流体力学笔记

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Chapter 1

Introduction

1.1 描述流体

密度ρ, 压力p, 温度T, 速度$\vec{u}$

1.2 稳态和非稳态流体

$$\frac{\partial}{\partial t} = 0$$

1.3 二维流动

two components (plane flow)

two coordinates (e.g. : $\partial / \partial \phi = 0$)

1.4 流线

At a **fixed** time

$$\frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z} = const$$

1.5 物质导数和迁移 material derivative and advection

$$
\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} \\
&= \frac{\partial f}{\partial t} + u_x\frac{\partial f}{\partial x} + u_y\frac{\partial f}{\partial y} + u_z\frac{\partial f}{\partial z} \\
&= \frac{\partial f}{\partial t} + (\vec{u}\cdot\vec{\nabla})f
\end{aligned}
\tag{1.1}
$$

其中：

$\frac{df}{dt}$ is material derivative

$\frac{\partial f}{\partial t}$ is local rate

$(\vec{u}\cdot\vec{\nabla})f$ is advection rate

1.6 拉格朗日描述和欧拉描述

If f is $\vec{u}$:

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = \frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}$$

左边为拉格朗日描述，右边为欧拉描述

1.7 不可压缩流体（incompressible fluid）

$$\oint \vec{u}\cdot\vec{n}\ ds = 0$$

微分形式

$$\vec{\nabla}\cdot\vec{u} = 0$$

1.8 mass conservation 质量守恒

对于一个流体微团, 其质量变化率为:

$$\frac{\partial}{\partial t}\int \rho dV = -\oint \rho\ \vec{u}\cdot\vec{n}\ ds = -\int \nabla\cdot(\rho\vec{u})dV$$

右边为流体微团质量的变化率; 右边为流体微团的面积微元的法向上, 物质流出/流入速率, 变成微分形式：

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

等式左边第二项展开：

$$\begin{aligned} \frac{\partial \rho}{\partial t} + (\vec{u} \cdot \vec{\nabla})\rho + \rho(\vec{\nabla} \cdot \vec{u}) = 0 \\ \frac{d\rho}{dt} + \rho(\vec{\nabla} \cdot \vec{u}) = 0 \end{aligned} \tag{1.2}$$

换个角度看质量守恒(连续方程), 在拉格朗日视角, 一个流体微元的体积为$\delta\tau = \delta x \delta y \delta z$, 质量为$\delta m = \rho\delta\tau$; 在运动过程中, 该流体微元质量总是守恒的.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta m) = \frac{\mathrm{d}}{\mathrm{d}t}(\rho\delta\tau) = 0$$

展开, 得到:

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{1}{\delta\tau}\frac{\mathrm{d}}{\mathrm{d}t}(\delta\tau) = 0$$

其中, $\frac{1}{\delta\tau}\frac{\mathrm{d}}{\mathrm{d}t}(\delta\tau)$ 为流体膨胀速率与原来体力比, 表征流体的体膨胀速度., 写开变为:

$$\frac{1}{\delta\tau}\frac{\mathrm{d}}{\mathrm{d}t}(\delta\tau) = \nabla \cdot u$$

整理得到连续方程. 对于不可压缩流体：

$$\begin{aligned} \rho(\vec{\nabla} \cdot \vec{u}) = 0 \\ \frac{d\rho}{dt} = 0 \end{aligned} \tag{1.3}$$

1.9 Vorticity 涡度/旋度

$$\vec{\omega} = \vec{\nabla} \times \vec{u}$$

由斯托克斯公式$\iint_s \nabla \times f \cdot n\mathrm{d}s = \oint_l f \cdot dl$对速度环量和涡度积分：

$$\oint \vec{u} d\vec{l} = \int \vec{\nabla} \times \vec{u}\ d\vec{s} = \int \vec{\omega}\ d\vec{s}$$

Ω为流体运动的角速度，积分得到：

$$\Omega R \cdot 2\pi R = \omega \cdot \pi R^2$$

即：

$$\omega = 2\Omega$$

涡度是局地旋转角速度的两倍。

1.10 potential flow 势场

当$\vec{\omega} = 0$：

$$\vec{u} = \vec{\nabla}\psi$$

1.11 算符：梯度，散度和旋度

Chapter 2

Inviscid fluid

2.1 Euler equation

momentum equation：

$$
\begin{aligned}
\frac{\partial}{\partial t}\int \rho u_i dV &= -\oint \rho u_i(\boldsymbol{u}\cdot\boldsymbol{n})ds - \oint p n_i d\boldsymbol{s} + \int \rho f_i dV \\
&= -\int \boldsymbol{\nabla}(\rho u_i \boldsymbol{u})dV - \int \frac{\partial p}{\partial x_i}dV + \int \rho f_i dV
\end{aligned}
\tag{2.1}
$$

这个式子运用了雷诺输运方程. 左边的式子是一个流体微团在u_i方向上的动量随着时间的变化率。计算可知左边第一项为动量的量纲。右边第一项为流体的流出率，表征流体微团因为流进流出导致的动量变化。$(u\cdot n)ds$表示在一个很小的面元ds上的流体速度u在其法向上的大小，表征流体微团在方向n的扩散速度，可以表征流体的动量守恒.

有：

$$
\frac{\partial}{\partial t}(\rho u_i) + \boldsymbol{\nabla}(\rho u_i \boldsymbol{u}) = -\frac{\partial p}{\partial x_i} + \rho f_i
$$

其中：

$$
\begin{aligned}
LHS &= \rho\frac{\partial u_i}{\partial t} + u_i\frac{\partial \rho}{\partial t} + \rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})u_i + u_i\boldsymbol{\nabla}(\rho\boldsymbol{u}) \\
&= \rho\left(\frac{\partial u_i}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})u_i\right) \\
&= \rho\frac{du_i}{dt}
\end{aligned}
$$

对所有分量：

$$\frac{d\boldsymbol{u}}{dt} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \boldsymbol{f}$$

2.2 Close the equation

流体方程的参数：ρ $\vec{u}$ p 一共五个参数。

连续方程：

$$\frac{d\rho}{dt} + \rho(\vec{\nabla}\cdot\vec{u}) = 0$$

欧拉方程：

$$\frac{d\boldsymbol{u}}{dt} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \boldsymbol{f}$$

