MATB41 Final Review

1. lines, Planes and curves

- The vector equation of the line $\vec{l} = a + t\vec{v}$
  $= (a_1, a_2, a_3) + t[v_1, v_2, v_3]$, $t \in \mathbb{R}$

- Parametric equation of line in $\mathbb{R}^3$: $x = a_1 + tv_1$
  $y = a_2 + tv_2$
  $z = a_3 + tv_3$, $t \in \mathbb{R}$

Ex ① Find equation of the plane passes $(1,1,-1)$ and $\perp$ to the line: $x = 1+t$, $y = 1-3t$, $z = -7t$

$n = [1,-3,-7]$, let (x,y,z) be on the plane

then $[x-1, y-1, z+1] \cdot [1,-3,-7] = 0$

$x - 1 - 3y + 3 - 7z - 7 = 0$

$\therefore$ equation of plane $x - 3y - 7z = 5$

② $\vec{n_1} = [-2,6,14]$, $\vec{n_2} = [1,-3,-7]$ $\Rightarrow \vec{n_1} \parallel \vec{n_2}$ | $\vec{n_1} = [2,3,1]$, $\vec{n_2} = [1,-1,1]$ $\Rightarrow \vec{n_1} \perp \vec{n_2}$ | $\vec{n_1} = [2,3,1]$, $\vec{n_2} = [1,-2,3]$ $\Rightarrow \theta = \arccos \dfrac{\vec{n_1} \cdot \vec{n_2}}{\|\vec{n_1}\|\|\vec{n_2}\|}$

③ Find equation of plane passes $A(-1,1,2)$, $B(2,0,-3)$ and $C(2,-1,2)$

$\vec{V} = B - A = (3,-1,-5)$

$\vec{W} = C - A = (3,-2,0)$

$$\vec{n} = \vec{V} \times \vec{W} = \begin{vmatrix} i & j & k \\ 3 & -1 & -5 \\ 3 & -2 & 0 \end{vmatrix} = i\begin{vmatrix} -1 & -5 \\ -2 & 0 \end{vmatrix} - j\begin{vmatrix} 3 & -5 \\ 3 & 0 \end{vmatrix} + k\begin{vmatrix} 3 & -1 \\ 3 & -2 \end{vmatrix}$$

$= 10i - 15j - 3k$

$= [-10, -15, -3]$

$\therefore -10x - 15y - 3z = d$

and $10(-1) - 15(1) - 3(2)$

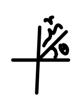
$= 10 - 15 - 6$

$= -11 = d$

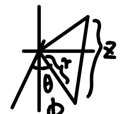
$\therefore -10x - 15y - 3z = -11$

- tangent vector of c at a: $x - a = c'(t_0)(t - t_0)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix} = \begin{bmatrix} \vdots \end{bmatrix}(t - :) \Rightarrow \begin{cases} x = \\ y = \\ z = \end{cases}$$

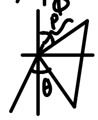
- Polar (r, θ) 
- Polar to Cartesian: $x = r\cos\theta$, $y = r\sin\theta$
- Cartesian to Polar: $\theta = \arctan\frac{y}{x}$, $r = \sqrt{x^2+y^2}$

- Cylindrical (r, θ, z)



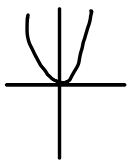
$x = r\cos\theta$ $\quad \theta = \arctan \frac{y}{x}$,
$y = r\sin\theta$ $\quad r = \sqrt{x^2+y^2}$
$z = z$

- Spherical (ρ, θ, ϕ)

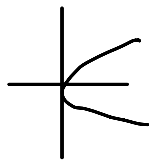


$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$ $\quad \rho = \sqrt{x^2+y^2+z^2}$

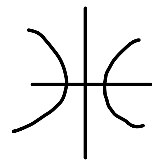
2. Vector and function



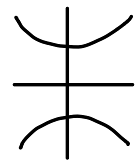
$y = x^2$



$x = y^2$



$x^2 - y^2 = 1$



$y^2 - x^2 = 1$

3. Limits

δ-ε proof:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < \|(x,y) - (0,0)\| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Prove $\lim_{(x,y)\to(0,0)} \frac{2x^2y^2}{x^2+y^2} = 0$

WTS: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < \|(x,y)-(0,0)\| < \delta \Rightarrow \left|\frac{2x^2y^2}{x^2+y^2}\right| < \varepsilon$

