Chapter 8

Induction

Consider the following three statements, each involving a general positive integer n:

(1) The sum of the first n odd numbers is equal to n^2.

(2) If $p > -1$ then $(1+p)^n \geq 1+np$.

(3) The sum of the internal angles in an n-sided polygon is $(n-2)\pi$.

[A *polygon* is a closed figure with straight edges, such as a triangle (3 sides), a quadrilateral (4 sides), a pentagon (5 sides), etc.]

We can check that these statements are true for various specific values of n. For instance, (1) is true for $n=2$ as $1+3=4=2^2$, and for $n=3$ as $1+3+5=9=3^2$; statement (2) is true for $n=1$ as $1+p \geq 1+p$, and for $n=2$ as $(1+p)^2 = 1+2p+p^2 \geq 1+2p$; and (3) is true for $n=3$ as the sum of the angles in a triangle is π, and for $n=4$ as the sum of the angles in a quadrilateral is 2π.

But how do we go about trying to prove the truth of these statements for *all* values of n?

The answer is that we use the following basic principle. In it we denote by $P(n)$ a statement involving a positive integer n; for example, $P(n)$ could be any of statements (1), (2) or (3) above.

Principle of Mathematical Induction

Suppose that for each positive integer n we have a statement $P(n)$. If we prove the following two things:

(a) *$P(1)$ is true;*

(b) *for all n, if $P(n)$ is true then $P(n+1)$ is also true;*

then $P(n)$ is true for all positive integers n.

The logic behind this principle is clear: by (a), the first statement $P(1)$ is true. By (b) with $n=1$, we know that $P(1) \Rightarrow P(2)$, hence $P(2)$ is true. By (b) with $n=2$, $P(2) \Rightarrow P(3)$, hence $P(3)$ is true; and so on.

should clarify matters.

Example 8.1

Let us try to prove statement (1) above using the Principle of Mathematical Induction. Here $P(n)$ is the statement that the sum of the first n odd numbers is n^2. In other words:

$$P(n) : 1+3+5+\cdots+2n-1 = n^2 .$$

We need to carry out parts (a) and (b) of the principle.

(a) $P(1)$ is true, since $1 = 1^2$.

(b) Suppose $P(n)$ is true. Then

$$1+3+5+\cdots+2n-1 = n^2 .$$

Adding $2n+1$ to both sides gives

$$1+3+5+\cdots+2n-1+2n+1 = n^2+2n+1 = (n+1)^2 ,$$

which is statement $P(n+1)$. Thus, we have shown that $P(n) \Rightarrow P(n+1)$.

We have now established parts (a) and (b). Hence by the Principle of Mathematical Induction, $P(n)$ is true for all positive integers n.

The phrase "Principle of Mathematical Induction" is quite a mouthful, and we usually use just the single word "induction" instead.

Example 8.2

Now let us prove statement (2) above by induction. Here, for n a positive integer $P(n)$ is the statement

$$P(n) : \text{ if } p > -1 \text{ then } (1+p)^n \geq 1+np .$$

For (a), observe $P(1)$ is true, as $1+p \geq 1+p$.

For (b), suppose $P(n)$ is true, so $(1+p)^n \geq 1+np$. Since $p > -1$, we know that $1+p > 0$, so we can multiply both sides of the inequality by $1+p$ (see Example 4.3) to obtain

$$(1+p)^{n+1} \geq (1+np)(1+p) = 1+(n+1)p+np^2 .$$

Since $np^2 \geq 0$, this implies that $(1+p)^{n+1} \geq 1+(n+1)p$, which is statement $P(n+1)$. Thus we have shown $P(n) \Rightarrow P(n+1)$.

Therefore, by induction, $P(n)$ is true for all positive integers n.

Next we attempt to prove the statement (3) concerning n-sided polygons. There is a slight problem here. If we naturally enough let $P(n)$ be statement (3), then $P(n)$ makes sense only if $n \geq 3$; $P(1)$ and $P(2)$ make no sense, as there is no such thing as a 1-sided or 2-sided polygon. To take care of such a situation, we need a slightly modified Principle of Mathematical Induction:

Principle of Mathematical Induction II

Let k be an integer. Suppose that for each integer $n \geq k$ we have a statement $P(n)$. If we prove the following two things:

(a) $P(k)$ is true;

(b) for all $n \geq k$, if $P(n)$ is true then $P(n+1)$ is also true;

then $P(n)$ is true for all integers $n \geq k$.

