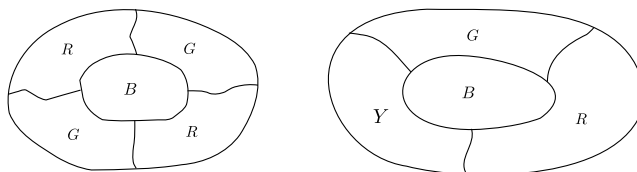


The Four Color Theorem

December 12, 2011

The Four Color Theorem is one of many mathematical puzzles which share the characteristics of being easy to state, yet hard to prove. Very simply stated, the theorem has to do with coloring maps.

***Given a map of countries, can every map be colored
(using different colors for adjacent countries)
in such a way so that you only use four colors?***



Through a little experimentation, you will quickly see for yourself that not every map needs four colors. The maps shown above need 3 and 4 colors, respectively. After several hours of banging your head against a wall trying to think of a map that needs more than four colors, you might even convince yourself that no such map exists. But this is not a proof!

In 1852, Francis Guthrie, a student at University College London, was coloring a map of the counties of England when he noticed that at least four colors were required so that no two counties which shared a border had the same color. He conjectured that four colors would suffice to color any map, and this later became known as the Four Color Problem.

Throughout history, many mathematicians have offered various insights, reformulations, and even proofs of the theorem. Since first being stated in 1852, the theorem was finally considered “proved” in 1976. Requiring over

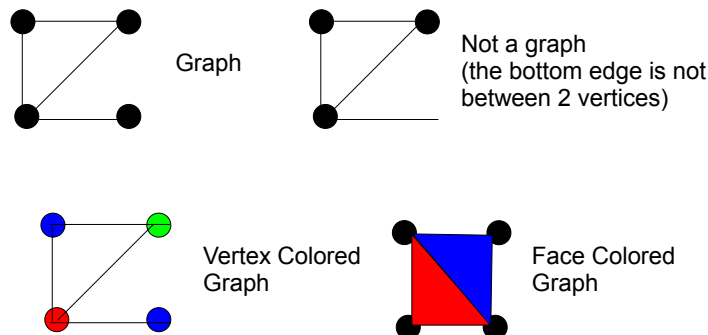
1,000 hours of computing time to check irreducibility conditions in graphs caused the proof to receive a cool reaction from many in the mathematical community. Though the Four Color Theorem is now considered solved, at the time the computer-assisted proof sparked a philosophical debate about the role of human minds and computing machines in mathematics.

To be completely honest, I am unsatisfied with the current proof of the Four Color Theorem. This is not because I doubt its accuracy, but because the method of proof does not shed any light on why the theorem is true! In my opinion, a good mathematical proof gives us insight as to why certain relationships exist. This paper will look at some of the first purported proofs and reformulations of the theorem into other branches of mathematics, and some of the simplifications/progress that were made at the very beginning. I want to apologize to the reader here and warn you that this paper has been over researched and under written. There are many, many historically significant contributions that are being omitted here. Most of these omissions are those that in some way formed the logical foundations and proof strategy for Haken and Appel's 1976 computer assisted proof. Also missing from this paper is a discussion of some of the philosophical questions it raised Haken and Appel's proof raised, and countless many restatements of the Four Color Problem. If the paper somehow manages to delight your curiosity, the references at the end are very extensive and are a good place to start to learn more about the history of the problem and its countless restatements.

This paper starts with the assumption that we would like a "better" proof, somehow, of the Four Color Theorem that gives us some insight on why it is true. The purpose of this paper is to understand some of the first "proofs" (which were not always correct), where the difficulties in those proofs arose, and what reformulations of the Four Color Problem they created. I should point out here that entire books have been written that contain theorem after theorem, all equivalent statements to the Four Color Theorem. What I (not so) secretly hope to do is to inspire the reader, after learning some of the very first approaches to explaining the Four Color Theorem, to try and understand more deeply why Four Color Theorem is true and provide the rest of us with the explanation we crave!

Before we begin, we need to start with some common vocabulary. Maps (which you are familiar with) are an example of a special type of mathematical object called a graph. Graphs are made out of vertices (points)

and *edges* (lines). Edges must always go between 2 vertices (points). On a map, we can think of the boundary lines of countries as edges. Graphs also contain *faces* (regions enclosed by the edges). The following diagrams are some examples and non-examples of different types of graphs.



