- $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ where $\Delta x = \frac{b-a}{n}$ $x_i^* = a + (\Delta x)i$

- The Darboux Definition

  Let $a, b \in \mathbb{R}$, $a<b$. Suppose f is bounded on $[a,b]$, let $p=\{x_i\}_{i=0}^{n}$ be any partition of $[a,b]$

  Then for each $i = 1, \dots, n$

$$m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$$

$$M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$$

$$U(f,p) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

$$L(f,p) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

- f is integrable on $[a,b] \Leftrightarrow \sup\{L(f,p) \mid p \text{ is any partition of } [a,b]\}$
  $= \inf\{U(f,p) \mid p \text{ is any partition of } [a,b]\}$

- Integrability Reformulation

  f is integrable on $[a,b] \Leftrightarrow \forall \varepsilon > 0, \exists\, p$ partition of $[a,b]$ s.t. $U(f,p) - L(f,p) < \varepsilon$

After midterm

- Determine if an improper integral conv or div
  ① By definition Let $A\to\infty$ then $\lim_{A\to\infty}\int_{\#}^{A}\#$
  ② Comparison Theorem (CT)
  - If $0\le g(x)\le f(x)$ $\forall x\in[a,\infty)$ and $\int_a^\infty f(x)dx$ conv, then $\int_a^\infty g(x)dx$ conv
  - if $0\le h(x)\le g(x)$ $\forall x\in[a,\infty)$ and $\int_a^\infty h(x)$ div, then $\int_a^\infty g(x)dx$ div

Proof: Given f, g are cont on $[a,\infty)$

Suppose $0\le g(x)\le f(x)$ $\forall x\in[a,\infty)$ ①

$\int_a^\infty f(x)dx$ converges ②

WTS $\int_a^\infty g(x)dx$ converges

$\Updownarrow$

WTS $\lim_{A\to\infty}\int_a^A g(x)dx$ exists

Let $A\in[a,\infty)$ be arbitrary

① $\Rightarrow$ $0\le g(x)\le f(x)$ $\forall x\in[a,A]$

$\Rightarrow \int_a^A 0\,dx\le\int_a^A g(x)dx\le\int_a^A f(x)dx$, by int properties

$\Leftrightarrow 0\le\int_a^A g(x)dx\le\int_a^A f(x)dx$ $\forall A\ge a$

$\Rightarrow \lim_{A\to\infty}0\le\lim_{A\to\infty}\int_a^A g(x)dx\le\lim_{A\to\infty}\int_a^A f(x)dx$

$\Leftrightarrow 0\le\underbrace{\lim_{A\to\infty}\int_a^A g(x)dx}_{\text{is an area accumulation func. Note it's increasing as } A\to\infty}\le\underbrace{\int_a^\infty f(x)dx}_{\text{conv by ② } =\#}$

$\therefore$ This limit exists.

$\therefore \int_a^\infty g(x)dx$ conv

■

Given h, g are cont on $[a,\infty)$

Suppose $0\le h(x)\le g(x)$ $\forall x\in[a,\infty)$ ①

$\int_a^\infty g(x)dx$ diverges ②

WTS $\int_a^\infty g(x)dx$ diverges

$\Updownarrow$

WTS $\lim_{A\to\infty}\int_a^A g(x)dx$ DNE

Let $A\in[a,\infty)$ be arbitrary

① $\Rightarrow$ $0\le h(x)\le g(x)$ $\forall x\in[a,A]$

$\Rightarrow \int_a^A 0\,dx\le\int_a^A h(x)\,dx\le\int_a^A g(x)\,dx$, by int properties

$\Leftrightarrow 0\le\int_a^A h(x)\,dx\le\int_a^A g(x)\,dx$ $\forall A\ge a$

$\Rightarrow \lim_{A\to\infty}0\le\lim_{A\to\infty}\int_a^A h(x)\,dx\le\lim_{A\to\infty}\int_a^A g(x)\,dx$

$\Leftrightarrow 0\le\underbrace{\lim_{A\to\infty}\int_a^A h(x)\,dx}_{\text{div by ② = DNE}}\le\underbrace{\int_a^\infty g(x)\,dx}_{\text{is an area accumulation func. Note it's increasing as } A\to\infty}$

