Homework 3

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1. By the Bayes Rule,

$$\begin{aligned} f(\theta|\boldsymbol{x}) &\propto f_{\boldsymbol{X}}(\boldsymbol{x}|\theta)\pi(\theta) \\ &= e^{-n\theta}\theta^{\sum_{i=1}^n x_i}\frac{1}{\prod_{i=1}^n x_i!}\frac{1}{\Gamma(a)b^a}\theta^{a-1}e^{-\theta/b}I(\theta>0) \\ &= e^{-\theta(n+1/b)}\theta^{\sum_{i=1}^n x_i+a-1}\frac{1}{\prod_{i=1}^n x_i!}\frac{1}{\Gamma(a)b^a}I(\theta>0) \end{aligned}$$

The Gamma kernel shows $\theta \sim \text{Gamma}(\sum_{i=1}^n x_i + a, \frac{b}{nb+1})$.

Thus the Bayes estimator of θ under squared error loss is $\delta^\pi(\boldsymbol{x}) = E(\theta|\boldsymbol{x}) = (\sum_{i=1}^n x_i + a)\frac{b}{nb+1}$.

2.

a. Since $X_1, X_2, ..., X_n$ are iid $(\theta, 1)$, the likelihood function is

$$L(\theta|\boldsymbol{x}) = (\frac{1}{\sqrt{2\pi}})^n e^{-\frac{1}{2}\sum_{i=1}^n(x_i-\theta)^2}$$

Thus,

$$\begin{aligned} \log L(\theta|\boldsymbol{x}) &= -n\log\sqrt{2\pi} - \frac{1}{2}\sum_{i=1}^n(x_i-\theta)^2 \\ \frac{\partial}{\partial\theta}\log L(\theta|\boldsymbol{x}) &= \sum_{i=1}^n x_i - n\theta \end{aligned}$$

$\frac{\partial}{\partial\theta}\log L(\theta|\boldsymbol{x}) = 0$ only when $\theta = \frac{1}{n}\sum_{i=1}^n x_i = \bar{x}$. Note that $\log L(\theta|\boldsymbol{x})$ is monotonic increasing when $-\infty < \theta < \bar{x}$ and monotonic decreasing when $\bar{x} < \theta < \infty$. Thus,

$$\hat{\theta} = \begin{cases} \bar{X} & \text{if } \theta_0 \geq \bar{X} \\ \theta_0 & \text{if } \theta_0 < \bar{X} \end{cases}$$

The LRT is

$$\lambda(\boldsymbol{x}) = \frac{\sup_{\theta\in\Theta_0} L(\theta|\boldsymbol{x})}{\sup_{\theta\in\Theta} L(\theta|\boldsymbol{x})} = \begin{cases} 1 & \text{if } \theta_0 \geq \bar{X} \\ e^{-\frac{n}{2}(\bar{x}-\theta_0)^2} & \text{if } \theta_0 < \bar{X} \end{cases}$$

Given a number $c \in [0,1]$, the rejection region is

$$\begin{aligned}
\lambda(\boldsymbol{x}) &\leq c \\
e^{-\frac{n}{2}(\bar{x}-\theta_0)^2} &\leq c \\
\sqrt{n}|\bar{x}-\theta_0| &\geq \sqrt{-2\log c} \\
\sqrt{n}(\bar{x}-\theta_0) &\geq \sqrt{-2\log c} \qquad \text{(Because } \theta_0 < \bar{x}\text{)}
\end{aligned}$$

Let $c' = \sqrt{-2\log c}$, we have the rejection region $R = \{\boldsymbol{x} : \sqrt{n}(\bar{x}-\theta_0) \geq c'\}$.

b. The power function of this test is

$$\begin{aligned}
\beta(\theta) &= P_\theta(\sqrt{n}(\bar{x}-\theta_0) \geq c') \\
&= P_\theta(\sqrt{n}(\bar{x}-\theta) \geq c' - \sqrt{n}(\theta-\theta_0)) \\
&= 1 - F_Z(c' - \sqrt{n}(\theta-\theta_0))
\end{aligned}$$

where F_Z is the CDF of standard Normal distribution.

