第十二讲　数列中的不等式

例1. 已知数列$\{a_n\}$满足$\frac{1}{2}<a_1<\frac{2}{3}$，$a_{n+1}=a_n\left(2-a_{n+1}\right)$（$n=1,2,\cdots$）.

求证：对任意正整数n，均有$n+\frac{1}{2}<\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}<n+2$.

证：由$a_{n+1}=a_n\left(2-a_{n+1}\right)$可得$a_{n+1}=\frac{2a_n}{a_n+1}$.

利用不动点法可得$\frac{a_{n+1}-1}{a_{n+1}}=\frac{1}{2}\cdot\frac{a_n-1}{a_n}$，即$\frac{1}{a_{n+1}}-1=\frac{1}{2}\left(\frac{1}{a_n}-1\right)$.

注意$\frac{1}{2}<\frac{1}{a_1}-1<1$，故$\left(\frac{1}{a_1}-1\right)+\left(\frac{1}{a_2}-1\right)+\cdots+\left(\frac{1}{a_n}-1\right)\geq\left(\frac{1}{a_1}-1\right)>\frac{1}{2}$；

$\left(\frac{1}{a_1}-1\right)+\left(\frac{1}{a_2}-1\right)+\cdots+\left(\frac{1}{a_n}-1\right)=\left(1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}}\right)\left(\frac{1}{a_1}-1\right)<2\left(\frac{1}{a_1}-1\right)<2$.

整理可得$n+\frac{1}{2}<\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}<n+2$.

例2. 数列$\{a_n\}$满足$a_1=1$，$a_2=\frac{1}{4}$，$a_{n+1}=\frac{(n-1)a_n}{n-a_n}$（$n=2,3,\cdots$）.

（1）求数列的通项公式；

（2）求证：对任意正整数n，均有$\sum_{k=1}^{n}a_k^2<\frac{7}{6}$.

（1）解：由$a_{n+1}=\frac{(n-1)a_n}{n-a_n}$变形得$\frac{1}{a_{n+1}}-1=\frac{n}{n-1}\left(\frac{1}{a_n}-1\right)$，

由累乘法可得$\frac{1}{a_n}-1=(n-1)\left(\frac{1}{a_2}-1\right)$，于是求出通项$a_n=\frac{1}{3n-2}$.

（2）证：将通项公式代入，可得：

$$\sum_{k=1}^{n}a_k^2=1+\frac{1}{4^2}+\frac{1}{7^2}+\cdots+\frac{1}{(3n-2)^2}<1+\frac{1}{2\times5}+\frac{1}{5\times8}+\cdots+\frac{1}{(3n-4)(3n-1)}=1+\frac{1}{3}\left(\frac{1}{2}-\frac{1}{3n-1}\right)<\frac{7}{6}.$$

例3. 数列$\{a_n\}$定义如下：$a_1=2$，$a_{n+1}=a_n^2-a_n+1(n=1,2,\cdots)$.

求证：$1-\frac{1}{2000^{2000}}<\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{2000}}<1$

证：由$a_{n+1}=a_n^2-a_n+1$变形得$\frac{1}{a_n}=\frac{1}{a_n-1}-\frac{1}{a_{n+1}-1}$

代入得$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{2000}}=\frac{1}{a_1-1}-\frac{1}{a_{2000}-1}=1-\frac{1}{a_{2000}-1}<1$.

要证不等式左边，只要证明$a_{2000}>2000^{2000}+1$即可．下证当$n\geq2$时，$a_n\geq2^{2^{n-2}}+1$.

当$n=2,3$时显然成立.

若$n=k$时成立．则$n=k+1$时，$a_{k+1}=a_k^2-a_k+1\geq\left(a_k-1\right)^2+1\geq\left(2^{2^{k-2}}\right)^2+1=2^{2^{k-1}}+1$，命题成立.

于是得到$a_{2000}\geq2^{2^{1998}}+1>2^{\left(2^{10}\right)^{199}}+1>2^{1000^{199}}>2^{22000}+1=\left(2^{11}\right)^{2000}+1>2000^{2000}+1$.

综上即可得证.

例4. 已知数列$\{a_n\}$的各项均为非负实数，且$a_n^2-a_n+a_{n+1}\le 0$（$n=1,2,\cdots$）.

求证：对所有不小于 2 的正整数 n，均有$a_n\le\dfrac{1}{n+2}$.

