the 2019 Exam Assessing Readiness (👂)

Wait, who are you?

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Directions:

- Try to complete as much of the 👂 as you can in one time block, spending no more than 4 hours on it. It's okay if you don't finish.
- You are not allowed to use outside sources (including other humans and the internet) in your work on the 👂. Cheating is not cool. Seriously. There's an honor pledge at the end of the 👂.
- Please show your work right on the 👂; sentences are a bonus. Actually, sentences are a really useful way of communicating your understanding, so *hint:* use them.
- When you're done, take your 👂, and any additional pages you want to include, and either
  - scan this to one PDF file and send the result to `director@mathily.org`, or
  - mail it all to... MathILy c/o dr. sarah-marie belcastro
231 W. Franklin St.
Holyoke, MA 01040-3150.
- Questions? Contact dr. sarah-marie belcastro at `director@mathily.org`.

☆☺☆☆☆☆☺☆☺☆☆☆☺☆☆☆☆☆☆☺☆☺☆☺☆☺

Gnirts are strings of ☆s and ☺s, with but a single rule: No two ☺s can be next to each other.

The 2-gnirts are ☆☆, ☆☺, ☺☆. There are three 2-gnirts.

The 3-gnirts are ☆☆☆, ☆☆☺, ☆☺☆, ☺☆☆, ☺☆☺. There are 5 3-gnirts.

The 4-gnirts are ☆☆☆☆, ☆☆☆☺, . . .

. . . , ☺☆☆☺, ☺☆☺☆. There are 8 4-gnirts.

How many 1-gnirts are there? 2.

How many 5-gnirts are there? How about 6-gnirts?

There are 13 5-gnirts and 21 6-gnirts.

Explain how to produce a list of 4-gnirts using lists of shorter gnirts.

4-gnirts can be produced by using 2-gnirts and 3-gnirts:

2-gnirts: ☆☆ ☆☺ ☺☆

3-gnirts: ☆☆☆ ☆☆☺ ☆☺☆ ☺☆☆ ☺☆☺

Connect only when the end of 3-gnirts is the same as the beginning of 2-gnirts. Thus, there are 8 lines in total, and thus 8 4-gnirts.

How many k-gnirts are there? Explain.

For a list of of k-gnirts, there are $a_k = a_{k-1} + a_{k-2}$ gnirts.

By solving the characteristic equation $x^2 - x - 1 = 0$

we get $x_1 = \frac{1+\sqrt5}{2}$, $x_2 = \frac{1-\sqrt5}{2}$.

Thus, since $x_1 \neq x_2$,

plugging them in the formula of general term $a_n = Ax_1^{n-1} + Bx_2^{n-1}$, we get
$$\begin{cases} a_1 = A + B = 2. \\ a_2 = \frac{1+\sqrt5}{2}A + \frac{1-\sqrt5}{2}B = 3. \end{cases}$$

solving the equation set, we get
$$\begin{cases} A = \frac{5+2\sqrt5}{5} \\ B = \frac{5-2\sqrt5}{5}. \end{cases}$$

Thus,
$$a_k = \left(\frac{5+2\sqrt5}{5}\right)\left(\frac{\sqrt5+1}{2}\right)^{k-1} + \left(\frac{\sqrt5-2\sqrt5}{5}\right)\left(\frac{-\sqrt5+1}{2}\right)^{k-1}$$

Rearranging Gnirt-lists

We can write the 2-gnirts as a list
where each 2-gnirt differs from the next by only one symbol: ☆☺ ☆☆ ☺☆

(It's silly, but we can write the 1-gnirts this way as well: ☆ ☺)

Do the same for the the 3-gnirts: ☆O☆ ☆☆☆ ☆☆O O☆O ☺☆☆

How does the 2-gnirt list come into play here?

Since there are three 2-gnirts: ☆☆. ☆O and O☆. however, ☆O and O☆ ~~are~~ differ in two. so they can not be put next to each other, So they must be ~~and~~ at two ends of 2-gnirts, with ☆☆ in the middle. Finally, check to see ☆O, ☆☆, O☆ is true.

Try it for the 4-gnirts:

☆O☆☆ ☆O☆O ☆☆☆☺ ☆☆☆☆ ☆☆O☆ ☺☆☺☆ O☆☆☆ ☺☆☆☺

How is this list related to previous gnirt lists?

Based on the ~~3-gnirts~~ 3-gnirts, the 4-gnirts are produced by adding a ☆ to each of the 3-gnirts; and adding a ☺ if the symbol in 3-gnirts end with ☆.

