Rademacher Average and Covering Number

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Recall

We want to find a bound for $R^{\text{true}}(f) - R^{\text{emp}}(f)$ where f is from a function class $\mathcal{F}$.

If $|\mathcal{F}| = N$ is finite, the bound can be written as

$$P\left[\sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f)) \leq \sqrt{\frac{\log N + \log\frac{1}{\delta}}{2m}}\right] \geq 1-\delta$$

If $|\mathcal{F}|$ is infinite, we need to find a finite measure for the capacity of $\mathcal{F}$. Growth Function and VC Dimension are two classes of such measure and they are closely related.

Growth Function: The maximum number of ways into which m points can be classified by $\mathcal{F}$, denoted as $S_{\mathcal{F}}(m)$.

Shattering: We say $\mathcal{F}$ shatters an m-point dataset if $S_{\mathcal{F}} = 2^m$. That is, for an arbitrary way of classifying m points, there always exists one f in $\mathcal{F}$ that can generate it.

VC Dimension: The VC Dimension of a function class $\mathcal{F}$ is the largest h such that $S_{\mathcal{F}} = 2^h$, i.e., the maximum number of points that $\mathcal{F}$ can shatter.

Recall

VC Theorem: The bound for $R^{\text{true}}(f) - R^{\text{emp}}(f)$ in terms of Growth Function is,

$$P\left[\sup_{f\in\mathcal{F}}\left(R^{\text{true}}(f) - R^{\text{emp}}(f)\right) \leq 2\sqrt{2\frac{\log S_{\mathcal{F}}(2m) + \log\frac{4}{\delta}}{m}}\right] \geq 1-\delta$$

Sauer's Lemma shows an upper bound of the growth function. Supposing the VC Dimension of $\mathcal{F}$ is h, and there are m points to be classified, then the growth function can be written as

$$S_{\mathcal{F}}(m) \leq \sum_{i=0}^{h}\binom{m}{i}$$

In particular, when $m \geq h$, we further have

$$S_{\mathcal{F}}(m) \leq (\frac{em}{h})^h$$

By combining Sauer's Lemma and VC Theorem we get

$$P\left[\sup_{f\in\mathcal{F}}\left(R^{\text{true}}(f) - R^{\text{emp}}(f)\right) \leq 2\sqrt{2\frac{h\log\frac{2em}{h} + \log\frac{4}{\delta}}{m}}\right] \geq 1-\delta$$

Other Types of Capacity Measure

The capacity measure is a metric for the learning ability (complexity/expressiveness/richness...) of a function class $\mathcal{F}$.

Some types of Capacity Measure:

- Growth Function and VC dimension
- VC entropy
- Covering Numbers
- Rademacher Average

Growth Function and VC dimension are independent of data distribution. The bound based on them may be loose for most distributions.

The other 3 measures are data dependent, which means they can generate tighter bounds. We will focus on Rademacher Average and Covering Numbers.

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Rademacher Average

One way to measure complexity is to see how functions from the class $\mathcal{F}$ can classify random noise.

Suppose the dataset $S = \{(x_1, y_1), (x_2, y_2), ..., (x_m, y_m)\}$ has m samples where $x_i \in \mathbb{R}^n$ and $y_i \in \{-1, 1\}$ (Note that y_i is not in $\{0, 1\}$ anymore). Let each $f \in \mathcal{F}$ be a binary classification function such that $f(x_i) \in \{-1, 1\}$ for $i = 1, 2, ..., m$.

We can write the empirical risk function as

$$
\begin{aligned}
R^{\text{emp}}(f) &= \frac{1}{m} \sum_{i=1}^{m} \mathbf{1}_{f(x_i) \neq y_i} \\
&= \frac{1}{m} \sum_{i=1}^{m} \begin{cases} 1 & (f(x_i), y_i) = (1, -1) \text{ or } (-1, 1) \\ 0 & (f(x_i), y_i) = (1, 1) \text{ or } (-1, -1) \end{cases} \\
&= \frac{1}{m} \sum_{i=1}^{m} \frac{1 - y_i f(x_i)}{2} \\
&= \frac{1}{2} - \frac{1}{2m} \sum_{i=1}^{m} y_i f(x_i)
\end{aligned}
$$

The term $\frac{1}{m} \sum_{i=1}^{m} y_i f(x_i)$ can be interpreted as the correlation of the predictions $f(x_i)$ with the labels y_i. Since $\frac{1}{m} \sum_{i=1}^{m} y_i f(x_i) = 1 - 2R^{\text{emp}}(f)$, the greater $\frac{1}{m} \sum_{i=1}^{m} y_i f(x_i)$ makes the smaller $R^{\text{emp}}(f)$, which means $\mathcal{F}$ has stronger ability to learn.

Rademacher Average

Thus our goal is to find an $f \in \mathcal{F}$ satisfying

$$\underset{f \in \mathcal{F}}{\operatorname{argmax}} \frac{1}{m} \sum_{i=1}^{m} y_i f(x_i)$$

Now assume the labels satisfy a specific distribution, i.e., each label is a Rademacher random variable.

Definition (Rademacher Variable): A random variable σ is called Rademacher random variable if $P(\sigma = 1) = P(\sigma = -1) = 1/2$.

Let $\sigma_1, \sigma_2, ..., \sigma_m$ be iid Rademacher random variables. Replacing $y_1, y_2, ..., y_m$ by $\sigma_1, \sigma_2, ..., \sigma_m$, we get

$$\underset{f \in \mathcal{F}}{\operatorname{argmax}} \frac{1}{m} \sum_{i=1}^{m} \sigma_i f(x_i)$$

Instead of select the $f \in \mathcal{F}$ that correlates best with labels, this selects the $f \in \mathcal{F}$ that correlates best with random noise variables σ_i. To measure how well $\mathcal{F}$ correlate with random noise, we take the expectation of this correlation over the random variables σ_i:

Definition (Empirical Rademacher Average): For a class $\mathcal{F}$ of functions, on a given dataset S, the Empirical Rademacher Average is defined as

$$\hat{\mathcal{R}}_S(\mathcal{F}) = \mathbb{E}_{\boldsymbol{\sigma}} \sup_{f \in \mathcal{F}} (\frac{1}{m} \sum_{i=1}^{m} \sigma_i f(x_i))$$

Rademacher Average

By our definition of Empirical Rademacher Average $\hat{\mathcal{R}}_S(\mathcal{F}) = \mathbb{E}_{\boldsymbol{\sigma}} \sup_{f \in \mathcal{F}}(\frac{1}{m}\sum_{i=1}^{m} \sigma_i f(x_i))$, we have $0 \leq \hat{\mathcal{R}}_S(\mathcal{F}) \leq 1$. Consider two extreme cases:

- When $S_m(\mathcal{F}) = 1$, we only have one group of f and can only generate one way of classification. In this case $\hat{\mathcal{R}}_S(\mathcal{F}) = 0$ since the max term disappears.
- When $S_m(\mathcal{F}) = 2^m$, $\mathcal{F}$ shatters S. In this case $\hat{\mathcal{R}}_S(\mathcal{F}) = 1$.

