Maclaurin and Taylor Polynomials

Polynomials

First observe that if $p(x)$ is an nth degree polynomial and we know the value and first n derivatives of $p(x)$ at $x=0$, then we know the polynomial. To see this, let

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n.$$

Then

$$\begin{aligned} p'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \ldots + n a_n x^{n-1} \\ p''(x) &= 2a_2 + 6a_3 x + \ldots + n(n-1) a_n x^{n-2} \\ \vdots \quad & \quad \vdots \\ p^{(n)}(x) &= n! a_n \end{aligned}$$

So

$$p(0) = a_0, \quad p'(0) = a_1, \quad \ldots \quad p^{(k)}(0) = k! a_k, \quad \ldots \quad p^{(n)}(0) = n! a_n.$$

That is,

$$\boxed{a_k = \frac{p^{(k)}(0)}{k!}, \quad k \in \mathbb{N}}$$

Example 1 *Find the cubic polynomial with value and successive derivatives at $x=0$ equal to $1, 0, -2$ and 12 respectively.*

Let $P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

$a_0 = 1$
$a_1 = 0$
$2a_2 = -2$.
$6a_3 = 12$.

$a_0 = 1$
$a_1 = 0$
$a_2 = -1$
$a_3 = 2$.

$P_3(x) = 1 - x^2 + 2x^3$.

Maclaurin Polynomials

Given any function f having derivatives of order up to and including n at $x=0$, we may construct a polynomial p that agrees with f in the following way:

$$p(0) = f(0), p'(0) = f'(0), \ldots, p^{(n)}(0) = f^{(n)}(0).$$

Such a polynomial is called the nth degree Maclaurin polynomial for the function f. We conclude that the nth degree Maclaurin polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ for the function f, has coefficients a_k satisfying

$$\boxed{a_k = \frac{f^{(k)}(0)}{k!}, \quad k \in \mathbb{N}}$$

Example 2 *Find the cubic Maclaurin polynomial for $f(x) = e^x$.*

$f^0(x) = e^x.$

$f'(x) = e^x$ $\qquad x = 0.$

$f^2(x) = e^x$ $\qquad \Rightarrow f^{(n)}(0) = 1.$

$\vdots$

$f^3(x) = e^x.$

$a_0 = 1.$

$a_1 = 1$

$a_2 = \frac{1}{2}$ $\qquad \Rightarrow P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3.$

$a_3 = \frac{1}{6}.$

Example 3 *Find the fourth degree Maclaurin polynomial for $f(x) = \cos x$.*

$$\begin{aligned} P_4(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \\ &= \frac{P^0(0)}{0!} + \dots + \frac{P^4(0)}{4!}x^4 \\ &= 1 + 0 + \frac{(-1)}{2}x^2 + 0 + \frac{1}{24}x^4 \\ &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4. \end{aligned}$$

Taylor Polynomials

The Taylor polynomial is a generalization of the Maclaurin polynomial to allow for any centre. Here we observe that if $p(x)$ is an nth degree polynomial and we know the value and first n derivatives of $p(x)$ at the centre $x = a$, then we know the polynomial. This time we express $p(x)$ in powers of $(x-a)$, this greatly simplifies the algebra, that is we let

$$p(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots + a_n(x-a)^n.$$

$Q(x) = P(x+a)$

let ~~$y = x-a$~~

Example 4 *Show that on this occasion*

$$\boxed{a_k = \frac{p^{(k)}(a)}{k!}, \quad k \in \mathbb{N}}$$

$Q(x)$

$= a_0 + a_1 x + \dots + a_n x^n.$

$\Rightarrow a_k = \frac{Q^{(k)}(0)}{k}$

So given any function f having derivatives of order up to and including n at $x = a$, we may construct a polynomial p that agrees with f in the following way:

$$p(a) = f(a), p'(a) = f'(a), \ldots, p^{(n)}(a) = f^{(n)}(a).$$

Such a polynomial is called the nth degree Taylor polynomial for the function f with centre $x = a$.

Example 5 *Conclude that the nth degree Taylor polynomial for the function f with centre $x = a$ is*

$$\boxed{p(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k, \quad n \in \mathbb{N}}$$

$$p(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n.$$

$$= \frac{p^{(0)}(a)}{0!} + \cdots + \frac{p^{(n)}(a)}{n!}(x-a)^n.$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^x.$$

Example 6 *Find the quadratic Taylor polynomial for $f(x) = e^x$ with centre $x = 1$.*

$$p(x) = \frac{a_0}{0!} + \cancel{\frac{a_1(x-a)}{1!} + \frac{a_2(x-a)^2}{2!}}$$

$$= f^{(0)}(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2$$

$$= e^a + e^a(x-1) + \frac{e^a}{2}(x-1)^2.$$

$$= \cancel{e} + ex \cancel{-e} + \frac{e}{2}x^2 \cancel{-} \cancel{ex} + \frac{e}{2}$$

$$= \frac{e}{2}x^2 + \frac{e}{2}$$

Often we replace x by $a + h$ in the above polynomial to obtain the following polynomial in h

$$\boxed{p(a+h) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}h^k, \quad n \in \mathbb{N}}$$

Example 7 *Expand* $\sin(\pi + h)$ *as a 5th degree Taylor polynomial for the sine function with centre* π*. Show that your result is consistent with the trigonometric identity* $\sin(\pi + h) = -\sin(h)$.

$\cancel{P_5(h) = \sin(\pi + h)}$

$\cancel{P_5(a + h) = \sin(\pi + h)}$

$$P_5(\pi + h) = a_0 + a_1(h) + a_2(h^2) + \cdots + a_5(h^5)$$

$$= \frac{f^{(0)}(\pi)}{0!} + \frac{f^{(1)}(\pi)}{1!}h + \frac{f^{(2)}(\pi)}{2!}h^2 + \cdots + \frac{f^{(5)}(\pi)}{5!}h^5$$

$$= 0 + (-h) + 0 + \cdot \frac{1}{6}h^3 + 0 - \frac{1}{5!}h^5$$

$$= -h + \frac{1}{6}h^3 - \frac{1}{5!}h^5.$$

$\cancel{P_5(h) = a_0 + a_1(h-\pi) + a_2(h-\pi)^2 + \cdots + a_5(h-\pi)^5}$

$\cancel{= \frac{f^{(0)}(\pi)}{0!} + \frac{f'(\pi)}{1!}(h-\pi) + \cdots + \frac{f^{(5)}(\pi)}{5!}(h-\pi)^5}$

$$P_5(h) = \frac{f^{(0)}(0)}{0!} + \cdots \frac{f^{(5)}(0)}{5!}h^5$$

$$= h - \frac{1}{6}h^3 + \frac{1}{5!}h^5$$

$\Rightarrow P_5(\pi + h) = -P_5(h)$

$\Rightarrow \sin(\pi + h) = -\sin(h)$