若$\rho = \mathrm{const}$，则上述方程封闭。若该流体是可压缩的，引入状态方程：

$$p = \rho RT$$

2.3 Bernoulli theorem

steady flow : $\partial u/\partial t = 0, \rho = \mathrm{const}.$

$$\vec{u}\cdot\vec{\nabla}u = -\frac{1}{\rho}\vec{\nabla}p + \vec{g} \tag{2.2}$$

$$\vec{u}\cdot\vec{\nabla}u = \vec{\nabla}(u^2/2) - \vec{u}\times\vec{\omega} \tag{2.3}$$

$$-\nabla(p/\rho + u^2/2 + gz) + u\times\omega = 0 \tag{2.4}$$

记$H = p/\rho + u^2/2 + gz$:

$$-\nabla H + u\times\omega = 0$$

两边同时乘u :

$$u\cdot\nabla H = 0 \partial H/\partial s = 0$$

因此

$$H = \frac{u^2}{2} + \frac{p}{\rho} + gz = const$$

2.4 Vorticity equation

$$\frac{\partial u}{\partial t} = -\nabla H + u \times \omega$$

$$\frac{\partial \omega}{\partial t} = \nabla \times (u \times \omega) = (\omega \cdot \nabla)u - (u \cdot \nabla)\omega \frac{d\omega}{dt} = (\omega \cdot \nabla)u$$

涡度形变：

$$(\omega \cdot \nabla)u \simeq \omega_z \frac{\partial u_z}{\partial z}$$

2.5 Kelvin theorems

$$\Gamma = \oint u \cdot dl = \int \omega \cdot ds$$

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\int_{s(t)} \omega \cdot ds = \int_{s(t)} \left[\frac{\partial \omega}{\partial t} - \nabla \times (u \times \omega)\right] ds = 0$$

$$\frac{d}{dt}(d\vec{l}) = \vec{u}(\vec{x} + d\vec{l}) - \vec{u}(\vec{x}) = (d\vec{l} \cdot \nabla)\vec{u}(\vec{x})$$

$$\frac{d\omega}{dt} = (\omega \cdot \nabla)u$$

如果ω方向与l方向平行, 那之后将会一直平行. 开尔文定理：

$$\frac{d\Gamma}{dt} = 0$$

也就是说涡度强度不随时间变化, 或者说无外力下无旋流体不可能获得涡度.

Chapter 3

Viscous fluid

3.1 Velocity decompostion

$$d\vec{u} = (d\vec{r} \cdot \vec{\nabla})\vec{u}$$

$$\begin{aligned} du_i &= dx\frac{\partial u_i}{\partial x} + dy\frac{\partial u_i}{\partial y} + dz\frac{\partial u_i}{\partial z} && (3.1)\\ &= \sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j} dx_j && (3.2)\\ &= \frac{\partial u_i}{\partial x_j} dx_j && (3.3) \end{aligned}$$

Einstein summation 爱因斯坦求和约定：相同指标求和

一个张量可以写成对称部分和反称部分

$$\begin{aligned} du_i &= \frac{\partial u_i}{\partial x_j} dx_j && (3.4)\\ &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) dx_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) dx_j && (3.5) \end{aligned}$$

其中对称部分和反称部分记为：

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

e_{ij} : strain tensor, symmetric about i, j. (对称的)

ξ_{ij} : rotation tensor, anti-symmetric about i, j. (反称的) 我们发现这刚好和涡度对应上.

因此：

$$du_i = e_{ij}dx_j + \xi_{ij}dx_j$$

对法向形变求和：

$$\begin{aligned} e_{ii} &= e_{xx} + e_{yy} + e_{zz} \\ &= \frac{1}{2}(\frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x}) + \cdots \\ &= \vec{\nabla} \cdot \vec{u} \end{aligned}$$

形变张量的主对角元是速度散度。散度实际上就是面元的膨胀率. 除了法向形变, 还有剪切形变. 公式[3.1]中当$i \neq j$ 时为剪切形变. 表征度量流体畸变程度的物理量.

ξ_{ij} 表征旋转量，是反称的。

$$\xi_{ij} = \frac{1}{2}\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

3.2 Constitutive relation

σ_{ij} : stress tensor; 从无粘流体到粘性流体:

$$\oint -p\vec{n} \cdot ds \longrightarrow \oint \sigma \cdot \hat{n}\ ds$$

而σ_{ij}分解为：

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

应力张量由压力和剪切组成。对于无粘流体，剪切不起作用：

$$(\sigma \cdot \hat{n})_i = -p\delta_{ij}n_j = -pn_i$$

对于粘性流体:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla\tau + f$$

其中, 广义牛顿粘性假设告诉我们 :

$$\tau_{ij} = 2\mu e_{ij} + \mu'(\nabla \cdot u)\delta_{ij}\mu' = -\frac{2}{3}\mu$$

3.3 Navier-Stokes Equation

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p + \nu\nabla^2 u + (\nu + \nu')\nabla(\nabla \cdot u) + f$$

其中, $\nu = \frac{\mu}{\rho}$为运动学粘性系数, 不可压缩流体：

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p + \nu\nabla^2 u + f$$

对于广义牛顿粘性假设的流体:

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{u} + \frac{1}{3}\nu\nabla(\nabla \cdot \vec{u}) + \vec{f}$$

3.4 Boundary Condition

1. fixed boundary 对于固定边界, 法向速度分量为：$\vec{u} \cdot \hat{n}$；切向速度分量为：$\vec{u} - \vec{u} \cdot \hat{n}$.

对于无粘流体：

$$\vec{u} \cdot \hat{n} = 0$$

对于粘性流体，流体固定于边界时：

$$\vec{u} = 0 \quad and \quad \begin{cases} \vec{u} \cdot \hat{n} = 0 \\ \vec{u} - \vec{u} \cdot \hat{n} = 0 \end{cases}$$

流体对边界没有应力时：

$$\begin{cases} \vec{u} \cdot \hat{n} = 0 \\ \frac{\partial}{\partial n}(\vec{u} - \vec{u} \cdot \hat{n}) = 0 \end{cases} \tag{3.6}$$

2. free boundary 运动学边界：

$$u_z = \frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + u_x \frac{\partial \eta}{\partial x} + u_y \frac{\partial \eta}{\partial y}$$

动力学边界：

$$p_- = p_+$$

即在边界上得压力相等。

3.5 Normalisation

无量纲化方程

对于无外力的不可压缩的N-S方程：

$$\frac{\partial \vec{u}}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u$$

距离尺度：l_0；时间尺度t_0；速度尺度：l_0/t_0; 压力尺度：$(l_0/t_0)^2\rho$. 把尺度因子代入[3.5]:

$$\frac{l_0/t_0}{t_0} \frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{(l_0/t_0)^2}{l_0} \tilde{u} \cdot \tilde{\nabla} \tilde{u} = -\frac{1}{\rho} \frac{(l_0/t_0)^2 \rho}{l_0} \tilde{\nabla} \tilde{p} + \quad \nu \frac{l_0/t_0}{\ell_0^2} \tilde{\nabla}^2 \tilde{u}$$

整理得：

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u} \cdot \nabla \tilde{u} = -\tilde{\nabla} \tilde{p} + \frac{\nu t_0}{\ell_0^2} \tilde{\nabla}^2 \tilde{u}$$

其中：

$$\frac{\nu t_0}{l_0^2} = \frac{\nu}{l_0 u_0} = \frac{1}{R_e}$$

R_e称为雷诺数。[3.5]公式则化简为：

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u} \cdot \nabla \tilde{u} = -\tilde{\nabla} \tilde{p} + \frac{1}{R_e} \tilde{\nabla}^2 \tilde{u}$$

从这个方程我们可以这样看：

$$R_e \simeq \frac{|u \cdot \nabla u|}{|\nu \nabla^2 u|} \simeq \frac{u_0^2 / l_0}{\nu u_0 / l_0^2} = \frac{l_0 u_0}{\nu}$$

雷诺数实际上表征惯性力和粘滞力之比.

