Problem 4

Let $S = \{1, 2, 3, \ldots, n\}$ be a set of n elements. S is closed under the binary operation $\diamond$.

Definition 1. $\diamond$ is a **binary operation** on a set S if $\diamond: \ S \times S \to S$.

Definition 2. S is **closed** under a binary operation $\diamond$ if $x \diamond y \in S \quad \forall x, y \in S$.

Definition 3. The operation $\diamond$ is **unital** if

$$1 \diamond x = x \diamond 1 = x, \quad \forall\, x \in S. \tag{0.1}$$

Definition 4. The operation $\diamond$ is **sandwiching** if

$$(x \diamond y) \diamond (z \diamond w) = (x \diamond z) \diamond (y \diamond w), \quad \forall\, x, y, z, w \in S. \tag{0.2}$$

Definition 5. The operation $\diamond$ is **commutative** if

$$x \diamond y = y \diamond x, \quad \forall\, x, y \in S. \tag{0.3}$$

Proposition 1 (Question a)**.** *If the binary operation $\diamond$ on S is both unital and sandwiching, it must be commutative.*

Proof. Let $x, y \in S$.
According to the unital axiom,

$$x \diamond 1 = x \diamond 1 = x$$
$$y \diamond 1 = y \diamond 1 = y$$

Use the sandwiching axiom,

$$(1 \diamond x) \diamond (y \diamond 1) = (x \diamond 1) \diamond (y \diamond 1)$$
$$(1 \diamond y) \diamond (x \diamond 1) = (x \diamond y) \diamond (1 \diamond 1)$$
$$y \diamond x = x \diamond y$$

which means that $\diamond$ is commutative over S. ∎

Definition 6. The operation $\diamond$ is **associative** if

$$x \diamond (y \diamond z) = (x \diamond y) \diamond z, \quad \forall\, x, y, z \in S. \tag{0.4}$$

Proposition 2 (Question b)**.** *If the binary operation $\diamond$ on S is both unital and sandwiching, it is also associative.*

Proof. Let $x, y, z \in S$. According to the sandwiching axiom,

$$(y \diamond x) \diamond (z \diamond 1) = (y \diamond z) \diamond (x \diamond 1)$$

On the other hand, because of the unital axiom,

$$(y \diamond x) \diamond (z \diamond 1) = (y \diamond x) \diamond z,$$
$$(y \diamond z) \diamond (x \diamond 1) = (y \diamond z) \diamond x$$

Therefore,

$$(y \diamond x) \diamond z = (y \diamond z) \diamond x$$

According to Proposition 1, $\diamond$ is commutative, so

$$x \diamond (y \diamond z) = (x \diamond y) \diamond z$$

which is the required property of associativity. $\square$

Definition 7. The operation $\diamond$ is **self-distributive** if

$$x \diamond (y \diamond z) = (x \diamond y) \diamond (x \diamond z), \quad \forall\, x, y, z \in S. \tag{0.5}$$

Proposition 3 (Question c)**.** *Define $\bullet$ to be a new operation on S. If $\bullet$ is both unital and self-distributive, then it must be associative.*

Proof.

$$\begin{aligned} x \bullet y &= x \bullet (y \bullet 1) \\ &= (x \bullet y) \bullet (x \bullet 1) \\ &= (x \bullet y) \bullet x \end{aligned}$$

On the other hand,

$$\begin{aligned} x \bullet y &= x \bullet (1 \bullet y) \\ &= (x \bullet 1) \bullet (x \bullet y) \\ &= x \bullet (x \bullet y) \end{aligned}$$

Therefore, we obtain

$$x \bullet y = (x \bullet y) \bullet x = x \bullet (x \bullet y),$$

Now, since $\bullet$ is also self-distributive, consider for $x, y, z \in S$,

$$\begin{aligned} x \bullet (y \bullet z) &= (x \bullet y) \bullet (x \bullet z) \\ &= [(x \bullet y) \bullet x] \bullet [(x \bullet y) \bullet z] \\ &= (x \bullet y) \bullet [(x \bullet y) \bullet z] \\ &= (x \bullet y) \bullet z \end{aligned}$$

which is the property of associativity that we desire. □

Proposition 4. *If a binary operation $\diamond$ closed on S is associative and commutative, it is also sandwiching.*

Proof. By associative and commutative we have

$$(x \diamond y) \diamond (z \diamond w) = x \diamond y \diamond z \diamond w = x \diamond z \diamond y \diamond w = (x \diamond z) \diamond (y \diamond w), \quad \forall\ x, y, z, w \in S$$

Hence it is sandwiching. □

Proposition 5. *Let $\diamond$ be a binary operation closed on the set S. Among the five properties:*

unital, sandwiching, commutative, associative, self-distributive, *except self-distributive $\Rightarrow$ sandwiching, any other pairs of properties cannot be deduced from one another. For example, if $\diamond$ is unital, it may not be sandwiching, commutative, associative, or self-distributive.*

Proof. Let $\diamond$ be a binary operation on the set S. We look at several counterexamples.

