Differential Equations (Math 285)

H50 Determine a fundamental system of solutions for Bessel's ODE with $p = \frac{1}{2}$,

$$y'' + \frac{1}{t}\,y' + \left(1 - \frac{1}{4t^2}\right) y = 0,$$

using the „Ansatz" $z = \sqrt{t}\,y$.

H51 Determine the general solution of the following ODE's (two answers suffice):

a) $(2t+1)y'' + (4t-2)y' - 8y = (6t^2+t-3)\mathrm{e}^t, \quad t > -1/2;$

b) $t^2(1-t)y'' + 2t(2-t)y' + 2(1+t)y = t^2, \quad 0 < t < 1;$

c) $(t^2-4t+4)y'' + (3t-6)y' + 2y = t^2+1, \quad t > 2.$

Hints: The associated homogeneous ODE in a) has a solution of the form $y(t) = \mathrm{e}^{\alpha t}$ and that in b) a solution of the form $y(t) = t^\beta$ with constants α, β. In both cases a particular solution of the inhomogeneous ODE can be determined by reducing it to a first-order system and using variation of parameters (though this may not be the most economic solution). The ODE in c) is an inhomogeneous Euler equation in disguise.

H52 Prove Leibniz's rule for the n-th derivative of a product: If $f, g\colon I \to \mathbb{R}$ are n-times differentiable then so is $F = fg$, and

$$\mathrm{D}^n F = \sum_{k=0}^{n} \binom{n}{k} (\mathrm{D}^k f)(\mathrm{D}^{n-k} g).$$

H53 *On Hermite Polynomials*

In the lecture the Hermite polynomials $\mathrm{H}_n(X) \in \mathbb{R}[X]$ are defined by $\mathrm{H}_n(t) = (-1)^n \mathrm{e}^{t^2} \mathrm{D}^n[\mathrm{e}^{-t^2}]$ for $t \in \mathbb{R}$ ($n = 0, 1, 2, \dots$).

a) Show that $t \mapsto \mathrm{H}_n(t)$ is a polynomial function, justifying the definition.

b) Show that $\deg \mathrm{H}_n(X) = n$ and the leading coefficient of $\mathrm{H}_n(X)$ is 2^n.

c) Show that $\mathrm{H}_n(X)$ satisfies the recurrence relation $\mathrm{H}_{n+1}(X) = 2X\,\mathrm{H}_n(X) - 2n\,\mathrm{H}_{n-1}(X)$, and compute $\mathrm{H}_n(X)$ for $n \leq 6$.

d) Show that $t \mapsto \mathrm{H}_n(t)$ solves Hermite's differential equation $y'' - 2ty' + 2ny = 0$. *Hint:* The equation is equivalent to $Ly = 0$, where $L = \mathrm{D}^2 - 2t\mathrm{D} + 2n\,\mathrm{id}$. Express $L\bigl[\mathrm{H}_n(t)\bigr]$ in terms of $\mathrm{D}^n[\mathrm{e}^{-t^2}]$, $\mathrm{D}^{n+1}[\mathrm{e}^{-t^2}]$, $\mathrm{D}^{n+2}[\mathrm{e}^{-t^2}]$, and rewrite the latter using $\mathrm{D}^{n+2}[\mathrm{e}^{-t^2}] = \mathrm{D}^{n+1}[-2t\,\mathrm{e}^{-t^2}]$.

H54 *On Legendre Polynomials (optional exercise)*

In the lecture the Legendre polynomials $\mathrm{P}_n(X) \in \mathbb{R}[X]$ were defined by $\mathrm{P}_n(t) = \frac{1}{2^n n!}\mathrm{D}^n\left[(t^2-1)^n\right]$, $n = 0, 1, 2, \dots$.

a) Compute $\mathrm{P}_n(X)$ for $n \leq 6$.

b) Show that

$$\int_{-1}^{1} \mathrm{P}_m(t)\mathrm{P}_n(t) = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2}{2n+1} & \text{if } m = n. \end{cases}$$

Hint: Use partial integration and the fact that $(t^2-1)^n$ has a zero of multiplicity n at $t = \pm 1$. For the case $m = n$ it may be helpful to recall from Calculus III that $\int_0^{\pi/2} \sin^{2n+1} t \,\mathrm{d}t = \frac{(2n)(2n-2)\cdots 4\cdot 2}{(2n+1)(2n-1)\cdots 5\cdot 3}$.

c) Show that $\mathrm{P}_n(X)$ has n distinct zeros $\alpha_1^{(n)} < \alpha_2^{(n)} < \cdots < \alpha_n^{(n)}$ in $[-1, 1]$.

d) Suppose $n \in \mathbb{Z}^+$ and $x_1, \dots, x_n \in \mathbb{R}$ are such that $-1 \leq x_1 < x_2 < \cdots < x_n \leq 1$. Show that there are uniquely determined constants ("weights") $c_1, \dots, c_n \in \mathbb{R}$ such that

$$\int_{-1}^{1} f(t)\,\mathrm{d}t \approx c_1 f(x_1) + \cdots + c_n f(x_n) \tag{GQ_n}$$

is exact for all polynomial functions $f(t)$ of degree $\leq n-1$.

e) Show that for the particular choice $x_i = \alpha_i^{(n)}$, cf. c), Formula (GQ_n) is exact for all polynomial functions $f(t)$ of degree $\leq 2n-1$.

