Differential Equations (Math 285)

H12 Determine the general solution of the following ODE's in terms of $y(0)$ (three answers suffice).

a) $\mathrm{d}y/\mathrm{d}t = e^{y+t}$;
b) $\mathrm{d}y/\mathrm{d}t = ty + y + t$;
c) $\mathrm{d}y/\mathrm{d}t = (\cos t)y + 4\cos t$;
d) $\mathrm{d}y/\mathrm{d}t = t^m y^n$ $(m, n \in \mathbb{Z})$.

H13 For the following ODE's, solve the corresponding IVP with $y(0) = 1$.

a) $\mathrm{d}y/\mathrm{d}t = -4ty$;
b) $\mathrm{d}y/\mathrm{d}t = t\,y^3$;
c) $(1+t)\mathrm{d}y/\mathrm{d}t = 4y$.

H14 Determine all maximal solutions of $t^2\,y' = y^2$ and decide for which points $(t_0, y_0) \in \mathbb{R}^2$ the IVP $t^2\,y' = y^2 \wedge y(t_0) = y_0$ has no solution/exactly one solution/more than one solution.

H15

a) According to `worldometers.info`, the world's population on Ju1y 1, 2020 was about 7.79 billion, with a 1.05 % increase since July 1, 2019. Use this data to determine a new logistic model for the world's population growth, and compare with that of the lecture. What is the limiting population according to the new model?

b) Show that the graph of $y(t) = a/(d\,\mathrm{e}^{-at} + b)$ $(a, b, d > 0)$ is point-symmetric to its inflection point.

   Hint: A superb way to solve this exercise is to observe that the mirror image of a solution curve w.r.t. its inflection point represents a solution as well and use the uniqueness of solutions of associated IVP's.

H16

a) Explain how to adapt the analysis of the harvesting equation in the lecture to $y' = ay^2 + by + c$ with $a, b, c \in \mathbb{R}$ and $a > 0$.

b) Sketch the solution curves of (i) $y' = y^2 - y + 1$, (ii) $y' = y^2 + 2y + 1$, (iii) $y' = y^2 + y - 2$ without actually computing solutions. Steady-state solutions and inflection points (if any) should be drawn exactly.

H17 The ODE $y' = a(t)y - b(t)y^n$, $n \in \mathbb{R} \setminus \{0, 1\}$ is called *Bernoulli's differential equation.*

a) Show that for an appropriate choice of $\beta \in \mathbb{R}$ the substitution $z = y^\beta$ turns Bernoulli's differential equation into a linear 1st-order ODE (which can be solved by the usual methods).

b) Solve the IVP $y' = 4y - y^3 \wedge y(0) = 1$ by the method suggested in a).

c) Investigate the asymptotic stability of the steady-state solutions of the ODE in b).

H18 *Optional exercise*

a) Show that the general (real) solution of $y'' = y$ is $y(x) = c_1 e^x + c_2 e^{-x}$, $c_1, c_2 \in \mathbb{R}$.

Hint: For a solution y the functions $y + y'$ and $y - y'$ satisfy linear 1st-order ODE's.

b) For $x \in \mathbb{R}$ let

$$F(x) = \int_0^\infty \frac{\cos(xt)}{t^2+1}\,\mathrm{d}t\,.$$

Show that

$$F'(x) = -\frac{\pi}{2} + \int_0^\infty \frac{\sin(xt)}{t(t^2+1)}\,\mathrm{d}t \quad \text{for } x > 0.$$

Hint: Differentiate F under the integral sign and use $\int_0^\infty \sin(xt)/t\,\mathrm{d}t = \int_0^\infty \sin(t)/t\,\mathrm{d}t = \pi/2$ for $x > 0$.

c) Show that F solves $y'' = y$ on $(0, \infty)$.

d) Determine F from a), c) and $F(0)$, $F'(0+)$, and use the result to evaluate the integral

$$\int_0^\infty \frac{\cos t}{t^2+1}\,\mathrm{d}t\,.$$

H19 *Optional exercise*

The task of this exercise is to show the Cauchy-Hadamard formula

$$R = \frac{1}{L}, \quad L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

(with the conventions $1/0 = \infty$, $1/\infty = 0$) for the radius of convergence R of a (complex) power series $\sum_{n=0}^\infty a_n(z-a)^n$. Here $L = \limsup_{n\to\infty} x_n \in [-\infty, +\infty]$ (*limit superior*) denotes the largest accumulation point of a real sequence (x_n), i.e., for every $\epsilon > 0$ there are only finitely many indexes n satisfying $x_n \geq L + \epsilon$ but no real number $L' < L$ has this property (with suitable modifications for $L = \pm\infty$).

a) If $L = \infty$ (i.e., $\sqrt[n]{|a_n|}$ is unbounded), show that $\sum_{n=0}^\infty a_n(z-a)^n$ converges only for $z = a$.

b) If $L = 0$ (i.e., $\sqrt[n]{|a_n|}$ converges to zero), show that $\sum_{n=0}^\infty a_n(z-a)^n$ converges for all $z \in \mathbb{C}$.

c) If $0 < L < \infty$, show that $\sum_{n=0}^\infty a_n(z-a)^n$ converges for $|z-a| < 1/L$ and diverges for $|z-a| > 1/L$.

