Lecture 19: Finding Interval Estimators

Mathematical Statistics II, MATH 60062/70062

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Reference: Casella & Berger, 9.2.2-9.2.4

Interval estimation methods

Methods of finding interval estimators

- Inverting a test statistic
- Using pivotal quantities
- Pivoting a CDF
- Bayesian credible intervals

Methods of evaluating interval estimators

- Coverage probability
- Interval length

Recap: Inverting a test statistic

Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ using the size α test function

$$\phi(\boldsymbol{x}) = I(\boldsymbol{x} \in R_{\theta_0}) = \begin{cases} 1 & \boldsymbol{x} \in R_{\theta_0} \\ 0 & \boldsymbol{x} \in R_{\theta_0}^c, \end{cases}$$

where R_{θ_0} is the rejection region and $P_{\theta_0}(\boldsymbol{X} \in R_{\theta_0}) = \alpha$. Let $A_{\theta_0} = R_{\theta_0}^c$ be the **acceptance region** of this level α test. For each $\boldsymbol{x} \in \mathcal{X}$, define a set $C(\boldsymbol{x})$ in the parameter space by

$$C(\boldsymbol{x}) = \{\theta_0 : \boldsymbol{x} \in A_{\theta_0}\}.$$

Clearly, $\theta_0 \in C(\boldsymbol{x}) \iff \boldsymbol{x} \in A_{\theta_0}$. Based on the duality between **hypothesis testing** and **confidence intervals**,

$$P_{\theta_0}(\theta_0 \in C(\boldsymbol{X})) = P_{\theta_0}(\boldsymbol{X} \in A_{\theta_0}) = 1 - P_{\theta_0}(\boldsymbol{X} \in R_{\theta_0}) = 1 - \alpha.$$

The same argument holds for all $\theta_0 \in \Theta$. Therefore,

$$C(\boldsymbol{X}) = \{\theta \in \Theta : \boldsymbol{x} \in A_\theta\}$$

is a $1 - \alpha$ confidence set.

Recap: Pivotal quantities

A random variable $Q = Q(\boldsymbol{X}, \theta)$ is a **pivotal quantity** (or **pivot**) if the distribution of Q is independent of θ.

If $Q = Q(\boldsymbol{X}, \theta)$ is pivot, then a $1 - \alpha$ confidence interval can be found by setting

$$1 - \alpha = P_\theta(a \leq Q(\boldsymbol{X}, \theta) \leq b),$$

where a and b are quantiles of the distribution of Q that satisfy the condition.

This is equivalent to inverting the acceptance region of a size α test of $H_0 : \theta = \theta_0$

$$A_{\theta_0} = \{\boldsymbol{x} : a \leq Q(\boldsymbol{x}, \theta_0) \leq b\},$$

to obtain

$$C(\boldsymbol{x}) = \{\theta_0 : a \leq Q(\boldsymbol{x}, \theta_0) \leq b\}$$

and $C(\boldsymbol{X})$ is a $1 - \alpha$ confidence set for θ.

Recap: Normal pivotal intervals

Suppose $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$, where $-\infty < \mu < \infty$ and $\sigma^2 > 0$. Both parameters are unknown. Set $\boldsymbol{\theta} = (\mu, \sigma^2)$.

We know that

$$Q_1 = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}.$$

The distribution of Q_1 is free of $\boldsymbol{\theta}$, so it is a pivot. Therefore

$$\begin{aligned} 1 - \alpha &= P_{\boldsymbol{\theta}}\left(-t_{n-1,\alpha/2} \leq \frac{\bar{X} - \mu}{S/\sqrt{n}} \leq t_{n-1,\alpha/2}\right) \\ &= P_{\boldsymbol{\theta}}\left(\bar{X} - t_{n-1,\alpha/2}\frac{S}{\sqrt{n}} \leq \mu \leq \bar{X} + t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}\right), \end{aligned}$$

which gives a $1 - \alpha$ confidence interval for μ,

$$C_1(\boldsymbol{X}) = \left(\bar{X} - t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}, \bar{X} + t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}\right).$$