Hint: Long division of $f(t)$ by $\mathrm{P}_n(t)$.

f) Determine (GQ_n) for $n = 1, 2, 3$ and the special choice $x_i = \alpha_i^{(n)}$.

Due on Fri April 21, 4 pm

Exercises H51 and H53 d) require knowledge about 2nd-order linear ODE's that will be provided in the lecture on Mon April 17. The optional exercise can be handed in until Fri April 28, 4 pm.

Solutions (prepared by Li Menglu and TH)

50 Using the Ansatz $z = \sqrt{t}\,y$, we have

$$\begin{aligned}
\frac{dz}{dt} &= \sqrt{t}\,y' + \frac{1}{2\sqrt{t}}\,y \\
\frac{d^2z}{dt^2} &= \sqrt{t}\,y'' + \frac{1}{\sqrt{t}}\,y' + \left(-\frac{1}{4}\right)t^{-\frac{3}{2}}y \\
\Longrightarrow \quad 4t^{\frac{3}{2}}\frac{d^2z}{dt^2} &= 4t^2y'' + 4ty' - y
\end{aligned}$$

Rewrite the ODE and substituting the above expression, we obtain

$$\begin{aligned}
& & 4t^2y'' + 4ty' + (4t^2-1)y = 0 \\
\Longleftrightarrow & & 4t^{\frac{3}{2}}z'' + 4t^2y = 0 \\
\Longleftrightarrow & & 4t^{\frac{3}{2}}(z + z'') = 0 \\
\Longleftrightarrow & & z'' + z = 0
\end{aligned}$$

The characteristic equation of $z'' + z = 0$ is $r^2 + 1 = 0$, so that $r_1 = \mathrm{i}$, $r_2 = -\mathrm{i}$.

$$\begin{aligned}
\therefore z(t) &= c_1\cos t + c_2\sin t, \\
\therefore y(t) &= \frac{c_1\cos t}{\sqrt{t}} + \frac{c_2\sin t}{\sqrt{t}}.
\end{aligned}$$

Thus a fundamental system of solutions for the ODE is $\frac{\cos t}{\sqrt{t}}$, $\frac{\sin t}{\sqrt{t}}$.

51 a) Let $y_1(t) = e^{\alpha t}$, so that $y_1'(t) = \alpha e^{\alpha t}$, $y''(t) = \alpha^2 e^{\alpha t}$. Substituting these into the associated homogeneous ODE, we get

$$(2t+1)\alpha^2 + (4t-2)\alpha - 8 = 0 \Rightarrow (2\alpha^2 + 4\alpha)t + \alpha^2 - 2\alpha - 8 = 0$$

$$\therefore \alpha = -2 \implies y_1(t) = e^{-2t} \text{ is a solution.}$$

Setting $y_2(t) = u(t)e^{-2t}$ and substituting this into the ODE (cf. "order reduction" in the lecture), we get for $u'(t)$ the 1st-order linear ODE

$$u''(t) + \left[2\frac{-2e^{-2t}}{e^{-2t}} + \frac{4t-2}{2t+1}\right]u'(t) = 0 \iff u''(t) + \left(\frac{4t-2}{2t+1} - 4\right)u'(t) = 0.$$

$$\therefore u'(t) = e^{\int -\frac{4t-2}{2t+1} + 4dt} = e^{2t}(2t+1)^2 \implies u(t) = \frac{4t^2+1}{2}e^{2t} \implies y_2(t) = \frac{4t^2+1}{2}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2t} & 2t^2 + \frac{1}{2} \\ -2e^{-2t} & 4t \end{vmatrix} = (2t+1)^2e^{-2t} \neq 0$$

$\Longrightarrow y_1(t)$, $y_2(t)$ form a fundamental system of solutions of the homogeneous ODE.