Remember that the dataset S is sampled from an unknown distribution D. The expectation of the empirical Rademacher average of all $S \sim D$ is called the Rademacher Average:

Definition (Rademacher Average): For a class $\mathcal{F}$ of functions, the Rademacher Average is defined as

$$\mathcal{R}(\mathcal{F}) = \mathbb{E}_S(\hat{\mathcal{R}}_S(\mathcal{F})) = \mathbb{E}_{\boldsymbol{\sigma},S} \sup_{f \in \mathcal{F}}(\frac{1}{m}\sum_{i=1}^{m} \sigma_i f(x_i))$$

Rademacher Average

In $\mathcal{R}(\mathcal{F}) = \mathbb{E}_{\boldsymbol{\sigma},S} \sup_{f\in\mathcal{F}}(\frac{1}{m}\sum_{i=1}^{m}\sigma_i f(x_i))$ we define $f \in \{-1, 1\}$. Generally, f can be any function of $\mathbb{R}^d \to \mathbb{R}$, this gives the general definition of Rademacher Average (See reference [2]).

Define $g(x_i) = \mathbf{1}_{f(x_i)\neq y_i} \in \{0, 1\}$, and the set of all gs is a function class $\mathcal{G}$. Define $\mathcal{R}(\mathcal{G}) = \mathbb{E}_{\boldsymbol{\sigma},S} \sup_{g\in\mathcal{G}}(\frac{1}{m}\sum_{i=1}^{m}\sigma_i g(x_i))$. Then we have the following relationship between $\mathcal{R}(\mathcal{F})$ and $\mathcal{R}(\mathcal{G})$:

$$\begin{aligned}
\mathcal{R}(\mathcal{G}) &= \mathbb{E}_{\boldsymbol{\sigma},S} \sup_{g\in\mathcal{G}}(\frac{1}{m}\sum_{i=1}^{m}\sigma_i g(x_i)) = \mathbb{E}_{\boldsymbol{\sigma},S} \sup_{f\in\mathcal{F}}(\frac{1}{m}\sum_{i=1}^{m}\sigma_i \mathbf{1}_{f(x_i)\neq y_i}) \\
&= \mathbb{E}_{\boldsymbol{\sigma},S} \sup_{f\in\mathcal{F}}(\frac{1}{m}\sum_{i=1}^{m}\sigma_i \frac{1 - y_i f(x_i)}{2}) \\
&= \mathbb{E}_{\boldsymbol{\sigma},S} \left[\frac{1}{2m}\sum_{i=1}^{m}\sigma_i + \sup_{f\in\mathcal{F}}(\frac{1}{2m}\sum_{i=1}^{m}(-\sigma_i y_i) f(x_i))\right] \\
&= \mathbb{E}_{\boldsymbol{\sigma},S} \sup_{f\in\mathcal{F}}(\frac{1}{2m}\sum_{i=1}^{m}\sigma_i f(x_i)) = \frac{1}{2}\mathcal{R}(\mathcal{F})
\end{aligned}$$

Rademacher Bound Theorem

Theorem (Rademacher Bound): Let $g(x_i) = \mathbf{1}_{f(x_i)\neq y_i} \in \{0,1\}$, $\mathcal{R}(\mathcal{G}) = \mathbb{E}_{\boldsymbol{\sigma},S} \sup_{g\in\mathcal{G}}(\frac{1}{m}\sum_{i=1}^{m}\sigma_i g(x_i))$ be the Rademacher Average of $\mathcal{F}$, then

$$P\left[\sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f)) \leq 2\mathcal{R}(\mathcal{G}) + \sqrt{\frac{\log\frac{1}{\delta}}{2m}}\right] \geq 1-\delta$$

Remember that $R^{\text{true}}(f)$ is the true risk, $R^{\text{emp}}(f) = \frac{1}{m}\sum_{i=1}^{m}\mathbf{1}_{f(x_i)\neq y_i}$ is the empirical risk and $R^{\text{true}}(f) = E(R^{\text{emp}}(f))$.

To prove this bound, I will first introduce some basics about conditional expectation and supremum, then talk about McDiarmid's Inequality, Symmetrization and Lemma of Rademacher Average. Finally I will put these components together.

Basics: Conditional Expectation

- Conditional Expectation: Suppose we have a bivariate random vector (X, Y) with joint PMF $P(X, Y)$. We call $E_X(X|Y = y)$ the expectation of X given that $Y = y$. Sometimes we just simplify the notation as $E(X|Y)$. Formally,

$$E(X|Y) = \sum_x xP(X = x|Y = y) = \sum_x x\frac{P(X = x, Y = y)}{P(Y = y)}$$

Note that $E_X(X|Y = y)$ is a function of y.

Properties:
(1) If Y is independent of X, then $E(X|Y) = E(X)$.
(2) Let $f(Z)$ be a function of a random variable Z, then $E(f(Z)|Z) = f(Z)$.

- Law of Total Expectation:

$$E(X) = E_Y(E_X(X|Y))$$

Basics: Supremum

- Let $t \in \mathcal{T}$ be a variable, f and g be two functions of t, then

$$\sup\{f(t) + g(t)\} \leq \sup\{f(t)\} + \sup\{g(t)\}$$

Proof: Since $f(t) \leq \sup\{f(t)\}$ and $g(t) \leq \sup\{g(t)\}$, we have $f(t) + g(t) \leq \sup\{f(t)\} + \sup\{g(t)\}$ for any t, thus $\sup\{f(t) + g(t)\} \leq \sup\{f(t)\} + \sup\{g(t)\}$.

