Exercise

1 Complete the following table, indicating whether the relations on the given sets are reflexive, symmetric or transitive:

Set	Relation	Reflexive	Symmetric	Transitive
(*a*) {boys}	'is the brother of'	F	T	F
(*b*) {children}	'is the brother of'			
(*c*) {straight lines in the plane}	'is perpendicular to'			
(*d*) {straight lines in the plane}	'is parallel to or coincident with'			
(*e*) {triangles in the plane}	'is similar to'			
(*f*) {triangles in the plane}	'is congruent with'			
(*g*) $\mathbb{Z}$	xRy if and only if $xy > 0$			
(*h*) $\mathbb{Z}$	xRy if and only if $\lvert x - y\rvert$ is even			
(*i*) $\mathbb{Z}$	xRy if and only if $(x^2 - y^2)$ is an even integer			
(*j*) $\mathbb{Z}$	xRy if and only if $(x^2 - y^2)$ is an odd integer			
(*k*) $\mathbb{Z}$	xRy if and only if $(x - y)$ is divisible by 2 or 3			
(*l*) $\mathbb{Z}$	xRy if and only if $x < y$			
(*m*) $\mathbb{Z}^+$	xRy if and only if x is a factor of y			
(*n*) $\mathbb{Z}^+$	xRy if and only if $2x > y$			
(*o*) $\mathbb{Z}^+$	xRy if and only if $(x - y)$ is divisible by 5			
(*p*) $\mathbb{Z}^+$	xRy if and only if x has the same number of digits as y			
(*q*) $\mathbb{Z}^+$	xRy if and only if $\frac{x}{y}$ is an integer			

2 Describe the equivalence classes for each of the equivalence relations in question 1.

3 Invent other examples of sets and relations which are
(*a*) symmetric and transitive, but not reflexive
(*b*) reflexive and symmetric, but not transitive
(*c*) reflexive and transitive, but not symmetric
(*d*) reflexive, symmetric and transitive

4 (*a*) A relation R_1 is defined on the set

$$S_1 = \{(a, b): a, b \in \mathbb{Z} \text{ and } b \neq 0\}$$

by $(p, q)R_1(r, s)$ if and only if $ps = rq$.
Prove that this is an equivalence relation, and describe the equivalence classes.
(*b*) A relation R_2 is defined on the set

$$S_2 = \{(a, b): a, b \in \mathbb{Z}^+\}$$

by $(p, q)R_2(r, s)$ if and only if $p + s = q + r$.
Prove that this is an equivalence relation, and describe the equivalence classes.

5 S is the set of matrices $\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}\right\}$.

(*a*) If $\mathbf{P} = \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$, find $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{P} = \mathbf{AQ}$ and $\mathbf{Q} = \mathbf{BP}$, and verify that $\mathbf{A}^{-1} = \mathbf{B}$.
(*b*) A relation is now defined on the set S by '$\mathbf{P}$ is related to $\mathbf{Q}$ if and only if there exists a non-singular matrix $\mathbf{A}$, such that $\mathbf{P} = \mathbf{AQ}$'. Is this relation an equivalence relation?

6 A relation R is defined on the set of 2×1 matrices as follows: '$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} R \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ if you can find a value of $\lambda \in \mathbb{R}$ which will make the statement $\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ true.'
(*a*) Show that this relation is an equivalence relation.
(*b*) Find three 2×1 matrices each related to $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$.
(*c*) Describe fully the equivalence class $\left[\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right]$.

7 Let S be the following set of matrices:

$$S = \left\{\begin{bmatrix} 2 - a & 1 - a \\ a - 1 & a \end{bmatrix} : a \in \mathbb{R}\right\}.$$

A relation R is defined on the set of points in the plane by

$$(x_1, x_2)R(y_1, y_2)$$

if and only if $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{A}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ for some $\mathbf{A} \in S$.
(*a*) Show that the relation is an equivalence relation.
(*b*) Describe the set of points in the equivalence class containing the point (0, 2).

8 How many distinct equivalence relations can be defined on the set $\{x, y, z\}$?

1 (*a*) F T F	(*b*) F F F	(*c*) F T F	(*d*) T T T	(*e*) T T T
(*f*) T T T	(*g*) F T T	(*h*) T T T	(*i*) T T T	(*j*) F T F
(*k*) T T F	(*l*) F F T	(*m*) T F T	(*n*) T F F	(*o*) T T T
(*p*) T T T	(*q*) T F T			

2 (*d*) Parallel lines
(*e*) Similar triangles
(*f*) Congruent triangles
(*h*) {odd integers}, {even integers}
(*i*) {odd integers}, {even integers}
(*o*) Congruent classes mod 5
(*p*) Numbers of same length

3 (*a*) Any non-empty set with a void relation
(*b*) Non-zero elements of $\mathbb{Z}$ with $xRy \Leftrightarrow x$ and y have same sign or same parity
(*c*) $\mathbb{R}$ with $xRy \Leftrightarrow x \leqslant y$
(*d*) $\mathbb{C}$ with $z_1 R z_1 \Leftrightarrow |z_1| = |z_2|$

5 (*a*) $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{7}{2} & -3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{7}{10} & -\frac{1}{10} \end{bmatrix}$ (*b*) Yes

6 (*b*) $\begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (*c*) $\left\{\begin{bmatrix} \mu \\ 3 \end{bmatrix}\right\}, \mu \in \mathbb{R}$

7 (*b*) The line $x + y = 2$

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