# The Density of the F Distribution

Stat 305 Spring Semester 2006

The purpose of this document is to determine the pdf of the $F_{m,n}$ distribution. Recall that the $F_{m,n}$ distribution is the ratio of two (scaled) independent $\chi^2$ random variables, the first having $m$ degrees of freedom and the second having $n$ degrees of freedom.

**Proposition 1** *If $X$ is $F_{m,n}$, then*

$$f_X(x) = \frac{\Gamma\left(\frac{m+n}{2}\right) m^{m/2} n^{n/2} x^{(m/2)-1}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)(n+mx)^{(m+n)/2}},\ x > 0.$$

**Proof.** Suppose that $X = (U/m)/(V/n)$ where $U$ and $V$ are independent $\chi^2$ random variables, $U$ having $m$ degrees of freedom and $V$ having $n$ degrees of freedom. For convenience, let $c = m/2$ and $d = m/2$. To compute the distribution of $X$, we need the joint pdf of $U$ and $V$. First note that

$$\begin{aligned} f_U(u) &= \frac{1}{\Gamma(c)2^c} u^{c-1} e^{-u/2},\ u > 0 \text{ and} \\ f_V(v) &= \frac{1}{\Gamma(d)2^d} v^{d-1} e^{-v/2},\ v > 0. \end{aligned}$$

Therefore, since $U$ and $V$ are independent by assumption,

$$\begin{aligned} f_{U,V}(u,v) &= \left(\frac{1}{\Gamma(c)2^c} u^{c-1} e^{-u/2}\right)\left(\frac{1}{\Gamma(d)2^d} v^{d-1} e^{-v/2}\right) \\ &= \frac{1}{\Gamma(c)\Gamma(d)2^{c+d}} u^{c-1} v^{d-1} e^{-u/2} e^{-v/2},\ u, v > 0. \end{aligned}$$

We will first find the distribution of the random variable $U/V$ by using the cdf method. Specifically,

$$F_{U/V}(x) = P(U/V \le x) = P(U \le xV).$$

Putting $U$ on the horizontal axis and $V$ on the vertical axis, we can graph the inequality $U \le xV$. See Figure 1. Integrating first with respect to $u$ and then with respect to $v$, we obtain

$$P(U \le xV) = \frac{1}{\Gamma(c)\Gamma(d)2^{c+d}} \int_0^\infty \left(\int_0^{xv} u^{c-1} e^{-u/2} du\right) v^{d-1} e^{-v/2} dv.$$

We will solve this integral without actually integrating anything! Differentiate the integral with respect to $x$ moving the $d/dx$ past the first integral. We get

$$\begin{aligned} f_{U/V}(x) = F'_{U/V}(x) &= \frac{1}{\Gamma(c)\Gamma(d)2^{c+d}} \int_0^\infty \left[(vx)^{c-1} e^{-vx/2} v\right] v^{d-1} e^{-v/2} dv \\ &= \frac{x^{c-1}}{\Gamma(c)\Gamma(d)2^{c+d}} \int_0^\infty v^{c+d-1} e^{-[(x+1)/2]v} dv. \end{aligned}$$

(Note that the extra $v$ in the first line comes from the application of the Chain Rule to the inner integral.)

