Homework #4

Ruixin Guo

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1.

Let $Z_i \sim \mathcal{N}(0,1)$, then $Z_i^2 \sim \chi_1^2$, $E(Z_i^2) = 1$, $Var(Z_i^2) = 2$. Let $\bar{Z}^2 = \frac{1}{n}\sum_{i=1}^n Z_i^2$, by Central Limit Theorem,

$$\sqrt{n}\frac{\bar{Z}^2 - 1}{\sqrt{2}} \xrightarrow{d} \mathcal{N}(0,1)$$
$$\bar{Z}^2 \xrightarrow{d} \mathcal{N}(1, \frac{2}{n})$$

Since $X_n = n\bar{Z}^2$, we have $X_n \xrightarrow{d} \mathcal{N}(n, 2n)$.

2.

a.

Since $X_1, ..., X_n$ are an iid sample from Exponential distribution with mean θ, the variance is θ^2. By Central Limit Theorem,

$$\sqrt{n}\frac{\bar{X} - \theta}{\theta} \xrightarrow{d} \mathcal{N}(0,1)$$
$$\bar{X} \xrightarrow{d} \mathcal{N}(\theta, \frac{\theta^2}{n})$$

b.

By Delta Method,

$$g(\bar{X}) \xrightarrow{d} \mathcal{N}(g(\theta), \frac{[g'(\theta)]^2\theta^2}{n})$$

c.

We can let $[g'(\theta)]^2 = \frac{1}{\theta^2}$ so that the variance $\frac{[g'(\theta)]^2\theta^2}{n} = \frac{1}{n}$ does not depend on θ. Thus, $g'(\theta) = \frac{1}{\theta}$, $g(\theta) = \ln\theta$.

3.

Given that $X_i \sim \text{Expo}(\frac{1}{\theta})$, the PDF of X_i is $f_{X_i}(x_i) = \frac{1}{\theta}e^{-\frac{x_i}{\theta}}$. Since $T(\mathbf{X}) = \bar{X} = \frac{1}{n}(X_1 + X_2 + ...X_n)$, for each $T(\mathbf{X}) = t$, let $t = \frac{1}{n}\sum_{i=1}^n x_i$, then

$$f_{\mathbf{X}}(\boldsymbol{x}|\theta) = \prod_{i=1}^n \frac{1}{\theta}e^{-\frac{x_i}{\theta}} = \frac{1}{\theta^n}e^{-\frac{1}{\theta}\sum_{i=1}^n x_i} = \frac{1}{\theta^n}e^{-\frac{n}{\theta}t}$$

By Factorization Theorem, let $g(t|\theta) = \frac{1}{\theta^n} e^{-\frac{n}{\theta}t}$, $h(\boldsymbol{x}) = 1$, we have $f_{\mathbf{X}}(\boldsymbol{x}|\theta) = g(t|\theta)h(\boldsymbol{x})$. Thus, $T(\mathbf{X}) = \bar{X}$ is a sufficient statistic.

4.

Given that $X_i \sim \mathcal{N}(\mu, \sigma^2)$, the PDF of X_i is $f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)$. Let $\boldsymbol{\theta} = (\mu, \sigma^2)$, we have

$$f_{\mathbf{X}}(\boldsymbol{x}|\boldsymbol{\theta}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left(-\frac{\sum_{i=1}^n x_i^2 - 2\mu\sum_{i=1}^n x_i + \mu^2}{2\sigma^2}\right)$$

By Factorization Theorem, let $g(\sum_{i=1}^n x_i, \sum_{i=1}^n x_i^2|\boldsymbol{\theta}) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left(-\frac{\sum_{i=1}^n x_i^2 - 2\mu\sum_{i=1}^n x_i + \mu^2}{2\sigma^2}\right)$, $h(\boldsymbol{x}) = 1$, we have $f_{\mathbf{X}}(\boldsymbol{x}|\boldsymbol{\theta}) = g(\sum_{i=1}^n x_i, \sum_{i=1}^n x_i^2|\boldsymbol{\theta})h(\boldsymbol{x})$. Thus, $\boldsymbol{T}(\mathbf{X}) = (\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$ is a sufficient statistic for (μ, σ^2).

5.

First we prove $X_1, ..., X_n$ are iid from a scale family. Let $f_Z(\frac{x}{\sigma}) = \frac{1}{2}I(x \in \mathbb{R})e^{-\frac{|x|}{\sigma}}$, then $f_X(x|\sigma) = \frac{1}{\sigma}f_Z(\frac{x}{\sigma})$, thus X_is are from a scale family.

Then we prove $S(\boldsymbol{X}) = \frac{\sum_{i=1}^k X_i}{\sum_{i=1}^n X_i}$ is scale-invariant statistic. For any $c > 0$, $S(c\boldsymbol{X}) = \frac{\sum_{i=1}^k cX_i}{\sum_{i=1}^n cX_i} = \frac{c\sum_{i=1}^k X_i}{c\sum_{i=1}^n X_i} = S(\boldsymbol{X})$. Thus $S(\boldsymbol{X})$ is scale invariant.

Therefore, $S(\boldsymbol{X})$ is ancillary.