Differential Equations (Math 285)

H32 Use the phase line to investigate the stability of the equilibrium solutions of the following autonomous ODE's.

a) $y' = 2(1-y)(1-\mathrm{e}^y)$; b) $y' = (1-y^2)(4-y^2)$; c) $y' = \sin^2 y$.

H33 For the following ODE's $y' = f(y)$, use the Existence and Uniqueness Theorem to determine the points $(t_0, y_0) \in \mathbb{R}^2$ such that the initial value problem $y' = f(y) \wedge y(t_0) = y_0$ has a unique solution near (t_0, y_0). Then solve the ODE, sketch the integral curves, and compare with your prediction.

a) $y' = |y|$; b) $y' = \sqrt{|y - y^2|}$.

H34 Use Picard-Lindelöf iteration to compute the solution $\phi = (\phi_1, \phi_2)^\mathsf{T}$ of the system

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}$$

with initial condition $\phi(0) = (1, 0)^\mathsf{T}$.

H35 Suppose that $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous and satisfies locally a Lipschitz condition, and that

$$f(-t, y) = -f(t, y) \quad \text{for all } (t, y) \in \mathbb{R}^2.$$

Show that any solution $\phi\colon [-r, r] \to \mathbb{R}$, $r > 0$, of $y' = f(t, y)$ is its own mirror image with respect to the y-axis.

H36 *From a previous final exam*

Consider the differential equation

$$(3xy + 2y^2)\,\mathrm{d}x + (3x^2 + 6xy + 3y^2)\,\mathrm{d}y = 0. \tag{DF}$$

a) Show that $(0, 0)$ is the only singular point of (DF).

b) Transform (DF) into an exact equation and determine the general solution in implicit form.

c) Is every point of $\mathbb{R}^2$ on a unique integral curve of (DF)?

Due on Fri Mar 31, 4 pm

The phase line of an autonomous ODE (required for H32) will be discussed in the lecture on Mon Mar 27 (cf. also [BDM17], Ch. 2.5).

Solutions

32 a) Setting $y' = 2(1-y)(1-e^y) = 0$ gives the two equilibrium solutions $y_1 = 0$, $y_2 = 1$. The graph of y' versus y is shown below. So $y_1 = 0$ is an asymptotically stable stable

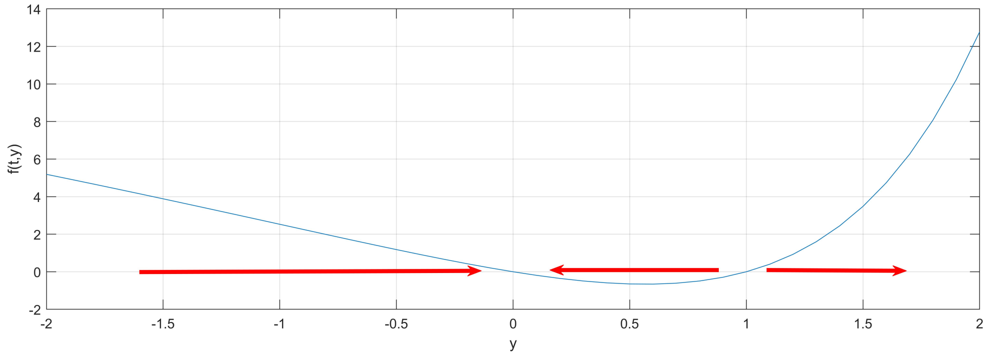

Figure 1: H32 a)

equilibrium, while $y_2 = 1$ is an unstable equilibrium.

b) Setting $y' = (1-y^2)(4-y^2) = 0$ gives the four equilibria $y_1 = -2, y_2 = -1, y_3 = 1, y_4 = 2$. The graph of y' versus y is shown below. So $y_1 = -2$, $y_3 = 1$ are asymptotically

