GRAPH

3.1. Introduction.

3.2. Graphs: definitions.

def 1. A graph $G=(V,E)$ consists of 2 sets: V, a non-empty set of vertices/nodes/points; an E, a set of unordered pairs of diff. elmts of V, called edges/arcs/sides.

Note

- A graph w/ no direction assigned to its edges is undirected.
- A graph where all pairs of adjacent vertices are connected by only one edge are simple graphs.

Def 2.

1. Vertices A and B in an undirected graph G are called adjacent if $u=\{A,B\}$ is an edge in G. The edge u is said to be incident w/ and connect A and B.

   A, B are called the endpoints of $\{A,B\}$.

   Two edges are said to be adjacent if they have a vertex in common.
2. If an edge has only one ~~ed~~ endpoint, then the edge joins the vertex to itself and is called a loop.
3. If 2 edges have the same endpoints, they are called multiple edges or parallel edges.
4. The degree of a vertex in an undirected graph is the # of edge incident w/ it. (A loop contributes 2 degrees).

   denoted by deg(a).

   - A vertex w/ deg 0 is isolated and a vertex w/ deg 1 is pendant.
   - odd / even vertices: odd/even degrees.

Def. 3

A simple graph $G=(V,E)$ is a graph w/ no loops or parallel edges.

If there are more than 2 edges, → multigraph.

Thrm 1. (the handshaking Thrm)

$G=(V,E)$. $|E|=e$.

$$\sum_{v\in V} \deg(v) = 2e.$$

NOTE: Applies to multigraph also.

Thrm 2.

An undirected graph can only have an even number of odd vertices.
(proof using thrm 1).

Def 4. (Subgraphs)

Given graph $G = (V, E)$. then $G_1 = (V_1, E_1)$ is a subgraph of G if $V_1 \subseteq V$, $E_1 \subseteq E$. and $V_1 \neq \emptyset$.

Def 5. (Union)

$$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$$

Some special Graphs.

Def 6. (Digraphs)

A directed graph / digraph $G = (V, E)$ has:
V, a non-empty set of vertices;
and E, a set of ordered pairs of diff. elmts of V called edges.

Note: Each directed arc has an initial vertex and a terminal vertex.
e.g. $e_1 = (b, a)$. $b \xrightarrow{e_1} a$

Def 7. (Deg in digraphs)

In-degree of a vertex v, $\deg^-(v)$, is the # of edges, is the # of edges w/ v as their terminal vertex.
Out-degree of v. $\deg^+(v)$, ... initial vertex.

Thrm 3

In a digraph $G = (V, E)$.
$$|E| = \sum_{v \in V} \deg^+(v) = \sum_{v \in V} \deg^-(v).$$

def 8. (Complete graphs)

A simple graph $G = (V, E)$ is called a complete graph if $\forall a, b \in V$, $\exists$ an edge $\{a, b\}$.
A complete graph w/ n vertices is denoted by K_n.

Thrm 4.

# of edges in a complete graph K_n is given by $|K_n| = \frac{n(n-1)}{2}$.

Def 9. (complement)

The complement of simple graph G, denoted G', contains the same set of vertices as G. and all edges not in G.

Remark:

- $G \cup G' = K_n$.
- K_n' is a null graph.

Def 10. (Bipartite graphs)

A simple graph $G=(V,E)$ is bipartite if $\exists$ V_1, V_2 s.t. $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$. (often called a partition) and all the edges are of the form $\{X, Y\}$ s.t. $X \in V_1$, $Y \in V_2$.

- A bipartite graph is complete if V_1 is adjacent to all V_2.
- Notation: $K_{m,n}$. $|V_1| = m$, $|V_2| = n$.

3.3. Graph Representation.

Adjacency Matrices.

def 11.

The adjacency matrix A_G of a simple graph $G=(V,E)$ w/ n vertices is an $n \times n$ matrix w/ 1 or 0 s.t.

