(45) \\ \therefore |A| = 45$$

② $A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -6 & 1 \end{bmatrix}$ find $\text{adj}(A)$ and A^{-1} $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$

$$a_{11}' = 1 \begin{vmatrix} 1 & 5 \\ -2 & 1 \end{vmatrix} = 31 \quad a_{12}' = - \begin{vmatrix} 0 & 5 \\ -2 & 1 \end{vmatrix} = -10 \quad a_{13}' = 2$$

$$a_{21}' = 9 \quad a_{22}' = 3 \quad a_{23}' = 0 \\ a_{31}' = 17 \quad a_{32}' = -5 \quad a_{33}' = 1$$

$$\therefore A' = \begin{bmatrix} 31 & -10 & 2 \\ 0 & 3 & 0 \\ 17 & -5 & 1 \end{bmatrix} \quad \text{adj}(A) = (A')^T = \begin{bmatrix} 31 & -9 & 7 \\ 0 & 3 & 0 \\ 17 & -5 & 1 \end{bmatrix}$$

$$\text{adj}(A) \cdot A = \begin{bmatrix} 31 & 0 & 0 \\ 0 & 3 & 0 \\ 17 & -5 & 1 \end{bmatrix} = \det(A) \cdot I$$

$$\therefore \det(A) = 3 \therefore A^{-1} = \frac{1}{3} \begin{bmatrix} 31 & -9 & 7 \\ 0 & 3 & 0 \\ 17 & -5 & 1 \end{bmatrix}$$

• use Cramer's rule to solve

$$\begin{array}{l} x_1 + x_2 + x_3 = 0 \\ 2x_1 - x_2 = 11 \\ x_2 + 4x_3 = 3 \end{array}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 11 \\ 3 \end{bmatrix}$$

$$|A| = - \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 4 \end{vmatrix} + 4 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{vmatrix} = 2 - 12 = -10 \neq 0$$

$$B_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & 4 \end{bmatrix} \quad |B_1| = 30 \Rightarrow x_1 = \frac{-10}{-10} = 3$$

$$B_2 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 3 & 4 \end{bmatrix} \quad |B_2| = 50 \Rightarrow x_2 = \frac{50}{-10} = 5 \quad \therefore x = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 3 & 1 & 3 \end{bmatrix} \quad |B_3| = -20 \Rightarrow x_3 = \frac{-20}{-10} = 2$$

③ if $\vec{u}, \vec{v} \in \mathbb{R}^3$

- a) $u \cdot v = \|u\| \|v\| \Rightarrow u = kv, k > 0$
- b) $u \times v = 0 \Rightarrow u = kv$
- c) $\|(u \times v)\| = \|u\| \|v\| \Rightarrow u \perp v$
- d) $(u \times v) \cdot u = 0 \Rightarrow \text{nothing}$
- e) $(u \times v) \times u = 0 \Rightarrow u = kv$

④ show that $A_{mn} \Rightarrow \det(rA) = r^n \det(A)$

Let B be rA $|B| = r|A_{m+1}| = r^2 |A_{m+2}| = r^n |A|$

⑤ if A_{m+1} and B_{m+1} and $\det(A) = 3$ $\det(B) = 5$ what is the determinant
of $(3A)^{-1} 3CB^{-1} (2A)B^{-1} (\frac{1}{5}B)A (3A^{-1})$

$$\det(3A^{-1}) = \frac{1}{\det(3A)} = \frac{1}{3^4 \det(A)} = \frac{1}{3^5}$$

$$\det(3CB^{-1}) = 3^4 \det(B^{-1}) = 3^4 \frac{1}{\det(CB)} = \frac{3^4}{5}$$

$$\det((2A)B^{-1}) = \det(2A) \det(B^{-1}) = 2^4 \det(A) \frac{1}{\det(CB)} = \frac{3 \cdot 2^4}{5}$$

$$\det(\frac{1}{5}B)A = \det(\frac{1}{5}B) \det(A) = (\frac{1}{5})^4 5 \times 3$$

$$\det(3A^{-1}) = 3^4 \frac{1}{\det(A)} = 3^3$$

⑥ Let U_{mn} s.t $U^T U = I$ show $|\det(U)| = 1$

since $\det(U) = \det(U^T)$ then $1 = \det(I) = \det(U^T U) = \det(U^T) \det(U) = (\det(U))^2$

$\therefore \det(U) = \pm 1 \therefore |\det(U)| = 1$

⑦ Let $A\vec{x} = \vec{b}$ be a system where all entries in A and \vec{b} are integers.

