Homework #3

MATH 60061/70061: Mathematical Statistics I

Please submit your answers as a **PDF** file to Blackboard by **9:15 a.m. on November 16**. Please show your work and write legibly. Your grade will be based on the correctness of your answers and the clarity with which you express them.

1. (10 points) Calculate $P(|X - \mu_X| \geq k\sigma_X)$ for $X \sim \text{Unif}(0,1)$ and $X \sim \text{Exp}(\lambda)$, and compare your answers to the bound from Chebychev's Inequality.

2. (10 points) For any two positive random variables U and V with neither a constant multiple of the other, show that
$$E\left(\frac{U}{V}\right) E\left(\frac{V}{U}\right) > 1.$$

3. (30 points) Let $X_1, \ldots, X_n$ be a random sample, where the sample mean $\bar{X}$ and the sample variance S^2 are calculated in the usual way.

    a. Show that
    $$S^2 = \frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (X_i - X_j)^2.$$

    Assume now that the X_i's have a finite fourth moment, and denote $\theta_1 = E(X_i)$, $\theta_j = E(X_i - \theta_1)^j$, $j = 2, 3, 4$.

    b. Show that
    $$\text{Var}(S^2) = \frac{1}{n}\left(\theta_4 - \frac{n-3}{n-1}\theta_2^2\right).$$

    c. Find $\text{Cov}(\bar{X}, S^2)$ in terms of $\theta_1, \ldots, \theta_4$. Under what conditions is $\text{Cov}(\bar{X}, S^2) = 0$?

4. (20 points) Establish the following recursion relations for means and variances. Let $\bar{X}_n$ and S_n^2 be the mean and variance, respectively, of $X_1, \ldots, X_n$. Then suppose another observation X_{n+1}, becomes available. Show that

    a. $\bar{X}_{n+1} = \frac{X_{n+1} + n\bar{X}_n}{n+1}$.

    b. $nS_{n+1}^2 = (n-1)S_n^2 + \frac{n}{n+1}(X_{n+1} - \bar{X}_n)^2$.

5. (15 points) Let $X_1, \ldots, X_n$ be a random sample from $\mathcal{N}(\mu_X, \sigma_X^2)$, $Y_1, \ldots, Y_m$ be a random sample from $\mathcal{N}(\mu_Y, \sigma_Y^2)$, X_i's and Y_j's be independent, and S_X^2 and S_Y^2 be the sample variances based on X_i's and Y_j's, respectively. Then the ratio
$$\frac{S_X^2/S_Y^2}{\sigma_X^2/\sigma_Y^2} = \frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2}$$

is said to have the F distribution with degrees of freedom $n-1$ and $m-1$ (denoted by $F_{n-1,m-1}$). Show that a random variable has the $F_{p,q}$ distribution if its PDF is given by

$$f(x) = \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \left(\frac{p}{q}\right)^{p/2} \frac{x^{(p/2)-1}}{(1+(p/q)x)^{(p+q)/2}}, \quad 0 < x < \infty.$$

6. (15 points) Let $X_{(1)}, \ldots, X_{(n)}$ be the order statistics of a random sample $X_1, \ldots, X_n$ from $\mathrm{Unif}(0,1)$. Find the distribution of $X_1/X_{(1)}$.