# Recurrence Relations #1

1. (a) Find the values of $u_2, u_3, u_4, u_5$ for these recursively defined sequences:
   i. $u_n = 2u_{n-1} - 3, \quad u_1 = 4;$
   ii. $u_n = \frac{1}{2}u_{n-1}, \quad u_1 = 64;$
   iii. $u_n = -\frac{1}{3}u_{n-1}, \quad u_1 = 81.$

   (b) For each of these sequences find $\lim_{n\to\infty} u_n$.

| | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $\lim_{n\to\infty} u_n$ |
|---|---|---|---|---|---|
| i | 5 | 7 | 11 | 19 | $\infty$ |
| ii | 32 | 16 | 8 | 4 | 0 |
| iii | -9 | 3 | -1 | $\frac{1}{3}$ | 0 |

2. Find a recurrence relation for each of these sequences:

   (a) $2, 5, 8, 11, 14, \ldots;$ $\quad u_n = u_{n-1} + 3$

   (b) $4, 9, 19, 39, 79, \ldots;$ $\quad u_n = 2u_{n-1} + 1$

   (c) $1, 1, 2, 3, 5, 8, 13, \ldots.$ $\quad u_n = u_{n-1} + u_{n-2}.$

3. Find $u_n$ as a function of $n$ for these recursively defined sequences:

   (a) $u_n = 2u_{n-1} - 1, \quad u_1 = 3;$ $\quad u_n = \alpha 2^n + 1 = 2^n + 1.$

   (b) $u_{n+1} = 5u_n + 8, \quad u_1 = 8.$ $\quad u_n = \alpha 5^n - 2 = 2\times 5^n - 2.$

4. Determine the limit, if it exists, for these recursively defined sequences:

   (a) $u_n = 0.8u_{n-1} + 20, \quad u_1 = 40;$ $\quad u_n = \alpha\left(\frac{4}{5}\right)^n + 100 = -75\left(\frac{4}{5}\right)^n + 100. \Rightarrow \lim_{n\to\infty} u_n = 100.$

   (b) $u_n = 60 - 0.5u_{n-1}, \quad u_1 = 4.$ $\quad u_n = \alpha\left(-\frac{1}{2}\right)^n + 40 = 72\left(-\frac{1}{2}\right)^n + 40 \Rightarrow \lim_{n\to\infty} u_n = 40.$

5. A retired teacher has \$150 000 invested in a pension fund. The fund earns 5% interest per annum. At the end of each year the teacher withdraws \$15 000 to cover living expenses for the following year.

   (a) Calculate the value of the fund just after the \$15 000 has been withdrawn for the first four years.

   (b) Find a recurrence relation for the value of the fund at the end of the $n^{\text{th}}$ year.

   (c) How many years will the fund last?

(b) $u_1 = 150{,}000$

$u_n = 1.05u_{n-1} - 15{,}000$

$0.05k = 15{,}000$

$k = 15{,}000 \times 200.$

(c) $u_n = \alpha(1.05)^n -$

Answers to selected exercises:
1. limits: $\infty, 0, 0$ $\quad$ 2. $u_n = u_{n-1}+3$, $u_n = 2u_{n-1}+1$, $u_n = u_{n-1}+u_{n-2}$ $\quad$ 3. $u_n = 2^n+1$, $u_n = 2\times 5^n - 2$
4. 100, 40 $\quad$ 5. $u_n = 1.05u_{n-1} - 15\,000$, 14 years

Do **not** write solutions on this page.

**12.** [Maximum mark: 18]

On the day of her birth, 1st January 1998, Mary's grandparents invested $\$x$ in a savings account. They continued to deposit $\$x$ on the first day of each month thereafter.
The account paid a fixed rate of $0.4\%$ interest per month. The interest was calculated on the last day of each month and added to the account.
Let $\$A_n$ be the amount in Mary's account on the last day of the $n$th month, immediately after the interest had been added.

(a) Find an expression for $A_1$ and show that $A_2 = 1.004^2x + 1.004x$. [2]

(b) (i) Write down a similar expression for $A_3$ and $A_4$.

(ii) Hence show that the amount in Mary's account the day before she turned $10$ years old is given by $251(1.004^{120} - 1)x$. [6]

(c) Write down an expression for $A_n$ in terms of $x$ on the day before Mary turned $18$ years old showing clearly the value of $n$. [1]

(d) Mary's grandparents wished for the amount in her account to be at least $\$20\,000$ the day before she was $18$. Determine the minimum value of the monthly deposit $\$x$ required to achieve this. Give your answer correct to the nearest dollar. [4]

(e) As soon as Mary was $18$ she decided to invest $\$15\,000$ of this money in an account of the same type earning $0.4\%$ interest per month. She withdraws $\$1000$ every year on her birthday to buy herself a present. Determine how long it will take until there is no money in the account. [5]

