

1. Let  $f(x) = \tan x$ . Observe that  $f(0) = f(\pi)$  but there is no  $c \in ]0, \pi[$  such that  $f'(c) = 0$ . Explain why this does not contradict Rolle's theorem.

Because Rolle's thrm requires that  $f(x)$  is continuous over the interval  $[0, \pi]$ .

However, for  $f(x) = \tan x$ ,  $f(x)$  is <sup>not</sup> continuous at  $f(\frac{\pi}{2})$ .  
Thus, Rolle's thrm doesn't apply.

2. Let  $f(x) = x + |x|$ . Prove that  $f$  is continuous but not differentiable at  $x = 0$ .

Proof: (a)  $f$  is continuous at  $x = 0$ .

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} f(x) = 0$$

$$\therefore \lim_{x \rightarrow 0} f(x) = 0 = f(0)$$

and thus conti.

(b)  $f$  is not differentiable at  $x = 0$ .

$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h + |h|}{h}$$

$$\lim_{h \rightarrow 0^+} \frac{h + |h|}{h} = 2$$

$$\lim_{h \rightarrow 0^-} \frac{h + |h|}{h} = 0$$

$$\text{since } \lim_{h \rightarrow 0^+} \frac{h + |h|}{h} \neq \lim_{h \rightarrow 0^-} \frac{h + |h|}{h}$$

3. Use the mean value theorem to prove the inequality  $|\sin a - \sin b| \leq |a - b|$  for all  $a, b \in \mathbb{R}$ . the limit does not exist.

proof: Let  $f(x) = \sin x$ .

We know that  $f(x)$  is conti. and differentiable on  $\mathbb{R}$  by construction.

Then according to MVT,

$\exists c \in \mathbb{R}$  s.t

$$f'(c) = \frac{\sin a - \sin b}{a - b}, \quad \forall a, b \in \mathbb{R}$$

since for  $f(x) = \sin x$ ,

$$f'(x) = (\sin x)' = \cos x$$

and  $-1 \leq \cos x \leq 1, \quad \forall x \in \mathbb{R}$ .

therefore,

$$-1 \leq f'(c) \leq 1$$

Therefore,  $f$  is not differentiable at  $x = 0$ .  $\square$

which means

$$-1 \leq \frac{\sin a - \sin b}{a - b} \leq 1$$

$$\frac{|\sin a - \sin b|}{|a - b|} \leq 1$$

Thus,  $|\sin a - \sin b| \leq |a - b|$

$\forall a, b \in \mathbb{R}$ .

$\square$

4. In  $\triangle ABC$ ,  $a = 9$ ,  $b = 6$  and  $c = 12$ . A circle with centre  $A$  and radius 4 meets sides  $[AB]$  and  $[AC]$  at  $E$  and  $F$  respectively. The secant  $(EF)$  meets  $(BC)$  at  $D$ . Use Menelaus's theorem to calculate the length  $CD$ .

According to Menelaus's thrm,

$$\frac{BC}{CA} \cdot \frac{AF}{Fc} \cdot \frac{CD}{DB} = -1$$

$$\frac{8}{4} \cdot \frac{4}{2} \cdot \left(-\frac{CD}{CD+9}\right) = -1$$

$$\frac{CD}{CD+9} = \frac{1}{4}$$

$$4CD = CD+9$$

$$3CD = 9$$

$$\boxed{CD = 3}$$

5. Verify that  $f(x) = 2x^4 - 3x^2 - x + 5$  satisfies the hypotheses of the mean value theorem on the interval  $[0, 1]$  and find all numbers  $c$  that satisfy the conclusion of the mean value theorem.

$$f'(x) = 8x^3 - 6x - 1,$$

which means  $f(x)$  is differentiable on  $]0, 1[$ .

and therefore continuous

on  $\Phi [0, 1]$ .

Thus, according to MVT,

$\exists c \in \mathbb{R}$ , s.t.

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$= \frac{3 - 5}{1}$$

$$= -2.$$

$$8c^3 - 6c - 1 = -2.$$

$$8c^3 - 6c + 1 = 0.$$

using technology.

$$c_1 \approx 0.174$$

$$c_2 \approx 0.766.$$

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1. What value should be assigned to  $k$  to make the function  $f(x) = \begin{cases} x^2 - 1, & x < 3, \\ 2kx, & x \geq 3, \end{cases}$  continuous at  $x = 3$ .

