Differential Equations (Math 285)

H20 Find integrating factors for the following ODE's and determine their integral curves.

a) $\mathrm{e}^x(x+1)\,\mathrm{d}x + (y\,\mathrm{e}^y - x\,\mathrm{e}^x)\,\mathrm{d}y = 0$;

b) $y(y+2x+1)\,\mathrm{d}x - x(2y+x-1)\,\mathrm{d}y = 0$.

H21 An ODE $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ is said to be *homogeneous* if M and N are homogeneous functions of the same degree, i.e., there exists $d \in \mathbb{R}$ such that $M(\lambda x, \lambda y) = \lambda^d M(x,y)$ and $N(\lambda x, \lambda y) = \lambda^d N(x,y)$ for all x, y, and λ.

a) Show that the substitution $z = y/x$ (or $z = x/y$) transforms any homogeneous ODE into a separable ODE.

b) Solve the following ODE's in implicit form (answering two of (i)–(iii) suffices):

(i) $(x+y)\,\mathrm{d}x - (x+2y)\,\mathrm{d}y = 0$; (ii) $(x-2y)\,\mathrm{d}x + y\,\mathrm{d}y = 0$;
(iii) $(x^2+y^2)\,\mathrm{d}x + 3xy\,\mathrm{d}y = 0$; (iv) $(x-y-1)\,\mathrm{d}x + (x+4y-6)\,\mathrm{d}y = 0$.

H22 Analyze the alternative model $\mathrm{d}y/\mathrm{d}t = ay - by^2 - Ey$ $(a, b, E > 0)$ for harvesting a population (individuals are removed at a rate proportional to the current size of the population). Which rates E are sustainable? How to choose E in order to maximize the *yield* Ey in the long run?

H23 a) Assuming that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, show that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$ without resorting to the evaluation of $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$.
Hint: Add the two series.

b) Show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$.

H24 Evaluate the two series

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^4} \quad \text{for } x \in \mathbb{R},$$

in a way similar to the evaluation of $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$ in the lecture, and use this in turn to evaluate the series

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} \pm \cdots,$$
$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \cdots$$

H25 *Optional exercise*

For $s \in \mathbb{C}$ consider the *binomial series*

$$\mathrm{B}_s(z) = \sum_{n=0}^{\infty} \binom{s}{n} z^n = \sum_{n=0}^{\infty} \frac{s(s-1)\cdots(s-n+1)}{1\cdot 2\cdots n}\, z^n.$$

a) Show that for $s \notin \{0,1,2,\dots\}$ the binomial series has radius of convergence $R = 1$.

b) Show that $\mathrm{B}_s(x) = (1+x)^s$ for $s \in \mathbb{C}$ and $-1 < x < 1$.
Hint: $x \mapsto (1+x)^s = \mathrm{e}^{s\ln(1+x)}$ is a solution of the IVP $y' = \frac{s}{1+x}\, y$, $y(0) = 1$. Show that the same is true of $x \mapsto \mathrm{B}_s(x)$; cf. also [Ste16], Ch. 11.10, Ex. 85.

c) Show $\mathrm{B}_s(z) = (1+z)^s$ for $s, z \in \mathbb{C}$ with $|z| < 1$.
Hint: Probably the easiest way to solve this part is to use the same idea as in b): Show that $z \mapsto \mathrm{B}_s(z)$ and $z \mapsto (1+z)^s = \mathrm{e}^{s\log(1+z)}$ both satisfy $y' = \frac{s}{1+z}\, y$ for $|z| < 1$ and $y(0) = 1$, and that the solution of this complex IVP is unique. Since we haven't discussed complex differentiation and ODE's in any depth, it is important that you justify carefully every step of your solution.

Due on Fri Mar 17, 4 pm

Exercise H25 can be handed in until Fri Mar 24, 4 pm. Exercise H24 is based on material from the lecture, which will only be discussed next week (on Mon Mar 13).

