HL Math Notes

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I. Sequence & Series.

- A sequence may be specified by:
  a) first few terms + ellipsis
  b) Recursive defn. ex. $u_n = u_{n-1} + 2$. $u_1 = 2$
  c) general term. ex. $u_n = n^2$.
- Fibonacci # : (general term)

  $$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$
- defn. (n^{th} partial sum)
  the sum of the 1^{st} n terms of a series, denoted S_n, is called the n-th partial sum of the series.

II. Logarithms.

- $\log_a a^x = x$. $a^{\log_a x} = x$.

III. Combinatorics.

- $\tau(n) = $ # of factors of n.
- $^nP_x = \frac{n!}{(n-x)!}$

  $^nC_r = \binom{n}{r}$
- 长方形个数 in $m \times n$ 网格: $\sum_{i=1}^{m} i \sum_{i=1}^{n} i = \frac{(m+1)(m)}{2} \cdot \frac{(n+1)(n)}{2}$
- Bionomial Thrm.

  $$(a+b)^n = \sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^{m}.$$
- Pascal's triangle:

perfect square.

```
                1
              1   1
            1   2   1            → 121 = 11^2
          1   3   3   1          → 1331 = 11^3
        1   4   6   4   1        → 14641 = 11^4
      1   5   10  10  5   1
    1 + 6 + 15 + 20 + 15 + 6 + 1 = 2^n
  1   7   21  35  35  21  7   1
1   8   28  56  70  56  28  8   1
  . . .       . . .       . . .
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   ①     ②
⑥    n    ③
   ⑤     ④
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$n^3 = ②④⑥ + ①③⑤ + n.$

(图形点亦需边时才清楚)

- $\sum_{m=0}^{n} \binom{n}{m} = 2^n.$

proof 1. : $\binom{n}{0}$: bit string $\overbrace{00\ldots0}^{n}$. $\frac{n!}{n!}$

$\binom{n}{1}$: bit string $\overbrace{10\ldots\ldots0}^{n}$ $\frac{n!}{(n-1)!\,1!}$

$\ldots \qquad \ldots$

$\Rightarrow \sum_{m=0}^{n} \binom{n}{m} = \overbrace{\%_1\ \%_1\ \cdots\ \%_1}^{n} = 2^n.$

proof 2 : $(1+1)^n = 2^n = \sum_{m=0}^{n} \binom{n}{m}$

- $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$

proof 1 : Algebraic

proof 2 : 在n个元素中，有一special elmt x,
选a个元素的子集，有$\binom{n}{a}$种可能.
其中，包含x的是$\binom{n-1}{a-1}$，钉死一个x,
不包含x的是$\binom{n-1}{a}$.
以上两种情况互补. □.

IV. Complex number

- $\mathrm{Re}\,z\,(2+3i) = 2.$ $\mathrm{Im}\,z\,(2+3i) = 3.$
- conjugate of $z = a+bi$: $z^* = a-bi.$

- $z + z^* = 2a$

  $z - z^* = 2bi$.

  $(a+bi)(c+di) = ac - bd + (ad+bc)i$.

  $z \cdot z^* = a^2+b^2$.

  $$\frac{a+bi}{c+di} = \frac{ac+bd+(bc-ad)i}{c^2+d^2}$$

- find the square rt of $a+bi$:

  let $z = c+di$. $z^2 = a+bi$.

