Lecture 15: Uniformly Most Powerful Tests

Mathematical Statistics II, MATH 60062/70062

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Reference: Casella & Berger, 8.3.2

Recap: Monotone likelihood ratio

A family of PDFs/PMFs $\{g_T(t \mid \theta) : \theta \in \Theta\}$ for a univariate random variable T has a **monotone likelihood ratio (MLR)** if for all $\theta_2 > \theta_1$, the ratio

$$\frac{g_T(t \mid \theta_2)}{g_T(t \mid \theta_1)}$$

is a nondecreasing function of t on $\{t : g_T(t \mid \theta_1) > 0 \text{ or } g_T(t \mid \theta_2) > 0\}$.

Note: If $T \sim g_T(t \mid \theta) = h(t)c(\theta)e^{w(\theta)t}$ and $w(\theta)$ is a nondecreasing function of θ, then $\{g_T(t \mid \theta) : \theta \in \Theta\}$ has an MLR.

Recap: Karlin-Rubin Theorem

Consider testing

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0.$$

Suppose that T is a sufficient statistic for θ and the family $\{g_T(t \mid \theta) : \theta \in \Theta\}$ has an MLR. Then the test that rejects H_0 if and only if $T > t_0$ is a UMP level α test, where

$$\alpha = P_{\theta_0}(T > t_0).$$

Similarly, when testing

$$H_0 : \theta \geq \theta_0 \quad \text{versus} \quad H_1 : \theta < \theta_0,$$

the test that rejects H_0 if and only if $T < t_0$ is a UMP level α test, where $\alpha = P_{\theta_0}(T < t_0)$.

Bernoulli/Binomial UMP test

Suppose $X_1, \ldots, X_n$ are iid $\mathrm{Bern}(\theta)$, where $0 < \theta < 1.$ Consider testing

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0.$$

We know that

$$T = \sum_{i=1}^{n} X_i$$

is sufficient for θ and $T \sim \mathrm{Bin}(n, \theta)$, and $\{g_T(t \mid \theta) : 0 < \theta < 1\}$ has an MLR.

Therefore, by the Karlin-Rubin Theorem the UMP level α test is

$$\phi(t) = I(t > t_0),$$

where t_0 satisfies

$$\alpha = P_{\theta_0}(T > t_0) = \sum_{t=\lfloor t_0 \rfloor + 1}^{n} \binom{n}{t} \theta_0^t (1 - \theta_0)^{n-t}.$$

Normal UMP test

Suppose $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma_0^2)$, where $-\infty < \mu < \infty$ and σ_0^2 is known. Consider testing

$$H_0 : \mu \leq \mu_0 \quad \text{versus} \quad H_1 : \mu > \mu_0.$$

We know that $T(\boldsymbol{X}) = \bar{X}$ is sufficient for μ and $T \sim \mathcal{N}(\mu, \sigma_0^2/n)$, and $\{g_T(t \mid \mu) : -\infty < \mu < \infty\}$ has an MLR (exercise).

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By the Karlin-Rubin Theorem, the UMP level α test is

$$\phi(t) = I(t > t_0),$$

where t_0 satisfies

$$\alpha = P_{\mu_0}(T > t_0) = P\left(\frac{\bar{X} - \mu_0}{\sigma_0/\sqrt{n}} > \frac{t_0 - \mu_0}{\sigma_0/\sqrt{n}}\right) = 1 - F_Z\left(\frac{t_0 - \mu_0}{\sigma_0/\sqrt{n}}\right)$$

$$\implies t_0 = \frac{z_\alpha \sigma_0}{\sqrt{n}} + \mu_0.$$

Thus, the UMP level α test function for H_0 versus H_1 is

$$\phi(\boldsymbol{x}) = I\left(\bar{x} > \frac{z_\alpha \sigma_0}{\sqrt{n}} + \mu_0\right)$$

and the corresponding power function is

$$\begin{aligned}
\beta(\mu) &= E_\mu[\phi(\boldsymbol{X})] \\
&= P_\mu\left(\bar{X} > \frac{z_\alpha \sigma_0}{\sqrt{n}} + \mu_0\right) \\
&= P_\mu\left(\frac{\bar{X} - \mu}{\sigma_0/\sqrt{n}} > z_\alpha + \frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}}\right) \\
&= 1 - F_Z\left(z_\alpha + \frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}}\right).
\end{aligned}$$

