**FM2** Assignment #31 Name: Maggie.

$\frac{10}{10}$ Excellent!

1. Find the ~~interval of~~ values where converge convergence for the ~~power~~ geometric series $x + \dfrac{x^2}{1+x} + \dfrac{x^3}{(1+x)^2}\ldots$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{(1+x)^{n-1}}$$

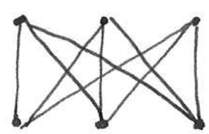
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(1+x)^n}\cdot\frac{(1+x)^{n-1}}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{1+x}\right| < 1. \Rightarrow x > -\frac{1}{2}$$

when $x = -\frac{1}{2}$, $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{(1+x)^{n-1}} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{2}$, which diverges.

i.e. interval of convergence is $]-\frac{1}{2}, \infty[$

2. Draw a simple connected non-planar graph with three or more vertices for which $e \le 3v - 6$.

$K_{3,3}$: $e = 9$; $v = 6$. $9 \le 18 - 6 = 12$.



it's non-planar.

3. Prove that a non-Abelian group cannot be isomorphic to an Abelian group.

Proof by contradiction:

Let $G$ be a non-Abelian group and $H$ an Abelian group.

let $f: G \to H$ be an isomorphism.

Since $G$ is no-Abelian, $\exists\, a, b \in G$ such that

$ab \neq ba$. and since $f$ is a bijection,

$a \mapsto f(a)$, $b \mapsto f(b)$, and $f(a), f(b) \in H$.

Since $H$ is abelian, $f(a)f(b) = f(b)f(a)$

$\Leftrightarrow f(ab) = f(ba)$. $\Leftrightarrow ab = ba$,

which is a contradiction.

Therefore, such an isomorphism does not exist. $\square$.

/6

4. The discrete random variable $X$ has the Poisson distribution with mean $\mu$.

(a) Denoting $P(X = k)$ by $p_k$, show that $p_{k+1} = \frac{\mu}{k+1} \cdot p_k$

$$p_k = \frac{e^{-\mu}\mu^k}{k!}$$

$$p_{k+1} = \frac{e^{-\mu}\mu^{k+1}}{(k+1)!} = \frac{e^{-\mu}\mu^k \cdot \mu}{k!\,(k+1)} = p_k \cdot \frac{\mu}{k+1}.$$

(b) If $\mu = 7.8$, determine the modal value of $X$.

We want to find $k$ such that $p_k$ is largest.

$p_{k+1} = \frac{7.8}{k+1} \cdot p_k$.

$p_{k+1} > p_k$ until $\frac{7.8}{k+1} = 1$.

$\Rightarrow k = 6.8 \approx 7$.

Therefore,

$p_7 = \frac{e^{-7.8}\, 7.8^7}{7!} = \underline{0.143 \text{ (3 s.f.)}}$.

5. Solve the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ given $a_0 = 2$, $a_1 = 7$. Hence find the least $n$ for which $a_n > 1\,000\,000$.

AQE:

$r^2 - r - 2 = 0$

$r_1 = -1$, $r_2 = 2$.

So $a_n = \alpha(-1)^n + \beta(2)^n$

Substituting $a_0 = 2$, ~~gives~~ $a_1 = 7$ gives

$$\begin{cases} 2 = \alpha + \beta \\ 7 = -\alpha + 2\beta \end{cases} \Rightarrow \begin{cases} \alpha = -1 \\ \beta = 3 \end{cases}$$

So $a_n = (-1)^{n+1} + 3(2)^n$.

Using technology, $a_{19} = 1.57E6 > 1 \times 10^6$ is the ~~when n =~~ least such term.

4

**FM2** Assignment #32

Name: Maggie . 12 TS Excellent!

1. There are nine men at a party. By considering an appropriate graph, explain why it is impossible for each man to shake hands with exactly five other men.

Let each vertex be a man and each edge be a hand-shake.

Then $V = 9$, and $\deg(v) = 5$.