For the rest of the paper, we will use the following words interchangeably:

Graph = Map
Vertex = Point
Edge = Boundary
Face = Country

Now that we have a common vocabulary we can begin exploring some ideas about why the Four Color Theorem is true. In 1879, Alfred Kempe published a proof of the Four Color Theorem that stood unchallenged for 11 years, until an error was found by Percy Heawood in 1890. In his paper he proves the Five Neighbour Theorem and then describes a method for colouring maps. A summary of the method, below, was taken from the book “Four Colors Suffice” by Robin Wilson.

Five Neighbor Theorem Every graph has a country with five or fewer neighbors.

This is a suprising statement, but challenge yourself to draw a counterexample! The proof is very nice and uses a counting argument that start with Euler’s formula for a graph: $V - E + F = 2$. Unfortunately, for time’s sake, we won’t include the proof here. Another way to think about what the Five Neighbor Theorem means is that it tells us that in any map we can always

find a region that is 2-sided (2 neighbors), 3-sided (the country looks like a triangle), 4-sided (the country looks like a square), or 5-sided (the country looks like a pentagon).

Kempe's Map Coloring Algorithm

Step 1 Locate a country with 5 or fewer neighbours.

Step 2 Cover the country with a blank piece of paper (a patch) of the same shape but slightly larger.

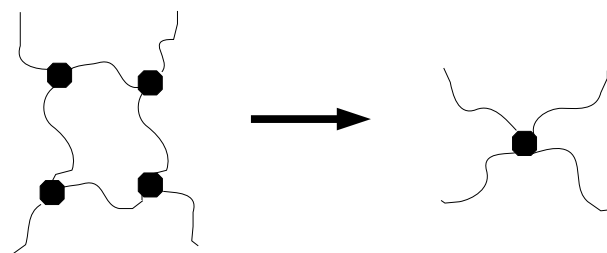
Step 3 Extend all the boundaries that meet this patch and join them together at a single point within the patch (as in the figure below) which amounts to shrinking the country to a point. It has the effect of reducing the number of countries by 1.

Step 4 Repeat the above procedure with the new map, continuing until there is just one country left; the whole map is now patched out.

Step 5 Colour the single remaining country with any of the four colors.

Step 6 Reverse the process, stripping off the patches in reverse order, until the original map is restored. At each stage, colour the restored country with any available colour until the entire map is coloured with four colours.

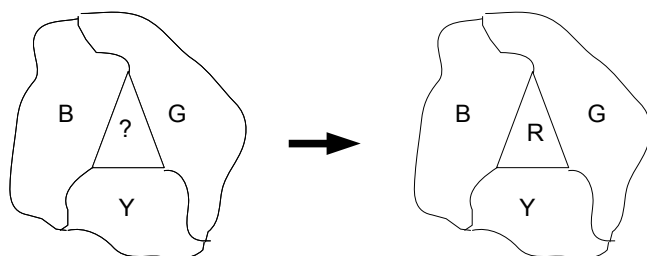
The following picture is example of Step 3 of Kempe's method, where the boundary of a 4-sided region is shrunk to a point.



In his paper, Kempe attempts to explain why during the last stage a fourth color is always available when a country is restored. Remember that each

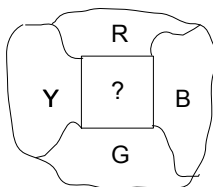
restored country has 5 or fewer neighbors. Notice that if the restored country has 3 or fewer neighbors, then a fourth color is clearly always available. Think about it to yourself while considering the picture below.

Case 1: The restored country is a triangle

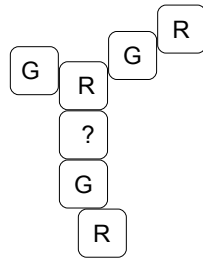


Case 2: The restored country is a square

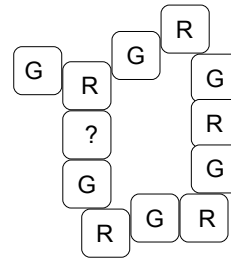
Now consider what happens if you restore a country that had 4 neighbors. One of the following 3 situations might occur. If the neighbors of the square are colored with 3 or fewer colors, then there is a color available to use for the square. So let's focus on what happens if the neighbors of the square have been colored with 4 colors.



Kempe describes the following method to color the square. Pick non-adjacent colors from the square's neighbors. We will use red and green from the picture above. Follow a chain of those colors through the rest of the map. Either the chain will form a link or it won't, as in the picture below.