Nonexistence of UMP test

Using the Karlin-Rubin Theorem, we can find UMP level α tests for

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0.$$

or

$$H_0 : \theta \geq \theta_0 \quad \text{versus} \quad H_1 : \theta < \theta_0.$$

Unfortunately, with a *two-sided* H_1 ($H_1 : \theta \neq \theta_0$), UMP tests do not exist.

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Suppose $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma_0^2)$, where $-\infty < \mu < \infty$ and σ_0^2 is known. Consider testing

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0.$$

If there exists a UMP test, then for all $\mu \neq \mu_0$ the power function of the test should be greater than the power function of any other level α test.

Nonexistence of UMP test

It is possible to find UMP tests when H_1 is *one-sided*.

- For $H_0' : \mu \le \mu_0$ versus $H_1' : \mu > \mu_0$, the UMP level α test function is
$$\phi'(\boldsymbol{x}) = I\left(\bar{x} > \frac{z_\alpha \sigma_0}{\sqrt{n}} + \mu_0\right)$$
and the corresponding power function is
$$\beta'(\mu) = 1 - F_Z\left(z_\alpha + \frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}}\right).$$
- For $H_0'' : \mu \ge \mu_0$ versus $H_1'' : \mu < \mu_0$, the UMP level α test function is
$$\phi''(\boldsymbol{x}) = I\left(\bar{x} < -\frac{z_\alpha \sigma_0}{\sqrt{n}} + \mu_0\right)$$
and the corresponding power function is
$$\beta''(\mu) = F_Z\left(-z_\alpha + \frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}}\right).$$

Note that both are also size (and level) α tests for H_0 versus H_1 because

$$\sup_{\mu \in \Theta_0} \beta'(\mu) = \beta'(\mu_0) = 1 - F_Z(z_\alpha) = \alpha$$

and

$$\sup_{\mu \in \Theta_0} \beta''(\mu) = \beta''(\mu_0) = F_Z(-z_\alpha) = \alpha.$$

Therefore,

- $\phi'(\boldsymbol{x})$ is UMP level α when $\mu > \mu_0$
- $\phi''(\boldsymbol{x})$ is UMP level α when $\mu < \mu_0$.

Since $\phi'(\boldsymbol{x}) \neq \phi''(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathcal{X}$, no UMP test exists for H_0 versus H_1.

Unbiased tests

When no UMP level α test within the class of all tests, we could further restrict our attention to a smaller class, the class of unbiased tests.

Consider testing

$$H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \in \Theta_0^c.$$

A test with power function $\beta(\theta)$ is **unbiased** if $\beta(\theta_1) \geq \beta(\theta_0)$ for all $\theta_1 \in \Theta_0^c$ and for all $\theta_0 \in \Theta_0$. In other words, the power is always larger in the alternative parameter space than it is in the null parameter space.

Uniformly most powerful unbiased tests

The **uniformly most powerful unbiased (UMPU)** level α test has power function that satisfies

$$\beta(\theta) \geq \beta^*(\theta)$$

for all $\theta \in \Theta_0^c$, where $\beta^*(\theta)$ is the power function of any other unbiased level α test.

