Differential Equations (Math 285)

H59 For each of the following ODE's, find two linearly independent real solutions.

a) $4xy'' + 3y' - 3y = 0, \quad x \lessgtr 0;$

b) $x^2y'' - x(1+x)y' + y = 0, \quad x \lessgtr 0;$

c) $x^2y'' + xy' + (1+x)y = 0, \quad x > 0.$

H60 Consider the ODE

$$xy'' + 3y' - 3y = 0, \quad x > 0.$$

a) Show that the roots of the indicial equation are $r = 0$ and $r = -2$.

b) Find a solution $y_1(x) = \sum_{n=0}^{\infty} a_n x^n$.

c) Find a second solution $y_2(x) = a\, y_1(x) \ln x + x^{-2}\left(1 + \sum_{n=1}^{\infty} c_n x^n\right)$.

H61 Do Exercises 5, 6, 9 in [BDM17], Ch. 5.7 (Exercises 6, 7, 10 in the 11th US edition). Optionally also show that $\mathrm{Y}_0'(x) = -\mathrm{Y}_1(x)$ for $x > 0$; see p. 236 (p. 238 in the 11th US edition) for the definition of $\mathrm{Y}_1(x)$. The solution $y_2(x)$ appearing in the definition of $\mathrm{Y}_1(x)$ is the same as that you obtain in Exercise 9 (resp., Exercise 10).

H62 The Γ function is defined for $x > 0$ by $\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$, and for non-integral $x < 0$ by choosing an integer $n > -x$ and setting

$$\Gamma(x) := \frac{\Gamma(x+n)}{x(x+1)\cdots(x+n-1)}.$$

a) Show that $\Gamma(x)$ is well-defined for $x < 0$, $x \notin \mathbb{Z}$, and satisfies $\Gamma(x+1) = x\,\Gamma(x)$ for all $x \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$.
Hint: Recall from Calculus III that $\Gamma(x+1) = x\,\Gamma(x)$ for $x > 0$.

b) Show $\lim_{x \to -n} \frac{1}{\Gamma(x)} = 0$ for $n \in \mathbb{N} = \{0, 1, 2, \dots\}$.
This shows that $1/\Gamma$ can be continuously extended to $\mathbb{R}$ by defining $1/\Gamma(-n) := 0$ for $n \in \mathbb{N}$.

c) The Bessel function of order $\nu \in \mathbb{R}$ is defined as (cf. the lecture)

$$\mathrm{J}_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{\nu+2m} m!\,\Gamma(\nu+m+1)}\, x^{\nu+2m} \quad \text{for } x \in \mathbb{R},$$

cf. b) for the definition of $1/\Gamma(\nu+m+1)$.
Show $\quad \mathrm{J}_{-\nu} = (-1)^\nu \mathrm{J}_\nu \quad$ for $\nu \in \mathbb{N}$.
Hint: Show first that the coefficients of x^n in the expansion of $\mathrm{J}_{-\nu}(x)$ are zero for $n < \nu$.

H63 *Optional Exercise*

For $x \in \mathbb{R} \setminus \{0\}$, $\nu \in \mathbb{R}$ show:

a) $\mathrm{J}_{\nu+1}(x) = \dfrac{2\nu}{x}\,\mathrm{J}_\nu(x) - \mathrm{J}_{\nu-1}(x)$;

b) $\mathrm{J}'_\nu(x) = -\mathrm{J}_{\nu+1}(x) + \dfrac{\nu}{x}\,\mathrm{J}_\nu(x)$.

Remark: a) Provides a recurrence relation to determine J_ν for $\nu \in \mathbb{N}$ from J_0, J_1. The Neumann functions Y_ν, $\nu \in \mathbb{N}$, satisfy the same recurrence relation and provide a 2nd solution of $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, which is linearly independent of J_ν. Thus in order to determine Y_ν for $\nu \in \mathbb{N}$ (the only case of interest) it suffices to know Y_0 and Y_1.

H64 *Optional Exercise*

Show $\quad \mathrm{J}_0(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(x\sin\theta)\mathrm{d}\theta \quad$ for $x \in \mathbb{R}$.

Due on Fri May 5, 4 pm

The optional exercises can be handed in one week later.

Solutions (prepared by Li Menglu and TH)

59 a) Rewriting the ODE as

$$y'' + \frac{3}{4x}\, y' - \frac{3}{4x}\, y = 0,$$

we see that $x = 0$ is a regular singular point and

$$p_0 = \lim_{x \to 0} x \frac{3}{4x} = \frac{3}{4}, \quad q_0 = \lim_{x \to 0} x^2 \frac{-3}{4x} = 0$$

. $\Longrightarrow$ The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 - \frac{1}{4}r = 0$$

.

$\Longrightarrow$ The exponents at the singularity $x = 0$ are $r_1 = 0$, $r_2 = \frac{1}{4}$. Since $r_1 - r_2$ is not an integer, there must be solutions $y_1(x)$, $y_2(x)$ on $(0, \infty)$ of the form

$$y_1(x) = 1 + \sum_{n=0}^{\infty} a_n x^n, \quad y_2(x) = x^{\frac{1}{4}} \left(1 + \sum_{n=0}^{\infty} a_n x^n \right).$$

In terms of the rational functions $a_n(r)$ defined in the lecture and textbook, the coefficients of $y_1(x)$, $y_2(x)$ are $a_n = a_n(0)$ and $a_n = a_n(1/4)$, respectively. (We use 'a_n' for both, in order to be compatible with the notation used in [BDM17], Theorem 5.6.1.)