- Switching Supremum and Expectation: Let $f(X, y)$ be any function of two variables X and y, and $y \in \mathcal{Y}$, then

$$\sup_{y \in \mathcal{Y}} \mathbb{E} f(X, y) \leq \mathbb{E} \sup_{y \in \mathcal{Y}} f(X, y)$$

Proof: Since $f(X, y) \leq \sup_{y \in \mathcal{Y}} f(X, y)$ for any X, we take expectation with respect to X on both sides, then $\mathbb{E} f(X, y) \leq \mathbb{E} \sup_{y \in \mathcal{Y}} f(X, y)$. Since this inequality holds for any y, we have $\sup_{y \in \mathcal{Y}} \mathbb{E} f(X, y) \leq \mathbb{E} \sup_{y \in \mathcal{Y}} f(X, y)$.

McDiarmid's Inequality

McDiarmid's Inequality: Define the function $F : \mathcal{X}^m \to \mathbb{R}$. Let $(x_1, x_2, ..., x_m) \in \mathcal{X}^m$. Suppose for any i, replacing x_i by any x_i', the following equality holds:

$$\sup_{x_i' \in \mathcal{X}} |F(x_1, ..., x_i, ..., x_m) - F(x_1, ..., x_i', ..., x_m)| \leq c$$

where c is a constant. Then for all $\epsilon > 0$,

$$P(|F - \mathbb{E}(F)| \geq \epsilon) \leq 2 \exp\left(-\frac{2\epsilon^2}{mc^2}\right)$$

Proof: Use the marginal sequence, suppose

$$V_i = \mathbb{E}(F|x_1, x_2, ..., x_i) - \mathbb{E}(F|x_1, x_2, ..., x_{i-1})$$

for $i = 1, 2, ..., m$. Then

$$\mathbb{E}(V_i) = \mathbb{E}(\mathbb{E}(F|x_1, x_2, ..., x_i)) - \mathbb{E}(\mathbb{E}(F|x_1, x_2, ..., x_{i-1})) = \mathbb{E}(F) - \mathbb{E}(F) = 0$$

$$\sum_{i=1}^{m} V_i = \mathbb{E}(F|x_1, x_2, ..., x_m) - \mathbb{E}(F) = F - \mathbb{E}(F)$$

$$
\begin{aligned}
\sup V_i - \inf V_i = & \left[\sup_x \mathbb{E}(F|x_1, x_2, ..., x_{i-1}, x) - \mathbb{E}(F|x_1, x_2, ..., x_{i-1})\right] - \\
& \left[\inf_x \mathbb{E}(F|x_1, x_2, ..., x_{i-1}, x) - \mathbb{E}(F|x_1, x_2, ..., x_{i-1})\right] \\
= & \sup_x \mathbb{E}(F|x_1, x_2, ..., x_{i-1}, x) - \inf_x \mathbb{E}(F|x_1, x_2, ..., x_{i-1}, x) \\
= & \sup_{x_u, x_l} \mathbb{E}(F|x_1, x_2, ..., x_{i-1}, x_u) - \mathbb{E}(F|x_1, x_2, ..., x_{i-1}, x_l) \\
= & \sup_{x_u, x_l} \sum_{x_{i+1}, \ldots, x_m} [F(x_1, ..., x_{i-1}, x_u, x_{i+1}, ..., x_m | x_1, ..., x_{i-1}) - \\
& F(x_1, ..., x_{i-1}, x_l, x_{i+1}, ..., x_m | x_1, ..., x_{i-1})] P(x_{i+1}, ..., x_m) \\
\leq & c \sum_{x_{i+1}, \ldots, x_m} P(x_{i+1}, ..., x_m) = c
\end{aligned}
$$

By Azuma's Inequality (see Appendix),

$$
P(|F - \mathbb{E}(F)| \geq \epsilon) = P(|\sum_{i=1}^{m} V_i| \geq \epsilon) \leq 2 \exp\left(-\frac{2\epsilon^2}{mc^2}\right)
$$

Symmetrization

Let $g(x_i) = \mathbf{1}_{f(x_i) \neq y_i} \in \{0, 1\}$. Since $R^{\text{emp}}(f) = \frac{1}{m} \sum_{j=1}^{m} g(x_j)$, we can consider $R^{\text{emp}}(f)$ as a function of $g(x_1), g(x_2), ..., g(x_m)$. Define $R^{\text{emp,i}}(f) = \frac{1}{m}(\sum_{j=1, j\neq i}^{m} g(x_j)$ $+g(x_i'))$, that is, $g(x_i)$ is replaced by $g(x_i')$. Then the following inequality holds:

$$|\sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f)) - \sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp,i}}(f))| \leq \sup_{f\in\mathcal{F}} |R^{\text{emp,i}}(f) - R^{\text{emp}}(f)| \quad (*)$$

To prove this, suppose f^* achieves $\sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f))$, and $\sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f)) > \sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp,i}}(f))$, then

$$\begin{aligned}
&\sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f)) - \sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp,i}}(f)) \qquad\qquad (\diamond)\\
&= (R^{\text{true}}(f^*) - R^{\text{emp}}(f^*)) - \sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp,i}}(f))\\
&\leq (R^{\text{true}}(f^*) - R^{\text{emp}}(f^*)) - (R^{\text{true}}(f^*) - R^{\text{emp,i}}(f^*))\\
&= R^{\text{emp,i}}(f^*) - R^{\text{emp}}(f^*) \leq |R^{\text{emp,i}}(f^*) - R^{\text{emp}}(f^*)| \leq \sup_{f\in\mathcal{F}} |R^{\text{emp,i}}(f) - R^{\text{emp}}(f)|
\end{aligned}$$

When $\sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f)) \leq \sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp,i}}(f))$, switching the first and second terms in $(\diamond)$ and we can get the same result, thus the inequality $(*)$ holds.

