Final Exam

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1.

a.

Let $X = X_1^2 + X_2^2 + ... + X_n^2$, $Y = X_i^n$ for $i = 1, 2, ..., n$, then $X \sim \chi_n^2$, $Y \sim \chi_1^2$, $X - Y \sim \chi_{n-1}^2$. The joint PDF of X and Y is

$$\begin{aligned} f_{X,Y}(x,y) &= P(X = x, Y = y) \\ &= P(X = x | Y = y)P(Y = y) \\ &= P(X - Y = x - y)P(Y = y) \\ &= \frac{1}{\Gamma(\frac{n-1}{2})} \left(\frac{1}{2}(x-y)\right)^{\frac{n-1}{2}} e^{-\frac{1}{2}(x-y)} \frac{1}{x-y} \frac{1}{\Gamma(\frac{1}{2})} \left(\frac{1}{2}y\right)^{\frac{1}{2}} e^{-\frac{1}{2}y} \frac{1}{y} \end{aligned}$$

Let $U = X$, $V = Y/X$. Then

$$f_{X,Y}(x,y) = f_{U,V}(x, \frac{y}{x}) | \begin{vmatrix} 1 & 0 \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} | = f_{U,V}(x, \frac{y}{x}) \frac{1}{x}$$

Let $u_1 = x$, $v_1 = y/x$, then $x = u_1$, $y = u_1 v_1$. Thus,

$$\begin{aligned} f_{U,V}(u_1, v_1) &= f_{X,Y}(u_1, v_1) u_1 \\ &= \frac{1}{\Gamma(\frac{n-1}{2})} \left(\frac{1}{2}u_1(1-v_1)\right)^{\frac{n-1}{2}} e^{-\frac{1}{2}u_1(1-v_1)} \frac{1}{u_1(1-v_1)} \frac{1}{\Gamma(\frac{1}{2})} \left(\frac{1}{2}u_1 v_1\right)^{\frac{1}{2}} e^{-\frac{1}{2}u_1 v_1} \frac{1}{u_1 v_1} u_1 \\ &= \frac{1}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} \left(\frac{1}{2}\right)^{\frac{n}{2}} u_1^{\frac{n}{2}} v_1^{\frac{1}{2}} (1-v_1)^{\frac{n-1}{2}} e^{-\frac{1}{2}u_1} \frac{1}{u_1 v_1 (1-v_1)} \end{aligned}$$

Since $U \sim \chi_n^2$, the PDF of U is $f_U(u_1) = \frac{1}{\Gamma(\frac{n}{2})} \left(\frac{1}{2}u_1\right)^{\frac{n}{2}} e^{-\frac{1}{2}u_1} \frac{1}{u_1}$. Therefore, the conditional PDF of V given U is

$$\begin{aligned} f_{V|U}(v_1 | U = u_1) &= \frac{f_{U,V}(u_1, v_1)}{f_U(u_1)} \\ &= \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} v_1^{\frac{1}{2}} (1-v_1)^{\frac{n-1}{2}} \frac{1}{v_1(1-v_1)} \\ &= \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} v_1^{-\frac{1}{2}} (1-v_1)^{\frac{n-3}{2}} \end{aligned}$$

Note that $f_{V|U}(v_1|U=u_1)$ does not dependent on u_1. Thus, $f_V(v_1) = f_{V|U}(v_1|U=u_1)$, and $f_{U,V}(u_1,v_1) = f_U(u_1)f_V(v_1)$. Therefore, U and V are independent.

b.

$$f_V(v_1) = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} v_1^{-\frac{1}{2}}(1-v_1)^{\frac{n-3}{2}}$$

2.

a.

Because $X|Q \sim \text{Bin}(n,q)$, $E(X|Q) = nq$. Because $q \sim \text{Unif}(0,1)$, $E(q) = \frac{1}{2}$. By the law of total expectation,

$$E(X) = E(E(X|Q)) = E(nq) = \frac{n}{2}$$

b.

By the law of total variance,

$$Var(X) = E(Var(X|Q)) + Var(E(X|Q)) = E(np(1-p)) + Var(np) = \frac{1}{6}n + \frac{1}{12}n^2$$

3.

a.

Since $X_1, ..., X_5$ are iid $\mathcal{N}(-1,4)$. The distribution of $\bar{X}$ is $\mathcal{N}(-1, \frac{4}{5})$.

Since $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$, and χ^2_{n-1} equals to Gamma$(\frac{n-1}{2}, \frac{1}{2})$. Let $T = (n-1)S^2/\sigma^2$, the PDF of T is

$$f_T(x) = \frac{1}{\Gamma(\frac{n-1}{2})}(\frac{1}{2}x)^{\frac{n-1}{2}}e^{-\frac{1}{2}x}\frac{1}{x}$$

Let $S^2 = \frac{\sigma^2}{n-1}$, the PDF of S^2 is

$$f_{S^2}(x) = \frac{\sigma^2}{n-1}f_T(\frac{\sigma^2}{n-1}x) = \frac{\sigma^2}{n-1}\frac{1}{\Gamma(\frac{n-1}{2})}(\frac{1}{2}\frac{\sigma^2}{n-1}x)^{\frac{n-1}{2}}e^{-\frac{1}{2}\frac{\sigma^2}{n-1}x}\frac{1}{\frac{\sigma^2}{n-1}x} = \frac{1}{\Gamma(\frac{n-1}{2})}(\frac{1}{2}\frac{\sigma^2}{n-1}x)^{\frac{n-1}{2}}e^{-\frac{1}{2}\frac{\sigma^2}{n-1}x}\frac{1}{x}$$

which is the same as the PDF of Gamma$(\frac{n-1}{2}, \frac{\sigma^2}{2(n-1)})$. Given that $n=5, \sigma^2=4$, the distribution of S^2 is Gamma$(2, \frac{1}{2})$.

b.

