Differential Equations (Math 285)

H55 Using power series, solve each of the following initial-value problems:

a) $t(2-t)y'' - 6(t-1)y' - 4y = 0, \quad y(1) = 1,\ y'(1) = 0;$

b) $y'' + (t^2 + 2t + 1)y' - (4 + 4t)y = 0, \quad y(-1) = 0,\ y'(-1) = 1.$

H56 a) Find 2 linearly independent solutions of $\quad y'' + t^3y' + 3t^2y = 0.$

b) Find the first 5 nonzero terms in the Taylor series expansion about $t = 0$ of the solution $y(t)$ of the initial value problem

$$y'' + t^3y' + 3t^2y = \mathrm{e}^t, \quad y(0) = y'(0) = 0.$$

H57 *A Problem from Sunday's Lecture*

Suppose (α_n) and (u_n) are sequences of nonnegative real numbers satisfying

$$\begin{aligned} \alpha_n &\leq \sum_{k=0}^{n-1} \frac{M(k+1)}{n(n-1)}\alpha_k \quad (n \geq 2), \\ u_n &= \sum_{k=0}^{n-1} \frac{M(k+1)}{n(n-1)}u_k \quad (n \geq 2), \\ u_0 &= \alpha_0,\ u_1 = \alpha_1 \end{aligned}$$

for some constant $M > 0$.

a) Show $\alpha_n \leq u_n$ for all n.

b) Show $\quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1.$

Hint: Express u_{n+1} in terms of u_n.

c) Is the sequence (u_n) (and hence (α_n) as well) necessarily bounded from above?

H58 *Optional Exercise*

Compute the Taylor series of $z \mapsto 1/(z^2 + 1)$ at $a = 1$ and $a = 1 + \mathrm{i}$.

Hint: Proceed as for $z \mapsto 1/(1 - z)$ in the lecture and then use partial fractions.

Due on Fri Apr 28, 4 pm

An in-depth discussion of power series solutions of 2nd-order linear ODE's (helpful for H55 and H56) will be provided in the lecture on Sun April 23.

The optional Exercise 58 can be handed in until Fri May 5, 4 pm. Exercise H57 c) is also optional, but should be handed in together with H57 a), b) on Fri April 28.

Solutions (prepared by Li Menglu and TH)

55 a) We look for a solution in the form of a power series about $t_0 = 1$. The series has the form

$$y(t) = \sum_{n=0}^{\infty} a_n (t-1)^n.$$

The point $t_0 = 1$ is an ordinary point of the differential equation, so the power series solution will be analytic at this point. Moreover, since the coefficient functions $p(t) = \frac{-6(t-1)}{t(2-t)}$, $q(t) = \frac{-4}{t(2-t)}$ of the corresponding explicit ODE have their singularities, viz. $t = 0$ and $t = 2$, at distance 1 from t_0, the radius of convergence of the power series will be at least 1, and $y(t)$ will solve the ODE on $(-1, 1)$.

Differentiating the equation term by term, we obtain that

$$y'(t) = \sum_{n=1}^{\infty} a_n n (t-1)^{n-1},$$

$$y''(t) = \sum_{n=2}^{\infty} a_n n(n-1)(t-1)^{n-2}.$$

Substituting the above series into the original equation gives

$$t(2-t)\sum_{n=2}^{\infty} a_n n(n-1)(t-1)^{n-2} - 6(t-1)\sum_{n=1}^{\infty} a_n n(t-1)^{n-1} - 4\sum_{n=0}^{\infty} a_n (t-1)^n = 0.$$

Rewrite the series so that they display the same generic term and using $t(2-t) = 1-(t-1)^2$ gives

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(t-1)^n - \sum_{n=2}^{\infty} a_n n(n-1)(t-1)^n - 6\sum_{n=1}^{\infty} a_n n(t-1)^n -$$
$$- 4\sum_{n=0}^{\infty} a_n (t-1)^n = 0,$$

which can be simplified to

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n^2+5n+4)a_n\right](t-1)^n = 0.$$

Hence the coefficients a_n must satisfy the recurrence relation

$$a_{n+2} = \frac{n^2+5n+4)}{(n+2)(n+1)}a_n = \frac{n+4}{n+2}a_n, \quad n = 0, 1, 2, 3, 4, \ldots$$

According to the initial conditions,

$$a_0 = y(1) = 1, \quad a_1 = y'(1) = 0.$$

The solution is $a_{2k+1} = 0$ for $k = 0, 1, 2, \ldots$ and

$$a_{2k} = \frac{2k+2}{2k}a_{2k-2} = \cdots = \frac{2k+2}{2k}\,\frac{2k}{2k-2}\cdots\frac{4}{2}a_0 = \frac{2k+2}{2} = k+1 \quad \text{for } k = 0, 1, 2, \ldots.$$

Substituting these coefficients into the original series, the solution of the IVP is

$$y(t) = \sum_{k=0}^{\infty} (k+1)(t-1)^{2k}, \quad -1 < t < 1.$$

The radius of convergence of this power series is obviously 1.

