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Welcome to the
2019 Hampshire College
Summer Studies in Mathematics
INTERESTING TEST

This is a link to the article College Admissions and the Stability of Marriage, the basis for IT's 2nd problem.

You should try to limit the time you spend on the IT to 7 or 8 hours, indicating as afterthoughts later ideas (which can be sent separately). Send IT in just before you run out of ideas to try. We don't want this application to become a career or to interfere with other studies.

Enjoy the IT; expect much of IT to be unfamiliar, but don't expect to complete all of IT. Email me if you suspect that there is an error on the IT or if a problem remains incomprehensible; do not suffer in silence. Please send questions about the IT to: dckNS@hampshire.edu with the subject line: "[*your name*]'s IT Question [*date*]."

Let us know if you can't finish within 17 days. This is not a rigid deadline--just let us know that your candidacy is still active with a note to sgoff@hampshire.edu with the subject line: "[*your name*]'s IT update [*date*]".

Let your reasoning and computations show. Use the provided spaces, but feel free to use the backs of pages and additional paper as needed. Feel obligated to use additional space if the alternative is tiny-fonted cramped writing. Leave space between lines. We do not need to see your scratch work, but you're welcome to tell us about efforts that didn't work out. Revise and neaten your work. The harder it is to decipher your writing or to find strands of logic or complete sentences, the less inclined we'll be to spend 6 weeks doing it with you this summer. Spend more time doing your work than digitizing it. Feel free to add generalizations, speculations, and questions.

One reason you've applied to HCSSiM is that you enjoy sharing mathematics—but please do not discuss these questions. Instead, encourage those with whom you'd like to share IT ideas to apply to the Summer Studies promptly. As you know: Applying is fast, free, fun, obligationless, and doable on line even while (especially if) other options are being explored; not applying unnecessarily closes doors, means not seeing the Interesting Test, and may cause drowsiness and boredom.

We're not testing your Internet search skills. (In fact, we are not very concerned with how much you know—there's no need to flaunt prior knowledge.) Do not use the Internet for IT. You are obliged to tell us if you have encountered in the past or stumble upon relevant information online.

Paper submissions are still fine. Send well-labeled things to:

David C. Kelly or Susan Goff
HCSSiM, Box NS
Hampshire College
893 West St.
Amherst, MA 01002-3359

Important advice: When you send things electronically the email's subject line and the name of each and every attachment should identify you, an approximate date, and hint about the content. Please combine things into as few pdfs (e.g., one) as possible when you send your IT work to sgoff@hampshire.edu and use the subject line: "[*your name*]'s 2019 IT work, [*date*]".

Give us more feedback, please, when you have completed your work on the 2019 HCSSiM Interesting Test:

How and how long did you work on the IT? One afternoon.

Which, if any, of the problems had you considered before? None.

Which, if any, did you particularly enjoy? Tiling. with tetrominoes.

Particularly not enjoy?

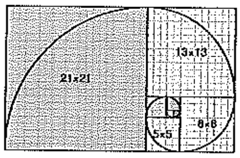
Other comments:

Thanks.

Welcome again to the

2019 HAMPSHIRE COLLEGE SUMMER STUDIES IN MATHEMATICS INTERESTING TEST

FIBONACCI SQUARES $\qquad f_1 = f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n > 2$.

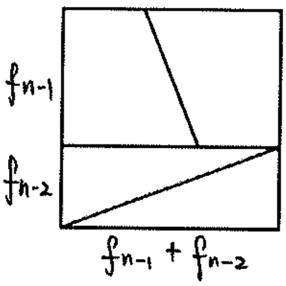


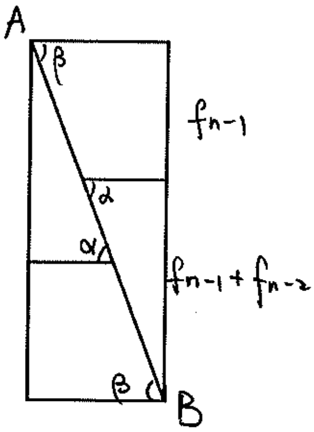
$$\sum_{k=1}^{n} f_k^2 = 1 + 1 + 4 + 9 + 16 + \cdots + f_n^2 = \underline{\quad f_n f_{n+1} \quad}.$$

$f_{n-1}f_{n+1} - f_n^2 = \underline{\quad (-1)^n \quad}$. $\qquad f_{n-2}f_{n+2} - f_n^2 = \underline{\quad (-1)^{n+1} \quad}$.

A square with side f_n is partitioned into triangles and trapezoids which, it seems, can be rearranged to form a rectangle or an octogon with different areas.

In terms of the f_ks, what are dimensions of the triangles and trapezoids that create the paradox? Please explain the paradox; and specify the property (or properties) of Fibonacci numbers that you think are relevant to creating it.





