ZJU-UIUC Institute
Prof. Thomas Honold

Spring Semester 2023
Homework 2

Differential Equations (Math 285)

H6 Solve the initial value problem $y' + 4y = 8\mathrm{e}^{-4t} + 20,\ y(0) = 0$ and determine $y_\infty = \lim_{t\to\infty} y(t)$ for the solution.

H7 Solve $y' - 2y = \mathrm{e}^{ct},\ y(0) = 1$ and graph the solution for

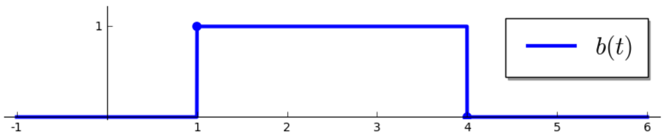
a) $c = 2$; b) $c = 2.01$.

What do you observe?

H8 The Heaviside function $\mathrm{u}\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathrm{u}(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

Express $b(t)$ (cf. picture) in terms of $\mathrm{u}(t)$, solve the initial value problem $y' + 2y = b(t)$, $y(0) = 0$, and determine y_∞ (cf. H6).



H9 a) Write the following complex numbers in polar form:

(i) $\sqrt{3}\,\mathrm{i} + 1$; (ii) $\sqrt{3}\,\mathrm{i} - 1$; (iii) $\mathrm{i} - \sqrt{3}$.

b) Determine the general solution of the following ODE's:

(i) $y' + y = \cos\bigl(\sqrt{3}\,t\bigr)$; (ii) $y' - y = \cos\bigl(\sqrt{3}\,t\bigr)$;

(iii) $y' - \sqrt{3}\,y = \cos t + \sin t$.

c) Suppose $A\colon I \to \mathbb{C},\ t \mapsto A_1(t) + \mathrm{i}\,A_2(t)$ is differentiable (i.e., $A_1 = \operatorname{Re} A$ and $A_2 = \operatorname{Im} A$ are differentiable). Show that $I \to \mathbb{C},\ t \mapsto \mathrm{e}^{A(t)}$ is differentiable as well, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{A(t)} = A'(t)\mathrm{e}^{A(t)}.$$

Hint: Start with $\mathrm{e}^{A(t)} = \mathrm{e}^{A_1(t)+\mathrm{i}\,A_2(t)} = \mathrm{e}^{A_1(t)}\mathrm{e}^{\mathrm{i}\,A_2(t)} = \mathrm{e}^{A_1(t)}\cos A_2(t) + \mathrm{i}\,\mathrm{e}^{A_1(t)}\sin A_2(t)$.

H10 a) Show that in the 3rd model $mv' = mg - kv^2$ for a falling object released at height s_0 the terminal velocity v_T of the object at time of impact is given by

$$v_T = \sqrt{\frac{mg}{k}} \cdot \sqrt{1 - \mathrm{e}^{-2ks_0/m}}.$$

Hint: Consider the velocity as a function $v(s)$ of the distance s traveled. Show that $y(s) = v(s)^2$ satisfies the ODE $my' = 2mg - 2ky$.

b) The limiting velocity of a falling basketball ($m = 620\,\mathrm{g}$) has been estimated at $20\,\mathrm{m/s}$. Using this data, graph v_T as a function of s_0. For which heights s_0 does the basketball reach $50\,\%$, $90\,\%$, and $99\,\%$ of its limiting velocity?

H11 a) Let $f_\lambda(t) = \mathrm{e}^{\lambda t}$ for $\lambda \in \mathbb{R}$. Show that $\{f_\lambda; \lambda \in \mathbb{R}\}$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$.

