Differential Equations (Math 285)

H37 Determine a real fundamental system of solutions for the following ODE's:

a) $y'' - 4y' + 4y = 0$;
b) $y''' - 2y'' - 5y' + 6y = 0$;
c) $y''' - 2y'' + 2y' - y = 0$;
d) $y''' - y = 0$;
e) $y^{(4)} + y = 0$;
f) $y^{(8)} + 4y^{(6)} + 6y^{(4)} + 4y'' + y = 0$.

Four answers suffice.

H38 Determine the general real solution of

a) $y'' + 3y' + 2y = 2$;
b) $y'' + y' - 12y = 1 + t^2$;
c) $y'' - 5y' + 6y = 4t\mathrm{e}^t - \sin t$;
d) $y''' - 2y'' + y' = 1 + \mathrm{e}^t \cos(2t)$;
e) $y^{(4)} + 2y'' + y = 25\mathrm{e}^{2t}$;
f) $y^{(n)} = t\mathrm{e}^t$, $n \in \mathbb{N}$.

Four answers suffice.

H39 a) Suppose $\phi\colon \mathbb{R} \to \mathbb{C}$ solves a homogeneous linear ODE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0, \quad a_i \in \mathbb{C}, \tag{H}$$

but no such ODE of order $< n$. Show that ϕ, ϕ', ϕ'', $\dots$, $\phi^{(n-1)}$ form a fundamental system of solutions of (H).

b) Find a fundamental system of solutions of the form ϕ, ϕ', ϕ'', ϕ''' for the ODE $y^{(4)} - y^{(3)} - y' + y = 0$.

H40 Do three of the four Exercises 4, 6, 14, 16 in the previous edition of our Calculus textbook [Ste16], Ch. 17.3.
You may need to study the relevant material in [Ste16], Ch. 17, or [BDM17], Ch. 3.7, 3.8 first.

H41 *Optional Exercise*
For the following functions ϕ_i, find the homogeneous linear ODE $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$ ($a_i \in \mathbb{C}$) of smallest order having ϕ_i as a solution; cf. H42.

a) $\phi_1(t) = 2\sin t - 3\cos(3t)$;
b) $\phi_2(t) = \sin t \cos(3t)$;
c) $\phi_3(t) = -1 + t\mathrm{e}^{-2t}\cos t$;
d) $\phi_4(t) = \mathrm{e}^t + t^{1949} + t^{2019}$.

H42 *Optional Exercise*

Suppose that $y\colon \mathbb{R} \to \mathbb{C}$ solves some homogeneous linear ODE $a(D)y = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$ with coefficients $a_i \in \mathbb{C}$ (i.e., y is an exponential polynomial). Show:

a) There is a unique monic polynomial $m(X) \in \mathbb{C}[X]$ of smallest degree satisfying $m(\mathrm{D})y = 0$.

b) If $b(X) \in \mathbb{C}[X]$ satisfies $b(\mathrm{D})y = 0$ then $m(X)$ divides $b(X)$.

Hint: There is a link with the annihilator polynomials (periods) discussed in Math 257; see the section on companion matrices in `lecture19-24_handout.pdf`.

H43 *Optional Exercise*

This exercise shows that characteristic polynomials $a(X)$ of homogeneous linear ODEs $a(\mathrm{D})y = 0$, respectively, homogeneous linear recurrence relations $a(\mathrm{S})\mathbf{y} = \mathbf{0}$ are characteristic polynomials in the sense of Linear Algebra.

a) Show that the (complex) solution space V of $a(\mathrm{D})y = 0$ is D-invariant, and that the characteristic polynomial of the restriction $\mathrm{D}|_V$ is equal to $a(X)$.

b) Show that the (complex) solution space V of $a(\mathrm{S})\mathbf{y} = \mathbf{0}$ is S-invariant, and that the characteristic polynomial of the restriction $\mathrm{S}|_V$ is equal to $a(X)$.

Hint: In Math 257 we have characterized endomorphisms of finite-dimensional vector spaces whose minimum polynomial equals the characteristic polynomial; see the section on companion matrices in `lecture19-24_handout.pdf`. Do b) first, which is easier.

H44 *Optional Exercise*

In the lecture we have found that the ODE $y'' - y' - y = 1$ and its discrete "analogue" $y_{i+2} - y_{i+1} - y_i = 1$ both have the constant function $y(t) \equiv -1$ as a solution (of course, with different domains $\mathbb{R}$ resp. $\mathbb{N}$). Is this a pure coincidence or an instance of a more general correspondence between the continuous and discrete case?

Hint: It may help to identify the discrete analogue of the exponential function e^t first.

H45 *Optional Exercise*

a) Determine (directly or using the theory you have learned in Discrete Mathematics) the homogeneous linear recurrence relation of order (degree) 3 satisfied by the squares sequence $(n^2)_{n\in\mathbb{N}} = (0, 1, 4, 9, 16, \dots)$.

b) Suppose $\mathbf{x} = (x_0, x_1, x_2 \dots)$ satisfies $a(\mathrm{S})\mathbf{x} = \mathbf{0}$. Show that the sequence of partial sums $\mathbf{y} = (x_0, x_0 + x_1, x_0 + x_1 + x_2, \dots)$ satisfies $(\mathrm{S} - 1)a(\mathrm{S})\mathbf{y} = \mathbf{0}$.

c) Derive an explicit formula for $s_n = \sum_{k=1}^{n} k^2$ by representing the sequence as solution of a 4th-order homogeneous linear recurrence relation and solving this recurrence relation. (The solution is of course well-known, but the exercise provides a conceptual approach which also works for higher powers and other sequences.)

Due on Fri Apr 7, 4 pm

Solution methods for inhomogenenous linear ODE's with constant coefficients (required for H38 and part of H40) will be discussed in the lecture on Mon April 3.

The optional exercises can be handed in until Fri Apr 14, 4 pm.