1. Consider the case the binary operation $\diamond : (x, y) \mapsto x$.
   $\diamond$ is sandwiching because $\forall x, y, z, w \in S$,

$$(x \diamond y) \diamond (z \diamond w) = x \diamond z = x,$$

$$(x \diamond z) \diamond (y \diamond w) = x \diamond y = x,$$

$$\text{Thus, } (x \diamond y) \diamond (z \diamond w) = (x \diamond z) \diamond (y \diamond w).$$

   $\diamond$ is associative because $\forall x, y, z \in S$,

$$x \diamond (y \diamond z) = x \diamond y = x$$

$$(x \diamond y) \diamond z = x \diamond z = x,$$

$$\text{Thus, } x \diamond (y \diamond z) = (x \diamond y) \diamond z$$

   $\diamond$ is self-distributive because $\forall x, y, z \in S$,

$$x \diamond (y \diamond z) = x \diamond y = x$$

$$(x \diamond y) \diamond (x \diamond z) = x \diamond x = x,$$

$$\text{Thus, } x \diamond (y \diamond z) = (x \diamond y) \diamond (x \diamond z)$$

   However, $\diamond$ is not unital because $\forall x \in S$,

$$x \diamond 1 = x \neq 1 = 1 \diamond x.$$

   Nor is $\diamond$ commutative, because $\forall x, y \in S$,

$$x \diamond y = x \neq y = y \diamond x.$$

Therefore, from this counterexample, we can conclude that

$$\text{sandwiching} \not\Rightarrow \text{unital}; \quad \text{sandwiching} \not\Rightarrow \text{commutative};$$
$$\text{associative} \not\Rightarrow \text{unital}; \quad \text{associative} \not\Rightarrow \text{commutative};$$
$$\text{self-distributive} \not\Rightarrow \text{unital}; \quad \text{self-distributive} \not\Rightarrow \text{commutative};$$

2. Consider the case $\diamond : (x,y) \mapsto \begin{cases} x+y \mod n, \text{ if } n \not| (x+y) \\ 1, \text{ if } n|(x+y) \end{cases}$.
   Since modular addition is inherently associative and commutative, by Proposition 4, $\diamond$ is also sandwiching.
   However, $\diamond$ not unital, as $x+1 = 1+x \neq x \mod n \quad \forall x \in S$
   Nor is $\diamond$ self-distributive, because $x+(y+z) \neq 2x+y+z = (x+y)+(x+z)$.
   Therefore, we conclude that

$$\text{sandwiching} \not\Rightarrow \text{unital}; \quad \text{sandwiching} \not\Rightarrow \text{self-distributive};$$
$$\text{associative} \not\Rightarrow \text{unital}; \quad \text{associative} \not\Rightarrow \text{self-distributive};$$
$$\text{commutative} \not\Rightarrow \text{unital}; \quad \text{commutative} \not\Rightarrow \text{self-distributive};$$

3. Consider when $\diamond : (x,y) \mapsto \begin{cases} |x-y|, \text{ if } x \neq y \\ 1, \text{ if } x = y \end{cases}$.
   In this case, only the commutative axiom is satisfied.
   Therefore, we conclude that commutativity alone cannot deduce any other properties for a binary operation.

4. Consider when $\diamond : (x,y) \mapsto \begin{cases} \left\lceil \frac{x}{y} \right\rceil, \text{ if } x \neq 1 \\ y, \text{ if } x = 1 \end{cases}$.
   In this case, $\diamond$ is unital because $1 \diamond x = x \diamond 1 = 1$. However, it does not satisfy any other axioms.
   We hence conclude that unital axiom alone cannot deduce any other properties for a binary operation.

5. Consider when $\diamond : (x,y) \mapsto xy \mod n$.
   In this case, $\diamond$ is unital because $1 \diamond x = x \diamond 1 = x$;
   Since multiplication is associative and commutative, by Proposition 4, it is sandwiching. However, it is not self-distributive.

We hence conclude that none of the axioms alone can deduce self-distributivity.

6. Consider when $\diamond : (x,y) \mapsto \begin{cases} \frac{x+y}{2}, \text{ if } x+y \equiv 0 \mod 2 \\ x, \text{ if } x+y \equiv 1 \mod 2 \end{cases}$.

   If $x+y \equiv 1 \mod 2$, we have shown that the operation maps(x,y) to x is sandwiching and self-distributive in Counterexample 1.