3.6 Kinetic Energy Equation

u点乘不可压缩流体的N-S方程, 并对该流体微元体积分：

$$\frac{\mathrm{d}}{\mathrm{dt}} \int \frac{u^2}{2} \mathrm{d}V = -\int u \cdot (u \cdot \nabla u) dV - \frac{1}{\rho} \int u \cdot \nabla p \mathrm{d}V + \int u \cdot (\nabla \cdot \tau) \mathrm{d}V$$

由于：

$$\begin{aligned}
\int u \cdot (u \cdot \nabla u) dV &= \int u_i u_j \frac{\partial u_i}{\partial x_j} \mathrm{d}V \\
&= \int u_j \frac{\partial}{\partial x_j} (\frac{u^2}{2}) \mathrm{d}V \\
&= \int \frac{\partial}{\partial x_j} (u_j u^2 / 2) \mathrm{d}V \\
&= \oint \frac{u^2}{2} u_j n_j \mathrm{dS} \\
&= 0
\end{aligned} \tag{3.7}$$

因此：

$$\frac{\mathrm{d}}{\mathrm{dt}} \int \frac{u^2}{2} \mathrm{d}V = -2\nu \int e_{ij} e_{ij} \mathrm{dV}$$

能量耗散与粘性有关。

3.7 Vorticity Equation

对不可压缩的N-S方程[3.3] , 两边取速度旋度：

$$\frac{\partial \omega}{\partial t} = \nabla \times (u \times \omega) + \nu \nabla^2 \omega$$

因为由粘性项，与Kelvin定理[2.5]不一致，涡线重联。

3.8 Microscopic Physics

有概率密度函数$f(r, v, t)$

$$N = \int f \mathrm{d}^3 r \mathrm{d}^3 v$$

玻尔兹曼分布律:

$$\frac{\partial f}{\partial t} + v \cdot \nabla f + a \cdot \nabla_v f = C$$

其中

$$\rho(r, t) = \int m f \mathrm{d}^3 v$$

$$\rho < v >= \int m f v \mathrm{d}^3 v$$

公式[3.8]对时间求偏导, 得到质量守恒. 能量守恒和动量守恒也是这个方式.

Chapter 4

Applications

4.1 Pressure Driven Flow

plan posieuille flow

对于稳态流体$du/dt = 0$, 只受到压力，两边平板固定，$p_1 > p_2$

边界条件:

$$u_x = u; u_y = u_z = 0$$

考虑不可压流体：

$$\nabla \cdot u = 0 \longrightarrow \quad \frac{\partial u}{\partial x} = 0 \longrightarrow \quad u(y, z, t) = u(y)$$

流体速度与x, z方向无关, 只与y有关. 得到：

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} = 0 - \frac{1}{\rho}\frac{\partial p}{\partial y} = 0 - \frac{1}{\rho}\frac{\partial p}{\partial z} = 0$$

得到：

$$\nu\frac{d^2 u}{dy^2} = \frac{1}{\rho}\frac{dp}{dx}$$

设两平板之间的高度差为h, 解得：

$$u = -\frac{h^2}{2\mu}\frac{dp}{dx}\frac{y}{h}(1-\frac{y}{h})$$

容易发现速度最大值在$y = h/2$上 .

4.2 Plane Couette Flow

Boundary driven flow

边界条件：

$$u(h) = U u(0) = 0$$

由于两边压强为0，即$dp/dx = 0$, 代入N-S方程：

$$\mu\frac{d^2u}{dy^2} = \frac{dp}{dx}\mu\frac{d^2u}{dy^2} = 0$$

解得：

$$u = U\frac{y}{h}$$

当两边加上压力时：

$$u = U\left[\frac{y}{h} - \frac{h^2}{2\mu}\frac{dp}{dx}\frac{y}{h}(1-\frac{y}{h})\right]$$

4.3 Taylor-Couette Flow

在一个稳态不可压缩的流体在旋转的圆柱上，边界条件为：

$$u_R = 0, \quad u_z = 0, \quad u_\theta = u(R), \quad p = p(R)$$

在柱坐标上，压力梯度与离心势平衡：

$$\frac{\mathrm{d}p}{\mathrm{d}R} = \rho\frac{u^2}{R}$$

代入柱坐标下的拉普拉斯算子$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2} + \frac{\partial}{\partial z^2}$：

$$\frac{1}{R}\frac{d}{dR}(r\frac{du}{dR})-\frac{u}{R^2}=0$$

假设$u\propto R^{\alpha}$, 当$\alpha=\pm1$时，

$$u=c_1R+\frac{c_2}{R}$$

$$\Omega=c_1+\frac{c_2}{R^2}$$

对于Kepler运动：$\Omega\propto R^{-1.5}$

4.4 Rotating Flow

对于旋转流动，在随动坐标中展开：

$$\frac{du}{dt}\longrightarrow\frac{du}{dt}+\Omega\times(\Omega\times r)+2\Omega\times u$$

其中右边第二项为离心项，第三项为科里奥利力项

$$\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\vec{\nabla}\vec{u}=-\frac{1}{\rho}\vec{\nabla}p+\vec{g}-\vec{\Omega}\times(\vec{\Omega}\times\vec{r})+2\vec{u}\times\vec{\Omega}$$

其中离心势写为：

$$\vec{\Omega}\times(\vec{\Omega}\times\vec{r})=\nabla(\frac{1}{2}|\vec{\Omega}\times\vec{r}|^2)$$

方程则变为：

$$\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\vec{\nabla}\vec{u}=-\frac{1}{\rho}\vec{\nabla}p+\vec{g}-\nabla(\frac{1}{2}|\vec{\Omega}\times\vec{r}|^2)+2\vec{u}\times\vec{\Omega}$$

