Calculus

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Calculus

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1 Sequences, Limits and Improper Integrals

Introduction

Important concepts regarding sequences, series and limits were covered in previous textbook chapters on the core syllabus. It would be helpful to go back and read through the first four sections of Chapter 4, especially the material on infinite geometric series in Section 4.4. The first section in Chapter 13 includes an informal approach to limits of functions and also covers properties of limits. Central to any discussion about sequences, series and limits is the concept of a function. Thus, it may also prove worthwhile to review some of the fundamental ideas, terminology and notation for functions covered in the first section of Chapter 2.

Arithmetic and geometric series, both finite and infinite, were discussed in Chapter 4. Much of the material in this chapter and the next two chapters is directly or indirectly involved with infinite series. As you will see, infinite series are mathematically interesting and have very useful applications. Our treatment of series in this option topic will require a more formal approach than taken in Chapter 4. In order to develop a more thorough treatment of infinite series, we must first consider **infinite sequences** of numbers.

Sequences and series are closely related, so you need to be careful to apply these words correctly. A sequence is an ordered list of numbers commonly written out with commas separating the numbers. A series is a sum of a sequence. The finite sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ is an ordered list whereas the closely related finite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ is a sum that is precisely equal to the number $\frac{15}{8}$.

1.1 Infinite sequences

Sequences occur in many areas of mathematics. For example, the positive even numbers less than or equal to 10 form a sequence:

2, 4, 6, 8, 10.

This sequence is **finite** because the list of numbers ends with a specific number, 10 in this case. If a sequence does not end, it is **infinite**. We will be focusing on infinite sequences, so from now on if we use the word 'sequence' it is understood that we are referring to an infinite sequence.

Definition of a sequence

A sequence of numbers is a discrete function whose domain is the set of positive integers, $\mathbb{Z}^+$.

From the definition it is understood that an infinite sequence is a rule that associates a number to each positive integer. The number associated with the integer n is called the nth term of the sequence. Instead of using the familiar function notation $f(n)$ to represent the value (term) of a sequence f for a certain positive integer n, it is customary to use a subscripted letter, such as a_n or u_n. Hence, we will denote a sequence by $\{a_1, a_2, a_2, \ldots, a_n, \ldots\}$, or more simply with the notation $\{a_n\}$, $n \in \mathbb{Z}^+$. It follows that a_n is an **explicit formula** (sometimes called a closed formula) that is a function whose domain, n, is the set of positive integers and generates the value of the nth term of a sequence. The notation $\{a_n\}$ represents all the terms of a sequence, not just a single term. For example, for the sequence formed by the reciprocals of the positive integers, we can write $\{a_n\} = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ and $a_n = \frac{1}{n}$.

Example 1 – Listing the terms of a sequence

a) The terms of the sequence $\{a_n\} = \left\{1 - \frac{1}{n}\right\}$ are $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$.

b) The terms of the sequence $\{b_n\} = \left\{\frac{(-1)^{n+1}}{n}\right\}$ are $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$.

c) The terms of the sequence $\{c_n\} = \left\{\frac{2^{n-1}}{(n-1)!}\right\}$ are $1, \frac{2}{1}, \frac{4}{2}, \frac{8}{6}, \frac{16}{24}, \frac{32}{120}, \ldots$.

The first six terms of the sequence $\{c_n\}$ can be simplified to $1, 2, 2, \frac{4}{3}, \frac{2}{3}, \frac{4}{15}, \ldots$. This highlights the fact that although it is often helpful to view some of the initial terms in an infinite sequence, knowing the explicit formula for the value of the nth term is even more useful. (**Note:** Evaluating the first term in the sequence $\{c_n\}$ required using the definition that $0! = 1$.)

Example 2 – A sequence defined by a recursive formula

It is not necessary for a sequence to be defined by an explicit formula, as in Example 1. The sequence $\{a_n\}$ defined by

$$a_1 = a_2 = 1, \; a_{n+2} = a_{n+1} + a_n \; \text{ for } n \geqslant 1$$

is a sequence that we saw in Chapter 4 of the book. The rule giving a_{n+2} in terms of a_{n+1} and a_n is an example of a recursion formula. It produces the famous Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, \ldots.$$

Although a bit complicated, an explicit formula exists for the nth term of the Fibonacci sequence. In general, the rules for sequences and series in this chapter will be explicit rather than recursive. See Chapter 4 of the book for discussion of explicit and recursive formulae for sequences.

Our foremost concern with a sequence $\{a_n\}$ is whether a_n has a limit L as n approaches infinity ($n \to \infty$). If it does, we say that $\{a_n\}$ **converges** to L; otherwise we say that $\{a_n\}$ **diverges.**

Since a sequence is a type of function, it seems appropriate that in our investigation of limits of sequences, we can apply the same ideas from our work with limits of functions in Chapter 13 of the book. A function f whose domain is the half-open interval $[1, \infty[$ can be converted into a sequence by restricting its domain to the integers in that interval, i.e. the positive integers $\mathbb{Z}^+$. Conversely, given a sequence $\{a_n\}$, it is often possible to define a function f on $[1, \infty[$ such that $f(n) = a_n$ for each integer $n > 0$. Thus, if it was established that $\lim_{x\to\infty} f(x) = L$, it would necessarily follow that $\lim_{n\to\infty} a_n = L$. Therefore, results obtained in Chapter 13 of the book for limits of functions are available for our work with limits of sequences. In our development of the derivative through a limit process, we stated an informal definition of a limit of a function and five properties of limits (Section 13.1).

Our earlier informal definition of a limit of a function said that if $f(x)$ becomes *arbitrarily close* to a unique finite number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L.

In Section 13.1 of the book, we used some algebraic techniques combined with some informal reasoning to find limits of rational functions. It seems reasonable to conjecture that for a sequence $\{a_n\}$ if the value of a_n matches a function f at every positive integer, and $f(x)$ approaches a limit L as $x \to \infty$, then the sequence will converge to the same limit L.

A sequence that has a limit **converges**, whereas a sequence that does not have a limit **diverges**.

Limit of a sequence theorem

Suppose that $f(x)$ is a function defined for all $x \geqslant k$, $k \in \mathbb{Z}^+$, and $\{a_n\}$ is a sequence such that $a_n = f(n)$ when $n \geqslant k$. If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.

Also, in Section 13.1 we presented a set of five properties for limits of functions. All of these can be translated into properties for limits of sequences. We list here the set of five corresponding properties of limits of sequences and an additional important property on the limit of a rational power of a sequence.

The converse of the limit of a sequence theorem is not true. That is, a convergent sequence does not imply that the associated real variable function must also converge.

Properties of limits of sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$, and c is any real number, then:

1. Constant sequence: $\lim_{n\to\infty} c = c$
2. Scalar multiple of a sequence: $\lim_{n\to\infty} (c \cdot a_n) = cL$
3. Sum or difference of sequences: $\lim_{n\to\infty} (a_n \pm b_n) = L \pm K$
4. Product of sequences: $\lim_{n\to\infty} (a_n \cdot b_n) = LK$
5. Quotient of sequences: $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{L}{K}, \quad K \neq 0$
6. Rational power of a sequence: $\lim_{n\to\infty} (a_n)^p = L^p,\ p \in \mathbb{Q}$

These six properties of limits of sequences can be stated in words as follows:

1. The limit of a constant is equal to the constant.
2. The limit of a constant times a sequence is the constant times the limit of the sequence.

3. The limit of a sum/difference of sequences is the sum/difference of the limits of the sequences.
4. The limit of a product of sequences is the product of the limits of the sequences.
5. The limit of a quotient of sequences is the quotient of the limits of the sequences (given that the limit of the sequence in the denominator is not zero).
6. The limit of a rational power of a sequence is the rational power of the limit of the sequence.

In Chapter 13 of the book we reasoned informally that function values for functions in the form $f(x) = \frac{1}{x^k}$, where k is a rational number, approach zero as x goes to zero, i.e. $\lim\limits_{x\to\infty} \frac{1}{x^k} = 0, k \in \mathbb{Q}$. Thus, it makes sense that the result from Example 3, $\lim\limits_{n\to\infty} \left\{\frac{1}{n}\right\} = 0$, combined with property 6 for limits of sequences above, leads to the following intuitive rule for the limit of certain sequences.

If $r > 0,\ r \in \mathbb{Q}$, then $\lim\limits_{n\to\infty} \frac{1}{n^r} = 0$. **Note:** This rule is equivalent to $\lim\limits_{n\to\infty} n^r = 0$ if $r < 0$.

Example 3

Determine whether the sequence $\left\{\frac{3n^2+5n-1}{2n^2+1}\right\}$ is convergent or divergent.

Solution

In Example 4, part d) of Section 13.1, we found $\lim\limits_{x\to\infty} \frac{3x^2+5x-1}{2x^2+1}$ to be equal to $\frac{3}{2}$ as follows:

$$\lim_{x\to\infty} \frac{3x^2+5x-1}{2x^2+1} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2}+\frac{5x}{x^2}-\frac{1}{x^2}}{\frac{2x^2}{x^2}+\frac{1}{x^2}}$$

Dividing numerator and denominator by largest power of x, i.e. x^2.

$$= \lim_{x\to\infty} \frac{3+\frac{5}{x}-\frac{1}{x^2}}{2+\frac{1}{x^2}}$$

$$= \frac{\lim\limits_{x\to\infty} 3 + \lim\limits_{x\to\infty} \frac{5}{x} - \lim\limits_{x\to\infty} \frac{1}{x^2}}{\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty} \frac{1}{x^2}}$$

Applying $\lim\limits_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ and $\lim\limits_{x\to a}[f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$.

$$= \frac{3+0-0}{2+0}$$

Applying $\lim\limits_{x\to\infty} \frac{1}{x^k} = 0, k \in \mathbb{Q}$.

Hence, $\lim_{x\to\infty}\frac{3x^2+5x-1}{2x^2+1}=\frac{3}{2}$.

Therefore, from the limit of a sequence theorem above, we can conclude that the sequence $\left\{\frac{3n^2+5n-1}{2n^2+1}\right\}$ is convergent and it converges to $\frac{3}{2}$.

In our discussion of the end behaviour of rational functions in Section 3.4 of the book, the following limit results were hinted at. We state them here because by means of the limit of a sequence theorem they can also be applied in finding limits of sequences with rules that are rational functions, such as the sequence in Example 3.

Limits of rational functions

Let R be the rational function given by

$$R(x)=\frac{f(x)}{g(x)}=\frac{a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+\ldots+b_1x+b_0}$$

1. If $n<m$, then $\lim_{x\to\infty}R(x)=0$.
2. If $n=m$, then $\lim_{x\to\infty}R(x)=\frac{a_n}{b_m}$.
3. If $n>m$, then $\lim_{x\to\infty}R(x)=\infty$, i.e. does not exist.

Another useful limit theorem for functions that can be rewritten for sequences is the squeeze theorem from Section 13.2 where we used it to prove that $\lim_{x\to 0}\frac{\sin x}{x}=1$.

The squeeze theorem for sequences

If $a_n\leqslant b_n\leqslant c_n$ for all n such that $n\geqslant N$, $N\in\mathbb{Z}^+$, and $\lim_{n\to\infty}a_n=\lim_{n\to\infty}c_n=L$ then $\lim_{n\to\infty}b_n=L$.
See Figure 1.2 below.

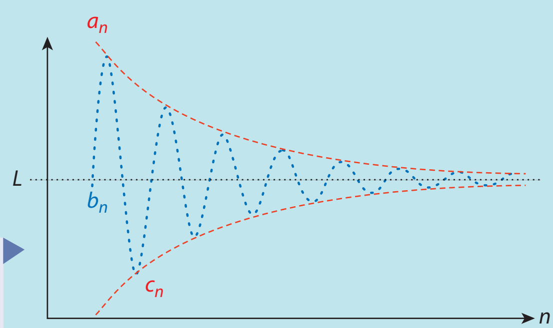


Figure 1.1 The sequences $\{a_n\}$ and $\{c_n\}$, both with limit of L, 'squeezing' the sequence $\{b_n\}$.

Note that the terms of sequence $\{b_n\}$ do not need to lie between $\{a_n\}$ and $\{c_n\}$ for all values of n. The requirement is that there must be some value of n for which all of the terms of $\{b_n\}$ beyond this value must lie between $\{a_n\}$ and $\{c_n\}$. This is illustrated in Example 5.

Example 4 – Applying the squeeze theorem

Show that each of the sequences converges, and find its limit.

a) $\left\{\frac{1}{2^n}\right\}$ b) $\left\{\frac{\cos n}{n}\right\}$

Solution

a) Because $2^n > 0$ and $2^n > n$ for all positive integers n, it follows that $0 \leqslant \frac{1}{2^n} \leqslant \frac{1}{n}$ for all integers $n \geqslant 1$. It is the case that $\lim\limits_{n\to\infty}\frac{1}{n} = 0$. Therefore, $\lim\limits_{n\to\infty}\frac{1}{2^n} = 0$ because $\lim\limits_{n\to\infty} 0 = \lim\limits_{n\to\infty}\frac{1}{n} = 0$ and the sequence $\left\{\frac{1}{2^n}\right\}$ converges to zero.

b) Because $-1 \leqslant \cos x \leqslant 1$ for all real numbers x, it follows that $\frac{-1}{n} \leqslant \frac{\cos n}{n} \leqslant \frac{1}{n}$ for all integers $n \geqslant 1$. Therefore, $\lim\limits_{n\to\infty}\frac{\cos n}{n} = 0$ because $\lim\limits_{n\to\infty}\left(-\frac{1}{n}\right) = \lim\limits_{n\to\infty}\frac{1}{n} = 0$ and the sequence $\left\{\frac{\cos n}{n}\right\}$ converges to zero.

Example 5 – Applying the squeeze theorem for an alternating sequence

Consider the infinite sequence $\left\{\frac{(-1)^n}{n!}\right\}$.

a) Write out the first six terms of the sequence.

b) Use the squeeze theorem to show that the sequence converges to 0.

Solution

a) The first six terms of the sequence are $-1, \frac{1}{2}, -\frac{1}{6}, \frac{1}{24}, -\frac{1}{120}, \frac{1}{720}$.
The sequence clearly alternates between positive and negative terms.

b) In order to apply the squeeze theorem, we need to find two convergent sequences that converge to 0 for which all terms for $n \geqslant N$ of the sequence $\left\{\frac{(-1)^n}{n!}\right\}$ will be between. Two sequences that will work in this case are $\left\{-\frac{1}{2^n}\right\}$ and $\left\{\frac{1}{2^n}\right\}$, both of which converge to 0.
The first six terms of these two sequences, respectively, are

$$-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, -\frac{1}{16}, -\frac{1}{32}, -\frac{1}{64} \text{ and } \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}.$$

Observe that for $n = 1, 2$ and 3, the terms of $\left\{\frac{(-1)^n}{n!}\right\}$ are *not* between $\left\{-\frac{1}{2^n}\right\}$ and $\left\{\frac{1}{2^n}\right\}$; however they are for $n \geqslant 4$. That is,

$$-\frac{1}{2^n} \leqslant \frac{(-1)^n}{n!} \leqslant \frac{1}{2^n}, n \geqslant 4.$$

Therefore, by the squeeze theorem it follows that the sequence $\left\{\frac{(-1)^n}{n!}\right\}$ converges to zero.

The squeeze theorem can also be used to prove that the sequence of absolute values for the sequence in Example 5, $\left\{\left|\frac{(-1)^n}{n!}\right|\right\} = \left\{\frac{1}{n!}\right\}$, also converges to 0 since the inequality $0 \leqslant \frac{1}{n!} \leqslant \frac{1}{2^n}$ is true for all $n \geqslant 4$. In fact, there is a very useful theorem that states that if the absolute value sequence converges to 0, then the original sequence consisting of positive and/or negative terms also converges to 0. It is often more efficient to consider the sequence of absolute values and then apply the following theorem to the original sequence.

Absolute value theorem

For the sequence $\{a_n\}$, if $\lim\limits_{n\to\infty}|a_n| = 0$ then $\lim\limits_{n\to\infty} a_n = 0$.

Proof of the absolute value theorem is fairly straightforward. Consider the two sequences $\{|a_n|\}$ and $\{-|a_n|\}$; one with all positive terms and one with all negative terms. Because both of these sequences converge to 0 and $-|a_n| \leqslant a_n \leqslant |a_n|$ we can conclude by means of the squeeze theorem that $\{a_n\}$ must also converge to 0.

The converse of the absolute value theorem is not true. That is, if $\lim\limits_{n\to\infty} a_n = 0$ it does not necessarily follow that $\lim\limits_{n\to\infty}|a_n| = 0$.

The sequence $\left\{\frac{1}{2^n}\right\}$, equivalent to $\left\{\left(\frac{1}{2}\right)^n\right\}$, in Example 4 part a) is a geometric sequence with a common ratio, r, equal to $\frac{1}{2}$. It was shown to converge to zero. For what values of r, other than $\frac{1}{2}$, is the geometric sequence $\{r^n\}$ convergent? Figure 1.2 shows the graphs of geometric sequences, $\{r^n\}$, for different positive values of r.

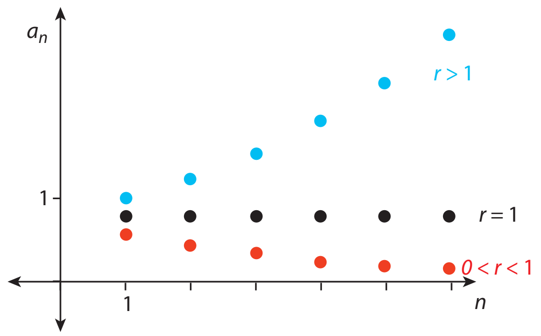


Figure 1.2 Graph of the sequence $\{r^n\}$ for different positive values of r.

When $r > 1$ the sequence $\{r^n\}$ increases without bound, i.e. for $r > 1$, $\lim\limits_{n\to\infty} r^n = \infty$. Visually it appears that for $0 < r < 1$, $\lim\limits_{n\to\infty} r^n = 0$. In Example 4, part a), we used the squeeze theorem to prove that $\lim\limits_{n\to\infty} r^n = 0$ when $r = \frac{1}{2}$. We can use a similar argument to show that $\lim\limits_{n\to\infty} r^n = 0$ for any value of r in the interval $0 < r < 1$.

Note that the graph of a real-valued function, e.g. $f(x) = 2^x$, $x \in \mathbb{R}$, is a continuous smooth curve; however, the graph of a sequence, e.g. $\{a_n\} = \{2^n\}$, $n \in \mathbb{Z}^+$, is discrete points because the domain consists of only positive integers.

Thus, we have $\lim\limits_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$

What about negative values of r? Figure 1.3 (below) shows the graphs of geometric sequences, $\{r^n\}$, for different negative values of r. There is no graph of $\{r^n\}$ for $r = -1$. In this case, the terms would oscillate infinitely between 1 and -1, and clearly the sequence does not converge to any number.

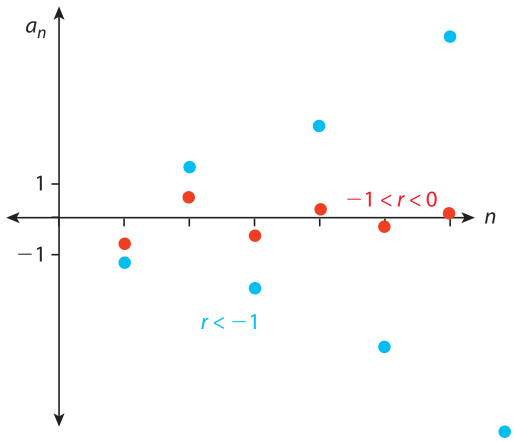


Figure 1.3 Graph of the sequence $\{r^n\}$ for different negative values of r.

Clearly, when $r < -1$ the sequence $\{r^n\}$ alternates between positive and negative values that increase without bound. Thus, for $r < -1$, $\lim\limits_{n\to\infty} r^n$ does not exist. Considering $-1 < r < 0$ we can also write the inequality as $0 < |r| < 1$. Additionally, $\lim\limits_{n\to\infty}|r^n| = \lim\limits_{n\to\infty}|r|^n$. Using the result above that $\lim\limits_{n\to\infty} r^n = 0$ for $0 < r < 1$, and since $0 < |r| < 1$, we can conclude that $\lim\limits_{n\to\infty}|r|^n = 0$. Therefore, by the absolute value theorem and the obvious fact that $\lim\limits_{n\to\infty} 0^n = 0$ it is true that $\lim\limits_{n\to\infty} r^n = 0$ for the interval $-1 < r < 1$, which is equivalent to $|r| < 1$. It is also obvious that $\lim\limits_{n\to\infty} 1^n = 1$. Thus the sequence $\{r^n\}$ is convergent for the interval $-1 < r \leqslant 1$ and divergent for other values of r. This result is summarized as follows.

Convergence of geometric sequences theorem

For $r \in \mathbb{R}$ and $n \in \mathbb{Z}^+$ the geometric sequence $\{r^n\}$ is convergent for $-1 < r \leqslant 1$ such that

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & r = 1 \end{cases}$$

Example 6 – The factorial function and exponential functions

Show that the sequence $\left\{\dfrac{x^n}{n!}\right\}$ converges to 0 for any real number x.

Solution

If $x < 0$, then the terms of the sequence will be alternately positive and negative. With the intention of applying the absolute value theorem, all that needs to be shown is that $\lim\limits_{n\to\infty}\dfrac{|x|^n}{n!} = 0$. This takes a bit of work. We start by choosing some positive integer N such that $N > |x|$. It follows that the sequence $\left\{\left(\dfrac{|x|}{N}\right)^n\right\}$ is geometric. Because $N > |x|$ then $\dfrac{|x|}{N} < 1$

and it must follow that $\lim_{n\to\infty}\left(\frac{|x|}{N}\right)^n = 0$. We now focus our attention on all of the values of n such that $n > N$. For these values of n, we can write the following:

$$\frac{|x|^n}{n!} = \frac{|x|^n}{1\times 2\times 3\ldots N\times \underbrace{(N+1)(N+2)\ldots n}_{(n-N)\text{ factors}}} \leqslant \frac{|x|^n}{N!N^{n-N}} = \frac{|x|^n N^N}{N!N^n} = \frac{N^N}{N!}\left(\frac{|x|}{N}\right)^n$$

Hence, $0 \leqslant \frac{|x|^n}{n!} \leqslant \frac{N^N}{N!}\left(\frac{|x|}{N}\right)^n$. The expression $\frac{N^N}{N!}$ is a constant and will not change as n changes. We know that $\lim_{n\to\infty}\left(\frac{|x|}{N}\right)^n = 0$, so applying the property $\lim_{n\to\infty}(c\cdot a_n) = c\lim_{n\to\infty} a_n$ we get $\lim_{n\to\infty}\frac{N^N}{N!}\left(\frac{|x|}{N}\right)^n = \frac{N^N}{N!}\lim_{n\to\infty}\left(\frac{|x|}{N}\right)^n = \frac{N^N}{N!}(0) = 0$.

Thus, $0 \leqslant \frac{|x|^n}{n!} \leqslant \frac{N^N}{N!}\left(\frac{|x|}{N}\right)^n = 0$ and we can conclude that $\lim_{n\to\infty}\frac{|x|^n}{n!} = 0$.

Therefore, by the absolute value theorem $\lim_{n\to\infty}\frac{x^n}{n!} = 0$, the sequence $\left\{\frac{x^n}{n!}\right\}$ converges to 0 for any real value of x.

Because we have shown that for any number x, $\lim_{n\to\infty}\frac{x^n}{n!} = 0$, we can conclude that the factorial function increases faster than any exponential function.

1.2 L'Hôpital's rule

We have one more important theorem to consider that is an essential tool for helping to determine the limits of certain functions, and consequently the limits of certain sequences.

With limits of rational functions in Chapter 13 of the book, we were sometimes confronted with an expression of indeterminate form, commonly in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. We handled these by performing some algebraic manipulations and applying limit theorems, as illustrated in Example 5 of Chapter 13. Not all limits can be managed in such a way. The following theorem specifically addresses limits of rational expressions that are of indeterminate form.

L'Hôpital's rule

Let f and g be functions whose derivative can be found at any value in an open interval $]a,b[$, except possibly at some value c where $a < c < b$. Assume that $g'(x) \neq 0$, except possibly at c. Suppose that $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$; or $\lim_{x\to c} f(x) = \pm\infty$ and $\lim_{x\to c} g(x) = \pm\infty$. (That is, the expression $\frac{f(x)}{g(x)}$ is in indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$.)

Then $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$ provided the limit on the right side exists (or is infinite).

L'Hôpital's rule first appeared in 1696 in a mathematical textbook entitled *L'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* (Analysis of the Infinitely Small for the Understanding of Curves). The textbook was written by the French nobleman and mathematician Guillaume de L'Hôpital (1661–1704) and is considered the first textbook on differential calculus.

Although the method for evaluating limits of indeterminate forms presented here is attributed to L'Hôpital, it was actually first developed by the Swiss mathematician Johann Bernoulli (1667–1748). In fact, most of the mathematics in L'Hôpital's groundbreaking textbook is widely considered to be the work of Johann Bernoulli. L'Hôpital did acknowledge Bernoulli's contributions in the preface to the textbook. Nevertheless, the name of L'Hôpital is forever associated with the rule.

When you are applying l'Hôpital's rule make sure that you differentiate the numerator and denominator *separately*. **Do *not* use the quotient rule for differentiation.**

L'Hôpital's rule states simply that, given the right conditions, the limit of a quotient of functions is equal to the limit of the quotient of their derivatives. It is important to first verify the conditions regarding the limits of f and g before applying l'Hôpital's rule.

Example 7 – Applying l'Hôpital's rule

For each limit, use your GDC to conjecture a result, and then find the limit using l'Hôpital's rule.

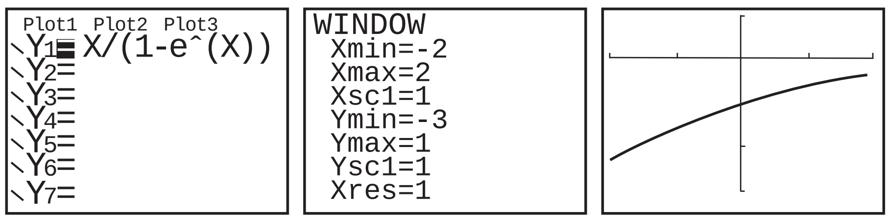
a) $\lim\limits_{x\to 0}\dfrac{x}{1-e^x}$

b) $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{\sec x}{1+\tan x}$

c) $\lim\limits_{x\to 0}\left(e^x+x\right)^{\frac{1}{x}}$

Solution

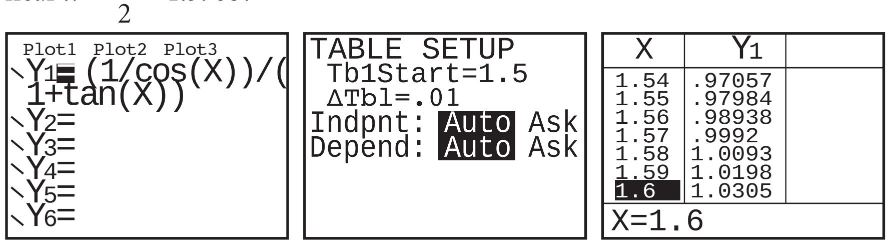
a) To visualize $\lim\limits_{x\to 0}\dfrac{x}{1-e^x}$ we graph $f(x)=\dfrac{x}{1-e^x}$ as shown in the GDC images below.



Although $x=0$ is not in the domain of f, the graph appears to pass through the point $(0,-1)$ implying that $\lim\limits_{x\to 0}\dfrac{x}{1-e^x}=-1$. Since $\lim\limits_{x\to 0}x=0$ and $\lim\limits_{x\to 0}\left(1-e^x\right)=0$, $\lim\limits_{x\to 0}\dfrac{x}{1-e^x}$ is in the indeterminate form $\dfrac{0}{0}$, and l'Hôpital's rule applies. Differentiating the numerator and denominator separately and evaluating the limit gives

$\lim\limits_{x\to 0}\dfrac{x}{1-e^x}=\lim\limits_{x\to 0}\dfrac{1}{-e^x}=\dfrac{1}{-1}=-1.$

b) Instead of viewing a graph of $f(x)=\dfrac{\sec x}{1+\tan x}$ to conjecture a value for $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{\sec x}{1+\tan x}$, let's use the GDC to construct a table of function values near $x=\dfrac{\pi}{2}\approx 1.5708$.



The values in the table show that the function appears to be approaching 1 from either direction.

The values of $\sec x$ vanish to $+\infty$ when $x \to \frac{\pi}{2}$ from the left $\left(\text{i.e. } x \to \frac{\pi}{2}^-\right)$ and vanish to $-\infty$ when $x \to \frac{\pi}{2}^+$.

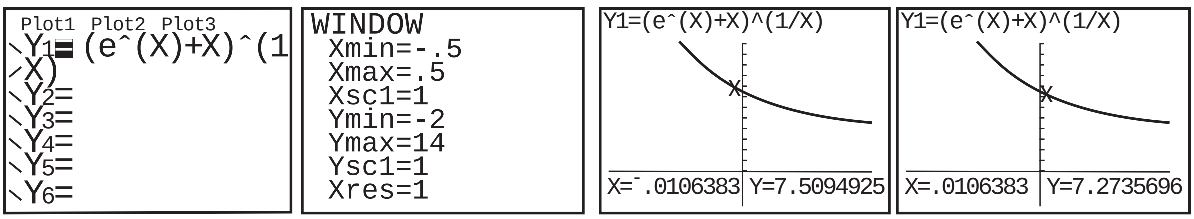
Similarly, $\lim_{x \to \frac{\pi}{2}^-} (1 + \tan x) = +\infty$ and $\lim_{x \to \frac{\pi}{2}^+} (1 + \tan x) = -\infty$. So when approaching $\frac{\pi}{2}$ from the left we have $\frac{+\infty}{+\infty}$, and $\frac{-\infty}{-\infty}$ when approaching from the right. L'Hôpital's rule also applies to one-sided limits. Applying the rule to the right-hand limit gives

$$\lim_{x \to \frac{\pi}{2}^+} \frac{\sec x}{1 + \tan x} = \lim_{x \to \frac{\pi}{2}^+} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \to \frac{\pi}{2}^+} \sin x = 1.$$

$\left(\textbf{Note:} \ \frac{\sec x \tan x}{\sec^2 x} \text{ simplifies to } \sin x.\right)$

The left-hand limit is also 1; therefore the two-sided limit is equal to 1, i.e. $\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{1 + \tan x} = 1$.

c) To visualize $\lim_{x \to 0} \left(e^x + x\right)^{1/x}$ we graph $f(x) = \left(e^x + x\right)^{1/x}$ as shown in the GDC images below.



Tracing on the graph indicates that as $x \to 0$ the function approaches a value between 7.2735 and 7.5094. The exact value of the limit is not clear.

We observe that $\lim_{x \to 0} \left(e^x + x\right)^{1/x}$ is in the indeterminate form 1^∞. However, by taking the logarithm of both sides of $f(x) = \left(e^x + x\right)^{1/x}$ and then taking the limit we can change the indeterminate form to $0/0$, to which we can apply l'Hôpital's rule.

$$\ln[f(x)] = \ln\left[\left(e^x + x\right)^{1/x}\right] = \frac{1}{x}\ln\left(e^x + x\right) = \frac{\ln\left(e^x + x\right)}{x}$$

Thus, $\ln[f(x)] = \frac{\ln\left(e^x + x\right)}{x}$, and taking the limit as $x \to 0$ of both sides produces $\lim_{x \to 0} \ln[f(x)] = \lim_{x \to 0} \frac{\ln\left(e^x + x\right)}{x}$

Right side in the form 0/0; apply l'Hôpital's rule.

$$= \lim_{x \to 0} \frac{\frac{e^x + 1}{e^x + x}}{1} = \frac{e^0 + 1}{e^0 + 0} = 2.$$

Hence, $\lim_{x \to 0} \ln[f(x)] = 2$.

Since $f(x) = (e^x + x)^{1/x}$, then $\lim\limits_{x\to 0}(e^x + x)^{1/x} = \lim\limits_{x\to 0} f(x)$

$= \lim\limits_{x\to 0} e^{\ln f(x)}$ Applying the rule $e^{\ln a} = a$.

$= \lim\limits_{x\to 0} e^2$. Using result $\lim\limits_{x\to 0} \ln[f(x)] = 2$.

Therefore, $\lim\limits_{x\to 0}(e^x + x)^{1/x} = e^2$.

$e^2 \approx 7.389$ (to 4 s.f.), so the limit is within the range estimated from the graph on the GDC.

If, after applying L'Hôpital's rule, the quotient of the derivatives remains in indeterminate form, the rule can be applied more than once.

L'Hôpital's rule should not be applied if the limit is not in indeterminate form. For example, consider the following limit: $\lim\limits_{x\to 0}\dfrac{\sin x}{x+1}$. The limit is *not* indeterminate, because $\dfrac{\sin(0)}{0+1} = \dfrac{0}{1}$. Hence, the application of L'Hôpital's rule produces an incorrect result. L'Hôpital's rule gives the following result: $\lim\limits_{x\to 0}\dfrac{\sin x}{x+1} = \lim\limits_{x\to 0}\dfrac{\cos x}{1} = \dfrac{\cos(0)}{1} = \dfrac{1}{1} = 1$. The correct result can be obtained simply from direct substitution: $\lim\limits_{x\to 0}\dfrac{\sin x}{x+1} = \dfrac{\sin(0)}{0+1} = \dfrac{0}{1} = 0$.

Example 8 – Repeated use of L'Hôpital's rule

Find $\lim\limits_{x\to 1}\dfrac{1 - x + \ln x}{x^3 - 3x + 2}$.

Solution

Substituting $x = 1$ into the rational expression gives $\dfrac{1 - 1 + \ln 1}{1 - 3\cdot 1 + 2} = \dfrac{0}{0}$. Thus the limit is in the indeterminate form $\dfrac{0}{0}$ and L'Hôpital's rule is applied:

$$\lim_{x\to 1}\frac{1-x+\ln x}{x^3-3x+2} = \lim_{x\to 1}\frac{-1+\frac{1}{x}}{3x^2-3}$$

Substituting $x = 1$ again gives the indeterminate form $\dfrac{0}{0}$, so L'Hôpital's rule is applied a second time, producing an expression that can be evaluated for $x = 1$:

$$\lim_{x\to 1}\frac{1-x+\ln x}{x^3-3x+2} = \lim_{x\to 1}\frac{-1+\frac{1}{x}}{3x^2-3} = \lim_{x\to 1}\frac{-\frac{1}{x^2}}{6x} = -\frac{1}{6}$$

Example 9 – Using l'Hôpital's rule to determine convergence of a sequence

Determine if the sequence $\{a_n\} = \left\{\dfrac{n^2+1}{3^n}\right\}$ converges. If it does, find its limit.

Solution

Consider the function $f(x) = \dfrac{x^2+1}{3^x}$, $x \in \mathbb{R}$, and its limit as $x \to \infty$.

Since $\lim_{x\to\infty} \frac{x^2+1}{3^x}$ is in indeterminate form of ∞/∞, we can apply l'Hôpital's rule.

$$\lim_{x\to\infty} \frac{x^2+1}{3^x} = \lim_{x\to\infty} \frac{2x}{(\ln 3)3^x}$$

But this limit is still in indeterminate form of ∞/∞, so we apply l'Hôpital's rule a second time.

$$\lim_{x\to\infty} \frac{x^2+1}{3^x} = \lim_{x\to\infty} \frac{2x}{(\ln 3)3^x} = \lim_{x\to\infty} \frac{2}{(\ln 3)^2 3^x} = 0$$

Because the value of a_n matches the value of $f(x)$ for every positive integer, we can apply the limit of a sequence theorem and conclude that $\lim_{n\to\infty} \frac{n^2+1}{3^n} = 0.$

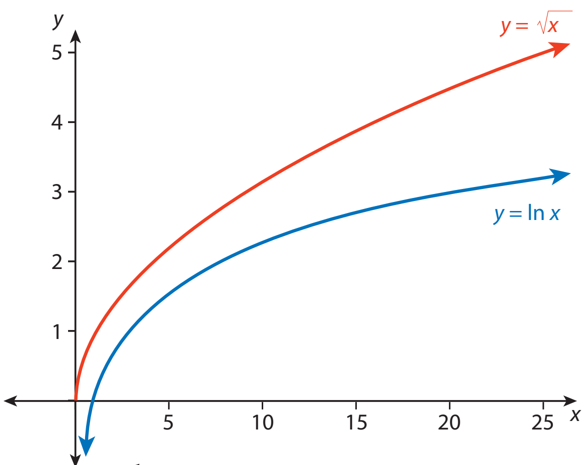
Therefore, the sequence $\left\{\frac{n^2+1}{3^n}\right\}$ converges to 0.

Example 10

Which sequence grows faster, $\{\ln n\}$ or $\{\sqrt{n}\}$?

Solution

We can gain some insight into this question by graphing the real-valued functions $y = \ln x$ and $y = \sqrt{x}$. The graph below implies that the sequence $\{\sqrt{n}\}$ grows faster than $\{\ln n\}$; that is, the infinite sequence $\left\{\frac{\ln n}{\sqrt{n}}\right\}$ converges to 0. Using l'Hôpital's rule to show that the limit of the function $f(x) = \frac{\ln x}{\sqrt{x}}$ is 0 as $x \to \infty$ will prove this result.



$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0 \text{ because } \lim_{x\to\infty} \frac{1}{x^k} = 0,\ k \in \mathbb{Q}^+$$

Therefore, the sequence $\left\{\frac{\ln n}{\sqrt{n}}\right\}$ converges to 0, and we can conclude that $\left\{\sqrt{n}\right\}$ grows faster than $\{\ln n\}$.

1.3 Improper integrals

Previously we have defined the definite integral, $\int_a^b f(x)\,dx$, for a function f that is continuous (i.e. no 'gaps' in the domain) for the finite, bounded, interval $a \leqslant x \leqslant b$. In this section, we will look at ways of evaluating integrals where either one or both of the limits of integration (i.e. a and b) are infinite, or the function f has an infinite discontinuity in the interval $a \leqslant x \leqslant b$. An integral having either one of these characteristics is called an **improper integral**.

Let's look at an integral where one of the limits is infinite.

Example 11

Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$ or show that it diverges.

Solution

We can replace the infinite limit of integration with a variable, say the variable b, and then take the limit of the integral as b approaches infinity.

Taking the limit as $b \to \infty$ gives $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(\int_1^b \frac{1}{x^2}\,dx\right) = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b$

$= \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 0+1 = 1.$

Therefore, $\int_1^\infty \frac{1}{x^2}\,dx = 1.$

This result can be interpreted as indicating that the area under the curve $y = \frac{1}{x^2}$ from one to infinity is finite and is exactly equal to 1 (see Figure 1.4 below).

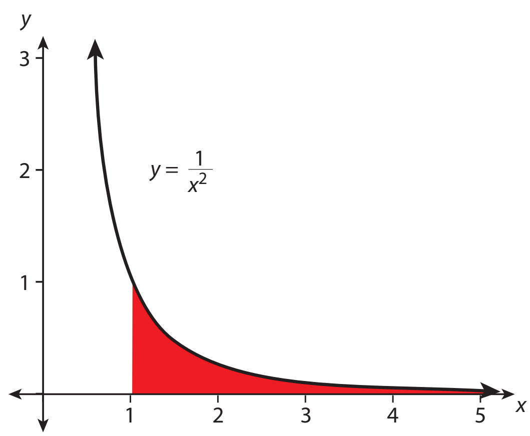


Figure 1.4 Area under the curve $y = \frac{1}{x^2}$ from 1 to ∞.

Certainly, not all improper integrals converge to a finite value.

Example 12

Evaluate $\int_1^{\infty} \frac{1}{x}\,dx$ or show that it diverges.

Solution

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty}\left(\int_1^{b} \frac{1}{x}\,dx\right) = \lim_{b\to\infty}\left[\ln x\right]_1^b = \lim_{b\to\infty}\left(\ln b - \ln 1\right) = \lim_{b\to\infty}\left(\ln b\right) = \infty$$

[or 'limit does not exist']

Therefore, the integral diverges. The area under the curve $y = \frac{1}{x}$ from 1 to infinity is infinite.

$\int \frac{1}{x}\,dx = \ln|x|$, but note that in Example 12 the absolute value is omitted because the integral is being evaluated from 1 to ∞, i.e. only positive numbers.

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists (as a finite number as in Example 11), and is called **divergent** if the limit does not exist (as in Example 12).

Example 13 – Using l'Hôpital's rule to evaluate an improper integral

Determine whether the integral $\int_1^{\infty} \frac{x}{e^x}\,dx$ converges or diverges; and if it converges, find its value.

Solution

We can rewrite the integral as a limit, $\int_1^{\infty} \frac{x}{e^x}\,dx = \lim_{b\to\infty}\int_1^{b} \frac{x}{e^x}\,dx$;

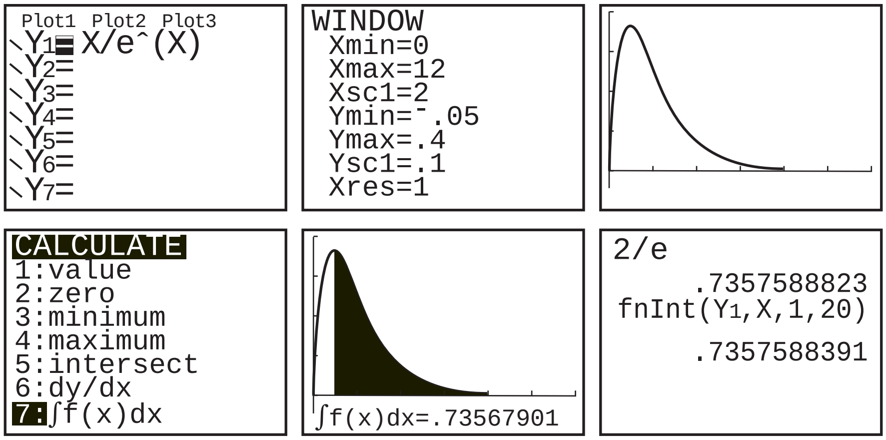
and now need to apply integration by parts to evaluate the integral.

Let $\begin{array}{ll} u = x & dv = e^{-x}dx \\ du = dx & v = -e^{-x} \end{array}$; then $\lim_{b\to\infty}\int_1^{b} \frac{x}{e^x}\,dx = \lim_{b\to\infty}\left[\left[-xe^{-x}\right]_1^b + \int_1^b e^{-x}\,dx\right]$

$$\begin{aligned}
&= \lim_{b\to\infty}\left[-xe^{-x} - e^{-x}\right]_1^b \\
&= \lim_{b\to\infty}\left[-(x+1)e^{-x}\right]_1^b \\
&= \lim_{b\to\infty}\left(\frac{-(b+1)}{e^b} + \frac{2}{e}\right) \\
&= \lim_{b\to\infty}\left(\frac{-(b+1)}{e^b}\right) + \frac{2}{e} \\
&= 0 + \frac{2}{e}
\end{aligned}$$

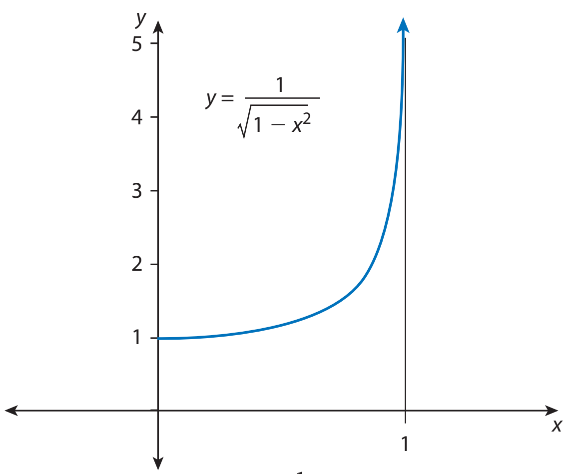
Therefore, $\int_1^{\infty} \frac{x}{e^x}\,dx = \frac{2}{e} \approx 0.735\,758\,8823$ (to ten significant figures).

The GDC images on the next page confirm our result. Note that even with an upper limit of just $x = 12$ the definite integral (computed on graph screen) agrees to three decimal places with the value of the 'improper' integral with an infinite upper limit; and when the upper limit is 20 (computed on home screen) the values agree to six decimal places. The integral converges at a fairly quick rate.



What is an infinite discontinuity? A function f has an infinite discontinuity at $x = c$ if either $\lim_{x \to c} f(x) = \infty$ or $\lim_{x \to c} f(x) = -\infty$ such that $x \to c$ from the right or left. For example, on the interval $0 \leqslant x \leqslant 1$, the function $f(x) = \dfrac{1}{\sqrt{1-x^2}}$ has an infinite discontinuity at $x = 1$ because $\lim_{x \to 1^-} \dfrac{1}{\sqrt{1-x^2}} = \infty$ (note: $x \to 1$ from the left) which can be observed in the graph in Figure 1.5.

Figure 1.5



The region under the curve $y = \dfrac{1}{\sqrt{1-x^2}}$ in the interval $0 \leqslant x \leqslant 1$ is unbounded – and would, at first thought, have an infinite area. However, the unbounded region has a finite area and we can find the exact area as follows.

Example 14

Find the area, if possible (not possible if it's infinite), under the curve $y = \dfrac{1}{\sqrt{1-x^2}}$ in the interval $0 \leqslant x \leqslant 1$.

Solution

We can replace the limit of integration where the infinite discontinuity occurs with a variable, say the variable b, and then take the limit of the integral as b approaches the value of x where the discontinuity occurs (approaching 1 from the left, in this case).

(Recall that the anti-derivative of $\dfrac{1}{\sqrt{1-x^2}}$ is $\arcsin x$.)

$$\text{Area} = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1^-}\left(\int_0^b \frac{1}{\sqrt{1-x^2}}\,dx\right) = \lim_{b\to 1^-}\left[\arcsin x\right]_0^b = \lim_{b\to 1^-}\left(\arcsin b - \arcsin 0\right)$$

$$= \lim_{b\to 1^-}\left(\arcsin b - 0\right) = \lim_{b\to 1^-}\left(\arcsin b\right) = \arcsin(1) = \frac{\pi}{2}$$

Therefore, the unbounded region under the curve $y = \dfrac{1}{\sqrt{1-x^2}}$ in the interval $0 \leqslant x \leqslant 1$ has a finite area of exactly $\dfrac{\pi}{2}$.

