Mathematics in Games
—Impartial Combinatorial Game Theory

How to find a winning strategy to all impartial combinatorial game?

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Catalogue

Outline

First lesson—Go play some games

Objectives:

1.Learn about basic concepts;
2.Learn the presentation and notation used in game theory;
3.Identify different combinatorial games in daily life.

Content:

1.1 a simple take-away game as introduction.
1.2 What is a combinatorial game?
    Definition of combinatorial games;
    Difference between impartial and partizan games;
    Difference between normal and misere play rule.
1.3 The game graph
    Figure of a game graph;
    Definition of terminal position and height.
1.4 N-position and P-position
    Definition of N, P-positions;
    Theorem: every position in impartial combinatorial game is either P or N;
    Exercise: identifying N,P positions in simple take-away games.
1.5 Application—solving the game
    Provide an exact example of how to use what we learn in this lesson to solve a take-away game.

Exercise:

Identifying combinational games.
Solve a take-away game, and an Empty-and-Divide game.

Second lesson—Nim!

Objectives:

1. Learning the additions of games;
2. Applying the addition of games in Nim games;
3. Proving several theorems and construct an incomplete Nim table;
4. Learn MEX rule to construct the whole Nim table;
5. Using the Nim table to solve games.

Content:

2.1 What is the rule of a Nim game?

Explain the rule;

Play one round together.

Exercise: prove that a Nim game is an impartial combinatorial game

2.2 Adding games

Definition 2.1

Addition of games and N/P positions

Theorem 2.1 & 2.2

2.3 Nim table

Theorem: 2.3

Working out part of the Nim table;

2.4 Nim-Sum

Definition 2.2

Binary system

Corollaries

Theorem 2.4

2.5 MEX rule

Definition 2.3

Theorem 2.5

Application of MEX rule to compute the whole Nim table.

2.6 Using the Nim table.

Work out an exercise together hat uses Nim table to solve.

Exercise:

Focus on the practice of calculating Nim sums as well as applying MEX rules.

Third lesson—playing games all at once!

Objectives:

6.Learn the equivalence of games;
7.Learn to play Nim with any number of piles;
8.Learn how to play all impartial combinatorial games using nim-value;
9.Summary on what we learned;
10.Provide the ultimate question.

Content:

3.1 Equivalence of games
  Definition 3.1
  Theorem 3.1
3.2 Playing any piles of Nim game
  Example of a 5 pile Nim game
3.3 Sprague-Grundy theorem
  Definition 3.2 (nim-value);
  Theorem 3.2 (Sprague-Grundy)
  Example of using SG theorem on a take-away game.
3.4 summary—what we have learned
3.5 what else we can explore

Exercise:

One final BIG question related to the application of all what we learned in three lessons.

First lesson—Go play some games

1.1 Introduction—a simple take-away game.

Let's start the lesson by jumping right into the water and play a game.
The game rule is:

(1) There are two players, Alice and Bob;
(2) There is a pile of 13 M&Ms;
(3) Alice and Bob can eat one, two, or three M&Ms from the pile every time;
(4) Alternate play;
(5) The first one who cannot move loses.

This is a game called **take-away game**.
How can we win this game? Or which player would you rather be, going first or second?
Think about it, play a few rounds with your friends, before you read on.

Now let's analyze the game together from the end back to the beginning. This method is known as **backward induction** (Ferguson 3).

We know that if you are left with 0 M&M, you lose because you cannot move anymore;
Also, if there are 1, 2, 3 M&Ms left for you, you can just eat all of them and win the game;
Then suppose there are 4 M&Ms left, but no matter how many you take, you can only move to 1, 2, 3 M&Ms and you lose;
Now with 5, 6, 7 M&Ms, you can always move to 4 M&Ms, so the next player will absolutely lose the game, while you can win the game;
With 8 M&Ms left, you can only move to 5, 6, 7 M&Ms, and the next player can win, while you lose;
If this induction continues, we may conjecture that the positions with 0, 4, 8, 12, … M&Ms are positions to be avoided for you, but you want to move to them in order to win the game. However, as you can see, this method is a little messy, and we will introduce **game graph** in Section 1.3 as a more clear representation.

We may now analyze the game with 13 M&Ms. Since 13 is not divisible by 4, you want to move first and take one M&M to make it 12, which is a multiple of 4.

1.2 What is a combinatorial game?

The take-away game we played in Section 1.1 is a typical example of an impartial combinatorial game, but we need a more precise definition for impartial combinatorial games.

Definition 1.1 (Ferguson 4)

A combinatorial game is a game that satisfies all the following conditions:

(1) **Two players**. There should only be two players. Solitaire is not a combinatorial game because there is only one player.

(2) **Alternate.** The two players alternate their turns as we did in the take-away game.

(3) **No ties**. Chess can actually have a draw, so according to this, chess is not a combinatorial game.

(4) **Finite**. The game ends in a finite number of moves. We may not be able to say how many time that is, but we know it doesn't go on forever. Under the **normal play rule**, the last player to move wins. Under the **misere play rule** the last player to move loses.

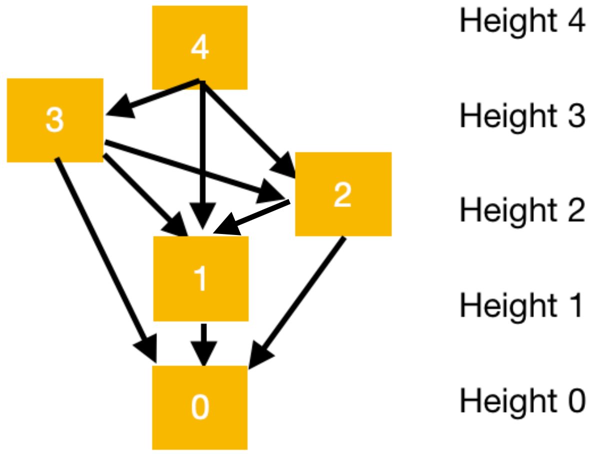
(5) **No distinction**. There should be no distinction between the players, that is if both players have the same options of moving from each position, the game is called **impartial**; otherwise, the game is called **partizan**.

A little summary: 2 players, alternate, no distinction, no ties, finite.

1.3 Games Played on Graphs

Here comes a much clearer way to represent the game we played in Section 1.3: **Game graph**. To construct a game graph, we give an equivalent graph for every impartial combinatorial game that shows all the possibilities the game can proceed starting from that position. The **vertices** of the graph are positions of the game, and the **edges** are moves (Ferguson 14).

Here is an example showing another take-away game, but only with four M&Ms to begin with.



