Midterm Exam #2

MATH 60062/70062: Mathematical Statistics II

March 24, 2022

- Please turn off your phone.
- Print your name clearly at the top of this page.
- This is a closed-book and closed-notes exam.
- This exam contains 4 questions. There are 100 points in total.
- You have 75 minutes to complete the exam.
- Please show your work and explain all of your reasoning.
- You must work by yourself. Do not communicate in any way with others.

1. (15 points) Give full definitions for the following concepts:

   a. Power function of a hypothesis test

   b. Level α test

   c. Uniformly most powerful (UMP) level α test

   d. Monotone likelihood ratio

   e. Unbiased test

   Solution:

   a. The power function of a hypothesis test with rejection region R is the function of θ defined as
   $$\beta(\theta) = P_\theta(\mathbf{X} \in R).$$

   b. For $0 \le \alpha \le 1$, a test with power function $\beta(\theta)$ is a level α test if
   $$\sup_{\theta \in \Theta_0} \beta(\theta) \le \alpha,$$
   where Θ_0 is some subset of the parameter space under H_0.

   c. Let $\mathcal{C}$ be a class of level α tests for testing
   $$H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \in \Theta_0^c.$$
   A test in class $\mathcal{C}$, with power function $\beta(\theta)$, is a uniformly most powerful (UMP) level α test if
   $$\beta(\theta) \ge \beta^*(\theta)$$
   for all $\theta \in \Theta_0^c$, where $\beta^*(\theta)$ is the power function of any other test in $\mathcal{C}$.

   d. A family of PDFs/PMFs $\{g_T(t \mid \theta) : \theta \in \Theta\}$ for a univariate random variable T has a monotone likelihood ratio (MLR) if for all $\theta_2 > \theta_1$, the ratio
   $$\frac{g_T(t \mid \theta_2)}{g_T(t \mid \theta_1)}$$
   is a nondecreasing function of t on $\{t : g_T(t \mid \theta_1) > 0 \text{ or } g_T(t \mid \theta_2) > 0\}$.

   e. Consider testing
   $$H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \in \Theta_0^c.$$
   A test with power function $\beta(\theta)$ is unbiased if $\beta(\theta_1) \ge \beta(\theta_0)$ for all $\theta_1 \in \Theta_0^c$ and for all $\theta_0 \in \Theta_0$.

2. (15 points) Prove the sufficiency part of the Neyman-Pearson Lemma. Consider testing

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_1.$$

The PDFs/PMFs of $\boldsymbol{X} = (X_1, \ldots, X_n)$ corresponding to θ_0 and θ_1 are $f_{\boldsymbol{X}}(\boldsymbol{x} \mid \theta_0)$ and $f_{\boldsymbol{X}}(\boldsymbol{x} \mid \theta_1)$, respectively. Define the test function

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \frac{f_{\boldsymbol{X}}(\boldsymbol{x}|\theta_1)}{f_{\boldsymbol{X}}(\boldsymbol{x}|\theta_0)} > k \\ 0 & \text{if } \frac{f_{\boldsymbol{X}}(\boldsymbol{x}|\theta_1)}{f_{\boldsymbol{X}}(\boldsymbol{x}|\theta_0)} < k, \end{cases}$$

for $k \geq 0$, where $\alpha = P_{\theta_0}(\boldsymbol{X} \in R) = E_{\theta_0}[\phi(\boldsymbol{X})]$. Show that $\phi(\boldsymbol{x})$ is a most powerful level α test.

Solution:

First, note that $\phi(\boldsymbol{x})$ is a size α test (and of course, a level α test),

$$E_{\theta_0}[\phi(\boldsymbol{X})] = \alpha.$$

For any other level α test with test function $0 \leq \phi^*(\boldsymbol{x}) \leq 1$ that satisfies $E_{\theta_0}[\phi^*(\boldsymbol{X})] \leq \alpha$,

$$E_{\theta_0}[\phi(\boldsymbol{X}) - \phi^*(\boldsymbol{X})] = E_{\theta_0}[\phi(\boldsymbol{X})] - E_{\theta_0}[\phi^*(\boldsymbol{X})] \geq 0.$$

Consider the function

$$b(\boldsymbol{x}) = [\phi(\boldsymbol{x}) - \phi^*(\boldsymbol{x})][f_{\boldsymbol{X}}(\boldsymbol{x} \mid \theta_1) - k f_{\boldsymbol{X}}(\boldsymbol{x} \mid \theta_0)].$$