Solutions

20 a) According to the equation,

$$M = \mathrm{e}^x(x+1), \quad N = ye^y - xe^x,$$
$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

This is not an exact ODE and an appropriate integrating factor is needed. Computing the following equation, we find that

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{(x+1)e^x}{(x+1)e^x} = 1 = g(y)$$

Thus, there exists a suitable integrating factor $\mu(y)$ that is a function of y only, and μ satisfies the differential equation

$$\mu'(y) = -\mu(y)g(y) = -\mu(y).$$

Hence,

$$\mu(y) = e^{-y}$$

is a suitable integrating factor.

Multiplying the original equation by this integrating factor, we obtain

$$e^{x-y}(x+1)\,\mathrm{d}x + (y - xe^{x-y})\,\mathrm{d}y = 0,$$
$$M' = e^{x-y}(x+1), N' = y - xe^{x-y},$$
$$\frac{\partial M'}{\partial y} = -xe^{x-y} = \frac{\partial N'}{\partial x}.$$

This is an exact ODE.
Therefore, there exists a function φ so that

$$\frac{\partial \varphi}{\partial x} = M' = e^{x-y}(x+1), \quad \frac{\partial \varphi}{\partial y} = N' = y - xe^{x-y}.$$

Integrating the first equation with respect to x, we obtain that

$$\varphi(x,y) = \int e^{x-y}(x+1)dx + h(y) = xe^{x-y} + h(y).$$

Substituting $\varphi(x,y)$ into the second equation, we find that

$$\frac{\partial \varphi}{\partial y} = -xe^{x-y} + h'(y) = y - xe^{x-y} = N',$$

so $h'(y) = y$ and $h(y) = \frac{1}{2}y^2$. Thus the solution is given implicitly by

$$F(x,y) = xe^{x-y} + \frac{1}{2}y^2 = C, \quad C \in \mathbb{R}.$$

b) According to the equation,

$$M = y(y+2x+1), \quad N = -x(2y+x+1),$$
$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

This is not an exact ODE and an appropriate integrating factor is needed. Computing the following expression,

$$\frac{1}{Ny - Mx}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-3xy(x+y)}[(2y+2x+1)-(-2y-2x+1)] = \frac{4}{-3xy},$$

which depends only on xy, we can conclude that there exists a suitable integrating factor $\mu(xy)$ that is a function of xy only, and (subsitute $s = xy$) μ satisfies the differential equation

$$\mu'(s) = \mu(s)g(s) = \frac{4}{-3s}\,\mu(s).$$

Hence,

$$\mu(s) = e^{\int -\frac{4}{3s}\,\mathrm{d}s} = s^{-\frac{4}{3}},$$

and the integrating factor is $\mu(xy) = (xy)^{-\frac{4}{3}}$. Multiplying the original equation by this integrating factor, we obtain

$$(xy)^{-\frac{4}{3}}y(y+2x+1)\,\mathrm{d}x - (xy)^{-\frac{4}{3}}x(2y+x-1)\,\mathrm{d}y = 0$$
$$M' = (xy)^{-\frac{4}{3}}y(y+2x+1), N' = -(xy)^{-\frac{4}{3}}x(2y+x-1)$$
$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$

This is an exact ODE.
Therefore, there exists a function φ so that

$$\frac{\partial \varphi}{\partial x} = M' = (xy)^{-\frac{4}{3}}y(y+2x+1),$$
$$\frac{\partial \varphi}{\partial y} = N' = -(xy)^{-\frac{4}{3}}x(2y+x-1).$$

Integrating the first equation with respect to x, we obtain

$$\varphi(x,y) = \int (xy)^{-\frac{4}{3}}y(y+2x+1)dx + h(y) = (y^{-\frac{1}{3}})[-3(y+1)x^{-\frac{1}{3}} + 3x^{\frac{2}{3}}] + h(y).$$

Substituting $\varphi(x,y)$ into the second equation, we find that

$$\frac{\partial \varphi}{\partial y} = -3x^{-\frac{1}{3}}y^{-\frac{1}{3}} + (y+1)x^{-\frac{1}{3}}y^{-\frac{4}{3}} - x^{\frac{2}{3}}y^{-\frac{4}{3}} + h'(y) = N',$$

so $h'(y) = 0$ and $h(y) = C$. Thus the solution is given implicitly by

$$F(x,y) = -3x^{-\frac{1}{3}}y^{-\frac{1}{3}} - 3x^{-\frac{1}{3}}y^{\frac{2}{3}} + 3x^{\frac{2}{3}}y^{-\frac{1}{3}} = C, \quad C \in \mathbb{R}.$$