Due on Fri Mar 10, 4 pm

The optional exercises can be handed in until Fri Mar 17, 4 pm. Complex power series (relevant for H19) will be discussed in the lecture on Fri Mar 10. For Exercises H18 b)–d) you need to study the material on uniform convergence of improper parameter integrals in `lecture11-13_handout.pdf`, which won't be discussed in the Math 285 lecture. Exercise H18 a) is very instructive (also quite easy), and is recommended for every student.

Solutions

12 a)

$$e^{-y}\,\mathrm{d}y = e^t\,\mathrm{d}t$$

Integrating both sides of the equation, we get

$$\int_{y(0)}^{y} e^{-r}\,\mathrm{d}r = \int_0^t e^s\,\mathrm{d}s$$
$$-e^{-y} + e^{-y(0)} = e^t - 1$$

Finally, we obtain

$$y(t) = -\ln(e^{-y(0)} + 1 - e^t), \quad t < \ln(1 - e^{-y(0)}).$$

Remark: When determining the solution, one can also use indefinite integration $\int e^{-y}\,\mathrm{d}y = e^t\,\mathrm{d}t + C$ and determine C in terms of $y(0)$. This applies to the subsequent exercises as well.

b) Rewrite the ODE in the form of $y' = a(t)y + b(t)$:

$$y' = (t+1)y + t$$

According to the particular solution formula,

$$\begin{aligned}
y_p(t) &= e^{\frac{t^2}{2}+t}\int_0^t s\,e^{-(\frac{s^2}{2}+s)}\,\mathrm{d}s \\
&= e^{\frac{t^2}{2}+t}\left(\int_0^t (s+1)e^{-(\frac{s^2}{2}+s)}\,\mathrm{d}s - \int_0^t e^{-(\frac{s^2}{2}+s)}\,\mathrm{d}s\right) \\
&= -e^{\frac{t^2}{2}+t}\left(e^{-(\frac{t^2}{2}+t)} - 1\right) - e^{\frac{1}{2}}\int_0^t e^{-\left(\frac{s^2}{2}+s+\frac{1}{2}\right)}\,\mathrm{d}s \\
&= e^{\frac{t^2}{2}+t} - 1 - e^{\frac{1}{2}}\int_0^t e^{-\left(\frac{s+1}{\sqrt{2}}\right)^2}\,\mathrm{d}s,
\end{aligned}$$

and the “homogeneous solution” is

$$y_h(t) = y(0)\,e^{\frac{t^2}{2}+t}$$

Since $y_p(0) = 0$, the general solution in terms of $y(0)$ is

$$y(t) = y(0)e^{\frac{t^2}{2}+t} - 1 - e^{\frac{1}{2}}\int_0^t e^{-\left(\frac{s+1}{\sqrt{2}}\right)^2}\,\mathrm{d}s\,.$$

Remark: It is not necessary to rewrite the integrand occuring in $y_p(t)$ in the particular form shown above, but at least this shows the relation with the incomplete Gauss integral (or the so-called error function). The simple answer is $y(t) = y_p(t) + y_h(t)$, $t \in \mathbb{R}$, with y_p, y_h as above.

c) According to the particular solution formula,

$$
\begin{align}
y_p(t) &= e^{\sin(t)} \int_0^t 4\cos(s) e^{-\sin(s)} \,\mathrm{d}s \tag{1}\\
&= -4e^{\sin(t)}(e^{-\sin(t)} - 1) \tag{2}\\
&= 4e^{\sin(t)} - 4, \tag{3}
\end{align}
$$

and the “homogeneous solution” is

$$y_h(t) = y(0)e^{\sin(t)}.$$

The general solution is then

$$y(t) = (y(0) + 4)e^{\sin(t)} - 4.$$

The general form $y(t) = C\mathrm{e}^{\sin t} - 4$, $C \in \mathbb{R}$, also follows from the observation that $y(t) \equiv -4$ is a particular solution.

d) There is the constant solution $y = 0$, and for $y \neq 0$ we can separate:

$$\frac{dy}{y^n} = t^m \,\mathrm{d}t\,.$$

Integrating both sides, we get

$$
\begin{gathered}
\int_{y(0)}^{y} \frac{1}{r^n} \,\mathrm{d}r = \int_0^t s^m \,\mathrm{d}s \\
-\frac{1}{(n-1)y^{n-1}} + \frac{1}{(n-1)y(0)^{n-1}} = \frac{t^{m+1}}{m+1}, \quad (n \neq 1, \quad m \neq -1).
\end{gathered}
$$

Then, we obtain the general solution

$$y(t) = \left[(n-1)\left(\frac{1}{(n-1)y(0)^{n-1}} - \frac{t^{m+1}}{m+1}\right)\right]^{-\frac{1}{n-1}} \quad (n \neq 1, \quad m \neq -1).$$

Next, we deal with the special cases:

i) $n = 1, \quad m = -1$

$$
\begin{gathered}
\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}t}{t} \\
\ln|y| = \ln|t| + C
\end{gathered}
$$

Finally, we obtain, with a different parameter $C' \in \mathbb{R}$,

$$y(t) = C't, \quad t \in (-\infty, 0) \text{ or } t \in (0, +\infty).$$

$y(0)$ is not defined in this case.