For all $\boldsymbol{x} \in \mathcal{X}$, $b(\boldsymbol{x}) \geq 0$, as in the following conditions:

- When $f_{\boldsymbol{X}}(\boldsymbol{x} \mid \theta_1) - k f_{\boldsymbol{X}}(\boldsymbol{x} \mid \theta_0) > 0$, $\phi(\boldsymbol{x}) = 1$ and $b(\boldsymbol{x}) \geq 0$.
- When $f_{\boldsymbol{X}}(\boldsymbol{x} \mid \theta_1) - k f_{\boldsymbol{X}}(\boldsymbol{x} \mid \theta_0) < 0$, $\phi(\boldsymbol{x}) = 0$ and $b(\boldsymbol{x}) \geq 0$.
- When $f_{\boldsymbol{X}}(\boldsymbol{x} \mid \theta_1) - k f_{\boldsymbol{X}}(\boldsymbol{x} \mid \theta_0) = 0$, $b(\boldsymbol{x}) = 0$.

Therefore,

$$[\phi(\boldsymbol{x}) - \phi^*(\boldsymbol{x})] f_{\boldsymbol{X}}(\boldsymbol{x} \mid \theta_1) \geq k[\phi(\boldsymbol{x}) - \phi^*(\boldsymbol{x})] f_{\boldsymbol{X}}(\boldsymbol{x} \mid \theta_0)$$

Integrating both sides, we get

$$E_{\theta_1}[\phi(\boldsymbol{X}) - \phi^*(\boldsymbol{X})] \geq k E_{\theta_0}[\phi(\boldsymbol{X}) - \phi^*(\boldsymbol{X})] \geq 0$$

$$\implies E_{\theta_1}[\phi(\boldsymbol{X})] \geq E_{\theta_1}[\phi^*(\boldsymbol{X})].$$

This shows that $\phi(\boldsymbol{x})$ is more powerful than $\phi^*(\boldsymbol{x})$. Since $\phi^*(\boldsymbol{x})$ is an arbitrary level α test, we know $\phi(\boldsymbol{x})$ is most powerful level α.

3. (35 points) Suppose that $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$, where $-\infty < \mu < \infty$ and $\sigma^2 > 0$. Both parameters are unknown. Consider testing

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0.$$

Let $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$, where $\bar{X}$ is the sample mean. The size α one-sample two-sided t-test rejects H_0 when

$$|\bar{x} - \mu_0| \geq t_{n-1,\alpha/2}\sqrt{s^2/n}.$$

a. (20 points) Show that the test can be derived as a likelihood ratio test.

b. (15 points) Show that the p-value for this two-sided t-test is

$$p(\boldsymbol{x}) = 2P\left(T_{n-1} \geq \left|\frac{\bar{x} - \mu_0}{s/\sqrt{n}}\right|\right).$$

Solution:

a. Set $\boldsymbol{\theta} = (\mu, \sigma^2)$. The likelihood function is

$$L(\boldsymbol{\theta} \mid \boldsymbol{x}) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2},$$

and the log-likelihood function is

$$\log L(\boldsymbol{\theta} \mid \boldsymbol{x}) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2.$$

To find the unrestricted maximum likelihood estimator (MLE):

$$\frac{\partial}{\partial\mu}\log L(\boldsymbol{\theta} \mid \boldsymbol{x}) = \frac{1}{\sigma^2}\sum_{i=1}^n (x_i - \mu) = 0$$
$$\frac{\partial}{\partial\sigma^2}\log L(\boldsymbol{\theta} \mid \boldsymbol{x}) = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^n (x_i - \mu) = 0.$$

Solving the above equations gives the unrestricted MLE

$$\hat{\boldsymbol{\theta}} = \begin{pmatrix} \bar{X} \\ \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2 \end{pmatrix}.$$

To find the restricted MLE (under $H_0 : \mu = \mu_0$):

$$-\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^n (x_i - \mu_0) = 0$$

$$\implies \hat{\boldsymbol{\theta}}_0 = \begin{pmatrix} \mu_0 \\ \frac{1}{n}\sum_{i=1}^n (X_i - \mu_0)^2 \end{pmatrix}.$$

Therefore, the likelihood ratio test (LRT) statistic is

$$\lambda(\boldsymbol{x}) = \frac{L(\hat{\boldsymbol{\theta}}_0 \mid \boldsymbol{x})}{L(\hat{\boldsymbol{\theta}} \mid \boldsymbol{x})} = \left[\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{\sum_{i=1}^n (x_i - \mu_0)^2}\right]^{n/2}$$