Definition 1.2 (Mathcamp 2)

Terminal position: A terminal position in a game is a position from which there are no valid moves.The terminal position in take-away games is 0

Height: The height of a position is the length of the longest path in its game graph. (Terminal positions have height 0.) 4, for example, 4 M&Ms have a height of 4, as shown in the graph.

Since combinatorial games are finite, we are sure a game graph will always contain only a finite number of vertices and no loops. However, I do not draw the graph for 13 M&Ms because that will be too huge to draw, and we will have easier ways to deal with those with relatively large-number games in Section 1.5.

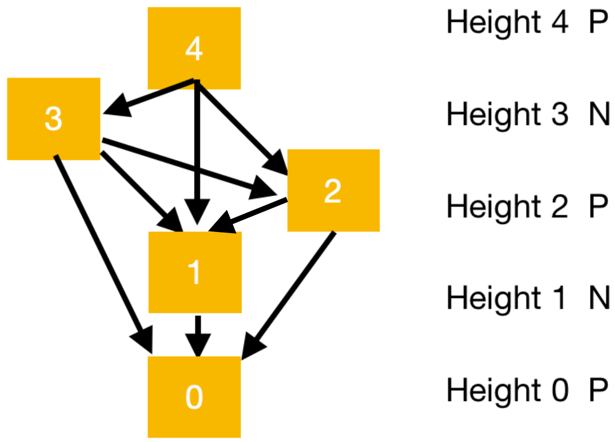
1.4 N-position and P-position

Let's go back to the take-away game, we conjectured that 0,4,8,12,16,... are positions that are winning for your opponent, or what we call **P**revious player (the player who just moved) and that 1, 2, 3, 5, 6, 7, 9, 10, 11, . . . are winning for the **N**ext player to move, which is you. The former positions are called **P-positions**, and the latter ones are **called N-positions** (Ferguson 4-5).

Intuitively, N-positions are ones from which you want to go first; P-positions are ones from which you want to go second.

In impartial combinatorial games, one can find which positions are P- positions and which are N-positions by using the game graph as we introduced in Section 1.3.

We go back to the graph:



The terminal position must be P, so the former one, which goes to a P, should be N, and so on and so forth. So 4 M&Ms, where we start, is indeed a P-position.
Now you may wonder why this method works. Before proceeding to the proofs below, you can first think about it yourself.

In order to prove the method works, we need to prove the following statements (Mathcamp):

(1) A position from which you can move to a P-position is an N-position;
(2) A position from which the only moves are to N-positions is a P-position.
(3) Every position in an impartial combinatorial game is either N or P.

Proof for (1):
Think it backward: you don't know yet the position you are at right now, but the position you are moving to is a P-position, and you become the previous player. By definition of P, you have a winning strategy from that P-position.
Now we go back to your original position. Since you can move to a P-position, you have a winning strategy, so your current position is N.

Proof for (2):
Similarly, if you are now the next player and you can only move to N-positions, this means that whatever you do, the next player, you opponent, can win the game. This means your current position is a P-position.

Proof for (3):
<u>Proof by strong mathematical induction:</u>
Let P(n): every position of height n in an impartial game is either N or P.

First, consider Height n = 0. A position of height 0 is a terminal position, which means the Previous wins, so it's a P-position. P(0) is true.
Now, we want to show that if P(1), P(2), …, P(k) are true, then P(k+1) is also true.
Assume P(1), P(2), …, P(k) are true.
All moves from k+1 must be within the range of Height i, with $0 \leq I \leq k$, and since all Height i have positions either P or N, we can thus determine the position for k+1 should be either P or N. (according to the two statements we just proved (1) and (2))
Thus, P(k+1) is also true.
Therefore, because the truth of P(1), P(2), …, P(k) implies the truth of P(k+1), by strong mathematical induction, we prove that P(n) is true for all nonnegative integer n, i.e. every position in an imperial combinatorial game is either N or P. ∎

1.5 Application—solving a game

Let us now consider another take-away game.
This time, Alice and Bob are allowed to take 1, 2, or 5 M&Ms at a time.
So, let us analyze it.
This time, instead of drawing the full game graph for each position, which will be too big and too messy, let's just make a table for the different positions, which is easier to keep track of each position.
This is the table I constructed.

x	0	1	2	3	4	5	6	7	8	9	10	11	…
Position	P	N	N	P	N	N	P	N	N	P	N	N	…

Here is how it is constructed:
First, 0, the terminal position, must be P;
1, 2, 5 are N-positions, since they can be moved to 0, a P-position.
3 must be a P-position since it can only move to 1 or 2, which are all N-positions;
4 must then be N-positions as they can be moved to the P-position 3;
6 is a P-position because it can only move to 5, 4, or 1, which are all N-positions;
7, 8, and 11 must be N-positions because they can all move to the P-position 6;
9 is a P-position because it can only move to 8, 7, or 4, which are all N-positions;
10 is a N-position as it can move to the P-position 9;
…
We notice that the pattern PNN of length 3 repeats forever.

Now if there are 100 M&Ms in total, who will win the game, with Alice goes first and Bob second?

The P-positions are multiples of 3. Since 100 is not a multiple of 3, 100 is a N-position. Thus, Alice, if play optimally, can win the game by always moving to another N-position.
That's how you analyze a simple impartial combinatorial game.

1.6 Exercises.

1. Identify which of the following games are impartial combinatorial games and which ones are not, and explain your decision.
   (1) Aeroplane chess
   (2) Notakto
   (3) Dots-and-boxes

   Note: Notakto is played on a finite number of empty 3-by-3 boards. Then, each player takes turns placing an X on the board(s) in a vacant space (a space not occupied by a X already on the board). If a board has a three-in-a-row, the board is dead and it cannot be played on anymore. When one player creates a three-in-a-row and there are no more boards to play on, that player loses (Notakto).

   The game of dots-and-boxes starts with an empty grid of dots. Usually two players take turns adding a single horizontal or vertical line between two unjoined adjacent dots. A player who completes the fourth side of a 1×1 box earns one point and takes another turn. (A point is typically recorded by placing a mark that identifies the player in the box, such as an initial.) The game ends when no more lines can be placed. The winner is the player with the most points (Dots-and-Boxes).

2. Another Take-away game: analyze the take away game with 203 M&Ms in total and each player can take either 1, 2, 3, or 4 M&Ms; construct a position table, and tell whether you should go first or second if you want to win the game.

3. We have a new game called Empty and Divide: There are two piles of M&Ms. Initially, one pile has m M&Ms and the other has n M&Ms. Such a position is denoted by (m,n), where $m > 0$ and $n > 0$. The two players alternate moving. A move consists of eating all the M&Ms in one pile, and dividing the other into two new piles with at least one M&M in each pile. Whoever cannot move loses, i.e. the terminal position is (1,1).