ii) $n = 1, \quad m \neq -1$

$$\frac{\mathrm{d}y}{y} = t^m dt$$

Integrating both sides, we get

$$\int_{y(0)}^{y} \frac{1}{r}\,\mathrm{d}r = \int_0^t s^m\,\mathrm{d}s,$$

$$\ln|y| - \ln|y(0)| = \frac{t^{m+1}}{m+1}.$$

Finally, noting that $y(t)$ and $y(0)$ must have the same sign, we obtain

$$y(t) = y(0)e^{\frac{t^{m+1}}{m+1}}, \quad t \in \mathbb{R}.$$

.

iii) $n \neq 1, \quad m = -1$

$$\frac{\mathrm{d}y}{y^n} = \frac{\mathrm{d}t}{t}$$

Integrate both sides, we get

$$-\frac{1}{(n-1)y^{n-1}} = \ln|t| + C$$

Finally, we obtain

$$y(t) = (-(n-1)(\ln|t| + C))^{-\frac{1}{n-1}}, \quad t < -e^{-C} \text{ or } t > e^{-C}.$$

$y(0)$ is not defined in this case.

13 a) $dy/dt = -4ty$

This is a homogeneous linear ODE, so we get

$$y(t) = Ce^{-2t^2}$$

Plugging into the IVP $y(0) = 1$, we can obtain the solution as

$$y(t) = e^{-2t^2}, \quad t \in \mathbb{R}.$$

b) $dy/dt = ty^3$

This is a separable ODE, so we can write

$$\frac{dy}{y^3} = tdt$$

$$\int_1^y \frac{1}{r^3}\,dr = \int_0^t s\,ds$$

The solution is

$$y(t) = (1 - t^2)^{-\frac{1}{2}}, \quad -1 < t < 1.$$

c) $(1+t)dy/dt = 4y$
Rewrite the ODE as

$$y' = \frac{4}{t+1}y.$$

We use the "homogeneous solution formula" to get

$$y(t) = Ce^{4\ln|t+1|} = C(t+1)^4.$$

Plugging into the IVP $y(0) = 1$, we obtain the solution as

$$y(t) = (t+1)^4, \quad t \in \mathbb{R}.$$

14 We can rewrite this separable ODE as

$$\frac{dy}{y^2} = \frac{dt}{t^2} \quad (y, t \neq 0)$$

Integrating both sides of the above equation, we get

$$\int_{y_0}^{y} \frac{1}{\eta^2}\,\mathrm{d}\eta = \int_{t_0}^{t} \frac{1}{\tau^2}\,\mathrm{d}\tau$$
$$-\frac{1}{y} + \frac{1}{y_0} = -\frac{1}{t} + \frac{1}{t_0}$$

which gives

$$y(t) = \frac{t_0y_0}{\frac{t_0y_0}{t} - (y_0 - t_0)} = \frac{(t_0y_0)t}{t_0y_0 - (y_0 - t_0)t} = \frac{t}{1 - Ct} \quad \text{with} \quad C := \frac{y_0 - t_0}{t_0y_0}.$$

Removal of the coordinate axes splits the (t, y)-plane into 4 quadrants ("small rectangles" in the parlance of the lecture), and every solution that is contained entirely in one of these quadrants must be of this form.
For $t = 0$ we must have $y(t) = 0$ (from $t^2y' = y^2$).
There is the constant solution $y(t) = 0$, $t \in \mathbb{R}$.

1) $t_0 = y_0 \neq 0$

$$y(t) = \begin{cases} t, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

And $y(0) = 0$ fits with the expression $y(t) = t$, so we can write the solution as $y(t) = t$, $t \in \mathbb{R}$.
There is only one solution.

2) $t_0 = y_0 = 0$
Every solution of the ODE defined at $t = 0$ must satisfy $y(0) = 0$ (see above). The non-constant (maximal) solutions are

$$y(t) = \frac{t}{1 - Ct}, \quad C \in \mathbb{R}.$$

with domain $\mathbb{R}$ if $C = 0$, $(-\infty, 1/C)$ if $C > 0$, and $(1/C, +\infty)$ if $C < 0$. In particular there are an infinite number of solutions.

3) $t_0 = 0, y_0 \neq 0$
This IVP contradicts $y(0) = 0$. Therefore, there is no solution.

4) $t_0 \neq 0, y_0 = 0$
The solution is

$$y(t) = 0$$

Therefore, there is only one solution.

5) $0 \neq t_0 \neq y_0 \neq 0$

$$y(t) = \frac{t}{1 - \frac{y_0 - t_0}{t_0 y_0}\, t}, \quad t \neq \frac{t_0 y_0}{y_0 - t_0}$$

Therefore, there is only one solution.