For the inhomogeneous ODE in explicit form we have $b(t) = \frac{6t^2+t-3}{2t+1}\,e^t$. Using variation of parameters for the order-reduced 2×2 system, we need to extract the first coordinate function of

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\int\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ b \end{pmatrix}\mathrm{d}t = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\int\frac{1}{W}\begin{pmatrix} -y_2b \\ y_1b \end{pmatrix}\mathrm{d}t\,.$$

We have

$$\int \frac{1}{W}\begin{pmatrix} -y_2 b \\ y_1 b \end{pmatrix} \mathrm{d}t = \int \begin{pmatrix} -\frac{(2t^2+\frac{1}{2})(6t^2+t-3)}{(2t+1)^3}\,\mathrm{e}^{3t} \\ \frac{(6t^2+t-3)}{(2t+1)^3}\,\mathrm{e}^{t} \end{pmatrix} \mathrm{d}t$$
$$= \begin{pmatrix} \frac{-12t^3+8t^2+5t-4}{6(2t+1)^2}\,\mathrm{e}^{3t} \\ \frac{3t+2}{(2t+1)^2}\,\mathrm{e}^{t} \end{pmatrix}.$$

Remark: For the integration step we have used a computer algebra program. If $r(t)$ is any rational function (quotient of two polynomials) and $a \in \mathbb{C}$ then $r(t)\mathrm{e}^{at}$ can be integrated in finite terms iff there exists a rational function R such that $r(t) = R'(t) + a\,R(t)$; if this is the case then $\int r(t)\mathrm{e}^{at} = R(t)\mathrm{e}^{at}$. (This result is due to Liouville.) Only few rational functions $r(t)$ have this property. In the two cases under consideration one can find $R(t)$ with some effort by using the „Ansatz“ $R = u/v^2$, $v(t) = 2t+1$, which the special form of the integrand suggests. The details are omitted.

$$\begin{aligned}
\Longrightarrow y_p(t) &= \mathrm{e}^{-2t}\,\frac{-12t^3+8t^2+5t-4}{6(2t+1)^2}\,\mathrm{e}^{3t} + \frac{4t^2+1}{2}\,\frac{3t+2}{(2t+1)^2}\,\mathrm{e}^{t} \\
&= \frac{-12t^3+8t^2+5t-4+36t^3+24t^2+9t+6}{6(2t+1)^2}\,\mathrm{e}^{t} \\
&= \frac{24t^3+32t^2+14t+2}{6(2t+1)^2}\,\mathrm{e}^{t} = \left(t+\tfrac{1}{3}\right)\mathrm{e}^{t}
\end{aligned}$$

is a particular solution of the inhomogeneous ODE, and its general solution is

$$y(t) = c_1 e^{-2t} + c_2(4t^2+1) + \left(t+\tfrac{1}{3}\right)\mathrm{e}^{t}.$$

Remark: A much quicker (but in a way dirty) solution is the following. Using the differential operator

$$L = (2t+1)\mathrm{D}^2 + (4t-2)\mathrm{D} - 8\,\mathrm{id},$$

the inhomogeneous ODE can be written as $L[y] = (6t^2+t-3)\mathrm{e}^t$. It is clear that L maps the space of exponential polynomials of the special form $p(t)\mathrm{e}^t = (p_0 + p_1 t + \dots + p_d t^d)\mathrm{e}^t$ into itself. Thus we might hope for a particular solution of this form. When determining the images under L of the first few exponential monomials,

$$\begin{aligned}
L[\mathrm{e}^t] &= (2t+1)\mathrm{e}^t + (4t-2)\mathrm{e}^t - 8\mathrm{e}^t = (6t-9)\mathrm{e}^t, \\
L[t\,\mathrm{e}^t] &= (2t+1)(t+2)\mathrm{e}^t + (4t-2)(t+1)\mathrm{e}^t - 8t\,\mathrm{e}^t = (6t^2-t)\mathrm{e}^t, \\
&\ \vdots
\end{aligned}$$

we find that

$$6t^2+t-3 = L[t\,\mathrm{e}^t] + \tfrac{1}{3}L[\mathrm{e}^t] = L\left[t\,\mathrm{e}^t + \tfrac{1}{3}\,\mathrm{e}^t\right] = L\left[\left(t+\tfrac{1}{3}\right)\mathrm{e}^t\right].$$

This gives the same particular solution as above. (The general solution is determined in the same way as above.)

b) Let $y_1(t) = t^\beta$, so that $y_1'(t) = \beta t^{\beta-1}$, $y_1''(t) = \beta(\beta-1)t^{\beta-2}$. Substituting these into the associated homogeneous ODE, we get

$$t^2(1-t)\beta(\beta-1)t^{\beta-2} + 2t(2-t)\beta t^{\beta-1} + 2(1+t)t^\beta = 0 \Rightarrow \left[\beta^2+3\beta+2+\left(-\beta^2-\beta+2\right)t\right]t^\beta = 0$$

$$\therefore \beta = -2 \Rightarrow y_1(t) = t^{-2}.$$

Set $y_2(t) = u(t)t^{-2}$ and substitute this into the ODE, we can get

$$u''(t) + \left[2\frac{-2t^{-3}}{t^{-2}} + \frac{2t(2-t)}{t^2(1-t)}\right] u'(t) = 0 \Rightarrow u''(t) + \frac{2}{1-t}u'(t) = 0$$

$$\therefore u'(t) = e^{\int \frac{2}{t-1}dt} = (t-1)^2 \Rightarrow u(t) = \frac{(t-1)^3}{3} \Rightarrow y_2(t) = \frac{(t-1)^3}{3t^2}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{-2} & \frac{(t-1)^3}{3t^2} \\ -2t^{-3} & (t-1)^2t^{-2} - \frac{2}{3}(t-1)^3t^{-3} \end{vmatrix} = (t-1)^2t^{-4} \neq 0$$

