Lecture 18: Test Conversion and Pivotal Quantities

Mathematical Statistics II, MATH 60062/70062

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Reference: Casella & Berger, 9.2.1-9.2.2

Recap: Interval estimate and interval estimator

An **interval estimate** of a real-valued parameter θ is any pair of functions

$$L(\boldsymbol{x}) = L(x_1, \ldots, x_n)$$
$$U(\boldsymbol{x}) = U(x_1, \ldots, x_n),$$

satisfying $L(\boldsymbol{x}) \leq U(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathcal{X}$. When $\boldsymbol{X} = \boldsymbol{x}$ is observed, the inference

$$L(\boldsymbol{x}) \leq \theta \leq U(\boldsymbol{x})$$

is made. The random version of the interval $[L(\boldsymbol{X}), U(\boldsymbol{X})]$ is called an **interval estimator**.

Sometimes a **one-sided interval** estimate may be formed.

- If $L(\boldsymbol{x}) = -\infty$, then the interval is $(-\infty, U(\boldsymbol{x})]$ with the assertion that $\theta \leq U(\boldsymbol{x})$.
- If $U(\boldsymbol{x}) = \infty$, then the interval is $[L(\boldsymbol{x}), \infty)$ with the assertion that $\theta \geq L(\boldsymbol{x})$.

Interval estimation methods

Methods of finding interval estimators

- Inverting a test statistic
  - There is a strong duality between hypothesis testing and confidence intervals.
- Using pivotal quantities
- Pivoting a CDF
- Bayesian credible intervals

Methods of evaluating interval estimators

- Coverage probability
- Interval length

Inverting a test statistic

Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ using the level α test function

$$\phi(\boldsymbol{x}) = I(\boldsymbol{x} \in R_{\theta_0}) = \begin{cases} 1 & \boldsymbol{x} \in R_{\theta_0} \\ 0 & \boldsymbol{x} \in R_{\theta_0}^c, \end{cases}$$

where $P_{\theta_0}(\boldsymbol{X} \in R_{\theta_0}) = E_{\theta_0}[\phi(\boldsymbol{X})] = \alpha$. The rejection region depends on the value of θ_0—hence the notation R_{θ_0}.

Let $A_{\theta_0} = R_{\theta_0}^c$ be the "acceptance region" of this level α test, that is, the set of $\boldsymbol{x} \in \mathcal{X}$ that do *not* lead to H_0 being rejected.

For each $\boldsymbol{x} \in \mathcal{X}$, define a set $C(\boldsymbol{x})$ in the parameter space by

$$C(\boldsymbol{x}) = \{\theta_0 : \boldsymbol{x} \in A_{\theta_0}\}.$$

From the definition $C(\boldsymbol{x}) = \{\theta_0 : \boldsymbol{x} \in A_{\theta_0}\}$, clearly,

$$\theta_0 \in C(\boldsymbol{x}) \iff \boldsymbol{x} \in A_{\theta_0}.$$

Therefore, based on the duality,

$$\begin{aligned} P_{\theta_0}(\theta_0 \in C(\boldsymbol{X})) &= P_{\theta_0}(\boldsymbol{X} \in A_{\theta_0}) \\ &= 1 - P_{\theta_0}(\boldsymbol{X} \in R_{\theta_0}) \\ &= 1 - \alpha. \end{aligned}$$

The same argument holds for all $\theta_0 \in \Theta$. Therefore,

$$C(\boldsymbol{X}) = \{\theta \in \Theta : \boldsymbol{x} \in A_\theta\}$$

is a $1 - \alpha$ confidence set.

Inverting a two-sided test

Suppose $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$, where $-\infty < \mu < \infty$ and $\sigma^2 > 0$. Both parameters are unknown.

