Third lesson—playing games all at once!

<u>Outline</u>

Objectives:
1.Learn the equivalence of games;
2.Learn to play Nim with any number of piles;
3.Learn how to play all impartial combinatorial games using nim-value;
4.Summary on what we learned;
5.Provide the ultimate question.

Content:
3.1 Equivalence of games
      Definition 3.1
      Theorem 3.1
3.2 Playing any piles of Nim game
      Example of a 5 pile Nim game
3.3 Sprague-Grundy theorem
      Definition 3.2 (nim-value);
      Theorem 3.2 (Sprague-Grundy)
      Example of using SG theorem on a take-away game.
3.4 summary—what we have learned
3.5 what else we can explore

Exercise:
One final BIG question related to the application of all what we learned in three lessons.

3.1 Equivalence of games

Before we proceed to learning how to play multiple different games together, we first need to the know the concept of the equivalence of games.

Definition 3.1

Let G_1 and G_2 be two games, and let g_1 and g_2 be positions of G_1 and G_2, respectively. Then g_1 is equivalent to g_2 ($g_1 \approx g_2$) if, for any position h in any game H, $g_1 + h$ has the same N/P position as $g_2 + h$. (Crash Course 15)

That means two games are equivalent if they produce the same result in addition of games.

Theorem 3.1

$g_1 \approx g_2$ if and only if $g_1 + g_2$ is a P-position. (Crash Course 15)

Proof:

$g_1 \approx g_2 \Rightarrow g_1 + g_2$ is a P-position.

According to definition of equivalence, for any position in any game H, $g_1 + h$ has the same N/P position as $g_2 + h$. Let h be g_2.

Thus, $g_1 + g_2$ has the same N/P position as $g_2 + g_2$.

And since $g_2 + g_2$ is a P-position (Theorem 2.1), $g_1 + g_2$ is also a P-position.

$g_1 + g_2$ is a P-position $\Rightarrow g_1 \approx g_2$

If $g_1 + g_2$ is a P-position, and since adding a P-position has no effect on the result position (Theorem 2.3), $g_1 + h$ must have the same position has $(g_1 + h) + (g_1 + g_2)$.

Since the addition of games is commutative and associative,

$(g_1 + h) + (g_1 + g_2) = (g_2 + h) + (g_1 + g_1)$.

Again, since $g_1 + g_1$ is a P-position, according to Theorem 2.3, $(g_2 + h)$ has the same N/P position as $(g_2 + h) + (g_1 + g_1)$, which equals to $(g_1 + h) + (g_1 + g_2)$, which has the same position as $g_1 + h$, so $g_1 \approx g_2$.

Therefore, $g_1 \approx g_2$ if and only if $g_1 + g_2$ is a P-position. //

Corollary 3.1

If $g_1 \approx g_2$ and h is any position, then $g_1 + h \approx g_2 + h$.

Proof:

$(g_1 + h) + (g_2 + h) = (g_1 + g_2) + (h + h)$.

Since both $(g_1 + g_2)$ and $(h + h)$ are P-positions (Theorem 2.2), $(g_1 + g_2) + (h + h)$ is also a P-position, and so is $(g_1 + h) + (g_2 + h)$.

According to Theorem 3.1, $g_1 + h \simeq g_2 + h$. //

Corollary 3.2

$g \simeq 0$ if and only if $g + 0$ is a P-position.

Proof:

According to Theorem 3.1, $g_1 \simeq g_2$ if and only if $g_1 + g_2$ is a P-position.

When $g_1 = g$ and $g_2 = 0$,

$g \simeq 0$ if and only if $g + 0$ is a P-position. //

Corollary 3.3

Two nimheaps are equivalent if and only if they are the same size.

Proof:

Let X, Y denote two nimheaps, $X \geq Y$.

$X = Y \Rightarrow X \simeq Y$

If $X = Y$, $X \oplus Y$ is a P=position according to Theorem 2.2, so $X \simeq Y$.

$X \simeq Y \Rightarrow X = Y$, i.e. $X \neq Y \Rightarrow X$ not equivalen to Y

Since $X > Y$, $X \oplus Y$ can be moved to $Y \oplus Y$, a P-position, so $X \oplus Y$ itself is a N-position.