In conclusion, the IVP $t^2 y' = y^2 \wedge y(t_0) = y_0$ has

1) no solution when $t_0 = 0,\ y_0 \neq 0$;

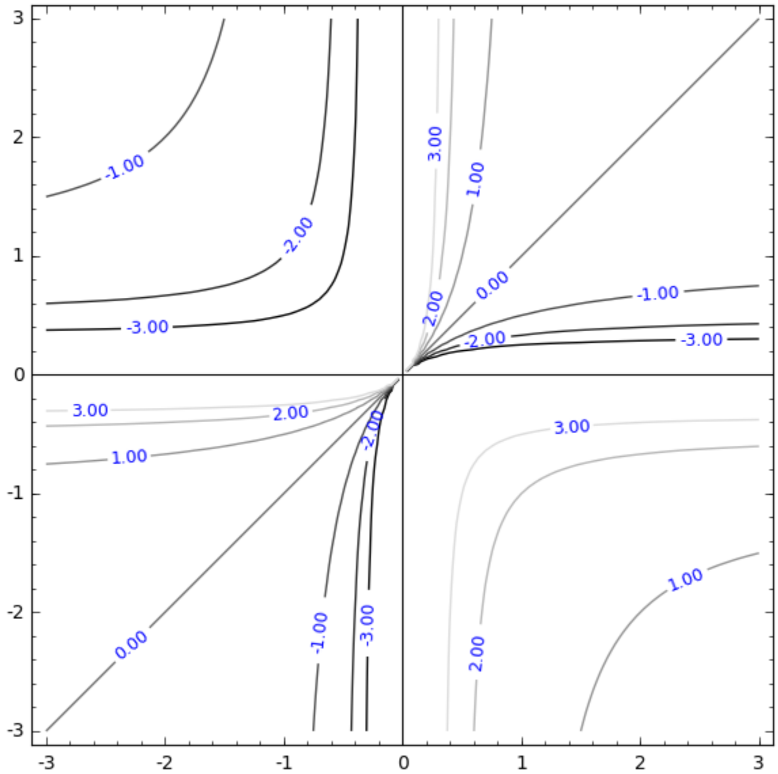
2) infinitely many (maximal) solutions when $t_0 = y_0 = 0$;

3) exactly one (maximal) solution otherwise.

Moreover, the maximal solutions are

$$\begin{aligned}
y(t) &= 0, && t \in \mathbb{R};\\
y(t) &= t, && t \in \mathbb{R};\\
y_C^-(t) &= \frac{t}{1 - Ct}, && t \in (-\infty, 1/C);\\
y_C^+(t) &= \frac{t}{1 - Ct}, && t \in (1/C, +\infty).
\end{aligned}$$

The 3rd and 4th type of solutions exist for any real number $C \neq 0$.
A better picture can be obtained by solving $y = t/(1 - Ct)$ for C, which gives $C = (y - t)/(ty)$ and shows that $F(t, y) = (y - t)/(ty)$ provides a first integral for the given ODE.



From the contour plot of F you can see that all non-constant solutions that are defined at $t = 0$ share the same tangent at $(0, 0)$. This also follows from $\frac{\mathrm{d}}{\mathrm{d}t}\frac{t}{1-Ct} = \frac{1}{(1-Ct)^2}$. Solutions $y_C^+(t)$ with $C > 0$ fill the 4th quadrant, solutions $y_C^-(t)$ with $C > 0$ fill the region above $y = t$ in the 1st and 3rd quadrant, etc. All these properties can also be derived with some effort from the formulas.

Remark: With the Existence and Uniqueness Theorems now at hand, we can easily get a complete qualitive picture. Rewriting $t^2 y' = y^2$ as $y^2\,\mathrm{d}t - t^2\,\mathrm{d}y = 0$, we see that the origin $(t_0, y_0) = (0, 0)$ is the only singular point, and hence that through any other point there passes precisely one integral curve (solution curve). For points $(0, y_0)$ with $y_0 \neq 0$ this is the curve $t = 0$, which cannot be seen from $t^2 y' = y^2$, because it can be parametrized only as $t(y)$.

15 a) The new logistic model is $y(t) = \frac{a}{d' \mathrm{e}^{-a(t-2020)} + b}$ with $a = 0.029$ (natural reproduction rate of humans) and b, d' determined from $\frac{y'(2020)}{y(2020)} = a - b\,y(2020) = 0.0105$, $y(2020) = \frac{a}{d'+b} = 7.79 \times 10^9$. The solution is $b = \frac{37}{1558} \times 10^{-10}$, $d' = \frac{21}{1558} \times 10^{-10}$, so that

$$y(t) = \frac{0.029 \times 10^{10}}{\frac{21}{1558}\,\mathrm{e}^{-0.029(t-2020)} + \frac{37}{1558}}$$

and the limiting population is $a/b \approx 12.2$ billion people.

b) With the Hint, we want to prove that the mirror image of a solution curve w.r.t. its inflection point represents a solution as well and use the uniqueness of solutions of associated IVP's.

The function of the mirror image is

$$g(t) = \frac{a}{b} - \frac{a}{de^{-a(2t_h - t)} + b}$$

where $t_h = (\ln d - \ln b)/a$.
First, we prove that $g(t)$ is a solution to the ODE $y' = ay - by^2$.

$$g'(t) = -\frac{a^2 de^{a(2t_h - t)}}{(de^{a(2t_h-t)} + b)^2}$$

and

$$\begin{aligned}
ag(t) - bg^2(t) &= \frac{a^2}{b} - \frac{a^2}{de^{-a(2t_h-t)} + b} - b\Big(\frac{a^2}{b^2} - \frac{2a^2}{b(de^{-a(2t_h-t)} + b)} + \frac{a^2}{(de^{-a(2t_h-t)} + b)^2}\Big) \\
&= \frac{-a^2 de^{-a(2t_h-t)} - a^2 b + 2a^2 b - a^2 b}{(de^{-a(2t_h-t)} + b)^2} \\
&= -\frac{a^2 de^{-a(2t_h-t)}}{(de^{-a(2t_h-t)} + b)^2}
\end{aligned}$$

Thus, $g'(t) = ag(t) - bg^2(t)$, which means $g(t)$ is also a solution to the ODE $y' = ay - by^2$.
Then, we will use the uniqueness of the solution of th IVP $y' = ay - by^2 \wedge y(t_h) = a/2b$. Since the original solution curve has the inflection point $(t_h, a/2b)$, it shares the same IVP with the mirror image $g(t)$. The logistic equation has a unique solution for any given IVP, so $y(t) = g(t)$.