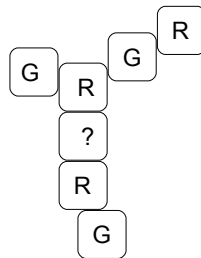
Graph A



Graph B

Graph A

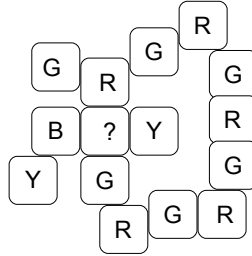
In Graph A, we see that the chain we made of countries alternating red/green countries does not link up to the green country. This means we can switch the colors red and green in the chain without affecting the rest of the colors in the map, as shown below. Now we've recolored the neighbors of the square using only 3 colors, leaving us with a fourth color to use on the square.



Graph A

Graph B

In Graph B, however, we can't do this! Since the two countries link up, switching the colors won't help us save a color around the square. And now comes the clever part: Since the red and green neighbors of the square are connected, this means the blue and yellow are not! Both countries cannot link up at the same time. Try to visualize this with the diagram below.

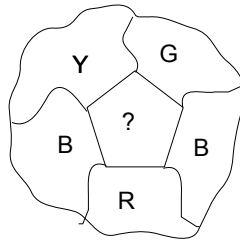


Graph B

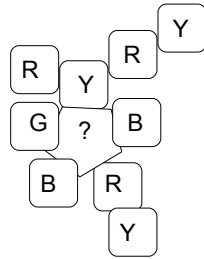
In this case, we can switch the blue and yellow colors in our chain without messing up the coloring in the rest of our map. This leaves us with a square whose neighbors are colored blue, red, and green, letting us color the square yellow. Now that we've considered both cases, we've shown that there is a way to color a map with a restored square with 4 colors.

Case 3: The restored country is a pentagon

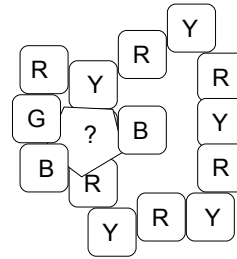
Let's now turn our attention to the case when the restored country is a pentagon. If the neighbors of the pentagon are colored with less than 4 colors, we have a fourth color available to use on the pentagon. So let's focus on a pentagon whose neighbors are colored with 4 colors.



In his paper, Kempe applies the same techniques he used when the restored country was a square, like above. He starts by looking at 2 non-adjacent countries and forms a chain of countries of alternating colors. Either the chain becomes a link or it does not, like below. Let's look at the yellow and red countries.

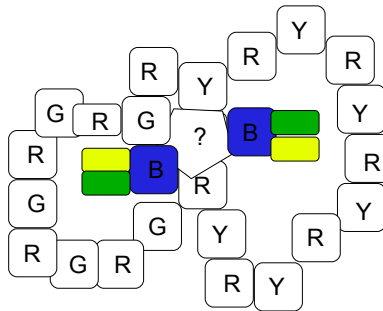


Graph A

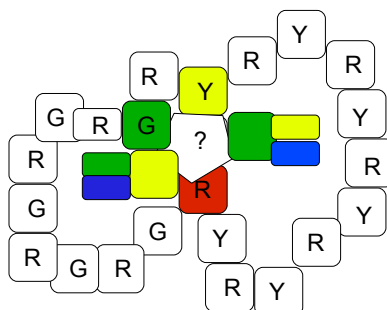


Graph B

If the chain is not a link, like in Graph A, we're done. We can switch the yellow and red colors of the chain and end up only using 3 colors to color the neighbors of the pentagon. This leaves us with a fourth color available for the pentagon. If the chain does turn into a link, though, go ahead and pick two other non-adjacent colors and look at the chain they form. We'll use red and green below. Now again, either the red and green countries will form a link or they will not form a link. If the red/green chain does not form a link, we can switch those colors and color the neighbors of the pentagon with only 3 colors. This is just like the cases above. Let's consider what happens if the red/green chain also forms a link.



If the red/green chain does form a link, however, think about the picture above. Focus on the color blue, which is repeated twice around the pentagon. Notice the blue/yellow/green parts inside the right and left links. The blue/yellow parts on both sides are separated from each other, and the blue/green parts are separated from each other. If we go through and interchange both colors below, we've colored the neighbors of the pentagon with 3 colors, leaving a fourth color to use on the pentagon.