$\Longrightarrow y_1(t)$, $y_2(t)$ form a fundamental system of solutions of the homogeneous ODE.

For the determination of a particular solution of the inhomogeneous ODE we proceed as before, setting $b(t) = \frac{t^2}{t^2(1-t)} = \frac{1}{1-t}$.

$$\int \frac{1}{W}\begin{pmatrix} -y_2 b(t) \\ y_1 b(t) \end{pmatrix} \mathrm{d}t = \int \frac{t^4}{(t-1)^2}\begin{pmatrix} \frac{(t-1)^2}{3t^2} \\ \frac{1}{t^2(1-t)} \end{pmatrix} \mathrm{d}t = \int \begin{pmatrix} \frac{1}{3}t^2 \\ -\frac{t^2}{(t-1)^3} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{1}{9}t^3\right) \\ -\ln(t-1) + \frac{2}{t-1} + \frac{1}{2(t-1)^2} \end{pmatrix}$$

$$y_p(t) = t^{-2}\frac{1}{9}t^3 + \frac{(t-1)^3}{3t^2}\left(-\ln(t-1) + \frac{2}{t-1} + \frac{1}{2(t-1)^2}\right)$$

$$= \frac{2t^3 + 12t^2 - 21t + 9 - 6(t-1)^3\ln(t-1)}{18t^2}$$

In the numerator of $y_p(t)$ we can subtract $2(t-1)^3 + 11$ to change it into $18t^2 - 27t - 6(t-1)^3\ln(t-1)$, since this amounts to adding a linear combination of $y_1(t)$ and $y_2(t)$ to $y_p(t)$. This leaves the simpler function $t \mapsto 1 - \frac{3}{2t} - \frac{(t-1)^3}{3t^2}\log(t-1)$. The general solution of the inhomogeneous ODE is then

$$y(t) = c_1t^{-2} + c_2\frac{(t-1)^3}{t^2} + 1 - \frac{3}{2t} - \frac{(t-1)^3}{3t^2}\log(t-1).$$

c) Writing the associated homogeneous ODE as $(t-2)^2y'' + 3(t-2)y' + 2y = 0$ and setting $t-2 = x$ we get $x^2y'' + 3xy' + 2y = 0$, which is apparently an Euler equation with $\alpha = 3$, $\beta = 2$. The indicial equation is $r^2 + (\alpha-1)r + \beta = r^2 + 2r + 2 = 0$. It has roots $r_1 = -1 + \mathrm{i}$, $r_2 = -1 - \mathrm{i}$, and hence a complex fundamental system of solutions of the (untransformed) homogeneous ODE on $(2, \infty)$ is

$$z_1(t) = (t-2)^{-1+\mathrm{i}} = \mathrm{e}^{\ln(t-2)(-1+\mathrm{i})},$$
$$z_2(t) = (t-2)^{-1-\mathrm{i}} = \mathrm{e}^{\ln(t-2)(-1-\mathrm{i})}.$$

A real fundamental system of solutions—strictly speaking, this is not required—is

$$y_1(t) = \mathrm{Re} z_1(t) = (t-2)^{-1}\cos\ln(t-2),$$
$$y_2(t) = \mathrm{Im} z_1(t) = (t-2)^{-1}\sin\ln(t-2).$$

Since the associated differential operator $L = (t-2)^2\mathrm{D}^2 + 3(t-2)\mathrm{D} + 2\,\mathrm{id}$ maps the space P_2 of (real, say) quadratic polynomials into itself, it is reasonable to guess that there must be a particular solution of the form $y_p(t) = a(t-2)^2 + b(t-2) + c = ax^2 + bx + c$, $x = t-2$. Substituting $y' = 2ax + b$, $y'' = 2a$ into the inhomogeneous ODE, we obtain

$$\begin{aligned} 10ax^2 + 5bx + 2c = x^2 + 4x + 5 &\implies a = \frac{1}{10},\ b = \frac{4}{5},\ c = \frac{5}{2} \\ &\implies y_p(t) = \frac{1}{10}x^2 + \frac{4}{5}x + \frac{5}{2} = \frac{1}{10}t^2 + \frac{2}{5}t + \frac{13}{10}. \end{aligned}$$

The general real (or complex) solution of $(t-2)^2y'' + 3(t-2)y' + 2y = t^2 + 1$ on $(2, \infty)$ is therefore

$$y(t) = c_1(t-2)^{-1}\cos\ln(t-2) + c_2(t-2)^{-1}\sin\ln(t-2) + \frac{1}{10}t^2 + \frac{2}{5}t + \frac{13}{10}$$

with $c_1, c_2 \in \mathbb{R}$ (resp., $c_1, c_2 \in \mathbb{C}$). For the complex solution we could have used the complex fundamental system $z_1(t)$, $z_2(t)$ instead.