   For example, if m=3 and n=4, you can either eat up all the 3 M&Ms in the first pile and divide the 4 M&Ms into two piles of either (1, 3) or (2, 2); or you can eat all the 4 M&Ms in the second pile and divide the 3 M&Ms in the first pile into (1, 2).

   (1) Show that this is indeed an impartial combinatorial game;
   (2) If m=3 and n=7, is it better to go first or second?
   (3) What about m=2019 and n=9021, should you go first or second?

Second lesson—Nim!

2.1 What is the rule of a Nim game?

Now it is time to introduce our new game called Nim game.
There are three piles of M&Ms. Two players take turns to select one of the piles and eat any number of M&Ms (at least one) in it. The one who eat the last M&M wins. You can practice playing this game on the web at Nim Game—http://www.dotsphinx.com/nim/.

It is easy yet boring when there is only one pile of M&Ms, and you can easily know that for a pile of size X:
It is an N-position if $X > 0$. (the player can simply take all the M&Ms!)
It is a P-position if $X=0$.
However, when 3 piles are played together, it is much more complicated. We will discuss that in Section 2.3.

2.2 Adding games together

Before we can add the three piles of Nim game together, we may first wonder: what does it mean by adding games together? And how can we do that?

Definition 2.1 Given two impartial combinatorial games, G_1 and G_2, we define their sum, $G_1 + G_2$, to be an impartial game that:
(1) For a position g_1 of the game G_1 and a position g_2 of the game G_2. We call it $g_1 + g_2$;
(2) A valid move from $g_1 + g_2$ is either a valid move in G_1 from g_1 to some other position g_1' or a valid move in G_2 from g_2 to some other position g_2' . i.e. we can only move to $g_1' + g_2$ or $g_1 + g_2'$;
(3) A player loses the game $G_1 + G_2$ if it is their turn and they are unable to move in either game. (Crash Course 6)

For example, suppose G_1 is a take-away game and G_2 is an Empty-and-Divide game. The initial position G_1+G_2 is 9 (with valid moves of 1, 2, 4) + (3, 3).
Now we need to first choose a game to play: if we start with G1, the next position can either be 8 + (3, 3), 7 + (3, 3), or 5 + (3, 3); if we choose G2, the next position can only be 9 + (1, 2).

Continuing is this way, the next player select one of the games to play, until the game reaches the terminal position 0 + (1, 1).

Some Notes:

1. Addition of games is **commutative**. i.e. G1+G2 is equivalent to G2+G1;
2. Addition of games is **associative**. i.e. (G1+G2)+G3 is equivalent to G1+(G2+G3);

How do we know N/P positions of the games adding together?
There are two useful theorems (Crash Course 7):
Theorem 2.1 For any position g in an impartial game, g + g is always a P-position.
Theorem 2.2 If g is a P-position, then for any other position h in an impartial game, h + g has the same N/P position as h. i.e. N + P = N, P + P = P.

The proofs of these two theorems do not need much advanced math knowledge, but they does require some mathematical ways of thinking. So think it about that before moving on the see the proofs below.

Proof of Theorem 2.1:
Given two identical impartial games with same position g, a move in one game can always be copied by the next game. i.e. if the previous player moves to g′ + g, (it does not matter which one is chosen because the two games are identical), the next player can always find a move by moving to g′ + g′. Thus, the Previous player will eventually run out of move and loses. ∎

Proof of Theorem 2.2:
Case one: h is a P-position.
Then for both g and h, the second player has a winning strategy for both of the games individually. Now whatever game the first player move, either g′ or h′, the second player can find a winning strategy in the same game. Thus, the first player will finally run out of move and loses;
Case two: h is a N-position.
From the N-position, the first player can always move to a P-position, say h′(as proved in Lesson 1). So the game goes back to two P-position games g + h′ for the next player, which is the same as case one. Since case one is a P-position and the first player loses, the position that moves to such a P-position must be a N-position, in which the first player to move wins. ∎

2.3 Nim Table

Now that we know some basic rules of adding games now, let's look at 3-pile Nim game. In order to play it in a simpler way, we must need some theorems to simplify the problem.
We express the sum of three piles X, Y, Z by $X \oplus Y \oplus Z$. Still, order does not matter.

Theorem 2.3. For any two numbers X and Y , there exist at most one Z such that $X \oplus Y \oplus Z$ is a P-position. (Crash Course 9)
Proof by contradiction:
Suppose $X \oplus Y \oplus Z_1$ and $X \oplus Y \oplus Z_2$ are both P-positions with $Z_1 > Z_2$. Now the player can move from the Z_1 to Z_2 by taking $Z_1 - Z_2$ stones from Z_1. i.e. the player can move from a P-position to another P-position, which is impossible in am impartial game, contradiction.
Thus, the pile of Z, if exist, is unique. ∎
Because of the uniqueness of Z, given two numbers X and Y, we can always find a Z that makes $X \oplus Y \oplus Z$ a P-position. We can thus construct a table with Row x and Column Y.
Since we already know that $X \oplus X$ is a P-position, according to theorem 2.1, $X \oplus X \oplus 0$ must be a P-position, i.e triples of $0 \oplus 0 \oplus 0$, $1 \oplus 1 \oplus 0$, $2 \oplus 2 \oplus 0$, etc, are all P-positions. Thus, we get the incomplete Nim table below.

	0	1	2	3	4	5	6	7	…
0	0	1	2	3	4	5	6	7	…
1	1	0							
2	2		0						
3	3			0					
4	4				0				
5	5					0			
6	6						0		
7	7							0	
…	…								…

Now in order to fill out the entire table, we may try to find the value of Z when $1 \oplus 2 \oplus Z$ is a P-position.

First, we know that Z cannot be either 0 or 1 or 2, according to the rule that $X \oplus X \oplus 0$ must is a P-position and Z is unique.
Thus, we may conjecture that Z = 3. You may want to try to write out all the possibilities and to see if this is true.

Indeed, we find that no matter what the first player do, he/she always loses. So we can verify that $1 \oplus 2 \oplus Z$ is a P-position.

However, as we can see, calculating the Nim table one by one is time-consuming, and it becomes even more difficult when we go to larger numbers. Thus, we need another technique to help us construct the Nim table, which is what we are going to discuss in the next section, Nim sum.

