Differential Equations (Math 285)

H65 Find the Laplace transforms of

a) $1 + 2t + 3t^2$; b) e^{5t+3}; c) $\int_0^t \tau \sin\tau \,\mathrm{d}\tau$; d) $\sin^3 t$.

H66 Find inverse Laplace transforms of

a) $\dfrac{5}{s+6}$; b) $\dfrac{2s-1}{s^2+3}$; c) $\dfrac{1}{(s^2+1)(s^2+4)}$; d) $\dfrac{\mathrm{d}}{\mathrm{d}s}\,\dfrac{1-\mathrm{e}^{-5s}}{s}$;

e) $\ln\dfrac{s}{s-1}$; f) $\ln\dfrac{s^2+1}{(s-1)^2}$; g) $\dfrac{s+1}{s^2(s^2+1)}$; h) $\dfrac{\mathrm{e}^{-2s}-\mathrm{e}^{-4s}}{s}$;

i) $\operatorname{arccot}\dfrac{s}{\omega}$; j) $\dfrac{s^2-1}{(s^3+s^2-5s+3)(s^2-4)}$.

Six answers suffice.

H67 Solve the following initial value problems with the Laplace transform:

a) $y'' - 3\,y' + 2\,y = 6\,\mathrm{e}^{-t}, \quad y(0) = 9,\ y'(0) = 6$;
b) $y'' + 2\,y' - 3\,y = 6\sinh(2t), \quad y(0) = 0,\ y'(0) = 4$;
c) $y''' + y'' - 5\,y' + 3\,y = 6\sinh(2t), \quad y(0) = y'(0) = 0,\ y''(0) = 4$.

H68 Find the Laplace transform of the Bessel function J_0 in one of two ways (the other is optional):

a) From the power series of J_0 and termwise integration of the Laplace integral.
Hint: The power series expansion

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n, \quad \text{valid for } |x| < 1/4,$$

may help (but you should prove it first).

b) From the Bessel ODE of order $\nu = 0$.

H69 Do Exercise 24 in [BDM17], Ch. 6.3, and use the result to verify that $\mathcal{L}\{|\sin t|\} = \frac{1}{s^2+1}\coth\frac{\pi s}{2}$ for $\operatorname{Re}(s) > 0$; cp. also [BDM17], Ch. 6.3, Ex. 28.

H70 *Optional Exercise*

Suppose $F(s) = \mathcal{L}\{f(t)\}$ is defined for $\mathrm{Re}(s) > a$, $a \in [-\infty, \infty)$. Show that $\lim_{s\to+\infty} F(s) = 0$; cp. Exercise 24 in [BDM17], Ch. 6.1.

Hint: Use the uniform convergence of $\int_0^\infty f(t)\mathrm{e}^{-st}$ on $\mathrm{Re}(s) \geq a+1$ (resp., for $a = -\infty$ on $\mathrm{Re}(s) \geq 0$).

H71 *Optional Exercise*

a) Show that $\displaystyle\int_0^\infty \mathrm{e}^{-t} \ln t \,\mathrm{d}t = -\gamma = -0.577\ldots.$ For this recall that the Euler-Mascheroni constant γ was defined as $\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right)$

Hint: Relate the integral to the Gamma function. Gauss's formula

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^x}{x(x+1)\cdots(x+n)} \qquad (x \neq 0, -1, -2, \ldots),$$

which you don't need to prove, may help.

b) Use a) to find the Laplace transform of $t \mapsto \ln t$ and the inverse Laplace transform of $s \mapsto \frac{\ln s}{s}$ $(\mathrm{Re}\, s > 0)$.

Due on Fri May 12, 4 pm

The knowledge required for most of the exercises (except, maybe, H67 and H68) will be provided in the lecture on Sat May 6.

Exercises H70 and H71 can be handed in until Fri May 19, 4 pm.