Since $|R^{\text{emp}}(f) - R^{\text{emp,i}}(f)| = \frac{1}{m}|Z_i - Z_i'| \leq \frac{1}{m}$, we have

$$|\sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f)) - \sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp,i}}(f))| \leq \frac{1}{m}$$

This satisfies the condition of McDiarmid's inequality with $c = \frac{1}{m}$, thus

$$P\left[|\sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f)) - \mathbb{E}_S \sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f))| \geq \epsilon\right] \leq 2\exp\left(-2m\epsilon^2\right)$$

Since the McDiarmid's Bound is symmetric, for one-sided bound, we have

$$P\left[\sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f)) - \mathbb{E}_S \sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f)) \geq \epsilon\right] \leq \exp\left(-2m\epsilon^2\right)$$

Lemma of Rademacher Average

Lemma: Let $S = \{(x_1, y_1), (x_2, y_2), ..., (x_m, y_m)\}$ be a set of training samples, and $R^{\text{emp}}(f) = \frac{1}{m}\sum_{i=1}^{m} \mathbf{1}_{f(x_i)\neq y_i} = \frac{1}{m}\sum_{i=1}^{m} g(x_i)$, then

$$\mathbb{E}_S \sup_{f\in\mathcal{F}}[R^{\text{true}}(f) - R^{\text{emp}}(f)] \leq 2\mathbb{E}_{\boldsymbol{\sigma},S} \sup_{f\in\mathcal{F}}(\frac{1}{m}\sum_{i=1}^{m}\sigma_i g(x_i)) = 2\mathcal{R}(\mathcal{G})$$

Proof: Suppose we have a set of ghost samples $S' = \{(x'_1, y'_1), (x'_2, y'_2), ..., (x'_m, y'_m)\}$ and the empirical risk on the ghost samples is $R^{\text{emp}\prime}(f) = \frac{1}{m}\sum_{i=1}^{m} g(x'_i)$. Since $R^{\text{true}}(f) = \mathbb{E}_{S'}[R^{\text{emp}\prime}(f)|S]$ and $R^{\text{emp}}(f) = \mathbb{E}_{S'}[R^{\text{emp}}(f)|S]$. Then,

$$\begin{aligned}
\mathbb{E}_S \sup_{f\in\mathcal{F}}[R^{\text{true}}(f) - R^{\text{emp}}(f)] &= \mathbb{E}_S \sup_{f\in\mathcal{F}}[\mathbb{E}_{S'}[R^{\text{emp}\prime}(f)|S] - \mathbb{E}_{S'}[R^{\text{emp}}(f)|S]] \\
&= \mathbb{E}_S \sup_{f\in\mathcal{F}}[\mathbb{E}_{S'}[(R^{\text{emp}\prime}(f) - R^{\text{emp}}(f))|S]] \\
&\leq \mathbb{E}_S\mathbb{E}_{S'} \sup_{f\in\mathcal{F}}[(R^{\text{emp}\prime}(f) - R^{\text{emp}}(f))|S] \qquad (\diamond) \\
&= \mathbb{E}_{S,S'}\left[\sup_{g\in\mathcal{G}}\frac{1}{m}\sum_{i=1}^{m}(g(x'_i) - g(x_i))\right]
\end{aligned}$$

In $(\diamond)$ we use the property that $\sup_{y\in\mathcal{Y}}\mathbb{E}f(X, y) \leq \mathbb{E}\sup_{y\in\mathcal{Y}} f(X, y)$.

Now we introduce a set of Rademacher variables $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, ..., \sigma_m\}$. Each σ_i satisfies $P(\sigma_i = -1) = P(\sigma_i = 1) = \frac{1}{2}$.

$$
\begin{aligned}
\mathbb{E}_S \sup_{f \in \mathcal{F}} [R^{\text{true}}(f) - R^{\text{emp}}(f)] &\leq \mathbb{E}_{S,S'} \left[\sup_{g \in \mathcal{G}} \frac{1}{m} \sum_{i=1}^{m} (g(x_i') - g(x_i)) \right] \\
&= \mathbb{E}_{\boldsymbol{\sigma},S,S'} \left[\sup_{g \in \mathcal{G}} \frac{1}{m} \sum_{i=1}^{m} \sigma_i (g(x_i') - g(x_i)) \right] \qquad (*) \\
&\leq \mathbb{E}_{\boldsymbol{\sigma},S'} \left[\sup_{g \in \mathcal{G}} \frac{1}{m} \sum_{i=1}^{m} \sigma_i g(x_i') \right] + \mathbb{E}_{\boldsymbol{\sigma},S} \left[\sup_{g \in \mathcal{G}} \frac{1}{m} \sum_{i=1}^{m} -\sigma_i g(x_i) \right] \qquad (\star) \\
&= 2\mathbb{E}_{\boldsymbol{\sigma},S} \sup_{g \in \mathcal{G}} (\frac{1}{m} \sum_{i=1}^{m} \sigma_i g(x_i)) = 2\mathcal{R}(\mathcal{G})
\end{aligned}
$$

The equality $(*)$ uses the fact that, since $g(x_i')$ and $g(x_i)$ are interchangeable and $\sigma_i = -1$ or 1, multiplying σ_i will not change the distribution of $g(x_i') - g(x_i)$. The inequality $(\star)$ uses the fact $\sup\{f(t) + g(t)\} \leq \sup\{f(t)\} + \sup\{g(t)\}$.

Putting All Components Together

By Symmertrization we have the following one-sided bound:

$$P\left[\sup_{f\in\mathcal{F}}(R^{\textsf{true}}(f)-R^{\textsf{emp}}(f))-\mathbb{E}\sup_{f\in\mathcal{F}}(R^{\textsf{true}}(f)-R^{\textsf{emp}}(f))\geq\epsilon\right]\leq\exp\left(-2m\epsilon^2\right)$$