Consider testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ using the test function of the level α likelihood ratio test (LRT)

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \frac{|\bar{x}-\mu_0|}{s/\sqrt{n}} \geq t_{n-1,\alpha/2} \\ 0 & \text{otherwise.} \end{cases}$$

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$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \frac{|\bar{x}-\mu_0|}{s/\sqrt{n}} \geq t_{n-1,\alpha/2} \\ 0 & \text{otherwise.} \end{cases}$$

The acceptance region for the LRT is

$$A_{\mu_0} = \left\{ \boldsymbol{x} \in \mathcal{X} : \frac{|\bar{x} - \mu_0|}{s/\sqrt{n}} < t_{n-1,\alpha/2} \right\},$$

where

$$P_{\mu_0}(\boldsymbol{X} \in A_{\mu_0}) = P_{\mu_0}\left(\frac{|\bar{X} - \mu_0|}{S/\sqrt{n}} < t_{n-1,\alpha/2} \right) = 1 - \alpha.$$

Thus, by inverting the acceptance region for the test, a $1-\alpha$ confidence set for μ is expressed as

$$
\begin{aligned}
C(\boldsymbol{x}) &= \{\mu \in \mathbb{R} : \boldsymbol{x} \in A_\mu\} \\
&= \left\{\mu : -t_{n-1,\alpha/2} < \frac{\bar{x}-\mu}{s/\sqrt{n}} < t_{n-1,\alpha/2}\right\} \\
&= \left\{\mu : -t_{n-1,\alpha/2}\frac{s}{\sqrt{n}} < \bar{x}-\mu < t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}\right\} \\
&= \left\{\mu : \bar{x} - t_{n-1,\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}\right\}.
\end{aligned}
$$

The random version of this confidence set (interval) is

$$
\left(\bar{X} - t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}, \bar{X} + t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}\right).
$$

Correspondence between testing and interval estimation

Both procedures look for consistency between sample statistics and population parameters.

- The **hypothesis test** fixes the **parameter** and asks what **sample values (the acceptance region)** are consistent with that fixed value.
- The **confidence set** fixes the **sample value** and asks what **parameter values (the confidence interval)** make this sample value most plausible.

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There is *no* guarantee that the confidence set obtained from converting an acceptance region is an interval, but in most cases

- One-sided tests give one-sided intervals.
- Two-sided tests give two-sided intervals.

Inverting a one-sided test

Suppose $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$, where $-\infty < \mu < \infty$ and $\sigma^2 > 0$. Both parameters are unknown.

Consider testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu < \mu_0$ using the test function of the level α LRT (exercise):

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \frac{\bar{x}-\mu_0}{s/\sqrt{n}} \leq -t_{n-1,\alpha} \\ 0 & \text{otherwise.} \end{cases}$$

LRT is a one-sided test

The acceptance region for the LRT is

$$A_{\mu_0} = \left\{ \boldsymbol{x} \in \mathcal{X} : \frac{\bar{x} - \mu_0}{s/\sqrt{n}} > -t_{n-1,\alpha} \right\},$$

where

$$P_{\mu_0}(\boldsymbol{X} \in A_{\mu_0}) = P_{\mu_0}\left(\frac{\bar{X} - \mu_0}{S/\sqrt{n}} > -t_{n-1,\alpha} \right) = 1 - \alpha.$$

Inverting the acceptance region for the test gives a $1-\alpha$ confidence set for μ:

$$\begin{aligned}
C(\boldsymbol{x}) &= \{\mu \in \mathbb{R} : \boldsymbol{x} \in A_\mu\} \\
&= \left\{\mu : \frac{\bar{x}-\mu}{s/\sqrt{n}} > -t_{n-1,\alpha}\right\} \\
&= \left\{\mu : \mu < \bar{x} + t_{n-1,\alpha}\frac{s}{\sqrt{n}}\right\}.
\end{aligned}$$

Thus, the random version of this confidence set (one-sided interval) is

$$\left(-\infty, \bar{X} + t_{n-1,\alpha}\frac{S}{\sqrt{n}}\right).$$

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Pivotal quantities

A random variable $Q = Q(\boldsymbol{X}, \theta)$ is a **pivotal quantity** (or **pivot**) if the distribution of Q is independent of θ.

If $Q = Q(\boldsymbol{X}, \theta)$ is pivot, then a $1 - \alpha$ confidence interval can be found by setting

$$1 - \alpha = P_\theta(a \leq Q(\boldsymbol{X}, \theta) \leq b),$$

where a and b are quantiles of the distribution of Q that satisfy the condition (there are many of them).

Uniform pivotal interval

Suppose that $X_1, \ldots, X_n$ are iid $\mathrm{Unif}(0, \theta)$, where $\theta > 0$.

