Midterm Exam #2

MATH 60062/70062: Mathematical Statistics II

March 24, 2022

- Please turn off your phone.
- Print your name clearly at the top of this page.
- This is a closed-book and closed-notes exam.
- This exam contains 4 questions. There are 100 points in total.
- You have 75 minutes to complete the exam.
- Please show your work and explain all of your reasoning.
- You must work by yourself. Do not communicate in any way with others.

1. (15 points) Give full definitions for the following concepts:
    a. Power function of a hypothesis test
    b. Level α test
    c. Uniformly most powerful (UMP) level α test
    d. Monotone likelihood ratio
    e. Unbiased test

2. (15 points) Prove the sufficiency part of the Neyman-Pearson Lemma. Consider testing

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_1.$$

The PDFs/PMFs of $\boldsymbol{X} = (X_1, \ldots, X_n)$ corresponding to θ_0 and θ_1 are $f_{\boldsymbol{X}}(\boldsymbol{x} \mid \theta_0)$ and $f_{\boldsymbol{X}}(\boldsymbol{x} \mid \theta_1)$, respectively. Define the test function

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \frac{f_{\boldsymbol{X}}(\boldsymbol{x}|\theta_1)}{f_{\boldsymbol{X}}(\boldsymbol{x}|\theta_0)} > k \\ 0 & \text{if } \frac{f_{\boldsymbol{X}}(\boldsymbol{x}|\theta_1)}{f_{\boldsymbol{X}}(\boldsymbol{x}|\theta_0)} < k, \end{cases}$$

for $k \geq 0$, where $\alpha = P_{\theta_0}(\boldsymbol{X} \in R) = E_{\theta_0}[\phi(\boldsymbol{X})]$. Show that $\phi(\boldsymbol{x})$ is a most powerful level α test.

3. (35 points) Suppose that $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$, where $-\infty < \mu < \infty$ and $\sigma^2 > 0$. Both parameters are unknown. Consider testing

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0.$$

Let $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$, where $\bar{X}$ is the sample mean. The size α one-sample two-sided t-test rejects H_0 when

$$|\bar{x} - \mu_0| \geq t_{n-1,\alpha/2} \sqrt{s^2/n}.$$

a. (20 points) Show that the test can be derived as a likelihood ratio test.

b. (15 points) Show that the p-value for this two-sided t-test is

$$p(\boldsymbol{x}) = 2P\left(T_{n-1} \geq \left|\frac{\bar{x} - \mu_0}{s/\sqrt{n}}\right|\right).$$

4. (35 points) Suppose $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma_0^2)$, where $-\infty < \mu < \infty$ and σ_0^2 is known. Consider testing

$$H_0 : \mu \geq \mu_0 \quad \text{versus} \quad H_1 : \mu < \mu_0.$$

   a. (10 points) Find the UMP level α test.

   b. (10 points) Find the power function corresponding to the UMP level α test.

   c. (5 points) Is the UMP level α test unbiased for H_0 versus H_1?

   d. (5 points) Consider applying the test found in part (a) to test

$$H_0' : \mu = \mu_0 \quad \text{versus} \quad H_1' : \mu \neq \mu_0.$$

   Show that the test is a level α test for H_0' versus H_1'.

   e. (5 points) Is the test found in part (a) unbiased for $H_0' : \mu = \mu_0$ versus $H_1' : \mu \neq \mu_0$? Why?