Solutions (prepared by Liang Tingou and TH)

37 a) The characteristic polynomial is $a(X) = X^2 - 4X + 4$.
The only root is $x = 2$ with multiplicity 2.
So, a real fundamental system of solutions is e^{2t}, te^{2t}.

b) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^3 - 2X^2 - 5X + 6 \\ &= (x-1)(x+2)(x-3) \end{aligned}$$

The roots are $x_1 = -2$, $x_2 = 1$, $x_3 = 3$, all with multiplicity 1.
So, a real fundamental system of solutions is e^{-2t}, e^t, e^{3t}.

c) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^3 - 2X^2 + 2X - 1 \\ &= (X-1)\left(X - \frac{1-\sqrt{3}\,\mathrm{i}}{2}\right)\left(X - \frac{1+\sqrt{3}\,\mathrm{i}}{2}\right) \end{aligned}$$

The roots are $x_1 = 1$, $x_2 = \frac{1-\sqrt{3}\,\mathrm{i}}{2}$, $x_3 = \frac{1+\sqrt{3}\,\mathrm{i}}{2}$ with multiplicities 1.
So, a real fundamental system of solutions is e^t, $e^{\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right)$, $e^{\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)$.

d) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^3 - 1 \\ &= (X-1)\left(X - \frac{-1-\sqrt{3}\,\mathrm{i}}{2}\right)\left(X - \frac{-1+\sqrt{3}\,\mathrm{i}}{2}\right) \end{aligned}$$

The roots are 1, $\frac{-1-\sqrt{3}\,\mathrm{i}}{2}$, $\frac{-1+\sqrt{3}\,\mathrm{i}}{2}$ with multiplicities 1.
So, a real fundamental system of solutions is e^t, $e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)$, $e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$.

e) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^4 + 1 \\ &= X^4 + 2X^2 + 1 - 2X^2 \\ &= (X^2+1)^2 - (\sqrt{2}X)^2 \\ &= (X^2 + \sqrt{2}X + 1)(X^2 - \sqrt{2}X + 1) \\ &= \left(X - \frac{-\sqrt{2}-\sqrt{2}\,\mathrm{i}}{2}\right)\left(X - \frac{-\sqrt{2}+\sqrt{2}\,\mathrm{i}}{2}\right)\left(X - \frac{\sqrt{2}-\sqrt{2}\,\mathrm{i}}{2}\right)\left(X - \frac{\sqrt{2}+\sqrt{2}\,\mathrm{i}}{2}\right) \end{aligned}$$

The roots are $\frac{\pm\sqrt{2}\pm\sqrt{2}\,\mathrm{i}}{2}$ (all 4 combinations) with multiplicities 1.
So, a real fundamental system of solutions is $e^{-\frac{\sqrt{2}}{2}t}\cos\left(\frac{\sqrt{2}}{2}t\right)$, $e^{-\frac{\sqrt{2}}{2}t}\sin\left(\frac{\sqrt{2}}{2}t\right)$, $e^{\frac{\sqrt{2}}{2}t}\cos\left(\frac{\sqrt{2}}{2}t\right)$, $e^{\frac{\sqrt{2}}{2}t}\sin\left(\frac{\sqrt{2}}{2}t\right)$.

f) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^8 + 4X^6 + 6X^4 + 4X^2 + 1 \\ &= (X^2+1)^4 \\ &= (X-\mathrm{i})^4(X+\mathrm{i})^4 \end{aligned}$$

The roots are $x_1 = \mathrm{i}$ and $x_2 = -\mathrm{i}$, each with multiplicity 4.
So, a real fundamental system of solutions is
$\cos(t)$, $t\cos(t)$, $t^2\cos(t)$, $t^3\cos(t)$, $\sin(t)$, $t\sin(t)$, $t^2\sin(t)$, $t^3\sin(t)$.

38 a) Using the method for determining the solution of inhomogeneous linear ODE's stated in the lecture slides, we have $b(t) = 2 = 2e^{0t}$.
So, $\mu = 0$.
The characteristic polynomial is

$$\begin{aligned} a(X) &= X^2 + 3X + 2 \\ &= (X+1)(X+2) \end{aligned}$$

The roots are $x_1 = -1$ and $x_2 = -2$ with multiplicities 1.
So, e^{-t}, e^{-2t} form a fundamental system of solutions.
Since $\mu = 0$ has multiplicity 0, there exists a particular solution of the form $y_p(t) = c_0$.
Substituting it into the ODE gives

$$2c_0 = 2,$$

which gives $c_0 = 1$. (Alternatively, $y_p(t) = \frac{1}{a(\mu)}b(t) = \frac{2}{a(0)} = 1$.)
So, the general real solution is $y(t) = c_1e^{-t} + c_2e^{-2t} + 1$, $c_i \in \mathbb{R}$.

b) $b(t) = 1 + t^2 = (1+t^2)e^{0t}$
So, $\mu = 0$.
The characteristic polynomial is

$$\begin{aligned} a(X) &= X^2 + X - 12 \\ &= (X+4)(X-3) \end{aligned}$$

The roots are $x_1 = -4$ and $x_2 = 3$ with multiplicity 1.
So, e^{-4t}, e^{3t} form a fundamental system of solutions.
Since $\mu = 0$ has multiplicity 0, there exists a particular solution of the form $y(t) = c_0 + c_1t + c_2t^2$. Substituting it into the ODE gives

$$\begin{gathered} 2c_2 + 2c_2t + c_1 - 12(c_2t^2 + c_1t + c_0) = 1 + t^2, \\ -12c_2t^2 + (-12c_1 + 2c_2)t + (-12c_0 + c_1 + 2c_2) = 1 + t^2. \end{gathered}$$

The solution is

$$\begin{cases} c_2 = -\frac{1}{12} \\ c_1 = -\frac{1}{72} \\ c_0 = -\frac{85}{864} \end{cases}$$

Therefore, the general real solution is $y(t) = c_1e^{-4t} + c_2e^{3t} - \frac{85}{864} - \frac{1}{72}t - \frac{1}{12}t^2$, $c_i \in \mathbb{R}$.

c) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^2 - 5X + 6 \\ &= (X-2)(X-3) \end{aligned}$$

The roots are $x_1 = 2$ and $x_2 = 3$ with multiplicity 1. So, e^{2t}, e^{3t} form a fundamental system of solutions.
We now calculate particular solutions for $y'' - 5y' + 6y = 4te^t$ and $y'' - 5y' + 6y = -\sin(t)$.

