Lecture 20: Evaluation of Interval Estimators

Mathematical Statistics II, MATH 60062/70062

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Reference: Casella & Berger, 9.3.1

Size and coverage probability

When evaluating a confidence set, we want the set to have small **size** and large **coverage probability**.

- Coverage probability is typically measured by the **confidence coefficient**. If the coverage probability is not equal to $1 - \alpha$ for all $\theta \in \Theta$, we would like it to be as close as possible to the nominal $1 - \alpha$ level.
- If the set is an interval, the size is usually measured by the **interval length**. Shorter intervals are more informative.

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Finding an optimal interval using these criteria is often a constrained minimization problem, where for a specified coverage the goal is to find the confidence interval with the shortest length.

Optimizing length

Suppose $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$, where $-\infty < \mu < \infty$ and $\sigma^2 > 0$. Both parameters are unknown. Let $\boldsymbol{\theta} = (\mu, \sigma^2)$. Based on the pivot

$$Q = Q(\boldsymbol{X}, \boldsymbol{\theta}) = \frac{\bar{X} - \mu}{S/\sqrt{n}},$$

a $1 - \alpha$ confidence interval for μ can be obtained with

$$1 - \alpha = P_{\boldsymbol{\theta}}\left(a \leq \frac{\bar{X} - \mu}{S/\sqrt{n}} \leq b\right).$$

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Suppose $n = 25$ and $1 - \alpha = 0.95$. Some values of a and b that satisfy the condition are

- $a = t_{24,0.96} = -1.828$ and $b = t_{24,0.01} = 2.492$
- $a = t_{24,0.97} = -1.974$ and $b = t_{24,0.02} = 2.172$
- $a = t_{24,1} = -\infty$ and $b = t_{24,0.05} = 1.171$
- $a = t_{24,0.95} = -1.171$ and $b = t_{24,0} = \infty$

An infinite number of intervals satisfy this condition.

Optimal length with specified coverage probability

Suppose $Q = Q(\boldsymbol{X}, \theta)$ is a pivotal quantity and $P_\theta(a \leq Q \leq b) = 1 - \alpha$, where a and b are constants. Let $f_Q(q)$ be the unimodal PDF of Q. If

1. $\int_a^b f_Q(q)dq = 1 - \alpha$
2. $f_Q(a) = f_Q(b) > 0$
3. $a \leq q^* \leq b$, where q^* is the mode of $f_Q(q)$,

then $b - a$ is minimized relative to Q.

Proof: Let (a', b') be any interval with $b' - a' < b - a$. We will show that this implies

$$\int_{a'}^{b'} f_Q(q)dq < 1 - \alpha.$$

The result will be proved for $a' \leq a$, where two cases need to be considered, $b' \leq a$ and $b' > a$. The proof for $a' > a$ is similar.

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If $b' \leq a$, then $a' \leq b' \leq a \leq q^*$ and

$$\begin{aligned} \int_{a'}^{b'} f_Q(q)dq &\leq f_Q(b')(b' - a') \\ &\leq f_Q(a)(b' - a') \\ &< f_Q(a)(b - a) \\ &\leq \int_a^b f_Q(q)dq = 1 - \alpha, \end{aligned}$$

which completes the proof in the first case.

If $b' > a$, then $a' \le a < b' < b$, since if $b' \ge b$, $b' - a' \ge b - a$. In this case, we write

$$\begin{aligned}\int_{a'}^{b'} f_Q(q)dq &= \int_a^b f_Q(q)dq + \left[\int_{a'}^{a} f_Q(q)dq - \int_{b'}^{b} f_Q(q)dq\right] \\ &= (1-\alpha) + \left[\int_{a'}^{a} f_Q(q)dq - \int_{b'}^{b} f_Q(q)dq\right].\end{aligned}$$

Using the unimodality of f, the ordering $a' \le a < b' < b$, and $a \le q^* \le b$, we have

$$\int_{a'}^{a} f_Q(q)dq \le f_Q(a)(a - a') \quad \text{and} \quad \int_{b'}^{b} f_Q(q)dq \ge f_Q(b)(b - b').$$

Since $f_Q(a) = f_Q(b)$, the expression in square brackets is

$$\begin{aligned}\int_{a'}^{a} f_Q(q)dq - \int_{b'}^{b} f_Q(q)dq &\le f_Q(a)(a-a') - f_Q(b)(b-b') \\ &= f_Q(a)[(a-a') - (b-b')] \\ &= f_Q(a)[(b'-a') - (b-a)],\end{aligned}$$

which is negative. This implies $\int_{a'}^{b'} f_Q(q)dq < 1 - \alpha$ for $b' > a$, completing the proof in the second case.

Optimizing length

Suppose $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$, where $-\infty < \mu < \infty$ and $\sigma^2 > 0$. Both parameters are unknown. Let $\boldsymbol{\theta} = (\mu, \sigma^2)$. Based on the pivot

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If we choose $a = -t_{n-1,\alpha/2}$ and $b = t_{n-1,\alpha/2}$, then the conditions for optimal length are satisfied. Therefore,

$$\left(\bar{X} - t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}, \bar{X} + t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}\right)$$

has the shortest expected length among all $1 - \alpha$ confidence intervals based on Q.

Optimal length with specified coverage probability

Remark:

- The method works well for location families and location-scale families because the interval's length is proportional to $b - a$.
- When the interval's length is not proportional to $b - a$ (e.g., in scale families), then the method is not directly applicable. However, a modified version of the method might be applicable (e.g., see CB pp. 443-444).