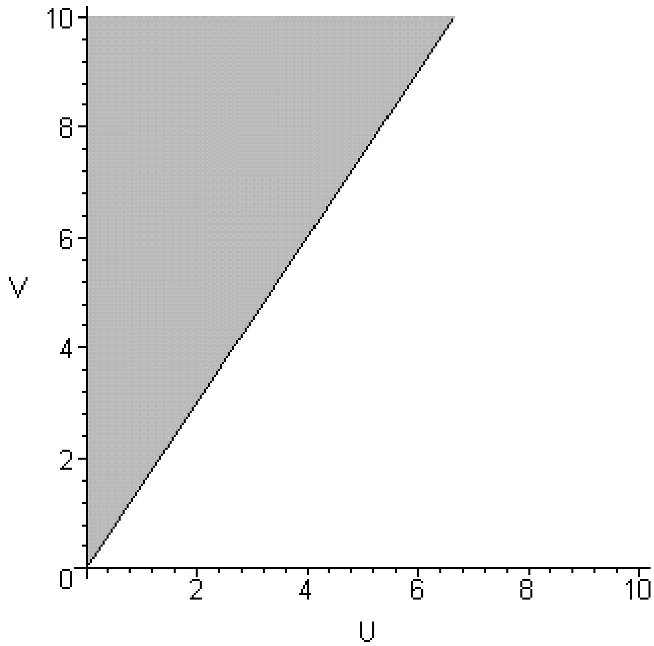



Figure 1: The inequality $U \leq xV$

Observe that

$$f(v) = \frac{\left(\frac{x+1}{2}\right)^{c+d}}{\Gamma(c+d)} v^{c+d-1} e^{-[(x+1)/2]v}$$

is the pdf of a Gamma random variable with parameters $\alpha = c+d$ and $\beta = (x+1)/2$. Since $\int_0^\infty f(v)dv = 1$,

$$\begin{aligned} f_{U/V}(x) &= \frac{x^{c-1}}{\Gamma(c)\Gamma(d)2^{c+d}} \cdot \frac{\Gamma(c+d)}{\left(\frac{x+1}{2}\right)^{c+d}} \\ &= \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)} \cdot \frac{x^{c-1}}{(x+1)^{c+d}}. \end{aligned}$$

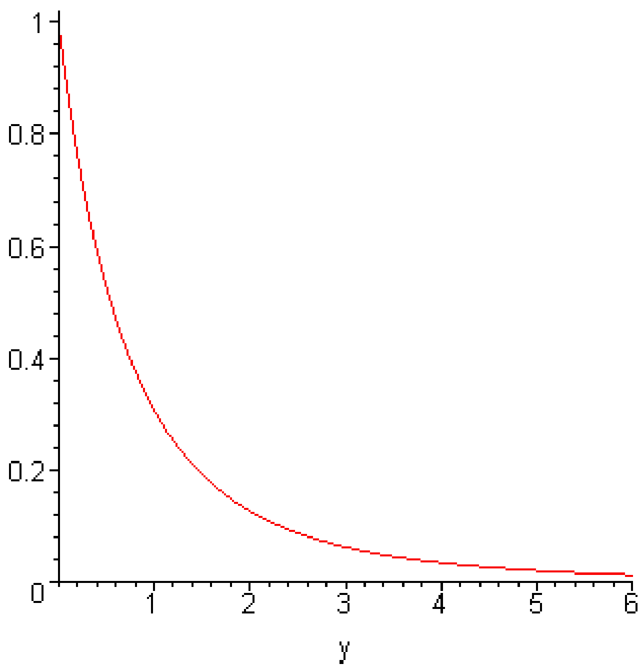
We now have that

$$\begin{aligned} f_X(x) = f_{(U/m)/(V/n)}(x) &= f_{(n/m)(U/V)}(x) \\ &= \frac{m}{n} f_{U/V}\left(\frac{m}{n}x\right) \\ &= \frac{m}{n} \cdot \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\left(\frac{m}{n}x\right)^{(m/2)-1}}{\left(\frac{m}{n}x+1\right)^{(m+n)/2}} \\ &= \frac{\Gamma\left(\frac{m+n}{2}\right) m^{m/2} n^{n/2} x^{(m/2)-1}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)(n+mx)^{(m+n)/2}}. \end{aligned}$$