i) $y'' - 5y' + 6y = 4te^t$
$\mu = 1$ has multiplicity 0.
So, the correct „Ansatz" is $y_1(t) = (c_0 + c_1 t)e^t$, $c_i \in \mathbb{R}$. Substituting it into the ODE gives

$$(2c_0 - 3c_1 + 2c_1 t)e^t = 4te^t,$$

which gives

$$\begin{cases} c_1 = 2 \\ c_0 = 3 \end{cases}$$

So, $y_1(t) = (3+2t)e^t$.

ii) $y'' - 5y' + 6y = -\sin(t)$
We consider the "complexified" ODE $y'' - 5y' + 6y = -e^{\mathrm{i}t}$. The imaginary part of any particular solution of the complex ODE will solve the real ODE.
$b(t) = -e^{\mathrm{i}t}$ gives $\mu = i$, which has multiplicity 0.
So, the complex ODE has a particular solution of the form $y_c(t) = a_0 e^{\mathrm{i}t}$, $a_0 \in \mathbb{C}$. Substituting it into the ODE gives

$$5a_0 e^{\mathrm{i}t} - 6\mathrm{i}e^{\mathrm{i}t} = -e^{\mathrm{i}t},$$

which gives $a_0 = -\frac{1}{10} - \frac{1}{10}\mathrm{i}$ and $y_c(t) = \frac{-1-\mathrm{i}}{10}\,\mathrm{e}^{\mathrm{i}t}$.
(Alternatively, $y_c(t) = \frac{-1}{a(\mathrm{i})}\,\mathrm{e}^{\mathrm{i}t} = \frac{-1}{\mathrm{i}^2-5\mathrm{i}+6}\,\mathrm{e}^{\mathrm{i}t} = \frac{-1}{5-5\mathrm{i}}\,\mathrm{e}^{\mathrm{i}t} = \frac{-1-\mathrm{i}}{10}\,\mathrm{e}^{\mathrm{i}t}$.)
Therefore, $y_2(t) = -\frac{1}{10}\big(\sin(t) + \cos(t)\big)$

In all, a particular solution of $y'' - 5y' + 6y = 4te^t - \sin t$ is $y_p(t) = (3+2t)e^t - \frac{1}{10}(\sin(t) + \cos(t))$. Therefore, the general real solution of this ODE is

$$y(t) = c_1 e^{2t} + c_2 e^{3t} + 3e^t + 2t\mathrm{e}^t - \tfrac{1}{10}\sin(t) - \tfrac{1}{10}\cos(t), \quad c_i \in \mathbb{R}.$$

d) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^3 - 2X^2 + X \\ &= X(X-1)^2. \end{aligned}$$

The roots are $x_1 = 0$ with multiplicity 1 and $x_2 = 1$ with multiplicity 2. So, 1, e^t, te^t form a fundamental system of solutions.
We now calculate particular solutions for $y''' - 2y'' + y' = 1$ and $y''' - 2y'' + y' = e^t\cos(2t)$.

i) $y''' - 2y'' + y' = 1$
$\mu = 0$ has multiplicity 1. So, $y_1(t) = c_0 t$. Substituting it into the ODE gives $c_0 = 1$, which gives $y_1(t) = t$. (This solution can also be found by just looking at the ODE.)

ii) $y''' - 2y'' + y' = e^t \cos(2t)$
We consider the "complexified" ODE $y''' - 2y'' + y' = e^{(1+2\mathrm{i})t}$. The real part of any particular solution of the complex ODE will solve the real ODE.
$b(t) = e^{(1+2\mathrm{i})t}$ gives $\mu = 1 + 2\mathrm{i}$, which has multiplicity 0.
So, the complex ODE has a solution of the form $y_c(t) = a_0 e^{(1+2\mathrm{i})t}$, $a_0 \in \mathbb{C}$. Substituting it into the ODE gives

$$a_0[(1+2\mathrm{i})^3 - 2(1+2\mathrm{i})^2 + (1+2\mathrm{i})]e^{(1+2\mathrm{i})t} = e^{(1+2\mathrm{i})t},$$

which gives $a_0 = -\frac{1}{20} + \frac{1}{10}\mathrm{i}$. (Alternatively, $a_0 = \frac{1}{a(1+2\mathrm{i})}$, which leads to the same result.)
So, $y_2(t) = -\frac{1}{20}\,\mathrm{e}^t \cos(2t)) - \frac{1}{10}\,\mathrm{e}^t \sin(2t)$.

Therefore, the general real solution of $y''' - 2y'' + y' = 1 + \mathrm{e}^t \cos(2t)$ is

$$y(t) = c_0 + c_1 \mathrm{e}^t + c_2 t \mathrm{e}^t + t - \frac{1}{20}\,\mathrm{e}^t \cos(2t)) - \frac{1}{10}\,\mathrm{e}^t \sin(2t), \quad c_i \in \mathbb{R}.$$

e) $b(t) = 25e^{2t}$ gives $\mu = 2$.
The characteristic polynomial is

$$\begin{aligned} a(X) &= X^4 + 2X^2 + 1 \\ &= (X^2+1)^2 \\ &= (X+i)^2(X-i)^2 \end{aligned}$$

The roots are $x_1 = \mathrm{i}$ and $x_2 = -\mathrm{i}$, both with multiplicity 2, which means $\mu = 2$ has multiplicity 0.
$\cos(t)$, $t\cos(t)$, $\sin(t)$, $t\sin(t)$ form a fundamental system of solutions.
There exists a particular solution of the form $y_p(t) = c_0 e^{2t}$. Substituting it into the ODE gives

$$c_0(2^4 + 2 \times 2^2 + 1)e^{2t} = 25e^{2t},$$

which gives $c_0 = 1$. (Alternatively, $c_0 = \frac{25}{a(2)} = \frac{25}{25} = 1$.)
Therefore, the general real solution is

$$y(t) = c_1 \cos(t) + c_2 t \cos(t) + c_3 \sin(t) + c_4 t \sin(t) + e^{2t}, \quad c_i \in \mathbb{R}.$$

f) $b(t) = te^t$ gives $\mu = 1$.
The characteristic polynomial is $a(X) = X^n$ with root 0 of multiplicity n. (For $n = 0$ there is no root, but the multiplicity is still correct.)
Then, 1, t, t^2, ..., t^{n-1} form a fundamental system of solutions.
A particular solution is given by $y(t) = (c_0 + c_1 t)e^t$.
Substituting it into the ODE, we get

$$(c_0 + nc_1 + c_1 t)e^t = te^t,$$

which gives $c_1 = 1$ and $c_0 = -n$.
So, the general real solution is

$$y(t) = (t-n)e^t + \sum_{i=0}^{n-1} a_i t^i, \quad a_i \in \mathbb{R}.$$

For $n = 0$ this is true as well.