i) $r_1 = 0$:

$$\begin{aligned} y_1 &= \sum_{n=0}^{\infty} a_n x^n \\ y_1' &= \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \\ x y_1'' &= x \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n \end{aligned}$$

Substituting these into the ODE, we get

$$\begin{aligned} & 4 \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n + 3 \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 3 \sum_{n=0}^{\infty} a_n x^n = 0 \\ \Longrightarrow \quad & 3a_1 - 3a_0 + \sum_{n=1}^{\infty} \left\{ \left[4n(n+1) + 3(n+1) \right] a_{n+1} - 3a_n \right\} x^n = 0 \\ \Longrightarrow \quad & a_1 = a_0 \quad \text{and} \quad a_{n+1} = \frac{3}{(4n+3)(n+1)} a_n \quad \text{for} \quad n \geq 1. \end{aligned}$$

Setting $a_0 = 1$, we have

$$\begin{aligned}
y_1(x) &= 1 + x + \frac{3}{7\cdot 2}\,x^2 + \frac{3^2}{7\cdot 2\cdot 11\cdot 3}\,x^3 + \cdots \\
&= 1 + x + \sum_{n=2}^{\infty} \frac{3^{n-1}}{7\cdot 11\cdots(4n-1)\cdot n!}\,x^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{3^{n}}{3\cdot 7\cdot 11\cdots(4n-1)\cdot n!}\,x^n \\
&= \sum_{n=0}^{\infty} \frac{3^{n}}{3\cdot 7\cdot 11\cdots(4n-1)\cdot n!}\,x^n,
\end{aligned}$$

using the convention that $\prod_{n=1}^{0}(4n-1) = 1$ ("empty product").

ii) $r_2 = \frac{1}{4}$:

$$\begin{aligned}
y_2 &= x^{\frac14}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\frac14} \\
y_2' &= \sum_{n=0}^{\infty}\left(n+\frac14\right) a_n x^{n-\frac34} = \sum_{n=-1}^{\infty}\left(n+\frac54\right) a_{n+1} x^{n+\frac14} \\
xy_2'' &= \sum_{n=0}^{\infty}\left(n+\frac14\right)\left(n-\frac34\right) a_n x^{n-\frac34} = \sum_{n=-1}^{\infty}\left(n+\frac54\right)\left(n+\frac14\right) a_{n+1} x^{n+\frac14}
\end{aligned}$$

Substituting these into the ODE, the coefficient of $x^{-3/4}$ vanishes by construction, and we get

$$\sum_{n=0}^{\infty}\left\{\left[4\left(n+\frac54\right)\left(n+\frac14\right)+3\left(n+\frac54\right)\right] a_{n+1} - 3a_n\right\} x^{n+\frac14} = 0$$

$$\implies \quad a_{n+1} = \frac{3\,a_n}{4\left(n+\frac54\right)\left(n+\frac14\right)+3\left(n+\frac54\right)} = \frac{3\,a_n}{(4n+5)(n+1)} \quad \text{for} \quad n \geq 0.$$

Setting $a_0 = 1$, we obtain

$$y_2(x) = x^{\frac14} + \sum_{n=1}^{\infty} \frac{3^n}{5\cdot 9\cdots(4n+1)\cdot n!}\,x^{n+\frac14} = \sum_{n=0}^{\infty} \frac{3^n}{5\cdot 9\cdots(4n+1)\cdot n!}\,x^{n+\frac14}.$$

As shown in the lecture, $y_1(x)$ and $y_2(x)$ are linearly independent. This is also clear from the fact that $y_1(x)$ is analytic at $x = 0$ and $y_2(x) = x^{1/4} \times$ "nonzero analytic" is not.

As discussed in the lecture (or see Theorem 5.6.1 in [BDM17], p. 227), a fundamental system of solutions on $(-\infty, 0)$ is obtained by replacing the fractional part x^r (if any) in the solutions by $(-x)^r = |x|^r$. This doesn't affect $y_1(x)$ ($y_1(x)$ is analytic on $\mathbb{R}$ and hence solves the ODE on $\mathbb{R}$), but $y_2(x)$ is changed to

$$y_2^-(x) = (-x)^{\frac14}\sum_{n=0}^{\infty} \frac{3^n}{5\cdot 9\cdots(4n+1)\cdot n!}\,x^n, \quad x \in (-\infty, 0).$$

b) Rewriting the ODE as

$$y'' - \left(1 + \frac{1}{x}\right) y' + \frac{1}{x^2}\, y = 0,$$

we see that $x = 0$ is a regular singular point with $p_0 = -1$, $q_0 = 1$.
$\Longrightarrow$ The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = (r-1)^2 = 0$$

.
$\Longrightarrow$ The exponents at the singularity $x = 0$ are $r_1 = r_2 = 1$. Thus there must be solutions $y_1(x)$, $y_2(x)$ on $(0, \infty)$ of the form

$$y_1(x) = 1 + \sum_{n=1}^{\infty} a_n x^{n+1}, \qquad y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^{n+1}.$$

i) $r_1 = 1$:

$$\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^{n+1}, \\
y_1' &= \sum_{n=0}^{\infty} (n+1) a_n x^n, \\
x(1+x) y_1' &= \sum_{n=0}^{\infty} (n+1) a_n x^{n+1} + \sum_{n=0}^{\infty} (n+1) a_n x^{n+2} \\
&= \sum_{n=0}^{\infty} (n+1) a_n x^{n+1} + \sum_{n=1}^{\infty} n a_{n-1} x^{n+1} \\
&= \sum_{n=0}^{\infty} \left[(n+1) a_n + n a_{n-1}\right] x^{n+1}, \qquad (a_{-1} := 0) \\
x^2 y_1'' &= x^2 \sum_{n=1}^{\infty} (n+1) n a_n x^{n-1} = \sum_{n=1}^{\infty} (n+1) n a_n x^{n+1} = \sum_{n=0}^{\infty} (n+1) n a_n x^{n+1}.
\end{aligned}$$

