Homework #2

MATH 60061/70061: Mathematical Statistics I

Please submit your answers as a **PDF** file to Blackboard by **9:15 a.m. on October 19**. Please show your work and write legibly. Your grade will be based on the correctness of your answers and the clarity with which you express them.

1. (15 points) This is a phenomenon called the *bias-variance tradeoff*, which arises throughout statistics and machine learning. Consider an experiment where we observe the value of a random variable X, and estimate the value of an unknown constant θ using some random variable $T = g(X)$ that is a function of X. The random variable T is called an *estimator*. Think of X as the data observed in the experiment, and θ as an unknown parameter related to the distribution of X. The *bias* of an estimator T for θ is defined as $b(T) = E(T) - \theta$. The *mean squared error* is the average squared error when using $T(X)$ to estimate θ:

$$\text{MSE}(T) = E(T - \theta)^2.$$

   Show that

$$\text{MSE}(T) = \text{Var}(T) + (b(T))^2.$$

2. (10 points) Let $Z \sim \mathcal{N}(0, 1)$ and let S be a random sign independent of Z, i.e., S is 1 with probability 1/2 and -1 with probability 1/2. Show that $SZ \sim \mathcal{N}(0, 1)$.

3. (20 points) Show that each of the following families is an exponential family.

   a. Normal family with either μ or σ known.

   b. Beta family with either a or b known or both known.

   c. Poisson family.

   d. Negative Binomial family with r known, $0 < p < 1$.

4. (10 points) Let X and Y be standardized random variables (i.e., marginally they each have mean 0 and variance 1) with correlation $\rho \in (-1, 1)$. Find a, b, c, d (in terms of ρ) such that $Z = aX + bY$ and $W = cX + dY$ are uncorrelated but still standardized.

5. (15 points) Show that $E((Y - E(Y|X))^2 \mid X) = E(Y^2 \mid X) - (E(Y \mid X))^2$, so these two expressions for $\text{Var}(Y \mid X)$ agree.

6. (30 points) Consider a Beta-Binomial hierarchical model

$$\begin{aligned} X \mid P &\sim \text{Bin}(n, P), \\ P &\sim \text{Beta}(a, b). \end{aligned}$$

   a. What is the unconditional expectation and variance of X?

b. Show that the marginal distribution of X is given by the *Beta-Binomial distribution*

$$P(X=x) = \binom{n}{x} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(x+a)\Gamma(n-x+b)}{\Gamma(a+b+n)}.$$

c. Show that the variance of X can be written as

$$\mathrm{Var}(X) = nE(P)(1-E(P)) + n(n-1)\mathrm{Var}(P).$$

The second term is often called "extra-Binomial" variation, showing how the hierarchical model has a variance that is larger than the Binomial alone.