# Recurrence Relations #2

2 -3

1. Find $u_3$ and $u_4$ for the sequence defined recursively by $u_n = u_{n-1} - u_{n-2}$ and $u_1 = 3,\ u_2 = 5$.

2. Show that $u_n = 4^n$ is a solution to the recurrence relation $u_n = 3u_{n-1} + 4u_{n-2}$. $4^n = 3\cdot 4^{n-1} + 4\cdot 4^{n-2}$

3. Which of the following sequences satisfy the recurrence relation $u_n = 2u_{n-1} - u_{n-2}$? $= 4^n$. ✓

   (a) $3, 6, 9, 12, 15 \ldots$ ✓

   (b) $2, 4, 8, 16, 32, \ldots$ ✗

   (c) $10, 10, 10, 10, 10, \ldots$ ✓

   $\frac{45-38}{7}$

4. Consider the sequence $(u_n)_{n\in\mathbb{N}}$ whose first six terms are $3, 5, 11, 21, 43, 85$.

   (a) Find a second order recurrence relation for the sequence. $u_n = u_{n-1} + 2u_{n-2}$.

   (b) Give a recursive definition for the sequence. $r^2 - r - 2 = 0$ $r_1 = 2,\ r_2 = -1$ $u_n = \alpha(2)^n + \beta(-1)^n$

   (c) Use the auxiliary equation technique to find $u_n$ as a function of $n$.

   $\begin{cases}\alpha+\beta=3\\ 2\alpha-\beta=5\end{cases}$ $\alpha = \frac{8}{3}$ $\beta = \frac{1}{3}$

   $u_n = \frac{8}{3}(2)^n + \frac{1}{3}(-1)^n$.

   $u_n = \frac{9}{7}(5)^n + \frac{19}{7}(-2)^n$

5. Solve the recurrence relation $u_n = 3u_{n-1} + 10u_{n-2}$ with initial terms $u_0 = 4$ and $u_1 = 1$.

   ~~$u_n$~~ $\alpha + \beta = 4$. $5\alpha - 2\beta = 1$ $\alpha = \frac{9}{7}$ $\beta = \frac{19}{7}$

6. Consider the sequence $(u_n)_{n\in\mathbb{Z}^+}$ whose first seven terms are $2, 5, 12, 29, 70, 169, 408$.

   $u_n = 2u_{n-1} + u_{n-2}$. $x^2 - 2x - 1 = 0$ $\frac{2\pm\sqrt{\ }}{2} = 1\pm\sqrt{2}$.

   (a) Find a second order recurrence relation for the sequence.

   (b) Use the auxiliary equation technique to find $u_n$ as a function of $n$. $\alpha + \beta = 2$

   $(1+\sqrt{2})\alpha + (1-\sqrt{2})\beta = 5$

7. Solve the recurrence relation $u_n = u_{n-1} + 2^n$ with $u_0 = 5$.

   $u_n = 5 + (2^n - 1)2 = \cancel{2^n} + 4. = 2^{n+1} + 3$

8. Suppose that $r^n$ and $q^n$ are both solutions to the recurrence relation $u_n = au_{n-1} + bu_{n-2}$. Prove that any linear combination of $r^n$ and $q^n$ is also a solution to the recurrence relation.

   $q^n = aq^{n-1} + bq^{n-2}$

   $r^n = ar^{n-1} + br^{n-2}$

   $sq^n + tr^n$

   $=$

9. The Student Store received a magical candy machine for Christmas. The first time you put a nickel in the machine 1 Smartie comes out. The second time 3 Smarties, the third time 4 Smarties, the fourth time 7 Smarties, the fifth time 11 Smarties, and so on.

   (a) Find a second order recurrence relation for the number of Smarties the machine produces with the $n^{\text{th}}$ nickel. $u_n = u_{n-1} + u_{n-2}$

   (b) Use the `sequence mode` of your GDC to find the number of Smarties the machine produces with the $17^{\text{th}}$ nickel.

10. Consider the number of $1 \times n$ tile designs that can be made using $1 \times 1$ tiles available in four colours and $1 \times 2$ tiles available in five colours.

    $u_n = 4u_{n-1} + 5u_{n-2}$.

    (a) Find a recursive definition for the number of these $1 \times n$ tile designs.

    (b) Write out the first six terms of this tile design sequence. $4, 21,$

    (c) Find a formula for the general term of the sequence.

    (d) Hence determine the minimum value of $n$ so the number of these $1 \times n$ tile designs exceeds 100 000.

$r^2 - 4r - 5 = 0$

$5, -1$.

$u_n = \alpha(5)^n + \beta(-1)^n$

$4 = 5\alpha - \beta$

$21 = 25\alpha + \beta$

$\alpha = \frac{25}{30} = \frac{5}{6}$

$\beta = \frac{1}{6}$

$u_n = \frac{1}{6}\left(5^{n+1} + (-1)^n\right)$

Answers to selected exercises:

1. $2, -3$ 3. y, n, y 4. $u_n = \frac{1}{3}(2^{n+3} + (-1)^n)$ 5. $u_n = \frac{19}{7}(-2)^n + \frac{9}{7}5^n$ 7. $u_n = 4^n + (-1)^n$

9. 3571 10. $u_n = \frac{1}{6}(5^{n+1} + (-1)^n)$, 8

**3.** [Maximum mark: 13]

A contagious virus affects the population of a small town with 5000 inhabitants.
Let $I_n$ denote the total number of people who have been infected by the end of the $n^{th}$ week.
In the first week there were 10 cases of infection and by the end of the second week there was a total of 22 cases. A proposed model is that the number of cases is increasing in such a way that the number of new cases in any week is 1.2 times the number of new cases in the previous week.