$\therefore$ This limit DNE

$\therefore \int_a^\infty g(x)dx$ div

■

if $0<p\le 1$, then $\frac{1}{x^p}\ge\frac{1}{x}$ for $x\in[1,\infty)$ and $\int_1^\infty \frac{1}{x^p}$ div
if $p>1$, then $\frac{1}{x^p}<\frac{1}{x}$ for $x\in[1,\infty)$ and $\int_1^\infty \frac{1}{x^p}$ conv to $\frac{1}{p-1}$ $\Big\}$ $[1,\infty]$

if $0<p<1$, then $\frac{1}{x^p}<\frac{1}{x}$ for $x\in[0,1]$ and $\int_0^1 \frac{1}{x^p}dx$ conv to $\frac{1}{1-p}$
if $p\ge 1$, then $\frac{1}{x^p}\ge\frac{1}{x}$ for $x\in[0,1]$ and $\int_0^1 \frac{1}{x^p}dx$ div $\Big\}$ $[0,1]$

e.g.

(i) Does $\int_3^\infty \underbrace{\frac{\cos^2(x)+1}{\sqrt{1+x^6}}}_{=g(x)}dx$ conv or div?

① for $x\in[3,\infty)$, $g(x)=\frac{\cos^2(x)+1}{\sqrt{1+x^6}}\ \frac{+}{+}>0$

② Find a good & explicit comparison
- bigger
- easier
- conv

$$\int_3^\infty \frac{\cos^2(x)+1}{\sqrt{1+x^6}}dx \overset{Rw}{\sim} \int_3^\infty \frac{\#}{\sqrt{x^6}}dx = \int_3^\infty \frac{\#}{x^3}dx \text{ conv}$$

for $x\in[3,\infty)$, $g(x)=\frac{\cos^2(x)+1}{\sqrt{1+x^6}}$

$\le \frac{1+1}{\sqrt{1+x^6}}$ max numerator b/c $\cos^2(x)\le 1$. $\forall x$

$=\frac{2}{\sqrt{1+x^6}}\le\frac{2}{\sqrt{x^6}}$ min denominator

$=\frac{2}{x^3}=f(x)$

Consider $\int_3^\infty f(x)dx$

$=\lim_{A\to\infty}\int_3^A \frac{2}{x^3}dx$

$=\lim_{A\to\infty} -x^{-2}\Big|_3^A$

$=\lim_{A\to\infty} -\left(\frac{1}{A^2}-\frac{1}{9}\right)$

$=\frac{1}{9}$ $\therefore$ lim exist

$\therefore \int_3^\infty f(x)dx$ conv

$\therefore$ By CT, $\int_3^\infty \frac{\cos^2(x)+1}{\sqrt{1+x^6}}dx$ conv. ■

(ii) Does $\int_0^1 \frac{\csc^2(x)}{x^{\frac{3}{2}}}dx$ con or div $=\int_0^1 \frac{1}{\sin^2(x)x^{\frac{3}{2}}}dx$

① For $x\in[0,1]$, $g(x)=\frac{1}{\sin^2(x)x^{\frac{3}{2}}}\ge 0$

② Find a good & explicit comparison
- smaller
- easier
- div

$Rw \sim \underbrace{\int_0^1 \frac{1}{x^{\frac{3}{2}}}dx}_{\text{div}}$

For $x\in[0,1]$, $g(x)\ \frac{1}{\sin^2(x)x^{\frac{3}{2}}}\ge\frac{1}{1\cdot x^{\frac{3}{2}}}$ max denominator b/c $0\le\sin^2(x)\le 1$ $\forall x$

$=x^{-\frac{3}{2}}=h(x)$

consider $\int_0^1 h(x)dx=\int_0^1 x^{-\frac{3}{2}}dx$

$=\lim_{A\to 0^+}\int_A^1 x^{-\frac{3}{2}}dx$

$=\lim_{A\to 0^+}\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}\Big|_A^1$