21 a) The homogeneous ODE can be represented as

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0.$$

Substituting $z = y/x$ and using the defining property of homogeneous ODE's, we obtain

$$\begin{aligned}
& & M(x,xz)\,\mathrm{d}x + N(x,xz)(x\,\mathrm{d}z + z\,\mathrm{d}x) &= 0\\
&\iff & x^d M(1,z)\,\mathrm{d}x + x^d N(1,z)(x\,\mathrm{d}z + z\,\mathrm{d}x) &= 0\\
&\iff & [M(1,z) + zN(1,z)]\,\mathrm{d}x + N(1,z)x\,\mathrm{d}z &= 0\\
&\iff & \frac{M(1,z) + zN(1,z)}{N(1,z)}\,\mathrm{d}x &= -x\,\mathrm{d}z\,.
\end{aligned}$$

Hence, the homogeneous ODE can be converted to the separable ODE

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{M(1,z) + zN(1,z)}{x\,N(1,z)}.$$

Alternative solution: We use the explicit form

$$y' = -\frac{M(x,y)}{N(x,y)} = -\frac{x^d\,M(1,y/x)}{x^d\,N(1,y/x)} = -\frac{M(1,y/x)}{N(1,y/x)} = f(y/x), \quad \text{say.}$$

The substitution $z = y/x$ gives

$$z' = \frac{y'x - y}{x^2} = \frac{f(z) - z}{x},$$

which is separable. (Inserting $f(z) = -M(1,z)/N(1,z)$ transforms this ODE into the ODE obtained above.)

Remark: The substitution $z = y/x$ requires $x \neq 0$. Some integral curves may be lost in this way, e.g., consider $x\,\mathrm{d}y - y\,\mathrm{d}x = 0$, which has the y-axis as an integral curve, but the transformed ODE $z' = 0$ doesn't reflect this.

b) (i) Substituting $z = y/x$ gives

$$(x+y)\,\mathrm{d}x - (x+2y)\,\mathrm{d}y = (x+xz)\,\mathrm{d}x - (x+2xz)(z\,\mathrm{d}x + x\,\mathrm{d}z) = 0$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1-2z^2}{(1+2z)x}$$

This is a separable ODE and can be solved in the usual way: There are the constant solutions $z = \pm\frac{1}{2}\sqrt{2}$, corresponding to $y = \pm\frac{1}{2}\sqrt{2}\,x$. Otherwise we get

$$\int \frac{1+2z}{1-2z^2}\,\mathrm{d}z = \int \frac{1}{x}\,\mathrm{d}x$$

$$\ln|x| = \frac{1}{2}\int \frac{1}{1-\sqrt{2}\,z}\,\mathrm{d}z + \frac{1}{2}\int \frac{1}{1+\sqrt{2}\,z}\,\mathrm{d}z + \int \frac{2z}{1-2z^2}\,\mathrm{d}z$$

$$\begin{aligned}
\ln|x| &= -\frac{1}{2\sqrt{2}}\ln\left|1-\sqrt{2}\,z\right| + \frac{1}{2\sqrt{2}}\ln\left|1+\sqrt{2}\,z\right| - \frac{1}{2}\ln\left|1-2z^2\right| + C\\
&= \ln\frac{\left|1+\sqrt{2}\,z\right|^{\frac{1}{2\sqrt{2}}-\frac{1}{2}}}{\left|1-\sqrt{2}\,z\right|^{\frac{1}{2\sqrt{2}}+\frac{1}{2}}} + C
\end{aligned}$$

$$|x| = \mathrm{e}^C\frac{\left|1+\sqrt{2}\,z\right|^{\frac{1}{4}\sqrt{2}-\frac{1}{2}}}{\left|1-\sqrt{2}\,z\right|^{\frac{1}{4}\sqrt{2}+\frac{1}{2}}} = \mathrm{e}^C\frac{|x|\left|x+\sqrt{2}\,y\right|^{\frac{1}{4}\sqrt{2}-\frac{1}{2}}}{\left|x-\sqrt{2}\,y\right|^{\frac{1}{4}\sqrt{2}+\frac{1}{2}}}$$

