Lecture 10: Conditional Expectation

Mathematical Statistics I, MATH 60061/70061

Thursday September 30, 2021

Reference: Casella & Berger, 4.4

Example: life expectancy

Fred is 30 years old, and he hears that the average life expectancy in his country is 80 years. Should he conclude that, *on average, he has 50 years of life left?*

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No, there is a crucial piece of information that he must condition on: the fact that he has lived to age 30 already.

Letting T be Fred's lifespan, we have

$$E(T) < E(T \mid T \geq 30).$$

- LHS: Fred's life expectancy at birth
- RHS: Fred's life expectancy given that he reaches age 30.

Geometric expectation redux

Let $X \sim \mathrm{Geom}(p)$. Interpret X as the number of Tails before the first Heads in a sequence of coin flips with probability p of Heads.

To get $E(X)$, we condition on the outcome of the first toss:

- If it lands Heads, then X is 0 and we are done.
- If it lands Tails, then we've wasted one toss and are back to where we started, by memorylessness.

Therefore,

$$\begin{aligned} E(X) &= E(X \mid \text{first toss } H) \cdot p + E(X \mid \text{first toss } T) \cdot q \\ &= 0 \cdot p + (1 + E(X)) \cdot q, \end{aligned}$$

which gives $E(X) = q/p$.

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$$\underbrace{TTTH}_{W_1}\underbrace{HHT}_{W_2}HHT\ldots$$

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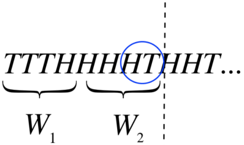

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By the Geometric interpretation, $W_1 - 1$ and $W_2 - 1$ are i.i.d. $\mathrm{Geom}(1/2)$, so $E(W_1) = E(W_2) = 1 + 1$ and $E(W_{HT}) = 4$.