Midterm Exam #2

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1.

a.

Since $Z \sim \mathcal{N}(0,1)$, the PDF of Z is $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, thus

$$P(|Z| > r) = 2\int_r^\infty f_Z(z)dz = \sqrt{\frac{2}{\pi}} \int_r^\infty e^{-\frac{z^2}{2}} dz = \sqrt{\frac{2}{\pi}} \frac{\int_r^\infty re^{-\frac{z^2}{2}} dz}{r} \leq \sqrt{\frac{2}{\pi}} \frac{\int_r^\infty ze^{-\frac{z^2}{2}} dz}{r} = \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{r^2}{2}}}{r}$$

b.

Since $X_1, X_2, ..., X_n$ are iid $\mathcal{N}(0,1)$ and $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$, we have $\bar{X}_n \sim \mathcal{N}(0, \frac{1}{n})$. Thus the PDF of $\bar{X}_n$ is $f_{\bar{X}_n}(z) = \frac{n}{\sqrt{2\pi}} e^{-\frac{z^2 n^2}{2}}$. Therefore,

$$P(|\bar{X}_n| > r) = 2\int_r^\infty f_{\bar{X}_n}(z)dz = \sqrt{\frac{2}{\pi}} \int_r^\infty ne^{-\frac{z^2n^2}{2}} dz = \sqrt{\frac{2}{\pi}} \frac{\int_r^\infty nre^{-\frac{z^2n^2}{2}} dz}{r} \leq \sqrt{\frac{2}{\pi}} \frac{\int_r^\infty nze^{-\frac{z^2n^2}{2}} dz}{r}$$
$$= \sqrt{\frac{2}{\pi}} \frac{1}{n} \frac{\int_{nr}^\infty nze^{-\frac{z^2n^2}{2}} d(zn)}{r} = \sqrt{\frac{2}{\pi}} \frac{1}{n} \frac{e^{-\frac{n^2r^2}{2}}}{r}$$

By Chebyshev's inequality,

$$P(|\bar{X}_n| > r) \leq \frac{E|\bar{X}_n|^2}{r^2} = \frac{1}{nr^2}$$

Assume that

$$\sqrt{\frac{2}{\pi}} \frac{1}{n} \frac{e^{-\frac{n^2r^2}{2}}}{r} < \frac{1}{nr^2} \tag{1}$$
$$\sqrt{\frac{2}{\pi}} re^{-\frac{n^2r^2}{2}} < 1$$

Let $g(r) = re^{-\frac{n^2r^2}{2}}$, $g'(r) = (1-n^2r^2)e^{-\frac{n^2r^2}{2}}$. $g'(r) = 0$ only when $r = \frac{1}{n}$. Thus $g(r)_{max} = g(\frac{1}{n}) = \frac{1}{n}e^{-\frac{1}{2}}$. Since $\sqrt{\frac{2}{\pi}} < 1$, $n \geq 1$ and $e^{-\frac{1}{2}} < 1$, it is obvious that $\sqrt{\frac{2}{\pi}}\frac{1}{n}e^{-\frac{1}{2}} < 1$, thus Inequality (1) holds. Therefore, the bound of $\bar{X}_n$ is tighter than the Chebyshev's bound.

2.

a.

Let $t = \sum_{i=1}^{n} x_i$ be the number of 1's among $x_1, x_2..., x_k$, the probability of k randoms variables containing t 1's is $\binom{t}{k} p^t (1-p)^{k-t}$. $\binom{t}{k}$ is the number of permutations, thus for each permutation the probability is $p^t(1-p)^{k-t}$.

Since $P \sim \text{Unif}(0,1)$, $f_P(p) = 1$. Thus

$$\begin{aligned} P(X_1 = x_1, X_2 = x_2, ..., X_k = x_k) &= \int_0^1 P(X_1 = x_1, X_2 = x_2, ..., X_k = x_k | p) f_P(p) dp \\ &= \int_0^1 p^t (1-p)^{k-t} dp \\ &= \frac{t!(k-t)!}{(k+1)!} \end{aligned}$$

b.

Marginally,

$$P(X_i = x_i) = \int_0^1 p^{x_i}(1-p)^{1-x_i} dp = \frac{x_i!(1-x_i)!}{2} = \frac{1}{2}$$

where $\frac{x_i!(1-x_i)!}{2} = \frac{1}{2}$ is because $x_i = 0$ or 1.

Thus,

$$\prod_{i=1}^{n} P(X_i = x_i) = \frac{1}{2^n}$$

Since

$$P(X_1 = x_1, X_2 = x_2, ..., X_n = x_n) = \frac{(\sum_{i=1}^{n} x_i)!(n - \sum_{i=1}^{n} x_i)!}{(n+1)!}$$

It is obvious that $P(X_1 = x_1, X_2 = x_2, ..., X_n = x_n) \neq \prod_{i=1}^{n} P(X_i = x_i)$, thus $X_1, X_2, ..., X_n$ are not independent.

3.

Since $X_1 \sim \text{Unif}(0,1)$, $X_{(1)} = \min\{X_1, X_2, ..., X_n\}$, $X_{(n)} = \max\{X_1, X_2, ..., X_n\}$. The CDF of $X_{(1)}$ is $F_{X_{(1)}}(x) = 1 - (1-x)^n$. The PDF of $X_{(1)}$ is $f_{X_{(1)}}(x) = n(1-x)^{n-1}$. The CDF of $X_{(n)}$ is $F_{X_{(n)}}(x) = x^n$. The PDF of $X_{(1)}$ is $f_{X_{(1)}}(x) = nx^{n-1}$.