$0 < \|(x,y)-(0,0)\| < \delta$ $\Rightarrow x^2+y^2 < \delta^2$

$\left|\frac{2x^2y^2}{x^2+y^2}\right| < \varepsilon$ $\Rightarrow \frac{2x^2y^2}{x^2+y^2} < \varepsilon$

$2 \cdot \underbrace{\frac{x^2}{x^2+y^2}}_{\leq 1} \cdot \underbrace{y^2}_{\leq x^2+y^2} \leq 2 \cdot 1 (x^2+y^2) < 2\delta^2 = \varepsilon$

Proof: $\forall \varepsilon > 0, \exists \delta > 0$, choose $\delta = \sqrt{\frac{\varepsilon}{2}}$ s.t. $0 < \|(x,y)-(0,0)\| < \delta \Rightarrow \left|\frac{2x^2y^2}{x^2+y^2}\right| = \frac{2x^2y^2}{x^2+y^2} \leq 2(x^2+y^2) < 2\delta^2 = \varepsilon$ ∎

$\lim_{x\to x_0} f(x) = f(x_0)$ means

a) $x_0 \in U$ (in domain)
b) $\lim_{x\to x_0} f(x) = L$
c) $f(x_0) = L$

$\lim_{(x,y)\to(a,b)} f(x,y) = L = f(a,b)$

$$\lim_{(x,y)\to(a,b)} xy = ab$$

WTS: $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < \|(x-a, y-b)\| < \delta \Rightarrow |xy - ab| < \varepsilon$

$\|(x-a, y-b)\| < \delta$

$|y-b| < \sqrt{(x-a)^2 + (y-b)^2} < \delta \le 1$

$|y| - |b| < 1$

$|y| < 1 + |b|$

Then $|xy - ab| = |xy - ab + ay - ay|$

$= |xy - ay + ay - ab|$

$\le |y(x-a)| + |a(y-b)|$

$\le |y||x-a| + |a||y-b|$

$\le (1+|b|)\sqrt{(x-a)^2+(y-b)^2} + |a|\sqrt{(x-a)^2+(y-b)^2}$

$= (1+|b|+|a|)\sqrt{(x-a)^2+(y-b)^2}$

$< (1+|b|+|a|)\delta$

$\therefore (1+|b|+|a|)\delta = \varepsilon$

$\delta = \dfrac{\varepsilon}{(1+|b|+|a|)}$

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^3}{x^8+y^2} = 0$$

$x^8 + y^2 \pm 2x^4y = (x^4 \pm y)^2 \ge 0$

$\Rightarrow x^8 + y^2 \pm 2x^4y \ge 0$

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

$$\lim_{(x,y)\to(1,0^+)} \frac{y\ln y}{x}$$

$\lim_{y\to 0^+} y\ln y = \dfrac{\ln y}{\frac{1}{y}} \overset{L.H}{=} \lim_{y\to 0^+} \dfrac{\frac{1}{y}}{-\frac{1}{y^2}} = -y = 0$

$\therefore = 0$

$\dfrac{\sin(t)}{t} = 1 \qquad \dfrac{x^2}{x^2+y^2} \le 1$

$|x| \le \sqrt{x^2+y^2}$

$|y| \le \sqrt{x^2+y^2}$

$x^2 + y^2 \pm 2xy = (x \pm y)^2$

$x^2 + y^2 \ge \mp 2xy$

$x^2 + y^2 \ge 2|xy|$

4. Differentiation

directional derivates of f at a in direction v is: $D_v(f(a)) = \lim_{t\to 0} \dfrac{f(a+tv) - f(a)}{t\|v\|} = \nabla f(a) \dfrac{v}{\|v\|}$

$f: \mathbb{R}^n \mapsto \mathbb{R}^m \quad Df(a) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x} & \frac{\partial f_n}{\partial y} & \frac{\partial f_n}{\partial z} \end{bmatrix} \qquad f(x,y,z) = (\underbrace{\cdots}_{f_1}, \underbrace{\cdots}_{f_2}, \underbrace{\cdots}_{f_n})$

$f: \mathbb{R}^n \mapsto \mathbb{R} \quad Df(a) = \left[\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}, \frac{\partial f}{\partial z}\right] = \nabla f(a)$

tangent line equation

$f: \mathbb{R}^3 \mapsto \mathbb{R}$ $\qquad \nabla f(x_0)\cdot(x-x_0)=0$ $\qquad L(x)=f(x_0)+\nabla f(x_0)\cdot(x-x_0)$