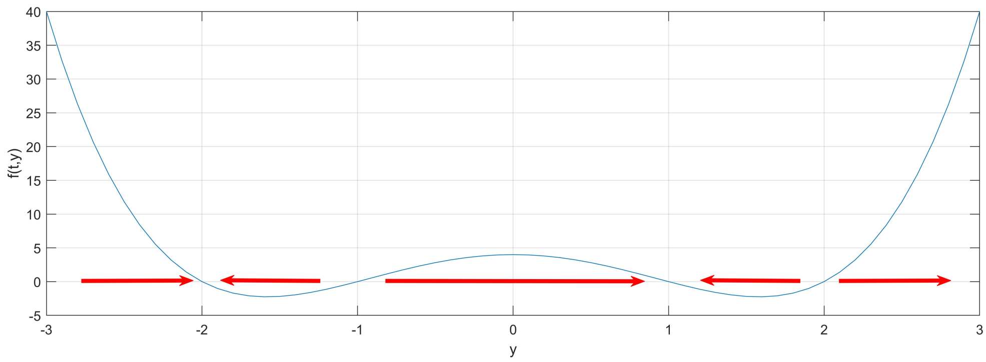

Figure 2: H32 b)

stable solutions, while $y_2 = -1, y_4 = 2$ are unstable solutions.

c) Setting $y' = \sin^2 y = 0$ gives infinitely many equilibrium solutions, viz. $y_k = k\pi$ ($k \in \mathbb{Z}$). The graph of y' versus y is shown below. So all equilibria are semistable (asymptotically stable from below, unstable from above).

33 Note that solutions of all three ODE's must have non-negative derivative and hence cannot decrease anywhere strictly.

a) The function $f(t, y) = |y|$ is continuous and trivially satisfies a Lipschitz condition with respect to y (with Lipschitz constant $L = 1$, since $|f(t, y_1) - f(t, y_2)| = |y_1 - y_2| \leq 1 \cdot |y_1 - y_2|$. Hence solutions exist and are unique everywhere. The general solution is

$$y_C(t) = \begin{cases} C\,\mathrm{e}^t & \text{if } C \geq 0, \\ C\,\mathrm{e}^{-t} & \text{if } C < 0, \end{cases}$$

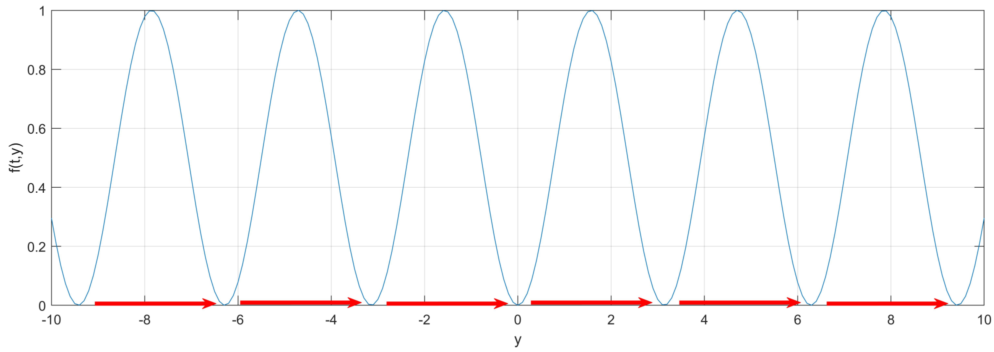

Figure 3: H32 c)

where C can be any real number. This follows by considering the three cases $y > 0$, $y = 0$, $y < 0$ separately.

b) $f(t, y) = \sqrt{|y - y^2|}$ is C^1 on the three plane regions $y < 0$, $0 < y < 1$, $y > 1$, and does not satisfy a Lipschitz condition with respect to y locally at points of the separating lines $y = 0$ and $y = 1$. The latter follows from the fact that the derivative $\frac{\partial f}{\partial y}$ is unbounded near $y = 0$ and $y = 1$. For example, for $0 < y < 1$ we have

$$|f(t, y) - f(t, 1)| = \left|\frac{\partial f}{\partial y}(t, \eta)\right| |y - 1| = \left|\frac{1 - 2\eta}{2\sqrt{\eta - \eta^2}}\right| |y - 1|$$

for some $\eta \in (y, 1)$, and for y (and hence η) close to 1 the factor $\left|\frac{1-2\eta}{2\sqrt{\eta-\eta^2}}\right|$ becomes arbitarily large.

The Existence and Uniqueness Theorem gives that solutions exist and are unique locally at points within the three regions. At points (t, y) with $y \in \{0, 1\}$ solutions are not unique as the following explicit solution shows.