The logic behind this is the same as explained before.

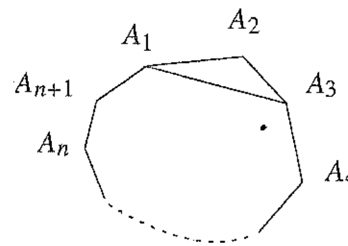
Example 8.3

Now we prove statement (3). Here we have $k = 3$ in the above principle, and for $n \geq 3$, $P(n)$ is the statement

$$P(n) : \text{the sum of the internal angles in an } n\text{-sided polygon is } (n-2)\pi.$$

For (a), observe that $P(3)$ is true, since the sum of the angles in a triangle is $\pi = (3-2)\pi$.

Now for (b). Suppose $P(n)$ is true. Consider an $(n+1)$-sided polygon with corners $A_1, A_2, \ldots, A_{n+1}$:



Draw the line A_1A_3. Then $A_1A_3A_4\ldots A_{n+1}$ is an n-sided polygon. Since we are assuming $P(n)$ is true, the internal angles in this n-sided polygon add up to $(n-2)\pi$. From the picture we see that the sum of the angles in the $(n+1)$-sided polygon $A_1A_2\ldots A_{n+1}$ is equal to the sum of those in $A_1A_3A_4\ldots A_{n+1}$ plus the sum of those in the triangle $A_1A_2A_3$, hence is

$$(n-2)\pi+\pi = ((n+1)-2)\pi .$$

We have now shown that $P(n) \Rightarrow P(n+1)$. Hence, by induction, $P(n)$ is true for all $n \geq 3$.

The next example also uses the slightly modified Principle of Mathematical Induction II. In it, for a positive integer n we define

$$n! = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1\,,$$

the product of all the integers between 1 and n. The symbol $n!$ is usually referred to as *n factorial*. By convention, we also define $0! = 1$.

Example 8.4
For which positive integers n is $2^n < n!$?

Answer Let $P(n)$ be the statement that $2^n < n!$. Observe that

$$2^1 > 1!,\ 2^2 > 2!,\ 2^3 > 3!,\ 2^4 < 4!,\ 2^5 < 5!\,,$$

so $P(1), P(2), P(3)$ are false, while $P(4), P(5)$ are true. Therefore, it seems sensible to try to prove $P(n)$ is true for all $n \geq 4$.

First, $P(4)$ is true, as observed above.

Now suppose n is an integer with $n \geq 4$, and $P(n)$ is true. Thus

$$2^n < n!$$

Multiplying both sides by 2, we get

$$2^{n+1} < 2(n!)\,.$$

Since $2 < n+1$, we have $2(n!) < (n+1)n! = (n+1)!$ and hence $2^{n+1} < (n+1)!$. This shows that $P(n) \Rightarrow P(n+1)$. Therefore, by induction, $P(n)$ is true for all $n \geq 4$.

Guessing the Answer

Some problems cannot immediately be tackled using induction, but first require some intelligent guesswork. Here is an example.

Example 8.5
Find a formula for the sum

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}\,.$$

Answer Calculate this sum for the first few values of n:

$$n=1:\ \frac{1}{1\cdot 2} = \frac{1}{2}\,,$$
$$n=2:\ \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}\,,$$
$$n=3:\ \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{3}{4}\,.$$

We intelligently spot a pattern in these answers and guess that the sum of n terms is probably $\frac{n}{n+1}$. Hence we let $P(n)$ be the statement

$$P(n):\ \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

and attempt to prove $P(n)$ true for all $n \geq 1$ by induction.

First, $P(1)$ is true, as noted above.

Now assume $P(n)$ is true, so

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}\,.$$

Adding $\frac{1}{(n+1)(n+2)}$ to both sides gives

$$\frac{1}{1\cdot 2} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$
$$= \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}\,.$$

Hence $P(n) \Rightarrow P(n+1)$. So, by induction, $P(n)$ is true for all $n \geq 1$.