52 The formula can easily be proved by induction on n. For $n = 0$ it is trivial, and for $n = 1$ it is $\mathrm{D}(fg) = (\mathrm{D}f)g + f(\mathrm{D}g)$, which is just the product rule of differentiation. Assuming that the formula holds for n, we obtain

$$\begin{aligned} \mathrm{D}^{n+1}F &= \mathrm{D}(\mathrm{D}^nF) \\ &= \mathrm{D}\left(\sum_{k=0}^{n}\binom{n}{k}(\mathrm{D}^kf)(\mathrm{D}^{n-k}g\right) && \text{(inductive hypothesis)} \\ &= \sum_{k=0}^{n}\binom{n}{k}\mathrm{D}\big((\mathrm{D}^kf)(\mathrm{D}^{n-k}g)\big) && \text{(linearity of differentiation)} \\ &= \sum_{k=0}^{n}\binom{n}{k}\big((\mathrm{D}^{k+1}f)(\mathrm{D}^{n-k}g) + (\mathrm{D}^kf)(\mathrm{D}^{n-k+1}g)\big) && \text{product rule} \\ &= \sum_{k=0}^{n}\binom{n}{k}(\mathrm{D}^{k+1}f)(\mathrm{D}^{n-k}g) + \sum_{k=0}^{n}\binom{n}{k}(\mathrm{D}^kf)(\mathrm{D}^{n-k+1}g) \\ &= \sum_{k=1}^{n+1}\binom{n}{k-1}(\mathrm{D}^kf)(\mathrm{D}^{n-(k-1)}g) + \sum_{k=0}^{n}\binom{n}{k}(\mathrm{D}^kf)(\mathrm{D}^{n-k+1}g) \\ &= (\mathrm{D}^{n+1}f)(\mathrm{D}^0g) + (\mathrm{D}^0f)(\mathrm{D}^{n+1}g) + \sum_{k=1}^{n}\left[\binom{n}{k-1} + \binom{n}{k}\right](\mathrm{D}^kf)(\mathrm{D}^{n+1-k}g) \\ &= \sum_{k=0}^{n+1}\binom{n+1}{k}(\mathrm{D}^kf)(\mathrm{D}^{n+1-k}g), \end{aligned}$$

since $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ for $1 \le k \le n$ and $\binom{n+1}{0} = \binom{n+1}{n+1} = 1$. Thus the formula holds also for $n+1$, and the proof by induction is complete.

53 a) If f is a polynomial function then

$$\mathrm{D}\left[f(t)\mathrm{e}^{-t^2}\right] = f'(t)\mathrm{e}^{-t^2} + f(t)\mathrm{e}^{-t^2}(-2t) = \big(f'(t) - 2t\,f(t)\big)\mathrm{e}^{-t^2} = F(t)\mathrm{e}^{-t^2}, \qquad \text{(H)}$$

where $F(t) = f'(t) - 2t\, f(t)$ is also a polynomial function. Starting with $f(t) = f_0(t) = 1$, it follows by induction that $\mathrm{D}^n[\mathrm{e}^{-t^2}] = f_n(t)\mathrm{e}^{-t^2}$ for some polynomial function f_n. Hence $\mathrm{H}_n(t) = (-1)^n \mathrm{e}^{t^2}[\mathrm{D}^n \mathrm{e}^{-t^2}] = (-1)^n f_n(t)$ is also a polynomial function.

b) From (H) we see that f_n has degree n and leading coefficient $(-2)^n$. Hence $\mathrm{H}_n(t)$ has degree n as well and leading coefficient 2^n.