$\sum_{v \in V} \deg(v) = 5 \times 9 = 45$. According to hand-shaking thrm,

$\sum_{v \in V} \deg(v) = 2e$, which is even, therefore, such graph doesn't exist.

2. For what values of $x$ is the series $\sum_{k=1}^{\infty} e^{kx}$ convergent? For these values of $x$, find the sum as a simple function of $x$.

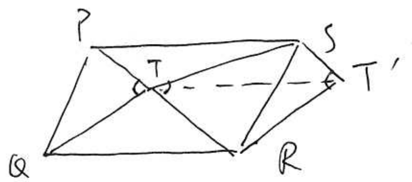
$\lim_{n \to \infty} \left| \frac{e^{(k+1)x}}{e^{kx}} \right| = \lim_{n \to \infty} \left| e^{x} \right| < 1. \Rightarrow x < 0.$

when $x = 0$ $\sum_{k=1}^{\infty} e^{k \cdot 0} = \sum_{k=1}^{\infty} 1$, which is not convergent.

$\sum_{k=1}^{\infty} e^{kx} = e^{x} + e^{2x} + e^{3x} + \dots$

$= \frac{e^{x}}{1-e^{x}}$ . $x < 0$.

3. Point $T$ lies inside parallelogram $PQRS$ so that $\angle PTQ + \angle RTS = 180°$. Show that $PT \times TR + ST \times TQ = PQ \times QR$.



Draw $ST' \parallel PT$,

$T'R \parallel TQ$.

Since $PQ \parallel SR$, $PQ = SR$.

$\Rightarrow \angle RST' = \angle QPT$.

$\angle SRT' = PQT$.

$\Rightarrow \triangle SRT' \cong \triangle PQT$.

$\Rightarrow PT \times TR + ST \times TQ$

$= ST' \times TR + ST \times T'R$.

Since $\angle PTQ + \angle RTS$

$= \angle ST'R + \angle RTS = 180°$.

So $T, S, T', R$ are con cyclic.

$\Rightarrow ST' \times TR + ST \times T'R = TT' \times SR$.

Since $ST' \parallel PT$, $ST' = PT$.

$TST'R$ is a parallelogram

$TT' = PS = QR$,

and since $SR = PQ$.

$\Rightarrow PT \times TR + ST \times TQ = TT' \times SR = PQ \times QR$

$\square$.

6

4. Is the series

$$\frac{1^1}{(101)!} + \frac{2^2}{(102)!} + \ldots + \frac{n^n}{(100+n)!} + \ldots \quad = \sum_{n=1}^{\infty} \frac{n^n}{(100+n)!}$$

convergent or divergent? Justify your answer.

$$\lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(101+n)!} \cdot \frac{(100+n)!}{n^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^n \cdot (n+1)}{(101+n) \cdot n^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{101+n} \left(1+\frac{1}{n}\right)^n \right|$$

$$= e > 1 \Rightarrow \text{divergent}.$$

✓

5. Let $S = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q},\ a^2 + b^2 \neq 0\}$. Prove that $(S, \times)$ is a group. Is $(S, \times)$ a group if $a, b \in \mathbb{R},\ a^2 + b^2 \neq 0$?

?

Proof:

Let $a+b\sqrt{2},\ c+d\sqrt{2} \in S$.

① $(a+b\sqrt{2})(c+d\sqrt{2})$

$= (ac+2bd) + (ad+bc)\sqrt{2} \in S$,

So it is closed.

② since $(a+b\sqrt{2})(1+0\sqrt{2})$

$= a+b\sqrt{2}$,

So $1+0\sqrt{2} \in S$ is the identity.

③ We want to find

$(a+b\sqrt{2})(c+d\sqrt{2}) = 1$

$$\Rightarrow \begin{cases} ac+2bd = 1 \\ ad+bc = 0 \end{cases}$$

$\Rightarrow d = -\frac{bc}{a}$.

So $ac + 2\left(-\frac{b^2c}{a}\right) = 1$.

$\Rightarrow a^2c - 2b^2c = a$.