If $|\det(A)| = 1$ pure x are all integers

since $|\det(A)| = 1 \Rightarrow \det(A) = \pm 1 \neq 0 \therefore$ solution is unique

$$x = \frac{\det(B_e)}{\det(A)} = \pm \det(B_e) \Rightarrow \text{integers only}$$

⑧ matrix sum of each row and column is 0. pure determinant of this is 0

If $x = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix}$, then there is a nonzero solution to $A\vec{x} = \vec{0} \therefore$ matrix is not invertible $\therefore \det(A) = 0$.

6. Eigenvalues and Eigenvectors

Lecture Notes

- Let $A_{n \times n}$, if $A\vec{v} = \lambda\vec{v}$ then λ is eigenvalue of A and \vec{v} is the eigenvectors of A

- λ is a solution to $|A - \lambda I| = 0$
- characteristic polynomial of A : $p(\lambda) = |A - \lambda I|$
- eigenspace of λ : $E_\lambda = \{\vec{v} \in \mathbb{R}^n \mid A\vec{v} = \lambda\vec{v}\}$
- $E_\lambda = \text{null}(A - \lambda I)$
- $T(\vec{v}) = \lambda\vec{v}$
- If $A^k\vec{v} = \lambda^k\vec{v}$, then λ^k is eigenvalue and \vec{v} is eigenvector of A .
- $A^k\vec{v} = \frac{1}{\lambda}\vec{v}$
- If A is invertible $\Leftrightarrow \lambda = 0$ is not an eigenvalue of A
- $P^{-1}AP$ is a diagonal matrix $\Rightarrow A$ is diagonalizable
- If $\vec{v}_1, \dots, \vec{v}_k$ are eigenvalue corresp. to distinct eigenvalue of $A \Rightarrow \vec{v}_1, \dots, \vec{v}_k$ are linearly independent
- $A_{n \times n}$ A is diagonalizable $\Leftrightarrow A$ has n linearly independent eigenvectors.
- If $A_{n \times n}$ has n distinct eigenvalues $\Rightarrow A$ is diagonalizable
- If $A_{n \times n}, B_{n \times n}$ s.t. $B = P^{-1}AP \Rightarrow A$ and B are similar
 - $\det(A) = \det(B)$
 - A is invertible $\Leftrightarrow B$ is invertible
 - $\text{rank}(A) = \text{rank}(B)$
 - $\text{nullity}(A) = \text{nullity}(B)$
 - $\det(A - \lambda I) = \det(B - \lambda I)$
- $(\lambda - \lambda_i)^{a_i}$ is a factor of $p(\lambda)$, but $(\lambda - \lambda_i)^{a_i+1}$ is not $\Rightarrow a_i$ is the algebraic multiplicity of λ_i
- $\dim(E_{\lambda_i})$ is the geometric multiplicity of λ_i
- $GM(\lambda_i) \leq AM(\lambda_i)$
- A is diagonalizable $\Leftrightarrow p(\lambda)$ is a product of linear factors and $AM(\lambda_i) = GM(\lambda_i)$
- $A = P^{-1}DP \Rightarrow A^k = P^{-1}D^kP$

• Cayley-Hamilton Theorem

$$\text{if } P(\lambda) = (2-\lambda)(\lambda^2 - \lambda - 2) = -\lambda^3 + 2\lambda^2 + \lambda - 2 = 0 \text{ (0 matrix)}$$

$$\text{then } P(A) = -A^3 + 2A^2 + A - 2I = 0$$

$$-A^3 + 2A^2 + A = 2I$$

$$A(-A^2 + 2A + I) = 2I$$

$$\frac{1}{2}A(-A^2 + 2A + I) = I$$

$$A^{-1} = \frac{1}{2}(-A^2 + 2A + I)$$

• Every symmetric matrix is diagonalizable

Assignments

① Determine if $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ is diagonalizable. If it is, find P and D s.t. $D = P^{-1}AP$