$f: \mathbb{R}^2 \mapsto \mathbb{R}$ $\qquad f(x_0)+\nabla f(x_0)(x-x_0)=z$ $\qquad L(x)=f(x_0)+\nabla f(x_0)\cdot(x-x_0)$

$z=\sin(x^2y)+2 \qquad (5,0,2)$

$f(x,y)=\sin(x^2y)+2 \qquad (2xy\cos(x^2y),\ x^2\cos(x^2y))$

$z=\underbrace{f(a_0)}_{2}+\nabla f(a_0)(x-x_0)=2+(0,25)(x-5,y)$

$z=2+25y$

Ex. Find the point on the surface $x^2+2y^2+3z^2=1$ where tangent line is // to $3x-y+3z=1$

$\nabla f=(2x,4y,6z)$

$\nabla f(x_0)=(2x_0,4y_0,6z_0)$

$(2x_0,4y_0,6z_0)=k(3,-1,3)$

$\begin{cases}2x_0=3k\\4y_0=-k\\6z_0=3k\end{cases} \Rightarrow \begin{cases}x_0=\frac{3k}{2}\\y_0=-\frac{k}{4}\\z_0=\frac{1}{2}k\end{cases}$

$\frac{9}{4}k^2+\frac{1}{8}k^2+\frac{3}{4}k^2=1$

$\frac{18}{8}k^2+\frac{1}{8}k^2+\frac{6}{8}k^2=1$

$\frac{25}{8}k^2=1$

$k^2=\frac{8}{25}$

$k=\pm\sqrt{\frac{8}{25}}$

$\therefore \left(\frac{3\sqrt{2}}{5}, -\frac{\sqrt{2}}{10}, \frac{\sqrt{2}}{5}\right)$

$\left(-\frac{3\sqrt{2}}{5}, \frac{\sqrt{2}}{10}, -\frac{\sqrt{2}}{5}\right)$

Let $f: U\subset\mathbb{R}^n \mapsto \mathbb{R}^m$, if all the partial derivative exist at a and

if $\lim\limits_{x\to a}\frac{\|f(x)-f(a)-Df(a)(x-a)\|}{\|x-a\|}=0$, then f is diff at a

Ex $f(x,y)=x^{\frac{1}{3}}y^{\frac{1}{3}}$ at $(0,0)$

$\frac{\partial f}{\partial x}=\frac{1}{3}y^{\frac{1}{3}}x^{-\frac{2}{3}}=\frac{1}{3}\sqrt[3]{y}\,\frac{1}{\sqrt[3]{x^2}}$

5. Higher Order Derivative

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=\sum_{n=0}^{\infty}\frac{1}{n!}\left[f^{(n)}(a)(x-a)^n\right]$$

$$=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots \qquad \frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}=1-\frac{x^2}{2!}+\frac{x^4}{4!}$$

$$(1+x)^a=\sum_{n=0}^{\infty}\binom{a}{n}x^n=1+\frac{a}{1!}x+\frac{a(a-1)}{2!}x^2+\frac{a(a-1)(a-2)}{3!}x^3$$

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

$$\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

$|x|<1$

$$F(x,y)=f(a,b)$$
$$+\frac{1}{1!}\left[(x-a)\frac{\partial f}{\partial x}(a,b)+(y-b)\frac{\partial f}{\partial y}(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b)+2(x-a)(y-b)\frac{\partial^2 f}{\partial x\partial y}(a,b)+(y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right]$$

at (a,b)

$$F(x,y)=f(a,b)+\frac{1}{1!}\left[(x-a)\frac{\partial f}{\partial x}(a,b)+(y-b)\frac{\partial f}{\partial y}(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b)+2(x-a)(y-b)\frac{\partial^2 f}{\partial x\partial y}(a,b)+(y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right]$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

$$(1+x)^a=\sum_{n=0}^{\infty}\binom{a}{n}x^n$$

$$\ln(1+x)=\sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

$$\tan^{-1}(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

Midterm Test

① find tangent line of $x=e^{-t}\cos t \quad y=e^{-t}\sin t \quad z=e^{-t}$ at $(1,0,1)$

then $\begin{cases} e^{-t}\cos t=1 \\ e^{-t}\sin t=0 \\ e^{-t}=1 \end{cases} \Rightarrow t=0$ then $\nabla f(x,y,z)=(-e^{-t}\cos t-e^{-t}\sin t,\ -e^{-t}\sin t+e^{-t}\cos t,\ -e^{-t})_{t=0}$