$=\lim_{A\to 0^+} -2\left(1-\frac{1}{\sqrt{A}}\right)$

$=+\infty$ $\therefore$ limit DNE

$\therefore \int_0^1 h(x)dx$ div

$\therefore \int_0^1 \frac{\csc^2(x)}{x^{\frac{3}{2}}}dx$ div by CT. ■

Definition: Given $\{a_n\}$ let $L \in \mathbb{R}$. We say $\{a_n\}$ converges to L

iff $\exists L \in \mathbb{R}, \forall \varepsilon > 0, \exists \overbrace{N > 0}^{\text{a large number}}$ s.t for all $n \in \mathbb{N}$

if $n > N$ then $|a_n - L| < \varepsilon$

Prove a sequence is conv we have two ways

① ε-N proof

② Bounded Monotone Convergent Theorem (BMCT)

e.g. of ①

(i) Prove $a_n = \frac{n^2-2}{n^2+2n+2}$ converges

wts $\exists L \in \mathbb{R}, \forall \varepsilon > 0, \exists N > 0$ s.t all for $n \in \mathbb{N}$, if $n > N$ then $|a_n - L| < \varepsilon$

Pf

choose $L = 1 \in \mathbb{R}$ (compute $\lim_{n\to\infty} a_n$)

Let $\varepsilon > 0$ be arbitrary

Choose $N = \frac{2}{\varepsilon} > 0$

Suppose $n > N$

Then $\left|\frac{n^2-2}{n^2+2n+2} - 1\right| = \left|\frac{n^2-2-n^2-2n-2}{n^2+2n+2}\right|$

$= \left|\frac{-2n-4}{n^2+2n+2}\right| = \frac{|-2|\cdot|n+2|}{|n^2+2n+2|} = \frac{2\,|n+2|}{|n^2+2n+2|} = \frac{2(n+2)}{n^2+2n+2}$, by alg. abs value property

WANT: $|a_n - L| \leq \frac{\#}{n^p}$, $P \in \mathbb{R}, P > 0$

We have $|a_n - L| = \frac{2(n+2)}{n^2+2n+2} < \frac{2(n+2)}{n^2+2n}$ (min denom) $= \frac{2\cancel{(n+2)}}{n\cancel{(n+2)}} = \frac{2}{n} < \frac{2}{N} = \frac{2}{\frac{2}{\varepsilon}} = \frac{1}{\left(\frac{1}{\varepsilon}\right)} = \varepsilon$ as needed ■

given: $n > N$

$\Rightarrow \frac{1}{n} < \frac{1}{N}$

$\Rightarrow \frac{2}{n} < \frac{2}{N}$

$\frac{2}{N} = \varepsilon$

$2 \cdot \frac{1}{N} = \varepsilon$

$\frac{1}{N} = \frac{\varepsilon}{2}$

$N = \frac{2}{\varepsilon}$

(ii)

Ex2 Prove that $\underbrace{\left\{\frac{2n-1}{n-3}\right\}}_{a_n}$ conv to $\underbrace{2}_{L}$

WTS $\forall \varepsilon>0, \exists N>0$, s.t. $\forall n \in \mathbb{N}$, if $n>N$ then $\left|\frac{2n-1}{n-3}-2\right|<\varepsilon$

Pf let $\varepsilon>0$ be arbitrary

choose $N=\max\{4, \frac{20}{\varepsilon}\}>0$

Suppose $n>N$

Then $\left|\frac{2n-1}{n-3}-2\right|=\left|\frac{2n-1-2n+6}{n-3}\right|=\left|\frac{5}{n-3}\right|=\frac{5}{|n-3|}=5\cdot\frac{1}{|n-3|}=5\cdot\frac{1}{|n(1-\frac{3}{n})|}=5\cdot\frac{1}{n}\cdot\underbrace{\frac{1}{|1-\frac{3}{n}|}}_{\star}$ $n=0, n=3$ are problem pts

$\star$ if $n>4$ (i.e. $N\geq 4$)

$\Rightarrow \frac{1}{n}<\frac{1}{4}$

$\Rightarrow \frac{3}{n}<\frac{3}{4}$

$\Rightarrow -\frac{3}{n}>-\frac{3}{4} \Rightarrow 1-\frac{3}{n}>1-\frac{3}{4}=\frac{1}{4}$ $\quad \therefore \left|1-\frac{3}{n}\right|>\frac{1}{4} \Rightarrow \frac{1}{|1-\frac{3}{n}|}<4$

Therefore $\left|\frac{2n-1}{n-3}-2\right|=5\cdot\frac{1}{n}\cdot\underbrace{\frac{1}{|1-\frac{3}{n}|}}_{<4}<5\cdot\frac{1}{n}\cdot 4=\frac{20}{n}<\frac{20}{N}\overset{\star}{\leq}\frac{20}{\left(\frac{20}{\varepsilon}\right)}=\varepsilon$

as needed ■

(iii) Prove $\{1+(-1)^n\}$ diverges

Assume $\{1+(-1)^n\}$ converges to some $L\in\mathbb{R}$

We have $\forall\varepsilon>0, \exists N>0$ s.t. $\forall n\in\mathbb{N}$, if $n>N$ then $|a_n-L|<\varepsilon$

