

1. Lines, Planes and Curves

The vector equation of the line $\vec{l} = \vec{a} + t\vec{v}$
 $= (a_1, a_2, a_3) + t[v_1, v_2, v_3] , t \in \mathbb{R}$

Parametric equation of line in \mathbb{R}^3 : $x = a_1 + t v_1$
 $y = a_2 + t v_2$
 $z = a_3 + t v_3 , t \in \mathbb{R}$

Ex ① Find equation of the plane passes $(1, 1, -1)$ and \perp to the line: $\begin{cases} x=1+t \\ y=1-t \\ z=-t \end{cases}$

$\vec{n} = [1, -3, -1]$, let (x, y, z) be on the plane

$$\text{then } [x-1, y-1, z+1] \cdot [1, -3, -1] = 0$$

$$x-1-3y+3-z-1=0$$

$$\therefore \text{equation plane } x-3y-z=5$$

$$\begin{array}{l} \text{② } \vec{n}_1 = [-2, 6, 14] \Rightarrow \vec{n}_1 \parallel \vec{n}_2 \quad | \quad \vec{n}_1 = [2, 3, 1] \\ \vec{n}_2 = [1, -3, -1] \quad | \quad \vec{n}_2 = [1, -1, 1] \Rightarrow \vec{n}_1 \perp \vec{n}_2 \quad | \quad \vec{n}_1 = [2, 3, 1] \\ \qquad \qquad \qquad \vec{n}_2 = [1, -2, 3] \Rightarrow \theta = \arccos \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \|\vec{n}_2\|} \end{array}$$

③ Find equation of plane passes $A(1, 1, 2)$, $B(2, 0, -3)$ and $C(2, -1, 2)$

$$\begin{aligned} \vec{v} &= B-A = (3, -1, -5) \quad \vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 3 & -1 & -5 \\ 3 & -2 & 0 \end{vmatrix} = i[(-1)(-5) - (-2)(-5)] + j[3(-5) - 3(-2)] \\ \vec{w} &= C-A = (3, -2, 0) \quad = [10i - 15j - 3k] \quad \text{and } 10(-1) - 15(1) - 3(2) \\ &= [10, -15, -3] \quad = 10 - 15 - 6 \\ &\therefore 10x - 15y - 3z = d \quad = -11 = d \quad \therefore 10x - 15y - 3z = -11 \end{aligned}$$

tangent vector of c at a : $x-a = c'(t_0)(t-t_0)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} (t-t_0) \Rightarrow \begin{cases} x = a_0 + t \\ y = b_0 + t \\ z = c_0 + t \end{cases}$$

Polar (r, θ)

Polar to Cartesian: $x = r \cos \theta, y = r \sin \theta$

Cartesian to Polar: $\theta = \arctan \frac{y}{x}, r = \sqrt{x^2 + y^2}$

Cylindrical (r, θ, z)

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \quad \theta = \arctan \frac{y}{x}, r = \sqrt{x^2 + y^2}$$

Spherical (ρ, θ, ϕ)

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases} \quad \rho = \sqrt{x^2 + y^2 + z^2}$$

2. Vector and function

$$y = x^2$$

$$x = y^2$$

$$x^2 - y^2 = 1$$

$$y^2 - x^2 = 1$$

3. Limits

$\delta - \varepsilon$ proof:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < \|(x,y) - (0,0)\| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

$$\text{Prove } \lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y^2}{x^2+y^2} = 0$$

$$\text{WTS: } \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < \|(x,y) - (0,0)\| < \delta \Rightarrow \left| \frac{2x^2y^2}{x^2+y^2} \right| < \varepsilon$$

$$\begin{aligned} &\Rightarrow 0 < \|(x,y) - (0,0)\| < \delta & \left| \frac{2x^2y^2}{x^2+y^2} \right| < \varepsilon \\ &\Rightarrow x^2 + y^2 < \delta^2 & \Rightarrow \frac{2x^2y^2}{x^2+y^2} < \varepsilon & 2 \cdot \frac{x^2}{x^2+y^2} \cdot y^2 \leq 2 \cdot 1 \cdot (x^2+y^2) \\ && \downarrow & \downarrow & \leq 2 \cdot \delta^2 = \varepsilon \\ && \leq 1 & \leq x^2+y^2 & \end{aligned}$$

$$\text{Proof: } \forall \varepsilon > 0, \exists \delta > 0, \text{choose } \delta = \sqrt{\frac{\varepsilon}{2}} \text{ s.t. } 0 < \|(x,y) - (0,0)\| < \delta \Rightarrow \left| \frac{2x^2y^2}{x^2+y^2} \right| = \frac{2x^2y^2}{x^2+y^2} \leq 2(x^2+y^2) < 2\delta^2 = \varepsilon$$

