1. The relation R is defined on $\mathbb{Z}$ by $x \; R \; y$ if 5 divides $x + y$. Prove that R is not an equivalence relation.

R is not reflexive, since for example $1 \not R 1$. So R cannot be an equivalence relation //

2. Let S be the set of positive irrational numbers together with the number 1. Does $(S, \times)$ form a group?

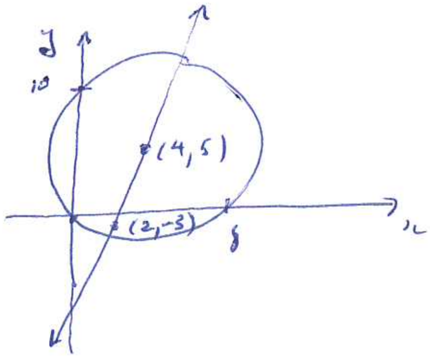
No. S is not closed under $\times$, since for example $\sqrt{2} \times \sqrt{2} = 2 \notin S$. //

3. Use the Maclaurin series for e^{-x} and $\sin 2x$ to evaluate the limit $\lim\limits_{x \to 0} \dfrac{1 - e^{-x}}{\sin 2x}$.

$$\frac{1-e^{-x}}{\sin 2x} = \frac{1-(1-x+O(x^2))}{2x + O(x^3)} = \frac{1+O(x)}{2+O(x^2)}$$

So $\lim\limits_{x \to 0} \dfrac{1-e^{-x}}{\sin 2x} = \dfrac{1}{2}$ //

4. A circle intersects the axes at $(0, 10)$, $(0, 0)$ and $(8, 0)$. A line through $(2, -3)$ cuts the circle in half. Find the y-intercept of the line.



$y - 5 = \frac{8}{2}(x - 4)$

So y-intercept is $(0, -11)$//

5. State the mean value theorem. If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be?

See notes.

$f(4) - f(1) \geqslant 2(4 - 1) \iff f(4) \geqslant 16$ //

6. Let $y = f(x)$ be the particular solution to the differential equation $y' = x^2 + y^2$ for which $f(1) = 2$. Use Euler's method starting at $x = 1$ with a step size of 0.1 to approximate $f(1.2)$. Set out your work in a table.

n	x_n	y_n	h	$h \times f(x_n, y_n)$
0	1	2	0.1	0.5
1	1.1	2.5	0.1	0.746
2	1.2	3.246		

So $f(1) \approx 3.246 = 3.25$ (3 s.f)//

7. Prove that a simple graph with more than one vertex contains two vertices of the same degree.

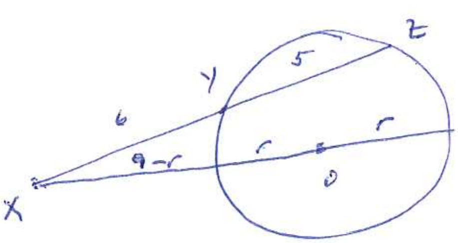
Suppose our graph G has $n > 1$ vertices and the degrees of the vertices are all different. Since G is simple the degrees must be $0, 1, 2, \ldots, n-1$. But this is a contradiction as the vertex of degree $n-1$ must be adjacent to all other vertices making impossible to have a vertex of degree 0. This contradiction proves the result //

8. Find a basis for the null space of the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 \end{pmatrix}$.

$$A \sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$\text{So } \operatorname{null}(A) = \left\langle \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\rangle \quad //$$

9. The circle $\mathscr{C}$ has centre O, the point X lies outside $\mathscr{C}$, the point Z lies on $\mathscr{C}$ and the secant $[XZ]$ cuts $\mathscr{C}$ at Y. If $XY = 6$, $YZ = 5$ and $XO = 9$, find the area of the circle.



By secant-secant theorem

$$6 \cdot 11 = (9-r)(9+r) = 81 - r^2$$

Hence $r^2 = 15$, and $A = 15\pi$ //

10. The function $f : \mathbb{Z}_{91} \to \mathbb{Z}_7 \times \mathbb{Z}_{13}$ with rule $f(x) = (x \text{ mod } 7, x \text{ mod } 13)$ is a bijection. Find $f^{-1}(1, 4)$.

$$\begin{cases} x \text{ mod } 7 = 1 & -① \\ x \text{ mod } 13 = 4 & -② \end{cases}$$

So $x \in \{4, 17, 30, 43, 56, 69, 81\}$ from ②.

Using ①, we conclude $x = 43$. Hence $f^{-1}(1,4) = 43$ //

11. Prove that the order of a non-Abelian group cannot be prime.

We prove the logically equivalent contrapositive, namely if the order of a group is prime then the group is cyclic.

Proof:

If the order of a group is prime then the group is cyclic by a corollary to Lagrange's theorem, but cyclic groups are Abelian. Hence the result //

12. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove that $\ker T$ is a subspace of $\mathbb{R}^n$.

See notes. //

13. Let $f : G \to G'$ be a homomorphism of groups whose respective identity elements are e and e'. Prove

(a) $f(e) = e'$;

See notes. //

(b) $f(a^{-1}) = [f(a)]^{-1}$.

See notes. //

14. Find the general solution of the differential equation $\frac{dy}{dx} + y \cot x = x$, $0 < x < \pi$. Give your answer in the form $y = f(x)$.

We spot I.F. $= \sin x$. So

$$(y \sin x)' = x \sin x$$

Integrating gives

$$y \sin x = \int x \sin x \, dx = -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + A$$

Hence $y = -x \cot x + 1 + A \csc x$ //

15. Prove that the intersection of two subgroups of a group is also a subgroup of that group.

Let $H, K \leq G$. We wish to prove $H \cap K \leq G$.