2.4 Nim-Sum

Definition 2.2 The Nim-sum of $(x_m x_{m-1} \cdots x_0)_2$ and $(y_m y_{m-1} \cdots y_0)_2$ is $(z_m z_{m-1} \cdots z_0)_2$, and we write $(x_m x_{m-1} \cdots x_0)_2 \oplus (y_m y_{m-1} \cdots y_0)_2 = (z_m z_{m-1} \cdots z_0)_2$, where for all k, $z_k = x_k + y_k \pmod 2$, i.e. $z_k = 1$ if and only if $x_k + y_k = 1$, otherwise, $z_k = 0$. (Ferguson 9)

Don't panic if you don't know what's going on here! Let's look at these notations one by one.
First, the $()_2$ is the binary representation. Every non-negative integer x can be expressed in the binary system in the form $x = x_m 2^m + x_{m-1} 2^{m-1} + \cdots + x_1 2 + x_0$ for some m, where each x_i is either zero or one. We use the notation $(x_m x_{m-1} \cdots x_1 x_0)_2$ to denote this representation of x to the base two.
For example, 27 = 16 + 8 + 2 + 1, so m=4, $x_4 = 1$, $x_3=1$, $x_2=0$, $x_1=1$, $x_0=1$. Thus, $27 = (11011)_2$.

Then, to calculate the Nim-sum of two integers, we first express the integers in binary system and use addition modulo 2 for every digit.
That is, for every digit,

$$0 + 0 = 0,$$
$$0 + 1 = 1,$$
$$1 + 0 = 1,$$
$$1 + 1 = 0.$$

For example, to compute $27 \oplus 13$, we first express them by

$27 = (11011)_2$

$13 = (1101)_2$

Since $(11011)_2 \oplus (1101)_2 = (10110)_2$,

we know that $27 \oplus 13 = 21$.

Below are some corollaries that may be useful:

Corollary 2.1 (Ferguson 10)

0 is an identity for addition, i.e. $0 \oplus x = x$;

Corollary 2.2 (Ferguson 10)

Every number is its own negative, i.e. $x \oplus x = 0$

Proof:

For every digit, either 1 or 0 in x,$0 + 0 = 0$, $1 + 1 = 0$.

Thus, every digit in x always adds up to zero, i.e. $x \oplus x = 0$. ∎

Corollary 2.3 (Ferguson 10)

$x \oplus y = x \oplus z$ implies $y=z$.

Proof:

For impartial games x, y and z,

If $x \oplus y = x \oplus z$,

then $x \oplus x \oplus y = x \oplus x \oplus z$.

Since $x \oplus x = 0$, $y=z$. ∎

Corollary 2.4

$X \oplus Y = Z$ if and only if $X \oplus Y \oplus Z = 0$

Proof:

If $X \oplus Y = Z$,

$X \oplus Y \oplus Z = Z \oplus Z = 0$

If $X \oplus Y \oplus Z = 0$,

$X \oplus Y \oplus Z = 0 = Z \oplus Z$,

$X \oplus Y = Z$. ∎

OK. Now we know how to calculate Nim-sums. But what has it to do with playing Nim games? The theorem below gives us the answer.

Theorem 2.4

A position, (X, Y, Z), in Nim is a P-position if and only if the Nim sum of its components is zero, i.e. $X \oplus Y \oplus Z = 0$. (Crash Course 14)

Proof by strong induction

Let P(n): be the sum of $X + Y + Z$, where n is a nonnegative integer.

First, when $n = 0$, $X = Y = Z = 0$.

(0, 0, 0) is the terminal position, and thus a P-position, so P(0) is true.

Now, we want to show that if P(0), P(1), P(2), …, P(k) are true, then P(k+1) is also true.

Assume P(0), P(1), P(2), …, P(k) are true.

Suppose first that $X \oplus Y \oplus Z = 0$. We need to show that in this case, (X, Y, Z) is a P-position.

A move from this position involves reducing the value of one of the numbers – say going from X to X', with X' < X. But any change will make $X' \oplus Y \oplus Z$ not equal to 0, according to Theorem 2.3.

Since $X' + Y + Z < k$, we can apply the inductive hypothesis to conclude that $X' \oplus Y \oplus Z$ is an N-position. This shows that all moves from $X \oplus Y \oplus Z$ are to N-positions, so $X \oplus Y \oplus Z$ itself is a P-position.

Now suppose $X \oplus Y \oplus Z$ do not equal 0. We want to show that this is an N-position by finding a move to a P-position.

Denote the binary digits of X by $X_1, X_2, \ldots X_n$. (Thus X_n is the 1's digit, X_{n-1} is the 2's digit, etc.) Find the smallest m such that $X_m+Y_m+Z_m = 1 \pmod 2$, so at least one of X_m, Y_m, and Z_m is not 0. Since X, Y and Z are identical, we assume $X_m = 1$.

Let X' be the integer with binary expansion

$$X' = \begin{cases} X_i, & \text{if } i < m \\ 0, & \text{if } i=m \\ Y_i + Z_i \pmod 2, & \text{if } i > m \end{cases}$$

By construction, $X' \oplus Y \oplus Z = 0$ and $X' < X$. By the inductive hypothesis, $X' \oplus Y \oplus Z$ is a P-position, so $X \oplus Y \oplus Z$ is an N-position.

Thus, the proposition is also true for P(k+1), and by strong Mathematical induction, the proposition is true for all positive integer n. ∎

By using both Corollary 2.4 and Theorem 2.4, we know that in order to know the N/P position of any given Nim game with position (X, Y, Z), we can simply get the answer by calculating the Nim sum of any of the two numbers.
So here is what we can get by calculating the Nim sum, and if this continues, we will be able to get the complete Nim table.

	0	1	10	11	100	101	110	111	…
0	0	1	10	11	100	101	110	111	…
1	1	0	11	10	101	100	111	110	…
10	10	11	0	1	110	111	100	101	…
11	11	10	1	0	111	110	101	100	…
100	100	101	110	111	0	1	10	11	…
101	101	100	111	110	1	0	11	10	…
110	110	111	100	101	10	11	0	1	…
111	111	110	101	100	11	10	1	0	…
…	…	…	…	…	…	…	…	…	…

By converting into decimal system, we get:

	0	1	2	3	4	5	6	7	…
0	0	1	2	3	4	5	6	7	…
1	1	0	3	2	5	4	7	6	…
2	2	3	0	1	6	7	4	5	…
3	3	2	1	0	7	6	5	4	…
4	4	5	6	7	0	1	2	3	…
5	5	4	7	6	1	0	3	2	…
6	6	7	4	5	2	3	0	1	…
7	7	6	5	4	3	2	1	0	…
…	…	…	…	…	…	…	…	…	…

Try to find some other interesting patterns of Nim table before moving on the the next section.

2.5 MEX rule

Now when we look at the Nim table again, we may find some other interesting patterns.

Definition 2.3
Let S be a set of nonnegative integers. MEX(S) is defined to be the smallest nonnegative integer that does not appear in S. (MEX stands for "minimal excluded".) (Crash Course 11)

For example,
MEX(1,2,5,4) = 0, because the smallest number that is not 1, 2, 5, 4 is 0;
MEX(2,1,6,0,3,5,8) = 4, because the smallest number that does not appear is 4;
MEX(0,87,23455,35)=1, because the smallest number does not in the list is 1.