对比左边的第二项和右边的最后一项

$$R_0=\frac{|\vec{u}\cdot\vec{\nabla}\vec{u}|}{|2\vec{u}\times\vec{\Omega}|}\simeq\frac{u_0^2/l_0}{2\Omega u_0}=\frac{u_0}{2\Omega l_0}R_0=\frac{u_0}{2\Omega l_0}$$

也就是惯性力项/科氏力项，这个数称为Rossby数. 在地球上，假设空气是不可压缩流体, 有势函数：

$$-\frac{1}{\rho}\vec{\nabla}p + \vec{g} - \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = \vec{\nabla}\psi$$

认为地球大气是准静态的，地球上的$R_0 << 1$，有$du/dt = 0$, 则有准地转平衡:

$$\nabla\psi + 2u \times \Omega = 0$$

由于势函数是无旋的，我们对方程[4.4]取旋度：

$$\nabla \times (2\vec{u} \times \vec{\Omega}) = 0$$

即：

$$2\vec{\Omega} \cdot \nabla u = 0\frac{\partial \vec{u}}{\partial l} = 0$$

即速度沿着$\vec{\Omega}$方向(旋转轴方向)不变。

4.5 Accretion Disk

吸积盘结构：黑洞吸积盘, AGN, 原行星盘, 行星环, 双星系统盘的洛希极限, X射线源

1. 估计重力势能

$$E_p = \frac{GMm}{a}/(mc^2) = \frac{GM}{ac^2}$$

2. 薄盘近似 在柱坐标中: (R, ϕ, z), 薄盘即为$h \ll R_d$, 因此物理量为:($\rho, p, T, v_R, v_\phi, v_z$) . 其中:

$$v_z \simeq 0 v_\phi \gg v_R \frac{\partial}{\partial \phi} = 0$$

也就是说在盘上垂直速度很小, 且以ϕ方向速度为主, 相比起来径向速度非常小. 这里引入引力势函数

$$\Psi = -GM(R^2 + z^2)^{-1/2}$$

在R方向和z方向的引力为:

$$F_R = -\partial\Psi/\partial R = -GM(R^2 + z^2)^{-3/2} \simeq -GMR^{-2}$$

$$F_z = -\partial\Psi/\partial z = -GMz(R^2+z^2)^{-3/2} \simeq -zGMR^{-3}$$

由于是稳态场, $\partial/\partial t = 0$, 在径向和垂直方向的运动方程为:

$$\begin{array}{c} v_R \frac{\partial v_R}{\partial R} - \frac{v_\phi^2}{R} = -\frac{1}{\rho_g}\frac{\partial p}{\partial R} - GM/R^2 \\ 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{GMz}{R^3} \end{array} \tag{4.1}$$

对于理想气体状态方程:

$$p = \frac{\rho_g}{m_H} k_B T$$

密度ρ_g的分布为一个高斯分布:

$$\rho_g \propto \exp\left(-\frac{GMm_H}{2k_BTR^3}z^2\right)$$

等温过程为:

$$-\frac{1}{\rho_g}\frac{k_BT}{m_H}\frac{d\rho_g}{dz} - \frac{GM}{R^3}z = 0$$

其中:

$$\begin{aligned} H_z = \quad & \left(2k_BTR^3/GMm_H\right)^{1/2} \quad \text{scale height} \\ & \rho_g \propto \exp\left(-z^2/H_z^2\right) \end{aligned} \tag{4.2}$$

通过垂直方向的运动方程, **对压力p的估计为**:

$$\frac{1}{\rho_g}\frac{p}{h} \simeq \frac{GMh}{R_d^3} \Rightarrow p \simeq \frac{GM\rho_g h^2}{R_d^3}$$

有了上面的公式, 我们对上面的公式进行量级估计:

$$v_R \ll v_\varphi; \quad v_R\frac{\partial v_R}{\partial R} \ll v_\varphi^2/R$$

得到:

$$\left(\frac{1}{\rho_g}\frac{p}{R_d}\right) / \left(\frac{GM}{R_d^2}\right) \simeq \left(\frac{1}{\rho_g}\frac{1}{R_d}\frac{GM\left(gh^2\right)}{R_d^3}\right) / \left(\frac{GM}{R_d^2}\right) = h^2/R_d^2 \ll 1$$

因此在薄盘中, 径向压力梯度远远小于重力, 压力梯度不是主要的, 但在厚盘中需要考虑.

离心项写为:

$$\frac{v_\varphi^2}{R} \simeq \frac{GM}{R^2} \quad \longrightarrow \quad v_\varphi = (GM/R)^{1/2}$$

在开普勒盘中, 角速度为:

$$\Omega = v_\varphi / R = (GM/R^3)^{1/2}$$

考虑角动量. 吸积盘面密度为

$$\sum = \int \rho dz$$

连续方程写为：

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$$

稳态, 在柱坐标中, r方向的连续方程为：

$$\frac{1}{R}\frac{\partial}{\partial R}(R\rho v_R) = 0$$

因此:

$$\int \frac{1}{R}\frac{\partial}{\partial R}(R\rho v_R) dz = 0$$

由于密度取决于z, 但是速度不取决于z, 改写为面密度:

$$\frac{d}{dR}(R\Sigma v_R) = 0$$

吸积率可以表示为单位面密度成径向速度的和, 因此吸积率为:

$$\dot{m} = -2\pi R \Sigma v_R$$

因此:

$$R\Sigma v_R = -\dot{m}/2\pi$$

计算角动量, 这时候的粘性力是重要的. 单位面积上的角动量为：

$$J = \Sigma R^2 \Omega$$

单位面积的粘滞力矩为：

$$\frac{\partial}{\partial t}\left(\sum R^2\Omega\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(R\cdot\sum R^2\Omega\cdot V_R\right) = G \qquad \text{(viscous torgue per unit area)}$$

力矩等于$\partial L/\partial t$; 第二项为散度算子在柱坐标中的展开. 其中角动量守恒:

$$\frac{\partial}{\partial t}\left(\sum R^2\Omega\right) = 0$$

因此力矩为：

$$\begin{gathered} G\cdot 2\pi R\cdot dR = d\Gamma \quad (\Gamma:\ \text{viscous torgue}) \\ G = \tfrac{1}{2\pi R}\tfrac{d\Gamma}{dR} \end{gathered} \tag{4.3}$$

有速度差就有速度剪切. 因此:

$$\frac{dv_\varphi}{dR} = \frac{d}{dR}(\Omega R) = \Omega + R\frac{d\Omega}{dR}$$

其中$\frac{d\Omega}{dR}$就是剪切. 粘滞力可以写为:

$$\text{viscous force} = \mu R\frac{d\Omega}{dR}$$

因此粘滞力矩可以写为:

$$\begin{aligned} \Gamma(R) &= \iint\left(\mu R\frac{d\Omega}{dR}\right)\cdot(Rd\varphi dz)\cdot R \\ &= R^3\frac{d\Omega}{dR}\cdot 2\pi\cdot\int\rho\nu dz \\ &= 2\pi\nu\sum R^3\frac{d\Omega}{dR} \end{aligned} \tag{4.4}$$

