Homework 4

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1. Let $\theta_2 > \theta_1$,

$$\frac{g_T(t|\theta_2)}{g_T(t|\theta_1)} = \frac{h(t)c(\theta_2)e^{w(\theta_2)t}}{h(t)c(\theta_1)e^{w(\theta_1)t}} = \frac{c(\theta_2)}{c(\theta_1)}e^{(w(\theta_2)-w(\theta_1))t}$$

Since g_T is an Exponential family, $c(\theta_2) \geq 0, c(\theta_1) \geq 0$. Since w is a nondecreasing function, $w(\theta_2) - w(\theta_1) > 0$, thus $\frac{g_T(t|\theta_2)}{g_T(t|\theta_1)}$ is increasing function of t. Therefore, this family has an MLR.

2. $T(\boldsymbol{X}) = \bar{X} \sim \mathcal{N}(\mu, \sigma_0^2/n)$. Thus, for any $\mu_2 > \mu_1$,

$$\frac{g_T(t|\mu_2)}{g_T(t|\mu_1)} = \frac{\frac{\sqrt{n}}{\sqrt{2\pi}\sigma_0}e^{-\frac{(t-\mu_2)^2}{2\sigma_0^2/n}}}{\frac{\sqrt{n}}{\sqrt{2\pi}\sigma_0}e^{-\frac{(t-\mu_1)^2}{2\sigma_0^2/n}}} = e^{-\frac{n}{2\sigma_0^2}[(t-\mu_2)^2-(t-\mu_1)^2]} = e^{-\frac{n}{2\sigma_0^2}[(\mu_2^2-\mu_1^2)+2(\mu_2-\mu_1)t]}$$

Since $\mu_2 > \mu_1$, $\frac{g_T(t|\mu_2)}{g_T(t|\mu_1)}$ is an increasing function of t. Thus the family of the distribution for $\bar{X}$ has an MLR.

3.

By Neyman-Pearson Lemma, supposing that the rejection region is $R : \{f_{\boldsymbol{X}}(\boldsymbol{x}|\theta = 2) > kf_{\boldsymbol{X}}(\boldsymbol{x}|\theta = 1)\}$ for some $k \geq 0$. Then the UMP level α test is

$$\begin{aligned}
\alpha &= P_{\theta=1}(\boldsymbol{x} \in R) \\
&= P_{\theta=1}(\frac{f_{\boldsymbol{X}}(\boldsymbol{x}|\theta = 2)}{f_{\boldsymbol{X}}(\boldsymbol{x}|\theta = 1)} > k) \\
&= P_{\theta=1}(\frac{k}{2^n} < \prod_{i=1}^n x_i < 1) \\
&= P_{\theta=1}(0 < -\sum_{i=1}^n \log x_i < n\log 2 - \log k)
\end{aligned}$$

When $\theta = 1$, $f_X(x|\theta = 1) = I(0 < x < 1)$, thus $x \sim \text{Unif}(0, 1)$. Let $y = -\log x$, $f_Y(y|\theta = 1) = f_X(e^{-y}|\theta = 1)e^{-y} = e^{-y}$, thus $y \sim \text{Expo}(1)$, $\sum_{i=1}^n y_i = -\sum_{i=1}^n \log x_i \sim \text{Gamma}(n, 1)$.

Let $k_{1-\alpha}$ be the cutoff point that $1 - \alpha = P_{\theta=1}(-\sum_{i=1}^n \log x_i > k_{1-\alpha})$, then the test that rejects H_0 if $P_{\theta=1}(-\sum_{i=1}^n \log x_i < k_{1-\alpha})$ is an UMP level α test.

4.

a. By Neyman-Pearson Lemma, supposing that the rejection region is $R : \{f_{\boldsymbol{X}}(\boldsymbol{x}|\theta = 2) > kf_{\boldsymbol{X}}(\boldsymbol{x}|\theta = 1)\}$ for some $k \geq 0$. Then the UMP level α test is

$$\alpha = P_{\theta=1}(\boldsymbol{x} \in R) = P_{\theta=1}(\frac{f_{\boldsymbol{X}}(\boldsymbol{x}|\theta = 2)}{f_{\boldsymbol{X}}(\boldsymbol{x}|\theta = 1)} > k) = P_{\theta=1}(\frac{\frac{1}{2^n}e^{-\sum_{i=1}^n x_i/2}}{e^{-\sum_{i=1}^n x_i}} > k)$$
$$= P_{\theta=1}(e^{\sum_{i=1}^n x_i/2} > 2^n k) = P_{\theta=1}(\sum_{i=1}^n x_i > 2(n\log 2 + \log k))$$