  $a+bi = c^2-d^2+2cdi$.

  $\therefore \begin{cases} a = c^2-d^2 \\ b = 2cd \end{cases} \Rightarrow c = \frac{b}{2d} \Rightarrow d^4 + ad^2 - \frac{b^2}{4} = 0$

  $\therefore \Delta = a^2+b^2$, $\begin{cases} d^2 = \dfrac{-a \pm \sqrt{a^2+b^2}}{2}. \\ c^2 = \dfrac{a \mp \sqrt{a^2+b^2}}{2}. \end{cases}$

- $\Delta < 0$

  $\Rightarrow x_{1,2} \in \mathbb{C} \setminus \mathbb{R}$, $x_1 = x_2^*$

- 2^{2018} 是几位数 ?

  $\lceil \log 2^{2018} \rceil = \lceil 2018 \log 2 \rceil = 608$.

V Polynomials

- Notation: $33/7 = 4 R 5$.
- Thrm.

  If $P(x)$ is a poly. $d(x)$ is a non-zero poly.

  $\exists!$ poly's $q(x)$ and $r(x)$, s.t.

  $p(x) = d(x) \cdot q(x) + r(x)$.

  and either $r(x) = 0$. or $\deg r(x) < \deg d(x)$.
- Remainder thrm:

  $P(x)$ 除以 $x-a$, 余数为 $P(a)$.
- fator thrm.

  If a is a rt of poly $p(x)$, then $x-a$ is a factor of $p(x)$.

- Rational Rt's thrm

在 $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ 中. $a_i \in \mathbb{Z}$, $(i = 0, \ldots n)$

若 $f(x)$ 有有理根 $\frac{p}{q}$. $p\,q$ 互质. 则 $p \mid a_0$, $q \mid a_n$.

* A candidate thrm.

Pf: multiply by q^n on both sides of the equation.

$$a_n\left(\frac{p}{q}\right)^n + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0$$

$$\Rightarrow a_n p^n = -(a_{n-1}p^{n-1}q + \cdots + a_1 p q^{n-1} + a_0 q^n).$$

$\because q \mid RHS$. $\therefore$ $\cancel{a_n p^n}$. $RHS \in \mathbb{Z}$.

$\therefore q \mid LHS$, 又. p, q 互质.

$\therefore q \mid a_n$.

同理, $p \mid a_0$.

EX. $z^3 - 3z^2 + 3z + i = 1$ 的 3 个解 在复数域中 作为一个三角形的顶点. 求三角形的面积.

Method 1:

The equation is equivalent to $(z-1)^3 = -i$.

$\cancel{\Rightarrow (z-1) = \sqrt[3]{-i}}$.

let $z - 1 = a + bi$. then $(a+bi)^3 = -i$.

待定系数法求 a, b, $\Rightarrow z = a + bi + 1$.

Method 2.

- In complex plane, $w\ (a, \alpha) \in \mathbb{C}$,

  $\sqrt[n]{w} \Rightarrow n$ 个 均匀分布在 a circle w/ $r = \sqrt[n]{a}$, centered at the origin.

  The solutions are $(\sqrt[n]{a}, \frac{\alpha}{n})$, $(\sqrt[n]{a}, \frac{\alpha}{n} + \frac{360}{n}) \ldots$

  $(\sqrt[n]{a}, \frac{\alpha}{n} + \frac{360}{n}(n-1))$

- in this question, $-i : (1, \frac{3}{2}\pi) \Rightarrow \underset{i}{(1, \frac{1}{2}\pi)}$ 为一解.

  $\Rightarrow$ 其余两解为 $(1, \frac{1}{2}\pi + \frac{2\pi}{3})$. $(1, \frac{1}{2}\pi + \frac{4\pi}{3})$. $\Rightarrow$ 正三角.

  $\because z = 1 + \sqrt[3]{-i}$, 平移, 体积不变. $\Rightarrow S = \frac{\sqrt{3}}{4}a^2 = \frac{3}{4}\sqrt{3}$.

