# Homework #1

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**1.**

Supposing that $S = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{3, 4, 5\}$.

a. $A \cup B = B \cup A = \{1, 2, 3, 4\}$, $A \cap B = B \cap A = \{2, 3\}$

b. $(A \cup B) \cup C = A \cup (B \cup C) = \{1, 2, 3, 4, 5\}$, $(A \cap B) \cap C = A \cap (B \cap C) = \{3\}$

c. $(A \cup B)^C = (A^C \cap B^C) = \{5\}$, $(A \cap B)^C = (A^C \cup B^C) = \{1, 4, 5\}$

**2.**

The sampling is with replacement and without order. The problem is analogy to putting $n$ identical balls into $n$ boxes $a_1, a_2, ..., a_n$, which is again analogy to inserting $n$ identical balls into a line of $n - 1$ barriers.

Thus the number of all possible samples is $\binom{2n-1}{n}$.

**3.**

a. Let $P(A)$ be the probability that A is the guilty party, $P(B)$ be the probability that B has rare blood. If B does not have rare blood, then A must be the guilty party, otherwise A has 1/2 probability. Therefore,

$$P(A) = P(A|B)P(B) + P(A|B^C)P(B^C) = 0.5 \cdot 0.1 + 1 \cdot 0.9 = 0.95$$

b. By given the new information, the probability that $B$ has rare blood is the conditional probability $P(B|A)$. Thus

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} = \frac{0.5 \cdot 0.1}{0.95} = \frac{1}{19}$$

**4.**

a. Since we pick chip without replacement, $P(X = 0) = \frac{5}{100}$, $P(X = 1) = \frac{95}{100} \cdot \frac{5}{99}$, $P(X = 2) =$

$\frac{95}{100} \cdot \frac{94}{99} \cdot \frac{5}{98} \ldots$ Thus the PMF is

$$\begin{aligned}
P(X = x) &= \frac{5}{100} \cdot \frac{95 \cdot 94 \ldots \cdot (96 - x)}{99 \cdot 98 \ldots \cdot (100 - x)} \\
&= \frac{1}{20} \frac{\frac{95!}{(95-x)!}}{\frac{99!}{(99-x)!}} \\
&= \frac{1}{20} \frac{\frac{95!}{(95-x)!x!}}{\frac{99!}{(99-x)!x!}} \\
&= \frac{1}{20} \frac{\binom{95}{x}}{\binom{99}{x}}
\end{aligned}$$

where the possible values of $X$ are $0, 1, 2, \ldots, 95$.

The expected value is

$$\begin{aligned}
E(X) &= \sum_{x=0}^{95} x \cdot \frac{1}{20} \frac{\binom{95}{x}}{\binom{99}{x}} \\
&= \sum_{x=0}^{95} \frac{95}{20} \frac{\binom{94}{x-1}}{\binom{99}{x}} \\
&=
\end{aligned}$$

b. Since we pick chips with replacement, $X$ follows geometric distribution $Geom(p)$, where $p = 95/100 = 0.95$ is the probability of picking a good chip.

The PMF is

$$P(X = x) = p^x(1 - p) = 0.95^x \cdot 0.05$$

where the possible values of $X$ are $0, 1, 2, \ldots, 95$.

The expected value is

$$E(X) = \sum_{x=1}^{\infty} xp^x(1 - p) = \frac{p}{1 - p} = \frac{0.95}{0.05} = 19$$

c. $X$ follows hypergeometric distribution $HGeom(g, b, n)$, where $g = 95$ is the number of good chips, $b = 5$ is the number of bad chips, $n = 20$ is the number of chips picked out.

The PMF is

$$P(X = x) = \frac{\binom{g}{x}\binom{b}{n-x}}{\binom{g+b}{n}} = \frac{\binom{95}{x}\binom{5}{20-x}}{\binom{100}{20}}$$

where the possible values of $X$ are $15, 16, 17, 18, 19, 20$.

The expected value is

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{n} \frac{x\binom{g}{x}\binom{b}{n-x}}{\binom{g+b}{n}} \\
&= \sum_{x=0}^{n} \frac{g\binom{g-1}{x-1}\binom{b}{n-x}}{\binom{g+b}{n}} \\
&= \frac{gn}{g+b} \sum_{x=0}^{n} \frac{\binom{g-1}{x-1}\binom{b}{n-x}}{\binom{g+b-1}{n-1}} \\
&= \frac{gn}{g+b} \qquad [\sum_{x=0}^{n} \frac{\binom{g-1}{x-1}\binom{b}{n-x}}{\binom{g+b-1}{n-1}} = 1, \text{Vandermonde's identity}] \\
&= \frac{95 \cdot 20}{95 + 5} \\
&= 19
\end{aligned}
$$

**5.**

a. Let $X$ be the number of wining times. $X \sim Bin(n, p)$, where $n = 200$ and $p = 1/400$. Thus the probability of wining at least twice is

$$
\begin{aligned}
P(X \geq 2) &= 1 - P(X < 2) \\
&= 1 - P(X = 0) - P(X = 1) \\
&= 1 - \binom{200}{0}(\frac{1}{400})^0(\frac{399}{400})^{200} - \binom{200}{1}(\frac{1}{400})^1(\frac{399}{400})^{199}
\end{aligned}
$$

When $n$ is very large and $p$ is very small, the binomial distribution $Bin(n, p)$ converges to Poisson distribution $Pois(\lambda)$ where $\lambda = np$. Thus, $\binom{n}{p}p^k(1-p)^{n-k} \approx \frac{\lambda^k e^{-\lambda}}{k!}$.

$$
P(X \geq 2) \approx 1 - \frac{0.5^0 \cdot e^{-0.5}}{0!} - \frac{0.5^1 \cdot e^{-0.5}}{1!} \approx 0.09
$$

b. The probability of winning at least twice while knowing wining at least once is the conditional probability $P(X \geq 2 | X \geq 1)$.

$$
\begin{aligned}
P(X \geq 2 | X \geq 1) &= \frac{P(X \geq 2, X \geq 1)}{P(X \geq 1)} \\
&= \frac{P(X \geq 2)}{P(X \geq 1)} \\
&= \frac{1 - \binom{200}{0}(\frac{1}{400})^0(\frac{399}{400})^{200} - \binom{200}{1}(\frac{1}{400})^1(\frac{399}{400})^{199}}{1 - \binom{200}{0}(\frac{1}{400})^0(\frac{399}{400})^{200}} \\
&\approx \frac{1 - \frac{0.5^0 \cdot e^{-0.5}}{0!} - \frac{0.5^1 \cdot e^{-0.5}}{1!}}{1 - \frac{0.5^0 \cdot e^{-0.5}}{0!}} \\
&\approx 0.23
\end{aligned}
$$