Exercise 1

For questions 1–15, determine if the sequence converges or diverges. If it converges, find the limit of the sequence.

1 $\left\{\dfrac{7}{\sqrt[3]{n}}\right\}$ **2** $\left\{\dfrac{2n^2+n+1}{n^2+1}\right\}$ **3** $\left\{\dfrac{5n-13}{n^3+5n}\right\}$

4 $\{\cos n\pi\}$ **5** $\left\{\dfrac{(-1)^{n+1}}{2n-1}\right\}$ **6** $\left\{\left(-\dfrac{4}{5}\right)^n\right\}$

7 $\left\{\dfrac{e^n}{n^2}\right\}$ **8** $\left\{\dfrac{3\sqrt{n^2+1}}{4\sqrt[3]{n^2}-1}\right\}$ **9** $\left\{\sqrt{\dfrac{2n}{n+1}}\right\}$

10 $\left\{1+\dfrac{(-1)^n}{n}\right\}$ **11** $\left\{\dfrac{n}{1+\sqrt{n}}\right\}$ **12** $\left\{\left(1+\dfrac{2}{n}\right)^{\frac{1}{n}}\right\}$

13 $\left\{\dfrac{3^n}{n!}\right\}$ **14** $\left\{\dfrac{\ln 2n}{\ln n}\right\}$ **15** $\left\{\left(\dfrac{|n|+1}{n}\right)\right\}$

16 Use the squeeze theorem to show that $\lim\limits_{n\to\infty}\dfrac{\sin 2n}{\sqrt{n}} = 0$.

17 Use the fact that $\lim\limits_{n\to\infty}\dfrac{1}{n^2} = 0$ to prove that $\lim\limits_{n\to\infty}\dfrac{n^2}{n^2+1} = 1$.

For questions 18–20, use l'Hôpital's rule to find the value of each limit.

18 $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$ **19** $\lim\limits_{x\to 1}\dfrac{x-1}{\sqrt{x^2+3}-2}$ **20** $\lim\limits_{x\to 1}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)$

21 Determine whether the sequence $\left\{n\sin\dfrac{\pi}{n}\right\}$ is convergent or divergent. If convergent, find its limit. (Hint: Rewrite $n\sin\dfrac{\pi}{n}$ as $\dfrac{\sin\frac{\pi}{n}}{\frac{1}{n}}$.)

In questions 22–27, evaluate the limit.

22 $\lim\limits_{x\to 1}\dfrac{x^2-1}{x^2-4x-3}$ **23** $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1-x}-1}{x}$

24 $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}$ **25** $\lim\limits_{x\to 0}\dfrac{\frac{1}{x}-\cot x}{x}$

26 $\lim\limits_{x\to\infty}\dfrac{\ln(x+1)}{\log_2 x}$ **27** $\lim\limits_{x\to 0}\dfrac{a^x-b^x}{x},\quad a>0, b>0$

28 Given $f(x) = (1 + x)^{1/x}$, find $\lim_{x\to\infty} f(x)$. (Hint: Start by taking the natural logarithm of both sides, converting the right side to the indeterminate form $\frac{0}{0}$. Then you can use l'Hôpital's rule.)

In questions 29–36, evaluate, or identify as divergent, the given integral.

29 $\int_0^1 \frac{1}{x^3}\,dx$

30 $\int_1^\infty \frac{1}{x^3}\,dx$

31 $\int_{-\infty}^\infty \frac{1}{x^2+1}\,dx$

32 $\int_0^\infty \frac{\sin x}{e^x}\,dx$

33 $\int_0^{\pi/2} \tan x\,dx$

34 $\int_0^\infty \frac{1}{1+e^x}\,dx$

35 $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$

36 $\int_0^k \frac{x}{\sqrt{k^2-x^2}}\,dx$

37 Consider the *unbounded* region lying between the graph of $y = \frac{1}{x}$ and the x-axis for $x \geqslant 1$.

a Find the area of this region, if possible.

b Find the volume, if possible, of the solid generated by rotating this unbounded region about the x-axis.

c Comment on your results for **a** and **b**.

Practice questions 1

1 Show that $\lim_{x\to 1} \frac{x^2-1}{\sqrt{1+x^2}-\sqrt{2}} = 2\sqrt{2}$.

2 Use L'Hôpital's rule to find $\lim_{x\to 0} \frac{x\cos x - e^x + 1}{\cos^2 x}$.

3 Determine whether the integral $\int_{-1}^0 \frac{e^x}{e^x-1}\,dx$ converges or diverges. If it converges, find its value.

4 Find the following.

a $\lim_{x\to 1} \frac{1-x^3}{2-\sqrt{x^2+3}}$

b $\lim_{x\to a} \frac{x-a}{x^3-a^3}$

5 Find the set of values of p for which the improper integral $\int_e^\infty \frac{\ln x}{x^p}\,dx$ converges.

6 Calculate each of the following limits.

a $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$

b $\lim_{x\to 0}\left(\frac{\arctan x}{x}\right)$

7 Show that $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = 2\sqrt{3}$.

8 Calculate each of the following one-sided limits.

a $\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{e^x-1}\right)$

b $\lim_{x\to \frac{\pi}{2}^-} (\tan x - \sec x)$

9 **a** **i** Find $I_n = \int_{-n}^{\alpha n} \frac{x}{1+x^2}\,dx$ where α is a positive constant and n is a positive integer.

ii Determine $\lim_{n\to\infty} I_n$.

b Using L'Hôpital's rule to find $\lim_{x\to 0}\left(\frac{\tan \beta x - \beta \tan x}{\sin \beta x - \beta \sin x}\right)$ where β is a non-zero constant and $\beta \neq \pm 1$.

10 Show that $\lim_{x\to 1^+}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \frac{1}{2}$.

11 Giving a reason, state whether the following argument is correct or incorrect.

Using L'Hôpital's rule, $\lim_{x\to\pi^-}\left(\frac{\sin x}{1-\cos x}\right) = \lim_{x\to\pi^-}\left(\frac{\cos x}{\sin x}\right) = -\infty$.

12 For what values of k do the following converge?

a $\int_0^1 x^k\,dx$ **b** $\int_1^\infty x^k\,dx$

13 Find $\lim_{x\to 0}\left(\frac{\ln\left(a^2+x^2\right)}{\ln\left(a-x^3\right)}\right)$, where a is a positive constant, not equal to 1.

14 Show that $\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$.

15 Find the value of each limit.

a $\lim_{x\to 0}\left(\frac{2+x^2-2\cos x}{e^x+e^{-x}-2\cos x}\right)$

b $\lim_{x\to 0}\left(\frac{e^x-1-x-\frac{1}{2}x^2}{x^3}\right)$

2 Series and Convergence

2.1 Infinite series

To start our study of infinite series in the option topic we consider using the terms of a sequence $\{a_n\}$ to form the sequence $\{s_n\}$ of **partial sums** of $\{a_n\}$ as follows:

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

We can use sigma notation to write the general expression for s_n:

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i$$

Here we have used the letter i as a subscript to indicate the ith term of a sequence; and have used the letter n as a subscript to indicate the nth partial sum. You need to be comfortable with using different letters for subscripts.

Definition of the sum of an infinite series

If the sequence of partial sums $\{s_n\} = \left\{\sum_{i=1}^{n} a_i\right\}$ converges, we say that its limit S is the sum of the infinite series $a_1 + a_2 + a_3 + \ldots$ and we write $S = \sum_{i=1}^{\infty} a_i$. If the sequence $\{s_n\}$ diverges then we say that the infinite series $\sum_{i=1}^{\infty} a_i$ also diverges.

As pointed out in Section 4.4 (of the textbook) in our discussion on infinite geometric series, the word 'sum' here is being used in a completely different way from how it is normally used. Ordinary addition of real numbers is a finite process; hence, it does not make sense to find the 'sum' of infinitely many terms. To be more precise, the 'sum' of an infinite series is a limit – that is, the limit of the partial sums for the series. We can write the sum as $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ but we must be careful not to assume that the '+' signs have the same properties to which we are accustomed. For example, as we will see, a rearrangement of the terms of a convergent series may change the value of its sum or even cause the series to diverge.

Example 1

a) Find the sum of the *finite* series $\sum_{n=1}^{6} (-1)^{n+1}$.

b) Consider the *infinite* series $\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - 1 + \ldots$.
Determine if the series converges to a sum or diverges.

Solution

a) Clearly, $\sum_{n=1}^{6}(-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - 1 = 0.$

We can make the further observation that if the number of terms in this finite series was *any* even number, not just six, the sum is always 0; and, if the number of terms is odd the sum is always 1. In either case, we can 'pair up' consecutive terms to get zero. For example,

$$\sum_{n=1}^{6}(-1)^{n+1} = (1-1)+(1-1)+(1-1) = 0+0+0 = 0, \text{ or}$$

$$\sum_{n=1}^{7}(-1)^{n+1} = (1-1)+(1-1)+(1-1)+1 = 0+0+0+1 = 1.$$

b) It is very tempting to use the same strategy of 'pairing up' consecutive terms in this manner

$$\sum_{n=1}^{\infty}(-1)^{n+1} = (1-1)+(1-1)+(1-1)+\ldots$$

to argue that the sum of this infinite series is 0. However, this is erroneous. Consider that if we leave out the first term and start 'pairing up' from the second term we will obtain a different sum. The associative property of addition is what allowed us to 'pair up' the numbers for the finite sum in part a). Although the associative property works for finite sums it is clear that it does *not* work for infinite sums. The sum of an infinite series is defined to be the limit of the sequence of partial sums.

For the sequence $\{s_n\} = \left\{\sum_{i=1}^{n}(-1)^{i+1}\right\}$, we have $s_1 = 1$, $s_2 = 0$, $s_3 = 1$, $s_4 = 0$, etc. Clearly this sequence is not converging to a limit. Therefore, the series has no sum and it diverges.

In studying infinite series, there are commonly two basic questions: Does a particular series converge or does it diverge? If it does converge, what is its sum?

Geometric series

There is one type of infinite series with which we are already familiar – and for which we know how to answer questions regarding convergence/divergence and computing sums; and this is **infinite geometric series** that we encountered in Chapter 4 of the textbook.

If a_1 represents the first term and r is the number that multiplies a term to obtain the next term in the series, then an infinite geometric series can be generalized as follows:

$$a_1 + a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^{n-1} + \ldots = \sum_{n=1}^{\infty}a_1r^{n-1},\ a_1 \neq 0$$

Let's consider three cases: $r = 1$, $r = -1$, and $r \neq \pm 1$.

If $r = 1$, then the nth partial sum is $s_n = a_1 + a_1 + a_1 + \ldots + a_1 = na_1$. Clearly the sequence of partial sums, $\{s_n\}$, will increase without bound and the geometric series diverges in this case.

If $r = -1$, then the nth partial sum is $s_n = a_1 - a_1 + a_1 - a_1 \ldots$. The sequence of partial sums, $\{s_n\}$, will behave in the same way as in Example 1 b) with $s_1 = a_1$, $s_2 = 0$, $s_3 = a_1$, $s_4 = 0$. The sequence of partial sums is not converging to a limit, so the geometric series also diverges for this case.

If $r \neq \pm 1$, then $s_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-1}$.

Multiplying through by r gives

$$rs_n = a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1} + a_1 r^n.$$

Subtracting the second equation from the first produces

$$s_n - rs_n = a_1 - a_1 r^n.$$

Factorizing yields $s_n(1-r) = a_1(1-r^n)$.

Thus, the nth partial sum is

$$s_n = \frac{a_1(1-r^n)}{1-r}.$$

We know from the theorem for convergence of geometric sequences in the previous section that if $|r| < 1$ then r^n converges to 0 as $n \to \infty$. We can apply this fact and some properties of limits to give the following result:

$$\lim_{n\to\infty} \frac{a_1(1-r^n)}{1-r} = \lim_{n\to\infty} \frac{a_1 - a_1 r^n}{1-r} = \lim_{n\to\infty} \frac{a_1}{1-r} - \lim_{n\to\infty} \frac{a_1 r^n}{1-r}$$
$$= \frac{a_1}{1-r} - \left(\frac{a_1}{1-r}\right)\lim_{n\to\infty} r^n = \frac{a_1}{1-r} - 0 = \frac{a_1}{1-r}$$

Therefore, if $|r| < 1$ then $\lim_{n\to\infty} s_n = \dfrac{a_1}{1-r}$.

This rigorously confirms a result that appeared in Chapter 4, and we state

Convergence of geometric series

The geometric series with common ratio r

$$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1}$$

converges to the sum $\dfrac{a_1}{1-r}$ if $|r| < 1$, and diverges if $|r| \geqslant 1$.

it again here. In this chapter, when we refer to a geometric series, it can be assumed that it is an infinite geometric series.

This result answers the two basic questions about geometric series. By identifying the value of the common ratio, r, we can determine which geometric series converge and which ones diverge; and for ones that converge we can easily compute the sum with the formula $S_\infty = \dfrac{a_1}{1-r}$.

For any geometric series, the interval $|r| < 1$, which can also be written as $-1 < r < 1$, is known as its **interval of convergence.**

It is essential to understand that for any series $\sum a_n$ there are two important sequences for us to consider: the sequence $\{s_n\}$ of its partial sums and the sequence $\{a_n\}$ of its terms.

Example 2

For each of the series, $\sum_{n=1}^{\infty} a_n$, below

(i) write the first four terms and find the limit (if it exists) of the sequence of its terms, $\lim_{n\to\infty} a_n$; and

(ii) write the first four terms of the sequence of its partial sums $\{s_n\}$ and find its limit (if it exists), i.e. the sum of the series.

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$ b) $\sum_{n=1}^{\infty} 2^{2n}5^{1-n}$

Solution

a) (i) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n} = -\frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \ldots$

The sequence of terms in the series is $\{a_n\} = \left\{\frac{(-1)^n}{3^n}\right\}$. This is a geometric sequence with $r = -\frac{1}{3}$ and because $-1 < -\frac{1}{3} < 1$ then it follows that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{3^n} = 0$.

(ii) The sequence of partial sums begins as follows:

$$s_1 = -\frac{1}{3}$$

$$s_2 = -\frac{1}{3} + \frac{1}{9} = -\frac{2}{9}$$

$$s_3 = -\frac{1}{3} + \frac{1}{9} - \frac{1}{27} = -\frac{7}{27}$$

$$s_4 = -\frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} = -\frac{20}{81}$$

Because the series is geometric such that $-1 < r < 1$, then the series converges to

$$\lim_{n\to\infty} s_n = \frac{a_1}{1-r} = \frac{-\frac{1}{3}}{1-\left(-\frac{1}{3}\right)} = \frac{-\frac{1}{3}}{\frac{4}{3}} = -\frac{1}{4}.$$

Therefore, the sum of the series is $-\frac{1}{4}$.

b) (i) $\sum_{n=1}^{\infty} 2^{2n}5^{1-n} = 4 + \frac{16}{5} + \frac{64}{25} + \frac{256}{125} + \ldots$

The series appears to be geometric with $r = \frac{4}{5}$. We can confirm this by simplifying the rule for the nth term:

$$2^{2n}5^{1-n} = \frac{(2^2)^n}{5^{n-1}} = \frac{4^1 4^{n-1}}{5^{n-1}} = 4\left(\frac{4}{5}\right)^{n-1}.$$

Hence, $\sum_{n=1}^{\infty} 2^{2n}5^{1-n} = \sum_{n=1}^{\infty} 4\left(\frac{4}{5}\right)^{n-1}$ and its clear that the series is geometric with $a_1 = 4$ and $r = \frac{4}{5}$. Because $-1 < r < 1$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(4\left(\frac{4}{5}\right)^{n-1}\right) = 4\lim_{n\to\infty}\left(\frac{4}{5}\right)^{n-1} = 4\cdot 0 = 0.$$

(ii) The sequence of partial sums begins as follows:

$$s_1 = 4$$

$$s_2 = 4 + \frac{16}{5} = \frac{36}{5} = 7.2$$

$$s_3 = 4 + \frac{16}{5} + \frac{64}{25} = \frac{244}{25} = 9.76$$

$$s_4 = 4 + \frac{16}{5} + \frac{64}{25} + \frac{256}{125} = \frac{1476}{125} = 11.808$$

Because the series is geometric such that $-1 < r < 1$, then the series converges to

$$\lim_{n\to\infty} s_n = \frac{a_1}{1-r} = \frac{4}{1-\left(\frac{4}{5}\right)} = \frac{4}{\frac{1}{5}} = 20.$$

Therefore, the sum of the series is 20.

It is obvious that any series whose sequence of terms does not converge to zero, i.e. $\lim_{n\to\infty} a_n \neq 0$, will have a sequence of partial sums that diverges. In such a case, the magnitude (positive or negative) of terms will increase, causing the sequence of partial sums to increase without bound. We established that both series in Example 2 are convergent and also that $\lim_{n\to\infty} a_n = 0$ for both series. It seems reasonable to conjecture that a necessary *and* sufficient condition for an infinite series $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ to converge to a finite quantity is that the sequence, $\{a_n\}$, of individual terms a_n converges to zero. Is it possible for the sequence of terms of a series to converge to zero but the series itself does not converge, i.e. does not have a sum?

Example 3

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$. Determine whether the series converges or diverges.

Solution

Clearly, the sequence of terms converges to zero, i.e. $\lim_{n\to\infty} \frac{1}{n} = 0$. To answer the question about convergence of the series we need to look at the sequence of partial sums. Our analysis begins by bracketing the terms in the following way:

$$s_n = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \ldots + \frac{1}{8}\right) + \left(\frac{1}{9} + \ldots + \frac{1}{16}\right) + \ldots + \left(\frac{1}{2^{n-1}+1} + \ldots + \frac{1}{2^n}\right) + \ldots$$

so that the final term in each bracketed group is the reciprocal of a power of two. Let's consider the sum of the first 2^n terms,

$$s_{2^n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} \ldots + \frac{1}{16}\right) + \ldots + \left(\frac{1}{2^{n-1}+1} + \ldots + \frac{1}{2^n}\right)$$

$$\geqslant 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right) + \ldots + \left(\frac{1}{2^{n-1}+1} + \ldots + \frac{1}{2^n}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2}$$

$$= 1 + n\left(\frac{1}{2}\right) = \frac{n+2}{2} \quad \Rightarrow \quad s_{2^n} \geqslant \frac{n+2}{2}$$

Clearly the sequence of these partial sums diverges, so s_{2^n} diverges.

Hence, the series $s_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is greater than a series that diverges, so it must also diverge.

Therefore, even though the sequence $\frac{1}{n} \to 0$ as $n \to \infty$, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. This series is called the **harmonic series** – and we will encounter it often.

The clever method used in Example 3 is attributed to a French scholar, Nicole Oresme (1323–1382), who was the first to mathematically prove that the harmonic series diverges. Considering the state of mathematics in the 14th century, Oresme was well ahead of his time by inventing a type of coordinate geometry and using the idea of a fractional exponent – three centuries before Descartes developed coordinate geometry and Newton first invented our modern notation for fractional exponents.

With regard to his proof of the divergence of the harmonic series, Oresme's ingenious strategy involved replacing groups of fractions in the harmonic series with smaller fractions that have a sum of $\frac{1}{2}$. The following shows the heart of his strategy:

$$1 + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} = \frac{2}{2}$$

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \left(\frac{1}{4} + \frac{1}{4}\right) = \frac{3}{2}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > \frac{3}{2} + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = \frac{4}{2}$$

$$1 + \frac{1}{2} + \ \ldots \ + \frac{1}{8} + \left(\frac{1}{9} + \frac{1}{10} + \ \ldots \ + \frac{1}{16}\right) > \frac{4}{2} + \left(\frac{1}{16} + \ \ldots \ + \frac{1}{16}\right) = \frac{5}{2}$$

This process can be continued indefinitely, so that, in general, for any positive integer n we have

$$1 + \frac{1}{2} + \frac{1}{3} + \ \ldots \ + \frac{1}{2^n} > \frac{n+1}{2}.$$

For example, if $n = 25$ then

$$1 + \frac{1}{2} + \frac{1}{3} + \ \ldots \ + \frac{1}{33\,554\,432} = 1 + \frac{1}{2} + \frac{1}{3} + \ \ldots \ + \frac{1}{2^{25}} > \frac{25+1}{2} = 13.$$

So Oresme's strategy shows that by taking enough terms of the harmonic series, we can guarantee that its sum will be greater than any finite number. Therefore, the series will diverge to infinity. It is interesting to note that although the harmonic series diverges, it does so very slowly. The sum of the harmonic series does not get above 10 until we have added 12 367 terms of the series!

The fact that the harmonic series diverges (Example 3) serves as a counterexample to our conjecture that $\lim_{n\to\infty} a_n = 0$ is both a necessary *and* sufficient condition for the series $\sum_{n=1}^{\infty} a_n$ to converge. It is true that convergence can only occur if $\lim_{n\to\infty} a_n = 0$ (i.e. a *necessary* condition), but $\lim_{n\to\infty} a_n = 0$ is NOT sufficient to guarantee convergence (i.e. not a *sufficient* condition). This leads to the following theorem.

***n*th term divergence test**

If $\lim_{n\to\infty} a_n$ does not exist, or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example 4 – Using the *n*th term divergence test

Determine, if possible, whether each of the following series converges or diverges.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2+1} = \frac{1}{2} + \frac{4}{5} + \frac{9}{10} + \frac{16}{17} + \ldots$ b) $\displaystyle\sum_{n=1}^{\infty} 3(-1)^{n+1} = 3 - 3 + 3 - 3 + \ldots$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n+1} = \frac{1}{2} + \frac{1}{5} + \frac{1}{14} + \frac{1}{41} + \ldots$ d) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3n!+1} = \frac{1}{4} + \frac{2}{7} + \frac{6}{19} + \frac{24}{73} + \ldots$

e) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1} = \frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \frac{4}{17} + \ldots$

Solution

a) $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{n^2/n^2}{n^2/n^2 + 1/n^2} = \lim_{n\to\infty} \frac{1}{1+1/n^2} = 1$

Therefore, by the *n*th term divergence test, the series is divergent.

b) $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(3(-1)^{n+1}\right)$ does not exist because the terms alternate between $+3$ and -3.

Therefore, by the *n*th term divergence test, the series is divergent.

c) Certainly, $3^n + 1 \to \infty$ as $n \to \infty$, so it follows that $\dfrac{1}{3^n+1} \to 0$ as $n \to \infty$. Hence, $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2}{3^n+1} = 2\lim_{n\to\infty} \frac{1}{3^n+1} = 0$. Since the limit of the *n*th term is 0, the *n*th term divergence test does not apply and we are not able to make a conclusion about convergence or divergence. We can make an educated guess that it will probably converge because it is very similar to the convergent geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n} = 2\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ with $r = \dfrac{1}{3}$. In the next section we will learn that it does in fact converge and recognizing that it is similar to a convergent geometric series is important.

d) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n!}{3n!+1} = \lim\limits_{n\to\infty} \dfrac{n!/n!}{3n!/n!+1/n!} = \dfrac{1}{3+0} = \dfrac{1}{3}$

Therefore, by the nth term divergence test, the series is divergent.

e) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n}{n^2+1} = \lim\limits_{n\to\infty} \dfrac{n/n^2}{n^2/n^2+1/n^2} = \dfrac{0}{1+0} = 0$

We cannot apply the nth term divergence test since the limit of the nth term is 0. We will find in the next section that this series behaves like the harmonic series, that is, even though the sequence of its terms converges to 0 the series itself diverges.

Before moving onto the next section and investigating more thoroughly the convergence of infinite series, we state below some important properties of convergent series that are direct consequences of the properties of limits of sequences in Section 1.2 of the previous chapter.

Properties of convergent series

Given that $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ are both convergent series, and c is a constant, then the following series are also convergent:

$\sum\limits_{n=1}^{\infty} ca_n$, $\sum\limits_{n=1}^{\infty} (a_n + b_n)$ and $\sum\limits_{n=1}^{\infty} (a_n - b_n)$.

(i) $\sum\limits_{n=1}^{\infty} ca_n = c\sum\limits_{n=1}^{\infty} a_n$

(ii) $\sum\limits_{n=1}^{\infty} (a_n + b_n) = \sum\limits_{n=1}^{\infty} a_n + \sum\limits_{n=1}^{\infty} b_n$

(iii) $\sum\limits_{n=1}^{\infty} (a_n - b_n) = \sum\limits_{n=1}^{\infty} a_n - \sum\limits_{n=1}^{\infty} b_n$

2.2 Convergence tests

In this section, we develop some more sophisticated tests for convergence. These tests will allow us to efficiently determine convergence for a wide range of series. In Example 4 we were thwarted from determining whether the series in parts c) and e) were convergent or divergent. In general, it is not easy to find the exact sum of a series. We have been able to find exact sums for certain geometric series and telescoping series because we were able to obtain a formula for the sequence of partial sums, s_n. In this section, our purpose is to develop some tests that will let us determine whether a series is convergent or divergent without the need for a formula for the sequence of partial sums. Although in some cases the convergence test being employed will help us to find the sum of a series (or at least an approximation for the sum), in general, it is limited to finding out about convergence of a series without finding the sum. We will study four useful convergence tests that apply to series whose terms are **non-negative** and a fifth test that will apply to alternating series.

Integral test

From our discussion about improper integrals in the previous section, you may feel that there is a relationship between the convergence of an improper integral and the convergence of a series. We can take the formula for the nth term a_n of a series $\sum_{n=1}^{\infty} a_n$ and replace n by x to write a function $f(x)$. The relationship between $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_1^{\infty} f(x)\,dx$ is explained in the following theorem.

The integral test for convergence

Let f be a function that is continuous, decreasing and positive for all $x \geqslant 1$ and $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ converges. In other words:

1) If $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

2) If $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

Before we can conduct a formal proof of the integral test we need to establish the definition of two words for which we have had a common-sense understanding up to now, and to state an important theorem.

Lower and upper bounds of a sequence

The number M is a **lower bound** of the sequence $\{a_n\}$ if $a_n \geqslant M$ for all positive integers n, and the number N is an **upper bound** of $\{a_n\}$ if $a_n \leqslant N$ for all positive integers n. A sequence $\{a_n\}$ is **bounded** if and only if it has a lower bound and an upper bound.

For the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, the sequence of its terms $\left\{\frac{1}{n}\right\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$ any number greater than or equal to 1 is an upper bound, and any number that is less than or equal to zero is a lower bound. For the sequence $\left\{\frac{1}{n}\right\}$ we can call 1 the **least upper bound** and 0 the **greatest lower bound**. Another characteristic of the sequence $\left\{\frac{1}{n}\right\}$ is that the terms are always decreasing and it is not surprising that the sequence converges to its greatest lower bound. In our discussion of one-to-one functions in Chapter 2 of the book, we used the word **monotonic** to describe a function that is either always increasing or always decreasing. Also for the harmonic series, we established that the sequence of its partial sums, $\{s_n\}$, is divergent by essentially showing that $\{s_n\}$ does not have an upper bound, and hence is not bounded. It is sensible to conjecture that a bounded monotonic sequence will be convergent.

Bounded sequence theorem

A monotonic sequence converges if and only if it is bounded.

Before we conduct a formal proof of this theorem, we state an important property of the real numbers with the following postulate.

Completeness postulate

In the real numbers, every non-empty set that has an upper bound has a least upper bound.

Proof of the bounded sequence theorem

We prove the theorem for the case when the monotonic sequence, call it $\{a_n\}$, is increasing. If it converges to some limit L then it is bounded below by the first term of the sequence a_1 and above by L and is therefore bounded. Conversely, if $\{a_n\}$ is bounded, then the completeness postulate guarantees that $\{a_n\}$ has a least upper bound L. We now need to show that $\{a_n\}$ must converge to L. Firstly, since L is an upper bound for $\{a_n\}$ then it follows that $a_n \leqslant L$ for all n. Also, since L is the least upper bound then $L - \varepsilon$ is not an upper bound for any $\varepsilon > 0$. Hence, there exists an integer N such that $L - \varepsilon < a_N$. Because $\{a_n\}$ is always increasing then $a_N \leqslant a_n$ whenever $n > N$. Therefore, $L - \varepsilon < a_n \leqslant L$ and consequently $L - \varepsilon < a_n < L + \varepsilon$ which is equivalent to $-\varepsilon < a_n - L < \varepsilon$ and $|a_n - L| < \varepsilon$. This satisfies the $\varepsilon - N$ definition for the limit of a sequence and completes the proof for an increasing sequence $\{a_n\}$. A parallel argument can be written to prove the theorem for a decreasing sequence $\{a_n\}$.

Proof of the integral test

The essential idea behind the proof is that the terms in a series $\sum_{n=1}^{\infty} a_n$ can be assigned to represent the area of ever decreasing rectangles of constant width and that the improper integral $\int_1^{\infty} f(x)\,dx$ is approximated by the sum of these rectangles. The total areas of the inscribed rectangles (Figure 2.1) and the circumscribed rectangles (Figure 2.2) are as follows:

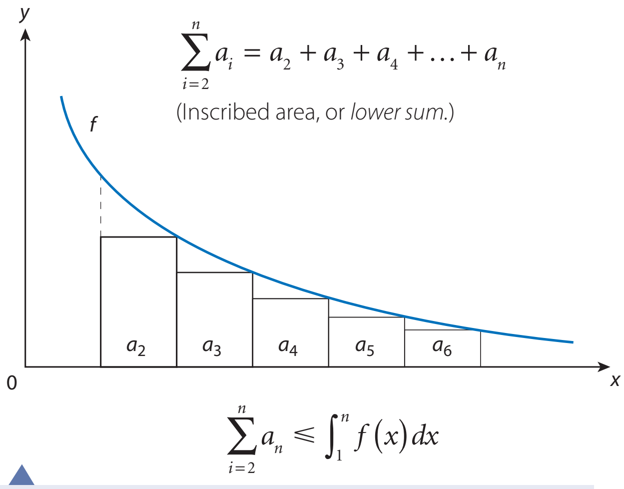


Figure 2.1 Inscribed rectangles gives lower sum.

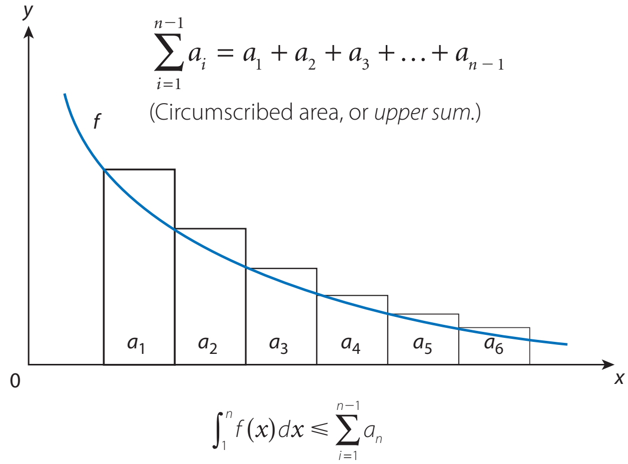


Figure 2.2 Circumscribed rectangles gives upper sum.

The exact area under the graph of f from $x = 1$ to n, i.e. the definite integral $\int_1^n f(x)\,dx$, lies between the inscribed and circumscribed areas. As Figures 2.1 and 2.2 illustrate,

$$\sum_{i=2}^{n} a_i \leqslant \int_1^n f(x)\,dx \leqslant \sum_{i=1}^{n-1} a_i.$$

Using the nth partial sum, $s_n = a_1 + a_2 + a_3 + \ldots + a_n$, we can write the inequality above as

$$s_n - a_1 \leqslant \int_1^n f(x)\,dx \leqslant s_{n-1}.$$

To prove part (1) we start by assuming $\int_1^n f(x)\,dx$ converges to L. Then it follows that for $n \geqslant 1$

$$s_n - a_1 \leqslant L$$

and consequently

$$s_n \leqslant L + a_1.$$

Hence, the sequence of partial sums $\{s_n\}$ is bounded and monotonic and it follows from the bounded sequence theorem that $\{s_n\}$ converges, and consequently the series $\sum_{n=1}^{\infty} a_n$ must also converge. For part (2) assume that the improper integral $\int_1^n f(x)\,dx$ diverges. Thus, $\int_1^n f(x)\,dx$ goes to infinity as $n \to \infty$, and given the inequality $s_{n-1} \geqslant \int_1^n f(x)\,dx$ it must follow that $\{s_n\}$ diverges which means that $\sum_{n=1}^{\infty} a_n$ also diverges.

Example 5 – Using the integral test

Determine the convergence or divergence of each series.

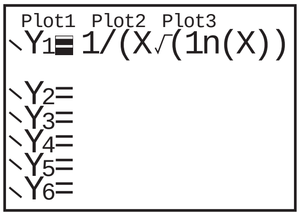
a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

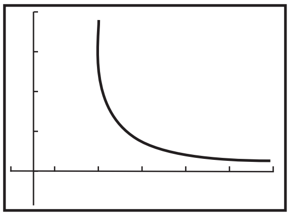
b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

c) $\displaystyle\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \ldots$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$ [Example 4 part e)]

Solution





a) Graphing the function $f(x) = \dfrac{1}{x\sqrt{\ln x}}$ on our GDC provides us with a quick confirmation that f is continuous, decreasing and positive for all $x \geqslant 2$, thereby satisfying the conditions for applying the integral test. Recalling techniques for improper integrals from the first section of this chapter, we now need to evaluate $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx$ to see if it converges to a finite number or diverges to infinity. For this integral we will also need to apply the technique of u-substitution.

$$\int \frac{1}{x\sqrt{\ln x}}\,dx = \int (\ln x)^{-\frac{1}{2}}\left(\frac{1}{x}dx\right) = \int u^{-\frac{1}{2}}\,du = 2u^{\frac{1}{2}} \quad \text{Let } u = \ln x, \text{ then } du = \frac{1}{x}dx.$$

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{b\to\infty}\int_2^{b} \frac{1}{x\sqrt{\ln x}}\,dx \qquad \text{Rewriting improper integral as a limit.}$$

$$= \lim_{b\to\infty}\left[2\sqrt{\ln x}\,\right]_2^b \qquad \text{Applying result from } u\text{-substitution.}$$

$$= \lim_{b\to\infty}\left[2\sqrt{\ln b} - 2\sqrt{\ln 2}\,\right]$$

$$= \infty$$

Therefore, the integral $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx$ diverges, and by the integral test the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ must also diverge.

b) For $f(x) = \frac{x}{e^x}$, it is clear that f is continuous, decreasing and positive for $x \geqslant 1$ because $e^x > 0$ and e^x grows faster than x; so the integral test applies. Using integration by parts:

$$\int \frac{x}{e^x}\,dx = \int xe^{-x}\,dx \qquad \text{Choose } u = x \Rightarrow du = dx \text{ and } dv = e^{-x} \Rightarrow v = -e^{-x}.$$

$$= -xe^{-x} - \int -e^{-x}\,dx \qquad \text{Substituting into formula } \int u\,dv = uv - \int v\,du.$$

$$= -xe^{-x} + \int e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x}$$

$$\int_1^{\infty} \frac{x}{e^x}\,dx = \lim_{b\to\infty}\left[-xe^{-x} - e^{-x}\right]_1^b \qquad \text{Rewriting improper integral as a limit.}$$

$$= \lim_{b\to\infty}\left[\left(-be^{-b} - e^{-b}\right) - \left(-e^{-1} - e^{-1}\right)\right]$$

$$= \lim_{b\to\infty}\left[-\frac{b+1}{e^b} + \frac{2}{e}\right] = -\lim_{b\to\infty}\left[\frac{b+1}{e^b}\right] + \lim_{b\to\infty}\left[\frac{2}{e}\right]$$

Applying l'Hôpital's rule to the first limit gives $\lim_{b\to\infty}\left[\frac{b+1}{e^b}\right] = \lim_{b\to\infty}\left[\frac{1}{e^b}\right] = 0.$

Therefore, $\int_1^{\infty} \frac{x}{e^x}\,dx = \frac{2}{e}.$

By the integral test, since the integral $\int_1^{\infty} \frac{x}{e^x}\,dx$ converges then the series $\sum_{n=1}^{\infty} \frac{n}{e^n}$ must also converge.

c) We need to find a rule for the nth term for the series that starts $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \ldots.$

Using some inductive reasoning we determine that the series expressed in summation notation is

As Example 5 a) illustrates, if the summation index for an infinite series starts at $n = k > 1$ rather than $n = 1$, the integral test can still be applied. The integral test can be modified as follows:

Let f be a function that is continuous, decreasing and positive for all $x \geqslant k$ such that $k > 1$ and $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ converges *if and only if* the improper integral $\int_k^{\infty} f(x)\,dx$ converges.

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1}=\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17}+\frac{1}{26}+\ldots+\frac{1}{n^2+1}+\ldots.$$

The function $f(x)=\dfrac{1}{x^2+1}$ satisfies the conditions of the integral test.

We need to recognize that the anti-derivative of $\dfrac{1}{x^2+1}$ is $\arctan x$ (a 'standard integral' in the IB formula booklet).

$$\int_1^{\infty}\frac{1}{x^2+1}dx=\lim_{b\to\infty}\int_1^{b}\frac{1}{x^2+1}dx$$

Rewriting improper integral as a limit.

$$=\lim_{b\to\infty}\left[\arctan x\right]_1^b$$
$$=\lim_{b\to\infty}\left[\arctan b-\arctan 1\right]$$
$$=\lim_{b\to\infty}\left[\arctan b\right]-\lim_{b\to\infty}\left[\frac{\pi}{4}\right]$$
$$=\frac{\pi}{2}-\frac{\pi}{4}$$

Therefore, $\displaystyle\int_1^{\infty}\frac{1}{x^2+1}dx=\frac{\pi}{4}$.

By the integral test, since the integral $\displaystyle\int_1^{\infty}\frac{1}{x^2+1}dx$ converges then the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ must also converge.

It is very important to know when using the integral test that the value of the improper integral is not equal to the sum of the series. The sum, expressed to ten significant figures, of the first 50 terms of the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+1}$, Example 5 d), is approximately 1.056 875 301; whereas $\displaystyle\int_1^{\infty}\frac{1}{x^2+1}dx=\frac{\pi}{4}\approx 0.7853981634$. Therefore, in general $\displaystyle\sum_{n=1}^{\infty}a_n\neq\int_1^{\infty}f(x)dx$.

d) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ was the series in Example 4 e) for which the nth term divergence test was inconclusive because $\displaystyle\lim_{n\to\infty}\frac{n}{n^2+1}=0$. The function $f(x)=\dfrac{x}{x^2+1}$ satisfies the conditions of the integral test. The method of u-substitution will be useful to evaluate the integral $\displaystyle\int\frac{x}{x^2+1}dx$.

Let $u=x^2+1$ and it follows that $du=2xdx\ \Rightarrow\ \frac{1}{2}du=xdx$.

Substituting gives

$$\int\frac{x}{x^2+1}dx=\int\frac{1}{u}\cdot\frac{1}{2}du=\frac{1}{2}\int\frac{1}{u}du=\frac{1}{2}\ln u=\frac{1}{2}\ln\left(x^2+1\right).$$

Using this result we have:

$$\int_1^{\infty}\frac{1}{x^2+1}dx=\lim_{b\to\infty}\int_1^{b}\frac{1}{x^2+1}dx$$

Rewriting improper integral as a limit.

$$=\frac{1}{2}\lim_{b\to\infty}\left[\ln\left(x^2+1\right)\right]_1^b$$
$$=\frac{1}{2}\lim_{b\to\infty}\left[\ln\left(b^2+1\right)-\ln 2\right]$$
$$=\infty$$

By the integral test, since the integral $\displaystyle\int_1^{\infty}\frac{x}{x^2+1}dx$ diverges then the series $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ must also diverge.

p-series

Before we move onto the next convergence test, we can use the integral test to give us important results for any series that is in the form shown below, known as a ***p*-series**.

$$\sum_{1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots \text{ where } p \text{ is a constant.}$$

If $p=1$, the p-series is the harmonic series which we know diverges. What about series for other values of p? The following example will lead to a simple test for the convergence of any p-series.

Example 6 – Convergence of *p*-series

For what values of p is the series $\sum_{1}^{\infty}\frac{1}{n^p}$ convergent?

Solution

Let's consider when $p<0,\ p=0,$ and $p>0$.

When $p<0$, then $\frac{1}{n^p}\to\infty$ as $n\to\infty$. For example, if $p=-3$ then $\frac{1}{n^{-3}}=n^3$; and clearly n^3 increases without bound as $n\to\infty$.

When $p=0$, then $\lim\limits_{n\to 0}\frac{1}{n^p}=\frac{1}{n^0}=1$.

In both of these cases, $\lim\limits_{n\to 0}\frac{1}{n^p}\neq 0$ so the p-series diverges by the nth term divergence test.

When $p>0$, the function $f(x)=\frac{1}{x^p}$ is continuous, decreasing and positive for $x\geqslant 1$ so we can use the integral test. We know from Example 3 in the previous section that the harmonic series $(p=1)$ diverges, so let's assume that $p\neq 1$ and investigate the improper integral $\int_1^\infty \frac{1}{x^p}\,dx$.

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx\\ &=\lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b\\ &=\left(\frac{1}{1-p}\right)\lim_{b\to\infty}\left[x^{-p+1}\right]_1^b\\ &=\left(\frac{1}{1-p}\right)\lim_{b\to\infty}\left[b^{-p+1}-1\right]\end{aligned}$$

If $p>1$, then $-p+1<0$ and consequently as $b\to\infty,\ b^{-p+1}\to 0$.

Hence, if $p>1$ then $\int_1^\infty \frac{1}{x^p}\,dx=\left(\frac{1}{1-p}\right)(-1)=\frac{1}{p-1}$. Therefore the integral converges and the series $\sum_{1}^{\infty}\frac{1}{n^p}$ must also converge.

If $p<1$, then $-p+1>0$ and consequently as $b\to\infty,\ b^{-p+1}\to\infty$. Hence, if $p<1$ then the integral $\int_1^\infty \frac{1}{x^p}\,dx$ diverges and so does the series $\sum_{1}^{\infty}\frac{1}{n^p}$.

The results from Example 6 are summarized below.

Convergence of p-series

The p-series $\sum_{1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$

(i) **converges** if $p > 1$, and (ii) **diverges** if $p \leqslant 1$.

Note: When $p = 1$ this is the harmonic series.

Comparison test

The integral test compares a series consisting of all positive terms with an integral as a means of testing the convergence of the series. It is possible to use a second series in a similar way. If each term of a series of positive terms is less than or equal to the corresponding term of a known convergent series of positive terms, then the series is convergent. We will call this the comparison test and can state it as follows.

In the statement of the comparison test, $n \geqslant N$ means *from some term onward*. That is, eventually for some term and forever afterwards the terms of the series $\sum_{n=1}^{\infty} b_n$ are always greater than the corresponding terms of the series $\sum_{n=1}^{\infty} a_n$. This is often expressed by saying that $\sum_{n=1}^{\infty} b_n$ *dominates* $\sum_{n=1}^{\infty} a_n$.

Comparison test

Given $0 \leqslant a_n \leqslant b_n$ for all $n \geqslant N$ for some integer N, it follows that

1 if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges;

2 if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.

Note: The comparison test can also be applied for the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ whenever there exists a positive constant c such that $0 \leqslant a_n \leqslant cb_n$ for all $n \geqslant N, N \in \mathbb{Q}^+$.

Before proving both parts of the comparison test, we will find it helpful to state a corollary to the bounded sequence theorem that we recall says the following: A monotonic (always decreasing or always increasing) sequence converges if and only if it is bounded. If all the terms of an infinite series are positive, the sequence of partial sums is increasing. Therefore, the following theorem follows directly from the bounded sequence theorem.

The comparison test significantly expands our ability to determine the convergence of a series with more complicated rules for the nth term. We achieve this by comparing a 'complicated' series to a 'simpler' series whose convergence or divergence is known.

Positive series convergence

A series of positive terms is convergent if and only if its sequence of partial sums has an upper bound.

Proof of comparison test

Proof of 1: Let $\{u_n\}$ and $\{v_n\}$ be sequences of the partial sums for the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, respectively. Because $\sum_{n=1}^{\infty} b_n$ is a series of positive terms that is convergent, it follows from the positive series convergence theorem that the sequence $\{v_n\}$ has an upper bound – let's call it B. Since $a_n \leqslant b_n$ for all $n \geqslant 1$, we can conclude that $a_n \leqslant b_n \leqslant B$ for all $n \geqslant 1$. Thus, B is an upper bound of the sequence $\{u_n\}$. Because the terms of the series $\sum_{n=1}^{\infty} a_n$ are all positive then it follows from the positive series convergence theorem that $\sum_{n=1}^{\infty} a_n$ is convergent.

Proof of 2: If $\sum_{n=1}^{\infty} a_n$ is divergent, then since $\{u_n\}$ is increasing $u_n \to \infty$. However, $b_n \geqslant a_n$, so $v_n \geqslant u_n$. It follows that $v_n \to \infty$ and, therefore, $\sum_{n=1}^{\infty} b_n$ must also diverge.

Example 7 – Using the comparison test

Determine the convergence or divergence of each series.

a) $\sum_{n=1}^{\infty} \frac{2}{3^n + 1}$ [Example 4 c)] b) $\sum_{n=1}^{\infty} \frac{1}{3+\sqrt{n}}$ c) $\sum_{n=0}^{\infty} \frac{1}{n!}$

Solution

a) We can compare the given series

$$\frac{2}{4} + \frac{2}{10} + \frac{2}{28} + \frac{2}{82} + \ldots + \frac{2}{3^n + 1} + \ldots$$

with the nth term of the geometric series

$$\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \ldots + \frac{2}{3^n} + \ldots$$

which converges because its common ratio is between one and negative one; $r = \frac{1}{3} < 1$.

It is clear that each term in the given series is less than its corresponding term in the geometric series. That is, $\frac{2}{3^n + 1} < \frac{2}{3^n}$ for all $n \geqslant 1$.

Therefore, by the comparison test since the series $\sum_{n=1}^{\infty} \frac{2}{3^n}$ converges the series $\sum_{n=1}^{\infty} \frac{2}{3^n + 1}$ must also converge.