For  $f(x)$  to be conti. at  $x=3$

$$\lim_{x \rightarrow 3} f(x) = f(3)$$

which means

$$\lim_{x^+ \rightarrow 3} f(x) = \lim_{x^- \rightarrow 3} f(x)$$

$$\text{then } \lim_{x \rightarrow 3} (x^2 - 1) = \lim_{x \rightarrow 3} (2kx).$$

$$3^2 - 1 = 2k(3)$$

$$8 = 6k$$

$$\boxed{k = \frac{4}{3}}$$

✓

2. Construct a function that is continuous on  $\mathbb{R}$  but fails to be differentiable at the four numbers 0, 1, 2, 3.

$$f(x) = \begin{cases} x+1, & x < 0 \\ ||1x-2|-2|-1|, & 0 \leq x \leq 3. \\ x-3, & x > 3. \end{cases}$$

3. Suppose  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a differentiable function with  $f'(x) > 0$  for all  $x \in \mathbb{R}$ . Prove that if  $a < b$  then  $f(a) < f(b)$ .

proof:

since  $f$  is differentiable on  $\mathbb{R}$ ,

it is also conti on  $\mathbb{R}$ .

then MVT applies.

If  $a < b$ ,

then  $\exists c \in ]a, b[$  s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

and since  $f'(x) > 0, \forall x \in \mathbb{R}$ ,

$$f'(c) > 0$$

thus

$$\frac{f(b) - f(a)}{b - a} > 0.$$

and since  $b - a > 0$ ,

$$f(b) - f(a) > 0$$

$$f(b) > f(a)$$

✓ □

✓

4. The third degree Taylor polynomial of  $\ln x$  about  $x = 1$  is  $a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3$ . Find the values of  $a_0, a_1, a_2$  and  $a_3$  and hence estimate  $\ln 1.2$ .

$$\begin{aligned} P_3(x) &= \frac{f^{(0)}(1)}{0!} + \frac{f^{(1)}(1)}{1!}(x-1) + \dots + \frac{f^{(3)}(1)}{3!}(x-1)^3 \\ &= \ln(1) + \frac{1}{1}(x-1) + \frac{(-1)(1)^2}{2!}(x-1)^2 + \frac{(-2)(1)^3}{3!}(x-1)^3 \\ &= 0 + x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \end{aligned}$$

Thus,  $a_0 = 0, a_1 = 1, a_2 = -\frac{1}{2}, a_3 = \frac{1}{3}$ .

$$\ln(1.2) \approx P_3(1.2)$$

$$= 1.2 - 1 - \frac{1}{2}(1.2-1)^2 + \frac{1}{3}(1.2-1)^3$$

$$= 0.2 - 0.5(0.2)^2 + \frac{1}{3}(0.2)^3$$

$$\approx 0.183 \text{ (3 s.f.)}$$

5. In the trapezium  $ABCD$ , the midpoints of the parallel sides  $[AB]$  and  $[CD]$  are  $M$  and  $N$  respectively. The sides  $[BC]$  and  $[AD]$  are not parallel. Show that the diagonals and the line segment  $[MN]$  are concurrent.

Lemma:

Prolong  $CA$  and  $DB$ ,

let them intersect at point  $P$ .

then  $P, M, N$  are collinear.

i.e.  $CN' = DN'$ ,  $N'$  is the midpoint of  $CD$ , which means

$N$  and  $N'$  coincide.

Since  $P, M, N'$  are collinear,

$P, M, N$  are collinear.  $\square$ .

Proof:

Suppose not, ~~i.e.~~

then connect and prolong  $PM$ ,

let  $PM$  intersect  $CD$  at  $N'$ .