Hint: Suppose there exists $r \in \mathbb{Z}^+$ and distinct numbers $\lambda_1, \dots, \lambda_r, c_1, \dots, c_r \in \mathbb{R}$ such that

$$c_1\mathrm{e}^{\lambda_1 t} + c_2\mathrm{e}^{\lambda_2 t} + \cdots + c_r\mathrm{e}^{\lambda_r t} = 0 \quad \text{for all } t \in \mathbb{R}. \tag{$\star$}$$

Assuming $\lambda_1 < \lambda_2 < \cdots < \lambda_r$ and $c_r \neq 0$, divide this equation by $\mathrm{e}^{\lambda_r t}$ and let $t \to +\infty$ to obtain a contradiction.

b) For $\lambda \in \mathbb{C}$ the functions $f_\lambda(t) = \mathrm{e}^{\lambda t}$ belong to the vector space $\mathbb{C}^{\mathbb{R}}$ of all complex-valued functions on $\mathbb{R}$ (with scalar multiplication by complex numbers). Show that $\{f_\lambda; \lambda \in \mathbb{C}\}$ is linearly independent in $\mathbb{C}^{\mathbb{R}}$.

Hint: The proof outlined in a) breaks down in the complex case. Instead differentiate the identity in $(\star)$ j times, $0 \leq j < r$, and set $t = 0$.

c) Let $c_\lambda(t) = \cos(\lambda t)$, $s_\lambda(t) = \sin(\lambda t)$. Show that $\{c_\lambda; \lambda \in \mathbb{R}, \lambda \geq 0\} \cup \{s_\lambda; \lambda \in \mathbb{R}, \lambda > 0\}$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$.

Due on Fri Mar 3, 4 pm

Exercises H11 b) and H11 c) are optional.

Solutions

6 According to the particular solution formula,

$$\begin{aligned} y_p(t) &= e^{-4t} \int_0^t (8e^{-4s} + 20)e^{4s}\, ds \\ &= (8t-5)e^{-4t} + 5 \end{aligned}$$

$$\begin{aligned} \implies y(t) &= Ce^{-4t} + y_p(t) \\ &= Ce^{-4t} + (8t-5)e^{-4t} + 5, \quad C \in \mathbb{R}. \end{aligned}$$

Plug the initial condition $y(0) = 0$ into the general solution:

$$y(0) = Ce^{-4*0} + (8*0-5)e^{-4*0} + 5 = 0$$

$$\implies C = 0$$

$$\implies y(t) = (8t-5)e^{-4t} + 5$$

$$y_\infty = \lim_{t\to\infty} \left[(8t-5)e^{-4t} + 5\right] = 5$$

. (It can also be seen directly that the particular solution $y_p(t)$ satisfies already $y_p(0) = 0$.)

7 a)

$$\because c = 2$$

$$\therefore y' = 2y + e^{2t}$$

According to the particular solution formula,

$$y_p(t) = e^{2t} \int_0^t e^{2s}e^{-2s}\, ds = te^{2t}$$

$$y(t) = te^{2t} + C_1e^{2t}$$

Plug the initial condition $y(0) = 1$ into the general solution

$$y(0) = 0 * e^{2*0} + C_1e^{2*0} = 1$$

$$\implies C_1 = 1$$

$$\implies y(t) = (t+1)e^{2t}$$

b)

$$\because c = 2.01$$

$$\therefore y' = 2y + e^{2.01t}$$

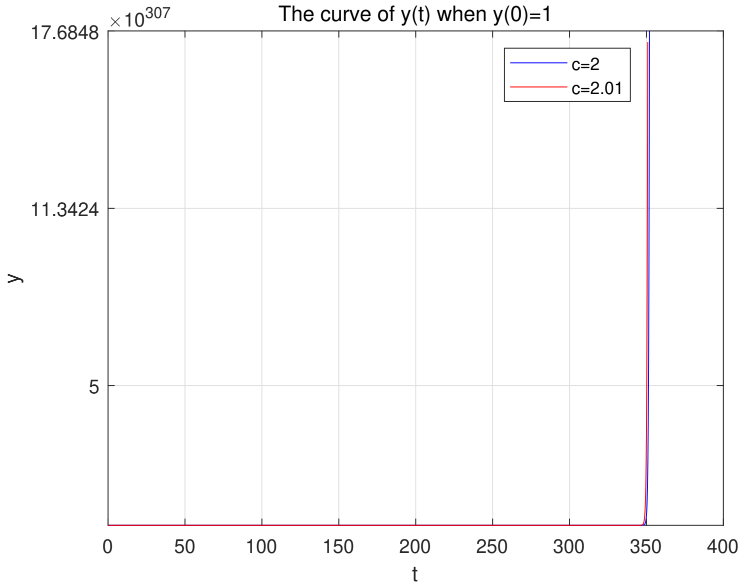
According to the particular solution formula,

