Lecture 06: Continuous Distributions

Mathematical Statistics I, MATH 60061/70061

Thursday September 16, 2021

Reference: Casella & Berger, 3.3

Minimum of independent Exponentials

Let $X_1, \ldots, X_n$ be independent with $X_j \sim \mathrm{Expo}(\lambda_j)$. Let $L = \min(X_1, \ldots, X_n)$. What is the distribution of L?

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The PDF and CDF of $\text{Expo}(\lambda_j)$ are

$$f(x) = \lambda e^{-\lambda_j x}, \quad x > 0$$

and

$$F(x) = 1 - e^{-\lambda_j x}, \quad x > 0.$$

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Considering the *survival function* of L, $P(L > t)$,

$$\begin{aligned} P(L > t) &= P(\min(X_1, \ldots, X_n) > t) = P(X_1 > t, \ldots, X_n > t) \\ &= P(X_1 > t) \cdots P(X_n > t) = e^{-\lambda_1 t} \cdots e^{-\lambda_n t} \\ &= e^{-(\lambda_1 + \cdots + \lambda_n)t}. \end{aligned}$$

So, $L \sim \mathrm{Expo}(\lambda_1 + \cdots + \lambda_n)$.

Sum of Exponential RVs and sum of Gamma RVs

Let $X_1, \ldots, X_n$ be i.i.d. $\text{Expo}(\lambda)$. What is the distribution of $X_1 + \cdots + X_n$?

Let $X_1, \ldots, X_n$ be independent with $X_j \sim \text{Gamma}(a_j, \lambda)$. What is the distribution of $X_1 + \cdots + X_n$?

A sum of i.i.d. $\mathrm{Expo}(\lambda)$ RVs is a Gamma RV

Let $X_1, \ldots, X_n$ be i.i.d. $\mathrm{Expo}(\lambda)$. Then

$$X_1 + \cdots + X_n \sim \mathrm{Gamma}(n, \lambda).$$

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Let $X_1, \dots, X_n$ be i.i.d. $\mathrm{Expo}(\lambda)$. Then

$$X_1 + \cdots + X_n \sim \mathrm{Gamma}(n, \lambda).$$

The $\mathrm{Expo}(\lambda)$ MGF is $\frac{\lambda}{\lambda - t}$ for $t < \lambda$, so the MGF of $X_1 + \cdots + X_n$ is

$$M_n(t) = \left(\frac{\lambda}{\lambda - t}\right)^n, \quad \text{for } t < \lambda.$$

Let $Y \sim \mathrm{Gamma}(n, \lambda)$. The MGF of Y is

$$\begin{aligned} E(e^{tY}) &= \int_0^\infty e^{ty} \frac{1}{\Gamma(n)} (\lambda y)^n e^{-\lambda y} \frac{dy}{y} \\ &= \frac{\lambda^n}{(\lambda - t)^n} \int_0^\infty \frac{1}{\Gamma(n)} e^{-(\lambda - t)y} ((\lambda - t)y)^n \frac{dy}{y}. \end{aligned}$$

The expression inside the integral is the $\mathrm{Gamma}(n, \lambda - t)$ PDF, assuming $t < \lambda$. Since PDFs integrate to 1, we have

$$E(e^{tY}) = \left(\frac{\lambda}{\lambda - t}\right)^n, \quad \text{for } t < \lambda.$$

A sum of independent Gamma RVs

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The $\mathrm{Gamma}(a_j, \lambda)$ MGF is $\left(\frac{\lambda}{\lambda - t}\right)^{a_j}$ for $t < \lambda$, so the MGF of $X_1 + \cdots + X_n$ is

$$M_n(t) = \left(\frac{\lambda}{\lambda - t}\right)^{(a_1 + \cdots + a_n)}, \quad \text{for } t < \lambda.$$

This is the MGF of $\mathrm{Gamma}(\sum_{i=1}^{n} a_i, \lambda)$