Solutions

65 a) $\mathcal{L}\{1+2t+3t^2\} = \mathcal{L}\{1\} + 2\,\mathcal{L}\{t\} + 3\,\mathcal{L}\{t^2\} = 1/s + 2/s^2 + 6/s^3$ for $\mathrm{Re}(s) > 0$;

b) $\mathcal{L}\{\mathrm{e}^{5t+3}\} = \mathrm{e}^3\mathcal{L}\{\mathrm{e}^{5t}\} = \mathrm{e}^3/(s-5)$ for $\mathrm{Re}(s) > 5$;

c) $\mathcal{L}\left\{\int_0^t \tau\sin\tau\,\mathrm{d}\tau\right\} = \frac{1}{s}\,\mathcal{L}\{t\sin t\} = -\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s}\,\mathcal{L}\{\sin t\} = -\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s^2+1} = -\frac{1}{s}\frac{-2s}{(s^2+1)^2} = \frac{2}{(s^2+1)^2}$. Alternatively (but more costly), evaluate the integral first using integration by parts, $\int_0^t \tau\sin\tau\,\mathrm{d}\tau = \sin t - t\cos t$, and then recall $\frac{1}{(s^2+1)^2} = \mathcal{L}\left\{\frac{1}{2}(\sin t - t\cos t)\right\}$ from the lecture.

d) From $\sin(3t) = \mathrm{Im}(\cos t + \mathrm{i}\sin t)^3 = 3\cos^2 t\sin t - \sin^3 t = 3\sin t - 4\sin^3 t$ we get $\mathcal{L}\left\{\sin^3 t\right\} = \mathcal{L}\left\{\frac{1}{4}(3\sin t - \sin(3t))\right\} = \frac{1}{4}\left(\frac{3}{s^2+1} - \frac{3}{s^2+9}\right) = \frac{6}{(s^2+1)(s^2+9)}$.

66 a) $\mathcal{L}^{-1}\left\{\frac{5}{s+6}\right\} = 5\,\mathcal{L}^{-1}\left\{\frac{1}{s+6}\right\} = 5\,\mathrm{e}^{-6t}$;

b) $\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+3}\right\} = 2\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+3}\right\} = 2\cos(\sqrt{3}\,t) - \frac{1}{\sqrt{3}}\sin(\sqrt{3}\,t)$;

c) $\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) \Longrightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\} = \frac{1}{3}\left(\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\right) = \frac{1}{3}\sin t - \frac{1}{6}\sin(2t)$;

d) $\frac{1-\mathrm{e}^{-5s}}{s} = \mathcal{L}\{H(t)-H(t-5)\} \Longrightarrow \frac{\mathrm{d}}{\mathrm{d}s}\frac{1-\mathrm{e}^{-5s}}{s} = \mathcal{L}\{-t\,H(t)+t\,H(t-5)\}$, i.e., $\mathcal{L}^{-1}\left\{\frac{\mathrm{d}}{\mathrm{d}s}\frac{1-\mathrm{e}^{-5s}}{s}\right\} = -t\,H(t) + t\,H(t-5)$;

e) We have

$$\ln\frac{s}{s-1} = \ln\frac{1}{1-1/s} = -\ln(1-1/s) = \frac{1}{s} + \frac{1}{2s^2} + \frac{1}{3s^3} + \frac{1}{4s^4} + \cdots$$

for $|s| > 1$, and hence

$$\begin{aligned}\mathcal{L}^{-1}\left\{\ln\frac{s}{s-1}\right\} &= 1 + \frac{t}{2} + \frac{t^2}{3\,2!} + \frac{t^3}{4\,3!} + \cdots = \sum_{n=1}^{\infty}\frac{t^{n-1}}{n!}\\ &= \frac{\mathrm{e}^t - 1}{t}.\end{aligned}$$

f) Let $F(s) = \ln\frac{s^2+1}{(s-1)^2} = \ln(s^2+1) - 2\ln(s-1)$ and $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

$$\begin{aligned}\Longrightarrow\qquad \mathcal{L}\{-t\,f(t)\} &= F'(s) = \frac{2s}{s^2+1} - \frac{2}{s-1} = \mathcal{L}\left\{2\cos t - 2\,\mathrm{e}^t\right\}\\ \Longrightarrow\qquad -t\,f(t) &= 2\cos t - 2\,\mathrm{e}^t\\ \Longrightarrow\qquad f(t) &= \frac{2\,\mathrm{e}^t - 2\cos t}{t} \qquad (t \geq 0)\end{aligned}$$

Since $\mathrm{e}^0 = \cos 0 = 1$ this is in fact an everywhere analytic function of t.