Theorem 2.5 — "the MEX rule". (Crash Course 11)
For all integers X, Y ≥ 0,

$$[X,Y] = \mathrm{MEX}\,(\{[X', Y] : X' < X\} \cup \{ [X, Y'] : Y' < Y\})$$

i.e. [X,Y] is the smallest integer that does not appear directly above or to the left of its position in the table.
Proof:
Given X, Y ≥ 0, let $M = \mathrm{MEX}\,(\{[X', Y] : X' < X\} \cup \{ [X, Y'] : Y' < Y\})$.
We want to show that (X, Y, M) is a P-position by showing that it can only move to N-positions.
Case one: X/Y is moved to X'/Y
By definition of MEX, M != [X',Y], (or M != [X, Y']).
Thus it is not a P-position, so it is an N-position.
Case two: M is moved to M'
By definition of MEX, $M' \in \{[X', Y] : X' < X\} \cup \{ [X, Y'] : Y' < Y\}$.
Then there must be some X' that M' = [X',Y].
Because (X', Y, M') is a P-position, (X, Y, M') must be a N-position.
Since all possible moves from (X, Y, M) are N-positions, (X, Y, M) is a P-position. ∎

2.6 Using the Nim table.

Now that we construct the whole Nim table, what can we do to play the Nim game? Here is an example:
Suppose we are playing a Nim game with three piles of M&Ms, each of size 5, 7, and 15. If I am generous enough to let you first choose if you want to go first or second, which one should you choose?

Let's work this out together.
First, according to the Nim table,

	0	1	2	3	4	5	6	7	…
0	0	1	2	3	4	5	6	7	…
1	1	0	3	2	5	4	7	6	…
2	2	3	0	1	6	7	4	5	…
3	3	2	1	0	7	6	5	4	…
4	4	5	6	7	0	1	2	3	…
5	5	4	7	6	1	0	3	**2**	…
6	6	7	4	5	2	3	0	1	…
7	7	6	5	4	3	2	1	0	…
…	…	…	…	…	…	…	…	…	…

Every pair in the table is a P-position, as we proved in Theorem 2.4. So we know that (5, 7, 2) is a P-position, and according to theorem 2.3, which states the uniqueness of each one of these three numbers, we know that (5, 7, 15) must be a N-position. (you can also work out the Nim-Sum if you are not convinced).
Thus, you must want to be the first player and start the game by moving this N-position to a P-position.
Since we know that (2, 5, 7) is a P-position, what you can do next is to eat 13 M&Ms from the 17 M&Ms pile, making it to be (2, 5, 7).

That's how you play a Nim game as a mathematician. :)
Do some exercise to make sure you have mastered this trick and go play with your friends (Don't tell them the strategy and impress them!!)

2.7 Exercises.

1. Calculate the Nim-sums below:

   (1) $2 \oplus 4 = ?$

   (2) $9 \oplus 8 = ?$

   (3) $100 \oplus 324 = ?$

2. Suppose you are playing a Nim game with three piles of M&Ms of size 2019, 215, and 9102, and still, I am so generous to let you choose whether you want to be the first or second, which one will you choose, and why?

3. Suppose you are playing a Nim game (again!) with three piles of M&Ms of size 9, 10 and 8, determine **all** possible move if you are the first player.

4. Now, still a Nim game, but you only know there is a nimheap of size 1, but you don't know the size of the other two piles yet. Without knowing the specific size of the other two piles, how can you win this game?

   * Not knowing specific size does not necessarily mean that you don't know anything about these two piles.

   * You may want to discuss this under different cases.

Third lesson—playing games all at once!

3.1 Equivalence of games

Before we proceed to learning how to play multiple different games together, we first need to the know the concept of the equivalence of games.

Definition 3.1
Let G_1 and G_2 be two games, and let g_1 and g_2 be positions of G_1 and G_2, respectively. Then g_1 is equivalent to g_2 ($g_1 \simeq g_2$) if, for any position h in any game H, $g_1 + h$ has the same N/P position as $g_2 + h$. (Crash Course 15)
That means two games are equivalent if they produce the same result in addition of games.

Theorem 3.1
$g_1 \simeq g_2$ if and only if g_1+g_2 is a P-position. (Crash Course 15)

This theorem will lead to a lot of corollaries in the later section, which are extremely crucial for us to work out how to converting different impartial games into Nim game and thus solve those games. Before we proceed to those, we may wonder how to prove this theorem. It is relatively easy by using the some of the theorems we conclude in lesson two.

Proof:
$g_1 \simeq g_2 \Rightarrow g_1 + g_2$ is a P-position.
According to definition of equivalence, for any position in any game H, $g_1 + h$ has the same N/P position as $g_2 + h$. Let h be g_2.
Thus, $g_1 + g_2$ has the same N/P position as $g_2 + g_2$.
And since $g_2 + g_2$ is a P-position (Theorem 2.1), $g_1 + g_2$ is also a P-position.
$g_1 + g_2$ is a P-position $\Rightarrow g_1 \simeq g_2$
If $g_1 + g_2$ is a P-position, and since adding a P-position has no effect on the result position (Theorem 2.3), $g_1 + h$ must have the same position has $(g_1 + h) + (g_1 + g_2)$.
Since the addition of games is commutative and associative,
$(g_1 + h) + (g_1 + g_2) = (g_2 + h) + (g_1 + g_1)$.

Again, since $g_1 + g_1$ is a P-position, according to Theorem 2.3, $(g_2 + h)$ has the same N/P position as $(g_2 + h) + (g_1 + g_1)$, which equals to $(g_1 + h) + (g_1 + g_2)$, which has the same position as $g_1 + h$, so $g_1 \simeq g_2$.

Therefore, $g_1 \simeq g_2$ if and only if g_1+g_2 is a P-position. ∎

Corollary 3.1

If $g_1 \simeq g_2$ and h is any position, then $g_1 + h \simeq g_2 + h$.

Proof:

$(g_1 + h) + (g_2 + h) = (g_1 + g_2) + (h + h)$.

Since both $(g_1 + g_2)$ and $(h + h)$ are P-positions (Theorem 2.2), $(g_1 + g_2) + (h + h)$ is also a P-position, and so is $(g_1 + h) + (g_2 + h)$.

According to Theorem 3.1, $g_1 + h \simeq g_2 + h$. ∎

Corollary 3.2

$g \simeq 0$ if and only if $g + 0$ is a P-position.

Proof:

According to Theorem 3.1, $g_1 \simeq g_2$ if and only if $g_1 + g_2$ is a P-position.