39 a) A linear dependency relation $c_0\phi(t)+c_1\phi'(t)+c_2\phi''(t)+\cdots+c_{n-1}\phi^{(n-1)}(t) = 0$, $t \in \mathbb{R}$, says that ϕ solves the ODE $(c_0/c_m)y+(c_1/c_m)y'+\cdots+(c_{m-1}/c_m)y^{(m-1)}+y^{(m)} = 0$, where $m = \max\{0 \leq i \leq n-1; c_i \neq 0\}$. Since this ODE has order smaller than n, under the given assumption this is impossible. Hence ϕ, ϕ', ϕ'', ..., $\phi^{(n-1)}$ are linearly independent in $\mathbb{C}^{\mathbb{R}}$.

Differentiating both sides of (H) shows that ϕ' is a solution of (H) as well. Repeating this argument then gives that all derivatives of ϕ are solutions. Since we know that the solution space S of (H) has dimension n, the n linearly independent solutions ϕ, ϕ', ϕ'', ..., $\phi^{(n-1)}$ must form a basis of S.

b) The characteristic polynomial of this ODE is $X^4 - X^3 - X + 1 = (X^3-1)(X-1) = (X^2+X+1)(X-1)^2$. Its roots are $\lambda_1 = 1$ of multiplicity 2, and $\lambda_{2/3} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$ of multiplicity 1. A fundamental system of solutions is therefore e^t, $t\,\mathrm{e}^t$, $\mathrm{e}^{(-1+\sqrt{3}\,\mathrm{i})t/2}$, $\mathrm{e}^{(-1-\sqrt{3}\,\mathrm{i})t/2}$. We need to find a linear combination of these functions which doesn't solve an ODE of the same type and order < 4. Using the known properties of polynomial differential operators $p(\mathrm{D})$ (cf. lecture) and H42b), it can be shown that any solution $\phi(t) = c_1\mathrm{e}^t + c_2 t\,\mathrm{e}^t + c_3\mathrm{e}^{(-1+\sqrt{3}\,\mathrm{i})t/2} + c_4\mathrm{e}^{(-1-\sqrt{3}\,\mathrm{i})t/2}$ with $c_2c_3c_4 \neq 0$ has this property. (Sketch of proof: $p(\mathrm{D})\phi(t) = c_1p(1)\mathrm{e}^t + c_2p(\mathrm{D})[t\mathrm{e}^t] + c_3p(\lambda_2)\mathrm{e}^{\lambda_2 t} + c_4p(\lambda_3)\mathrm{e}^{\lambda_3 t} = 0$ iff $c_1p(1)\mathrm{e}^t + c_2p(\mathrm{D})[t\mathrm{e}^t] = c_3p(\lambda_2)\mathrm{e}^{\lambda_2 t} = c_4p(\lambda_3)\mathrm{e}^{\lambda_3 t} = 0$ (since e^t, $t\mathrm{e}^t$, $\mathrm{e}^{\lambda_2 t}$, $\mathrm{e}^{\lambda_3 t}$ are linearly independent and $p(\mathrm{D})[t\mathrm{e}^t] \in \langle \mathrm{e}^t, t\mathrm{e}^t\rangle$) iff $p(X)$ is divisible by $(X-1)^2$, $X-\lambda_2$, and $X-\lambda_3$, which in turn implies that $p(\mathrm{D})$ is divisible by $(X-1)^2(X-\lambda_2)(X-\lambda_3) = X^4 - X^3 - X + 1$.) For example, we can take

$$\phi(t) = t\,\mathrm{e}^t + \frac{1}{2}\mathrm{e}^{(-1+\sqrt{3}\,\mathrm{i})t/2} + \frac{1}{2}\mathrm{e}^{(-1-\sqrt{3}\,\mathrm{i})t/2} = t\,\mathrm{e}^t + \mathrm{e}^{-t/2}\cos\left(\sqrt{3}\,t/2\right).$$

Remark: In Linear Algebra terms, a fundamental system of solutions of $a(\mathrm{D})y = 0$ of the form $y, y', \dots, y^{(n-1)}$, $n = \deg a(X)$, amounts to a representation of the solution space as D-cyclic span $\langle y, \mathrm{D}y, \dots, \mathrm{D}^{n-1}y\rangle$. Such a generator y always exists; cf. the solution to H43 a). A different proof, using polynomial arithmetic, can be inferred from the solution above.

40 1) Exercise 4

a) From Hooke's Law, the force required to stretch the spring is

$$k(0.25) = 13,$$

so $k = 13/0.25 = 52$ [N/m]. Adopting the standard units of measurement (N for forces, kg for masses, seconds (s) for time, m for lengths), we get the ODE

$$2\,\frac{d^2x}{dt^2} + 8\,\frac{dx}{dt} + 52\,x = 0$$

The characteristic polynomial of this ODE is $X^2 + 4X + 26$, with roots $x_{1/2} = -2 \pm \sqrt{22}\,\mathrm{i}$, and the solution is

$$x(t) = e^{-2t}\big(c_1 \cos(\sqrt{22}\,t) + c_2 \sin(\sqrt{22}\,t)\big).$$

Since $x(0) = 0$, we have $c_1 = 0$.

$$x'(t) = -2c_2 e^{-2t} \sin(\sqrt{22}\,t) + c_2\sqrt{22}\,e^{-2t} \cos(\sqrt{22}\,t)$$

Since $x'(0) = 0.5$, we have $c_2 = \frac{1}{2\sqrt{22}}$. So, the position (measured in m) at time t (measured in s) is

$$x(t) = e^{-2t}\frac{1}{2\sqrt{22}} \sin(\sqrt{22}\,t).$$

b)