$=(-1\cdot1-0,\ 0+1\cdot1,\ -1)=(-1,1,-1)$

$\therefore \begin{pmatrix} x\\y\\z \end{pmatrix}=\begin{pmatrix} 1\\0\\1 \end{pmatrix}+t\begin{pmatrix} -1\\1\\-1 \end{pmatrix}$ and $\Rightarrow \begin{cases} x=1-t \\ y=t \\ z=1-t \end{cases}$

② $\lim\limits_{(x,y)\to(0,0)} \frac{y\sin x}{x^2+y^2}$

restric $y=0 \Rightarrow \frac{0\sin x}{x^2+0}=\frac{0}{x^2}=0$

restric $y=x \Rightarrow \frac{x\sin x}{2x^2}=\frac12\frac{\sin x}{x}=\frac12\cdot1=\frac12$

$\therefore$ DNE

③ $\lim\limits_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2+xy}$ $\left(\begin{array}{l} x^3+y^3=(x+y)(x^2-xy+y^2) \\ x^3-y^3=(x-y)(x^2+xy+y^2) \end{array} \right)$

$=\lim\limits_{(x,y)\to(0,0)} \frac{(x-y)(x^2+xy+y^2)}{x^2+y^2+xy}=x-y=0-0=0$

④ δ-ε proof of $\lim\limits_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^4}=0$

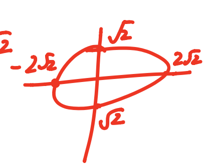
WTS: $\forall\varepsilon>0, \exists\delta>0$ s.t. $\|(x,y)-(0,0)\|<\delta \Rightarrow \left|\frac{xy^3}{x^2+y^4}\right|<\varepsilon$

$\|(x,y)-(0,0)\|<\delta$ $\quad \left|\frac{xy^3}{x^2+y^4}\right| < |y|\left|\frac12\right| < \frac12\sqrt{x^2+y^2} < \frac12\delta$

$\sqrt{x^2+y^2}<\delta$

⑤ equation of ellipse 

$x^2+4y^2=1 \Rightarrow \begin{cases} \left(\frac{x}{1}\right)^2=1 \\ \left(\frac{y}{2}\right)^2=1 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=\frac12 \end{cases}$ $\quad x^2+4y^2=8 \Rightarrow \begin{cases} \left(\frac{x}{2\sqrt2}\right)^2=1 \\ \left(\frac{y}{\sqrt2}\right)^2=1 \end{cases} \Rightarrow \begin{cases} x=2\sqrt2 \\ y=\sqrt2 \end{cases}$ $\quad \frac{x}{2\sqrt2}=1$



6 Express $\frac{\partial^2 z}{\partial r^2}$ in Cartesian given $\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta$

$$\frac{\partial^2 z}{\partial r^2}=\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\right)\frac{\partial}{\partial r}$$

$$=\underbrace{\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)}_{1}\cos\theta+\underbrace{\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)}_{2}\sin\theta$$

$$1 = \frac{\partial}{\partial x}\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial}{\partial y}\frac{\partial z}{\partial x}\frac{\partial y}{\partial r}=\frac{\partial^2 z}{\partial x^2}\cos\theta+\frac{\partial^2 z}{\partial xy}\sin\theta$$

$$2 = \frac{\partial}{\partial x}\frac{\partial z}{\partial y}\frac{\partial x}{\partial r}+\frac{\partial}{\partial y}\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial^2 z}{\partial xy}\cos\theta+\frac{\partial^2 z}{\partial xy}\sin\theta$$

$$=\left(\frac{\partial^2 z}{\partial x^2}\cos\theta+\frac{\partial^2 z}{\partial xy}\sin\theta\right)\cos\theta+\left(\frac{\partial^2 z}{\partial xy}\cos\theta+\frac{\partial^2 z}{\partial xy}\sin\theta\right)\sin\theta$$

$$=\frac{\partial^2 z}{\partial x^2}\cos^2\theta+\frac{\partial^2 z}{\partial xy}\sin\theta\cos\theta+\frac{\partial^2 z}{\partial xy}\cos\theta\sin\theta+\frac{\partial^2 z}{\partial y}\sin^2\theta$$