Remark: The computation can be simplified a little by using the observation that $y(t)$ solves $y' = ay - by^2$ iff $t \mapsto y(t - t_0)$, $t_0 \in \mathbb{R}$, does.

16 a) It should be noted that the analysis in the lecture used the notation $y' = ay - by^2 - h$, where $a, b, h > 0$. However, the parabola $f(y) = y' = ay^2 + by + c$, $a > 0$, is a vertically flipped version of that considered in the lecture. This discrepancy will lead to different behaviors of the solution curves.
The discriminant is $\Delta = b^2 - 4ac$. For $\Delta \geq 0$, there are the steady-state solutions

$$y_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$y_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

where $0 < y_1 \leq y_2$.

i) $c < b^2/4a$
If the initial condition $y(t_0)$ satisfies $y_1 < y(t_0) < y_2$, then $y(t)$ decreases and $\lim_{t\to\infty} y(t) = y_1$.
If $y(t_0) > y_2$, then $y(t)$ increases to ∞. If $y(t_0) < y_1$, then $y(t)$ increases and $\lim_{t\to\infty} y(t) = y_1$.

ii) $c = b^2/4a$
If $y(t_0) > -b/2a$, then $y(t)$ increases to ∞. If $y(t_0) < -b/2a$, then $y(t)$ increases and $\lim_{t\to\infty} y(t) = y_1$.

iii) $c > b^2/4a$
Regardless of the initial condition, $y(t)$ will increase to ∞.