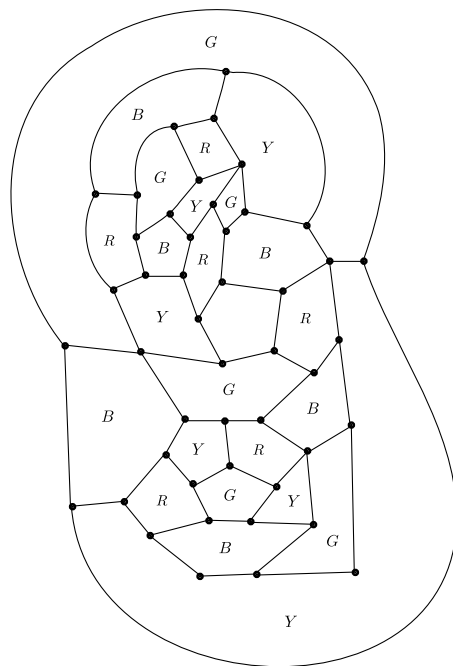


Since we have now dealt with all possible cases, we have proved the Four Color Theorem.

Or have we? Kempe's proof was accepted for 11 years before Heawood pointed out a flaw. Try going over the proof above one more time and see if you can find the flaw Heawood found in 1890. Read the first line of the following page only if you want a hint.

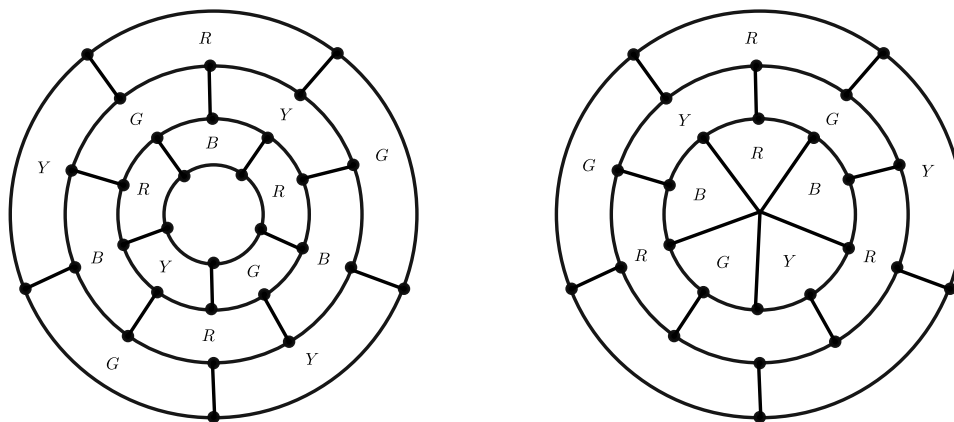
Is Kempe's treatment of the pentagon case as complete as it seems?

In 1891 Percy Heawood, a mathematician at Durham University in England, published a counterexample to Kempe's proof. Heawood's counterexample centers around the argument Kempe used for the case of the pentagon. In his final case, Kempe used 2 simultaneous changes of color to recolor the neighbors of the pentagon with just 3 colors. Either of those color substitutions can work, but BOTH color substitutions may not always work at the same time. The following map is an example of when Kempe's algorithm for coloring does not work. It is important to notice that the map is not a counterexample to the Four Color Theorem (the map can be four colored), but a counterexample to Kempe's method of proof.



Heawood's Map

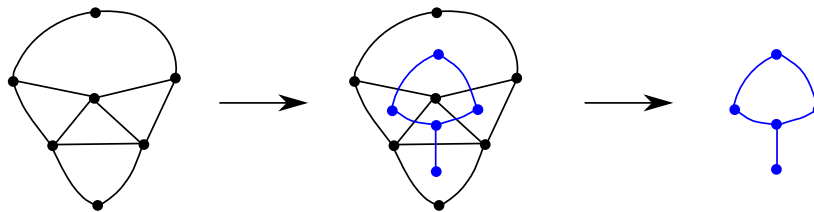
The diagram above is Heawood's original counterexample. Later, in 1896 and 1921, Charles Poussin and Alfred Errera respectively gave simpler and more visually appealing counterexamples. One thing to point out here is that while Kempe did not manage to prove the Four Color Theorem, his method of proof was sound. By slightly adjusting some of his arguments, you can complete the proof of the Five Color Theorem (that you need at most 5 colors to color any map such that no two neighboring countries are the same color).



