ZJU-UIUC Institute
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Spring Semester 2023
Homework 8

Differential Equations (Math 285)

H46 For $\alpha, \beta \in \mathbb{C}$ consider the explicit so-called Euler equation

$$y'' + \frac{\alpha}{t}\,y' + \frac{\beta}{t^2}\,y = 0 \qquad (t > 0). \tag{1}$$

a) Show that $\phi\colon \mathbb{R}^+ \to \mathbb{C}$ is a solution of (1) iff $\psi\colon \mathbb{R} \to \mathbb{C}$ defined by $\psi(s) = \phi(\mathrm{e}^s)$ is a solution of

$$y'' + (\alpha - 1)y' + \beta y = 0. \tag{2}$$

b) Using a), determine the general solution of (1) for $(\alpha, \beta) = (6, 4)$ and $(3, 1)$.

H47 The function e^t has no zero and satisfies $y' = y$. The function $\sin t$ has no zero in common with its derivative $\cos t$ and satisfies $y'' = -y$. Generalizing this observation, show that a nonzero C^n-function $f\colon I \to \mathbb{R}$ on an interval $I \subseteq \mathbb{R}$ of positive length satisfies an explicit (possibly time-dependent) homogeneous linear ODE of order n if and only if $y, y', \dots, y^{(n-1)}$ have no common zero.
Hint: For the if-part work with the function $t \mapsto f(t)^2 + f'(t)^2 + \dots + f^{(n-1)}(t)^2$.

H48 The solution to this exercise provides an easy method for computing $\mathrm{e}^{\mathbf{A}t}$ for a 2×2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbb{R}^{2\times 2}$ (or $\mathbb{C}^{2\times 2}$). We assume throughout the exercise that $\mathbf{A}$ is not a scalar multiple of the identity matrix $\mathbf{I}_2$.

a) Show $\mathbf{A}^2 = (bc - ad)\mathbf{I}_2 + (a + d)\mathbf{A}$.

b) Use a) to show that there exist uniquely determined functions $c_0, c_1\colon \mathbb{R} \to \mathbb{R}$ such that

$$\mathrm{e}^{\mathbf{A}t} = c_0(t)\mathbf{I}_2 + c_1(t)\mathbf{A} \quad \text{for } t \in \mathbb{R}. \tag{$\star$}$$

Further, show that c_0, c_1 are at least twice differentiable.

c) Show that c_0, c_1 solve the homogeneous linear ODE of order 2 with characteristic polynomial $X^2 - (a + d)X + ad - bc$ and satisfy the initial conditions $c_0(0) = 1$, $c_0'(0) = 0$ and $c_1(0) = 0$, $c_1'(0) = 1$.
Hint: Differentiate ($\star$) twice.

d) By solving the IVP's in c) determine $\mathrm{e}^{\mathbf{A}t}$ for $\mathbf{A} = \begin{pmatrix} 0 & 6 \\ 1 & 1 \end{pmatrix}$.

H49 Solve the initial value problem

$$\mathbf{y}' = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\mathbf{y} + \begin{pmatrix} t \\ \sin t \end{pmatrix}, \quad \mathbf{y}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Due on Fri Apr 14, 4 pm

The matrix eponential function and variation of parameters for 1st-order linear ODE systems (required for Exercises H48 and H49) will be discussed in the lecture on Mon April 10. You are advised to do H46 and H48 before the midterm, because Euler equations and simple (2×2) instances of the matrix exponential function can be the subject of a midterm question; cf. the online midterm samples.

Solutions

46 a) If $\psi(s)$ is a solution of

$$y'' + (\alpha - 1)y' + \beta y = 0,$$

we can use the variable substitution $\psi(s) = \phi(e^s)$ and get the first and second derivative of ψ as

$$\begin{aligned} \psi'(s) &= \phi'(e^s)e^s, \\ \psi''(s) &= [\phi'(e^s)e^s]' = \phi''(e^s)e^{2s} + \phi'(e^s)e^s. \end{aligned}$$

This gives

$$\begin{aligned} \psi(s)'' + (\alpha - 1)\psi(s)' + \beta\psi(s) = 0, \\ [\phi''(e^s)e^{2s} + \phi'(e^s)e^s] + (\alpha - 1)\phi'(e^s)e^s + \beta\phi(s) = 0. \end{aligned}$$

This simplifies to

$$e^{2s}\phi''(e^s) + \alpha e^s\phi'(e^s) + \beta\phi(e^s) = 0.$$

Since $t > 0$ is assumed, we can make the variable transformation $t = e^s$, i.e. $s = \ln(t)$, and obtain

$$\begin{aligned} t^2\phi''(t) + \alpha t\phi'(t) + \beta\phi(t) = 0, \\ \phi''(t) + \frac{\alpha}{t}\phi'(t) + \frac{\beta}{t^2}\phi(t) = 0. \end{aligned}$$

Therefore, the function $\phi(t)$ is a solution of Equation (1).