Explain how to construct this list in general (that is, for k-gnirts).

In order ~~too~~ to construct ~~f~~ the list for k-gnirts, we need to know the arrangement for (k−1)-gnirts, and since (k−1)-gnirst must already followed the rule. by adding one ☺ or ☆ at the end of each in (k−1)-gnirst, there must be a way that k-gnirts work also.

Why does it always work?

It will always work because every k-gnirts produced is based on (k−1)-gnirts. and using strong mathematical induction, we already know the truth of $k=1$, $k=2$, $k=3$ and $k=4$. by assuming the truth of all k-gnirts when $k \in [1, n]$, we know the truth of $k=n+1$. i.e. (k+1)-gnirts will also work, because it will only add one element based on k-gnirts and k-gnirts work.

Thus, we can know the rule works for all $k \in \mathbb{Z}$. and it will always work.

gnirt problems inspired by work of S. Klavžar, Slovenia

BRRR!

In honor of the recent very cold temperatures across much of the northern United States, we bring you boring recurrence relation rearrangements. Please fill in the blanks; we apologize for the temporary tedium.

$T_n = T_{n-2} + T_{n-4} + \boxed{T_{n-6}} + \cdots + T_2 + T_0.$

$T_{n-2} = \boxed{T_{n-4}} + T_{n-6} + \boxed{T_{n-8}} + \cdots + \boxed{T_2} + \boxed{T_0}.$

Therefore $T_n = \boxed{2}T_{\boxed{n-2}}.$

$Z_n = Z_{n-1} + 3Z_{n-3} + 3\boxed{Z_{n-5}} + \cdots + 3Z_1.$

$Z_{n-2} = \boxed{Z_{n-3}} + \boxed{3}\boxed{Z_{n-5}} + \boxed{3}\boxed{Z_{n-7}} + \cdots + \boxed{3}\boxed{Z_1}.$

Therefore $Z_n = \boxed{Z_{n-1}} + \boxed{2}\boxed{Z_{n-3}} + \boxed{Z_{n-2}}.$

Zero Magic

Fill in the cells of this square with $a, -a, b, -b, c, -c, d, -d$ so that

- All letters represent positive integers,
- No two of those integers are equal,
- Every row sums to 0, and
- Every column sums to 0.

c	a	-b
-d	0	d
b	-a	-c

What are the possible sums for the main diagonals? Explain.

0. Since every row and column sums up to 0, and there is a 0 in the center already, the numbers in the ~~row~~ middle row must be opposite, so is it for the middle column. Now in order to fill the four corners, we cannot put the opposite numbers in the same row/column. (otherwise the sum becomes the value in the middle and all numbers can't be 0). Thus, the opposite numbers must be on the same diagonal, which make the sum of main diagonals to be 0.

How many ways are there to choose a, b, c, d to fill in the square?

4! = 24 ways. because a,b,c,d are equilibrium. and can exchange with each other

There is an interesting relationship between two of a, b, c, d. Which two, and what is the relationship?

From the square above, we know that ~~c+d~~ c+b=d.
~~c+d=b and d-c=a~~. b-c=a

Fill in the cells of this square with integers that obey the rules given above.

3	-1	-2
-5	0	5
2	1	-3

Can you fill in the cells of this square with really different* values than the square on the left? Do so, or explain why it's not possible.

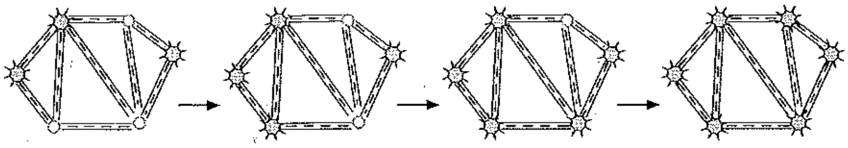
10	-4	-6
-16	0	16
6	4	-10

* "really different" could mean lots of things. We mean *all* of them.

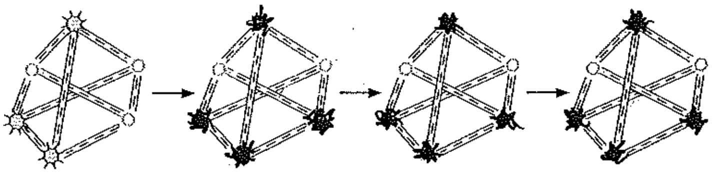
inspired by work of Abdollah Khodkar.