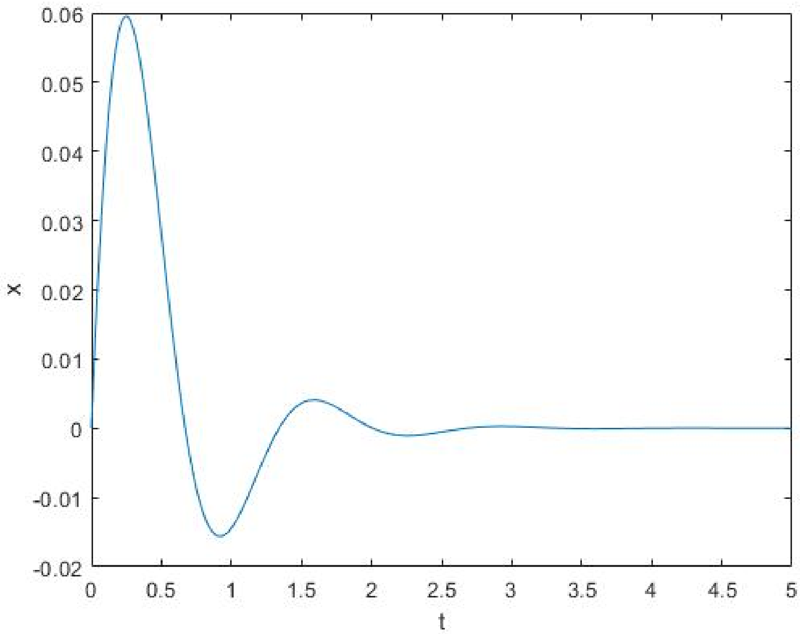

Figure 1: $x(t) = e^{-2t}\frac{1}{2\sqrt{22}} \sin(\sqrt{22}\,t)$

2) Exercise 6
The condition for critical damping is

$$c^2 = 4mk = 4 \times 2 \times 52 = 4^2 \times 26$$

So, $c = 4\sqrt{26}$ [N/m] will produce critical damping.

3) Exercise 14

a) By Kirchhoff's voltage law (and using the indicated standard units of measurement), we have

$$2\,\frac{d^2Q}{dt^2} + 24\,\frac{dQ}{dt} + 200\,Q = 12.$$

The (monic) characteristic polynomial of this ODE is

$$a(X) = X^2 + 12X + 100.$$

The roots are $X = -6 \pm 8\mathrm{i}$, so the solution of the associated homogeneous ODE (called *complementary equation* in [Ste16]) is

$$Q_c(t) = e^{-6t}\big(c_1 \cos(8t) + c_2 \sin(8t)\big).$$

For the method of undetermined coefficients we try a constant solution

$$\begin{aligned} Q_p(t) &= A, \\ Q_p'(t) = Q_p''(t) &= 0. \end{aligned}$$

Inserting this into the ODE gives $A = \frac{12}{200} = \frac{3}{50}$, so a particular solution is

$$Q_p(t) \equiv \frac{3}{50}$$

and the general solution is

$$\begin{aligned} Q(t) &= Q_c(t) + Q_p(t) \\ &= e^{-6t}\big(c_1 \cos(8t) + c_2 \sin(8t)\big) + \frac{3}{50}. \end{aligned}$$

The corresponding current is

$$\begin{aligned} I(t) &= \frac{dQ}{dt} \\ &= \mathrm{e}^{-6t}[(-6c_1 + 8c_2)\cos(8t) + (-8c_1 - 6c_2)\sin(8t)]. \end{aligned}$$

Imposing the initial conditions $Q(0) = 0.001$ and $I(0) = 0$, we get $c_1 = -0.059$, $c_2 = -0.04425$.
Thus, the formula for the charge is

$$Q(t) = \mathrm{e}^{-6t}\big(-0.059 \cos(8t) - 0.04425 \sin(8t)\big) + \frac{3}{50},$$

and the expression for the current is

$$I(t) = 0.7375\, \mathrm{e}^{-6t} \sin(8t).$$

b)

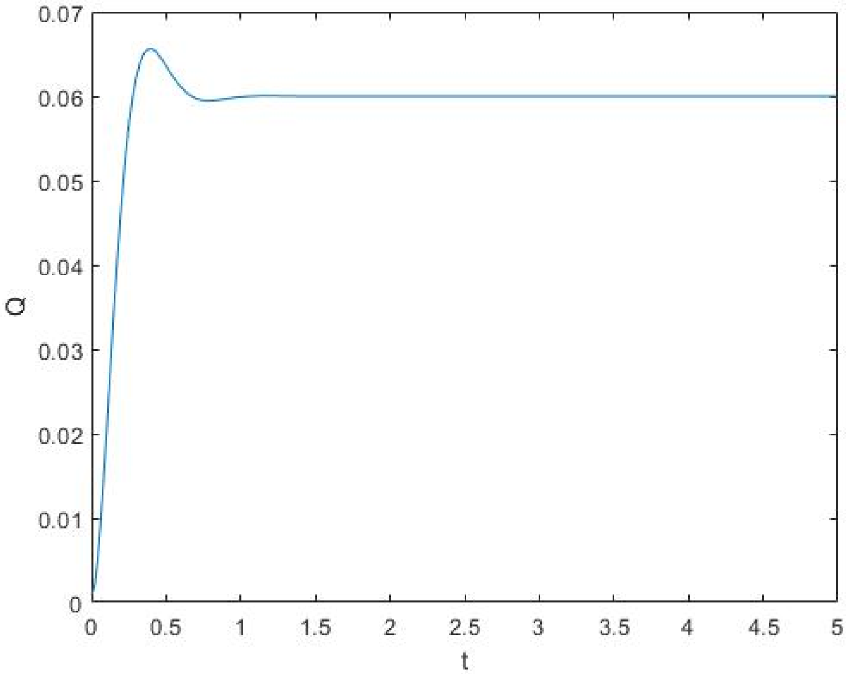

Figure 2: $Q(t) = \mathrm{e}^{-6t}\big(-0.059\cos(8t) - 0.04425\sin(8t)\big) + \frac{3}{50}$