Conversely, if $y(t) = \phi(t)$ is a solution of Equation (1), we can make the substitution $t = e^s$ to obtain a function $y(s) = \psi(s)$. The derivatives can be represented as

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}s}, \\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}s}) = \frac{1}{t^2}(\frac{\mathrm{d}^2y}{\mathrm{d}s^2} - \frac{\mathrm{d}y}{\mathrm{d}s}). \end{aligned}$$

Hence the Euler equation is converted to

$$\frac{\mathrm{d}y^2}{\mathrm{d}s^2} + (\alpha - 1)\frac{\mathrm{d}y}{\mathrm{d}s} + \beta y = 0$$

.

b) 1) When $(\alpha, \beta) = (6, 4)$, the Euler equation can be converted to

$$\frac{\mathrm{d}y^2}{\mathrm{d}s^2} + 5\,\frac{\mathrm{d}y}{\mathrm{d}s} + 4y = 0.$$

The corresponding characteristic equation is $X^2 + 5X + 4 = 0$, and hence the general solution of Equation (2) in this case is

$$y(s) = C_1e^{-s} + C_2e^{-4s}, \quad s \in \mathbb{R}.$$

Therefore, the solution of Equation (1) is

$$y(t) = \frac{C_1}{t} + \frac{C_2}{t^4}, \quad t > 0.$$

2) When $(\alpha, \beta) = (3, 1)$, the Euler equation can be converted to

$$\frac{\mathrm{d}y^2}{\mathrm{d}s^2} + 2\,\frac{\mathrm{d}y}{\mathrm{d}s} + y = 0.$$

The corresponding characteristic equation is $X^2 + 2X + 1 = (X + 1)^2 = 0$, and hence the general solution of Equation (2) in this case is

$$y(s) = C_1 e^{-s} + C_2 s e^{-s}, \quad s \in \mathbb{R}.$$

Therefore, the solution of Equation (1) is

$$y(t) = \frac{C_1}{t} + \frac{C_2 \ln t}{t}, \quad t > 0.$$

47 $\Longrightarrow$: Suppose, by contradiction, that $f^{(n)}(t) = a_0(t)f(t) + a_1(t)f'(t) + \cdots + a_{n-1}(t)f^{(n-1)}(t)$ for all $t \in I$ and $f(t_0) = f'(t_0) = \cdots = f^{(n-1)}(t_0) = 0$ for some $t_0 \in I$. Then both f and the all-zero function on I solve the IVP $y^{(n)} = a_0(t)y + a_1(t)y' + \cdots + a_{n-1}(t)y^{(n-1)} \wedge y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0$. The Uniqueness Theorem (for linear ODEs, say) then implies that $y \equiv 0$, which contradicts the assumption.
$\Longleftarrow$: Under the given assumption $g(t) = f(t)^2 + f'(t)^2 + \cdots + f^{(n-1)}(t)^2$ is zero-free on I, i.e., we can write

$$1 = \frac{g(t)}{g(t)} = \frac{f(t)^2}{g(t)} + \frac{f'(t)^2}{g(t)} + \cdots + \frac{f^{(n-1)}(t)^2}{g(t)}.$$

Multiplying this identity by $f^{(n)}(t)$ gives

$$f^{(n)}(t) = \frac{f^{(n)}(t)f(t)^2}{g(t)} + \frac{f^{(n)}(t)f'(t)^2}{g(t)} + \cdots + \frac{f^{(n)}(t)f^{(n-1)}(t)^2}{g(t)},$$

which is an explicit homogeneous linear ODE of order n for f with coefficient functions

$$a_0(t) = \frac{f^{(n)}(t)f(t)}{g(t)}, \quad a_1(t) = \frac{f^{(n)}(t)f'(t)}{g(t)}, \quad \ldots, \quad a_{n-1}(t) = \frac{f^{(n)}(t)f^{(n-1)}(t)}{g(t)}.$$