By definition, $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$, thus $\frac{\bar{X}+1}{S/\sqrt{5}} \sim t_4$.

c.

Let $Z_i = \frac{X_i-\mu}{\sigma} = \frac{X_i+1}{2}, i = 1,2,3,4,5$. Then $Z_1^2+Z_2^2 \sim \chi^2_2$, $Z_3^2+Z_4^2+Z_5^2 \sim \chi^2_3$. Since Z_1, Z_2, Z_3, Z_4, Z_5 are independent, $\frac{Z_1^2+Z_2^2}{Z_3^2+Z_4^2+Z_5^2} \sim F_{2,3}$.

4.

Since $Cov(X_i, X_j) = E(X_iX_j) - E(X_i)E(X_j)$, we can get $E(X_iX_j) = \mu^2$ when $|i-j| > 1$ and $E(X_iX_j) = \mu^2 + c$ when $|i-j| = 1$.

Suppose $\epsilon > 0$, by Markov inequality,

$$\begin{aligned}
P(|\bar{X} - \mu| > \epsilon) &= P((\bar{X} - \mu)^2 > \epsilon^2) \\
&\leq \frac{E(\bar{X} - \mu)^2}{\epsilon^2} \\
&= \frac{E(\sum_{i=1}^n X_i - n\mu)^2}{n^2\epsilon^2} \\
&= \frac{E(\sum_{i=1}^n X_i)^2 - 2n\mu E(\sum_{i=1}^n X_i) + n^2\mu^2}{n^2\epsilon^2} \\
&= \frac{E(\sum_{i=1}^n X_i)^2 - n^2\mu^2}{n^2\epsilon^2} \\
&= \frac{\sum_{i=1}^n E(X_i^2) + 2\sum_{i<j} E(X_iX_j) - n^2\mu^2}{n^2\epsilon^2} \\
&= \frac{n(\mu^2 + \sigma^2) + 2[(n-1)(\mu^2 + c) + \frac{(n-1)(n-2)}{2}\mu^2] - n^2\mu^2}{n^2\epsilon^2} \qquad (*) \\
&= \frac{n\sigma^2 + 2(n-1)c}{n^2\epsilon^2} \to 0
\end{aligned}$$

when $n \to \infty$. Thus, $\bar{X}$ converges in probability to μ.

The equality $(*)$ above is because $\sum_{i<j} E(X_iX_j)$ has $\frac{n(n-1)}{2}$ items, in which $n-1$ items satisfies $|i-j| = 1$ and the rest satisfies $|i-j| > 1$.

5.

Since $X_i \sim$ Expo(1), the PDF of X_i is $f_{X_i}(x) = e^{-x}$,. The CDF of $X_{(n)}$ is $F_{X_{(n)}}(x) = (1 - e^{-x})^n$.

$$\begin{aligned}
F_{Y_{(n)}}(y) &= P(Y_{(n)} \leq y) = P(X_{(n)} - \log n \leq y) = P(X_{(n)} \leq y + \log n) \\
&= [1 - e^{-(y + \log n)}]^n \\
&= [1 - \frac{e^{-y}}{n}]^n
\end{aligned}$$

Let $\frac{1}{m} = -\frac{e^{-y}}{n}$, then $F_{Y_{(n)}}(y) = \left([1 + \frac{1}{m}]^m\right)^{-e^{-y}}$.

$$\lim_{n\to\infty} [1 - \frac{e^{-y}}{n}]^n = \lim_{m\to\infty} \left([1 + \frac{1}{m}]^m\right)^{-e^{-y}} = e^{-e^{-y}}$$

Thus $Y_{(n)}$ converges in distribution and the limit distribution is $F_{Y_{(n)}}(y) = e^{-e^{-y}}$.

6.