$0 < y < 1$: $\mathrm{d}y/\sqrt{y - y^2} = 2\,\mathrm{d}y\,/\sqrt{1 - (2y - 1)^2} = 1 \implies \arcsin(2y - 1) = t + C \implies y = \frac{1}{2}\big(1 + \sin(t + C)\big) = \frac{1}{2}\big(1 + \cos(t + C')\big)$

$y > 1$: $\mathrm{d}y/\sqrt{y^2 - y} = 2\,\mathrm{d}y\,/\sqrt{(2y - 1)^2 - 1} = 1 \implies \operatorname{arcosh}(2y - 1) = t + C \implies y = \frac{1}{2}\big(1 + \cosh(t + C)\big)$

$y < 0$: $\mathrm{d}y/\sqrt{y^2 - y} = 2\,\mathrm{d}y\,/\sqrt{(1 - 2y)^2 - 1} = 1 \implies -\operatorname{arcosh}(1 - 2y) = t + C \implies y = \frac{1}{2}\big(1 - \cosh(-t + C')\big)$

Solutions from the 3 cases can be glued together at $y = 0$ and $y = 1$ to satisfy the same initial conditions as the constant solutions. One particular example is

$$y(t) = \begin{cases} \frac{1}{2}(1 - \cosh(-t - \pi) & \text{for } t \leq -\pi, \\ \frac{1}{2}(1 + \cos t) & \text{for } -\pi \leq t \leq 0, \\ \frac{1}{2}(1 + \cosh t) & \text{for } t \geq 0; \end{cases}$$

see Figure 4. When constructing solutions, there is more degree of freedom, e.g., we can make solutions follow the line $y = 0$ for a while, then branch off and flow into the line $y = 1$, follow this line for another while, etc.

