Midterm Exam #1

MATH 60062/70062: Mathematical Statistics II

February 24, 2022

- Please turn off your phone.
- Print your name clearly at the top of this page.
- This is a closed-book and closed-notes exam.
- This exam contains 5 questions. There are 100 points in total.
- You have 75 minutes to complete the exam.
- Please show your work and explain all of your reasoning.
- You must work by yourself. Do not communicate in any way with others.

1. (10 points) Give full definitions for the following concepts:
    a. Convergence in probability
    b. Statistic
    c. Sufficient statistic
    d. Uniformly minimum-variance unbiased estimator (UMVUE)
    e. Exponential family of distributions

2. (10 points) Suppose that $X_1, \ldots, X_n$ are iid $\text{Pois}(\theta)$,

$$f_X(x \mid \theta) = \frac{e^{-\theta}\theta^x}{x!}, \quad x = 0, 1, 2, \ldots$$

where the prior distribution on θ is $\text{Gamma}(a, b)$,

$$\pi(\theta) = \frac{1}{\Gamma(a)b^a}\theta^{a-1}e^{-\theta/b}I(\theta > 0)$$

where the values of a and b are known. Find the posterior distribution $f(\theta \mid \boldsymbol{X} = \boldsymbol{x})$.

3. (10 points) Prove the Cramér–Rao Inequality. Suppose $\boldsymbol{X} \sim f_{\boldsymbol{X}}(\boldsymbol{x} \mid \theta)$. Let $W(\boldsymbol{X})$ be any estimator satisfying the regularity condition

$$\frac{d}{d\theta} E_\theta[W(\boldsymbol{X})] = \int_{\mathcal{X}} \frac{\partial}{\partial\theta}[W(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x} \mid \theta)] d\boldsymbol{x}$$

and

$$\mathrm{Var}_\theta(W(\boldsymbol{X})) < \infty.$$

Show that

$$\mathrm{Var}_\theta(W(\boldsymbol{X})) \geq \frac{(\frac{d}{d\theta} E_\theta[W(\boldsymbol{X})])^2}{E_\theta\left[(\frac{\partial}{\partial\theta} \log f_{\boldsymbol{X}}(\boldsymbol{X} \mid \theta))^2\right]},$$

where the quantity on the RHS is called the Cramér–Rao Lower Bound (CRLB) on the variance of the estimator $W(\boldsymbol{X})$.

4. (50 points) Suppose that $X_1, \ldots, X_n$ are iid Gamma(α_0, β),

$$f_X(x \mid \beta) = \frac{1}{\Gamma(\alpha_0)\beta^{\alpha_0}} x^{\alpha_0 - 1} e^{-x/\beta}$$

where α_0 is known and $\beta > 0$. **Useful fact:** $\Gamma(z+1) = z\Gamma(z)$. If $X \sim$ Gamma(α_1, β) and $Y \sim$ Gamma(α_2, β) are independent, then $X + Y \sim$ Gamma$(\alpha_1 + \alpha_2, \beta)$.

a. (10 points) Show that $T = T(\boldsymbol{X}) = \sum_{i=1}^n X_i$ is complete and sufficient for β.

b. (10 points) Find the CRLB on the variance of unbiased estimator of β.

c. (10 points) Find the maximum likelihood estimator (MLE) of $\tau(\beta) = 1/\beta$.

d. (20 points) Find the UMVUE for $\tau(\beta) = 1/\beta$.

5. (20 points) Suppose $X_1, \ldots, X_n$ are iid Pois(θ), where $\theta > 0$. Consider the function

$$\tau(\theta) = P_\theta(X = 0) = e^{-\theta}.$$

a. (5 points) Show that $W = W(\mathbf{X}) = I(X_1 = 0)$ is an unbiased estimator of $\tau(\theta)$.

b. (15 points) Find the UMVUE for $\tau(\theta)$. **Hint:** Rao-Blackwell Theorem. **Useful fact:** If $X \sim$ Pois(θ_1) and $Y \sim$ Pois(θ_2) are independent, then $X + Y \sim$ Pois$(\theta_1 + \theta_2)$.