- $w : (a, \alpha) \in \mathbb{C}$: $\mathrm{abs}(w) = a$, $\arg(w) = a$.

- <u>Conjugate thrm</u>

$$(z+w)^* = z^* + w^*$$
$$(z \cdot w)^* = z^* w^*$$

corollary:

- $(kz)^* = kz^*$
- $(z-w)^* = z^* - w^*$
- $\left(\frac{z}{w}\right)^* = \frac{z^*}{w^*}$
- $[f(z)]^* = f(z^*)$

- <u>Conjugate pts thrm</u>

若 z 是 $f(x)$ 的一个根，则 z^* 也是.

pf: $f(z^*) = [f(z)]^* = [0]^* = 0$.

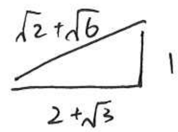
- <u>fundamental thrm of algebra</u>

a poly of deg $= n$ has exactly n complex roots (prf?), counting multiplicity.

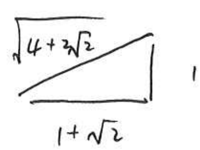
VI <u>Trignometry</u>

- Notation: $m(OQ)$: OQ的斜率
- 特殊角:

$\frac{\pi}{12}$:



$\frac{\pi}{8}$:



- ~~def~~ <u>defn</u>. (related pts)

(a, b) $(a, -b)$, $(-a, b)$ $(-a, -b)$ on $x^2+y^2=1$.

- reference angle's sin, cos, tan are the same as the related angle's, except their signs.

often denoted by θ^*

- $\sec\theta = \dfrac{1}{\cos\theta}$.

  $\csc\theta = \dfrac{1}{\sin\theta}$.

  $\Rightarrow \dfrac{\sin\theta}{1+\cos\theta} = \csc\theta - \cot\theta$.

- $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

  $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

  $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

  $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

  $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.

  $\tan(\alpha-\beta) = \dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$.

- sin : 奇.

  cos : 偶.

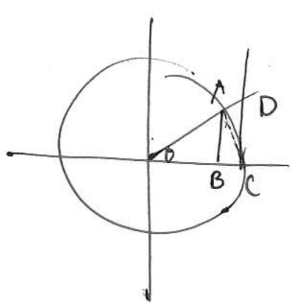
  tan : 奇.

- − θ (radiance) $= \dfrac{l}{r}$.

  $\Rightarrow l = \theta r$.

  − $S_{扇形} = \dfrac{1}{2} l r = \dfrac{1}{2}\theta r^2$.

  Ex. prove that $\sin\theta < \theta < \tan\theta$. $\theta \in (0, \frac{\pi}{2})$.



$\overset{\frown}{AC} = \theta r = \theta$.

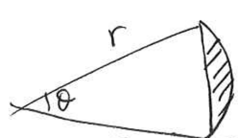
① $AB < AC < \overset{\frown}{AC}$

$\Rightarrow \sin\theta < \theta$.

② $S_{\triangle OCD} > S_{A\hat{O}C}$

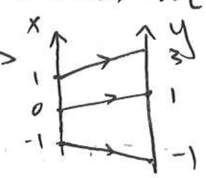
$\Rightarrow \frac{1}{2}\tan\theta > \frac{1}{2}\theta$.

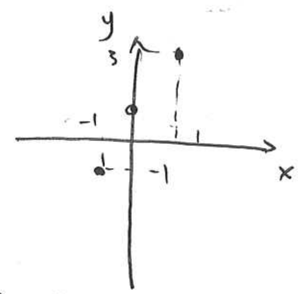
− $S_{弓形} = \frac{1}{2} r^2 (\theta - \sin\theta)$



VII Functions

- domain $\xrightarrow{f}$ codomain.
- range = {images} = $\{f(x) \mid x \in D,\ f: D \to C\}$.
- Domain = {preimage}.
- 当 domain. codamain 未明确. domain 即取能使 range 全部为实数的 最大实数内范围.
- $\mathbb{R} \setminus [-3,3] \iff]-\infty,-3[\cup]3,+\infty[$.
