Continuous Random Variables

1a. *[4 marks]*

The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} kx & 0 \leqslant x < 1 \\ kx^2 & 1 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}.$$

$\cancel{kx + kx^2 = 1}.$

$\cancel{kx^2 + kx - 1 = 0}..$

[illegible]

kx^2

Show that $k = \frac{6}{17}$.

$$\int_0^1 kx\,dx + \int_1^2 kx^2\,dx$$

$$= \left[\frac{k}{2}x^2\right]_0^1 + \left[\frac{k}{3}x^3\right]_1^2$$

$$\cancel{= \frac{k}{3} + 4k - \frac{k}{4} = 4k + \frac{k}{12} = 1}$$

$$= \frac{k}{2} + \frac{8}{3}k - \frac{k}{3} \qquad \cancel{48k + k = 12.}$$

$$= \frac{8}{3}k + \frac{k}{6} = \frac{17}{6}k = 1 \qquad \cancel{k = \frac{12}{49} =}$$

$$k = \frac{6}{17}. \checkmark$$

1b. *[6 marks]*

Find the cumulative distribution function of X.

$F(x) = \int_{-\infty}^{x}$

1c. *[3 marks]*

Find the median, m, of X.

1d. *[3 marks]*

Find $\mathrm{P}\left(|X - m| < 0.75\right)$.

2a. *[3 marks]*

A continuous random variable T has a probability density function defined by

$$f(t) = \begin{cases} \frac{t(4-t^2)}{4} & 0 \leqslant t \leqslant 2 \\ 0, & \text{otherwise} \end{cases}.$$

Find the cumulative distribution function $F(t)$, for $0 \leqslant t \leqslant 2$.

2b. *[2 marks]*

Sketch the graph of $F(t)$ for $0 \leqslant t \leqslant 2$, clearly indicating the coordinates of the endpoints.

2c. *[2 marks]*

Given that $P(T < a) = 0.75$, find the value of a.

3a. *[1 mark]*

A random variable X has probability density function

$$f(x)=\begin{cases}0 & x<0\\ \frac{1}{2} & 0\le x<1\\ \frac{1}{4} & 1\le x<3\\ 0 & x\ge 3\end{cases}$$

Sketch the graph of $y=f(x)$.

3b. *[5 marks]*

Find the cumulative distribution function for X.

3c. *[3 marks]*

Find the interquartile range for X.

4a. *[3 marks]*

The random variable *X* represents the lifetime in hours of a battery. The lifetime may be assumed to be a continuous random variable *X* with a probability density function given by $f(x)=\lambda e^{-\lambda x}$, where $x \geqslant 0$.

Find the cumulative distribution function, $F(x)$, of *X*.

4b. *[2 marks]*

Find the probability that the lifetime of a particular battery is more than twice the mean.

4c. *[3 marks]*

Find the median of *X* in terms of λ.

4d. *[2 marks]*

Find the probability that the lifetime of a particular battery lies between the median and the mean.

5a. *[3 marks]*

The continuous random variable *X* has probability density function *f* given by

$$f(x) = \begin{cases} 2x, & 0 \leqslant x \leqslant 0.5, \\ \frac{4}{3} - \frac{2}{3}x, & 0.5 \leqslant x \leqslant 2 \\ 0, & \text{otherwise.} \end{cases}$$

Sketch the function f and show that the lower quartile is 0.5.

5b. *[4 marks]*

(i) Determine $\mathrm{E}(X)$.

(ii) Determine $\mathrm{E}(X^2)$.

5c. *[5 marks]*

Two independent observations are made from X and the values are added.

The resulting random variable is denoted Y.

(i) Determine $\mathrm{E}(Y - 2X)$.

(ii) Determine $\mathrm{Var}\,(Y - 2X)$.

5d. *[7 marks]*

(i) Find the cumulative distribution function for X.

(ii) Hence, or otherwise, find the median of the distribution.

6a. *[4 marks]*

The continuous random variable X has probability density function

$$f(x) = \begin{cases} \mathrm{e}^{-x} & x \geqslant 0 \\ 0 & x < 0 \end{cases}.$$

The discrete random variable Y is defined as the integer part of X, that is the largest integer less than or equal to X.

Show that the probability distribution of Y is given by $\mathrm{P}(Y = y) = \mathrm{e}^{-y}(1 - \mathrm{e}^{-1}),\ y \in \mathbb{N}$.

