Homework 1

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1.

a. First we prove $T(X) = \prod_{i=1}^n X_i$ is a minimum sufficient statistic. Since

$$\frac{f_{\boldsymbol{X}(\boldsymbol{x}|\theta)}}{f_{\boldsymbol{X}(\boldsymbol{y}|\theta)}} = \frac{\theta^n(\prod_{i=1}^n x_i)^{\theta-1}}{\theta^n(\prod_{i=1}^n y_i)^{\theta-1}} = \frac{(\prod_{i=1}^n x_i)^{\theta-1}}{(\prod_{i=1}^n y_i)^{\theta-1}}$$

does not depend on θ only when $\prod_{i=1}^n x_i = \prod_{i=1}^n y_i$, $T(X) = \prod_{i=1}^n X_i$ is a minimum sufficient statistic.

Since $T(X)$ is a minimum sufficient statistic, it is a function of any sufficient statistic. Note that $T(X)$ is not a function of $\sum_{i=1}^n X_i$. Thus, $\sum_{i=1}^n X_i$ is not a sufficient statistic.

b. Let $y_i = -\ln x_i$. Since $0 < x_i < 1$, we have $y_i > 0$.

$$f_{Y_i}(y_i|\theta) = f_{X_i}(e^{-y_i}|\theta)|\left|\frac{de^{-y_i}}{dy_i}\right|| = \theta e^{-y_i(\theta-1)}e^{-y_i} = \theta e^{-\theta y_i}$$

Note that $y_i \sim \text{Expo}(\theta)$. Let $Y = \sum_{i=1}^n y_i = -\sum_{i=1}^n \ln x_i$, we know that $Y \sim \text{Gamma}(n, \theta)$. Thus,

$$E_\theta(g(Y)) = \int_0^\infty g(Y)\frac{1}{\Gamma(n)}e^{-\theta Y}(\theta Y)^n \frac{dY}{Y}$$

Thus $E_\theta(g(Y)) = 0$ only when $g(Y) = 0$. $Y = -\sum_{i=1}^n \ln X_i$ is a complete statistic for θ.

2.

a.

Since $X_i \sim \text{Unif}[0, \theta]$, the first population moment is $E(X_i) = \frac{\theta}{2}$. The first sample moment is $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$. Therefore, $\hat{\theta} = 2\bar{X} = \frac{2}{n}\sum_{i=1}^n X_i$.

b.

The likelihood function:

$$L(\theta|\boldsymbol{x}) = f_{\boldsymbol{X}}(\boldsymbol{x}|\theta) = \prod_{i=1}^n f_{\boldsymbol{X}}(x_i|\theta) = \prod_{i=1}^n \frac{1}{\theta}I(0 \le x_i \le \theta) = \frac{1}{\theta^n}I(\theta \ge x_{(n)})\prod_{i=1}^n I(x_i \ge 0)$$

I is an indicator function where $I(x) = 1$ if x is true and 0 otherwise. $x_{(n)} = \max\{x_1, x_2, ..., x_n\}$.

Note that $L(\theta|\boldsymbol{x})$ is maximized by taking $\theta = x_{(n)}$. Thus, $\hat{\theta} = X_{(n)}$.

3.

a.

$$f(\boldsymbol{X}|\theta) = \prod_{i=1}^{n} f(\boldsymbol{x_i}|\theta) = \theta^n (\prod_{i=1}^{n} x_i)^{-2}$$

Let $t = \prod_{i=1}^{n} x_i$. By Factorization Theorem, $f(\boldsymbol{X}|\theta) = f(t|\theta)h(\boldsymbol{x})$. Define $h(\boldsymbol{x}) = 1$ and $f(t|\theta) = \theta^n t^{-2}$, the factorization holds. Thus $T(\boldsymbol{X}) = \prod_{i=1}^{n} X_i$ is a sufficient statistic.

b. We cannot find a method of moments estimator because this distribution has neither mean nor variance.

