Sets.

Relations.

Groups.

# I. Sets

## 1.1. Basic set properties

cardinality : # of elmts in a set.

↳ $|A|$ or $n(A)$.

## 1. 3. Subset

$A \subseteq B \Leftrightarrow \forall x \in A \Rightarrow x \in B$.

| | |
|---|---|
| e.g. $\{\emptyset\} \in \{\{\emptyset\}\}$ | $\{x\} \in \{\{x\}, y, z\}$ |
| $\emptyset \subseteq \{\{\emptyset\}\}$ | $\{x\} \subset \{x, y, z\}$ |
| $\{\emptyset\} \not\subset \{\{\emptyset\}\}$ | $\{x\} \not\subset \{\{x\}, y, z\}$ |

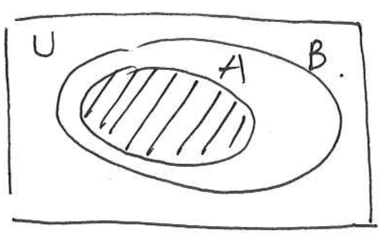
* $\{\emptyset\} \neq \emptyset$

## 1.2. Venn Diagrams

* ~ "contra-positive".

$A \Rightarrow B$. ( A inside B ).

$\neg B \Rightarrow \neg A$ ( $\forall$ elmt not in $B \Rightarrow$ not in $A$ )




## 1.4. Power Set

Def. $P(A) = \{ X \mid X \subseteq A \}$.

the set of all subsets of A.

Thrm

$|A| = n. \Rightarrow |P(A)| = 2^n$.

[ proof by binomial thrm ].

## 1.5. Operation on sets.

disjoint sets : $A \cap B = \emptyset$

property.

- Associativity.
- Distributivity.

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

- Commutative.

$\bigcup_{i=1}^{n} A_i$. $\bigcap_{i=1}^{n} A_i$.

## 1.6 Set Differences

Def. $A \setminus B = \{ x \mid x \in A \text{ and } x \notin B \}$.

$A' = \{x \mid x \in U \text{ and } x \notin A\}$

* $A \setminus B = A \cap B'$.

**symmetric difference**

$$A \triangle B = \{x \mid x \in (A \cup B) \text{ and } x \notin (A \cap B)\}.$$

$$= (A \cup B) \setminus (A \cap B)$$

$$= (A \setminus B) \cup (B \setminus A)$$

**De Morgan's laws**

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

**Disjoint:**

Intersection is empty

# II Relations.

## 2.1. Relations.

- The Cartesian product.

  **Def.** $A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$.

- Relations

  The Cartesian product is distributive over set intersection & union.

  Notation:

  - If $R$ is a relation, $(x,y) \in R \Leftrightarrow xRy$

- Equivalence relations

  **def.** A relation $R$ on a set $M$ is reflexive. iff $(x,x) \in R$. or, equivalently $xRx \ \forall x \in M$.

  e.g. $S = \{(a,b) \in \mathbb{N}^2 \mid ab \geq 0\}$. is reflexive.

  **def.** $R$ is symmetric iff :

  $\forall x,y \in M, \ (x,y) \in R \Rightarrow (y,x) \in R$.

  or equivalently.

  $xRy \Rightarrow yRx \ \forall x,y \in M$.

  e.g. $S = \{(a,b) \in \mathbb{N}^2 \mid ab \geq 0\}$ is symmetric.

If R \subset A x B, then R is a relation from A to B
If R = X x Y, then X is the domain of R and Y is the range.

If R is a relation from A to B then the domain of R is a subset of A and the range of R is a subset of B

**Def.** If R \subset A x A, then R is a relation in A.

$R$ on a set $M$ is antisymmetric iff $\forall x,y \in M$,

$(x,y) \in R$ and $(y,x) \in R \Rightarrow x = y$.

i.e. $\forall x,y \in M$,

$xRy$ and $yRx \Rightarrow x = y$.

e.g. $\rho = \{(x,y) \in \mathbb{R}^2 \mid x \geq y\}$.

**Def.** $R$ is transitive iff :

$\forall x,y,z \in M. \ (x,y) \in R. \ (y,z) \in R \Rightarrow (x,z) \in R$. ;

i.e. $\forall x,y,z \in M. \ xRy$ and $yRz \Rightarrow xRz$.

e.g. $\Upsilon = \{(x,y) \in \mathbb{R}^2 \mid x \geq y\}$.