Noting that $\sum_{i=1}^n (x_i - \mu_0)^2 = \sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2$, the condition that the LRT rejects H_0,

$$\lambda(\boldsymbol{x}) = \left[\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{\sum_{i=1}^n (x_i - \mu_0)^2} \right]^{n/2} \le c, \text{ for } 0 \le c \le 1,$$

can be written as

$$\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{\sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2} \le c^{2/n},$$

which is equivalent to

$$\frac{1}{1 + \frac{n(\bar{x} - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2}} \le c^{2/n} \iff \frac{n(\bar{x} - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \ge c^{-2/n} - 1.$$

By defining $S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$, the rejection condition is equivalent to

$$\frac{(\bar{x} - \mu_0)^2}{s^2/n} = \frac{n(\bar{x} - \mu_0)^2}{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2} \ge (n-1)(c^{-2/n} - 1)$$

That is,

$$\lambda(\boldsymbol{x}) \le c \iff \left| \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \right| \ge c',$$

where c' satisfies (because of the size α condition)

$$\begin{aligned}
\alpha &= \sup_{\boldsymbol{\theta} \in \Theta_0} P_{\boldsymbol{\theta}} \left(\left| \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \right| \ge c' \right) \\
&= \sup_{\boldsymbol{\theta} \in \Theta_0} P_{\boldsymbol{\theta}} \left(\frac{\bar{X} - \mu}{S/\sqrt{n}} \ge c' + \frac{\mu_0 - \mu}{S/\sqrt{n}} \text{ or } \frac{\bar{X} - \mu}{S/\sqrt{n}} \le -c' + \frac{\mu_0 - \mu}{S/\sqrt{n}} \right).
\end{aligned}$$

Since $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$, $\frac{\bar{X} - \mu}{S/\sqrt{n}}$ has a t_{n-1} distribution. Thus, the critical value c' is chosen to satisfy

$$\alpha = P(|T_{n-1}| \ge c'),$$

which gives $c' = t_{n-1,\alpha/2}$. Therefore, the LRT rejects H_0 when

$$\left| \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \right| \ge t_{n-1,\alpha/2} \iff |\bar{x} - \mu_0| \ge t_{n-1,\alpha/2} \sqrt{s^2/n}.$$

This is the size α one-sample two-sided t-test.

b. The test rejects H_0 for large values of

$$W = W(\boldsymbol{X}) = \left| \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \right|.$$

The null parameter space is

$$\Theta_0 = \{\boldsymbol{\theta} = (\mu, \sigma^2) : \mu = \mu_0, \sigma^2 > 0\}.$$

With observed value $w = W(\boldsymbol{x})$, the p-value for the test is

$$
\begin{aligned}
p(\boldsymbol{x}) &= \sup_{\boldsymbol{\theta}\in\Theta_0} P_{\boldsymbol{\theta}}(W(\boldsymbol{X}) \geq w) \\
&= \sup_{\boldsymbol{\theta}\in\Theta_0} P_{\boldsymbol{\theta}}\left(\left|\frac{\bar{X}-\mu_0}{S/\sqrt{n}}\right| \geq w\right) \\
&= \sup_{\boldsymbol{\theta}\in\Theta_0} P_{\boldsymbol{\theta}}\left(\frac{\bar{X}-\mu_0}{S/\sqrt{n}} \geq w \text{ or } \frac{\bar{X}-\mu_0}{S/\sqrt{n}} \leq -w\right) \\
&= \sup_{\boldsymbol{\theta}\in\Theta_0} P_{\boldsymbol{\theta}}\left(\frac{\bar{X}-\mu}{S/\sqrt{n}} \geq w + \frac{\mu_0-\mu}{S/\sqrt{n}} \text{ or } \frac{\bar{X}-\mu}{S/\sqrt{n}} \leq -w + \frac{\mu_0-\mu}{S/\sqrt{n}}\right) \\
&= P\left(T_{n-1} \geq w \text{ or } T_{n-1} \leq -w\right) \\
&= 2P\left(T_{n-1} \geq \left|\frac{\bar{x}-\mu_0}{s/\sqrt{n}}\right|\right).
\end{aligned}
$$

4. (35 points) Suppose $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma_0^2)$, where $-\infty < \mu < \infty$ and σ_0^2 is known. Consider testing

$$H_0 : \mu \geq \mu_0 \quad \text{versus} \quad H_1 : \mu < \mu_0.$$

a. (10 points) Find the UMP level α test.

b. (10 points) Find the power function corresponding to the UMP level α test.

c. (5 points) Is the UMP level α test unbiased for H_0 versus H_1?

d. (5 points) Consider applying the test found in part (a) to test

$$H_0' : \mu = \mu_0 \quad \text{versus} \quad H_1' : \mu \neq \mu_0.$$

Show that the test is a level α test for H_0' versus H_1'.

e. (5 points) Is the test found in part (a) unbiased for $H_0' : \mu = \mu_0$ versus $H_1' : \mu \neq \mu_0$? Why?

