Lecture 14: Sampling from Normal Distribution

Mathematical Statistics I, MATH 60061/70061

Tuesday October 26, 2021

Reference: Casella & Berger, 5.3

Change of variables in multiple dimensions

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a continuous random vector with joint PDF $f_{\boldsymbol{X}}$. Let $\boldsymbol{Y} = g(\boldsymbol{X})$, and mirror this by letting $\boldsymbol{y} = g(\boldsymbol{x})$. Suppose g is invertible, so we have $\boldsymbol{X} = g^{-1}(\boldsymbol{Y})$ and $\boldsymbol{x} = g^{-1}(\boldsymbol{y})$.

Suppose that all partial derivatives $\partial x_i / \partial y_j$ exist and are continuous, so we can form the **Jacobian matrix**

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{y}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}.$$

Also assume that the determinant of this Jacobian matrix is never 0. Then the joint PDF of $\boldsymbol{Y}$ is

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = f_{\boldsymbol{X}}\left(\boldsymbol{g}^{1}(\boldsymbol{y})\right) \cdot |\left|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{y}}\right||.$$

Inference about μ with a Normal random sample

Let $X_1, \ldots, X_n$ be a random sample from $\mathcal{N}(\mu, \sigma^2)$. We know

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1).$$

- If σ is known, we can use the above expression as a basis for inference about μ.
- If both μ and σ are unknown (as in most cases), we consider the quantity

  $$\frac{\bar{X} - \mu}{S/\sqrt{n}}$$

  as a basis for inference about μ.

t distribution

Note

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} = \frac{(\bar{X}-\mu)/(\sigma/\sqrt{n})}{\sqrt{S^2/\sigma^2}} = \frac{(\bar{X}-\mu)/(\sigma/\sqrt{n})}{\sqrt{[(n-1)S^2/\sigma^2]/(n-1)}}$$

- The numerator is a $\mathcal{N}(0,1)$ random variable.
- The denominator is a $\sqrt{\chi^2_{n-1}/(n-1)}$ random variable, independent of the numerator.

The quantity $(\bar{X}-\mu)/(S/\sqrt{n})$ is said to have t distribution with $n-1$ degrees of freedom. Equivalently, a random variable T has t distribution with p degrees of freedom, and we write $T \sim t_p$ if it has PDF is given by

$$f_T(t) = \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \frac{1}{(p\pi)^{1/2}} \frac{1}{(1+t^2/p)^{(p+1)/2}}, \quad -\infty < t < \infty.$$

t distribution

Suppose $U \sim \mathcal{N}(0,1)$, $V \sim \chi^2_p$, and U and V are independent. The joint PDF of (U, V) is

$$f_{U,V}(u,v) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \frac{1}{\Gamma(\frac{p}{2})2^{p/2}} v^{p/2-1} e^{-v/2},$$

for $-\infty < u < \infty$ and $v > 0$. Consider the bivariate transformation

$$\begin{aligned} T &= g_1(U,V) = \frac{U}{\sqrt{V/p}} \\ W &= g_2(U,V) = V. \end{aligned}$$

The support of (U, V) is the set $\{(u,v) : -\infty < u < \infty, v > 0\}$. The support of (T, W) is $\{(t,w) : -\infty < t < \infty, w > 0\}$. The transformation is one-to-one, so the inverse transformation exists and is given by

$$\begin{aligned} u &= g_1^{-1}(t,w) = t\sqrt{w/p} \\ v &= g_2^{-1}(t,w) = w. \end{aligned}$$

The Jacobian of the (inverse) transformation is $J = \sqrt{w/p}$.

The joint PDF of (T, W) is

$$\begin{aligned}
f_{T,W}(t,w) &= f_{U,V}(g_1^{-1}(t,w), g_2^{-1}(t,w))|J| \\
&= \frac{1}{\sqrt{2\pi}} e^{-(t\sqrt{w/p})^2/2} \frac{1}{\Gamma(\frac{p}{2})2^{p/2}} w^{p/2-1} e^{-w/2} |\sqrt{w/p}| \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{p}} e^{-(t\sqrt{w/p})^2/2} \frac{1}{\Gamma(\frac{p}{2})2^{p/2}} w^{\frac{p+1}{2}-1} e^{-w/2} \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{p}} \frac{1}{\Gamma(\frac{p}{2})2^{p/2}} w^{\frac{p+1}{2}-1} e^{-w(1+\frac{t^2}{p})/2}.
\end{aligned}$$

Therefore, the marginal PDF of T is

$$\begin{aligned} f_T(t) &= \int_0^\infty f_{T,W}(t,w)dw \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{p}}\frac{1}{\Gamma(\frac{p}{2})2^{p/2}}\int_0^\infty w^{\frac{p+1}{2}-1}e^{-w(1+\frac{t^2}{p})/2}dw, \end{aligned}$$

where the integrand is a Gamma kernel with parameters $a = (p+1)/2$ and $\lambda = (1+t^2/p)/2$. Therefore, the integral is

$$\Gamma\left(\frac{p+1}{2}\right)\left[\frac{1}{2}\left(1+\frac{t^2}{p}\right)\right]^{-(p+1)/2},$$

and marginal PDF of T is given by

$$\begin{aligned} f_T(t) &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{p}}\frac{1}{\Gamma(\frac{p}{2})2^{p/2}}\Gamma\left(\frac{p+1}{2}\right)\left[\frac{1}{2}\left(1+\frac{t^2}{p}\right)\right]^{-(p+1)/2} \\ &= \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\frac{1}{(p\pi)^{1/2}}\frac{1}{(1+t^2/p)^{(p+1)/2}}. \end{aligned}$$