48 a) This follows from the Cayley-Hamilton Theorem, which says $\chi_{\mathbf{A}}(X) = \mathbf{A}^2 - (a + d)\mathbf{A} + (ad - bc)\mathbf{I}_2 = \mathbf{0}$. Here is a direct proof:

$$\begin{aligned}
\mathbf{A}^2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & (a+d)b \\ (a+d)c & d^2 + bc \end{pmatrix} \\
&= (a + d) \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a^2 + bc - (a+d)a & 0 \\ 0 & d^2 + bc - (a+d)d \end{pmatrix} \\
&= (a + d)\mathbf{A} + (bc - ad)\mathbf{I}_2.
\end{aligned}$$

b) The relation $\mathbf{A}^2 = \alpha\mathbf{I}_2 + \beta\mathbf{A}$ ($\alpha = bc - ad$, $\beta = a + d$) can be used to express any power $\mathbf{A}^k$ as a linear combination of $\mathbf{I}_2$ and $\mathbf{A}$. It follows that

$$\sum_{k=0}^{n} \frac{t^k}{k!}\,\mathbf{A}^k = f_n(t)\mathbf{I}_2 + g_n(t)\mathbf{A}$$

for certain functions $f_n, g_n \colon \mathbb{R} \to \mathbb{R}$. Since the left-hand side converges to $\mathrm{e}^{\mathbf{A}t}$, so does the right-hand side, and hence the function sequences (f_n), (g_n) converge (point-wise) to functions f resp. g such that $\mathrm{e}^{\mathbf{A}t} = f(t)\mathbf{I}_2 + g(t)\mathbf{A}$ for $t \in \mathbb{R}$. This representation is unique, since $\mathbf{I}_2$ and $\mathbf{A}$ are assumed to be linearly independent.

From the lecture we know that $\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{\mathbf{A}t} = \mathbf{A}\mathrm{e}^{\mathbf{A}t}$. This can be iterated to yield $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\mathrm{e}^{\mathbf{A}t} = \mathbf{A}^2\mathrm{e}^{\mathbf{A}t}$ (and likewise $\frac{\mathrm{d}^k}{\mathrm{d}t^k}\,\mathrm{e}^{\mathbf{A}t} = \mathbf{A}^k\mathrm{e}^{\mathbf{A}t}$ for all $k \in \mathbb{N}$). Hence the functions f, g, which are the coordinate functions of $t \mapsto \mathrm{e}^{\mathbf{A}t}$ with respect to the "matrix basis" $\mathbf{I}, \mathbf{A}$, are twice differentiable as well (even of class C^∞).

Thus the assertion holds with $c_0 = f$, $c_1 = g$.

c) Using the observation made about the derivatives of $t \mapsto \mathrm{e}^{\mathbf{A}t}$ in b), we obtain

$$\begin{aligned} \mathrm{e}^{\mathbf{A}t} &= c_0(t)\mathbf{I}_2 + c_1(t)\mathbf{A}, \\ \mathbf{A}\mathrm{e}^{\mathbf{A}t} &= c_0'(t)\mathbf{I}_2 + c_1'(t)\mathbf{A}, \\ \mathbf{A}^2\mathrm{e}^{\mathbf{A}t} &= c_0''(t)\mathbf{I}_2 + c_1''(t)\mathbf{A} \end{aligned} \tag{$\star\star$}$$

for $t \in \mathbb{R}$. Together with a) this yields

$$\begin{aligned} \mathbf{0} &= \mathbf{A}^2\mathrm{e}^{\mathbf{A}t} - (a+d)\mathbf{A}\mathrm{e}^{\mathbf{A}t} + (ad-bc)\mathrm{e}^{\mathbf{A}t} \\ &= \big(c_0''(t) - (a+d)c_0'(t) + (ad-bc)c_0(t)\big)\mathbf{I}_2 + \big(c_1''(t) - (a+d)c_1'(t) + (ad-bc)c_1(t)\big)\mathbf{A} \end{aligned}$$

for all $t \in \mathbb{R}$, which can only hold if the coefficient functions of $\mathbf{I}_2$ and $\mathbf{A}$ vanish, i.e., c_0, c_1 solve the homogeneous linear ODE with characteristic polynomial $X^2 - (a+d)X + ad - bc$. The asserted initial conditions follow by substituting $t = 0$ into the 1st and 2nd equation of $(\star\star)$ and comparing coefficients of $\mathbf{I}_2, \mathbf{A}$.