When $g_1 = g$ and $g_2 = 0$,

$g \simeq 0$ if and only if $g + 0$ is a P-position. ∎

Corollary 3.3

Two nimheaps are equivalent if and only if they are the same size.

Proof:

Let X, Y denote two nimheaps, $X \geq Y$.

$X = Y \Rightarrow X \simeq Y$

If $X = Y$, $X \oplus Y$ is a P=position according to Theorem 2.2, so $X \simeq Y$.

$X \simeq Y \Rightarrow X = Y$, i.e. $X \neq Y \Rightarrow X$ not equivalen to Y

Since $X > Y$, $X \oplus Y$ can be moved to $Y \oplus Y$, a P-position, so $X \oplus Y$ itself is a N-position.

Thus, X is not equivalent to Y.

Therefore, $X = Y$ iff $X \simeq Y$. ∎

3.2 Playing any piles of Nim game

Do you remember the Nim table we constructed in Lesson 2?

	0	1	2	3	4	5	6	7	8	9	10	…
0	0	1	2	3	4	5	6	7	8	9	10	…
1	1	0	3	2	5	4	7	6	9	8	11	…
2	2	3	0	1	6	7	4	5	10	11	8	…
3	3	2	1	0	7	6	5	4	11	10	9	…
4	4	5	6	7	0	1	2	3	12	13	14	…
5	5	4	7	6	1	0	3	2	13	12	15	…
6	6	7	4	5	2	3	0	1	14	15	12	…
7	7	6	5	4	3	2	1	0	15	14	13	…
8	8	9	10	11	12	13	14	15	0	**1**	2	…
9	9	8	11	10	13	12	15	14	1	0	**3**	…
10	10	11	8	9	14	15	12	13	**2**	3	0	…
…	…	…	…	…	…	…	…	…	…	…	…	…

Now that we have learned the equivalence of games, we can understand that the Nim table is not only for solving 3-pile Nim games, but also for all Nim games, no matter how many piles there are. Because by saying $X \oplus Y \oplus Z$ is a P-position, we are really saying that $X \oplus Y \simeq Z$.

Let's look at some examples of how to play a Nim game with more than 3 piles:

Example 3.1

In a 4-pile Nim game with a position of (4, 7, 3, 8), as the first player, what can you do to win the game?

Solution:

According to Nim table,

$4 \oplus 7 \oplus 3 \oplus 8 \simeq 3 \oplus 3 \oplus 8$ (because $4 \oplus 7 = 3$)

$\simeq 8$ (because $3 \oplus 3 = 0$)

Thus, the position (4, 7, 3, 8) is a N-position, and since you happen to go first, you want to make it a P-position by taking all 8 M&Ms in the 8-pile.

Also, since the position is equivalent to 8, when this position is added to other impartial games, it behaves exactly like a Nimheap with 8 M&Ms.

Example #2:

Now you are playing the Nim game with 5 piles of (2, 10, 9, 4, 5), would you go first or second?

Solution:

$2 \oplus 10 \oplus 9 \oplus 4 \oplus 5 \simeq 8 \oplus 9 \oplus 4 \oplus 5$ (because $2 \oplus 10 = 8$)

$\simeq 1 \oplus 4 \oplus 5$ (because $8 \oplus 9 = 1$)

$\simeq 5 \oplus 5$ (because $1 \oplus 4 = 5$)

$\simeq 0$

Thus, this is a P-position and in order to win the game, you would choose to be the second player.

3.3 Sprague-Grundy theorem

Now it's time to know why we are so concerned with Nim game (we played that for the whole second lesson!). The answer lies in the theorem below:

Theorem 3.2 —Sprague-Grundy (Crash Course 16)

Any position g in an impartial game is equivalent to a nimheap. i.e., there is always some nonnegative integer X such that $g \simeq X$.

And X is known as the **nimvalue** of g, writes as $|g| = X$.

Proof by strong mathematical induction:

Let a g denote a position in an impartial game with height n.

We first check for n=0.

All terminal positions are P-positions and thus have nimvalue 0.

Now we want to show that if the statement is true for all height less than k, i.e. n = 1, n = 2, …, n = k, then the statement is also true for height k+1.

Assume for every position g with height less than k+1 is equivalent to a nimheap.

Let $S = \{h_1, h_2, . . . h_k\}$ be the set of positions to which can be moved to from g. Since all the elements of S have height less than k+1, each $h_i \in S$ is equivalent to some nimheap Xi. Then we claim that $g \simeq M$, where $M = MEX(X_1,\ldots,X_k)$.

We want to show that $g \oplus M$ is a P-position by showing that all moves from $g \oplus M$ are N-positions. There are two ways to move from g + M:

First, there can be a move in g to some $h_i \in S$. The resulting position is $h_i \oplus M$, which is equivalent to $X_i \oplus M$, which is an N-position unless $M = X_i$, which is impossible by the definition of MEX.
Also, there can be moves from M.
The resulting position is $g \oplus M_1$, for some $M_1 < M$.
By definition of MEX, $M_1 = X_i$ for some i.
Thus it is possible to move from $g \oplus M_1$ to $h_i \oplus M_1 \simeq X_i \oplus M_1 = M_1 \oplus M_1 \simeq 0$.
This means $g \oplus M_1$ is an N -position.
We have shown that all moves from $g \oplus M$ are to N-positions. This implies that $g \oplus M$ is a P-position, so $g \simeq X$, and the statement is true for height k+1.
Since the truth of all height less than k implies the truth of height k+1, by strong mathematical induction, the theorem is true for g with height of positive integer n. ∎

The reason why Sprague-Grundy is so important is that you can literally solve any impartial games with this theorem in mind and compute the nimvalue of every position using the MEX rule.
Then, you will find that you are actually playing various Nim games with n-piles.
Below is an example to illustrate how it works:

Again, we go back to where we start—a take-away game. We denote the game position by Tx.
Now we play three of it once, with positions of 11, 16, and 2019 M&Ms. Valid moves consist of eating 1, 2, or 4 M&Ms.
Still, it will be very convenient if we construct a Nim table like this:

X	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	…
Tx	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	…

Here is how it is constructed:
First, we know the terminal position $|T_0| = 0$;
Building on this, we have $|T_1| = \text{MEX}(\ |T_0|\ ) = \text{MEX}(0) = 1$
$|T_2| = \text{MEX}(\ |T_1|, |T_0|)$
$= \text{MEX}(0, 1)$
$= 2$
$|T_3| = \text{MEX}(\ |T_1|, |T_2|)$
$= \text{MEX}(1, 2)$
$= 0$

$|T_4| = \text{MEX}(\ |T_0|, |T_2|, |T_3|\ )$
$\quad = \text{MEX}(0, 2, 0)$
$\quad = 1$

…

Continuing in this way, we again find a cycle of 0, 1, 2 repeats.
Since $2019 \equiv 0 \pmod 3$
$|T_{2019}| = 0$

Thus, $11+15+2019 \simeq 2 \oplus 1 \oplus 0$
$\simeq 3 \oplus 0$
$\simeq 3$

Therefore, (11, 15, 2019) is a N-position, and you would want to go first and make it a P-position for the next player.