■

$\lim_{x \rightarrow x_0} f(x) = f(x_0)$ means

a) $x_0 \in U$ (in domain)

b) $\lim_{x \rightarrow x_0} f(x) = L$

c) $f(x_0) = L$

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L = f(a,b)$$

$$\lim_{(x,y) \rightarrow (a,b)} xy = ab$$

WLS: $\forall \varepsilon > 0 \exists \delta > 0$ sit $0 < \|(x-a, y-b)\| < \delta \Rightarrow |xy - ab| < \varepsilon$

$$\|(x-a, y-b)\| < \delta$$

$$|y-b| < \sqrt{(x-a)^2 + (y-b)^2} < \delta \leq 1$$

$$|y| - |b| < 1$$

$$|y| < |b| + 1$$

$$\lim_{(x,y) \rightarrow (1,0^+)} \frac{y \ln y}{x}$$

$$\lim_{y \rightarrow 0^+} y \ln y = \lim_{y \rightarrow 0^+} \frac{\ln y}{\frac{1}{y}} = \lim_{y \rightarrow 0^+} \frac{\frac{1}{y}}{-\frac{1}{y^2}} = -\infty$$

$$\therefore = 0$$

$$\lim |xy - ab| = |xy - ab + ay - ay|$$

$$= |xy - ay + ay - ab|$$

$$\leq |y(x-a)| + |acy - ab|$$

$$\leq |y||x-a| + |a||y-b|$$

$$\leq (|y| + |b|) \sqrt{(x-a)^2 + (y-b)^2} + |a| \sqrt{(x-a)^2 + (y-b)^2}$$

$$= (|y| + |b| + |a|) \sqrt{(x-a)^2 + (y-b)^2}$$

$$< (|y| + |b| + |a|) \delta$$

$$\therefore (|y| + |b| + |a|) \delta = \varepsilon$$

$$\delta = \frac{\varepsilon}{(|y| + |b| + |a|)}$$

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{x^2 + y^2} \leq 1$$

$$|x| \leq \sqrt{x^2 + y^2}$$

$$x^2 + y^2 \geq 2xy$$

$$x^2 + y^2 \geq 2|xy|$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 y^3}{x^4 + y^2} = 0$$

$$x^4 y^2 \pm 2x^4 y = (x^4 \pm y)^2 \geq 0$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2}$$

4. Differentiation

directional derivatives of f at a in direction v is: $D_v(f(a)) = \lim_{t \rightarrow 0} \frac{f(a+tv) - f(a)}{t\|v\|} = Df(a) \frac{v}{\|v\|}$

$$f: \mathbb{R}^n \rightarrow \mathbb{R}^m \quad Df(a) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial z_1} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial y_n} & \frac{\partial f_n}{\partial z_n} \end{bmatrix} \quad f(x, y, z) = \begin{pmatrix} \dots, \dots, \dots \\ f_1, f_2, f_3 \end{pmatrix}$$

$$f: \mathbb{R}^n \rightarrow \mathbb{R} \quad Df(a) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] = Df(a)$$

tangent line equation

$$f: \mathbb{R}^3 \rightarrow \mathbb{R} \quad \nabla f(x_0) \cdot (x - x_0) = 0 \quad L(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0)$$

$$f: \mathbb{R}^2 \rightarrow \mathbb{R} \quad f(x_0) + \nabla f(x_0)(x - x_0) = z \quad L(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0)$$

$$z = \sin(xy) + 2 \quad (5, 0, 2)$$

$$f(x, y) = \sin(xy) + 2 \quad (2xy \cos(xy), x^2 \cos(xy))$$

$$\frac{z}{2} = f(x_0) + \nabla f(x_0)(x - x_0) = 2 + (0, 25)(x - 5, y)$$

Ex. Find the point on the surface $x^2 + 2y^2 + 3z^2 = 1$ where tangent line is // to $3x - y + 3z = 1$