Substituting these into the ODE, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} (n+1) n a_n x^{n+1} - \sum_{n=0}^{\infty} \left[(n+1) a_n + n a_{n-1}\right] x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} \\
= &\sum_{n=0}^{\infty} (n^2 a_n - n a_{n-1}) x^{n+1} = 0.
\end{aligned}$$

$$\Longrightarrow \quad a_n = \frac{a_{n-1}}{n} \quad \text{for} \quad n \geq 1$$

Setting $a_0 = 1$, we obtain $a_n = 1/n!$ and

$$y_1(x) = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = x\, \mathrm{e}^x.$$

ii) For the determination of $y_2(x)$ we use the recurrence relation for $a_n(r)$ derived in the lecture; cf. also [BDM17], p. 223, Eq. (8). Since $F(r)=(r-1)^2$, $p_0=p_1=-1$, $q_0=1$ and all other coefficients p_i, q_i are zero, we have

$$\begin{aligned} a_n(r) &= -\frac{1}{F(r+n)}\sum_{k=0}^{n-1}\left[(r+k)p_{n-k}+q_{n-k}\right]a_k(r) \\ &= \frac{-1}{(r+n-1)^2}\,(r+n-1)p_1 a_{n-1}(r) = \frac{a_{n-1}(r)}{r+n-1} \qquad (n\geq 1). \end{aligned}$$

Setting $a_0(r)=1$, we get

$$\begin{aligned} a_1(r) &= \frac{1}{r}, \\ a_2(r) &= \frac{1}{r(r+1)}, \\ &\vdots \\ a_n(r) &= \frac{1}{r(r+1)(r+2)\cdots(r+n-1)}. \end{aligned}$$

$$\begin{aligned} \Longrightarrow\quad b_n(r) &:= a_n'(r) = \frac{a_n'(r)}{a_n(r)}\,a_n(r) \\ &= -\left(\frac{1}{r}+\frac{1}{r+1}+\cdots+\frac{1}{r+n-1}\right)\frac{1}{r(r+1)(r+2)\cdots(r+n-1)} \end{aligned}$$

$$\Longrightarrow\quad b_n = b_n(1) = -\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\frac{1}{n!} = -\frac{\mathrm{H}_n}{n!}$$

$$\Longrightarrow\quad y_2(x) = \left(\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}\right)\ln x - \sum_{n=1}^{\infty}\frac{\mathrm{H}_n}{n!}\,x^{n+1} = x\,\mathrm{e}^x\ln x - \sum_{n=1}^{\infty}\frac{\mathrm{H}_n}{n!}\,x^{n+1}$$

The linear independency of $y_1(x)$, $y_2(x)$ was shown in the lecture.
A fundamental system of solutions on $(-\infty,0)$ is formed by $y_1(x)$ and

$$y_2^-(x) = x\,\mathrm{e}^x\ln(-x) - \sum_{n=1}^{\infty}\frac{\mathrm{H}_n}{n!}\,x^{n+1}, \quad x\in(-\infty,0).$$

Remark: The coefficients b_n can also be determined by substituting the „Ansatz“

for $y_2(x)$ into the ODE. Writing $L = x^2\mathrm{D}^2 - x(x+1)\mathrm{D} + \mathrm{id}$, we obtain

$$\begin{aligned}
y_2(x) &= y_1(x)\ln x + \sum_{n\geq 0} b_n x^n,\\
y_2'(x) &= y_1'(x)\ln x + \frac{y_1(x)}{x} + \sum_{n\geq 1} n b_n x^{n-1},\\
y_2''(x) &= y_1''(x)\ln x + 2\,\frac{y_1'(x)}{x} - \frac{y_1(x)}{x^2} + \sum_{n\geq 2} n(n-1) b_n x^{n-2},\\
L\left[y_2(x)\right] &= L\left[y_1(x)\right]\ln x + 2x\,y_1'(x) - (x+2)y_1(x) + L\left[\sum_{n\geq 0} b_n x^n\right]\\
&= 0 + \underbrace{2x(x+1)\mathrm{e}^x - (x+2)x\mathrm{e}^x}_{=x^2\mathrm{e}^x} + \sum_{n=1}^{\infty}(n^2 b_n - n b_{n-1})x^{n+1}\\
&= \sum_{n=1}^{\infty}\left(n^2 b_n - n b_{n-1} + \frac{1}{(n-1)!}\right)x^{n+1}.
\end{aligned}$$

$L\left[y_2(x)\right] = 0$ is equivalent to an inhomogeneous linear recurrence relation for b_n, which has the particular solution $b_0 = 0$, $b_n = -\mathrm{H}_n/n!$ for $n \geq 1$ (as can be seen by introducing $B_n = n!b_n$, which satisfies $B_n - B_{n-1} = -1/n$).

c) Rewriting the ODE as

$$y'' + \frac{1}{x}y' + \left(\frac{1}{x^2} + \frac{1}{x}\right)y = 0,$$

we see that $x = 0$ is a regular singular point and

$$p_0 = \lim_{x\to 0} x\frac{x}{x^2} = 1, \quad q_0 = \lim_{x\to 0} x^2\frac{1+x}{x^2} = 1.$$

$\Longrightarrow$ The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 + 1 = 0.$$

