FM2 Assignment #6 Name: Maggie $\frac{10}{10}$ Excellent!

1. Prove, without recourse to Venn diagrams, that $A \setminus B$ and $B \setminus A$ are disjoint sets.

Proof by contradiction:

Suppose $(A\setminus B)\cup(B\setminus A) \neq \phi$.

then $\exists x$ s.t. $x \in (A\setminus B)$ and $x \in (B\setminus A)$.

Since $x \in (A\setminus B)$.

$x \in A$. $x \notin B$.

Since $x \in (B\setminus A)$

$x \in B$, $x \notin A$,

and there is a contradiction.

Therefore, $(A\setminus B)\cup(B\setminus A) = \phi$.

~~and~~ i.e. $A\setminus B$ and $B\setminus A$ are disjoint sets. $\square$.

2. Suppose that K is a proper subgroup of H and H is a proper subgroup of G. If $|K| = 42$ and $|G| = 420$, what are the possible orders of H?

$K \leq H \leq G$.

according to Lagrange's thrm,

$|K| \mid |H|$. $|H| \mid |G|$.

$42 \mid |H|$ $|H| \mid 420$.

So $|H|$ can only be

42, 42×2, 42×5, 42×10.

But since they are all proper subgroups,

The only possible orders of H are 84 and 210.

3. Suppose G is a finite group of order n and m is relatively prime to n. If $g \in G$ and $g^m = e$, prove that $g = e$.

proof. Since G is of order n,

according to Lagrange's thrm,

$|a| \mid |G|$. $\forall a \in G$.

and since $g^m = e$. $|G| = n$,

$|g| \mid m$, $|g| \mid n$.

~~So $m \mid n$.~~

and because $\gcd(m,n) = 1$.

~~$m = 1$.~~ $\gcd(|g|, n) = 1$.

Therefore, $|g| = 1$.

and thus $g = e$. $\square$.

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4. Determine the null space, nullity and rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$.

Let $Ax = \cancel{0}\, 0$.

It has AM:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 0 \end{array}\right).$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right).$$

{using technology}.

Let $x_3 = s$, $x_4 = t$.

$$\begin{cases} x_1 + (-1)x_3 + (-2)x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0. \end{cases}$$

$$\begin{cases} x_1 = s + 2t \\ x_2 = -2s - 3t \end{cases}.$$

$\therefore$ Null A is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}. \quad s, t \in \mathbb{R}.$$

which is the subspace spanned by

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$\therefore$ nullity$(A) = 2$.

Since ~~the~~ a basis for column space is $\left\{ \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix} \right\}$.

rank(A) = colume rank of $A = 2$.

5. Let T be a tree with $n > 1$ vertices. Use the handshaking lemma to prove that T has at least two leaves.

proof by contradiction.

Suppose there are n vertices, $n > 1$.

and it has less than two leaves,

i.e. there are at least $(n-1)$ vertices w/ ~~deg(v) > 1~~ degree more than 1.

According to handshaking lemma,

$$\sum_{v \in V} \deg(v) = 2e.$$

where V is the vertices in T. and e is the # of edges.

$$\therefore \sum_{v \in V} \deg(v) \geq (n-1)(2) + 1.$$

$$\Rightarrow e \geq \frac{2n-1}{2} = n - \frac{1}{2}.$$

and since for a tree w/ n vertices, there are exactly $n-1$ edges.

yet $n-1 < n - \frac{1}{2}$. which is a contradiction

Thus T has at least 2 leaves. $\square$

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FM2 Assignment #7 Name: Maggie $\frac{10}{10}$ Excellent!

1. Suppose $f\colon G \to G'$ is a group homomorphism with identities e and e' respectively. Prove that $f(e) = e'$.

proof: Since $f: G \to G'$

$$f(e \cdot a) = f(e) \cdot f(a) = f(a).$$

where $a \in G$.

post multiple by $f(a)^{-1}$, we get

$$f(e) = f(e) \cdot f(a) \cdot f(a)^{-1} = f(a) \cdot f(a)^{-1} = e'.$$

Thus,

$f(e) = e'$ □.

2. Show that the improper integral $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$ converges and find its value.

$$\int_0^\infty \frac{1}{1+x^2}dx$$

$$= \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}dx$$

$$= \lim_{b\to\infty} \left[\arctan x\right]_0^b$$

$$= \lim_{b\to\infty} [\arctan b - \arctan 0]$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}.$$

3. Suppose $\phi\colon \mathbb{Z}_{30} \to \mathbb{Z}_{30}$ is a homomorphism with $\ker(\phi) = \{0, 10, 20\}$. If $\phi(23) = 9$ find all elements that map to 9.

Let e denote the identity elmt in the first group.