Remark: Making the variable transformation $x = t - 1$ early on saves some writing (but otherwise leads to the same solution, of course).

b) We look for a solution in the form of a power series about $t_0 = -1$. The series has the form

$$y = \sum_{n=0}^{\infty} a_n (t+1)^n.$$

The point $t_0 = -1$ is an ordinary point of the differential equation, and the coefficient functions $p(t) = (t+1)^2$, $q(t) = -4(t+1)$ are polynomials. Hence the power series will have radius of convergence ∞ and $y(t)$ will be defined and solve the ODE on $\mathbb{R}$. Proceeding as before, we obtain

$$\sum_{n=2}^{\infty} a_n n(n-1)(t+1)^{n-2} + (t+1)^2 \sum_{n=1}^{\infty} a_n n(t+1)^{n-1} - 4(t+1) \sum_{n=0}^{\infty} a_n (t+1)^n = 0,$$

$$\sum_{n=2}^{\infty} a_n n(n-1)(t+1)^{n-2} + \sum_{n=1}^{\infty} a_n n(t+1)^{n+1} - 4 \sum_{n=0}^{\infty} a_n (t+1)^{n+1} = 0,$$

$$2a_2 + \sum_{n=0}^{\infty} \left[(n+3)(n+2)a_{n+3} + (n-4)a_n \right] (t+1)^{n+1} = 0.$$

Hence the coefficients a_n must satisfy

$$a_2 = 0, \quad a_{n+3} = -\frac{n-4}{(n+3)(n+2)}\, a_n \quad \text{for } n = 0, 1, 2, 3, \dots.$$

The initial conditions are

$$a_0 = y(-1) = 0, \quad a_1 = y'(-1) = 1.$$

Hence $a_0 = a_3 = a_6 = \cdots = 0$, $a_2 = a_5 = a_8 = \cdots = 0$,

$$\begin{aligned} a_4 &= -\frac{1-4}{(1+3)(1+2)}\, a_1 = \frac{3}{12} = \frac{1}{4}, \\ a_7 &= -\frac{4-4}{(4+3)(4+2)}\, a_4 = 0, \end{aligned}$$

and $a_{10} = a_{13} = \cdots = 0$ as well. Substituting these coefficients into the original series, the solution of the IVP is

$$y = (t+1) + \frac{1}{4}(t+1)^4, \quad t \in \mathbb{R}.$$

56 a) As in H55b) solutions at $t_0 = 0$ must be analytic and exist on the whole real line. The power series „Ansatz" $y(t) = \sum_{n=0}^{\infty} a_n t^n$ yields

$$\sum_{n=2}^{\infty} a_n n(n-1)t^{n-2} + t^3 \sum_{n=1}^{\infty} a_n n t^{n-1} + 3t^2 \sum_{n=0}^{\infty} a_n t^n = 0,$$

$$\sum_{n=-2}^{\infty} (n+4)(n+3)a_{n+4}t^{n+2} + \sum_{n=0}^{\infty} n a_n t^{n+2} + 3\sum_{n=0}^{\infty} a_n t^{n+2} = 0,$$

$$2a_2 + 6a_3 t + \sum_{n=0}^{\infty} \left[(n+4)(n+3)a_{n+4} + (n+3)a_n\right] t^{n+2} = 0.$$

Hence the coefficients a_n satisfy

$$a_2 = a_3 = 0, \quad a_{n+4} = -\frac{1}{n+4}\, a_n \quad \text{for } n = 0, 1, 2, 3, \dots.$$