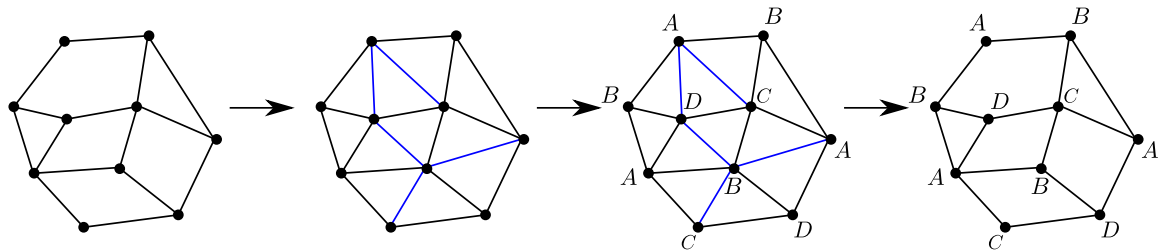
In 1880, Peter Tait gave an incomplete proof of the Four Color Theorem. In fact, over his lifetime Tait gave several incomplete or incorrect proofs of the Four Color Theorem. Let's go over 2 incomplete proofs he published in 1880 and see if you can complete them!

Tait's first proof - March 1880

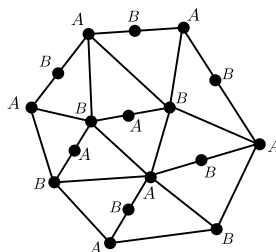
Tait's first approach was to take a map and draw what we call the "Dual Map", where a vertex is placed in every face and an edge is drawn between vertices that came from adjacent faces. Instead of trying to color adjacent countries with different colors, the problem now becomes to color adjacent vertices (vertices connected by an edge) with different colors. An example of constructing the dual map is shown below.



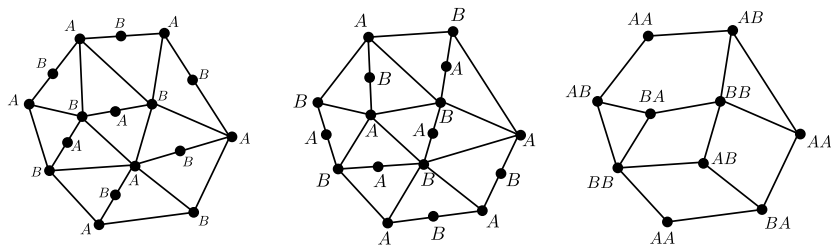
Tait would then add extra edges into his map in order to triangulate the diagram. We'll call this a "triangulated dual map". An example of how to triangulate a map is shown below. He pointed out that if he found a way to put 4 letters on the vertices of his triangulated dual map, so that no 2 adjacent vertices were labeled with the same letter, then he automatically had a lettering of his dual map. Now remember, the lettering of his dual map can be translated into a coloring of the countries of the original map. Consider the example below to convince yourself that a lettering of the triangulated dual map always gives a lettering of the dual map.



And Tait goes on further still! He now adds extra points to his “triangulated dual map” to make all the countries four-sided. Notice that this new map is no longer triangulated - the regions are squares, not triangles. The extra points are added into the middle of an edge as shown below.



Once every region is 4-sided, we can letter the vertices with alternating letters A and B. Tait writes in his proof to letter the vertices in this way and then says “Do the same thing again, another way”. What he means by this is to draw an exact copy of the triangulated dual graph (before we put the extra vertices in the middle of edges to make the regions into squares). Then put in the extra vertices again, but differently! In fact, you have to be careful to do it completely differently. This means that if you put a vertex in the middle of an edge in the first graph, then you can’t put a vertex on the corresponding edge in the second graph. Once you put the vertices in, label the graph with A’s and B’s again in an alternating manner. This gives a second lettering of the graph. Take a peak at the example below for an illustration of this process. Tait would then superimpose both graphs and omit the extra points (the ones he added to make the regions 4-sided). At the end, he would get the lettering on this graph shown in the last diagram below.



Tait then claimed that for every map, the resulting map gave a lettering of the vertices with one of AA, AB, BA, or BB such that any 2 connected

vertices had different letters. However, how can we be certain that such a lettering can always be carried out? Tait claimed that he had a few simple rules about how to make the second lettering (“Do the same thing again, another way”), but he never explained his rules. To this day, we can’t be sure why Tait’s method works. Can you think of an explanation?

One thing we can take away from Tait’s first proof is our first translation of the Four Color Theorem. Originally the problem was a statement about coloring countries of maps with 4 or less colors so that no two adjacent countries shared a color. Now we see that if we can letter the vertices of a cubic (3-valent) graph with 4 letters so that no 2 adjacent vertices share the same letter, we have proven the Four Color Theorem. Is this problem any easier to solve? Tait’s second proof gives another reformulation of the Four Color Problem.