b) i) $y' = y^2 - y + 1$

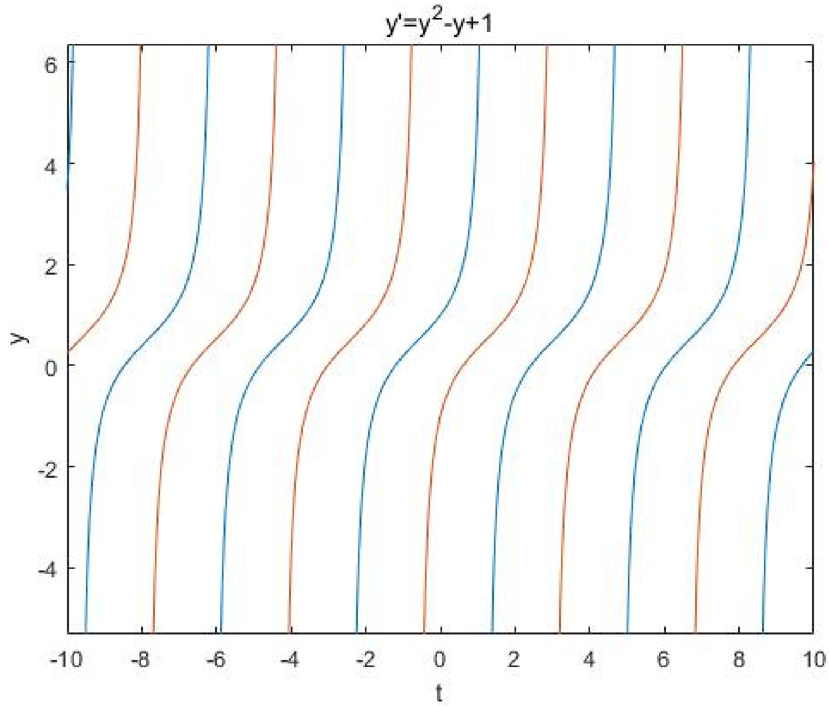


Figure 1: $y' = y^2 - y + 1$

ii) $y' = y^2 + 2y + 1$

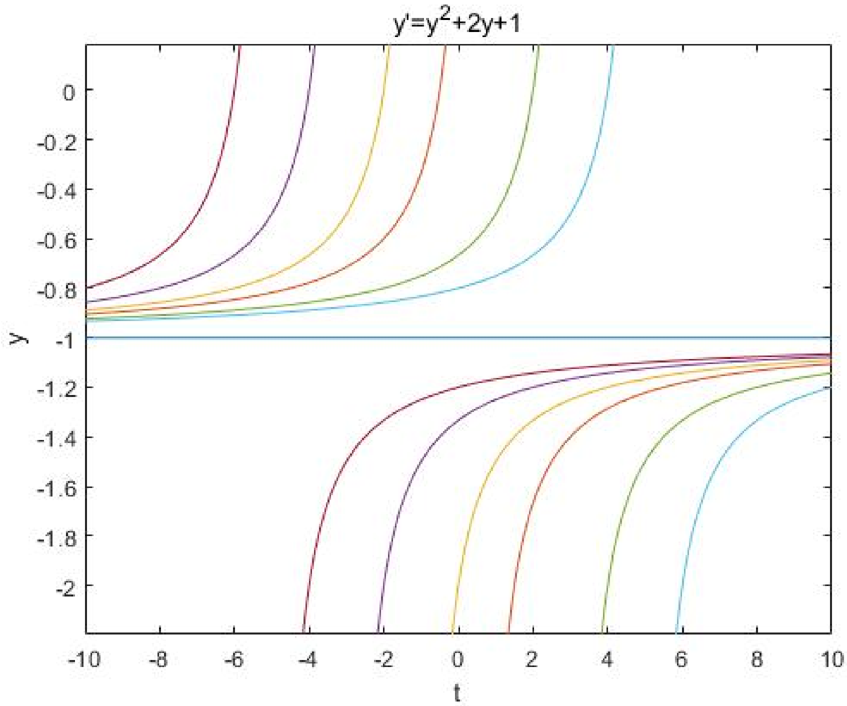


Figure 2: $y' = y^2 + 2y + 1$

iii) $y' = y^2 + y - 2$

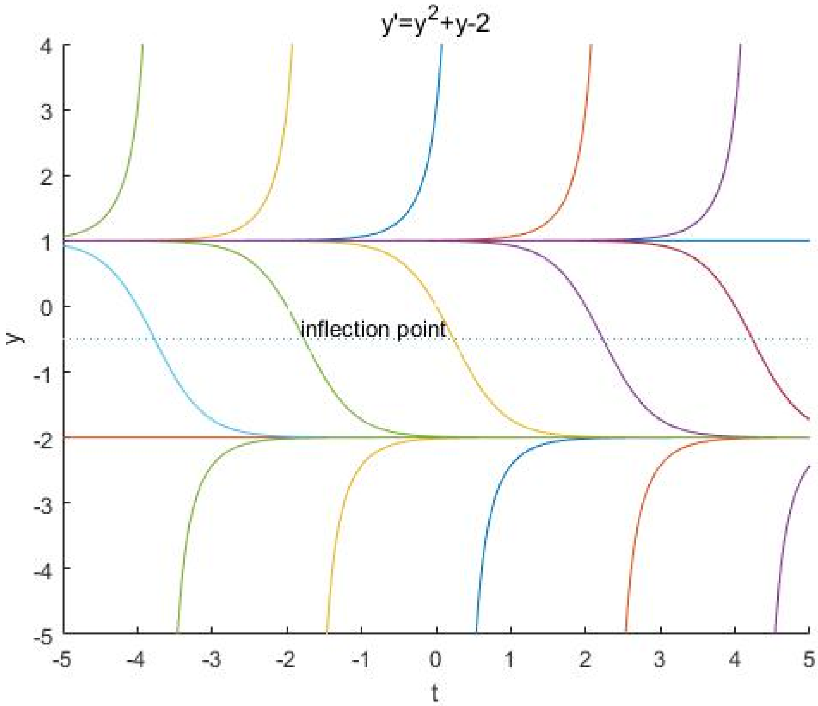


Figure 3: $y' = y^2 + y - 2$

17 a) We can write

$$\frac{dz}{dt} = \beta y^{\beta-1}\frac{dy}{dt}.$$

Then, we get

$$\frac{dy}{dt} = \frac{1}{\beta}y^{1-\beta}\frac{dz}{dt}.$$

Substituting the above expression into the ODE, we get

$$\begin{aligned} \frac{1}{\beta}y^{1-\beta}\frac{dz}{dt} &= a(t)y - b(t)y^n, \\ z' &= \beta a(t)y^{\beta} - \beta b(t)y^{n+\beta-1}, \\ z' &= \beta a(t)z - \beta b(t)y^{n+\beta-1}. \end{aligned}$$

Then, setting $\beta = 1-n$, we can obtain the 1st-order linear ODE

$$z' = \beta a(t)z - \beta b(t)$$

for $z(t) = y(t)^{1-n}$. Depending on n, the 1-1 correspondence between solutions of both ODEs may only hold for a smaller domain, e.g., for general $n > 1$ we need to restrict to $y > 0$ (except for certain integers n).

b) Setting $\beta = 1 - 3 = -2$, we can rewrite the ODE as

$$z' = -8z + 2.$$

The corresponding IVP is $z(0) = y(0)^{-2} = 1$.
Then, we can get its solution as

$$z(t) = \frac{3}{4}e^{-8t} + \frac{1}{4}.$$

Since $z = y^{\beta} = y^{-2}$,

$$y(t) = \pm z(t)^{-\frac{1}{2}} = \pm \left(\frac{3}{4}e^{-8t} + \frac{1}{4}\right)^{-\frac{1}{2}}$$

Because $y(0) = 1$, we eliminate the negative solution, leaving

$$y(t) = \left(\frac{3}{4}e^{-8t} + \frac{1}{4}\right)^{-\frac{1}{2}} = \frac{2}{\sqrt{1 + 3\,\mathrm{e}^{-8t}}}, \quad t \in \mathbb{R}.$$

c) The steady-state solution is $z(t) = 1/4$, corresponding to $y(t) = \pm 2$.
The general solution to the ODE in b) is $y(t) \equiv 0$ and the non-constant solutions

$$y_1(t) = -\left[\left(y^{-2}(0) - \frac{1}{4}\right)e^{-8t} + \frac{1}{4}\right]^{-\frac{1}{2}},$$

and

$$y_2(t) = \left[\left(y^{-2}(0) - \frac{1}{4}\right)e^{-8t} + \frac{1}{4}\right]^{-\frac{1}{2}}.$$

$\lim_{t\to\infty} y_1(t) = -2$, and $\lim_{t\to\infty} y_1(t) = 2$.