We can prove $X_{(1)}/X_{(n)}$ and $X_{(n)}$ to be independent by proving $F_{X_{(1)}/X_{(n)}, X_{(n)}}(x_1, x_2) = F_{X_{(1)}/X_{(n)}}(x_1) \cdot F_{X_{(n)}}(x_2)$.

$$\begin{aligned} F_{X_{(1)}/X_{(n)}}(x) &= \int_0^1 P(x_{(1)} < xt | X_{(n)} = t) f_{X_{(n)}}(t) dt \\ &= \int_0^1 [1 - (1-xt)^n] nt^{n-1} dt \\ &= 1 - \int_0^1 (1-xt)^n nt^{n-1} dt \\ &= 1 - \frac{n}{x^n} \int_0^x (1-xt)^n (xt)^{n-1} d(xt) \end{aligned}$$

$$
\begin{aligned}
F_{X_{(1)}/X_{(n)},X_{(n)}}(x_1,x_2) &= P(X_{(1)} \leq x_1 X_{(n)}, X_{(n)} \leq x_2) \\
&= \int_0^{x_2} \left(\int_0^{x_1 t} n(1-y)^{n-1} dy \right) n t^{n-1} dt \\
&= \int_0^{x_2} [1-(1-x_1 t)^n] n t^{n-1} dt \\
&= x_2^n - \int_0^{x_2} (1-x_1 t)^n n t^{n-1} dt \\
&= x_2^n - \frac{n}{x_1^n} \int_0^{x_1 x_2} (1-x_1 t)^n (x_1 t)^{n-1} d(x_1 t)
\end{aligned}
$$

(I am not sure whether $x_2^n - \frac{n}{x_1^n} \int_0^{x_1 x_2} (1-x_1 t)^n (x_1 t)^{n-1} d(x_1 t)$ equals to $[1-\frac{n}{x_1^n} \int_0^{x_1} (1-x_1 t)^n (x_1 t)^{n-1} d(x_1 t)] \cdot x_2^n$.)

4.

Since $X_i \sim \mathcal{N}(i, i^2)$, let $Z_i = \frac{X_i}{i} - 1$, $Z_i \sim \mathcal{N}(0,1)$.

a.

$$
(X_1 - 1)^2 + (\frac{X_2}{2} - 1)^2 + (\frac{X_3}{3} - 1)^2 \sim \chi_3^2
$$

b.

Let $\bar{X} = \frac{1}{3}(X_1 + \frac{X_2}{2} + \frac{X_3}{3}) - 1$ and $S^2 = \frac{1}{2}[(X_1 - 1 - \bar{X})^2 + (\frac{X_2}{2} - 1 - \bar{X})^2 + (\frac{X_3}{3} - 1 - \bar{X})^2]$, then

$$
\frac{\bar{X}}{S/\sqrt{3}} \sim t_2
$$

c.

Let $\bar{X}_a = \frac{1}{3}(X_1 + \frac{X_2}{2} + \frac{X_3}{3}) - 1$,

$S_a^2 = \frac{1}{2}[(X_1 - 1 - \bar{X}_a)^2 + (\frac{X_2}{2} - 1 - \bar{X}_a)^2 + (\frac{X_3}{3} - 1 - \bar{X}_a)^2]$,

$\bar{X}_b = \frac{1}{2}(X_1 + \frac{X_2}{2}) - 1$,

$S_b^2 = (X_1 - 1 - \bar{X}_b)^2 + (\frac{X_2}{2} - 1 - \bar{X}_b)^2$, then

$$
\frac{S_a^2}{S_b^2} \sim F_{2,1}
$$

5.

a.

Since $X_n \sim \text{Pois}(\frac{1}{n})$, $E(X_n) = \frac{1}{n}$, $Var(X_n) = \frac{1}{n}$.

$$\begin{aligned} P(|X_n - 0| > \epsilon) &= P((X_n - 0)^2 > \epsilon^2) \\ &\leq \frac{E(X_n - 0)^2}{\epsilon^2} \qquad \text{[Markov's Inequality]} \\ &= \frac{(E(X_n))^2 + Var(X_n)}{\epsilon^2} \\ &= \frac{\frac{1}{n^2} + \frac{1}{n}}{\epsilon^2} \to 0 \end{aligned}$$

when $n \to \infty$.

Hence X_n converges in probability to 0 as $n \to \infty$.

b.

Since $X_n \sim \text{Pois}(\frac{1}{n})$, we have $P(X_n = 0) = e^{-\frac{1}{n}}$.

$$P(|Y_n - 0| > \epsilon) = P(Y_n > \epsilon) = P(nX_n > \epsilon) = P(X_n > \frac{\epsilon}{n}) \leq P(X_n > 0) = 1 - e^{-\frac{1}{n}} \to 0$$

when $n \to \infty$. Thus Y_n converges in probability to 0 as $n \to \infty$.

6.

By the definition,

$$S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2 - (\bar{X}_n - \mu)^2$$

Let $U_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$, $V_n = \bar{X}_n - \mu$. By the Weak Law of Large Number, $U_n \xrightarrow{p} \sigma^2$, $V_n \xrightarrow{p} 0$. Thus $S_n^2 \xrightarrow{p} \sigma^2$.