Lecture 08: Joint Distributions

Mathematical Statistics I, MATH 60061/70061

Thursday September 23, 2021

Reference: Casella & Berger, 4.1-4.2

Example: chicken-egg

Suppose a chicken lays a random number of eggs, N, where $N \sim \mathrm{Pois}(\lambda)$. Each egg independently hatches with probability p and fails to hatch with probability $q = 1 - p$. Let X be the number of eggs that hatch and Y the number of eggs that do not hatch, so $X + Y = N$. What is the joint PMF of X and Y?

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By the LOTP,

$$\begin{aligned} P(X = i, Y = j) &= \sum_{n=0}^{\infty} P(X = i, Y = j \mid N = n) P(N = n) \\ &= P(X = i, Y = j \mid N = i + j) P(N = i + j) \end{aligned}$$

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$$
\begin{aligned}
P(X=i,Y=j) &= P(X=i \mid N=i+j)P(N=i+j) \\
&= \binom{i+j}{i} p^i q^j \cdot \frac{e^{-\lambda}\lambda^{i+j}}{(i+j)!} \\
&= \frac{e^{-\lambda p}(\lambda p)^i}{i!} \cdot \frac{e^{-\lambda q}(\lambda q)^j}{j!}
\end{aligned}
$$

The joint PMF factors into the product of the $\mathrm{Pois}(\lambda p)$ PMF (as a function of i) and the $\mathrm{Pois}(\lambda q)$ PMF (as a function of j).

1. X and Y are independent.
2. $X \sim \mathrm{Pois}(\lambda p)$ and $Y \sim \mathrm{Pois}(\lambda q)$.

Example: chicken-egg, continued

For a *fixed* number of eggs, knowing the number of hatched eggs would perfectly determine the number of unhatched eggs. But in this example, the number of eggs is *random*, following a Poisson, which makes X and Y *unconditionally independent*.

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The chicken-egg example gives the result as follows: if $N \sim \text{Pois}(\lambda)$ and $X \mid N = n \sim \text{Bin}(n, p)$, then $X \sim \text{Pois}(\lambda p)$, $Y = N - X \sim \text{Pois}(\lambda q)$, and X and Y are independent.

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This supplements the Binomial-Poisson result (see Lecture 5):

If $X \sim \mathrm{Pois}(\lambda p)$, $Y \sim \mathrm{Pois}(\lambda q)$, and X and Y are independent, then $N = X + Y \sim \mathrm{Pois}(\lambda)$ and $X \mid N = n \sim \mathrm{Bin}(n, p)$