Tait’s second proof - Later in 1880

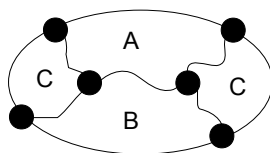
Tait starts off by reminding us that when trying to color a map, you can assume that the map you have is cubic. We showed earlier that there is a way to turn any map into a cubic map, and then reverse the process so that a coloring is preserved. Now, given a cubic map, Tait tries to color the edges instead of the countries! He writes the following claim:

***In a map where only 3 boundaries meet at each point,
(the boundaries may be coloured with 3 colours)
so that no two of the same colour are coterminous***

Coterminous colors are colors that meet up at the same vertex. So at every vertex, Tait wants each of the 3 edges to be labeled with a different color. In his paper, he describes the process of going from a map whose countries are colored to a map whose edges are colored, and vice versa. We’ll try to give a summary of his process below and illustrate it with examples. If you are convinced that the process of going between a cubic map whose countries are colored and a cubic map whose edges are colored, then it is important to notice that solving the Four Color Problem would be equivalent to proving Tait’s claim above. If one can show that in every cubic map we can always color the edges so that no edges that meet at the same vertex have the same color, then we’ve proved the Four Color Problem. Unfortunately, Tait does not prove his claim but brushes it aside as “A lemma easily proved”. Can you try and prove this lemma?

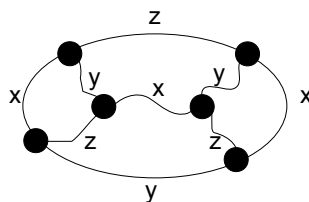
Turning a face coloring of a cubic map into an edge coloring

Consider a cubic map whose faces are colored with colors A, B, C, D, like in the example below. In our example, picked a map that only needs 3 colors to color the countries. We want to use the colors X, Y, Z for the edges. Notice that every edge in the graph separates 2 countries (of different colors). Tait lists a set of instructions that tell you how to put down the colors X, Y, and Z.



Edge Color	Conditions
X	If the edge separates countries colored A/B or C/D.
Y	If the edge separates countries colored A/C or B/D.
Z	If the edge separates countries colored A/D or B/C.

Now, using this set of instructions, try to color the edges of the graph above with X, Y, and Z.

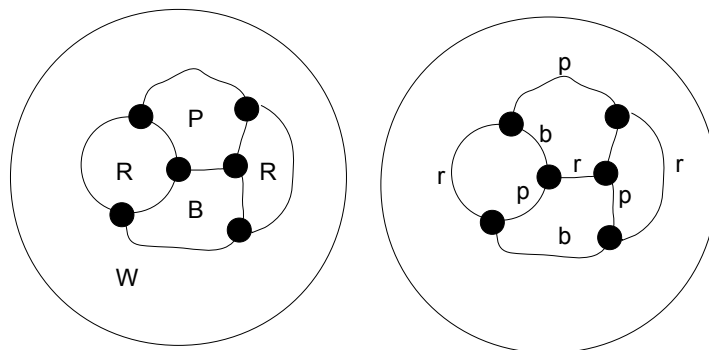


It is clear that this set of instructions is reversible. Given a map whose edges are colored, we could go ahead and start off by coloring some country with the color A. After that choice, we follow the instructions above (backwards) to choose colors for the rest of the countries. This coloring would be unique up to a permutation of the colors. There is an alternative way to reverse this process, though. This other reversal that Tait mentions later leads to another way to reformulate the Four Color Problem.

Another way to explain edge coloring

The following method to explain Tait's edge coloring instructions was to shown to me by Louis Kauffman. First we will change the names of our colors. Above we used the colors A, B, C, D for faces and X, Y, Z for edges. From now on we will use R, B, P, and W (for red, blue, and purple) on the faces, and r, b, and p on the edges. So capital letters for the country colors, and lowercase letters for the edge colors. When two countries are neighbors, we will “multiply” their colors to find what letter to color the boundary between the countries. The multiplication looks a lot like the result you get if you mixed cans of paints with those colors. Read the multiplication rules carefully and then follow along the process for the example below.