Take $\varepsilon=\underline{1}>0$

We have $\exists N>0$ s.t. $\forall n\in\mathbb{N}$, if $n>N$, then $|1+(-1)^n-L|<1$

Case 1 n is odd: if $n>N \Rightarrow |0-L|<1 \Rightarrow |L|<1$

$\Leftrightarrow -1<L<1$

Case 2 n is even: if $n>N \Rightarrow |2-L|<1 \Rightarrow -1<2-L<1$

$\Leftrightarrow -3<-L<-1$ $\quad$ so $L\in(-1,1)\cap(1,3)=\emptyset$

$\Leftrightarrow 1<L<3$ $\quad$ Contradiction $\quad \therefore \{1+(-1)^n\}$ diverges ■

(iv) $a_n = 8n^3 + n^2 - 2$ show div.

$\lim_{n \to \infty} a_n = \infty$ $\therefore$ seq div

WTS $\forall M > 0, \exists N > 0$ s.t. $\forall n \in \mathbb{N}, n > N \Rightarrow a_n > M$

Let $M > 0$ be arbitrary

choose $N = \left(\frac{M}{6}\right)^{\frac{1}{3}} > 0$

Suppose $n > N$,

Then $a_n = 8n^3 + n^2 \geq 8n^3 - 2$ b/c $n^2 \geq 0$

$\geq 8n^3 - 2n^3$ b/c $n^3 \geq 1$

$= 6n^3 > 6N^3$

$= 6 \cdot \left(\left(\frac{M}{6}\right)^{\frac{1}{3}}\right)^3$

$= M$, as wanted

■

e.g of ②

Prove the seq given $a_1 = \sqrt{6}$ and $a_{n+1} = \sqrt{6 + a_n}$. If $n \geq 1$, then $\{a_n\}$ converges.
(Use BMCT since $\{a_n\}$ is recursively defined)

WTS $\{a_n\}$ is bounded above and increasing

① WTS $\{a_n\}$ is bounded above

WTS $\exists M \in \mathbb{R}$ s.t $a_n \leq M$ $\forall n \in \mathbb{N}$

Investigate: $a_1 = \sqrt{6} < \sqrt{9} = 3$ we know $0 < 6 < 9 \Rightarrow \sqrt{6} < \sqrt{9}$

$a_2 = \sqrt{6 + a_1} = \sqrt{6 + \underbrace{\sqrt{6}}_{<3}} < \sqrt{6+3} = 3$

$a_3 = \sqrt{6 + a_2} = \sqrt{6 + \underbrace{\sqrt{6 + \sqrt{6}}}_{<3}} < \sqrt{6+3} = 3$

choose $M = 3 \in \mathbb{R}$

WTS $a_n \leq 3$ $\forall n \in \mathbb{N}$

Base Case: $a_1 = \sqrt{6}$ by def of $\{a_n\}$ $a_1 = \sqrt{6} < \sqrt{9} = 3$ b/c $0 < 6 < 9$ so $\sqrt{6} < \sqrt{9}$

Induction Step: WTS $\forall k \in \mathbb{N}$ $(\underbrace{P(k)}_{\text{holds}} \Rightarrow \underbrace{P(k+1)}_{\text{holds}})$

let $k \in \mathbb{N}$ be arbitrary, assume $a_k \leq 3$ (induction hypothesis)

WTS $a_{k+1} \leq 3$

Consider $a_{k+1} = \sqrt{6+\underbrace{a_k}_{\leq 3 \text{ by IH}}}$ by def of $\{a_n\}$

$\therefore a_{k+1} \leq \sqrt{6+3} = \sqrt{9} = 3$, as wanted

$\therefore$ By PMI (principle of mathematic induction), $\{a_n\}$ is bounded above by 3 ■

② WTS $\{a_n\}$ is increasing $\Leftrightarrow$ WTS $a_n \leq a_{n+1}$ $\forall n \in \mathbb{N}$