Let $\delta=\exp\left(-2m\epsilon^2\right)$, so that $\epsilon=\sqrt{\frac{\log\frac{1}{\delta}}{2m}}$, thus

$$P\left[\sup_{f\in\mathcal{F}}(R^{\textsf{true}}(f)-R^{\textsf{emp}}(f))-\mathbb{E}\sup_{f\in\mathcal{F}}(R^{\textsf{true}}(f)-R^{\textsf{emp}}(f))\geq\sqrt{\frac{\log\frac{1}{\delta}}{2m}}\right]\leq\delta$$

$$P\left[\sup_{f\in\mathcal{F}}(R^{\textsf{true}}(f)-R^{\textsf{emp}}(f))-\mathbb{E}\sup_{f\in\mathcal{F}}(R^{\textsf{true}}(f)-R^{\textsf{emp}}(f))\leq\sqrt{\frac{\log\frac{1}{\delta}}{2m}}\right]\geq1-\delta$$

$$P\left[\sup_{f\in\mathcal{F}}(R^{\textsf{true}}(f)-R^{\textsf{emp}}(f))\leq\mathbb{E}\sup_{f\in\mathcal{F}}(R^{\textsf{true}}(f)-R^{\textsf{emp}}(f))+\sqrt{\frac{\log\frac{1}{\delta}}{2m}}\right]\geq1-\delta$$

By Lemma of Rademacher Average, $\mathbb{E}\sup_{f\in\mathcal{F}}(R^{\textsf{true}}(f)-R^{\textsf{emp}}(f))\leq2\mathcal{R}(\mathcal{G})$, therefore

$$P\left[\sup_{f\in\mathcal{F}}(R^{\textsf{true}}(f)-R^{\textsf{emp}}(f))\leq2\mathcal{R}(\mathcal{G})+\sqrt{\frac{\log\frac{1}{\delta}}{2m}}\right]\geq1-\delta$$

The bound of Empirical Rademacher Average

We can also bound $\sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f))$ by Empirical Rademacher Average. Since $\hat{\mathcal{R}}_S(\mathcal{G}) = \mathbb{E}_{\boldsymbol{\sigma}} \sup_{g\in\mathcal{G}}(\frac{1}{m}\sum_{i=1}^m \sigma_i g(x_i))$, changing one x_i will change $\hat{\mathcal{R}}_S(\mathcal{G})$ by at most $\frac{1}{m}$. Then by McDiarmid's Inequality

$$P\left[\left|\hat{\mathcal{R}}_S(\mathcal{G}) - \mathcal{R}(\mathcal{G})\right| > \epsilon\right] \le 2\exp\left(-2m\epsilon^2\right)$$

$$P\left[\hat{\mathcal{R}}_S(\mathcal{G}) - \mathcal{R}(\mathcal{G}) > \epsilon\right] \le \exp\left(-2m\epsilon^2\right) \qquad (1)$$

Since

$$P\left[\sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f)) \le 2\mathcal{R}(\mathcal{G}) + \epsilon\right] \ge 1 - \exp\left(-2m\epsilon^2\right)$$

$$P\left[\sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f)) \ge 2\mathcal{R}(\mathcal{G}) + \epsilon\right] \le \exp\left(-2m\epsilon^2\right) \qquad (2)$$

By (1) and (2), we have

$$\begin{aligned}
&P\left[\sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f)) \ge 2\hat{\mathcal{R}}_S(\mathcal{G}) + 3\epsilon\right] \\
&\le P\left[\sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f)) \ge 2\mathcal{R}(\mathcal{G}) + \epsilon \text{ or } \hat{\mathcal{R}}_S(\mathcal{G}) - \mathcal{R}(\mathcal{G}) \ge \epsilon\right] \\
&\le P\left[\sup_{f\in\mathcal{F}}(R^{\text{true}}(f) - R^{\text{emp}}(f)) \ge 2\mathcal{R}(\mathcal{G}) + \epsilon\right] + P\left[\hat{\mathcal{R}}_S(\mathcal{G}) - \mathcal{R}(\mathcal{G}) \ge \epsilon\right] \le 2\exp\left(-2m\epsilon^2\right)
\end{aligned}$$

Let $\delta = 2\exp\left(-2m\epsilon^2\right)$, so that $\epsilon = \sqrt{\frac{\log\frac{2}{\delta}}{2m}}$, thus we have the following Empirical Rademacher Average bound:

$$P\left[\sup_{f\in\mathcal{F}}(R^{\textsf{true}}(f) - R^{\textsf{emp}}(f)) \geq 2\hat{\mathcal{R}}_S(\mathcal{G}) + 3\sqrt{\frac{\log\frac{2}{\delta}}{2m}}\right] \leq \delta$$

$$P\left[\sup_{f\in\mathcal{F}}(R^{\textsf{true}}(f) - R^{\textsf{emp}}(f)) \leq 2\hat{\mathcal{R}}_S(\mathcal{G}) + 3\sqrt{\frac{\log\frac{2}{\delta}}{2m}}\right] \geq 1-\delta$$

Massart's Lemma

Since we have

$$P\left[\sup_{f\in\mathcal{F}}\left(R^{\text{true}}(f)-R^{\text{emp}}(f)\right)\le 2\mathcal{R}(\mathcal{G})+\sqrt{\frac{\log\frac{1}{\delta}}{2m}}\right]\ge 1-\delta$$

The question is, how can we calculate $\mathcal{R}(\mathcal{G})$? The following lemma gives an upperbound:

Massart's Lemma: Let $A\subset\mathbb{R}^m$ be a finite set of points and $a=\{a_1,a_2,...,a_m\}\in A$. Let $r=\sup_{a\in A}\|a\|_2$, then

$$\mathbb{E}_{\boldsymbol{\sigma}}\left[\sup_{a\in A}\sum_{i=1}^m\sigma_i a_i\right]\le r\sqrt{2\log|A|}$$

Proof: Let $t>0$ be a number to be chosen later.

$$\exp\left(t\mathbb{E}_{\boldsymbol{\sigma}}\sup_{a\in A}\sum_{i=1}^m\sigma_i a_i\right)\le\mathbb{E}_{\boldsymbol{\sigma}}\exp\left(t\sup_{a\in A}\sum_{i=1}^m\sigma_i a_i\right)=\mathbb{E}_{\boldsymbol{\sigma}}\sup_{a\in A}\exp\left(\sum_{i=1}^m t\sigma_i a_i\right)$$

$$=\mathbb{E}_{\boldsymbol{\sigma}}\sup_{a\in A}\left(\prod_{i=1}^m\exp\left(t\sigma_i a_i\right)\right)\le\mathbb{E}_{\boldsymbol{\sigma}}\sum_{a\in A}\prod_{i=1}^m\exp\left(t\sigma_i a_i\right)=\sum_{a\in A}\prod_{i=1}^m\mathbb{E}_{\boldsymbol{\sigma}}\exp\left(t\sigma_i a_i\right)$$

$$\le\sum_{a\in A}\prod_{i=1}^m\exp\left(\frac{t^2a_i^2}{2}\right)=\sum_{a\in A}\exp\left(\frac{t^2(\sum_{i=1}^m a_i^2)}{2}\right)\le|A|\exp\left(\frac{t^2r^2}{2}\right)$$

The $\leq$ uses the Jenson's Inequality. The $\leq$ uses Hoeffding's Lemma, given that $-|a_i| \leq \sigma_i a_i \leq |a_i|$.