$\Rightarrow c = \frac{a}{a^2-2b^2}$. $d = \frac{b}{-a^2+2b^2}$

since $a, b \in \mathbb{Q}$, $a^2+b^2 \neq 0$.

such $c, d \in \mathbb{Q}$ exist $\forall\ a, b \in S$.

Therefore, $(S, \times)$ is a group.

If $a, b \in \mathbb{R}$,

$\frac{a}{a^2-2b^2}$, $\frac{b}{-a^2+2b^2} \in \mathbb{R}$,

$(ac+2bd)$, $(ad+bc) \in \mathbb{R}$

However, when $a^2=2b^2$

$a = \pm\sqrt{2}b$, ✓ inverse does not exist,

so it's not a group.

what if $a=\sqrt{2}$, $b=1$? Doesn't this have an inverse?

4

④ Associativity?

You should mention number multiplication is associative.

More than this, only one real number has no inverse under $\times$.

FM2 Assignment #33

Name: Maggie.

$\frac{10}{10}$ Excellent!

1. The heights in centimetres of a random sample of five Pearson students were 158, 184, 177, 166, 170. Calculate unbiased estimates of the mean and variance of the population of heights of all Pearson students.

$$\mu = \bar{x} = E(x) = 171$$

$$\sigma^2 = S_{n-1}^2 = \sum_{i=1}^{5} \frac{(x_i - \bar{x})^2}{n-1} = 100.$$

2. Consider the Mersenne number $M_n = 2^n - 1$. Prove that if $M_n$ is prime then $n$ is prime. Is the converse also true?

$(\Rightarrow)$ proof by contrapositive:

If $n = km$. $km \in \mathbb{Z}^+$, $k, m \neq 1$.

then $2^{km} - 1 = M_n$

$\Rightarrow M_n = (2^k - 1)(2^{k(m-1)} + 2^{k(m-2)} + \cdots + 2 + 1)$

Since $k, m > 1$, $2^k - 1 \geq 3$.

$2^{k(m-1)} + \cdots + 2 + 1 \geq 7$, so $M_n$ is not prime. //

$(\Leftarrow)$ converse not true.

A counterexample is when $n = 11$,

$2^n - 1 = 2^{11} - 1 = 2047$

$= 23 \times 89$.

3. Currently, the largest known prime number is the Mersenne prime $2^{82\,589\,933} - 1$.

(a) How many digits does this number have in base 16?

$2^{82589933} - 1$

$= (2-1)(\sum_{i=1}^{82589932} 2^i)$

$= (1+2+4+8)(\sum_{i=0}^{20647482} 16^i) + 16^{20647483}$

$= 15 \sum_{i=0}^{20647482} 16^i + 16^{20647483}$

Therefore, the will be

20647484 digits.

(b) How many digits does this number have in base 10?

let $2^{82589933} - 1 = N$.

from (a) we know

$1 + \lfloor \log_{16} N \rfloor = 20647484$.

so, $20647483 \leq \log_{16} N < 20647484$.

$\Leftrightarrow 24862047 \leq \log_{10} N < 24862048$

therefore,

$\lfloor \log_{10} N \rfloor + 1 = \underline{24862048}$,

which is the number of digits of $N$.

6

4. Pippin chocolates are packed in boxes of 25. The weight in grams of a Pippen chocolate is distributed $N(10, 2^2)$.

(a) What is the probability that the contents of a box of Pippin chocolates weighs more than 245 grams?

Let $X$ be the weight of a Pippen chocolate. $X \sim N(10, 2^2)$.

Then a box of chocolate $Y = \sum_{i=1}^{25} X_i$, $Y \sim N(250, 25 \times 2^2)$

$P(Y > 245) = \text{normalcdf}(245, \infty, 250, 10)$

$\approx 0.691$ (3 s.f.)

(b) What is the probability that the mean weight of the chocolates in a box is between 9.9 and 10.1 grams?