6b. *[8 marks]*

(i) Show that $G(t)$, the probability generating function of Y, is given by $G(t) = \frac{1-\mathrm{e}^{-1}}{1-\mathrm{e}^{-1}t}$.

(ii) Hence determine the value of $\mathrm{E}(Y)$ correct to three significant figures.

7a. *[3 marks]*

A discrete random variable U follows a geometric distribution with $p = \frac{1}{4}$.

Find $F(u)$, the cumulative distribution function of U, for $u = 1,\ 2,\ 3\ldots$

7b. *[2 marks]*

Hence, or otherwise, find the value of $P(U > 20)$.

7c. *[4 marks]*

Prove that the probability generating function of U is given by $G_u(t) = \frac{t}{4-3t}$.

7d. *[6 marks]*

Given that $U_i \sim \text{Geo}\left(\frac{1}{4}\right),\ i = 1,\ 2,\ 3$, and that $V = U_1 + U_2 + U_3$, find

(i) $\mathrm{E}(V)$;

(ii) $\mathrm{Var}(V)$;

(iii) $G_v(t)$, the probability generating function of V.

7e. *[4 marks]*

A third random variable W, has probability generating function $G_w(t) = \frac{1}{(4-3t)^3}$.

By differentiating $G_w(t)$, find $\mathrm{E}(W)$.

7f. *[3 marks]*

A third random variable W, has probability generating function $G_w(t) = \frac{1}{(4-3t)^3}$.

Prove that $V = W + 3$.

HW: Cont. Rand. Vari.

Maggie. Feb 05 2020

1 a. $\int_0^1 kx\,dx + \int_1^2 kx^2\,dx = 1.$

$\left[\frac{1}{2}kx^2\right]_0^1 + \left[\frac{1}{3}kx^3\right]_1^2 = 1$

$\frac{1}{2}k + \frac{8}{3}k - \frac{1}{3}k = 1$

$\frac{17}{6}k = 1 \Rightarrow k = \frac{6}{17}$ //.

b. ① $0 \le x < 1$.

$F(x) = \int_0^x \frac{6}{17}t\,dt = \left[\frac{3}{17}t^2\right]_0^x = \frac{3}{17}x^2.$

② $1 \le x \le 2$.

$F(x) = \int_1^x \frac{6}{17}t^2\,dt + \frac{3}{17}\ F(1).$

$= \left[\frac{2}{17}t^3\right]_1^x + \frac{3}{17}$

$= \frac{2}{17}x^3 + \frac{1}{17}$

$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ \frac{3}{17}x^2 & 0 \le x < 1 \\ \frac{2}{17}x^3 + \frac{1}{17} & 1 \le x \le 2 \\ 1 & x > 2. \end{cases}$

c. $F(m) = 0.5$

$\frac{2}{17}m^3 + \frac{1}{17} = \frac{1}{2}$

$4m^3 + 2 = 17$

$m^3 = \frac{15}{4}$

$m = 1.55$ (3 s.f.)

d. $P(|X-1.55|<0.75)$

$= P(x<2.3) - P(X<0.8)$

$= F(2.3) - F(0.8) = 1 - 0.119 = \underline{0.881}$ (3 s.f.)

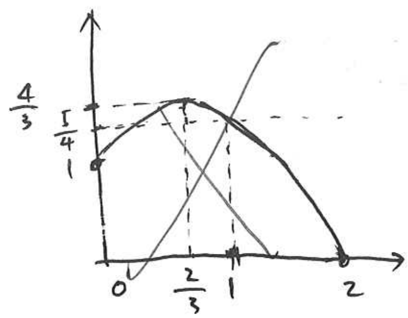
2a. $F(t) = \int_0^t \frac{x(4-x^2)}{4}\,dx = \int_0^t x - \frac{x^3}{4}\,dx = \left[\frac{x^2}{2} - \frac{1}{16}x^4\right]_0^t$

$= \frac{1}{4}\left[(4-x^2) + x(4-2x)\right]_0^t$ $\quad = \frac{t^2}{2} - \frac{t^4}{16}$

$= \left[\frac{4-x^2+4x-2x^2}{4}\right]_0^t$

$= \left[\frac{-3x^2+4x+4}{4}\right]_0^t = \cancel{\frac{-3t^2+4t+4}{4}}$.