If the initial condition is $y(0) = y_0 < 0$, then the solution will be $y_1(t)$, so $\lim_{t\to\infty} y(t) = -2$;
if the initial condition is $y(0) = y_0 > 0$, then the solution will be $y_2(t)$, so $\lim_{t\to\infty} y(t) = 2$.
This shows that both $y = -2$ and $y = 2$ are asymptotically stable.
The third steady state solution $y(t) \equiv 0$ is unstable. This follows from the cases $y(0) > 0$ and $y(0) < 0$ covered above.

18 a) We have

$$\begin{aligned}(y + y')' &= y' + y'' = y' + y = y + y',\\ (y - y')' &= y' - y'' = y' - y = -(y - y'),\end{aligned}$$

i.e., $z = y + y'$ satisfies $z' = z$ and $w = y - y'$ satisfies $w' = -w$. From the theory of 1st-order linear ODE's it follows that $z(x) = y(x)+y'(x) = c_1\mathrm{e}^x$, $w(x) = y(x)-y'(x) = c_2\mathrm{e}^{-x}$ for some $c_1, c_2 \in \mathbb{R}$. $\implies y(x) = \frac{1}{2}(c_1\mathrm{e}^x + c_2\mathrm{e}^{-x}) = (c_1/2)\mathrm{e}^x + (c_2/2)\mathrm{e}^{-x}$, which is of the required form.

b) From the lecture recall that F is continuous on $\mathbb{R}$ and can be differentiated under the integral sign for $x > 0$. Thus for $x > 0$ we have

$$\begin{aligned}F'(x) &= -\int_0^\infty \frac{t\sin(xt)}{t^2+1}\,\mathrm{d}t = -\int_0^\infty \frac{t^2\sin(xt)}{t(t^2+1)}\,\mathrm{d}t = -\int_0^\infty \frac{(t^2+1-1)\sin(xt)}{t(t^2+1)}\,\mathrm{d}t\\ &= -\int_0^\infty \frac{\sin(xt)}{t}\,\mathrm{d}t + \int_0^\infty \frac{\sin(xt)}{t(t^2+1)}\,\mathrm{d}t\,.\end{aligned}$$

The first integral is actually independent of x, since

$$\int_0^\infty \frac{\sin(xt)}{t} = \int_0^\infty \frac{\sin s}{(s/x)x}\,\mathrm{d}s = \int_0^\infty \frac{\sin s}{s}\,\mathrm{d}s, \qquad \text{(Subst. } s = xt,\ \mathrm{d}s = x\,\mathrm{d}t)$$

and has the value $\pi/2$, as we know from the Calculus III fnal exam.

c) Differentiating the expression in b) again under the integral sign, we obtain

$$F''(x) = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin(xt)}{t(t^2+1)}\,\mathrm{d}t = \int_0^\infty \frac{t\cos(xt)}{t(t^2+1)}\,\mathrm{d}t = \int_0^\infty \frac{\cos(xt)}{t^2+1}\,\mathrm{d}t = F(x).$$

This is justified, since

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin(xt)}{t(t^2+1)}\right| = \frac{|\cos(xt)|}{t^2+1} \leq \frac{1}{t^2+1} = \Phi(t),$$

which is independent of x and integrable over $(0,\infty)$.

d) According to a) and c) we have

$$\begin{aligned}F(x) &= c_1\mathrm{e}^x + c_2\mathrm{e}^{-x},\\ F'(x) &= c_1\mathrm{e}^x - c_2\mathrm{e}^{-x}\end{aligned}$$

for some $c_1, c_2 \in \mathbb{R}$ and $x > 0$. Since F is continuous in 0, the first identity holds also for $x = 0$ and gives $c_1 + c_2 = F(0) = \int_0^\infty \frac{\mathrm{d}t}{t^2+1} = \pi/2$.

Since

$$\left|\frac{\sin(xt)}{t(t^2+1)}\right| \le \frac{1}{t(t^2+1)} = \Phi(t),$$

which is independent of x and integrable over $(0,\infty)$, we get

$$F'(0+) = -\frac{\pi}{2} + \int_0^\infty \lim_{x\downarrow 0} \frac{\sin(xt)}{t(t^2+1)}\,\mathrm{d}t = -\frac{\pi}{2} + \int_0^\infty 0\,\mathrm{d}t = -\frac{\pi}{2}$$