g) We have

$$\frac{s+1}{s^2(s^2+1)} = \frac{1}{s(s^2+1)} + \frac{1}{s^2(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1}$$
$$\Longrightarrow \mathcal{L}\left\{\frac{s+1}{s^2(s^2+1)}\right\} = 1 - \cos t + t - \sin t.$$

h) $\mathcal{L}^{-1}\left\{\left(\mathrm{e}^{-2s} - \mathrm{e}^{-4s}\right)/s\right\} = \mathcal{L}^{-1}\left\{\mathrm{e}^{-2s}/s\right\} - \mathcal{L}^{-1}\left\{\mathrm{e}^{-4s}/s\right\} = H(t-2) - H(t-4)$.

i) From the lecture we know $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \operatorname{arccot} s$. Dilation in the domain gives

$$\mathcal{L}\left\{\frac{\sin(\omega t)}{\omega t}\right\} = \frac{1}{\omega}\operatorname{arccot}\frac{s}{\omega}. \quad \Longrightarrow \quad \mathcal{L}^{-1}\left\{\operatorname{arccot}\frac{s}{\omega}\right\} = \frac{\sin(\omega t)}{t}$$

j) We have

$$\frac{s^2-1}{s^3+s^2-5s+3} = \frac{s+1}{(s-1)(s+3)} = \frac{1}{2}\left(\frac{1}{s-1} + \frac{1}{s+3}\right).$$
$$\Longrightarrow \mathcal{L}^{-1}\left\{\frac{s^2-1}{s^3+s^2-5s+3}\right\} = \frac{1}{2}\left(\mathrm{e}^{t} + \mathrm{e}^{-3t}\right).$$

67 As usual, we denote the Laplace transform of $y(t)$ by $Y(s)$

a) Applying $\mathcal{L}$ to both sides of the equation and inserting the initial conditions gives

$$\begin{aligned} s^2 Y(s) - 9s - 6 - 3\big(sY(s) - 9\big) + 2Y(s) &= \frac{6}{s+1} \\ (s^2 - 3s + 2)Y(s) &= \frac{6}{s+1} + 9s - 21 = \frac{9s^2 - 12s - 15}{s+1} \\ Y(s) &= \frac{9s^2 - 12s - 15}{(s-1)(s-2)(s+1)} \end{aligned}$$

The partial fraction decomposition of $Y(s)$ is

$$Y(s) = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+1}$$

with

$$\begin{aligned} A &= (s-1)Y(s)\big|s = 1 = 9, \\ A &= (s-2)Y(s)\big|s = 2 = -1, \\ C &= (s+1)Y(s)\big|s = -1 = 1, \\ \Longrightarrow Y(s) &= \frac{9}{s-1} - \frac{1}{s-2} + \frac{1}{s+1} \\ \Longrightarrow y(t) &= \mathcal{L}^{-1}\big\{y(s)\big\} = 9\,\mathrm{e}^{t} - \mathrm{e}^{2t} + \mathrm{e}^{-t}. \end{aligned}$$

b) The Laplace transform of $\sinh t = \frac{1}{2}(\mathrm{e}^t - \mathrm{e}^{-t})$ is $F(s) = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) = \frac{1}{s^2-1}$, from which $\mathcal{L}\{\sinh(2t)\} = \frac{1}{2}F\left(\frac{s}{2}\right) = \frac{1/2}{(s/2)^2-1} = \frac{2}{s^2-4}$.

$$\begin{aligned}
\implies s^2\,Y(s) - 4 + 2s\,Y(s) - 3\,Y(s) &= \frac{12}{s^2-4}\\
(s^2+2s-3)Y(s) &= \frac{12}{s^2-4} + 4 = \frac{4\,s^2-4}{s^2-4}\\
Y(s) &= \frac{4(s^2-1)}{(s^2+2s-3)(s^2-4)} = \frac{4(s+1)}{(s+3)(s-2)(s+2)}
\end{aligned}$$

