Homework 1

Ruixin Guo

February 27, 2022

1.

Since $X_1, X_2, ..., X_n$ are iid Bern(θ), $\bar{X} = \theta$, thus $\bar{X}$ is an unbiased estimator of θ.

The variance of $\bar{X}$ is $Var(\bar{X}) = \frac{\sigma^2}{n} = \frac{\theta(1-\theta)}{n}$. Since

$$(\frac{d}{d\theta}E_\theta[\bar{X}])^2 = 1^2 = 1$$

$$\begin{aligned}
I_n(\theta) = nI_1(\theta) &= nE_\theta[(\frac{\partial}{\partial\theta}\log f_{X_1}(x|\theta))^2] \\
&= nE_\theta[(\frac{\partial}{\partial\theta}[x\log\theta + (1-x)\log(1-\theta)])^2] \\
&= nE_\theta[(\frac{x}{\theta} - \frac{1-x}{1-\theta})^2] \\
&= n[\theta\frac{1}{\theta^2} + (1-\theta)\frac{1}{(1-\theta)^2}] \\
&= \frac{n}{\theta(1-\theta)}
\end{aligned}$$

The CRLB of $\bar{X}$ is

$$\frac{(\frac{d}{d\theta}E_\theta[\bar{X}])^2}{I_n(\theta)} = \frac{\theta(1-\theta)}{n}$$

Thus $Var(\bar{X})$ attains CRLB, $\bar{X}$ is the UMVUE of θ.

2. Yes. Let $W(\boldsymbol{X})$ be an unbiased estimator of $\tau(\theta)$, $W(\boldsymbol{X})$ attains CRLB if and only if $\frac{\partial}{\partial\theta}\log L(\theta|\boldsymbol{x}) = a(\theta)(W(\boldsymbol{X}) - \tau(\theta))$ for some function $a(\theta)$. Since

$$\begin{aligned}
\frac{\partial}{\partial\theta}\log L(\theta|\boldsymbol{x}) &= \frac{\partial}{\partial\theta}\log\theta^n(\prod_{i=1}^n x_i)^{\theta-1} \\
&= \frac{\partial}{\partial\theta}(n\log\theta + (\theta-1)\log(\prod_{i=1}^n x_i)) \\
&= \frac{n}{\theta} + \log(\prod_{i=1}^n x_i)
\end{aligned}$$

Let $\tau(\theta) = \frac{n}{\theta}$, $W(\boldsymbol{X}) = -\log(\prod_{i=1}^n X_i)$ is the UMVUE of $\tau(\theta)$. Here we set $a(\theta) = -1$.

3.

a. The regularity condition of CRLB is

$$\frac{d}{d\theta}E_\theta[W(\boldsymbol{X})] = \int_{\mathcal{X}} \frac{\partial}{\partial\theta}[W(\boldsymbol{x})f_{\boldsymbol{X}}(\boldsymbol{x}|\theta)]d\boldsymbol{x}$$

Since X_is are iid Unif$(0,\theta)$, the range of each X_i is $[0,\theta]$. By Leibnitz's rule

$$\frac{d}{d\theta}E_\theta[W(\boldsymbol{X})] = \frac{d}{d\theta}\int_{x_i\in[0,\theta]} W(\boldsymbol{x})f_{\boldsymbol{X}}(\boldsymbol{x}|\theta)d\boldsymbol{x} = f_{\boldsymbol{X}}(\boldsymbol{\theta}|\theta) + \int_{x_i\in[0,\theta]} \frac{\partial}{\partial\theta}[W(\boldsymbol{x})f_{\boldsymbol{X}}(\boldsymbol{x}|\theta)]d\boldsymbol{x}$$

where $\boldsymbol{\theta}$ is a n-dimensional vector with each element be θ.

$f_{\boldsymbol{X}}(\boldsymbol{\theta}|\theta)$ means the probability that each element in $\boldsymbol{X}$ is θ. Thus $f_{\boldsymbol{X}}(\boldsymbol{\theta}|\theta) = \frac{1}{\theta^n}$. It is obvious that $\frac{d}{d\theta}E_\theta[W(\boldsymbol{X})] \neq \int_{\mathcal{X}} \frac{\partial}{\partial\theta}[W(\boldsymbol{x})f_{\boldsymbol{X}}(\boldsymbol{x}|\theta)]d\boldsymbol{x}$. Thus the Unif$(0,\theta)$ PDF does not satisfy the regularity conditions of CRLB.

b. By Factorization Theorem,

$$f_{\boldsymbol{X}}(\boldsymbol{x}|\theta) = \frac{1}{\theta^n}\prod_{i=1}^n I(0 < x_i < \theta) = \frac{1}{\theta^n}I(x_{(n)} < \theta)\prod_{i=1}^n I(x_i > 0)$$

