Midterm Exam #2

MATH 60061/70061: Mathematical Statistics I

November 9, 2020

This is a take-home exam. Please submit your answers as a **PDF** file to Blackboard by **11:59 p.m. on November 10**. Please show your work and write legibly. Your grade will be based on the correctness of your answers and the clarity with which you express them. Collaboration, copying, and cheating are not allowed.

Problems

1. (20 points) Let $Z \sim \mathcal{N}(0, 1)$.

   a. Show that
   $$P(|Z| > r) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-r^2/2}}{r}$$

   b. Suppose $X_1, \ldots, X_n$ are iid $\mathcal{N}(0, 1)$ random variables. Bound $P(|\bar{X}_n| > r)$ using the above inequality, where $\bar{X}_n$ is the sample mean. Compare to the Chebyshev's bound.

2. (20 points) The random variables $X_1, \ldots, X_n$ are *exchangeable* if any permutation of any subset of them of k ($k \leq n$) has the same distribution. Given P, let $X_1, \ldots, X_n$ be iid $\text{Bern}(P)$. Let $P \sim \text{Unif}(0, 1)$.

   a. Show that the marginal distribution of any k of the X's is the same as
   $$P(X_1 = x_1, \ldots, X_k = x_k) = \int_0^1 p^t (1-p)^{k-t} dp = \frac{t!(k-t)!}{(k+1)!},$$
   where $t = \sum_{i=1}^k x_i$. Hence, $X_1, \ldots, X_n$ are exchangeable.

   b. Show that, marginally,
   $$P(X_1 = x_1, \ldots, X_n = x_n) \neq \prod_{i=1}^n P(X_i = x_i),$$
   so $X_1, \ldots, X_n$ are exchangeable but not iid.

3. (15 points) Let $X_1, \ldots, X_n$ be iid $\text{Unif}(0, 1)$ random variables. Let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(n)}$ and $X_{(n)}$ are independent random variables.

4. (15 points) Let X_i, $i = 1, 2, 3$, be independent with $\mathcal{N}(i, i^2)$ distributions. For each of the following situations, use the X_i's to construct a statistic with the indicated distribution.

a. Chi-Squared distribution with 3 degrees of freedom.

b. t distribution with 2 degrees of freedom.

c. F distribution with 2 and 1 degrees of freedom.

5. (20 points) Let $\lambda_n = 1/n$ for $n = 1, 2, \ldots$. Let $X_n \sim \text{Pois}(\lambda_n)$.

   a. Show that X_n converges in probability to 0 as $n \to \infty$.

   b. Show that $Y_n = nX_n$ converges in probability to 0 as $n \to \infty$.

6. (10 points) Suppose $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$ random variables, where $-\infty < \mu < \infty$ and $\sigma^2 > 0$. Define

$$S_n^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2,$$

where $\bar{X}_n$ is the sample mean. Show that S_n^2 converges in probability to σ^2, as $n \to \infty$.