Third lesson—playing games all at once!

Outline

Objectives:

1.Learn the equivalence of games;
2.Learn to play Nim with any number of piles;
3.Learn how to play all impartial combinatorial games using nim-value;
4.Summary on what we learned;
5.Provide the ultimate question.

Content:

3.1 Equivalence of games
  Definition 3.1
  Theorem 3.1
3.2 Playing any piles of Nim game
  Example of a 5 pile Nim game
3.3 Sprague-Grundy theorem
  Definition 3.2 (nim-value);
  Theorem 3.2 (Sprague-Grundy)
  Example of using SG theorem on a take-away game.
3.4 summary—what we have learned
3.5 what else we can explore

Exercise:

One final BIG question related to the application of all what we learned in three lessons.

3.1 Equivalence of games

Before we proceed to learning how to play multiple different games together, we first need to the know the concept of the equivalence of games.

Definition 3.1

Let G_1 and G_2 be two games, and let g_1 and g_2 be positions of G_1 and G_2, respectively. Then g_1 is equivalent to g_2 ($g_1 \simeq g_2$) if, for any position h in any game H, $g_1 + h$ has the same N/P position as $g_2 + h$. (Crash Course 15)

That means two games are equivalent if they produce the same result in addition of games.

Theorem 3.1

$g_1 \simeq g_2$ if and only if $g_1 + g_2$ is a P-position. (Crash Course 15)

Proof:

$g_1 \simeq g_2 \Rightarrow g_1 + g_2$ is a P-position.

According to definition of equivalence, for any position in any game H, $g_1 + h$ has the same N/P position as $g_2 + h$. Let h be g_2.

Thus, $g_1 + g_2$ has the same N/P position as $g_2 + g_2$.

And since $g_2 + g_2$ is a P-position (Theorem 2.1), $g_1 + g_2$ is also a P-position.

$g_1 + g_2$ is a P-position $\Rightarrow g_1 \simeq g_2$

If $g_1 + g_2$ is a P-position, and since adding a P-position has no effect on the result position (Theorem 2.3), $g_1 + h$ must have the same position has $(g_1 + h) + (g_1 + g_2)$.

Since the addition of games is commutative and associative,

$(g_1 + h) + (g_1 + g_2) = (g_2 + h) + (g_1 + g_1)$.

Again, since $g_1 + g_1$ is a P-position, according to Theorem 2.3, $(g_2 + h)$ has the same N/P position as $(g_2 + h) + (g_1 + g_1)$, which equals to $(g_1 + h) + (g_1 + g_2)$, which has the same position as $g_1 + h$, so $g_1 \simeq g_2$.

Therefore, $g_1 \simeq g_2$ if and only if $g_1 + g_2$ is a P-position. //

Corollary 3.1

If $g_1 \simeq g_2$ and h is any position, then $g_1 + h \simeq g_2 + h$.

Proof:

$(g_1 + h) + (g_2 + h) = (g_1 + g_2) + (h + h)$.

Since both $(g_1 + g_2)$ and $(h + h)$ are P-positions (Theorem 2.2), $(g_1 + g_2) + (h + h)$ is also a P-position, and so is $(g_1 + h) + (g_2 + h)$.

According to Theorem 3.1, $g_1 + h \simeq g_2 + h$. //

Corollary 3.2

$g \simeq 0$ if and only if $g + 0$ is a P-position.

Proof:

According to Theorem 3.1, $g_1 \simeq g_2$ if and only if $g_1 + g_2$ is a P-position.

When $g_1 = g$ and $g_2 = 0$,

$g \simeq 0$ if and only if $g + 0$ is a P-position. //

Corollary 3.3

Two nimheaps are equivalent if and only if they are the same size.

Proof:

Let X, Y denote two nimheaps, $X \geq Y$.

$X = Y \Rightarrow X \simeq Y$

If $X = Y$, $X \oplus Y$ is a P=position according to Theorem 2.2, so $X \simeq Y$.