Let $g(x_{(n)}|\theta) = \frac{1}{\theta^n}I(x_{(n)} < \theta)$ and $h(\boldsymbol{x}) = \prod_{i=1}^n I(x_i > 0)$, we have $f_{\boldsymbol{X}}(\boldsymbol{x}|\theta) = g(x_{(n)}|\theta)h(\boldsymbol{x})$. Thus the Factorization Theorem holds, and $X_{(n)}$ is a sufficient statistic for θ.

c. Let $T(\boldsymbol{X}) = X_{(n)}$, and $F(X)$ be the CDF of Unif$(0,\theta)$, the PDF of T is

$$f_T(t|\theta) = \frac{d}{dt}(F(t))^n = \frac{d}{dt}(\frac{t}{\theta}I(0<t<\theta))^n = \frac{n}{\theta^n}t^{n-1}I(0<t<\theta)$$

The expectation of $g(T)$ is

$$E_\theta(g(T)) = \int_0^\theta g(t)f_T(t|\theta)dt = \int_0^\theta g(t)\frac{n}{\theta^n}t^{n-1}I(0<t<\theta)dt$$

Since $\frac{n}{\theta^n}t^{n-1}I(0<t<\theta) > 0$ when $t \in (0,\theta)$, $E_\theta(g(T)) = 0$ only when $g(T) = 0$ for all T. Thus T is a complete statistic.

d.

$$E_\theta(\frac{n+1}{n}X_{(n)}) = \frac{n+1}{n}E_\theta(X_{(n)}) = \frac{n+1}{n}\int_0^\theta \frac{n}{\theta^n}t^n I(0<t<\theta)dt = \frac{n+1}{n}\frac{n}{n+1}[\frac{t^{n+1}}{\theta^n}]_0^\theta = \theta$$

Thus $\frac{n+1}{n}X_{(n)}$ is an unbiased estimator of θ.

e. Since $T(\boldsymbol{X}) = X_{(n)}$ is a complete and sufficient statistic, by Rao-Blackwell Theorem, if we find a function $\phi(T)$ such that $E_\theta(\phi(T)) = \theta$, then $\phi(T)$ is the UMVUE. Obviously $\phi(T) = \frac{n+1}{n}X_{(n)}$ is the UMVUE.

5.

a. Because

$$I(X_1 = 0) = \begin{cases} 1 & \text{if } X_1 = 0 \\ 0 & \text{if } X_1 \neq 0 \end{cases}$$

We have $E_\theta(W(\boldsymbol{X})) = E_\theta(I(X_1 = 0)) = P_\theta(X_1 = 0) = e^{-\theta} = \tau(\theta)$. Thus $W(\boldsymbol{X})$ is an unbiased estimator of $\tau(\theta)$.

b.

First we prove Poisson distribution belongs to Exponential Family,

$$P(X = x|\theta) = \frac{\theta^x e^{-\theta}}{x!} = \frac{1}{x!} e^{-\theta} e^{x \log \theta}$$

Let $h(x) = \frac{1}{x!}$, $g(\theta) = e^{-\theta}$, $w_1(\theta) = \log \theta$ and $t_1(x) = x$. Then $P(X = x|\theta) = h(x)g(\theta)e^{w_1(x)t_1(\theta)}$. Thus Poisson distribution belongs to Exponential Family. Since $d = k = 1$, $T = T(\boldsymbol{X}) = \sum_{i=1}^n X_i$ is a complete and sufficient statistic of θ.

Since $W(\boldsymbol{X})$ is an unbiased estimator of $\tau(\theta)$, by Rao-Blackwell Theorem, $E_\theta(W|T)$ is the UMVUE of $\tau(\theta)$.

$$\begin{aligned} E_\theta(W|T) &= E(I(X_1 = 0)|\sum_{i=1}^n X_i = t) \\ &= P(X_1 = 0|\sum_{i=1}^n X_i = t) \\ &= \frac{P(X_1 = 0, \sum_{i=1}^n X_i = t)}{P(\sum_{i=1}^n X_i = t)} \\ &= \frac{P(X_1 = 0)P(\sum_{i=2}^n X_i = t)}{P(\sum_{i=1}^n X_i = t)} \end{aligned}$$

Note that $\sum_{i=1}^n X_i \sim \text{Pois}(n\theta)$ and $\sum_{i=2}^n X_i \sim \text{Pois}((n-1)\theta)$. Thus,

$$E_\theta(W|T) = \frac{e^{-\theta} \cdot [(n-1)\theta]^t e^{-(n-1)\theta}/t!}{[n\theta]^t e^{-n\theta}/t!} = (\frac{n-1}{n})^t$$

Therefore, $(\frac{n-1}{n})^{\sum_{i=1}^n X_i}$ is the UMVUE of $\tau(\theta)$.