Homework #2

MATH 60062/70062: Mathematical Statistics II

Please submit your answers as a **PDF** file to Blackboard by **9:15 a.m. on March 3**. Please show your work and write legibly. Your grade will be based on the correctness of your answers and the clarity with which you express them.

1. (15 points) Suppose that $X_1, \dots, X_n$ are iid Bern(θ). Show that the variance of $\bar{X}$ attains the Cramer-Rao Lower Bound, and hence $\bar{X}$ is the UMVUE for θ.

2. (15 points) Suppose $X_1, \dots, X_n$ are iid from a distribution whose PDF is

$$f_X(x \mid \theta) = \theta x^{\theta - 1}$$

   for $0 < x < 1$, $\theta > 0$. Is there a function of θ, say $\tau(\theta)$, for which there exists an unbiased estimator whose variance attains the Cramer-Rao Lower Bound? If so, find $\tau(\theta)$ and the UMVUE. If not, show why not.

3. (50 points) Suppose $X_1, \dots, X_n$ are iid Unif$(0, \theta)$, where $\theta > 0$.

   a. Show that the Unif$(0, \theta)$ PDF does not satisfy the regularity conditions needed for the Cramer-Rao Inequality to apply. Hint: Use Leibnitz's rule

$$\frac{d}{d\theta} \int_{a(\theta)}^{b(\theta)} f(x \mid \theta) dx = f(b(\theta) \mid \theta) b'(\theta) - f(a(\theta) \mid \theta) a'(\theta) + \int_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial \theta} f(x \mid \theta) dx$$

   b. Show that $T = T(\boldsymbol{X}) = X_{(n)}$ is a sufficient statistic for θ, where $X_{(n)}$ is the maximum order statistic.

   c. Show that $T = T(\boldsymbol{X}) = X_{(n)}$ is a complete statistic.

   d. Show that $\frac{n+1}{n} X_{(n)}$ is an unbiased estimator of θ.

   e. Find the UMVUE for θ.

4. (20 points) Suppose $X_1, \dots, X_n$ are iid Pois(θ), where $\theta > 0$. Consider the function

$$\tau(\theta) = P_\theta(X = 0) = e^{-\theta}.$$

   a. (5 points) Show that $W = W(\boldsymbol{X}) = I(X_1 = 0)$ is an unbiased estimator of $\tau(\theta)$.

   b. (15 points) Find the UMVUE for $\tau(\theta)$. Hint: Rao-Blackwell Theorem.