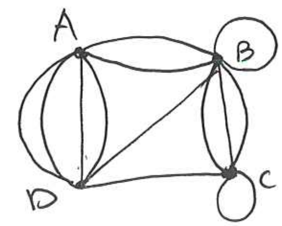
$$a_{i,j} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge.} \\ 0 & \text{otherwise} \end{cases}$$

Note: For a multigraph:

$$a_{i,j} = \begin{cases} k(i,j) & k = \#\text{ of edges btw } v_i, v_j \\ 0 & \text{otherwise.} \end{cases}$$

- Symmetric for a simple graph, (main diag = 0) but loops and multiple edges contribute to a non-diagonal.

e.g.



$$\Rightarrow A_G = \begin{pmatrix} 0 & 2 & 0 & 4 \\ 2 & 1 & 3 & 1 \\ 0 & 3 & 1 & 1 \\ 4 & 1 & 1 & 0 \end{pmatrix}$$

Remark:

- $A_{K_3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ (All 1s, except md).
- for complementary graph, all entries except main diagonal (always 0) are complementary 1 and 0.

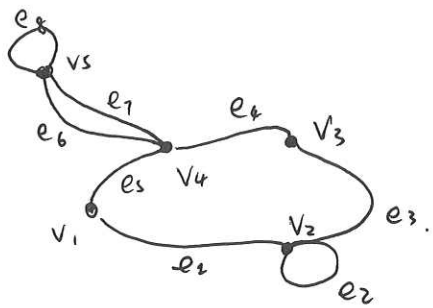
Incidence Matrices

Def 12

The incidence matrix I_G of a simple graph $G=(V,E)$ w/ n vertices and k edges is an $n \times k$ matrix w/ 1 an 0, s.t.

$$a_{i,j} = \begin{cases} 1 & \text{if } e_j \text{ is incident w/ } v_i \\ 0 & \text{otherwise.} \end{cases}$$

e.g.



$$\begin{array}{c|cccccccc} & e_1 & e_2 & e_3 & \cdots & & & & e_8 \\ \hline v_1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ v_2 & 1 & \boxed{1} & 1 & 0 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ v_5 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \boxed{1} \end{array}$$

Isomorphic graphs.

Def 13

Let $G=(V,E)$, $G'=(V',E')$ be two simple graphs. If $\exists$ a bijection $f: V \to V'$. s.t $\forall$ pair of vertices $v_i . v_j$ adjacent in G, vertices $f(v_i) . f(v_j)$ are adjacent in G'.

$\Rightarrow$ G and G' are <u>isomorphic</u>.

f is a graph <u>isomorphism</u>.

Note: when checking for isomaphism, use adjacency matrices, & check for deg. first.

Thrms

Let G and G' be iso. graphs. and $f: V \to V'$ a graph isosm. $\forall a \in V$, $\deg(a) = \deg(f(a))$

3.4. Paths. walks. and trails.

Def. 14. (walks)

A $v_0 - v_n$ <u>walk</u> in graph G is a finite alternating sequence $v_0, e_1, v_1, e_2, \dots e_{n-1}, v_{n-1}, e_n, v_n$.

of vertices and edges starting at vertex v_0 and ending at vertex v_n. and involving the n edges $e_i = \{v_{i-1}, v_i\}$, where $1 \le i \le n$.

* $v_0 . v_n$ do not have to be different.

The <u>length of a walk</u>, n, is the # of edges used in the sequence.

Note: A walk may repeate both edges and vertices.

Def. 15

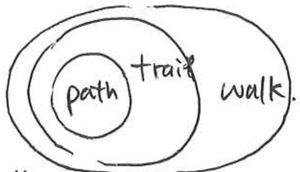
1. A <u>trail</u> is a walk in which no edge appears more than once.

   A trail begins and ends at the same vertex is called a <u>circuit</u>.

2. A walk where no vertex is visited more than once is called a <u>path</u>.

A path begin and end w/ same vertex is called a <u>cycle</u>.

* Every path is a trail; A trail can be a path only in a simple graph. → same for circuit & cycle.



Adjacency matrices and walks.

Def. (regular graph)

A regular graph is a graph where all vertices have the same degree.