The partial fraction decomposition of $Y(s)$ is

$$Y(s) = \frac{A}{s+3} + \frac{B}{s-2} + \frac{C}{s+2}$$

with

$$\begin{aligned}
A &= (s+3)Y(s)\big|s = -3 = -8/5,\\
A &= (s-2)Y(s)\big|s = 2 = 3/5,\\
C &= (s+2)Y(s)\big|s = -2 = 1,\\
\implies Y(s) &= -\frac{8/5}{s+3} + \frac{3/5}{s-2} + \frac{1}{s+2}\\
\implies y(t) &= -\frac{8}{5}\,\mathrm{e}^{-3t} + \frac{3}{5}\,\mathrm{e}^{2t} + \mathrm{e}^{-2t}.
\end{aligned}$$

c)

$$\begin{aligned}
s^3\,Y(s) - 4 + s^2\,Y(s) - 5s\,Y(s) + 3\,Y(s) &= \frac{12}{s^2-4}\\
Y(s) = \frac{4(s^2-1)}{(s^3+s^2-5s+3)(s^2-4)} &= \frac{4(s+1)}{(s-1)(s+3)(s-2)(s+2)}
\end{aligned}$$

The partial fraction decomposition of $Y(s)$ is (details omitted)

$$\begin{aligned}
Y(s) &= \frac{2}{5\,(s+3)} - \frac{1}{3\,(s+2)} - \frac{2}{3\,(s-1)} + \frac{3}{5\,(s-2)}.\\
\implies y(t) &= \frac{2}{5}\,\mathrm{e}^{-3t} - \frac{1}{3}\,\mathrm{e}^{-2t} - \frac{2}{3}\,\mathrm{e}^{t} + \frac{3}{5}\,\mathrm{e}^{2t}.
\end{aligned}$$

68 a) We have

$$\begin{aligned}
\mathrm{J}_0(t) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m(m!)^2}\,t^{2m} = \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m}\binom{2m}{m}\frac{t^{2m}}{(2m)!}.\\
\implies \mathcal{L}\{\mathrm{J}_0\} &= \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m}\binom{2m}{m}\frac{1}{s^{2m+1}} = \frac{1}{s}\sum_{m=0}^{\infty}\binom{2m}{m}\left(-\frac{1}{4s^2}\right)^m\\
&= \frac{1}{s}\frac{1}{\sqrt{1-4\left(-\frac{1}{4s^2}\right)}} \qquad \text{(using the hint)}\\
&= \frac{1}{\sqrt{s^2+1}}.
\end{aligned}$$

The computation is valid for $|s| > 1$, since the binomial series involved (see below) has radius of convergence 1; cf. the theorem about termwise integration of Laplace integrals in the lecture.

Finally we prove the asserted series expansion:

$$\begin{aligned}\binom{-1/2}{m} &= \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2m-1}{2}\right)}{m!} = (-1)^m\frac{1\cdot 3\cdot 5\cdots(2m-1)}{m!\,2^m}\\ &= (-1)^m\frac{(2m)!}{m!\,2^m\cdot 2\cdot 4\cdot 6\cdots(2m)} = (-1)^m\frac{(2m)!}{(m!)^2 4^m} = \frac{(-1)^m}{4^m}\binom{2m}{m},\end{aligned}$$

and therefore

$$\sum_{m=0}^{\infty}\binom{2m}{m}x^m = \sum_{m=0}^{\infty}(-1)^m 4^m\binom{-1/2}{m}x^m = \sum_{m=0}^{\infty}\binom{-1/2}{m}(-4x)^m = (1-4x)^{-1/2},$$

using the binomial series.

J_0 is the solution of the IVP $t\,y'' + y' + t\,y = 0$, $y(0) = 1$, $y'(0) = 0$. Writing $Y(s) = \mathcal{L}\big\{\mathrm{J}_0(t)\big\}$ and taking the Laplace transform on both sides gives

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}s}\left(s^2\,Y(s) - s\right) + s\,Y(s) - 1 - Y'(s) &= 0\\ -\left(s^2\,Y'(s) + 2s\,Y(s) - 1\right) + s\,Y(s) - 1 - Y'(s) &= 0\\ Y'(s) &= -\frac{s}{s^2+1}\,Y(s)\end{aligned}$$

$$\implies Y(s) = c\exp\int_0^s -\frac{1}{2}\ln(\sigma^2+1)\mathrm{d}\sigma = \frac{c}{\sqrt{s^2+1}} \quad \text{for some constant } c.$$

The constant c can be determined from

$$\mathcal{L}\big\{\mathrm{J}_0'(t)\big\} = s\,Y(s) - \mathrm{J}_0(0)$$

and the general fact that Laplace transforms tend to zero for $s \to \infty$. It follows that

$$c = \lim_{s\to\infty}\frac{cs}{\sqrt{s^2+1}} = \lim_{s\to\infty} s\,Y(s) = \mathrm{J}_0(0) = 1,$$

and hence $\mathcal{L}\big\{\mathrm{J}_0(t)\big\} = Y(s) = 1/\sqrt{s^2+1}$.

