Final Exam

MATH 60062/70062: Mathematical Statistics II

May 5, 2022

- Please turn off your phone.
- Print your name clearly at the top of this page.
- This is a closed-book and closed-notes exam.
- This exam contains 4 questions. There are 100 points in total.
- You have 75 minutes to complete the exam.
- Please show your work and explain all of your reasoning.
- You must work by yourself. Do not communicate in any way with others.

1. (15 points) Give full definitions for the following concepts:
    a. Coverage probability
    b. Confidence coefficient
    c. Pivotal quantity
    d. Consistent estimator
    e. Asymptotic relative efficiency

2. (35 points) Suppose that $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$, where $-\infty < \mu < \infty$ and $\sigma^2 > 0$. Both parameters are unknown. Consider testing

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0.$$

Let $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$, where $\bar{X}$ is the sample mean. The size α one-sample two-sided t-test rejects H_0 when

$$|\bar{x} - \mu_0| \geq t_{n-1,\alpha/2} \sqrt{s^2/n}.$$

a. (20 points) Show that the test can be derived as a likelihood ratio test.

b. (15 points) Find a $1 - \alpha$ confidence set for μ by inverting the two-sided t-test.

3. (35 points) Suppose $X_1, \dots, X_n$ are iid $\text{Beta}(\theta, 1)$, where $\theta > 0$.

   a. (5 points) Find the method of moments estimator of θ, $\hat{\theta}_{\text{MOM}}$.

   b. (10 points) Show that $\hat{\theta}_{\text{MOM}}$ satisfies

   $$\sqrt{n}(\hat{\theta}_{\text{MOM}} - \theta) \xrightarrow{d} \mathcal{N}\left(0, \frac{\theta(\theta+1)^2}{\theta+2}\right).$$

   Hint: Use Central Limit Theorem and Delta Method. **Useful fact:** For $Y \sim \text{Beta}(\alpha, \beta)$,

   $$f_Y(y \mid \theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}.$$

   The mean and variance of Y are $E[Y] = \frac{\alpha}{\alpha+\beta}$ and $\text{Var}[Y] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, respectively.

   c. (5 points) Find the maximum likelihood estimator of θ, $\hat{\theta}_{\text{MLE}}$.

   d. (10 points) Show that $\hat{\theta}_{\text{MLE}}$ satisfies

   $$\sqrt{n}(\hat{\theta}_{\text{MLE}} - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

   Hint: Use large sample results for MLEs.

   e. (5 points) What is the asymptotic relative efficiency (ARE) of $\hat{\theta}_{\text{MOM}}$ to $\hat{\theta}_{\text{MLE}}$? Graph the ARE as a function of θ, and summarize the graph in 1-3 sentences.

4. (15 points) Suppose $X_1, \ldots, X_n$ are iid Bern(p), where $0 < p < 1$. Derive a $1 - \alpha$ Wald confidence interval for

$$g(p) = \log\left(\frac{p}{1-p}\right),$$

the log odds of p.