d) According to a) the given matrix satisfies $\mathbf{A}^2 - \mathbf{A} - 6\mathbf{I}_2 = \mathbf{0}$.
$\Longrightarrow c_0, c_1$ solve $y'' - y' - 6y = 0$, which has characteristic polynomial $X^2 - X - 6 = (X+2)(X-3)$. The general solution of this ODE is $y(t) = a_1\,\mathrm{e}^{-2t} + a_2\,\mathrm{e}^{3t}$, with initial conditions $y(0) = a_1 + a_2$, $y'(0) = -2\,a_1 + 3\,a_2$. A short computation yields $c_0(t) = \frac{3}{5}\,\mathrm{e}^{-2t} + \frac{2}{5}\,\mathrm{e}^{3t}$, $c_1(t) = -\frac{1}{5}\,\mathrm{e}^{-2t} + \frac{1}{5}\,\mathrm{e}^{3t}$, and hence

$$\begin{aligned} \mathrm{e}^{\mathbf{A}t} &= \left(\frac{3}{5}\,\mathrm{e}^{-2t} + \frac{2}{5}\,\mathrm{e}^{3t}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \left(-\frac{1}{5}\,\mathrm{e}^{-2t} + \frac{1}{5}\,\mathrm{e}^{3t}\right)\begin{pmatrix}0 & 6\\ 1 & 1\end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix}3\,\mathrm{e}^{-2t} + 2\,\mathrm{e}^{3t} & -6\,e^{-2t} + 6\,\mathrm{e}^{3t} \\ -\mathrm{e}^{-2t} + \mathrm{e}^{3t} & 2\,\mathrm{e}^{-2t} + 3\,\mathrm{e}^{3t}\end{pmatrix}. \end{aligned}$$

49 For $\mathbf{A} = \left(\begin{smallmatrix}1 & 2\\ 3 & 6\end{smallmatrix}\right)$ the identity $\mathbf{A}^2 = 7\mathbf{A}$ is easily verified. It is a special case of the Cayley-Hamilton Theorem or of H48 a). From it we obtain $\mathbf{A}^3 = 7\mathbf{A}^2 = 7^2\mathbf{A}$, $\mathbf{A}^4 = 7^2\mathbf{A}^2 = 7^3\mathbf{A}$, etc., and in general $\mathbf{A}^k = 7^{k-1}\mathbf{A}$ for $k \in \mathbb{N}$ by induction.

$$\begin{aligned} \mathrm{e}^{\mathbf{A}t} &= \mathbf{I}_2 + t\,\mathbf{A} + \frac{t^2}{2!}\,\mathbf{A}^2 + \frac{t^3}{3!}\,\mathbf{A}^3 + \cdots \\ &= \mathbf{I}_2 + t\,\mathbf{A} + \frac{7t^2}{2!}\,\mathbf{A} + \frac{7^2t^3}{3!}\,\mathbf{A} + \cdots \\ &= \mathbf{I}_2 + \frac{\mathrm{e}^{7t}-1}{7}\,\mathbf{A} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}\frac{1}{7}(\mathrm{e}^{7t}-1) & \frac{2}{7}(\mathrm{e}^{7t}-1) \\ \frac{3}{7}(\mathrm{e}^{7t}-1) & \frac{6}{7}(\mathrm{e}^{7t}-1)\end{pmatrix} = \begin{pmatrix}\frac{1}{7}(\mathrm{e}^{7t}+6) & \frac{2}{7}(\mathrm{e}^{7t}-1) \\ \frac{3}{7}(\mathrm{e}^{7t}-1) & \frac{1}{7}(6\,\mathrm{e}^{7t}+1)\end{pmatrix}. \end{aligned}$$

Alternatively, we can proceed as in H48 d) to determine $\mathrm{e}^{\mathbf{A}t}$.
The particular solution $\mathbf{y}(t)$ of the inhomogeneous system satisfying $\mathbf{y}(0) = \mathbf{0}$ is of the form $\mathbf{y}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{c}(t)$ with