Two linearly independent solutions are obtained by setting $(a_0, a_1) = (1, 0)$ and $(0, 1)$, respectively, i.e.,

$$y_1(t) = 1 - \frac{t^4}{4} + \frac{t^8}{4\cdot 8} - \frac{t^{12}}{4\cdot 8\cdot 12} \pm \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k k!}\, t^{4k},$$

$$y_2(t) = t - \frac{t^5}{5} + \frac{t^9}{5\cdot 9} - \frac{t^{13}}{5\cdot 9\cdot 13} \pm \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{5\cdot 9\cdot 13 \cdots (4k+1)}\, t^{4k+1}.$$

b) The right-hand side of the equation can be expressed using Taylor series as

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Inserting this series into the ODE and using the initial conditions $a_0 = y(0) = 0$, $a_1 = y'(0) = 0$, changes the homogeneous recurrence relation in a) to the inhomogeneous recurrence relation $a_0 = a_1 = 0$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{6}$, and

$$a_{n+4} = -\frac{1}{n+4}\, a_n + \frac{1}{(n+4)!} \quad \text{for } n = 0, 1, 2, 3, \dots$$

The latter is obtained from equating coefficients at t^{n+2}, which gives $(n+4)(n+3)a_{n+4} + (n+3)a_n = \frac{1}{(n+2)!}$. The first few terms in the Taylor series expansion about $t = 0$ of the solution are then

$$y(t) = \frac{1}{2}\, t^2 + \frac{1}{6}\, t^3 + \frac{1}{24}\, t^4 + \frac{1}{120}\, t^5 - \frac{59}{6!}\, t^6 - \frac{119}{7!}\, t^7 - \frac{209}{8!}\, t^8 - \frac{335}{9!}\, t^9 + \frac{29737}{10!}\, t^{10} + \cdots.$$

(We asked for the "first 5 nonzero terms", because $a_6 = -\frac{59}{6!}$ disproves the apparent pattern $a_n = \frac{1}{n!}$, which holds for $n = 2, 3, 4, 5$.)

57 a) The assertion is trivially true for $n = 0, 1$. For $n \geq 2$ we may assume by induction that $\alpha_k \leq u_k$ for $0 \leq k < n$.

$$\Longrightarrow \alpha_n \leq \sum_{k=0}^{n-1} \frac{M(k+1)}{n(n-1)}\alpha_k \leq \sum_{k=0}^{n-1} \frac{M(k+1)}{n(n-1)} u_k = u_n.$$

b) We have

$$u_{n+1} = \frac{1}{(n+1)n}\sum_{k=0}^{n} M(k+1)u_k = \frac{1}{(n+1)n}\left(\sum_{k=0}^{n-1} M(k+1)u_k + M(n+1)u_n\right)$$
$$= \frac{n(n-1)u_n + M(n+1)u_n}{(n+1)n} = \frac{n(n-1)+M(n+1)}{(n+1)n}\,u_n \quad \text{for } n \geq 2.$$

It follows that

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{n(n-1)+M(n+1)}{(n+1)n} = \lim_{n\to\infty}\frac{n^2+(M-1)n+M}{n^2+n} = 1.$$

c) The answer is “No”. For $M \leq 2$ the sequence (u_n) remains bounded, but for $M > 2$ it diverges to $+\infty$ (except in the trivial case $u_0 = u_1 = 0$, in which $u_n = 0$ for all n).

The sum of the coefficients in the definition of u_n is $\frac{Mn(n+1)/2}{n(n-1)} = \frac{M(n+1)}{2(n-1)} \approx M/2$ for large n. For $M < 2$ the coefficient sum is ≤ 1 for large n, and one can prove by induction that (u_n) is bounded. (We had a similar example in the lecture.)

We will now show that if u_0, u_1 are not both zero and $M > 2$ then (u_n) is unbounded. Applying the formula for u_{n+1}/u_n repeatedly, we have

$$u_{n+1} = u_2\prod_{k=2}^{n}\frac{k(k-1)+M(k+1)}{(k+1)k}.$$

This says that the numbers u_n are the partial products of the infinite product

$$\prod_{n=2}^{\infty}\frac{n(n-1)+M(n+1)}{(n+1)n}.$$

It is known that an infinite product $\prod_{n=1}^{\infty}(1+b_n)$ with $b_n \geq 0$ converges (equivalently, is bounded) iff the series $\sum_{n=1}^{\infty} b_n$ converges. (In what follows we need only the implication $\Longrightarrow$, which is clear from $\prod_{k=1}^{n}(1+b_k) \geq 1 + \sum_{k=1}^{n} b_k$.) Since

$$\frac{n(n-1)+M(n+1)}{(n+1)n} = 1 + \frac{(M-2)n+M}{n^2+n} > 1 + \frac{(M-2)n+M-2}{n^2+n} = 1 + \frac{M-2}{n},$$

the divergence of the harmonic series implies for $M > 2$ that $\lim_{n\to\infty} u_n = \infty$ as well. (For $M = 2$ the fact about infinite products quoted above shows that (u_n) converges in $\mathbb{R}$, since this is true of the series $\sum_{n=1}^{\infty}\frac{1}{n^2+n}$.)