Thus, X is not equivalent to Y.

Therefore, $X = Y$ iff $X \simeq Y$. //

3.2 Playing any piles of Nim game

Do you remember the Nim table we constructed in Lesson 2?

	0	1	2	3	4	5	6	7	8	9	10	...
0	0	1	2	3	4	5	6	7	8	9	10	...
1	1	0	3	2	5	4	7	6	9	8	11	...
2	2	3	0	1	6	7	4	5	10	11	8	...
3	3	2	1	0	7	6	5	4	11	10	9	...
4	4	5	6	7	0	1	2	3	12	13	14	...
5	5	4	7	6	1	0	3	2	13	12	15	...
6	6	7	4	5	2	3	0	1	14	15	12	...
7	7	6	5	4	3	2	1	0	15	14	13	...
8	8	9	10	11	12	13	14	15	0	1	2	...
9	9	8	11	10	13	12	15	14	1	0	3	...
10	10	11	8	9	14	15	12	13	2	3	0	...
...	...	...	...	...	...	...	...	...	...	...	...	...

Now that we have learned the equivalence of games, we can understand that the Nim table is not only for solving 3-pile Nim games, but also for all Nim games, no matter how many piles there are. Because by saying $X \oplus Y \oplus Z$ is a P-position, we are really saying that $X \oplus Y \simeq Z$.

Let's look at some examples of how to play a Nim game with more than 3 piles:

Example 3.1

In a 4-pile Nim game with a position of (4, 7, 3, 8), as the first player, what can you do to win the game?

Solution:

According to Nim table,

$4 \oplus 7 \oplus 3 \oplus 8 \simeq 3 \oplus 3 \oplus 8$ (because $4 \oplus 7 = 3$)

$\simeq 8$ (because $3 \oplus 3 = 0$)

Thus, the position (4, 7, 3, 8) is a N-position, and since you happen to go first, you want to make it a P-position by taking all 8 M&Ms in the 8-pile.

Also, since the position is equivalent to 8, when this position is added to other impartial games, it behaves exactly like a Nimheap with 8 M&Ms.

Example #2:

Now you are playing the Nim game with 5 piles of (2, 10, 9, 4, 5), would you go first or second?

Solution:

$2 \oplus 10 \oplus 9 \oplus 4 \oplus 5 \approx 8 \oplus 9 \oplus 4 \oplus 5$ (because $2 \oplus 10 = 8$)

$\qquad\qquad\quad \approx 1 \oplus 4 \oplus 5$ (because $8 \oplus 9 = 1$)

$\qquad\qquad\quad \approx 5 \oplus 5$ (because $1 \oplus 4 = 5$)

$\qquad\qquad\quad \approx 0$

Thus, this is a P-position and in order to win the game, you would choose to be the second player.

3.3 Sprague-Grundy theorem

Now it's time to know why we are so concerned with Nim game (we played that for the whole second lesson!). The answer lies in the theorem below:

Theorem 3.2 —Sprague-Grundy (Crash Course 16)
Any position g in an impartial game is equivalent to a nimheap. i.e., there is always some nonnegative integer X such that $g \approx X$.
And X is known as the **nimvalue** of g, writes as $|g| = X$.

Proof by strong mathematical induction:
Let a g denote a position in an impartial game with height n.
We first check for n=0.
All terminal positions are P-positions and thus have nimvalue 0.
Now we want to show that if the statement is true for all height less than k, i.e. $n = 1$, $n = 2$, …, $n = k$, then the statement is also true for height k+1.
Assume for every position g with height less than k+1 is equivalent to a nimheap.
Let $S = \{h1, h2, \ldots hk\}$ be the set of positions to which can be moved to from g.
Since all the elements of S have height less than k+1, each $hi \in S$ is equivalent to some nimheap Xi. Then we claim that $g \approx M$, where $M = MEX(X_1,\ldots,X_k)$.

We want to show that $g \oplus M$ is a P-position by showing that all moves from $g \oplus M$ are N-positions. There are two ways to move from $g + M^*$:
First, there can be a move in g to some $h_i \in S$. The resulting position is $h_i \oplus M$, which is equivalent to $X_i \oplus M$, which is an N-position unless $M = X_i$, which is impossible by the definition of MEX.