Suppose $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma_0^2)$, where $-\infty < \mu < \infty$ and σ_0^2 is known. Consider testing

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0.$$

The UMPU level α test rejects H_0 if and only if

$$\frac{\bar{x} - \mu_0}{\sigma_0/\sqrt{n}} < -z_{\alpha/2} \quad \text{or} \quad \frac{\bar{x} - \mu_0}{\sigma_0/\sqrt{n}} > z_{\alpha/2}$$

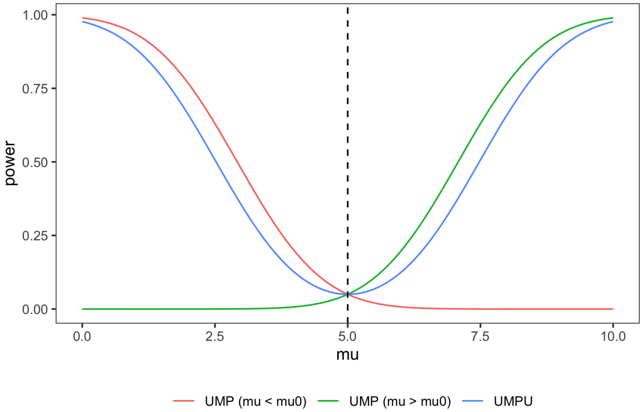
The rejection region of the UMPU level α test is

$$R = \left\{ \boldsymbol{x} \in \mathcal{X} : \left| \frac{\bar{x} - \mu_0}{\sigma_0/\sqrt{n}} \right| > z_{\alpha/2} \right\}.$$

The power function of the test is

$$\begin{aligned}
\beta(\mu) &= P_\mu(\boldsymbol{X} \in R) \\
&= P_\mu \left(\frac{\bar{X} - \mu_0}{\sigma_0/\sqrt{n}} > z_{\alpha/2} \text{ or } \frac{\bar{X} - \mu_0}{\sigma_0/\sqrt{n}} < -z_{\alpha/2} \right) \\
&= P \left(Z > z_{\alpha/2} + \frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}} \text{ or } Z < -z_{\alpha/2} + \frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}} \right) \\
&= 1 - F_Z \left(z_{\alpha/2} + \frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}} \right) + F_Z \left(-z_{\alpha/2} + \frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}} \right).
\end{aligned}$$

Power function of $\alpha = 0.05$ test with parameters $n = 10$, $\mu_0 = 5$, $\sigma_0^2 = 4$:



Appendix: Proof of Karlin-Rubin Theorem

Consider testing

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0.$$

Suppose that T is a sufficient statistic for θ and the family $\{g_T(t \mid \theta) : \theta \in \Theta\}$ has an MLR. Then the test that rejects H_0 if and only if $T > t_0$ is a UMP level α test, where

$$\alpha = P_{\theta_0}(T > t_0).$$

Lemma 1: If both $g(x)$ and $h(x)$ are nondecreasing functions of x, then

$$\mathrm{Cov}[g(X), h(X)] \geq 0.$$

Let X_1 and X_2 be iid with the same distribution as X. Then

$$\begin{aligned}
& E[(h(X_1) - h(X_2))(g(X_1) - g(X_2))] \\
&= E[h(X_1)g(X_1)] - E[h(X_1)g(X_2)] - E[h(X_2)g(X_1)] + E[h(X_2)g(X_2)] \\
&= \underbrace{E[h(X_1)g(X_1)] - E[h(X_1)]E[g(X_2)]}_{\mathrm{Cov}[g(X),h(X)]} \underbrace{-E[h(X_2)]E[g(X_1)] + E[h(X_2)g(X_2)]}_{\mathrm{Cov}[g(X),h(X)]}
\end{aligned}$$

Therefore

$$\mathrm{Cov}[g(X), h(X)] = \frac{1}{2} E[(h(X_1) - h(X_2))(g(X_1) - g(X_2))] \geq 0.$$

Lemma 2: Suppose the family $\{g_T(t \mid \theta) : \theta \in \Theta\}$ has an MLR. If $\phi(t)$ is a nondecreasing function of t, then $E_\theta[\phi(T)]$ is a nondecreasing function of θ.