69 We have

$$\begin{aligned}\int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t &= \int_0^T f(t)\mathrm{e}^{-st}\,\mathrm{d}t + \int_T^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t\\ &= \int_0^T f(t)\mathrm{e}^{-st}\,\mathrm{d}t + \int_0^\infty f(T+\tau)\mathrm{e}^{-s(T+\tau)}\mathrm{d}\tau\\ &\qquad\qquad (\text{Subst. } \tau = t - T,\ \mathrm{d}\tau = \mathrm{d}t)\\ &= \int_0^T f(t)\mathrm{e}^{-st}\,\mathrm{d}t + \mathrm{e}^{-sT}\int_0^\infty f(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau. \quad (\text{Since } f(T+\tau) = f(\tau))\end{aligned}$$

$$\implies \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t = \frac{1}{1-\mathrm{e}^{-sT}}\int_0^T f(t)\mathrm{e}^{-st}\,\mathrm{d}t$$

In the special case $f(t)=|\sin t|$ the smallest period is π, so that

$$\begin{aligned}\mathcal{L}\,|\sin t|\} &= \frac{1}{1-\mathrm{e}^{-\pi s}}\int_0^\pi \sin t\,\mathrm{e}^{-st}\,\mathrm{d}t = \frac{1}{1-\mathrm{e}^{-\pi s}}\left[\frac{\mathrm{e}^{-st}}{s^2+1}(-s\sin t-\cos t)\right]_0^\pi\\ &= \frac{\mathrm{e}^{-\pi s}+1}{1-\mathrm{e}^{-\pi s}}\,\frac{1}{s^2+1} = \frac{\mathrm{e}^{-\pi s/2}+\mathrm{e}^{\pi s/2}}{\mathrm{e}^{\pi s/2}-\mathrm{e}^{-\pi s/2}}\,\frac{1}{s^2+1} = \frac{1}{s^2+1}\coth\frac{\pi s}{2}.\end{aligned}$$

70 The assertion "$\lim_{s\to+\infty}F(s)=0$", which tacitly assumes $s\in\mathbb{R}$, can in fact be strenghtened to $\lim_{\mathrm{Re}(s)\to+\infty}F(s)=0$, as the subsequent proof shows. But the complex limit $\lim_{|s|\to\infty}F(s)$ need not exist, because near the line of convergence $F(s)$ may be unbounded.
Let $\epsilon>0$ be given. Since the Laplace integral converges uniformly for $\mathrm{Re}\,s\geq a+1$ ($\mathrm{Re}\,s\geq 0$), as shown in the lecture, we can find $R>0$ such that $\left|\int_R^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t\right|<\epsilon/2$ for all such s. Assuming that f is piecewise continuous, hence bounded on $[0,R]$, there exists $M>0$ such $|f(t)|\leq M$ for $t\in[0,R]$. Writing $s=x+\mathrm{i}y$, we then have

$$\left|\int_0^R f(t)\mathrm{e}^{-st}\,\mathrm{d}t\right|\leq\int_0^R|f(t)|\,\mathrm{e}^{-xt}\,\mathrm{d}t\leq M\int_0^R\mathrm{e}^{-xt}\,\mathrm{d}t=\frac{M(1-\mathrm{e}^{-xR})}{x}\leq\frac{M}{x},$$

provided that $x>0$. For $x>2M/\epsilon$ the right-hand side is $\epsilon/2$.

$$\begin{aligned}\Longrightarrow |F(s)| &= \left|\int_0^R f(t)\mathrm{e}^{-st}\,\mathrm{d}t+\int_R^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t\right|\\ &\leq\left|\int_0^R f(t)\mathrm{e}^{-st}\,\mathrm{d}t\right|+\left|\int_R^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t\right|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon,\end{aligned}$$

provided that $\mathrm{Re}(s)>\max\{a+1,0,2M/\epsilon\}$. This shows $\lim_{\mathrm{Re}(s)\to+\infty}F(s)=0$.

