FM2 Test #3 Name: Maggie. 88 ½

Very good

1. Is the group $(\mathbb{Z}_7^*, \otimes)$ cyclic? Justify your answer.

Yes.

$\mathbb{Z}_7^* = \{1, 2, 3, 4, 5, 6\}$.

which can be generated by ~~any element except 1~~ either 3 or 5:

$\langle 3 \rangle = \{3, 2, 6, 4, 5, 1\}$.

$\langle 5 \rangle = \{5, 4, 6, 2, 3, 1\}$.

2. Find $\lim_{n\to\infty} \frac{\pi}{2n}\left(1 + \cos\frac{\pi}{2n} + \cos\frac{2\pi}{2n} + \cdots + \cos\frac{(n-1)\pi}{2n}\right)$.

~~Let~~ Let $f(x) = \cos x$, which is continuous and thus ~~differentiable~~ integrable on $[0, \frac{\pi}{2}]$.

So the lower Riemann Sum L_n, ~~which is~~ equals to $\int_0^{\frac{\pi}{2}} f(x)\,dx$ as n approaches ∞.

i.e. $\lim_{n\to\infty} \frac{\pi}{2n}\left(1 + \cos\frac{\pi}{2n} + \cdots + \cos\frac{(n-1)\pi}{2n}\right) \cancel{= \int_0^{\frac{\pi}{2}} f(x)\,dx}$

$= \int_0^{\frac{\pi}{2}} f(x)\,dx = [\sin x]_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = \boxed{1}$.

3. For each of the following either explain why the graph cannot exist or draw a graph with the given property.

(a) A graph with degree sequence 3, 2, 2, 1, 1.

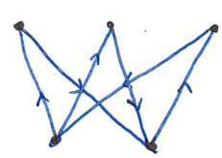
Since $\sum_{v\in V} \deg(v) = 2e$, where V is the set of vertices. and e is the number of edges.

yet in this case, $\sum_{v\in V} \deg(v) = 9$, which is not a multiple of 2.

So the graph does not exist.

(b) A complete bipartite graph on 5 vertices that has a Hamiltonian path and an Eulerian trail.

Hamiltonian path



Eulerian trail.

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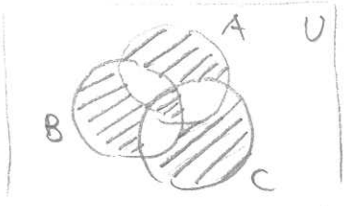
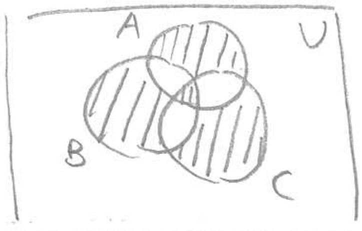
4. Draw Venn diagrams illustrating the sets $A \triangle (B \triangle C)$ and $(A \triangle B) \triangle C$. What is your conclusion?





$A \triangle (B \triangle C)$ $\qquad$ $(A \triangle B) \triangle C$

$\Rightarrow A \triangle (B \triangle C) = (A \triangle B) \triangle C$.

the operation of symmetric difference is associative.

5. The space $S = \langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \rangle$ is a subspace of $\mathbb{R}^3$. Find a Cartesian equation for S.

let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, c_1, c_2 are constants.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_1 + 3c_2 \\ 2c_1 + 2c_2 \\ 3c_1 + c_2 \end{pmatrix}.$$

Therefore, $x + z = 2y$.

which means, The space S is the plane $x - 2y + z = 0$.

6. Use the first three terms in the binomial expansion of $(1 + \frac{1}{8})^{1/3}$ to find an approximation to $\sqrt[3]{9}$. Give your answer as a fraction in simplest terms.

$$(1+\tfrac{1}{8})^{\frac{1}{3}} = \binom{\frac{1}{3}}{0} + \binom{\frac{1}{3}}{1}\frac{1}{8} + \binom{\frac{1}{3}}{2}\left(\frac{1}{8}\right)^2 + \cdots$$

$$\sqrt[3]{9} = 9^{\frac{1}{3}} = (1+8)^{\frac{1}{3}} = 2\left(\tfrac{1}{8}+1\right)^{\frac{1}{3}}$$