$$\nabla f = (2x, 4y, 6z)$$

$$\nabla f(x_0) = (2x_0, 4y_0, 6z_0)$$

$$\frac{9}{4}k^2 + \frac{1}{8}k^2 + \frac{3}{4}k^2 = 1$$

$$(2x_0, 4y_0, 6z_0) = k(3, -1, 3)$$

$$\frac{18}{8}k^2 + \frac{1}{8}k^2 + \frac{6}{8}k^2 = 1 \quad \therefore \left(\frac{3\sqrt{2}}{5}, -\frac{\sqrt{2}}{10}, \frac{\sqrt{2}}{3} \right)$$

$$\begin{cases} 2x_0 = 3k \\ 4y_0 = -k \\ 6z_0 = 3k \end{cases} \Rightarrow \begin{cases} x_0 = \frac{3k}{2} \\ y_0 = -\frac{k}{4} \\ z_0 = \frac{1}{2}k \end{cases}$$

$$\begin{aligned} \frac{25}{8}k^2 &= 1 \\ k^2 &= \frac{8}{25} \\ k &= \pm \sqrt{\frac{8}{25}} \end{aligned} \quad \left(-\frac{3\sqrt{2}}{5}, \frac{\sqrt{2}}{10}, -\frac{\sqrt{2}}{3} \right)$$

Let $f: U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, if all the partial derivative exist at a and

if $\lim_{x \rightarrow a} \frac{\|f(x) - f(a) - Df(a)(x-a)\|}{\|x-a\|} = 0$, then f is diff at a

Ex. $f(x, y) = x^{\frac{1}{3}}y^{\frac{1}{3}}$ at $(0, 0)$

$$\frac{\partial f}{\partial x} = \frac{1}{3}y^{\frac{1}{3}}x^{-\frac{2}{3}} = \frac{1}{3}\sqrt[3]{y} \frac{1}{3\sqrt[3]{x^2}}$$

5. Higher Order Derivative

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = \sum_{n=0}^{\infty} \frac{1}{n!} [f^{(n)}(a)(x-a)^n]$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n = 1 + \frac{a}{1!} x + \frac{a(a-1)}{2!} x^2 + \frac{a(a-1)(a-2)}{3!} x^3$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$|x| < 1$

$$F(xy) = f(a, b)$$

$$+ \frac{1}{1!} \left[(x+a) \frac{\partial f}{\partial x}(a, b) + (y+b) \frac{\partial f}{\partial y}(a, b) \right]$$

$$+ \frac{1}{2!} \left[(x+a)^2 \frac{\partial^2 f}{\partial x^2}(a, b) + 2(x+a)(y+b) \frac{\partial^2 f}{\partial x \partial y}(a, b) + (y+b)^2 \frac{\partial^2 f}{\partial y^2}(a, b) \right]$$

at (a, b)

$$F(x, y) = f(a, b) + \frac{1}{1!} \left[(x+a) \frac{\partial f}{\partial x}(a, b) + (y+b) \frac{\partial f}{\partial y}(a, b) \right]$$

$$+ \frac{1}{2!} \left[(x+a)^2 \frac{\partial^2 f}{\partial x^2}(a, b) + 2(x+a)(y+b) \frac{\partial^2 f}{\partial x \partial y}(a, b) + (y+b)^2 \frac{\partial^2 f}{\partial y^2}(a, b) \right]$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Midterm Test

① find tangent line of $x = e^{-t} \cos t$, $y = e^{-t} \sin t$, $z = e^{-t}$ at $(1, 0, 1)$

$$\begin{aligned} \text{then } & e^{-t} \cos t = 1 \\ & e^{-t} \sin t = 0 \Rightarrow t=0 \\ & e^{-t} = 1 \end{aligned} \quad \begin{aligned} \text{then } & f(x, y, z) = (-e^{-t} \cos t - e^{-t} \sin t, -e^{-t} \sin t + e^{-t} \cos t, -e^{-t})_{t=0} \\ & = (-1 - 0, 0 + 1, -1) = (-1, 1, -1) \end{aligned}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \text{ and } \Rightarrow \begin{cases} x = 1 - t \\ y = t \\ z = 1 - t \end{cases}$$