~~Since~~ $\cancel{\ker(\phi) + \phi(23) =}$

Because of homormorphism,

$$\phi(e) + \phi(23)$$

$$= \phi(e+23) = \phi(23) = 9.$$

And since

$$\phi(0) = \phi(10) = \phi(20) = \phi(e),$$

and they are the only ones that map to the identity in the second group,

$$\phi(23) = \phi(0+23) = \phi(23)$$
$$= \phi(10+23) = \phi(3)$$
$$= \phi(20+23) = \phi(13)$$
$$= 9.$$

Therefore,

The elmts 23, 3, 13 in the first group all map to 9.

4. By considering the permutations $\alpha = (12)$ and $\beta = (123)$ in S_4, show that $f\colon S_4 \to S_4$ defined by $f(p) = p \circ p$ is not a homomorphism.

$f(\alpha) = \alpha^2 = (12)(12) = e.$

$f(\beta) = \beta^2 = (123)(123) = (132).$

$f(\alpha)\cdot f(\beta) = e(132) = (132)$

$f(\alpha\cdot\beta) = f((12)(123))$

$= (12)(123)(12)(123)$

$= e.$

and $f(\alpha\cdot\beta) \neq f(\alpha)\cdot f(\beta)$

So $f: S_4 \to S_4$ defined by $f(p) = p \circ p$ is ~~to~~ not a homomorphism.

5. Consider the curve $y = x^3$. The tangent at a point P on the curve meets the curve again at Q. The tangent at Q meets the curve again at R. Denoting the x-coordinates of P, Q, R by x_1, x_2, x_3 respectively where $x_1 \neq 0$, show that x_1, x_2, x_3 form the first three terms of a divergent geometric sequence.

$P(x_1, x_1^3)$

$Q(x_2, x_2^3)$

$R(x_3, x_3^3)$

So the slope of PQ is

$$\frac{x_2^3 - x_1^3}{x_2 - x_1} = x_2^2 + x_2x_1 + x_1^2$$

and since PQ is tangent to $y=x^3$ at P,

the slope of PQ also equals to

$$3x_1^2.$$

$\therefore\ x_2^2 + x_2x_1 + x_1^2 = 3x_1^2$

$x_2^2 + x_2x_1 - 2x_1^2 = 0.$ ①

Similarly,

$x_3^2 + x_3x_2 - 2x_2^2 = 0.$ ②

Let $\frac{x_2}{x_1} = a$, $\frac{x_3}{x_2} = b$.

Then devide ①, ② by x_2^2,

we get $\begin{cases} 1 + \frac{1}{a} - 2\left(\frac{1}{a}\right)^2 = 0 \\ b^2 + b - 2 = 0 \end{cases}$

$\Rightarrow \begin{cases} a^2 + a - 2 = 0 \\ b^2 + b - 2 = 0 \end{cases}$

so, a, b are solutions to the equation

$$x^2 + x - 2 = 0.$$

$a_1 = -2$, $a_2 = 1$

Since $x_1 \neq x_2 \neq x_3$.

$a = b = -2$.

i.e. $\frac{x_2}{x_1} = \frac{x_3}{x_2} = -2$.

which means that x_1, x_2, x_3 forms a geometric sequence, and since $|-2| > 1$, it is divergent.

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FM2 Assignment #8 Name: Maggie. $\frac{10}{10}$ Excellent!

1. Give an example of a relation that is symmetric and transitive but not reflexive.

The relation R is defined on $\mathbb{Z}^+$ by xRy if the product xy is odd. $x, y \in \mathbb{Z}^+$.

Check:

① Symmetric because if xRy, xy is odd, then yx is odd, yRx.

② transitive because if xRy, yRz, xy, yz are odd.

So x, y, z are all odd. thus xz is odd, xRz.

③ Not reflexive because xRx is not true when x is even.

2. Evaluate the improper integral $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$.

let $u = \frac{x}{2}$. $du = \frac{dx}{2}$

$$\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{4-4u^2}} \cdot 2\,du = 2 \cdot \frac{1}{2} \int_0^1 \frac{1}{\sqrt{1-u^2}}\,du$$

$$= [\sin^{-1} u]_0^1 = [\sin^{-1} \tfrac{x}{2}]_0^2 = [\sin^{-1} 1 - \sin^{-1} 0] = \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}$$

3. Determine the kernel of the group homomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (4x+2y, 2x+y)$.

The idenity elmt in both groups are $(0,0)$

Thus $\ker(f) = \{(x,y) \mid 4x+2y=0,\ 2x+y=0,\ x,y \in \mathbb{R}\}$.

$= \{(x,y) \mid 2x+y=0,\ x,y \in \mathbb{R}\}$.

$= \{(x,-2x) \mid x \in \mathbb{R}\}$.

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4. The relation $\sim$ is defined on $\mathbb{R}^2$ by $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 y_1 = x_2 y_2$. Show that $\sim$ is an equivalence relation and graph the equivalence class $[(1,1)]$.