Relevant property:

(A B is not straight).

$$\tan\alpha = \frac{f_{n-1} + f_{n-2}}{f_{n-2}} = 1 + \frac{f_{n-1}}{f_{n-2}}$$

$$\tan\beta = \frac{2f_{n-1} + f_{n-2}}{f_{n-1}} = 2 + \frac{f_{n-2}}{f_{n-1}}.$$

However,

$1 + \frac{f_{n-1}}{f_{n-2}}$ and $2 + \frac{f_{n-2}}{f_{n-1}}$ are not always equal, creating the paradox.

Area of square:

$$(f_{n-1} + f_{n-2})^2 = f_{n-1}^2 + 2f_{n-1}f_{n-2} + f_{n-2}^2.$$

Area of rectangle:

$$(f_{n-1} + f_{n-2} + f_{n-1}) \times f_{n-1} = 2f_{n-1}^2 + f_{n-1}f_{n-2}.$$

Area of square = Area of rectangle

0.01
+.001
+.0002
+.00003
+.000005
+.0000008
+.00000013
+.000000021
+.0000000034
+ ...

$\Leftrightarrow$

$$-f_{n-1}^2 + f_{n-1}f_{n-2} + f_{n-2}^2 = 0$$

$$(f_{n-2} + f_{n-1})(f_{n-2} - f_{n-1}) + f_{n-1}f_{n-2} = 0$$

$$-f_n f_{n-3} + f_{n-1}f_{n-2} \neq 0$$

which is a paradox.

STABLE MARRIAGES

Based on *College Admissions and the Stability of Marriage* by L. S. Shapley and D. Gale. (Link on IT cover page.)

Explain why the Gale-Shapley algorithm with n men and n women terminates in at most $n^2 - 2n + 2$ stages.

Are there preference tables that will cause it to last that long for every n? YES.

Consider the worst case. (men ($\alpha, \beta, \gamma \ldots$) proposing to women ($A, B, C \ldots$)). Assume woman A is ranked last by every man. Except one men have different first choice. For every woman that is ranked first by a man, the man is ranked last for the woman.

However, there shall be two man who rank the same first choice, the woman rank these two men at the last two choices. and the man who is ranked last proposes to second choice. and the second choice ranks the man second last. so the man ranked last by this woman keep proposing, ...

when all men finish proposing to their second last choice, there are $1 + (n-2)n$ stages adding the last stage, there are $n^2 - 2n + 2$ stages in total.

For the preferences table on p.13, find the man-optimal set of marriages and the woman-optimal set of marriages:

$\alpha \leftrightarrow$ C, $\beta \leftrightarrow$ D, $\gamma \leftrightarrow$ A, $\delta \leftrightarrow$ B; A $\leftrightarrow$ γ, B $\leftrightarrow$ δ, C $\leftrightarrow$ α, D $\leftrightarrow$ β.

Back to the general case: Show that, when the women do the proposing, the resulting set of marriages is as bad as any stable set of marriages could be for each man.

Because it is women who propose first, women will first propose their preferred man, and men can only choose when women propose to them, and thus become passive, resulting in the worst senario for men.

Not all of the results about marriages generalize to college admissions: In this 3–college 4-student example, c_1 has a 2-student quota while c_2 and c_3 are 1-student colleges. Their rankings (from most to least favored) are:

$c_1 : s_1, s_2, s_3, s_4$; $c_2 : s_1, s_2, s_3, s_4$; $c_3 : s_3, s_1, s_2, s_4$;

$s_1 : c_3, c_1, c_2$; $s_2 : c_2, c_1, c_3$; $s_3 : c_1, c_3, c_2$; $s_4 : c_1, c_2, c_3$.

Show that when the colleges do the "proposing" (so that, on day 1, c_1 invites s_1 and s_2), the resulting assignment of students to colleges is (very) stable. There is, however, another stable arrangement favored by all the colleges.

first, C_1 "proposes" to S_1, S_2, C_2 to S_1. C_3 to S_3 $\Rightarrow$ S_1C_1, S_2C_1, S_3C_3.

Second, C_2 "proposes" to S_2. $\Rightarrow$ S_1C_1, S_2C_2, S_3C_3

third, C_1 "proposes" to S_3. $\Rightarrow$ S_1C_1, S_3C_1, S_2C_2

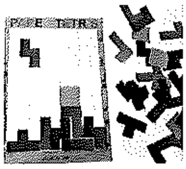
Then, C_3 to S_1 $\Rightarrow$ S_1C_3 S_3C_1 C_2C_2

Last, C_1 to S_4 $\Rightarrow$ S_4C_1 S_3C_1 S_1C_3 S_2C_2,

Another stable arrangement can be achieved by letting students do the "propose" and get S_1C_3, S_2C_2, S_3C_1, S_4C_1.