c) We have

$$\begin{aligned}
\mathrm{H}_{n+1}(t) &= (-1)^{n+1}\mathrm{e}^{t^2}\mathrm{D}^n\big[\mathrm{D}\mathrm{e}^{-t^2}\big] = (-1)^{n+1}\mathrm{e}^{t^2}\mathrm{D}^n\big[-2t\,\mathrm{e}^{-t^2}\big] = 2(-1)^n\mathrm{e}^{t^2}\mathrm{D}^n\big[t\,\mathrm{e}^{-t^2}\big] \\
&= 2(-1)^n\mathrm{e}^{t^2}\big(t\mathrm{D}^n[\mathrm{e}^{-t^2}] + n\mathrm{D}^{n-1}[\mathrm{e}^{-t^2}]\big) \qquad \text{(by Leibniz' formula)} \\
&= 2t(-1)^n\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}] + 2n(-1)^n\mathrm{e}^{t^2}\mathrm{D}^{n-1}[\mathrm{e}^{-t^2}] = 2t\,\mathrm{H}_n(t) - 2n\,\mathrm{H}_{n-1}(t).
\end{aligned}$$

This proves the recursion formula in view of the 1-1 correspondence between polynomials in $\mathbb{R}[X]$ and polynomial functions on $\mathbb{R}$.

Together with $\mathrm{H}_0(t) = (-1)^0\mathrm{e}^{t^2}(\mathrm{e}^{-t^2}) = 1$, $\mathrm{H}_1(t) = (-1)^1\mathrm{e}^{t^2}(-2t\,\mathrm{e}^{-t^2}) = 2t$ the recursion formula gives

$$\begin{aligned}
&\mathrm{H}_0(X) = 1, \\
&\mathrm{H}_1(X) = 2X, \\
&\mathrm{H}_2(X) = 2X(2X) - 2\cdot 1 = 4\,X^2 - 2, \\
&\mathrm{H}_3(X) = 2X(4X^2 - 2) - 4(2X) = 8\,X^3 - 12\,X, \\
&\mathrm{H}_4(X) = 2X(8X^3 - 12X) - 6(4X^2 - 2) = 16\,X^4 - 48\,X^2 + 12, \\
&\mathrm{H}_5(X) = 2X(16\,X^4 - 48\,X^2 + 12) - 8(8\,X^3 - 12\,X) = 32\,X^5 - 160\,X^3 + 120\,X, \\
&\mathrm{H}_6(X) = 2X(32X^5 - 160X^3 + 120X) - 10(16X^4 - 48X^2 + 12) = 64X^6 - 480\,X^4 + 720\,X^2 - 120.
\end{aligned}$$

d) We have

$$\begin{aligned}
L\big[\mathrm{H}_n(t)\big] &= (-1)^n\mathrm{D}^2\big[\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}]\big] - 2t(-1)^n\mathrm{D}\big[\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}]\big] + 2n(-1)^n\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}], \\
(-1)^n L\big[\mathrm{H}_n(t)\big] &= \mathrm{e}^{t^2}\mathrm{D}^{n+2}[\mathrm{e}^{-t^2}] + 2\,2t\mathrm{e}^{t^2}\mathrm{D}^{n+1}[\mathrm{e}^{-t^2}] + (2+4t^2)\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}] \\
&\quad - 2t\mathrm{e}^{t^2}\mathrm{D}^{n+1}[\mathrm{e}^{-t^2}] - 2t\,2t\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}] + 2n\,\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}], \\
(-1)^n\mathrm{e}^{-t^2}L\big[\mathrm{H}_n(t)\big] &= \mathrm{D}^{n+2}[\mathrm{e}^{-t^2}] + 2t\mathrm{D}^{n+1}[\mathrm{e}^{-t^2}] + 2(n+1)\mathrm{D}^n[\mathrm{e}^{-t^2}].
\end{aligned}$$

On the other hand, we also have

$$\mathrm{D}^{n+2}[\mathrm{e}^{-t^2}] = \mathrm{D}^{n+1}[-2t\,\mathrm{e}^{-t^2}] = -2t\mathrm{D}^{n+1}[\mathrm{e}^{-t^2}] - 2(n+1)\mathrm{D}^n[\mathrm{e}^{-t^2}].$$

$\Longrightarrow (-1)^n\mathrm{e}^{-t^2}L\big[\mathrm{H}_n(t)\big] = 0 \Longrightarrow L\big[\mathrm{H}_n(t)\big] = 0$, as desired.

54 a) Using $(X^2-1)^n = \sum_{k=0}^n \binom{n}{k}(-1)^k X^{n-2k}$, we obtain

$$\begin{aligned}
&\mathrm{P}_0(X) = 1, \\
&\mathrm{P}_1(X) = \tfrac{1}{2}(X^2-1)' = X, \\
&\mathrm{P}_2(X) = \tfrac{1}{8}(X^4 - 2\,X^2 + 1)'' = \tfrac{1}{8}(12\,X^2 - 4) = \tfrac{1}{2}(3\,X^2 - 1), \\
&\mathrm{P}_3(X) = \tfrac{1}{48}(X^6 - 3\,X^4 + 3\,X^2 - 1)''' = \tfrac{1}{2}(5\,X^3 - 3\,X), \\
&\mathrm{P}_4(X) = \cdots = \tfrac{1}{8}(35\,X^4 - 30\,X^2 + 3), \\
&\mathrm{P}_5(X) = \cdots = \tfrac{1}{8}(63\,X^5 - 70\,X^3 + 15\,X), \\
&\mathrm{P}_6(X) = \cdots = \tfrac{1}{16}(231\,X^6 - 315\,X^4 + 105\,X^2 - 5).
\end{aligned}$$