(a) Show that $I_n$ satisfies the recurrence relation $I_{n+2} - 2.2I_{n+1} + 1.2I_n = 0$. [2]

(b) State appropriate initial conditions. [1]

(c) Solve the recurrence relation to obtain an expression for $I_n$ in terms of $n$. [6]

(d) Hence find during which week the whole town will become infected. [2]

(e) State two limitations of the model. [2]

# Recurrence Relations #3

1. Show that $u_n = n2^n$ is a solution to the recurrence relation $u_n = 4u_{n-1} - 4u_{n-2}$.
2. Consider the sequence with recurrence relation $u_n = 4u_{n-1} - 4u_{n-2}$ and initial terms $u_0 = 1$ and $u_1 = 4$.
   (a) Find $u_2$ and $u_3$.
   (b) Use sequence mode and the table of your GDC to find the minimum value of $n$ such that $u_n > 500\,000$.
   (c) Use the auxiliary equation technique to find $u_n$ as a function of $n$.
   (d) Hence find $\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$.
3. Solve the recurrence relation $u_{n+2} = 2u_{n+1} - u_n$ with initial terms $u_1 = 5$ and $u_2 = 7$.
4. Solve the recurrence relation $u_n = 2u_{n-1} - 2u_{n-2}$ with $u_1 = -6$ and $u_2 = 0$. Leave your answer in complex form.
5. Let $z = \sqrt{3} + i$. Use De Moivre's theorem to find $z^9$. Hence write down $(z^*)^9$. Confirm your results with the GDC.
6. If $z = re^{i\theta}$ and $z + z^* = a + bi$ where $a, b \in \mathbb{R}$, find expressions for $a$ and $b$ in terms of $r$ and $\theta$.
7. Explain why $\alpha z^n + \alpha^*(z^*)^n$ must be a real number. Write $(1+i)i^{10} + (1-i)(-i)^{10}$ as a real number.
8. Solve the recurrence relation $u_{n+2} = u_{n+1} - u_n$ with $u_1 = 1$ and $u_2 = -1$. Give your answer in real form.
9. Two sequences $a_n$ and $b_n$ satisfy the recurrence relations $a_{n+1} = 3a_n + b_n$ and $b_{n+1} = 5a_n - b_n$ with initial conditions $a_1 = 6$ and $b_1 = -6$.
   (a) Find the value of $a_2$.
   (b) Show that $a_{n+2} = 2a_{n+1} + 8a_n$.
   (c) Solve the second order recurrence relation for $a_n$.
   (d) Hence solve for $b_n$.
10. Consider the second order recurrence relation $u_n = 4u_{n-1} - 5u_{n-2}$ with initial terms $u_0 = u_1 = 2$.
    (a) Show that $u_n = (1+i)(2+i)^n + (1-i)(2-i)^n$.
    (b) Show that $u_n$ may also be written as $u_n = 2^{3/2}5^{n/2}\cos(\frac{\pi}{4} + n\arctan(\frac{1}{2}))$.

Answers to selected exercises:
2. $u_{15} = 524\,288$, limit is 2  3. $u_n = 3 + 2n$  4. $u_n = -3(1+i)^n - 3(1-i)^n$  6. $a = 2r\cos\theta, b = 0$  7. $-2$
8. $u_n = 2\cos(n\pi/3)$  9. (a) 12 (b) $a_n = 4^n - (-2)^n$ (c) $b_n = 4^n + (-2)^n$

**1a.** *[1 mark]*

On the 1st March in a country there are 5000 environmentally contaminated sites requiring clean-up. By the 1st April 80 % of these 5000 contaminated sites are cleaned up but 200 new sites requiring clean-up are identified. This situation is assumed to recur every month. Jim sets up a first-degree recurrence relation that represents this information.

State Jim's first-degree recurrence relation for the number of sites, $u_n$, requiring clean-up after $n$ months in the form $u_n = Au_{n-1} + B$, where $A$ and $B$ are non-zero constants.

**1b.** *[1 mark]*

State the value of $u_0$.

**1c.** *[5 marks]*

Solve Jim's first-degree recurrence relation.

**1d.** *[5 marks]*

Jim now sets up a second-degree recurrence relation that gives information regarding environmental clean-up in a different country.
The second model is $d_n = 0.6d_{n-1} - 0.09d_{n-2}$ with initial conditions $d_0 = d_1 = 4000$.

Solve Jim's second-degree recurrence relation.