Lecture 08: Joint Distributions

Mathematical Statistics I, MATH 60061/70061

Thursday September 23, 2021

Reference: Casella & Berger, 4.1-4.2

Joint, marginal, and conditional distributions

The **joint distribution** of two random variables X and Y provides complete information about the probability of the vector (X, Y) falling into any subset of the plane.

The **marginal distribution** of X is the individual distribution of X, ignoring the value of Y.

The **conditional distribution** of X given $Y = y$ is the updated distribution for X after observing $Y = y$.

Joint CDF

The most general description of the joint distribution is the **joint CDF**.

The joint CDF of random variables X and Y is the function $F_{X,Y}$ given by

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y).$$

The joint CDF of n random variables is defined analogously.

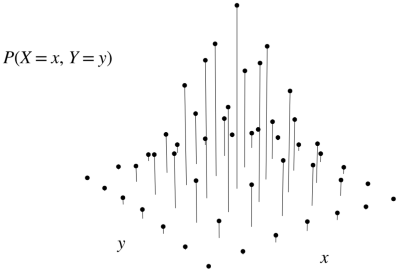
Joint PMF

The **joint PMF** of discrete random variables X and Y is the function $p_{X,Y}$ given by

$$p_{X,Y}(x, y) = P(X = x, Y = y).$$

We require valid joint PMFs to be nonnegative and sum to 1,

$$\sum_x \sum_y P(X = x, Y = y) = 1.$$

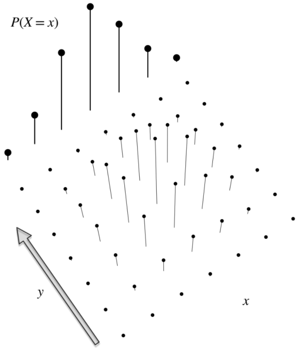


Marginal PMF

The **marginal PMF** of X is

$$P(X = x) = \sum_y P(X = x, Y = y).$$

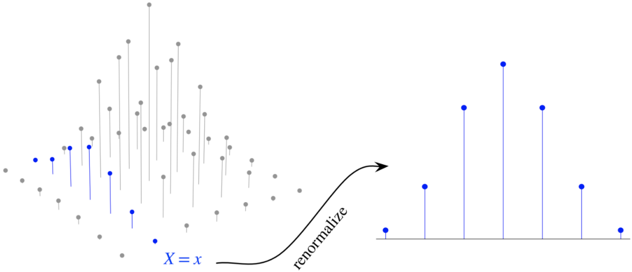
The marginal PMF of X is the PMF of X, viewing X individually rather than jointly with Y.



Conditional PMF

For discrete random variables X and Y, the **conditional PMF** of Y given $X = x$ is

$$P(Y = y \mid X = x) = \frac{P(X = x, Y = y)}{P(X = x)}.$$



Conditional PMF, continued

Conditional expectation of Y given $X = x$:

$$E(Y \mid X = x) = \sum_y y P(Y = y \mid X = x)$$

Bayes' rule:

$$P(Y = y \mid X = x) = \frac{P(X = x \mid Y = y) P(Y = y)}{P(X = x)}$$

Law of total probability:

$$P(X = x) = \sum_y P(X = x \mid Y = y) P(Y = y)$$

Independence of discrete random variables

Random variables X and Y are **independent** if for all x and y,

$$F_{X,Y}(x, y) = F_X(x)F_Y(y).$$

If X and Y are discrete, this is equivalent to the condition

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

for all x, y, or equivalently, for all x, y such that $P(X = x) > 0$

$$P(Y = y \mid X = x) = P(Y = y)$$

For independent random variables, the joint CDF/PMF factors into the product of the marginal CDFs/PMFs. In general, the marginal distributions do *not* determine the joint distribution.

Example: 2×2 table

A discrete joint distribution of two Bernoulli RVs X and Y.

In a hypothetical example, suppose we randomly sample an adult male from the US. Let X be the indicator of the sampled individual being a current smoker, and let Y be the indicator of his developing lung cancer at some point in his life.

Suppose the joint PMF is as follows:

	$Y=1$	$Y=0$
$X=1$	$\frac{5}{100}$	$\frac{20}{100}$
$X=0$	$\frac{3}{100}$	$\frac{72}{100}$

Example: 2×2 table

A discrete joint distribution of two Bernoulli RVs X and Y.

In a hypothetical example, suppose we randomly sample an adult male from the US. Let X be the indicator of the sampled individual being a current smoker, and let Y be the indicator of his developing lung cancer at some point in his life.

Suppose the joint PMF is as follows:

	$Y = 1$	$Y = 0$	**Total**
$X = 1$	$\frac{5}{100}$	$\frac{20}{100}$	$\frac{25}{100}$
$X = 0$	$\frac{3}{100}$	$\frac{72}{100}$	$\frac{75}{100}$
Total	$\frac{8}{100}$	$\frac{92}{100}$	$\frac{100}{100}$

The marginal distribution of X is $\text{Bern}(0.25)$ and the marginal distribution of Y is $\text{Bern}(0.08)$.

Example: 2×2 table, continued

In a hypothetical example, suppose we randomly sample an adult male from the US. Let X be the indicator of the sampled individual individual being a current smoker, and let Y be the indicator of his developing lung cancer at some point in his life.