$$\begin{aligned} P(\lambda) = |A - \lambda I| &= \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 0 & -2 & -\lambda - 2 \\ 0 & 3-\lambda & 1 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} \lambda & 0 \\ 0 & 0 \end{vmatrix} \\ &= (2-\lambda)(-\lambda(3-\lambda) + 2) = (2-\lambda)(\lambda^2 - 3\lambda + 2) = (2-\lambda)(\lambda-2)(\lambda-1) = -(2-\lambda)^2(\lambda-1) \\ &\therefore \lambda_1 = 2, \lambda_2 = 1 \end{aligned}$$

For $\lambda = 2$

$$A - 2I = \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{\lambda_1} = \text{sp}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

For $\lambda = 1$

$$A - I = \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{\lambda_2} = \text{sp}\left(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right)$$

$$\therefore \alpha(\lambda_1) = g(\lambda_1) \text{ and } \alpha(\lambda_2) = g(\lambda_2)$$

$\therefore A$ is diagonalizable

$$P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \quad P^{-1}AP = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

② If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find A^{100}

$$A^{100} = P^{-1}D^{100}P$$

$$|A - \lambda I| = (2-\lambda)(\lambda-1)(\lambda+1) \Rightarrow \lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$$

For each λ we have that

$$E_{\lambda_1} = \text{sp}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), E_{\lambda_2} = \text{sp}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), E_{\lambda_3} = \text{sp}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \Rightarrow P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{use } P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{100} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{100} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2^{100} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

③ consider $x_0=1, x_1=1, x_2=7, x_3=13, x_4=55$

$$\begin{aligned} x_{k+2} &= 6x_k + x_{k+1} \\ x_{k+1} &= x_{k+1} \\ 6x_k + x_{k+1} &= x_{k+2} \end{aligned}$$

$$\therefore A\vec{y}_k = \vec{y}_{k+1} \text{ where } \vec{y}_0 = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } A\vec{y}_0 = \vec{y}_1 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\therefore A^k \vec{y}_0 = \vec{y}_k \quad (\text{since } A \cdot A \vec{y}_1 = \vec{y}_2)$$

find A^{27} and we get \vec{y}_k

④ $x_0=1, x_1=1, x_2=7, x_3=25, x_4=103$

$$x_{k+2} = 3x_{k+1} + 4x_k$$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}}_y = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix}$$

$$\therefore A^{27} \vec{y}_0 = \vec{y}^{27}$$

$$\vec{y}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} = -\lambda(3-\lambda) - 4 = -3\lambda + \lambda^2 - 4 = \lambda^2 - 3\lambda - 4$$

$$\lambda_1 = 1, \lambda_2 = 4$$

$$\text{For } \lambda_1 = -1 \quad A + I = \begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_1} = \text{sp}(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix})$$

$$\text{For } \lambda_2 = 4 \quad A - 4I = \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix} \sim \begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_2} = \text{sp}(\begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix})$$

$$\therefore P = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}, P^{-1} = \frac{1}{5} \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\begin{aligned} A^{27} &= \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 4^{27} \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix} \frac{1}{5} \\ &= \frac{1}{5} \begin{bmatrix} 1+4^{27} & -4+4^{27} \\ -1+4^{27} & 4+4^{27} \end{bmatrix} \end{aligned}$$

1. Complex Numbers

Lecture Notes

$$\cdot |z| = r = \sqrt{a^2 + b^2}$$

$$\cdot \operatorname{Arg}(z) = \theta$$

$$\cdot z = a + ib = r \cos \theta + i r \sin \theta \text{ — polar form of } z$$

$$\cdot z_1 z_2 = |z_1| |z_2| (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

$$\cdot \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

$$\cdot \operatorname{Arg}(z^n) = n\theta$$

$$\cdot \overline{a+ib} = a - ib$$

$$\cdot |z| = \sqrt{z\bar{z}}$$

$$1. \bar{\bar{z}} = z \text{ and } |z|^2 = z\bar{z}.$$

$$2. \overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$$

$$3. \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$4. z^{-1} = \frac{\bar{z}}{|z|^2}$$

$$5. \overline{z_1/z_2} = \bar{z}_1/\bar{z}_2$$

$$6. Re(z) = \frac{z + \bar{z}}{2} \text{ and } Re(z) \leq |z|$$

$$7. |z_1 + z_2| \leq |z_1| + |z_2|.$$

$$\cdot z^n = r^n (\cos(n\theta) + i \sin(n\theta))$$

$$\cdot z^{\frac{1}{n}} = r^{\frac{1}{n}} \left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i \sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) \right)$$