Therefore, taking $\log$ on both sides, we get

$$\mathbb{E}_{\boldsymbol{\sigma}} \sup_{a \in A} \sum_{i=1}^{m} \sigma_i a_i \leq \inf_{t>0} \left(\frac{\log |A|}{t} + \frac{tr^2}{2} \right) = r\sqrt{2 \log |A|}$$

Let $A = \mathcal{F}$, $|A| = S_m(\mathcal{F})$, $a_i = f(x_i) \in \{-1, 1\}$, then $r = \sqrt{m}$. Therefore

$$\mathcal{R}(\mathcal{F}) = \mathbb{E}_{\boldsymbol{\sigma}, S} \sup_{f \in \mathcal{F}} \frac{1}{m} \sum_{i=1}^{m} \sigma_i f(x_i) \leq \frac{r}{m} \sqrt{2 \log S_m(\mathcal{F})} = \sqrt{\frac{2 \log S_m(\mathcal{F})}{m}}$$

By Sauer's Lemma, $S_m(\mathcal{F}) \leq (\frac{em}{h})^h$, where h is the VC dimension of $\mathcal{F}$, then

$$\mathcal{R}(\mathcal{G}) = \frac{1}{2} \mathcal{R}(\mathcal{F}) \leq \frac{1}{2} \sqrt{\frac{2 \log S_m(\mathcal{F})}{m}} \leq \frac{1}{2} \sqrt{\frac{2h \log \frac{em}{h}}{m}}$$

Thus the Rademacher Bound becomes

$$P \left[\sup_{f \in \mathcal{F}} (R^{\text{true}}(f) - R^{\text{emp}}(f)) \leq \sqrt{\frac{2h \log \frac{em}{h}}{m}} + \sqrt{\frac{\log \frac{1}{\delta}}{2m}} \right] \geq 1 - \delta$$

Comparing the Rademacher Bound and the Growth Function Bound, both bounds grow like $O(\sqrt{\frac{h \log(em/h)}{m}})$ with respect to m and h, so the Rademacher Bound will not be looser than the Growth Function Bound.

Actually Massart's Lemma gives a loose bound for $\mathcal{R}(\mathcal{G})$. Sometimes we bound $\mathcal{R}(\mathcal{G})$ by a constant instead of Massart's Lemma. For example, since $g(x_i) \in \{0, 1\}$

$$\mathcal{R}(\mathcal{G}) = \frac{1}{2}\mathcal{R}(\mathcal{F}) = \frac{1}{2}\mathbb{E}_{\boldsymbol{\sigma}, S} \sup_{f \in \mathcal{F}} \frac{1}{m} \sum_{i=1}^{m} \sigma_i f(x_i) \leq \frac{1}{2}$$

$\mathcal{R}(\mathcal{G})$ reaches its maximum when given any dataset S, there always exists an $f \in \mathcal{F}$ that perfectly classifies it. This gives another upperbound for $\mathcal{R}(\mathcal{G})$.

Contents

Covering Number

Recall that we have a dataset $S = \{(x_1, y_1), (x_2, y_2), ..., (x_m, y_m)\}$, and a function class $\mathcal{F}$. The functions $f \in \mathcal{F}$ are binary-valued, i.e., $f(x_i) \in \{-1, 1\}$. The Growth Function partitions $\mathcal{F}$ into finite number of groups because the cardinality of the set $\{-1, 1\}$ is finite. Since the cardinality of $\{-1, 1\}$ is 2 and we have m samples, we can partition $\mathcal{F}$ into at most 2^m groups.

However, when f is continuous, the candinarity of the value set will be infinite, thus the Growth Function will not work. To address this, we introduce another way to measure the capacity of $\mathcal{F}$ – Covering Number.

Let $(\mathcal{F}, d)$ be a metric space, and we define the distance of a function $f \in \mathcal{F}$ by L_p norm of m points:

$$\|f\|_{L_p(m)} = \left(\frac{1}{m} \sum_{i=1}^{m} |f(x_i)|^p \right)^{1/p}$$

The distance of two functions f and f' can be defined as

$$\|f - f'\|_{L_p(m)} = \left(\frac{1}{m} \sum_{i=1}^{m} |f(x_i) - f'(x_i)|^p \right)^{1/p}$$

Let's use L_1 norm to define the distance, then

$$\|f - f'\|_{L_1(m)} = \left(\frac{1}{m} \sum_{i=1}^{m} |f(x_i) - f'(x_i)| \right)$$

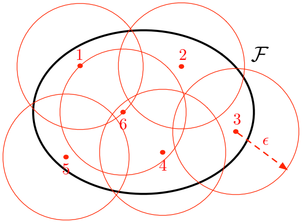
Covering Number

Denote $B(f_j, \epsilon)$ as a ball of radius ϵ around f_j, using the metric $L_p(m)$. $B(f_j, \epsilon)$ contains all the function in $\mathcal{F}$ that are within distance ϵ of f_j, i.e., $B(f_j, \epsilon) = \{f' : \|f - f'\|_{L_p(m)} < \epsilon\}$. Here $B(f_j, \epsilon)$ is considered as one group of functions, and we would like to know how many groups can cover $\mathcal{F}$. We say that $f_1, f_2, ..., f_N$ covers $\mathcal{F}$ at radius ϵ if:

$$\mathcal{F} \subset \bigcup_{j=1}^{N} B(f_j, \epsilon)$$

Definition (Covering Number): The covering number of $\mathcal{F}$ at radius $\mathcal{F}$ with respect to $L_p(m)$, denoted by $N(\mathcal{F}, \epsilon, L_p(m))$, is the minimum N such that $f_1, f_2, ..., f_N$ covers $\mathcal{F}$ at radius ϵ.