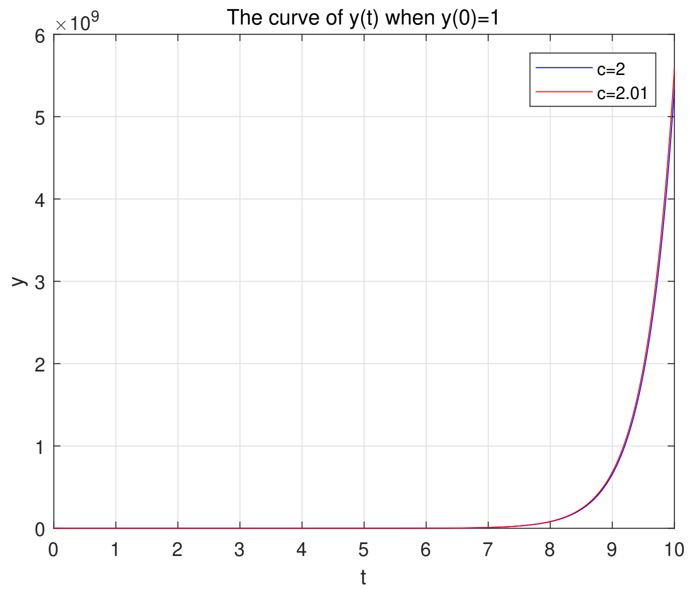
$$y_p(t) = e^{2t} \int_0^t e^{2.01s}e^{-2s}\, ds = 100e^{2.01t} - 100e^{2t}$$

$$y(t) = 100e^{2.01t} + (C_2 - 100)e^{2t}$$

Plug the initial condition $y(0) = 1$ into the general solution

$$y(0) = 100e^{2.01*0} + (C_2 - 100)e^{2*0} = 1$$
$$\implies C_2 = 1$$
$$\implies y(t) = 100e^{2.01t} - 99e^{2t}$$





Therefore, there is no significant difference between these two functions, provided t is not too large. On the other hand, we have

$$\frac{100\,\mathrm{e}^{2.01\,t} - 99\,\mathrm{e}^{2t}}{(t+1)\mathrm{e}^{2t}} = \frac{100\,\mathrm{e}^{0.01\,t} - 99}{(t+1)},$$

and the quotient grows exponentially. Hence for large t the solution of b) is significantly larger.

8 We have $b(t) = \mathrm{u}(t-1) - \mathrm{u}(t-4)$ for $t \in \mathbb{R}$ (this also holds at $t = 1$ and $t = 4$).

$$y' = -2y + b(t)$$
$$b(t) = \begin{cases} 0, & \text{if } t < 1 \text{ or } t \geq 4 \\ 1, & \text{if } 1 \leq t < 4 \end{cases}$$

When $t < 1$, $b(t) = 0$
$\Longrightarrow$ The equation is $y' + 2y = 0$, which is homogeneous.

$$\therefore y(t) = C_1 e^{-2t}, C_1 \in \mathbb{R}$$

$$y(0) = C_1 * e^0 = 0 \Longrightarrow C_1 = 0$$
$$\therefore y(t) = 0$$

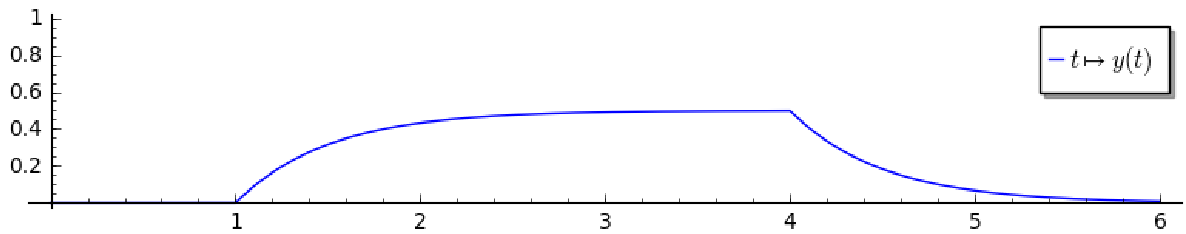
When $1 \leq t < 4$, $b(t) = 1$
$\Longrightarrow$ The equation is $y' + 2y = 1$, which is inhomogeneous and has $y_p(t) = \frac{1}{2}$ as particular solution.

