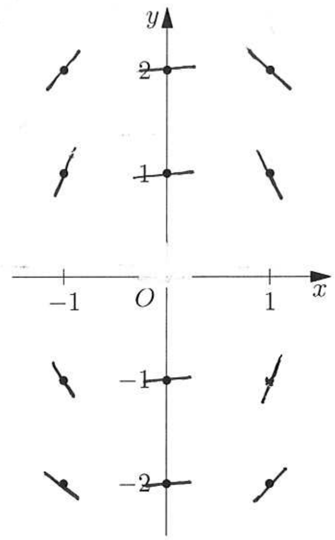
FM2 Differential Equations Name: Maggie.

$\frac{31}{32}$ Excellent!

1. Consider the differential equation $\frac{dy}{dx} = -\frac{2x}{y}$.

(a) Sketch the slope field for the differential equation at the twelve points indicated.



(b) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = -1$. Write an equation for the line tangent to the graph of f at $(1, -1)$ and use it to approximate $f(1.1)$.

$y = 2x - 3$

$f(1.1) \approx 2(1.1) - 3$

≈ -0.8

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = -1$.

$\int y\, dy = \int -2x\, dx$.

$\frac{1}{2}y^2 = -x^2 + C_1$

$y^2 + 2x^2 = C_2$, where $C_2 = 2C_1$

when $x = 1$.

$(-1)^2 + 2(1)^2 = C_2 = 3$

So $y^2 + 2x^2 = 3$. $(y \neq 0)$.

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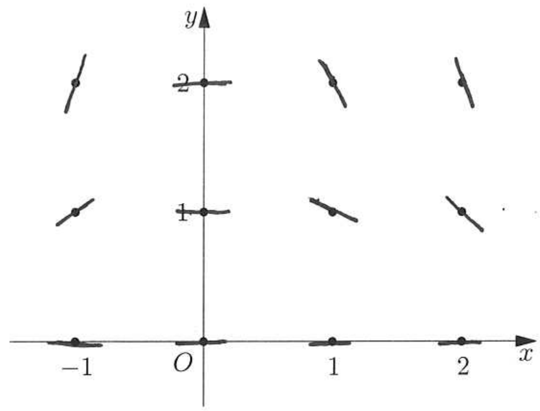
2. Consider the differential equation $\frac{dy}{dx} = -\frac{xy^2}{2}$.

(a) Sketch the slope field for the differential equation at the twelve points indicated.



(1, 0)

$\frac{+1\cdot 1}{2}$

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$-\frac{1\cdot 4}{2}$

$-\frac{2(1)}{2}$

$-\frac{2(4)}{2}$

(b) Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = 2$. Write an equation for the line tangent to the graph of f at $x = -1$.

$y = 2x + 4$

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.

$$\int \frac{1}{y^2}\,dy = -\frac{1}{2}\int x\,dx.$$

$$-y^{-1} = -\frac{1}{2}\cdot\frac{1}{2}x^2 - C_1$$

$$\frac{1}{y} = \frac{1}{4}x^2 + C_1$$

$$y = \frac{1}{\frac{1}{4}x^2 + C_1}$$

when $x = -1$

$$y = \frac{1}{\frac{1}{4} + C_1} = 2.$$

$$\frac{1}{4} + C_1 = \frac{1}{2}$$

$$C_1 = \frac{1}{4}$$

So $y = \frac{1}{\frac{1}{4}x^2 + \frac{1}{4}}$

$= \frac{4}{x^2+1}$

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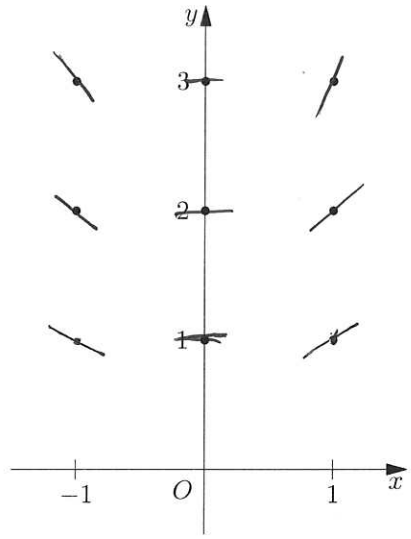
3. Consider the differential equation $\frac{dy}{dx} = \frac{xy}{2}$.

