Further Mathematics First term, Yr1

Review Package

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Notes

1 Sets, Relations, and Groups

- In order to prove A = B:
  1. Use operation rules
  2. Let $x \in B \Rightarrow x \in A \Rightarrow B \subseteq A$
     Similarly prove $A \subseteq B$
- For R to be reflexive, xRx needs to be true $\forall x \in S$
- To find the order of a permutation, use lcm.
- $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic only when $\gcd(m, n) = 1$.

2 Linear Algebra

- The characteristic equation:

  $$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

  In other words,

  $$\lambda^2 - \mathrm{tr}A + \det A = 0$$

- The 3-step subspace test:
  1. $\vec{0} \in S$;
  2. S is closed under addition
  3. S is closed under scalar multiplication
- The rank of a matrix is defined as the number of independent rows/columns.
- Does spanning imply linear independence?
  Answer: Because of spanning, the system $\mathbf{A}x = b$ must be consistent for all b, so rref($\mathbf{A}$) must have a leading 1 in each row and column, so there are no free variables. Therefore the system $\mathbf{A}x = 0$ has a unique solution $x = 0 \Rightarrow$ linearly independent.
- Using the discriminant $b^2 - 4ac$ of the conic to classify:
  Δellipse < 0
  Δparabola $= 0$
  Δhyperbola > 0

3 Calculus

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4 Number theory

Questions

1 Sets, Relations, and Groups

1. Prove: $A \cup B = A \iff B \subset A$
2. Prove De Morgan's law
3. Prove: $A \Delta B = A \cup B \iff A \cap B = \emptyset$
4. Prove that if G is a group then each element of G will appear exactly once in every row and every column of its Cayley table.
5. Prove that if G and H are isomorphic, G is cyclic $\iff$ H is cyclic.
6. Prove that there is a cyclic group of order n, $\forall n \in \mathbb{Z}^+$
7. Prove: for any $n \in \mathbb{Z}^+$, all cyclic groups of order n are isomorphic to each other.
8. Prove: given a non-empty subset H of G, $H \quad \leq G$ if $a * b^{-1} \in H \quad \forall a, b \in H$
9. Prove: if G and H are isomorphic then any subgroup of G will be isomorphic to some subgroup of H.
10. Prove: for a finite group G of order n, if $\exists$ an element $g \in G$ with order m where $2 \leq m \leq n$, then the set $H = \{e, g, g^2, \ldots, g^{m-1}\}$ forms a cyclic subgroup of G.
11. Prove: the order of a finite group is divisible by the order of any element.
12. Consider the set $J = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ under the binary operation multiplication.
    Consider $a + b\sqrt{2} \in G$, where $\gcd(a, b) = 1$.
    Show that the subset, G, of elements of J which have inverses, forms a group of infinite order. [QB]
13. The set $S_n = \{1, 2, 3, \ldots, n-2, n-1\}$, where n is a prime number greater than 2, and $\times_n$ denotes multiplication modulo n.
    (a) Show that there are no elements $a, b \in S_n$ such that $a \times_n b = 0$.
    (b) Show that, for $a, b, c \in S_n, a \times_n b = a \times_n c \Rightarrow b = c$.
    (c) Show that $G_n = \{S_n, \times_n\}$ is a group. Associativity may be assumed.
14. Prove that a cyclic group with exactly one generator cannot have more than two elements.

2 Linear Algebra

1. Describe the transformation T represented by the matrix PQ, where P is the 2×2 matrix for a reflection in the line $y = (tan\theta)x$, and Q is the 2×2 matrix for an anticlockwise rotation of θ about the origin. [QB]

2. Prove: let A be a symmetric matrix that is not a multiple of the identity matrix. Show that the eigenvectors of A are orthogonal. [NT]

3. Prove: if A is a 2×2 symmetric matrix, then A can be factorized as $A = QDQ^T$, where $Q^T = Q^{-1}$ is an orthogonal matrix. [NT]

4. Give a geometric description of the transformation with the matrix

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

   [QB]

