Homework #1

MATH 60061/70061: Mathematical Statistics I

Please submit your answers as a **PDF** file to Blackboard by **9:15 a.m. on September 28**. Please show your work and write legibly. Your grade will be based on the correctness of your answers and the clarity with which you express them.

1. (15 points) For any events A, B, and C defined on a sample space S, show that

   a. $A \cup B = B \cup A$, $A \cap B = B \cap A$. (Commutativity)

   b. $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$. (Associativity)

   c. $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$. (DeMorgan's Laws)

2. (15 points) Bootstrap samples arise in a widely used statistical method known as the *bootstrap*. Given $n \geq 2$ distinct numbers $(a_1, a_2, \ldots, a_n)$, a bootstrap sample is a sequence $(x_1, x_2, \ldots, x_n)$ formed from the a_j's by sampling with replacement with equal probabilities, where *order does not matter* (in the sense that it only matters how many times each a_j was chosen, not the order in which they were chosen). For example, if $n = 2$ and $(a_1, a_2) = (1, 3)$, then the possible bootstrap samples are $(1, 1)$, $(1, 3)$, and $(3, 3)$. How many possible bootstrap samples are there for $(a_1, a_2, \ldots, a_n)$?

3. (20 points) A crime is committed by one of two suspects, A and B. Initially, there is equal evidence against both of them. In further investigation at the crime scene, it is found that the guilty party had a blood type found in 10% of the population. Suspect A does match this blood type, whereas the blood type of Suspect B is unknown.

   a. Given this new information, what is the probability that A is the guilty party?

   b. Given this new information, what is the probability that B's blood type matches that found at the crime scene?

4. (30 points) You are assigned with a task to inspect the quality of 100 electronic chips in a box, of which 95 are good and 5 are bad. You consider 3 different approaches to perform the task.

   a. Approach 1: you randomly pick chips one at a time *without replacement* and test each chip until you have found a bad one. Let X be the number of good chips you have to test in order to find one that is bad. Find the probability mass function (PMF) and the expected value of X. The PMF should specify the range of possible values.

   b. Approach 2: you randomly pick chips one at a time, test each chip, then put it back to the box, until you have found a bad one. Let X be the number of good chips you have to test in order to find one that is bad. Find the PMF (including the range of possible values) and the expected value of X.

c. Approach 3: you randomly pick exactly 20 chips *without replacement* and then test each of these 20 chips. Let X denote the number of good chips among the 20 you have taken out. Find the PMF (including the range of possible values) and the expected value of X.

5. (20 points) A lottery has a $1/400$ chance of winning a prize. You play the lottery 200 times.

    a. What is the probability that you win at least twice? Give an exact answer (it is OK to present the answer in an unevaluated form), *and* a simple, suitable approximation.

    b. Suppose you know that you have won at least once. What is the probability that you have won at least twice? Use a suitable approximation to answer the question.