The Σ Notation

Before proceeding with the next example, we introduce an important notation for writing down sums of many terms. If $f_1, f_2, \ldots, f_n$ are numbers, we abbreviate the sum of all of them by

$$f_1 + f_2 + \cdots + f_n = \sum_{r=1}^{n} f_r\,.$$

(The symbol Σ is the Greek capital letter "sigma," so this is often called the "sigma notation.") For example, setting $f_r = \frac{1}{r(r+1)}$, we have

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \sum_{r=1}^{n} \frac{1}{r(r+1)}\,.$$

Thus, Example 8.5 says that

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1},$$

and Example 8.1 says

$$\sum_{r=1}^{n} (2r-1) = n^2.$$

Notice that if a, b, c are constants, then

$$\sum_{r=1}^{n} (af_r + bg_r + c) = a\sum_{r=1}^{n} f_r + b\sum_{r=1}^{n} g_r + cn, \tag{8.1}$$

since the left-hand side is equal to

$$\begin{aligned}&(af_1 + bg_1 + c) + \cdots + (af_n + bg_n + c)\\ &= a(f_1 + \cdots + f_n) + b(g_1 + \cdots + g_n) + (c + \cdots + c),\end{aligned}$$

which is the right-hand side.

The equation (8.1) is quite useful for manipulating sums. Here is an elementary example using it.

Example 8.6

Find a formula for $\sum_{r=1}^{n} r \ (= 1 + 2 + \cdots + n)$.

Answer Write $s_n = \sum_{r=1}^{n} r$. By Example 8.1, $\sum_{r=1}^{n} (2r-1) = n^2$, so using (8.1),

$$n^2 = \sum_{r=1}^{n} (2r-1) = 2\sum_{r=1}^{n} r - n = 2s_n - n.$$

Hence, $s_n = \frac{1}{2}n(n+1)$.

So we know the sum of the first n positive integers. What about the sum of the first n squares?

Example 8.7

Find a formula for $\sum_{r=1}^{n} r^2 \ (= 1^2 + 2^2 + \cdots + n^2)$.

Answer We first try to guess the answer (intelligently). The first few values $n = 1, 2, 3, 4$ give sums 1,5,14,30. It is not easy to guess a formula from these values, so yet a smidgeon more intelligence is required. The sum we are trying to find is the sum of n terms of a quadratic nature, so it seems reasonable to look for a formula for the sum which is a cubic in n, say $an^3 + bn^2 + cn + d$.

What should a, b, c, d be? Well, they have to fit in with the values of the sum for $n = 1, 2, 3, 4$ and hence must satisfy the following equations:

$$n = 1 : 1 = a + b + c + d \tag{8.2}$$
$$n = 2 : 5 = 8a + 4b + 2c + d \tag{8.3}$$
$$n = 3 : 14 = 27a + 9b + 3c + d \tag{8.4}$$
$$n = 4 : 30 = 64a + 16b + 4c + d \tag{8.5}$$

Subtracting (8.2) from (8.3), (8.3) from (8.4), and (8.4) from (8.5), we then obtain the equations $4 = 7a + 3b + c$, $9 = 19a + 5b + c$, $16 = 37a + 7b + c$. Subtraction of these gives $5 = 12a + 2b$, $7 = 18a + 2b$. Hence we get the solution

$$a = \frac{1}{3},\ b = \frac{1}{2},\ c = \frac{1}{6},\ d = 0.$$

Consequently, our (intelligent) guess is that

$$\sum_{r=1}^{n} r^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{1}{6}n(n+1)(2n+1).$$

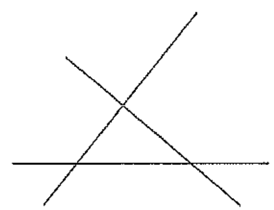
This turns out to be correct, and we leave it to the reader to prove it by induction. (It is set as Exercise 2 at the end of the chapter in case you forget.)

(Actually, there is a much better way of working out a formula for $\sum_{r=1}^{n} r^2$, given in Exercise 4 at the end of the chapter.)

The next example is a nice geometric proof by induction.

Example 8.8

Lines in the plane. If we draw a straight line in the plane, it divides the plane into two regions. If we draw another, not parallel to the first, the two lines divide the plane into four regions. Likewise, three lines, not all going through the same point, and no two of which are parallel, divide the plane into seven regions:



We can carry on drawing lines and counting the regions they form, which leads us naturally to a general question:

If we draw n straight lines in the plane, no three going through the same point, and no two parallel, how many regions do they divide the plane into?