Use derivative analysis for not-recursive funcs

Let $n \in \mathbb{N}$ be arbitrary

Consider $a_n^2 - a_{n+1}^2 = a_n^2 - (\sqrt{6+a_n})^2$, by def of $\{a_n\}$

$= a_n^2 - (6+a_n)$

$= a_n^2 - a_n - 6$

$= \underbrace{\underbrace{(a_n-3)}_{\leq 3}}_{\leq 0}\underbrace{(a_n+2)}_{>0}$

$\therefore a_n^2 - a_{n+1}^2 \leq 0$ $\forall n \in \mathbb{N}$

$\Rightarrow a_n^2 \leq a_{n+1}^2$

$\Rightarrow a_n \leq a_{n+1}$, as wanted

$\therefore \{a_n\}$ is $\uparrow$.

$\therefore$ By BMCT, $\{a_n\}$ converges ■

Proof of Conv property

Let $a_n \to A$ ① and $b_n \to B$ ②

(i) $a_n + b_n \to A+B$

WTS $\forall \varepsilon>0\ \exists N>0$ s.t. $\forall n\in\mathbb{N}$, if $n>N$ then $|a_n+b_n-A-B|<\varepsilon$

Let ε be arbitrary, choose $N=\max\{N_1,N_2\}>0$

① $\to \exists N_1>0$ s.t. $\forall n>N$, then $|a_n-A|<\frac{\varepsilon}{2}$

② $\to \exists N_2>0$ s.t. $\forall n>N$, then $|b_n-B|<\frac{\varepsilon}{2}$

Suppose $n>N$, then

$|a_n+b_n-A-B| = |(a_n-A)+(b_n-B)| \leq |a_n-A|+|b_n-B| < \varepsilon$ ∎

(ii) $ka_n \to kA$

WTS $\forall \varepsilon>0\ \exists N>0$ s.t. $\forall n\in\mathbb{N}$, if $n>N$ then $|ka_n-kA|<\varepsilon$

Let ε be arbitrary. Suppose $n>N$

① $\to \exists N_1>0$ s.t. $\forall n>N$, then $|a_n-A|<\frac{\varepsilon}{|k|}$

$|ka_n-kA| = |k||a_n-A| < |k|\cdot\frac{\varepsilon}{|k|} = \varepsilon$ ∎

(iii) $a_nb_n \to AB$

WTS $\forall \varepsilon>0\ \exists N>0$ s.t. $\forall n\in\mathbb{N}$, if $n>N$, then $|a_nb_n-AB|<\varepsilon$

Let ε be arbitrary, choose $N=\max\{N_1,N_2\}>0$

Since $b_n\to B$ then $\exists K\in\mathbb{R}^+$ s.t. $|b_n|<K$

① $\to \exists N_1>0$ s.t. $\forall n>N$, then $|a_n-A|<\frac{\varepsilon}{2K}$

② $\to \exists N_2>0$ s.t. $\forall n>N$, then $|b_n-B|<\frac{\varepsilon}{2A}$

$|a_nb_n-AB| = |b_n(a_n-A)+A(b_n-B)| \leq |b_n||a_n-A|+|A||b_n-B| < K|a_n-A|+|A||b_n-B| < \varepsilon$ ∎

(iv) $\frac{a_n}{b_n} = \frac{A}{B}$

WTS $\forall \varepsilon > 0\ \exists N > 0$ s.t $\forall n \in \mathbb{N}$, if $n > N$, then $\left|\frac{a_n}{b_n} - \frac{A}{B}\right| < \varepsilon$

Let ε be arbitrary, choose $N = \max\{N_1, N_2\} > 0$

① $\to \exists N_1 > 0$ s.t. $\forall n > N$, then $|a_n - A| < \frac{\varepsilon|B|}{4}$

② $\to \exists N_2 > 0$ s.t $\forall n > N$, then $|b_n - B| < \frac{\varepsilon B^2}{4|A|}$

$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| = \left|\frac{a_n}{b_n} - \frac{A}{b_n} + \frac{A}{b_n} - \frac{A}{B}\right| = \left|\frac{a_n - A}{b_n} + \frac{AB - Ab_n}{b_n B}\right| = \left|\frac{a_n - A}{b_n} + (B - b_n)\frac{A}{b_n B}\right| \leq \left|\frac{1}{b_n}\right||a_n - A| + \left|\frac{A}{b_n B}\right||B - B_n|$