The first population moment is

$$E(X) = \int_{\theta}^{\infty} x f(x|\theta) dx = \int_{\theta}^{\infty} \theta x^{-1} dx = [\theta \ln x]_{\theta}^{\infty} = \infty$$

The second population moment is

$$E(X^2) = \int_{\theta}^{\infty} x^2 f(x|\theta) dx = \int_{\theta}^{\infty} \theta dx = \infty$$

c. The likelihood function:

$$L(\theta|\boldsymbol{x}) = \prod_{i=1}^{n} f_{\boldsymbol{X}}(x_i|\theta) = \theta^n (\prod_{i=1}^{n} x_i)^{-2} \prod_{i=1}^{n} I(\theta \leq x_i) = \theta^n (\prod_{i=1}^{n} x_i)^{-2} I(\theta \leq x_{(1)})$$

where $x_{(1)} = \min\{x_1, x_2, ..., x_n\}$.

Taking the first derivative,

$$\frac{\partial}{\partial \theta} L(\theta|\boldsymbol{x}) = n\theta^{n-1} (\prod_{i=1}^{n} x_i)^{-2} I(\theta \leq x_{(1)})$$

Note that $\frac{\partial}{\partial \theta} L(\theta|\boldsymbol{x}) > 0$ when $0 < \theta \leq x_{(1)}$, which means that $L(\theta|\boldsymbol{x})$ is monotonic increasing. We maximize $L(\theta|\boldsymbol{x})$ by setting $\theta = x_{(1)}$. Thus, $\hat{\theta} = X_{(1)}$.

4.

a. By Bayes's Theorem,

$$f(\theta|\boldsymbol{x}) = \frac{f(\boldsymbol{x}|\theta)\pi(\theta)}{\int_{\Theta} f(\boldsymbol{x}|\theta)\pi(\theta) d\theta}$$

Since $X \sim \mathcal{N}(\theta, \sigma^2)$ and $\theta \sim \mathcal{N}(\mu, \tau^2)$, we have $f(\boldsymbol{x}|\theta) = \frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{(x-\theta)^2}{2\sigma^2}\}$ and $\pi(\theta) = \frac{1}{\sqrt{2\pi}\tau}\exp\{-\frac{(x-\mu)^2}{2\tau^2}\}$.

$$
\begin{aligned}
f(\theta|\boldsymbol{x})\pi(\theta) &= \frac{1}{2\pi\sigma\tau}\exp\{-\frac{1}{2}[\frac{(x-\theta)^2}{\sigma^2} - \frac{(\theta-\mu)^2}{\tau^2}]\} \\
&= \frac{1}{2\pi\sigma\tau}\exp\{-\frac{1}{2}[\frac{(\tau^2+\sigma^2)\theta^2 - 2(x\tau^2+\mu\sigma^2)\theta + (\tau^2x^2+\sigma^2\mu^2)}{\sigma^2\tau^2}]\} \\
&= \frac{1}{2\pi\sigma\tau}\exp\{-\frac{1}{2}[\frac{(\theta - \frac{x\tau^2+\mu\sigma^2}{\tau^2+\sigma^2})^2}{\frac{\sigma^2\tau^2}{\tau^2+\sigma^2}} + \frac{(x-\mu)^2}{\tau^2+\sigma^2}]\}
\end{aligned}
$$

$$
\begin{aligned}
\int_\Theta f(\boldsymbol{x}|\theta)\pi(\theta)d\theta &= \int_{-\infty}^{\infty}\frac{1}{2\pi\sigma\tau}\exp\{-\frac{1}{2}[\frac{(\theta - \frac{x\tau^2+\mu\sigma^2}{\tau^2+\sigma^2})^2}{\frac{\sigma^2\tau^2}{\tau^2+\sigma^2}} + \frac{(x-\mu)^2}{\tau^2+\sigma^2}]\}d\theta \\
&= \frac{1}{2\pi\sigma\tau}\exp\{-\frac{1}{2}\frac{(x-\mu)^2}{\tau^2+\sigma^2}\}\sqrt{2\pi}\frac{\sigma\tau}{\sqrt{\tau^2+\sigma^2}} \cdot \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\frac{\sigma\tau}{\sqrt{\tau^2+\sigma^2}}}\exp\{-\frac{1}{2}[\frac{(\theta - \frac{x\tau^2+\mu\sigma^2}{\tau^2+\sigma^2})^2}{\frac{\sigma^2\tau^2}{\tau^2+\sigma^2}}]\}d\theta \\
&= \frac{1}{2\pi\sigma\tau}\exp\{-\frac{1}{2}\frac{(x-\mu)^2}{\tau^2+\sigma^2}\}\sqrt{2\pi}\frac{\sigma\tau}{\sqrt{\tau^2+\sigma^2}}
\end{aligned}
$$