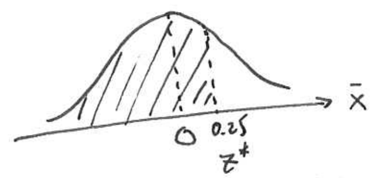
$\bar{x} + z^* \frac{\sigma}{\sqrt{n}} = 10.1$

$10 + z^* \frac{2}{\sqrt{25}} = 10.1$

$\Rightarrow \quad z^* = 0.25$

$P(\bar{x} < 0.25) = \text{normalcdf}(-\infty, 0.25, 0, 1)$

$= 0.599$ (3 s.f.).




$\Rightarrow P(9.9 < \bar{x} < 10.1)$

$= 0.599 \times 2 - 1$

$= 0.197$ (3 s.f.)

5. Consider the permutations (1 2) and (1 2 3) in the symmetric group $S_3$.

(a) Let $H = \langle (1\ 2) \rangle$.

i. Determine the left cosets of $H$ in $S_3$ giving your answers in cycle notation.

$H = \{e, (1\ 2)\}$

$eH = \{e, (1\ 2)\}$

$(1\ 2)H = \{(1\ 2), e\}$

$(1\ 3)H = \{(1\ 3), (1\ 2\ 3)\}$

$(2\ 3)H = \{(2\ 3), (2\ 1\ 3)\}$

$(1\ 2\ 3)H = \{(1\ 2\ 3), (1\ 3)\}$

$(1\ 3\ 2)H = \{(1\ 3\ 2), (2\ 3)\}$.

ii. Determine the right cosets of $H$ in $S_3$ giving your answers in cycle notation.

$He = \{e, (1\ 2)\}$

$H(1\ 2) = \{(1\ 2), e\}$

$H(1\ 3) = \{(1\ 3), (1\ 3\ 2)\}$

$H(2\ 3) = \{(2\ 3), (3\ 1\ 2)\}$.

$H(1\ 2\ 3) = \{(1\ 2\ 3), (2\ 3)\}$

$H(1\ 3\ 2) = \{(1\ 3\ 2), (1\ 3)\}$.

(b) Describe the group $\langle (1\ 2), (1\ 2\ 3) \rangle$.

$|\langle (1\ 2) \rangle| = 2$

$|\langle (1\ 2\ 3) \rangle| = 3$.

therefore $2 \mid |\langle (1\ 2), (1\ 2\ 3) \rangle|$

$3 \mid |\langle (1\ 2), (1\ 2\ 3) \rangle|$

$\Rightarrow 6 \mid |\langle (1\ 2), (1\ 2\ 3) \rangle|$.

and since $S_3 = 6$,

$|\langle (1\ 2), (1\ 2\ 3) \rangle| = 6$

$\Rightarrow \langle (1\ 2), (1\ 2\ 3) \rangle = S_3$.

i.e. $(1\ 2), (1\ 2\ 3)$ generates the symmetric group $S_3$.

4

FM2 Assignment #34 Name: Maggie. $\frac{9\frac{1}{2}}{10}$ Very good.

1. The weights of Pearson students are distributed normally. The weights in kilograms of a random sample of five Pearson students are 53, 72, 65, 58 and 61. Find a 95% confidence interval for the mean weight of all Pearson students.

$\bar{x} = 61.8$
$S = 7.19$
$n = 5$,
$df = 4$.

by calculating
$$\bar{x} \pm t^* \frac{S}{\sqrt{n}},$$
where $t^*$ is the critical value of the t-distribution on 4 degrees of freedom. at 95%. level of confident

$\Rightarrow [52.9, 70.7]$.

2. Prove that every simple planar graph contains a vertex whose degree is at most five.

proof by contradiction:

if $\deg(v) \geq 6 \quad \forall v \in V$, then by handshaking's thrm,
$$2e = \sum_{v \in V} \deg(v) \geq 6|V|.$$
$\Rightarrow e \geq 3|V|$.

However, since for a planar graph,
$$e \leq 3|V| - 6, < 3|V|,$$
the graph cannot be planar, which is a contradiction. $\square$.

$1\frac{1}{2}$

Also connected and $|V| \geq 3$. This should be considered in your proof.

3. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$ given $y(1) = 2$.

Let $y = vx$. then
$$\frac{dy}{dx} = \frac{x}{2y} + \frac{y}{2x}$$
$$x\frac{dv}{dx} = \frac{1}{2v} + \frac{v}{2} - v.$$
$$\int \frac{2v}{1-v^2}\,dv = \int \frac{1}{x}\,dx.$$
Let $u = 1 - v^2$,
$du = -2dv$,

So $-\int \frac{1}{u}\,du = \int \frac{1}{x}\,dx$
$$-\ln(1-v^2) = \ln x + C_1$$
$$\left[1 - \left(\frac{y}{x}\right)^2\right]^{-1} = C_2 x, \quad C_2 = e^{C_1}$$
$$\Rightarrow 1 - \frac{y^2}{x^2} = \frac{1}{C_2 x}$$
$$\Rightarrow y^2 = x^2 + Cx, \quad C = -\frac{1}{C_2}.$$
$$\Rightarrow y = \sqrt{x^2 + Cx}$$
since $y(1) = 2$.
$$2 = \sqrt{1 + C} \Rightarrow C = 3$$
So $y = \sqrt{x^2 + 3x}$.

$5\frac{1}{2}$

4. Let $G$ be a simple graph. Prove that $G$ and its complement $\bar{G}$ cannot both be disconnected.

Proof:

If $G$ is disconnected, we want to show that $\bar{G}$ is connected, ie. there exists a path btw any $v_1, v_2 \in V_{\bar{G}}$.

Case 1: $v_1, v_2$ are adjacent in $G$.

Since $v_1, v_2$ are connected, they must be in the same component of $G$.

Consider $v_3$ in another compotent of $G$, $v_3$ is not adjacent to $v_1$ or $v_2$.

So in $\bar{G}$, $v_3$ is adjacent to both $v_1$ and $v_2$. ✓

Hence there exists the path $v_1 v_3 v_2$ in $\bar{G}$ from $v_1$ to $v_2$, so $v_1$ and $v_2$ are connected in $\bar{G}$;

Case 2 $v_1, v_2$ are not adjacent in $G$.

So in $\bar{G}$, $v_1$ and $v_2$ have to be adjacent, and thus connected;

in either case, $v_1, v_2$ are connected in $\bar{G}$, $\forall v_1, v_2 \in V_{\bar{G}}$.

Since $\bar{\bar{G}} = G$, if one of $\bar{G}$ and $G$ is disconnected, the other one has to be connected, So they cannot be both disconnected. $\square$.

5. Suppose $\phi: \mathbb{Z}_{50} \to \mathbb{Z}_{15}$ is a group homomorphism with $\phi(7) = 6$.

(a) Write down $\phi(21)$.

$$\phi(21) = \phi(7+7+7) = 3\phi(7) = 18 = \underline{3}$$ ✓

(b) Find $\phi(1)$.

$$\phi(49) = 7\phi(7) = 42 = \underline{12}$$

$$\phi(1) = \phi(49^{-1}) = 12^{-1} = \underline{3}$$ ✓

(c) Find ran($\phi$).

Since $\phi(x) = x\phi(1)$, $x \in \mathbb{Z}_{50}$,

$\phi(x) = 3x \mod 15$

$\Rightarrow \operatorname{ran}(\phi) = \{0, 3, 6, 9, 12\}$ ✓

(d) Find ker($\phi$).

$\ker(\phi) = \{x \mid \phi(x) = 0, x \in \mathbb{Z}_{50}\}$.

$\Rightarrow 3x \equiv 0 \mod 15$

$x \equiv 0 \mod 5$ $\Rightarrow \ker(\phi) = \{0, 5, 10, \cdots, 45\}$.

(e) Solve the equation $\phi(x) = 12$, $x \in \mathbb{Z}_{50}$.

$3x \equiv 12 \mod 15$.

$x \equiv 4 \mod 5$ ✓

$x = 4, 9, 14, 19, 24, 29, 34, 39, 44$, or $49$.

4