Since $b_n \to B$, then at some point $|b_n| \geq \frac{|B|}{2} \Rightarrow \frac{1}{|b_n|} \leq \frac{2}{|B|}$

$\therefore \left|\frac{1}{b_n}\right||a_n - A| + \left|\frac{A}{b_n B}\right||B - B_n| = \left|\frac{1}{b_n}\right||a_n - A| + \frac{1}{|b_n|}\cdot\left|\frac{A}{B}\right|\cdot|B - B_n| < \frac{2}{|B|}|a_n - A| + \frac{2|A|}{B^2}|B_n - b_n| = \varepsilon$ ∎

Proof of if $\{a_n\}$ conv, then its limit is unique

Proof: Suppose $\{a_n\}$ conv to both L_1, L_2 (① ②) where $L_1, L_2 \in \mathbb{R}$

WTS $\forall \varepsilon > 0$, $|L_1 - L_2| < \varepsilon$

① $\Rightarrow \exists N_1 > 0$, if $n > N_1$, then $|a_n - L_1| < \frac{\varepsilon}{2}$

② $\Rightarrow \exists N_2 > 0$, if $n > N_2$, then $|a_n - L_2| < \frac{\varepsilon}{2}$

choose $N = \{N_1, N_2\}$, Suppose $n > N$

we have $|L_1 - L_2| = |L_1 - L_2 + a_n - a_n| = |-(a_n - L_1) + (a_n - L_2)| \leq |-(a_n - L_1)| + |a_n - L_2|$ by Δ inequality

$= |a_n - L_1| + |a_n - L_2|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$= \varepsilon$, ∎

Proof of BMCT

Suppose $\{a_n\}$ is bounded above ①

& $\{a_n\}$ is increasing ② WTS $\{a_n\}$ conv

WTS $\exists L\in\mathbb{R}, \forall \varepsilon>0 \exists N>0$ s.t. $\forall n\in\mathbb{N}$, if $n>N$, then $|a_n-L|<\varepsilon$

Define $A=\{a_n \mid n\in\mathbb{N}\}\subset\mathbb{R}$ and A is not empty since $a_1\in A$

and A is bounded above by ①

$\therefore$ Sup(A) exist

choose $L=\alpha\in\mathbb{R}$ let $\varepsilon>0$ be arbitrary

choose $N\in\mathbb{N}$ and $\alpha-\varepsilon<a_N$, suppose $n>N$

we have $a_n\leq\alpha<\alpha+\varepsilon$ b/c $\varepsilon>0$

and $\alpha-\varepsilon<a_N<a_n\leq\alpha<\alpha+\varepsilon$

by ②

$\therefore \alpha-\varepsilon<a_n<\alpha+\varepsilon$

$-\varepsilon<a_n-\alpha<\varepsilon$

$\therefore |a_n-\alpha|<\varepsilon$. ∎

Series

Ways to determine if a series conv or div

① Inspection and telecoping sum.

② GS Test

③ Div Test

④ Integral Test (IT)

⑤ P-Series Test

① Inspection and telescoping sum

e.g a) $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ conv or div?

$\ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln(n)$ $\quad S_n = \cancel{\ln(2)} - \ln(1) + \cancel{\ln(3)} - \cancel{\ln(2)} + \cancel{\ln(4)} - \cancel{\ln(3)} + \cdots + \ln(n+1) - \cancel{\ln(n)}$

$= \ln(n+1) - \ln(1)$

$= \ln(n+1)$

$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1) = \infty$ $\quad \therefore$ div

b) $\sum_{n=1}^{\infty} \frac{3}{n^2+3n}$

$S_n = \frac{3}{n^2+3n} = \frac{3}{n(n+3)}$ $\quad$ Pfd: $S_n = \frac{3}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3} = \frac{An+3A+Bn}{n(n+3)}$ $\therefore A=1 \quad B=-1$

$\therefore S_n = \frac{3}{n(n+3)} = \frac{1}{n} - \frac{1}{n+3}$ $\quad \therefore \sum_{n=1}^{n} S_n = 1 - \cancel{\frac{1}{4}} + \frac{1}{2} - \cancel{\frac{1}{5}} + \frac{1}{3} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{7}} + \cdots + \cancel{\frac{1}{n-4}} - \cancel{\frac{1}{n-1}} + \cancel{\frac{1}{n-3}} - \cancel{\frac{1}{n}} + \cancel{\frac{1}{n-2}} - \frac{1}{n+1} + \cancel{\frac{1}{n-1}} - \frac{1}{n+2} + \cancel{\frac{1}{n}} - \frac{1}{n+3}$

$= 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}$

$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+1}\right)$

$= 1 + \frac{1}{2} + \frac{1}{3} = \frac{3}{2} + \frac{1}{3} = \frac{9+2}{6} = \frac{11}{6}$ $\therefore$ Conv.