(a) Sketch the slope field for the differential equation at the nine points indicated.



(b) Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(0) = 3$. Use Euler's method starting at $x = 0$ with a step size of 0.1 to approximate $f(0.2)$. Set out your work in a table.

n.	x_n	y_n	h	$h * f(x_n, y_n)$.
0	0	3	0.1	0
1	0.1	3	0.1	0.015
2	0.2	3.015	0.1	

So $f(0.2) \approx 3.015$

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$. Use your solution to find $f(0.2)$.

$$\int \frac{1}{y}\, dy = \int \frac{x}{2}\, dx$$

$$\ln|y| = \frac{1}{4}x^2 + C_1$$

$$|y| = e^{\frac{1}{4}x^2} \cdot e^{C_1}$$

$$y = Ae^{\frac{1}{4}x^2}. \quad A = \pm e^{C_1}$$

when $x = 0$

$$y = A = 3.$$

So $y = 3e^{\frac{1}{4}x^2}$.

when $x = 0.2$

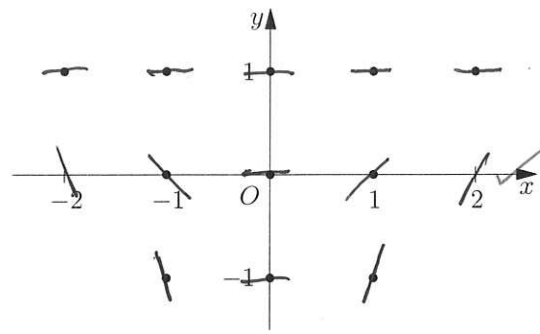
$$y = 3e^{\frac{1}{4} \cdot \frac{1}{25}}$$

$$= 3e^{\frac{1}{100}}$$

$$\approx 3.03 \text{ (3 s.f.)}$$

4. Consider the differential equation $\frac{dy}{dx} = x(y-1)^2$.

(a) Sketch the slope field for the differential equation at the eleven points indicated.



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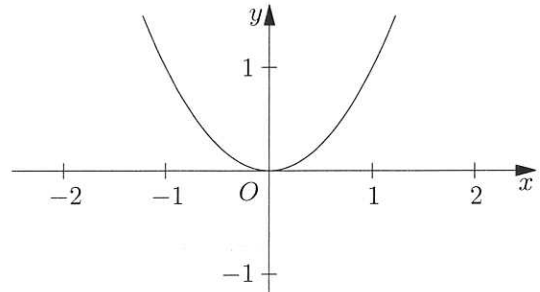
$-1,$

(b) Use the slope field for the given differential equation to explain why a solution could not have the graph shown below.



According to the ~~slope~~ slope field, when $y=1$, ~~the~~ the derivative / slope of the fcn is always 0, which is not the case in this graph.

So a solution could not have the graph ~~bet~~ above.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -1$.

$$\int \frac{1}{(y-1)^2}\,dy = \int x\,dx$$

$$-\frac{1}{(y-1)} = \frac{1}{2}x^2 + C_1$$

$$y-1 = -\frac{2}{x^2+2C_1}$$

$$y = -\frac{2}{x^2+C_2} + 1 . \quad C_2 = 2C_1 .$$

when $x=0$,

$$y = -\frac{2}{C_2} + 1 = -1$$

$$C_2 = 1$$

So $y = -\frac{2}{x^2+1} + 1$

$$= \frac{x^2-1}{x^2+1}$$

$(y \neq 1)$

$\frac{1}{y} = \frac{x^2+1}{x^2-1} = \frac{2}{x^2-1} + 1$

≥ -1

≥ -2

(d) Find the range of the solution found in part (c).