Since the rings of polynomials and polynomial functions over an infinite field are isomorphic, it doesn't matter whether we write $\mathrm{P}_n(X)$ or $\mathrm{P}_n(t)$. This applies also to derivatives, which for polynomials are defined formally to resemble the derivative of the corresponding polynomial function, i.e., $\left(\sum_{k=0}^{d} a_k X^k\right)' := \sum_{k=1}^{d} k a_k X^{k-1}$.

b) For integers $m, n \geq 0$ we have

$$\begin{aligned}
2^{n+m} n! m! \int_{-1}^{1} \mathrm{P}_m(t)\mathrm{P}_n(t)\,\mathrm{d}t &= \int_{-1}^{1} \mathrm{D}^m\left[(t^2-1)^m\right]\mathrm{D}^n\left[(t^2-1)^n\right]\mathrm{d}t \\
&= \mathrm{D}^{m-1}\left[(t^2-1)^m\right]\mathrm{D}^n\left[(t^2-1)^n\right]\Big|_{-1}^{1} - \int_{-1}^{1} \mathrm{D}^{m-1}\left[(t^2-1)^m\right]\mathrm{D}^{n+1}\left[(t^2-1)^n\right]\mathrm{d}t \\
&= -\int_{-1}^{1} \mathrm{D}^{m-1}\left[(t^2-1)^m\right]\mathrm{D}^{n+1}\left[(t^2-1)^n\right]\mathrm{d}t = \cdots \\
&= (-1)^m \int_{-1}^{1} \mathrm{D}^0\left[(t^2-1)^m\right]\mathrm{D}^{n+m}\left[(t^2-1)^n\right]\mathrm{d}t,
\end{aligned}$$

since for $0 \leq k \leq m-1$ the polynomial $\mathrm{D}^k\left[(t^2-1)^m\right]$ has a zero (in fact an $(m-k)$-fold zero) at $t = \pm 1$. (For $m = 0$ there are no intermediate steps, but the identity holds as well.)

If $m > n$ then $n + m > 2n$ and hence $\mathrm{D}^{n+m}\left[(t^2-1)^n\right] = 0$. It follows that $\int_{-1}^{1} \mathrm{P}_m(t)\mathrm{P}_n(t)\,\mathrm{d}t = 0$ for $m > n$. By symmetry this also holds for $m < n$, showing the claimed formula for $m \neq n$,

In the case $m = n$ we get, using $\mathrm{D}^{2n}\left[(t^2-1)^n\right] = (2n)!$,

$$\int_{-1}^{1} \mathrm{P}_n(t)^2\,\mathrm{d}t = \frac{(-1)^n(2n)!}{2^{2n}n!^2}\int_{-1}^{1}(t^2-1)^{2n}\,\mathrm{d}t = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\int_{-1}^{1}(1-t^2)^{2n}\,\mathrm{d}t\,.$$

Moreover, we have

$$\begin{aligned}
\int_{-1}^{1}(1-t^2)^{2n}\,\mathrm{d}t &= 2\int_{0}^{1}(1-t^2)^{2n}\,\mathrm{d}t \\
&= 2\int_{0}^{\pi/2}\sin^{2n+1}(\theta)\mathrm{d}\theta && \text{(Subst. } t = \cos\theta) \\
&= 2\,\frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n+1)} && \text{(cf. Calculus III)}
\end{aligned}$$

It follows that $\int_{-1}^{1} \mathrm{P}_n(t)^2\,\mathrm{d}t = \frac{2}{2n+1}$, as asserted.

c) In the case $n = 0$ the assertion is trivially true, so we can assume $n \geq 1$. Since $f(t) = (t^2-1)^n$ satisfies $f(-1) = f(1) = 0$, Rolle's Theorem implies that there exists $x \in (-1, 1)$ such that $f'(\xi) = 0$. Since $f'(\pm 1) = 0$, we can apply Rolle's Theorem again (provided that $n \geq 2$) and conclude that f'' has zeros $\xi_1 \in (-1, \xi)$ and $\xi_2 \in (\xi, 1)$. Continuing in this way, we find that $f^{(n)}(t) = 2^n n!\,\mathrm{P}_n(t)$ has n distinct zeros in $(-1, 1)$. The same is then true of $\mathrm{P}_n(t)$ itself, of course.