We use the 3-step subgroup test.

1. Suppose $x, y \in H \cap K$. Then $x, y \in H$ and $x, y \in K$.
   Hence $xy \in H$ and $xy \in K$. So $xy \in H \cap K$.

2. Since $e \in H$ and $e \in K$, we have $e \in H \cap K$

3. Suppose $x \in H \cap K$. Then $x \in H$ and $x \in K$. So
   $x^{-1} \in H$ and $x^{-1} \in K$. Hence $x^{-1} \in H \cap K$.

We conclude $H \cap K \leq G$ by the 3-step subgroup test. //

16. Determine the interval of convergence for the power series $1 + \frac{x+2}{3 \times 1} + \frac{(x+2)^2}{3^2 \times 2} + \frac{(x+2)^3}{3^3 \times 3} + \ldots$.

$\left|\frac{u_{n+1}}{u_n}\right| = \frac{n}{3(n+1)} \cdot |x+2| \to \frac{1}{3}|x+2|$ as $n \to \infty$.

So $R = 3$ with centre $x = -2$.

When $x = 1$, the series is harmonic and hence diverges.

When $x = -5$, the series is alternating harmonic and hence converges.

Therefore the interval of convergence is $[-5, 1[$ //

17. Find the equation of the line containing the major axis of the ellipse $2x^2 - 4xy + 5y^2 = 6$.

The matrix form is $(x\ y)\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 6$, or

$(x\ y)\frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 6$. Letting $\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$,

gives $(x')^2 + 6(y')^2 = 6$. So we must rotate the original ellipse through θ clockwise about the origin where $\tan\theta = \frac{1}{2}$ to align the major axis with the x-axis. Hence the required line is $y = \frac{1}{2}x$ //

18. Solve the differential equation $x^2\frac{dy}{dx} = y^2 + xy + 4x^2$ given $y = 2$ when $x = 1$. Give your answer in the form $y = f(x)$.

Dividing by x^2 gives $y' = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) + 4$. Letting $y = vx$ gives $v + v'x = v^2 + v + 4$. So

$$\int \frac{1}{4+v^2}\,dv = \int \frac{1}{x}\,dx, \text{ which gives}$$

$$\frac{1}{2}\arctan\frac{v}{2} = \ln|x| + c, \text{ or}$$

$$\arctan\frac{y}{2x} = \ln x^2 + 2c.$$

Substituting $x = 1$, $y = 2$ gives $2c = \frac{\pi}{4}$.

Hence $y = 2x\tan\left(\ln x^2 + \frac{\pi}{4}\right)$ //

19. The random variable X has probability generating function $G(t) = \dfrac{t}{2-t}$, mean μ and variance σ^2. Find $P(|X-\mu| < \sigma)$.

$$G'(t) = \frac{1\cdot(2-t) + t}{(2-t)^2} = \frac{2}{(2-t)^2}$$

$$G''(t) = \frac{4}{(2-t)^3}$$

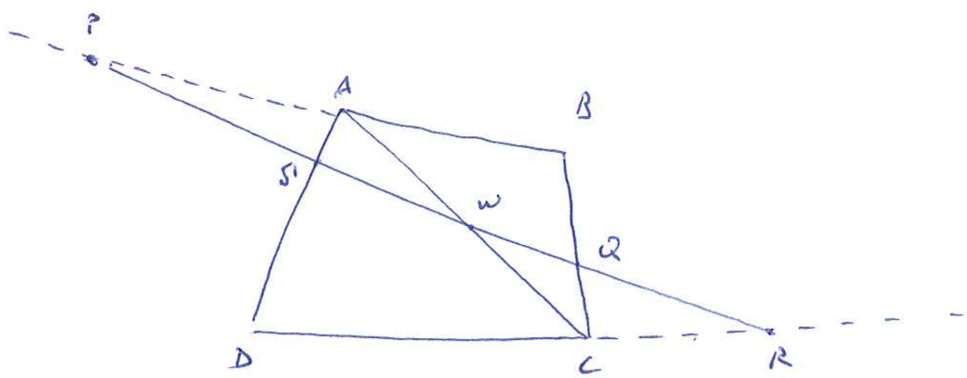
So $\mu = G'(1) = 2$ and $\sigma^2 = G''(1) + G'(1) - \mu^2 = 4 + 2 - 4 = 2.$

Now $G(t) = \dfrac{t/2}{1-t/2} = \frac{1}{2}t + \frac{1}{4}t^2 + \frac{1}{8}t^3 + \frac{1}{16}t^4 + \dots$

So $P(|x-\mu| < \sigma) = P(|x-2| < \sqrt{2}) = \frac{7}{8}$ //

20. The sides $[AB]$, $[BC]$, $[CD]$, $[AD]$ of quadrilateral $ABCD$ (produced if necessary) are cut by a transversal in the points P, Q, R and S, respectively. Prove that

$$\frac{AP}{PB} \times \frac{BQ}{QC} \times \frac{CR}{RD} \times \frac{DS}{SA} = 1.$$



Construct $[AC]$. By Menelaus's theorem in ΔADC and ΔCBA, we have

$$\frac{AS}{SD} \times \frac{DR}{RC} \times \frac{CW}{WA} = \frac{CQ}{QB} \times \frac{BP}{PA} \times \frac{AW}{WC} = 1$$

$$\Leftrightarrow \quad \frac{PB}{AP} \times \frac{QC}{BQ} \times \frac{RD}{CR} \times \frac{SA}{DS} = 1,$$ which proves the result //