- { arrow diagram / cartesian graph:





- vertical line test:
  用一条竖直的线划过 $f(x)$. 最多能与图象在一个时刻有一个交点.
- 奇: $f(-x) = -f(x)$. 偶: $f(x) = f(-x)$.
- ~~奇偶运算. 有奇结果即为奇~~.
  奇 + 奇 = 奇. 偶 + 偶 = 偶.
  奇 * 奇 = 偶 偶 × 偶 = 偶. 奇 × 偶 = 奇.
- Bijection: $n(D) = n(C)$.
  $C \to D$ 的反函数是函数.
  - injection 保证了 $f^{-1}(x)$ 是函数.
  - surjection 确定了定义域不会出问题.
- 关于 $y = x$ 对称的 fcn 的反函数与 $f(x)$ 相同.
  ⇒ self-inverse / involution.
- Domain Restriction.
  当 $f(x)$ not bij. 可 change domain to get $f^{-1}(x)$

$$
\begin{aligned}
\sin(2\arccos\theta) &= 2\sin(\arccos\theta)\cdot\theta \\
&= 2\theta\cdot\sqrt{1-\cos^2(\arccos\theta)} \\
&= 2\theta\sqrt{1-\theta^2}.
\end{aligned}
$$

- 圖像转换、

  $\begin{pmatrix} a \\ b \end{pmatrix}$: $\rightarrow a \quad \uparrow b$.

  - reflection: $(0,0) \rightarrow y = -f(-x)$

    $y = x \rightarrow x = f(y)$.

  - stretches:

    $$y = f(x) \xrightarrow{\begin{pmatrix} \frac{1}{b} & 0 \\ 0 & a \end{pmatrix}} y = af(bx)$$

    ?) $y = f(x) \xrightarrow{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} h \\ k \end{pmatrix}} y = bf\left(\frac{\cancel{a}x}{a} - h\right) + k$. or $\frac{x-h}{a}$?

- Sinusoids.

  $$y = a \sin/\cos[b(x \pm c)] + d$$

  period : $\frac{2\pi}{b}$.

  amplitude : a.

  phase shift : c.

- 找 bilinear fcn. 的 asymptotes:

  $$y = \frac{ax+b}{cx+d} \Rightarrow \begin{matrix} x: & x = -\frac{d}{c} \\ y: & y = \frac{a}{c} \end{matrix}$$

- $y = \dfrac{ax+b}{cx+d} = \dfrac{a}{c} + \dfrac{b - \frac{a}{c} d}{cx+d}$.

  $$y = \frac{1}{x} \xrightarrow{\begin{pmatrix} c & 0 \\ 0 & b - \frac{a}{c} \cdot d \end{pmatrix} \begin{pmatrix} -\frac{d}{c} \\ \frac{a}{c} \end{pmatrix}} \frac{y - \frac{a}{c}}{b - \frac{a}{c} \cdot d} = \frac{1}{\frac{x + \frac{d}{c}}{c}}$$

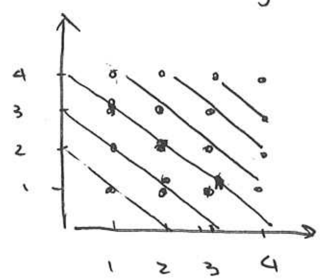
  $\Rightarrow$ 过点 $\left(0, \frac{b}{d}\right)$, $\left(-\frac{b}{a}, 0\right)$

  $\Rightarrow$ $bc - ad > 0$. 一三象限. 反之 二、四象限.

VIII Probability.

- U denote sample space.
- ex. 2 dice w/ 4 sides. sum. most likely = ?

→ lattice diagram.



$P(5) = \frac{1}{4}$.

- Ex. ABCD排列中. A.C 位置与ABCD中 有不同的概率是?

法1: Venn diagram.

$\cancel{n(A\cap C)}\ n(A\cup C) = n(A) + n(C) - n(A\cap C)$

法2: table.

	C	C'	
A	2	4	6
A'	4	14	18
	6	18	24

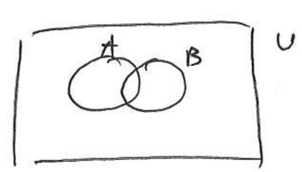
- independent event: $P(A\cap B) = P(A)\cdot P(B)$
- Ex. $P(A) = 0.5$. $P(B) = 0.6$. ind. events. ⇒ prove A', B' ind.

	B	B'	
A	0.3	0.2	0.5
A'	0.3	0.2	0.5
	0.6	0.4	1