3.4 summary—what we have learned

1. Identify impartial combinatorial games;
2. Decompose games into addition of single impartial games and analyze each individually;
3. Find the nimvalues for each of the game using the MEX rule and construct a Nim table;
4. Play different impartial games as if playing Nim.

3.5 what else we can explore

However, this lessons can only serve as an introduction to game theories.
Still, there are many other impartial combinatorial games, which can be found at this website www.math.ucla.edu/~tom/Game_Theory/Contents.html.

Moreover, impartial combinatorial games are just a small part of game theories. There are way more interesting topics under game theory like Zero-sum and Genial-sum games, which can also be explored in this website https://www.math.ucla.edu/~tom/Game_Theory/Contents.html.

3.6 Exercises.

Do you still remember the game of Empty-And-Divide we played at the first lesson? Let's look at it again, but now we play the addition of three empty-and-divide games: There are three games, two piles in each, a valid move consists of eating up all M&Ms in the chosen game and split the other into two new piles each with at least one M&M.
The one who cannot further split any pile loses, i.e. the one who is left with six piles of 1 M&M loses.

Now Alice has to move. She has the position of (2, 4), (5, 9) and (3, 6).
What should she do in next in order to win this game?

In another round, Alice is faced with the position (2, 4), (5, 4) and (2019, 9102). State whether that is a P or N position.

Explain your answer using what you learned in the three lessons.
You may want to construct a Nim table and use Sprague-Grundy theorem.

Answer key:

Lesson 1

1. (1) Aeroplane chess is not an impartial game because it involves chances. According to definition, an impartial game should have no distinction between two players, and they should have the same options. Thun, aeroplane chess is not an impartial combinatorial game.
   Note: any game involves dies is NOT a combinatorial game.

   (2) Notakto is an impartial combinatorial game because it satisfies all the requirements to be an impartial game, with two players alternate playing, no ties, with finite game. Lastly, the two players have no distinction in the moving, as all moving all available for each player.

   (3) Dots-and-boxes is an impartial combinatorial game because it satisfies all the requirements to be an impartial game, with two players alternate playing, no ties, with finite game. Lastly, the two players have no distinction in the moving, as all moving all available for each player.
   However, dots-and-boxes is a little bit different from other impartial games given its different way of determining who wins the game. (counting the boxes instead of just who moves last)

2. The table:

x	0	1	2	3	4	5	6	7	8	9	10	11	…
Position	P	N	N	N	N	P	N	N	N	N	P	N	…

   the pattern of PNNNN repeats forever.
   203 has a remainder of 3 when divided by 5, which is N, so 203 is a N-position, and we wish to go first.

3. (1) Empty-and-Devide is indeed an impartial combinatorial game because it satisfies all the requirements to be an impartial game, with two players alternate playing, no ties, no distinction, finite game.

(2) In order to solve the problem, I constructed the game table. However, I only show half of it because (m, n) and (n, m) do not make a difference:

(1,1) P					
(1,2) N	(2,2) N				
(1,3) P	(2,3) N	(3,3) P			
(1,4) N	(2,4) N	(3,4) N	(4,4) N		
(1,5) P	(2,5) N	(3,5) P	(4,5) N	(5,5) P	
(1,6) N	(2,6) N	(3,6) N	(4,6) N	(5,6) N	(6,6)N
(1,7) P	(2,7) N	(3,7) P	(4,7) N	(5,7) P	…

…

Thus, (3,7) will be a P-position, and it will be better to go second.

Keep constructing the table. You may find the pattern that whenever there is an even number present, it will be a N-position.

Here is why:

If you are at a position of (n, m):

Case one: at least one of n, m is an even number.

An even number can always be divided into a sum of two odd numbers. Your opponent will then have two odd numbers in front. The only choice for him/her is to split one of the two odd numbers into two piles with one even and one odd. Now it comes to your turn again, and you can keep dividing the even number into a sum of two odd numbers. If this continues, you will eventually get a pile with 2 M&Ms, and you can eat the other pile, and split the 2 M&Ms into (1,1), and your opponent has no moves available and loses.

Case two; n, m are both odd numbers.

n, m can only be divided into a pile with a sum of one odd and one even number. Then your opponent, as what you do in Case one, will keep splitting her even-number pile into two odd-number piles for you until you get (1,1) and loses.

(3) this question is relatively easy now as we have figure out the rules behind the game. Since both 2019 and 9021 are odd number, it is a P-position, and thus it will be better to go second.

Lesson 2

1. (1) $2 \oplus 4 = 6$

$$\begin{array}{r} 10\ (2) \\ +\ 100\ (4) \\ \hline 110\ (6) \end{array}$$

(2) $9 \oplus 8 = 1$

$$\begin{array}{r} 1001\ (9) \\ +\ 1000\ (8) \\ \hline 0001\ (1) \end{array}$$

(3) $100 \oplus 324 = 288$

$$\begin{array}{r} 1100100\ (100) \\ +\ 101000100\ (324) \\ \hline 100100000\ (288) \end{array}$$

2. If I were you, I would choose to go first. Here is why:
The Nim sum of 2019 and 215 is

$$\begin{array}{r} 11111100011\ (2019) \\ +\ 11010111\ (215) \\ \hline 11100110100\ (1844) \end{array}$$

Since $2019 \oplus 215 = 1844$!= 9102, (2019, 215, 9102) is a N-position.
In order to win, you should go first.

3. Before solving this game, we can fill in more of the Nim table below:

From the table, we can easily get that

$$9 \oplus 8 = 1;$$
$$9 \oplus 10 = 3;$$
$$8 \oplus 10 = 2.$$

Thus, we can either eat 9 M&Ms from 10 to make it 1, eat 5 M&Ms from 8 to make it 3, or eat 7 M&Ms from 9 to make it 2.