$$
\begin{aligned}
\mathbf{c}(t) &= \int_0^t \mathrm{e}^{-\mathbf{A}s}\,\mathbf{b}(s)\,\mathrm{d}s \qquad\qquad (\text{since } \mathbf{c}(0) = \mathbf{0})\\
&= \int_0^t \begin{pmatrix} \frac{1}{7}(\mathrm{e}^{-7s}+6) & \frac{2}{7}(\mathrm{e}^{-7s}-1) \\ \frac{3}{7}(\mathrm{e}^{-7s}-1) & \frac{1}{7}(6\,\mathrm{e}^{-7s}+1) \end{pmatrix} \begin{pmatrix} s \\ \sin s \end{pmatrix} \mathrm{d}s \\
&= \frac{1}{7}\int_0^t \left[\begin{pmatrix} 6 & -2 \\ -3 & 1 \end{pmatrix} + \mathrm{e}^{-7s} \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \right] \begin{pmatrix} s \\ \sin s \end{pmatrix} \mathrm{d}s \\
&= \frac{1}{7}\int_0^t \begin{pmatrix} 6s - 2\sin s + s\,\mathrm{e}^{-7s} + 2\sin s\,\mathrm{e}^{-7s} \\ -3s + \sin s + 3s\,\mathrm{e}^{-7s} + 6\sin s\,\mathrm{e}^{-7s} \end{pmatrix} \mathrm{d}s \\
&= \frac{1}{7} \begin{pmatrix} 3\,t^2 - \frac{1}{49}\,(7\,t+1)e^{(-7\,t)} - \frac{1}{25}\,(\cos(t) + 7\,\sin(t))e^{(-7\,t)} + 2\,\cos(t) - \frac{2376}{1225} \\ -\frac{3}{2}\,t^2 - \frac{3}{49}\,(7\,t+1)e^{(-7\,t)} - \frac{3}{25}\,(\cos(t) + 7\,\sin(t))e^{(-7\,t)} - \cos(t) - \frac{2376}{1225} \end{pmatrix}.
\end{aligned}
$$

$$
\implies \mathbf{y}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{c}(t) = \begin{pmatrix} \frac{3}{7}\,t^2 - \frac{1}{49}\,t + \frac{7}{25}\,\cos(t) + \frac{74}{8575}\,e^{(7\,t)} - \frac{1}{25}\,\sin(t) - \frac{99}{343} \\ -\frac{3}{14}\,t^2 - \frac{3}{49}\,t - \frac{4}{25}\,\cos(t) + \frac{222}{8575}\,e^{(7\,t)} - \frac{3}{25}\,\sin(t) + \frac{46}{343} \end{pmatrix}
$$

For the last two steps the computer algebra system SageMath was used.
Remark: There is a third way to determine a fundamental matrix of $\mathbf{y}' = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 6 \end{smallmatrix}\right)\mathbf{y}$ using the eigenvalues and eigenvectors of $\left(\begin{smallmatrix} 1 & 2 \\ 3 & 6 \end{smallmatrix}\right)$. This will be discussed later in the lecture. An adhoc solution, essentially equivalent to it, is the following: Since $\left(\begin{smallmatrix} 1 & 2 \\ 3 & 6 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, there is the constant solution $y_1(t) \equiv \left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right)$. Then one needs to guess that a non-constant solution of the form $y_2(t) = \mathrm{e}^{\lambda t}\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)$ exists. Substituting this into the ODE gives $\lambda\,\mathrm{e}^{\lambda t}\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right) = y_2'(t) = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 6 \end{smallmatrix}\right) y_2(t) = \mathrm{e}^{\lambda t}\left(\begin{smallmatrix} v_1+2v_2 \\ 3v_1+6v_2 \end{smallmatrix}\right)$. Comparing both sides, we obtain $v_1 + 2v_2 = \lambda\,v_1$, $3v_1 + 6v_2 = \lambda\,v_2$. Thus $\lambda v_2 = 3\lambda v_1$ and, since $\lambda \neq 0$, necessarily $v_2 = 3v_1$. Then, using the 1st equation, $\lambda = 7$ since $v_1 \neq 0$. Hence $\mathbf{y}_2(t) = \mathrm{e}^{7t}\left(\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right)$ is a solution, and $\mathbf{\Phi}(t) = \left(\begin{smallmatrix} 2 & \mathrm{e}^{7t} \\ -1 & 3\,\mathrm{e}^{7t} \end{smallmatrix}\right)$ is a fundamental matrix. From there we can either obtain $\mathrm{e}^{\mathbf{A}t}$ using the formula $\mathrm{e}^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}$ (cf. lecture) and proceed as above, or perform variation of parameters with the fundamental matrix $\mathbf{\Phi}(t)$ in place of $\mathrm{e}^{\mathbf{A}t}$ to obtain a solution of the inhomogeneous system.