We already showed that

$$Q = Q(\boldsymbol{X}, \theta) = \frac{X_{(n)}}{\theta} \sim \mathrm{Beta}(n, 1).$$

The distribution of Q is free of θ, so it is a pivot.

Uniform pivotal interval

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Let $b_{n,1,1-\alpha/2}$ and $b_{n,1,\alpha/2}$ be the lower and upper $\alpha/2$ quantiles of $\mathrm{Beta}(n, 1)$, respectively. We have

$$1-\alpha = P_\theta\left(b_{n,1,1-\alpha/2} \leq \frac{X_{(n)}}{\theta} \leq b_{n,1,\alpha/2}\right) = P_\theta\left(\frac{X_{(n)}}{b_{n,1,\alpha/2}} \leq \theta \leq \frac{X_{(n)}}{b_{n,1,1-\alpha/2}}\right)$$

Therefore,

$$\left(\frac{X_{(n)}}{b_{n,1,\alpha/2}}, \frac{X_{(n)}}{b_{n,1,1-\alpha/2}}\right)$$

is a $1 - \alpha$ confidence interval for θ.

Normal pivotal intervals

Suppose $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$, where $-\infty < \mu < \infty$ and $\sigma^2 > 0$. Both parameters are unknown. Set $\boldsymbol{\theta} = (\mu, \sigma^2)$.

We know that

$$Q_1 = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}.$$

The distribution of Q_1 is free of $\boldsymbol{\theta}$, so it is a pivot. Therefore

$$\begin{aligned} 1 - \alpha &= P_{\boldsymbol{\theta}}\left(-t_{n-1,\alpha/2} \leq \frac{\bar{X} - \mu}{S/\sqrt{n}} \leq t_{n-1,\alpha/2}\right) \\ &= P_{\boldsymbol{\theta}}\left(\bar{X} - t_{n-1,\alpha/2}\frac{S}{\sqrt{n}} \leq \mu \leq \bar{X} + t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}\right), \end{aligned}$$

which gives a $1 - \alpha$ confidence interval for μ,

$$C_1(\boldsymbol{X}) = \left(\bar{X} - t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}, \bar{X} + t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}\right).$$

We also know that

$$Q_2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}.$$

The distribution of Q_2 is free of $\boldsymbol{\theta}$, so it is a pivot. Therefore

$$\begin{aligned} 1-\alpha &= P_{\boldsymbol{\theta}}\left(\chi^2_{n-1,1-\alpha/2} \le \frac{(n-1)S^2}{\sigma^2} \le \chi^2_{n-1,\alpha/2}\right) \\ &= P_{\boldsymbol{\theta}}\left(\frac{(n-1)S^2}{\chi^2_{n-1,\alpha/2}} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{n-1,1-\alpha/2}}\right), \end{aligned}$$

which gives a $1-\alpha$ confidence interval for σ^2,

$$C_2(\boldsymbol{X}) = \left(\frac{(n-1)S^2}{\chi^2_{n-1,\alpha/2}}, \frac{(n-1)S^2}{\chi^2_{n-1,1-\alpha/2}}\right).$$

We also know that

$$Q_2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}.$$

The distribution of Q_2 is free of $\boldsymbol{\theta}$, so it is a pivot. Therefore

$$\begin{aligned} 1-\alpha &= P_{\boldsymbol{\theta}}\left(\chi^2_{n-1,1-\alpha/2} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{n-1,\alpha/2}\right) \\ &= P_{\boldsymbol{\theta}}\left(\frac{(n-1)S^2}{\chi^2_{n-1,\alpha/2}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{n-1,1-\alpha/2}}\right), \end{aligned}$$

which gives a $1-\alpha$ confidence interval for σ^2,

$$C_2(\boldsymbol{X}) = \left(\frac{(n-1)S^2}{\chi^2_{n-1,\alpha/2}}, \frac{(n-1)S^2}{\chi^2_{n-1,1-\alpha/2}}\right).$$

Suppose we wanted to find a confidence set for $\boldsymbol{\theta} = (\mu, \sigma^2)$ *simultaneously. Is* $C_1(\boldsymbol{X}) \times C_2(\boldsymbol{X})$ *a* $1-\alpha$ *confidence set for* $\boldsymbol{\theta}$*?*