$$=\frac{\partial^2 z}{\partial x^2}\cos^2\theta+\frac{\partial^2 z}{\partial xy}\sin\theta\cos\theta+\frac{\partial^2 z}{\partial xy}\cos\theta\sin\theta+\frac{\partial^2 z}{\partial y}\sin^2\theta$$

$$=\frac{\partial^2 z}{\partial x^2}\cos^2\theta+2\frac{\partial^2 z}{\partial xy}\sin\theta\cos\theta+\frac{\partial^2 z}{\partial xy}\sin^2\theta$$

$$=\frac{\partial^2 z}{\partial x^2}\frac{x^2}{r^2}+2\frac{\partial^2 z}{\partial xy}\frac{xy}{r^2}+\frac{\partial^2 z}{\partial xy}\frac{y^2}{r^2}=\frac{\partial^2 z}{\partial x^2}\frac{x^2}{x^2+y^2}+2\frac{\partial^2 z}{\partial xy}\frac{xy}{x^2+y^2}+\frac{\partial^2 z}{\partial xy}\frac{y^2}{x^2+y^2}$$

$$=\frac{1}{x^2+y^2}\left(x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial xy}+y^2\frac{\partial^2 z}{\partial xy}\right)$$

7 Find equation of plane passes $P(-2,3,-1)$ and contains $x=1-t, y=2+3t, z=-3+2t$

then we have another point which is $(1,2,-3)$

then the $(1,2,-3)-(-2,3,-1)$ is a line on plane $=(3,-1,-2)$

$n=[3,-1,-2]\cdot[-1,3,2]=[-4,4,-8]$

then $-4x+4y-8z=d$ and $-4(-2)+4(3)-8(-1)=8+12+8=28$

$\therefore -4x+4y-8z=28$

6. Extrema of real valued functions

$$f(x,y) = x^2+y^2-xy = [x,y]\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$f(x,y,z) = x^2+2xy+6xz+3y^2+2yz+z^3$$

$$= [x,y,z]\begin{bmatrix} 1 & 2 & 6 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Find critical points : $\frac{\partial f}{\partial x_i} = 0$ or undefined

2nd derivative test for function with two variables

$$\left.\begin{matrix} f_x = 0 \\ f_y = 0 \end{matrix}\right\} \Rightarrow \text{c. pt } (\ ),(\ )\dots$$

$$D(x_0,y_0) = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - \left(f_{xy}(x_0,y_0)\right)^2$$

if $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) > 0 \Rightarrow$ L. min

if $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) < 0 \Rightarrow$ L. max

if $D(x_0,y_0) < 0 \Rightarrow$ saddle point

if $D(x_0,y_0) = 0 \Rightarrow$ inconclusive

2nd derivative test for function with more than two variables

① find c.pts ($\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y} = 0 = \frac{\partial f}{\partial z} = 0$)

② $Hf\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \cdots \\ \cdots & \cdot \end{bmatrix}$

③ $Hf(x_0,y_0,z_0) = \begin{bmatrix} \ddots \end{bmatrix} \Rightarrow \left.\begin{matrix} \lambda_1 = \cdots \\ \lambda_2 = \cdots \\ \lambda_3 = \cdots \end{matrix}\right\}$ all eigenvalue is $+ \Rightarrow$ min

all eigenvalue is $- \Rightarrow$ max

det $\neq 0 \Rightarrow$ saddle

det $= 0 \Rightarrow$ degenerate type

A closed and bounded set in $\mathbb{R}^n$ is said to be compact

Ex. shortest distance from a point (a,b,c) to (x,y,z) = $d=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$

Example 6.8: Find the global maximum and minimum value of $f(x, y) = x^2 + y^2 - 2x + 2y + 5$ on the set $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$. P967 Briggs

Lagrande: in D $x^2+y^2<4$

in ∂D $\underbrace{x^2+y^2}_{g(x,y)}=4$

$$\mathcal{L}(x,y,\lambda)=f(x,y)-\lambda(g(x,y)-4)$$

$$=x^2+y^2-2x+2y+5-\lambda[x^2+y^2-4]$$

[since if $\lambda=1$ we have no solution]