If the covering number is finite, we can approximately represent $\mathcal{F}$ by a finite set of functions that cover $\mathcal{F}$. This is another a finite measure for the capacity of $\mathcal{F}$.



Discretization Theorem

We can bound the empirical Rademacher Average with the following theorem:

Theorem (Discretication): Let $-1 \leq f \leq 1$ for all $f \in \mathcal{F}$, then for any $S = \{x_1, x_2, ..., x_m\}$ we have

$$\hat{\mathcal{R}}_S(\mathcal{F}) \leq \inf_{\epsilon \geq 0} \{\epsilon + \sqrt{\frac{2\log(2N(\mathcal{F}, \epsilon, L_1(m)))}{m}}\}$$

Proof: Fix $S = \{x_1, x_2, ..., x_m\}$ and $\epsilon > 0$. Let V be the minimal set of ϵ-balls that covers $\mathcal{F}$, thus $|V| = N(\mathcal{F}, \epsilon, L_1(m))$. For any $f \in \mathcal{F}$, define $f' \in V$ such that $\|f - f'\|_{L_1(m)} < \epsilon$. Then

$$\begin{aligned}
\hat{\mathcal{R}}_S(\mathcal{F}) &= \mathbb{E}_{\boldsymbol{\sigma}} \sup_{f \in \mathcal{F}} (\frac{1}{m} \sum_{i=1}^{m} \sigma_i f(x_i)) \\
&\leq \mathbb{E}_{\boldsymbol{\sigma}} \sup_{f \in \mathcal{F}} (\frac{1}{m} \sum_{i=1}^{m} \sigma_i (f(x_i) - f'(x_i))) + \mathbb{E}_{\boldsymbol{\sigma}} \sup_{f \in \mathcal{F}} (\frac{1}{m} \sum_{i=1}^{m} \sigma_i f'(x_i)) \\
&\leq \epsilon + \mathbb{E}_{\boldsymbol{\sigma}} \sup_{f \in \mathcal{F}} (\frac{1}{m} \sum_{i=1}^{m} \sigma_i f'(x_i)) \\
&\leq \epsilon + \sqrt{\frac{2\log(2|V|)}{m}} \qquad \text{[Massart's Lemma]} \\
&\leq \epsilon + \sqrt{\frac{2\log(2N(\mathcal{F}, \epsilon, L_1(m)))}{m}}
\end{aligned}$$

Note that this inequality holds for any $\epsilon \geq 0$, so we can add an $\inf_{\epsilon \geq 0}$ on the RHS.

Covering Theorem

The following theorem gives a generalization error bound using Covering Number:

Theorem (Covering): Let $R^{\text{emp}}(f) = \frac{1}{m}\sum_{i=1}^{m}(f(x_i) - y_i)^2$ and $R^{\text{true}}(f) = \mathbb{E}[R^{\text{emp}}(f)]$. Suppose there exists $M > 0$ such that $|f(x_i) - y_i| \leq M$ for any x_i. For any $f_1, f_2 \in \mathcal{F}$, define the L_∞ measure as $\|f_1 - f_2\|_{L_\infty} = \max_x |f_1(x) - f_2(x)|$. Then for any $\epsilon > 0$,

$$P\left[\sup_{f\in\mathcal{F}} \left|R^{\text{true}}(f) - R^{\text{emp}}(f)\right| \geq \epsilon\right] \leq 2N(\mathcal{F}, \frac{\epsilon}{8M}, L_\infty) \exp\left(-\frac{m\epsilon^2}{2M^4}\right)$$

Proof:
Let $L_S(f) = R^{\text{true}}(f) - R^{\text{emp}}(f)$, we show that for all $f_1, f_2 \in \mathcal{F}$ and any dataset S, the following inequality holds: $|L_S(f_1) - L_S(f_2)| \leq 4M\|f_1 - f_2\|_{L_\infty}$.

This is because

$$\begin{aligned}
|L_S(f_1) - L_S(f_2)| &\leq |R^{\text{true}}(f_1) - R^{\text{true}}(f_2)| + |R^{\text{emp}}(f_1) - R^{\text{emp}}(f_2)| \\
&= \left|\mathbb{E}_{x,y}[(f_1(x) - y)^2 - (f_2(x) - y)^2]\right| + \frac{1}{m}\sum_{i=1}^{m}\left|(f_1(x_i) - y_i)^2 - (f_2(x_i) - y_i)^2\right|
\end{aligned}$$

Since for any x, y,

$$\begin{aligned}
\left|(f_1(x) - y)^2 - (f_2(x) - y)^2\right| &= |(f_1(x) - f_2(x))[(f_1(x) - y) + (f_2(x) - y)]| \\
&\leq \|f_1 - f_2\|_{L_\infty} 2M
\end{aligned}$$

We have $|L_S(f_1) - L_S(f_2)| \leq 4M\|f_1 - f_2\|_{L_\infty}$.

Assume $\mathcal{F}$ can be covered by $B_1, B_2, ..., B_k$ such that $\mathcal{F} \subset \bigcup_{i=1}^{k} B_i$, then

$$P\left[\sup_{f\in\mathcal{F}} \left|R^{\text{true}}(f) - R^{\text{emp}}(f)\right| \geq \epsilon\right] = P\left[\bigcup_{i=1}^{k} \sup_{f\in B_i} \left|R^{\text{true}}(f) - R^{\text{emp}}(f)\right| \geq \epsilon\right]$$
$$\leq \sum_{i=1}^{k} P\left[\sup_{f\in B_i} \left|R^{\text{true}}(f) - R^{\text{emp}}(f)\right| \geq \epsilon\right] \qquad (*)$$

Let f_i be the center of the ball B_i and the radius of B_i is $\frac{\epsilon}{8M}$, then for all $f \in B_i$, we have $\|f - f_i\|_{L_\infty} \leq \frac{\epsilon}{8M}$, thus

$$|L_S(f) - L_S(f_i)| \leq 4M\|f - f_i\|_{L_\infty} \leq \frac{\epsilon}{2}$$

Therefore, let the events $A : |L_S(f)| \geq \epsilon$ and $B : |L_S(f_i)| \geq \frac{\epsilon}{2}$, we must have $A \Rightarrow B$. Thus for any $f \in B_i$

$$P\left[\left|R^{\text{true}}(f) - R^{\text{emp}}(f)\right| \geq \epsilon\right] \leq P\left[\left|R^{\text{true}}(f_i) - R^{\text{emp}}(f_i)\right| \geq \frac{\epsilon}{2}\right]$$