**Def.** $R$ on set $M$ is an equivalence relation if it is reflexive, symmetric and transitive.

# Equivalence classes

**def:** if $R$ is an equivalence relation on a set $A$ for $a \in A$, the set $[a] = \{x \in A \mid xRa\}$ of elmts of $A$ which are equiv. to $a$

is called the equiv. class of $a$ w/ respect to $R$, (R-equiv. class of $a$).

**Thrm**

if $R$ is an equiv. relation on a set $A$, then any 2 equiv. classes. $[a]$ and $[b]$ are either disjoint, or if they have any elmt ~~is~~ in common then they must be equal..

i.e. the 3 statements are equiv.:

1. $aRb$. 2. $[a]=[b]$. 3. $[a] \cap [b] \neq \emptyset$

$\Rightarrow$ Remark:

$[a] \neq [b]$ iff. $[a] \cap [b] = \emptyset$.

**def.** A partition of a set $A$ is a collection of ① non-empty, ② disjoint subsets of $A$ that are ③ mutually exhaustive.

i.e. a collection of $n$ non-empty subsets of $A$ s.t.

$A_i \cap A_j = \emptyset$. $\forall i \neq j$. and $\bigcup_{i=1}^{n} A_i = A$.

$\Rightarrow$ $[a_i] \neq \emptyset$

$\bigcup_{i=1}^{n} [a_i] = A$.

$[a_i] \cap [a_j] = \emptyset$. $\forall i \neq j$.

i.e. equiv. relation created a partition of the set $A$ whose subsets are the equiv. classes.

**Thrm**

if $R$ is an equiv. relation on a set $A$, then the equiv. classes of $R$ induce a partition of set $A$.

Proof:

1: the equiv. classes form a partition of set $A$.

2. A partition of set $A$ forms an equiv. relation on set $A$.

## Congruence

Residue classes of modulo n with remainder x

**def.** $a \equiv b \pmod m \Leftrightarrow m \mid (a-b)$

$a \not\equiv b \pmod m \Leftrightarrow m \nmid (a-b)$

m is the modulus of congruence.

**Thrm**

Let $m \in \mathbb{Z}^+$. Then congruence modulo m is an equiv. relation.

## 2.2. Functions.

**def** if A and B are non-empty sets. a function from A to B is a relation f from A to B. s.t. $\forall x \in A$, there is a unique elmt $y \in B$ w/ $(x,y) \in f$.

Note: $f: A \to B$.

$\Leftrightarrow f: x \mapsto y$. $x \in A$, $y \in B$.

( $\to$ for sets ; $\mapsto$ for elmts )

- A is domain ; B is codomain.
- y is the image of x under f. / x is mapped to $y = f(x)$ by the function f.
- x : input / preimage

  y : output.

**def.** the subset of B defined by $\{ f(a) \mid a \in A \}$.
is the image of A. and is denoted $f(A)$.
i.e. the image of A is the subset of B that consists of the images of all elmts of A.

**Remark** If $f(A) = B$, then B is the range of f.

**def.** $f: A \to B$ is a surjection
$\Leftrightarrow \forall y \in B$, $\exists$ at least one $x \in A$. s.t. $f(x) = y$.

**def.** $f: A \to B$ is an injection
$\Leftrightarrow \forall x_1, x_2 \in A$, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
(distinct inputs of f produce dist. outputs)

Remark:

The def is equiv. to :

1. $\forall x_1, x_2 \in A$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.
   (Contra positive).
2. $\forall y \in f(A)$. $\exists!$ $x \in A$. s.t. $f(x) = y$.
3. $\forall y \in$ Codomain. $\exists$ at most one $x \in A$ ...

Note: if f is injective,
$n(A) \le n(B)$. or. $|A| \le |B|$.

**Def.** $f : A \to B$ is a <u>bijection</u>

$\Leftrightarrow$ $f$ is both an injection and surjection.