The conditions about not going through the same point and not being parallel may seem strange, but in fact they are very natural: if you draw lines at random, it is very unlikely that two will be parallel or that three will pass through the same point — so you could say the lines in the question are "random" lines. Technically, they are said to be *lines in general position.*

The answers to the question for $n = 1,2,3,4$ are 2,4,7,11. Even from this flimsy evidence you have probably spotted a pattern — the difference between successive terms seems to be increasing by 1 each time. Can we predict a formula from this pattern? Yes, of course we can: the number of regions for one line is two, for two lines is $2+2$, for three lines is $2+2+3$, for four lines is $2+2+3+4$; so we predict that the number of regions for n lines is

$$2+2+3+4+\cdots+n\,.$$

This is just $1+\Sigma_{r=1}^{n} r$, which by Example 8.6 is equal to $1+\frac{1}{2}n(n+1)$.

Let us therefore attempt to prove the following statement $P(n)$ by induction: the number of regions formed in the plane by n straight lines in general position is $\frac{1}{2}(n^2+n+2)$.

First, $P(1)$ is true, as the number of regions for one line is 2, which is equal to $\frac{1}{2}(1^2+1+2)$.

Now suppose $P(n)$ is true, so n lines in general position form $\frac{1}{2}(n^2+n+2)$ regions. Draw in an $(n+1)^{th}$ line. Since it is not parallel to any of the others, this line meets each of the other n lines in a point, and these n points of intersection divide the $(n+1)^{th}$ line into $n+1$ pieces. Each of these pieces divides an old region into two new ones. Hence, when the $(n+1)^{th}$ line is drawn, the number of regions increases by $n+1$. (If this argument is not clear to you, try drawing a picture to illustrate it when $n=3$ or 4.) Consequently, the number of regions with $n+1$ lines is equal to $\frac{1}{2}(n^2+n+2)+n+1$. Check that this is equal to $\frac{1}{2}((n+1)^2+(n+1)+2)$.

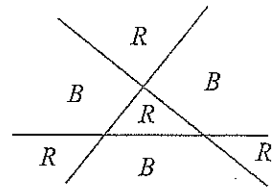
We have now shown that $P(n) \Rightarrow P(n+1)$. Hence, by induction, $P(n)$ is true for all $n \geq 1$.

Induction is a much more powerful method than you might think. It can often be used to prove statements that do not actually explicitly mention an integer n. In such instances, one must imaginatively design a suitable statement $P(n)$ to fit in with the problem and then try to prove $P(n)$ by induction. In the next two examples this is fairly easy to do. The next chapter, however, will be devoted to an example of a proof by induction where the statement $P(n)$ lies a long way away from the initial problem.

Example 8.9

Some straight lines are drawn in the plane, forming regions as in the previous example. Show that it is possible to colour each region either red or blue in such a way that no two neighbouring regions have the same colour.

For example, here is such a colouring when there are three lines:



How do we design a suitable statement $P(n)$ for this problem? This is very simple: just take $P(n)$ to be the statement that the regions formed by n straight lines and the plane can be coloured in the required way.

Actually, the proof of $P(n)$ by induction is so neat and elegant that I would hate to deprive you of the pleasure of thinking about it, so I leave it to you. (It is set as Exercise 14 at the end of the chapter in case you forget.)

Prime Factorization

In the next example, we prove a very important result about the integers. First we need a definition:

DEFINITION A prime number *is a positive integer p such that $p \geq 2$ and the only positive integers dividing p are 1 and p.*

You are probably familiar to some extent with prime numbers. The first few are $2,3,5,7,11,13,17,19,23,29,\ldots$.

The important result we shall prove is the following:

PROPOSITION 8.1
Every positive integer greater than 1 is equal to a product of prime numbers.

In the proposition, the number of primes in a product must be allowed to be 1, since a prime number itself is a product of one prime. If n is a positive integer, we call an expression $n = p_1 \dots p_k$, where $p_1, \dots, p_k$ are prime numbers, a *prime factorization* of n. Here are some examples of prime factorizations:

$$30 = 2 \cdot 3 \cdot 5, \quad 12 = 2 \cdot 2 \cdot 3, \quad 13 = 13 .$$

A suitable statement to attempt to prove by induction is easy to design: for $n \geq 2$, let $P(n)$ be the statement that n is equal to a product of prime numbers.