   If $x+y \equiv 0 \mod 2$, $\diamond$ is sandwiching because

$$(x \diamond y) \diamond (z \diamond w) = (\frac{x+y}{2}) \diamond (\frac{z+w}{2}) = \frac{x+y+z+w}{4}, \text{ and}$$

$$(x \diamond z) \diamond (y \diamond w) = (\frac{x+z}{2}) \diamond (\frac{y+w}{2}) = \frac{x+y+z+w}{4}$$
$$\Rightarrow (x \diamond y) \diamond (z \diamond w) = (x \diamond z) \diamond (y \diamond w)$$

   It is also self-distributive because

$$x \diamond (y \diamond z) = x \diamond (\frac{y+z}{2}) = \frac{x + \frac{y+z}{2}}{2}, \text{ and}$$

$$(x \diamond y) \diamond (x \diamond z) = (\frac{x+y}{2}) \diamond (\frac{x+z}{2}) = \frac{x + \frac{y+z}{2}}{2}$$
$$\Rightarrow x \diamond (y \diamond z) = (x \diamond y) \diamond (x \diamond z)$$

   However, it does not satisfy associativity, because

$$x \diamond (y \diamond z) = \frac{x + \frac{y+z}{2}}{2},$$

$$(x \diamond y) \diamond z = \frac{\frac{x+y}{2} + z}{2}$$
$$\Rightarrow x \diamond (y \diamond z) \neq (x \diamond y) \diamond z$$

   We hence conclude that

$$\text{sandwiching} \nRightarrow \text{associative; self-distributive} \nRightarrow \text{associative.}$$

7. Lastly we are left to show that associative $\nRightarrow$ sandwiching. Since

$$(x \diamond y) \diamond (z \diamond w) = [(x \diamond y) \diamond z] \diamond w = [x \diamond (y \diamond z)] \diamond w$$

$$(x \diamond z) \diamond (y \diamond w) = [(x \diamond z) \diamond y] \diamond w = [x \diamond (z \diamond y)] \diamond w$$

If we find a non-commutative associative binary operation, it is not sandwiching. One of them is matrix multiplication.

Consider when $\diamond : (x,y) \mapsto$sum of entries of $\begin{pmatrix} x & x+1 \\ x+2 & x+3 \end{pmatrix} \begin{pmatrix} y & y+1 \\ y+2 & y+3 \end{pmatrix} \mod n$.

Since these matrices are associative but not commutative, we know that the sum of the entries resulting matrix for xy must be different from that of yx. Since it is not commutative, $\diamond$ is not sandwiching, i.e. associative $\not\Rightarrow$ sandwiching.

Therefore, drawing conclusion from the seven examples above, we have shown that any pairs of properties cannot be deduced from one another, except for self-distributive $\Rightarrow$ sandwiching. □

Conjecture 1. *I conjecture that self-distributive $\Rightarrow$ sandwiching for a binary operation $\diamond$ closed on S, i.e. $\forall x,y,z,w \in S$,*

$$\begin{cases} x \diamond (y \diamond z) = (x \diamond y) \diamond (x \diamond z) \\ (x \diamond y) \diamond (z \diamond w) \neq (x \diamond z) \diamond (y \diamond w) \end{cases}.$$

This conjecture holds for all the examples we discussed above. Therefore, I am still working on finding a proof or a counterexample for this conjecture.

Definition 8. The operation $\diamond$ is **identical** if

$$x \diamond x = x. \tag{0.6}$$

Proposition 6. *If a binary operation $\diamond$ is both identical and sandwiching, it is also self-distributive.*

Proof. By identical and sandwiching we have:

$$x \diamond (y \diamond z) = (x \diamond x)(y \diamond z) = (x \diamond y)(x \diamond z)$$

Therefore, $\diamond$ is self-distributive. □

Proposition 7. *If a binary operation $\diamond$ is both unital and self-distributive, it is also identical.*

Proof. By unital and self-distributive we have:

$$x = x \diamond 1 = x \diamond (1 \diamond 1) = (x \diamond 1)(x \diamond 1) = x \diamond x$$

Therefore, $\diamond$ is identical. $\square$

Definition 9. The operation $\diamond$ is **absorptive** if

$$x \diamond (x \diamond y) = x \diamond y. \tag{0.7}$$

Proposition 8. *If a binary operation $\diamond$ is both unital and self-distributive, it is also absorptive.*

Proof. By unital and self-distributive we have:

$$x \diamond y = x \diamond (1 \diamond y) = (x \diamond 1) \diamond (x \diamond y) = x \diamond (x \diamond y)$$

Therefore, $\diamond$ is absorptive. $\square$

Proposition 9. *If a binary operation $\diamond$ is both commutative and identical, it is also absorptive.*

Proof. By commutative and identical we have:

$$x \diamond (x \diamond y) = (x \diamond x) \diamond y = x \diamond y$$

Therefore, $\diamond$ is absorptive. $\square$

As a summary of the properties we have explored, I have the following proposition:

Proposition 10. *If a binary operation $\diamond$ is also , sandwiching, and self-distributive, $\diamond$ is commutative, associative, absorptive, and identical.*