71 a) In Calculus III it was shown that $\Gamma(x)=\int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$ can be differentiated termwise to yield $\Gamma'(x)=\int_0^\infty\ln t\,t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$. It follows that $\int_0^\infty\ln t\,\mathrm{e}^{-t}\,\mathrm{d}t=\Gamma'(1)$, and we are thus left to show that $\Gamma'(1)=-\gamma$.

Writing $\Gamma_n(x)=\frac{n!\,n^x}{x(x+1)\cdots(x+n)}$, we have

$$\begin{aligned}\frac{\Gamma_n'(x)}{\Gamma_n(x)} &= \ln n-\frac{1}{x}-\frac{1}{x+1}-\cdots-\frac{1}{x+n},\\ \frac{\Gamma_n'(1)}{\Gamma_n(1)} &= \ln n-1-\frac{1}{2}-\cdots-\frac{1}{n+1}\\ &= \ln\frac{n}{n+1}+\ln(n+1)-1-\frac{1}{2}-\cdots-\frac{1}{n+1}\to-\gamma\quad\text{for } n\to\infty.\end{aligned}$$

Since $\Gamma(1)=1$, it remains to show that $\Gamma_n'(1)/\Gamma_n(1)\to\Gamma'(1)/\Gamma(1)$ for $n\to\infty$.

To this end we rewrite the expression for the logarithmic derivative of Γ as follows:

$$\begin{aligned}\frac{\Gamma_n'(x)}{\Gamma_n(x)} &= \ln n-\sum_{k=1}^n\frac{1}{k}-\frac{1}{x}+\sum_{k=1}^n\left(\frac{1}{k}-\frac{1}{x+k}\right)\\ &= \ln n-\sum_{k=1}^n\frac{1}{k}-\frac{1}{x}+x\sum_{k=1}^n\frac{1}{k(x+k)}\to-\gamma-\frac{1}{x}+x\sum_{k=1}^\infty\frac{1}{k(x+k)}\end{aligned}$$

for $n \to \infty$. Since

$$\frac{1}{k(x+k)} \le \frac{1}{k^2} \quad \text{for } x \ge 0,$$

the convergence is uniform on every interval $[r, R]$ with $0 < r < R$. Since $\frac{\Gamma_n'(x)}{\Gamma_n(x)} = \frac{\mathrm{d}}{\mathrm{d}x} \ln \Gamma_n(x)$, we can apply the Differentiation Theorem to the function sequence $(\ln \Gamma_n)$ and conclude that $\ln \Gamma$ is differentiable with derivative

$$\frac{\Gamma'(x)}{\Gamma(x)} = \frac{\mathrm{d}}{\mathrm{d}x} \ln \Gamma(x) = \lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}x} \ln \Gamma_n(x) = \lim_{n\to\infty} \frac{\Gamma_n'(x)}{\Gamma_n(x)} = -\gamma - \frac{1}{x} + x \sum_{k=1}^{\infty} \frac{1}{k(x+k)}.$$

Setting $x = 1$ finishes the proof.

The formula for $\Gamma'(x)/Gamma(x)$ was derived for $x > 0$, but holds in fact for all $x \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, as one can show with a refined argument.

b) We have

$$\begin{aligned}
\mathcal{L}\{\ln t\} &= \int_0^\infty \ln t \, \mathrm{e}^{-st} \, \mathrm{d}t \\
&= \frac{1}{s} \int_0^\infty (\ln \tau - \ln s) \mathrm{e}^{-\tau} \mathrm{d}\tau && (\text{Subst. } \tau = st,\ \mathrm{d}\tau = s\,\mathrm{d}t) \\
&= \frac{1}{s} \int_0^\infty \ln \tau \, \mathrm{e}^{-\tau} \mathrm{d}\tau - \frac{\ln s}{s} \int_0^\infty \mathrm{e}^{-\tau} \mathrm{d}\tau \\
&= -\frac{\gamma}{s} - \frac{\ln s}{s}. && (\text{using a)})
\end{aligned}$$

From this it follows that $\mathcal{L}\{\gamma + \ln t\} = \gamma/s + \mathcal{L}\{\ln t\} = -\ln s/s$, i.e.

$$\mathcal{L}^{-1}\left\{\frac{\ln s}{s}\right\} = -\gamma - \ln t.$$