Clearly $P(2)$ is true, as $2 = 2$ is a prime factorization of 2. However, it is not clear at all how to go about showing that $P(n) \Rightarrow P(n+1)$. In fact this cannot be done, since the primes in the prime factorization of n do not occur in the factorization of $n+1$.

However, all is not lost. We shall use the following, apparently stronger, principle of induction.

Principle of Strong Mathematical Induction
Suppose that for each integer $n \geq k$ we have a statement $P(n)$. If we prove the following two things:

(a) *$P(k)$ is true;*

(b) *for all n, if $P(k), P(k+1), \dots, P(n)$ are all true, then $P(n+1)$ is also true;*

then $P(n)$ is true for all $n \geq k$.

The logic behind this principle is not really any different from that behind the old principle: by (a), $P(k)$ is true. By (b), $P(k) \Rightarrow P(k+1)$, hence $P(k+1)$ is true. By (b) again, $P(k), P(k+1) \Rightarrow P(k+2)$, hence $P(k+2)$ is true, and so on.

[In fact, the Principle of Strong Induction is actually implied by the old principle. To see this, let $Q(n)$ be the statement that all of $P(k), \dots, P(n)$ are true. Suppose we have proved (a) and (b) of Strong Induction. Then by (a), $Q(k)$ is true, and by (b), $Q(n) \Rightarrow Q(n+1)$. Hence, by the old principle, $Q(n)$ is true for all $n \geq k$, and therefore so is $P(n)$.]

Let us now apply Strong Induction to prove Proposition 8.1.

Proof of Proposition 8.1 For $n \geq 2$, let $P(n)$ be the statement that n is equal to a product of prime numbers. As we have already remarked, $P(2)$ is true.

Now for part (b) of Strong Induction. Suppose that $P(2), \dots, P(n)$ are all true. This means that every integer between 2 and n has a prime factorization. Now consider $n+1$. If $n+1$ is prime, then it certainly has a prime factorization (as a product of 1 prime). If $n+1$ is not prime, then by the definition of a prime number, there is an integer a dividing $n+1$ such that $a \neq 1$ or $n+1$. Writing $b = \frac{n+1}{a}$, we then have

$$n+1 = ab \quad \text{and} \quad a, b \in \{2, 3, \dots, n\} .$$

By assumption, $P(a)$ and $P(b)$ are both true, i.e., a and b have prime factorizations. Say

$$a = p_1 \dots p_k, \quad b = q_1 \dots q_l ,$$

where all the p_i and q_i are prime numbers. Then

$$n+1 = ab = p_1 \dots p_k q_1 \dots q_l .$$

This is an expression for $n+1$ as a product of prime numbers.

We have now shown that $P(2), \dots, P(n) \Rightarrow P(n+1)$. Therefore, $P(n)$ is true for all $n \geq 2$, by Strong Induction.

Example 8.10
Suppose we are given a sequence of integers $u_0, u_1, u_2, \dots, u_n, \dots$ such that $u_0 = 2, u_1 = 3$ and

$$u_{n+1} = 3u_n - 2u_{n-1}$$

for all $n \geq 1$. (Such an equation is called a *recurrence relation* for the sequence.) Can we find a formula for u_n?

Using the equation with $n = 1$, we get $u_2 = 3u_1 - 2u_0 = 5$; and likewise $u_3 = 9, u_4 = 17, u_5 = 33, u_6 = 65$. Is there an obvious pattern? Yes, a reasonable guess seems to be that $u_n = 2^n + 1$.

So let us try to prove by Strong Induction that $u_n = 2^n + 1$. If this is the statement $P(n)$, then $P(0)$ is true, as $u_0 = 2^0 + 1 = 2$. Suppose $P(0), P(1), \dots, P(n)$ are all true. Then $u_n = 2^n + 1$ and $u_{n-1} = 2^{n-1} + 1$. Hence from the recurrence relation,

$$u_{n+1} = 3\left(2^n + 1\right) - 2\left(2^{n-1} + 1\right) = 3 \cdot 2^n - 2^n + 1 = 2^{n+1} + 1 ,$$

which shows $P(n+1)$ is true. Therefore, $u_n = 2^n + 1$ for all n, by Strong Induction.