The joint PDF of $\boldsymbol{X} = (X_1, X_2, ..., X_n)$ is

$$f_{\boldsymbol{X}}(x|\mu, \lambda) = \prod_{i=1}^n \left(\frac{\lambda}{2\pi x_i^3}\right)^{\frac{1}{2}} \exp\left[-\frac{\lambda}{2\mu^2} \sum_{i=1}^n \frac{(x_i - \mu)^2}{x_i}\right] = \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \frac{1}{\prod_{i=1}^n x_i^{\frac{3}{2}}} \exp\left[-\frac{\lambda}{2\mu^2}(\sum_{i=1}^n x_i - 2n\mu + \sum_{i=1}^n \frac{1}{x_i})\right]$$

Let $T_1 = \sum_{i=1}^n x_i, T_2 = \sum_{i=1}^n \frac{1}{x_i}$. Let $h(\boldsymbol{x}) = \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \frac{1}{\prod_{i=1}^n x_i^{\frac{3}{2}}}$, $g(T_1, T_2|\mu, \lambda) = \exp\left[-\frac{\lambda}{2\mu^2}(\sum_{i=1}^n x_i - 2n\mu + \sum_{i=1}^n \frac{1}{x_i})\right]$, then $f_{\boldsymbol{X}}(x|\mu, \lambda) = h(\boldsymbol{x})g(T_1, T_2|\mu, \lambda)$. By Factorization Theorem, T_1, T_2 are sufficient statistic of μ, λ.

Note that $\bar{X}, T$ are one-to-one functions of T_1, T_2, i.e., $\bar{X} = \frac{T_1}{n}, T = n/(\frac{1}{n}T_2 - \frac{n}{T_1})$. Thus $(\bar{X}, T)$ are sufficient statistics.

Let $g(\bar{X}, T)$ be a function of $\bar{X}, T$. Let Ω be the set of all possible $(\bar{X}, T)$s. For any μ, λ,

$$E_{\mu,\lambda}(g(\bar{X}, T)) = \iint_{\Omega} g(\bar{X}, T) \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \frac{1}{\prod_{i=1}^n x_i^{\frac{3}{2}}} \exp\left[-\frac{\lambda}{2\mu^2}(\sum_{i=1}^n x_i - 2n\mu + \sum_{i=1}^n \frac{1}{x_i})\right] d\bar{X}\, dT$$
$$= \iint_{T_1, T_2 > 0} g(\bar{X}, T) \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \frac{1}{\prod_{i=1}^n x_i^{\frac{3}{2}}} \exp\left[-\frac{\lambda}{2\mu^2}(\sum_{i=1}^n x_i - 2n\mu + \sum_{i=1}^n \frac{1}{x_i})\right] \frac{1}{n(\frac{1}{n}T_2 - \frac{n}{T_1})^2} dT_1\, dT_2$$

The second equation is because $d\bar{X}\, dT = | \begin{vmatrix} \frac{1}{n} & 0 \\ -\frac{n^2}{T_1^2(\frac{1}{n}T_2 - \frac{n}{T_1})^2} & -\frac{1}{(\frac{1}{n}T_2 - \frac{n}{T_1})^2} \end{vmatrix} |\, dT_1\, dT_2 = \frac{1}{n(\frac{1}{n}T_2 - \frac{n}{T_1})^2} dT_1\, dT_2$.

Since $\left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} > 0$, $\frac{1}{\prod_{i=1}^n x_i^{\frac{3}{2}}} > 0$, $\exp\left[-\frac{\lambda}{2\mu^2}(\sum_{i=1}^n x_i - 2n\mu + \sum_{i=1}^n \frac{1}{x_i})\right] > 0$, $\frac{1}{n(\frac{1}{n}T_2 - \frac{n}{T_1})^2} > 0$. In order to make $E_{\mu,\lambda}(g(\bar{X}, T)) = 0$, we must have $g(\bar{X}, T) = 0$. Thus $(\bar{X}, T)$ are complete statistics.

In conclusion, $(\bar{X}, T)$ are sufficient and complete statistics.

7.

Let the random variables $U_1, U_2, ..., U_n$ be iid from Unif$(0, 1)$, then for $i = 1, 2, ..., n$, the order statistics $U_{(i)} \sim$ Beta$(i, n - i + 1)$, $E(U_{(i)}) = \frac{i}{n+1}$.

Since $X_1, X_2, ..., X_n$ are iid from Unif$(0, \theta)(\theta > 0)$, for $i = 1, 2, ..., n$, $X_{(i)} = \theta U_{(i)}$, $E(X_{(i)}) = \frac{i\theta}{n+1}$.

By the law of total expectation,

$$E(\frac{X_{(1)} + X_{(2)}}{X_{(n)}}) = E\left(E(\frac{X_{(1)} + X_{(2)}}{X_{(n)}}|X_{(n)} = x)\right)$$
$$= E\left(\frac{1}{x}E(X_{(1)} + X_{(2)}|X_{(n)} = x)\right)$$
$$= E\left(\frac{1}{x}[E(X_{(1)}|X_{(n)} = x) + E(X_{(2)}|X_{(n)} = x)]\right)$$

Note that by given $X_{(n)} = x$, $X_{(1)}, X_{(2)}, ..., X_{(n-1)}$ are equivalent to $n - 1$ order statistics on Unif$(0, x)$. Thus, $E(X_{(1)}|X_{(n)} = x) = \frac{x}{n}$, $E(X_{(2)}|X_{(n)} = x) = \frac{2x}{n}$. Therefore,

$$E(\frac{X_{(1)} + X_{(2)}}{X_{(n)}}) = E\left(\frac{1}{x}[\frac{x}{n} + \frac{2x}{n}]\right) = E(\frac{3}{n}) = \frac{3}{n}$$