Lecture 04: Moments and Moment Generating Functions

Mathematical Statistics I, MATH 60061/70061

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Reference: Casella & Berger, 2.3

Moment generating function

A moment generating function is a function that encodes the **moments** of a distribution.

The **moment generating function** (MGF) of a random variable X is $M(t) = E(e^{tX})$, as a function of t, if this is finite on some open interval $(-a, a)$ containing 0. Otherwise we say the MGF of X does not exist.

Properties of MGF

- MGF of a *sum of independent random variables*: If X and Y are independent, then the MGF of $X+Y$ is the product of the individual MGFs:

$$M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX})E(e^{tY}) = M_X(t)M_Y(t).$$

- MGF of *location-scale transformation*: If X has MGF $M_X(t)$, then the MGF of $a+bX$ is

$$E(e^{t(a+bX)}) = e^{at}E(e^{btX}) = e^{at}M_X(bt).$$

Bernoulli and Geometric MGFs

- For $X \sim \mathrm{Bern}(p)$,

$$M(t) = E(e^{tX}) = e^t p + e^0 q = pe^t + q.$$

  The MGF is defined on the entire real line.

- For $X \sim \mathrm{Geom}(p)$,

$$M(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} q^k p = p \sum_{k=0}^{\infty} (qe^t)^k = \frac{p}{1 - qe^t}$$

  for $qe^t < 1$, i.e., for t in $(-\infty, \log(1/q))$.

Binomial and Negative Binomial MGFs

Binomial MGF: The MGF of a $\mathrm{Bern}(p)$ R.V. is $pe^t + q$, so the MGF of a $\mathrm{Bin}(n, p)$ R.V. is

$$M(t) = (pe^t + q)^n.$$

Negative Binomial MGF: The MGF of a $\mathrm{Geom}(p)$ R.V. is $\frac{p}{1-qe^t}$ for $qe^t < 1$, so the MGF of a $\mathrm{NBin}(r, p)$ R.V. is

$$M(t) = \left(\frac{p}{1 - qe^t}\right)^r, \text{ for } qe^t < 1.$$

Sum of independent Poissons

The MGF of $X \sim \mathrm{Pois}(\lambda)$:

$$E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} \frac{e^{-\lambda}\lambda^k}{k!}$$

Sum of independent Poissons

The MGF of $X \sim \mathrm{Pois}(\lambda)$:

$$E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

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Let $Y \sim \mathrm{Pois}(\mu)$ be independent of X. The MGF of $X + Y$ is

$$E(e^{tX})E(e^{tY}) = e^{\lambda(e^t-1)} e^{\mu(e^t-1)} = e^{(\lambda+\mu)(e^t-1)},$$

which is the $\mathrm{Pois}(\lambda + \mu)$ MGF.

Since the MGF determines the distribution, it must be the case that $X + Y \sim \mathrm{Pois}(\lambda + \mu)$.

Normal MGF

The MGF of a standard Normal R.V. Z is

$$M_Z(t) = E(e^{tZ}) = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz.$$

After *completing the square*, we have

$$M_Z(t) = e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z-t)^2/2} dz = e^{t^2/2},$$

since the $\mathcal{N}(t, 1)$ PDF integrates to 1.

Thus, the MGF of $X = \mu + \sigma Z \sim \mathcal{N}(\mu, \sigma^2)$ is

$$M_X(t) = e^{\mu t} M_Z(\sigma t) = e^{\mu t} e^{(\sigma t)^2/2} = e^{\mu t + \frac{1}{2}\sigma^2 t^2}.$$

Sum of independent Normals

If we have $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ independently, what is the distribution of $X_1 + X_2$?

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The MGF of $X_1 + X_2$ is

$$
\begin{aligned}
M_{X_1+X_2}(t) &= M_{X_1}(t) M_{X_2}(t) \\
&= e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} \cdot e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2} \\
&= e^{(\mu_1+\mu_2)t + \frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2},
\end{aligned}
$$

which is the $\mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ MGF.

Since the MGF determines the distribution, it must be the case that $X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Sum of independent random variables is Normal

If X_1 and X_2 are independent and $X_1 + X_2$ is Normal, can we say something about the distributions of X_1 and X_2?

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Craḿer's theorem: If X_1 and X_2 are independent and $X_1 + X_2$ is Normal, then X_1 and X_2 must be Normal.

Proving the Craḿer's theorem in full generality is difficult. We will consider a special case when X_1 and X_2 are i.i.d. with $M(t)$.

Without loss of generality, assume $X_1 + X_2 \sim \mathcal{N}(0, 1)$, and then its MGF is

$$e^{t^2/2} = E(e^{t(X_1+X_2)}) = E(e^{tX_1})E(e^{tX_2}) = (M(t))^2,$$

so $M(t) = e^{t^2/4}$, which is the $\mathcal{N}(0, 1/2)$ MGF. Thus, $X_1, X_2 \sim \mathcal{N}(0, 1/2)$.