$- \frac{\cancel{2}}{9} \cdot \frac{1}{\cancel{2} \cdot 64}$

$$\approx 2\left[\binom{\frac{1}{3}}{0} + \binom{\frac{1}{3}}{1}\frac{1}{8} + \binom{\frac{1}{3}}{2}\left(\frac{1}{8^2}\right)\right].$$

$$= 2\left(1 + \frac{1}{3}\cdot\frac{1}{8} + \frac{\frac{1}{3}(-\frac{2}{3})}{2 \cdot 64}\right)$$

$$= 2\left(1 + \frac{1}{24} - \frac{1}{576}\right)$$

$$= 2\left(\frac{576 + 24 - 1}{576}\right)$$

$$= \frac{599}{288}.$$

7. Determine the rank, nullity and null space of the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \end{pmatrix}$.

Let $Ax=0$. which has AM:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 5 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & -3 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

{using technology}.

let $x_3 = s$,
$x_4 = t$,
$x_5 = r$,

$$\begin{cases} x_1 + (-1)x_3 - 2x_4 - 3x_5 = 0 \\ x_2 + 2x_3 + 3x_4 + 4x_5 = 0. \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = s + 2t + 3r \\ x_2 = -2s - 3t - 4r. \end{cases}$$

So Null A is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = s\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + r\begin{pmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad s,t,r \in \mathbb{R}.$$

which is the space spanned by

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$ ✓

So it has nullity $= 3$, ✓

and therefore ~~rank~~

rank = colume number − nullity
$= 5 - 3 = 2$. ✓

8. Use the fourth degree Maclaurin polynomial for $\cos x$ to show that $\pi/3$ approximately satisfies the equation $x^4 - 12x^2 + 12 = 0$. Hence calculate an approximate value for π expressing your answer as a surd.

The forth degree Maclaurin poly for $\cos x$ is:

$$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cancel{- \frac{x^6}{6!}}$$

when $x = \frac{\pi}{3}$.

$\cos(\frac{\pi}{3}) = \frac{1}{2}$.

$$\frac{1}{2} \approx P_4(\tfrac{\pi}{3}) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cancel{\frac{x^6}{6!}}$$

which is equivalent to

$$x^4 - 12x^2 + 12 = 0.$$

So $\frac{\pi}{3}$ approximately satisfies the equation.

$\sqrt{96}$
$= 4\sqrt{6}$

let $x^2 = y$. $y \geq 0$.

then $y^2 - 12y + 12 = 0$.

$$y = \frac{12 \pm \sqrt{144 - 48}}{2}$$

~~and $y \geq 0$.~~

so $y_1 = 6 + 2\sqrt{6}$.
$y_2 = 6 - 2\sqrt{6}$.

we take $y_2 = 6 - 2\sqrt{6}$

as $\cos(\frac{\pi}{3}) = \cos(\frac{5}{3}\pi)$.

and $\frac{\pi}{3} < \frac{5}{3}\pi$.

So $\pi = 3(\frac{\pi}{3})$

$\approx 3\sqrt{(6 - 2\sqrt{6})}$

$= 18 - 6\sqrt{6}$.

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9. Suppose $f : G \to H$ is a group homomorphism. Prove $\ker(f) \leq G$. "Quote any result you use".

proof. we use the 3-step subgroup test to show that $\ker(f) \leq G$.

(i) let $a, b \in \ker(f)$.

i.e. $f(a) = f(b) = e_H$.

then $f(ab) = f(a) * f(b)$

$= e_H * e_H$

$= e_H$.

because of homomorphism,

so $f(ab) \in \ker(f)$. and

$\ker(f)$ is closed.

(ii) since

~~$f(e_G) = e_H$,~~

$f(e_G \cdot a) = f(a)$

$= f(e_G) \cdot f(a)$.

post multiply by $f^{-1}(a)$, we get

$e_H = f(e_G)$.

so $e_G \in \ker(f)$.

so identity axiom holds.

(iii) $f(a \cdot a^{-1}) = f(a) \cdot f(a^{-1})$

~~$= f(a) \cdot f(a^{-1})$~~ ~~$f(e_G) = e_H$.~~

~~$\Rightarrow f^{-1}(a) = f(a^{-1})$.~~

$= e_H \cdot f(a^{-1})$.

on the other hand.

$f(a \cdot a^{-1}) = f(e_G) = e_H$.

so $\overset{e_H}{f(a)} \cdot f(a^{-1}) = e_H$.

i.e. $f(a^{-1}) = e_H$.

so $a^{-1} \in \ker(f)$. $\forall a \in \ker(f)$

Therefore, by the 3-step subgroup test, $\ker(f) \leq G$.