$\beta(\theta)$ is an increasing function of θ because

$$\frac{\partial}{\partial\theta}\beta(\theta) = \sqrt{n} f_Z(c' - \sqrt{n}(\theta-\theta_0)) > 0$$

Thus,

$$\sup_{\theta \leq \theta_0} \beta(\theta) = \inf_{\theta > \theta_0} \beta(\theta) = \beta(\theta_0)$$

Thus for any $\theta' > \theta_0$ and $\theta'' \leq \theta_0$, $\beta(\theta') \geq \beta(\theta'')$ always holds. The LRT is an unbiased test.

3.

a. Since Θ_0 only has one element θ_0,

$$\begin{aligned}
\sup_{\theta\in\Theta_0} \beta(\theta) &= \beta(\theta_0) \\
&= P_{\theta_0}(|\bar{X}-\theta_0| > t_{n-1,\alpha/2} S/\sqrt{n}) \\
&= P_{\theta_0}\left(\frac{\bar{X}-\theta_0}{S/\sqrt{n}} > t_{n-1,\alpha/2}\right) + P_{\theta_0}\left(\frac{\bar{X}-\theta_0}{S/\sqrt{n}} < -t_{n-1,\alpha/2}\right)
\end{aligned}$$

Since $\frac{\bar{X}-\theta_0}{S/\sqrt{n}} \sim t_{n-1}$, the above equation equals to $\alpha/2 + \alpha/2 = \alpha$. Thus it is a size α test.

b. The LRT is

$$\begin{aligned}
\lambda(\boldsymbol{x}) &= \frac{\sup_{\theta\in\Theta_0} L(\theta|\boldsymbol{x})}{\sup_{\theta\in\Theta} L(\theta|\boldsymbol{x})} \\
&= \frac{(\frac{1}{\sqrt{2\pi}\sigma})^n e^{-\frac{\sum_{i=1}^n (x_i-\theta_0)^2}{2\sigma^2}}}{(\frac{1}{\sqrt{2\pi}\sigma})^n e^{-\frac{\sum_{i=1}^n (x_i-\bar{x})^2}{2\sigma^2}}} \\
&= e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta_0)^2}
\end{aligned}$$

Let the rejection region be $\lambda(\boldsymbol{x}) > c$ where $0 < c < 1$, then

$$\begin{aligned}
e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta_0)^2} &> c \\
-\frac{n}{2\sigma^2}(\bar{x}-\theta_0)^2 &> \log c \\
-\frac{n^2}{2(n-1)S^2}(\bar{x}-\theta_0)^2 &> \log c \\
\frac{(\bar{x}-\theta_0)^2}{S^2/n} &> -\frac{2(n-1)}{n}\log c \\
\frac{|\bar{x}-\theta_0|}{S/\sqrt{n}} &> \sqrt{-\frac{2(n-1)}{n}\log c}
\end{aligned}$$

Let $\sqrt{-\frac{2(n-1)}{n}\log c} = t_{n-1,\alpha/2}$, i.e., $c = e^{-\frac{n}{2(n-1)}t^2_{n-1,\alpha/2}}$, we get the test.

4.

a. The power function of the first test:

$$\beta(\theta) = P_\theta(\boldsymbol{X} \in R) = P_\theta(x_1 = 0) = e^{-\theta}$$

The power function of the second test:

$$\begin{aligned}
\beta(\theta) &= P_\theta(\boldsymbol{X} \in R) \\
&= P_\theta(x_1 = 0, x_2 = 0) + P_\theta(x_1 = 1, x_2 = 0) + P_\theta(x_1 = 0, x_2 = 1) \\
&= e^{-2\theta} + \theta e^{-2\theta} + \theta e^{-2\theta} \\
&= (1 + 2\theta)e^{-2\theta}
\end{aligned}$$

b. The size of the first test:

$$\alpha = \sup_{\theta\in\Theta_0} \beta(\theta) = \beta(3) = e^{-3}$$

For the second test, since $\beta'(\theta) = -4\theta e^{-2\theta} < 0$ when $\theta \geq 3$, $\beta(\theta)$ is a decreasing function when $\theta \geq 3$. Thus the size is

$$\alpha = \sup_{\theta\in\Theta_0} \beta(\theta) = \beta(3) = 7e^{-6}$$

c. Since $e^{-3} < 0.05$ and $7e^{-6} < 0.05$, both ϕ_1 and ϕ_2 are level $\alpha = 0.05$ test.