Contagion

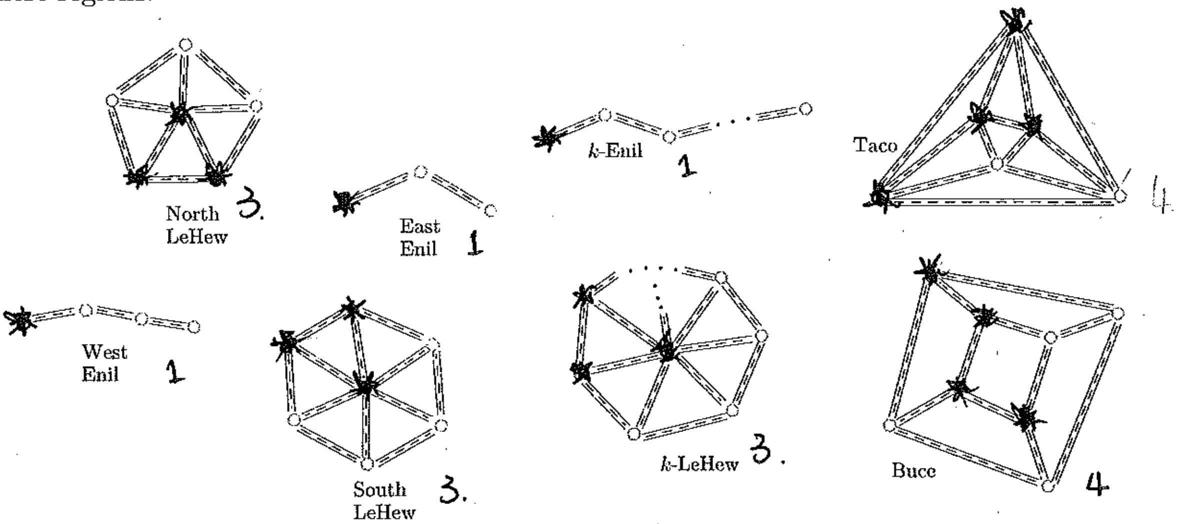
Pricklyrasp plague† spreads in an interesting way: When all cities connected by roads to a plague-ridden city—except one—are also plague-ridden, that last city will become infected with Pricklyrasp plague as well. For example, in the region of Chartra, here's how the plague spread:



Show what will happen in the region of Spekh:



Economical bioterrorists want to choose the smallest number of cities to infect while still eventually overwhelming the region with Pricklyrasp plague. Please find that smallest number for each of these regions.



Economical bioterrorists *must* infect at least as many cities in a region R as the smallest number of cities that can infect the whole region., called $L(R)$.

In any region, the greatest number of cities that economical bioterrorists *might* need to infect is the total number of cities minus one, called $G(R)$.

Give an example of a region where economical bioterrorists must infect more than $L(R)$ cities, but less than $G(R)$ cities.

So the the economical bioterrorists must infect 4 cities, which is less than $G(R) = 5$

Give an example of a region where $L(R) = G(R)$.

When the region only have one city

O

OR, it is a triangle

, so that $L(R) = G(R) = 2$.

† Pricklyrasp plague is characterized by a prickly feeling in the elbows accompanied by a raspy voice.

inspired by a 2008 AIM Working Group project.

More Gnirtiness

Please fill in this table:

number	gnirt	sum
1	☺	1
2	☺☆	2
3	☺☆☆	3
4	☺☆☺	$3+1$
5	☺☆☆☆	5
6	☺☆☆☺	$5+1$
7	☺☆☺☆	$5+2$
8	☺☆☆☆☆	8
9	☺☆☆☆☺	$8+1$
10	☺☆☆☺☆	$8+2$

number	gnirt	sum
11	☺☆☺☆☆	$8+3$
12	☺☆☺☆☺	$8+3+1$
13	☺☆☆☆☆☆	13
14	☺☆☆☆☆☺	$13+1$
15		
16		
17		
18		
19		
20		

(1) Which numbers have a single term in the sum? when the number $a_n = a_{n-1} + a_{n-2}$, $a_1 = 1$, $a_2 = 2$.
And what property do the corresponding gnirts have? i.e., 1, 2, 3, 5, 8, 13, 21 ...

They all have a ☺ at and only at the first, with the rest all to be ☆. i.e. the corresponding gnirts are in the form of ☺$\underbrace{☆\dots☆}_{n-1}$

(2) Describe how to produce the sum and gnirt associated with the number 59.

