Homework #3

MATH 60062/70062: Mathematical Statistics II

Please submit your answers as a **PDF** file to Blackboard by **9:15 a.m. on March 17**. Please show your work and write legibly. Your grade will be based on the correctness of your answers and the clarity with which you express them.

1. (10 points) Suppose that $X_1, \ldots, X_n$ are iid Pois(θ),

$$f_X(x \mid \theta) = \frac{e^{-\theta}\theta^x}{x!}, \quad x = 0, 1, 2, \ldots$$

   where the prior distribution on θ is Gamma(a, b),

$$\pi(\theta) = \frac{1}{\Gamma(a)b^a}\theta^{a-1}e^{-\theta/b}I(\theta > 0)$$

   where the values of a and b are known. Find the Bayes estimator of θ under squared error loss.

2. (25 points) Suppose that $X_1, \ldots, X_n$ are iid $\mathcal{N}(\theta, 1)$, where $-\infty < \theta < \infty$. Consider testing

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0.$$

   a. (15 points) Show that the likelihood ratio test (LRT) rejection region can be expressed as

$$R = \left\{ \boldsymbol{x} : \sqrt{n}(\bar{x} - \theta_0) \geq c \right\}.$$

   Do not use the result in Slide 5 of Lecture 13 without a proof.

   b. (10 points) A test with power function $\beta(\theta)$ is *unbiased* if $\beta(\theta') \geq \beta(\theta'')$ for every $\theta' \in \Theta_0^c$ and $\theta'' \in \Theta_0$. Is the LRT an unbiased test?

3. (30 points) Suppose that $X_1, \ldots, X_n$ are iid $\mathcal{N}(\theta, \sigma^2)$, where $-\infty < \theta < \infty$ and $\sigma^2 > 0$. Both parameters are unknown. Consider testing

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0.$$

   a. Let $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Show that the test that rejects H_0 when

$$|\bar{X} - \theta_0| > t_{n-1,\alpha/2}\sqrt{S^2/n}$$

   is a test of size α.

   b. Show that the test in part (a) can be derived as an LRT.

4. (35 points) Suppose X_1 and X_2 are iid Pois(θ), where $\theta > 0$, and consider testing

$$H_0 : \theta \geq 3 \quad \text{versus} \quad H_1 : \theta < 3.$$

Consider two tests: one rejects H_0 when $x_1 = 0$,

$$\phi_1 = \phi_1(x_1, x_2) = I(x_1 = 0),$$

and the other rejects H_0 when $x_1 + x_2 \leq 1$,

$$\phi_2 = \phi_2(x_1, x_2) = I(x_1 + x_2 \leq 1).$$

a. (15 points) Find the power function of each test.

b. (15 points) What is the size of each test?

c. (5 points) Are ϕ_1 and ϕ_2 level $\alpha = 0.05$ tests?