$\Longrightarrow$ The exponents at the singularity $x = 0$ are $r_1 = \mathrm{i}$, $r_2 = -\mathrm{i}$. Thus there must be solutions $y_1(x)$, $y_2(x)$ on $(0,\infty)$ of the form

$$\begin{aligned}
y_1(x) &= x^{\mathrm{i}}\sum_{n=0}^{\infty} a_n x^n = \mathrm{e}^{\mathrm{i}\ln x}\sum_{n=0}^{\infty} a_n x^n,\\
y_2(x) &= x^{-\mathrm{i}}\sum_{n=0}^{\infty} a_n x^n = \mathrm{e}^{-\mathrm{i}\ln x}\sum_{n=0}^{\infty} a_n x^n.
\end{aligned}$$

This time we first determine the functions $a_n(r)$ from the recurrence relation and then substitute $r = \pm\mathrm{i}$. Since $p_1 = 0$, $q_1 = 1$, the recurrence relation for $a_n(r)$ is

$$a_n(r) = -\frac{a_{n-1}(r)}{F(r+n)} = -\frac{a_{n-1}(r)}{(r+n)^2 + 1}.$$

$$
\begin{aligned}
\Longrightarrow \quad a_1(r) &= -\frac{a_0(r)}{(r+1)^2+1} = -\frac{1}{(r+1)^2+1},\\
a_2(r) &= \frac{1}{[(r+1)^2+1][(r+2)^2+1]},\\
&\vdots\\
a_n(r) &= \frac{(-1)^n}{[(r+1)^2+1][(r+2)^2+1]\cdots[(r+n)^2+1]},\\
\Longrightarrow y_1(x) &= e^{i\ln x}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{[(1+i)^2+1][(2+i)^2+1]\cdots[(n+i)^2+1]}\right),\\
y_2(x) &= e^{i\ln x}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{[(1-i)^2+1][(2-i)^2+1]\cdots[(n-i)^2+1]}\right).
\end{aligned}
$$

Two linearly independent real solutions $y_1^*(x)$, $y_2^*(x)$ are obtained by extracting real and imaginary part of $y_1(x)$, say.

$$
\begin{aligned}
y_1^*(x) &= \cos(\ln x)\left(1-\frac{x}{5}-\frac{x^2}{40}+\frac{3x^3}{520}\mp\cdots\right)-\sin(\ln x)\left(\frac{2x}{5}-\frac{3x^2}{40}+\frac{7x^3}{1560}\mp\cdots\right),\\
y_2^*(x) &= \sin(\ln x)\left(1-\frac{x}{5}-\frac{x^2}{40}+\frac{3x^3}{520}\mp\cdots\right)+\cos(\ln x)\left(\frac{2x}{5}-\frac{3x^2}{40}+\frac{7x^3}{1560}\mp\cdots\right).
\end{aligned}
$$

60 a) Rewriting the ODE as

$$y''+\frac{3}{x}y'-\frac{3}{x}=0,$$

we see that $x=0$ is a regular singular point and

$$p_0=\lim_{x\to 0}x\frac{3}{x}=3,\quad q_0=\lim_{x\to 0}x^2\frac{-3}{x}=0$$

.

$\Longrightarrow$ The indicial equation is

$$r^2+(p_0-1)r+q_0=r^2+2r=0$$

.

$\Longrightarrow$ The exponents at the singularity $x=0$ are $r_1=0$, $r_2=-2$.

b) $r_1=0$:

$$
\begin{aligned}
y_1 &= \sum_{n=0}^{\infty}a_n x^n,\\
y_1' &= \sum_{n=1}^{\infty}n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n,\\
xy_1'' &= \sum_{n=2}^{\infty}n(n-1)a_n x^{n-1} = \sum_{n=1}^{\infty}(n+1)n a_{n+1}x^n = \sum_{n=0}^{\infty}(n+1)n a_{n+1}x^n
\end{aligned}
$$

Substituting these into the ODE, we get

$$\sum_{n=0}^{\infty} \{[n(n+1)+3(n+1)]\,a_{n+1} - 3a_n\}\,x^n = 0.$$

$$\implies \quad a_{n+1} = \frac{3}{(n+1)(n+3)} a_n, \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Setting $a_0 = 1$ gives

$$y_1(x) = \sum_{n=0}^{\infty} \frac{2 \cdot 3^n}{n!(n+2)!} x^n.$$

c) $r_2 = -2$:

$$y_2(x) = a\, y_1(x) \ln x + x^{-2} \left(1 + \sum_{n=1}^{\infty} c_n x^n\right)$$

with

$$a = \lim_{r \to -2} (r+2) a_2(r), \quad c_n = \frac{\mathrm{d}}{\mathrm{d}r} \left[(r+2)a_n(r)\right]\big|_{r=-2}.$$

Since $p_1 = 0$, $q_1 = -3$ (and all other relevant p_i, q_i are zero), the recurrence relation for $a_n(r)$ is

$$a_n(r) = -\frac{-3\,a_{n-1}(r)}{F(r+n)} = \frac{3\,a_{n-1}(r)}{(r+n)(r+n+2)}.$$

Together with $a_0(r) = 1$ this leads to

$$\begin{aligned}
a_n(r) &= \frac{3^n}{[(r+1)(r+2)\cdots(r+n)][(r+3)(r+4)\cdots(r+n+2)]}, \\
a_N(r) = a_2(r) &= \frac{3^2}{(r+1)(r+2)(r+3)(r+4)}, \\
a &= \lim_{r \to -2} \frac{3^2}{(r+1)(r+3)(r+4)} = -\frac{9}{2}, \\
c_1 &= \left(\frac{3(r+2)}{(r+1)(r+3)}\right)\bigg|_{r=-2} = -3, \\
c_n &= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{3^n}{[(r+1)(r+3)(r+4)\cdots(r+n)][(r+3)(r+4)\cdots(r+n+2)]}\right]\bigg|_{r=-2} \\
&= \frac{3^n}{(n-2)!n!}\left(-1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n-2} + 1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \\
&= \frac{3^n}{(n-2)!n!}\left(\mathrm{H}_n + \mathrm{H}_{n-2} - 1\right) \qquad \text{for } n \geq 2.
\end{aligned}$$