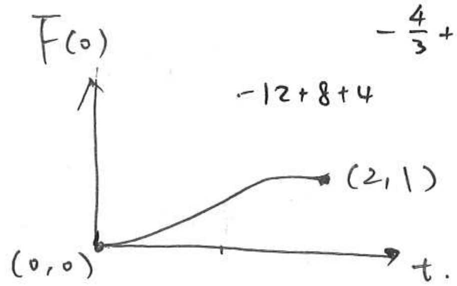
b



$\frac{-3\frac{4}{9} + \frac{2}{3}\cdot 4 + 4}{4}$

$-\frac{4}{3} + \frac{8}{3}$ $\quad \frac{4}{3} + 4$

$-12+8+4$ $\quad \frac{4}{3}$



c. $F(a) = 0.75$

$\cancel{\frac{-3a^2+4a+4}{4} = 0.75}$

$\cancel{-3a^2+4a+4 = a}$.

$\frac{t^2}{2} - \frac{t^4}{16} = \frac{3}{4}$

$8t^2 - t^4 = 12$

$(t^2)^2 - 8t^2 + 12 = 0$

$t^2 = 6$, $t^2 = 2$.

-1 $\quad$ 1

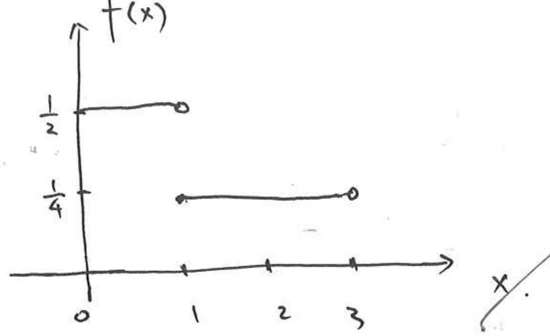
3 $\quad$ 1

1 $\quad$ 3 $\quad$ -2

1 $\quad$ 4 $\quad$ -6

$\cancel{t}$ $a = \sqrt{2}$.

3a.



b. ① $0 \le x < 1$

$$F(x) = \int_0^x \frac{1}{2}\,dt = \frac{1}{2}x$$

② $1 \le x < 3$

$$F(x) = \int_1^x \frac{1}{4}\,dt + \frac{1}{2} = \frac{1}{4}x + \frac{1}{4}$$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x & 0 \le x < 1 \\ \frac{1}{4}x + \frac{1}{4} & 1 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

c.

$F(Q_1) = 0.25$

$\frac{1}{2}Q_1 = \frac{1}{4}$

$Q_1 = 0.5$

$F(Q_3) = 0.75$

$\frac{1}{4}Q_3 + \frac{1}{4} = \frac{3}{4}$

$Q_3 = 2.$

$\Rightarrow IQR = \underline{1.5}$

$x \xrightarrow{D} 1.$

$\lambda e^{-\lambda x} \xrightarrow{I} -e^{-\lambda x}$

4a.

$$F(x) = \int_0^x \lambda e^{-\lambda t}\,dt.$$

$$= \left[\lambda \cdot \frac{1}{-\lambda} e^{-\lambda t}\right]_0^x$$

$$= -e^{-\lambda x} + 1$$

$\lambda x \to \lambda$

$e^{-\lambda x} \to -\frac{e^{-\lambda x}}{\lambda}$

b.

$$E(x) = \int_0^\infty x \cdot \lambda e^{-\lambda x}\,dx = \left[-xe^{-\lambda x}\right]_0^\infty - \int_0^\infty -e^{-\lambda x}\,dx.$$

$$= \left[-xe^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x}\,dx$$

$$= \left[-\frac{e^{-\lambda x}}{\lambda} - xe^{-\lambda x}\right]_{-\infty}^{\infty} = \left[-e^{-\lambda x}\left(\frac{1}{\lambda}+x\right)\right]_0^\infty \quad ?$$

$$= \frac{1}{\lambda}.$$

c. $F(m) = \frac{1}{2}$

$$1 - e^{-\lambda m} = \frac{1}{2}$$

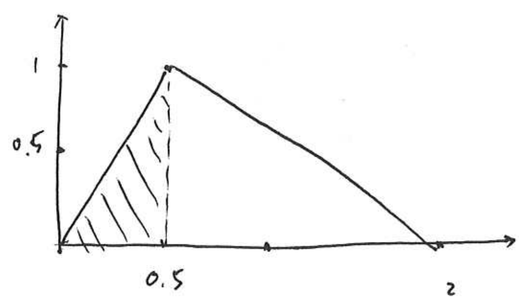
$$e^{-\lambda m} = \frac{1}{2}$$

$$-\lambda m = \ln\frac{1}{2}$$

$$m = -\frac{\ln(\frac{1}{2})}{\lambda.} = \frac{\ln 2}{\lambda.}$$

d.