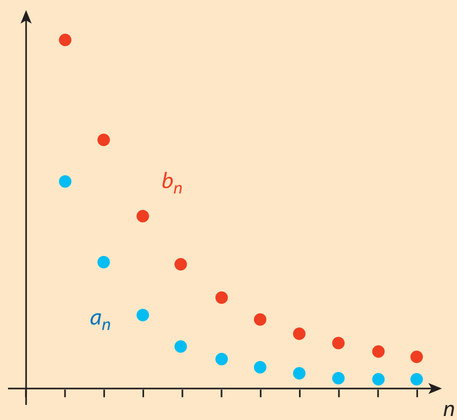


Figure 2.3

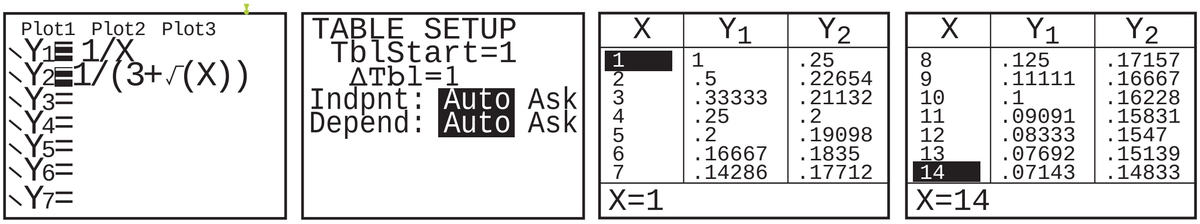
Note that part **1** and part **2** of the comparison test require that $0 \leqslant a_n \leqslant b_n$. You can think of $\sum a_n$ as the 'lower' series and $\sum b_n$ as the 'higher' series (see Figure 2.3). Thus, in a very informal sense the two parts of the comparison test say:

1. If the 'higher' series converges, then the 'lower' series must also converge.
2. If the 'lower' series diverges, then the 'higher' series must also diverge.

The 'higher' series *dominates* the 'lower' series.

b) The series $\sum_{n=1}^{\infty} \frac{1}{3+\sqrt{n}}$ is similar to the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ which diverges because $p = \frac{1}{2} \leqslant 1$. If we compare the given series to this p-series we see that $\frac{1}{3+\sqrt{n}} < \frac{1}{\sqrt{n}}$ for all $n \geqslant 1$. However, the comparison test provides no conclusive result in this case where a series is *dominated* by a divergent series. Suspecting that the given series does in fact diverge we need to find a divergent series that the given series dominates. Let's compare it to the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. Remember, to satisfy the comparison test it is not necessary for $a_n \leqslant b_n$ to be true for all integers $n \geqslant 1$ but for all integers $n \geqslant N$ where N is some positive integer.

Our GDC is a handy tool to quickly compare the terms of the given series to the harmonic series. The screen images below show values for the first 14 terms of the two series in a table.



How could we prove that the inequality $\frac{1}{n} < \frac{1}{3+\sqrt{n}}$ is true for $n \geqslant 6$? Try doing so by proving the inequality $3+\sqrt{n} < n$ for $n \geqslant 6$ by mathematical induction.

In Example 7 c), we know that the sum of the infinite geometric series $\sum_{n=0}^{\infty} 2\left(\frac{1}{2}\right)^n$ is $S_\infty = \frac{a_1}{1-r} = \frac{2}{1-\frac{1}{2}} = 4$. Thus the sum $\sum_{n=0}^{\infty} \frac{1}{n!}$ must be less than 4. In fact, we will learn in the next section that this sum is exactly the number e. That is,

$$e = 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\ldots+\frac{1}{n!}+\ldots$$

For the first five terms the terms in the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ are greater than those for $\sum_{n=1}^{\infty} \frac{1}{3+\sqrt{n}}$. However, it appears from the sixth term onwards that this reverses, that is,

$$\frac{1}{n} < \frac{1}{3+\sqrt{n}} \quad \text{for } n \geqslant 6.$$

Therefore, by the comparison test the series $\sum_{n=1}^{\infty} \frac{1}{3+\sqrt{n}}$ diverges.

c) Consider the first few terms of the given series:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1+1+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\ldots$$

Now consider the first few terms of the convergent geometric series with $a_1 = 2$ and $r = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} 2\left(\frac{1}{2}\right)^n = 2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$$

It appears that the terms of $\sum_{n=0}^{\infty} \frac{1}{n!}$ are less than or equal to the corresponding terms of the convergent geometric series for all $n \geqslant 1$. Recall that in Example 6 of the previous chapter we proved that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any real number x. From that we concluded that the factorial function increases faster than any exponential function. Hence, $\frac{1}{n!} \leqslant 2\left(\frac{1}{2}\right)^n$ for $n \geqslant 1$. Therefore, by the comparison test the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

Limit comparison test

In order for the comparison test to provide us with a conclusive result on the convergence or divergence of a series, the series being tested must be dominated by ('lower' than) a convergent series, or it must dominate ('higher' than) a divergent series. If these conditions are not met then the comparison test (sometimes called the direct comparison test) cannot be used. For example, consider the series $\sum_{n=1}^{\infty} \frac{2}{3^n - 1}$ that is nearly identical to the series $\sum_{n=1}^{\infty} \frac{2}{3^n + 1}$ that we proved is convergent in Example 7 a).

We strongly expect $\sum_{n=1}^{\infty} \frac{2}{3^n - 1}$ to also converge. However, the inequality $\frac{2}{3^n - 1} > \frac{2}{3^n}$ shows that the series dominates the convergent geometric series $\sum_{n=1}^{\infty} \frac{2}{3^n}$ so the comparison test does not apply. In a case like this another form of the comparison test, known as the **limit comparison test**, can be used. This test can be particularly useful in comparing a series to a p-series or a geometric series.

Limit comparison test

Given $a_n > 0$ and $b_n > 0$ for all $n \geqslant N$ for some integer N, it follows that:

1. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L$, where L is finite and positive, then the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
2. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ also converges.
3. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ also diverges.

If applying the limit comparison test you get $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, this does ***not*** imply that the series $\sum_{n=1}^{\infty} a_n$ also diverges.

Proof

1. Let k and m be positive numbers such that $k < L < m$. Since $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L$ then there is a positive integer N, where $N > n$, such that
   $k < \frac{a_n}{b_n} < m.$
   It follows that
   $$kb_n < a_n < mb_n.$$
   If the series $\sum_{n=1}^{\infty} b_n$ converges then from the properties of series, the series

$\sum_{n=1}^{\infty} mb_n$ must also converge. Since $\sum_{n=1}^{\infty} mb_n$ dominates $\sum_{n=1}^{\infty} a_n$ then by the comparison test $\sum_{n=1}^{\infty} a_n$ must converge. Likewise, if the series $\sum_{n=1}^{\infty} b_n$ diverges then the series $\sum_{n=1}^{\infty} kb_n$ must also diverge, and since $\sum_{n=1}^{\infty} a_n$ dominates $\sum_{n=1}^{\infty} kb_n$ then by the comparison test $\sum_{n=1}^{\infty} a_n$ must diverge.

The proofs of parts 2 and 3 are left as exercises.

Example 8 – Using the limit comparison test

Determine the convergence or divergence of each series.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n - 1}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{\sqrt{n}}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+7n}{3n^6 - n^3}$ d) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Solution

a) As mentioned above, this series resembles the convergent geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n}$. Thus, we evaluate the following limit.

$$\lim_{n\to\infty} \frac{\frac{2}{3^n-1}}{\frac{2}{3^n}} = \lim_{n\to\infty} \frac{3^n}{3^n - 1}$$

$$= \lim_{n\to\infty} \frac{3^n/3^n}{3^n/3^n - 1/3^n}$$

$$= \lim_{n\to\infty} \frac{1}{1 - 1/3^n}$$

$$= 1$$

Since the limit is finite and positive and $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n}$ converges then by the limit comparison test the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n - 1}$ must also converge.

b) The given series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n+1}$ is similar to $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$ which is a p-series best written as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$. Since $p = \frac{2}{3} \leqslant 1$ we know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges. We then evaluate the following limit.

$$\lim_{n\to\infty} \frac{\frac{\sqrt[3]{n}}{n+1}}{\frac{1}{n^{2/3}}} = \lim_{n\to\infty} \frac{n^{1/3} \cdot n^{2/3}}{n+1}$$

$$= \lim_{n\to\infty} \frac{n}{n+1}$$
$$= 1$$

Since the limit is finite and positive and $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges then by the limit comparison test the series $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n+1}$ must also diverge.

c) As we saw in part b), it is possible to find a suitable p-series for comparison purposes by disregarding all but the highest powers of n in the numerator and denominator. Hence, for the given series $\sum_{n=1}^{\infty} \frac{n^2+7n}{3n^6-n^3}$ we can compare the series to $\sum_{n=1}^{\infty} \frac{n^2}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n^4}$ which is a convergent p-series.

$$\lim_{n\to\infty} \frac{\frac{n^2+7n}{3n^6-n^3}}{\frac{1}{n^4}} = \lim_{n\to\infty} \frac{n^4\left(n^2+7n\right)}{3n^6-n^3}$$
$$= \lim_{n\to\infty} \frac{n^6+7n^5}{3n^6-n^3}$$
$$= \lim_{n\to\infty} \frac{n^6/n^6+7n^5/n^6}{3n^6/n^6-n^3/n^6}$$
$$= \frac{1+0}{3-0}$$
$$= \frac{1}{3}$$

Since the limit is finite and positive and $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges then by the limit comparison test the series $\sum_{n=1}^{\infty} \frac{n^2+7n}{3n^6-n^3}$ must also converge.

d) Remember that in Section 13.2 of the book we proved $\lim_{n\to\infty} \frac{\sin x}{x} = 1$ by means of the squeeze theorem. So we can use the limit comparison theorem and compare the given series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ to the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

Hence, $\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1.$

Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges then $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ also diverges.

Ratio test

In a geometric series, the ratio of adjacent terms is constant. This can be expressed as

$$\frac{a_1 r^{n+1}}{a_1 r^n} = r.$$

We know that a geometric series converges if and only if this ratio is between -1 and 1. In other types of series, the ratio of adjacent terms does not remain constant but it can still give us helpful information about whether or not the series converges, as indicated in the following theorem.

Ratio test

Let $\sum_{n=1}^{\infty} a_n$ be a series with non-zero terms, and with

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

Then **1** the series *converges* if $L < 1$
2 the series *diverges* if $L > 1$
3 the test is *inconclusive* if $L = 1$.

Proof

1. For the case when $L < 1$, there must be a number r with $0 < r < 1$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| \leqslant r$$

for all n sufficiently large. Suppose that there exists some integer N such that $\frac{a_{n+1}}{a_n} \leqslant r$ if $n \geqslant N$.

Then $$\left|\frac{a_{N+1}}{a_N}\right| \geqslant r \quad \Rightarrow \quad |a_{N+1}| \leqslant r|a_N|$$

$$\left|\frac{a_{N+2}}{a_{N+1}}\right| \geqslant r \quad \Rightarrow \quad |a_{N+2}| \leqslant r|a_{N+1}| \leqslant r^2|a_N|$$

and so on. Thus,

$$|a_N| + |a_{N+1}| + |a_{N+2}| + \ldots \leqslant a_N|1 + r + r^2 + \ldots|$$

This shows that for $n \geqslant N$ the series $\sum_{n=1}^{\infty} a_n$ is dominated by the geometric series $a_N \sum_{n=1}^{\infty} r^{n-1}$. Because $0 < r < 1$ this geometric series converges and by the comparison test $\sum_{n=1}^{\infty} a_n$ must also converge.

2. For the case when $L > 1$, it must be true that $|a_{n+1}| > |a_n|$ for all n sufficiently large. Therefore, $\lim_{n\to\infty} a_n \neq 0$ and the series $\sum_{n=1}^{\infty} a_n$ must diverge by the nth term divergence test.

3. Applying the ratio test to the general p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ gives

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^p = 1.$$

We know that a p-series converges if $p > 1$ and diverges if $p \leqslant 1$. Hence, this shows that if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ then it is possible to have a series that is either convergent or divergent. Therefore, the ratio test is inconclusive if $L = 1$.

The ratio test is particularly useful for testing series involving exponential expressions or expressions with factorials, as illustrated in the following example.

Example 9 – Using the ratio test

Determine the convergence or divergence of each series.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^3 3^{n+1}}{4^n}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Solution

a) All the terms of the given series are positive so we can do without the absolute value signs.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+1)^3 3^{n+2}}{4^{n+1}}}{\dfrac{n^3 3^{n+1}}{4^n}}$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)^3}{n^3} \cdot \frac{3^{n+2}}{3^{n+1}} \cdot \frac{4^n}{4^{n+1}} \right)$$

$$= \lim_{n\to\infty} \frac{3(n+1)^3}{4n^3}$$

$$= \frac{3}{4} < 1$$

Therefore, by the ratio test the series $\displaystyle\sum_{n=0}^{\infty} \frac{n^3 3^{n+1}}{4^n}$ converges.

b) Again, the series has only positive terms so we can write the ratio test without absolute signs.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+1)^{n+1}}{(n+1)!}}{\dfrac{n^n}{n!}}$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} \right)$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n$$

$$= e > 1$$

Therefore, by the ratio test the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges.

When applying the ratio test to series involving quotients of expressions with factorials, it is often necessary to perform simplification steps similar to those we did in Example 9:

$$\frac{n!}{(n+1)!} = \frac{\cancel{n!}}{(n+1)\cancel{n!}} = \frac{1}{n+1}$$

Although the ratio test worked in Example 9 part b) we could have used the nth term divergence test to prove that the series diverges by considering the following:

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} = 1 + \frac{2^2}{2} + \frac{3^3}{6} + \frac{4^4}{24} + \ldots \text{ and for the } n\text{th term } a_n = \frac{n \cdot n \cdot n \cdot \ldots \cdot n}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n} \geqslant n$$

Thus as $n \to \infty$ the terms do not approach 0 and the series diverges by the nth term divergence test. It is often the case that we can determine whether or not a series converges by more than one test. The summary at the end of this section gives some tips on how to find the most efficient test to apply for a certain series.

As we will learn even further in the next section, the ratio test is useful in answering questions about convergence, as in the following example.

Example 10

For what values of x will the series $\sum_{n=1}^{\infty} \frac{2^n}{nx^n}$ converge?

Solution

Applying the ratio test gives the following inequality to solve.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{2^{n+1}}{(n+1)x^{n+1}}}{\dfrac{2^n}{nx^n}}\right| < 1$$

$$\lim_{n\to\infty}\left|\frac{2^n 2}{(n+1)x^n x}\cdot\frac{nx^n}{2^n}\right| < 1$$

$$\lim_{n\to\infty}\left|\frac{2n}{x(n+1)}\right| < 1$$

$$\frac{1}{|x|}\cdot\lim_{n\to\infty}\left|\frac{2n}{n+1}\right| < 1$$

$$\frac{2}{|x|} < 1$$

$$|x| > 2$$

Therefore, the series converges for any values of x such that $x < -2$ or $x > 2$.

Note that when $x < -2$ the terms of $\sum_{n=1}^{\infty} \frac{2^n}{nx^n}$ will alternate between positive and negative. The ratio test is very useful in analyzing series with alternating terms that we will take a closer look at in the next section.

2.3 Alternating series and absolute convergence

Although we have encountered series whose terms alternate between positive and negative, the four convergence tests we have established – with the exception of the ratio test – apply only to series with positive terms.

We have encountered series with some negative terms, for example these alternating series:

$$-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}+\ldots+\frac{(-1)^n}{3^n}+\ldots \qquad \text{[Example 2 a)]}$$

$$3-3+3-3+3-3+\ldots+3(-1)^{n+1}+\ldots \qquad \text{[Example 4 b)]}$$

Series such as these, having terms that are alternately positive and negative, are called **alternating series.** The first series above is a geometric series with $r=-\frac{1}{3}$, so it converges to $\dfrac{-\frac{1}{3}}{1-\left(-\frac{1}{3}\right)}=-\frac{1}{4}$, and the second series diverges by the nth term divergence test. But not all alternating series will be geometric nor satisfy the conditions of the nth term divergence test. The following test can be used to determine the convergence of a wider range of alternating series that satisfy certain conditions.

These examples show that an alternating series can be written in one of two forms: $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ where the first term is positive, or $\sum_{n=1}^{\infty}(-1)^n a_n$ where the first term is negative such that a_n is a positive number. The alternating series test is stated using the first form. Since $(-1)\left[(-1)^n a_n\right]=(-1)^{n+1}a_n$ then the test also holds true for the second form by applying the property of convergent series: if $\sum_{n=1}^{\infty}a_n$ converges, and c is some real constant, then $\sum_{n=1}^{\infty}ca_n$ also converges.

Alternating series test

The alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n=a_1-a_2+a_3-a_4+\ldots \qquad (a_n>0)$$

converges if both of the following conditions are satisfied.

1. $\lim_{n\to\infty}a_n=0$
2. $a_{n+1}\leqslant a_n$ for all $n\geqslant N$, for some positive integer N.

Proof

Consider the sequence of partial sums $\{s_n\}$ for an alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ such that $a_{n+1}\leqslant a_n$ and $\lim_{n\to\infty}a_n=0$. Figure 2.4 shows a graph of a few terms of $\{s_n\}$.

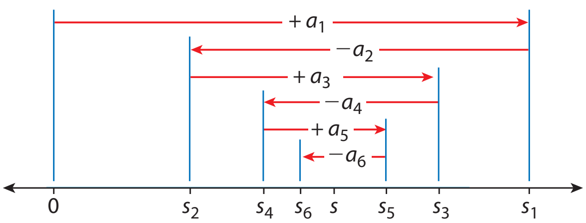


Figure 2.4 Convergence of the partial sums of an alternating series to their limit s.

Observe how the alternating signs of the terms of the series cause the partial sums to be alternately larger and smaller. As n increases the points corresponding to the nth partial sum 'jump' back and forth on either side of their limit s, gradually closing in as the value of the terms go to zero. Also observe that as these 'jumps' get smaller and smaller (because $a_{n+1} \leqslant a_n$), the odd-numbered terms in the sequence of partial sums, $\{s_{2n+1}\}$, form a decreasing sequence and the even-numbered terms, $\{s_{2n}\}$, form an increasing sequence. Furthermore, the decreasing sequence of odd-numbered terms has s_2 as a lower bound and thus, by the bounded sequence theorem, has a limit – call it L_1. Similarly, the sequence of even-numbered terms has s_1 as an upper bound and must also have a limit – call it L_2. If we can show that $L_1 = L_2$ then the series converges to a unique limit $s = L_1 = L_2$. With L_1 the limit of the sequence of odd-numbered terms in the sequence of partial sums and L_2 the limit of the even-numbered terms, it follows that

$$s_{2n} \leqslant L_2 \leqslant L_1 \leqslant s_{2n+1}$$

and

$$s_{2n+1} - s_{2n} = a_{2n+1}$$

Since

$$\lim_{n\to\infty} a_{2n+1} = 0$$

then it must follow that

$$L_1 = L_2$$

Therefore, the sequence of partial sums of the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converges.

When applying the alternating series test, it is best to verify condition (1) first. If condition (1) fails, that is, $\lim_{n\to\infty} a_n \neq 0$, then the series diverges by the nth term divergence test. If $\lim_{n\to\infty} a_n = 0$ but condition (2) fails then the alternating series test is inconclusive. There is another condition implied that must be met – that the series is truly (eventually) alternating. If this is not obvious by inspection then the easiest way to verify that a series written in the form $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ or $\sum_{n=1}^{\infty}(-1)^{n} a_n$ is alternating is to show that $a_n > 0$ for all $n \geqslant N$, for some positive integer N.

Example 11 – Using the alternating series test

Determine the convergence or divergence of each series.

a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$ b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 2n}{3n-1}$ c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$

Solution

a) The series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$ is alternating because $\dfrac{n}{n^2+1} > 0$ for $n \geqslant 1$.
Condition (1) is easily verified.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{n^2+1} = \lim_{n\to\infty}\frac{n/n^2}{n^2/n^2+1/n^2} = \frac{0}{1+0} = 0$$

Now, let's attempt to satisfy condition (2) by proving the inequality $a_{n+1} \leqslant a_n$ for $a_n = \dfrac{n}{n^2+1}$.

$$\frac{n+1}{(n+1)^2+1} \leqslant \frac{n}{n^2+1}$$

$$(n+1)(n^2+1) \leqslant n\left[(n+1)^2+1\right]$$

Cross-multiplying; both denominators are positive.

$$n^3+n^2+n+1 \leqslant n^3+2n^2+2n$$

$$1 \leqslant n^2+n$$

$$n(n+1) \geqslant 1$$

Since $n \geqslant 1$, then the inequality $n(n+1) \geqslant 1$ is true. Hence, $a_{n+1} \leqslant a_n$ and condition (2) is satisfied. Therefore, the series $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\dfrac{n}{n^2+1}$ converges by the alternating series test.

b) The series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n 2n}{3n-1}$ is alternating since $\dfrac{2n}{3n-1} > 0$ for all $n \geqslant 1$, but

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\dfrac{2n}{3n-1} = \dfrac{2}{3} \neq 0$ so condition (1) is not satisfied.

Applying the nth term divergence test, we need to find the limit of the nth term as $n \to \infty$.

$$\lim_{n\to\infty}\frac{(-1)^n 2n}{3n-1} = \lim_{n\to\infty}(-1)^n \cdot \lim_{n\to\infty}\frac{2n}{3n-1}$$

$\lim\limits_{n\to\infty}\dfrac{2n}{3n-1} = \dfrac{2}{3}$ but $\lim\limits_{n\to\infty}(-1)^n$ does not exist (Example 1 a)), so

$\lim\limits_{n\to\infty}\dfrac{(-1)^n 2n}{3n-1}$ does not exist. Therefore, the series diverges by the nth term divergence test.

c) $a_n = \dfrac{\ln n}{n} > 0$ for all integers $n \geqslant 2$, so the series is alternating.

Checking condition (1) we can evaluate the following limit using l'Hôpital's rule because it has the indeterminate form $\dfrac{\infty}{\infty}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{\frac{d}{dx}(\ln n)}{\frac{d}{dx}(n)} = \lim_{n\to\infty}\frac{\frac{1}{n}}{1} = 0$$

Hence, condition (1) is satisfied.

For condition (2) we must show that the sequence given by $a_n = \dfrac{\ln n}{n}$ is decreasing. It is not obvious whether this is true so we consider the derivative of the related function $f(x) = \dfrac{\ln x}{x}$.

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2} = \frac{1-\ln x}{x^2} < 0 \text{ for all } x > e$$

Hence, f is decreasing for $x > e$ which means that $f(n+1) < f(n)$, so it follows that $a_{n+1} \leqslant a_n$ for $n \geqslant 3$.

Therefore, both conditions of the alternating series test have been satisfied and the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$ is convergent.

Take another look at Figure 2.4 that was used in the proof of the alternating series test. Recalling that s is the limit of the partial sums, notice that $|s - s_3| < a_4$, $|s - s_4| < a_5$, $|s - s_5| < a_6$, etc. Furthermore, note that s is always between any two consecutive partial sums. This provides us with the means to estimate the error when we use the partial sum s_n to approximate the sum of an alternating series.

Alternating series estimation theorem

Suppose that $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is a convergent alternating series that satisfies both conditions of the alternating series test and has an unknown sum of s. When estimating s with the sum of the first n terms, the absolute value of the remainder R_n (i.e. the amount of error) is less than or equal to the first unused term. That is,

$$|R_n| = |s - s_n| \leqslant a_{n+1}.$$

In other words, the error generated in estimating the sum with the nth partial sum does not exceed the value of the $n + 1$ term.

Note that the **alternating series estimation theorem** does **not** give a formula for the precise value of the error, but rather a *bound* for the error. Also note that this rule for the bound of the error when estimating s with s_n only applies to alternating series that satisfy the condition of the **alternating series test**.

Proof

As previously mentioned, the sum, s, of a convergent alternating series is always between any two consecutive partial sums. That is,

$$s_n \leqslant s \leqslant s_{n+1}, \text{ if } n \text{ is even and } s_{n+1} \leqslant s \leqslant s_n, \text{ if } n \text{ is odd.}$$

Whether n is even or odd, it follows that

$$|s - s_n| \leqslant |s_{n+1} - s_n|.$$

Given that

$$a_{n+1} = |s_{n+1} - s_n|$$ Remember $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is an alternating series, so $a_{n+1} > 0$.

$$|s - s_n| \leqslant a_{n+1}$$

and therefore the proof is complete.

Example 12

Show that $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$ converges, and find the sum of the series with error less than 0.0001.

Solution

Since $\lim_{n\to\infty}\frac{1}{n^4} = 0$ and $\frac{1}{(n+1)^4} \leqslant \frac{1}{n^4} \Rightarrow n^4 \leqslant (n+1)^4$ is true for all $n \geqslant 1$, the series satisfies both conditions of the alternating series test and therefore converges.

We know from the alternating series estimation theorem that the sum of the first nine terms will give an estimate for the sum with an error of at most

$$a_{9+1} = \frac{1}{10^4} = 0.0001.$$

Our GDC computes the ninth partial sum to be

$$s_9 = -1 + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{8^4} - \frac{1}{9^4} \approx -0.947\,092\,5924.$$

This estimate of the sum of the series is accurate to three decimal places because an error of less than 0.0001 does not affect the third decimal place.

Therefore, the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ is $s \approx -0.947$, correct to three decimal places.

```
-1+1/2^4-1/3^4+1
/4^4-1/5^4+1/6^4
-1/7^4+1/8^4-1/9
^4
        -.947095924
```

Example 13

Determine the convergence or divergence of the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Solution

Applying the alternating series test we have

1 $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

and

2 $a_{n+1} \leqslant a_n \quad \Rightarrow \quad \frac{1}{n+1} \leqslant \frac{1}{n} \quad \Rightarrow \quad n \leqslant n+1$ which is true for all n.

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the alternating series test.

Absolute and conditional convergence

In the next section, we will learn that the alternating harmonic series converges to exactly $\ln 2$.

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots + \frac{(-1)^{n+1}}{n} + \ldots$$

But more relevant to this section is that the result of Example 13 illustrates an important point to investigate further. We know that the harmonic series (a p-series with $p = 1$) diverges. However, if we take the harmonic series and change the sign of alternate terms to get the alternating harmonic series (Example 13), the positive and negative terms offset one another to produce a series that converges even though the series consisting of only positive terms diverges. The same situation is true of the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. You may recall that in Example 5 d) we used the integral test to prove that this series diverges. However, in Example 11 a) of this section we showed that the corresponding alternating series

Absolute and conditional convergence

Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive and negative terms that is convergent.

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ is said to be **absolutely convergent**.

If $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ is said to be **conditionally convergent**.

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ converges. In contrast, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ (Example 12) converges and so does the corresponding series with positive terms $\sum_{n=1}^{\infty} \frac{1}{n^4}$ (a p-series with $p = 4 > 1$). The difference between these two situations requires us to define two types of convergence when considering the convergence of a series with positive and negative terms as occurs with any alternating series.

We have seen then that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ (alternating harmonic series) and $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ are both **conditionally convergent** because for each the series composed of their terms all made positive diverges. Whereas $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ is **absolutely convergent** because its corresponding series of positive terms also converges. You may wonder if it is possible for a series with positive terms, $\sum_{n=1}^{\infty} |a_n|$, to converge, but for a related series with some (or all) of the terms changed to negative, $\sum_{n=1}^{\infty} a_n$, to diverge. The answer is no, and we state the following theorem.

Absolute convergence theorem

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges, and therefore $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

The **absolute convergence theorem** essentially says that it is not possible to take a *convergent* series with *only positive* terms and change some of them to negative to create a new series that is divergent. However, as the alternating harmonic series demonstrates, it is possible to take a convergent series with *positive and negative* terms and change them all to positive to create a new series that is divergent.

Proof

It is true that $0 \leqslant a_n + |a_n| \leqslant 2|a_n|$ because by the definition of absolute value $|a_n|$ is either a_n or $-a_n$. A given condition for the theorem is that $\sum_{n=1}^{\infty} |a_n|$ converges, so $\sum_{n=1}^{\infty} 2|a_n|$ also converges. Therefore, by the comparison test $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges. Since $a_n = (a_n + |a_n|) - |a_n|$, it follows from properties for convergent series that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$ where both series on the right converge. Therefore, $\sum_{n=1}^{\infty} a_n$ must converge. Q.E.D.

When trying to determine if an alternating series is absolutely convergent, conditionally convergent, or divergent, it is most effective to first check if the limit of the nth term is zero. If it is not then the series diverges, and you are finished. If the nth term divergence test is inconclusive then check whether the related series of positive terms converges (using any of the four tests for positive series). If it converges, then by the absolute convergence theorem, the series is absolutely convergent and you are finished. If it diverges, then test the alternating series using the alternating series test. It is inefficient to start by first applying the alternating series test.

Example 14

Classify each series as absolutely convergent, conditionally convergent, or divergent.

a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}$

b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n^2}$

c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(\frac{1}{n}\right)$

Solution

a) $\displaystyle\lim_{n\to\infty}\frac{(-1)^n}{n!}=0$, so result of nth term divergence test is inconclusive. We next consider the corresponding series with only positive terms $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$. Recall that in Example 7 c), we used the comparison test to show $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$ converges. We now apply the alternating series test to $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}$. Knowing $\displaystyle\lim_{n\to\infty}\frac{1}{n}=0$, and since $0<\dfrac{1}{n!}<\dfrac{1}{n}$ for all $n\geqslant 1$ then $\displaystyle\lim_{n\to\infty}\frac{1}{n!}=0$. Thus $\displaystyle\lim_{n\to\infty}a_n=0$. We now need to show that $\dfrac{1}{(n+1)!}\leqslant\dfrac{1}{n!}\Rightarrow n!\leqslant(n+1)!$. Rewriting $(n+1)!$ as $n!(n+1)$ gives $n!\leqslant n!(n+1)$ which is clearly true for all $n\geqslant 1$. Thus $a_{n+1}\leqslant a_n$, and we have satisfied both conditions of the alternating series test. Hence, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}$ converges and converges absolutely because $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n!}\right|=\sum_{n=1}^{\infty}\frac{1}{n!}$ also converges.

b) We can apply the nth term divergence test to show that $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n^2}$ diverges. Recall that the absolute value theorem stated that if $\displaystyle\lim_{n\to\infty}|a_n|=0$ then $\displaystyle\lim_{n\to\infty}a_n=0$. From this we can also say that if $\displaystyle\lim_{n\to\infty}|a_n|\neq 0$ then $\displaystyle\lim_{n\to\infty}a_n\neq 0$. We apply l'Hôpital's rule twice to prove that $\displaystyle\lim_{n\to\infty}\frac{2^n}{n^2}\neq 0$.

$$\lim_{n\to\infty}\frac{2^n}{n^2}=\lim_{n\to\infty}\frac{2^n\ln 2}{2n}=\lim_{n\to\infty}\frac{2^n(\ln 2)^2}{2}=\infty\text{ (does not exist)}$$

Therefore, by the nth term divergence test $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n^2}$ diverges.

c) In Example 8 d) we compared the series $\displaystyle\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$ to the divergent harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ and using the limit comparison test showed

• **Note:** For Example 14 a), we could have been more efficient by applying the absolute convergence theorem since we have previously used the comparison test to show that $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n!}\right|=\sum_{n=1}^{\infty}\frac{1}{n!}$ converges.

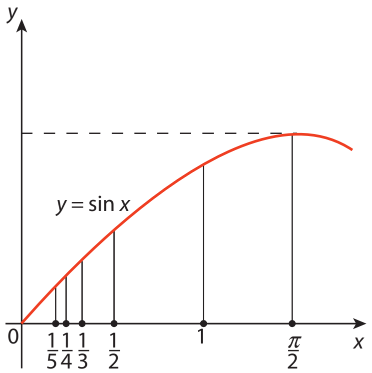


Figure 2.5

that $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ diverges. We now turn our attention to the given series and first need to confirm whether it is an alternating series. Since $\sin\left(\frac{1}{n}\right) > 0$ for all $n \geqslant 1$ then $\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(\frac{1}{n}\right)$ is an alternating series and we can apply the test for alternating series.

The graph shown in Figure 2.5 provides confirmation that not only $\lim_{n\to\infty}\sin\left(\frac{1}{n}\right) = 0$, but also that $\sin\left(\frac{1}{n+1}\right) < \sin\left(\frac{1}{n}\right)$ for all $n \geqslant 1$. Thus the series satisfies the alternating series test and converges. Since the corresponding series of positive terms, $\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$, diverges, $\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(\frac{1}{n}\right)$ converges conditionally.

Rearrangements of conditionally convergent series

The distinction between absolute and conditional convergence is important in many applications of infinite series. It seems perfectly logical that it is possible to rearrange a finite number of terms in an infinite series without affecting the sum. However, if we rearrange an infinite number of terms in an infinite series, the sum is unchanged only if the series is absolutely convergent. An extraordinary characteristic of series that are conditionally convergent is that their terms can be rearranged to form a divergent series, and even rearranged to form a series that converges to *any* predetermined sum. This is a direct consequence of the fact that the sum of an infinite series is defined to be the limit of the sequence of its partial sums. As mentioned previously, this means that operations (such as the associative property) that are valid for finite sums are not valid for infinite sums.

We can illustrate this paradoxical behaviour with the alternating harmonic series that is conditionally convergent. As stated earlier without explanation (next section), the sum of the alternating harmonic series is ln 2.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\ldots=\ln 2 \qquad (1)$$

Consider the following series:

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\ldots \qquad (2)$$

(2) consists of a rearrangement of the same terms as in (1). It is plausible to expect that the sum of the series in (2) is also ln 2.

Let's continue by dividing (1) by 2, giving:

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\ldots=\frac{1}{2}\ln 2 \qquad (3)$$

Now we add (3) and (1):

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\ldots=\ln 2 \qquad (1)$$

$$\frac{1}{2}\qquad-\frac{1}{4}\qquad+\frac{1}{6}\qquad-\frac{1}{8}\qquad+\frac{1}{10}\qquad-\ldots=\frac{1}{2}\ln 2 \qquad (3)$$

The result is

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\ldots=\frac{3}{2}\ln 2 \qquad (4)$$

where the terms are arranged the same as in (2), but the sum is not what we expected. So which is correct, (1) or (4)? The answer is that they are both correct. Although both (1) and (4) are series containing the same terms, by rearranging the terms we have manipulated how the partial sums are formed which affects the limit of the partial sums and, consequently, affects the sum of the series.

Table 2.1 Tests for infinite series.

Test	Converges	Diverges	Notes
nth term divergence test $\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	Can only be used to show divergence
Geometric series $\sum_{n=0}^{\infty} a_1 r^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geqslant 1$	$S_\infty = \frac{a_1}{1-r}$
p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leqslant 1$	Harmonic series when $p = 1$
Integral test $\sum_{n=1}^{\infty} a_n;\ a_n = f(n)$ f is continuous, positive and decreasing	$\int_1^\infty f(x)dx$ converges	$\int_1^\infty f(x)dx$ diverges	$s_n + \int_{n+1}^\infty f(x)dx$ and $s_n + \int_n^\infty f(x)dx$ are bounds for estimation of sum by s_n
Comparison test $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ $0 \leqslant a_n \leqslant b_n$	$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges	$\sum_{n=1}^{\infty} a_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} b_n$ diverges	Useful for series similar to p-series or geometric series
Limit comparison test $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ $a_n > 0, b_n > 0$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L$; if $0 < L < \infty \Rightarrow$ both behave the same $L = 0 \Rightarrow$ if b_n converges then a_n converges $L = \infty \Rightarrow$ if b_n diverges then a_n diverges		Useful if not able to show $0 \leqslant a_n \leqslant b_n$ for direct comparison
Ratio test $\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$	Inconclusive if $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = 1$
Alternating series test $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$	$\lim_{n\to\infty} a_n = 0$ and $0 < a_{n+1} < a_n$		s_n as estimate of sum remainder: $\lvert R_n\rvert < a_{n+1}$

Guidelines for testing series for convergence

Important questions to consider:

1. Is $\lim_{n\to\infty} a_n = 0$? If not, the series diverges by the nth term divergence test.
2. Is the series geometric, or similar to a geometric series? If similar, apply one of the comparison tests.
3. Is the series a p-series, or similar to a p-series? If similar, apply one of the comparison tests.
4. Consider $a_n = f(n)$. Is f a continuous, positive, decreasing function and is it possible to integrate f? If so, try integral test.
5. Does a_n involve n in a product or power, or has an expression with factorials? If so, try the ratio test.
6. Is the series an alternating series? If so, try the alternating series test. Remember that if $\sum |a_n|$ is convergent then $\sum a_n$ is absolutely convergent. Testing $\sum |a_n|$ makes more tests available.

Exercise 2

1 Using properties of convergent series and geometric series, find the sum of each of the series.

a $\displaystyle\sum_{n=0}^{\infty} \frac{7^n}{2^{3n}}$ **b** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{2}{3^n}\right)$ **c** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5^n + 3\left(2^{3n}\right)}{9^n}\right)$

In questions 2–9, write the first four terms of the infinite series and determine whether the series is convergent or divergent. If the series is convergent, find its sum.

2 $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

3 $\displaystyle\sum_{n=1}^{\infty} \frac{3}{4^{n-1}}$

4 $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{1}{n}\right)$

5 $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{3}{2^n}$

6 $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^n}$

7 $\displaystyle\sum_{n=1}^{\infty} \cos(n\pi)$

8 $\displaystyle\sum_{n=1}^{\infty} \frac{2n+3}{5n+6}$

9 $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

10 a Find $\int xe^{-x}dx$ by using the method of integration by parts.

b Use the integral test to determine whether the series $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$ is convergent or divergent.

11 Use the integral test to determine whether the series is convergent or divergent.

a $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ **b** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

12 Show that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^2+3}$ diverges by both of the following methods.

a Using the comparison test, compare the series to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$.

b Using the limit comparison test, compare the series to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

13 Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n3^n}$ converges by **a** the comparison test, and **b** the ratio test.

14 Give an example to show that the converse of the nth term divergence test is false. That is, find a series that diverges even though $\lim_{n\to\infty} a_n = 0$.

15 Use the ratio test to show that $\sum_{n=0}^{\infty} \frac{n^{10}}{10^n}$ converges.

In questions 16–29, determine the convergence or divergence of the series.

16 $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

17 $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$

18 $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

19 $\sum_{n=0}^{\infty} \frac{2^n}{3^n+1}$

20 $\sum_{n=0}^{\infty} \frac{n!}{(n+2)!}$

21 $\sum_{n=1}^{\infty} \frac{2^n}{n+1}$

22 $\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{2n+1}$

23 $\sum_{n=1}^{\infty} \frac{n^3}{(\ln 2)^n}$

24 $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

25 $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^n$

26 $\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{n^2+n}$

27 $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}$

28 $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

29 $\sum_{n=0}^{\infty} \frac{1}{e^n}$

30 Use the integral test to determine whether $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ converges or diverges.

31 Find the sum of the following infinite series.

$$\frac{5}{1\times 2}+\frac{5}{2\times 3}+\frac{5}{3\times 5}+\ldots$$

32 For each series, use the sum of the first four terms to approximate the sum of the series. State an upper bound for the error of the approximation.

a $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)^2}$

b $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

33 **a** Express $n^2 + 2n + 2$ in the form $(n + a)^2 + b$ where a and b are integers.

b Use the integral test to determine whether $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+2}$ converges or diverges.

34 Determine whether $\sum_{n=1}^{\infty} \frac{\arctan n}{n}$ converges or diverges by comparing the series to $\sum_{n=1}^{\infty} \frac{1}{n}$ and applying the limit comparison test.

35 Use the alternating series estimation theorem to determine the minimum number of terms of the series $1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\ldots$ so that an approximation of the sum has an error less than 0.000 05.

36 Give an example of a series that is conditionally convergent. That is, a series that is convergent but not absolutely convergent.

In questions 37–42, determine whether each series converges absolutely, converges conditionally, or diverges.

37 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

38 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$

39 $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{(n+1)^2}$

40 $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

41 $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2}$

42 $\sum_{n=1}^{\infty} \frac{n(-3)^n}{4^{n-1}}$

43 Describe how the terms of the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ can be rearranged so that its sum is 1.

44 What is the minimum number of terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ needed to approximate the sum of the series correct to three decimal places?

45 Prove parts 2 and 3 of the limit comparison test.

Practice questions 2

1 For each infinite series below, determine whether or not the infinite series converges or diverges. Clearly state/explain your reasoning.

a $1 + \frac{1}{1.1} + \frac{1}{1.21} + \frac{1}{1.331} + \frac{1}{1.4641} + \ldots + \frac{1}{(1.1)^{n-1}} + \ldots$

b $e + e^{\frac{1}{2}} + e^{\frac{1}{3}} + e^{\frac{1}{4}} + \ldots$

c $3 + \frac{3}{8} + \frac{3}{27} + \frac{3}{64} + \ldots + \frac{3}{n^3} + \ldots$

2 For each infinite series below, use the indicated convergence test to determine whether the infinite series converges or diverges.

a $\frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \ldots + \frac{n^3}{n!} + \ldots$ [Ratio test]

b $\frac{1}{1 \times 3} + \frac{2}{3 \times 5} + \frac{3}{5 \times 7} + \frac{4}{7 \times 9} + \ldots + \frac{n}{(2n-1)(2n+1)} + \ldots$ [Integral test]

3 By using the Limit Comparison Test, prove that the *general* harmonic series $\sum_{n=1}^{\infty} \frac{1}{an+b}$ diverges for any $a > 0$ and $b > 0$.

4 Test the convergence or divergence of the following infinite series, indicating the tests used to arrive at your conclusion.

a $\sum_{k=1}^{\infty} \frac{k+1}{3^k}$ **b** $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$ **c** $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2+1}$

5 Determine whether the series $\sum_{n=1}^{\infty} \left(\frac{1}{n^{1+\frac{1}{n}}} \right)$ converges.

6 **a** Describe how the integral test is used to show that a series is convergent. Clearly state all the necessary conditions.

b Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges.

7 Find the range of values of x for which the following series is convergent.

$$\sum_{n=0}^{\infty} \frac{x^n}{n+1}$$

8 Determine whether the alternating series $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n+1}$ converges conditionally, converges absolutely or diverges.

9 Use the integral test to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$.

10 Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

a Show that the series is convergent.

b **i** Express $\frac{1}{n(n+2)}$ in partial fractions.

ii Hence find $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

11 Find the interval of convergence of the series $\sum_{n=1}^{\infty} \sin \frac{\pi}{n} x^n$.

12 Determine whether each of the following series converges or diverges.

a $\sum_{n=1}^{\infty} \frac{e}{\sqrt{n}}$ **b** $\sum_{n=1}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ **c** $\sum_{n=1}^{\infty} \frac{2^{3n-1}}{3^{2n+4}}$

13 Show that the series $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ is convergent but not absolutely convergent.

14 Use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{3n^2+1}$ is convergent.

15 Consider the infinite series $-1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{2} - \frac{1}{\sqrt{5}} + \cdots$

a Show that the series converges.

b Determine if the series converges absolutely or conditionally.

3 Power Series

3.1 Power series

Have you ever considered how your calculator computes values for certain functions? For functions such as $f(x) = 3x^2 - 5x + 8$, $g(x) = \dfrac{x^4 + 2x}{-4x^3 + x^2 - 6}$, and $h(x) = \sqrt{7x - 3}$ the method of evaluation is fairly straightforward because these are **algebraic functions**. As explained in Chapter 3 of the book, algebraic functions can be expressed as a finite number of sums, differences, multiples, quotients and radicals involving x^n. Polynomial functions, rational functions and functions involving radicals are examples of algebraic functions. But how does your calculator compute values for a function such as e^x? This is an example of a **transcendental function**. A transcendental function is non-algebraic, i.e. it *cannot* be expressed as a finite number of sums, differences, multiples, quotients and radicals involving x^n. Other familiar transcendental functions include the trigonometric and logarithmic functions.

Except for Example 10 in the previous chapter, all the series we have encountered thus far contained terms consisting of constants. A power series is essentially a polynomial function of infinite degree expressed in terms of a single variable (we will always use x).

Let's return to the primary question we wish to investigate. How does your calculator compute the values of transcendental functions, such as e^x? The manufacturers of the calculator had to decide on a computational algorithm. What computational method could be programmed into a calculator to evaluate e^x for a certain value of x?

The answer lies in the fact that the calculator is summing up a type of series with variable terms, called a **power series**, that is representing e^x. In this section we will see that the power series for the function $f(x) = e^x$ is $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots$. A calculator can only display a finite number of digits and thus it only sums enough terms to produce the necessary degree of accuracy. For example, suppose we wanted to use this series (we'll investigate its derivation later) to evaluate e^2 to three significant figures.

A power series is a very useful mathematical tool that can be used to represent a range of very important functions.

$$e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \ldots$$

Shown below are successively more accurate approximations for the value of e^2 by summing the terms of the power series $\sum_{n=0}^{k} \frac{x^n}{n!}$ for $k = 2, 3, \ldots, 9$.

Once we get past the ninth term in the series we are no longer adding enough to change the first three digits. Thus, the first nine terms of the series are sufficient to give an approximation of e^2 accurate to three significant figures.

```
e^(2)
          7.389056099
```

$$e^2 \approx 1 + 2 = 3$$

$$e^2 \approx 1 + 2 + \frac{2^2}{2!} = 5$$

$$e^2 \approx 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} = 6\tfrac{1}{3} = 6.\overline{3}$$

$$e^2 \approx 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} = 7$$

$$e^2 \approx 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} = 7\tfrac{4}{15} = 7.2\overline{6}$$

$$e^2 \approx 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} = 7\tfrac{16}{45} = 7.3\overline{5}$$

$$e^2 \approx 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \frac{2^7}{7!} = 7\tfrac{8}{21} = 7.3\overline{809523} \approx 7.38$$

$$e^2 \approx 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \frac{2^7}{7!} + \frac{2^8}{8!} = 7\tfrac{122}{315} = 7.3\overline{873015} \approx 7.39$$

$$e^2 \approx 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \frac{2^7}{7!} + \frac{2^8}{8!} + \frac{2^9}{9!} = 7\tfrac{1102}{2835} = 7.38871252205\ldots \approx 7.39$$

A calculator (see screen image above) computes to an accuracy of ten significant figures the value of e^2 to be 7.389 056 099. It certainly appears that the series $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ is converging to e^2. For any given value of x, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is an infinite series. This leads to an important question: For what values of x does the power series converge?