- Ind. 在图中的意义:

$P(A|B) = PA$, as $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$

$\frac{n(A\cap B)}{n(B)} = \frac{n(A)}{n(U)}$.



- mutually exclusive: $P(A\cap B) = 0$. } complementary
  exhaustive: $P(A\cup B) = 1$

- $P(A|B) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{n(A\cap B)}{n(B)}$

$\Rightarrow \quad P(A\cap B) = P(B|A)\cdot P(A) = P(A|B)\cdot P(B)$

- <u>Bayes' thrm</u>.

$$P(B_i|A) = \frac{P(B_i)\cdot P(A|B_i)}{\sum_{a=1}^{n} P(B_a)\cdot P(A|B_a)} . \qquad i \le n,\ i \in \mathbb{Z}^+.$$

简: $\quad P(B|A) = \dfrac{P(B)}{P(A)} \cdot P(A|B)$

IX. Calculus.

- $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$.

$\left(\dfrac{1}{f(x)}\right)' = \dfrac{-f'(x)}{f^2(x)}$ $\qquad\qquad \dfrac{d}{dx} = 1$

- stationary pt : $f'(x) = 0$.

turning pt : local minimum

'' max ''

horizontal pt of inflection. rising falling

- $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.

$\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = 0 \quad \Leftarrow \quad \dfrac{1-\cos}{x} = \dfrac{1-\cos^2 x}{x(1+\cos x)}$

- 二阶导数.

$f''(x) > 0$. concave up.

$f''(x) < 0$ concave down.

	$f' > 0$	$f' < 0$.
$f'' < 0$		
$f'' > 0$		

Ex. A spherical raindrop absorbs water at a rate proportional to its surface area. 求水滴半径增长速度.

$$\frac{dv}{dt} \propto S \Rightarrow \frac{dv}{dt} = k \cdot S.$$

$$\frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt}.$$

$$\frac{dv}{dr} = \left(\frac{4}{3}\pi r^3\right)' = 4\pi r^2 = S.$$

$$k \cdot S = S \cdot \frac{dr}{dt}.$$

$$\Rightarrow \frac{dr}{dt} = k. \quad r \text{ 匀速增加}$$

Ex prove $(1+\frac{1}{n})^n = e$.

$f(x) = \ln x$. $f'(x) = \frac{1}{x}$. $f'(1) = 1$

$$f'(x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h}$$

$$f'(1) = \lim_{h\to 0} \frac{1}{h}\ln(1+h)$$

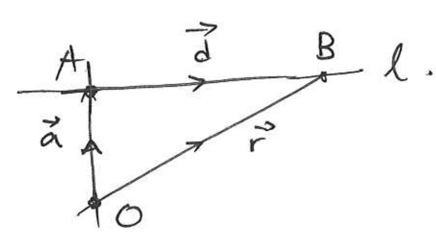
$$= \lim_{h\to 0} \ln(1+h)^{\frac{1}{h}} = 1$$

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = 1.$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e.$$

X. Vector

- 结合律不适用: $\vec{v}(\vec{w}\cdot\vec{u}) \neq (\vec{v}\cdot\vec{w})\cdot\vec{u}$.
- $\vec{v}\cdot\vec{w} = \cos\theta\,|\vec{v}|\,|\vec{w}|$ 用余弦定理推.
  即 $\vec{v}$ 在 $\vec{w}$ 方向上的投影.
- vector equation of a line.
  $\vec{r} = \vec{a} + t\vec{d}$.



意义: $x = a_1 + td_1$
$y = a_2 + td_2$

vector equation 是 l 上的点的 position vector 的全集,
i.e. A pt satisfies vector equation $\Leftrightarrow$ on l.