To check for equivalence relation:

① if $x_1 y_1 = x_2 y_2$.

then $x_2 y_2 = x_1 y_1$.

So $\sim$ is symmetric

② Since $x_1 y_1 = x_1 y_1$,

$\sim$ is reflexive.

③ if $x_1 y_1 = x_2 y_2$.

$x_2 y_2 = x_3 y_3$.

then $x_1 y_1 = x_2 y_2 = x_3 y_3$

i.e. $(x_1, y_1) \sim (x_3, y_3)$

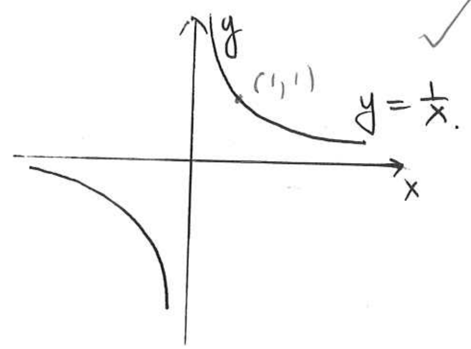
So $\sim$ is transitive.

when $x_1 = y_1 = 1$.

$x_2 y_2 = 1 \times 1 = 1$.

So $[(1,1)] = \{(x,y) \mid xy = 1.\ x, y \in \mathbb{R}\}$

Thus, it is the graph of



5. The convergent sequence $u_n = \dfrac{e^n + 2^n}{2e^n}$ has limit L. Find the smallest value of n for which $|u_n - L| < 0.001$.

$$\lim_{n \to \infty} \frac{e^n + 2^n}{2e^n} = \frac{\infty}{\infty},$$ which is of inderterminate form.

We apply L'Hôpital's rule,

$$\lim_{n \to \infty} \frac{(e^n + 2^n)'}{(2e^n)'} = \frac{ne^{n-1}}{2ne^{n-1}} = \frac{1}{2}.$$

Thus, $$\lim_{n \to \infty} \frac{e^n + 2^n}{2e^n} = \frac{1}{2} = L.$$

$$\left| \frac{e^n + 2^n}{2e^n} - \frac{1}{2} \right| < 0.001.$$

$$\left| \frac{2^n}{e^n} \right| < 0.002.$$

$$n > \log_{\frac{2}{e}} 0.002.$$

$$n > 20.3.$$

So the smalest value of n is 21.

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FM2 Assignment #9 Name: Maggie.

$\frac{10}{10}$ Excellent!

1. Give two reasons why the set of odd integers under addition is not a group.

① It is not closed.
Because the sum of two odd numbers ~~are~~ is even, which does not belong to odd integers.

② It does not have an identity.
~~∵~~ the only possible identity is 0.
yet 0 is even.

2. Construct the Cayley table for $(\mathbb{Z}_{12}^*, \otimes)$.

	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

3. The space $S = \langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix} \rangle$ is a subspace of $\mathbb{R}^3$. Find a Cartesian equation for S.

$$S = \left\{ c_1\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\2\\3\end{pmatrix} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\}.$$

$$= \left\{ \begin{pmatrix}c_1+c_2\\c_1+2c_2\\c_1+3c_2\end{pmatrix} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\}.$$

Let $x = c_1+c_2$. $y = c_1+2c_2$. $z = c_1+3c_2$.

then $x-2y+z=0$.

Thus S is the plane in $\mathbb{R}^3$ $x-2y+z=0$

4. A sequence is defined recursively by $u_1 = 1$ and $u_{n+1} = \dfrac{1}{1+u_n}$. Assuming the sequence is convergent find its limit.

Since it is assumed that the sequence is convergent,

~~We~~ when n approaches infinity, $u_{n+1} = u_n$.

Better to write $\lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} u_n = L$ say.

i.e. $\dfrac{1}{1+u_n} = u_n$.

$\therefore u_n = \dfrac{-1+\sqrt{5}}{2}$ or $\dfrac{-1-\sqrt{5}}{2}$

Since $u_1 = 1$. and $u_{n+1} = \dfrac{1}{1+u_n}$.

u_n can not be negative.

so $u_n = \dfrac{-1+\sqrt{5}}{2}$ as n approach ∞.

which means the limit is $\dfrac{-1+\sqrt{5}}{2}$.

5. Prove that the set of 3×3 matrices with real entries of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ is a group under matrix multiplication.

proof: ① Matrix Multiplication is associative.

② The set is closed

$$\begin{pmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a_2 & b_2 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_1+a_2 & b_2+a_1c_2+b_1 \\ 0 & 1 & c_1+c_2 \\ 0 & 0 & 1 \end{pmatrix}$$

Since a_1+a_2, $b_2+a_1c_1+b_1$, $c_1+c_2 \in \mathbb{R}$.

The set is closed.