因此力矩可以写为：

$$G(R)=\frac{1}{R}\frac{d}{dR}\left(\nu\sum R^3 d\Omega/dR\right)$$

$$\frac{1}{R}\frac{d}{dR}\left(\sum R^3\Omega V_R\right)=\frac{1}{R}\frac{d}{dR}\left(\nu\Sigma R^3\frac{d\Omega}{dR}\right)$$

$$\sum R^3\Omega v_R=\nu\sum R^3\frac{d\Omega}{dR}+C$$

当没有剪切的时候, $\frac{d\Omega}{dR}=0$, 确定系数C为:

$$C=\sum R_*^3\Omega_* v_{R_*}=-\dot{m}/2\pi\cdot R_*^2\Omega_*$$

代回去, 得到:

$$-\dot{m}/2\pi\cdot R^2\Omega=\nu\sum R^3\frac{d\Omega}{dR}-\dot{m}/2\pi\cdot R_*^2\Omega_*$$

$$\Omega=\left(GM/R^3\right)^{1/2}$$

解得:

$$\nu\Sigma=\frac{\dot{m}}{3\pi}\left(1-\left(\frac{R_*}{R}\right)^{1/2}\right)$$

因此**粘滞系数正比于吸积率**. 接下来估计能量和角动量的变化率.

能量变化率为:

$$\begin{aligned}\frac{dE}{dt}&=-\int_{\text{(per unit area)}}\mu(Rd\Omega/dR)^2dz\\&=-\nu\sum R^2(d\Omega/dR)^2\\&=-\frac{3GM\dot{m}}{4\pi R^3}\left(1-\left(\frac{R_*}{R}\right)^{1/2}\right)\end{aligned}\tag{4.5}$$

角动量为:

$$\begin{aligned}L&=\int_{R_*}^{\infty}-\frac{dE}{dt}2\pi RdR\\&=GM\dot{m}/2R_*\\&=\text{ half of }\frac{GM}{R_*}\dot{m}\end{aligned}\tag{4.6}$$

从上面的分析我们知道

$$L \longrightarrow \dot{m} \longrightarrow \nu$$

因此吸积率越大, 粘滞力越大, 角动量越大, 盘越不稳定.

3. 薄盘演化 综合上面的式子, 得到diffusion equation :

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left(R^{1/2}\frac{\partial}{\partial R}\left(\nu \Sigma R^{1/2}\right)\right)$$

$$\sum \propto \exp\left(-\frac{R^2}{12\nu t}\right)$$

这就是不稳定薄盘的演化方程.

4.6 Stellar Structure

compressible fluid and spherical symmetric

$$-\vec{\nabla} p + \rho \vec{g} = 0 - \frac{dp}{dr} - \rho g = 0$$

重力g表示为：

$$g = \frac{G}{r^2}\int_0^r 4\pi r'^2 \rho dr'$$

也就是说对球壳dr'积分，得到重力g.

在球坐标中表示：

$$\frac{dp}{dr} + \rho \frac{G}{r^2}\int_0^r 4\pi r'^2 dr' = 0$$

对r求导：

$$\frac{d}{dr}\left(\frac{dp}{dr}r^2\frac{1}{\rho}\right) + 4\pi G\rho r^2 = 0$$

整理得：

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{1}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho$$

another approach

$$-\vec{\nabla}p + \rho\vec{g} = 0 \quad , \quad \vec{g} = -\vec{\nabla}\Phi$$

对两边求微分，有：

$$\vec{\nabla}(\frac{1}{\rho}\vec{\nabla}p) = \vec{\nabla}\vec{g} = -\vec{\nabla}^2\Phi$$

引力势能与引力的关系：

$$\oint \vec{g}d\vec{s} = \int -\nabla^2\phi dV$$

积分得到：

$$\frac{G\rho V}{r^2}4\pi r^2 = \nabla^2\phi V$$

得到：

$$\nabla^2\phi = 4\pi G\rho$$

polytropic gas:

多方关系：

$$p = \kappa\rho^\gamma = \kappa\rho^{(1+1/n)}$$

其中n为多方指数。对于每单位体积的能量：

$$u = np$$

对于恒星对流区，以H原子为主，因此：

$$\gamma = 5/3; \quad n = 1.5$$

对于辐射区以光子为主：

$$\gamma = 4/3; \quad n = 3$$

边界条件为：isothermal边界

$$p \propto \rho; \quad n \to \infty$$

incompressible边界

$$p \propto \rho, \quad n \to 0$$

方程为：

$$K\left(1+\frac{1}{n}\right)\frac{1}{r^2}\frac{d}{dr}\left[r^2\rho^{(-1+1/n)}\frac{d\rho}{dr}\right] + 4\pi G\rho = 0$$

设：

$$a = \frac{4\pi G}{K(1+n)}\rho_c^{1-1/n} z = aru = (\rho/\rho_c)^{1/n}$$

得到：

$$\frac{1}{z^2}\frac{d}{dz}(z^2\frac{du}{dz}) + u^n = 0$$

4.7 Boundary Layer

边界层是把靠近边界的地区考虑粘滞力,

$$\begin{aligned}
&\partial u_x/\partial x + \partial u_y/\partial y = 0 \\
&u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right) \\
&u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 U_y}{\partial y^2}\right)
\end{aligned} \tag{4.7}$$

设边界层为δ, 平板长度为l, 化简得到:

$$\delta \simeq \left(\frac{\nu l}{u_x}\right)^{1/2} \simeq Re^{-1/2} l$$

当雷诺数越大时, 边界层越小. 同时也可以知道:

$$Re^{-1/2} \ll 1$$

对u_y方向也同理. 最后化简得到的方程为：

$$\begin{cases} \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \\ u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u_x}{\partial y^2} \\ 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{cases}$$

Prandtl's equation

Chapter 5

Gas Dynamics

5.1 thermodynamics

energy per unit volume:

$$u = np$$

pdytropic index:

$$\gamma = 1 + \frac{1}{n}$$

energy per unit mass

$$\varepsilon = u/p = c_V T$$

c_V: specific heat per unit mass

$$p = N k_B T$$

N: number desity, $\rho = m_p N$

$$p = \frac{\rho}{m_p} k_B T = \rho \frac{k_B N_A}{m_p N_A} T$$

there: $k_B N_A = R = 8.314$; $m_p N_A = \mu$ mass per mole

$$p = \rho \frac{R}{\mu} T$$

$$c_V T = u/\rho = np/\rho = n \frac{R}{\mu} T$$

有：

$$c_V = nR/\mu = \frac{1}{\gamma - 1} \frac{R}{\mu}$$

带入压力方程：

$$p = (\gamma - 1)\rho c_V T$$

the first law of thermodynamics:

$$d\varepsilon = Tds + pd(\frac{1}{\rho})$$

带入p：

$$c_V dT = Tds + (\gamma - 1)c_V T \frac{1}{\rho} d\rho d\ln T = \frac{1}{c_V} ds + (\gamma - 1) d\ln\rho$$

