Homework #3

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1.

By the Chebychev's Inequality,

$$P(|X-\mu_X| \geq k\sigma_X) \leq \frac{1}{k^2}$$

If $X \sim \text{Unif}(0,1)$, then $\mu_X = 1/2$, $\sigma_X^2 = 1/12$. Thus

$$P(|X-\frac{1}{2}| \geq \frac{k}{2\sqrt{3}}) = \begin{cases} 1-\frac{k}{\sqrt{3}} & 0 \leq k < \sqrt{3} \\ 0 & k \geq \sqrt{3} \end{cases}$$

It can be verified that $1-\frac{k}{\sqrt{3}} < \frac{1}{k^2}$ for $0 \leq k < \sqrt{3}$ and $0 < \frac{1}{k^2}$ for $k \geq \sqrt{3}$. The Chebychev's bound is more loose than the bound derived above.

If $X \sim \text{Exp}(\lambda)$, then $\mu_X = 1/\lambda$, $\theta_X = 1/\lambda^2$. When $k \geq 1$,

$$P(|X-1/\lambda| \geq k/\lambda) = \int_{(k+1)/\lambda}^{\infty} \lambda e^{-\lambda x} dx = e^{-(k+1)}$$

When $0 \leq k < 1$,

$$P(|X-1/\lambda| \geq k/\lambda) = 1 - \int_{(1-k)/\lambda}^{(k+1)/\lambda} \lambda e^{-\lambda x} dx = 1 + e^{-(k+1)} - e^{-(1-k)} = 1 + e^{-1}(e^{-k} - e^{k})$$

When $k \geq 1$, let $e^{-(k+1)} > 1/k^2$, taking the logarithm of both sides gives $k+1 \geq 2\log k$. Let $g(k) = k+1-2\log k$, then $g'(k) = 1-2/k$. When $g'(k) = 0$, $k = 2$. Thus $g(k) \geq g(2) = 3-2\log 2 > 0$, the inequality $e^{-(k+1)} \leq 1/k^2$ holds when $k \geq 1$.

When $0 \leq k < 1$, $g(k) = e^{-k} - e^{k}$ is monotonically decreasing, thus $g(k) \leq g(0) = 0$, $1+e^{-1}(e^{-k}-e^{k}) \leq 1$. Since $1/k^2 > 1$ when $0 \leq k < 1$, $1+e^{-1}(e^{-k}-e^{k}) < 1/k^2$ holds.

2. Let $X = U/V$, then $E(U/V)E(V/U) = E(X)E(1/X)$. Let $g(X) = 1/X$, $g(x)$ is a convex function, thus $E(g(X)) \geq g(E(X))$ (Jensen's inequality). Since U, V are not constant multiple of each other, X is not a constant, thus $E(g(X)) > g(E(X))$. Therefore, $E(X)E(1/X) > E(X)(1/E(X)) = 1$.

3.

a. Since $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})$,

$$
\begin{aligned}
\frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_i - X_j)^2 &= \frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}[(X_i - \bar{X}) - (X_j - \bar{X})]^2 \\
&= \frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}[(X_i - \bar{X})^2 + (X_j - \bar{X})^2 - 2(X_i - \bar{X})(X_j - \bar{X})] \\
&= \frac{1}{2n(n-1)}\{\sum_{i=1}^{n}\sum_{j=1}^{n}[(X_i - \bar{X})^2 + (X_j - \bar{X})^2] - 2\sum_{i=1}^{n}\sum_{j=1}^{n}(X_i - \bar{X})(X_j - \bar{X})\} \\
&= \frac{1}{2n(n-1)}\{\sum_{i=1}^{n}[n(X_i - \bar{X})^2 + \sum_{j=1}^{n}(X_j - \bar{X})^2] - 2\sum_{i=1}^{n}(X_i - \bar{X})[\sum_{j=1}^{n}(X_j - \bar{X})]\} \\
&= \frac{1}{2n(n-1)}\{\sum_{i=1}^{n}[n(X_i - \bar{X})^2 + (n-1)S^2] - 2\sum_{i=1}^{n}(X_i - \bar{X})\cdot 0\} \\
&= \frac{1}{2n(n-1)}2n(n-1)S^2 \\
&= S^2
\end{aligned}
$$

b.