证：当$n=2$时，由$a_1^2-a_1+a_2\le 0$可得$a_2\le -a_1^2+a_1=-\left(a_1-\dfrac{1}{2}\right)^2+\dfrac{1}{4}\le\dfrac{1}{4}$，命题成立.

若$n=k$成立，$n=k+1$时，由$a_k^2-a_k+a_{k+1}\le 0$可得$a_{k+1}\le -a_k^2+a_k$，

由归纳假设，$a_k\le\dfrac{1}{k+2}<\dfrac{1}{2}$，故$a_{k+1}\le -\dfrac{1}{(k+2)^2}+\dfrac{1}{k+2}<\dfrac{1}{k+3}$，命题成立

综上，原命题得证.

例5. 给定实数 a，使得$0<a<1$，数列$\{a_n\}$满足$a_1=1+a$，$a_{n+1}=\dfrac{1}{a_n}+a$（$n=1,2,\cdots$）.

求证：对任意正整数 n，均有$a_n>1$.

证一：加强命题，证明$1<a_n<\dfrac{1}{1-a}$.

当$n=1$时，$1<1+a<\dfrac{1}{1-a}$，命题成立.

若$n=k$时成立，则$n=k+1$时，$a_{k+1}=\dfrac{1}{a_k}+a<\dfrac{1}{1}+a=1+a<\dfrac{1}{1-a}$，$a_{k+1}=\dfrac{1}{a_k}+a>\dfrac{1}{\frac{1}{1-a}}+a=1$.

综上即可得证.

证二：当$n=1$时，$a_1=1+a>1$，命题成立.

当$n=1$时，$a_2=\dfrac{1}{a_1}+a=\dfrac{1}{1+a}+a=\dfrac{1+a+a^2}{1+a}>1$，命题成立.

若$n=k$时命题成立，则$n=k+2$时：

$$a_{k+2}=\frac{1}{a_{k+1}}+a=\frac{1}{\frac{1}{a_k}+a}+a=\frac{(1+a^2)a_k+a}{a\cdot a_k+1}=\frac{1+a^2}{a}-\frac{1}{a^2\cdot a_n+a}>\frac{1+a^2}{a}-\frac{1}{a^2+a}=\frac{a^3+a^2+a}{a^2+a}>1.$$

综上即可得证.

例6. 已知数列$\{a_n\}$满足$a_1=4$，$a_{n+1}=\sqrt{2a_n+3}$（$n=1,2,\cdots$），

求证：对任意正整数 n，均有$3-\left(\dfrac{2}{3}\right)^{n-1}\le a_n\le 3+\left(\dfrac{2}{3}\right)^{n-1}$.

证：本题即证$|a_{n+1}-3|\le\left(\dfrac{2}{3}\right)^{n-1}$. 由条件$|a_{n+1}-3|=\left|\sqrt{2a_n+3}-3\right|=\dfrac{|2a_n-6|}{\sqrt{2a_n+3}+3}\le\dfrac{2}{3}|a_n-3|$.

从而$|a_n-3|\le\left(\dfrac{2}{3}\right)^{n-1}|a_1-3|=\left(\dfrac{2}{3}\right)^{n-1}$. 综上即可得证.

例7. 对任意正整数 n，求证：$\displaystyle\sum_{k=1}^{n}\frac{1}{3^k+(-2)^k}<\frac{7}{6}$.

证：记$a_k=\dfrac{1}{3^k+(-2)^k}$，本题即证$S_n=\displaystyle\sum_{k=1}^{n}a_k<\frac{7}{6}$.

$$a_{2k}+a_{2k+1}=\frac{1}{3^{2k}+2^{2k}}+\frac{1}{3^{2k+1}-2^{2k+1}}=\frac{3^{2k}+2^{2k}+3^{2k+1}-2^{2k+1}}{(3^{2k}+2^{2k})(3^{2k+1}-2^{2k+1})}=\frac{4\cdot 3^{2k}-2^{2k}}{3^{4k+1}+(3^{2k}-2^{2k+1})2^{2k}}<\frac{4}{3^{2k+1}}.$$

故$S_n<a_1+(a_2+a_3)+(a_4+a_5)+\cdots=1+\dfrac{4}{27}+\dfrac{16}{243}+\cdots=1+\dfrac{4}{27}\cdot\dfrac{1}{1-\frac{1}{9}}=\dfrac{7}{6}$.