We first list all the number in the sequence before 59:

1, 2, 3, 5, 8, 13, 21, 34, 55, ...

Since $59 = 55 + 3 + 1$, and 55, 3, 1 are the 9th, 3rd, and 1st number respectively, we know that the gnirt has ☺s at 9th, 3rd and 1st; ☺☆☆☆☆☆☺☆☺. ($55+3+1$)

(3) Given any number n, describe how to produce the sum and gnirt associated with the number n.

Express n using the sum of numbers (as large as possible) in the sequence, $a_n = a_{n-1} + a_{n-2}$, $a_1 = 1$ $a_2 = 2$; find the position of each of the numbers in the sequence, and, counting from right to left, put ☺ at each of those numbers and fill the rest with ☆.

(4) What sum corresponds to the non-gnirt ☺☺☆? How about ☺☆☺☺?

(5) What numbers are associated to ☺☺☆ and ☺☆☺☺?

(6) Why are we glad that gnirts never have two ☺s next to each other?

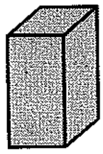
(4) ☺☺☆ corresponds to 3; and ☺☆☺☺ corresponds to $5+2+1 = 8$.

(5) 1, 2 are associated to ☺☺☆, and 5, 2, 1 are associated to ☺☆☺☺

(6) Since 3 can be expressed by ☺☆☆, and 8 can be expressed as ☺☆☆☆☆. So if there are two ☺s next to each other, there will be no repetition. Also, when two ☺s are together, they shall move one position forward and become one ☺, and there initial positions become ☆s.

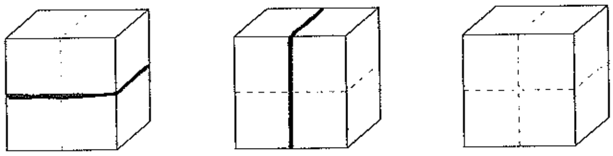
e.g. ☺☺☆ → ~~☺☺~~ ☺☆☆☆ ; ☺☆☺☺ → ☺☺☆☆ → ☺☆☆☆☆.

Baby Carrots in Lunchboxes

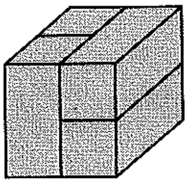


At Precision Vegetable School, every student has a lunchbox for hir lasercut $2 \times 1 \times 1$ baby carrots.

First graders are assigned $2 \times 2 \times 1$ lunchboxes. Please demonstrate all possible ways a first-grader can pack hir 2 lasercut baby carrots into hir lunchbox.



Second graders are assigned $2 \times 2 \times 2$ lunchboxes. Here is one way for a second-grader to pack hir 4 lasercut baby carrots into hir lunchbox.

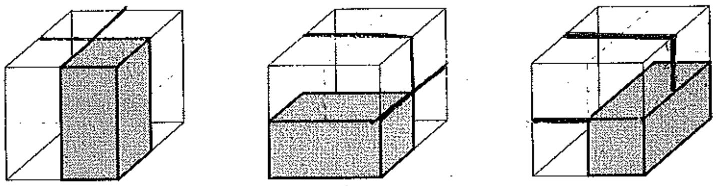


How many lasercut baby carrots may a kth-grader eat at lunch?

$$\frac{2 \times 2 \times k}{2 \times 1 \times 1} = 2k.$$

A kth grader shall eat 2k carrots.

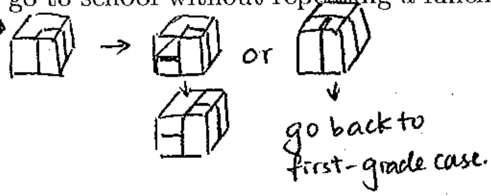
Please complete these partial packings.



How many days can a second-grader go to school without repeating a lunchbox carrot packing?

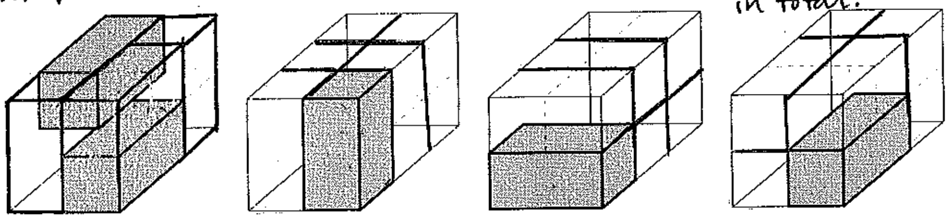
9 days.
For the three starting position, we each have two ways to put the next carrot!

