1 SEQUENCES, LIMITS, & PROPER INTEGRALS

1.1 Infinite Sequences

- $\{a_n\} = \{\frac{1}{n}\} = 1, \frac{1}{2}, \frac{1}{3}, \dots$
- { Explicit formula e.g. ↑
  recursive formula. e.g. $a_{n+2} = a_{n+1} + a_n$. $n \geq 1$.
- If $\{a_n\}$ has a limit L as $n \to \infty$, $\{a_n\}$ converges to L; otherwise, diverges.

Thrm 1 (Limit of a sequence thrm)

$f(x)$ defined $\forall x \geq k$, $k \in \mathbb{Z}^+$. $\{a_n\}$ is a sequence s.t $a_n = f(n)$ when $n \geq k$. If $\lim_{x\to\infty} f(x) = L$. then $\lim_{n\to\infty} a_n = L$.

- Operation involving limit is the same. as the ordinary operation.

Thrm 2 (Squeeze thrm).

if $a_n \leq b_n \leq c_n$ $\forall n$. s.t. $n \geq N$, $N \in \mathbb{Z}^+$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

How to prove? $\lim_{n\to\infty} \frac{\sin x}{x} = 1$.

Thrm 3 (Absolute value thrm).

If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

? what is an counterexample * converse not true.

Ex 1.1 show that $\{\frac{x^n}{n!}\}$ converges to 0 $\forall x \in \mathbb{R}$.

1.2 L'Hopital's Rule

Thrm 1 (L'Hopital's Rule).

Let f and g be fcn's whose derivative can be found at any value in an open interval $]a, b[$, except possibly at some value c where $a < c < b$. Assume that $g'(x) \neq 0$, except possibly at c. Suppose that $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$; or

$\lim_{x\to c} f(x) = \pm\infty$ and $\lim_{x\to c} g(x) = \pm\infty$.

(i.e. $\frac{f(x)}{g(x)}$ is in indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$).

Then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$ provided the limit on the right side exists.

Ex. 2.1 which sequence grows faster, $\{\ln n\}$, or $\{\sqrt{n}\}$?

1.3 Improper Integrals

- Improper: one of the limits is infinite.

Ex 3.1 Evaluate $\int_1^\infty \frac{x}{e^x} dx$.

$$\int_1^\infty \frac{x}{e^x} dx = \lim_{b\to\infty} \int_1^b \frac{x}{e^x} dx.$$ let $u = x$, $dv = e^{-x} dx$, $du = dx$, $v = -e^{-x}$

$$\lim_{b\to\infty} \int_1^b \frac{x}{e^x} dx = \lim_{b\to\infty} \left[\left[-xe^{-x}\right]_1^b + \int_1^b e^{-x} dx \right]$$

$$= \lim_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]_1^b$$

$$= \lim_{b\to\infty} \left(\frac{-(b+1)}{e^b}\right) + \frac{2}{e} = \frac{2}{e}.$$

2 SERIS AND CONVEGENCE

2.1 Infinite Series

Defn 1.1 if the seq. of partial sums $\{S_n\} = \{\sum_{i=1}^{n} a_i\}$ converges, its limit $S = \sum_{i=1}^{\infty} a_i$. If $\{S_n\}$ diverges, $\sum_{i=1}^{\infty} a_i$ diverges.

· Geometric Series

Thrm 1.1 if $\lim_{n\to\infty} a_n$ does not exist, or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges. [nth term div. test].

* Harmonic Series :

  even though the sequence $\frac{1}{n} \to 0$, as $n \to \infty$.
  the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

* Properties of convergent series:

  $\sum_{n=1}^{\infty} a_n$. $\sum_{n=1}^{\infty} b_n$ convergent,

  $\Rightarrow$ $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

  $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$. convergent.

2.2 Convergence tests

I. Integral test.

f be a fcn. ① cont. ② $\downarrow$, ③ positive $\forall x \geq 1$. and $a_n = f(n)$.

the $\sum_{n=1}^{\infty} a_n$ conv. $\Leftrightarrow$ $\int_1^{\infty} f(x)dx$ conv.

defn 2.1. (lower + upper bounds)

The # M is a lower bound of $\{a_n\}$ if $a_n \geq M$ $\forall n \in \mathbb{Z}^+$.
The # N is an upper bound of $\{a_n\}$ if $a_n \leq N$ $\forall n \in \mathbb{Z}^+$.
A seq. $\{a_n\}$ is bounded $\Leftrightarrow$ $\exists$ both M & N.

* Monotonic : a fcn. always either ↑ or ↓.

Thrm 2.1. (Bounded Seq. thrm).

A monotonic seq. converges $\Leftrightarrow$ it is bounded.

Postulate. 2.1 (Completeness postulate).

In the real #s, every non-empty set that has an upper bound has a least upper bound.

II p-series

the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p} + \cdots$

(i) converges if $p > 1$

(ii) Diverges if $p \leq 1$.

(iii) Harmonic if $p = 1$.

III Comparison test

Given $0 \leq a_n \leq b_n$, $\forall n \geq N$ for some $N \in \mathbb{Z}$, then

(i) if $\sum_{n=1}^{\infty} b_n$ conv. $\rightarrow \sum_{n=1}^{\infty} a_n$ conv.

(ii) if $\sum_{n=1}^{\infty} a_n$ div. $\rightarrow \sum_{n=1}^{\infty} b_n$ div.

* Test applies whenever $\exists\ c \in \mathbb{R}^+$. s.t. $0 \leq a_n \leq c b_n\ \forall n \geq N \in \mathbb{Q}^+$.

Thrm 2.2 (Positive. series convergence).