Since  $AB \parallel CD$ ,

$$\angle PAM = \angle PCN'$$

$$\angle APM = \angle CPN'$$

$$\text{So } \triangle APM \sim \triangle CPN'$$

Similarly,  $\triangle BPM \sim \triangle DPN'$ .

$$\text{So } \frac{AM}{CN'} = \frac{PM}{PN'} = \frac{BM}{DN'}$$

Since  $M$  is the mid point of  $AB$ ,

$$\frac{AM}{BM} = 1.$$

$$\text{then } \frac{CN'}{DN'} = \frac{AM}{BM} = 1.$$

Proof: Since  $P, M, N$  are collinear,

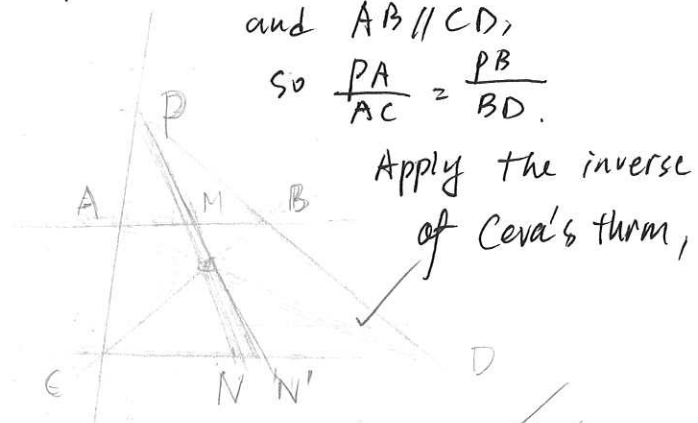
and  $AB \parallel CD$ ,

$$\text{So } \frac{PA}{AC} = \frac{PB}{BD}.$$

Apply the inverse of Ceva's thm,

$$\begin{aligned} \text{Since } \frac{PA}{AC} \cdot \frac{CN}{ND} \cdot \frac{DB}{BP} &= 1 \\ &= \frac{PB}{BD} \cdot \frac{CN}{ND} \cdot \frac{DB}{BP} = 1 \end{aligned}$$

$PN, CB, DA$  are concurrent.  $\square$ .



1. Use l'Hôpital's rule to evaluate  $\lim_{x \rightarrow 1} \frac{\arctan x - \pi/4}{x - 1}$ .

$$\lim_{x \rightarrow 1} \frac{\arctan x - \frac{\pi}{4}}{x - 1} = \frac{0}{0},$$

Using l'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\arctan x - \frac{\pi}{4}}{x - 1} &= \lim_{x \rightarrow 1} \frac{(\arctan x - \frac{\pi}{4})'}{(x - 1)'} \\ &= \lim_{x \rightarrow 1} \frac{1}{1 + x^2} = \frac{1}{2}. \end{aligned}$$

2. Let  $A = (-1, 0)$  and  $B = (1, 0)$ . Find the locus of a point  $P$  that moves so that  $PA^2 + PB^2 = 10$ .

$$\frac{PA^2}{10} + \frac{PB^2}{10} = 1.$$

Let  $P = (x, y)$ . ~~PA =~~

$$\text{then } (x + 1)^2 + y^2 + (x - 1)^2 + y^2 = 10.$$

$$2x^2 + 2y^2 = 8$$

$$\frac{x^2}{4} + \frac{y^2}{4} = 1 \quad x^2 + y^2 = 4 = 2^2.$$

So the locus of  $P$   
is a circle at the  
origin with radius = 2.

3. The third degree Taylor polynomial for the function  $f$  centred at 1 is  $4 - (x - 1) + 3(x - 1)^2 - 5(x - 1)^3$ .

- (a) Write down the value of  $f''(1)$ .

$$\frac{f''(1)}{2!} = 3. \quad f''(1) = 6.$$

- (b) Approximate  $f'(1.2)$ .

$$f'(x) = -1 + 6(x - 1) - 15(x - 1)^2.$$

$$f'(1.2) = -1 + 6(0.2) - 15(0.2)^2$$

$$\begin{aligned} &= -1 + 1.2 - 0.6 \\ &= -0.4. \end{aligned}$$

4. The sequence  $\{u_n\}$  is defined recursively by  $u_1 = 2$  and  $u_{n+1} = \frac{1}{2}(u_n + 4)$ . Use mathematical induction to show that  $\{u_n\}$  is an increasing sequence bounded above by 4. What is the limit of the sequence?

proof by induction:

①  $u_1 = 2$ .  $u_2 = \frac{1}{2}(u_1 + 4) = 3$ .

~~$u_2 > u_1$~~ .  $u_1 < u_2 < 4$ .