Before addressing this question, let's give a proper definition for a power series.

Performing such computations entirely by hand would be immensely tedious (and prone to error). However, this is not an impediment for an electronic computing device like a GDC. As we will see, the computation process is made more efficient by means of a formula that determines the number of terms required for a power series to produce a value to a given accuracy.

Definition of power series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots + a_n(x-c)^n + \ldots$$

is called a **power series centred at c**, where c is a constant and a_n is the rule that determines each of the coefficients $a_0, a_1, a_2, \ldots$. Note that we have $(x-c)^0 = 1$ even when $x = c$.

For any power series centred at $c = 0$, we have

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots.$$

Radius of convergence

At each value of x, a power series becomes a series of constants. In the previous section we gave a great deal of attention to such series, investigating whether they converge or diverge. The issue of convergence is very important for power series because for each value of x for which a power series converges, the series represents the number that is the sum of the series. Therefore, **a power series defines a function**. The function $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ has as its domain all values of x for which the power series converges. It is evident that every power series is convergent for $x = c$. Some power series are only convergent at $x = c$ (see Example 3). Far more useful power series will converge for a finite interval with the same centre as the series (see Example 1), or converge for all x (see Example 2).

Example 1

For the general power series $\sum_{n=0}^{\infty} a_n (x-c)^n$, if we let $a_n = 1$ for all n and 'centre' the series at $c = 0$, we get the geometric series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$ having first terms $a_1 = 1$, and common ratio $r = x$.

The sum formula for geometric series assures us that this series converges to $\frac{1}{1-x}$ when $-1 < x < 1$, and consequently diverges when $|x| \geqslant 1$.
Therefore, we can write

$$1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \frac{1}{1-x}, \qquad -1 < x < 1.$$

The expression on the right side of this equation defines a function whose domain is $x \in \mathbb{R}$, $x \neq 1$. The expression on the left side defines a function whose domain is the interval $-1 < x < 1$. The equation can only be true where both sides are defined, so its domain is $-1 < x < 1$, equivalent to $|x| < 1$. On this domain, the given power series is a **polynomial representation** of the function $f(x) = \frac{1}{1-x}$ (Figure 3.1, on next page). A power

series is best regarded as an attempt to describe a function *locally*, near where it is 'centred', i.e. near the value of c. To illustrate this point, partial sums of the series $\sum_{n=0}^{\infty} x^n$ with 3, 6 and 9 terms have been graphed in Figure 3.2. Figure 3.3 shows the same three partial sums along with $f(x) = \frac{1}{1-x}$ focused on the interval $-1 < x < 1$.

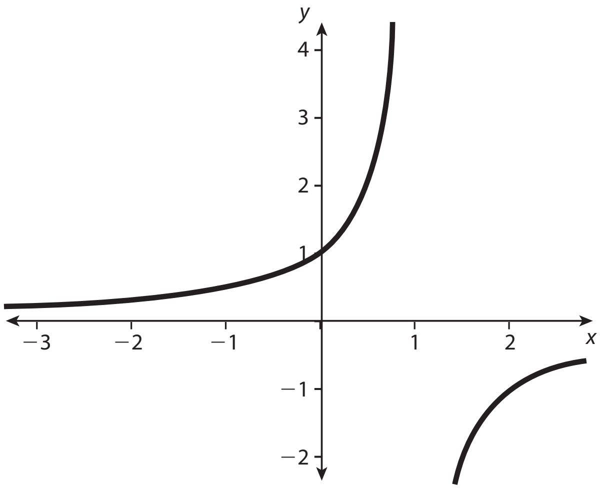


Figure 3.1 Graph of $y = \dfrac{1}{1-x}$.

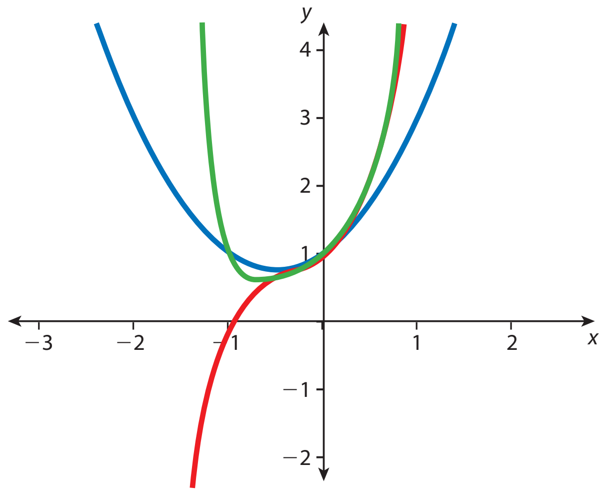


Figure 3.2 Graphs of the partial sums $1 + x + x^2$, $1 + x + x^2 + x^3 + x^4 + x^5$ and $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8$.

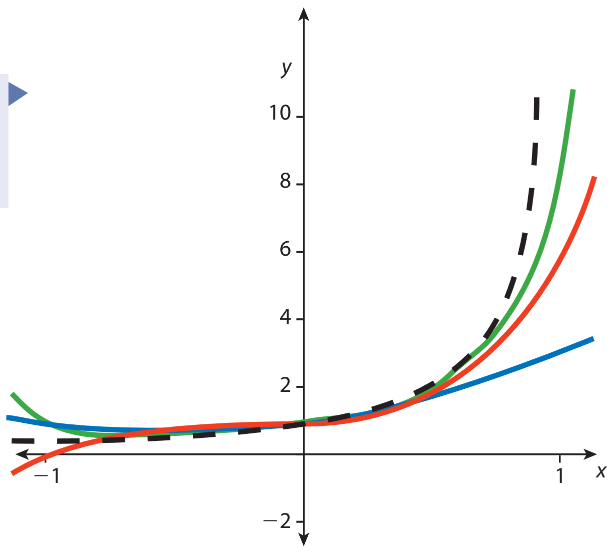


Figure 3.3 The partial sums $1 + x + x^2$, $1 + x + x^2 + x^3 + x^4 + x^5$, $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8$ and $\dfrac{1}{1-x}$ (dashed).

Observe how in the interval $-1 < x < 1$ the graph of a partial sum of $\sum_{n=0}^{\infty} x^n$ gets closer to that of the graph of $f(x) = \dfrac{1}{1-x}$ as the number of terms increase, but are not close outside this interval.

Example 2

We've demonstrated that the power series $\sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots + \dfrac{x^5}{n!} + \ldots$ represents the function $f(x) = e^x$. Find the values of x for which this power series converges.

Solution

Example 10 in the previous chapter illustrated that the ratio test is effective for answering this kind of question. Applying the ratio test gives the following inequality to solve.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| < 1$$

$$\lim_{n\to\infty}\left|\frac{x^n x}{(n+1)n!}\cdot\frac{n!}{x^n}\right| < 1$$

$$\lim_{n\to\infty}\left|\frac{x}{n+1}\right| < 1$$

$$|x|\cdot\lim_{n\to\infty}\left|\frac{1}{n+1}\right| < 1$$

$$|x|\cdot 0 < 1$$

$$0 < 1$$

Therefore, the power series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ is convergent for all real values of x.

This means that as we compute partial sums of more and more terms for $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ they will become ever more accurate representations of the function $f(x) = e^x$ over all real numbers, and not just constrained to a finite interval as occurred in the previous example. This is illustrated by comparing graphs of $f(x) = e^x$ (Figure 3.4) to graphs of partial sums of $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ with 4, 5 and 10 terms in Figure 3.5.

Figure 3.4 Graph of $y = e^x$.

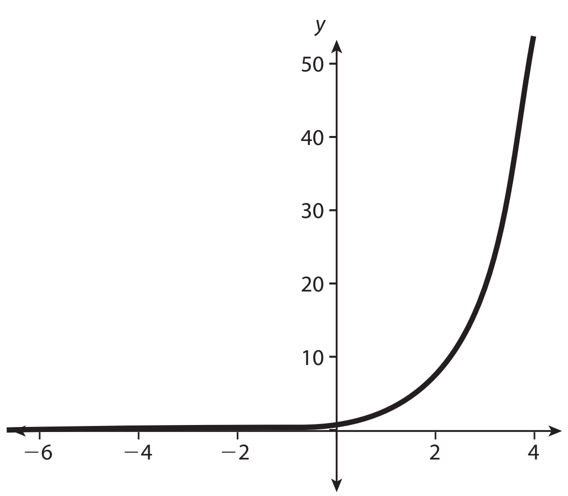
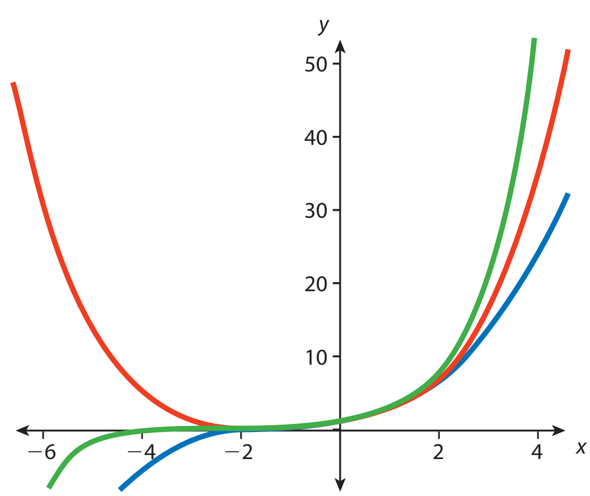


Figure 3.5 The partial sums $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$, $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$ and $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!}$.



Example 3

Find the values of x for which $\sum_{n=0}^{\infty} n!x^n$ is convergent.

Solution

Again, applying the ratio test gives:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\ &= \lim_{n\to\infty}\left|x(n+1)\right| \\ &= |x|\lim_{n\to\infty}(n+1) \\ &= \infty, \quad x \neq 0\end{aligned}$$

Hence, the series diverges for all values of x, $x \neq 0$. We need to check the series when $x = 0$.

$$\sum_{n=0}^{\infty} n!0^n = 1 + 0 + 0 + \ldots = 1$$

Therefore, the power series $\sum_{n=0}^{\infty} n!x^n$ converges only at its centre, at $x = 0$.

As illustrated in the preceding three examples, the domain of a power series can be a single point, an interval of the real numbers centred at c, or all real numbers. The following theorem (which we present without proof) states that these are the only possibilities.

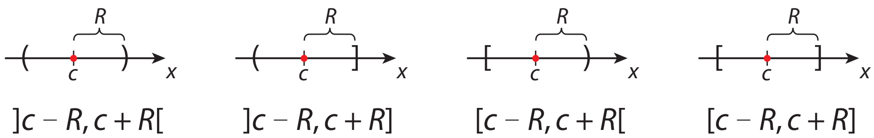
Convergence of a power series theorem

For a given power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ exactly one of the following is true:

1. The series converges only when $x = c$.
2. The series converges for all real values of x.
3. There is a positive number R such that the series converges for $|x - c| < R$, and diverges for $|x - c| > R$. The series may or may not converge at either of the endpoints $x = c - R$ and $x = c + R$ so we need to check for convergence at each of the endpoints.

The set of all values of x for which a given power series is convergent is called the **interval of convergence** of the power series. The number R of possibility **3** in the theorem is called the **radius of convergence** of the power series. If possibility **1** occurs, then $R = 0$; and if possibility **2** occurs, then $R = \infty$. For possibility **3**, there remains the question of what happens at the endpoints of the interval. Each endpoint must be tested separately to determine if the series converges or diverges for that value of x. Thus, if $0 < R < \infty$ the interval of convergence can be one of four different kinds, as illustrated in Figure 3.6.

Figure 3.6 Intervals of convergence.



In Example 1 we showed that the interval of convergence for the power series $\sum_{n=0}^{\infty} x^n$ is $-1 < x < 1$, so $R = 1$. But we did not check the endpoints. At $x = -1$, $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \ldots$; and at $x = 1$, $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 1^n = 1 + 1 + 1 + \ldots$. Both of these series diverge confirming that the interval of convergence is, in fact, $-1 < x < 1$. In Example 2, $R = \infty$; and, in Example 3, $R = 0$.

As we did in Examples 2 and 3, it is best to use the ratio test to determine the radius of convergence for a power series. The ratio test will fail when x is an endpoint of the interval of convergence, so you will need to check endpoints with one of the other tests from the previous chapter.

Example 4

Find the radius of convergence and interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{2n(3^n)}.$$

Solution

We apply the ratio test:

$$\lim_{n\to\infty}\left|\frac{(x-4)^{n+1}}{2(n+1)(3^{n+1})}\cdot\frac{2n(3^n)}{(x-4)^n}\right| = \lim_{n\to\infty}\left|\frac{2n(x-4)}{3(2n+2)}\right|$$

$$= \lim_{n\to\infty}\left(\frac{2n}{6n+6}\right)|x-4|$$

$$= \frac{1}{3}|x-4|$$

Hence, the series converges for $\frac{1}{3}|x-4| < 1$ or $|x-4| < 3$.

The radius of convergence is $R = 3$. The series is centred at $c = 4$, so the series converges for

$$4 - 3 < x < 4 + 3 \text{ or } 1 < x < 7.$$

At $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{(-3)^n}{2n(3^n)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$, which converges by the alternating series test.

At $x = 7$, the series is $\sum_{n=1}^{\infty} \frac{(3)^n}{2n(3^n)} = \sum_{n=1}^{\infty} \frac{1}{2n}$, which diverges by limit comparison with the harmonic series. Therefore, the interval of convergence is $1 \leqslant x < 7$, or $x \in [1, 7[$.

Why are we interested in power series? One reason is that power series share many of the desirable properties of polynomials. In particular, polynomial functions are generally much easier to differentiate and integrate. The power series representation of a particular function may enable us to perform some difficult operations on the function that would otherwise be quite difficult, for example differentiation and integration of the function.

3.2 Maclaurin and Taylor series

In the preceding part of this chapter we were given the power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}$$

without any indication about its origin. In this section we will devise a general method for constructing a power series.

As we stated in our work for Example 1, 'a power series is best regarded as an attempt to describe a function *locally*, near where it is "centred", i.e. near the value of c', and that 'a power series is essentially a **polynomial function** of infinite degree'. With theses ideas in mind, we propose the following.

To find a polynomial function P that represents another function f, begin by choosing a number c in the domain of f at which f and P have the same value. This is the first requirement – that $P(c) = f(c)$. Thus, the graphs of f and P will pass through the same point, $(c, f(c))$. The approximating polynomial P is said to be *expanded about c* or *centred at c*. Of course, there will be many polynomial functions that will have the same value as f at $x = c$. We can make the graph of P further resemble that of f near the point they share by a second requirement: P has the same slope of f at $x = c$, that is, $P'(c) = f'(c)$. We can continue to improve how well P mimics the behaviour of f near c by additionally requiring that $P''(c) = f''(c)$, $P^{(3)}(c) = f^{(3)}(c)$, and so on.

Deriving the power series for $f(x) = e^x$ centred at 0

We start by finding a first degree polynomial $P_1(x) = a_0 + a_1 x$ whose value and slope agree with the value and slope of $f(x) = e^x$ at $x = 0$, that is, $P(0) = f(0)$ and $P'(0) = f'(0)$.

$$f(0) = e^0 = 1 \qquad P_1(0) = a_0 + a_1(0) = 1 \;\Rightarrow\; a_0 = 1$$

$$f'(x) = e^x \;\Rightarrow\; f'(0) = e^0 = 1 \qquad P_1'(x) = a_1 \;\Rightarrow\; P_1'(0) = a_1 = 1 \;\Rightarrow\; a_1 = 1$$

Therefore, $P_1(x) = 1 + x$.

Now to find the second degree polynomial approximation $P_2(x) = a_0 + a_1 x + a_2 x^2$ we require that $P''(0) = f''(0)$, knowing that $a_0 = 1$ and $a_1 = 1$.

$$f''(x) = e^x \;\Rightarrow\; f''(0) = e^0 = 1 \qquad P_2'(x) = a_1 + 2a_2 x$$

$$P_2''(x) = 2a_2 \;\Rightarrow\; P_2''(0) = 2a_2 = 1 \;\Rightarrow\; a_2 = \frac{1}{2}$$

Therefore, $P_2(x) = 1 + x + \frac{1}{2}x^2$.

We continue and find the third degree polynomial $P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ and require that $P^{(3)}(0) = f^{(3)}(0)$ (third derivatives are equal).

$$f^{(3)}(x) = e^x \Rightarrow f^{(3)}(0) = e^0 = 1 \quad P_3'(x) = a_1 + 2a_2x + 3a_3x^2$$
$$P_3''(x) = 2a_2 + 6a_3x$$
$$P_3^{(3)}(x) = 6a_3 \Rightarrow P_3^{(3)}(0) = 6a_3 = 1 \Rightarrow a_3 = \frac{1}{6}$$

Therefore, $P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$.

From the function values for $f(x) = e^x$ (six significant figures), $P_2(x)$ and $P_3(x)$ displayed in Table 3.1, we can make two observations: (**1**) the accuracy of the approximating polynomial improves as $x \to 0$; and we would expect, in general, as $x \to c$; and (**2**) the higher the degree of the approximating polynomial (more terms of the partial sum of the power series), the better the polynomial represents the function f.

x	−1.0	−0.2	−0.1	0	0.1	0.2	1.0
$f(x) = e^x$	0.367 879	0.818 731	0.904 837	1	1.105 17	1.221 40	2.718 28
$P_2(x)$	0.5	0.82	0.905	1	1.105	1.22	2.5
$P_3(x)$	$0.\overline{3}$	$0.818\overline{6}$	$0.9048\overline{3}$	1	$1.1051\overline{6}$	$1.221\overline{3}$	$2.\overline{6}$

Table 3.1 Comparing $P_2(x) = 1 + x + \frac{1}{2}x^2$, $P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$, and $f(x) = e^x$.

Let's apply the procedure one more time to find $P_4(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$.

$$f^{(4)}(x) = e^x \Rightarrow f^{(4)}(0) = e^0 = 1 \quad P_4'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$$
$$P_4''(x) = 2a_2 + 6a_3x + 12a_4x^2$$
$$P_4^{(3)}(x) = 6a_3 + 24a_4x$$
$$P_4^{(4)}(x) = 24a_4 \Rightarrow P_4^{(4)}(0) = 24a_4 = 1 \Rightarrow a_3 = \frac{1}{24}$$

Therefore, $P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$.

By now we can see the pattern for $P_n(x)$.

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!}$$

We complete the derivation of the power series for $f(x) = e^x$ by determining its interval of convergence by means of the ratio test and checking interval endpoints, if they exist. In this way we derive the result that was presented to us at the start of this chapter. That is,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}.$$

General result

We now wish to generalize the method for finding the power series representation for a given function $f(x)$. For expansions about an arbitrary value of c, it is convenient to write the polynomial in the standard form from the definition.

$$P_n(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \ldots + a_n(x - c)^n$$

where $P_n(x)$ is the nth partial sum of the infinite series

$$\sum_{n=0}^{\infty} a_n(x - c)^n.$$

We will streamline the procedure that we used for deriving the power series for $f(x) = e^x$. Repeated differentiation of P_n produces

$$P_n'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \ldots + na_n(x-c)^{n-1}$$
$$P_n''(x) = 2a_2 + 2(3a_3)(x-c) + \ldots + n(n-1)a_n(x-c)^{n-2}$$
$$P_n^{(3)}(x) = 2(3a_3) + \ldots + n(n-1)(n-2)a_n(x-c)^{n-3}$$
$$\vdots$$
$$P_n^{(n)}(x) = n(n-1)(n-2)\ldots(2)(1)a_n$$

Evaluating P_n and its first n derivatives at $x = c$, we obtain the following:

$$P_n(c) = a_0 \qquad P_n'(c) = a_1 \qquad P_n''(c) = 2a_2 \qquad \ldots \qquad P_n^{(n)}(c) = n!a_n$$

Because the value of f and its first n derivatives must agree with the value of P_n and its first n derivatives at $x = c$, it follows that:

$$a_0 = f(c) \qquad a_1 = f'(c) \qquad a_2 = \frac{f''(c)}{2!} \qquad \ldots \qquad a_n = \frac{f^{(n)}(c)}{n!}$$

Consequently, we assert that if f has a power series representation centred at $x = c$, that is, if

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n, \quad |x-c| < R$$

then its coefficients are given by the formula

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

Substituting this formula for a_n into $\sum_{n=0}^{\infty} a_n(x-c)^n$ we determine that if a function f has a power series representation centred at $x = c$, then it will take the form as defined below.

Definition of Taylor series and Maclaurin series

If a function f has derivatives of all orders at $x = c$, then the series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots$$

is called the **Taylor series of the function f centred at c.** As often occurs, if $c = 0$, then the series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

and is called the **Maclaurin series for f.**

For the simplicity of notation we agree to define the 'zeroth' derivative, $f^{(0)}$, to be the function f itself.

Although the English mathematician Brook Taylor (1685–1731) did not invent the process of using polynomial approximations for transcendental functions (others such as Leibniz, Johann Bernoulli and Abraham de Moivre had already used series that we would call Taylor series), Taylor was the first to provide a comprehensive study of their importance and applications in a textbook he wrote in 1715. Similarly, the Scottish mathematician Colin Maclaurin (1698–1746) did not invent the series named for him. He also wrote a textbook (in 1742) that greatly furthered applications of power series, so much so that Taylor series centred at $x = 0$ became known as Maclaurin series. Isaac Newton (1642–1727) was an early innovator with power series, especially his work in writing binomial expressions as series. A contemporary of Newton's was another brilliant Scottish mathematician, James Gregory (1638–1675). Gregory published power (Maclaurin) series for tan x, sec x, arctan x and arcsec x ten years before Maclaurin was born. Apparently Taylor was not aware of Gregory's mathematical work when writing his 1715 textbook.

Example 5

Find the Maclaurin series for $f(x) = \sin x$ and its interval of convergence.

Solution

To use the definition of a Maclaurin series we need to repeatedly differentiate $f(x) = \sin x$, and evaluate each derivative at $x = 0$, sufficiently to establish a pattern for the power series.

$$\begin{aligned} f^{(0)}(x) &= f(x) = \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1 \end{aligned}$$

The pattern that emerges repeats in blocks of four. Thus, the power series is

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots \\ &= 0 + 1 \cdot x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots \end{aligned}$$

Therefore, the Maclaurin series for $f(x) = \sin x$ is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

Using the ratio test, we have

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| &= \lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right| \\ &= \left| x^2 \right| \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} \\ &= \left| x^2 \right| \cdot 0 \\ &= 0 < 1 \end{aligned}$$

Hence, the power series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for $\sin x$ converges for all real values of x.

Finding a Taylor series centred at *c* for a function *f*(*x*)

Step 1: Compute several derivatives of f. The 'zeroth' derivative, $f^{(0)}(x)$, is $f(x)$ itself.

Step 2: Evaluate the derivatives at $x = c$, and try to identify a pattern for the values of $f^{(n)}(c)$.

Step 3: Knowing that the rule for coefficients of a Taylor series is $a_n = \dfrac{f^{(n)}(c)}{n!}$, write the formula for the Taylor series, substituting into $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$.

Step 4: Using the ratio test determine the radius of convergence for the series.

Step 5: If $0 < R < \infty$, use an appropriate convergence test to check for convergence/divergence at both of the interval endpoints $x = c - R$ and $x = c + R$. State the interval of convergence.

Finding a Taylor (or Maclaurin) series by this five-step process can prove to be difficult, primarily finding a pattern for the values of $f^{(n)}(c)$. We now establish some very useful properties of power series, giving us 'short cuts' for finding Taylor series from a known Taylor series.

3.3 Operations with power series

The power and versatility of representing functions with power series is due largely to the fact that they retain many of the properties of 'finite' polynomials. Two of the properties that make polynomials particularly useful in calculus are the ease with which they can be differentiated and integrated term by term. It is natural to ask whether the same holds true for power series. The answer is provided by the next theorem, which we state without proof.

Differentiation and integration of power series

If R is the radius of convergence of the power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

then

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n (x-c)^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(a_n (x-c)^n\right) = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1}$$

for $c - R < x < c + R$

and

$$\int f(x)\,dx = \int\left(\sum_{n=0}^{\infty} a_n (x-c)^n\right)dx = \sum_{n=0}^{\infty}\left(\int a_n (x-c)^n\right)dx = \sum_{n=0}^{\infty} \frac{1}{n+1}(x-c)^{n+1}$$

for $c - R < x < c + R$.

The above theorem means that term-by-term differentiation or integration of a power series produces a power series that converges, respectively, to the derivative or integral of the function represented by the original series, given that we are in the interval of convergence of the original series. This

The radius of convergence of the series produced by differentiating or integrating a power series is equivalent to that of the original series. However, the interval of convergence may change due to convergence/divergence changing at one or both of the endpoints.

gives us another way to derive power series representations from a known power series, as the next example illustrates.

Example 6

Find the Maclaurin series for $f(x) = \cos x$.

Solution

We could apply the formula for a Maclaurin series as we did in Example 5 but it will be more efficient to simply differentiate the power series that we already established for $f(x) = \sin x$.

$$\begin{aligned}\cos x = \frac{d}{dx}(\sin x) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)\\ &= \sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)\\ &= \sum_{n=0}^{\infty}\frac{d}{dx}\left((2n+1)\frac{(-1)^n x^{2n}}{(2n+1)!}\right)\\ &= \sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{(-1)^n x^{2n}}{(2n)!}\right)\\ &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\end{aligned}$$

Note: $(2n+1)! = (2n+1)(2n)!$

The power series for $\sin x$ converges for all x, so the same is true for this power series for $\cos x$.

Therefore,

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots+(-1)^n\frac{x^{2n}}{(2n)!}+\dots \text{ for } x\in\mathbb{R}.$$

For convenience, we state the operations with power series using series centred at 0, but they also apply for series centred at c, e.g. $\sum_{n=0}^{\infty} a_n(x-c)^n$.

A further beneficial property of power series is that algebraic manipulations valid for polynomials also work for power series. These are described below.

It is important to note that the operations for power series stated here can change the interval of convergence for the resulting series. In general, the interval of convergence of the resulting series for properties **1** and **2** will be the same as the original series; and for properties **3**, **4** and **5** it will be the intersection of the intervals of convergence of the two original series.

Properties of power series

Given the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ and constant k, the following hold true:

1 $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$

2 $f(x^k) = \sum_{n=0}^{\infty} a_n (x^k)^n = \sum_{n=0}^{\infty} a_n x^{kn}$

3 $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

4 $f(x)\cdot g(x) = \sum_{n=0}^{\infty} (a_n b_n x)^n$

5 $\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty}\left(\frac{a_n}{b_n}\right)x^n, b_n \neq 0$

Example 7

a) Use the fact that the Maclaurin series for $f(x) = \frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$ to find the Maclaurin series for $g(x) = \frac{1}{1+x}$.

b) Use the result from a) and integration of power series to find the Maclaurin series for $h(x) = \ln(1+x)$.

Solution

a) We can create the power series for $g(x) = \frac{1}{1+x}$ by simply substituting $-x$ for x into the power series for $f(x) = \frac{1}{1-x}$. Thus,

$$g(x) = f(-x) = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

and we can write this result as

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + \ldots \text{ for } -1 < x < 1.$$

Note: $\sum_{n=0}^{\infty}(-1)^n x^n$ diverges at both endpoints so the interval of convergence is the same as for the original series $\sum_{n=0}^{\infty} x^n$.

b) Knowing that $\int \frac{1}{1+x}\,dx = \ln(1+x)$, we can integrate the power series for $\frac{1}{1+x}$ to obtain the power series for $\ln(1+x)$.

$$\begin{aligned}\ln(1+x) &= \int \frac{1}{1+x}\,dx = \int\left(\sum_{n=0}^{\infty}(-1)^n x^n\right)dx \\ &= \sum_{n=0}^{\infty}\int\left((-1)^n x^n\right)dx \\ &= \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n+1}\right)x^{n+1} \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}\end{aligned}$$

Starting the index at $n = 1$ rather than $n = 0$.

Note: At $x = -1$, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ becomes the divergent harmonic series and at $x = 1$ it becomes the convergent alternating harmonic series, so the interval of convergence changes to $-1 < x \leqslant 1$.

Therefore, we can write the result as

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots \text{ for } -1 < x \leqslant 1.$$

Knowing that $\int \frac{1}{1+x^2}\,dx = \arctan x$, we can use the same approach used in Example 7 b) to show that the Maclaurin series for $\arctan x$ is

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + \ldots \text{ for } -1 \leqslant x \leqslant 1.$$

We now have derived Maclaurin series for several important functions. We list these below with their corresponding intervals of convergence. These series can be used to construct other series.

Except for the Maclaurin series for $\frac{1}{1-x}$, the Maclaurin series listed here are given in the Mathematics HL Formula booklet.

Selected Maclaurin series

Function	Interval of convergence
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$	$-1 < x < 1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$	$-\infty < x < \infty$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$	$-1 < x \leqslant 1$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$	$-\infty < x < \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$	$-\infty < x < \infty$
$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$	$-1 \leqslant x \leqslant 1$

Example 8

a) Find the Maclaurin series for $f(x) = e^{x^2}$.

b) Hence, find a series for $\int e^{x^2}\,dx$.

c) Use the first four terms of the series in b) to approximate the value of $\int_0^1 e^{x^2}\,dx$.

Solution

a) We prefer not to derive a Taylor series (centred at 0, in this case) by the direct method used to find the series for $\sin x$ in Example 5, if there is an alternative approach. To find the Maclaurin series for e^{x^2} we can simply substitute x^2 in for x in the Maclaurin series for e^x.

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots \text{ for } -\infty < x < \infty$$

This series converges for all real values of x because the original series for e^x did so.

b) We can take the result from a) and integrate it term by term.

$$\int e^{x^2}\,dx = \int\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots + \frac{x^{2n}}{n!} + \ldots\right)dx$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} + \frac{x^7}{7\cdot 3!} + \ldots + \frac{x^{2n+1}}{(2n+1)n!} + \ldots$$

That is,

$$\int e^{x^2}\,dx = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)n!}$$

and because the series for e^{x^2} converged for all real values of x, then this series does also.

c) The first four terms of the series for $\int e^{x^2}\,dx$ are

$$\sum_{n=0}^{3}\frac{x^{2n+1}}{(2n+1)n!} = x + \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} + \frac{x^7}{7\cdot 3!} = x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42}.$$

We use this to approximate the value of the definite integral $\int_0^1 e^{x^2}\,dx$.

$$\int_0^1 e^{x^2}\,dx \approx \left[x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42}\right]_0^1$$

$$= 1 + \frac{1}{3} + \frac{1}{10} + \frac{1}{42}$$

$$= \frac{51}{35} = 1\frac{16}{35} \approx 1.457\,142\,857 \text{ (to ten significant figures)}$$

Also, to ten significant figures, a GDC computes the value of $\int_0^1 e^{x^2}\,dx$ to be 1.462651746.

The percentage difference between the value from the partial sum of the series for e^{x^2} and the calculator value is only about 0.377%.

We are unable to find a function that is an anti-derivative of e^{x^2} with the calculus techniques covered in this course. Helping us to integrate such functions is a very useful application of power series.

Continuing with the idea of using power series for approximation purposes, recall the discussion at the start of this chapter. As in the last example, when a power series is used (by your GDC, for example) to compute an approximate value for a function it does so with a suitable partial sum of the power series that, by definition, will be a polynomial. Given the definition of a Taylor series

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$$

it follows that $f(x)$ is the limit of the partial sums. In our derivation of the power series for $f(x) = e^x$ centred at 0 we found four partial sums that were polynomials of degree 1, 2, 3 and 4.

Taylor polynomials

In general the nth partial sum of a Taylor series is

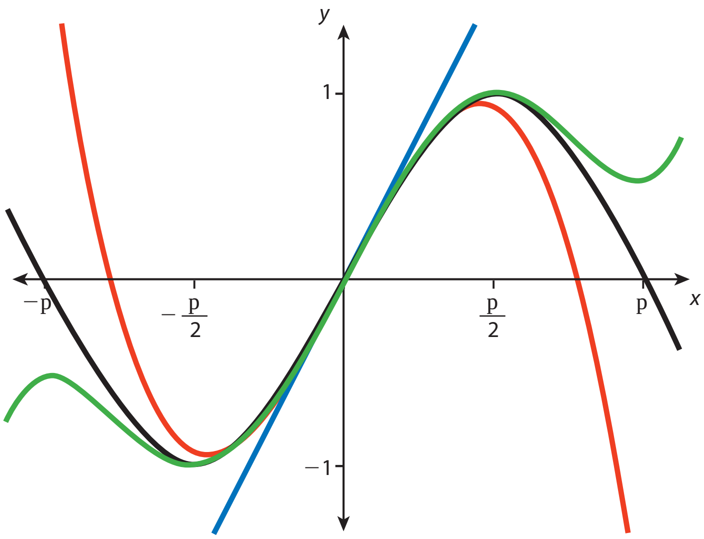
$$P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k$$

which is a polynomial, typically called a **Taylor polynomial**, of degree n.

For instance, the result for Example 5 shows that the Taylor polynomials for $f(x) = \sin x$ centred at 0 for $n = 1, 3$ and 5 are:

$$P_1(x) = x \qquad P_3(x) = x - \frac{x^3}{3!} \qquad P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Figure 3.7 The sine function (in black) and the three Taylor polynomials $P_1(x) = x$, $P_3(x) = x - \frac{x^3}{3!}$ and $P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$.



Since the error term for a Taylor polynomial results from 'cutting off' the series after a certain number of terms to a finite polynomial it is occasionally referred to as the **truncation error**.

In general, $f(x)$ is the limit of the sequence of its Taylor polynomials. That is,

$$f(x) = \lim_{n\to\infty} P_n(x).$$

Because any particular Taylor polynomial, $P_n(x)$, is an approximation to $f(x)$ we can write

$$f(x) = P_n(x) + R_n(x)$$

where $R_n(x)$ is the **remainder**, also called the **error term**.

We are now in a position to present the following important theorem that not only gives a formula for the Taylor polynomial for approximating a function f but also two formulae for computing the error term.

Although the Lagrange form cannot compute the error exactly, it does provide an efficient and accurate way to find the maximum error. By finding the value of b between c and x (inclusive) that maximizes $f^{(n+1)}(b)$, thereby maximizing $\frac{f^{(n+1)}(b)}{(n+1)!}(x-c)^{n+1}$, we can compute the maximum error from using an nth degree Taylor polynomial centred at c to approximate $f(x)$ for a specific value of x.

Taylor's theorem

If a function f has derivatives of all orders in an open interval I centred at c, then for each positive integer n and for each x in I,

$$f(x) = P_n(x) + R_n(x)$$

where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$ is the ***n*th degree Taylor polynomial** centred at $x = c$.

The **error term**, $R_n(x)$, can be computed with the following formula:

$$R_n(x) = \frac{f^{(n+1)}(b)}{(n+1)!}(x-c)^{n+1}$$ where b lies between c and x, inclusive (**Lagrange form**)

Example 9

a) Find the fifth degree Taylor polynomial centred at $x = 1$ for $f(x) = \ln x$.

b) Use this polynomial to approximate $\ln(1.3)$.

c) Use the Lagrange error term to determine an upper bound to the error in this approximation.

d) How many terms of the Taylor series centred at $c = 1$ are needed to approximate $\ln(1.3)$ so that the error is less than $0.000001 = 1.0 \times 10^{-6}$.

Solution

a) We can construct the Taylor series centred at $x = 1$ by substituting $x - 1$ for x in the Maclaurin series for $\ln(1 + x)$.

Since $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ for $-1 < x \leqslant 1$, then

$$\ln(1+(x-1)) = \ln x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots \text{ for } 0 < x \leqslant 2.$$

Hence, the fifth degree Taylor polynomial centred at $x = 1$ is

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{4}.$$

b) $\ln(1.3) \approx (0.3) - \frac{(0.3)^2}{2} + \frac{(0.3)^3}{3} - \frac{(0.3)^4}{4} + \frac{(0.3)^5}{4} = 0.262\,461$

c) For the Lagrange form of the error term we have $x = 1.3$, $n = 5$, $c = 1$. By computing a few derivatives of $f(x) = \ln x$

$$f'(x) = \frac{1}{x},\ f''(x) = -\frac{1}{x^2},\ f^{(3)}(x) = \frac{2}{x^3},\ f^{(4)}(x) = -\frac{6}{x^4}$$

we can see that the nth derivative is $f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{x^n}$. Therefore, the sixth derivative at $x = b$ is $f^{(6)}(b) = -\frac{5!}{b^6}$. We need to estimate its largest magnitude, that is, $\left|-\frac{5!}{b^6}\right| = \frac{120}{b^6}$ as b ranges from 1 to 1.3. Clearly it has a maximum when $b = 1$. Therefore,

$$R_5(1.3) \leqslant \frac{120}{(5+1)!}(1.3-1)^{5+1} = \frac{0.087\,48}{720} = 0.000\,1215.$$

The error is no larger than 0.000 1215. Given our approximation $\ln(1.3) \approx 0.262\,461$, we can say with absolute certainty that the exact value of $\ln(1.3)$ is somewhere between 0.262 3395 and 0.262 5825.

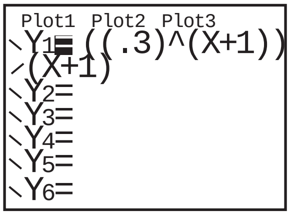
d) The magnitude of the $(n+1)$st derivative is given by

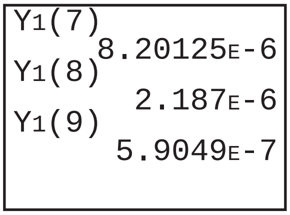
$\left|f^{(n+1)}(b)\right| = \left|(-1)^n\frac{n!}{b^{n+1}}\right| = \frac{n!}{b^{n+1}}$. As in the previous part, we have $x = 1.3$

and $c = 1$, so $1 \leqslant b \leqslant 1.3$; and the $(n+1)$st derivative will be largest when $b = 1$. Therefore,

$$\max\left[R_n(1.3)\right] = \frac{n!}{(n+1)!}(1.3-1)^{n+1} = \frac{(0.3)^{n+1}}{n+1} < 1.0 \times 10^{-6}.$$

By trial and error (see GDC images below), we determine that the smallest value of n that satisfies this inequality is $n = 9$. So, the ninth degree Taylor polynomial would give us an error of less than 1.0×10^{-6} when approximating $\ln(1.3)$.





Example 10

a) Express the indefinite integral $\int \frac{\sin x}{x}\,dx$ as an infinite series.

b) Evaluate the definite integral $\int_0^1 \frac{\sin x}{x}\,dx$ accurate to three decimal places.

Solution

a)
$$\int \frac{\sin x}{x}\,dx = \int \frac{\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}}{x}\,dx = \int \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n+1)!}\,dx$$
$$= \sum_{n=0}^{\infty}(-1)^n \int \frac{x^{2n}}{(2n+1)!}\,dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!}$$
$$= x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \ldots$$

b)
$$\int_0^1 \frac{\sin x}{x}\,dx = \left[\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!}\right]_0^1$$
$$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)(2n+1)!}$$
$$= 1 - \frac{1}{18} + \frac{1}{600} - \frac{1}{29400} + \ldots$$

We can use the alternating series estimation theorem that says that the error generated in using the nth partial sum of an alternating series to approximate its sum does not exceed the value of the $(n + 1)$st term. Which is the first term to have a magnitude so that adding/subtracting it will not change the third decimal point of the sum?

$a_3 = \frac{1}{600} = 0.001\bar{6}$, $a_4 = \frac{1}{29400} \approx 0.000034$. Hence, the first three terms will give us an estimate with a maximum error of about 0.000 034, sufficient to guarantee an estimate accurate to three decimal places.

Therefore,

$$\int_0^1 \frac{\sin x}{x}\,dx \approx 1 - \frac{1}{18} + \frac{1}{600} \approx 0.946$$

A GDC confirms our result.

```
fnInt(sin(X)/X,X
,0,1)
      .9460830704
```

Exercise 3

In questions 1–14, determine the radius of convergence and interval of convergence of the power series.

1 $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$

2 $\sum_{n=1}^{\infty} n(x-2)^n$

3 $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

4 $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n-1)!}$

5 $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

6 $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}$

7 $\sum_{n=0}^{\infty} (-1)^n (x-1)^n$

8 $\sum_{n=2}^{\infty} \frac{\ln n}{n} x^n$

9 $\sum_{n=0}^{\infty} n!x^n$

10 $\sum_{n=1}^{\infty} \frac{3^n x^n}{n4^n}$

11 $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n!)} x^n$

12 $\sum_{n=0}^{\infty} \frac{\sqrt{n}x^n}{3^n}$

13 $\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$

14 $\sum_{n=0}^{\infty} \frac{(-1)^n n!(x-4)^n}{3^n}$

15 Given $k > 0$, express the interval of convergence of $\sum_{n=0}^{\infty} (kx)^n$ in terms of k.

16 Find a power series for $\frac{1}{1+x}$ by replacing x with $-x$ in the geometric series for $\frac{1}{1-x}$ (Example 1). Also determine the interval of convergence for the series.

17 **a** Find the Maclaurin series for e^{-x^2} and determine its radius of convergence.

b Express $\int e^{-x^2}\,dx$ as a Maclaurin series and determine its radius of convergence.

c Use the first five terms of the series from **b** to estimate $\int_0^1 e^{-x^2}\,dx$. Show that the error for this estimate is less than 0.001.

18 Use multiplication or division of power series to find the first three non-zero terms in the Maclaurin series for each function.

a $f(x) = x\sin x$

b $g(x) = \tan x$ $\quad\left[\text{Hint: } \tan x = \frac{\sin x}{\cos x}\right]$

c $f(x) = e^x \ln(1-x)$

19 Find a Maclaurin series for $\frac{1}{(1-x)^2}$ by differentiating the Maclaurin series for $\frac{1}{1-x}$. State the interval of convergence for the series.

20 **a** Find a power series for x^2e^{-x}.

b By differentiating term by term the power series in **a**, show that

$$\sum (-2)^{n+1}\frac{n+2}{n!} = 4.$$

21 a Write down the seventh degree Taylor polynomial for sin x.

b Use this polynomial to estimate $\sin\left(\frac{\pi}{12}\right)$.

c Use the Lagrange error term to determine an upper bound to the error in this approximation.

22 Determine all the values of x for which the series $\sum_{n=1}^{\infty} \frac{2^n x^n}{\ln(n+1)}$ converges.

23 Find the first four non-zero terms of the Taylor series centred at $x = 1$ for the function $f(x) = (x - 1)e^x$.

24 Find the Maclaurin series for $\ln\left(\frac{1+x}{1-x}\right)$.

25 a Find the Maclaurin series for the function $f(x) = \frac{1}{1+x^2}$.

b Hence by integrating each term, show that

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + \ldots.$$

c Show that this series converges for $-1 \leqslant x \leqslant 1$, and thus show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

d Use the first six terms of the series to estimate the value of ⊠ Compute the maximum error for this estimate.

26 The function f is defined by $f(x) = \frac{e^x + e^{-x}}{2}$.

a Obtain an expression for $f^{(n)}(x)$, the nth derivative of $f(x)$ with respect to x.

b Hence, derive the Maclaurin series for $f(x)$ up to and including the term in x^4.

c Use the result to find a rational approximation to $f\left(\frac{1}{2}\right)$.

d Use the Lagrange error term to determine an upper bound to the error in this approximation.

27 Estimate the range of values of x for which the Maclaurin approximation

$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$ is accurate to within 0.005.

28 Find a power series for xe^x and then integrate the resulting series term by term from 0 to 1 to show that $\sum_{n=1}^{\infty} \frac{1}{n!(n+2)} = \frac{1}{2}$.

29 Find a power series for $\sec^2 x$. (Hint: consider the derivative of tan x.)

30 Compute the Taylor series for each of the following functions $f(x)$ centred at the given point c.

a $f(x) = e^x, \ c = 2$

b $f(x) = \sin(x^3), \ c = 0$

c $f(x) = \frac{1}{(1-x)^3}, \ c = 0$

d $f(x) = (x-1)^3 \ln x, \ c = 1$

31 Use series to evaluate each limit.

a $\lim_{x \to 0} \frac{1 - \cos x}{e^x - 1 - x}$

b $\lim_{x \to 0} \frac{x - \arctan x}{x^3}$

Practice questions 3

1 Find the fourth degree Taylor polynomial centred at $x = 0$ for the function $f(x) = \ln(\cos x), 0 \leqslant x < \frac{\pi}{2}$.

2 **a** Find the Maclaurin series for $\sin^2 x$ up to the term containing x^4.

b Hence, write down a series for $\cos^2 x$ up to the term containing x^4.

3 Find the first three non-zero terms in the Maclaurin expansion of $e^x \sin x$.

4 Find the first four terms of the Maclaurin series for the function $f(x) = e^{3x}$.

5 Find the Maclaurin series of $\sec x$ up to the term containing x^4.

6 **a** Write down the fourth degree Taylor polynomial centred at $x = 0$ for e^x.

b Determine the fourth degree Taylor polynomial centred at $x = 0$ for e^{x^2}.

c Hence, or otherwise, find the fourth degree Taylor polynomial centred at $x = 0$ for e^{x+x^2}.

7 Find the Maclaurin series for $\ln(2 + 3x)$ and find an expression for the error term $R_n(x)$.

8 **a** Find the fourth degree Taylor polynomial centred at $x = 0$ for the function $f(x) = \sqrt{4 + x}$.

b Find an expression for the error term $R_4(x)$, and find an error bound when $x = 0.1$.

9 Using the formula for the Lagrange form of the error term, determine how many terms of the Maclaurin series for $\cos x$ are needed to approximate $\cos(5°)$ correct to six decimal places. State this approximate value.

10 **a** Find the Maclaurin series for e^{-x^2}.

b Use the first three terms of this series to approximate $\int_0^1 e^{-x^2}\,dx$.

c Find an upper bound for the error in this approximation.

11 **a** Find the Maclaurin series for $\frac{1}{1+x^2}$. Express the result as an equation in the form $\frac{1}{1+x^2} = \sum_{n=1}^{\infty} a_n$ where a_n is in terms of x and n.

b By integrating both sides of the equation found in **a**, show that $\arctan x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1}$.

c Show that this series converges for $-1 \leqslant x \leqslant 1$.

d Hence, find the exact value to which the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ converges.