Thus,

$$
f(\theta|\boldsymbol{x}) = \frac{f(\boldsymbol{x}|\theta)\pi(\theta)}{\int_\Theta f(\boldsymbol{x}|\theta)\pi(\theta)d\theta} = \frac{1}{\sqrt{2\pi}\frac{\sigma\tau}{\sqrt{\tau^2+\sigma^2}}}\exp\{-\frac{1}{2}[\frac{(\theta - \frac{x\tau^2+\mu\sigma^2}{\tau^2+\sigma^2})^2}{\frac{\sigma^2\tau^2}{\tau^2+\sigma^2}}]\}
$$

Hence, $f(\theta|\boldsymbol{x})$ is a Normal distribution with mean $E(\theta|\boldsymbol{x}) = \frac{x\tau^2+\mu\sigma^2}{\tau^2+\sigma^2} = \frac{\tau^2}{\tau^2+\sigma^2}x + \frac{\sigma^2}{\tau^2+\sigma^2}\mu$ and variance $Var(\theta|\boldsymbol{x}) = \frac{\sigma^2\tau^2}{\tau^2+\sigma^2}$.

b. Since $X_1, X_2, ..., X_n$ are random variables iid from $\mathcal{N}(\theta, \sigma^2)$,

$$
\begin{aligned}
f(\theta|\boldsymbol{x}) &\propto f(\boldsymbol{x}|\theta)\pi(\theta) \\
&= (\frac{1}{\sqrt{2\pi}\sigma})^n\exp\{-\frac{\sum_{i=1}^n(x_i-\theta)^2}{2\sigma^2}\}\frac{1}{\sqrt{2\pi}\tau}\exp\{-\frac{(\theta-\mu)^2}{2\tau^2}\} \\
&= \frac{1}{(\sqrt{2\pi})^{n+1}\sigma^2\tau}\exp\{-\frac{1}{2}[\frac{\sum_{i=1}^n(x_i-\theta)^2}{\sigma^2} + \frac{(\theta-\mu)^2}{\tau^2}]\} \\
&= \frac{1}{(\sqrt{2\pi})^{n+1}\sigma^2\tau}\exp\{-\frac{1}{2}[\frac{(\tau^2n+\sigma^2)\theta^2 - 2\theta(\tau^2\sum_{i=1}^n x_i + \sigma^2\mu) + \tau^2\sum_{i=1}^n x_i^2 + \sigma^2\mu^2}{\sigma^2\tau^2}]\} \\
&= \frac{1}{(\sqrt{2\pi})^{n+1}\sigma^2\tau}\exp\{-\frac{1}{2}[\frac{(\theta - \frac{\tau^2\sum_{i=1}^n x_i+\sigma^2\mu}{\tau^2n+\sigma^2})^2}{\frac{\sigma^2\tau^2}{\tau^2n+\sigma^2}} + \frac{\frac{\tau^2\sum_{i=1}^n x_i^2+\sigma^2\mu^2}{\tau^2n+\sigma^2} - (\frac{\tau^2\sum_{i=1}^n x_i+\sigma^2\mu}{\tau^2n+\sigma^2})^2}{\frac{\sigma^2\tau^2}{\tau^2n+\sigma^2}}]\}
\end{aligned}
$$

The Normal kernel shows that the posterior distribution of θ is $\mathcal{N}(\frac{\tau^2\sum_{i=1}^n X_i+\sigma^2\mu}{\tau^2n+\sigma^2}, \frac{\sigma^2\tau^2}{\tau^2n+\sigma^2})$.