Dividing by $|x|$ and raising the equation to the 4th power, which only changes the constant, gives

$$\left|x-\sqrt{2}\,y\right|^{2+\sqrt{2}} = C'\left|x+\sqrt{2}\,y\right|^{2-\sqrt{2}}, \quad C'>0.$$

This is probably the best form of the solution curves we can get. It includes the two special solutions $y=\pm\frac{1}{2}\sqrt{2}\,x$ as boundary cases $C'=0$ and $C'=\infty$.

(ii) Substituting $z=y/x$ into the equation gives

$$\begin{aligned}(x-2y)\,\mathrm{d}x+y\,\mathrm{d}y &= (x-2xz)\,\mathrm{d}x+xz(z\,\mathrm{d}x+x\,\mathrm{d}z)=0,\\ \frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{-z^2+2z-1}{zx} = -\frac{(z-1)^2}{zx}.\end{aligned}$$

This is a separable ODE, hence for $z\neq 1 (z=1$ gives the solution $y=x)$ we can continue as usual:

$$\int\frac{-z}{1-2z+z^2}\,\mathrm{d}z = \int\frac{1}{x}\,\mathrm{d}x$$

$$\ln|x| = \int\frac{1-z-1}{1-2z+z^2}\,\mathrm{d}z = \int\frac{\mathrm{d}z}{1-z} - \int\frac{\mathrm{d}z}{(1-z)^2} = \frac{1}{z-1} - \ln|z-1| + C$$

Therefore, the implicit solution is

$$\ln|y-x| - \frac{x}{y-x} = C, \quad C\in\mathbb{R},$$

complemented by the additional solution curve $y=x$.

(iii) Substituting $z=y/x$ into the equation gives

$$\begin{aligned}(x^2+y^2)\,\mathrm{d}x+3xy\,\mathrm{d}y &= (x^2+(xz)^2)\,\mathrm{d}x+3x^2z(z\,\mathrm{d}x+x\,\mathrm{d}z)=0,\\ \frac{\mathrm{d}z}{\mathrm{d}x} &= -\frac{1+4z^2}{3xz}.\end{aligned}$$

This is a separable ODE without constant solutions, hence equivalent to

$$\begin{aligned}\int\frac{z}{1+4z^2}\,\mathrm{d}z+\int\frac{1}{3x}\,\mathrm{d}x &= 0,\\ \frac{1}{3}\ln|x|+\frac{1}{8}\ln\left|1+4z^2\right| &= C,\\ 8\ln|x|+3\ln\left|1+4z^2\right| &= C',\\ x^8\left(1+4y^2/x^2\right)^3 &= \mathrm{e}^{C'},\\ x^2(x^2+4y^2)^3 &= C'', \quad C''>0. \qquad \text{(S)}\end{aligned}$$

This is the desired implicit solution, except for the integral curve $x=0$, which is a solution of $(x^2+y^2)\,\mathrm{d}x+3xy\,\mathrm{d}y=0$ (check it!) but missed by the substitution $z=y/x$, as mentioned earlier. However, if we allow in (S) also $C''=0$ then this solution is included.

(iv) This ODE is not homogeneous, but it can be transformed into a homogeneous ODE by a translation $x = u + a$, $y = v + b$. Since $x - y - 1 = x + 4y - 6 = 0$ has the solution $(x, y) = (2, 1)$, the point $(2, 1)$ is singular and the corresponding translation is $x = u + 2$, $y = v + 1$ (clear, since it removes the constants). Using $\mathrm{d}x = \mathrm{d}u$, $dy = \mathrm{d}v$, the transformed ODE is then

$$\big(u + 2 - (v + 1) - 1\big)\,\mathrm{d}u + \big(u + 2 + 4(v + 1) - 6\big) = (u - v)\,\mathrm{d}u + (u + 4v)\,\mathrm{d}v = 0.$$