We also know that

$$Q_2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}.$$

The distribution of Q_2 is free of $\boldsymbol{\theta}$, so it is a pivot. Therefore

$$\begin{aligned} 1-\alpha &= P_{\boldsymbol{\theta}}\left(\chi^2_{n-1,1-\alpha/2} \le \frac{(n-1)S^2}{\sigma^2} \le \chi^2_{n-1,\alpha/2}\right) \\ &= P_{\boldsymbol{\theta}}\left(\frac{(n-1)S^2}{\chi^2_{n-1,\alpha/2}} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{n-1,1-\alpha/2}}\right), \end{aligned}$$

which gives a $1-\alpha$ confidence interval for σ^2,

$$C_2(\boldsymbol{X}) = \left(\frac{(n-1)S^2}{\chi^2_{n-1,\alpha/2}}, \frac{(n-1)S^2}{\chi^2_{n-1,1-\alpha/2}}\right).$$

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$$C_2(\boldsymbol{X}) = \left(\frac{(n-1)S^2}{\chi^2_{n-1,\alpha/2}}, \frac{(n-1)S^2}{\chi^2_{n-1,1-\alpha/2}}\right).$$

Suppose we wanted to find a confidence set for $\boldsymbol{\theta} = (\mu, \sigma^2)$ simultaneously. Is $C_1(\boldsymbol{X}) \times C_2(\boldsymbol{X})$ a $1-\alpha$ confidence set for $\boldsymbol{\theta}$?

Confidence set with Bonferroni adjustment

To find a confidence set for $\boldsymbol{\theta} = (\mu, \sigma^2)$ simultaneously, the Cartesian product of $C_1(\boldsymbol{X})$ and $C_2(\boldsymbol{X})$, $C_1(\boldsymbol{X}) \times C_2(\boldsymbol{X})$, is *not* a $1 - \alpha$ confidence set for $\boldsymbol{\theta}$.

By Bonferroni's Inequality,

$$
\begin{aligned}
P_{\boldsymbol{\theta}}(\boldsymbol{\theta} \in C_1(\boldsymbol{X}) \times C_2(\boldsymbol{X})) &\geq P_{\boldsymbol{\theta}}(\mu \in C_1(\boldsymbol{X})) + P_{\boldsymbol{\theta}}(\sigma^2 \in C_2(\boldsymbol{X})) - 1 \\
&= (1 - \alpha) + (1 - \alpha) - 1 \\
&= 1 - 2\alpha.
\end{aligned}
$$

So, $C_1(\boldsymbol{X}) \times C_2(\boldsymbol{X})$ is a $1 - 2\alpha$ confidence set for $\boldsymbol{\theta}$.

P(C1 \cup C2) = P(C1) + P(C2) - P(C1 \cap C2) <= 1
P(C1 \cap C2) >= P(C1) + P(C2) - 1
P(C1 \cup C2) >= P(C1) + P(C2) - 1

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So, $C_1(\boldsymbol{X}) \times C_2(\boldsymbol{X})$ is a $1-2\alpha$ confidence set for $\boldsymbol{\theta}$.

Bonferroni adjustment: Adjust the confidence coefficient for $C_1(\boldsymbol{X})$ and $C_2(\boldsymbol{X})$ individually to be $1-\alpha/2$. The Cartesian product of the adjusted $1-\alpha/2$ sets for μ and σ^2 is a $1-\alpha$ confidence set.

Confidence set without adjustment

Consider the quantity

$$Q = Q(\boldsymbol{X}, \boldsymbol{\theta}) = \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2 + \frac{(n-1)S^2}{\sigma^2}.$$

Confidence set without adjustment

Consider the quantity

Chi_1 Chi_{n-1}

$$Q = Q(\boldsymbol{X}, \boldsymbol{\theta}) = \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2 + \frac{(n-1)S^2}{\sigma^2}.$$

Since $Q \sim \chi^2_n$, it is a pivot. Therefore, we can write

$$1 - \alpha = P_{\boldsymbol{\theta}}(Q \leq \chi^2_{n,\alpha}) = P_{\boldsymbol{\theta}}\left(\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2 + \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{n,\alpha}\right),$$

which gives a $1 - \alpha$ confidence set for $\boldsymbol{\theta}$

$$C(\boldsymbol{X}) = \{\boldsymbol{\theta} : Q(\boldsymbol{X}, \boldsymbol{\theta}) \leq \chi^2_{n,\alpha}\}.$$