$X \simeq Y \Rightarrow X = Y$, i.e. $X \neq Y \Rightarrow X$ not equivalen to Y

Since $X > Y$, $X \oplus Y$ can be moved to $Y \oplus Y$, a P-position, so $X \oplus Y$ itself is a N-position.

Thus, X is not equivalent to Y.

Therefore, $X = Y$ iff $X \simeq Y$. //

3.2 Playing any piles of Nim game

Do you remember the Nim table we constructed in Lesson 2?

	0	1	2	3	4	5	6	7	8	9	10	…
0	0	1	2	3	4	5	6	7	8	9	10	…
1	1	0	3	2	5	4	7	6	9	8	11	…
2	2	3	0	1	6	7	4	5	10	11	8	…
3	3	2	1	0	7	6	5	4	11	10	9	…
4	4	5	6	7	0	1	2	3	12	13	14	…
5	5	4	7	6	1	0	3	2	13	12	15	…
6	6	7	4	5	2	3	0	1	14	15	12	…
7	7	6	5	4	3	2	1	0	15	14	13	…
8	8	9	10	11	12	13	14	15	0	**1**	2	…
9	9	8	11	10	13	12	15	14	1	0	**3**	…
10	10	11	8	9	14	15	12	13	**2**	3	0	…
…	…	…	…	…	…	…	…	…	…	…	…	…

Now that we have learned the equivalence of games, we can understand that the Nim table is not only for solving 3-pile Nim games, but also for all Nim games, no matter how many piles there are. Because by saying $X \oplus Y \oplus Z$ is a P-position, we are really saying that $X \oplus Y \simeq Z$.

Let's look at some examples of how to play a Nim game with more than 3 piles:

Example 3.1

In a 4-pile Nim game with a position of (4, 7, 3, 8), as the first player, what can you do to win the game?

Solution:

According to Nim table,

$4 \oplus 7 \oplus 3 \oplus 8 \simeq 3 \oplus 3 \oplus 8$ (because $4 \oplus 7 = 3$)

$\simeq 8$ (because $3 \oplus 3 = 0$)

Thus, the position (4, 7, 3, 8) is a N-position, and since you happen to go first, you want to make it a P-position by taking all 8 M&Ms in the 8-pile.

Also, since the position is equivalent to 8, when this position is added to other impartial games, it behaves exactly like a Nimheap with 8 M&Ms.

Example #2:
Now you are playing the Nim game with 5 piles of (2, 10, 9, 4, 5), would you go first or second?
Solution:

$$\begin{aligned} 2 \oplus 10 \oplus 9 \oplus 4 \oplus 5 &\simeq 8 \oplus 9 \oplus 4 \oplus 5 \text{ (because } 2 \oplus 10 = 8) \\ &\simeq 1 \oplus 4 \oplus 5 \text{ (because } 8 \oplus 9 = 1) \\ &\simeq 5 \oplus 5 \text{ (because } 1 \oplus 4 = 5) \\ &\simeq 0 \end{aligned}$$

Thus, this is a P-position and in order to win the game, you would choose to be the second player.

3.3 Sprague-Grundy theorem

Now it's time to know why we are so concerned with Nim game (we played that for the whole second lesson!). The answer lies in the theorem below:

Theorem 3.2 —Sprague-Grundy (Crash Course 16)
Any position g in an impartial game is equivalent to a nimheap. i.e., there is always some nonnegative integer X such that $g \simeq X$.
And X is known as the **nimvalue** of g, writes as $|g| = X$.