Remark: In fact we have $f'(t) = 2nt(t^2-1)^{n-1}$ and hence $\xi = 0$. The derivative $f^{(k)}$, $1 \leq k \leq n$, has k simple zeros (and no further zero) in $(-1, 1)$, since it has degree $2n - k$ and zeros of multiplicity $n - k$ at $t = \pm 1$.

d) Since both sides of (GQ_n) are linear, (GQ_n) is exact for all polynomials of degree $\leq n-1$ provided it is exact for $1, t, t^2, \ldots, t^{n-1}$. This is the case iff

$$c_1 x_1^k + \cdots + c_n x_n^k = \int_{-1}^{1} t^k \,\mathrm{d}t = \begin{cases} \frac{2}{k+1} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd,} \end{cases}$$

for $k = 0, 1, \ldots, n-1$. The coefficient matrix of this linear system of equations for $c_1, \ldots, c_n$ is Vandermonde, hence invertible, and this shows that their is a unique assignment of weights haing this property.

e) Long division of f by P_n shows that their exist polynomials q, r with

$$f(t) = q(t)\mathrm{P}_n(t) + r(t) = c\,q(t)(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n) + r(t) \tag{LD}$$

and $\deg r \leq n-1$ or $r = 0$. The constant c is equal to $2^{-n}\binom{2n}{n}$ (the leading coefficient of P_n), and we have written $\alpha_i = \alpha_i^{(n)}$.

If $\deg f \leq 2n-1$ then $\deg q \leq n-1$, and hence q is a linear combination of the Legendre polynomials $\mathrm{P}_0, \mathrm{P}_1, \ldots, \mathrm{P}_{n-1}$. On account of b) this implies

$$\int_{-1}^{1} q(t)\mathrm{P}_n(t)\,\mathrm{d}t = 0,$$

$$\begin{aligned} \int_{-1}^{1} f(t)\,\mathrm{d}t &= \int_{-1}^{1} q(t)\mathrm{P}_n(t)\,\mathrm{d}t + \int_{-1}^{1} r(t)\underset{\cdot}{t} = \int_{-1}^{1} r(t)\,\mathrm{d}t \\ &= c_1 r(\alpha_1) + c_2 r(\alpha_2) + \cdots + c_n r(\alpha_n) && \text{(by d))} \\ &= c_1 f(\alpha_1) + c_2 f(\alpha_2) + \cdots + c_n f(\alpha_n). && \text{(cf. (LD))} \end{aligned}$$

f) For $n = 1$ we have $\mathrm{P}_1(X) = X$.
$\Longrightarrow \alpha_1 = 0$, $c_1 = c_1\alpha_1^0 = \int_{-1}^{1} \mathrm{d}t = 2$, and the approximation is

$$\int_{-1}^{1} f(t)\,\mathrm{d}t \approx 2\,f(0)$$

For $n = 2$ we have $\mathrm{P}_1(X) = \frac{1}{2}(3X^2 - 1)$.
$\Longrightarrow \alpha_1 = -\frac{1}{3}\sqrt{3}$, $\alpha_2 = \frac{1}{3}\sqrt{3}$, $c_1 + c_2 = \int_{-1}^{1} \mathrm{d}t = 2$, $c_1\alpha_1 + c_2\alpha_2 = \int_{-1}^{1} t\,\mathrm{d}t = 0$
The solution is $c_1 = c_2 = 1$, and hence the approximation is

$$\int_{-1}^{1} f(t)\,\mathrm{d}t \approx f\left(-\tfrac{1}{3}\sqrt{3}\right) + f\left(\tfrac{1}{3}\sqrt{3}\right).$$

For $n = 3$ we have $\mathrm{P}_1(X) = \frac{1}{2}(5X^3 - 3X)$.
$\Longrightarrow \alpha_1 = -\frac{1}{5}\sqrt{15}$, $\alpha_2 = 0$, $\alpha_3 = \frac{1}{5}\sqrt{15}$. The system in d) is

$$\begin{array}{ccccccc} c_1 & + & c_2 & + & c_3 & = & 2, \\ \alpha_1 c_1 & + & \alpha_2 c_2 & + & \alpha_3 c_3 & = & 0, \\ \alpha_1^2 c_1 & + & \alpha_2^2 c_2 & + & \alpha_3^2 c_3 & = & \frac{2}{3}, \end{array}$$

which is solved by $c_1 = c_3 = \frac{5}{9}$, $c_2 = \frac{8}{9}$. (Since $\alpha_3 = -\alpha_1$, the 2nd equation gives $c_1 = c_3$, and then the 3rd equation gives $c_1 = 1/3\alpha_1^2 = 5/9$.) Hence the approximation in this case is

$$\int_{-1}^{1} f(t)\,\mathrm{d}t \approx \frac{5}{9} f\left(-\tfrac{1}{5}\sqrt{15}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\tfrac{1}{5}\sqrt{15}\right).$$