58 $\underline{a = 1:}$

$$\frac{1}{z^2+1} = \frac{1}{(z-\mathrm{i})(z+\mathrm{i})} = \frac{1}{2\mathrm{i}}\left(\frac{1}{z-\mathrm{i}} - \frac{1}{z+\mathrm{i}}\right)$$
$$= \frac{1}{2\mathrm{i}}\left(\frac{1}{z-1+1-\mathrm{i}} - \frac{1}{z-1+1+\mathrm{i}}\right)$$
$$= \frac{1}{2\mathrm{i}}\left[\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{(1-\mathrm{i})^{n+1}} - \sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{(1+\mathrm{i})^{n+1}}\right]$$
$$= \sum_{n=0}^{\infty} b_n(z-1)^n$$

with

$$b_n = \frac{(-1)^n}{2^{(n+1)/2}} \frac{(\mathrm{e}^{\mathrm{i}\pi/4})^{n+1} - (\mathrm{e}^{-\mathrm{i}\pi/4})^{n+1}}{2\mathrm{i}} = \frac{(-1)^n}{2^{(n+1)/2}} \sin\frac{(n+1)\pi}{4}$$

$$= \begin{cases} 2^{-n/2-1} & \text{if } n = 8k,\ 8k+2, \\ -2^{-(n+1)/2} & \text{if } n = 8k+1, \\ 0 & \text{if } n = 8k+3,\ 8k+7, \\ -2^{-n/2-1} & \text{if } n = 8k+4,\ 8k+6, \\ 2^{-(n+1)/2} & \text{if } n = 8k+5. \end{cases}$$

This can also be written as

$$\frac{1}{z^2+1} = \sum_{k=0}^{\infty} \frac{(z-1)^{8k}}{16^k} \left(\frac{1}{2} - \frac{(z-1)}{2} + \frac{(z-1)^2}{4} - \frac{(z-1)^4}{8} + \frac{(z-1)^5}{8} - \frac{(z-1)^6}{16} \right).$$

and shows the known fact that $\sum_{n=0}^{\infty} b_n(z-1)^n$ has radius of convergence $\sqrt{2}$ (the distance from 1 to the singularities $\pm\mathrm{i}$ of $1/(z^2+1)$).
<u>$a = 1+\mathrm{i}$</u>: Since $\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left(\frac{1}{z-a} - \frac{1}{z-b}\right)$, we obtain

$$\begin{aligned} \frac{1}{z^2+1} &= \frac{1}{(z-1-\mathrm{i}+1)(z-1-\mathrm{i}+1+2\mathrm{i})} = \frac{\mathrm{i}}{2}\left(\frac{1}{z-1-\mathrm{i}+1} - \frac{1}{z-1-\mathrm{i}+1+2\mathrm{i}} \right) \\ &= \frac{\mathrm{i}}{2}\left[\sum_{n=0}^{\infty} (-1)^n (z-1-\mathrm{i})^n - \sum_{n=0}^{\infty} (-1)^n \frac{(z-1-\mathrm{i})^n}{(1+2\mathrm{i})^{n+1}} \right] \\ &= \sum_{n=0}^{\infty} c_n (z-1-\mathrm{i})^n \end{aligned}$$

with

$$c_n = \frac{(-1)^n\,\mathrm{i}}{2}\left(1 - \frac{(1-2\mathrm{i})^{n+1}}{5^{n+1}} \right) = \frac{(-1)^n\,\mathrm{i}}{2}\left(1 - \frac{\left(\frac{1-2\mathrm{i}}{\sqrt{5}}\right)^{n+1}}{5^{(n+1)/2}} \right).$$

Since $\left|\frac{1-2\mathrm{i}}{\sqrt{5}}\right| = 1$, the last representation shows $c_n \simeq (-1)^n\,\mathrm{i}/2$ for $n \to \infty$, implying the known fact that $\sum_{n=0}^{\infty} c_n(z-1-\mathrm{i})^n$ has radius of convergence 1 (the distance from $1+\mathrm{i}$ to the nearest singularity i of $1/(z^2+1)$).