(Any Color) X (White) = The Original Color
 (Red) X (Blue) = Purple
 (Red) X (Purple) = Blue
 (Blue) X (Purple) = Red

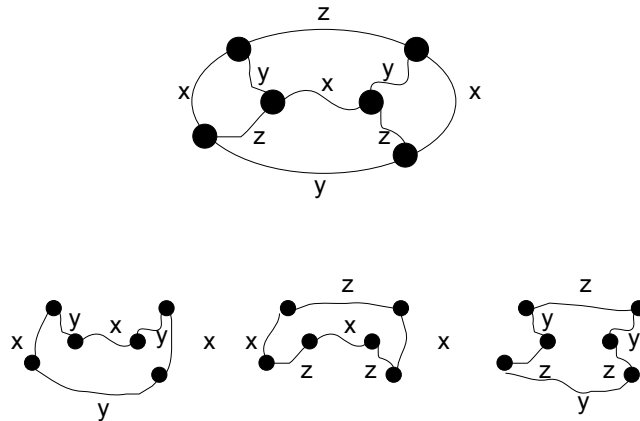


This explanation is much simpler to follow than Tait's original.

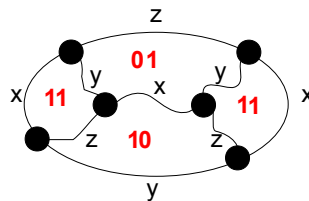
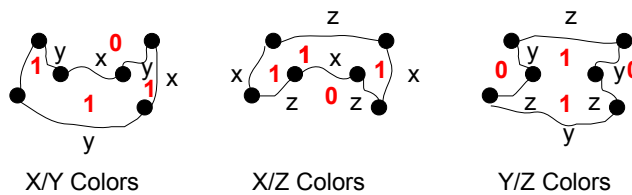
Another curious way to go from edge-coloring to a face-coloring

Some steps in this process should remind us of steps Tait used in his first proof above. They involve doing something “two different ways” and superimposing the resulting graphs on top of each other. For this reversal, start by choosing two different colors, say X and Y. Then go to your graph and draw the paths made from alternating the colors X and Y, like in the example below. Doing this always results in drawing disjoint cycles on your

graph. Take a minute to think about why that's true!



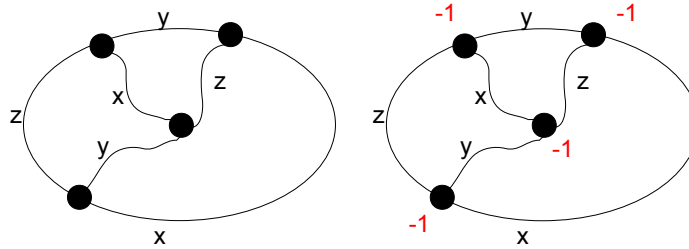
Do this process for each pair of colors, like above. Then, in each picture, draw a 1 for each country inside a circle and a zero for each country outside a circle. Now take any two of the pictures and superimpose them (put one graph on top of the other). We get the original graph back with each face labeled with 11, 10, 01, 00. This is a way to label the countries with 4 colors, and no 2 adjacent countries have the same colour! There is an explanation for why this always works, but it is beyond the scope of this paper. Can you try and work it out for yourself? In the example below, we pick the first 2 graphs on the left to superimpose.



Heawood strikes again!

Another reformulation of the Four Color Theorem from Tait's second proof

Percy Heawood, the first mathematician to notice the error in Kempe's original proof of the Four Color Problem, spent more than 60 years trying to solve the theorem. In 1898, Heawood found a reformulation of the Four Color Problem that was based on Tait's ideas in his second proof - coloring edges in cubic graphs. Heawood would start with a cubic graph whose edges were 3 colored, and then assign the numbers 1 and -1 to each vertex. Write down 1 if the colors X, Y, Z appear clockwise around a vertex, and -1 if the colors X, Y, Z appear counterclockwise around a vertex. Let's look at what happens when we assign numbers to our example from Tait's second proof.

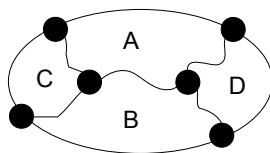


Take a moment to think about why, with 3 letters, they always appear clockwise or counterclockwise around a vertex. Try writing down all the ways you can put the letters X,Y, and Z in order, and see what happens when they you loop that ordering up in a circle.

In his paper, Heawood showed that the sum of the numbers at around each country is always divisible by 3. Reversing this argument, we get the following reformulation of the Four Color Problem.

Solving the Four Color Problem is equivalent to showing that in a cubic map you can always assign the numbers 1 and -1 at each vertex so that the sum around a face is divisible by 3.