$$y = \frac{x^2-1}{x^2+1}$$

$\Leftrightarrow \quad yx^2 + y = x^2 - 1$

$(y-1)\cancel{x}x^2 + y + 1 = 0$

$\Delta = 0 - 4(y-1)(y+1) \geq 0 .$

$\Leftrightarrow \quad y^2 - 1 \leq 0$

$-1 \leq y \leq 1$

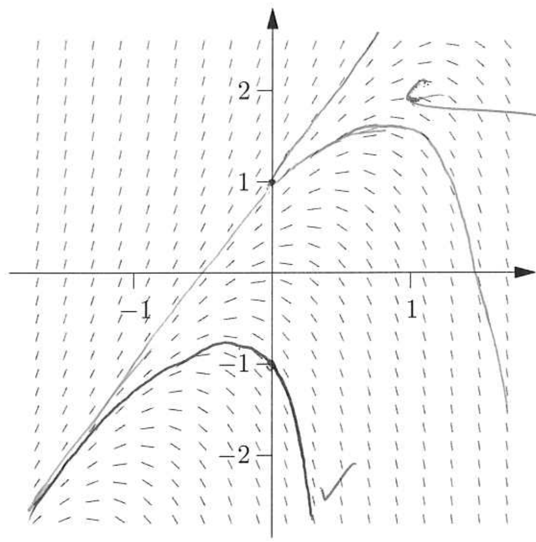
since $\int \left(\frac{1}{y-1}\right)^2 dy$ exists,

$y \neq 1$.

So $\boxed{-1 \leq y < 1}$ this is the range.

5. Consider the differential equation $\frac{dy}{dx} = 2y - 4x$.

(a) The slope field for this differential equation is provided. Sketch the solution curve that passes through the point $(0, 1)$ and sketch the solution curve that passes through the point $(0, -1)$.



4.8
-0.4
4.4

×

(b) Let f be the function that satisfies the differential equation with the initial condition $f(0) = 2$. Use Euler's method starting at $x = 0$ with a step size of 0.1 to approximate $f(0.2)$. Set out your work in a table.

n	x_n	y_n	h	$h \times f(x_n, y_n)$
0	0	2	0.1	0.4
1	0.1	2.4	0.1	0.44
2	0.2	2.84	0.1	

$\Rightarrow f(0.2) \approx 2.84$.

(c) Find the value of b for which $y = 2x + b$ is a solution to the differential equation. Justify your answer.

$y = 2x + b$

$\Rightarrow \frac{dy}{dx} = 4x + 2b - 4x$

$= 2b$.

$\frac{d}{dx}\, 2x + b = 2b$

$2 = 2b$

$b = 1$

$y = 2x + 1$

It is a local maximum.

because ① at $(0,0)$, $\frac{dy}{dx} = 0$, which means it is a local extreme.

② $\frac{dy}{dx} = -2e^{2x} + 2$. ~~is~~ decreasing. which is a concave function, So it is a concave function, which only has local maximum

(d) Let g be the function that satisfies the differential equation with the initial condition $g(0) = 0$. Does the graph of g have a local extremum at the point $(0, 0)$? If so, is the point a local maximum or a local minimum? Justify your answer.

$\frac{dy}{dx} = 2y - 4x$

$\Rightarrow \frac{dy}{dx} - 2y = -4x$.

~~since $\frac{d}{dx} e^{2x} - 2e^{2x}$~~

since $\frac{d}{dx} e^{2x} - 2e^{2x}$

$= 2e^{2x} - 2e^{2x} = 0$.

and one solution is

$y = 2x + 1$,

$y = ke^{2x} + 2x + 1, k \in \mathbb{R}$.

when $x = 0$.

$y = k + 1 = 0$.

$k = -1$

So $y = -e^{2x} + 2x + 1$.

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