$$\therefore y(t) = C_2 e^{-2t} + \frac{1}{2}, C_2 \in \mathbb{R}$$
$$y(1) = C_2 e^{-2*1} + \frac{1}{2} = 0 \Longrightarrow C_2 = -\frac{1}{2}e^2 \qquad \text{(Continuity at } t = 1)$$
$$\therefore y(t) = -\frac{1}{2}e^{2-2t} + \frac{1}{2}$$

When $t \geq 4$, $b(t) = 0$
$\Longrightarrow$ The equation is again $y' + 2y = 0$.

$$\therefore y(t) = C_3 e^{-2t}, C_3 \in \mathbb{R}$$

$$y(4) = C_3 e^{-2*4} = \frac{1}{2}(1 - e^{-6}) \Longrightarrow C_3 = \frac{1}{2}(e^8 - e^2) \qquad \text{(Continuity at } t = 4)$$
$$\therefore y(t) = \frac{1}{2}(e^8 - e^2)e^{-2t}$$
$$y_\infty = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{1}{2}(e^8 - e^2)e^{-2t} = 0$$



9 a) (i) $\sqrt{3}\,i + 1 = \sqrt{3+1}\,e^{i\arctan(\sqrt{3})} = 2\,e^{i\frac{\pi}{3}}$

(ii) $\sqrt{3}\,i - 1 = \sqrt{3+1}\,e^{i(\pi+\arctan(-\sqrt{3}))} = 2\,e^{i\frac{2\pi}{3}}$

(iii) $i - \sqrt{3} = \sqrt{3+1}\,e^{[\pi+i\arctan(-\frac{\sqrt{3}}{3})]} = 2\,e^{i\frac{5\pi}{6}}$

b) (i) Complexifying this ODE leads to $z' = -z + e^{i\sqrt{3}\,t}$. If $z(t)$ solves the complex ODE, $y_p(t) = \operatorname{Re} z(t)$ will be a particular solution of $y' + y = \cos(\sqrt{3}\,t)$.
Since $(e^{i\sqrt{3}\,t})' = i\sqrt{3}\,e^{i\sqrt{3}\,t}$, it is reasonable to guess that there exists a particular solution of the form $z(t) = C\,e^{i\sqrt{3}\,t}$ with $C \in \mathbb{C}$.

$$z'(t) = Ci\sqrt{3}e^{i\sqrt{3}t} = -(Ce^{i\sqrt{3}t}) + e^{i\sqrt{3}t} \iff Ci\sqrt{3} = 1 - C \iff C = \frac{1}{1+i\sqrt{3}}$$

$$\implies z(t) = \frac{1-i\sqrt{3}}{4}(\cos(\sqrt{3}\,t) + i\sin(\sqrt{3}\,t))$$

$$\therefore y_p(t) = \operatorname{Re} z(t) = \frac{1}{4}\cos(\sqrt{3}t) + \frac{\sqrt{3}}{4}\sin(\sqrt{3}t)$$

So the general (real) solution of $y' + y = \cos(\sqrt{3}\,t)$ is

$$y(t) = C_1 e^{-t} + \frac{1}{4}\cos(\sqrt{3}\,t) + \frac{\sqrt{3}}{4}\sin(\sqrt{3}\,t), \quad C_1 \in \mathbb{R}.$$

(ii) Here we have $z' = z + e^{i\sqrt{3}\,t}$ and can use the same "Ansatz" as in (i).