Proof by strong mathematical induction:
Let a g denote a position in an impartial game with height n.
We first check for n=0.
All terminal positions are P-positions and thus have nimvalue 0.
Now we want to show that if the statement is true for all height less than k, i.e. $n = 1$, $n = 2$, …, $n = k$, then the statement is also true for height k+1.
Assume for every position g with height less than k+1 is equivalent to a nimheap.
Let $S = \{h1, h2, . . . hk\}$ be the set of positions to which can be moved to from g.
Since all the elements of S have height less than k+1, each $hi \in S$ is equivalent to some nimheap Xi. Then we claim that $g \simeq M$, where $M = MEX(X_1,\ldots,X_k)$.
We want to show that $g \oplus M$ is a P-position by showing that all moves from $g \oplus M$ are N-positions. There are two ways to move from $g + M^*$:

First, there can be a move in g to some $h_i \in S$. The resulting position is $h_i \oplus M$, which is equivalent to $X_i \oplus M$, which is an N-position unless $M = X_i$, which is impossible by the definition of MEX.
Also, there can be moves from M.
The resulting position is $g \oplus M_1$, for some $M_1 < M$.
By definition of MEX, $M_1 = X_i$ for some i.
Thus it is possible to move from $g \oplus M_1$ to $h_i \oplus M_1 \simeq X_i \oplus M_1 = M_1 \oplus M_1 \simeq 0$.
This means $g \oplus M_1$ is an N -position.
We have shown that all moves from $g \oplus M$ are to N-positions. This implies that $g \oplus M$ is a P-position, so $g \simeq X$, and the statement is true for height k+1.
Since the truth of all height less than K=1 implies the truth of height k+1, by strong mathematical induction, the theorem is true for g with height of positive integer n. //

The reason why Sprague-Grundy is so important is that you can literally solve any impartial games with this theorem in mind and compute the nimvalue of every position using the MEX rule.
Then, you will find that you are actually playing various Nim games with n-piles.
Below is an example to illustrate how it works:

Again, we go back to where we start—a take-away game. We denote the game position by Tx.
Now we play three of it once, with positions of 11, 16, and 2019 M&Ms. Valid moves consist of eating 1, 2, or 4 M&Ms.
Still, it will be very convenient if we construct a Nim table like this:

X	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**…**
Tx	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	…

Here is how it is constructed:
First, we know the terminal position $|T_0| = 0$;
Building on this, we have $|T_1| = MEX(|T_0|) = MEX(0) = 1$
$|T_2| = MEX(|T_1|, |T_0|) = MEX(0, 1) = 2$
$|T_3| = MEX(|T_1|, |T_2|) = MEX(1, 2) = 0$
$|T_4| = MEX(|T_0|, |T_2|, |T_3|) = MEX(0, 2, 0) = 1$
…
Continuing in this way, we again find a cycle of 0, 1, 2 repeats.

Since $2019 \equiv 0 \pmod 3$
$|T_{2019}| = 0$

Thus, $11+15+2019 \simeq 2 \oplus 1 \oplus 0$
$\simeq 3 \oplus 0$
$\simeq 3$

Therefore, (11, 15, 2019) is a N-position, and you would want to go first and make it a P-position for the next player.

3.4 summary—what we have learned

1. Identify impartial combinatorial games;
2. Decompose games into addition of single impartial games and analyze each individually;
3. Find the nimvalues for each of the game using the MEX rule and construct a Nim table;
4. Play different impartial games as if playing Nim.

3.5 what else we can explore

However, this lessons can only serve as an introduction to game theories.
Still, there are many other impartial combinatorial games, which can be found at this website www.math.ucla.edu/~tom/Game_Theory/Contents.html.

Moreover, impartial combinatorial games are just a small part of game theories.
There are way more interesting topics under game theory like Zero-sum and Genial-sum games, which can also be explored in this website https://www.math.ucla.edu/~tom/Game_Theory/Contents.html.

3.6 Exercises.

Do you still remember the game of Empty-And-Divid we played at the first lesson? Let's look at it again, but now we play the addition of three empty-and-divide games: There are three games, two piles in each, a valid move consists of eating up all M&Ms in the chosen game and split the other into two new piles each with at least one M&M.
The one who cannot further split any pile loses, i.e. the one who is left with six piles of 1 M&M loses.