$\mathrm{Var}(S^2) = E(S^2)^2 - (E(S^2))^2 = E(S^4) - (E(S^2))^2$. Given that $\theta_1 = E(X_i), \theta_j = E(X_i - \theta_1)^j$. Let $Z_i = X_i - \theta_1$, since $X_1, X_2, ..., X_n$ are independent, we have $E(Z_i Z_j) = 0$ $(i \neq j)$ and $E(Z_i^2) = \theta_2$.

$$
\begin{aligned}
E(S^2) &= E(\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2) \\
&= E(\frac{1}{n-1}(\sum_{i=1}^{n}(X_i - \theta_1)^2 - n(\bar{X} - \theta_1)^2)) \\
&= E(\frac{1}{n-1}(\sum_{i=1}^{n}(X_i - \theta_1)^2 - \frac{1}{n}(\sum_{i=1}^{n}(X_i - \theta_1))^2)) \\
&= \frac{1}{n(n-1)}(n\sum_{i=1}^{n}E(X_i - \theta_1)^2 - E(\sum_{i=1}^{n}(X_i - \theta_1))^2) \\
&= \frac{1}{n}\sum_{i=1}^{n}E(X_i - \theta_1)^2 \\
&= \theta_2
\end{aligned}
$$

$$
\begin{aligned}
E(S^4) &= \frac{1}{n^2(n-1)^2}E((n\sum_{i=1}^{n}(X_i - \theta_1)^2 - (\sum_{i=1}^{n}(X_i - \theta_1))^2)^2 \\
&= \frac{1}{n^2(n-1)^2}[n^2E(\sum_{i=1}^{n}Z_i^2)^2 - 2nE((\sum_{i=1}^{n}Z_i^2)(\sum_{i=1}^{n}Z_i)^2) + E(\sum_{i=1}^{n}Z_i)^4] \\
&= \frac{1}{n^2(n-1)^2}[n^2(n\theta_4 + n(n-1)\theta_2^2) - 2n(n\theta_4 + n(n-1)\theta_2^2) + (n\theta_4 + 3n(n-1)\theta_2^2)] \\
&= \frac{1}{n(n-1)}[(n-1)\theta_4 + (n^2 - 2n + 3)\theta_2^2]
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\mathrm{Var}(S^2) &= E(S^4) - (E(S^2))^2 \\
&= \frac{1}{n(n-1)}[(n-1)\theta_4 + (n^2 - 2n + 3)\theta_2^2] - \theta_2^2 \\
&= \frac{1}{n}(\theta_4 - \frac{n-3}{n-1}\theta_2^2)
\end{aligned}$$

c. $\mathrm{Cov}(\bar{X}, S^2) = E(\bar{X}S^2) - E(\bar{X})E(S^2)$ where $E(\bar{X}) = \theta_1$ and $E(S^2) = \theta_2$.

$$\begin{aligned}
E(\bar{X}, S^2) &= E(\frac{1}{n-1}\bar{X}\sum_{i=1}^{n}(X_i - \bar{X})^2) \\
&= \frac{1}{n-1}[E((\bar{X} - \theta_1)(\sum_{i=1}^{n}(X_i - \theta_1)^2 - n(\bar{X} - \theta_1)^2)) + E(\theta_1(\sum_{i=1}^{n}(X_i - \theta_1)^2 - n(\bar{X} - \theta_1)^2))] \\
&= \frac{1}{n-1}[E((\bar{X} - \theta_1)(\sum_{i=1}^{n}(X_i - \theta_1)^2)) - nE(\bar{X} - \theta_1)^3 + (n-1)\theta_1\theta_2] \\
&= \frac{1}{n-1}[\frac{1}{n}E((\sum_{i=1}^{n} Z_i)(\sum_{i=1}^{n} Z_i^2)) - \frac{1}{n^2}E(\sum_{i=1}^{n} Z_i)^3 + (n-1)\theta_1\theta_2] \\
&= \frac{1}{n-1}[\theta_3 - \frac{1}{n}\theta_3 + (n-1)\theta_1\theta_2] \\
&= \frac{\theta_3}{n} + \theta_1\theta_2
\end{aligned}$$

Thus,

$$\mathrm{Cov}(\bar{X}, S^2) = E(\bar{X}S^2) - E(\bar{X})E(S^2) = \frac{\theta_3}{n} + \theta_1\theta_2 - \theta_1\theta_2 = \frac{\theta_3}{n}$$

$\mathrm{Cov}(\bar{X}, S^2) = 0$ if and only if $\theta_3 = 0$.