$$z'(t) = Ci\sqrt{3}e^{i\sqrt{3}t} = (Ce^{i\sqrt{3}t}) + e^{i\sqrt{3}t} \iff Ci\sqrt{3} = 1 + C \iff C = \frac{1}{-1+i\sqrt{3}}$$

$$\implies z(t) = -\frac{1+i\sqrt{3}}{4}(\cos(\sqrt{3}t) + i\sin(\sqrt{3}t))$$

$$\therefore y_p(t) = \operatorname{Re} z(t) = -\frac{1}{4}\cos(\sqrt{3}t) + \frac{\sqrt{3}}{4}\sin(\sqrt{3}t)$$

So the general solution of $y' - y = \cos(\sqrt{3}\,t)$ is

$$y(t) = C_1 e^{t} - \frac{1}{4}\cos(\sqrt{3}\,t) + \frac{\sqrt{3}}{4}\sin(\sqrt{3}\,t), \quad C_1 \in \mathbb{R}.$$

(iii)

$$y' - \sqrt{3}\,y = \cos t + \sin t \iff y' = \sqrt{3}\,y + \sqrt{2}\sin\left(\frac{\pi}{4} + t\right)$$

Complexifying this ODE leads to $z' = \sqrt{3}\,z + \sqrt{2}\,e^{i(\frac{\pi}{4}+t)} = \sqrt{3}\,z + (1+i)\mathrm{e}^{it}$ and, since we have complexified a sine, this time $y_p(t) = \operatorname{Im} z(t)$ will give a particular solution of the original ODE. Using the "Ansatz" $z(t) = C\,e^{it}$, $C \in \mathbb{C}$, we obtain

$$z'(t) = Ci\,e^{it} = \sqrt{3}\,C\,\mathrm{e}^{it} + (1+i)e^{it} \iff Ci = \sqrt{3}\,C + 1 + i$$

$$\iff C = \frac{1+i}{-\sqrt{3}+i} = \frac{(1+i)(-\sqrt{3}-i)}{4} = \frac{1-\sqrt{3}}{4} - \frac{1+\sqrt{3}}{4}i$$

$$\implies y_p(t) = \operatorname{Im}\left[\left(\frac{1-\sqrt{3}}{4} - \frac{1+\sqrt{3}}{4}i\right)\mathrm{e}^{it}\right]$$

$$= -\frac{1+\sqrt{3}}{4}\cos t + \frac{1-\sqrt{3}}{4}\sin t$$

$$= -\frac{\sqrt{2}}{4}\cos\left(\frac{\pi}{4} + t\right) - \frac{\sqrt{6}}{4}\sin\left(\frac{\pi}{4} + t\right)$$

So the general solution of $y' - \sqrt{3}\,y = \cos t + \sin t$ is

$$\begin{aligned} y(t) &= C_1 e^{\sqrt{3}\,t} - \frac{1+\sqrt{3}}{4}\cos t + \frac{1-\sqrt{3}}{4}\sin t \\ &= C_1 e^{\sqrt{3}\,t} - \frac{\sqrt{2}}{4}\cos\left(\frac{\pi}{4}+t\right) - \frac{\sqrt{6}}{4}\sin\left(\frac{\pi}{4}+t\right), \quad C_1 \in \mathbb{R}. \end{aligned}$$

c) Using the rules for differentiating real-valued functions (in particular, the rule $\frac{\mathrm{d}}{\mathrm{d}t}\, e^{A_1(t)} = A_1'(t)\mathrm{e}^{A_1(t)}$, which is an instance of the chain rule), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{A_1(t)}\cos A_2(t)\right] &= A_1'(t)\mathrm{e}^{A_1(t)}\cos A_2(t) - \mathrm{e}^{A_1(t)}\sin A_2(t)\, A_2'(t), \\ \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{A_1(t)}\sin A_2(t)\right] &= A_1'(t)\mathrm{e}^{A_1(t)}\sin A_2(t) + \mathrm{e}^{A_1(t)}\cos A_2(t)\, A_2'(t), \end{aligned}$$

and hence

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{A(t)} &= \left(A_1'(t)\mathrm{e}^{A_1(t)}\cos A_2(t) - \mathrm{e}^{A_1(t)}\sin A_2(t)A_2'(t)\right) + \mathrm{i}\left(A_1'(t)\mathrm{e}^{A_1(t)}\sin A_2(t) + \mathrm{e}^{A_1(t)}\cos A_2(t)A_2'(t)\right) \\ &= \left(A_1'(t) + \mathrm{i}\,A_2'(t)\right)\left(\mathrm{e}^{A_1(t)}\cos A_2(t) + \mathrm{i}\,\mathrm{e}^{A_1(t)}\sin A_2(t)\right) \\ &= A'(t)\mathrm{e}^{A(t)}. \end{aligned}$$

Of course, this also proves that $t \mapsto \mathrm{e}^{A(t)}$ is differentiable.

