Solution : Paper 3

1 (a) By Euclidean Algorithm,

$581 = 259 \times 2 + 63$

$259 = 63 \times 4 + 7$

$63 = 7 \times 9 + 0$, so $\gcd(581, 259) = \gcd(63, 7) = \boxed{7}$

(b) $259x + 581y = 7 = 259 - 63 \times 4$

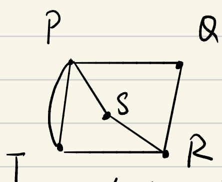
$= 259 + 259 \times 8 - 581 \times 4$

$= 259 \times 9 + 581(-4)$

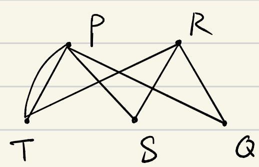
Thus we find the particular solution $\begin{cases} x = 9 \\ y = -4 \end{cases}$

$\Rightarrow \begin{cases} x = 9 + 83t \\ y = -4 - 37t, \ t \in \mathbb{Z} \end{cases}$

2. (a)



(planar)



(Bipartite)

(b) G is not simple, because there are multiple edges connecting P and T.
G is connected because there is at least one path connecting any two vertices in G.
G is bipartite because it can be divided into {P, R} and {T, S, Q}
such that any vertice is adjacent only to vertices in the other set.

(c) Eulerian trail: TPQRTPSR ; it exists because there are exactly 2 vertices of odd degree in G.

(d) Since not all vertices are of even degree, there can't be an Eulerian circuit.

(e) Since $3v - e \geq 6$ for a planar graph.
$v = 5$, $e \leq 9$ $\Rightarrow$ add 3 edges at most.

3 (a) $2^8 \equiv (2^3)^2 \cdot 2^2 \equiv (-1)^2 \cdot 2^2 \equiv 4 \pmod 9$
it doesn't satisfy because a is not relatively prime to $p-1$.

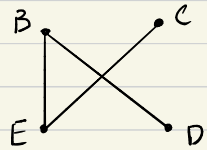
(ii) $2^{45} \equiv (2^3)^{15} \equiv (-1)^{15} \equiv 8 \pmod 9 \Rightarrow k = 8$.

(b) $3x \equiv 4 \pmod 5 \Rightarrow x \equiv 3 \pmod 5 \Rightarrow x \in \{\ldots 103, 108, \cdots, 198, \cdots\}$
$5x \equiv 6 \pmod 7 \Rightarrow x \equiv 4 \pmod 7 \Rightarrow x \in \{\ldots, 102, 109, \cdots, 193, 200, \cdots\}$
$\Rightarrow x = 123, 158, 193$.

4.(a) By NNA, we obtain AECDBA, with weight = $\underline{59}$

(b)

edge	weight
CE	10
BD	13
BE	14



$10+13+14=37$. Therefore, by DVA, the lower bound is $37+11+9=\underline{57}$

5(a) The characteristic equation: $x^2-5x+6=0 \Rightarrow x_1=2,\ x_2=3$

$\Rightarrow u_n = a(2)^n + b(3)^n$. $u_1=u_2=3$

$\Rightarrow \begin{cases} 2a+3b=3 \\ 4a+9b=3 \end{cases} \Rightarrow \begin{cases} a=3 \\ b=-1 \end{cases} \Rightarrow u_n = 3(2)^n-(3)^n$

(b) Base case: when $n=1$, $V_1 = 2^1(2-1)=2$

when $n=2$, $V_2 = 2^2(2\times 2-1)=12$.

Induction case: assume that the statement is true for all $n\le k$. We want to show the truth of the statement for $n=k+1$. i.e. $V_{n+1} = 2^{n+1}(2(n+1)-1)$

$$\begin{aligned} LHS &= 4V_n - 4V_{n-1} \\ &= 4\cdot 2^n(2n-1) - 4\cdot 2^{n-1}(2n-3) \\ &= 2^{n+3}n - 2^{n+2} - 2^{n+2}n + 3\cdot 2^{n+1} \\ &= 2^{n+3}n - 2^{n+2}(n+1) + 3\cdot 2^{n+1} \\ &= 2^{n+1}(4n-2n-2+3) \\ &= 2^{n+1}(2n+1) = RHS. \end{aligned}$$

Therefore, since the truth of $n\le k$ implies that for $n=k+1$, by Strong Mathematical Induction, $V_n = 2^n(2n-1)\quad \forall n\in \mathbb{N}^+$ $\square$