② $\lim_{(x,y) \rightarrow (0,0)} \frac{y \sin x}{x^2+y^2}$

$$\text{restrict } y=0 \Rightarrow \frac{0 \sin x}{x^2+0} = \frac{0}{x^2} = 0$$

$$\text{restrict } y=x \Rightarrow \frac{x \sin x}{2x^2} = \frac{\frac{1}{2} \sin x}{x} = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

$\therefore \text{DNE}$

$$③ \lim_{(x,y) \rightarrow (0,0)} \frac{x^3-y^3}{x^2+y^2+xy} \quad \begin{cases} x^3+y^3 = (x+y)(x^2-xy+y^2) \\ x^3-y^3 = (x-y)(x^2+xy+y^2) \end{cases}$$

$$\stackrel{\text{num}}{(x,y) \rightarrow (0,0)} \frac{(x-y)(x^2+xy+y^2)}{x^2+xy+y^2+xy} = x-y = 0-0 = 0$$

④ $\delta-\varepsilon$ proof of $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2} = 0$

$$\text{wts: } \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } |(x,y) - (0,0)| < \delta \Rightarrow \left| \frac{xy}{x^2+y^2} \right| < \varepsilon$$

$$\begin{aligned} & |(x,y) - (0,0)| < \delta \\ & \sqrt{x^2+y^2} < \delta \quad \left| \frac{xy}{x^2+y^2} \right| < |y| \left| \frac{x}{x^2+y^2} \right| < \frac{1}{\delta} \sqrt{x^2+y^2} < \frac{1}{\delta} \delta = 1 \end{aligned}$$

⑤ equation of ellipse

$$x^2+4y^2=1 \Rightarrow \begin{cases} \left(\frac{x}{1}\right)^2=1 \\ \left(\frac{y}{\frac{1}{2}}\right)^2=1 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=\pm\frac{1}{2} \end{cases} \quad x^2+4y^2=8 \Rightarrow \begin{cases} \left(\frac{x}{2\sqrt{2}}\right)^2=1 \\ \left(\frac{y}{\frac{1}{2}\sqrt{2}}\right)^2=1 \end{cases} \Rightarrow \begin{cases} x=2\sqrt{2} \\ y=\pm\frac{1}{2}\sqrt{2} \end{cases}$$

⑥ Express $\frac{\partial^2 z}{\partial r^2}$ in Cartesian given $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos\theta + \frac{\partial z}{\partial y} \sin\theta$

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \left(\frac{\partial z}{\partial x} \cos\theta + \frac{\partial z}{\partial y} \sin\theta \right) \frac{\partial}{\partial r} \quad (1) = \frac{\partial}{\partial x} \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x^2} \cos\theta + \frac{\partial^2 z}{\partial xy} \sin\theta \\ &= \underbrace{\frac{\partial}{\partial r} \left(\frac{\partial z}{\partial x} \right)}_{(1)} \cos\theta + \underbrace{\frac{\partial}{\partial r} \left(\frac{\partial z}{\partial y} \right)}_{(2)} \sin\theta \quad (2) = \frac{\partial}{\partial x} \frac{\partial z}{\partial y} \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial xy} \cos\theta + \frac{\partial^2 z}{\partial y^2} \sin\theta \\ &= \left(\frac{\partial^2 z}{\partial x^2} \cos^2\theta + \frac{\partial^2 z}{\partial xy} \sin\theta \cos\theta + \frac{\partial^2 z}{\partial xy} \cos\theta \sin\theta + \frac{\partial^2 z}{\partial y^2} \sin^2\theta \right) \cos\theta + \left(\frac{\partial^2 z}{\partial xy} \cos\theta + \frac{\partial^2 z}{\partial y^2} \sin\theta \right) \sin\theta \\ &= \frac{\partial^2 z}{\partial x^2} \cos^2\theta + \frac{\partial^2 z}{\partial xy} \sin\theta \cos\theta + \frac{\partial^2 z}{\partial xy} \cos\theta \sin\theta + \frac{\partial^2 z}{\partial y^2} \sin^2\theta \\ &= \frac{\partial^2 z}{\partial x^2} \cos^2\theta + \frac{\partial^2 z}{\partial xy} \sin\theta \cos\theta + \frac{\partial^2 z}{\partial xy} \cos\theta \sin\theta + \frac{\partial^2 z}{\partial y^2} \sin^2\theta \\ &= \frac{\partial^2 z}{\partial x^2} \cos^2\theta + 2 \frac{\partial^2 z}{\partial xy} \sin\theta \cos\theta + \frac{\partial^2 z}{\partial y^2} \sin^2\theta \\ &= \frac{\partial^2 z}{\partial x^2} \frac{x^2}{r^2} + 2 \frac{\partial^2 z}{\partial xy} \frac{xy}{r^2} + \frac{\partial^2 z}{\partial y^2} \frac{y^2}{r^2} = \frac{\partial^2 z}{\partial x^2} \frac{x^2}{x^2+y^2} + 2 \frac{\partial^2 z}{\partial xy} \frac{xy}{x^2+y^2} + \frac{\partial^2 z}{\partial y^2} \frac{y^2}{x^2+y^2} \\ &= \frac{1}{x^2+y^2} \left(x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial xy} + y^2 \frac{\partial^2 z}{\partial y^2} \right)\end{aligned}$$