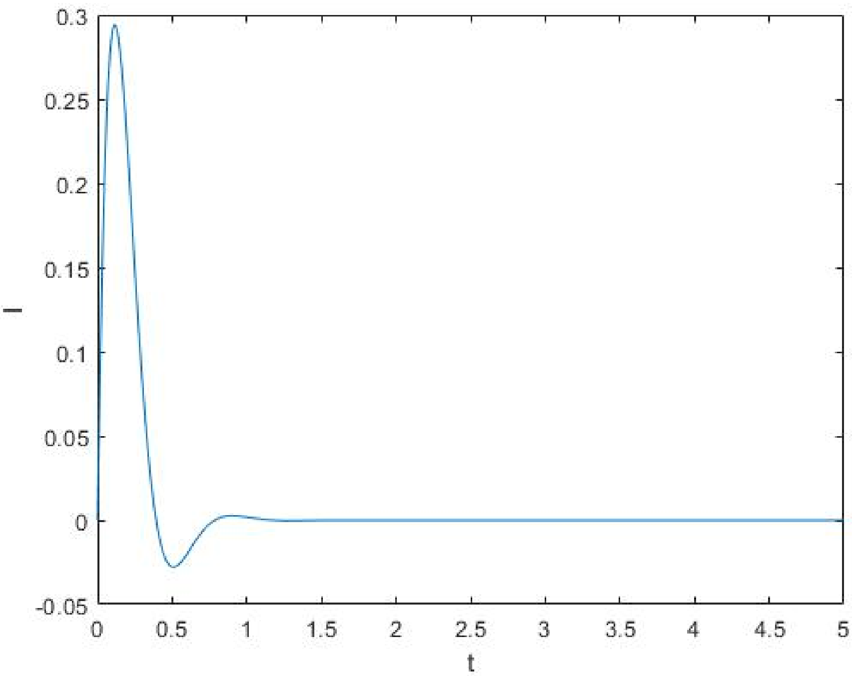

Figure 3: $I(t) = 0.7375\,\mathrm{e}^{-6t}\sin(8t)$

4) Exercise 16

(a) The ODE becomes

$$2\,\frac{d^2Q}{dt^2} + 24\,\frac{dQ}{dt} + 200\,Q = 12\sin(10t). \tag{1}$$

We need to re-compute a particular solution, which can be taken of the form $Q_p(t) = A\cos(10t) + B\sin(10t)$. (This equivalent to the complexification „Ansatz“. Let

$$\begin{aligned} Q_p(t) &= A\cos(10t) + B\sin(10t), \\ Q_p'(t) &= -10A\sin(10t) + 10B\cos(10t), \\ Q_p''(t) &= -100A\cos(10t) - 100B\sin(10t). \end{aligned}$$

Substituting this into Equation (1), we have

$$\begin{aligned} 2(-100A\cos(10t) - 100B\sin(10t)) + 24(-10A\sin(10t) + 10B\cos(10t)), \\ +200(A\cos(10t) + B\sin(10t)) = 12\sin(10t), \\ 240B\cos(10t) - 240A\sin(10t) = 12\sin(10t). \end{aligned}$$

So, $A = -0.05$, $B = 0$, which gives $Q_p(t) = -0.05\cos(10t)$.
The general solution is

$$\begin{aligned} Q(t) &= Q_c(t) + Q_p(t) \\ &= e^{-6t}(c_1\cos(8t) + c_2\sin(8t)) - 0.05\cos(10t) \end{aligned}$$

And

$$\begin{aligned} I(t) &= \frac{dQ}{dt} \\ &= e^{-6t}[(-6c_1 + 8c_2)\cos(8t) + (-8c_1 - 6c_2)\sin(8t)] + 0.5\sin(10t) \end{aligned}$$

Imposing the initial conditions $Q(0) = 0.001$ and $I(0) = 0$, we get $c_1 = 0.051$, $c_2 = 0.03825$.
Thus, the formula for the charge is

$$Q(t) = e^{-6t}(0.051\cos(8t) + 0.03825\sin(8t)) - 0.05\cos(10t)$$

(b)

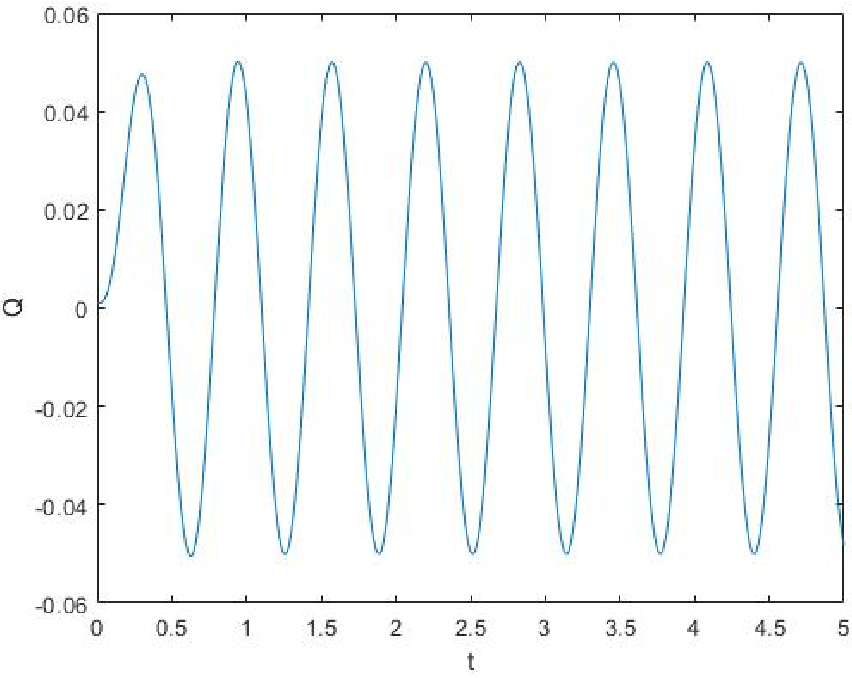

Figure 4: $Q(t) = e^{-6t}(0.051\cos(8t) + 0.03825\sin(8t)) - 0.05\cos(10t)$

41 a) $(\mathrm{D}^2+1)\sin t = (\sin t)'' + \sin t = 0$, $(\mathrm{D}^2+9)\cos(3t) = \cos(3t)'' + 9\cos(3t) = -9\cos(3t) + 9\cos(3t) = 0$,
$\Longrightarrow (\mathrm{D}^2+1)(\mathrm{D}^2+9)[2\sin t - 3\cos(3t)] = 0$, since $(\mathrm{D}^2+1)(\mathrm{D}^2+9)$ annihilates both $\sin t$ and $\cos(3t)$ and hence any linear combination of these functions. Using the same argument as in the solution of H39b), if follows that $(X^2+1)(X^2+9) = X^4+10X^2+9$ is the monic polynomial of smallest degree annihilating ϕ_1, and hence the "monic" ODE of smallest order having ϕ_1 as solution is $y^{(4)} + 10y'' + 9y = 0$.