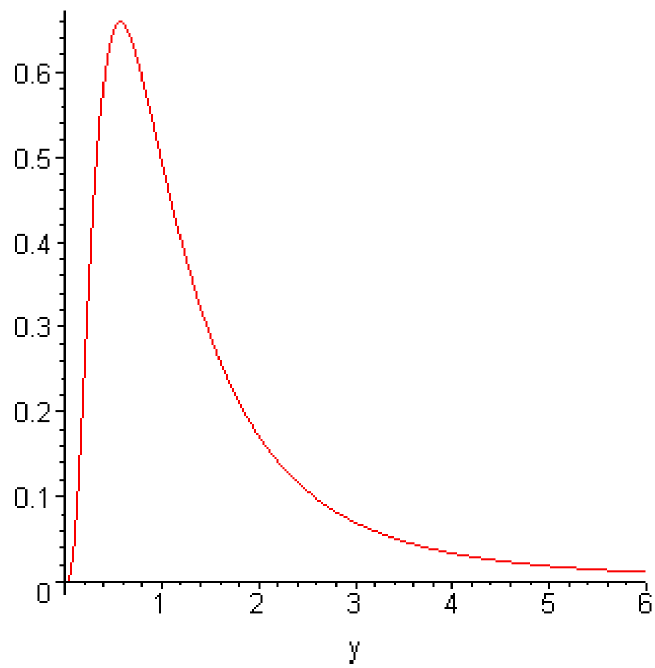
This completes the proof. ■

Using Maple, we can plot the graphs of $F_{m.n}$ using the following commands.

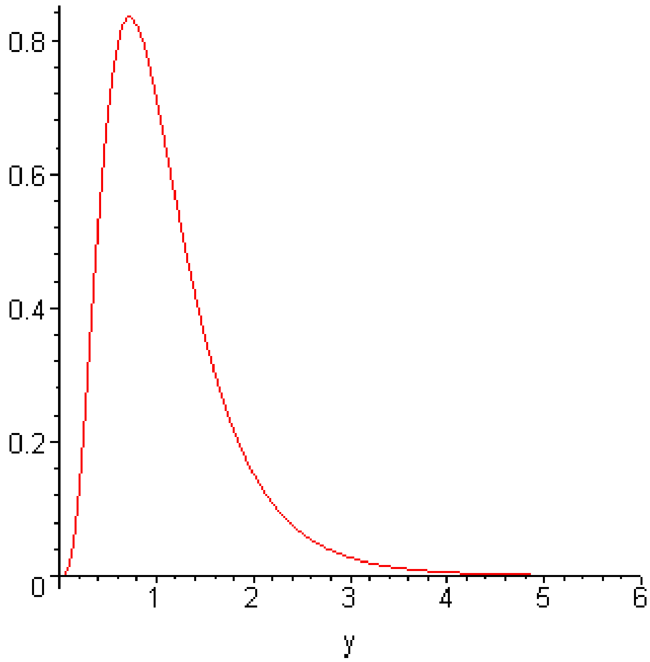
```
f := (m, n, x)− > (GAMMA((m + n)/2) ∗ mˆ(m/2) ∗ nˆ(n/2) ∗ xˆ((m/2) − 1))/
(GAMMA(m/2) ∗ GAMMA(n/2) ∗ (n + m ∗ x)ˆ((n + m)/2));

plot(f(2, 5, x), x = 0..6);
```