⑦ Find equation of plane passes $P(-2, 3, -1)$ and contains $x=1-t, y=2+3t, z=-3+2t$

then we have another point which is $(1, 2, -3)$

then the $(1, 2, -3) - (-2, 3, -1)$ is a line on plane $= (3, -1, -2)$

$$\eta = [3, -1, -2] \cdot [1, 3, 2] = [-4, 4, -8]$$

then $-4x + 4y - 8z = d$ and $-4(-2) + 4(3) - 8(-1) = 8 + 12 + 8 = 28$

$$\therefore -4x + 4y - 8z = 28$$

6. Extrema of real valued functions

$$f(x, y) = x^2 + y^2 - xy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$f(x, y, z) = x^2 + 2xy + 6xz + 3y^2 + 2yz + z^2$$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Find critical points: $\frac{\partial f}{\partial x_i} = 0$ or undefined

2nd derivative test for function with two variables

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Rightarrow C. pt (c_1, c_2, \dots)$$

$$D(x_0, y_0) = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

if $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0 \Rightarrow L. \min$

if $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0 \Rightarrow L. \max$

if $D(x_0, y_0) < 0 \Rightarrow$ saddle point

if $D(x_0, y_0) = 0 \Rightarrow$ inconclusive

2nd derivative test for function with more than two variables

$$\text{① find c.p.s } \left(\frac{\partial f}{\partial x_i} = 0 = \frac{\partial f}{\partial y_j} = 0 = \frac{\partial f}{\partial z_k} = 0 \right)$$

$$\text{② } H_f \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \cdots \\ \cdots & \ddots \end{bmatrix}$$

$$\text{③ } H_f(x_0, y_0, z_0) = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \Rightarrow \begin{cases} \lambda_1 = \dots \\ \lambda_2 = \dots \\ \lambda_3 = \dots \end{cases} \begin{array}{l} \text{all eigenvalue is } + \Rightarrow \min} \\ \text{all eigenvalue is } - \Rightarrow \max} \\ \det \neq 0 \Rightarrow \text{saddle} \\ \det = 0 \Rightarrow \text{degenerate type} \end{array}$$

A closed and bounded set in \mathbb{R}^n is said to be compact

Ex. Shortest distance from a point (a, b, c) to $(x, y, z) = d = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$

Example 6.8: Find the global maximum and minimum value of $f(x, y) = x^2 + y^2 - 2x + 2y + 5$ on the set $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$. P967 Briggs

$$\begin{aligned} \text{Lagrange: } & \text{ in } D \quad x^2+y^2 \leq 4 \\ & \text{in } \partial D \quad \underline{x^2+y^2=4} \\ & \quad g(x, y) \end{aligned}$$

$$L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - 4)$$

$$= x^2 + y^2 - 2x + 2y + 5 - \lambda[x^2 + y^2 - 4] \quad \left[\text{since if } \lambda = 0 \right]$$

$$L_x = 2x - 2 - 2\lambda x = 0 \rightarrow 2(1-\lambda)x = 2 \quad \left[\begin{array}{l} \text{we have no} \\ \text{solution} \end{array} \right]$$