The boundary of $C(\boldsymbol{X})$ is

$$
\begin{aligned}
Q(\boldsymbol{x}, \boldsymbol{\theta}) = \chi^2_{n,\alpha} &\iff \left(\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right)^2 + \frac{(n-1)s^2}{\sigma^2} = \chi^2_{n,\alpha} \\
&\iff (\mu - \bar{x})^2 = \frac{\chi^2_{n,\alpha}}{n}\left[\sigma^2 - \frac{(n-1)s^2}{\chi^2_{n,\alpha}}\right],
\end{aligned}
$$

which is a parabola in $\Theta = \{\boldsymbol{\theta} = (\mu, \sigma^2) : -\infty < \mu < \infty, \sigma^2 > 0\}$. The parabola has vertex at

$$
\left(\bar{x}, \frac{(n-1)s^2}{\chi^2_{n,\alpha}}\right)
$$

and it opens upward. The confidence set is the interior of the parabola.

The boundary of $C(\boldsymbol{X})$ is

$$\begin{aligned} Q(\boldsymbol{x}, \boldsymbol{\theta}) = \chi^2_{n,\alpha} &\iff \left(\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right)^2 + \frac{(n-1)s^2}{\sigma^2} = \chi^2_{n,\alpha} \\ &\iff (\mu - \bar{x})^2 = \frac{\chi^2_{n,\alpha}}{n}\left[\sigma^2 - \frac{(n-1)s^2}{\chi^2_{n,\alpha}}\right], \end{aligned}$$

which is a parabola in $\Theta = \{\boldsymbol{\theta} = (\mu, \sigma^2) : -\infty < \mu < \infty, \sigma^2 > 0\}$. The parabola has vertex at

$$\left(\bar{x}, \frac{(n-1)s^2}{\chi^2_{n,\alpha}}\right)$$

and it opens upward. The confidence set is the interior of the parabola.

For other confidence sets for $\boldsymbol{\theta}$, see Arnold and Shavelle (1998), Joint Confidence Sets for the Mean and Variance of a Normal Distribution, *The American Statistician*, 52(2): 133:140.

Find pivots

In location-scale families, there are many pivotal quantities. A few important pivots are

Family	Form of PDF	Parameter	Pivot example
Location	$f(x-\mu)$	μ	$\bar{X}-\mu$
Scale	$\frac{1}{\sigma}f\left(\frac{x}{\sigma}\right)$	σ	$\frac{\bar{X}}{\sigma}$
Location-scale	$\frac{1}{\sigma}f\left(\frac{x-\mu}{\sigma}\right)$	μ, σ	$\frac{\bar{X}-\mu}{S}$

In general, **differences** are pivotal in location problems, while **ratios** are pivotal for scale problems.

Pivoting a continuous CDF

The CDF of any random variable is from a Unif(0, 1) distribution.
Thus CDF itself is a pivot.

Let $T(\boldsymbol{X})$ be the statistic with continuous CDF $F_T(t \mid \theta)$. Then

$$F_T(T \mid \theta) \sim \mathrm{Unif}(0, 1).$$

Suppose $\alpha_1 + \alpha_2 = \alpha$. Suppose for all $t \in \mathcal{T}$, the functions $\theta_L(t)$ and $\theta_U(t)$ are defined as follows:

- When $F_T(t \mid \theta)$ is a **decreasing** function of θ,
  - $F_T(t \mid \theta_U(t)) = \alpha_1$
  - $F_T(t \mid \theta_L(t)) = 1 - \alpha_2$.
- When $F_T(t \mid \theta)$ is a **increasing** function of θ,
  - $F_T(t \mid \theta_U(t)) = 1 - \alpha_2$
  - $F_T(t \mid \theta_L(t)) = \alpha_1$.

Then the random interval $(\theta_L(T), \theta_U(T))$ is a $1 - \alpha$ confidence interval for θ.

Exponential interval estimator

Suppose $X_1, \ldots, X_n$ are iid with population PDF

$$f_X(x \mid \theta) = \begin{cases} e^{-(x-\theta)}, & x \geq \theta \\ 0, & x < \theta, \end{cases}$$

where $-\infty < \theta < \infty$.