Solution:

a. The sample mean $T(\boldsymbol{X}) = \bar{X}$ is a sufficient statistic for θ, since by the Factorization Theorem

$$\begin{aligned} f_{\boldsymbol{X}}(\boldsymbol{x} \mid \mu) &= \left(\frac{1}{\sqrt{2\pi\sigma_0^2}}\right)^n e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^n (x_i-\mu)^2} \\ &= \left(\frac{1}{\sqrt{2\pi\sigma_0^2}}\right)^n e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^n (x_i-\bar{x})^2} e^{-\frac{1}{2\sigma_0^2} n(\bar{x}-\mu)^2}. \end{aligned}$$

The sufficient statistic $T(\boldsymbol{X}) = \bar{X}$ is distributed as $\mathcal{N}(\mu, \sigma_0^2/n)$, and the family of the distributions has a monotone likelihood ratio, since for $\mu_2 > \mu_1$,

$$\begin{aligned} \frac{g_T(t \mid \mu_2)}{g_T(t \mid \mu_1)} &= e^{-\frac{n}{2\sigma_0^2}[(t-\mu_2)^2 - (t-\mu_1)^2]} \\ &= e^{\frac{n}{\sigma_0^2} t(\mu_2-\mu_1)} e^{-\frac{n}{2\sigma_0^2}(\mu_2^2-\mu_1^2)} \end{aligned}$$

is an increasing (nondecreasing) function of t. By the Karlin-Rubin Theorem, the UMP level α test is

$$\phi(t) = I(t < t_0),$$

where t_0 satisfies

$$\begin{aligned} \alpha = P_{\mu_0}(T < t_0) = P\left(\frac{\bar{X} - \mu_0}{\sigma_0/\sqrt{n}} < \frac{t_0 - \mu_0}{\sigma_0/\sqrt{n}}\right) = F_Z\left(\frac{t_0 - \mu_0}{\sigma_0/\sqrt{n}}\right) \\ \implies t_0 = -\frac{z_\alpha \sigma_0}{\sqrt{n}} + \mu_0. \end{aligned}$$

Thus, the UMP level α test is

$$\phi(\boldsymbol{x}) = I\left(\bar{x} < -\frac{z_\alpha \sigma_0}{\sqrt{n}} + \mu_0\right).$$

That is, the test rejects H_0 if and only if

$$\bar{x} < -\frac{z_\alpha \sigma_0}{\sqrt{n}} + \mu_0.$$

b. The corresponding power function is

$$\begin{aligned}
\beta(\mu) &= E_\mu[\phi(\boldsymbol{X})] \\
&= P_\mu\left(\bar{X} < -\frac{z_\alpha \sigma_0}{\sqrt{n}} + \mu_0\right) \\
&= P_\mu\left(\frac{\bar{X} - \mu}{\sigma_0/\sqrt{n}} < -z_\alpha + \frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}}\right) \\
&= F_Z\left(-z_\alpha + \frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}}\right),
\end{aligned}$$

where F_Z is the standard Normal CDF.

c. F_Z is a decreasing function of μ. Therefore, the test is unbiased because

$$\beta(\mu_1) \geq \beta(\mu_2)$$

for all $\mu_1 < \mu_0$ (under H_1) and for all $\mu_2 \geq \mu_0$ (under H_0).

d. When applying the test $\phi(\boldsymbol{x})$ to $H_0' : \mu = \mu_0$ versus $H_1' : \mu \neq \mu_0$,

$$\sup_{\mu=\mu_0} \beta(\mu) = \beta(\mu_0) = F_Z(-z_\alpha) = \alpha.$$

Therefore, the test is of level α (and size α) for $H_0' : \mu = \mu_0$ versus $H_1' : \mu \neq \mu_0$.

e. No, it's not unbiased for $H_0' : \mu = \mu_0$ versus $H_1' : \mu \neq \mu_0$, since the power function for μ under H_0', $\beta(\mu_0)$ can be greater than that for μ under H_1', $\beta(\mu_1)$ when $\mu_1 < \mu_0$ (part of the alternative parameter space).