$\mathcal{L}_x=2x-2-2\lambda x=0 \rightarrow 2(1-\lambda)x=2$

$\mathcal{L}_y=2y+2-2\lambda y=0 \rightarrow 2(1-\lambda)y=-2$

$\mathcal{L}_\lambda=x^2+y^2-4=0$

as $\lambda\neq 1$ $\begin{cases} x=\frac{1}{1-\lambda} \\ y=-\frac{1}{1-\lambda}\end{cases} \Rightarrow \begin{cases} x=\sqrt{2} \\ y=-\sqrt{2} \\ \lambda=1-\frac{1}{\sqrt{2}}\end{cases}$ and $\begin{cases} x=-\sqrt{2} \\ y=\sqrt{2} \\ \lambda=1+\frac{1}{\sqrt{2}}\end{cases}$

$\therefore$ C.pts are $(\sqrt{2},-\sqrt{2})$, $(-\sqrt{2},\sqrt{2})$

Procedure:

1. Constructing a new function L: $\mathbb{R}^{n+1} \rightarrow \mathbb{R}$ by $\mathcal{L}(x, \lambda) = f(x) - \lambda (g(x) - C)$.
2. Finding all the critical points of $\mathcal{L}$ about λ and the constrained critical points of f.
3. Evaluating all the constrained critical points of f. the largest is the maximum value of f and the smallest is the minimum value of f.

Example 6.12: Find extrema values of $f(x, y, z) = 3x - y - 3z$ subject to the constraints $\underbrace{x + y - z}_{g_1} = \underbrace{0}_{C_1}$ and $\underbrace{x^2 + 2z^2}_{g_2} = \underbrace{1}_{C_2}$.

set $\mathcal{L}(x,y,z,\lambda_1,\lambda_2)$

$$=f(x,y,z)-\lambda_1(g_1-C_1)-\lambda_2(g_2-C_2)$$

$$\therefore \mathcal{L}=3x-y-3z-\lambda_1(x+y-z)-\lambda_2(x^2+2z^2-1)$$

$\mathcal{L}_x=3-\lambda_1-2x\lambda_2=0$ $\quad 4-2x\lambda_2=0 \Rightarrow x=\frac{2}{\lambda_2}$

$\mathcal{L}_y=-1-\lambda_1=0$ $\quad \Rightarrow \lambda_1=-1$

$\mathcal{L}_z=-3+\lambda_1-4z\lambda_2=0$ $\quad \Rightarrow -4-4z\lambda_2=0 \Rightarrow z=\frac{-1}{\lambda_2}$

$\mathcal{L}_{\lambda_1}=x+y-z=0$

$\mathcal{L}_{\lambda_2}=x^2+2z^2-1=0$

plug x,z into the last equation to obtain

$$\left(\frac{2}{\lambda_2}\right)^2+2\left(\frac{-1}{\lambda_2}\right)^2-1=0$$

$$\frac{6}{\lambda_2^2}-1=0 \Rightarrow \lambda_2=\pm\sqrt{6}$$

$$\therefore x=\pm\frac{2}{\sqrt{6}} \quad z=\mp\frac{1}{\sqrt{6}}$$

then $x+y-z=0 \Rightarrow y=z-x=\mp\frac{3}{\sqrt{6}}$

so two c.pts $\left(\frac{2}{\sqrt{6}},-\frac{3}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$ & $\left(-\frac{2}{\sqrt{6}},\frac{3}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$

$$f\left(\frac{2}{\sqrt{6}},-\frac{3}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)=3\cdot\frac{2}{\sqrt{6}}+\frac{3}{\sqrt{6}}+\frac{3}{\sqrt{6}}=2\sqrt{6}\text{ . max}$$

$$f\left(-\frac{2}{\sqrt{6}},\frac{3}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)=-2\sqrt{2}\text{ min}$$

7. Multiple Integrals

y-simple: $\int_b^a \int_{y=\cdots}^{y=\cdots} \bigcirc \, dy\,dx$

x-simple: $\int_b^a \int_{x=\cdots}^{x=\cdots} \bigcirc \, dx\,dy$

8. The Change of Variables in Multiple Integrals

change of variables in double integral

$$\iint_D dA = \iint_{D^*} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA^*$$

$$\iint_D f(x,y)\,dx\,dy = \int_b^a \int_{h_1(\theta)}^{h_2(\theta)} f(x(r,\theta), y(r,\theta))\,|r|\,dr\,d\theta$$

change of variables in triple integral

$$\iiint_D dA = \iiint_{D^*} \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dA^*$$

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$$

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$