Lecture 07: Exponential Families

Mathematical Statistics I, MATH 60061/70061

Tuesday September 21, 2021

Reference: Casella & Berger, 3.4

Exponential families

A family of PDFs or PMFs indexed by $\boldsymbol{\theta}$ is called an **exponential family** if it can be expressed as

$$f(x \mid \boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta}) \exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\right), \quad \boldsymbol{\theta} \in \boldsymbol{\Theta},$$

where $\boldsymbol{\Theta}$ is the set of all values of $\boldsymbol{\theta}$ (parameter space), $h(x) \geq 0$ and $t_1(x), \ldots, t_k(x)$ are real-valued functions of observation x (not depending on $\boldsymbol{\theta}$), and $c(\boldsymbol{\theta}) \geq 0$ and $w_1(\boldsymbol{\theta}), ..., w_k(\boldsymbol{\theta})$ are functions of the possibly vector-valued $\boldsymbol{\theta}$ (not depending on x).

Note that the expression for f may not be unique.

Exponential families

Many common families introduced in the previous lectures are exponential families. These include

- Continuous families: Normal, Gamma, Beta
- Discrete families: Binomial, Poisson, Negative Binomial

To verify that a family of PDFs or PMFs is an exponential family, we must identify the functions $h(x)$, $c(\boldsymbol{\theta})$, $w_i(\boldsymbol{\theta})$, and $t_i(x)$ and show that the family has the form given above.

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If $f(x \mid \boldsymbol{\theta})$, $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ is an exponential family, then

$$\{x : f(x \mid \boldsymbol{\theta}) > 0\} = \{x : h(x) > 0\}$$

which does not depend on $\boldsymbol{\theta}$ values.

This fact can be used to show a family is *non-exponential*, i.e., if $\{x : f(x \mid \boldsymbol{\theta}) > 0\}$ depends on $\boldsymbol{\theta}$, then $f(x \mid \boldsymbol{\theta})$, $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, is not an exponential family.

Consider the family of two parameters Exponential distributions with PDFs

$$f(x \mid \boldsymbol{\theta}) = \begin{cases} \lambda^{-1} e^{-(x-\mu)/\lambda} & x > \mu \\ 0 & x \leq \mu \end{cases} \qquad \mu \in \mathbb{R},\ \lambda > 0$$

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It is not an exponential family because

$$\{x : f(x \mid \boldsymbol{\theta}) > 0\} = \{x : x > \mu\}$$