5a.



$\frac{0.5 \times 1}{2} = \frac{1}{4}$ ✓

b. $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_0^{0.5} 2x^2 + \int_{0.5}^{2} \frac{4}{3}x - \frac{2}{3}x^2$$

$$= \left[\frac{2}{3}x^3\right]_0^{0.5} + \left[\frac{2}{3}x^2 - \frac{2}{9}x^3\right]_{0.5}^{2}$$

$$= \frac{2}{3}\cdot\frac{1}{8} + \frac{8}{3} - \frac{2}{9}\cdot 8 - \frac{2}{3}\cdot\frac{1}{4} + \frac{2}{9}\cdot\frac{1}{8}.$$

$= \frac{1}{12} + \frac{8}{3} - \frac{16}{9} - \frac{1}{6} + \frac{1}{36}$

$= \frac{3 + 8\times12 - 64 - 6 + 1}{36} = \frac{96 - 70 + 4 = 100 - 70}{36}$

$= \frac{5}{6}.$

$E(x^2) = \int_0^{0.5} 2x^3\,dx + \int_{0.5}^{2} \frac{4}{3}x^2 - \frac{2}{3}x^3.$

$= \left[\frac{1}{2}x^4\right]_0^{0.5} + \left[\frac{4}{9}x^3 - \frac{1}{6}x^4\right]_{0.5}^{2}$

$= \frac{1}{2}\left(\frac{1}{16}\right) + \frac{4}{9}(8) - \frac{1}{6}(16) - \frac{4}{9}\left(\frac{1}{8}\right) + \frac{1}{6}\left(\frac{1}{16}\right)$

$= \frac{1}{32} + \frac{32}{9} - \frac{8}{3} - \frac{1}{18} + \frac{1}{96}$

$= \frac{1}{32} + \frac{64 - 48 - 1}{18} + \frac{1}{96}$ $\quad \frac{64-1}{18} = \frac{63}{18} - \frac{48}{18} = \frac{15}{18} = \frac{5}{6}$

$= \frac{1}{32} + \frac{5}{6} + \frac{1}{96} = \cancel{\frac{3+160+2}{192}} = \cancel{\frac{165}{192}} =$

$= \frac{3 + 80 + 1}{96} = \frac{84}{96} = \frac{7}{8}$

5c. $E(Y-2X) = E(Y) - 2E(x) = \cancel{-E(x) = -\frac{5}{6}}\ 0$

$Var(Y-2x) = Var(Y) + 4Var(x)$

$= \frac{7}{2} - \frac{25}{18} + \left(\frac{7}{8} - \frac{25}{36}\right) \times 4$

$\cancel{= 7 - \frac{25+30}{36} = 7 - \frac{55}{36}.}$

$= \frac{7}{2} - \frac{25}{18} + \frac{7}{2} - \frac{25}{9}$

$= 7 - \frac{25+50}{18} = 7 - \frac{75}{18}\ \frac{25}{6} =$ $\quad 42 - 25 = \frac{17}{6}.$

Var(Y) = 2Var(x)

$\Rightarrow Var(Y-2X) = 6\,Var(x) = \frac{13}{12}.$

d. $F(x) = \int_0^x 2t\,dt = \left[t^2\right]_0^x = x^2.$ $\quad 0 \le x \le \frac{1}{2}.$

$F(x) = \int_{0.5}^x \frac{4}{3} - \frac{2}{3}t\,dt + \left(\frac{1}{4}\right) = \left[\frac{4}{3}t - \frac{1}{3}t^2\right]_{0.5}^x + \frac{1}{4}$

$= \frac{4}{3}x - \frac{1}{3}x^2 - \frac{2}{3} + \frac{1}{12} + \frac{1}{4}.$ $\quad \frac{-8+1+3}{12}$