Suppose $\theta_2 > \theta_1$. Because $\{g_T(t \mid \theta) : \theta \in \Theta\}$ has an MLR,

$$\frac{g_T(t \mid \theta_2)}{g_T(t \mid \theta_1)}$$

is a nondecreasing function of t for $\theta_2 > \theta_1$. By Lemma 1, we know

$$\mathrm{Cov}_{\theta_1}\left[\phi(T), \frac{g_T(T \mid \theta_2)}{g_T(T \mid \theta_1)}\right] \geq 0$$

$$\begin{aligned}
&\implies \underbrace{E_{\theta_1}\left[\phi(T)\frac{g_T(T \mid \theta_2)}{g_T(T \mid \theta_1)}\right]}_{E_{\theta_2}[\phi(T)]} \geq E_{\theta_1}[\phi(T)]\,\underbrace{E_{\theta_1}\left[\frac{g_T(T \mid \theta_2)}{g_T(T \mid \theta_1)}\right]}_{1} \\
&\implies E_{\theta_2}[\phi(T)] \geq E_{\theta_1}[\phi(T)].
\end{aligned}$$

Now, consider $\phi(t) = I(t > t_0)$, where t_0 is fixed. Clearly, $\phi(t)$ is a nondecreasing function of t. From Lemma 2, we know that

$$E_\theta[\phi(T)] = E_\theta[I(T > t_0)] = P_\theta(T > t_0)$$

is a nondecreasing function of θ.

Consider testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$, we have shown that the power function

$$\beta(\theta) = P_\theta(T > t_0)$$

is a nondecreasing function of θ.

In the Karlin-Rubin Theorem, the condition $\alpha = P_{\theta_0}(T > t_0)$ must be satisfied, where

$$\alpha = \sup_{\theta \leq \theta_0} \beta(\theta) = \beta(\theta_0) = P_{\theta_0}(T > t_0).$$

This means that $\phi(t) = I(t > t_0)$ is a size α (and therefore level α) test function. All that remains is to show that this test is uniformly most powerful among level α tests.

Let $\phi^*(\boldsymbol{x})$ be any other level α test for

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0.$$

Fix $\theta_1 > \theta_0$ and consider testing

$$H_0^* : \theta = \theta_0 \quad \text{versus} \quad H_1^* : \theta = \theta_1$$

instead. Because $\phi^*(\boldsymbol{x})$ is a level α test for H_0 versus H_1,

$$E_{\theta_0}[\phi^*(\boldsymbol{X})] \leq \sup_{\theta \leq \theta_0} E_\theta[\phi^*(\boldsymbol{X})] \leq \alpha.$$

This means that $\phi^*(\boldsymbol{x})$ is also a level α test for H_0^* versus H_1^*.

Consider the ratio

$$\frac{g_T(t \mid \theta_1)}{g_T(t \mid \theta_0)}$$

and define

$$k = \inf_{t \in \mathcal{T}} \frac{g_T(t \mid \theta_1)}{g_T(t \mid \theta_0)},$$

where $\mathcal{T} = \{t : t > t_0 \text{ and either } g_T(t \mid \theta_1) > 0 \text{ or } g_T(t \mid \theta_0) > 0\}$. It follows that

$$T > t_0 \iff \frac{g_T(t \mid \theta_1)}{g_T(t \mid \theta_0)} > k.$$

By the Neyman-Pearson Lemma with a sufficient statistic T, we know that $\phi(t)$ is the most powerful level α test for H_0^* versus H_1^*. That is

$$E_{\theta_1}[\phi(T)] \geq E_{\theta_1}[\phi^*(\boldsymbol{X})].$$

Because $\theta_1 > \theta_0$ was chosen arbitrarily,

$$E_\theta[\phi(T)] \geq E_\theta[\phi^*(\boldsymbol{X})]$$

holds for all $\theta > \theta_0$.