Substituting $z = v/u$ into the equation gives

$$\begin{aligned}(1 - z)\,\mathrm{d}u + (1 + 4z)(z\,\mathrm{d}u + u\,\mathrm{d}z) &= 0,\\ (4z^2 + 1)\,\mathrm{d}u + u(1 + 4z)\,\mathrm{d}z &= 0,\\ \frac{\mathrm{d}z}{\mathrm{d}u} &= -\frac{1 + 4z^2}{u(1 + 4z)}.\end{aligned}$$

This is a separable ODE without constant solutions and hence equivalent to

$$\begin{aligned}\int \frac{1}{u}\,\mathrm{d}u + \int \frac{1 + 4z}{1 + 4z^2}\,\mathrm{d}z &= 0\\ ln|u| + \frac{1}{2}\ln\left(1 + 4z^2\right) + \frac{1}{2}\arctan(2z) &= C,\\ 2\ln|u| + \ln\left(1 + 4z^2\right) + \arctan(2z) &= 2C,\\ \ln(u^2 + 4v^2) + \arctan(2v/u) &= 2C,\\ \ln\big((x - 2)^2 + 4(y - 1)^2\big) + \arctan\left(\frac{2(y - 1)}{x - 2}\right) &= 2C = C', \quad C' \in \mathbb{R}.\end{aligned}$$

This is the desired implicit solution.

22 This problem is also discussed in [BDM17], Ch. 2.5, Exercise 19.
According to the model $dy/dt = ay - by^2 - Ey = (a - E)y - by^2$,

$$\Delta = (a - E)^2 \geqslant 0$$

1) When $E > a$, we have $y_1 = (a - E)/b < 0$, $y_2 = 0$.
   If the initial population $y_0 = y(t_0)$ is a positive number, then $\lim_{t\to\infty} y(t) = 0$; cf. the discussion of the harvesting equation in the lecture. In this case, the harvesting is not sustainable.

2) When $E = a$, we have $y_1 = y_2 = 0$.
   Again this implies $\lim_{t\to\infty} y(t) = 0$ if $y_0 > 0$, so the harvesting is not sustainable either.

3) When $E < a$, we have $y_1 = 0$, $y_2 = (a - E)/b > 0$. If $y_0 > 0$, then in the long run

$$\lim_{t\to\infty} y(t) = y_2 = \frac{a - E}{b}.$$

   Hence,

$$\lim_{t\to\infty}\big(E\,y(t)\big) = \frac{E(a - E)}{b}$$

   The eventual yield in this case, $Y = E\,y_2 = E(a-E)/b$, defines a parabola. Therefore, Y is maximized when $E = a/2$, with maximum value $a^2/4b$.

Therefore, E should equal $a/2$ in order to maximize the yield Ey in the long run.

23 a) Using the hint and subtracting the two series, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}&=2\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\cdots\right)\\&=\frac{1}{2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}.\\\implies\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{12}.\end{aligned}$$

b) Adding the two series, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)=\frac{\pi^2}{6}+\frac{\pi^2}{12}=\frac{\pi^2}{4}.$$
$$\implies\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}$$

24 The two series converge uniformly on $\mathbb{R}$ due to the inequalities

$$\left|\frac{\sin(nx)}{n^3}\right|\leqslant\frac{1}{n^3},$$
$$\left|\frac{\cos(nx)}{n^4}\right|\leqslant\frac{1}{n^4},$$

and the known fact that $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent when $p>1$. But for termwise differentiability, which we want to employ, we need only the pointwise convergence and the uniform convergence of the series of derivatives.
Now let us evaluate the first series. Its series of derivatives is

$$\sum_{n=1}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin(nx)}{n^3}=\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2}$$

From the lecture slides, this series converges uniformly and evaluates to

$$\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2}=\frac{(x-\pi)^2}{4}-\frac{\pi^2}{12}$$

Hence we can apply the Differentiation Theorem to conlcude that

$$\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^3}=\sum_{n=1}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin(nx)}{n^3}=\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2}=\frac{(x-\pi)^2}{4}-\frac{\pi^2}{12}.$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^3}=\int\left(\frac{(x-\pi)^2}{4}-\frac{\pi^2}{12}\right)\mathrm{d}x=\frac{(x-\pi)^3}{12}-\frac{\pi^2x}{12}+C$$