12 **a** Find the first four terms in the Maclaurin series for both of the following functions:

$f(x) = \frac{1}{1+x}$ and $g(x) = \frac{1}{1-x}$

b Rewrite the function $h(x) = \frac{x+1}{x^2 - 5x + 6}$ as the sum of two fractions.

c Hence, find the first four terms in the Maclaurin series for the function $h(x) = \frac{x+1}{x^2 - 5x + 6}$.

13 **a** Write down the function represented by the power series $\sum_{n=1}^{\infty} x^{n-1}$ where the interval of convergence is $-1 < x < 1$.

b Using the result from **a** and the identity $\frac{1}{x} = \frac{-1}{1-(x+1)}$, find the Taylor series centred at $x = -1$ for the function $f(x) = \frac{1}{x}$. State the interval of convergence.

14 **a** Find the Maclaurin series of the function $g(x) = \sin(x^2)$ using the series expansion of $\sin x$, that is, $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

b Using the Maclaurin series of $g(x) = \sin(x^2)$, evaluate the definite integral

$$\int_0^1 \sin(x^2)\,dx$$

correct to four decimal places.

15 **a** Find a Maclaurin series expansion for $f(x) = \ln(1 + x)$, for $0 \leqslant x < 1$.

b R_n is the error term in approximating $f(x)$ by taking the sum of the first $(n + 1)$ terms of its Maclaurin series. Prove

$$|R_n| \leqslant \frac{1}{n+1},\ (0 \leqslant x < 1).$$

16 Find the range of values of x for which the following series is convergent.

$$\sum_{n=0}^{\infty} \frac{x^n}{n+1}$$

17 Consider the function f defined by

$f(x) = \arcsin x$, for $|x| \leqslant 1$.

The derivatives of $f(x)$ satisfy the equation

$(1 - x^2) f^{(n+2)}(x) - (2n + 1)\, x f^{(n+1)}(x) - n^2 f^{(n)}(x) = 0$, for $n \geqslant 1$.

The coefficient of x^n in the Maclaurin series for $f(x)$ is denoted by a_n. You may assume that the series contains only odd powers of x.

a **i** Show that, for $n \geqslant 1$, $(n + 1)(n + 2)\, a_{n+2} = n^2 a_n$.

ii Given that $a_1 = 1$, find an expression for a_n in terms of n, valid for odd $n \geqslant 3$.

b Find the radius of convergence of this Maclaurin series.

c Find an approximate value for π by putting $x = \frac{1}{2}$ and summing the first three non-zero terms of this series. Give your answer to **four** significant figures.

18 Find the interval of convergence of the series $\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n}\right) x^n$.

19 **a** **i** State the domain and range of the function $f(x) = \arcsin(x)$.

ii Determine the first two non-zero terms in the Maclaurin series for $f(x)$.

b Use the small angle approximation

$$\cos(y) \approx 1 - \frac{y^2}{2} + \frac{y^4}{24}$$

to find a series for $\cos(\arcsin(x))$ up to and including the term in x^4.

c **i** Find the Maclaurin series for $(p + qx^2)^r$ up to and including the term in x^4 where $p, q, r \in \mathbb{R}$.

ii Find values of p, q and r such that your series in **c i** is identical to your answer to **b**. Comment on this result.

20 **a** Find the value of $\lim_{x\to 1}\left(\dfrac{\ln x}{\sin 2\pi x}\right)$.

b By using the series expansions for e^{x^2} and $\cos x$ evaluate $\lim_{x\to 1}\left(\dfrac{1-e^{x^2}}{1-\cos x}\right)$.

21 The function f is defined by $f(x) = \ln(1 + \sin x)$.

a Show that $f''(x) = \dfrac{-1}{1+\sin x}$.

b Determine the Maclaurin series for $f(x)$ as far as the term in x^4.

c Deduce the Maclaurin series for $\ln(1 - \sin x)$ as far as the term in x^4.

d By combining your two series, show that $\ln \sec x = \dfrac{x^2}{2} + \dfrac{x^4}{12} + \ldots$.

e Hence, or otherwise, find $\lim_{x\to 0}\dfrac{\ln \sec x}{x\sqrt{x}}$.

22 **a** Find the first three terms of the Taylor series centred at $x = \frac{1}{2}$ for the function $f(x) = \sin(\pi x)$.

b Hence, find an approximate value to $\sin\left(\dfrac{\pi}{2} + \dfrac{\pi}{8}\right)$, correct to three significant figures.

4 Calculus

Introduction

Many important ideas of differential and integral calculus have been presented and explained earlier in both the core syllabus (textbook) and in this option topic. Although we endeavoured to provide thorough explanations for the calculus methods developed and applied earlier in this course, this chapter will attempt to 'fill in the gaps' with regard to some important theorems that provide the theoretical foundation for much of the calculus ideas and methods previously encountered. We have made extensive use of derivatives and integrals to analyze functions, but this has mostly been done in an intuitive way while bypassing some of the fundamental theorems that make these analytical methods possible. In this chapter, we will look back at several fundamental ideas in calculus and present some important theorems. We will make use of material already covered in the textbook – in particular, some content from Chapter 16 (Integral Calculus). It will be very helpful to study this chapter in conjunction with the relevant parts of Chapter 16 that will be mentioned here.

4.1 Continuity and differentiability

The main difference between calculus and other branches of mathematics lies in the idea of a **limit** and the intimately related concept of **continuity**. We have made use of limits, continuity and the important concept of **differentiability** in the calculus topic in the core syllabus (Chapters 13, 15 and 16) and in this option topic. Our approach thus far has been informal and has relied on a visual interpretation of the graphs of functions. In Chapter 13 of the textbook (Section 13.3), a margin note stated the following:

Geometrically speaking a function is **continuous** if there is no break in its graph; and a function is **differentiable** (i.e. a derivative exists) at any point where its graph is 'smooth'.

In the first section of Chapter 3 Algebraic Functions, Equations and Inequalities, it was demonstrated that one of the properties of all polynomial functions is that they are continuous for all real numbers, i.e. the graph of a polynomial function never has a 'gap' or a 'hole' in it. Continuity is such a common feature of many familiar functions (such as polynomial functions) that to understand and recognize it we should look at some functions that lack this property, i.e. some **discontinuous** functions.

Consider the function $f(x) = \dfrac{x^2 + x - 6}{x^2 - x - 2}$. By factoring the numerator and denominator the function can also be expressed as $f(x) = \dfrac{(x-2)(x+3)}{(x-2)(x+1)}$.

The graph of f (Figure 4.1) clearly shows that there is a 'gap' at $x = -1$ and a 'hole' at $x = 2$. Thus at the points where $x = -1$ and $x = 2$ the function f is not continuous. It is **discontinuous**. It seems reasonable to say that the function is continuous everywhere else since the graph appears to have no other 'gaps' or 'holes'.

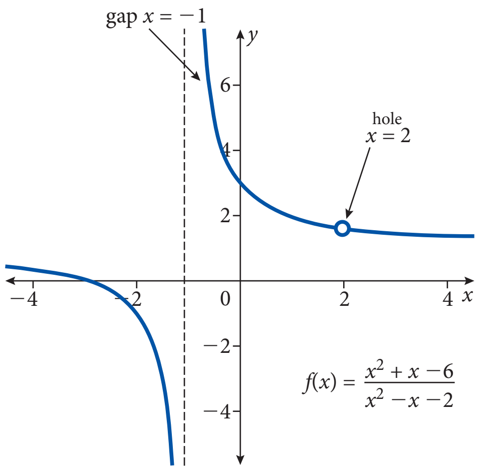
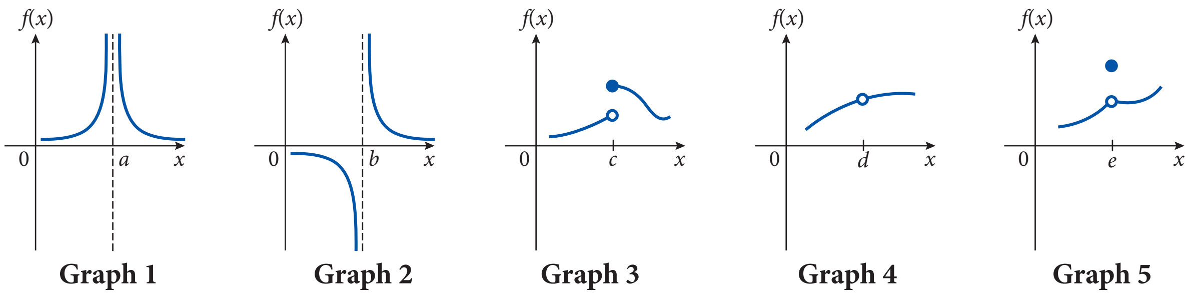


Figure 4.1

Figure 4.2 below shows examples of graphs of five functions that have different types of discontinuities. Respectively, the functions shown have points of discontinuity at $x = a$, b, c, d and e.

Figure 4.2



The functions in Graphs 1 and 2 have vertical asymptotes at $x = a$ and $x = b$, so the functions are not defined for these values of x (as seen in the graph of the function f in Figure 4.1). This can be referred to as an **infinite discontinuity**.

The function in Graph 3 illustrates what can be described as a **step discontinuity**, where it is defined at $x = c$. However, the graph shows that a small change in x produces a 'jump' in the value of $f(x)$ so the function is not continuous at $x = c$.

The type of discontinuity seen in Graphs 4 and 5 is the same as the 'hole' that occurred at $x = 2$ in the graph of f in Figure 4.1. This type of discontinuity is often called a **removable discontinuity** because it can be *removed* by simply redefining the value of the function at the particular point where the 'hole' occurs.

We now need to develop a precise definition of continuity from the observations made in the preceding examples. From the examples, it is clear that the definition needs to incorporate the following two ideas:

1 Continuity is a *local* matter. In other words, a function can be continuous at some points and discontinuous at other points. Therefore, continuity cannot be defined for an *entire* function.

We must define continuity *at a point*.

2 A function f is continuous at a point $x = c$ of its domain if $f(x)$ is near $f(c)$ when x is near c.

The second of these ideas is close to the definition we're looking for, but the idea of 'near' is not mathematically precise. In order to do so, we need to apply the formal concept of a limiting value. We also need to distinguish between a function being continuous at a point and a function being continuous at all points in a certain interval.

The functions in the Graphs 1, 2 and 4 in Figure 4.2 are discontinuous respectively at $x = a$, b and d because they do not satisfy the first condition for the definition of continuity. The function in Graph 4 is discontinuous at $x = c$ because it does not satisfy the second condition. In order for the limit of the function as x approaches c to exist, it must be true that the limit of the function as x approaches c from the left (**one-sided limit** from the left) equals the limit of the function as x approaches c from the right (**one-sided limit** from the right), i.e. $\lim_{x\to c} f(x)$ exists if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$. The function in Graph 5 is discontinuous at $x = e$ because it does not satisfy the third condition for the definition of continuity at a point.

Definition of continuity

1 Continuity at a point:
A function f is continuous at a point where $x = c$, if and only if the following three conditions are satisfied.

i $f(c)$ exists **ii** $\lim_{x\to c} f(x)$ exists **iii** $\lim_{x\to c} f(x) = f(c)$

2 Continuity on an interval:
A function f is continuous on an interval of x-values, if and only if it is continuous at each value of x in that interval. At the endpoints of a closed interval (i.e. endpoints included in the interval), only the one-sided limits need to equal the function value.

Example 1

Consider the piece-wise function f, which is defined as follows.

$$f(x) = \begin{cases} |x| + 3 & \text{for } x < 1 \\ ax^2 + bx & \text{for } x \geqslant 1 \end{cases}$$

Find the values of a and b, such that f is continuous for all real numbers.

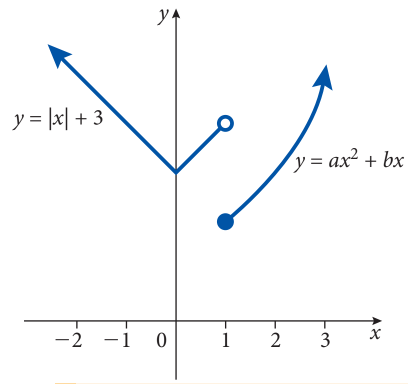
Solution

We know that:

for $x < 1$, the graph of f will be the typical 'v' shape of an absolute value function with a vertex at (0, 3)

for $x \geqslant 1$, the graph of f will be a parabola.

Although we do not know the values of a and b we can make a rough sketch of f (shown on the right). (Diagram not to scale)



We see that f satisfies all three conditions for continuity at all points except at $x = 1$. At this point, it satisfies the first condition, i.e. $f(1)$ exists, because $f(1) = a + b$. However, whether the second and third conditions are met depends on the values of a and b. The limit of $|x| + 3$ as x approaches 1 from the left is equal to $|1| + 3 = 4$. The limit of $ax^2 + bx$ as x approaches 1 from the right is equal to $a + b$. Thus, f will be continuous at all points if $a + b = 4$. Therefore, f will be continuous for all real numbers for any pair of values of a and b whose sum is 4.

For the purpose of consistency all intervals in this chapter are expressed using inequalities. For example, the closed interval $a \leqslant x \leqslant b$ could also be written as $x \in [a, b]$; and the open interval $a < x < b$ could also be written as $\in\ ]a, b[$. See Section 1.1 of the textbook for notation overview.

An important property of functions that are continuous on an interval or intervals – and that makes them especially useful in various mathematical applications – is a property expressed in the following theorem.

The intermediate value theorem

If a function f is continuous on the closed interval $a \leqslant x \leqslant b$ and N is a number between $f(a)$ and $f(b)$, then a number $x = c$ must exist such that $f(c) = N$.

It is important to mention that the intermediate value theorem guarantees the existence of at least one number c in the closed interval $a \leqslant x \leqslant b$. Of course, there may be more than one number c such that $f(c) = N$.

It is beyond the scope of this course to give a proof of the intermediate value theorem.

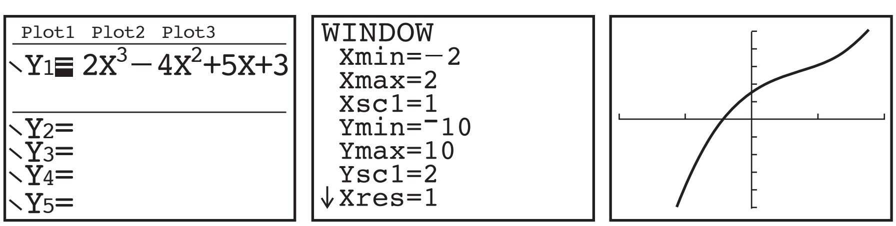
Example 2

Use the intermediate value theorem to show that the polynomial function $f(x) = 2x^3 - 4x^2 + 5x + 3$ has a zero in the closed interval $-1 \leqslant x \leqslant 0$.

Solution

The function f is a polynomial function so it is continuous for $x \in \mathbb{R}$, and hence also continuous on the closed interval $-1 \leqslant x \leqslant 0$. With reference to the intermediate value theorem, we take $a = -1$, $b = 0$ and $N = 0$.

Since $f(-1) = 2(-1)^3 - 4(-1)^2 + 5(-1) + 3 = -2 - 4 - 5 + 3 = -8 < 0$ and $f(0) = 2(0)^3 - 4(0)^2 + 5(0) + 3 = 3 > 0$, it follows that $f(-1) < 0 < f(0)$. We can now apply the intermediate value theorem to conclude that there must be at least one number c in the interval $-1 \leqslant x \leqslant 0$ such that $f(c) = 0$ as shown in the GDC screen images below.



Of course, the intermediate value theorem is useful when access to a GDC is not allowed. The GDC images above are provided simply to confirm the result obtained from the intermediate value theorem.

The intermediate value theorem is an example of what is often referred to as an **existence theorem**. The theorem guarantees that a number *exists* with a certain property, but it does *not* provide a method for finding the value of the number. The following theorem is also an existence theorem where continuity of a function, or lack of it, plays an important role. It guarantees the existence, under certain conditions, of a solution to an extreme value (minimum/maximum) problem. Again, we will present this theorem without a formal proof.

The extreme value theorem

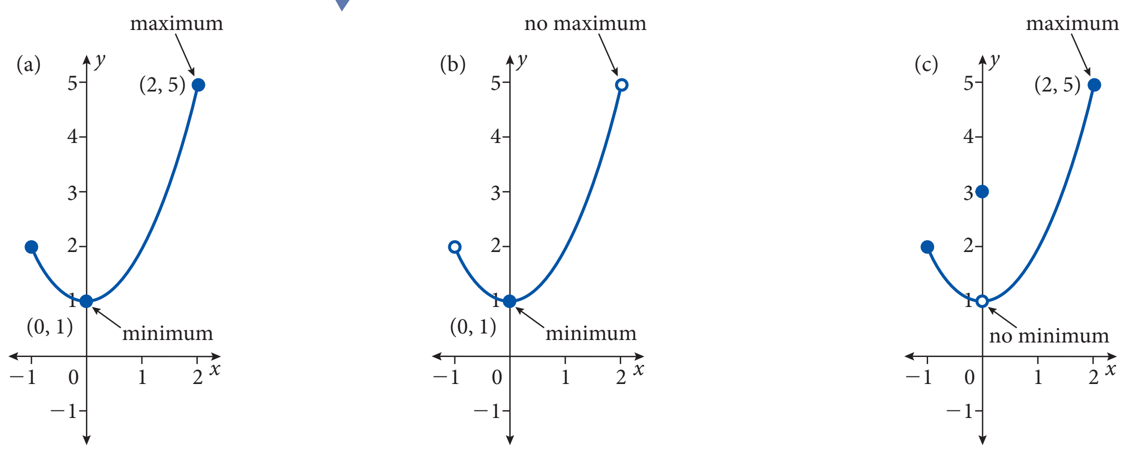
If a function f is continuous on a closed interval, then f has an absolute maximum and an absolute minimum on the closed interval.

The functions graphed in Figure 4.3 below illustrate some possibilities for a function having a maximum or a minimum on an interval. In (a), the function $y = x^2 + 1$ has both a maximum and a minimum on the **closed** interval $-1 \leqslant x \leqslant 2$. The maximum at the point $(2, 5)$ is an example of an extreme value (maximum in this case) that occurs at an endpoint. In (b), the function $y = x^2 + 1$ on the **open** interval $-1 < x < 2$ has a minimum but no maximum. In (c), the function is:

$$y = \begin{cases} x^2 + 1 & \text{for } x \neq 0 \\ 3 & \text{for } x = 0 \end{cases}$$

It is on the closed interval $-1 \leqslant x \leqslant 2$. It has a maximum but no minimum because of the **discontinuity** at $x = 0$.

Figure 4.3



$y = x^2 + 1, -1 \leqslant x \leqslant 2$

$y = x^2 + 1, -1 < x < 2$

$$y = \begin{cases} x^2 + 1 & \text{for } x > 0 \\ 3 & \text{for } x \neq 0 \end{cases}, -1 \leqslant x \leqslant 2$$

In (a) of Figure 4.3, since the function is continuous on a closed interval the extreme value theorem guarantees that an absolute minimum and an absolute maximum must exist.

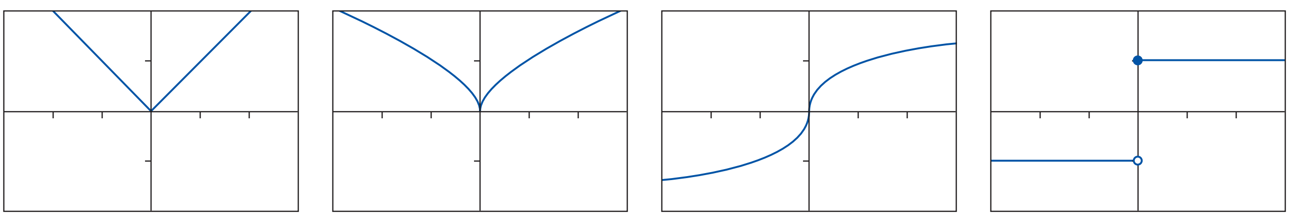
As already mentioned, the fact that a function is differentiable at a point (i.e. a derivative exists for a function at a point) was described informally in Chapter 13 to be related to the 'smoothnesss' of the graph of the function. Recall the definition of the derivative of a function f from Section 13.2. The derivative at a point $x = c$, $f'(c)$, is given by $f'(c) = \lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$

provided that this limit exists. The key phrase in this definition is 'provided that this limit exists'. The limit exists if the left-hand and right-hand limits are equal. Substituting $x - c$ for h in the limit definition for the derivative gives $\lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h} = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$.

This alternative limit form for the derivative is useful for determining whether or not a function is differentiable at a particular point where $x = c$. Thus, to show that a function f is *not* differentiable at $x = c$ we must show that the two one-sided limits (as x approaches c from either direction) for the definition of the derivative are *not* equal; that is, show $\lim\limits_{x \to c^+} \dfrac{f(x) - f(c)}{x - c} \neq \lim\limits_{x \to c^-} \dfrac{f(x) - f(c)}{x - c}$.

Graphically speaking, this means that a function f will not have a derivative at a point$(c, f(c))$ where the slopes of the secant lines fail to approach the same value as x approaches c from the right and from the left. This agrees with the previous informal description that a function is differentiable at a point where the graph of the function is 'smooth'. Also, a function will not be differentiable at a point of discontinuity because a discontinuity will cause one or both of the one-sided limits to be non-existent. The four graphs in Figure 4.4 illustrate four different types of situations where a function fails to be differentiable at a point.

Figure 4.4



at $x = 0$: (a) a sharp 'corner' (b) a sharp point or 'cusp' (c) a vertical tangent (d) a point of discontinuity

Each of the four functions shown in Figure 4.4 fail to have a derivative (i.e. not differentiable) at $x = 0$. A brief rationale is given for each.

- Function (a): The left-hand derivative and the right-hand derivative are not equal at $x = 0$. As x approaches 0 from the left the derivative approaches the value of -1, and as x approaches 0 from the right the derivative approaches the value of $+1$.
- Function (b): Both the left-hand derivative and the right-hand derivative do not exist at $x = 0$. As x approaches 0 from the left the derivative (slope of tangent) approaches $-\infty$, and as x approaches 0 from the right the derivative approaches $+\infty$.

- Function (c): Both the left-hand derivative and the right-hand derivative do not exist at $x = 0$. The derivative (slope of tangent) approaches $+\infty$ as x approaches 0 from both sides.
- Function (d): The function is discontinuous at $x = 0$ which will cause one or both of the one-sided derivatives to be non-existent. The function shown in (d) can be expressed in piecewise form as

$$f(x) = \begin{cases} -1 & \text{for } x < 0 \\ 1 & \text{for } x \geqslant 0 \end{cases}.$$

Using the form of the limit definition of the derivative given earlier, $\lim\limits_{x \to c^-} \dfrac{f(x) - f(c)}{x - c}$, we can show that the left-hand derivative does not exist at $x = 0$.

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-1-1}{x} = \lim_{x \to 0^-} \frac{-2}{x} = \infty \text{ (increases without bound)}$$

Definition of differentiability

A function f is *differentiable* at a point where $x = c$ if the derivative, $f'(c)$, exists.

Example 3

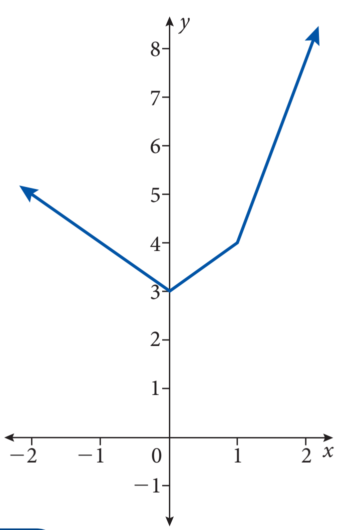
Consider the piece-wise function f from Example 1:

$$f(x) = \begin{cases} |x| + 3 & \text{for } x < 1 \\ ax^2 + bx & \text{for } x \geqslant 1 \end{cases}$$

a) Example 1 concluded that f is continuous for all real numbers if $a + b = 4$. Let $a = \frac{1}{2}$ and $b = \frac{7}{2}$. For these values of a and b, are there any values of x where f is not differentiable?

b) Find the values of a and b, such that f is differentiable for all x where $x \geqslant 0$.

Solution



a) From the graph below, the two points on the graph of f where $f'(c)$ may not exist (i.e. where the graph is not 'smooth') is at $x = 0$ and at $x = 1$. Let's consider both points separately.

For $x = 0$: The portion of f that is an absolute value function, $y = |x| + 3$, can be treated as a piecewise function – let's call it $g(x)$.

$$g(x) = \begin{cases} -x + 3 & \text{for } x \leqslant 0 \\ x + 3 & \text{for } x \geqslant 0 \end{cases}$$

We compute the derivatives of $y = -x + 3$ and $y = x + 3$.

$g(x) = -x + 3$	$g(x) = x + 3$
$g'(x) = -1$	$g'(x) = 1$
$g'(0) = -1$	$g'(0) = 1$

The left-hand derivative does not equal the right-hand derivative when $x = 0$. Thus, the function is **not differentiable** (does not have a derivative) at $x = 0$.

For $x = 1$: Left of $x = 1$ is the function $y = x + 3$ and right of $x = 1$ is the function $y = \frac{1}{2}x^2 + \frac{7}{2}x$.

We compute the left-hand and right-hand derivatives at $x = 1$.

left-hand derivative:	right-hand derivative:
$y = x + 3$	$y = \frac{1}{2}x^2 + \frac{7}{2}x$
$y' = 1$	$y' = x + \frac{7}{2}$
$y'(1) = 1$	$y'(1) = 1 + \frac{7}{2} = \frac{9}{2}$

The left-hand derivative does not equal the right-hand derivative when $x = 1$. Thus, the function is **not differentiable** at $x = 1$.

Therefore, the function $f(x) = \begin{cases} |x| + 3 & \text{for } x < 1 \\ \frac{1}{2}x^2 + \frac{7}{2}x & \text{for } x \geqslant 1 \end{cases}$

is not differentiable at $x = 0$ and at $x = 1$.

b) In order for f to be differentiable at $x = 1$ the left-hand and right hand derivatives must be equal at $x = 1$.

left-hand derivative:	right-hand derivative:
$y = x + 3$	$y = ax^2 + bx$
$y' = 1$	$y' = 2ax + b$
$y'(1) = 1$	$y'(1) = 2a + b = 1$

From Example 1, we know that $a + b = 4$ in order for $f(x)$ to be continuous at $x = 1$. Thus, solving simultaneous equations $a + b = 4$ and $2a + b = 1$ gives $a = -3$ and $b = 7$.

From the four functions graphed in Figure 4.4 and Example 3, we can conjecture that continuity of a function at a point does not imply that the function will also be differentiable at that point. However, differentiability does imply continuity, which is stated in the next theorem.

Differentiability implies continuity

If a function f is **differentiable** at a point $x = c$, then f is also **continuous** at $x = c$.

Proof

To prove that f is continuous at $x = c$ we must show that the three conditions of the definition of continuity are satisfied. That is, we must show that **i** $f(c)$ exists, **ii** $\lim_{x \to c} f(x)$ exists, and **iii** $\lim_{x \to c} f(x) = f(c)$.

i From the hypothesis of the *differentiability implies continuity* property, f is differentiable at $x = c$ so it must follow that $f(c)$ exists. From the definition of the derivative $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$. It follows that $f(c)$ must exist otherwise this limit has no meaning.

ii and **iii** We can use the product rule for limits (Section 13.1 in textbook) which states that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = K$, then

$\lim_{x\to a}[f(x)\cdot g(x)] = L\cdot K$ and knowing that $\lim_{x\to c}(x-c) = 0$ and that

$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = f'(c)$ to perform the following:

$$\begin{aligned}\lim_{x\to c}[f(x)-f(c)] &= \lim_{x\to c}\left[(x-c)\cdot\frac{f(x)-f(c)}{x-c}\right]\\ &= \lim_{x\to c}(x-c)\cdot\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\\ &= 0\cdot f'(x)\\ &= 0\end{aligned}$$

This result helps to produce the following:

$$\begin{aligned}\lim_{x\to c} f(x) &= \lim_{x\to c}[f(x)-f(c)+f(c)]\\ &= \lim_{x\to c}[f(x)-f(c)] + \lim_{x\to c} f(c)\\ &= 0 + \lim_{x\to c} f(c)\end{aligned}$$

Thus, $\lim_{x\to c} f(x)$ exists, and it is equal to $\lim_{x\to c} f(c)$. Therefore, all three conditions of the definition of continuity are satisfied and the theorem is proved.

One consequence of the property that differentiability implies continuity is proof that all polynomial functions are continuous for all real numbers.

The property that if a function is differentiable at a point then it must also be continuous at that point can be symbolized by writing: differentiable → continuous. It is worthwhile to point out that both the converse and the inverse of this property are false.

Converse: continuous → differentiable … *false*
Inverse: not differentiable → not continuous … *false*

Both of these false statements were illustrated in Example 3. However, the contrapositive of the property is true. That is …

Contrapositive: not continuous → not differentiable … *true*

In other words, if a function *f* is not continuous at a point then *f* is also not differentiable at that point. The property 'differentiable → continuous' and its contrapositive 'not continuous → not differentiable' provide an effective way to prove that a function is continuous or not differentiable at a particular point.

Example 4

Consider the function $g(x) = \dfrac{x^2-2x-3}{x-3}$.

a) Show that g is continuous at $x = 4$.

b) Show that g is not differentiable at $x = 3$.

Solution

a) In order to show that g is continuous at $x = 4$, we just need to show that a derivative exists for g at $x = 4$.

$$g'(x) = \frac{(x-3)(2x-2)-(x^2-2x-3)(1)}{(x-3)^2} = \frac{x^2-6x+9}{(x-3)^2}$$
$$= \frac{(x-3)^2}{(x-3)^2} = 1$$

for all values of x except $x = 3$.

Thus, $g'(4) = 1$ and g is differentiable at $x = 4$. Since differentiability implies continuity then f is continuous at $x = 4$ … Q.E.D.

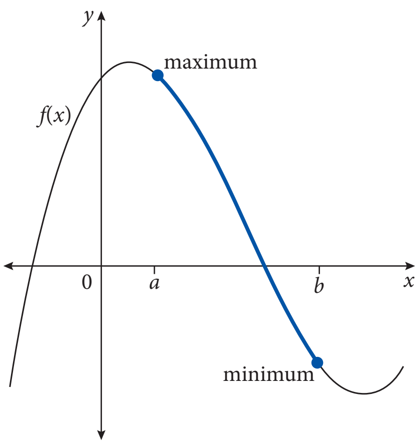
b) To prove that g is not differentiable we need to show that g is not continuous at $x = 3$.

The given function is equivalent to $g(x) = \frac{(x+1)(x-3)}{x-3}$. It's clear that g has a removable discontinuity at $x = 3$. Applying the contrapositive of the property that 'differentiability implies continuity' proves that since g is discontinuous at $x = 3$, then it is also not differentiable at $x = 3$.

4.2 Rolle's theorem and the mean value theorem

The extreme value theorem presented earlier in this chapter states that a function that is continuous on a closed interval must have both a minimum and a maximum on the interval. As mentioned, this is an example of an existence theorem. The theorem tells us that if a function satisfies a certain condition, then at least one minimum and at least one maximum must exist. The function does not tell us where these extreme values are located. Both of these extreme values could occur at the endpoints of the closed interval as illustrated in Figure 4.5. **Rolle's theorem**, named after the French mathematician Michel Rolle (1652–1719), is an existence theorem that states conditions that guarantee when a function must have at least one extreme value in the interior of a closed interval (i.e. an open interval).

Figure 4.5
$f(x)$ on closed interval $a \leqslant x \leqslant b$



Essentially what Rolle's theorem says is that between consecutive zeros of a function there must be at least one location where the derivative of the function is zero. Geometrically speaking, this means that between two zeros there must be at least one place where the graph of the function has a horizontal tangent.

Rolle's theorem

Let f be a function such that:

i it is continuous on the closed interval $a \leqslant x \leqslant b$;

ii it is differentiable on the open interval $a < x < b$;

iii $f(a) = 0$ and $f(b) = 0$.

Then there must exist a number c in the open interval $a < x < b$ such that $f'(c) = 0$.

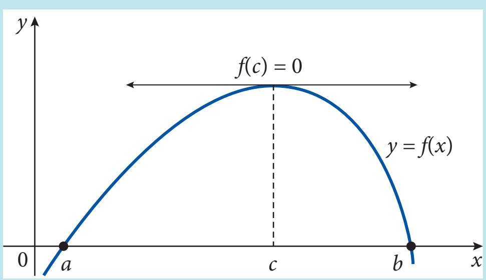
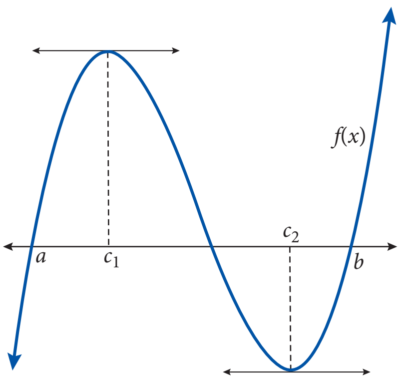


Figure 4.6



It is possible for a continuous function f to have more than location in the open interval $a < x < b$ where the derivative of f is zero. This is illustrated in Figure 4.6 where there is a horizontal tangent at $x = c_1$ and also at $x = c_2$. Thus, both $f'(c_1) = 0$ and $f'(c_2) = 0$.

Rolle's theorem is a special case of a more powerful existence theorem known as the **mean value theorem**. Recall the discussion in Section 2 of Chapter 13 (Differential Calculus I: Fundamentals) demonstrating that the derivative of a function (slope of tangent line) gives the **instantaneous rate of change** of the function at a point and that the slope of the secant line through two points gives the average rate of change between the two points. Over a particular interval in the domain of a function, the mean value theorem connects the average rate of change of the function with instantaneous rate of change of the function at a point within the interval. Although the mean value theorem can be used as an effective tool in solving certain problems, its importance lies in the fact that it has been used to prove several other important theorems in calculus. The theorem was briefly presented in the first section of Chapter 16 (Integral Calculus) where it was used to help establish the general rule for finding anti-derivatives (indefinite integrals) of functions. The mean value theorem plays an important role in the development of the fundamental theorem of calculus that is presented briefly at the end of this option topic chapter – and was thoroughly discussed in Section 16.4 (Area and definite integral) of the textbook.

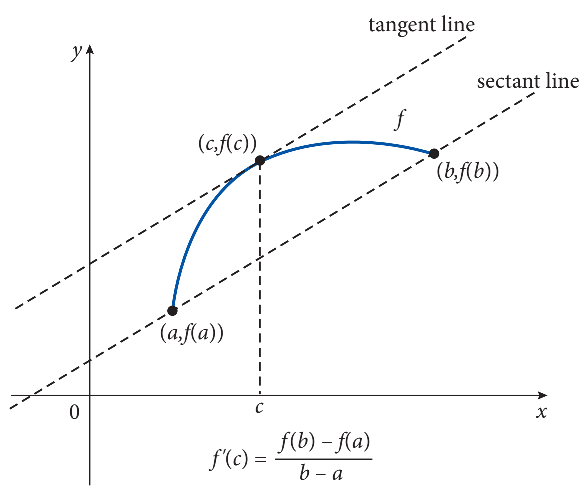
The mean value theorem

Let f be a function such that:

i it is continuous on the closed interval $a \leqslant x \leqslant b$
ii it is differentiable on the open interval $a < x < b$.
Then there must exist a number c in the open interval $a < x < b$ such that

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$. (See Figure 4.7)

Figure 4.7



The theorem presented on this page is sometimes referred to as the mean value theorem **for derivatives** to contrast it with another theorem involving the average (mean) value of a continuous function over an interval that is usually referred to as the mean value theorem **for integrals**. The word 'mean' in the theorem on this page refers to the average rate of change (slope of secant line) of function f in the interval $a \leqslant x \leqslant b$.

A geometric interpretation of the mean value theorem – as illustrated in Figure 4.7 – guarantees the existence of at least one tangent line to a function

f in the interval $a < x < b$ that is parallel to the secant line through the points $(a, f(a))$ and $(a, f(b))$. This is demonstrated in Example 5 below.

Example 5

Consider the function $f(x) = 6 - \frac{9}{x}$ over the open interval $1 < x < 9$. Find all values of c in this interval at which the conclusion of the mean value theorem is true. For any resulting value of c, verify the result by graphing f, the secant line through $(1, f(1))$ and $(9, f(9))$, and the tangent through $(c, f(c))$.

Solution

Firstly, $f(x)$satisfies the required conditions of the mean value theorem because the only point where f is not continuous and not differentiable is at $x = 0$ and f is being considered only over the interval $1 < x < 9$. Now

need to find any value of c that satisfies $f'(c) = \frac{f(9) - f(1)}{9 - 1}$. Given that $f'(x) = \frac{9}{x^2}$, then $\frac{9}{c^2} = \frac{5 - (-3)}{8} \Rightarrow \frac{9}{c^2} = 1 \Rightarrow c^2 = 9 \Rightarrow c = \pm 3$. Thus, $c = 3$.

Equation of secant line through (1, –3) and (9, 5):

$$\text{slope} = \frac{-3 - 5}{1 - 9} = \frac{-8}{-8} = 1$$

$$y - y_1 = m(x - x_1) \Rightarrow y - (-3) = 1(x - 1) \Rightarrow y + 3 = x - 1$$

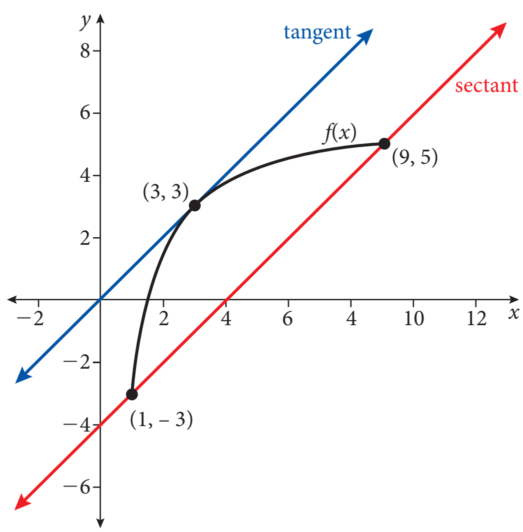
equation of secant line: $y = x - 4$

Equation of tangent line through $(3, f(3))$:

$f(3) = 6 - \frac{9}{3} = 3$; point of tangency is (3, 3); $f'(3) = \frac{9}{3^2} = 1$

equation of tangent line: $y - 3 = 1(x - 3) \Rightarrow y = x$

Graph of f, secant line and tangent line:



The graph visually confirms the result in that the secant line and tangent line are parallel.

As mentioned, the mean value theorem can also be interpreted in terms of rates of change. The theorem guarantees the existence of at least one point in the open interval $a < x < b$ at which the instantaneous rate of change is equal to the average rate of change over the closed interval $a \leqslant x \leqslant b$. Example 6 illustrates the use of the mean value theorem in the context of rates of change.

Example 6

Two motion detectors that can measure the instantaneous rate of change of a toy car moving along a straight track are positioned 5 metres apart. As the toy car passes the first detector, its velocity is measured at 17 metres/minute. Fifteen seconds later the toy car passes the second detector and its velocity is measured at 19 metres/minute. Show that the velocity of the toy car must have been 20 metres/minute at some moment during the fifteen seconds that it traveled between the two detectors.

Solution

Since the instantaneous rates measured by the two detectors are measured in metres per minute – and that 15 seconds = $\frac{1}{4}$ minute – the motion of the toy car is being considered over the interval $0 < t < \frac{1}{4}$ with t in minutes. It makes sense to set the distance s in metres to be zero for $t = 0$, i.e. $s(0) = 0$; and then $s\left(\frac{1}{4}\right) = 5$ since the detectors are 5 metres apart. Thus, the average velocity for the toy car during the quarter minute that it took to travel 5 metres is given by

$$\text{average velocity} = \frac{s\left(\frac{1}{4}\right) - s(0)}{\frac{1}{4} - 0} = \frac{5 - 0}{\frac{1}{4}} = 20\,\text{metres/minute}$$

Assuming that the distance function $s(t)$ is differentiable over the interval, we can apply the mean value theorem to conclude that the toy car must have been traveling at a velocity of 20 metres/minute for at least one instant during the time it moved between the two detectors.

4.3 Riemann sums and the fundamental theorems of calculus

At the start of Section 16.4 (Area and definite integral) in the textbook we developed an informal, but logical, explanation for the area under a continuous function over a certain interval to be equal to the definite integral where the limits of integration are the endpoints of the interval. Critical to this explanation is the process of finding the sum of sets of rectangles of decreasing width to form better and better approximations of the area under the curve for a particular interval. Although the name is not used in Section 16,4, the sum of an infinite set of rectangles for the purpose of computing the area under a curve is called a **Riemann sum**. The discussion in Section 16.4 also presented two important theorems in calculus that are usually referred to as the **first fundamental theorem of calculus** and the **second fundamental theorem of calculus**. Before studying this section in the calculus option topic, it is very important that you go back and carefully read all of Section 16.4 in the textbook. What follows here is a review and brief description of material on the definite integral, Riemann sums and the fundamental theorems of calculus that are relevant to this HL option topic.

Riemann sums

In Section 16.4 we used the limits of sums of rectangles to define what we mean by the phrase *the area under a curve*. Figure 4.8 shows how we approximate this area with rectangles and also shows the notation we've chosen to use. The area being approximated is for the interval $a \leqslant x \leqslant b$. The interval is partitioned into n sub-intervals of equal width Δx. We then draw n rectangles each having a width of Δx and a height of $f(x_i^*)$ where x_i^* is an arbitrary point within the ith sub-interval.

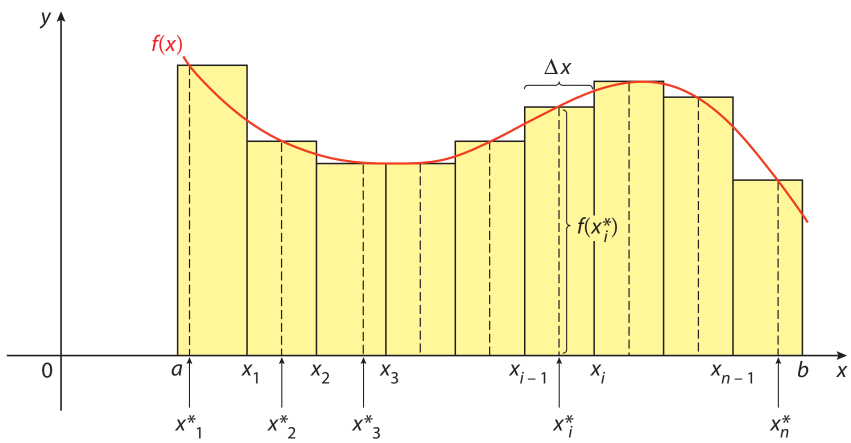


Figure 4.8

We learned that if we let the number of sub-intervals n (or rectangles) go to infinity – and simultaneously the width Δx go to zero – that the limit of the sum of the rectangles is equal to the area under the curve. This result is written as

$$\text{area} = \lim_{n\to\infty}\left[f\left(x_1^*\right)\Delta x + f\left(x_2^*\right)\Delta x + \ldots + f\left(x_n^*\right)\Delta x\right] = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

The sum $\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$ is called a **Riemann sum** and is named after the German mathematician Bernhard Riemann (1826–1866). As we have encountered previously when computing areas with definite integrals in

Chapter 16, if the region whose area we are computing is below the x-axis then the 'heights' of the rectangles, i.e. $f(x_1^*)$, will be negative. Area is defined to be a positive value. Rather than changing the definition of area, mathematicians decided to call a Riemann sum a definite integral rather than an area.

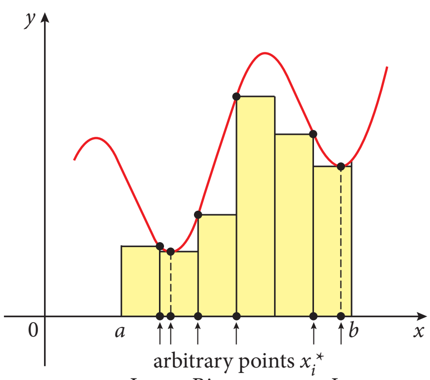
Therefore – as explained in Section 16.4 – the limit of a Riemann sum for a continuous function $f(x)$ on the interval $a \leqslant x \leqslant b$ is defined to be the definite integral of $f(x)$ from a to b; that is $\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \int_a^b f(x)\,dx$.

As you should understand at this point of your study of advanced mathematics, this is an enormously significant result in the development of calculus.

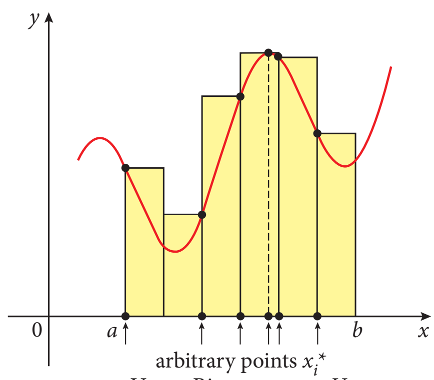
Although in forming a Riemann sum the widths of the rectangles does not need to be constant, most graphical illustrations of using rectangles to approximate the area of a region between a function and the x-axis (i.e. a Riemann sum) do use a constant width – as shown in Figure 4.8.

We will not prove it here, but it turns out that when forming a Riemann sum, it is not necessary for the rectangles to have a constant width. The width of the ith rectangle is denoted as Δx_i. As long as the function f is continuous and integrable over the given interval and the number of rectangles goes to infinity ($n \to \infty$) – thereby causing $\Delta x_i \to 0$ – then the limit of any Riemann sum will be equal to the definite integral $\int_a^b f(x)\,dx$.

It is possible to choose the location of each arbitrary point x_i^* located within the ith sub-interval so that height of the rectangle $f(x_i^*)$ is the lowest or highest in each sub-interval, as illustrated in Figure 4.9. The sum of the areas of the rectangles that are all the lowest possible is referred to as a **lower Riemann sum** (denoted L_n) and the sum of the area of the rectangle that are all the highest possible is referred to as an **upper Riemann sum** (denoted U_n).