10 a)

$$mv' = mg - kv^2 \iff m\frac{\mathrm{d}v}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = mg - kv^2 \iff mv\frac{\mathrm{d}v}{\mathrm{d}s} = mg - kv^2$$

Assuming $y(s) = v(s)^2$, we have $\frac{\mathrm{d}y}{\mathrm{d}s} = 2v\frac{\mathrm{d}v}{\mathrm{d}s}$.
By substituting this into the equation $mv\frac{\mathrm{d}v}{\mathrm{d}s} = mg - kv^2$, we get

$$\begin{aligned} my' &= 2mg - 2ky \\ \implies y' &= -\frac{2k}{m}y + 2g \end{aligned}$$

According to the general solution formula,

$$y(s) = Ce^{-\frac{2k}{m}s} + e^{-\frac{2k}{m}s}\int_0^s 2ge^{\frac{2k}{m}\lambda}\,d\lambda = (C - \frac{mg}{k})e^{-\frac{2k}{m}s} + \frac{mg}{k}$$

$$\because v(0) = 0$$

$$\therefore y(0) = C - \frac{mg}{k} + \frac{mg}{k} = 0 \implies C = 0$$

$$\implies y = \frac{mg}{k}(1 - e^{-\frac{2k}{m}s})$$

$$\implies v = \sqrt{y} = \sqrt{\frac{mg}{k}}\sqrt{1 - e^{-\frac{2k}{m}s}}$$

$$\implies v_T = v(s_0) = \sqrt{\frac{mg}{k}}\sqrt{1 - e^{-\frac{2k}{m}s_0}}$$

Remark: The general solution of $y' = -\frac{2k}{m}y + 2g$ can also be obtained using the observation that $y \equiv mg/k$ is a particular (constant) solution.

b) When $m = 620\,\text{g}$,

$$v_l = \lim_{s\to\infty} \sqrt{\frac{mg}{k}}\sqrt{1 - e^{-\frac{2k}{m}s}} = 20$$

$$\implies \sqrt{\frac{mg}{k}} = 20$$

Assuming that $g = 10\,\text{m/s}^2$, we have $\frac{2k}{m} = \frac{1}{20}$

$$\implies v_T = 20\sqrt{1 - e^{-\frac{1}{20}s_0}}$$

(i)

$$v_T = 50\%\, v_l \implies \sqrt{1 - e^{-\frac{1}{20}s_0}} = 50\% \implies s_0 = -20\ln\frac{3}{4}$$

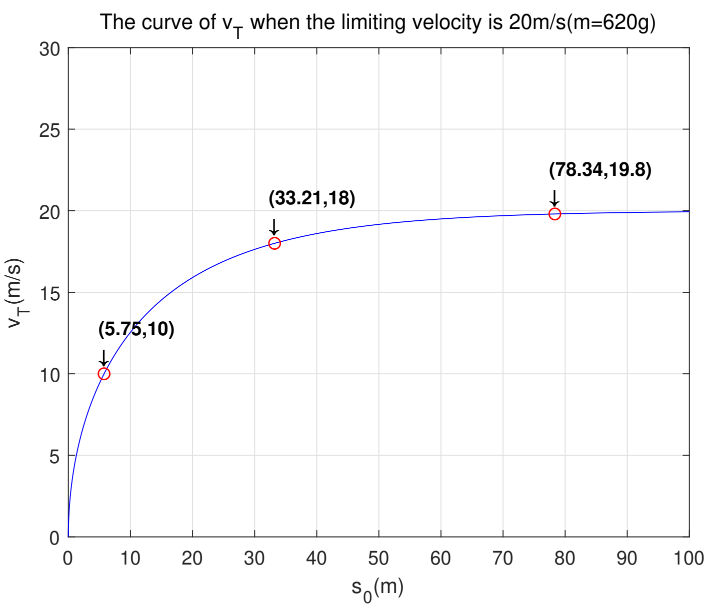
(ii)

$$v_T = 90\%\, v_l \implies \sqrt{1 - e^{-\frac{1}{20}s_0}} = 90\% \implies s_0 = -20\ln\frac{19}{100}$$