得到:

$$ds = c_V d\ln\frac{T}{\rho^\gamma} s = c_V \ln\frac{p}{\rho^\gamma} + s_0$$

adiabatic: $ds = 0$

5.2 Sound Wave

由于

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho u) = 0 \frac{\partial u}{\partial t} + u\cdot\nabla = -\frac{1}{\rho}\nabla p$$

微扰：

$$\rho = \rho_0 + \rho_1 \quad p = p_0 + p_1 \quad u = 0 + u$$

代入线性微扰：

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot u_1 = 0 \frac{\partial u}{\partial t} = -\frac{1}{\rho_0} \nabla p_1$$

解得：

$$\frac{\partial^2 \rho_1}{\partial t^2} - \nabla^2 p_1 = 0$$

记$dp/d\rho = C_s^2$, $p_1 = dp/d\rho \times \rho_1$：

$$\frac{\mathrm{d}^2 \rho_1}{\mathrm{d}t^2} - C_s^2 \nabla^2 \rho_1 = 0$$

代入波动解：

$$\rho_1 = \hat{\rho} e^{i(kx - \omega t)}$$

解得：

$$-\omega^2 + C_s^2 k^2 = 0 \quad \rightarrow \quad \omega = \pm C_s k$$

相速度：

$$\vec{c}_p = \frac{\omega}{k} = \pm c_s \hat{k}$$

群速度：

$$\vec{c}_g = \vec{\nabla}_k \omega = \pm c_s \hat{k}$$

将波动解代入[5.2]：

$$-\mathrm{i}\omega \hat{u} = \frac{1}{\rho_0} \mathrm{i} \vec{k} \hat{p}$$

则: $\hat{u} = \hat{p}$，两个方向平行

多方过程：

$$p = \kappa\rho^\gamma$$

得到：

$$c_s = (\gamma\frac{p}{\rho})^{1/2} = (\gamma\frac{R}{\mu}T)^{1/2}$$

当$\gamma = 1$, 即为等温过程时：

$$c_s = (\frac{p}{\rho})^{1/2}$$

5.3 Shock

5.4 Blast Wave

5.5 Galatic Jet

5.6 Sphere Wind and Accretion

Chapter 6

Waves

6.1 water wave

unperturbed:

$$-\nabla p_0 + \rho g = 0$$

浮力定律：

$$p_0 = p_a + \rho g(h - y)$$

微扰欧拉方程：

$$\frac{\partial u_i}{\partial t} + u_i \nabla u_i = -\frac{1}{\rho} \nabla p_1$$

其中$u_i \nabla u_i = 0$ (微扰)，有

$$\frac{\partial u_i}{\partial t} = -\frac{1}{\rho} \nabla p_1$$

取旋：

$$\frac{\partial \omega_1}{\partial t} = 0$$

速度分解：$u_1 = v_1 + v_2$,

$$\nabla \times v_1 = 0 \quad \frac{\partial v_1}{\partial t} \neq 0 \nabla \times v_2 = 0 \quad \frac{\partial v_2}{\partial t} = 0 \ \ (no\ waves)$$

对于上面的公式，v_1旋度为0，有波动行为，有速度势ϕ

$$v_1 = \nabla \phi$$

$$\nabla v_1 = 0$$

$$\nabla^2 \phi = 0$$

kinematic B. C. : $\partial\phi/\partial y = d\xi/dt = \partial\xi/\partial t + \partial\phi/\partial x \cdot \partial\xi/\partial x$

dynamic B. C. : $p_0 + p_1 = p_a$ and $p_a = \rho g \xi$

At $y = h$:

$$\frac{\partial^2 \phi}{\partial t} + g\frac{\partial \phi}{\partial y} = 0$$

波动解的形式为: $\phi = \hat{\phi} \exp i(kx - \omega t)$:

解得

$$\omega^2 = gk \tanh(kh)$$

当为深水波时:$kh \gg 1$, 有:

$$\begin{aligned} &\omega^2 = gk \\ &c_p = \tfrac{\omega}{k} = \sqrt{g/k} \\ &c_g = \tfrac{\partial\omega}{\partial k} = \tfrac{1}{2}\sqrt{g/k} \end{aligned} \tag{6.1}$$

其中, c_p为相速度, c_g为群速度. 记波动速度为c, 一个波动周期为γ, 则有: $\omega = 2\pi/\gamma = kc_p$; 一个周期内有多少波动周期称为波数, 波数为: $k = 2\pi/\lambda$; 波的移动速度又称为相速度, 而当不同频率的波叠加时的波包移动速度即为群速度c_g .

6.2 Internal wave

1. Internal Gravity wave (g mode) 流体密度是线性的 :

$$\rho_0 = \bar{\rho} + \frac{d\rho_0}{dz} z$$

对于线性部分的连续方程：

$$\frac{d\rho_0}{dt} = 0; \quad \nabla \cdot u = 0$$

$$\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho + \rho \cdot \nabla u = 0$$

欧拉方程:

$$\bar{\rho} \frac{du}{dt} = -\nabla p + \rho g$$

对于扰动部分展开，我们注意到$u_1 \cdot \nabla u_1$为二阶微扰, 因此连续方程和欧拉方程可写为:

$$\frac{\partial \rho_1}{\partial t} + u_z \frac{d\rho_0}{dz} = 0, \qquad \nabla \cdot u = 0 \bar{\rho} \frac{\partial u_1}{\partial t} = -\nabla p_1 + \rho_1 g, \quad -\nabla p_0 + \rho_0 g = 0$$

联立得到方程:

$$\frac{\partial^2}{\partial t^2} \nabla^2 u_z^2 + N^2 \nabla_\perp^2 u_z = 0 N^2 = -\frac{g}{\bar{\rho}} \frac{d\rho_0}{dz}$$

其中$\nabla_\perp = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, 设波动解为:

$$u_z = \hat{u}_z \exp i(k \cdot x - \omega t)$$

解得:

$$\omega = \pm N \frac{k_\perp}{k}$$

其中$k^2 = k_x^2 + k_y^2 + k_z^2 = k_\perp^2 + k_z^2$

群速度和相速度为:

$$c_p = \frac{\omega}{k} \hat{k} c_g = \nabla_k \omega c_g \cdot c_p = 0$$

也就是说群速度垂直于相速度.