Lower Riemann sum, L_n



Upper Riemann sum, U_n

The lower sum is a lower bound for the value of the definite integral and the upper sum is an upper bound, i.e. $L_n \leqslant \int_a^b f(x)\,dx \leqslant U_n$. The lower and upper sums will approach the same limit as $n \to \infty$ (and $\Delta x_i \to 0$) causing the value of the definite integral to be squeezed (recall the Squeeze theorem from the second section of Chapter 13) to this common limit, i.e. the definite integral.

Riemann sum and definition of a definite integral

If $\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$ is any Riemann sum, such that a closed interval $a \leqslant x \leqslant b$ is divided into n sub-intervals where the ith sub-interval has an arbitrary point x_i^* within it and has width Δx_i, and a function f is continuous and integrable on the same interval, then

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i = \int_a^b f(x)\,dx \cdot$$

Continuity implies integrability

If a function f is continuous over the closed interval $a \leqslant x \leqslant b$, then f is also integrable over $a \leqslant x \leqslant b$.

One of the prerequisites for the definite integral of a function over a certain interval being defined as the limit of a Riemann sum is that the function be continuous and integrable (i.e. can be integrated) over the interval. In the first section of this chapter we thoroughly described and defined continuity of a function, but have not done so for integrability of a function. Fortunately, it can be proved that if a function is continuous over an interval then it must also be integrable over the interval. We will not present a proof because it is beyond the scope of this course.

Example 7

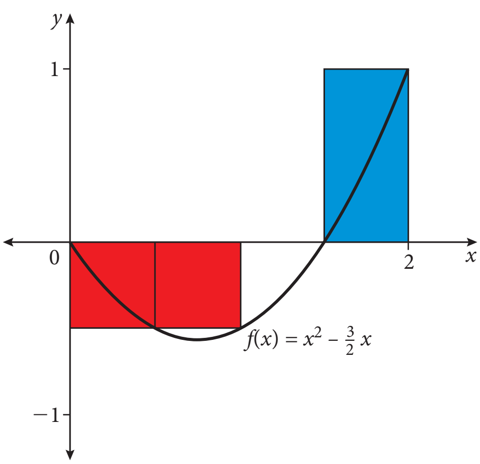
a) Evaluate the Riemann sum for $f(x) = x^2 - \frac{3}{2}x$ for the closed interval $0 \leqslant x \leqslant 2$ divided into 4 sub-intervals of equal width by evaluating the heights of the 4 rectangles at the right endpoint of each sub-interval. Comment on the result.

b) Using the same information from a), find the Riemann sum for f, but now dividing the interval into 6 sub-intervals. Comment on the result.

c) Using integration rules from earlier in the course, evaluate the exact value of the definite integral $\int_0^2 \left(x^2 - \frac{3}{2}x\right)dx$. Comment on the result.

Solution

a)



Given that $n = 4$, then the width of each sub-interval is $\Delta x = \frac{2-0}{4} = \frac{1}{2}$.

The values of the endpoints of each of the 4 sub-intervals are $x_1 = \frac{1}{2}$, $x_2 = 1$, $x_3 = \frac{3}{2}$ and $x_4 = 2$. Thus the Riemann sum is:

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{4}\left(x_i^2 - \frac{3}{2}x_i\right)\cdot\frac{1}{2}$$

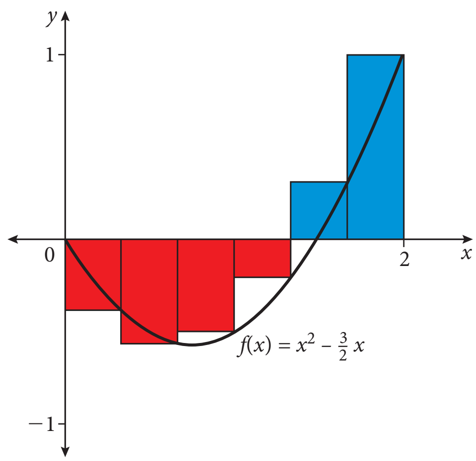
$$= \frac{1}{2}\cdot\left[\left(\frac{1}{2}\right)^2 - \frac{3}{2}\left(\frac{1}{2}\right) + (1)^2 - \frac{3}{2}(1) + \left(\frac{3}{2}\right)^2 - \frac{3}{2}\left(\frac{3}{2}\right) + (2)^2 - \frac{3}{2}(2)\right]$$

$$= \frac{1}{2}\cdot\left[-\frac{1}{2} - \frac{1}{2} + 0 + 1\right]$$

$$= 0$$

Clearly, the Riemann sum does not represent a sum of areas of rectangles. As shown in the figure above, the Riemann sum is the sum of the areas of the blue rectangles (above the x-axis) *minus* the sum of the red rectangles (below the x-axis). With the rectangles shown in the figure, it appears that the value of zero for the Riemann sum is an overestimate because the portion of the blue rectangle outside the region below the curve seems to be larger than the portion between the curve and the x-axis for the sub-interval $1 \leqslant x \leqslant \frac{3}{2}$.

b)



Given that $n = 6$, then the width of each sub-interval is $\Delta x = \frac{2-0}{6} = \frac{1}{3}$.

The values of the endpoints of each of the 6 sub-intervals are $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 1$, $x_4 = \frac{4}{3}$, $x_5 = \frac{5}{3}$, and $x_6 = 2$. Thus, the Riemann sum is:

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{6}\left(x_i^2 - \frac{3}{2}x_i\right)\cdot\frac{1}{3}$$

$$= \frac{1}{3}\cdot\left[\begin{array}{l}\left(\frac{1}{3}\right)^2 - \frac{3}{2}\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)^2 - \frac{3}{2}\left(\frac{2}{3}\right) + (1)^2 - \frac{3}{2}(1) \\ + \left(\frac{4}{3}\right)^2 - \frac{3}{2}\left(\frac{4}{3}\right) + \left(\frac{5}{3}\right)^2 - \frac{3}{2}\left(\frac{5}{3}\right) + (2)^2 - \frac{3}{2}(2)\end{array}\right]$$

$$= \frac{1}{3} \cdot \left[-\frac{7}{18} - \frac{5}{9} - \frac{1}{2} - \frac{2}{9} + \frac{5}{18} + 1 \right]$$
$$= \frac{1}{3} \cdot -\frac{7}{18}$$
$$= -\frac{7}{54}$$

From the figure above – showing the red rectangles that contribute negatively and the blue rectangles that contribute positively to the Riemann sum – it appears that the result of $-\frac{7}{54}$ is a much better approximation than the result in a) for the exact value of the Riemann sum. This should be expected because the number of rectangles has increased from 4 to 6.

c) $\int_0^2 \left(x^2 - \frac{3}{2}x \right) dx = \left[\frac{1}{3}x^3 - \frac{3}{4}x^2 \right]_0^2 = \left[\frac{1}{3}(2)^3 - \frac{3}{4}(2)^2 \right] - 0 = \frac{8}{3} - 3 = -\frac{1}{3}$

Therefore, the limit of the Riemann sum as $n \to \infty$ is exactly $-\frac{1}{3}$. The result of $-\frac{7}{54}$ in b) is a better estimate than the result in a) of 0 of the exact value of the definite integral. By computing the definite integral for the portion of the curve above the x-axis we can determine the exact area of the two regions bounded by the curve and the x-axis.

$$\int_{\frac{3}{2}}^{2} \left(x^2 - \frac{3}{2}x \right) dx = \left[\frac{1}{3}x^3 - \frac{3}{4}x^2 \right]_{\frac{3}{2}}^{2}$$
$$= \left[\frac{1}{3}(2)^3 - \frac{3}{4}(2)^2 \right] - \left[\frac{1}{3}\left(\frac{3}{2}\right)^3 - \frac{3}{4}\left(\frac{3}{2}\right)^2 \right]$$
$$= -\frac{1}{3} - \left(-\frac{9}{16} \right) = \frac{11}{48}$$

Thus, the area of the bounded region above the x-axis is $\frac{11}{48}$ and consequently the area of the bounded region below the x-axis is $\frac{9}{16}$. Since the region below the x-axis has a negative value for the definite integral the exact result of $-\frac{1}{3}$ is confirmed by $\frac{11}{48} - \frac{9}{16} = \frac{11}{48} - \frac{27}{48} = -\frac{16}{48} = -\frac{1}{3}$.

Although the same notation is used for both, it is important to understand that a definite integral is **not** the same thing as an indefinite integral. A definite integral is a *number* while in contrast an indefinite integral is a *family of functions*.

Example 8

Express the following limit as a definite integral on the interval $0 \leqslant x \leqslant \pi$ where x_i is an arbitrary point in the ith sub-interval and Δx_i is the width of the ith sub-interval.

$$\lim_{n \to \infty} \sum_{i=1}^{n} \left(2x_i + \cos x_i \right) \Delta x_i$$

Solution

Comparing the limit $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}(2x_i + \cos x_i)\Delta x_i$ to the limit in the definition of a definite integral, we can see that $f(x) = 2x + \cos x$. Since the endpoints of the closed interval are $a = 0$ and $b = \pi$, then

$$\lim_{n\to\infty}\sum_{i=1}^{n}(2x_i + \cos x_i)\Delta x_i = \int_0^{\pi}(2x + \cos x)dx$$

Fundamental theorems of calculus

Look again at the computational work done in part c) of Example 7.

$$\int_0^2\left(x^2 - \frac{3}{2}x\right)dx = \left[\frac{1}{3}x^3 - \frac{3}{4}x^2\right]_0^2 = \left[\frac{1}{3}(2)^3 - \frac{3}{4}(2)^2\right] - 0 = \frac{8}{3} - 3 = -\frac{1}{3}$$

In Chapter 16, we learned methods of finding the anti-derivative (indefinite integral) of a function. In the work above, we had to know that the anti-derivative of x^2 is $\frac{1}{3}x^3$ and anti-derivative of $\frac{3}{2}x$ is $\frac{3}{4}x^2$. But how do we know the method for computing the numerical value of the definite integral? This method for computing a definite integral is given in the **second fundamental theorem of calculus** that was presented in the latter part of Section 16.4. This theorem follows from the **first fundamental theorem of calculus** that was also presented in Section 16.4 and is a consequence of the definition of the definite integral using Riemann sums. Collectively the two theorems are often referred to as **<u>the</u> fundamental theorem of calculus.** The development of these two theorems was thoroughly explained in Section 16.4 so there is no need to reproduce that discussion here. However, it is important that you go back and read that section of the textbook again. We consolidate the two theorems into one below.

The fundamental theorem of calculus

If a function f is continuous (and hence integrable) over the closed interval $a \leqslant x \leqslant b$, then both of the following statements are true.

1 If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.

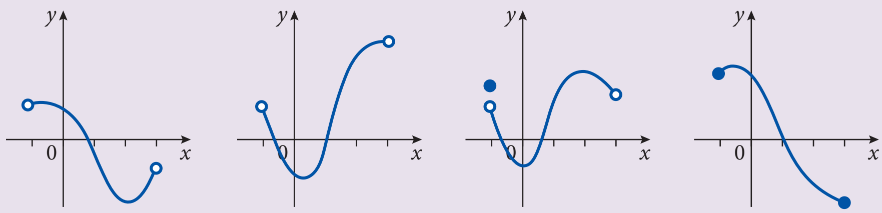
2 $\int_a^b f(x)dx = F(b) - F(a)$, where F is an anti-derivative of f, i.e. $\frac{d}{dx}[F(x)] = f(x)$.

The first part of the theorem can also be written as $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$.

Thus, we can see that this part of the theorem very importantly establishes the fact that integration and differentiation are inverse processes. The second part of the theorem makes use of this fact resulting in the method for evaluating definite integrals. By showing that such dissimilar objects as the derivative and the integral are so closely intertwined, the fundamental theorem of calculus is certainly one of the major achievements in the development of mathematics and certainly the most important theorem in calculus.

Exercise 4

1 Given that a function g is continuous on the closed interval $-1 \leqslant x \leqslant 3$, which of the following could be a graph of g?



A **B** **C** **D**

2 Consider the piece-wise function f defined as follows.

$$f(x) = \begin{cases} |x| + 2 & \text{for } x < 2 \\ ax^2 + bx & \text{for } x \geqslant 2 \end{cases}$$

Find the value(s) of b such that f is continuous for all real numbers.

3 State, in terms of a, the interval(s) on which the function g is continuous.

$$g(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a} & \text{for } x \neq a \\ 2a & \text{for } x = a \end{cases}$$

4 Consider the function f defined below.

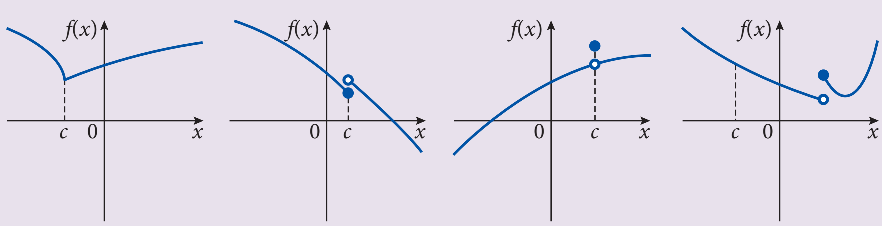
$$f(x) = \begin{cases} x^2 + x + 1 & \text{for } x \leqslant 1 \\ 2x + 1 & \text{for } x > 1 \end{cases}$$

At the point where $x = 1$, determine:

a whether f is continuous

b whether f is differentiable.

5 State whether each function graphed below is continuous or differentiable at $x = c$.



A **B** **C** **D**

6 Find the value of a and the value of b, such that the function g is differentiable at $x = 2$.

$$g(x) = \begin{cases} ax^3 & \text{for } x \leqslant 2 \\ b(x-3)^2 + 10 & \text{for } x > 2 \end{cases}$$

7 Consider the function h defined below.

$$h(x)=\begin{cases}3x & \text{for } x\leqslant 1\\ ax^2+b & \text{for } x>1\end{cases}$$

a Find the relationship between a and b, such that h is continuous for all real numbers?

b Find the value of a and the value of b, such that h is both continuous and differentiable for all real numbers.

8 If $f(x) = x^3 - 3x^2 + x - 1$, find the point x_0 at which $f'(x)$ has its mean value in the interval $1 < x < 4$.

9 Consider the function $f(x) = x^2 + 1$ over the open interval $1 < x < 3$. Find the value of c in this interval at which the conclusion of the mean value theorem is true. For any resulting value of c, verify the result by graphing f, the secant line through $(1, f(1))$ and$(3, f(3))$, and the tangent through $(c, f(c))$.

10 If $g(x) = \cos x$, find the point x_0 where $g'(x)$ has its mean value in the interval $0 \leqslant x \leqslant \frac{\pi}{2}$.

11 Explain why the mean value theorem does not apply to the function $x^{\frac{2}{3}}$ on the interval $-1 \leqslant x \leqslant 8$.

12 The speed limit along a highway is 60 km per hour. Two police officers positioned 13 km from each other along the highway were monitoring the speed of cars. A car passed the first police officer and was recorded as travelling at 56 km per hour. 12 minutes later, the car passed the second officer who measured the car's velocity as 59 km per hour. Show work and give an explanation confirming whether or not the car broke the speed limit on the portion of highway between the two police officers.

13 Use the mean value theorem to show that $e^x \geqslant x + 1$ for $x > 0$.

14 Consider the portion of the function $f(x) = 2x - x^2$ that is above the x-axis, i.e. $y > 0$. Find the mean value of this function.

15 Use Rolle's theorem to show that the equation $x^3 + 2x + b = 0$, where b is a constant, cannot have more than one real zero.

For the functions in questions 16 and 17, find the value of c in the given interval at which the conclusion of the mean value theorem is true.

16 $f(x) = x^3 - 5x^2 - 3x,\ 0 < x < \frac{\pi}{2}$

17 $g(x) = \sqrt{1 - \sin x}\ , 0 < x < \frac{\pi}{2}$

18 Find the Riemann sum for the function $f(x) = 2x - x^2$ over the interval $0 \leqslant x \leqslant 2$. Use four sub-intervals. The arbitrary point for each sub-interval is the right endpoint of the sub-interval.

19 Find the lower and upper Riemann sums for the function $g(x) = x^2 + 3$ over the interval $0 \leqslant x \leqslant 2$, partitioning the interval into 4 sub-intervals.

In questions 20–22, express the limit as a definite integral on the given interval where x_i is an arbitrary point in the ith sub-interval and Δx_i is the width of the ith sub-interval.

20 $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \sqrt{x_i + 6}\ \Delta x_i,\ 0 \leqslant x \leqslant 4$

21 $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{e^{x_i}}{x_i - 2} \Delta x_i,\ 3 \leqslant x \leqslant 5$

22 $\lim_{n\to\infty}\sum_{i=1}^{n}(3-\sin x_i)\,\Delta x_i,\ 0 \leqslant x \leqslant 11$

23 Consider each of the integrals below.

a $\int_2^6 x^3\,dx,\ n=4$

b $\int_0^\pi \sin x\,dx,\ n=3$

c $\int_{-2}^2 2^x\,dx,\ n=8$

i Estimate the definite integral (3 significant figures) by finding the value of the Riemann sum with n sub-intervals. Use the midpoint of each sub-interval as the arbitrary point for each sub-interval.

ii Find the exact value of the definite integral using the fundamental theorem of calculus (part 2).

iii State whether the estimate from **i** was an overestimate or underestimate and the percentage error for the estimate found in **i** compared to the exact value found in **ii**.

5 Differential Equations

Introduction

Equations involving an unknown function and its derivative(s) are called differential equations and frequently occur in mathematical models of real-life phenomena. Differential equations come in a great variety of forms, and many different procedures – analytic, graphical and numerical – exist for finding their solutions. The last section of Chapter 16 in the textbook (Section 16.9) is an optional section on differential equations. It provides an introduction to differential equations and also covers an analytic solution method for a certain class of differential equations (separable equations). In this chapter, we will explore differential equations further by considering two more classes of differential equations. Analytic methods are not always successful in solving a differential equation, so we will also investigate a graphical approach and a useful numerical method for approximating the solution to a differential equation.

There is a brief introduction to differential equations in Section 16.9 of the textbook. You are strongly encouraged to read through this section before working through this chapter.

A **differential equation** is an equation that relates an independent variable (commonly x or t), a dependent variable (usually y), and one or more derivatives of an unknown function $y = f(x)$ [or $y = f(t)$]. The general form of a differential equation (with independent variable x) can be written as

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ky}{dx^k}\right) = 0$$

where the largest k for which $\dfrac{d^ky}{dx^k}$ occurs in the equation is called the **order** of the differential equation.

Here are some examples:

1 $x\dfrac{dy}{dx} + y\dfrac{dy}{dx} - y = 0$ first order differential equation $F\left(x, y, \dfrac{dy}{dx}\right) = 0$

2 $\dfrac{dy}{dx} + \dfrac{y^2 - y}{x^2} = 0$ first order differential equation $F\left(x, y, \dfrac{dy}{dx}\right) = 0$

3 $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 5y = 0$ second order differential equation $F\left(x, y, \dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}\right) = 0$

4 $\dfrac{dy}{dx} + y\sin x - e^{\cos x} = 0$ first order differential equation $F\left(x, y, \dfrac{dy}{dx}\right) = 0$

5 $2\dfrac{d^3y}{dx^3} + (\ln x)\left(\dfrac{dy}{dx}\right)^2 + 4xy = 0$ third order differential equation $F\left(x, y, \dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}\right) = 0$

For this course, we only study **first order differential equations**, such as equations **1**, **2** and **4** above. In a first order differential equation, the first derivative, $\frac{dy}{dx}$, of the unknown function can be isolated on one side of the equation. Hence, a simpler general form for first order differential equations is

$$\frac{dy}{dx} = F(x, y)$$

where $\frac{dy}{dx}$ is expressed as a function in terms of x and y. Note that the first order differential equations **1**, **2** and **4** can all be re-written in this form. For example,

1. $x\frac{dy}{dx} + y\frac{dy}{dx} - y = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x+y}$

The **solution** of a differential equation is the (initially unknown) function $y = f(x)$ whose derivative is $\frac{dy}{dx}$. Consider the differential equation

$$\frac{dy}{dx} = \frac{1}{x+1}, \quad x \neq -1.$$

Every solution of this equation is an anti-derivative of $\frac{1}{x+1}$.

$$y = \int \frac{1}{x+1}\,dx = \ln|x+1| + C, \quad x \neq -1$$

So the solution of the differential equation $\frac{dy}{dx} = \frac{1}{x+1}$ is the **explicitly defined function** $y = \ln|x+1| + C$ where C is an arbitrary constant. This is called a **general solution** because it is not a single function, but an infinite 'family' of functions dependent on the constant C. Figure 5.1 shows a few members of this family.

A differential equation may use symbols for the independent and dependent variables other than x and y. For the sake of simplicity, we will use x and y while we are developing theory and solution methods for differential equations. Also note that we are using F ('large F') to represent a two-variable function that when set equal to $\frac{dy}{dx}$ is the differential equation, and f ('small f') represents the unknown function whose slope at the point (x, y) is $\frac{dy}{dx}$.

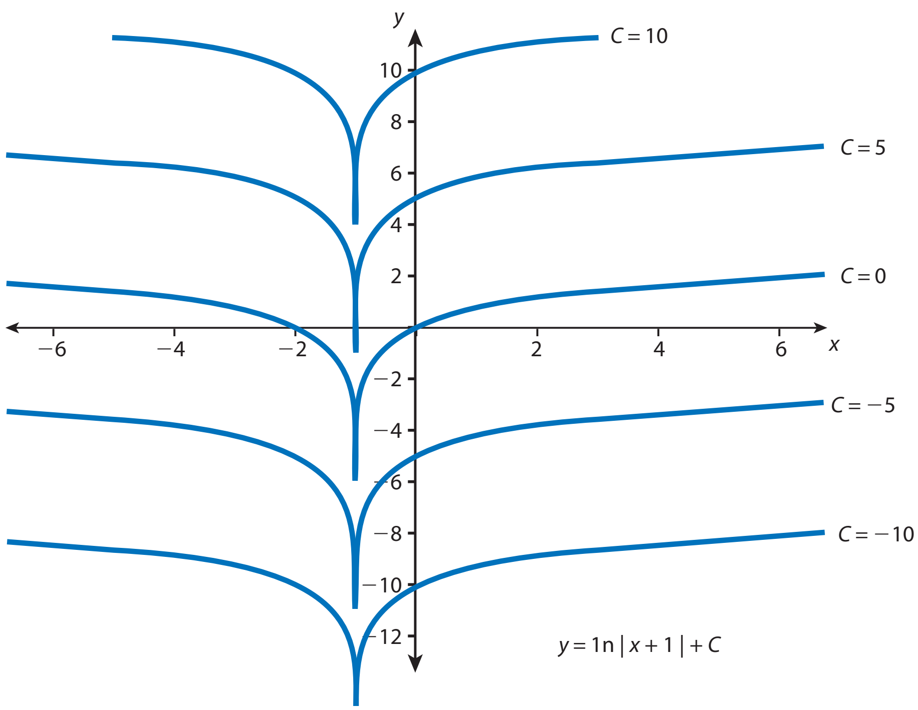


Figure 5.1

In general, we wish to find the **explicit solution** of a differential equation written in the form $y = f(x)$ where f is a known function. However, it is sometimes not possible to solve for y. In such a case we must settle for an **implicit solution** written in the form $g(y) = f(x)$ where g and f are known functions and $g(y) \neq y$.

In contrast, when we are given some **initial conditions** that allow us to evaluate a particular value for C we obtain a single function that we call a **particular solution** of the differential equation. For example, if we are given the initial conditions that $y = 5$ when $x = 0$ then we can solve for C, giving $C = 5$ and the particular solution of $y = \ln|x+1| + 5$.

Sometimes the solution of a differential equation will be expressed as an **implicitly defined function.** For example, the general solution to equation **1** is

$$\ln y = \frac{x}{y} + C.$$

It is an equation relating x and y and *implies* a function exists that defines y as a function of x.

To verify that this is a solution to **1**, we differentiate – applying implicit differentiation and the product rule:

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\frac{x}{y} + C\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left(xy^{-1}\right) + \frac{d}{dx}(C)$$

$$\frac{1}{y}\frac{dy}{dx} = y^{-1} + x\left(-y^{-2}\frac{dy}{dx}\right) + 0$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx}$$

$$y^2\left(\frac{1}{y}\frac{dy}{dx}\right) = y^2\left(\frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx}\right)$$

$$x\frac{dy}{dx} + y\frac{dy}{dx} - y = 0$$

Therefore, for any real number C the function $\ln y = \frac{x}{y} + C$ is a solution, in implicit form, to the differential equation $x\frac{dy}{dx} + y\frac{dy}{dx} - y = 0$. This means that the coordinates x and y of any point on the curve $\ln y = \frac{x}{y} + C$ combined with the value of the derivative $\frac{dy}{dx}$ at that point will solve the equation $x\frac{dy}{dx} + y\frac{dy}{dx} - y = 0$.

The only type of first order differential equation covered in Section 16.9 of the textbook is a class of differential equations referred to as **separable equations**. We solved these using a technique called **separation of variables.** One of our key goals in this chapter is to develop an analytic solution method for each of two further classes of first order differential equations. Before we delve into the details of these analytic methods, we examine a useful graphical method for helping us to sketch the function, or family of functions, that solves a differential equation.

5.1 Slope fields

Often the primary objective when solving a first order differential equation is to find an explicit solution. However, many differential equations used in mathematical models cannot be solved by means of an analytic method. For such equations, we must resort to graphical and/or numerical methods. Carried out by hand or by technology, a graphical method provides us with rough qualitative information about the graph of a solution to a differential equation.

A first order differential equation in the form $\frac{dy}{dx} = F(x, y)$ specifies the slope of the **solution curve** $y = f(x)$ at each point in the xy-plane where F is defined. We can use this fact to draw a short line segment whose slope is $F(x, y)$ at any point (x, y) in the plane. A plot of these line segments showing the slope (or direction) of the solution curve is a called a **slope field** (or direction field) for the first order differential equation. As a rule, the segments are drawn at representative points evenly spaced in both directions. Figure 5.2 shows a slope field for the equation $\frac{dy}{dx} = x - y$.

Although it looks fairly simple, the differential equation $\frac{dy}{dx} = x - y$ is not easy to solve. It can be solved analytically with one of the techniques that we develop later in the chapter. It is an example of a first order **linear** differential equation, and its general solution is $y = Ce^{-x} + x - 1$.

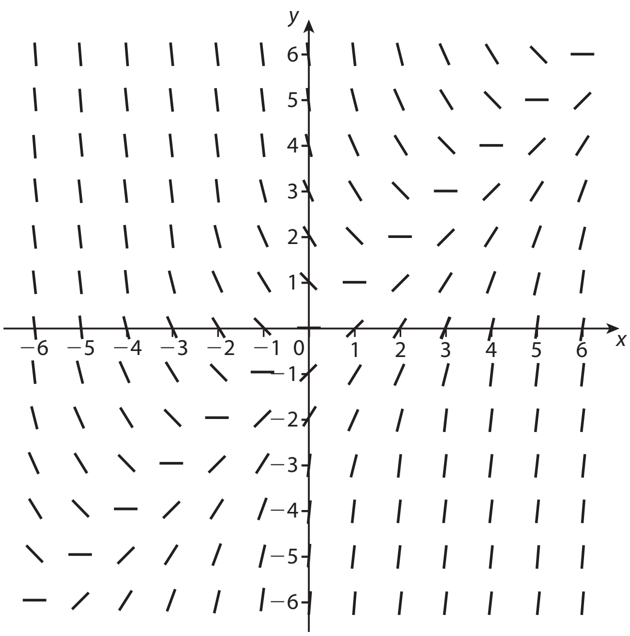
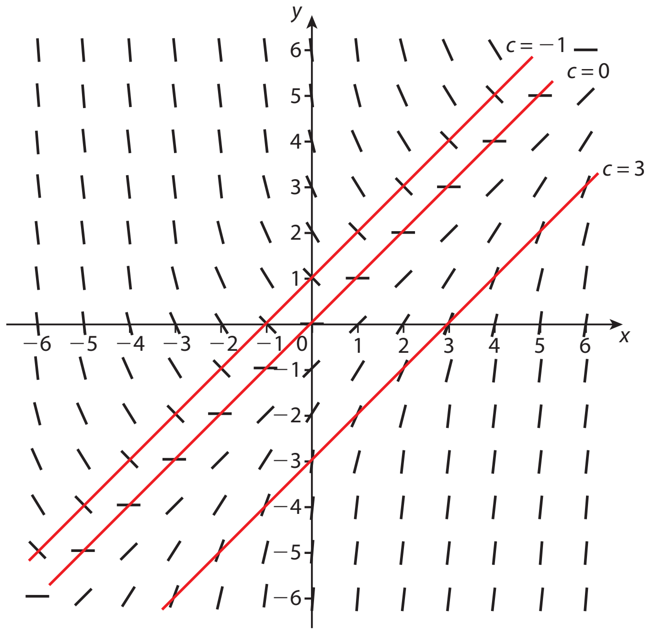


Figure 5.2 Slope field for $\frac{dy}{dx} = x - y$.

As you can imagine, it can be quite tedious to draw a slope field by hand. In practice, slope fields are easily generated by suitable graphing technology. However, there is a method that simplifies the process of doing it by hand. Rather than compute $\frac{dy}{dx}$ for a large number of x and y values, we look for points where $\frac{dy}{dx}$ has the same value. For some constant c, the graph of the equation $F(x, y) = c$ is a line, called an **isocline**, along which all the short line segments of a slope field have the same slope c. For the differential equation $\frac{dy}{dx} = x - y$, the isoclines are $x - y = c$. Figure 5.3, shows (in red) the isoclines for $c = -1$, 0 and 3. By first tracing in a few isoclines, we can create a slope field by easily drawing multiple line segments along it all having the same slope.

Figure 5.3 Slope field and three isoclines for $\dfrac{dy}{dx} = x - y$.



'Isocline' comes from 'iso-' meaning equal and '-cline' meaning slope. Be aware that isoclines themselves do not give any direct information about solution curves for the differential equation. They serve to ease the process of drawing a slope field. It is recommended that you draw isoclines lightly in pencil, and preferably dashed.

Solutions to a differential equation can be sketched by drawing in curves that are at each point tangent to the line segment at that point. Thus, a family of solution curves can be produced. To use a slope field to sketch a particular solution all we need to know is one point (an initial condition) that the solution curve passes through.

Isoclines are not always straight lines. Isoclines are analogous to contour lines on a map indicating land of equal elevation. Consider the differential equation $\dfrac{dy}{dx} = x^2 - y$ that has isoclines that are parabolas with equations of the form $y = x^2 - c$. When isocline curves are not lines, it is more difficult to use them to sketch a slope field.

Example 1

a) Draw a slope field for $\dfrac{dy}{dx} = -\dfrac{x}{y}$ on the xy-plane such that $-5 \leqslant x \leqslant 5$ and $-5 \leqslant y \leqslant 5$. Sketch some sample solution curves. What shape are they?

b) Confirm that both $y = \sqrt{c^2 - x^2}$ and $y = -\sqrt{c^2 - x^2}$, where c is a constant, are each a general solution of the equation.

Solution

a) Rather than evaluating $\dfrac{dy}{dx} = -\dfrac{x}{y}$ for a large number of x and y values, we establish some isoclines by looking for points where $-\dfrac{x}{y}$ has a constant value.

If $\dfrac{dy}{dx} = -\dfrac{x}{y} = 0$ then $x = 0$. Hence, the y-axis is an isocline where all the line segments are horizontal.

If $y = 0$ (x-axis), then $\frac{dy}{dx}$ is undefined. Hence, the x-axis is an isocline where all the line segments are vertical (undefined slope).
If $\frac{dy}{dx} = -\frac{x}{y} = 1$ then $y = -x$ is an isocline where all the line segments have a slope of 1.
If $\frac{dy}{dx} = -\frac{x}{y} = -1$ then $y = x$ is an isocline where all the line segments have a slope of -1.

If necessary, we can continue in this manner and establish further isoclines, such as:

$y = 2x$ is an isocline where all the line segments have a slope of $-\frac{1}{2}$.
$y = -\frac{1}{2}x$ is an isocline where all the line segments have a slope of 2.
In fact, any line passing through the origin will be an isocline for the slope field for $\frac{dy}{dx} = -\frac{x}{y}$.
The resulting slope field – showing six lightly drawn isoclines – is shown below in Figure 5.4.

Drawing curves parallel to the line segments gives a family of solution curves that appear to be circles. Three members of the family are drawn in Figure 5.5.

Figure 5.4

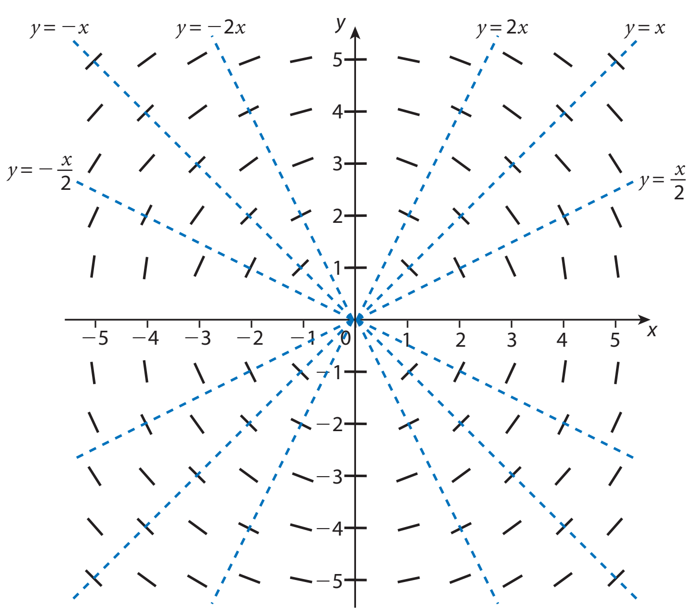
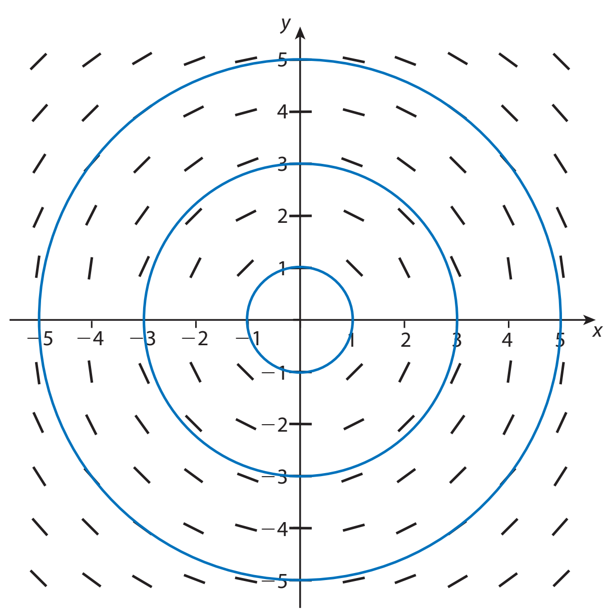


Figure 5.5



b) Checking that $y = \sqrt{c^2 - x^2}$ is a solution, we compute $\frac{dy}{dx}$ on the left side and substitute $\sqrt{c^2 - x^2}$ for y on the right side.

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{d}{dx}\left(\sqrt{c^2 - x^2}\right) = -\frac{x}{\sqrt{c^2 - x^2}}$$

$$\frac{1}{2}\left(c^2 - x^2\right)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{c^2 - x^2}}$$

$$-\frac{x}{\sqrt{c^2 - x^2}} = -\frac{x}{\sqrt{c^2 - x^2}} \qquad \text{Q.E.D.}$$

Checking that $y = -\sqrt{c^2 - x^2}$ is a solution.

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{d}{dx}\left(-\sqrt{c^2 - x^2}\right) = -\left(\frac{x}{-\sqrt{c^2 - x^2}}\right)$$

$$-\frac{1}{2}\left(c^2 - x^2\right)^{-\frac{1}{2}}(-2x) = \frac{x}{\sqrt{c^2 - x^2}}$$

$$\frac{x}{\sqrt{c^2 - x^2}} = \frac{x}{\sqrt{c^2 - x^2}} \qquad \text{Q.E.D.}$$

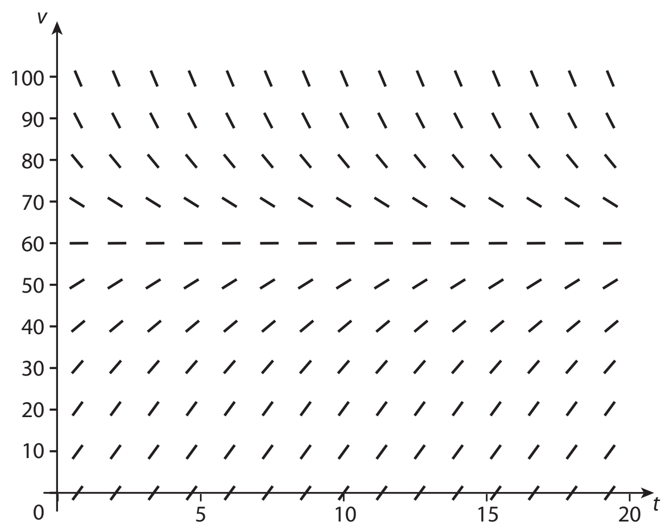
The solution $y = \sqrt{c^2 - x^2}$ is the family of curves consisting of the upper half of each circle, and the solution $y = -\sqrt{c^2 - x^2}$ is the family of curves consisting of the lower half of each circle.

Example 2

A model for the velocity v, in metres per second, at time t seconds of a 75 kg skydiver falling from an aeroplane is given by the equation

$$\frac{dv}{dt} = 10 - \frac{v^2}{360}.$$

Figure 5.6



a) From the direction field shown in Figure 5.6, sketch the solution curves with the following initial conditions:

(i) $v(0) = 0$, (ii) $v(0) = 35$, and (iii) $v(0) = 90$.

b) Explain why the value $v = 60$ is called the **terminal velocity** for this situation.

Solution

a) Solutions to $\frac{dv}{dt} = 10 - \frac{v^2}{360}$ satisfying $v(0) = 0$, $v(0) = 35$ and $v(0) = 90$ are sketched in Figure 5.7 below.

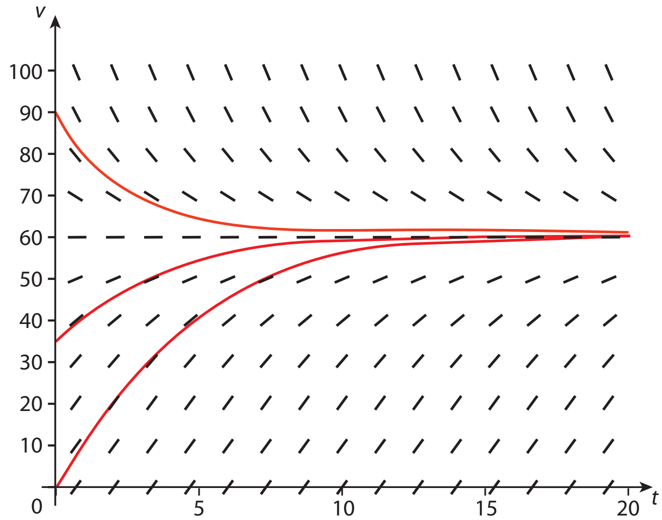
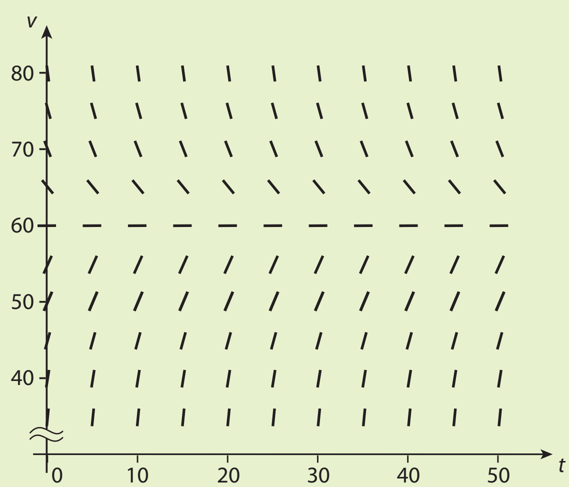


Figure 5.7

b) From the slope field it appears that all solutions have a limiting value of 60 as t goes to infinity. Due to increasing air resistance the skydiver reaches a maximum velocity, or terminal velocity, of 60 metres per second.

Note that the scales on the axes for the slope fields in Figures 5.2, 5.3, 5.4 and 5.5 are equal. Thus, the short line segments accurately depict the true slope for solution curves. The scales are not equal on the axes in Figures 5.6 and 5.7, so the line segments do not give a true indication of the slope. However, this is not an error. Sometimes, it is necessary to have unequal scales in order to show an appropriate interval of values for the independent and dependent variables. Figure 5.8 shows a portion of the same slope field given in Figures 5.6 and 5.7 but with equal scales on the axes.

Figure 5.8



5.2 Separable equations

A class of first order differential equations introduced in Section 16.9 of the textbook which can be solved analytically using integration is the class of **separable equations**. These are differential equations $\frac{dy}{dx} = F(x, y)$ that can be rewritten so that the variables x and y (along with their differentials dx and dy) are on opposite sides of the equation. For a first order differential equation where this separation of variables can be accomplished, the function $F(x, y)$ can be factored into a product of two functions – one involving only the independent variable (e.g. x) and the other involving only the dependent variable (e.g. y). That is,

$$\frac{dy}{dx} = F(x, y) = p(x)q(y).$$

Although there are two integrals in the equation $\int \frac{1}{q(y)} dy = \int p(x) dx + C$, only one constant of integration is needed. We could add a constant to both sides but they could then be combined into one constant.

Separable equation

A first order differential equation is considered separable if it can be written in the form

$$\frac{dy}{dx} = p(x)q(y).$$

The variables can then be separated by writing the equation in the form

$$\frac{1}{q(y)} dy = p(x) dx$$

and integrating both sides gives

$$\int \frac{1}{q(y)} dy = \int p(x) dx + C$$

which leads to a general solution.

It is not always obvious whether or not a differential equation is separable. Some algebraic manipulation is needed to confirm that the differential equation can, in fact, be written in the form $\frac{dy}{dx} = p(x)q(y)$. For example, $\frac{dy}{dx} = \frac{3}{xy} - \frac{x^2}{y}$ is separable because it can be written as $\frac{dy}{dx} = \frac{1}{y}\left(\frac{3}{x} - x^2\right)$; and $\frac{\tan x}{y}\frac{dy}{dx} = \frac{2}{\ln y}$ is also separable because it can be written as $\frac{dy}{dx} = \frac{2y}{\ln y}\cot x$. However, the equations $\frac{dy}{dx} = x^2 + y^2$ and $\frac{dy}{dx} = 1 + xy$ are *not* separable.

Example 3

Find the general solution of the differential equation

$$x^2 y \frac{dy}{dx} = x + 1, \quad x > 0, \; y > 0.$$

Solution

The equation is separable because algebraic rearrangements can be performed to write the equation as

$$\frac{dy}{dx} = \frac{1}{y}\left(\frac{x+1}{x^2}\right)$$

which is in the form $\frac{dy}{dx} = p(x)q(y)$ with $p(x) = \frac{x+1}{x^2}$ and $q(x) = \frac{1}{y}$.

We now separate the variables and integrate, giving:

$$y\,dy = \frac{x+1}{x^2}\,dx$$

$$\int y\,dy = \int\left(\frac{1}{x} + \frac{1}{x^2}\right)dx$$

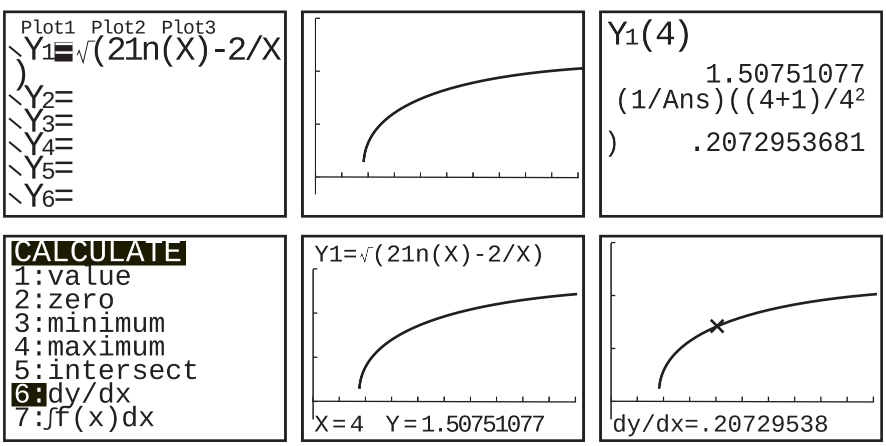
$$\frac{1}{2}y^2 = \ln x - \frac{1}{x} + C$$

$$y = \sqrt{2\ln x - \frac{2}{x} + C}$$

This is the general solution of $x^2 y\frac{dy}{dx} = x+1$, $x > 0$, $y > 0$ in explicit form.

With some thinking we can use our GDC to help confirm this result. $\frac{dy}{dx} = \frac{1}{y}\left(\frac{x+1}{x^2}\right)$ is the rule that gives us the slope of the graph of the function $y(x)$ at any point (x, y). In the GDC screen images below we enter the function $y = \sqrt{2\ln x - \frac{2}{x} + C}$; choose a value for x ($x = 4$, for example); use the GDC to find an approximate value for the corresponding y-coordinate; use the rule for $\frac{dy}{dx}$ to find the slope at that point; and then check to see if the same value for $\frac{dy}{dx}$ is given when evaluating it on the graph screen.

The GDC can also draw the tangent line at $x = 4$ and display its equation, confirming that the slope of the function at $x = 4$ is approximately 0.207 295 38, agreeing with the value computed by $\frac{dy}{dx} = \frac{1}{y}\left(\frac{x+1}{x^2}\right)$.



Here is an applied problem involving a separable differential equation.

Example 4

The rate of decay of a substance y at any time t is directly proportional to the amount of y and also directly proportional to the amount of another substance x. The constant of proportionality is $-\frac{1}{2}$ and the value of x at any time t is given by $x = \dfrac{4}{(1+t)^2}$.

a) Given the initial conditions that $y = 10$ when $t = 0$, find y as an explicit function of t.

b) Determine the amount of the substance remaining as t becomes very large.