(iii)

$$v_T = 99\%\, v_l \implies \sqrt{1 - e^{-\frac{1}{20}s_0}} = 99\% \implies s_0 = -20\ln\frac{199}{10000}$$

The graph of $v_T(s_0) = 20\sqrt{1 - e^{-\frac{1}{20}s_0}}$ as a function of s_0 is shown below, with three points indicating the corresponding s_0 for which the basketball reaches 50%, 90%, and 99% of its limiting velocity.



Since we have used an approximation of g with only 1 significant digit, we cannot expect the values of s_0 to be more accurate. All we can say is that the basketball reaches 50%, 90%, 99% of its limiting velocity for heights of approximately 6 m, 30 m, 80 m, respectively.

11 a) Dividing ($\star$) by $\mathrm{e}^{\lambda_r t}$ and solving for c_r gives

$$c_r = -c_1 \mathrm{e}^{(\lambda_1-\lambda_r)t} - c_2 \mathrm{e}^{(\lambda_2-\lambda_r)t} - \cdots - c_{r-1}\mathrm{e}^{(\lambda_{r-1}-\lambda_r)t}. \tag{1}$$

Since $\lambda_i - \lambda_r < 0$ for $1 \le i \le r-1$, we have $\lim_{t\to+\infty} \mathrm{e}^{(\lambda_i-\lambda_r)t} = 0$ for $1 \le i \le r-1$. Hence the right-hand side of (1) tends to zero for $t \to +\infty$, while the left-hand side is a non-zero constant. This obvious contradiction proves that the functions f_λ, $\lambda \in \mathbb{R}$, are linearly independent over $\mathbb{R}$.

b) In the complex case $\lambda_i = \alpha_i + \mathrm{i}\beta_i$ ($\alpha_i, \beta_i \in \mathbb{R}$) we have

$$\mathrm{e}^{\lambda_i t} = \mathrm{e}^{\alpha_i t + \mathrm{i}(\beta_i t)}, \quad \left|\mathrm{e}^{\lambda_i t}\right| = \mathrm{e}^{\alpha_i t}.$$

Assuming that $\alpha_r > \alpha_i$ for $1 \le i \le r-1$ and $c_r \ne 0$, we can still divide ($\star$) by $\mathrm{e}^{\alpha_r t}$ and obtain a contradiction in a similar way. But, since different λ_i's may have the same real part, this is not sufficient for a proof of linear independence.

However, we can argue as follows: Differentiating ($\star$) $r-1$-times gives the system of identities

$$\begin{array}{rcccrcl}
c_1\mathrm{e}^{\lambda_1 t} & + & \cdots & + & c_r\mathrm{e}^{\lambda_r t} & = & 0,\\
c_1\lambda_1\mathrm{e}^{\lambda_1 t} & + & \cdots & + & c_r\lambda_r\mathrm{e}^{\lambda_r t} & = & 0,\\
c_1\lambda_1^2\mathrm{e}^{\lambda_1 t} & + & \cdots & + & c_r\lambda_r^2\mathrm{e}^{\lambda_r t} & = & 0,\\
 & & & & & \cdots & \\
c_1\lambda_1^{r-1}\mathrm{e}^{\lambda_1 t} & + & \cdots & + & c_r\lambda_r^{r-1}\mathrm{e}^{\lambda_r t} & = & 0.
\end{array}$$

Setting $t = 0$ gives for $\mathbf{c} = (c_1, \dots, c_r)$ the linear system of equations $\mathbf{Ac} = \mathbf{0}$ with coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_r^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{r-1} & \lambda_2^{r-1} & \dots & \lambda_r^{r-1} \end{pmatrix}.$$