The Chromatic Polynomial is an equation that tells you how many ways there are to color a graph with λ many colors. This polynomial was introduced by George Birkhoff, a professor at Princeton University, in 1912. Finally, an American! To be fair, there are many contributions involving the Four Color Problem that are outside the scope of this paper, and Birkhoff was not the first American to attempt a solution or some progress with it. Birkhoff's contributions lead to an interesting reformulation of the problem into algebra. There is no easy definition of the chromatic polynomial, though it has been calculated for certain types of maps. Before stating the various formulas, let's go through a basic example to see how the polynomial is calculated. Imagine you start with the following graph and want to see how many ways there are to color the graph with λ -many colors (λ could equal 1, 2, 3, etc., depending how many colors you wanted to use for the countries). When we color a map, we always want to make sure we use different colors for adjacent countries. Let's say we want to color the map below with λ many colors. The letters in the countries, A, B, C, D, are just the names of those countries now, not their colors!



Let's start off by coloring country A. Since we haven't used any colors yet, we can color A with any of our λ -many colors. Since B is a neighbor of A, we can color B with $\lambda - 1$ colors (we can't use the same color we did for A). Since C and D are not neighbors of each other, they can have the same color. But since they are both neighbours of A and B, we can't use the colors that we picked for A and B to color C and D. So there are $\lambda - 2$ many colors we can use for C and D.

$$\begin{aligned} \text{Total Number of ways to color the map} &= (\lambda)(\lambda - 1)(\lambda - 2)^2 \\ P(\lambda) &= \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda \end{aligned}$$

If we wanted to see how many ways there were to color this map with 4 colors, we can substitute $\lambda = 4$ into the equation above.

$$\begin{aligned} P(4) &= (4)^4 - 5(4)^3 + 8(4)^2 - 4(4) \\ P(4) &= 48 \end{aligned}$$

So there are 48 different ways to color our example with 4 colors. We now have a reformulation of the Four Color Theorem to the following statement:

***If $P(4)$ is a positive number
(for any map)
then all maps can be Four-Colored***

The chromatic polynomial has been calculated for the following classes of graphs:

Complete Graph K_n	$P(\lambda) = (\lambda)(\lambda - 1)\dots(\lambda - (n - 1))$
Tree with n vertices	$P(\lambda) = (\lambda)(\lambda - 1)^{n-1}$
Cycle C_n	$P(\lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$
Petersen Graph	$(\lambda)(-1)(\lambda - 2)(\lambda^7 - 12\lambda^6 + 57\lambda^5 - 230\lambda^4 + 539\lambda^3 + 775\lambda - 352)$

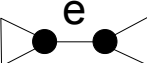


As you might guess from the table above, one curiosity about the Chromatic Polynomial is that the coefficients always alternate in sign. This was proved by Birkhoff's doctoral student Hassler Whitney in 1930. Whitney later further contributed to this area by creating a "skein relation" for the chromatic polynomial. A skein relation can be thought of as a formula (that is shown with pictures) that gives you recursive instruction for doing a computation. Whitney's formula relies on one of the results we talked about earlier - that if we draw a dual graph for a map, a vertex coloring of the dual graph is equivalent to a face coloring of the original graph.

I find the skein relation for the Chromatic Polynomial particularly interesting because of my intetests in Knot Theory. In Knot Theory we have several polynomials that are defined by recursive skein relations. We apply the relations to a knot diagram and end up with a polynomial as a way to try and tell 2 knots apart. 2 knots are different when they have different polynomials. Some knot polynomials that have definitions given by skein relations are the Jones, Kauffman, Conway, and HOMPFly polynomials.

Here we can mention an interesting connection between graph theory and knot theory - the Chromatic Polynomial for graphs and the Jones Polynomial for knots are both specializations of something called the Tutte Polynomial. We'll come back to this connection later on.

Z_G = The number of ways to vertex color G with q colors.

$$Z_{\text{diag1}} = Z_{\text{diag2}} - Z_{\text{diag3}}$$

$G - e$
 G/e

$$Z_{\bullet} = q \quad Z_{\bullet} \cup G = q Z_G$$

Whitney then gave the following reformulation of the Four Color Problem.

***To prove the Four Color Theorem
show this recursive process is non-zero
for graphs without an isthmus.***

For a long time, it was an unanswered question in Knot Theory whether or not the Jones Polynomial detected the unknot. Another way to phrase this is people wanted to show that the Jones Polynomial was non-zero for all knots except the unknot. The phrasing of this question is very similar to showing that the Chromatic Polynomial is non-zero for all knots without an isthmus. And remember that the Jones Polynomial and the Chromatic Polynomial are related; they are both specializations of the Tutte Polynomial. Could we develop some of the theory we have for the Jones Polynomial (that it arises as a differential in a homology theory) for the Chromatic and Tutte polynomials and see if we can piggyback a proof of the Four Color Theorem this way? The question is very hard to understand, and probably even harder to answer!