Solution

a) The rate of decay of substance y is proportional to the product xy, and with the constant of proportionality having a value of $-\frac{1}{2}$ and $x = \dfrac{4}{(1+t)^2}$, this gives:

$$\frac{dy}{dt} = -\frac{1}{2}\left(\frac{4}{(1+t^2)}\right)y$$

$$\frac{1}{y}\,dy = \frac{-2}{(1+t)^2}\,dt$$ Separating variables.

$$\int \frac{1}{y}\,dy = -2\int \frac{1}{(1+t)^2}\,dt$$ Integrating both sides.

$$\ln y = \frac{2}{1+t} + C$$

$$y = e^{\frac{2}{1+t}+C}$$ Exponentiating; using e as the base.

$$y = e^C e^{\frac{2}{1+t}}$$

$$y = Ae^{\frac{2}{1+t}}$$ Let $A = e^c$, a convenient form for the arbitrary constant.

Solve for A knowing that initially $y = 10$ when $t = 0$:

$$10 = Ae^{\frac{2}{1+0}} \quad \Rightarrow \quad 10 = Ae^2 \quad \Rightarrow \quad A = 10e^{-2}$$

Substituting gives:

$$y = 10e^{-2}e^{\frac{2}{1+t}} \quad \Rightarrow \quad y = 10e^{\frac{2}{1+t}-2} \quad \Rightarrow \quad y = 10e^{\frac{-2t}{1+t}}$$

b) As $t \to \infty$, $\dfrac{-2t}{1+t} \to -2$; thus, as $t \to \infty$, $y \to 10e^{-2} \approx 1.36$

Example 5

Solve the differential equation $x\ dx + e^{x+y}\cos y\ dy = 0$.

Solution

As it is the equation cannot be written in the variables separable form $\dfrac{dy}{dx} = p(x)q(y)$. Since $e^{x+y} = e^x e^y$ we can make it so by multiplying both sides of the equation by e^{-x} and doing some rearrangement.

$$xe^{-x}dx + e^y\cos y\,dy = 0 \quad \Rightarrow \quad e^y\cos y\,dy = -xe^{-x}dx \;\Rightarrow\; \frac{dy}{dx} = -xe^{-x}\left(\frac{1}{e^y\cos y}\right)$$

Separating the variables and integrating both sides gives:

$$\int e^y\cos y\,dy = -\int xe^{-x}\ dx$$

$$\frac{e^y}{2}(\sin y + \cos y) = xe^{-x} + e^{-x} + C$$ Using integration by parts on both sides.

Therefore, the implicit function $\dfrac{e^y}{2}(\sin y + \cos y) = e^{-x}(x+1) + C$ is the general solution.

To finish this section we will find an explicit solution by the method of separation of variables for a relatively straightforward first order differential equation, but one whose solution will prove useful in developing another solution method.

Example 6

Find the general solution to the differential equation $\dfrac{dy}{dx} = -2xy$.

Solution

$$\frac{1}{y}dy = -2x\,dx$$ Separating variables; note loss of solution where $y = 0$.

$$\int\frac{1}{y}dy = -\int 2x\,dx$$ Integrating both sides.

$$\ln|y| = -x^2 + C_1$$

$$e^{\ln|y|} = e^{-x^2 + C_1}$$ Exponentiate both sides to solve for y.

$$|y| = e^{C_1}e^{-x^2}$$

$$|y| = C_2e^{-x^2}$$ e^{C_1} is a positive constant; let $e^{C_1} = C_2$ and $C_2 > 0$.

If $y > 0$, then $|y| = y$ and the solution becomes

$$y = C_2e^{-x^2}.$$

If $y < 0$, then $|y| = -y$ and the solution becomes

$$y = -C_2e^{-x^2}.$$

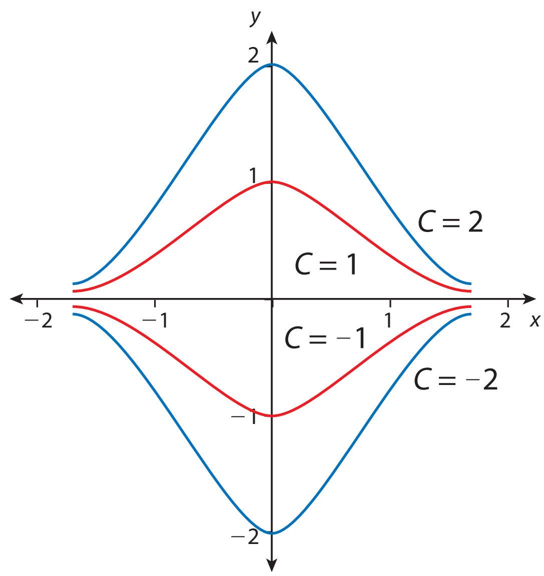
We can include both of these solutions, and also the 'lost' solution $y = 0$, by giving the general solution as

$$y = Ce^{-x^2}$$

with no restrictions on the constant C.

It is helpful for our review to recognize that the explicit solution $y = Ce^{-x^2}$ for Example 6 defines a 'family' of curves in the xy-plane. Some of these curves, with the corresponding value of C, have been graphed in Figure 5.9. In order to determine a specific curve from this 'family' we must impose an initial condition on the solution.

Figure 5.9



5.3 First order linear differential equations – use of integrating factor

As mentioned previously, a first order differential equation is called such because the first derivative $\frac{dy}{dx}$ appears in the equation. A differential equation is called *linear* when both $\frac{dy}{dx}$ and y appear only to the first power. The standard form for a *first order* ***linear*** *differential equation* is

$$\frac{dy}{dx} + P(x)y = Q(x).$$

We wish to develop a method to solve first order linear differential equations of this form (which could also be written as $y' + P(x)y = Q(x)$).

We start by considering a simple case when $Q(x) = 0$, so

$\frac{dy}{dx} + P(x)y = Q(x)$ becomes

$$\frac{dy}{dx} + P(x)y = 0.$$

This equation is variables separable, giving us

$$\frac{1}{y}\frac{dy}{dx} = -P(x).$$

This equation can be integrated in the same way as in Example 6 to give

$$\ln|y| = -\int P(x)\,dx + C_1$$

and following the same steps as in Example 6, we get

$$y = Ce^{-\int P(x)\,dx}$$

which is a general solution for the linear differential equation

$\frac{dy}{dx} + P(x)y = 0.$

However, we wish to find a general solution to the more general first order linear differential equation $\frac{dy}{dx} + P(x)y = Q(x)$ where $Q(x)$ is not necessarily zero. By applying the product rule and implicit differentiation we observe that

$$\begin{aligned}\frac{d}{dx}\left(ye^{\int P(x)\,dx}\right) &= \frac{dy}{dx}e^{\int P(x)\,dx} + yP(x)e^{\int P(x)\,dx}\\ &= e^{\int P(x)\,dx}\left(\frac{dy}{dx} + P(x)y\right).\end{aligned}$$

Thus, if we multiply both sides of $\frac{dy}{dx} + P(x)y = Q(x)$ by the factor $e^{\int P(x)\,dx}$ (called an **integrating factor**), we get

$$e^{\int P(x)\,dx}\left(\frac{dy}{dx} + P(x)y\right) = e^{\int P(x)\,dx}Q(x).$$

From the working above, we can substitute $\frac{d}{dx}\left(ye^{\int P(x)\,dx}\right)$ for $e^{\int P(x)\,dx}\left(\frac{dy}{dx} + P(x)y\right)$, yielding

$$\frac{d}{dx}\left(ye^{\int P(x)\,dx}\right) = e^{\int P(x)\,dx}Q(x).$$

Integrating both sides gives

$$ye^{\int P(x)\,dx} = \int e^{\int P(x)\,dx}Q(x)\,dx + C.$$

We can now solve for y, giving

$$y = e^{-\int P(x)\,dx}\left[\int e^{\int P(x)\,dx}Q(x)\,dx + C\right].$$

Solution to first order linear differential equations

Given a first order linear differential equation in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

the general solution is

$$y = e^{-\int P(x)dx}\int e^{\int P(x)dx}Q(x)\,dx + Ce^{-\int P(x)dx}$$

where C is an arbitrary constant.

Although the expression for the general solution given above looks quite complicated, the **basic steps for solving a first order linear differential equation** by means of an integrating factor are relatively simple.

Step 1: Make sure the equation is in the standard form $\frac{dy}{dx} + P(x)y = Q(x)$.

Step 2: Compute the integrating factor $e^{\int P(x)dx}$ by finding $\int P(x)\,dx$.

Step 3: Multiply both sides of the equation by the integrating factor.

Step 4: Integrate both sides of the equation. The left side will be $e^{\int P(x)dx}\left(\frac{dy}{dx} + P(x)y\right)$ which is equivalent to $\frac{d}{dx}\left(ye^{\int P(x)dx}\right)$ and the integral of this expression is $ye^{\int P(x)dx}$.

Step 5: Obtain an explicit solution for y by dividing both sides by the integrating factor $e^{\int P(x)dx}$.

When computing the integrating factor $e^{\int P(x)dx}$, it is standard practice to omit the constant of integration from the indefinite integral of $P(x)$.

Let's illustrate the five basic solution steps with an example.

Example 7

Find the general solution of $x\frac{dy}{dx} - 2y = x^2$.

Solution

1. $\frac{x}{x}\frac{dy}{dx} - \frac{2y}{x} = \frac{x^2}{x}$ Divide both sides by x to get equation into standard form.

$\frac{dy}{dx} - \left(\frac{2}{x}\right)y = x$ Standard form $\frac{dy}{dx} + P(x)y = Q(x)$; $P(x) = -\frac{2}{x}$ and $Q(x) = x$.

2. Integrating factor: $e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|} = e^{-\ln x^2} = \frac{1}{e^{\ln x^2}} = \frac{1}{x^2}$

3. $\frac{1}{x^2}\left[\frac{dy}{dx} - \left(\frac{2}{x}\right)y\right] = \frac{1}{x^2}(x)$ Multiply both sides by integrating factor.

$\frac{1}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = \frac{1}{x}$

4. $\displaystyle\int\left(\frac{1}{x^2}\frac{dy}{dx}-\frac{2y}{x^3}\right)dx=\int\frac{1}{x}\,dx$ Integrate both sides with respect to x.

$y\left(\dfrac{1}{x^2}\right)=\ln|x|+C$ $\quad \dfrac{d}{dx}\left[y\left(\dfrac{1}{x^2}\right)\right]=\dfrac{1}{x^2}\dfrac{dy}{dx}-\dfrac{2y}{x^3}$, by product rule and implicit differentiation.

5. Therefore, $y=x^2\ln|x|+Cx^2$ is the general solution.

It is appropriate to call the differential equation $\dfrac{dy}{dx}+P(x)y=Q(x)$ **linear** because $\dfrac{dy}{dx}=-P(x)y+Q(x)$ is a **linear function** of y.

Example 8

Find the particular solution of

$$\left(x^2+1\right)\frac{dy}{dx}+xy=(1-2x)\sqrt{x^2+1}$$

given that $y=2$ when $x=1$.

Solution

1. $\displaystyle\frac{x^2+1}{x^2+1}\frac{dy}{dx}+\frac{xy}{x^2+1}=\frac{(1-2x)\sqrt{x^2+1}}{x^2+1}$

$\displaystyle\frac{dy}{dx}+\left(\frac{x}{x^2+1}\right)y=\frac{1-2x}{\sqrt{x^2+1}}$ Standard form with $P(x)=\dfrac{x}{x^2+1}$, $Q(x)=\dfrac{1-2x}{\sqrt{x^2+1}}$.

2. Integrating factor:

$$\int P(x)\,dx=\int\frac{x}{x^2+1}\,dx=\frac{1}{2}\ln\left(x^2+1\right)=\ln\sqrt{x^2+1}\ \Rightarrow\ e^{\ln\sqrt{x^2+1}}=\sqrt{x^2+1}$$

3. $\displaystyle\sqrt{x^2+1}\frac{dy}{dx}+\sqrt{x^2+1}\left(\frac{x}{x^2+1}\right)y=\frac{\sqrt{x^2+1}(1-2x)}{\sqrt{x^2+1}}$ Multiply both sides by integrating factor.

$$\sqrt{x^2+1}\frac{dy}{dx}+\left(\frac{x}{\sqrt{x^2+1}}\right)y=1-2x$$

4. $\displaystyle\int\left[\sqrt{x^2+1}\frac{dy}{dx}+\left(\frac{x}{\sqrt{x^2+1}}\right)y\right]dx=\int(1-2x)\,dx$ Integrate both sides.

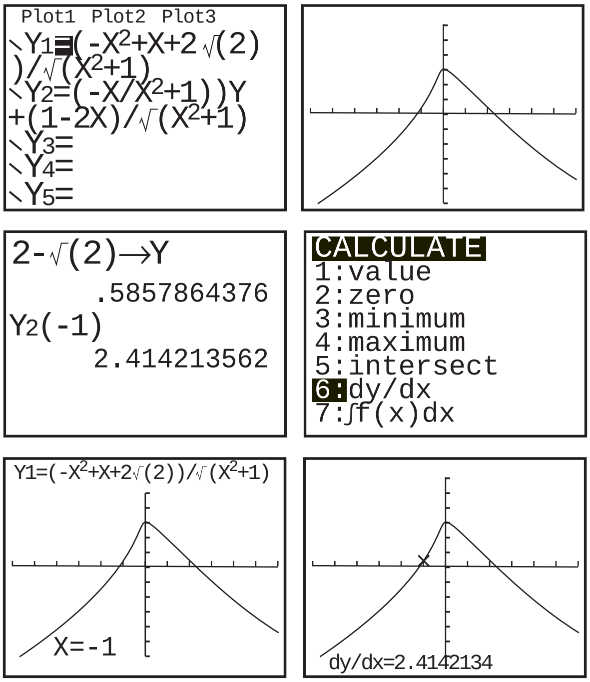
$$y\sqrt{x^2+1}=x-x^2+C$$

5. $\displaystyle y=\frac{-x^2+x+C}{\sqrt{x^2+1}}$ Divide both sides by integrating factor.

To solve for C, we substitute $y=2$ and $x=1$.

$$2=\frac{-1+1+C}{\sqrt{1+1}}\ \Rightarrow\ C=2\sqrt{2}$$

Therefore, the particular solution is $y=\dfrac{-x^2+x+2\sqrt{2}}{\sqrt{x^2+1}}$.



Once again, with a bit of effort, we can add some confidence to our result for Example 8 by using our GDC to graph the solution curve and then check to see if the original differential equation accurately describes its behaviour (shape).

Enter the solution curve for Y_1 and enter the differential equation in the form $\frac{dy}{dx} = -\left(\frac{x}{x^2+1}\right)y + \frac{1-2x}{\sqrt{x^2+1}}$ for Y_2. Turn Y_2 'off' (un-highlight) so that it is not graphed; only the solution curve is graphed. Choose a value for x that is in the graph window – say, $x = -1$; and evaluate the corresponding y-value for a point on the solution curve.
$y = \frac{-(-1)^2 - 1 + 2\sqrt{2}}{\sqrt{(-1)^2+1}} = \frac{-2+2\sqrt{2}}{\sqrt{2}} = 2 - \sqrt{2}$; point $\left(-1, 2-\sqrt{2}\right)$ is on the solution curve. After setting y equal to $2-\sqrt{2}$, use Y_2 to find the value of $\frac{dy}{dx}$ at $\left(-1, 2-\sqrt{2}\right)$. Check that this value for the slope of the curve at $\left(-1, 2-\sqrt{2}\right)$, found to be approximately 2.414 213 562, agrees with the value found on the graph window. Both methods of finding $\frac{dy}{dx}$ at $\left(-1, 2-\sqrt{2}\right)$, from the differential equation and from the solution to the differential equation, give the same value, thus supporting our particular solution to the differential equation.

Example 9

In the earlier section on slope fields, we displayed a slope field (Figure 5.2) for the differential equation $\frac{dy}{dx} = x - y$. Find the general solution to this equation.

Solution

The equation first appears that it may be separable, but it cannot be expressed in the form $\frac{dy}{dx} = p(x)q(y)$. It is a first order linear differential equation because it can be rearranged to $\frac{dy}{dx} + y = x$ which puts it into the standard form $\frac{dy}{dx} + P(x)y = Q(x)$ such that $P(x) = 1$ and $Q(x) = x$. The integrating factor is $e^{\int dx} = e^x$, and multiplying through by this gives

$$e^x\frac{dy}{dx} + e^x y = e^x x$$

and continuing with the steps for solving a first order linear differential equation yields

$$\int\left(e^x\frac{dy}{dx}+e^x y\right)dx=\int e^x x\,dx$$

$$ye^x=e^x x-e^x+C \qquad \text{Using integration by parts on the right side.}$$

Thus, the general solution is $y=x-1+Ce^{-x}$. Figure 5.10 shows the same slope field displayed in Figure 5.2 for $\frac{dy}{dx}=x-y$ along with the graphs of three different solution curves generated from the general solution for $C=1,\frac{1}{10}$ and -4.

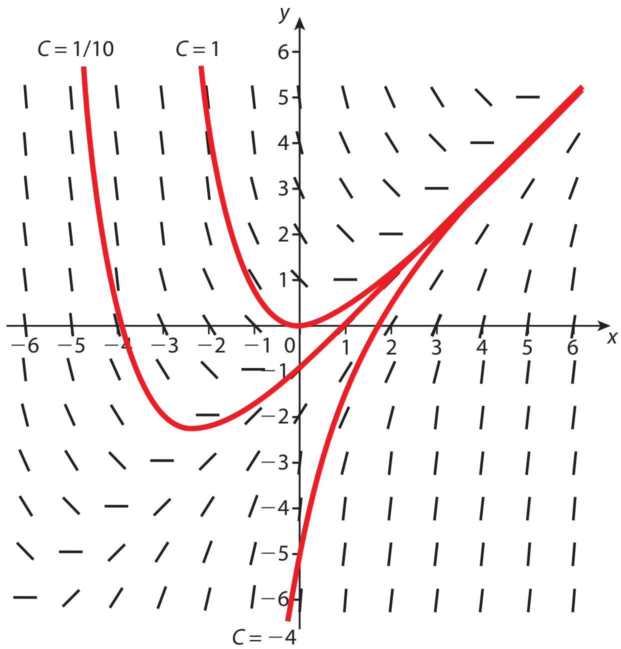


Figure 5.10

An analytic method for solving differential equations, such as those for separable equations and first order linear equations, demand fluency with a range of integration techniques, and differentiation – as the next example nicely illustrates.

Example 10

Find the particular solution to $(1+\sin x)\frac{dy}{dx}-y\cos x=(1+\sin x)^4$ given $y(0)=1$.

Solution

Dividing through by $1+\sin x$, the equation becomes

$$\frac{dy}{dx}-\left(\frac{\cos x}{1+\sin x}\right)y=(1+\sin x)^3.$$

The integrating factor is $e^{\int\frac{-\cos x}{1+\sin x}dx}=e^{-\ln(1+\sin x)}=e^{\ln\left[(1+\sin x)^{-1}\right]}=\frac{1}{1+\sin x}$.

Multiplying both sides by the integrating factor gives

$$\frac{1}{1+\sin x}\frac{dy}{dx}-\left(\frac{\cos x}{(1+\sin x)^2}\right)y=(1+\sin x)^2.$$

Our experience of differentiating functions and familiarity with the solution pattern for first order differential equations, informs us that the left side is equal to the derivative of $\frac{y}{1+\sin x}$.

$$\frac{d}{dx}\left(\frac{y}{1+\sin x}\right)=\sin^2 x+2\sin x+1$$

We now integrate both sides. The integral of the left is simply $\frac{y}{1+\sin x}$ and integrating each term on the right is straightforward except for $\sin^2 x$. We need to take the double-angle identity $\cos 2x = 1-2\sin^2 x$ and rearrange it to give us $\sin^2 x=\frac{1}{2}-\frac{1}{2}\cos 2x$.

$$\int\frac{d}{dx}\left(\frac{y}{1+\sin x}\right)dx=\int\left(\frac{1}{2}-\frac{1}{2}\cos 2x+2\sin x+1\right)dx$$

$$\frac{y}{1+\sin x}=\frac{x}{2}-\frac{1}{4}\sin 2x-2\cos x+x+C$$

$$y=(1+\sin x)\left(\frac{3x}{2}-\frac{1}{4}\sin 2x-2\cos x+C\right)$$

Given $y(0)=1$, it follows that $1=(1+0)(0-0-2+C) \quad\Rightarrow\quad C=3$

Therefore, the particular solution is

$$y=\frac{1}{4}(1+\sin x)(6x-\sin 2x-8\cos x+12).$$

5.4 Homogeneous differential equations

When a first order differential equation is not separable nor linear, it may still be possible to transform it by an appropriate substitution into an equation that we can solve analytically. One situation where this will always work is when the first order differential equation is **homogeneous**.

Homogeneous first order differential equations

The differential equation $\frac{dy}{dx}=F(x,y)$ is **homogeneous** if the right side can be expressed as a function of the ratio $\frac{y}{x}$ alone, that is, $\frac{dy}{dx}=F\left(\frac{y}{x}\right)$.

The function F can be written as a function of $\frac{y}{x}$ if it can be expressed as a quotient of two **homogeneous functions of the same degree.** In general, a two-variable function is homogeneous of degree n if the sum of the powers of x and y in *each* term is n. For example: $g(x,y)=2x^2+xy-5y^2$

is homogeneous of degree 2; and $h(x, y) = 3y^3 - xy^2$ is homogeneous of degree 3. The function $m(x, y) = 4x^2y^2 - x^3y^2$ is *not* homogeneous.

Thus, if we solve for $\frac{dy}{dx}$ and get it to be equal to a quotient in the form $\frac{M(x, y)}{N(x, y)}$, where M and N are homogeneous functions of the same degree, then the equation is a homogeneous differential equation. The equation $\frac{dy}{dx} = \frac{M(x, y)}{N(x, y)}$ can be written as a function of $\frac{y}{x}$, i.e. $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$, by dividing through both $M(x, y)$ and $N(x, y)$ by x^n, where n is the degree of M and N. Two examples are given below.

1. $\frac{dy}{dx} = \frac{6xy}{x^2 - y^2}$ is a homogeneous differential equation because both the numerator, $6xy$, and the denominator, $x^2 - y^2$, are homogeneous functions of degree 2. The right side can be expressed in terms of $\frac{y}{x}$ by dividing numerator and denominator by x^2.

$$\frac{dy}{dx} = \frac{\frac{6xy}{x^2}}{\frac{x^2 - y^2}{x^2}} = \frac{6\left(\frac{y}{x}\right)}{1 - \left(\frac{y}{x}\right)^2}$$

2. $\frac{dy}{dx} = \frac{3y^3 - xy^2}{x^3 + x^2y - xy^2}$ is a homogeneous differential equation because both the numerator, $3y^3 - xy^2$, and the denominator, $x^3 + x^2y - xy^2$, are homogeneous functions of degree 3. We divide numerator and denominator by x^3 to get

$$\frac{dy}{dx} = \frac{\frac{3y^3}{x^3} - \frac{xy^2}{x^3}}{\frac{4x^3}{x^3} + \frac{x^2y}{x^3} - \frac{2xy^2}{x^3}} = \frac{3\left(\frac{y}{x}\right)^3 - \left(\frac{y}{x}\right)^2}{4 + \frac{y}{x} - 2\left(\frac{y}{x}\right)^2}$$

Once a homogeneous differential equation is written in the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ it can be solved analytically by making the substitution $y = vx$ $\left(\text{or } v = \frac{y}{x}\right)$ where v is a differentiable function of x. As we will see, this substitution transforms the differential equation into a separable equation for which we have a solution method.

Example 11

Find the particular solution for $xy^2\frac{dy}{dx} = x^3 + y^3$ given $y = 3$ when $x = 1$.

Solution

Dividing both sides by xy^2 reveals that the differential equation is homogeneous because both numerator and denominator on the right side are homogeneous functions of degree 3.

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$$

Dividing both numerator and denominator by x^3 expresses the derivative in terms of $\frac{y}{x}$.

$$\frac{dy}{dx} = \frac{\frac{x^3}{x^3} + \frac{y^3}{x^3}}{\frac{xy^2}{x^3}} = \frac{1 + \left(\frac{y}{x}\right)^3}{\left(\frac{y}{x}\right)^2}$$

We now let $y = vx$ which means that $\frac{dy}{dx} = v + x\frac{dv}{dx}$ by means of the product rule. Substituting v for $\frac{y}{x}$ and $v + x\frac{dv}{dx}$ for $\frac{dy}{dx}$ produces

$$v + x\frac{dv}{dx} = \frac{1 + v^3}{v^2}$$

which is a separable equation for the variables x and v because it can be written in the form $\frac{dy}{dx} = p(x)q(v)$, as shown below:

$$x\frac{dv}{dx} = \frac{1+v^3}{v^2} - v \quad \Rightarrow \quad x\frac{dv}{dx} = \frac{1+v^3}{v^2} - \frac{v^3}{v^2} \quad \Rightarrow \quad x\frac{dv}{dx} = \frac{1}{v^2} \quad \Rightarrow \quad \frac{dv}{dx} = \left(\frac{1}{x}\right)\left(\frac{1}{v^2}\right)$$

Separating the variables and integrating:

$$v^2 dv = \frac{1}{x}dx$$

$$\int v^2\, dv = \int \frac{1}{x}dx$$

$$\frac{1}{3}v^3 = \ln|x| + C$$

If $y = 3$ when $x = 1$, then $v = \frac{y}{x} = \frac{3}{1} = 3$, and substituting gives

$9 = \ln 1 + C \quad \Rightarrow \quad C = 9$. Thus,

$$\frac{1}{3}v^3 = \ln|x| + 9 \quad \Rightarrow \quad v^3 = 3\ln|x| + 27$$

Substituting $\frac{y}{x}$ back in for v gives:

$$\left(\frac{y}{x}\right)^3 = 3\ln|x| + 27$$

$$y^3 = x^3\left(3\ln|x| + 27\right)$$

Therefore, the particular solution is $y = x\left(3\ln|x| + 27\right)^{\frac{1}{3}}$.

Using Example 11 as a guide we can outline the **basic steps for solving a first order homogeneous differential equation.**

Step 1: Confirm that, or rearrange it so that, the equation is in the form

$\dfrac{dy}{dx} = \dfrac{M(x,y)}{N(x,y)}$, where M and N are homogeneous functions of the same degree.

Step 2: Divide both $M(x,y)$ and $N(x,y)$ by x^n, where n is the degree of M and N, so that the equation is in the form $\dfrac{dy}{dx} = F\left(\dfrac{y}{x}\right)$.

Step 3: Let $y = vx$ from which it follows that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ and substitute v for $\dfrac{y}{x}$ and $v + x\dfrac{dv}{dx}$ for $\dfrac{dy}{dx}$ transforming the equation into a separable equation in terms of v and x.

Step 4: By applying the technique of separation of variables, find a solution in terms of v and x.

Step 5: Substitute $\dfrac{y}{x}$ back in for v and write the solution in terms of y and x.

Do not forget to perform *Step 5* – substituting $\dfrac{y}{x}$ back in for v – because you must express your final solution in terms of y and x.

Example 12

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$ where $x > 0,\ y > 0$.

a) Use the substitution $y = vx$ to show that $x\dfrac{dv}{dx} = \dfrac{1+v^2}{1-v}$.

b) Hence, find the general solution of the differential equation, giving your answer in the form $C = f(x,y)$.

Solution

a) 1. The equation is already in the form $\dfrac{dy}{dx} = \dfrac{M(x,y)}{N(x,y)}$ where, in this case, M and N are homogeneous of degree 1.

2. Divide numerator and denominator by x.

$$\frac{dy}{dx} = \frac{\frac{x}{x} + \frac{y}{x}}{\frac{x}{x} - \frac{y}{x}} = \frac{1 + \frac{y}{x}}{1 - \frac{y}{x}}$$

3. Letting $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Substituting v for $\dfrac{y}{x}$ and $v + x\dfrac{dv}{dx}$ for $\dfrac{dy}{dx}$, gives:

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

$$x\frac{dv}{dx} = \frac{1+v}{1-v} - \frac{v(1-v)}{1-v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{1-v} \qquad \text{Q.E.D.}$$

b) 4. Separating the variables and integrating, yields

$$\int \frac{1-v}{1+v^2}\,dv = \int \frac{1}{x}\,dx.$$

To integrate the left side we split up the fraction:

$$\int \frac{1}{1+v^2}\,dv - \int \frac{v}{1+v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\arctan v - \frac{1}{2}\ln\left(1+v^2\right) = \ln x + C$$

5. Substituting $\frac{y}{x}$ back in for v gives

$$\arctan\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \ln x + C.$$

Solving for C:

$$\arctan\left(\frac{y}{x}\right) - \left[\ln\left(1+\frac{y^2}{x^2}\right)^{\frac{1}{2}} + \ln x\right] = C$$

$$\arctan\left(\frac{y}{x}\right) - \ln\left(x\sqrt{1+\frac{y^2}{x^2}}\right) = C$$

$$C = \arctan\left(\frac{y}{x}\right) - \ln\sqrt{x^2+y^2}$$

This is the general solution such that y is an implicit function of x.

Example 13

a) Show that $\frac{d}{dx}\left[\ln\left(x+\sqrt{1+x^2}\right)\right] = \frac{1}{\sqrt{1+x^2}}$.

b) Show that the solution curve that satisfies the differential equation $x\frac{dy}{dx} = y + \sqrt{x^2+y^2}$ with initial conditions $y(0) = -1$ is the parabola $y = \frac{x^2}{4} - 1$. [Hint: Use the result from a) to integrate the separable equation that is in terms of v and x.]

Solution

a) $$\frac{d}{dx}\left[\ln\left(x+\sqrt{1+x^2}\right)\right] = \frac{1}{x+\sqrt{1+x^2}}\left[\frac{d}{dx}\left(x+\left(1+x^2\right)^{\frac{1}{2}}\right)\right]$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left[1+\frac{1}{2}\left(1+x^2\right)^{-\frac{1}{2}}(2x)\right]$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left[1+\frac{x}{\sqrt{1+x^2}}\right]$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left[\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}+\frac{x}{\sqrt{1+x^2}}\right]$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)$$

$$= \frac{1}{\sqrt{1+x^2}} \qquad \text{Q.E.D.}$$

b) First, divide both sides by x to give $\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$. The term $\sqrt{x^2+y^2}$ has a degree of 1, so both numerator and denominator are homogeneous functions of degree 1. Now divide numerator and denominator on right side by x in order to write the equation in the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

$$\frac{dy}{dx} = \frac{\frac{y}{x}+\frac{\sqrt{x^2+y^2}}{\sqrt{x^2}}}{\frac{x}{x}} = \frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}$$

Letting $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substituting v for $\frac{y}{x}$ and $v + x\frac{dv}{dx}$ for $\frac{dy}{dx}$, gives:

$$v + x\frac{dv}{dx} = v + \sqrt{1+v^2} \quad \Rightarrow \quad x\frac{dv}{dx} = \sqrt{1+v^2}$$

Separating the variables and integrating:

$$\int \frac{1}{\sqrt{1+v^2}}\,dv = \int \frac{1}{x}\,dx$$

From part a) we know that $\frac{d}{dx}\left[\ln\left(x+\sqrt{1+x^2}\right)\right] = \frac{1}{\sqrt{1+x^2}}$. Therefore, $\int \frac{1}{\sqrt{1+v^2}}\,dv = \ln\left(v+\sqrt{1+v^2}\right)+C$. Using this result gives:

$$\ln\left(v+\sqrt{1+v^2}\right) = \ln|x| + \ln C$$ Setting arbitrary constant to $\ln C$.

$$e^{\ln\left(v+\sqrt{1+v^2}\right)} = e^{\ln|x|+\ln C}$$ Exponentiating both sides using base of e.

$$v+\sqrt{1+v^2} = Cx$$

$$\left(\sqrt{1+v^2}\right)^2 = (Cx-v)^2$$

$$1+v^2 = C^2x^2 - 2Cxv + v^2$$

$$1 = C^2x^2 - 2Cxv$$

$$1 = C^2x^2 - 2Cx\left(\frac{y}{x}\right)$$ Substituting $\frac{y}{x}$ back in for v.

$$2Cy = C^2x^2 - 1$$

$$y = \frac{1}{2}Cx^2 - \frac{1}{2C}$$

Solve for C given the initial condition $y(0) = -1$:

$$-1 = 0 - \frac{1}{2C} \quad \Rightarrow \quad C = \frac{1}{2}$$

Hence,

$$y = \frac{1}{2}\left(\frac{1}{2}\right)x^2 - \frac{1}{2\left(\frac{1}{2}\right)}.$$

Therefore, the particular solution curve is the parabola $y = \frac{1}{4}x^2 - 1$.

5.5 Euler's method

We have established three analytic methods for solving different types of first order differential equations: separable equations, linear equations (integrating factor) and homogeneous equations (substitution $y = vx$). Also, earlier in this chapter we saw how a slope field is an effective graphical method that provides a rough idea about the behaviour of solutions to a differential equation, especially for an equation that we are not able to solve analytically. To roughly sketch a particular solution to a differential equation using a slope field, we need to know a point (initial condition) that the solution curve passes through in order to have a 'starting point' from which to sketch a curve that will be parallel to the short line segments drawn at representative points that indicate the slope of any solution. Several of the examples in this chapter have found particular solutions to what is referred to as an **initial-value problem** that is stated in the form

$$\frac{dy}{dx} = F(x, y), \quad y(x_0) = y_0.$$

Consider the initial-value problem

$$\frac{dy}{dx} = x + y, \quad y(2) = 0.$$

Figure 5.11 shows the slope field for the differential equation $\frac{dy}{dx} = x + y$.

An approximation to the particular solution can be sketched by drawing a smooth curve through the point $(2, 0)$ that follows the slopes in the slope field, as shown in the figure.

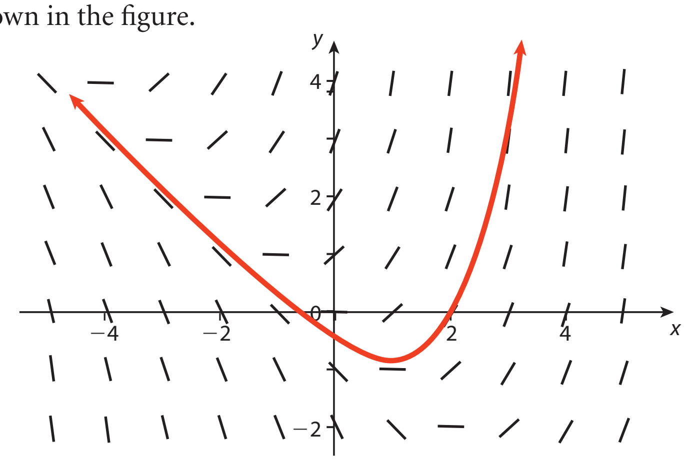


Figure 5.11 Slope field for $\frac{dy}{dx} = x + y$ and sketch of solution passing through (2, 0).

Let $y(x)$ represent the solution curve. To approximate a value of y for a specific value of x, for example y when $x = 3$, we could make an educated guess from the sketch of y made with the aid of the slope field. But if we want a more accurate approximation then we need to use a more refined method. The simplest numerical method is called **Euler's method**, after the prolific eighteenth-century mathematician who first devised this computational method to help him calculate the orbit of our Moon.

Euler's method uses the basic idea behind the construction of slope fields to find numerical approximations to solutions of differential equations. Let's illustrate the method with the initial-value problem that we have just been considering, namely:

$$\frac{dy}{dx} = x + y, \quad y(2) = 0$$

We know from the differential equation that the slope of the solution curve is 2 at the point (2, 0) because $\frac{dy}{dx} = x + y = 2 + 0 = 2$. Hence, the line tangent to the solution curve at (2, 0) has the equation: $y - 0 = 2(x - 2) \quad \Rightarrow \quad y = 2x - 4$. We can use this tangent line as a rough approximation to the solution curve (see Figure 5.12). This approximation clearly becomes less accurate as we move away from the point of tangency (2, 0).

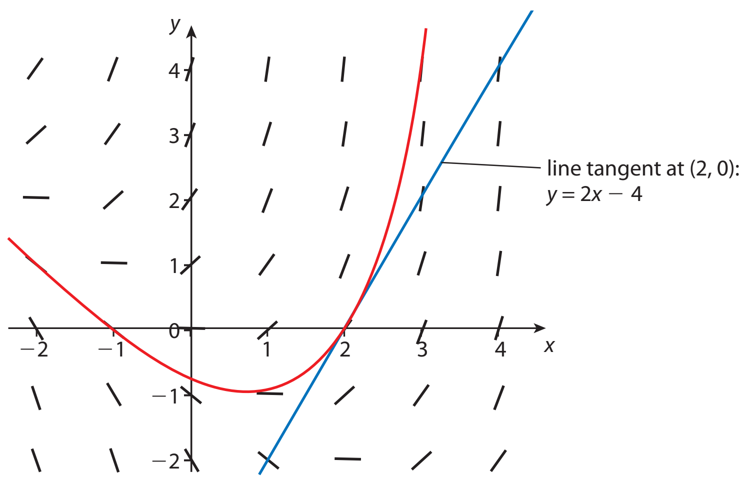


Figure 5.12

Euler's method improves this approximation by moving a short horizontal distance (the **step size** h) along this tangent line and then change direction according to the slope field. In this way we build an approximation to the curve by attaching little line segments together, each having the slope of the solution curve at its starting point.

In general, after being presented with an initial value problem:
$\frac{dy}{dx} = F(x, y),\ y(x_0) = y_0$ we choose a step size h. Starting at the point (x_0, y_0), for the interval $x_0 \leqslant x \leqslant x_0 + h$, we approximate the solution curve with the tangent line, i.e. the line with slope $F(x_0, y_0)$. This takes us

as far as the point (x_1, y_1), whose coordinates are calculated as follows (see Figure 5.13):

$$x_1 = x_0 + h,\ y_1 = y_0 + hF(x_0, y_0)$$

Now we are at the starting point of the second line segment (x_1, y_1). We repeat the process, with the next line segment having slope $F(x_1, y_1)$. This takes us to the next point (x_2, y_2) on the Euler approximation where $x_2 = x_1 + h$ and $y_2 = y_1 + hF(x_1, y_1)$.

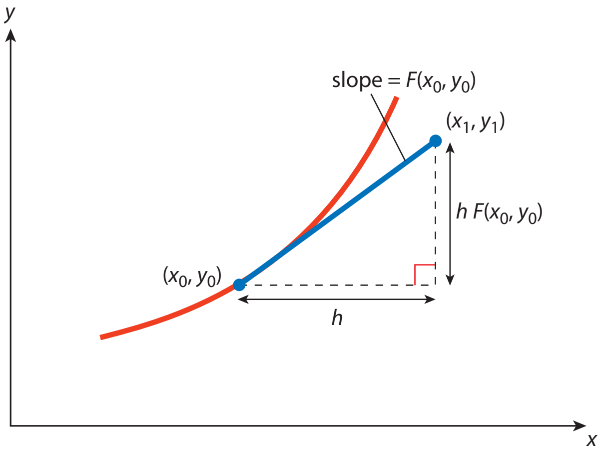
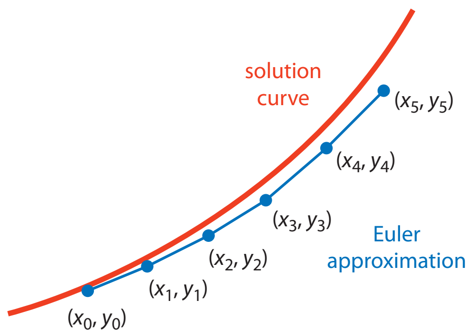


Figure 5.13 Euler's method starts at (x_0, y_0) on the solution curve and moves along a segment with slope $F(x_0, y_0)$ to define a new point (x_1, y_1) such that $x_1 = x_0 + h$ and $y_1 = y_0 + hF(x_0, y_0)$. The process is repeated with the new point.

Repeating this process we get an approximation to the solution curve consisting of line segments joining the points (x_0, y_0), (x_1, y_1), (x_2, y_2), etc. Each computed value y_n is an estimate of the corresponding 'true solution' y at $x = x_n$. The accuracy of the estimates depends on the choice of the step size h and the overall number of steps (iterations). Decreasing the step size while increasing the number of steps leads to increasingly more accurate estimates for solution values.

Figure 5.14 Further iterations of Euler's method build an approximation to the solution curve.



Euler's numerical method

For the differential equation $\frac{dy}{dx} = F(x, y)$ with the initial condition $y(x_0) = y_0$, the recursive formulae for generating the coordinates of the unknown $(n + 1)$st point (x_{n+1}, y_{n+1}) from the known nth point (x_n, y_n) on the approximate solution curve (Euler approximation) are:

$$x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + hF(x_n, y_n) \qquad \text{for } n = 0, 1, 2, \ldots, N$$

where h, the step size, is a constant; and N is the total number of steps (iterations).

Let's now apply Euler's method to answer a question posed earlier for the initial-value problem presented at the start of this section.

Example 14

For the differential equation $\frac{dy}{dx} = x + y$ such that $y(2) = 0$, use Euler's method with a step value of 0.2 to find an approximate value of y when $x = 3$, giving your answer to two decimal places.

Solution

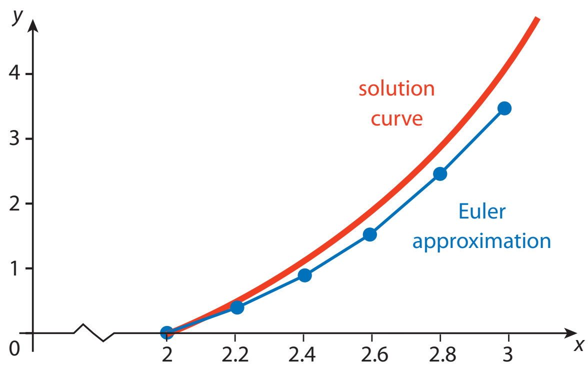


Figure 5.15

We use Euler's method to build an approximation to the 'true' solution curve starting at $x = 2$ and finishing at $x = 3$ by piecing together five short segments (Figure 5.15). We are given that $h = 0.2$, $x_0 = 2$, $y_0 = 0$ and $F(x, y) = x + y$. Using the appropriate formulae for x_n and y_n and iterating five times, we have:

$x_1 = x_0 + h = 2 + 0.2 = 2.2$

$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.2(2 + 0) = 0.4$

$x_2 = x_1 + h = 2.2 + 0.2 = 2.4$
$y_2 = y_1 + hF(x_1, y_1) = 0.4 + 0.2(2.2 + 0.4) = 0.92$

$x_3 = x_2 + h = 2.4 + 0.2 = 2.6$
$y_3 = y_2 + hF(x_2, y_2) = 0.92 + 0.2(2.4 + 0.92) = 1.584$

$x_4 = x_3 + h = 2.6 + 0.2 = 2.8$
$y_4 = y_3 + hF(x_3, y_3) = 1.584 + 0.2(2.6 + 1.584) = 2.4208$

$x_5 = x_4 + h = 2.8 + 0.2 = 3$
$y_5 = y_4 + hF(x_4, y_4) = 2.4208 + 0.2(2.8 + 2.4208) = 3.46496$

This process leads to an approximate (three decimal places) value of $y \approx 3.46$ when $x = 3$.

Because we will perform most of the calculations for each iteration on our GDC, it is often sufficient to simply display relevant results for each iteration in a table, as shown below.

n	x_n	y_n	$hF(x_n, y_n)$	x_{n+1}	y_{n+1}
0	2	0	0.4	2.2	0.4
1	2.2	0.4	0.52	2.4	0.92
2	2.4	0.92	0.664	2.6	1.584
3	2.6	1.584	0.8368	2.8	2.4208
4	2.8	2.4208	1.044 16	3.0	3.464 96

The first order differential equation in Example 14 is linear and hence can be solved by means of an integrating factor. Given $y(2) = 0$ the particular solution is $y = 3e^{x-2} - x - 1$. To three significant figures, the 'true' value of $y(3)$ is approximately 5.15. Thus, our approximation of 3.46 has an error of approximately 16.6%. Using a program on our GDC or a spreadsheet, we could easily decrease the step size (and increasing the number of steps) in order to improve the accuracy of the approximation. For example, if we used a step size of $h = 0.01$ (requiring 100 iterations) we would get an estimate of 5.11 (3 s.f.), reducing the error to less than 1%.

A numerical method like Euler's is especially useful when applied to a differential equation that cannot be solved by any known analytic methods, as we will do in the next example.

Example 15

Given that $\dfrac{dy}{dx} = \dfrac{x+1}{xy+2}$ and $y = 1$ when $x = 0$, use Euler's method with step size $h = 0.25$ to approximate the value of y when $x = 1$. Give the approximation to three significant figures.

Solution

We have that $x_0 = 0$, $y_0 = 1$, $h = 0.25$ and $F(x, y) = \dfrac{x+1}{xy+2}$. Thus the recursive formula for y_n is:

$$y_{n+1} = y_n + hF(x, y) = y_n + (0.25)\frac{x_n+1}{x_n y_n + 2} \quad \Rightarrow \quad y_{n+1} = y_n + \frac{x_n+1}{4x_n y_n + 8}$$

$n = 0$: $\quad x_1 = x_0 + h = 0 + 0.25 = 0.25$

$$y_1 = y_0 + \frac{x_0+1}{4x_0y_0+8} = 1 + \frac{0+1}{4(0)(1)+8} = \frac{9}{8} = 1.125$$

$n = 1$: $\quad x_2 = x_1 + h = 0.25 + 0.25 = 0.5$

$$y_2 = y_1 + \frac{x_1+1}{4x_1y_1+8} = 1.125 + \frac{0.25+1}{4(0.25)(1.125)+8} \approx 1.261986$$

$n = 2$: $\quad x_3 = x_2 + h = 0.5 + 0.25 = 0.75$

$$y_3 = y_2 + \frac{x_2 + 1}{4x_2y_2 + 8} = 1.261986 + \frac{0.5 + 1}{4(0.5)(1.261986) + 8} \approx 1.404518$$

$n = 3$: $\quad x_4 = x_3 + h = 0.75 + 0.25 = 1$

$$y_4 = y_3 + \frac{x_3 + 1}{4x_3y_3 + 8} = 1.404518 + \frac{0.75 + 1}{4(0.75)(1.404518) + 8} \approx 1.547801$$

Therefore, the approximate value of y when $x = 1$ is $y \approx 1.55$.

