FM2 Assignment #12 Name: Maggie. $\frac{10}{10}$ Excellent!

1. What value should be assigned to k to make the function $f(x) = \begin{cases} x^2 - 1, & x < 3, \\ 2kx, & x \geq 3, \end{cases}$ continuous at $x = 3$.

For $f(x)$ to be conti. at $x=3$

$\lim_{x \to 3} f(x) = f(3)$

which means

$\lim_{x^+ \to 3} f(x) = \lim_{x^- \to 3} f(x)$

then $\lim_{x \to 3} (x^2-1) = \lim_{x \to 3} (2kx)$.

$3^2 - 1 = 2k(3)$

$8 = 6k$

$\boxed{k = \frac{4}{3}}$.

2. Construct a function that is continuous on $\mathbb{R}$ but fails to be differentiable at the four numbers $0, 1, 2, 3$.

$$f(x) = \begin{cases} x+1 & x < 0 \\ \big|\,\big||x-2|-2\big|-1\big|. & 0 \leq x \leq 3. \\ x-3 & x > 3. \end{cases}$$

3. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f'(x) > 0$ for all $x \in \mathbb{R}$. Prove that if $a < b$ then $f(a) < f(b)$.

proof:

since f is differentiable on $\mathbb{R}$,
it is also conti on $\mathbb{R}$.
then MVT applies.
If $a < b$.
the $\exists\, c \in]a,b[$. s.t.

$f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

and since $f'(x) > 0.\ \forall x \in \mathbb{R}$,
$f'(c) > 0$

thus

$\dfrac{f(b)-f(a)}{b-a} > 0$.

and since $b - a > 0$.

$f(b) - f(a) > 0$

$f(b) > f(a)$.

$\square$.

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4. The third degree Taylor polynomial of $\ln x$ about $x = 1$ is $a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3$. Find the values of a_0, a_1, a_2 and a_3 and hence estimate $\ln 1.2$.

$$P_3(x) = \frac{f^{(0)}(1)}{0!} + \frac{f^{(1)}(1)}{1!}(x-1) + \cdots + \frac{f^{(3)}(1)}{3!}(x-1)^3$$

$$= \ln(1) + \frac{1}{1}(x-1) + \frac{(-1)(1)^{-2}}{2!}(x-1)^2 + \frac{(2)(1)^{-3}}{3!}(x-1)^3$$

$$= 0 + x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

Thus, $a_0 = 0$, $a_1 = 1$, $a_2 = -\frac{1}{2}$, $a_3 = \frac{1}{3}$.

$$\ln(1.2) \approx P_3(1.2)$$

$$= 1.2 - 1 - \frac{1}{2}(1.2-1)^2 + \frac{1}{3}(1.2-1)^3$$

$$= 0.2 - 0.5(0.2)^2 + \frac{1}{3}(0.2)^3$$

$$\approx 0.183 \text{ (3 s.f.)}$$

5. In the trapezium $ABCD$, the midpoints of the parallel sides $[AB]$ and $[CD]$ are M and N respectively. The sides $[BC]$ and $[AD]$ are not parallel. Show that the diagonals and the line segment $[MN]$ are concurrent.

Lemma:

Prolong CA and DB, let them intersect at point P. then P, M, N are collinear.

proof:

Suppose not, ~~i.e.~~ then connect and prolong PM, let PM intersect CD at N'.

since $AB \parallel CD$.

$\angle PAM = \angle PCN'$

$\angle APM = \angle DPN'$

So $\triangle APM \sim \triangle CPN'$

similarly, $\triangle BPM \sim \triangle DPN'$.

So $\frac{AM}{CN'} = \frac{PM}{PN'} = \frac{BM}{DN'}$

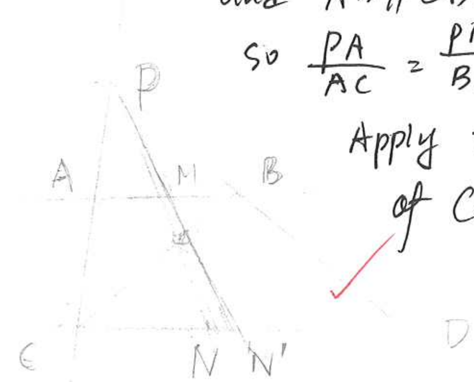
Since M is the midpoint of AB, $\frac{AM}{BM} = 1$.

then $\frac{CN'}{DN'} = \frac{AM}{BM} = 1$.

i.e. $CN' = DN'$, N' is the midpoint of CD, which means N and N' coincide.

Since P, M, N' are collinear, P, M, N are collinear. $\square$.

proof: Since P, M, N are collinear and $AB \parallel CD$, So $\frac{PA}{AC} = \frac{PB}{BD}$.



Apply the inverse of Ceva's thrm,

Since $\frac{PA}{AC} \cdot \frac{CN}{ND} \cdot \frac{DB}{BP}$

$= \frac{\cancel{PB}}{\cancel{BD}} \cdot \frac{CN}{ND} \cdot \frac{\cancel{DB}}{\cancel{BP}} = 1$

PN, CB, DA are concurrent. $\square$.

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