Lecture 09: Covariance and Correlation

Mathematical Statistics I, MATH 60061/70061

Tuesday September 28, 2021

Reference: Casella & Berger, 4.5

Key properties of covariance

1. $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$.
2. $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$.
3. $\mathrm{Cov}(X, c) = 0$ for any constant c.
4. $\mathrm{Cov}(aX, Y) = a\mathrm{Cov}(X, Y)$ for any constant a.
5. $\mathrm{Cov}(X + Y, Z) = \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z)$.
6. $\mathrm{Cov}(X + Y, Z + W) =$
   $\mathrm{Cov}(X, Z) + \mathrm{Cov}(X, W) + \mathrm{Cov}(Y, Z) + \mathrm{Cov}(Y, W)$.
7. $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y)$.
8. $\mathrm{Var}(X - Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\mathrm{Cov}(X, Y)$.
9. For n random variables $X_1, \ldots, X_n$,

   $$\mathrm{Var}(X_1 + \cdots + X_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) + 2\sum_{i<j} \mathrm{Cov}(X_i, X_j).$$

Example: Exponential max and min

Let X and Y be i.i.d. $\mathrm{Expo}(1)$ random variables. Find the correlation between $\max(X, Y)$ and $\min(X, Y)$.

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Let $M = \max(X, Y)$ and $L = \min(X, Y)$. By the memoryless property and results for min Exponentials, we know $L \sim \mathrm{Expo}(2)$, $M - L \sim \mathrm{Expo}(1)$, and $M - L$ is independent of L. Therefore,

$$\begin{aligned}\mathrm{Cov}(M, L) &= \mathrm{Cov}(M - L + L, L) = \mathrm{Cov}(M - L, L) + \mathrm{Cov}(L, L)\\ &= \mathrm{Var}(L) = \frac{1}{4},\end{aligned}$$

$$\mathrm{Var}(M) = \mathrm{Var}(M - L + L) = \mathrm{Var}(M - L) + \mathrm{Var}(L) = 1 + \frac{1}{4} = \frac{5}{4},$$

$$\mathrm{Corr}(M, L) = \frac{\mathrm{Cov}(M, L)}{\sqrt{\mathrm{Var}(M)\mathrm{Var}(L)}} = \frac{\frac{1}{4}}{\sqrt{\frac{5}{4} \cdot \frac{1}{4}}} = \frac{1}{\sqrt{5}}.$$