2. Internal Inertial Wave

6.3 Planetary Wave

0. Shallow Water $h << L$ and $\nabla u = 0$, in other word : $u_z/u_x \approx h/L << 1$

so u_x and u_y is independent of z

对于大气, 海洋: 浅水波, 有$h \ll L$, 对于不可压流体：

$$\nabla \cdot u = 0$$

$$\frac{u_z}{u_x} \simeq \frac{h}{L} \ll 1$$

$$h = h_0 + h_1(x, y, t)$$

假设u_x, u_y与z无关, 非扰动项:

$$p_0 = p_a + \rho g(h_0 - z)$$

扰动项有:

$$p_1 = \rho g h_1$$

$$\frac{\partial p_1}{\partial x} = \rho g \frac{\partial h_1}{\partial x} \frac{\partial p_1}{\partial y} = \rho g \frac{\partial h_1}{\partial y}$$

因此

$$u_z = \frac{dh}{dt} = \frac{\partial h}{\partial t} + u_x \frac{\partial h}{\partial x} + u_y \frac{\partial h}{\partial y}$$

由于与z无关, 有:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = -\frac{\partial u_z}{\partial z}$$

两边对z积分, 得到:

$$u_z = -h\nabla_{//}u$$

$$\frac{dh}{dt} - u_z = 0\frac{dh}{dt} + h\nabla\cdot u_{//} = 0$$

展开, 得到:

$$\frac{\partial h}{\partial t} + \nabla\cdot(hu_{//}) = 0$$

1. Kelvin Wave mean flow : $u_{x_0} = U$

$$\begin{cases} \frac{\partial u_x}{\partial t} + U\frac{\partial u_x}{\partial x} = -\frac{1}{\rho}\frac{\partial p_1}{\partial x} = -g\frac{\partial h_1}{\partial x} \\ \frac{\partial u_y}{\partial t} + U\frac{\partial u_y}{\partial x} = -\frac{1}{\rho}\frac{\partial p_1}{\partial y} = -g\frac{\partial h_1}{\partial y} \\ \frac{\partial h_1}{\partial t} + h_0\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + U\frac{\partial h_1}{\partial x} = 0 \end{cases}$$

有形式解：

$$u_x = \hat{u}_x e^{i(k_x x + k_y y - \omega t)} u_y = \hat{u}_y e^{i(k_x x + k_y y - \omega t)} h_1 = \hat{h} e^{i(k_x k + k_y y - \omega t)}$$

即:

$$A\begin{pmatrix} \hat{u}_x \\ \hat{u}_y \\ \hat{u}_z \end{pmatrix} = 0$$

$$A = \begin{pmatrix} -i\omega + iUk_x & 0 & igk_x \\ 0 & -i\omega + i0k_x & igk_y \\ ih_0k_x & ih_0k_y & -i\omega + ivk_x \end{pmatrix}$$

当该方程组有解时, $\det(A) = 0$

解得:

$$\omega = Uk_x \pm (gh_0(k_x^2 + k_y^2))^{1/2}$$

2. Poincare Wave

$$\vec{\Omega} = \Omega[\cos(\theta\hat{e}_y) + \sin(\theta\hat{e}_z)]$$

and

$$2u \times \Omega = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ 2u_x & 2u_y & 2u_z \\ 0 & \Omega\cos\theta & \Omega\sin\theta \end{vmatrix} = \begin{bmatrix} 2\Omega\sin(\theta u_y) - 2\Omega\cos(\theta u_z) \\ -2\Omega\sin(\theta u_x) \\ 2\Omega\cos(\theta u_x) \end{bmatrix} \tag{6.2}$$

Neglect $\cos\theta$ terms on high latitude region :

$$\begin{aligned} 2u \times \Omega &= 2\Omega\sin\theta_y\hat{e}_x - 2\Omega\sin\theta u_x\hat{e}_y \\ &= f\ u_y\hat{e}_x - f\ u_x\hat{e}_y \end{aligned} \tag{6.3}$$

Corilis number: $f = 2\Omega\sin\theta$

$$\begin{cases} \frac{\partial u_x}{\partial t} = -g\frac{\partial h_1}{\partial x} + fu_y \\ \frac{\partial u_y}{\partial t} = -g\frac{\partial h_1}{\partial x} - fu_x \\ \frac{\partial h_1}{\partial t} + h_0(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}) = 0 \end{cases}$$

we assume the form of the solution :

$$\begin{cases} u_x = \hat{u}_x \exp i(k_x x + k_y y - \omega t) \\ u_y = \hat{u}_y \exp i(k_x x + k_y y - \omega t) \\ h_1 = \ \hat{h}_1 \exp i(k_x x + k_y y - \omega t) \end{cases}$$

带入上述方程组：

$$A = \begin{bmatrix} -i\omega & -f & igk_x \\ f & -i\omega & igk_y \\ ih_0k_x & ih_0k_y & -i\omega \end{bmatrix}$$

即:

$$A \begin{bmatrix} \hat{u}_x \\ \hat{u}_y \\ \hat{h}_1 \end{bmatrix} = 0$$

由于$\det(A) = 0$，有：

$$
\begin{aligned}
&\begin{vmatrix} -i\omega & -f & igk_x \\ f & -i\omega & igk_y \\ ih_0k_x & ih_0k_y & -i\omega \end{vmatrix} \\
&= -i\omega \begin{vmatrix} -i\omega & igk_y \\ ih_0k_y & -i\omega \end{vmatrix} + f \begin{vmatrix} f & igk_y \\ ih_0k_x & -i\omega \end{vmatrix} + igk_x \begin{vmatrix} f & -i\omega \\ ih_0k_x & ih_0k_y \end{vmatrix} \\
&= i\omega^3 - i\omega f^2 - i\omega g h_0 k \\
&= 0
\end{aligned} \tag{6.4}
$$

解得：

$$\omega^2 = f^2 + gh_0k$$

即：

$$\omega = \pm\sqrt{f^2 + gh_0k}$$

3. Rossby Wave 科氏力:

$$f = 2\Omega \sin\theta$$

and

$$Rd\theta = dy$$

β平面近似:

$$f = f_0 + \beta y$$

其中:

$$f_0 = 2\Omega \sin\theta \beta = \partial f/\partial y$$

综合上以上式子,

$$\beta = 2\Omega \cos\theta / R$$

因此

$$\begin{aligned} &\tfrac{\partial U_x}{\partial t} = -g\tfrac{\partial h_1}{\partial x} + f u_y : \partial/\partial y \\ &\omega^3 - \left(g h_0 k^2 + f_0^2\right)\omega - \beta g h_0 k_x = 0 \end{aligned} \tag{6.5}$$

对于运动缓慢的大气长波:

$$\omega \simeq -\frac{\beta k_x}{k^2 + f_0^2/g h_0}$$

x方向的相速度:

$$c_{px} = \frac{\omega}{k}\frac{k_x}{k} < 0$$

因此Rossy wave是向西方向传播的.