$$L_y = 2y + 2 - 2\lambda y = 0 \rightarrow 2(1-\lambda)y = -2$$

$$\begin{aligned} L_\lambda = x^2 + y^2 - 4 &= 0 \\ \text{as } x \neq 1 &\quad \left\{ \begin{array}{l} x = \sqrt{2} \\ y = -\sqrt{2} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} x = -\sqrt{2} \\ y = \sqrt{2} \end{array} \right. \\ \left\{ \begin{array}{l} x = \frac{1}{\sqrt{2}} \\ y = -\frac{1}{\sqrt{2}} \end{array} \right. &\quad \left\{ \begin{array}{l} y = \sqrt{2} \\ \lambda = 1 + \frac{1}{\sqrt{2}} \end{array} \right. \quad \therefore \text{C.p.s are } (\sqrt{2}, -\sqrt{2}) \\ &\quad (-\sqrt{2}, \sqrt{2}) \end{aligned}$$

Procedure:

1. Constructing a new function $L: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ by $L(x, \lambda) = f(x) - \lambda(g(x) - C)$.
2. Finding all the critical points of L about λ and the constrained critical points of f .
3. Evaluating all the constrained critical points of f . the largest is the maximum value of f and the smallest is the minimum value of f .

Example 6.12: Find extrema values of $f(x, y, z) = 3x - y - 3z$ subject to the constraints

$$\begin{array}{ll} x + y - z = 0 & \text{and } x^2 + 2z^2 = 1 \\ g_1 & g_2 \end{array}$$

Set $L(x, y, z, \lambda_1, \lambda_2)$

$$= f(x, y, z) - \lambda_1(g_1 - C_1) - \lambda_2(g_2 - C_2)$$

$$\therefore L = 3x - y - 3z - \lambda_1(x + y - z) - \lambda_2(x^2 + 2z^2 - 1)$$

$$L_x = 3 - \lambda_1 - 2\lambda_2 = 0 \quad 4 - 2\lambda_2 = 0 \Rightarrow \lambda_2 = \frac{2}{\lambda_1}$$

$$L_y = -1 - \lambda_1 = 0 \quad \Rightarrow \lambda_1 = -1$$

$$L_z = -3 + \lambda_1 - 4\lambda_2 = 0 \Rightarrow -3 - 1 - 4\lambda_2 = 0 \Rightarrow \lambda_2 = -\frac{1}{4}$$

$$L_{\lambda_1} = x + y - z = 0$$

$$L_{\lambda_2} = x^2 + 2z^2 - 1 = 0$$

plug x, z into the last equation to obtain

$$\left(\frac{2}{\lambda_1}\right)^2 + 2\left(\frac{-1}{4}\right)^2 - 1 = 0$$

$$\frac{6}{\lambda_1^2} - 1 = 0 \Rightarrow \lambda_1 = \pm\sqrt{6}$$

$$\therefore x = \pm\frac{2}{\sqrt{6}}, z = \mp\frac{1}{\sqrt{6}}$$

$$\text{then } x + y - z = 0 \Rightarrow y = \pm\frac{1}{\sqrt{6}}$$

so two c.p.s $(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}})$ & $(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}})$

$$f\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) = 3 \cdot \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} - 3 \cdot \frac{1}{\sqrt{6}} = 2\sqrt{6} \text{ max}$$

$$f\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = -2\sqrt{6} \text{ min}$$

7. Multiple Integrals

$$y\text{-simple: } \int_b^a \int_{y=-}^{y=+} f(x,y) dx dy$$

$$x\text{-simple: } \int_b^a \int_{x=-}^{x=+} f(x,y) dy dx$$

8. The Change of Variables in Multiple Integrals

Change of variables in double integral

$$\iint_D dA = \iint_{D^*} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA^*$$

$$\iint_D f(x,y) dxdy = \int_b^a \int_{h(u)}^{h(u)} f(x(u,\theta), y(u,\theta)) h'(u) d\theta du$$

Change of variables in triple integral

$$\iiint_D dV = \iiint_{D^*} \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dV^*$$

$$\iiint_w f(x,y,z) dxdydz = \iiint_{w^*} f(r\cos\theta, r\sin\theta, z) r dr d\theta dz$$

$$\iiint_w f(x,y,z) dxdydz = \iiint_{w^*} f(p\sin\phi\cos\theta, p\sin\phi\sin\theta, p\cos\phi) \underbrace{p\sin\phi}_r dp d\theta d\phi$$