b) $\phi_2(t) = \frac{1}{4\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t}\right)\left(\mathrm{e}^{3\mathrm{i}t} + \mathrm{e}^{-3\mathrm{i}t}\right) = \frac{1}{4\mathrm{i}}\left(\mathrm{e}^{4\mathrm{i}t} - \mathrm{e}^{-4\mathrm{i}t} - \mathrm{e}^{2\mathrm{i}t} + \mathrm{e}^{-2\mathrm{i}t}\right)$
The corresponding minimum polynomial is $(X-4\mathrm{i})(X+4\mathrm{i})(X-2\mathrm{i})(X+2\mathrm{i}) = (X^2+16)(X^2+4) = X^4+20X^2+64$, and the desired ODE is $y^{(4)} + 20y'' + 64y = 0$.

c) $\phi_3(t) = -\mathrm{e}^{0\,t} + \frac{1}{2}\,t\,\mathrm{e}^{(-2+\mathrm{i})t} + \frac{1}{2}\,t\,\mathrm{e}^{(-2-\mathrm{i})t}$
The corresponding minimum polynomial is $X(X+2-\mathrm{i})^2(X+2+\mathrm{i})^2 = X(X^2+4X+5)^2 = X^5+8\,X^4+26\,X^3+40\,X^2+25\,X$, and the desired ODE is $y^{(5)}+8y^{(4)}+26y^{(3)}+40y''+25y' = 0$.

d) The desired minimal ODE is $(\mathrm{D}-1)\mathrm{D}^{2020}y = (\mathrm{D}^{2021} - \mathrm{D}^{2020})y = y^{(2021)} - y^{(2020)} = 0$.

42 It is clear that there is a nonzero polynomial $m(X)$ of smallest degree satisfying $m(\mathrm{D})y = 0$, and division of $m(X)$ by the leading coefficient shows that $m(X)$ can be taken as a monic polynomial.
Now suppose $b(\mathrm{D})y = 0$. Long division of $b(X)$ by $m(X)$ gives polynomials $q(X), r(X) \in \mathbb{C}[X]$ with $\deg r(X) < \deg m(X)$ (possibly $r(X) = 0$) and $b(X) = q(X)m(X) + r(X)$.

Substituting D shows $r(\mathrm{D})y = b(\mathrm{D})y - q(\mathrm{D})m(\mathrm{D})y = 0$. By minimality of $m(X)$ this is possible only if $r(X) = 0$; $\Longrightarrow m(X)$ divides $b(X)$ in $\mathbb{C}[X]$.
This proves b). For the proof of a) suppose that $m_1(X) \in \mathbb{C}[X]$ is another monic polynomial of smallest degree satisfying $m_1(\mathrm{D})y = 0$. Then b) gives $m(X) \mid m_1(X)$. Since $m(X)$ and $m_1(X)$ have the same degree, they must be constant multiples of each other. The constant must be 1 because $m(X)$, $m_1(X)$ are both monic. Thus $m_1(X) = m(X)$, and $m(X)$ is unique.

43 a) As observed repeatedly, $a(\mathrm{D})y = 0$ implies $a(\mathrm{D})\mathrm{D}y = a(\mathrm{D})y' = 0$. Thus V is D-invariant.

Let $f = \mathrm{D}|_V$. Then $a(f) = 0$ because of the ODE, and hence the minimum polynomial $\mu_f(X)$ of f divides $a(X)$. If there exists a solution $y \in V$ such that $y, y', \dots, y^{(n-1)}$ are linearly independent (i.e., the D-cyclic subspace of V generated by y is equal to V), we can conclude as in b) that $\chi_f(X) = a(X)$. The solution in b) also indicates how to find such a function: Using the EUT, prescribe initial conditions $y(0) = y'(0) = \dots = y^{(n-2)}(0) = 0$, $y^{(n-1)}(0) = 1$ or, in vectorial form, $\bigl(y(0), y'(0), \dots, y^{(n-1)}(0)\bigr) = (0, \dots, 0, 1)$. Then $y^{(k)}$ has initial values $\bigl(y^{k)}(0), y^{(k+1)}(0), \dots, y^{(k+n-1)}(0)\bigr) = (\underbrace{0, \dots, 0}_{n-1-k}, 1, *, *, \dots)$.

Since the vector of initial values determines the solution (by the EUT), one sees as in b) that $y, y', \dots, y^{(n-1)}$ are linearly independent.

b) $a(\mathrm{S})\mathbf{y} = \mathbf{0}$ implies $a(\mathrm{S})\mathrm{S}\mathbf{y} = \mathrm{S}a(\mathrm{S})\mathbf{y} = \mathrm{S}\mathbf{0} = \mathbf{0}$. Thus V is S-invariant.

Let $f = \mathrm{S}|_V$. Then $a(f) = 0$ because of the recurrence relation, and hence the minimum polynomial $\mu_f(X)$ of f divides $a(X)$. If there exists a sequence $\mathbf{y} \in V$ such that $\mathbf{y}$, $\mathrm{S}\mathbf{y}$, $\dots$, $\mathrm{S}^{(n-1)}\mathbf{y}$ are linearly independent (i.e., the S-cyclic subspace of V generated by $\mathbf{y}$ is equal to V), we can conclude that $\mu_f(X) = a(X)$, and then in turn $\chi_f(X) = a(X)$, because $\deg a(X) = n = \dim V$. A sequence $\mathbf{y} \in V$ with this property is easily found: Just use the last member $\mathrm{e}_{n-1} = (\underbrace{0, \dots, 0}_{n-1}, 1, *, *, \dots)$ of the canonical basis of V. (The remaining entries of e_{n-1} are computed from the first n entries and the recurrence relation.) The matrix formed by the first n coordinates of e_{n-1}, $\mathrm{S}\mathrm{e}_{n-1}$, $\dots$, $\mathrm{S}^{n-1}\mathrm{e}_{n-1}$ is triangular with 1's on the main diagonal, showing that these sequences are linearly independent.