Chapter 7

Instabililty

7.1 Gravitational Instability

微扰方程：

$$\begin{cases} \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla u_1 = 0 \\ \rho_0 \frac{\partial u_1}{\partial t} = -\nabla p_1 - \rho_0 \nabla \phi_1 \\ \nabla^2 \phi = 4\pi G \rho_1 \end{cases} \tag{7.1}$$

代入$p_1 = c_s^2 \rho_1$以及$c_s^2 = dp/d\rho$

7.2 Rotating Flow Instability

7.3 Convective Instabililty

Chapter 8

Turbulence

8.1 Concepts from phenomenology

动量方程：

$$\frac{\partial \vec{u}}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho}\nabla p + \nu \nabla^2 u$$

涡度方程：

$$\frac{\partial \vec{\omega}}{\partial t} + u \cdot \nabla \omega = \omega \cdot \nabla u + \nu \nabla^2 \omega$$

其中$\omega \cdot \nabla u$为拉伸项。

8.2 Reynolds Average and Turbulent Viscosity

设：

$$u = \bar{u} + u'$$

动量方程：

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial}{\partial x_j}(\frac{\partial u_i}{\partial x_j})$$

代入微扰假设：

$$\frac{\partial \bar{u}_i}{\partial t}+\bar{u}_j\frac{\partial \bar{u}_i}{\partial x_j}=\overline{-u_j'\frac{\partial u_i'}{\partial x_j}}-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i}+\nu\frac{\partial}{\partial x_j}(\frac{\partial \bar{u}_i}{\partial x_j})$$

8.3 Cascade and Scaling Law

Largest Scale : l ,τ_l, u_l

Any Other Scale : λ , τ_λ , u_λ

各个尺度下能量传输速率相等：

$$\varepsilon \sim u_l^2/\tau_l \sim u_\lambda^2/\tau_\lambda$$

由于$\tau_l \simeq u_l/l$, 代入上式，整理得：

$$u_\lambda/u_l \sim (\lambda/l)^{1/3}\tau_\lambda/\tau_\lambda \sim (\lambda/l)^{2/3}$$

粘滞系数得量纲为：$[m^2s^{-1}]$ ，即$\nu_l \sim u_l \cdot l$ ：

$$\nu_\lambda/\nu_l \sim \frac{u_\lambda \lambda}{u_l l} \sim (\frac{\lambda}{l})^{4/3}$$

湍流粘滞系数ν_t为：

$$\nu_t \sim \frac{\int_0^l \nu_\lambda d\lambda}{\int_0^l d\lambda} \sim \nu_l$$

若外力强迫是时间周期尺度，产生共振：

$$\nu_t \sim \nu_\lambda|_{\tau_\lambda = T}$$

由于：

$$\nu_\lambda/\nu_l \sim (\lambda/l)^{4/3} \sim (\tau_\lambda/\tau_l)^2$$

当$\lambda \sim T$：

$$\nu_t/\nu_l \sim (T/\tau_l)^2$$

得出：

$$\nu_t \sim \nu_l (T/\tau_l)^2$$

tidal turbulence $\nu_t << \nu_l$，对于 smallest scale：η, τ_η, u_η：

$$\varepsilon \sim \frac{u_l^2}{\tau_l} \sim \frac{u_\eta^2}{\tau_\eta} \sim \nu \tau_\eta^{-2} \quad and \quad [\varepsilon] = m^2 s^{-3}$$

$$\frac{u_\eta^2}{\tau_\eta} \sim \nu \tau_\eta^{-2} \to u_\eta \cdot \eta/\nu \sim 1$$

$$u_l^2/\tau_l \sim \nu \tau_\eta^{-2} \to \eta/l \sim (u_l \cdot l/\nu)^{-3/4} \sim \mathrm{Re}_l^{-3/4}$$

因此：

$$u_\eta/u_l \sim \mathrm{Re}_l^{-1/4} \tau_\eta/\tau_l \sim \mathrm{Re}_l^{-1/2}$$

Example: 设有云朵：$l \sim 10^3\mathrm{m}$, $u_l \sim 1\mathrm{m/s}$, $\nu \sim 10^{-5}\mathrm{m^2s^{-1}}$

$$\begin{aligned} \mathrm{Re}_l &\sim \frac{u_l \cdot l}{\nu} \sim 1 \times 10^3/10^{-5} = 10^8 \\ \eta &\sim \mathrm{Re}_l^{-3/4} \cdot l = (10^8)^{-3/4} \times 10^3\mathrm{m} = 10^{-3}\mathrm{m} \\ u_\eta &\sim \mathrm{Re}_l^{-1/4} \cdot u_l = (10^8)^{-1/4} \times 1\mathrm{m/s} = 10^{-2}\mathrm{m/s} \end{aligned} \tag{8.1}$$

8.4 Energy Spectrum

Chapter 9

Mangeto-hydrodynamics (MHD)

Introduction to MHD

9.1 Equation

在低速流动：$u << c$

流体参考系和实验室参考系的关系：

$$E' = E + U \times BB' = B$$

法拉第定律：

$$\nabla \times E = \frac{\partial B}{\partial t} \rightarrow B_0/t_0 \sim E_0/l_0 \rightarrow E_0/U_0$$

$$\mu\epsilon E/t_0 \quad / \quad B/l_0 \sim (u_0/c)^2 << 1$$

位移电流小，去掉

欧姆定律：

$$j' = \sigma E' = \sigma(E + U \times B)$$

$$\nabla \times B = \mu j$$

$$\partial B/\partial t = -\nabla \times E$$

η：磁扩散系数

磁感应项+磁扩散项

$$R_m = \frac{\nabla \times (U \times B)}{\eta \nabla^2 B} = \frac{U_0 B/l_0}{\eta B/l_0^2} = u_0 l_0/\eta$$

不可压缩流体：速度散度=0

$\eta = 0$：阿尔文磁冻结定理（$R_m \to \infty$）

磁力线冻结-->阿尔文波 1942

$$\rho(\frac{\partial u}{\partial t} + u \cdot \nabla u) = -\nabla p$$

洛伦兹项：应力张量，正+剪切

切向投影s

法向投影n-->张力

理想磁流体：没有粘性和磁扩散，

不可压，微扰

磁力线与群速度方向一样

群速度：v_a阿尔文速度

u/v_a=M_a：磁马赫数

可压缩：磁声波 $c_g = \sqrt{c_s^2 + v_a^2}$