5. (a) List the 6 elements of S_3 in cycle form;
   (b) Show that it is not Abelian;
   (c) Hence deduce that S_n is not Abelian for $n \geq 3$. [QB]

6. (a) Write down the matrices $\mathbf{M_1}$, $\mathbf{M_2}$ representing the permutations (1 2), (2 3), respectively;
   (b) Find $\mathbf{M_1 M_2}$ and state the permutation represented by the matrix;
   (c) Find $\det(\mathbf{M_1}), \det(\mathbf{M_2})$, and deduce the value of $\det(\mathbf{M_1 M_2})$. [QB]

7. Deduce that every permutation can be written as a product of cycles of length 2. [QB]

8. (a) Show that the set of position vectors of points whose coordinates satisfy $x - y - z = 0$ forms a vector subspace, V, of $\mathbb{R}^3$.
   (b) Determine an orthogonal basis for V of which one member is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$
   (c) Augment this basis with an orthogonal vector to form a basis for $\mathbb{R}^3$

9. Prove, using mathematical induction, that

$$\mathbf{B}^n = 8^{n-2}\mathbf{B}^2 \quad \text{for } n \in \mathbb{Z}^+, n \geq 3.$$

10. The non-zero vectors v_1, v_2, v_3 form an orthogonal set of vectors in $\mathbb{R}^3$.
    (a) By considering $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$, show that v_1, v_2, v_3 are linearly independent.
    (b) Explain briefly why v_1, v_2, v_3 form a basis of vectors in $\mathbb{R}^3$

11. The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is defined by $\mathbf{X} \mapsto \mathbf{X}$, where $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are all non-zero.
    Show that f is a bijection if $\mathbf{A}$ is non-singular.

12. Suppose that $m = ab$ where a, b are unequal prime numbers greater than 2. Show that $\{S, +_m\}$ has two proper subgroups and identify them.

13. The set S contains the eight matrices of the form $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ where a, b, c can each take one of the values +1 or -1.

    (a) Show that any matrix of this form is its own inverse.

    (b) Giving a reason, state whether or not this group is cyclic.

14. Given that the elements of a 2×2 symmetric matrix are real, show that

    (a) the eigenvalues are real;

    (b) the eigenvectors are orthogonal if the eigenvalues are distinct.

15. A linear transformation satisfies $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$. Find $T\begin{pmatrix} 4 \\ 7 \end{pmatrix}$, and the matrix for T. [NT]

3 Calculus

1. Find the general solution of the differential equation $\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}$, where $xy \neq 0$.

    (a) Find the particular solution passing through the point $(1, \sqrt{2})$.

    (b) Sketch the particular solution.

    (c) The graph of the solution only contains points with $|x| > a$. Find the exact value of $a, a > 0$.

2. Find the range of values of n for which $\int_1^\infty x^n dx$ exists.

4 Number theory

1. Prove that the number 14641 is the fourth power of an integer in any base greater than 6.

Answer Key

1 Sets, Relations, and Groups

14. Hint: if a is a generator of a group then so is a^{-1}

2 Linear Algebra

1. T is a reflection in the line $y = (tan\frac{1}{2}\theta)x$

6. (b) (1 3 2)

   (c) 1

7. Hint: note that a cycle of length $k \geq 2$ can be written as a product of k-1 transpositions as follows:
   $(a_1 \dots a_{k-1} a_k) = (a_1 a_k)(a_1 a_{k-1}) \dots (a_1 a_2)$.
   Every permutation is a product of cycles. Since every cycle is a product of cycles of length 2, every permutation can be written as a product of cycles of length 2.

10. Hint: multiply both sides by v_1, we can deduce that $\alpha_1 = 0$.

12. $\{1, a, 2a, \dots, (b-1)a\}$
    $\{1, b, 2b, \dots, (a-1)b\}$

15. Hint: use the fundamental linear properties.

3 Calculus

1.

4 Number theory

1. 14641 (base $a > 6$) $= a^4 + 4a^3 + 6a^2 + 4a + 1 = (a+1)^4$