44 The discrete analogue of the exponential function, which satisfies $\mathrm{D}\mathrm{e}^t = \mathrm{e}^t$, is the sequence $\mathbf{e} = (1, 1, 1, \dots)$, since $\mathrm{S}\mathbf{x} = \mathbf{x}$ iff $\mathbf{x}$ is a constant sequence. Thus the solution $y_i = -1$ in the discrete case is rather the analogue of $-\mathrm{e}^t$ than of the constant function $y(t) \equiv -1$ on $\mathbb{R}$, and the observed phenomenon is merely a coincidence.
In fact we should think of $y_{i+2} - y_{i+1} - y_i = 1$ as the discrete analogue of $y'' - y' - y = \mathrm{e}^t$, which has the solution $y(t) = -\mathrm{e}^{-t}$. This is so, because the right-hand side of $y_{i+2} - y_{i+1} - y_i = 1$ is actually a sequence, viz. $(1, 1, 1, \dots)$. In order to complete the picture, consider the recurrence relation

$$y_{i+2} - y_{i+1} - y_i = \delta_{i0} = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i \geq 1. \end{cases}$$

This is the true discrete analogue of $y'' - y' - y = 1$. It has the solution $y_0 = -1$, $y_i = 0$ for $i \geq 1$, i.e., $\mathbf{y} = (-1, 0, 0, \dots) = -(\delta_{i0})$, which is the true discrete analogue of $y(t) \equiv -1$.

The characteristic polynomial $a(X) = X^2 - X - 1$ of the ODE/recurrence relation satisfies $a(0) = a(1) = -1$. This explains why in both cases the coefficient -1 appears: $\mathrm{D}1 = \mathrm{D}\mathrm{e}^{0\,t} = a(0)1 = -1$, $\mathrm{D}\mathrm{e}^t = a(1)\mathrm{e}^t = -\mathrm{e}^t$. Thus $a(\mathrm{D})y = 1$ is solved by $y(t) \equiv -1$ and $a(\mathrm{D})y = \mathrm{e}^t$ by $y(t) = -\mathrm{e}^t$. The corresponding discrete analogues are: $a(\mathrm{S})\mathbf{y} = (\delta_{i0})$ is solved by $\mathbf{y} = -(\delta_{i0})$ and $a(\mathrm{S})\mathbf{y} = \mathbf{e}$ by $\mathbf{y} = -\mathbf{e}$. The full story is told in `lecture23-27` (in the section "The View from the Top"): The exponential generating function map egf identifies solutions of the linear recurrence relation $a(\mathrm{S})\mathbf{y} = \mathbf{b}$ with solutions of the linear ODE $a(\mathrm{D})y = \mathrm{egf}(\mathbf{b})$.

45 a) Students of Discrete Mathematics have learned that the homogeneous linear recurrence relation with characteristic polynomial $(X-1)^3$, i.e., $y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = 0$ has the sequences (1), (n), (n^2) as basic solutions. Alternatively, we can write

$$\begin{array}{rclllll} n^2 &=& n^2 \\ (n+1)^2 &=& n^2 &+& 2n &+& 1 \\ (n+2)^2 &=& n^2 &+& 4n &+& 4 \\ (n+3)^2 &=& n^2 &+& 6n &+& 9 \end{array}$$

and solve $c_3(n+3)^2 + c_2(n+2)^2 + c_1(n+1)^2 + c_0n^2 = (c_0 + c_1 + c_2 + c_3)n^2 + 2c_1 + 4c_2 + 6c_3)n + (c_1 + 4c_2 + 9c_3) = 0$. The solution is $(c_0, c_1, c_2, c_3) = (-c, 3c, -3c, c)$, $c \in \mathbb{R}$, leading to the same recurrence relation.

b) We have

$$\begin{aligned}(\mathrm{S} - 1)\mathbf{y} = \mathrm{S}\mathbf{y} - \mathbf{y} &= (x_0 + x_1, x_0 + x_1 + x_2, \dots) - (x_0, x_0 + x_1, \dots) \\ &= (x_1, x_2, \dots) = \mathrm{S}\mathbf{x}\end{aligned}$$

and hence

$$(\mathrm{S} - 1)a(\mathrm{S})\mathbf{y} = a(\mathrm{S})(\mathrm{S} - 1)\mathbf{y} = a(\mathrm{S})\mathrm{S}\mathbf{x} = \mathrm{S}a(\mathrm{S})\mathbf{x} = \mathrm{S}\mathbf{0} = \mathbf{0}.$$

c) From a), b) it follows that (s_n) satisfies a linear recurrence relation with characteristic polynomial $(X - 1)^4$ and hence $s_n = c_0 + c_1n + c_2n^2 + c_3n^3$ for some $c_0, c_1, c_2, c_3 \in \mathbb{R}$. Plugging in the initial values $s_0 = 0$, $s_1 = 1$, $s_2 = 1 + 4 = 5$, $s_3 = 1 + 4 + 9 = 14$ gives the linear system

$$\begin{array}{rcrcrcrcr} c_0 & & & & & & &=& 0, \\ c_0 &+& c_1 &+& c_2 &+& c_3 &=& 1, \\ c_0 &+& 2c_1 &+& 4c_2 &+& 8c_3 &=& 5, \\ c_0 &+& 3c_1 &+& 9c_2 &+& 27c_3 &=& 14. \end{array}$$

Solving the resulting system for c_1, c_2, c_3 we find

$$\left.\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 5 \\ 3 & 9 & 27 & 14 \end{array}\right. \longrightarrow \left.\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 2 & 6 & 3 \\ 0 & 6 & 24 & 11 \end{array}\right. \longrightarrow \left.\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 2 & 6 & 3 \\ 0 & 0 & 6 & 2 \end{array}\right.$$

and the solution $c_3 = 1/3$, $c_2 = (3 - 6c_3)/2 = 1/2$, $c_2 = 1 - c_2 - c_3 = 1/6$, so that

$$s_n = \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n = \tfrac{1}{6}n(n + 1)(2n + 1).$$