4.

a.

$$\begin{aligned}
\bar{X}_{n+1} &= \frac{1}{n+1}\sum_{i=1}^{n+1} X_i \\
&= \frac{1}{n+1}(\sum_{i=1}^{n} X_i + X_{n+1}) \\
&= \frac{1}{n+1}(n\bar{X}_n + X_{n+1})
\end{aligned}$$

b.

$$
\begin{aligned}
nS_{n+1}^2 &= n\cdot\frac{1}{n}\sum_{i=1}^{n+1}(X_i-\bar{X}_{n+1})^2 \\
&= \sum_{i=1}^{n+1}[(X_i-\bar{X}_n)-(\bar{X}_{n+1}-\bar{X}_n)]^2 \\
&= \sum_{i=1}^{n+1}[(X_i-\bar{X}_n)^2-2(X_i-\bar{X}_n)(\bar{X}_{n+1}-\bar{X}_n)+(\bar{X}_{n+1}-\bar{X}_n)^2] \\
&= \sum_{i=1}^{n}(X_i-\bar{X}_n)^2+(X_{n+1}-\bar{X}_n)^2-2[\sum_{i=1}^{n}(X_i-\bar{X}_n)+(X_{n+1}-\bar{X}_n)](\bar{X}_{n+1}-\bar{X}_n) \\
&\quad+\sum_{i=1}^{n+1}(\bar{X}_{n+1}-\bar{X}_n)^2 \\
&= (n-1)S_n^2+(X_{n+1}-\bar{X}_n)^2-2(X_{n+1}-\bar{X}_n)(\bar{X}_{n+1}-\bar{X}_n)+(n+1)(\bar{X}_{n+1}-\bar{X}_n)^2 \\
&= (n-1)S_n^2+(X_{n+1}-\bar{X}_n)^2-\frac{2}{n+1}(X_{n+1}-\bar{X}_n)^2+\frac{1}{n+1}(X_{n+1}-\bar{X}_n)^2 \\
&= (n-1)S_n^2+\frac{n}{n+1}(X_{n+1}-\bar{X}_n)^2
\end{aligned}
$$

The 6th equation above uses the results in **a.**. Since $\bar{X}_{n+1}=\frac{1}{n+1}(n\bar{X}_n+X_{n+1})$,

$$
\bar{X}_{n+1}-\bar{X}_n=\frac{1}{n+1}(X_{n+1}-\bar{X}_n)
$$

5.

Let $U=(n-1)S_X^2/\sigma_X^2$, $V=(m-1)S_Y^2/\sigma_Y^2$. Then $U\sim\chi_{n-1}^2$, $V\sim\chi_{m-1}^2$, thus

$$
\begin{aligned}
f_U(x) &= \frac{1}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}}x^{(n-1)/2-1}e^{-x/2} \\
f_V(x) &= \frac{1}{\Gamma(\frac{m-1}{2})2^{(m-1)/2}}x^{(m-1)/2-1}e^{-x/2}
\end{aligned}
$$

Let $T=\frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2}=\frac{m-1}{n-1}\frac{U}{V}$. By definition, $T\sim F_{n-1,m-1}$. Let $p=n-1$, $q=m-1$, then $T\sim F_{p,q}$. The CDF of T is

$$
\begin{aligned}
F_T(x) &= F_{(q/p)(U/V)}(x)=P((q/p)(U/V)<x)=P(U<(p/q)Vx) \\
&= \int_0^\infty P(U<(p/q)vx|V=v)f_V(v)dv \\
&= \int_0^\infty\left(\int_0^{(p/q)vx}\frac{1}{\Gamma(\frac{p}{2})2^{p/2}}y^{p/2-1}e^{-y/2}dy\right)\frac{1}{\Gamma(\frac{q}{2})2^{q/2}}v^{q/2-1}e^{-v/2}dv \\
&= \frac{1}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})2^{(p+q)/2}}\int_0^\infty\left(\int_0^{(p/q)vx}y^{p/2-1}e^{-y/2}dy\right)v^{q/2-1}e^{-v/2}dv
\end{aligned}
$$