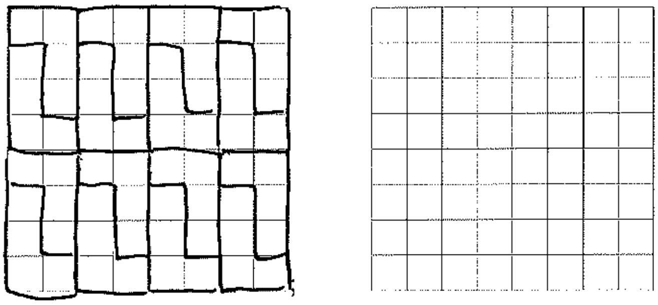
What do you think about the Gale-Shapley paper?

TILING WITH TETROMINOES



Too the right, some tetrominoes are playing.
How many different tetrominoes are there? 7.

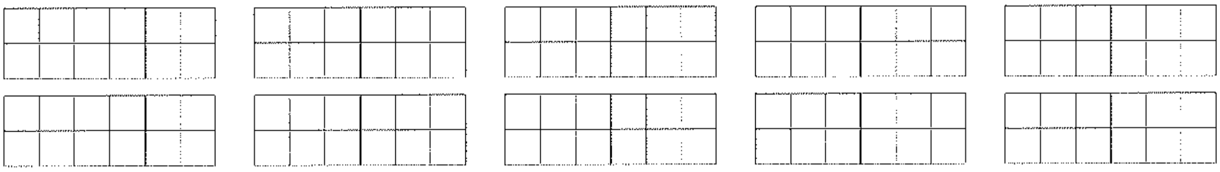
A *fault-free* tiling? Can an 8×8 grid can be tiled with L-tetrominoes (and reflections thereof) so that every horizontal or vertical line separating the grid into two non-empty pieces cuts through at least one tetromino? YES.
(What is the difference between an 8×8 grid and and an 8×8 checkerboard?)



Can an 8×8 checkerboard can be tiled with 15 L-tetrominoes and one 2×2 square?
No.

Let $T(N)$ = the number of ways a $2 \times N$ grid can be tiled with tetrominoes.

$T(1) =$ 0, $T(2) =$ 1, $T(3) =$ 0, $T(4) =$ 4, $T(5) =$ 0;



and $T(6) =$ 7. (Hint: $5.17 \leq T(6) \leq 11$).

Can you determine $T(N)$?
(If you want to define T(0), feel free to let IT know what you want it to be.)

Case 1. N is odd.

$T(N) = 0$.
because an odd number can only be sum of ~~two~~ one odd and one even number.
and since $T(1) = 0$,
$T(3) = \cancel{T(1)+T(2)}\ T(1+2)$ can only be 0.
and we can thus induce for all odd N, $\boxed{T(N) = 0}$.

Case 2. N is even.

$T(2) = 0$.
$T(4) = 4$.
However, because the is a repetition for $2\,T(2)$,

• we will only count 3 ways for $T(4)$.
Every even number N can be represented as $N = 4(a) + 2(b)$. $a, b \in \mathbb{Z}$, $a, b \geq 0$.
i.e. every $2 \times N$ grid canbe split into several 2×2 and 2×4 grids.

$$b = \frac{N-4a}{2}, \quad 0 \leq a \leq \frac{n}{4}.$$

$$a_0 = 0,\ a_1 = 1 \ \ldots\ a_k = \begin{cases} \frac{N}{4} & (N = 4k) \\ \frac{N-2}{4} & (N = 4k+2) \end{cases}$$

$$T(n) = \sum_{i=0}^{k} \frac{\left(a_i + \frac{N-4a_i}{2}\right)!}{a_i!\left(\frac{N-4a_i}{2}\right)!} \times 3^{a_i}$$

$$= \sum_{i=0}^{k} \frac{(N-2a_i)!}{a_i!\left(\frac{N-4a_i}{2}\right)!} \times 3^{a_i}.$$

1																1
1								2								1
1				3				2				3				1
1		4		3		5		2		5		3		4		1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1
⋮																

THE TABLE

Before seeing if you can determine without a computer the 17th entry in the 2019th row of this table or the number of times 2019 appears in that row, and before deciding if it would be possible, interesting, or fun to write a computer program to answer those questions, write down some ($\geq 3, \leq 10$) more reasonable questions about the table. Try to answer (or get partial results for) a few of those questions. See if you can formulate a conjecture that you don't now have a proof for or counterexample for, but which you might want to investigate further later.

1. What is the pattern?
   Every row produce the sum of two numbers that are adjacent in the former row.
2. What is the number of entries for n^{th} row?
   $2^{n-1}+1$.
3. How many 5 will there be in the 2019th row?
   4.

Further investigation:
What will the 3 dimensional table look like?

Before submitting your work, please share some comments on the IT. There are prompts on the cover page. Thanks.