Lecture 11: Inequalities

Mathematical Statistics I, MATH 60061/70061

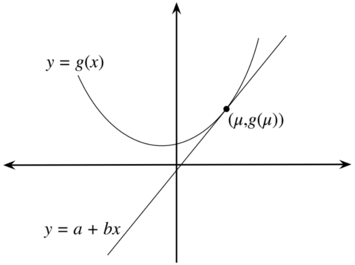
Tuesday October 12, 2021

Reference: Casella & Berger, 3.6, 4.7

Jensen: an inequality for convexity

Let X be a random variable.

- If g is a convex function, then $E(g(X)) \geq g(E(X))$.
- If g is a concave function, then $E(g(X)) \leq g(E(X))$.



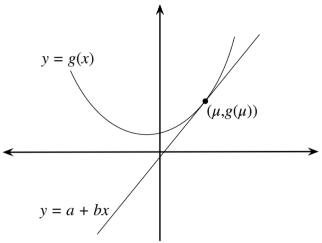
A function g whose domain is an interval I is *convex* if

$$g(px_1+(1-p)x_2) \leq pg(x_1)+(1-p)g(x_2)$$

for all $x_1, x_2 \in I$ and $p \in (0,1)$.

A function g is *concave* if $-g$ is convex.

Jensen: an inequality for convexity, continued



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If g is convex, then all lines tangent to g lie below g. Let $\mu = E(X)$ and consider the tangent point $(\mu, g(\mu))$. Denoting the tangent line by $a + bx$, we have $g(x) \geq a + bx$ for all x, so $g(X) \geq a + bX$. Taking the expectation of both sides:

$$E(g(X)) \geq a + bE(X) = a + b\mu = g(\mu) = g(E(X)).$$

If g is concave, then $h = -g$ is convex. The inequality for g is reversed for the concave case.

Example: inequality for means

If $a_1, \ldots, a_n$ are positive numbers, define

$$\begin{aligned}
a_A &= \frac{1}{n}(a_1 + a_2 + \ldots a_n) && \text{[arithmetic mean]} \\
a_G &= [a_1 a_2 \cdot \cdots \cdot a_n]^{1/n} && \text{[geometric mean]} \\
a_H &= \frac{1}{\frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)} && \text{[harmonic mean]}
\end{aligned}$$

Using Jensen's inequality, find the relationship between arithmetic mean, geometric mean, and harmonic mean.

Example: inequality for means

Let X be a random variable with range $a_1, \ldots, a_n$ and $P(X = a_i) = 1/n$, $i = 1, \ldots, n$. Since $\log x$ is a *concave* function, **Jensen's inequality** shows that $E(\log X) \leq \log(EX)$; hence

Example: inequality for means

Let X be a random variable with range $a_1, \ldots, a_n$ and $P(X = a_i) = 1/n$, $i = 1, \ldots, n$. Since $\log x$ is a *concave* function, **Jensen's inequality** shows that $E(\log X) \leq \log(EX)$; hence

$$\log a_G = \frac{1}{n}\sum_{i=1}^{n} \log a_i = E(\log X) \leq \log(EX) = \log\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right) = \log a_A,$$

so $\log a_G \leq \log a_A$.

Use again the fact that $\log x$ is concave

$$\log\frac{1}{a_H} = \log\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i}\right) = \log\left(E\frac{1}{X}\right) \geq E\left(\log\frac{1}{X}\right) = -E(\log X).$$

Since $E(\log X) = \log a_G$, it then follows that $\log(1/a_H) \geq \log(1/a_G)$, or $a_G \geq a_H$.

Chebychev's inequality is usually quite conservative:

$$P\left(\frac{|X-\mu|}{\sigma} \geq 1\right) \leq 1$$
$$P\left(\frac{|X-\mu|}{\sigma} \geq 2\right) \leq 1/4$$
$$P\left(\frac{|X-\mu|}{\sigma} \geq 3\right) \leq 1/9$$

For $X \sim \mathcal{N}(\mu, \sigma^2)$, the three bounds are far from those suggested by the 68-95-99.7 rule.