③ Since $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ which belongs to the set taking $a=b=c=0$

$= \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$.

the set has identity $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

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④ Using Linear Row Reduction:

$$\left(\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & -b+ac \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

we find the inverse $\begin{pmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$ belongs to the set.

Therefore, we've verified that the set $\left\{\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| a,b,c \in \mathbb{R}\right\}$ is a group under multiplication $\square$.

FM2 Assignment #10 Name: Maggie. $\frac{10}{15}$ Excellent!

1. The functions $i\colon x \to x$, $f\colon x \to 1/x$, $g\colon x \to -x$, $h\colon x \to -1/x$, form a group under composition. Construct the Cayley table for this group and state to which well-known group the given group is isomorphic.

	i	f	g	h.
i	i	f	g	h.
f	f	i	h	g.
g	g	h	i	f
h.	h.	g	f	i.

the group is isomorphic to V_4 (or $\mathbb{Z}_2 \times \mathbb{Z}_2$).

2. Consider the function $f(x) = \begin{cases} |x-2|+1, & x<2 \\ ax^2+bx, & x \geq 2 \end{cases}$. If f and f' are both continuous at $x=2$, find a and b.

$f(2) = 4a + 2b. = |2-2|+1 = 1.$

$f'(2) = 2a(2) + b = 4a + b = (2-x+1)' = -1$

$\therefore \begin{cases} 4a+2b=1 \\ 4a+b=-1 \end{cases}$

$\therefore \begin{cases} a = -\frac{3}{4} \\ b = 2. \end{cases}$

3. Suppose $f\colon G \to G'$ and $g\colon G' \to G''$ are group homomorphisms. Prove $g \circ f$ is a homomorphism from G to G''.

Proof. since f is a homomorphism,

$f(ab) = f(a)f(b)$, $a, b \in G$.

since g is a homomorphism,

$g(f(a)f(b)) = g(f(a))\, g(f(b))$

$= g\circ f(a) \cdot g\circ f(b)$. $f(a)f(b) \in G'$

on the other hand.

$g(f(a)f(b)) = g(f(ab))$

$= g\circ f(ab)$.

$g\circ f(a) . g\circ f(b) \in G''$.

Therefore,

$g\circ f(ab) = g\circ f(a) \cdot g\circ f(b)$

which means,

$g\circ f$ is a homomorphism from G to G''. $\square$

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4. Suppose $f: G \to G'$ is a group homomorphism. Prove $\mathrm{ran}(f) \leq G'$.

~~proof~~: since $f: G \to G'$, $\mathrm{ran}(f) \subseteq G'$. and $f(e) = e'$. where e is the identity in G. e' is in G'. so $\mathrm{ran}(f)$ is non-empty

we use the 3-step subgroup test to show $\mathrm{ran}(f) \leq G'$.

① let $a', b' \in \mathrm{ran}(f)$.

~~$a'b' =$~~ $\exists\, a, b \in G$, s.t.

$f(a) = a'$, $f(b) = b'$,

thus $f(ab) = f(a) f(b) = a'b'$.

Therefore, $a'b' \in G'$.

and $\mathrm{ran}(f)$ is closed. ✓

② Since $f(e) = e'$.

$e' \in \mathrm{ran}(f)$ ✓

identity axiom is verified.

③ $f(a^{-1}) = f(a)^{-1} = (a')^{-1}$

where a^{-1} is the inverse of $a \in G$.

thus, $\exists\, (a')^{-1}\ \forall a' \in G'$. ✓

and inverse axiom is verified.

Therefore, we conclude that ✓

$\mathrm{ran}(f) \leq G'$. $\square$.

5. The relation $\sim$ on $\mathbb{R}^2$ is defined by $(a, b) \sim (c, d)$ if $d - b = 2(c - a)$. Show that $\sim$ is an equivalence relation and describe the equivalence classes geometrically.

To show that $\sim$ is an equiv. relation

① since ~~a~~

$b - b = 2(a - a) = 0$.

$(a, b) \sim (a, b)$.

and $\sim$ is reflexive.

② If $(a,b) \sim (c,d)$

$(d - b) = 2(c - a)$

then

$(b - d) = -2(c - a)$

$= 2(a - c)$

thus $(c, d) \sim (a, b)$

and $\sim$ is symmetric.

③ If $(a,b) \sim (c,d)$

$(c,d) \sim (e,f)$

$\begin{cases} d - b = 2(c - a) & ① \\ f - d = 2(e - c) & ② \end{cases}$

① + ②, we get

$f - b = 2(e - a)$

which means

$(a, b) \sim (e, f)$.

and thus $\sim$ is transitive.

Therefore, $\sim$ is an equiv. relation.

Geometrically speaking, the equiv. classes are the sets of points on the lines of slope 2 in $\mathbb{R}^2$ ~~an the Cartesian plane.~~ ✓

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