Now Alice has to move. She has the position of (2, 4), (5, 9) and (3, 6).
What should she do in next in order to win this game?

In another round, Alice is faced with the position (2, 4), (5, 4) and (2019, 9102). What should she do this time in order to win the game?

Explain your answer using what you learned in the three lessons.
You may want to construct a Nim table and use Sprague-Grundy theorem.

Answer key:

Before we start answering those questions, we need to first identify the impartial games, which we have done in Lesson one, Exercise 3. Now that we already know it consists of three Empty-and-Divide game, we want to analyze them individually first, find the nimvalues by using MEX rule and construct the Nim table.

Here is the nim table I constructed:

	1	2	3	4	5	6	7	8	9	…
1	0	1	0	2	0	1	0	3	0	…
2	1	1	2	2	1	1	3	3	1	…
3	0	2	0	2	0	3	0	3	0	…
4	2	2	2	2	3	3	3	3	2	…
5	0	1	0	3	0	1	0	3	0	…
6	1	1	3	3	1	1	3	3	1	…
7	0	3	0	3	0	3	0	3	0	…
8	3	3	3	3	3	3	3	3	4	…
9	0	1	0	2	0	1	0	4	0	…
…	…	…	…	…	…	…	…	…	…	…

How is it calculated? It follows the same principle as the take-away example in Section 3.3:
First, we know that (1, 1) is the terminal position, which is a P-position, and thus $|(1, 1)| = 0$;
Then since the only move from (1, 2) and (2, 2) is (1, 1),
$|(1, 2)| = |(2, 2)| = \text{MEX}(\ |(1, 1)|\ ) = \text{MEX}(0) = 1$;
$|(1, 3)| = \text{MEX}(\ |(1, 2)|\ ) = \text{MEX}(1) = 0$;
$|(2, 3)| = \text{MEX}(\ |(1, 1)|, |(1, 2)|\ ) = \text{MEX}(0, 1) = 2$;
…

1. (2, 4), (5, 9) and (3, 7):

According to Sprague Grundy Theorem, every position, namely (X, Y) is equivalent to a nimheap:

$|(2, 4)| = 2$

$|(5, 9)| = 0$

$|(3, 6)| = 3$

Thus, $(2, 4) + (5, 9) + (3, 6) \simeq 2 \oplus 3$

In order to win, Alice should move to a P-position, namely $2 \oplus 2$.

So Alice can move (3, 6) to (2, 4) by eating all 3 M&Ms and split the 6 M&Ms into two piles of 2 and 4.

2. (2,4), (5, 4) and (2019, 9102):

Following the same principle as the first question:

$|(2, 4)| = 2$

$|(5, 4)| = 3$

$|(2019, 9102)| = ?$

Now we have some trouble here. At this stage, we want to find some pattern in the Nim table.

Here is the algorithm I find (there may be alternatives):

For a position (X, Y), let nonnegative integer n be the nimvalue of (X, Y), i.e. $(X, Y) \simeq n$.

Let there be nonnegative integers k_1, k_2 and positive integers b_1, b_2 that

$X = 2^{n+1} \bullet k_1 + b_1$

$Y = 2^{n+1} \bullet k_2 + b_2$

Where $0 < b_1 \leq 2^n$, $0 < b_2 \leq 2^n$, and n has the minimum value to satisfy the two equations.

An example when X = 7, Y = 8:

Since $X = 7 = 2 \bullet 3 + 1$, $n_1 = 0$;

And since $Y = 8 = 2^3$, and in order to express it in the form $2^{n+1} \bullet k_2 + b_2$, 2^{n+1} must be greater than 2^3, the smallest of which is $2^4 = 16$, so $Y = 0 \bullet 16 + 8$, $n_2 = 3$.

Since $\max(n_1, n_2) = \max(1, 3) = 3$,

We further adjust k1, b1:

$X = 2^4 \bullet 0 + 7$,

$Y = 2^4 \bullet 0 + 8$.