Thus

$$y_2(x) = -\left(\sum_{n=0}^{\infty} \frac{3^{n+2}}{n!(n+2)!} x^n\right) \ln x + x^{-2} - 3x^{-1} + \sum_{n=2}^{\infty} \frac{3^n(\mathrm{H}_n + \mathrm{H}_{n-2} - 1)}{n!(n-2)!} x^{n-2}.$$

Remark: If $y(x)$ solves $xy'' + 3y' - 3y = 0$ then $z(x) := x\,y(x/3)$ solves the ODE in H59c). This is suggested by the form of $y_1(x)$ and can be proved easily. Thus we can save the computation in c) and obtain directly $y_2^*(x) = z(3x)/(3x)$, where z denotes the 2nd solution of H59c). The coefficient of $y_2^*(x)$ at x^{-2} is $1/9$, so that $y_2(x) = 9\,y_2^*(x) + c\,y_1(x)$ for some $c \in \mathbb{R}$. Although it is not necessary for the solution, one can check that $c = \frac{9}{2}\ln 3$.

61 a) Exercise 5

Using the ratio test, we get

$$\begin{aligned}\lim_{m\to+\infty}\left|\frac{a_{m+1}}{a_m}\right| &= \lim_{m\to+\infty}\left|\frac{\frac{(-1)^{m+1}x^{2(m+1)}}{2^{2(m+1)}((m+1)!)^2}}{\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}}\right| \\ &= \lim_{m\to+\infty}\left|\frac{-x^2}{4(m+1)^2}\right| \\ &= \lim_{m\to+\infty}\frac{x^2}{4(m+1)^2} \\ &= 0 \\ &< 1\end{aligned}$$

for all $x \neq 0$.

So, the series for $\mathrm{J}_0(x)$ converges absolutely for all x.

b) Exercise 6

Using the ratio test, we get

$$\begin{aligned}\lim_{m\to+\infty}\left|\frac{a_{m+1}}{a_m}\right| &= \lim_{m\to+\infty}\left|\frac{\frac{(-1)^{m+1}x^{2(m+1)}}{2^{2(m+1)}(m+2)!(m+1)!}}{\frac{(-1)^m x^{2m}}{2^{2m}(m+1)!m!}}\right| \\ &= \lim_{m\to+\infty}\left|\frac{-x^2}{4(m+2)(m+1)}\right| \\ &= \lim_{m\to+\infty}\frac{x^2}{4(m+2)(m+1)} \\ &= 0 \\ &< 1\end{aligned}$$

for all $x \neq 0$.

So, the series for $\mathrm{J}_1(x)$ converges absolutely for all x.

It follows that we can obtain the derivative of $\mathrm{J}_0(x)$ everywhere by term-wise differentiation:

$$\begin{aligned}\mathrm{J}_0'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} \\ &= \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m-1}}{2^{2m-1}m!(m-1)!} \\ &= \sum_{m=0}^{\infty}\frac{(-1)^{m+1}x^{2m+1}}{2^{2m+1}(m+1)!m!} \\ &= -\frac{x}{2}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m+1)!m!} \\ &= -\mathrm{J}_1(x),\end{aligned}$$

as claimed.

c) Exercise 9
First, we want to show that $a_1(-1) = a_1'(-1) = 0$.
Equation (24) in Ch. 5.7 gives

$$a_1(r)((r+1)^2 - 1)x^{r+1} = 0.$$

Hence $a_1(r) = 0$ for $r \notin \{-2, 0\}$, and in particular $a_1(-1) = a_1'(-1) = 0$. (Alternatively, look at the recurrence relation $a_n(r) = -F(r+n)^{-1} \sum_{k=0}^{n-1} [(r+k)p_{n-k} + q_{n-k}]\, a_k(r)$, which for $n = 1$ reduces to $a_1(r) = -\frac{1}{r(r+2)}\, [rp_1 + q_1]\, a_0(r) = \frac{0}{r(r+2)}$, since for the Bessel equation $p_1 = q_1 = 0$.) Next,

$$c_1(-1) = \frac{\mathrm{d}}{\mathrm{d}r}[(r+1)a_1(r)]\bigg|_{r=-1} = 0.$$

Then, from equation (25) in Ch. 5.7 or using the said general recurrence relation for $a_n(r)$, we get

$$a_n(r) = \frac{-a_{n-2}(r)}{(r+n-1)(r+n+1)} \quad \text{for } n \geq 2.$$

Since $a_1(r) = 0$, this gives $a_n(r) = 0$ for all odd n wherever $a_n(r)$ is defined (i,e., $r \notin \{0, -2, -4, \ldots, -n-1\}$), and hence $c_n(-1) = \frac{\mathrm{d}}{\mathrm{d}r}\left[(r+1)a_n(r)\right]\big|_{r=-1} = 0$ for all odd n. For even n the recurrence relation gives by induction

$$\begin{aligned}
a_2(r) &= \frac{-a_0(r)}{(r+1)(r+3)} = -\frac{1}{(r+1)(r+3)}, \\
a_4(r) &= \frac{-a_2(r)}{(r+3)(r+5)} = \frac{1}{(r+1)(r+3)(r+3)(r+5)}, \\
&\vdots \\
a_{2m}(r) &= \frac{-1}{(r+2m-1)(r+2m+1)} \cdot \frac{-1}{(r+2m-3)(r+2m-1)} \cdots \frac{-1}{(r+1)(r+3)} \\
&= \frac{(-1)^m}{(r+1)(r+3)\cdots(r+2m-1)(r+3)(r+5)\cdots(r+2m+1)}.
\end{aligned}$$