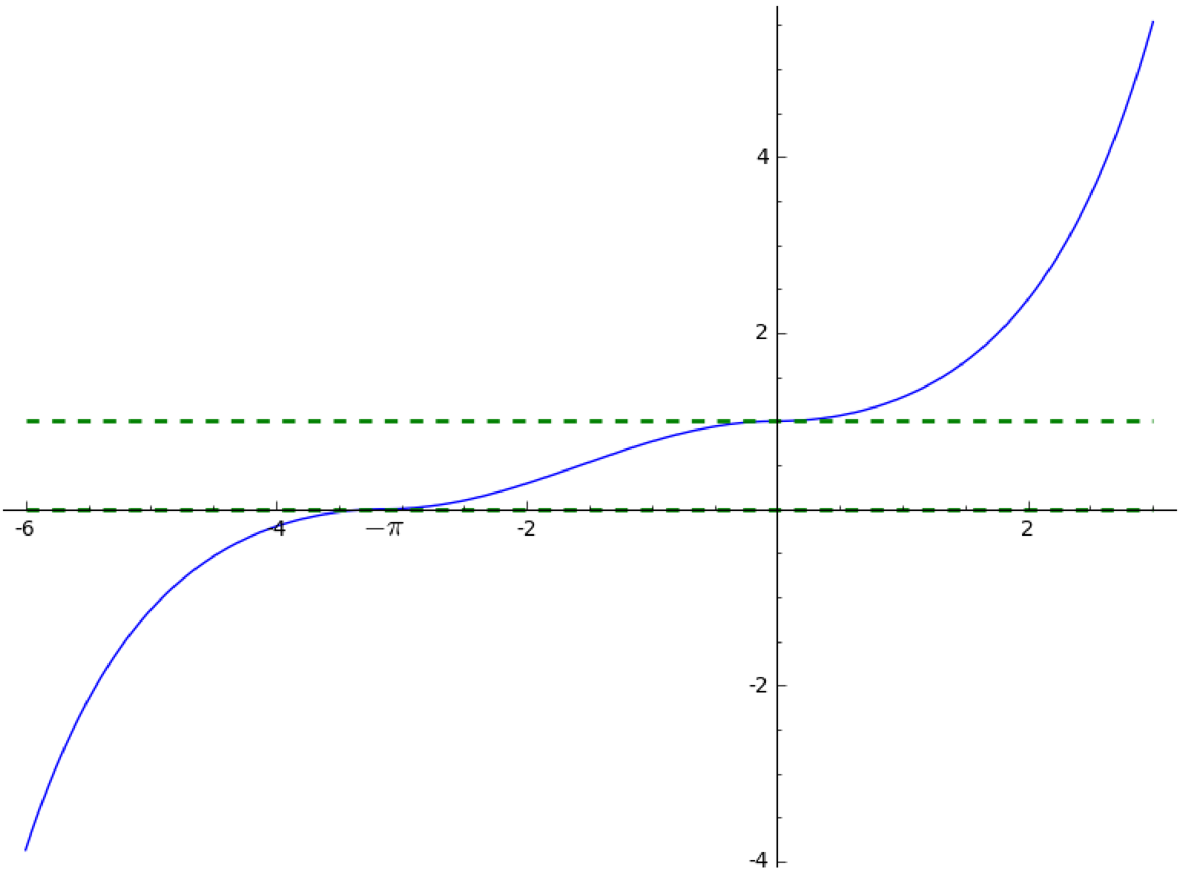

Figure 4: The solution $y(t)$ from H33b)

34 According to Picard-Lindelöf iteration, we have

$$\phi_{k+1}(t) = y_0 + \int_0^t f(s, \phi_k(s))\, ds, \quad k = 0, 1, 2, \dots$$

Note that in this case $\phi_k(t)$ and y_0 are vectors in $\mathbb{R}^2$, and the notation used is somewhat inconsistent with "$\phi = (\phi_1, \phi_2)^\mathsf{T}$" in the statement of the exercise (but preferred for its simplicity).
Since $\phi_0(t) = \begin{pmatrix}1\\0\end{pmatrix} = y_0$, we have

$$\begin{aligned}
\phi_1(t) &= y_0 + \int_0^t f(s, \phi_0(s))\, ds = \begin{pmatrix}1\\0\end{pmatrix} + \int_0^t \begin{pmatrix}0\\1\end{pmatrix} ds = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\t\end{pmatrix} = \begin{pmatrix}1\\t\end{pmatrix},\\
\phi_2(t) &= y_0 + \int_0^t f(s, \phi_1(s))\, ds = \begin{pmatrix}1\\0\end{pmatrix} + \int_0^t \begin{pmatrix}-s\\1\end{pmatrix} ds = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}\frac{t^2}{2}\\t\end{pmatrix} = \begin{pmatrix}1-\frac{t^2}{2}\\t\end{pmatrix},\\
\phi_3(t) &= y_0 + \int_0^t f(s, \phi_2(s))\, ds = \begin{pmatrix}1\\0\end{pmatrix} + \int_0^t \begin{pmatrix}-s\\1-\frac{s^2}{2}\end{pmatrix} ds = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}-\frac{t^2}{2}\\t-\frac{t^3}{6}\end{pmatrix} = \begin{pmatrix}1-\frac{t^2}{2}\\t-\frac{t^3}{6}\end{pmatrix},\\
\phi_4(t) &= y_0 + \int_0^t f(s, \phi_3(s))\, ds = \begin{pmatrix}1\\0\end{pmatrix} + \int_0^t \begin{pmatrix}\frac{s^3}{6}-s\\1-\frac{s^2}{2}\end{pmatrix} ds = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}\frac{t^4}{24}-\frac{t^2}{2}\\t-\frac{t^3}{6}\end{pmatrix} = \begin{pmatrix}1-\frac{t^2}{2}+\frac{t^4}{24}\\t-\frac{t^3}{6}\end{pmatrix},\\
&\ \,\vdots
\end{aligned}$$

$$\begin{aligned}
\phi_{2k-1}(t) &= \begin{pmatrix}1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\dots+(-1)^{k-1}\frac{t^{2k-2}}{(2k-2)!}\\ t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\dots+(-1)^{k-1}\frac{t^{2k-1}}{(2k-1)!}\end{pmatrix},\\
\phi_{2k}(t) &= \begin{pmatrix}1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\dots+(-1)^{k}\frac{t^{2k}}{2k!}\\ t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\dots+(-1)^{k-1}\frac{t^{2k-1}}{(2k-1)!}\end{pmatrix}.
\end{aligned}$$

$$\implies \phi(t) = \begin{pmatrix}\sum_{k=0}^\infty (-1)^k \frac{t^{2k}}{2k!}\\ \sum_{k=0}^\infty (-1)^k \frac{t^{2k+1}}{(2k+1)!}\end{pmatrix} = \begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$$

35 Consider the function $\psi(t) = \phi(-t)$, also defined for $t \in [-r, r]$. We have $\psi(0) = \phi(0) = y_0$, say, and

$$\psi'(t) = -\phi'(-t) = -f\bigl(-t, \phi(-t)\bigr) = f\bigl(t, \phi(-t)\bigr) = f\bigl(t, \psi(t)\bigr).$$

Hence both ϕ and ψ solve the IVP $y' = f(t, y) \wedge y(0) = y_0$. Since f satisfies the assumptions in the Existence and Uniqueness Theorem(s), it follows that $\phi = \psi$, i.e., $\phi(t) = \phi(-t)$ for $t \in [-r, r]$. This is the indicated symmetry property.
Remark: It is sufficient to assume that f satisfies locally a Lipschitz condition with respect to y, which is weaker than "Lipschitz condition per se".