**Remark:**

surjection : $|A| \geq |B|$.

injection : $|A| \leq |B|$

$\Rightarrow$ bijection : $|A| = |B|$

**Def.** $i_A : A \to A$ defined by $i_A(x) = x$. $\forall x \in A$ is known as the <u>identity function</u>.

## Composition of functions

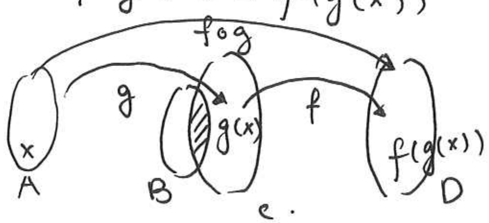
* the outputs of $g$ must be elmts of domain of $f$.

i.e. the range of $g$ is a subset of the domain of $f$.

**Def.** $g : A \to B$. $f : C \to D$. $g(A) \subseteq C$.

<u>$f \circ g$</u> $: A \to D$ defined by

$$f \circ g(x) = f(g(x))$$




**Note:**

the composition is not commutative, but is associative.

i.e. $(f \circ g) \circ h = f \circ (g \circ h)$.

## Inverse Functions

**def.** $R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in A \times B)$

$\Leftrightarrow$ $y R^{-1} x$ iff. $xRy$.

**Note:** Reflection w/ respect to line $y=x$.

(first bisector / identity line).

**Note:** The inverse relation of $f$ may not be a function.

If it is, it's the inverse function of $f$, denoted by $f^{-1}$. $f^{-1}(y) = x$ when $f(x) = y$.

**Thrm** ~~Note~~: the function have to be a bijection in order to have an inverse function.

**Note:** $f \circ f^{-1}(y) = y \Rightarrow f \circ f^{-1} = i_B$.

$f^{-1} \circ f(x) = x \Rightarrow f^{-1} \circ f = i_A$.

This is a method to test whether 2 functions are inverses.

# III Groups (I)

## 3.1. Binary Operations.

### Definition 1

A <u>binary operation</u> on a set $A$ is a function from $A \times A$ into $A$. Thus, it is a rule $*$ which assigns to every ordered pair $(a,b) \in A \times A$ exactly one element $c \in A$; denoted by $a * b = c$.

**Remark:**

- The rule for the operation must be well-defined: must assign to every ordered pair $(a,b)$ exactly one elmt $c$.
- $c \in A$. (<u>closure</u> property)

### properties of binary operations

**def 2.**

A binary operation $*$ on a set $G$ is:

- associative $\Leftrightarrow$ $\forall a,b,c \in G$,

  $a*(b*c) = (a*b)*c$.

- commutative. $\Leftrightarrow$ $a*b = b*a$
- distributive over $\triangle$ $\Leftrightarrow$

  $a*(b \triangle c) = (a*b) \triangle (a*c)$

### operation (Cayley) tables

**Note:** To see if a group is commutative, check if the table is symmetric about the main diagonal..

### the identity elmt.

**def 3.** A elmt $e$ in a set $S$ is an identity elmt for an operation $\triangle$ defined over $S$ if $e \triangle a = a \triangle e = a$. $\forall a \in S$.

(there's also right & left-identity)

**Thrm 1**

if an operation $\circ$ admits a left-identity $e_1$ and a right identity $e_2$, then these identities are equal.

**Thrm 2.**

If $*$ on a set $S$ admits an identity elmt $e$, then this elmt is unique.

### the inverse elmt.

**Def. 4** $a^{-1} \triangle a = a \triangle a^{-1} = e$. $\forall a \in S$.

$\uparrow$ left $\qquad$ $\uparrow$ right.

**Thrm 3** If, for an associative operation $\circ$, an elmt $a$ admits a left-inverse $a'$ and a right $a''$, then $a' = a''$.

**Thrm 4** If an operation $*$ defined on a set $S$ has an identity elmt. $e$, then every invertible elmt admits a unique inverse.

## Cancellation Rules

**Thrm 5.**

if $a*b = a*c$, then $b=c$.
if $b*a = c*a$, then $b=c$.

# 3.2 Groups.

**def 5.** $(G, *)$:

1. Closure: $a*b \in G$.
2. Associativity: $(a*b)*c = a*(b*c)$.
3. identity: $\exists\, e$ st. $a*e = e*a = a$.
4. Inverses: $\exists\, b$. s.t. $a*b = b*a = e$.

$*$ Abelian / Commutative: $a*b = b*a$.