So,

$$\begin{aligned}
c_{2m}(-1) &= \frac{\mathrm{d}}{\mathrm{d}r}[(r+1)a_{2m}(r)]\bigg|_{r=-1} \\
&= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{(-1)^m}{(r+3)^2(r+5)^2\cdots(r+2m-1)^2(r+2m+1)}\right)\bigg|_{r=-1} \\
&= \left[\left(-\frac{2}{r+3} - \frac{2}{r+5} - \frac{2}{r+2m-1} - \frac{1}{r+2m+1}\right)(r+1)a_{2m}(r)\right]\bigg|_{r=-1} \\
&= \left(-1 - \frac{1}{2} - \cdots - \frac{1}{m-1} - \frac{1}{2m}\right)\frac{(-1)^m}{2^2 4^2 \cdots (2m-2)^2(2m)} \\
&= -\frac{1}{2}(\mathrm{H}_{m-1} + \mathrm{H}_m)\frac{(-1)^m}{2^{2m-1}m!(m-1)!} \\
&= \frac{(-1)^{m+1}(\mathrm{H}_{m-1} + \mathrm{H}_m)}{2^{2m}m!(m-1)!} \qquad \text{for } m = 1, 2, \ldots
\end{aligned}$$

Finally, we need to compute

$$\begin{aligned} a &= \lim_{r\to -1}(r+1)a_2(r) \\ &= \lim_{r\to -1}\left(\frac{-1}{r+3}\right) \\ &= -\frac{1}{2}. \end{aligned}$$

According to the theory (e.g., Th. 5.6.1 in Ch. 5.6), a 2nd solution of Bessel's equation of order one is

$$\begin{aligned} y_2(x) &= -\frac{1}{2}y_1(x)\ln|x| + \frac{1}{|x|}\left(1-\sum_{m=1}^{\infty}\frac{(-1)^m(\mathrm{H}_m+\mathrm{H}_{m-1})}{2^{2m}m!(m-1)!}x^{2m}\right) \\ \text{`} &= -\mathrm{J}_1(x)\ln|x| + \frac{1}{|x|}\left(1-\sum_{m=1}^{\infty}\frac{(-1)^m(\mathrm{H}_m+\mathrm{H}_{m-1})}{2^{2m}m!(m-1)!}x^{2m}\right), \quad x\neq 0. \end{aligned}$$

For this note that $y_1(x)$ denotes the analytic solution normalized by $a_0 = y_1'(0) = 1$, so that $\mathrm{J}_1(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{2^{2m+1}m!(m+1)!}x^{2m+1} = \frac{x}{2}+\cdots = \frac{1}{2}y_1(x)$.
The corresponding Neumann function is then

$$\begin{aligned} \mathrm{Y}_1(x) &= \frac{2}{\pi}\left[-y_2(x)+(\gamma-\ln 2)\mathrm{J}_1(x)\right] \\ &= \frac{2}{\pi}\left[\left(\ln\frac{x}{2}+\gamma\right)\mathrm{J}_1(x) - \frac{1}{x} + \sum_{m=1}^{\infty}\frac{(-1)^m(\mathrm{H}_m+\mathrm{H}_{m-1})}{2^{2m}m!(m-1)!}x^{2m-1}\right]. \end{aligned}$$

Finally we show that $\mathrm{Y}_0'(x) = -\mathrm{Y}_1(x)$.

$$\begin{aligned} \mathrm{Y}_0'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{2}{\pi}\left[\left(\ln\frac{x}{2}+\gamma\right)\mathrm{J}_0(x) + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}\mathrm{H}_m}{2^{2m}(m!)^2}x^{2m}\right] \\ &= \frac{2}{\pi}\left[\frac{\mathrm{J}_0(x)}{x} + \left(\ln\frac{x}{2}+\gamma\right)\mathrm{J}_0'(x) + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}\mathrm{H}_m}{2^{2m-1}m!(m-1)!}x^{2m-1}\right] \\ &= \frac{2}{\pi}\left[-\left(\ln\frac{x}{2}+\gamma\right)\mathrm{J}_1(x) + \frac{1}{x} + \sum_{m=1}^{\infty}\left(\frac{(-1)^m}{2^{2m}(m!)^2} + \frac{(-1)^{m+1}\mathrm{H}_m}{2^{2m-1}m!(m-1)!}\right)x^{2m-1}\right] \\ &= \frac{2}{\pi}\left[-\left(\ln\frac{x}{2}+\gamma\right)\mathrm{J}_1(x) + \frac{1}{x} - \sum_{m=1}^{\infty}\frac{\frac{(-1)^{m-1}}{m}+(-1)^m 2\mathrm{H}_m}{2^{2m}m!(m-1)!}x^{2m-1}\right] \\ &= -\mathrm{Y}_1(x), \end{aligned}$$

since $2\mathrm{H}_m - \frac{1}{m} = \mathrm{H}_m + \mathrm{H}_{m-1}$.