Exercises for Chapter 8

1. Prove by induction that it is possible to pay, without requiring change, any whole number of roubles greater than 7 with banknotes of value 3 roubles and 5 roubles.

2. Prove by induction that $\Sigma_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

(extra) → Deduce formulae for

$$1\cdot 1+2\cdot 3+3\cdot 5+4\cdot 7+\cdots+n(2n-1) \quad\text{and}\quad 1^2+3^2+5^2+\cdots(2n-1)^2 .$$

3. (a) Work out 1, $1+8$, $1+8+27$ and $1+8+27+64$. Guess a formula for $\Sigma_{r=1}^{n} r^3$ and prove it.

   (b) Check that $1=0+1$, $2+3+4=1+8$ and $5+6+\cdots+9=8+27$. Find a general formula for which these are the first three cases. Prove your formula is correct.

4. Here is another way to work out $\Sigma_{r=1}^{n} r^2$. Observe that $(r+1)^3-r^3=3r^2+3r+1$. Hence

$$\sum_{r=1}^{n}(r+1)^3-r^3 = 3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + n .$$

   The left-hand side is equal to

$$(2^3-1^3)+(3^3-2^3)+(4^3-3^3)+\cdots$$
$$+\left((n+1)^3-n^3\right) = (n+1)^3-1 .$$

   Hence we can work out $\Sigma_{r=1}^{n} r^2$.

   Carry out this calculation, and check that your formula agrees with that in Exercise 2.

   Use the same method to work out formulae for $\Sigma_{r=1}^{n} r^3$ and $\Sigma_{r=1}^{n} r^4$.

5. Prove the following statements by induction:

   (a) For all integers $n \geq 0$, the number $5^{2n}-3^n$ is a multiple of 11.

   (b) For any integer $n \geq 1$, the integer 2^{4n-1} ends with an 8.

   (c) The sum of the cubes of three consecutive positive integers is always a multiple of 9.

   (d) If $x \geq 2$ is a real number and $n \geq 1$ is an integer, then $x^n \geq nx$.

   (e) If $n \geq 3$ is an integer, then $5^n > 4^n + 3^n + 2^n$.

6. The *Lucas sequence* is a sequence of integers $l_1, l_2, \ldots, l_n, \ldots$, such that $l_1 = 1, l_2 = 3$ and

$$l_{n+1} = l_n + l_{n-1}$$

   for all $n \geq 1$. So the sequence starts $1,3,4,7,11,18,\ldots$

   Find the pattern for the remainders when l_n is divided by 3. (*Hint:* Consider the first 8 remainders, then the next 8, and so on; formulate a conjecture for the pattern and prove it by induction.)

   Is L_{2000} divisible by 3?

7. The *Fibonacci sequence* is a sequence of integers $f_1, f_2, \ldots, f_n, \ldots$, such that $f_1 = 1, f_2 = 1$ and

$$f_{n+1} = f_n + f_{n-1}$$

   for all $n \geq 1$. Prove by strong induction that for all n,

$$f_n = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right) ,$$

   where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

8. I just worked out $(2+\sqrt{3})^{50}$ on my computer and got the answer

   39571031999226139563162735373.99999999999999999999999999974728...

   Why is this so close to an integer?

   (*Hint:* Try to use the idea of the previous question by constructing a suitable sequence.)

9. Prove that if $0 < q < \frac{1}{2}$, then for all $n \geq 1$,

$$(1+q)^n \leq 1 + 2^n q .$$

10. (a) Prove that for every integer $n \geq 2$,

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n} \geq \frac{7}{12} .$$

   (b) Prove that for every integer $n \geq 1$,

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} \leq 2\sqrt{n} .$$

11. Just for this question, count 1 as a prime number. A well-known result in number theory says that for every integer $x \geq 3$, there is a prime number p such that $\frac{1}{2}x < p < x$. Using this result and strong induction, prove that every positive integer is equal to a sum of primes, all of which are different.

12. Here is a "proof" by induction that any two positive integers are equal (e.g., $5 = 10$):

   First, a definition: if a and b are positive integers, define $\max(a,b)$ to be the larger of a and b if $a \neq b$, and to be a if $a = b$. [For instance, $\max(3,5) = 5$, $\max(3,3) = 3$.] Let $P(n)$ be the statement: "if a and b are positive integers such that $\max(a,b) = n$, then $a = b$." We prove $P(n)$