Check that b1, b2 is smaller than 8, with n be the minimum.
Thus, the nimvalue n for (X, Y) is 3.

Now we go back to the question, according to this algorithm, we can find the nimvalue for (2019, 9102).
$2019 = 2 \cdot 1009 + 1$, $n1 = 0$
$9102 = 4 \cdot 2275 + 2$, $n2 = 1$.
So max (n1, n2) = 1.
Thus,
$|(2, 4)| = 2$
$|(5, 4)| = 3$
$|(2019, 9102)| = 1$.
And $(2, 4) + (5, 4) + (2019, 9102) \simeq 2 \oplus 3 \oplus 1 \simeq 0$,
Which means it is a P-position. Ouch.
However, there is still something Alice can do. She now needs to give this P-position back to Bob, i.e. she wants to keep the nimvalues to be 2, 3, and 1, respectively. meanwhile, she wants to make sure that Bob cannot keep the nimvalues same.
We find immediately that it is impossible to spit (2, 4) or (5, 4) so that they keep the nimvalues, so she has to split (2019, 9102) so that n = 1.
Let the new position be (X', Y'), and suppose n =1.
Case 1: Alice eats the pile 2019, and split 9012.
According to the algorithm,
$X' = 4k1 + b1$
$Y' = 4k2 + b2$
Where k1, k2 are nonnegative integers and b1, b12 = 1 or 2.
Thus, $X' + Y' = 4(k1 + k2) + b1 + b2 = 9102$
Since b1, b2 can only be 1 or 2, the only possibility is that
$k1 + k2 = 2275$
$b1 + b2 = 2$.
So b1 =1, b2 = 1.
Then X' and Y' are two odd numbers.
However, odd numbers can always be expressed in the form 2k+1, where k is nonnegative integer.
Thus, n should be 0 instead of 1, where is a contradiction.
Therefore, Alice cannot split 9012.

Case 2: Alice eats the pile 9012, and split 2019.
According to the algorithm,

$X' = 4k1 + b1$

$Y' = 4k2 + b2$

Where k1, k2 are nonnegative integers and b1, b12 = 1 or 2.

Thus, $X' + Y' = 4(k1 + k2) + b1 + b2 = 2019$

Since b1, b2 can only be 1 or 2, the only possibility is that

$k1 + k2 = 504$

$b1 + b2 = 3$.

WLOG, $b1 = 1$, $b2 = 2$.

Now we want to prove that when it comes to Bob's turn, he cannot give a P-position with nimvalues of 1, 2, and 3 back to Alice.

Case 2.1: Bob eats X' and split Y'.

Since $Y' = 4k2 + 2$,

Similar to Case 1, we know that $n \neq 1$, which is a contradiction.

Thus, Bob cannot keep the nimvalue the same.

Case 2.2 Bob eats Y' and split X'.

Since $X' = 4k1 + 1$, yet we can no longer further split the remaining 1 M&M into two piles that are not 0 (otherwise it is no longer 4), nor can we rewrite X' to be $4(k1 - 1) +5$ which will make the two remainders to be (2, 3) or (1, 4), (yet the remainder cannot exceed 2!), we know that in this case, Bob cannot keep the nimvalue for this pile to be 1 anyhow.

Therefore, in no scenario can Bob return the P-position with nimvalues of 1, 2, and 3 to Alice.

Therefore, Alice can win this game by eating up all 9102 M&Ms in the pile (2019, 9102) and split the 2019 M&Ms into two piles of $(4k1+1, 4k2 + 2)$, where k1, k2 are nonnegative integers that $k1 + k2 = 504$.

Reference:

Ferguson, Thomas. “Game Theory, Second Edition, 2014.” *Game Theory*, Mathematics Department, UCLA, 2014, www.math.ucla.edu/~tom/Game_Theory/Contents.html.

Mathcamp. “Crash Course on Combinatorial Game Theory” 2019. https://www.mathcamp.org/2019/cgt.pdf