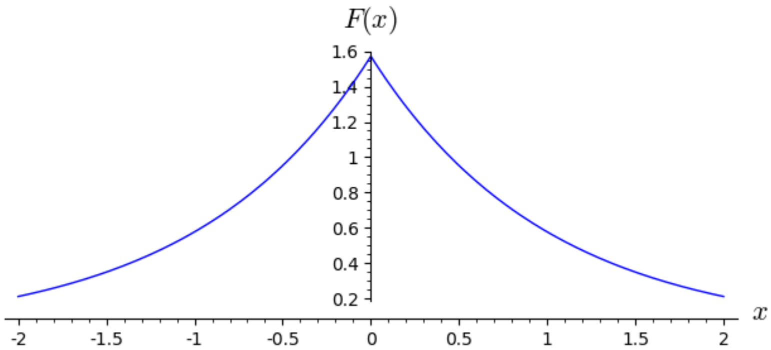
On the other hand, $F'(0+) = \lim_{x\downarrow 0}(c_1\mathrm{e}^x - c_2\mathrm{e}^{-x}) = c_1 - c_2$, so that $c_1 - c_2 = -\pi/2$. It follows that $c_1 = 0$, $c_2 = \pi/2$. Hence $F(x) = (\pi/2)\mathrm{e}^{-x}$ for $x \ge 0$ and

$$\int_0^\infty \frac{\cos t}{t^2+1}\,\mathrm{d}t = F(1) = \frac{\pi}{2\mathrm{e}}.$$

Remarks: This exercise is based on a video from the Youtube channel "Flammable Maths", who's author Jens Fehlau has shot several nice videos with quite nontrivial evaluations of interesting integrals.
Since F is even, we have $F(x) = (\pi/2)\mathrm{e}^{-|x|}$ for $x \in \mathbb{R}$. At $x = 0$ the function F is not differentiable, although the right-hand side of the integral representation

$$F'(x) = -\int_0^\infty \frac{t\sin(xt)}{t^2+1}\,\mathrm{d}t, \qquad \text{valid for } x \ne 0,$$

evaluates to zero at $x = 0$.



Numerically, $\pi/(2\mathrm{e}) \approx 0.5778636748954609$. This differs only slightly from the Euler-Mascheroni constant $\gamma = \lim_{n\to\infty}(1 + 1/2 + 1/3 + \cdots + 1/n - \ln(n)) \approx 0.5772156649015329$, so that perhaps someone who computes the integral $\int_0^\infty \frac{\cos t}{t^2+1}\,\mathrm{d}t$ numerically but doesn't know about the exact evaluation is mislead to conjecture that it has the value γ.

19 First a remark on the cases $L = \pm\infty$. If (x_n) is unbounded then (and only then) for every $R \in \mathbb{R}$ there exist infinitely many indexes n such that $x_n > R$, and hence it is natural to call $+\infty$ an accumulation point of (x_n) and set $L = +\infty$ in this case. On the other hand, if (x_n) diverges to $-\infty$ then (and only then) for every $R \in \mathbb{R}$ there exist only finitely many indexes n such that $x_n > R$, but of course infinitely many indexes n such that $x_n < R$, and hence it is natural to call $-\infty$ an accumulation point of (x_n) and set $L = -\infty$ in this case, since there is no other accumulation point. The case $L = -\infty$ doesn't occur for nonnegative sequences like $x_n = \sqrt[n]{|a_n|}$.

a) Suppose the power series converges for some $z_1 \neq a$ and set $r = |z_1 - a|$, which is then > 0. Since $\sum a_n(z_1 - a)^n$ converges, there exists a constant $M > 1$ such that $|a_n(z_1 - a)^n| = |a_n| r^n \leq M$ for all n. Hence

$$\sqrt[n]{|a_n|} \leq \frac{\sqrt[n]{M}}{r} \leq \frac{M}{r} \quad \text{for all } n,$$

contradicting the unboundedness of $\sqrt[n]{|a_n|}$.

b) Assume $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0$. Then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n| < \epsilon^n$ for $n > N$. Now let $z \in \mathbb{C} \setminus \{a\}$ be arbitary and $r = |z - a|$, i.e., $|a_n(z - a)^n| = |a_n| r^n$. Setting $\epsilon = 1/(2r)$ and denoting by N the corresponding response, we get

$$|a_n| r^n \leq \left(\frac{1}{2r}\right)^n r^n = \frac{1}{2^n} \quad \text{for } n > N.$$

Since $\sum 2^{-n}$ converges, the series $\sum a_n(z-a)^n$ converges absolutely by the comparison test. In particular $\sum a_n(z - a)^n$ converges for all $z \in \mathbb{C}$ (including $z = a$, of course).

c) Suppose first that $z \neq a$ satisfies $r = |z - a| < 1/L$. Then $L < 1/r$, and hence there exist $\theta \in (0, 1)$ and $N \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} \leq \theta/r$ for all $n > N$. (The number θ need only satisfy $L < \theta/r < 1/r$, i.e., $\theta \in (rL, 1)$. Then there can be only finitely many n such that $\sqrt[n]{|a_n|} > \theta/r$.) From this we obtain $|a_n| r^n \leq \theta^n$ for $n > N$ and can use the comparison test with the convergent series $\sum \theta^n$ to conclude that $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges.

Next suppose $r = |z - a| > 1/L$. Then $1/r < L$, and hence $\sqrt[n]{|a_n|} > 1/r$ for infinitely many n. Thus $|a_n| r^n > 1$ for infinitely many n, implying the divergence of $\sum_{n=0}^{\infty} a_n(z-a)^n$. (Since convergence requires $|a_n| r^n \to 0$.)