For example,



Thus, for , there are three possibilities; Using the same method, we can know there are 9 possibilities in total.

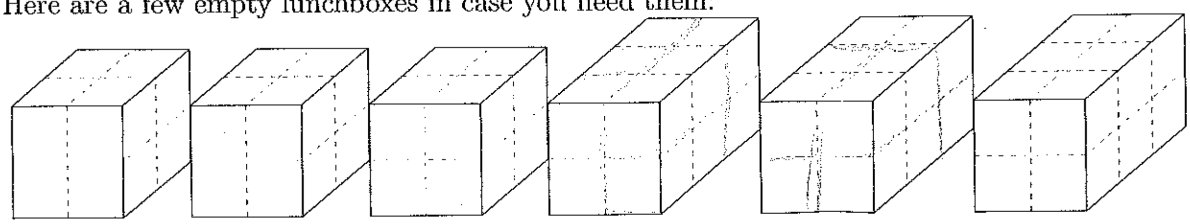
Third graders are assigned $2 \times 2 \times 3$ lunchboxes. Please complete these partial packings.

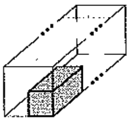


How many days can a third-grader go to school without repeating a lunchbox carrot packing?

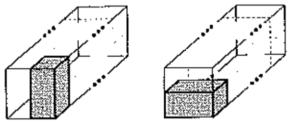
32 days. By discussing the three cases of ,  and , and using the conclusions drew from first and second graders, there are 12 possibilities for  and , and 8 possibilities for .

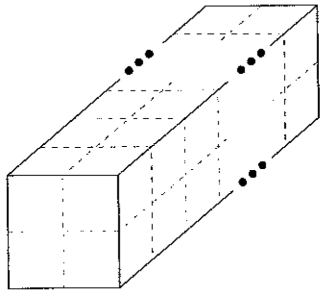
Here are a few empty lunchboxes in case you need them:





Baby Carrots: The Upper Grades





A kth-grade student at Precision Vegetable School is assigned a $2 \times 2 \times k$ lunchbox for hir lasercut $2 \times 1 \times 1$ baby carrots.

Write C_k for the number of school days a kth-grader can go to school without repeating a lunchbox carrot packing.

Find and explain an expression for C_k in terms of information about younger students.

$$C_k = 2C_{k-1} + 5C_{k-2} + 4\sum_{i=1}^{k-3} C_i$$

For the starting position, we still have three cases:

Case 1:

Then to fill Block A, we have two choices: either or

If it is , then the rest shall have C_{k-1} ways because it is now transformed to a $2\times2\times(k-1)$ lunchbox.

If it is , in order to fill the first $2\times2\times1$ layer, it can only be again, we have two choices now: or and again converts the problem to a $2\times2\times(k-2)$ lunchbox. If this continues, we know for this case, there are $\sum_{i=1}^{k-1} C_i$ ways in total.

Case 2: this case is the same as case one, as it can come back to it if it is turned $90°$ anticlockwise.

Case 3. in order to fill the first $2\times2\times2$, there must be either or but these two case are identical if one of them turn $90°$, so we can only consider one.
Also, this case can also be transformed to $2\times2\times(k-2)$, $2\times2\times(k-3)$. Thus, for each there are $2(\sum_{i=1}^{k-2} C_i + C_{k-2})$ ways, and delete the repeated case of , we have in total $2\sum_{i=1}^{k-2} C_i$.

Therefore, adding the three cases together, we have $C_k = 2C_{k-1} + 5C_{k-2} + 4\sum_{i=1}^{k-3} C_i$.

For what grade does C_k exceed the length of the school year? Does that happen for an earlier or later grade than it becomes unreasonable to eat all the assigned carrots at lunch?

A school year should be around 180 days. $C_k > 180$ when $k \geq 5$. It seems to be resonable to eat 10 carrots for lunch, so it happens earlier.

Now that you're done with the 🖑, give a little bit of feedback: Had you seen any of this material before? Did you think any of the problems, or parts thereof, were particularly engaging or excellent? Other thoughts to share?

I have not seen any of these before.
I found the problems of Gnints really interesting as there are so many different ways to approach the same problem.

Honesty Pledge: Sign below to indicate that you did not collaborate, give help, or receive help from any sources other than the MathILy director. (And you did not lend anyone your 🖑.)

Your signature:	Ruiyan Maggie Huang.
Print your name, too:	Ruiyan Maggie Huang