② Now we suppose that  $\{u_n\}$  is increasing and  $u_n < 4$ .

then  $u_{n+1} = \frac{1}{2}(u_n + 4)$

~~since  $u_{n+1} > u_n$~~

since  $\frac{1}{2}(u_n + 4) < \frac{1}{2}(8) = 4$

$u_{n+1} < 4$ , which

and  $u_{n+1} - u_n$

$= 4 - \frac{1}{2}u_n > 0$ .

So  $u_{n+1} > u_n$ ,

which means the sequence is increasing.

5. The function  $f$  has derivatives of all orders for all real numbers. The third degree Taylor polynomial for  $f$  centred at 2 is  $7 - 9(x-2)^2 - 3(x-2)^3$ . If  $|f^{(4)}(x)| \leq 6$  for all  $x$  in the open interval  $]0, 2[$ , show that  $f(0)$  must be negative.

proof: According to Taylor's thm,

$$f(x) = 7 - 9(x-2)^2 - 3(x-2)^3 + R_3$$

where  $R_3 = \frac{f^{(4)}(c)}{4!} (x-2)^4$

for some  $c \in ]1, x[$

$c$  between 2 and  $x$ .

let  $x = 0$ .

then  $f(0) = 7 - 9(-2)^2 - 3(-2)^3 + \frac{f^{(4)}(c)}{4!} (-2)^4$  for some  $c \in ]0, 2[$ .

and since  $|f^{(4)}(x)| \leq 6 \quad \forall x \in ]0, 2[$ .

$-6 \leq f^{(4)}(c) \leq 6$ .

thus  $-5 - 4 \leq f(0) \leq -5 + 4 < 0$ .

Therefore,  $f(0)$  must be negative.  $\square$

Therefore, since the truth of  $\{u_n\}$  implies the truth of  $\{u_{n+1}\}$  by induction,

$\{u_n\}$  is an increasing sequence bounded by 4.

Then according to Monotonic Sequence thm for convergence, the sequence has a limit, say  $L$ .

so  $\lim_{n \rightarrow \infty} u_{n+1} = L$ .

on the other hand,  $\lim_{n \rightarrow \infty} u_{n+1} = \frac{1}{2}(u_n + 4)$

so  $L = \frac{1}{2}(L + 4)$

$L = 4$ .

So the limit of the sequence is 4.  $\square$

4

1. Write the Maclaurin series for  $e^x$ ,  $\cos x$  and  $\sin x$  in sigma notation.

$$e^x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad k \in \mathbb{N}$$

*unnecessary as  $k$  is indicated by the index.*

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} (-1)^{\frac{k}{2}} \frac{x^k}{k!}, \quad k \in \mathbb{N}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} (-1)^{\frac{k-1}{2}} \frac{x^k}{k!}, \quad k \in \mathbb{N}$$

*what if  $k=1$ ?*

2. Find the  $n$ -th degree Taylor polynomial for  $\frac{1}{x}$  about  $x=1$  and write the polynomial in sigma notation.

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(1)}{k!} (x-1)^k$$

$$= \sum_{k=0}^n \frac{(-1)^k (k!) (1)^{-k-1}}{k!} (x-1)^k$$

$$= \sum_{k=0}^n (1-x)^k$$

3. Use the alternating series estimation theorem to find an interval centre 0 throughout which  $\cos x$  can be approximated by  $1 - \frac{1}{2}x^2$  to three decimal places.

The  $n^{\text{th}}$  Maclaurin poly for  $\cos x$  is

$$P_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^{\frac{n}{2}} \frac{x^n}{n!}$$

~~So the absolute difference~~

$$\left| \cos x - \left(1 - \frac{1}{2}x^2\right) \right| < \left| \frac{x^4}{4!} \right| < 0.5 \times 10^{-3}$$

By the Alternating series estimation thm,

$$\left| \cos x - \left(1 - \frac{1}{2}x^2\right) \right| < \frac{x^4}{4!} < 0.5 \times 10^{-3}$$

$$x^4 < 0.002$$

$$\text{so } -0.331 < x < 0.331$$

the interval is  $]-0.331, 0.331[$ .