36 a) $M(x, y) = 3xy + 2y^2 = y(3x + 2y)$, $N(x, y) = 3x^2 + 6xy + 3y^2 = 3(x + y)^2$ have no common zero except $(0, 0)$. $\implies$ $(0, 0)$ is the only singular point.

b) We have

$$M_y - N_x = 3x + 4y - (6x + 6y) = -3x - 2y = M(-1/y).$$

Thus $(M_y - N_x)/M$ depends only on y, and there is an integrating factor of the form $g(y)$.

The integrability condition $(gM)_y = (gN)_x$ then becomes $g'M + gM_y = gN_x$, i.e.,

$$g' = \frac{g(N_x - M_y)}{M} = \frac{g}{y}.$$

The solution of this ODE is $g(y) = cy$, so that we can take $g(y) = y$.
$\Longrightarrow$ On $\mathbb{R}^2 \setminus x$-axis the ODE $(3xy + 2y^2)\,\mathrm{d}x + (3x^2 + 6xy + 3y^2)\,\mathrm{d}y = 0$ is equivalent to the exact ODE

$$\bigl(3xy^2 + 2y^3\bigr)\,\mathrm{d}x + \bigl(3x^2y + 6xy^2 + 3y^3\bigr)\,\mathrm{d}y = 0.$$

An antiderivative f of the corresponding exact differential is determined in the usual way by "partial integration" with respect to x, say.

$$\begin{aligned} f(x,y) &= \tfrac{3}{2}\,x^2y^2 + 2\,xy^3 + g(y),\\ f_y(x,y) &= 3\,x^2y + 6\,xy^2 + g'(y) \overset{!}{=} 3x^2y + 6xy^2 + 3y^3 \end{aligned}$$

$\Longrightarrow g'(y) = 3\,y^3 \Longrightarrow g(y) = \tfrac{3}{4}\,y^4 + C \Longrightarrow f(x,y) = \tfrac{3}{2}\,x^2y^2 + 2\,xy^3 + \tfrac{3}{4}\,y^4 + C$
The general implicit solution of the exact ODE is then given by (in slightly simplified form and with a different C)

$$6\,x^2y^2 + 8\,xy^3 + 3\,y^4 = C, \quad C \in \mathbb{R}.$$

Solutions with $C < 0$ don't exist and for $C = 0$ the x-axis is obtained, since $6\,x^2y^2 + 8\,xy^3 + 3\,y^4 = y^2(6x^2 + 8xy + 3y^2)$ and the quadratic has discriminant $8^2 - 4 \cdot 6 \cdot 3 = -8 < 0$.

Since the x-axis (equivalently, the function $y(x) \equiv 0$) is a solution of (DF), multiplication by y hasn't introduced any new solution, and $6\,x^2y^2 + 8\,xy^3 + 3\,y^4 = C$, $C \geq 0$ solves (DF) as well.

c) Yes. This is implicit in the preceding discussion. Intersection points of integral curves must be singular, so that the only candidate for such a point is the origin. But the corresponding contour of f, the 0-contour, consists of a single integral curve, viz. the x-axis.

Remark: Part c) serves as an illustration for the fact that at singular points virtually anything can happen. Here it is due to the fact that at $(0,0)$ the partial derivatives of f up to order 3 vanish. In the lecture we have seen an example of a singular point being on exactly two integral curves (Example 8 of the introduction, `lecture1-3_handout.pdf, Slides 31 ff`), and another one with a singular point contained in infinitely many integral curves (the example at the end of `lecture17-18_handout.pdf`). Singular points contained in no integral curve are also possible: For exact equations $\omega = \mathrm{d}f = 0$ this happens at a strict local extremum of f, where the corresponding contour reduces to a single point (at least locally).