② GS Test $\sum_{n=0}^{\infty} ar^n$

For a GS $\sum_{n=1}^{\infty} ar^n$, if $|r|<1$, then conv to sum $\frac{a_1}{1-r}$

if $|r| \geq 1$, then div

e.g

a) $\sum_{n=2}^{\infty} \frac{\pi(\sqrt{2})^n}{3^{n-1}}$ $\quad$ aside: $3^{n-1} = 3^n \div 3 = 3^n \times \frac{1}{3}$

$S_n = \frac{\pi\sqrt{2}^n}{3^{n-1}} = \frac{\pi\sqrt{2}^n}{3^n \times \frac{1}{3}} = 3\pi\left(\frac{\sqrt{2}}{3}\right)^n$

$\therefore r = \frac{\sqrt{2}}{3}$ and $|r|<1$

$\therefore$ conv $= \frac{a}{1-r}$

③ Div Test

if $\lim_{n\to\infty} a_n \neq 0$, then div

④ IT. Given $\sum a_n$, if $a_n = f(n)$ $\forall n \in \mathbb{N}$ and $f(x)$ is cont, positive, decreasing on $[1,\infty]$ then $\sum_{n=1}^{\infty} a_n \Longleftrightarrow \int_1^{\infty} f(x)dx$

div $\Leftrightarrow$ div

conv $\Leftrightarrow$ conv

eg $\sum_{n=0}^{\infty} \frac{1}{1+4n^2}$ conv or div?

Let $f(n) = \frac{1}{1+4n^2}$, then $f(x) = \frac{1}{1+4x^2}$. Consider $f(x)$ on $[0,\infty)$

$\because$ $f(x) > 0$ and $f'(x) \leq 0$

$\therefore$ $f(x)$ is positive, $\downarrow$ and cont on $[0,\infty)$

Consider $\int_0^{\infty} f(x)dx = \lim_{A\to\infty} \int_0^A \frac{1}{1+4x^2}dx = \lim_{A\to\infty} \frac{\arctan(2x)}{2}\Big|_0^A = \frac{\pi}{4}$ $\therefore$ limit exist $\therefore \int_0^{\infty} f(x)dx$ conv

by IT. series conv.

⑤ P-series Test

Given $\sum_{n=1}^{\infty} \frac{1}{n^p}$. if $p>1$ then conv

if $0<p\leq 1$ then div

⑥ CT for series

if $0 \leq a_n \leq b_n$ $\forall n \in \mathbb{N}$, b_n conv $\Rightarrow a_n$ conv

if $0 \leq c_n \leq a_n$ $\forall n \in \mathbb{N}$, c_n conv $\Rightarrow a_n$ conv

⑦ LCT* Let $\sum a_n$ $\sum b_n$ be series. Suppose $a_n b_n > 0$ $\forall n \in \mathbb{N}$

Defined $\lim_{n\to\infty} \frac{a_n}{b_n} = p$

① if $0<p<\infty \Rightarrow$ both conv or div

② if $p=0$ and $\sum b_n$ conv $\Rightarrow \sum a_n$ conv

③ if $p=\infty$ and $\sum b_n$ div $\Rightarrow \sum a_n$ div

⑧ Alternating Series Test (AST)

Given $\sum (-1)^n b_n$, $b_n > 0$

if ① $b_n \geq b_{n+1} > 0 \ \forall n \in \mathbb{N}$

② $\lim_{n\to\infty} b_n = 0$ (Check this first)

then conv

e.g

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ cc? Ac? div?

consider $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ $p=1 \Rightarrow$ div

consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$

① $\lim_{n\to\infty} \frac{1}{n} = 0$

② let $n \in \mathbb{N}$ be arbitrary

we know $0 < n < n+1$

$\Rightarrow \frac{1}{n} > \frac{1}{n+1}$

$\therefore$ conv

$\therefore$ cc

⑨ Ratio Test

Let $\sum a_n$ be a series with $a_n \in \mathbb{R} - \{0\}$

Define $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ if $\rho < 1 \Rightarrow \sum a_n$ Ac and conv