$*$ order $|G|$ $\begin{cases} \text{finite} \\ \text{infinite.} \end{cases}$

**Thrm 6** (Latin square property)

$\forall a, b \in (G, *)$, $\exists!\ c$. s.t. $a*c = b$.

**Thrm 7**

If $a$ and $b$ are elmts of $(G, *)$, then.

1. $(a^{-1})^{-1} = a$
2. $(a*b)^{-1} = b^{-1} * a^{-1}$

**Note**

$a^0 = e$.

## Congruence revisited

**Def 6**

$[a] = \{x \mid x \in \mathbb{Z} \text{ and } x \equiv a \pmod n\}$
$a \in \mathbb{Z},\ n \in \mathbb{Z}^+$.

**Theorem 8**

$a \equiv b \pmod n \iff [a] = [b]$

**Thrm 9** There are $n$ diff. congruence classes.

**Def 7.** The set of all congruence classes modulo $n$ is denoted $\mathbb{Z}_n = \{[0], [1] \dots [n-1]\}$.

**Thrm 10**

Let $a, b, c, d \in \mathbb{Z}$. $m \in \mathbb{Z}^+$.

$a \equiv b \pmod{m}$. $c \equiv d \pmod{m}$

$\Rightarrow$ 1. $a \pm c \equiv b \pm d \pmod{m}$

2. $ac \equiv bd \pmod{m}$

**Remark:** if $\gcd(c, m) = 1$,

then $ac \equiv bc \pmod{m}$

$\Rightarrow a \equiv b \pmod{m}$.

**Thrm 11**

$[a] = [b]$. $[c] = [d]$ in $\mathbb{Z}_n$.

$\Rightarrow [a+c] = [b+d]$. $[ac] = [bd]$.

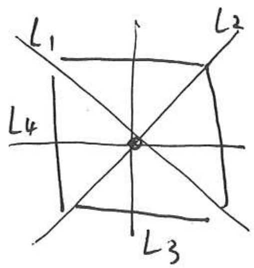
**Def 8.**

In $\mathbb{Z}_n$: $[a] + [c] = [a+c]$

$[a][c] = [ac]$.

## More Groups.

**Symmetries of a square.**



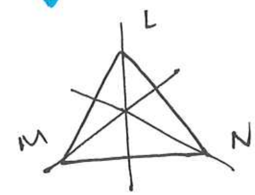

$(D_4, \circ)$. w/ operation

$D_4 = \{e, r, r^2, r^3, L_1, L_2, L_3, L_4\}$.

**Symmetries of an equilateral triangle.**

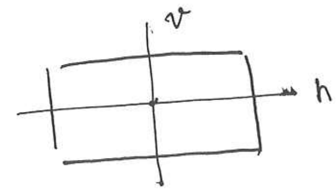
$(D, \circ)$.

$D = \{I, R, R^2, L, M, N\}$




**Symmetries of a rectangle.**

$(\{e, r, h, v\}, \circ)$.




## 3.3. Permutations

**Def 9**

A Permutation on a set $S$ is a bijection $\alpha: S \to S$. The set of all permutations on $S$ is denoted $S_n$.

If $\alpha, \beta \in S_n$. we simplify $\alpha \circ \beta$ as $\alpha\beta$.

**Remark** if $\alpha, \beta, \gamma \in S_n$:

1. $\alpha\beta \in S_n$.
2. $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.
3. $\alpha^{-1} \in S_n$.
4. The identity fuction $e$ is in $S_n$.

# IV Group (II)

## 4.1. Introduction.

**Def. 1** $a \in (G, *)$ has finite order if $a^m = e$ for some $m \in \mathbb{Z}^+$.

$\Rightarrow$ order of elmt $a$ is $|a|$.

$a \in (G, *)$ has infinite order if $a^m \neq e$ for every $m \in \mathbb{Z}^+$.

**Thrm 1**

$a \in (G, \cdot)$.

1. if $a$ has finite order $n$, then $a^m = e$ $\Leftrightarrow$ $n \mid m$.
2. $a^p = a^q$ $\Leftrightarrow$ $p \equiv q \pmod n$
3. If $a$ has infinite order, $a^i \neq a^j$ when $i \neq j$.