62 a) To show that $\Gamma(x)$ is well-defined for $x<0$, $x\notin\mathbb{Z}$, we only need to show that different choices of $n > -x$ don't affect the value of $\Gamma(x)$ as specified in the exercise.

$$\begin{aligned}
\Gamma(x) &= \frac{\Gamma(x+n)}{x(x+1)\cdots(x+n-1)} \\
&= \frac{\Gamma(x+n+1)}{x(x+1)\cdots(x+n-1)(x+n)} && \text{(since } \Gamma(x+n+1) = (x+n)\Gamma(x+n)\text{)} \\
&= \frac{\Gamma(x+n+2)}{x(x+1)\cdots(x+n-1)(x+n)(x+n+1)} && \text{(same reasoning)} \\
&= \cdots
\end{aligned}$$

So, as $n > -x$ varies, the result of $\Gamma(x)$ remains the same, which means that $\Gamma(x)$ is well-defined.
Then, we prove that $\Gamma(x+1) = \Gamma(x)$. For $x > 0$ this was shown in Calculus III, so it remains to consider the case $x < 0$, $x \notin \mathbb{Z}$. Choose $n \in \mathbb{N}$ such that $x + n > 0$. Then in the definition of $\Gamma(x+1)$ we can use $n - 1$, since $x + 1 + (n-1) = x + n > 0$.

$$\begin{aligned}
\implies \Gamma(x+1) &= \frac{\Gamma(x+1+n-1)}{(x+1)(x+2)\cdots(x+1+(n-1)-1)} \\
&= \frac{\Gamma(x+n)}{(x+1)(x+2)\cdots(x+n-1)} \\
&= x\,\frac{\Gamma(x+n)}{x(x+1)(x+2)\cdots(x+n-1)} \\
&= x\,\Gamma(x)
\end{aligned}$$

For $n = 1$, which is possible only if $-1 < x < 0$, the definition of $\Gamma(x)$ reduces to $\Gamma(x) = \frac{\Gamma(x+1)}{x}$ and the functional equation holds as well. This case is included in the above computation, provided the first denominator is interpreted as 1 (empty product).

b) For x close to $-n$ we have $x + n + 1 > 0$. Hence a) gives

$$\lim_{x \to -n} \frac{1}{\Gamma(x)} = \lim_{x \to -n} \frac{x(x+1)\cdots(x+n)}{\Gamma(x+n+1)}.$$

Since $\Gamma(1) = 1$, the limit evaluates to

$$\lim_{x \to -n} \frac{1}{\Gamma(x)} = \frac{(-n)(-n+1)\cdots(0)}{1} = 0.$$

c) First, we have

$$\mathrm{J}_{-\nu}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{-\nu+2m} m!\,\Gamma(-\nu+m+1)}\, x^{-\nu+2m}.$$

From b), we know that $1/\Gamma(-n) = 0$ for $n \in \mathbb{N}$. So, the coefficients of $x^{-\nu+2m}$ are zero for $m < \nu$. Then

$$
\begin{aligned}
\mathrm{J}_{-\nu}(x) &= \sum_{m=\nu}^{\infty} \frac{(-1)^m}{2^{-\nu+2m} m!\,\Gamma(-\nu+m+1)}\, x^{-\nu+2m} \\
&= \sum_{m=\nu}^{\infty} \frac{(-1)^m}{2^{\nu+2(m-\nu)} m!\,\Gamma((m-\nu)+1)}\, x^{\nu+2(m-\nu)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+\nu}}{2^{\nu+2n} (n+\nu)!\,\Gamma(n+1)}\, x^{\nu+2n} && (\text{let } n = m-\nu) \\
&= (-1)^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{\nu+2n}(n+\nu)!n!}\, x^{\nu+2n} \\
&= (-1)^{\nu} \mathrm{J}_{\nu}(x).
\end{aligned}
$$

63 For $\nu \in \mathbb{N}$ the function $\mathrm{J}_\nu(x)$ was defined in the lecture as the analytic solution of Bessel's equation of order ν normalized by setting the coefficient of x^ν (first nonzero coefficient) equal to $\frac{1}{2^\nu \nu!}$. It can also be derived using Frobenius' method as follows (not part of the exercise):

$$
\begin{aligned}
0 &= x^2 y'' + xy' + (x^2-\nu^2)y \\
&= \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n) x^{r+n} + (x^2-\nu^2)\cdot \sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty} a_n [(r+n)^2-\nu^2] x^{r+n} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n} \\
&= a_0(r^2-\nu^2)x^r + a_1[(r+1)^2-\nu^2]x^{r+1} + \sum_{n=2}^{\infty} \left\{ [(r+n)^2-\nu^2]a_n + a_{n-2} \right\} x^{r+n}
\end{aligned}
$$

For $r = \nu$ there are solutions with arbitrary a_0. These must satisfy $a_n = 0$ for all odd n and $[(\nu+n)^2-\nu^2]a_n + a_{n-2} = n(n+2\nu)a_n + a_{n-2}0$ for all even $n \geq 2$. By induction,

$$
\begin{aligned}
a_{2m} &= -\frac{a_{2m-2}}{2m(2m+2\nu)} = \cdots = \frac{(-1)^m a_0}{[2m(2m-2)\cdots 2]\,[(2m+2\nu)(2m-2+2\nu)\cdots(2+2\nu)]} \\
&= \frac{(-1)^m a_0}{2^{2m} m! (\nu+1)(\nu+2)\cdots(\nu+m)}.
\end{aligned}
$$

Choosing $a_0 = \frac{1}{2^\nu \nu!}$, we get

$$
\mathrm{J}_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{\nu+2m}}{2^{\nu+2m} m! (\nu+m)!}.
$$