$\rho > 1 \Rightarrow \sum a_n$ div

$\rho = 1 \Rightarrow$ we don't know

⑩ Power Series

A series in the form of $\sum_{n=0}^{\infty} C_n (x-a)^n$

eg. for what $x \in \mathbb{R}$ does $\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n 4^n}$ conv?

i.e what is the interval of conv

$I = \{x \in \mathbb{R}, \sum c_n (x-a)^n \text{ conv}\}$

① RT

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n (x-a)^n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-2)^{n+1}}{(n+1)4^{n+1}} \cdot \frac{n4^n}{(-1)^n (x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^n(-1)}{(-1)^n} \cdot \frac{4^n}{4^n \cdot 4} \cdot \frac{n}{n+1} \cdot \frac{(x-2)^{n+1}}{(x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left| (-1) \cdot \frac{1}{4} \cdot \frac{n}{n+1} \cdot (x-2) \right|$$

$$= \frac{1}{4}|x-2| \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \frac{|x-2|}{4}$$

$\therefore$ by RT $\frac{|x-2|}{4} < 1 \Rightarrow$ conv

$|x-2| < 4$ radius

② check end pts $x = a+R,\ x = a-r$

a) $x = a+r = 2+4 = 6$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (4)^n}{n 4^n} = \frac{(-1)^n}{n} \quad \checkmark$$

b) $x = a-r = 2-4 = -2$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (-4)^n}{n 4^n} \quad \frac{1}{n}$$

Series Test summary

① Inspection and telescoping ($\ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln(n)$) Conv to # or div

② GS Test $\sum_{n=1}^{\infty} ar^n$. if $|r|<1 \Rightarrow$ conv to $\frac{a}{1-r}$ a is the first nonzero term in GS
if $|r| \geq 1 \Rightarrow$ div Conv to # or div

③ Div test
if $\lim_{n\to\infty} a_n \neq 0 \Rightarrow$ div div

④ Integral Test
Given $\sum a_n$, $a_n = f(n)$. if $f(x)$ cont, +, ↓ on $[1, \infty)$
Then $\int_1^{\infty} f(x)dx$ conv $\Rightarrow \lim_{n\to\infty} \sum a_n$ Conv conv or div
$\int_1^{\infty} f(x)dx$ div $\Rightarrow \lim_{n\to\infty} \sum a_n$ div

⑤ P-series Test
$\sum_{n=1}^{\infty} \frac{1}{n^p}$ if $p>1 \Rightarrow$ conv conv or div
if $0<p\leq 1 \Rightarrow$ div

⑥ CT $\sum a_n, \sum b_n, \sum c_n$
if $0 \leq a_n \leq b_n$ $\forall n \in \mathbb{N}$ and b_n conv $\Rightarrow a_n$ Conv conv or div
if $0 \leq c_n \leq a_n$ $\forall n \in \mathbb{N}$ and c_n div $\Rightarrow a_n$ div

⑦* LCT Suppose $a_n b_n > 0$ $\forall n \in \mathbb{N}$
Defined $\lim_{n\to\infty} \frac{a_n}{b_n} = p$
① if $0<p<\infty \Rightarrow$ both conv or div Conv or div
② if $p=0$ and $\sum b_n$ conv $\Rightarrow \sum a_n$ Conv
③ if $p=\infty$ and $\sum b_n$ div $\Rightarrow \sum a_n$ div

⑧ Alternating Series Test (AST)
Given $\sum (-1)^n b_n$, $b_n > 0$ Conv
if ① $b_n \geq b_{n+1} > 0$ $\forall n \in \mathbb{N}$
② $\lim_{n\to\infty} b_n = 0$ (Check this first)
then Conv

⑨ Ratio Test
Let $\sum a_n$ be a series with $a_n \in \mathbb{R} - \{0\}$
Defind $p = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ if $p<1 \Rightarrow \sum a_n$ AC and conv
$p>1 \Rightarrow \sum a_n$ div
$p=1 \Rightarrow$ we don't know
if $p=0 \Rightarrow$ infinite radius and interval is $\mathbb{R}$
if $p=\infty \Rightarrow$ radius is 0 and $I=\{a\}$

$$P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$