$= -\frac{1}{3}x^2 + \frac{4}{3}x - \frac{1}{3}$ $\quad \frac{1}{2} \le x \le 2.$

$F(\cancel{0.5}\,m) = -\frac{1}{3}m^2 + \frac{4}{3}m - \frac{1}{3} = \frac{1}{2}$

$2m^2 - 8m + 2 + 3 = 0.$

$2m^2 - 8m + 5 = 0$ $\quad \frac{1}{2} \quad \frac{1}{5}$

$\frac{8 \pm \sqrt{64-40}}{4} = \frac{8 \pm \sqrt{24}}{4} = \frac{4 \pm \sqrt{6}}{2}$

$\frac{4-\sqrt{6}}{2} = \underline{0.775}$

6a.

$F(x) = \int_0^x e^{-t}\,dt = \left[-e^{-t}\right]_0^x = -e^{-x}$

$G(y) = P(Y \le y) = P(\lfloor X \rfloor \le y).$

$= P(X \le y + \{x\})$

$P(Y=y) = \int_y^{y+1} e^{-x}\,dx.$

$= \left[-e^{-x}\right]_y^{y+1}$

$= \left[-e^{-y+1} + e^{-y}\right]$

$= e^{-y}(1-e^{-1}).$

b. $G(t) = 1-e^{-1} + e^{-1}(1-e^{-1})t + e^{-2}(1-e^{-1})t^2 + \dots$

$= 1-e^{-1} \cdot \dfrac{1}{1-e^{-1}t}$ ✓

$\cancel{E(Y)} = G'(t) = (1-e^{-1})(1-e^{-1}t)^{-2} \cdot (+e^{-1})$

$=$

$E(Y) = G'(1) = (1-e^{-1})(e^{-1})(1-e^{-1})^{-2}$

$= \dfrac{e^{-1}}{1-e^{-1}} = \underline{0.582}$.

7a. $P(U=u) = q^{u-1}p = \left(\frac{3}{4}\right)^{u-1}\left(\frac{1}{4}\right)$.

$F(u) = P(U \le u) = \cancel{\int \left(\frac{3}{4}\right)^{x-1}\left(\frac{1}{4}\right) dx} \quad \sum_{x=1}^{u} \left(\frac{3}{4}\right)^{x-1}\left(\frac{1}{4}\right)$.

$= \dfrac{1}{4} \cdot \dfrac{1-\left(\frac{3}{4}\right)^{u}}{1-\frac{3}{4}} = 1-\left(\frac{3}{4}\right)^{u}$.

b. $P(U>20) = 1-P(U \le 20)$

$= 1 - 1 + \left(\frac{3}{4}\right)^{20} = \left(\frac{3}{4}\right)^{20}$.

c. $G_U(t) = \sum_{x=1}^{\infty} pq^{x-1}t^x$

$= \frac{1}{4}\sum_{x=1}^{\infty}\left(\frac{3}{4}\right)^{x-1}t^x = \frac{1}{4}\sum_{x=1}^{\infty}\frac{4}{3}\left(\frac{3}{4}t\right)^x$

$= \frac{1}{3}\cdot\frac{3}{4}t\,\dfrac{1}{1-\frac{3}{4}t} = \frac{1}{4}t\left(\dfrac{1}{1-\frac{3}{4}t}\right) = \dfrac{t}{4-3t}$

d. $E(V) = 3E(U) = 3G_U'(1) = 3(4) = 12$.

$$Var(V) = 3Var(U) = \frac{\frac{3}{4}}{\frac{1}{16}} \times 3 = 36.$$

$$G_v(t) = G_u^3(t) = \frac{t^3}{(4-3t)^3}$$

e. $G_w'(t) = -3(4-3t)^{-4}(-3)$

$G_w'(1) = -3(1)^{-4} = -3(-3) = 9$.

f. ~~$W = 3K$, $G_K(t) = \frac{1}{4-3t} = \frac{\frac{1}{4}}{1-\frac{3}{4}t}$.~~

$G_3(t) = 0 + 0t + 0t^2 + t^3 + \ldots$

$\cancel{= 3\left(\frac{1}{1-t}\right) = \frac{3}{1-t}}$

$= t^3$.

$$G_{W+3} = t^3\left(\frac{1}{4-3t}\right)^3$$

$$= \left(\frac{t}{4-3t}\right)^3 = G_v.$$