Exercise 5

1 Solve the differential equation $\frac{dy}{dx} = \frac{xy}{\sqrt{1+x^2}}$. Given that $y = 1$ when $x = 0$, express y as an explicit function of x.

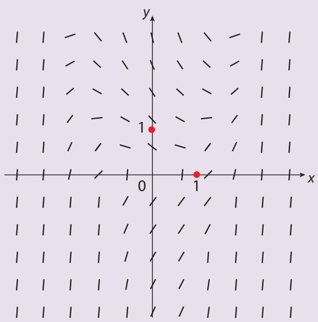
2 Find the particular solution to the differential equation $\frac{dy}{dx} = \sin x \cos^2 y$ given that $y = \frac{\pi}{4}$ when $x = \frac{\pi}{2}$.

3 The solution curve to the differential equation $x\frac{dy}{dx} = y(3 - y)$ passes through the point (2, 2). Find y as an explicit function of x.

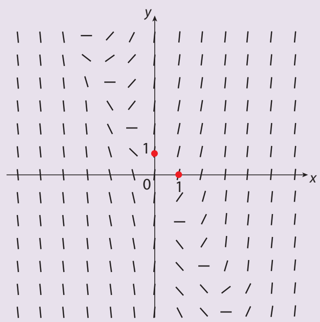
4 Show that the general solution to the differential equation $x\frac{dy}{dx} = y \ln x$ is $y = Cx^{\ln\sqrt{x}}$.

5 Match each slope field with its differential equation, listed below.

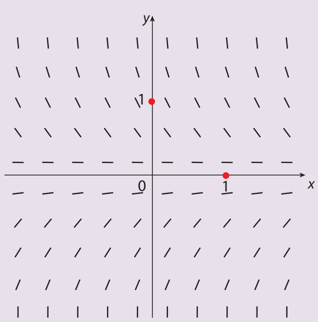
a



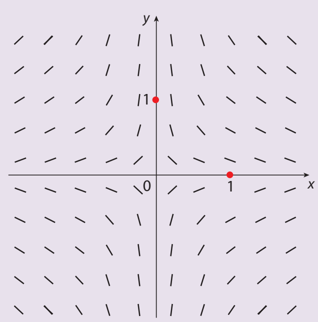
b



c



d



i $\frac{dy}{dx} = -2y$ $\qquad$ **ii** $\frac{dy}{dx} = x^2 - y$

iii $\frac{dy}{dx} = -\frac{y}{x}$ $\qquad$ **iv** $\frac{dy}{dx} = 2x + y$

6 All radioactive substances decay at a rate proportional to the amount of the substance that exists at any time. The half-life of radium is 1620 years. How much (accurate to 3 significant figures) of a 10-gram specimen of radioactive radium will remain after 25 years?

7 Solve the following separable differential equations.

a $\dfrac{dy}{dx} = \dfrac{2x}{y}$ **b** $\dfrac{dy}{dx} = \dfrac{y^2}{x^2}$ **c** $x^2\dfrac{dy}{dx} = y^2 - y$

d $x\dfrac{dy}{dx} = \tan y$ **e** $\dfrac{dy}{dx} = xy$ **f** $\sqrt{x^2+1}\dfrac{dy}{dx} = \dfrac{x}{y}$

g $\dfrac{dy}{dx} = \dfrac{y^2-1}{e^x}$ **h** $\ln y\dfrac{dy}{dx} = 1$

8 Using the method of separation of variables, show that an implicit solution for the differential equation $\dfrac{dy}{dx} = \dfrac{xy+y}{xy+x}$ is $ye^y = Axe^x$ where A is an arbitrary constant.

9 Find the general solution, in explicit form, to the differential equation $y\dfrac{dy}{dx} = \cos x$. Comment on the possible values of the constant C.

10 The equation for the rate of change of the population (in thousands), p, of a certain species is given by

$$\frac{dp}{dt} = 5p - 2p^2.$$

a Sketch the slope field.

b If the initial population is 4000 (that is, $p(0) = 4$), then what appears to be the limiting value of the population (that is, $\lim\limits_{t\to\infty} p(t)$)?

c If $p(0) = 0.5$, what is $\lim\limits_{t\to\infty} p(t)$?

d Comment on the long-term behaviour of the species' population growth.

11 Solve the initial-value problem:

$$\frac{dy}{dx} = \frac{2x+\sec^2 x}{2y}, \quad y(0) = -5$$

12 Consider the initial-value problem:

$$(1+x^2)\frac{dy}{dx} + 1 + y^2 = 0, \quad y(0) = -1$$

a Show that the implicit solution can be expressed as $\arctan y + \arctan x = \dfrac{\pi}{4}$.

b Use the formula for $\tan(A+B)$ to find the explicit solution.

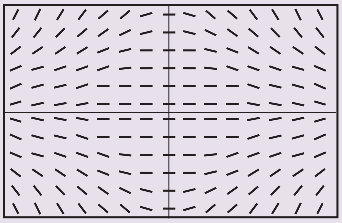
13 Solve the initial-value problem:

$$(1+x^2)\frac{dy}{dx} = 1 + y^2, \quad y(2) = 3$$

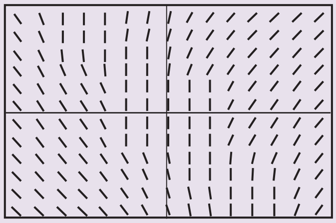
Write the solution in explicit form, expressing y in terms of x.

14 Match each slope field with its differential equation, listed below.

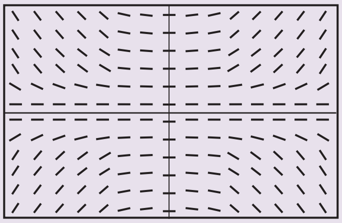
a



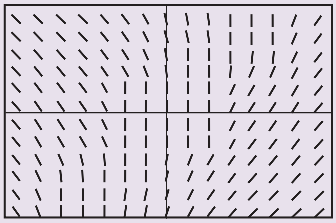
b



c



d



i $\dfrac{dy}{dx} = \dfrac{5}{x+y}$ **ii** $\dfrac{dy}{dx} = \dfrac{5}{x-y}$

iii $\dfrac{dy}{dx} = -\dfrac{xy}{10}$ **iv** $\dfrac{dy}{dx} = \dfrac{xy}{10}$

15 a Use the method of partial fractions to express $\dfrac{1}{x^2 - x - 2}$ as the sum of two fractions.

b Consider the differential equation $\dfrac{dy}{dx} = \dfrac{y^2}{x^2 - x - 2}$, $x > 2$ such that $y = 1$ when $x = 5$. Show that the solution is $2e^{\frac{3-3y}{y}} = \dfrac{x+1}{x-2}$.

16 Consider the differential equation $(1 - x^2)\dfrac{dy}{dx} + 2xy = 2x$.

a Find the general solution in the form $y = f(x)$ by the method of separation of variables.

b Write the differential equation in the standard form for a first order linear differential equation, $\dfrac{dy}{dx} + yP(x) = Q(x)$, and find the general solution by means of an integrating factor.

17 Solve each of the following first order linear differential equations.

a $\dfrac{dy}{dx} + \left(\dfrac{2}{x}\right)y = 6x^3$ **b** $\dfrac{dy}{dx} - xy = x$

c $\dfrac{dy}{dx} - \dfrac{y}{x} = x^3$ **d** $\dfrac{dy}{dx} + y \sin x = e^{\cos x}$

e $\dfrac{dy}{dx} - 3x^2 y = e^{x^3}$ **f** $x\dfrac{dy}{dx} = x + y$

18 Solve the first order linear differential equation $\tan x \dfrac{dy}{dx} + y = \sec x$ giving your answer in the form $y = f(x)$.

19 Consider the initial-value problem:

$$\frac{dy}{dx} - \frac{xy}{1 - x^2} = 1,\ y(0) = 1$$

a Show that the differential equation is a first order linear equation by writing it in the form $\dfrac{dy}{dx} + yP(x) = Q(x)$.

b Show that the integrating factor is $\sqrt{1-x^2}$.

c By using the substitution $x = \sin u$, show that

$$\int\sqrt{1-x^2}\,dx = \frac{x\sqrt{1-x^2}}{2} + \frac{\arcsin x}{2} + C.$$

d Find the solution to the initial-value problem expressed in the form $y = f(x)$.

20 **a** Show that $\int \tan x\,dx = -\ln|\cos x|$.

b Show that $\frac{dy}{dx} = 1 + y\tan x$ is a first order linear differential equation.

c Find the general solution of $\frac{dy}{dx} = 1 + y\tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$.

21 Find the particular solution to the differential equation $\frac{dy}{dx} = \frac{x^2\ln x - y}{x}$ given that $y = 1$ when $x = 1$.

22 Find the general solution, in explicit form, to the differential equation

$$x^2\frac{dy}{dx} - x^3 + xy = 0.$$

23 Find the general solution to the first order homogenous differential equation

$$\frac{dy}{dx} = \frac{3y-x}{3x-y}.$$

Write the answer in the form $C = f(x, y)$.

24 Solve each of the following first order homogeneous differential equations.

a $\frac{dy}{dx} = \frac{y}{x+1}$ **b** $\frac{dy}{dx} = \frac{x+2y}{x}$

c $x\frac{dy}{dx} = 2x + 3y$ **d** $\frac{dy}{dx} = -\frac{2x^2+y^2}{2xy+3y^2}$

e $xy\frac{dy}{dx} = x^2 - y^2$ **f** $x(y-x)\frac{dy}{dx} = y(x+y)$

25 Consider the differential equation $\frac{dy}{dx} = \frac{x+2y}{3y-2x}$, for $x > 0$.

a Use the substitution $y = vx$ to show that $v + x\frac{dv}{dx} = \frac{1+2v}{3v-2}$.

b Hence, find the solution of the differential equation, given that $y = 0$ when $x = 1$.

26 Use the substitution $y = vx$ to show that the general solution to the differential equation

$y^2 - x^2 + xy\frac{dy}{dx} = 0$ is $2x^2y^2 - x^4 = C$, where C is a constant.

27 Consider the initial-value problem:

$$x\frac{dy}{dx} = y + \sqrt{x^2 - y^2},\quad y(1) = 1$$

a Use the substitution $y = vx$ to show that $v + x\frac{dv}{dx} = v + \sqrt{1-v^2}$.

b Hence, show that the solution is $\arcsin\left(\frac{y}{x}\right) = \ln|x| + \frac{\pi}{2}$.

28 Consider the differential equation $\frac{dy}{dx} = \frac{y^2+y}{x}$.

a Find the general solution.

b Given that $y = 1$ when $x = 1$, find a particular solution solved explicitly for y.

c Use Euler's method with step size $h = 0.2$ to approximate the solution at $x = 1.2, 1.4, 1.6$ and 1.8.

d Compute the percentage error for each of the approximate solutions found in **c** compared to the solution for the same value of x found using the explicit solution found in **b**.

29 Given that $\frac{dy}{dx} = xy^2$ and $y = 1$ at $x = 0$, use Euler's method with 5 steps to approximate the value of y at $x = 1$.

30 Use Euler's method with step size $h = 0.1$ to approximate the value of y when $x = 1$ for the differential equation $\frac{dy}{dx} = e^{xy}$ given that the solution curve passes through the point $(0, 1)$.

31 Use the substitution $y = vx$ to find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}.$$

32 Given that $\frac{dy}{dx} = x\sqrt{y}$ and $y = 4$ when $x = 1$, use Euler's method with step size $h = 0.1$ to approximate the solution at $x = 1.1, 1.2, 1.3, 1.4$ and 1.5.

33 Consider the initial-value problem:

$$\frac{dy}{dx} = x - y, \quad y(0) = 0$$

a Show that the solution is $y = e^{-x} + x - 1$.

b Use Euler's method with 5 steps to find an approximate value of y when $x = 1$.

c Use Euler's method with 10 steps to find another approximation for $y(1)$.

d Compare the approximate values for $y(1)$ found in **b** and **c** to the actual value using the solution $y = e^{-x} + x - 1$. Comment.

Practice questions 5

1 Find the general solution to the differential equation $\frac{dy}{dx} = e^x\left(1 + y^2\right)$.

2 Show that the general solution to the differential equation $\frac{dy}{dx} = e^{x-y}$ is $y = \ln(Ce^x)$.

3 Find the general solution to the differential equation $\frac{dy}{dx} = -xy$.

4 The rate, in degrees Celsius per minute, at which the temperature of a cup of tea decreases is given by $-k(\alpha - 20)$ where α is the temperature in degrees Celsius and k is a constant. When $t = 0$ minutes $\alpha = 70°$, and when $t = 10$ minutes $\alpha = 50°$.

Find an equation for the temperature in terms of time t.

5 A curve that satisfies the differential equation $\frac{dy}{dx} = xy\sin x$ goes through the point $\left(\frac{\pi}{2}, 1\right)$. Show that the equation of the curve is $y = e^{\sin x - x\cos x - 1}$.

6 Consider the differential equation $x\dfrac{dy}{dx} - 3y = x^4$.

a Find the general solution.

b Given that $y = 2$ when $x = 1$, find the particular solution in explicit form.

7 Given that $y = 2$ when $x = 1$, solve the following differential equation explicitly for y.

$$y\frac{dy}{dx} - 3x = x^4$$

8 Find the general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{y}{x^2}$, $x \neq 0$.

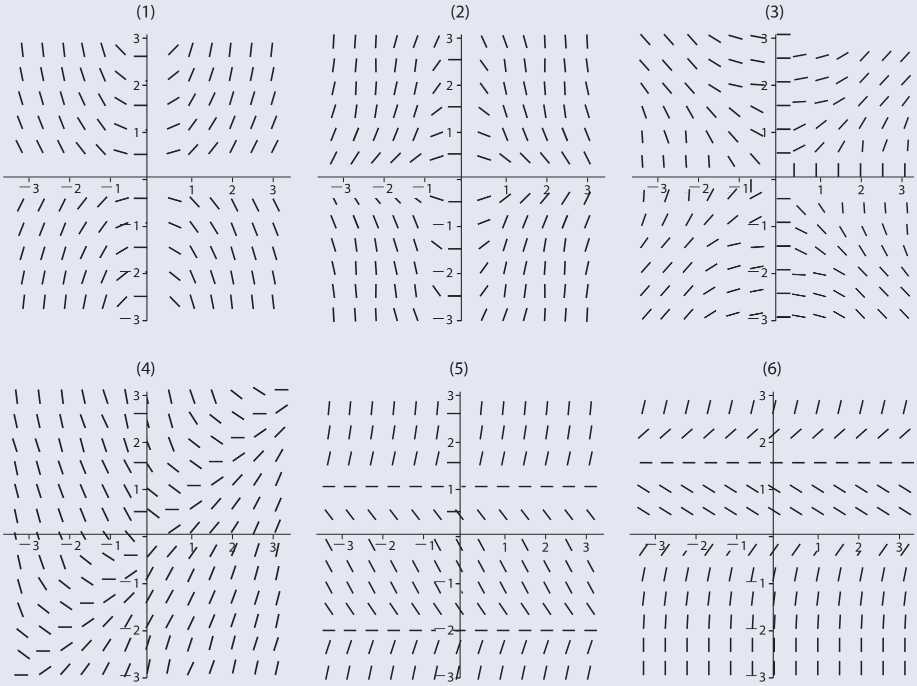
9 Solve $\dfrac{dy}{dx} + \dfrac{1}{x}y = \cos x$, $x \neq 0$, giving your answer in the form $y = f(x)$.

10 Consider the differential equation $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$.

a Find the general solution.

b Given that $y = 20$ when $x = 4$, find an explicit solution for y in terms of x.

11 Match each of the differential equations with its direction field.



a $\dfrac{dy}{dx} = y(y - 1.5)$ **b** $\dfrac{dy}{dx} = xy$ **c** $\dfrac{dy}{dx} = -xy$

d $\dfrac{dy}{dx} = \dfrac{x}{y}$ **e** $\dfrac{dy}{dx} = x - y$ **f** $\dfrac{dy}{dx} = (y-1)(y+2)$

12 Find an equation for the curve that passes through the point $\left(\dfrac{\pi}{6}, 0\right)$ and for which the slope of the curve at any point (x, y) on the curve is $\dfrac{2y+4}{\tan x}$.

13 For all positive values of x the slope of a curve at the point (x, y) is given by $\dfrac{y}{x^2 + x}$. The point $P(3, 6)$ lies on this curve. Find:

a the equation of the normal to the curve at P.

b the equation of the curve where y is expressed in terms of x.

14 Consider the differential equation $x\dfrac{dy}{dx} + 2y = x^2 - x + 1$.

a Show that an integrating factor for solving the differential equation is x^2.

b Given that $y = \dfrac{1}{2}$ when $x = 1$, solve the differential equation. Give the answer in the form $y = f(x)$.

15 Consider the differential equation $\dfrac{dy}{dx} = \dfrac{3y^2 + x^2}{2xy}$, for $x > 0$.

a Use the substitution $y = vx$ to show that $v + x\dfrac{dv}{dx} = \dfrac{3v^2 + 1}{2v}$.

b Hence, find the solution of the differential equation given that $y = 2$ when $x = 1$.

16 Consider the differential equation $x^2\dfrac{dy}{dx} = y^2 + 5xy + 5x^2$ such that $y = -2$ when $x = 1$. Using the substitution $y = vx$, show that the solution to the differential equation is $y = x\tan\left(\ln x + \dfrac{\pi}{4}\right) - x$.

17 Consider the differential equation $\dfrac{dy}{dx} = x^2 + y^2$ where $y = 2$ when $x = 0$.

a Use Euler's method with step length 0.25 to find an approximate value of y when $x = 1$.

b Write down, giving a reason, whether your approximate value for y is greater or less than the actual value of y.

18 Solve the differential equation $(x - y)\dfrac{dy}{dx} + x + y = 0$ given that $y = 0$ when $x = e$. Give the answer in the form $y = f(x)$.

19 Given that $\dfrac{dy}{dx} = \dfrac{y + 2}{xy + 1}$ and $y = 1$ when $x = 0$, use Euler's method with interval $h = 0.5$ to find an approximate value of y when $x = 1$.

20 a Show that the solution for the differential equation $\dfrac{dy}{dx} = \sec^2 x$ is $y = \tan x + c$.

b Consider the differential equation $(\cos x)\dfrac{dy}{dx} + (\sin x)\,y = 2\cos^3 x \sin x - 1$.

Write the differential equation in the form $\dfrac{dy}{dx} + P(x)\,y = Q(x)$, and find the integrating factor.

c Given $0 \leqslant x < \dfrac{\pi}{2}$ and $y = 3\sqrt{2}$ when $x = \dfrac{\pi}{4}$, show that the solution to the differential equation in (b) is $y = -\dfrac{1}{2}\cos x \cos 2x - \sin x + 7\cos x$.

21 Consider the differential equation $xy\dfrac{dy}{dx} = 3x^2 + y^2$ such that $x > 0$ and $y > 0$.

Given that $y = 2$ when $x = 1$, show that the solution to the differential equation is $y = 6x^2 \ln x + 4x^2$.

22 Consider the differential equation $\dfrac{dy}{dx} - 2y = \sin x$ with boundary condition $y = 1$ when $x = 0$.

Use four steps of Euler's method starting at $x = 0$, with interval $h = 0.1$, to find an approximate value for y when $x = 0.4$.

23 **a** Use integration by parts to show that

$$\int \sin x \cos x\, e^{-\sin x}\, dx = -e^{-\sin x}(1 + \sin x) + C.$$

Consider the differential equation $\frac{dy}{dx} - y\cos x = \sin x \cos x$.

b Find an integrating factor.

c Solve the differential equation given that $y = -2$ when $x = 0$. Give your answer in the form $y = f(x)$.

24 **a** Sketch on graph paper the slope field for the differential equation $\frac{dy}{dx} = x - y$ at the points (x, y) where $x \in \{0, 1, 2, 3, 4\}$ and $y \in \{0, 1, 2, 3, 4\}$. Use a scale of 2 cm for 1 unit on both axes.

b On the slope field sketch the curve that passes through the point (0, 3).

c Solve the differential equation to find the equation of this curve. Give your answer in the form $y = f(x)$.

25 Given that $\frac{dy}{dx} - 3e^x = y^2$ and $y = 2$ when $x = 0$, use Euler's method with a step length of 0.2 to find an approximation for the value of y when $x = 1$. Give all intermediate values with maximum possible accuracy.

26 Solve the differential equation $x\frac{dy}{dx} + 2y = \sqrt{1 + x^2}$ given that $y = 1$ when $x = \sqrt{3}$.

27 A curve that passes through the point (1, 2) is defined by the differential equation

$$\frac{dy}{dx} = 2x(1 + x^2 - y).$$

a **i** Use Euler's method to get an approximate value of y when $x = 1.3$, taking steps of 0.1. Show intermediate steps to four decimal places in a table.

ii How can a more accurate answer be obtained using Euler's method?

b Solve the differential equation, giving your answer in the form $y = f(x)$.

Answers

Chapter 1

Exercise 1

1 Converges to 0
2 Converges to 2
3 Converges to 0
4 Diverges
5 Converges to 0
6 Converges to 0
7 Diverges
8 Diverges
9 Converges to $\sqrt{2}$
10 Converges to 1
11 Diverges
12 Converges to 1
13 Converges to 0
14 Converges to 1
15 Converges to 1
16–17 Proof
18 $\frac{1}{2}$
19 2
20 $\frac{1}{2}$
21 Converges to ρ
22 -1
23 $-\frac{1}{3}$
24 $\frac{1}{6}$
25 $\frac{1}{3}$
26 $\ln 2$
27 $\ln\left(\frac{a}{b}\right)$
28 1
29 Divergent
30 $\frac{1}{2}$
31 ρ
32 $\frac{1}{2}$
33 Divergent
34 $\ln 2$
35 2
36 k
37 a) Area increases without bound, i.e. infinite
b) ρ units^3
c) The area of the region is infinite; however, the volume of the solid created by rotating the region about the x-axis is finite.

Practice questions 1

1 Proof
2 $\frac{1}{2}$
3 Diverges
4 a) 6 b) $\frac{1}{3a^2}$
5 $p > 1$
6 a) 0 b) 1
7 Proof
8 a) $\frac{1}{2}$ b) 0
9 a) (ii) $I_n = \frac{1}{2}\ln\left(\frac{1+\alpha^2 n^2}{1+n^2}\right)$
(ii) $\lim_{n\to\infty} I_n = \frac{1}{2}\ln\left(\alpha^2\right)$ or $\ln\alpha$
b) -2
10 Proof
11 Incorrect; $\frac{\sin x}{1-\cos x}$ is not of indeterminate form when $x = \pi$; $\lim_{x\to\pi^-}\left(\frac{\sin x}{1-\cos x}\right) = 0$.
12 a) $k > -1$ b) $k < -1$
13 2
14 Proof
15 a) 1 b) $\frac{1}{6}$

Chapter 2

Exercise 2

1 a) 8 b) -1 c) 25
2 $\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}} + \frac{4}{\sqrt{17}} + \ldots$; diverges by nth term divergence test
3 $3 + \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \ldots$; converges to 4
4 $0 + \ln\frac{1}{2} + \ln\frac{1}{3} + \ln\frac{1}{4} + \ldots$; diverges by nth term divergence test
5 $\frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \frac{3}{16} + \ldots$; converges to 1
6 $\frac{1}{3} + \frac{2}{9} + \frac{2}{9} + \frac{8}{27} + \ldots$; diverges by nth term divergence test
7 $-1+1-1+1-\ldots$; diverges by nth term divergence test
8 $\frac{5}{11} + \frac{7}{16} + \frac{3}{7} + \frac{11}{26} + \ldots$; diverges by nth term divergence test
9 $\frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} + \ldots$; converges to $\frac{1}{e-1}$
10 a) $\int xe^{-x}\,dx = -e^{-x}(x+1) + C$
b) $\int_1^\infty xe^{-x}\,dx = \frac{2}{e}$ and therefore the series is convergent.
11 a) Divergent b) Convergent
12–13 Proof
14 For $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, $\lim_{n\to\infty} a_n = 0$ but it is a p-series with $p = \frac{1}{2} \leqslant 1$ so the series diverges.
15 Proof
16 Converges
17 Diverges
18 Converges
19 Converges
20 Converges
21 Diverges
22 Diverges
23 Diverges
24 Diverges
25 Diverges
26 Converges

27 Diverges **28** Converges

29 Converges **30** Diverges

31 5

32 a) $S_4 = \dfrac{10\,016}{11\,025} \approx 0.908\,48$; error $< \dfrac{1}{81}$

b) $S_4 = 0.095\,308\overline{3}$; error $< 0.000\,006$

33 a) $(n+1)^2+1$

b) $\displaystyle\int_1^\infty \frac{1}{(x+1)^2+1}\,dx = \lim_{b\to\infty}\left[\arctan(x+1)\right]_1^b = \frac{\pi}{2} - \arctan(2)$

$= \arctan\left(\dfrac{1}{2}\right)$; since $\displaystyle\int_1^\infty \frac{1}{(x+1)^2+1}\,dx$ converges to

$\arctan\left(\dfrac{1}{2}\right)$, then $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+2n+2}$ must also converge.

34 Diverges

35 11 terms

36 $\displaystyle\sum_{n=1}^\infty \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ is conditionally convergent.

37 Converges absolutely **38** Converges conditionally

39 Diverges **40** Converges conditionally

41 Converges absolutely **42** Converges absolutely

43 $1 + \dfrac{1}{3} - \dfrac{1}{2} + \dfrac{1}{5} - \dfrac{1}{4} + \dfrac{1}{7} + \dfrac{1}{9} - \dfrac{1}{8} + \ldots$; the sum of this series is 1. The terms of the alternating harmonic series are rearranged such that consecutive positive terms are added until the sum is greater than 1, then consecutive negative terms are added until the sum is less than 1, and so on. Note that the difference between the partial sums and 1 is less than the last term used, so the series converges to 1.

44 7 terms **45** Proof

Practice questions 2

1 a) Converges; geometric series with $r = \dfrac{1}{1.1}$, so $|r| < 1$.

b) Diverges by nth term divergence test.

c) Converges; comparison test, compare to p-series with $p = 3$.

2 a) Converges

b) Diverges

3 Proof

4 a) Series converges by the ratio test.

b) Series converges by the integral test.

c) Series converges by the alternating series test.

5 Diverges by comparison with the harmonic series.

6 a) Integral test for $\sum a_n$: Let $a_n = f(n)$ where $f(x)$ is a continuous, positive and decreasing function for all $x \geqslant N$, where N is some positive integer. Then the series $\displaystyle\sum_{n=N}^\infty a_n$ and the integral $\displaystyle\int_N^\infty f(x)\,dx$ both diverge or both converge. That is, if the integral is finite then $\sum a_n$ is finite, and if the integral is infinite then $\sum a_n$ is infinite.

b) Diverges by the integral test.

7 Ratio test gives interval of convergence as $-1 \leqslant x < 1$.

8 Converges conditionally.

9 Proof

10 a) Proof

b) (i) $\dfrac{1}{n(n+2)} = \dfrac{1}{2n} - \dfrac{1}{2(n+2)}$

(ii) $\displaystyle\sum_{n=1}^\infty \frac{1}{n(n+2)} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

11 $-1 \leqslant x < 1$

12 a) Diverges

b) Converges

c) Converges

13 Proof

14 Proof

15 a) Proof

b) Converges conditionally.

Chapter 3

Exercise 3

1 $R = 1;\ -1 \leqslant x < 1$ **2** $R = 1;\ 1 < x < 3$

3 $R = 2;\ 2 \leqslant x < 4$ **4** $R = \infty;\ x \in \mathbb{R}$

5 $R = 1;\ -1 \leqslant x \leqslant 1$ **6** $R = 1;\ 1 \leqslant x \leqslant 3$

7 $R = 1;\ 0 < x < 2$ **8** $R = 1;\ -1 \leqslant x < 1$

9 $R = 0;\ x = 0$ **10** $R = \dfrac{4}{3};\ -\dfrac{4}{3} \leqslant x < \dfrac{4}{3}$

11 $R = 4;\ -4 < x < 4$ **12** $R = 3;\ -3 \leqslant x \leqslant 3$

13 $R = e;\ -e < x < e$ **14** $R = 0;\ x = 4$

15 $-\dfrac{1}{k} < x < \dfrac{1}{k}$

16 $\displaystyle\sum_{n=0}^\infty (-1)^n x^n; -1 < x < 1$

17 a) $e^{-x^2} = 1 - x^2 + \dfrac{x^4}{2!} - \dfrac{x^6}{3!} + \ldots + (-1)^n \dfrac{x^{2n}}{n!} + \ldots$; $R = \infty$

b) $\displaystyle\int e^{-x^2}dx = \int\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)\,n!} + \ldots\right)$

$= x - \dfrac{x^3}{3\cdot 1!} + \dfrac{x^5}{5\cdot 2!} - \dfrac{x^7}{7\cdot 3!} + \ldots + (-1)^n \dfrac{x^{2n+1}}{(2n+1)n!} + \ldots$;

radius of convergence is also $R = \infty$.

c) $\displaystyle\int_0^1 e^{-x^2}\,dx \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} = \frac{5651}{7560} \approx 0.747$;

error $< a_6 = \dfrac{1}{11\cdot 5!} = 0.000\,\overline{75} < 0.001$

18 a) $x^2 - \dfrac{x^4}{3!} + \dfrac{x^6}{5!}$ b) $x + \dfrac{1}{3}x^3 + \dfrac{2}{15}x^5$

c) $x - \dfrac{1}{2}x^2 + \dfrac{7}{6}x^3$

19 $\displaystyle\sum_{n=0}^\infty nx^{n-1}$ for $-1 < x < 1$

20 a) $\displaystyle\sum_{n=0}^\infty (-1)^n \frac{x^{n+2}}{n!}$ b) Proof

21 a) $\sin x \approx x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!}$

b) $\sin\left(\frac{\pi}{12}\right) \approx 0.258\,819$

c) Error $< 1.4165 \times 10^{-10}$

22 $-\frac{1}{2} < x < \frac{1}{2}$

23 $(x-1)e + (x-1)^2 e + \frac{(x-1)^3}{2} e + \frac{(x-1)^4}{6} e$

24 $\sum_{n=1}^{\infty} \frac{2}{(2n-1)} x^{2n-1} = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \ldots$

25 a) $\sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \ldots$

b) Proof c) Proof d) $\pi \approx 2.976$; error $< 0.142\,86$

26 a) $f^{(n)}(x) = \frac{e^x + (-1)^n e^{-x}}{2}$

b) $f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \ldots$

c) $f\left(\frac{1}{2}\right) \approx \frac{433}{384} = 1.127\,604\,1\bar{6}$

d) Error $< 0.000\,136$

27 $-1.59 < x < 1.59$

28 $xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \ldots$

29 $\sec^2 x = 1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \ldots$

30 a) $\sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n$

b) $\sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!}$

c) $-\frac{1}{2} \sum_{n=0}^{\infty} (n+1) n x^{n-1}$

d) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+4}$

31 a) $\lim_{x \to 0} \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \ldots}{\frac{x^2}{2!} + \frac{x^3}{3!} + \ldots} = 1$ b) $\lim_{x \to 0} \frac{x - \arctan x}{x^3} = \frac{1}{3}$

Practice questions 3

1 $\ln(\cos x) \approx -\frac{x^2}{2} - \frac{x^4}{12}$

2 a) $\sin^2 x \approx x^2 - \frac{x^4}{3}$ b) $\cos^2 x \approx 1 - x^2 + \frac{x^4}{3}$

3 $e^x \sin x \approx x + x^2 + \frac{x^3}{3}$

4 $e^{3x} \approx 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2}$

5 $\sec x \approx 1 + \frac{x^2}{2} + \frac{5x^4}{24}$

6 a) $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

b) $e^x \approx 1 + x^2 + \frac{x^4}{2!}$

c) $e^x \approx 1 + x + \frac{3x^2}{2} + \frac{7x^3}{6} + \frac{25x^4}{24}$

7 $\ln(2+3x) = \ln 2 + \frac{3}{2}x - \left(\frac{3}{2}\right)^2 \frac{x^2}{2} + \left(\frac{3}{2}\right)^3 \frac{x^3}{3} - \left(\frac{3}{2}\right)^4 \frac{x^4}{4} + \ldots;$

$R_n(x) = \frac{(-1)^n 3^{n+1}}{(n+1)(2+3c)^{n+1}} x^{n+1}$

8 a) $\sqrt{4+x} \approx 2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} - \frac{5x^4}{16\,384}$

b) $R_4(x) = \frac{7}{256(4+x)^{9/2}} x^5$; since $2^9 < (4+0.1)^{9/2}$ then

$0 \leqslant R_4(x) \leqslant \frac{7}{256 \cdot 2^9} (0.1)^5 < 5.34 \times 10^{-10}$

9 2 terms needed; 0.996 195

10 a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots$

b) $\int_0^1 e^{-x^2} dx \approx \frac{23}{30}$

c) Error $< \frac{e}{42}$

11 a) $\frac{1}{1+x^2} = \sum_{n=1}^{\infty} (-1)^{n+1} x^{2n-2}$

b) Proof c) Proof d) $\frac{\pi}{4}$

12 a) $\frac{1}{1+x} \approx 1 - x + x^2 - x^3 + \ldots$ and

$\frac{1}{1-x} \approx 1 + x + x^2 + x^3 + \ldots$

b) $\frac{-3}{x-2} + \frac{4}{x-3}$

c) $\frac{x+1}{x^2-5x+6} \approx \frac{1}{6} + \frac{11x}{36} + \frac{49x^2}{216} + \frac{179x^3}{1296} + \ldots$

13 a) $\frac{1}{1-x}$

b) $\sum_{n=1}^{\infty} \left[-(x+1)^{n-1}\right] = -1 - (x+1) - (x+1)^2 - (x+1)^3 - \ldots,$

$-2 < x < 0$

14 a) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$ b) 0.3103

15 a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ b) Proof

16 Ratio test gives interval of convergence as $-1 \leqslant x < 1$.

17 a) (i) Proof

(ii) $a_n = \frac{1^2 \times 3^2 \times \ldots (n-2)^2}{n!}$, for odd $n \geqslant 3$

b) $R = 1$

c) $\pi \approx 3.139$

18 $-1 \leqslant x < 1$

19 a) (i) Domain $[-1, 1]$, range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\arcsin x = x + \frac{x^3}{6} + \ldots$

b) $\cos(\arcsin x) = 1 - \frac{x^2}{2} - \frac{x^4}{8}$

c) (i) $p^r \left(1 + \frac{q}{p} x^2\right)^r = p^r \left(1 + r\frac{q}{p} x^2 + \frac{r(r-1)}{2} \frac{q^2}{p^2} x^4\right)$

(ii) $p = 1, q = -1,\ r = \frac{1}{2}$; hence, the series in b) and c) is $(1 - x^2)^{1/2}$ since $\cos(\arcsin x) = \cos\left(\arccos\sqrt{1-x^2}\right) = (1-x^2)^{1/2}$.

20 a) $\frac{1}{2\pi}$ b) -2

21 a) Proof

b) $\ln(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \dots$

c) $\ln(1 - \sin x) = -x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} - \dots$

d) Proof

e) 0

22 a) $\sin(\pi x) \approx 1 - \frac{\pi^2 (x - \frac{1}{2})^2}{2!} + \frac{\pi^4 (x - \frac{1}{2})^4}{4!} - \dots$

b) 0.924

Chapter 4

Exercise 4

1 D

2 $b = 2 - 2a = 2(1 - a)$

3 $x < -a, x > -a$

4 a) Continuous at $x = 1$.

b) Not differentiable at $x = 1$.

5 a) Continuous, not differentiable.

b) Neither

c) Neither

d) Continuous and differentiable.

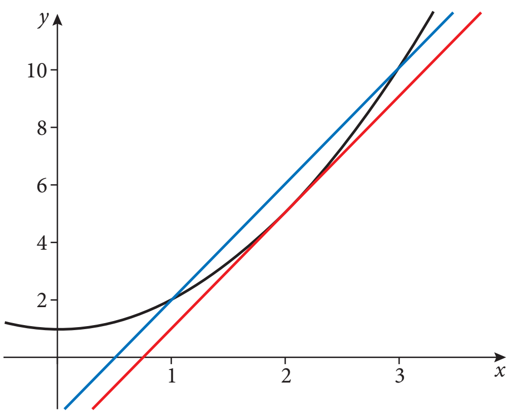
6 $a = \frac{5}{7}, b = -\frac{30}{7}$

7 a) $a + b = 3$

b) $a = \frac{3}{2}, b = \frac{3}{2}$

8 $x_0 = 1 + \sqrt{3}$

9 $c = 2$ (see graph)



10 $x_0 \approx 0.690$

11 $y = x^{2/3}$ is not differentiable at $x = 0$.

12 Along the 13 km portion of the highway the car's average speed was $\frac{13\text{ km}}{12\text{ min}} = \frac{13\text{ km}}{\frac{1}{5}\text{ hr}} = 65\,\frac{\text{km}}{\text{hr}}$. According to Mean Value Theorem, there was at least one instant in the 13 km portion when the car was travelling at $65\,\frac{\text{km}}{\text{hr}}$. This confirms that the car did break the speed limit.

13 Proof

14 $\frac{2}{3}$

15 Proof

16 $c = \frac{7}{3}$

17 $c \approx 0.670$

18 1.25

19 Lower sum $= \frac{31}{4}$; upper sum $= \frac{39}{4}$.

20 $\int_0^4 \sqrt{x+6}\,dx$

21 $\int_3^5 \frac{e^x}{x-2}\,dx$

22 $\int_0^{\pi} (3 - \sin x)\,dx$

23 a) (i) 316

(ii) 320

(iii) Underestimate; 1.25% error.

b) (i) $\frac{2\pi}{3} \approx 2.09$

(ii) 2

(iii) Overestimate; approx. 4.72% error.

c) (i) 5.38

(ii) $\frac{15}{4\ln 2}$

(iii) Underestimate; approx. 0.499% error.

Chapter 5

Exercise 5

1 $y = \frac{e^{\sqrt{1+x^2}}}{e}$

2 $y = \arctan(1 - \cos x)$

3 $y = \frac{3x^3}{x^3 + 4}$

4 Proof

5 (i) c (ii) a (iii) d (iv) b

6 24.7 grams

7 a) $2x^2 - y^2 = C$

b) $y = \frac{x}{1 - Cx}$

c) $\ln(y-1) - \ln y + C_1 = -\frac{1}{x}$ or $\frac{y}{y-1} = C_2 e^{1/x}$

d) $x = C_1 \sin y$ or $y = \arcsin(C_2 x)$

e) $y = Ce^{x^2/2}$

f) $y^2 = 2\sqrt{x^2+1} + C$

g) $\ln\sqrt{\frac{y-1}{y+1}} = e^x + C$

h) $x = y\ln y - y + C$

8 $\int \frac{y+1}{y}\,dy = \int \frac{x+1}{x}\,dx \Rightarrow \int\left(1 + \frac{1}{y}\right)dy = \int\left(1 + \frac{1}{x}\right)dx$

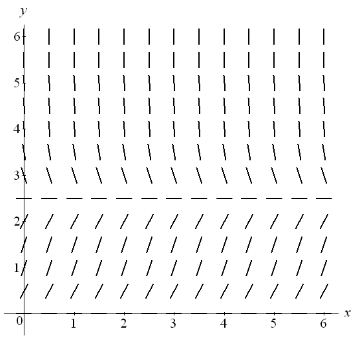
$\Rightarrow y + \ln|y| = x + \ln|x| + C$

$e^{y+\ln y} = e^{x+\ln x+C} \quad\Rightarrow\quad e^{\ln y}e^{y} = e^{\ln x}e^{x}e^{C} \quad\Rightarrow\quad ye^{y} = Axe^{x}$

9 $y = \pm\sqrt{2\sin x + C}$

The constant C cannot be completely arbitrary because $2\sin x + C \geqslant 0$. If $C < -1$, then $2\sin x + C$ will always be negative, regardless of the value of x. If $C > 1$, then $2\sin x + C$ will always be positive. If $-1 \leqslant C \leqslant 1$, then whether $2\sin x + C$ is positive or negative will depend on the value of x.

10 a)



b) $\frac{5}{2}$ c) $\frac{5}{2}$

d) Regardless of the initial value of the population, as time increases, the population stabilizes at 2 500.

11 $y = -\sqrt{x^2 + \tan x + 25}$

12 a) Proof

b) $y = \dfrac{x+1}{x-1}$

13 $y = \dfrac{7x+1}{7-x}$

14 (i) b (ii) d (iii) c (iv) a

15 a) $\dfrac{1}{3(x-2)} - \dfrac{1}{3(x+1)}$

b) proof

16 a) $y = C\left(x^2 - 1\right) + 1$

b) $\dfrac{dy}{dx} + \left(\dfrac{2x}{1-x^2}\right)y = \dfrac{2x}{1-x^2}$; integrating factor is $\left|\dfrac{1}{1-x^2}\right|$; leads to same solution as in part a)

17 a) $y = x^4 + \dfrac{C}{x^2}$ b) $y = Ce^{x^2/2} - 1$

c) $y = \dfrac{1}{3}x^4 + Cx$ d) $y = xe^{\cos x} + Ce^{\cos x}$

e) $y = xe^{x^3} + Ce^{x^3}$ f) $y = x\ln|x| + Cx$

18 $y = x\csc x + C\csc x$

19 a)–c) Proof

d) $y = \dfrac{x}{2} + \dfrac{\arcsin x}{2\sqrt{1-x^2}} + \dfrac{1}{\sqrt{1-x^2}}$

20 a)–b) Proof

c) $y = \tan x + C\sec x$

21 $y = \dfrac{1}{3}x^2 + \dfrac{C}{x}$

22 $y = \dfrac{1}{3}x^2\ln x - \dfrac{1}{9}x^2 + \dfrac{10}{9x}$

23 $C = \dfrac{y-x}{(y+x)^2}$

24 a) $y = Cx + C$ b) $y = Cx^2 - x$

c) $y = Cx^3 - x$ d) $2x^3 + 3xy^2 + 3y^3 = C$

e) $y^2 = \dfrac{x^2}{2} - \dfrac{C}{x^2}$ f) $y = x\ln(Cxy)$

25 a) Proof b) $x^2 + 4xy - 3y^2 - 1 = 0$

26 Proof

27 Proof

28 a) $\left|\dfrac{y}{y+1}\right| = C|x|$ b) $\left|\dfrac{y}{y+1}\right| = \dfrac{1}{2}|x|$

c)

x_n	y_n
1.2	1.400
1.4	1.960
1.6	2.789
1.8	4.110

d)

x_n	approx. y_n	exact y_n	% error
1.2	1.400	1.5	$6.\overline{6}$
1.4	1.960	$2.\overline{3}$	16
1.6	2.789	4	30.3
1.8	4.110	9	54.3

29 $y \approx 1.5405$ at $x = 1$

30 $y \approx 5.9584$ at $x = 1$

31 $y^2 = Cx^3 - x^2$

32

x_n	y_n
1.1	4.2
1.2	4.425 43
1.3	4.677 87
1.4	4.959 04
1.5	5.270 81

33 a) Proof

b) $y(1) \approx 0.327\,68$

c) $y(1) \approx 0.348\,678\,4401$

d) Actual value to 10 s.f. is $y(1) \approx 0.367\,879\,4412$; using more steps (and a smaller step size) gives a better approximation.

Practice questions 5

1 $y = \arctan\left(e^x + C\right)$

2 Proof

3 $y = Ce^{-\frac{1}{2}x^2}$

4 $\alpha = 20 + 50e^{-\frac{t}{10}\ln\frac{5}{3}}$

5 Proof

6 a) $y = (x+c)x^3$ b) $y = (x+1)x^3$

7 $y = \sqrt{\frac{2x^5}{5} + \frac{6x^2}{5} + \frac{3}{5}}$

8 $y = Ce - \frac{1}{x}$

9 $y = \frac{C}{x} + \frac{\sin x}{x} - \cos x$

10 a) $y = \frac{C}{x} + \frac{x^3}{4}$ b) $y = \frac{16}{x} + \frac{x^3}{4}$

11 a) 6 b) 1 c) 2 d) 3 e) 4 f) 5

12 $y = 8\sin^2 x - 2$

13 a) $y = -2x + 12$ b) $y = \frac{8x}{x+1}$

14 a) Proof b) $y = \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{2}x + \frac{1}{12x^2}$

15 a) Proof b) $5x = \frac{y^2}{x^2} + 1 \left(\text{or } y = x\sqrt{5x-1}\right)$

16 Proof

17 a) $y \approx 5.32$

b) Less than actual value; $\frac{dy}{dx} > 0$ so solution curve is curving upward; short segments from Euler's method to approximate solution curve will be below the actual solution curve.

18 $y = x - \sqrt{2x^2 - e^2}$

19 $y \approx 3.5$

20 a) Proof

b) $\frac{dy}{dx} + (\tan x)\, y = 2\cos^2 x \sin x - \sec x$; integrating factor is sec x.

c) Proof

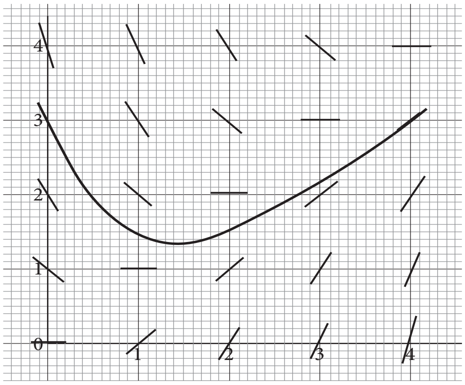
21 Proof

22 $y \approx 2.14$

23 a) Proof b) $e^{-\sin x}$

c) $y = -\sin x - 1 - e^{\sin x}$

24 a)–b)



c) $y = x - 1 + 4e^{-x}$

25

n	x_n	y_n
0	0	2
1	0.2	3.4
2	0.4	6.444841655
3	0.6	15.64713326
4	0.8	65.70696043
5	1	930.5232147

26 $yx^2 = \frac{1}{3}\left(1 + x^2\right)^{\frac{3}{2}} + \frac{1}{3}$

27 a) (i) $y(1.3) \approx 2.14$ (ii) Decrease the step size

b) $y = x^2 + e^{1-x^2}$