Let A, B, C be three events. If $A \Rightarrow C$ and $B \Rightarrow C$, then $A \cup B \Rightarrow C$, thus $P(A \cup B) \leq P(C)$. We have

$$P\left[\sup_{f\in B_i} \left|R^{\text{true}}(f) - R^{\text{emp}}(f)\right| \geq \epsilon\right] = P\left[\bigcup_{f\in B_i} \left|R^{\text{true}}(f) - R^{\text{emp}}(f)\right| \geq \epsilon\right]$$
$$\leq P\left[\left|R^{\text{true}}(f_i) - R^{\text{emp}}(f_i)\right| \geq \frac{\epsilon}{2}\right]$$

Since $R^{\textsf{emp}}(f_j) = \frac{1}{m}\sum_{i=1}^{m}(f_j(x_i) - y_i)^2 \le M^2$, use Hoeffding Bound, we have

$$P\left[\left|R^{\textsf{true}}(f_i) - R^{\textsf{emp}}(f_i)\right| \ge \frac{\epsilon}{2}\right] \le 2\exp\left(-\frac{2m(\epsilon/2)^2}{(M^2)^2}\right) = 2\exp\left(-\frac{m\epsilon^2}{2M^4}\right)$$

Since k is the minimum number of the balls B_i, and each B_i is with radius $\frac{\epsilon}{8M}$, we have $k = N(\mathcal{F}, \frac{\epsilon}{8M}, L_\infty)$.

Putting all things together, we get

$$P\left[\sup_{f\in\mathcal{F}}\left|R^{\textsf{true}}(f) - R^{\textsf{emp}}(f)\right| \ge \epsilon\right] \le 2N(\mathcal{F}, \frac{\epsilon}{8M}, L_\infty)\exp\left(-\frac{m\epsilon^2}{2M^4}\right)$$

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Appendix: Azuma's Inequality

Azuma's Inequality is a generalization of Hoeffding's Inequality.

Azuma's Inequality: Suppose $F : \mathcal{X}^m \to \mathbb{R}$ is a function of variables $x_1, x_2, ..., x_m$. Let $Z_i = \mathbb{E}(F|x_1, x_2, ..., x_i)$ for $i \geq 1$ and $Z_0 = \mathbb{E}(F)$. Let $V_i = Z_i - Z_{i-1}$ be the marginal sequence, and assume $\sup V_i - \inf V_i \leq c$ for $i = 1, 2, ..., m$, where c is a constant. Then

$$P\left[Z_m - Z_0 \geq \epsilon\right] = P\left[\sum_{i=1}^{m} V_i \geq \epsilon\right] \leq \exp\left(-\frac{2\epsilon^2}{mc^2}\right)$$

Proof: Let $t > 0$ be a value to be determined

$$P\left[\sum_{i=1}^{m} V_i \geq \epsilon\right] \leq e^{-t\epsilon}\mathbb{E}\left[e^{t\sum_{i=1}^{m} V_i}\right] \qquad \text{[Chernoff Bound]}$$

$$= e^{-t\epsilon}\mathbb{E}\left[\mathbb{E}\left[e^{t\sum_{i=1}^{m} V_i}\middle| x_1, x_2, ..., x_{m-1}\right]\right] \qquad \text{[Law of the total expectation]}$$

$$= e^{-t\epsilon}\mathbb{E}\left[e^{t\sum_{i=1}^{m-1} V_i}\mathbb{E}\left[e^{tV_m}\middle| x_1, x_2, ..., x_{m-1}\right]\right]$$

The last equality is because $\sum_{i=1}^{m-1} V_i$ can be considered as a function of $x_1, x_2, ..., x_{m-1}$. Let h be a function of X, we have

$$\mathbb{E}_X[\mathbb{E}_Y[h(X)Y|X]] = \mathbb{E}_X[h(X)\mathbb{E}_Y[Y|X]]$$

Appendix: Azuma's Inequality

Recall the Hoeffding Lemma: Let X be any random variable such that $a \le X \le b$, for all $t \in \mathbb{R}$, $\mathbb{E}[e^{tX}] \le \exp(\frac{t^2(b-a)^2}{8})$. Therefore,

$$\mathbb{E}\left[e^{tV_m}\middle| x_1, x_2, ..., x_{m-1}\right] \le e^{\frac{t^2c^2}{8}}$$

Induct the rest part $e^{t\sum_{i=1}^{m-1} V_i}$ by the following formula:

$$\mathbb{E}\left[e^{t\sum_{i=1}^{k} V_i}\right] = \mathbb{E}\left[e^{t\sum_{i=1}^{k-1} V_i}\mathbb{E}\left[e^{tV_k}\middle| x_1, x_2, ..., x_{k-1}\right]\right]$$

for $k = m-1, m-2, ..., 1$, we get

$$P\left[\sum_{i=1}^{m} V_i \ge \epsilon\right] \le e^{-t\epsilon}e^{\frac{mt^2c^2}{8}} = \exp\left(-t\epsilon + \frac{mt^2c^2}{8}\right)$$

The above inequality holds for any t. Let $t = \frac{4\epsilon}{mc^2}$, we get $\inf_t(-t\epsilon + \frac{mt^2c^2}{8}) = -\frac{2\epsilon}{mc^2}$, thus

$$P\left[\sum_{i=1}^{m} V_i \ge \epsilon\right] \le \exp\left(-\frac{2\epsilon}{mc^2}\right)$$

Appendix: Azuma's Inequality

Since the Chernoff Bound is symmetric, for two-sided bound, we have

$$P\left[\left|\sum_{i=1}^{m} V_i\right| \geq \epsilon\right] \leq 2\exp\left(-\frac{2\epsilon^2}{mc^2}\right)$$

Note that Azuma's Inequality does not assume $V_1, V_2, ..., V_m$ to be independent. In fact $V_1, V_2, ..., V_m$ are can be dependent (see [8]), we can not apply Hoeffding's Inequality to them. That is the reason we use Azuma's Inequality instead.