4. Show that the power series  $\sum_{n=1}^{\infty} \frac{n^2}{n!} (x-1)^n$  converges for all  $x \in \mathbb{R}$ .

proof: we apply the ratio test. let  $a_n = \frac{n^2}{n!} (x-1)^n$ .

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{(n+1)!} (x-1)^{n+1} \cdot \frac{n!}{n^2 (x-1)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| (x-1) \left( \frac{n+1}{n^2} \right) \right| \\ &= \lim_{n \rightarrow \infty} \left| (x-1) \left( \frac{1}{n} + \frac{1}{n^2} \right) \right| \\ &= \lim_{n \rightarrow \infty} |x-1| \cdot 0 = 0 \quad \forall x \in \mathbb{R}. \end{aligned}$$

and since

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1.$$

the series

$$\sum_{n=1}^{\infty} \frac{n^2}{n!} (x-1)^n$$

converges.  $\square$ .

5. Let  $f(x) = e^x \sin x$ . Show that  $f''(x) = 2(f'(x) - f(x))$ . Hence find the fifth degree Maclaurin polynomial for  $f$ .

$$f'(x) = e^x \sin x + e^x \cos x.$$

$$f'''(x) = 2(f''(x) - f'(x))$$

$$\begin{aligned} f''(x) &= e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x \\ &= 2e^x \cos x. \end{aligned}$$

$$\begin{aligned} 2(f'(x) - f(x)) &= (e^x \sin x + e^x \cos x - e^x \sin x) \cdot 2 \\ &= 2e^x \cos x. \end{aligned}$$

$$\text{So } f''(x) = 2(f'(x) - f(x)).$$

$$u_n = f^{(n)}(0).$$

$$\text{So } u_n = 2(u_{n-1} - u_{n-2});$$

$$u_0 = 0, u_1 = 1.$$

$$\Rightarrow u_5 = -4.$$

Thus, the fifth deg. Mac. poly for  $f$  is

$$P_5(x) = \sum_{k=0}^5 \frac{f^{(k)}(0)}{k!} x^k.$$

$$= 0 + x + \frac{2x^2}{2!} + \frac{2x^3}{3!} + \frac{2x^4}{4!} + \frac{2x^5}{5!}$$

$$= x + x^2 + \frac{2}{3!}x^3 + \frac{2}{4!}x^4 + \frac{2}{5!}x^5.$$

$$0 - \frac{1}{30}x^5$$

3 1/2



9<sup>h</sup> 12<sup>h</sup> Very faint

1. Write down the coefficient of  $x^6$  in the Maclaurin series for  $\cos 2x$ . Hence determine the coefficient of  $x^6$  in the Maclaurin series for  $\cos^2 x$  giving your answer as a fraction in lowest terms.

coefficient of  $x^6$  for  $\cos 2x$ :

$$\frac{f^{(6)}(0)}{6!} = \frac{-64 \cos(0)}{6!} = \boxed{-\frac{4}{45}}$$

Since  $\cos 2x = 2\cos^2 x - 1$

$$\cos^2 x = \frac{\cos 2x + 1}{2}$$

and thus the coefficient of  $x^6$  for  $\cos^2 x$  is

$$-\frac{4}{45} \cdot \frac{1}{2} = \boxed{-\frac{2}{45}}$$

2. Find the radius of convergence for the power series  $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n2^{n-1}}$ .

Observe that when  $x = 1$ ,

$$\text{the series becomes } \sum_{n=1}^{\infty} \frac{(-2)^n}{n2^{n-1}} = \sum_{n=1}^{\infty} (-1)^n \frac{2}{n},$$

which converges by applying alternating series test; so  $R \geq 2$ .

Also, when  $x = 5$ , the series becomes

$$\sum_{n=1}^{\infty} \frac{2^n}{n2^{n-1}} = \sum_{n=1}^{\infty} \frac{2}{n}, \text{ which diverges, so } R \leq 2.$$

Therefore,  $\boxed{R=2}$ .