The constant C can be determined by setting $x = 0$:

$$0 = \sum_{n=1}^{\infty} \frac{\sin(n\,0)}{n^3} = \frac{(-\pi)^3}{12} + C \implies C = \frac{\pi^3}{12}.$$

Hence, the first series can be expressed as

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} = \frac{(x-\pi)^3}{12} - \frac{\pi^2 x}{12} + \frac{\pi^3}{12}.$$

Plugging in $x = \pi/2$ yields, on account of $\sin\big(2k(\pi/2)\big) = \sin(k\pi) = 0$, $\sin\big((2k+1)\pi/2\big) = (-1)^k$,

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} &= \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n^3} = \frac{(\pi/2-\pi)^3}{12} - \frac{\pi^2(\pi/2)}{12} + \frac{\pi^3}{12} = \pi^3\left(-\frac{1}{96} - \frac{1}{24} + \frac{1}{12}\right) \\ &= \frac{\pi^3}{32}.\end{aligned}$$

The series of derivatives of the second series, up to sign, is exactly this series. As shown at the beginning, $\sum_{n=1}^{\infty} \sin(nx)/n^3$ converges uniformly on $\mathbb{R}$. Hence we can apply the Differentiation Theorem again and conclude that

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^4} = \sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\cos(nx)}{n^4}\right] = \sum_{n=1}^{\infty}\left[-\frac{\sin(nx)}{n^3}\right] = -\frac{(x-\pi)^3}{12} + \frac{\pi^2 x}{12} - \frac{\pi^3}{12}.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^4} = -\frac{(x-\pi)^4}{48} + \frac{\pi^2 x^2}{24} - \frac{\pi^3 x}{12} + B, \quad B \in \mathbb{R}.$$

In order to determine the constant B, we evaluate the integral of this function over a full period in two ways (as in the lecture). Using the Integration Theorem, we have

$$\int_0^{2\pi} \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^4}\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_0^{2\pi} \frac{\cos(nx)}{n^4}\,\mathrm{d}x = 0$$

. On the other hand,

$$\begin{aligned}\int_0^{2\pi} \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^4}\,\mathrm{d}x &= \int_0^{2\pi}\left(-\frac{(x-\pi)^4}{48} + \frac{\pi^2 x^2}{24} - \frac{\pi^3 x}{12} + B\right)\mathrm{d}x \\ &= \left[-\frac{(x-\pi)^5}{240} + \frac{\pi^2 x^3}{72} - \frac{\pi^3 x^2}{24} + Bx\right]_{x=0}^{2\pi} \\ &= \left(-\frac{\pi^5}{240} + \frac{\pi^2(2\pi)^3}{72} - \frac{\pi^3(2\pi)^2}{24} + B(2\pi) - \frac{\pi^5}{240}\right) = -\frac{46\,\pi^5}{720} + B(2\pi) = 0.\end{aligned}$$

$$\implies B = \frac{23\pi^4}{720}$$

Hence, the first series can be expressed as

$$\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^4}=-\frac{(x-\pi)^4}{48}+\frac{\pi^2x^2}{24}-\frac{\pi^3x}{12}+\frac{23\pi^4}{720}.$$

Finally, we evaluate the series $\sum_{n=1}^{\infty}1/n^4$.
Substituting $x=0$ into the second series, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\sum_{n=1}^{\infty}\frac{\cos(n\,0)}{n^4}=-\frac{\pi^4}{48}+\frac{23\pi^4}{720}=\frac{\pi^4}{90}.$$

Remark: Continuing in this way, one can obtain closed-form expressions for $\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^{2p-1}}$ and $\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^{2p}}$ for all positive integers p, and use this to evaluate the series $\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2p-1}}$ $(p=2,3,4,\dots)$ and $\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ $(p=1,2,3,\dots)$. The answers are of the form $a_p\,\pi^{2p-1}$, respectively, $b_p\,\pi^{2p}$ with certain numbers $a_p,b_p\in\mathbb{Q}$ ($\to$ Euler and Bernoulli numbers). About the values of the series $\sum_{n=1}^{\infty}\frac{1}{n^{2p-1}}$ $(p=2,3,4,\dots)$ much less is known. (Essentially the only thing known is that for $p=2$ the value is an irrational number.)