The conditional probability of $Y = 1$ given $X = 1$ is

$$P(Y = 1 \mid X = 1) = \frac{P(X = 1, Y = 1)}{P(X = 1)} = \frac{5/100}{25/100} = 0.2$$

The conditional probability of $Y = 1$ given $X = 0$ is

$$P(Y = 1 \mid X = 0) = \frac{P(X = 0, Y = 1)}{P(X = 0)} = \frac{3/100}{75/100} = 0.04$$

X and Y are *not* independent.

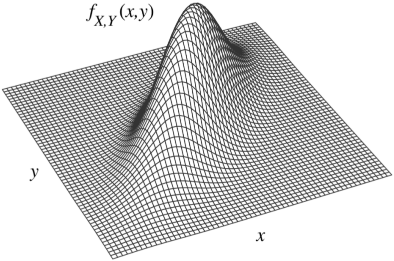
Joint PDF

If X and Y are continuous random variables with joint CDF $F_{X,Y}$, their **joint PDF** is the derivative of the joint CDF with respect to x and y:

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y).$$

We require valid joint PDFs to be nonnegative and integrate to 1:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1.$$



Marginal PDF

For continuous random variables X and Y with joint PDF $f_{X,Y}$, the **marginal PDF** of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy.$$

This is the PDF of X, viewing X individually rather than jointly with Y.

Marginalization works analogously with any number of variables. For example,

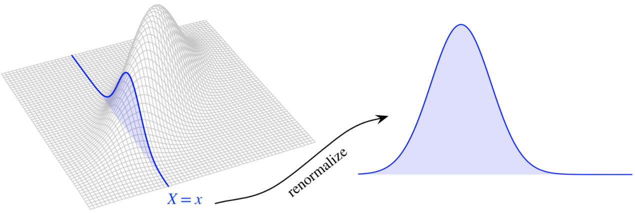
$$f_{X,W}(x,w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z,W}(x,y,z,w)dydz.$$

Conditional PDF

For continuous random variables X and Y with joint PDF $f_{X,Y}$, the **conditional PDF** of Y given $X = x$ is

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)},$$

for all x with $f_X(x) > 0$. This is considered as a function of y for fixed x.



Continuous form of Bayes' rule and LOTP

For continuous random variables X and Y, we have the following continuous form of Bayes' rule:

$$f_{Y|X}(y \mid x) = \frac{f_{X|Y}(x \mid y) f_Y(y)}{f_X(x)}, \text{ for } f_X(x) > 0.$$

And we have the following continuous form of the law of total probability:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x \mid y) f_Y(y) dy.$$

Independence of continuous random variables

Random variables X and Y are **independent** if for all x and y,

$$F_{X,Y}(x, y) = F_X(x)F_Y(y).$$

If X and Y are continuous with joint PDF $f_{X,Y}$, this is equivalent to the condition

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

for all x, y, or equivalently, for all x, y such that $f_X(x) > 0$

$$f_{Y|X}(y \mid x) = f_Y(y).$$

Example: Uniform on a square

Let (X, Y) be a completely random point in the square $\{(x, y) : x, y \in [0, 1]\}$, in the sense that the joint PDF of X and Y is constant over the square and 0 outside of it:

$$f_{X,Y}(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

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Marginally, X and Y are $\mathrm{Unif}(0, 1)$:

$$f_X(x) = \int_0^1 f_{X,Y}(x, y)dy = \int_0^1 1dy = 1,$$

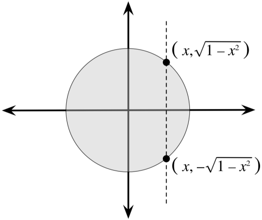
and similarly for f_Y.

The joint PDF factors into the product of the marginal PDFs, so X and Y are independent.

Example: Uniform on the unit disk

Let (X, Y) be a completely random point in the unit disk $\{(x, y) : x^2 + y^2 \leq 1\}$, with joint PDF:

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$



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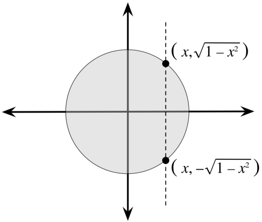

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The marginal distribution of X is now

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2}{\pi}\sqrt{1-x^2}, \quad -1 \leq x \leq 1.$$

Similarly, $f_Y(y) = \frac{2}{\pi}\sqrt{1-y^2}$.

X and Y are not independent: larger values of $|X|$ restrict Y to be in a smaller range.

2D LOTUS

Let g be a function from $\mathbb{R}^2$ to $\mathbb{R}$. If X and Y are discrete, then

$$E(g(X,Y)) = \sum_x \sum_y g(x,y) P(X = x, Y = y).$$

If X and Y are continuous with joint PDF $f_{X,Y}$, then

$$E(g(X,Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy.$$

Example: expected distance between two Uniforms

Let X and Y be i.i.d. $\mathrm{Unif}(0,1)$ random variables. Find $E(|X-Y|)$.

By 2D LOTUS,

$$\begin{aligned}
E(|X-Y|) &= \int_0^1 \int_0^1 |x-y| dx dy \\
&= \int_0^1 \int_y^1 (x-y) dx dy + \int_0^1 \int_0^y (y-x) dx dy \\
&= 2\int_0^1 \int_y^1 (x-y) dx dy \\
&= 1/3.
\end{aligned}$$