3. Use the substitution  $y = 1/x$  and L'Hôpital's rule to evaluate  $\lim_{y \rightarrow \infty} y - \sqrt{1+y^2}$ . Confirm your answer using a series approach.

$$\begin{aligned} \textcircled{1} \quad & \lim_{y \rightarrow \infty} (y - \sqrt{1+y^2}) \\ &= \lim_{x \rightarrow 0^+} \left( \frac{1}{x} - \sqrt{1 + \frac{1}{x^2}} \right) \\ &= \lim_{x \rightarrow 0^+} \left( \frac{1}{x} - \frac{\sqrt{x^2 + 1}}{|x|} \right) \\ &= \lim_{x \rightarrow 0^+} \frac{1 - \sqrt{1+x^2}}{x} = \frac{0}{0}. \end{aligned}$$

By L'Hôpital's rule,

$$\begin{aligned} & \lim_{x \rightarrow 0^+} \frac{(1 - \sqrt{1+x^2})'}{x'} \\ &= \lim_{x \rightarrow 0^+} \frac{2x \cdot \frac{1}{2}(1+x^2)^{-\frac{1}{2}}}{1} \\ &= 0 \end{aligned}$$

② Series Approach.

the  $n$ th degree polynomial for  $f(x) = (1+x^2)^{\frac{1}{2}}$ .

is  $P_n(x) = \binom{\frac{1}{2}}{0} + \binom{\frac{1}{2}}{1}x^2 + \dots + \binom{\frac{1}{2}}{n}x^{2n}$ .

$$\text{Now } \frac{1 - \sqrt{x^2 + 1}}{x} = \frac{1 - (1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \dots)}{x}$$

$$= \frac{1 - (1 + O(x^2))}{x}$$

$$= O(x).$$

$$\text{So } \lim_{y \rightarrow \infty} (y - \sqrt{1+y^2})$$

$$= \lim_{x \rightarrow 0} \left( \frac{1 - \sqrt{x^2 + 1}}{x} \right)$$

$$= \lim_{x \rightarrow 0} O(x) = 0.$$

4. Find the interval of convergence for the power series  $\sum_{n=1}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ .

Observe that when  $x = \frac{1}{3}$ , the series becomes

$$\sum_{n=1}^{\infty} \frac{(-3)^n (\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n+1}}, \text{ which converges by alternating}$$

Series test, and thus radius  $R \geq \frac{1}{3}$ ;

Also, when  $x = -\frac{1}{3}$ , the series becomes

$$\sum_{n=1}^{\infty} \frac{(-3)^n (-\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1^n}{\sqrt{n+1}}, \text{ which diverges.}$$

and thus  $R \leq \frac{1}{3}$

Therefore,  $R = \frac{1}{3}$ , and the interval of convergence  
is  $[-\frac{1}{3}, \frac{1}{3}]$ .

5. Prove in a conditionally convergent series both the series of positive terms and the series of negative terms diverge.

Proof: Suppose  $\sum_{n=1}^{\infty} a_n$  is a series w/ positive and negative terms that is conditionally convergent.

i.e.  $\sum_{n=1}^{\infty} a_n$  is convergent, yet  $\sum_{n=1}^{\infty} |a_n|$  is divergent.

The series of positive terms is  $\sum_{n=1}^{\infty} \frac{a_n + |a_n|}{2}$ ;

and the series of negative terms is  $\sum_{n=1}^{\infty} \frac{a_n - |a_n|}{2}$ .

Now, since  $\sum_{n=1}^{\infty} a_n$  converges,  $|a_n|$  is bounded  $\forall n$ , by  $\beta$  say.

$$\text{then } \sum_{n=1}^{\infty} \frac{a_n + |a_n|}{2} \leq \sum_{n=1}^{\infty} \left( \frac{\beta}{2} + \frac{|a_n|}{2} \right) = \frac{n\beta}{2} + \sum_{n=1}^{\infty} \frac{|a_n|}{2}.$$

$$\text{and } \sum_{n=1}^{\infty} \frac{a_n - |a_n|}{2} \leq \sum_{n=1}^{\infty} \left( \frac{\beta}{2} - \frac{|a_n|}{2} \right).$$

but since  $\sum_{n=1}^{\infty} |a_n|$  diverges, and  $\frac{\beta}{2}$  is a constant,

both  $\sum_{n=1}^{\infty} \left( \frac{\beta}{2} + \frac{|a_n|}{2} \right)$  and  $\sum_{n=1}^{\infty} \left( \frac{\beta}{2} - \frac{|a_n|}{2} \right)$  diverge.

Therefore, both the series of positive and negative terms diverge.  $\square$ .