**Remark:**

1. if $|a| = n$. $n = kr$ . $r > 0$

   $\Rightarrow |a^r| = k$.
2. if $a^x = a^y$. $x \neq y$

   $\Rightarrow$ $a$ must have a finite order.

## 4.2. Subgroups

**Def 2.**

If a non-empty subset $H$ of $G$ is itself a group under the same operation,

$\begin{cases} H \subset G, & \text{proper subgroup} \\ H \subseteq G. & \text{subgroup.} \end{cases}$

- trivial subgroup:

  $(\{e\}, *)$.

  Also, $(G, *)$.

  Aside from these two, all are proper.

**Thrm 2.**

$X = \{x^k \mid k \in \mathbb{Z}\} \subseteq (G, *)$ for $x \in G$.

$X$ is the cyclic subgroup generated by $x$. $x$ is the generator of the subgroup.

# Subgroup tests.

Thrm 3

$H \leq G$ iff $ab^{-1} \in H$. $\forall a, b \in H$.

[ proof by using identity. inverse. closure axiom ]

Thrm 4

$H \leq G$ iff.

1. $ab \in H$. $\forall a, b \in H$, and — closure.
2. $a^{-1} \in H$. $\forall a \in H$ — Inverse.

Thrm 5 (Finite Subgroup test)

$H \leq G$. ($H$ is finite.) if.

$H$ is closed under operation of $G$.

# The center of a group

$C(G) = \{a \in G : ag = ga \ \forall g \in G\}$.

Thrm $C(G)$ is a ~~st~~ subgroup of $G$.

⇒ Permutation Group on S

$S_n$ is the symmetric Group on n elmts.

**Remark:**

There are $n!$ permutations of a set of n objects.

# Notation

## Two-row notation (array notation)

e.g. $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix}$.

* Each member of the first row is mapped onto the corresponding member in 2nd row.

## Product of permutations

* $\alpha\beta$: 先$\beta$再$\alpha$.
* $\alpha\beta \neq \beta\alpha$.
* $e = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$.
* $\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}$.

## Cycle notation

$\alpha = (1\ 5\ 3\ 2)(4)$.

* permutations that do not move any items are written as $(1)$.

## product of permutations using cycle notation.

e.g. $(1\ 3\ 4\ 2)(3\ 6\ 4\ 5)(1\ 6\ 2\ 3)$.

$1 \to 6 \to 4 \to 2$. $\therefore 1 \to 2$

$2 \to 3 \to 6 \to 6$. $\therefore 2 \to 6$

$6 \to 2 \to 2 \to 1$. $\therefore 6 \to 1$ $\quad \therefore (1\ 2\ 6)$

$3 \to 1 \to 1 \to 3$. $\therefore 3 \to 3$. $\therefore$ 3 is fixed.

$4 \to 4 \to 5 \to 5$. $\therefore 4 \to 5$

$5 \to 5 \to 3 \to 4$. $\therefore 5 \to 4$ $\quad \therefore (4\ 5)$.

$\Rightarrow (1\ 2\ 6)(4\ 5)$

## Inverse of a permutation.

- $\alpha = (1\ 5\ 7\ 3)(4\ 6\ 8)$.

  $\Rightarrow \alpha^{-1} = (8\ 6\ 4)(3\ 7\ 5\ 1)$. → Cycle form.

- Swap $R_1$ and $R_2$, rearrange the new $R_1$. → Array form.

## Inverse of a product of permutations

* permutation is a function.

Thm

$$(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}.$$

* Cancellation law is valid.

i.e. $\alpha\beta = \alpha\gamma \iff \beta = \gamma$.

* Order of a permutation.

Def For any permutation $\alpha$, $\exists\, n \in \mathbb{Z}^{+}$.
s.t. $\alpha^{n} = e$. the smallest $n$ is the order.
denoted. $\mathrm{ord}(\alpha) = n$.

Thrm

The order of a permutation written in disjoint cycle form is the lcm of length of the cycles.

e.g. $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix} = (1\,3\,2)(4\,5)$.

$\mathrm{ord}(\alpha) = \mathrm{lcm}(2, 3) = 6$.

## Summary of properties of permutations.

- If disjoint cycle form of $\alpha$ has no number in common w/ the disjoint cycle form of $\beta$. the $\alpha \cdot \beta$ commute.