Final Exam

MATH 60061/70061: Mathematical Statistics I

December 14, 2020

This is a take-home exam. Please submit your answers as a **PDF** file to Blackboard by **11:59 p.m. on December 15**. Please show your work and write legibly. Your grade will be based on the correctness of your answers and the clarity with which you express them. Collaboration, copying, and cheating are not allowed.

Problems

1. (15 points) Consider iid random variables $X_i \sim \mathcal{N}(0,1)$, $i = 1, \dots, n$.

   a. (10 points) Show that the random variables $U = X_1^2 + \cdots + X_n^2$ and $V_i = X_i^2/U$ are independent, for every $i = 1, \dots, n$.

   b. (5 points) Find the PDF of V_i.

2. (10 points) An epidemiologist is studying the prevalence of a certain disease. She draws a county at random from the United States, and then randomly selects n people from that county. Let X be the number of those people who have the disease. Let Q be the proportion of people in that county with the disease, which is a random variable since it varies from county to county. Given $Q = q$, X is distributed as a Binomial distribution, $X|Q = q \sim \text{Bin}(n, q)$. Suppose that the random variable Q has a continuous Uniform distribution on $[0, 1]$. The distribution about X can then be summarized as a hierarchical model:

$$\begin{aligned} Q &\sim \text{Unif}(0,1) \\ X|Q = q &\sim \text{Bin}(n,q). \end{aligned}$$

   a. Find the unconditional expectation of X, $E(X)$.

   b. Find the unconditional variance of X, $\text{Var}(X)$.

3. (15 points) Suppose that $X_1, \dots, X_5$ is an iid sample of size 5 from a Normal distribution with mean $\mu = -1$ and variance $\sigma^2 = 4$, i.e., $X_i \sim \mathcal{N}(-1, 4)$ for $i = 1, \dots, 5$.

   a. What is the sampling distribution of $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$? What is the sampling distribution of $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$?

   b. Find a statistic as a function of $X_1, \dots, X_5$ that has a t_4 sampling distribution.

   c. Find a statistic as a function of $X_1, \dots, X_5$ that has an $F_{2,3}$ sampling distribution.

4. (15 points) Suppose that $X_1, \ldots, X_n$ are identically distributed with $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2 < \infty$, and that

$$\text{Cov}(X_i, X_j) = \begin{cases} c & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 0. \end{cases}$$

Show that $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ converges in probability to μ.

5. (15 points) Suppose that $X_1, \ldots, X_n$ are iid Exponential random variables with mean 1. Let

$$Y_n = X_{(n)} - \log n,$$

where $X_{(n)}$ is the nth order statistic. Show that Y_n converges in distribution and find the limiting distribution.

6. (15 points) Suppose that $X_1, \ldots, X_n$ is an iid sample from

$$f_X(x \mid \mu, \lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left[-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right],$$

where $x > 0$. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $T = n/(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_i} - \frac{1}{\bar{X}})$. Show that $(\bar{X}, T)$ is sufficient and complete.

7. (15 points) Suppose that $X_1, \ldots, X_n$ are an iid sample from $\text{Unif}(0, \theta)$, where $\theta > 0$. The order statistics are denoted by $X_{(1)}, \ldots, X_{(n)}$. Find $E\left(\frac{X_{(1)}+X_{(2)}}{X_{(n)}}\right)$.