The first order statistic, $T = T(\boldsymbol{X}) = X_{(1)}$, is a sufficient statistic for θ with CDF

$$F_T(t \mid \theta) = \begin{cases} 0, & t \leq \theta \\ 1 - e^{-n(t-\theta)}, & t > \theta. \end{cases}$$

Because $F_T(T \mid \theta) \sim \mathrm{Unif}(0, \theta)$, we can write

$$\begin{aligned}
1 - \alpha &= P_\theta(\alpha/2 \leq F_T(T \mid \theta) \leq 1 - \alpha/2) \\
&= P_\theta(\alpha/2 \leq 1 - e^{-n(T-\theta)} \leq 1 - \alpha/2) \\
&= P_\theta \left(T + \frac{1}{n} \log\left(\frac{\alpha}{2}\right) \leq \theta \leq T + \frac{1}{n} \log\left(1 - \frac{\alpha}{2}\right) \right).
\end{aligned}$$

Therefore,

$$\left(T + \frac{1}{n} \log\left(\frac{\alpha}{2}\right), T + \frac{1}{n} \log\left(1 - \frac{\alpha}{2}\right) \right)$$

is a $1 - \alpha$ confidence interval for θ.

Pivoting a discrete CDF

The discrete case is handled in the same way as in the continuous case except that the integrals are replaced by sums.

Let $T(\boldsymbol{X})$ be a discrete statistic with CDF $F_T(t \mid \theta)$. Suppose $\alpha_1 + \alpha_2 = \alpha$. Suppose for all $t \in \mathcal{T}$, the functions $\theta_L(t)$ and $\theta_U(t)$ are defined as follows:

- When $F_T(t \mid \theta)$ is a **decreasing** function of θ,
  - $P(T \leq t \mid \theta_U(t)) = \alpha_1$
  - $P(T \geq t \mid \theta_L(t)) = \alpha_2$.
- When $F_T(t \mid \theta)$ is a **increasing** function of θ,
  - $P(T \geq t \mid \theta_U(t)) = \alpha_1$
  - $P(T \leq t \mid \theta_U(t)) = \alpha_2$.

Then the random interval $(\theta_L(T), \theta_U(T))$ is a $1 - \alpha$ confidence interval for θ.

Poisson interval estimator

Suppose $X_1, \ldots, X_n$ are iid $\mathrm{Pois}(\theta)$, where $\theta > 0$.

Recall that $T = \sum_{i=1}^n X_i$ is sufficient for θ and $T \sim \mathrm{Pois}(n\theta)$. To find a $1 - \alpha$ confidence interval for θ, with an observed value $T = t_0$, we set

$$P_\theta(T \leq t_0) = \alpha/2 \quad \text{and} \quad P_\theta(T \geq t_0) = \alpha/2$$

and solve each equation for θ.

Poisson interval estimator

Suppose $X_1, \ldots, X_n$ are iid $\mathrm{Pois}(\theta)$, where $\theta > 0$.

Recall that $T = \sum_{i=1}^n X_i$ is sufficient for θ and $T \sim \mathrm{Pois}(n\theta)$. To find a $1-\alpha$ confidence interval for θ, with an observed value $T = t_0$, we set

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and solve each equation for θ.

See CB Example 9.2.15 for a closed-form expression for the confidence set using a relationship between the Poisson and Gamma distributions.

Bayesian intervals

In the Bayesian framework, all inference is carried out using the posterior distribution $f(\theta \mid \boldsymbol{x})$.

Since the posterior $f(\theta \mid \boldsymbol{x})$ is a legitimate probability distribution of θ, we can calculate probabilities involving θ directly by using this distribution.

For any set $\mathcal{A} \subset \mathbb{R}$, the **credible probability** associated with $\mathcal{A}$ is

$$P(\theta \in \mathcal{A} \mid \boldsymbol{X} = \boldsymbol{x}) = \int_{\mathcal{A}} f(\theta \mid \boldsymbol{x}) d\theta.$$

If the credible probability is $1 - \alpha$, $\mathcal{A}$ is called a $1 - \alpha$ **credible set**.

If $f(\theta \mid \boldsymbol{x})$ is discrete, we replace integrals with sums.

Interpretations of credible and confidence intervals

- $1 - \alpha$ **credible interval**: The probability that θ is inside the interval is $1 - \alpha$.
- $1 - \alpha$ **confidence interval**: Under the same condition, if we perform the experiment over and over again and calculate a $1 - \alpha$ confidence interval each time, then $100(1 - \alpha)$ percent of these intervals would contain the true value of θ.

Multiple intervals