Since each $x_i \sim$ Expo(1), $\sum_{i=1}^n x_i \sim$ Gamma$(n, 1)$. Supposing that k_α is the cutoff point such that $\alpha = P_{\theta=1}(\sum_{i=1}^n x_i > k_\alpha)$. Then the test that rejects H_0 if $\sum_{i=1}^n X_i > k_\alpha$ is an UMP level α test.

b. By definition,

$$f_{\boldsymbol{X}}(\boldsymbol{x}|\theta) = \frac{1}{\theta^n}e^{-\frac{1}{\theta}\sum_{i=1}^n x_i}\prod_{i=1}^n I(x_i > 0)$$

Let $T = \sum_{i=1}^n X_i$, $g(t|\theta) = \frac{1}{\theta^n}e^{-\frac{1}{\theta}\sum_{i=1}^n x_i}$, $h(\boldsymbol{x}) = \prod_{i=1}^n I(x_i > 0)$. By Factorization Theorem, $f_{\boldsymbol{X}}(\boldsymbol{x}|\theta) = g(t|\theta)h(\boldsymbol{x})$. Thus T is a sufficient statistic.

c. Since $X_i \sim$ Expo$(\frac{1}{\theta})$, $T(\boldsymbol{X}) = \sum_{i=1}^n X_i \sim$ Gamma$(n, \frac{1}{\theta})$. For any $\theta_2 > \theta_1$,

$$\frac{g_T(t|\theta_2)}{g_T(t|\theta_2)} = \frac{\frac{1}{\Gamma(n)\theta_2^n}x^{n-1}e^{-\frac{1}{\theta_2}x}}{\frac{1}{\Gamma(n)\theta_1^n}x^{n-1}e^{-\frac{1}{\theta_1}x}} = (\frac{\theta_1}{\theta_2})^n e^{(\frac{1}{\theta_1}-\frac{1}{\theta_2})x}$$

Since $\theta_2 > \theta_1 \implies \frac{1}{\theta_1} - \frac{1}{\theta_2} > 0$, $\frac{g_T(t|\theta_2)}{g_T(t|\theta_2)}$ is an increasing function of t. Thus T has a MLR. Let k_α be the cutoff point that $\alpha = P_{\theta_0}(T > t_\alpha)$. By Karlin-Rubin Theorem,

$$\alpha = P_{\theta_0}(T > t_0) \implies t_0 = k_\alpha$$

Thus the test that rejects H_0 if $T > k_\alpha$ is an UMP level α test.

5.

Let $T(\boldsymbol{X}) = \sum_{i=1}^n X_i$. Since each $X_i \sim$ Pois(θ), $T \sim$ Pois$(n\theta)$.

Since

$$f_{\boldsymbol{X}}(\boldsymbol{x}|\theta) = \frac{\theta^{\sum_{i=1}^n x_i}e^{-n\theta}}{\prod_{i=1}^n x_i!} = \frac{\theta^t e^{-n\theta}}{\prod_{i=1}^n x_i!}$$

Let $g(t|\theta) = \theta^t e^{-n\theta}$, $h(\boldsymbol{x}) = \frac{1}{\prod_{i=1}^n x_i!}$. By Factorization Theorem, $f_{\boldsymbol{X}}(\boldsymbol{x}|\theta) = g(t|\theta)h(\boldsymbol{x})$. Thus T is a sufficient statistic.

For any $\theta_2 > \theta_1$,

$$\frac{g_T(t|\theta_2)}{g_T(t|\theta_1)} = \frac{\frac{(n\theta_2)^t e^{-n\theta_2}}{t!}}{\frac{(n\theta_1)^t e^{-n\theta_1}}{t!}} = (\frac{\theta_2}{\theta_1})^t e^{n(\theta_1-\theta_2)}$$

Since $\theta_2 > \theta_1$, $\frac{g_T(t|\theta_2)}{g_T(t|\theta_1)}$ is a decreasing function of t. Thus T has a MLR.

By Karlin-Rubin Theorem,

$$\alpha = P_{\theta_0}(T > t_0) = \sum_{t=t_0}^{\infty} \frac{(n\theta_0)^t e^{-n\theta}}{t!}$$

Since Poisson is a discrete distribution, let $t' = \min\{t_0 | \sum_{t=t_0}^{\infty} \frac{(n\theta_0)^t e^{-n\theta}}{t!} \leq \alpha\}$, then the test that rejects H_0 if $T > t'$ is a UMP level α test.