Now $\mathbf{A}$ is a Vandermonde matrix and hence invertible; cf. any Linear Algebra book. (One can also compute the determinant of $\mathbf{A}$ recursively: Subtract Column 1 from the remaining columns and then expand along the first row. This leaves an $(n-1)\times(n-1)$ determinant, which has the factor $(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\cdots(\lambda_r-\lambda_1)$ (since the columns have the factors $\lambda_j - \lambda_1$). After taking the factors out, the Vandermonde form (with 2nd row $(\lambda_2, \dots, \lambda_r)$) can be restored using suitable elementary row operations. By induction it then follows that $\det(\mathbf{A}) = \prod_{1\le i<j\le r}(\lambda_j - \lambda_i)$, which is obviously nonzero.)

It follows that $c_1 = c_2 = \cdots = c_r = 0$, i.e., the functions f_λ, $\lambda \in \mathbb{C}$, are linearly independent.

c) Let $\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_t \in \mathbb{R}$ and $a_1, \dots, a_r, b_1, \dots, b_t \in \mathbb{R}$ be such that $0 \le \lambda_1 < \cdots < \lambda_r$, $0 < \mu_1 < \cdots < \mu_t$, and

$$a_1 c_{\lambda_1} + \cdots + a_r c_{\lambda_r} + b_1 s_{\mu_1} + + \cdots + b_t s_{\mu_t} = 0 \quad \text{in} \quad \mathbb{R}^{\mathbb{R}}. \tag{2}$$

Since $\cos(\lambda x) = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\lambda x} + \mathrm{e}^{-\mathrm{i}\lambda x})$, $\sin(\lambda x) = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\lambda x} - \mathrm{e}^{-\mathrm{i}\lambda x})$, we have $c_\lambda = \frac{1}{2}(f_{\mathrm{i}\lambda} + f_{-\mathrm{i}\lambda})$, $s_\lambda = \frac{1}{2\mathrm{i}}(f_{\mathrm{i}\lambda} - f_{-\mathrm{i}\lambda})$. Inserting this into (2) gives a complex linear combination of the functions f_λ, which is equal to zero. By Part b), all the coefficients must be zero.

If $\lambda_1 = 0$ then, since $c_0(t) = 1 = f_0(t)$, the function f_0 appears in the complex linear combination with coefficient a_0, and hence $a_0 = 0$.

If λ_1 is not equal to any of the numbers $\mu_1, \ldots, \mu_t$, then both $f_{\mathrm{i}\lambda_1}$, $f_{-\mathrm{i}\lambda_1}$ appear in the complex linear combination with coefficient $a_1/2$, showing that $a_1 = 0$.

Arguing similarly for $\lambda_2, \ldots, \lambda_r, \mu_1, \ldots, \mu_t$, we see that the only remaining case is $\lambda_i = \mu_j$ for some i, j. W.l.o.g. we may assume $\lambda_1 = \mu_1 = \lambda$. Then the coefficient of $f_{\pm\mathrm{i}\lambda}$ in the complex linear combination is clearly the same as in

$$a_1 c_{\lambda_1} + b_1 s_{\mu_1} = \frac{a_1}{2}\left(f_{\mathrm{i}\lambda} + f_{-\mathrm{i}\lambda}\right) + \frac{b_1}{2\mathrm{i}}\left(f_{\mathrm{i}\lambda} - f_{-\mathrm{i}\lambda}\right) = \frac{a_1 - \mathrm{i}b_1}{2}\, f_{\mathrm{i}\lambda} + \frac{a_1 + \mathrm{i}b_1}{2}\, f_{-\mathrm{i}\lambda}.$$

It follows that $\frac{a_1 - \mathrm{i}b_1}{2} = \frac{a_1 + \mathrm{i}b_1}{2} = 0$ and hence $a_1 = b_1 = 0$.

In all we have shown that (2) implies $a_1 = \cdots = a_r = b_1 = \cdots = b_t = 0$, i.e., the functions c_λ, $\lambda \geq 0$, and s_λ, $\lambda > 0$, are collectively linearly independent.