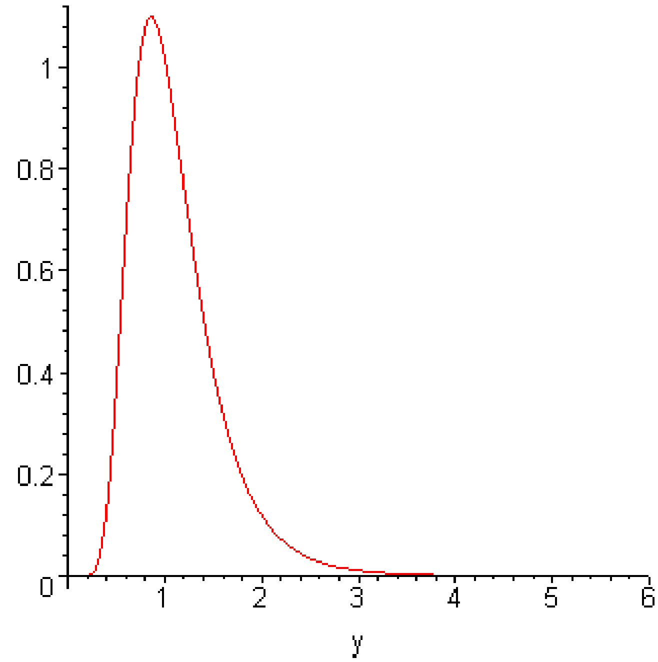

$m = 2$, $n = 5$



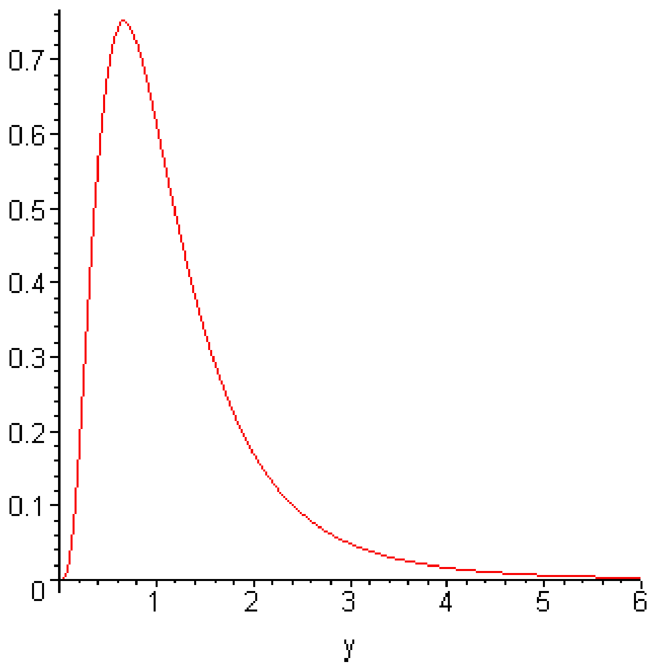

$m = 10$, $n = 5$



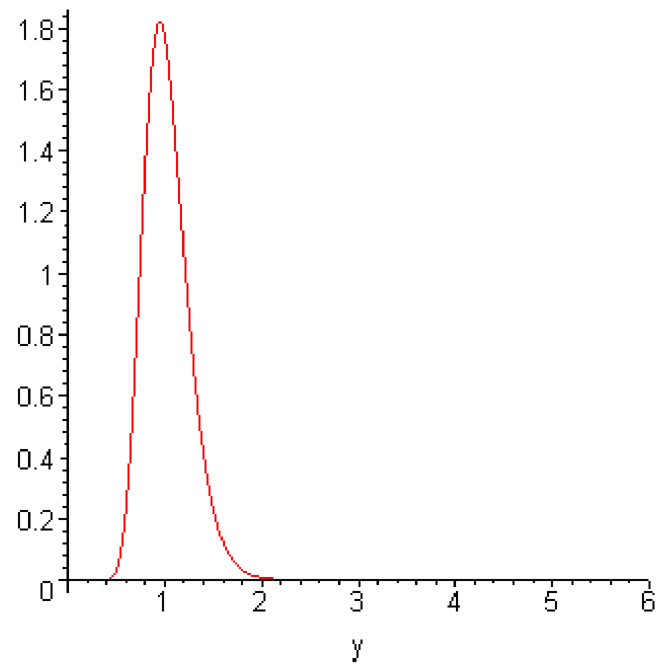

$m = 10$, $n = 20$




$m = 40$, $n = 20$




$m = 10$, $n = 10$




$m = 80$, $n = 80$