	0	1	2	3	4	5	6	7	8	9	10	…
0	0	1	2	3	4	5	6	7	8	9	10	…
1	1	0	3	2	5	4	7	6	9	8	11	…
2	2	3	0	1	6	7	4	5	10	11	8	…
3	3	2	1	0	7	6	5	4	11	10	9	…
4	4	5	6	7	0	1	2	3	12	13	14	…
5	5	4	7	6	1	0	3	2	13	12	15	…
6	6	7	4	5	2	3	0	1	14	15	12	…
7	7	6	5	4	3	2	1	0	15	14	13	…
8	8	9	10	11	12	13	14	15	0	**1**	2	…
9	9	8	11	10	13	12	15	14	1	0	**3**	…
10	10	11	8	9	14	15	12	13	**2**	3	0	…
…	…	…	…	…	…	…	…	…	…	…	…	…

4. Suppose we have the position of (X, Y, 1), where X and Y are nonnegative integers.

Case one: X is even, i.e the binary number of X ends in 0.
Then $X \oplus 1$ must end in 1, because 0+1=1.
Thus, $X \oplus 1 = X + 1$.
If $Y = X + 1$, then (X, Y, 1) is a P-position and you should go second; otherwise, it is a N-position and you should go first.

Case two: X is odd, i.e. the binary number of X ends in 1.
Then $X \oplus 1$ must end in 0, because $1 + 1 = 0$.
Thus, $X \oplus 1 = X - 1$.
If $Y = X - 1$, then (X, Y, 1) is a P-position and you should go second; otherwise, it is a N-position and you should go first.

Lesson 3

Before we start answering those questions, we need to first identify the impartial games, which we have done in Lesson one, Exercise 3. Now that we already know it consists of three Empty-and-Divide game, we want to analyze them individually first, find the nimvalues by using MEX rule and construct the Nim table.

Here is the nim table I constructed:

	1	2	3	4	5	6	7	8	9	…
1	0	1	0	2	0	1	0	3	0	…
2	1	1	2	2	1	1	3	3	1	…
3	0	2	0	2	0	3	0	3	0	…
4	2	2	2	2	3	3	3	3	2	…
5	0	1	0	3	0	1	0	3	0	…
6	1	1	3	3	1	1	3	3	1	…
7	0	3	0	3	0	3	0	3	0	…
8	3	3	3	3	3	3	3	3	4	…
9	0	1	0	2	0	1	0	4	0	…
…	…	…	…	…	…	…	…	…	…	…

How is it calculated? It follows the same principle as the take-away example in Section 3.3:
First, we know that (1, 1) is the terminal position, which is a P-position, and thus $|(1, 1)| = 0$;
Then since the only move from (1, 2) and (2, 2) is (1, 1),
$|(1, 2)| = |(2, 2)| = \text{MEX}(\ |(1, 1)|\ ) = \text{MEX}(0) = 1$;
$|(1, 3)| = \text{MEX}(\ |(1, 2)|\ ) = \text{MEX}(1) = 0$;
$|(2, 3)| = \text{MEX}(\ |(1, 1)|, |(1, 2)|\ ) = \text{MEX}(0, 1) = 2$;
…

1. (2, 4), (5, 9) and (3, 7):

According to Sprague Grundy Theorem, every position, namely (X, Y) is equivalent to a nimheap:

$|(2, 4)| = 2$

$|(5, 9)| = 0$

$|(3, 6)| = 3$

Thus, $(2, 4) + (5, 9) + (3, 6) \simeq 2 \oplus 3$

In order to win, Alice should move to a P-position, namely $2 \oplus 2$.

So Alice can move (3, 6) to (2, 4) by eating all 3 M&Ms and split the 6 M&Ms into two piles of 2 and 4.

2. (2,4), (5, 4) and (2019, 9102):

Following the same principle as the first question:

$|(2, 4)| = 2$

$|(5, 4)| = 3$

$|(2019, 9102)| = ?$

Now we have some trouble here. At this stage, we want to find some pattern in the Nim table.

Here is the algorithm I find (there may be much more easier alternatives and try to find that out :)

For a position (X, Y), let nonnegative integer n be the nimvalue of (X, Y), i.e. $(X, Y) \simeq n$.

Let there be nonnegative integers k_1, k_2 and positive integers b_1, b_2 that

$X = 2^{n+1} \bullet k_1 + b_1$

$Y = 2^{n+1} \bullet k_2 + b_2$

Where $0 < b_1 \leq 2^n$, $0 < b_2 \leq 2^n$, and n has the minimum value to satisfy the two equations.

An example when X = 7, Y = 8:

Since $X = 7 = 2 \bullet 3 + 1$, $n_1 = 0$;

And since $Y = 8 = 2^3$, and in order to express it in the form $2^{n+1} \bullet k_2 + b_2$, 2^{n+1} must be greater than 2^3, the smallest of which is $2^4 = 16$, so $Y = 0 \bullet 16 + 8$, $n_2 = 3$.

Since $\max(n_1, n_2) = \max(1, 3) = 3$,

We further adjust k1, b1:

$X = 2^4 \bullet 0 + 7$,

$Y = 2^4 \cdot 0 + 8$.

Check that b1, b2 is smaller than 8, with n be the minimum.

Thus, the nimvalue n for (X, Y) is 3.

Now we go back to the question, according to this algorithm, we can find the nimvalue for (2019, 9102).

$2019 = 2 \cdot 1009 + 1$, n1 = 0

$9102 = 4 \cdot 2275 + 2$, n2 = 1.

So max (n1, n2) = 1.

Check to see that $2019 = 4 \cdot 504 + 3$, which do not satisfy $d1 \leq 2$.

So we have to try n = 2,

$2019 = 8 \cdot 252 + 3$, which is OK.

$9102 = 8 \cdot 1137 + 6$, which does not satisfy $d2 \leq 4$;

Now we try n = 3,

$2019 = 16 \cdot 126 + 3$, which is OK,

$9102 = 16 \cdot 568 + 14$, which does not satisfy $d2 \leq 8$;

Then we try n = 4,

$2019 = 32 \cdot 63 + 3$, which is OK,

$9102 = 32 \cdot 284 + 14$, which is OK.

Thus, n = 4.

Thus,

$|(2, 4)| = 2$

$|(5, 4)| = 3$

$|(2019, 9102)| = 4$.

And $(2, 4) + (5, 4) + (2019, 9102) \simeq 2 \oplus 3 \oplus 4 \simeq 1 \oplus 4$, which is a N-position.

Here is an alternative inspired by Xingyu Nie:

Given a position (a, b), represent the position (a-1, b-1) using binary number.

Then using the logical operator OR, which follows

$0 + 0 = 0$

$1 + 0 = 1$

$0 + 1 = 1$

$1 + 1 = 1$

We add $(a-1)_2$ and $(b-1)_2$ together to get a sum, say $(x)_2$.

$(x)_2 = x_n \cdot 2^n + x_{n-1} \cdot 2^{n-1} + \ldots + x_1 \cdot 2 + x_0 \cdot 1$, where x_i are either 1 or 0, and n is a nonnegative integer.

Then the nimvalue is equally to i when x_i is 0 for the smallest i.

This method is essentially the same as the first method, but in a much clearer presentation.

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