25 a) For $s\notin\{0,1,2,\dots\}$ we have $\binom{s}{n}\neq 0$ for all n and

$$\frac{\binom{s}{n}}{\binom{s}{n+1}}=\frac{n+1}{s-n}=\frac{1+1/n}{s/n-1}\to -1\quad\text{for } n\to\infty.$$

$\implies R=\lim_{n\to\infty}\left|\binom{s}{n}/\binom{s}{n+1}\right|=1$ (ratio test).

b) The ODE $y'=\frac{s}{1+x}\,y$ is 1st-order linear, and hence all associated IVP's have a unique solution. Since $\mathrm{B}_s(0)=\binom{s}{0}=1$, we must have $\mathrm{B}_s(x)=(1+x)^s$ for $-1<x<1$, provided we can show that $x\mapsto\mathrm{B}_s(x)$ solves $y'=\frac{s}{1+x}\,y$ as well.

For $|z|<1$ and hence in particular for $-1<x<1$ we can differentiate $\mathrm{B}_s(x)$ termwise:

$$\mathrm{B}_s'(x)=\sum_{n=1}^{\infty}n\binom{s}{n}x^{n-1}=\sum_{n=1}^{\infty}s\binom{s-1}{n-1}x^{n-1}.$$

$$\begin{aligned}\implies (1+x)\mathrm{B}_s'(x)&=\sum_{n=1}^{\infty}s\binom{s-1}{n-1}x^{n-1}(1+x)=s\left(\sum_{n=1}^{\infty}\binom{s-1}{n-1}x^{n-1}+\sum_{n=1}^{\infty}\binom{s-1}{n-1}x^{n}\right)\\&=s\left(\sum_{n=0}^{\infty}\binom{s-1}{n}x^{n}+\sum_{n=0}^{\infty}\binom{s-1}{n-1}x^{n}\right)\\&=s\sum_{n=0}^{\infty}\left[\binom{s-1}{n-1}+\binom{s-1}{n}\right]x^n\\&=s\sum_{n=0}^{\infty}\binom{s}{n}x^n=s\,\mathrm{B}_s(x),\end{aligned}$$

as claimed.

c) The computation in b) is valid for all $z \in \mathbb{C}$ with $|z| < 1$, showing that $\mathrm{B}_s'(z) = \frac{s}{1+z}\,\mathrm{B}_s(z)$ for such z. The extension of the chain rule to complex differentiation (proved as in the real case) gives

$$\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{s\log(1+z)} = \mathrm{e}^{s\log(1+z)}\,\frac{\mathrm{d}}{\mathrm{d}z}\,[s\log(1+z)] = \mathrm{e}^{s\log(1+z)}\,\frac{s}{1+z}.$$

Thus both functions are solutions of $y' = \frac{s}{1+z}\,y$, $y(0) = 1$. For the uniqueness proof we consider the function $f(z) = \mathrm{B}_s(z)\,\mathrm{e}^{-s\log(1+z)}$, defined for $|z| < 1$. Using the product rule for differentiation of complex functions, we have

$$\begin{aligned} f'(z) &= \mathrm{B}_s'(z)\,\mathrm{e}^{-s\log(1+z)} + \mathrm{B}_s(z)\,\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{-s\log(1+z)} \\ &= \frac{s}{1+z}\mathrm{B}_s(z)\,\mathrm{e}^{-s\log(1+z)} + \mathrm{B}_s(z)\,\mathrm{e}^{-s\log(1+z)}\,\frac{-s}{1+z} = 0, \end{aligned}$$

and of course $f(0) = 1$. This implies $f(z) \equiv 1$ (since, e.g., $\mathrm{Re}f$ and $\mathrm{Im}f$ are real 2-variable functions with vanishing differential and hence must be constant), and hence $\mathrm{B}_s(z) = \mathrm{e}^{s\log(1+z)}$ for $|z| < 1$.