A series of + terms is conv. $\Leftrightarrow$ its sequence of partial sums has an upper bound.

IV limit Comparison test.

Given $a_n > 0$ and $b_n > 0\ \forall n \geq N$. for some $N \in \mathbb{Z}$, then

(i) if $\lim_{n \to \infty} \frac{a_n}{b_n} = L$, where L is finite + positive. then the 2 Series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

(ii) if $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$. $\sum_{n=1}^{\infty} b_n$ con. $\rightarrow \sum_{n=1}^{\infty} a_n$ Conv.

(iii) if $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$. $\sum_{n=1}^{\infty} b_n$ div. $\rightarrow \sum_{n=1}^{\infty} a_n$ div.

(*) Useful when comparing a series to a p-series/ geo-series.

V Ratio test

Let $\sum_{n=1}^{\infty} a_n$ be a series w/ non-zero terms, and w/

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L. \quad \text{then the series.}$$

(i) conv. if $L<1$.

(ii) div. if $L>1$.

(iii) inconclusive if $L=1$.

* Useful for series involving exponential expressions or expressions w/ factorials.

(2.3) Alternating Series and absolute conv.

VI Alternating Series test

The alternating series.

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots \qquad (a_n > 0)$$

conv. if both of the following conditions are satisfied.

1. $\lim_{n\to\infty} a_n = 0$
2. $a_{n+1} \le a_n \quad \forall\, n \geq N \in \mathbb{Z}^+$.

Thrm 3.1 (Alternating Series estimation theorem).

$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is a convergent alternating series that satisfies both conditions of the alternating series test, then.

$$|R_n| = |S - S_n| \le a_{n+1}.$$

Absolute and conditional convergence.

defn 3.1. Absolute conv. :

$$\sum_{n=1}^{\infty} a_n \text{ conv.} \qquad \sum_{n=1}^{\infty} |a_n| \text{ conv.}$$

Conditional conv.

$$\sum_{n=1}^{\infty} a_n \text{ conv} . \qquad \sum_{n=1}^{\infty} |a_n| \text{ div.}$$

Thrm 3.1 (Absolute convergence thrm).

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges, and therefore $\sum_{n=1}^{\infty} a_n$ is absolutely conv.

* It's impossible to take a convergent series w/ only positive terms and change some of them to negative. to create a new div. series.

③ POWER SERIES.

3.1 power series

defn 1.1 (transcendental fcn).

non-algebraic. i.e. cannot be expressed as a finite # of sums, differences, multiples, quotients and radicals involving x^n.

defn 1.2 (Power Series)

if x is a vari. then an infinite series of the form.

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + a_2 (x-c)^2 + \dots + a_n (x-c)^n + \dots$$

is a power series centered at c.

Note that $(x-c)^0 = 1$ even when $x=c$.

Radius of convergence.

* A power series defines a fcn.
  The fcn. $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ has its domain all values of x for which the power series conv.
* A power series is best regarded as an attempt to describe a fcn locally, near where it is "centered".

Thrm 1.1 (Convergence of a power series thrm)

For a given power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ exactly one of the following is true:

(i) the series conv. only when $x=c$.

(ii) the series conv. for all real values of x.

(iii) $\exists R \in \mathbb{R}^+$ s.t series conv. for $|x-c| < R$, and div. for $|x-c| > R$. The series may or may not conv. at either of the endpoints $x = c-R$ and $x = c+R$.

(3.2) Maclaurin and Taylor series

Defn 2.1. (Taylor Series and Mac laurin series)

If a fcn. f has derivative at all orders at $x=c$. the the series

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ is called the taylor series centered at c.

As often occurs, if $c=0$, then the series becomes

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$, which is the Maclaurin Series for f.

(3.3) Operations w/ power series

Differentiation & integration of power series.

If R is the radius of conv. of the power series.

$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$. then

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n (x-c)^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}(a_n (x-c)^n)$$

$$= \sum_{n=0}^{\infty} na(x-c)^{n-1}. \text{ for } c-R<x<c+R.$$

and

$$\int f(x)\,dx = \int \left(\sum_{n=0}^{\infty} a_n (x-c)^n\right) dx = \sum_{n=0}^{\infty}\left(\int a_n (x-c)^n\right) dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1}(x-c)^{n+1}. \quad \text{for } c-R<x<c+R.$$

properties of power series

Given the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ and k,

(i) $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$.

(ii) $f(x^k) = \sum_{n=0}^{\infty} a_n x^{kn}$.

(iii) $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$.

(iv) $f(x) \cdot g(x) = \sum_{n=0}^{\infty} (a_n b_n x)^n$

(v) $\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} \left(\frac{a_n}{b_n}\right) x^n$, $b_n \neq 0$

$f(x) = \sum x^n$

$g(x) = \sum 2x^n$

$2^n x^n$

defn. Taylor polynomials

In general the nth partial sum of a Taylor series is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

which is the taylor poly of deg. n.

* In general, $f(x) = \lim_{n\to\infty} P_n(x)$.

but $f(x) = P_n(x) + R_n(x)$, where $R_n(x)$ is the remainder, or error term.

Thrm 3.1. (Taylor's thrm).

If fcn f has derivatives of all orders in an open interval I centered at c, then for each $n \in \mathbb{Z}^+$ and $x \in I$.

$$f(x) = P_n(x) + R_n(x) \quad \text{where}$$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$ is the nth deg. Taylor poly centered at $x=c$.

The error term, $R_n(x) = \frac{f^{(n+1)}(b)}{(n+1)!}(x-c)^{n+1}$, where b is btw x and c, inclusive (Lagrange form)