Also, there can be moves from M.

The resulting position is $g \oplus M_1$, for some $M_1 < M$.

By definition of MEX, $M_1 = X_i$ for some i.

Thus it is possible to move from $g \oplus M_1$ to $h_i \oplus M_1 \simeq X_i \oplus M_1 = M_1 \oplus M_1 \simeq 0$.

This means $g \oplus M_1$ is an N -position.

We have shown that all moves from $g \oplus M$ are to N-positions. This implies that $g \oplus$ M is a P-position, so $g \simeq X$, and the statement is true for height k+1.

Since the truth of all height less than K=1 implies the truth of height k+1, by strong mathematical induction, the theorem is true for g with height of positive integer n. //

The reason why Sprague-Grundy is so important is that you can literally solve any impartial games with this theorem in mind and compute the nimvalue of every position using the MEX rule.

Then, you will find that you are actually playing various Nim games with n-piles. Below is an example to illustrate how it works:

Again, we go back to where we start—a take-away game. We denote the game position by Tx.

Now we play three of it once, with positions of 11, 16, and 2019 M&Ms. Valid moves consist of eating 1, 2, or 4 M&Ms.

Still, it will be very convenient if we construct a Nim table like this:

X	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
Tx	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	...

Here is how it is constructed:

First, we know the terminal position $|T_0| = 0$;

Building on this, we have $|T_1| = \text{MEX}(|T_0|) = \text{MEX}(0) = 1$

$|T_2| = \text{MEX}(|T_1|, |T_0|) = \text{MEX}(0, 1) = 2$

$|T_3| = \text{MEX}(|T_1|, |T_2|) = \text{MEX}(1, 2) = 0$

$|T_4| = \text{MEX}(|T_0|, |T_2|, |T_3|) = \text{MEX}(0, 2, 0) = 1$

…

Continuing in this way, we again find a cycle of 0, 1, 2 repeats.

Since $2019 \equiv 0 \pmod 3$

$|T_{2019}| = 0$

Thus, $11+15+2019 \simeq 2 \oplus 1 \oplus 0$

$$\simeq 3 \oplus 0$$

$$\simeq 3$$

Therefore, (11, 15, 2019) is a N-position, and you would want to go first and make it a P-position for the next player.

3.4 summary—what we have learned

1. Identify impartial combinatorial games;
2. Decompose games into addition of single impartial games and analyze each individually;
3. Find the nimvalues for each of the game using the MEX rule and construct a Nim table;
4. Play different impartial games as if playing Nim.

3.5 what else we can explore

However, this lessons can only serve as an introduction to game theories.
Still, there are many other impartial combinatorial games, which can be found at this website www.math.ucla.edu/~tom/Game_Theory/Contents.html.

Moreover, impartial combinatorial games are just a small part of game theories. There are way more interesting topics under game theory like Zero-sum and Genial-sum games, which can also be explored in this website https://www.math.ucla.edu/~tom/Game_Theory/Contents.html.

3.6 Exercises.

Do you still remember the game of Empty-And-Divid we played at the first lesson? Let's look at it again, but now we play the addition of three empty-and-divide games: There are three games, two piles in each, a valid move consists of eating up all M&Ms in the chosen game and split the other into two new piles each with at least one M&M.
The one who cannot further split any pile loses, i.e. the one who is left with six piles of 1 M&M loses.

Now Alice has to move. She has the position of (2, 4), (5, 9) and (3, 6).
What should she do in next in order to win this game?

In another round, Alice is faced with the position (2,4), (6, 43) and (2019, 9102).
What should she do this time in order to win the game?

What about four Empty-and-Divid playing together?
Try to analyze the position of (2, 4), (6, 9), (7, 8) and (3,5).

Explain your answer using what you learned in the three lessons.
You may want to construct a Nim table and use Sprague-Grundy theorem.

Reference:

Ferguson, Thomas. "Game Theory, Second Edition, 2014." *Game Theory*, Mathematics Department, UCLA, 2014, www.math.ucla.edu/~tom/ Game_Theory/Contents.html.

Mathcamp. "Crash Course on Combinatorial Game Theory" 2019. https:// www.mathcamp.org/2019/cgt.pdf