Assignments

(1) Find all solution to $z^8=1$

$$\begin{aligned} z^8=1 &= \cos(0)+i\sin(0) \\ &= \cos(0+2k\pi)+i\sin(0+2k\pi) \\ z^{\frac{1}{8}} &= \left| \left(\cos\left(\frac{2k\pi}{8}\right) + i\sin\left(\frac{2k\pi}{8}\right) \right) \right| \end{aligned}$$

$$\begin{aligned} k=0 \quad z_0 &= \cos(0)+i\sin(0)=1 \\ z_1 &= \cos\left(\frac{2\pi}{8}\right)+i\sin\left(\frac{2\pi}{8}\right)=\dots \\ &\dots \end{aligned}$$

(2) find all complex number x such that $x^4=1$

$$\begin{aligned} x^4=1 &= \cos(0+2k\pi)+i\sin(0+2k\pi) \\ z^{\frac{1}{4}} &= \left| \left(\cos\left(\frac{2k\pi}{4}\right) + i\sin\left(\frac{2k\pi}{4}\right) \right) \right| \end{aligned}$$

$$\begin{aligned} z_0 &= 1 \\ z_1 &= \cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right) \\ z_2 &= \cos(\pi)+i\sin(\pi) \\ z_3 &= \cos\left(\frac{3\pi}{2}\right)+i\sin\left(\frac{3\pi}{2}\right) \end{aligned}$$

Review seminar

① Find all complex numbers s.t. $x^4 = -1$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right) \right)$$

and based on given $n=4, r=1, \theta=\pi$

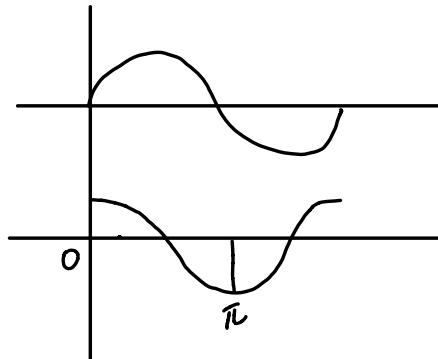
$$\therefore z^{\frac{1}{4}} = \left| \cos\left(\frac{\pi + 2k\pi}{4}\right) + i \sin\left(\frac{\pi + 2k\pi}{4}\right) \right|$$

$$k=0 \quad z^{\frac{1}{4}} = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$k=1 \quad z^{\frac{1}{4}} = \cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$k=2 \quad z^{\frac{1}{4}} = \cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

$$k=3 \quad z^{\frac{1}{4}} = \cos\left(\frac{7\pi}{4}\right) + i \sin\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$



②

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ prove } a+d = \lambda_1 + \lambda_2$$

$$\begin{aligned} p(\lambda) = |A - \lambda I| &= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc \\ &= ad - a\lambda - d\lambda + \lambda^2 - bc \\ &= \lambda^2 - \lambda(a+d) - (ad+bc) \end{aligned}$$

since λ_1 and λ_2 are root of $p(\lambda)$

$$\begin{aligned} \therefore p(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2) \\ &= \lambda^2 - \lambda \underbrace{(\lambda_1 + \lambda_2)}_{\lambda_1 + \lambda_2} + \lambda_1 \lambda_2 \end{aligned}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \lambda_1, \lambda_2 \quad \text{prove } \det(A) = \lambda_1 \lambda_2$$

$$p(\lambda) = \lambda^2 - \lambda(a+d) + (ad+bc)$$

$$\text{and } p(\lambda) = \lambda^2 - \lambda_1 \underbrace{(a+d)}_{\det(A)} + \underbrace{(ad+bc)}_{\det(A)}$$

$$= \lambda_1^2 - \lambda_1 \lambda_2 - \lambda_1 \lambda_2 + \det(A)$$