Then, we solve the exercise:

a)

$$
\begin{aligned}
\frac{2\nu}{x}\mathrm{J}_\nu(x) - \mathrm{J}_{\nu-1}(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m x^{\nu+2m-1}\nu}{2^{\nu+2m-1}m!(\nu+m)!} - \sum_{m=0}^{\infty} \frac{(-1)^m x^{\nu+2m-1}}{2^{\nu+2m-1}m!(\nu+m-1)!} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m x^{\nu+2m-1}}{2^{\nu+2m-1}m!(\nu+m-1)!}\left(\frac{\nu}{\nu+m} - 1\right) \\
&= \sum_{m=1}^{\infty} \frac{(-1)^{m+1} x^{\nu+2m-1}}{2^{\nu+2m-1}(m-1)!(\nu+m)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+2} x^{\nu+2n+1}}{2^{\nu+2n+1}n!(\nu+n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{\nu+1+2n}}{2^{\nu+1+2n}n!(\nu+1+n)!} \\
&= \mathrm{J}_{\nu+1}(x)
\end{aligned}
$$

b) The Bessel functions may be differentiated termwise to yield

$$
\begin{aligned}
\mathrm{J}_\nu'(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{\nu+2m}\, m!\, \Gamma(m+\nu+1)}\, x^{\nu+2m} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m(\nu+2m)}{2^{\nu+2m}\, m!\, \Gamma(m+\nu+1)}\, x^{\nu+2m-1} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m\nu}{2^{\nu+2m} m!\, \Gamma(m+\nu+1)}\, x^{\nu+2m-1} + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^{\nu+2m-1}(m-1)!\, \Gamma(m+\nu+1)}\, x^{\nu+2m-1} \\
&= \frac{\nu}{x}\,\mathrm{J}_\nu(x) + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{2^{\nu+2m+1} m!\, \Gamma(m+\nu+2)}\, x^{\nu+2m+1} \\
&= \frac{\nu}{x}\,\mathrm{J}_\nu(x) - \mathrm{J}_{\nu+1}(x).
\end{aligned}
$$

64 First, $\cos(x\sin\theta)$ can be written as

$$
\cos(x\sin(\theta)) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} \sin^{2m}\theta.
$$

If $x \in \mathbb{R}$ is kept fixed, this represents a function series $\sum_{m=0}^{\infty} f_m(\theta)$, which converges uniformly on $[0,\pi]$ by Weierstrass' test. Hence the series can be integrated termwise, and we obtain

$$
\frac{1}{\pi}\int_0^\pi \cos(x\sin(\theta))\mathrm{d}\theta = \frac{1}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} \int_0^\pi \sin^{2m}\theta\,\mathrm{d}\theta \tag{$\star$}
$$

The integral $\int_0^\pi \sin^{2m}\theta\,\mathrm{d}\theta = 2\int_0^{\pi/2} \sin^{2m}\theta\,\mathrm{d}\theta$ has been evaluated in Calculus III (or see our Calculus textbook [Ste12/16], Ch. 7.1, Exercise 50):

$$
\begin{aligned}
\int_0^\pi \sin^{2m}\theta\,\mathrm{d}\theta &= \frac{(2m-1)(2m-3)\cdots 1}{2m(2m-2)\cdots 2}\,\pi \\
&= \frac{(2m)!}{(2m)^2(2m-2)^2\cdots 2^2}\,\pi = \frac{(2m)!}{2^{2m}(m!)^2}\,\pi.
\end{aligned}
$$

Substituting this into ($\star$), we get

$$\begin{aligned}
\frac{1}{\pi}\int_0^{\pi}\cos(x\sin(\theta))\mathrm{d}\theta &= \frac{1}{\pi}\sum_{m=0}^{\infty}\left[\frac{(-1)^m}{(2m)!}x^{2m}\,\frac{(2m)!}{2^{2m}(m!)^2}\,\pi\right] \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} \\
&= \mathrm{J}_0(x)
\end{aligned}$$

for $x \in \mathbb{R}$.
Alternative solution: J_0 is the unique solution on $\mathbb{R}$ of the IVP $x^2y'' + xy' + x^2y = 0$, $y(0) = 1$. This follows from the fact that Y_0 is not defined at $x = 0$. The right-hand side $f(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\mathrm{d}\theta$ satisfies $f(0) = \frac{1}{\pi}\int_0^{\pi}\cos(0)\mathrm{d}\theta = \frac{1}{\pi}\int_0^{\pi}1\,\mathrm{d}\theta = 1$. Since $[0,\pi]$ is compact and the integrand $g(x,\theta) = \cos(x\sin\theta)$ has continuous partial derivatives up to order two (in fact up to any order), we can differentiate twice under the integral sign to obtain

$$\begin{aligned}
f'(x) &= \frac{1}{\pi}\int_0^{\pi} -\sin(x\sin\theta)\sin\theta\,\mathrm{d}\theta, \\
f''(x) &= \frac{1}{\pi}\int_0^{\pi} -\cos(x\sin\theta)\sin^2\theta\,\mathrm{d}\theta.
\end{aligned}$$

It follows that

$$\begin{aligned}
x^2\big(f(\theta) + f''(\theta)\big) &= \frac{1}{\pi}\int_0^{\pi} x^2\cos(x\sin\theta)\cos^2\theta\,\mathrm{d}\theta \\
&= \frac{1}{\pi}\left([(x\cos\theta)\sin(x\sin\theta)]_0^{\pi} + \int_0^{\pi} x\sin\theta\sin(x\sin\theta)\mathrm{d}\theta\right) \\
&= \frac{x}{\pi}\int_0^{\pi}\sin\theta\sin(x\sin\theta)\mathrm{d}\theta = -x\,f'(x).
\end{aligned}$$

Thus f solves the same IVP as J_0 and hence must be equal to J_0.