Thrm6

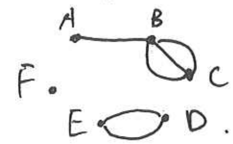
The # of walks of length n from vertex v_i to v_j is given by the (i,j)th entry of A_G^n, $n \in \mathbb{Z}^+$.

Def 16.

Let V be a non-empty set of vertices and E be a non-empty set of edges.

The graph $G=(V,E)$ is called a <u>connected graph</u> if ∃ a path btw any 2 vertices from the set V.

e.g.



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note: disconnected subgraph has only 0s in a row / column.

Properties of connected graphs.

property 1.

Let $G=(V,E)$ be a simple connected graph. $a, b \in V$ are not adjacent. If G' is formed by adding the edge ab to G, then G_1 has a cycle that contains the edge ab.

property 2.

when an edge is ~~no~~ removed from a cycle in a connected graph, the result is still a connected graph.

Eulerian graphs.

Def. 17 Let $G=(V,E)$ be a connected graph. A trail where every edge of G appears.

only once is an <u>Eulerian trail</u>. A circuit ... <u>Eulerian circuit</u>.

A connected graph w/ an Eulerian circuit is called an <u>Eulerian graph</u>.

Thrm 7.

Let $G=(V,E)$ be a connected graph. G has an Eulerian circuit iff. every vertex has an even deg.

Thrm 8.

Let $G=(V,E)$ be a connected graph. G has an Eulerian trail but not circuit $\Leftrightarrow$ G has exactly 2 vertices of odd deg.

Hamiltonian graphs.

Def 18.

Let $G=(V,E)$ be a connected path. A path that contains all vertices of G is called a <u>Hamiltonian path</u>. A cycle ... <u>Hamiltonian cycle</u>.

A connected graph .... <u>H~ graph</u>.

Thrm 9 (Dirac's theorem)

Let $G=(V,E)$ be a simple connected graph. If $|V|=n$, $n \geq 3$, and $\forall$ vertex $A \in V$, $\deg(A) \geq \frac{n}{2}$, then the graph G has a H cycle.

Thrm 10 (Ore's thrm)

$\forall$ pair of non-adjacent vertices $A, B \in V$. $\deg(A) + \deg(B) \geq n$. $\rightarrow$ H cycle.

Bipartite graphs - negative tests.

G is a b. graph w/ V_1 and V_2 subsets of vertices.

Let subset 1 have m vertices and subset 2, n vs.

- If $m \neq n$, G cannot have a Ham. cycle.
- if m and n differ by 2 or more, no H path.

3.5 Planar Graphs.

Def. 19

A <u>planar graph</u> is a graph that can be represented by a diagram w/o edge cross.

$\rightarrow$ <u>plane diagram</u> / planar representation / embedding.

Euler's Formula.

Thrm 11 (Euler's formula)

Let $G=(V,E)$ be a connected planar simple graph (multigraph) where $|V|=v$. $|E|=e$. and f is # of faces of the graph's planar embedding, then

$$v-e+f=2.$$

Thrm 12

If G is a connected simple planar graph w/ e edges and $v>2$ vertices.

then $e \leq 3v-6$.

proof: $\begin{cases} 2+e-v=f \\ 2e \geq 3f. \end{cases}$

Thrm 13

If G is a ..., and no circuits of length 3, then $e \leq 2v-4$.

Note:

K_5. $K_{3,3}$ is not planar, all graphs containing K_5, $K_{3,3}$ as subgroups are also not planar. Also, graph with subgroup that can be obtained from K_5, $K_{3,3}$ using certain operation, is not planar.

Homeomorphic graphs

def elementary subdivision:

remove $\{A,B\}$. add $\{A,C\}$, $\{B,C\}$.

Graphs are homeomorphic if obtainable from the same graph using elemt. subdivision.

Thrm 14 (Kuratowski's theorem)

A graph $G=(V,E)$ is not planar $\Leftrightarrow$ it contains a subgraph homeo to K_5 or $K_{3,3}$.