The PDF of T is

$$
\begin{aligned}
f_T(x) &= \frac{d}{dx}F_T(x) \\
&= \frac{1}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})2^{(p+q)/2}}\int_0^\infty (p/q)v((p/q)vx)^{p/2-1}e^{-(p/q)vx/2}v^{q/2-1}e^{-v/2}dv \\
&= \frac{1}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})2^{(p+q)/2}}\left(\frac{p}{q}\right)^{p/2} x^{p/2-1}\int_0^\infty v^{(p+q)/2-1}e^{-v[(p/q)x+1]/2}dv
\end{aligned}
$$

Let $v_0 = [(p/q)x+1]v$. Let $f(v_0) = \frac{1}{\Gamma(\frac{p+q}{2})2^{(p+q)/2}}v_0^{(p+q)/2-1}e^{-v_0/2}$ be the PDF of a Chi-Squared distribution. $\int_0^\infty f(v_0)dv_0 = 1$. Thus

$$
\begin{aligned}
\int_0^\infty v^{(p+q)/2-1}e^{-v[(p/q)x+1]/2}dv &= \Gamma(\frac{p+q}{2})2^{(p+q)/2}[(p/q)x+1]^{(p+q)/2}\int_0^\infty f(v_0)dv_0 \\
&= \Gamma(\frac{p+q}{2})2^{(p+q)/2}\frac{1}{[(p/q)x+1]^{(p+q)/2}}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f_T(x) &= \frac{1}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})2^{(p+q)/2}}\left(\frac{p}{q}\right)^{p/2} x^{p/2-1}\Gamma(\frac{p+q}{2})2^{(p+q)/2}\frac{1}{[(p/q)x+1]^{(p+q)/2}} \\
&= \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}\left(\frac{p}{q}\right)^{p/2}\frac{x^{p/2-1}}{[(p/q)x+1]^{(p+q)/2}}
\end{aligned}
$$

In summary, a random variable $T \sim F_{p,q}$ has PDF $f_T(x)$. Since a PDF defines a distribution uniquely, a random variable T which has PDF $f_T(x)$ will follow the distribution $F_{p,q}$.

6.

Since $X_1 \sim \text{Unif}(0,1)$, $X_{(1)} = \min\{X_1, X_2, ..., X_n\}$. The PDF of $X_{(1)}$ is $f_{X_{(1)}}(x) = n(1-x)^{n-1}$.

Thus the CDF of $X_1/X_{(1)}$ is

$$
\begin{aligned}
F_{X_1/X_{(1)}}(x) &= P(X_1/X_{(1)} < x) = P(X_1 < xX_{(1)}) \\
&= \int_0^1 P(X_1 < xt | X_{(1)} = t)f_{X_{(1)}}(t)dt \\
&= \int_0^{1/x} xt \cdot f_{X_{(1)}}(t)dt + \int_{1/x}^1 f_{X_{(1)}}(t)dt \\
&= xn\int_0^{1/x} t(1-t)^{n-1}dt + n\int_{1/x}^1 (1-t)^{n-1}dt \\
&= xn\left[\frac{-nt-1}{n(n+1)}(1-t)^n\right]_0^{1/x} + n\left[\frac{(1-t)^n}{-n}\right]_{1/x}^1 \\
&= \frac{x}{n+1}[1-(1-\frac{1}{x})^{n+1}]
\end{aligned}
$$

where $x \geq 1$.

The 3rd equation is because $P(X_1 < xt | X_{(1)} = t) = xt$ when $xt \leq 1 \Rightarrow t \leq 1/x$ and $P(X_1 < xt | X_{(1)} = t) = 1$ when $xt > 1 \Rightarrow t > 1/x$. $x \geq 1$ is because $X_1 \geq X_{(1)} \Rightarrow X_1/X_{(1)} \geq 1$.