10. Evaluate $\lim_{x \to 0} (1 + \sin x)^{1/x}$. $= e$.

$$\lim_{x \to 0} (1 + \sin x)^{\frac{1}{x}}$$

$$= \lim_{x \to 0} \left(1 + \binom{\frac{1}{x}}{1} \sin x + \binom{\frac{1}{x}}{2} \sin^2 x + \cdots + \binom{\frac{1}{x}}{\frac{1}{x}} \sin^{\frac{1}{x}} x \right).$$

$$\cancel{= \lim_{x \to 0} \left(1 + \frac{1}{x} \sin x + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2} \sin^2 x + \cdots + \right.}$$

$$= \lim_{x \to 0} \sum_{k=0}^{\frac{1}{x}} \binom{\frac{1}{x}}{k} \sin^k x.$$

$$\sum_{n=0}^{\infty} \binom{y}{n}$$

$$= \lim_{x \to 0} \sum_{k=0}^{\frac{1}{x}} \frac{\left(\frac{1}{x}\right)!}{\left(\frac{1}{x} - k\right)!\, k!} \sin^k x.$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{y!}{(y-n-1)!\,(n+1)!} \cdot \frac{(y-n)!\,n!}{y!} \right|$$

$$= \left| \frac{y-n}{n+1} \right|$$

0

$$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{x}\right)!}{\left(\frac{1}{x} - k\right)!\, k!} \sin^k x.$$

$y - n > n + 1$

$(\ln x)'$ $1 + \sin x$

$= \frac{1}{x}$.

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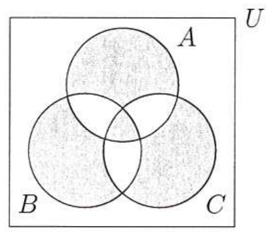
Solutions to FM2 Test #3

1. Since $\mathbb{Z}_7^* = \langle 3 \rangle$, $\mathbb{Z}_7^*$ is cyclic. (The only other generator of $\mathbb{Z}_7^*$ is 5.)

2. We recognize this limit as $\lim\limits_{n\to\infty} L_n$ for the integral $\int_0^{\pi/2} \cos x\,dx$, which evaluates to 1.

3. (a) Not possible as a graph has an even number of vertices of odd degree. (b) $K_{3,2}$ fulfills the criteria.

4. The Venn diagrams for $A \triangle (B \triangle C)$ and $(A \triangle B) \triangle C$ are the same. The common result is illustrated below. Hence the operation of symmetric difference is associative on sets.



5. $x - 2y + z = 0$.

6. First observe $\sqrt[3]{9} = (8+1)^{1/3} = 2(1+\frac{1}{8})^{1/3}$. The first three terms of this binomial expansion give

$$(1+\tfrac{1}{8})^{1/3} \approx 1 + \frac{1}{3}\cdot\frac{1}{8} + \frac{\frac{1}{3}\cdot\frac{-2}{3}}{2!}\cdot\frac{1}{64} = \frac{599}{576}.$$

We conclude $\sqrt[3]{9} \approx \dfrac{599}{288}$.

7. Using the GDC to find rref(A) we conclude rank$(A) = 2$ and therefore by the rank-nullity theorem nullity$(A) = 3$. The null space is

$$\langle \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \rangle.$$

8. The fourth degree Maclaurin polynomial for $\cos x$ is $P_4(x) = 1 - x^2/2! + x^4/4!$. Now $\cos(\pi/3) = 0.5$. So $P_4(\pi/3) \approx 0.5$. That is $\pi/3$ approximately satisfies the equation

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} = 0.5 \Leftrightarrow x^4 - 12x^2 + 12 = 0.$$

Calculation gives the appropriate root as $\sqrt{6 - 2\sqrt{6}}$. Hence $\pi \approx 3\sqrt{6 - 2\sqrt{6}}$.

9. See class notes.

10. Notice that this limit has the indeterminate form 1^∞. The standard approach for such a limit is to use logarithms. Letting $y = (1+\sin x)^{1/x}$ gives $\ln y = \dfrac{\ln(1+\sin x)}{x}$. Now using l'Hôpital's rule we have

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\frac{\cos x}{1+\sin x}}{1} = 1.$$

So our required limit is e.