The Normal Distribution

$X \sim N(169, 4^2)$

1. The average height of army recruits in Mongolia is 169 cm with a standard deviation of 4 cm. If the heights are normally distributed, find the probability that a recruit selected at random is between 160 and 180 cm in height. How many recruits in a group of 500 could be expected to be greater than 165 cm in height?

   0.985 ; 421

2. The IB examination scores for geography appear to be reasonably bell shaped with a mean of 60 and a standard deviation of 10. The chief examiner decides to grade "on the curve" giving 5% 7s and 1s, 10% 6s and 2s, 20% 5s and 3s, and the remaining 30% 4s. Find the 3–4 grade boundary and the 6–7 grade boundary.

   $X \sim N(60, 10^2)$ 56 ; 76

3. Mensa is an international high-IQ society that welcomes people from every walk of life whose IQ is in the top 2% of the population. What is the minimum IQ to be a member of Mensa? What IQ will you need to be in the brainiest quarter of Mensa members? (Take the mean and standard deviation to be 100 and 15 respectively.)

   ~~130~~ ; 131 ; 138.

4. Moreberry Jams fill their $\frac{1}{2}$ kg jam jars with a machine that delivers 505.4 g on average with a standard deviation of 2.2 g. If 3000 jars are filled in a day, estimate how many will be underweight.

   $Y = (X_1 + X_2) \sim N(1010.8, 9.68)$ 397

5. Les Slater takes 43 minutes on average to make it to first period class with a standard deviation of 3 minutes. How long should Les allow to be sure of being punctual 99% of the time?

   ~~Not~~ $X \sim N(43, 3^2)$ 50 min

6. Amy Rican took SAT math and verbal tests in the course of applying for admission to college. The mean and standard deviation for the population of math scores were 470 and 100 respectively, whereas for the verbal test the population mean and standard deviation were 430 and 100 respectively. Amy's test results were 620 for math and 600 for verbal. Find her SAT z-scores and decide whether her math or verbal score will be viewed more favourably by college admission.

   $X \sim N(470, 100)$ $Y \sim N(430, 100)$ $Z_x = \frac{620-470}{100} = 1.5$ $Z_y = \frac{600-430}{100}$ 1.7

7. A machine filling cereal packets produces a standard deviation of 4.5 g in its process. If only 1 out of 200 packets filled weighs more than 263 g, estimate the mean weight of packets filled.

   $X \sim N(\mu, 4.5^2)$ $P(X > 263) = \frac{1}{200}$ $P(Z > \frac{263-\mu}{4.5}) = \frac{1}{200}$ $\mu = \cancel{147}$ 251

8. Mrs Trotter catches the same train to work every morning. She timed its arrival on 100 successive days and found it arrived after 8.43 am on 1 day out of 100 and before 8.32 am on 9 days out of 100. Estimate to the nearest minute the mean and standard deviation of the arrival times.

   $X \sim N(\mu, \sigma^2)$

   $P(X > 43) = \frac{1}{100}$.

   $P(X < 32) = \frac{9}{100}$.

   $\Rightarrow$ ~~$P(Z > \frac{43-\mu}{})$~~ $\begin{cases} 43 - \mu = \sigma(2.33) \\ 32 - \mu = \sigma(-1.34) \end{cases}$

   $\begin{cases} \mu = 36.0 \\ \sigma = 3.00 \end{cases}$

9. A mathematics teacher told four of his students their test scores in a playful manner. He told them that the distribution of the class's scores was approximately normal with mean 70 and standard deviation 9, and he gave them the following information about their own scores.

   Jack: "Your score is the upper quartile."

   Jill: "Your raw score is 88."

   José: "Your standard score is -1.4."

   Juan: "Only 10% of the class had higher scores than yours."

   Fill in the missing scores in the following table.

   $X \sim N(70, 9^2)$

Name	Raw score	Standard score	Percentile
Jack	76.1	~~0.67~~ 0.6745	75
Jill	88	2	97.7
José	57.4	-1.4	8.08
Juan	81.5	1.28	90

$P(Z < \frac{15-\mu}{\sigma}) - P(Z < \frac{10-\mu}{\sigma}) =$

$P(Z < \frac{20-\mu}{\sigma}) - P(Z < \frac{15-\mu}{\sigma}) =$

10. If $X \sim N(\mu, \sigma^2)$ with $P(10 < X < 15) = 0.1$ and $P(15 < X < 20) = 0.2$, find μ and σ.

    $P\left(\frac{10-\mu}{\sigma} < Z < \frac{15-\mu}{\sigma}\right) = 0.1$

Answers to selected problems:
1. 0.985, ≈ 421 2. 76.4, 56.1 3. 131, 139 4. ≈ 21 5. 50 min 6. $z_m = 1.5$, $z_v = 1.7$, verbal
7. 251 g 8. 8.36 am, 3 min 9. 76.1, 0.674, 2, 97.7, 57.4, 8.08, 81.5, 1.28 10. 23.1, 7.50

Exercise

1. The maximum load a lift can carry is 450 kg. The weights of men are normally distributed with mean 60 kg and standard deviation 10 kg. The weights of women are normally distributed with mean 55 kg and standard deviation 5 kg. Find the probability that the lift will be overloaded by 5 men and 2 women, if their weights are independent. (L)

2. The mass, X g, of a grade A apple sold in a supermarket is a normally distributed random variable having mean 212 g and standard deviation 12 g. The mass, Y g, of a grade B apple sold on a market stall is a normally distributed random variable having mean 150 g and standard deviation 20 g. Find, to three decimal places,
   (i) $P(X < 230)$,
   (ii) the probability that a random sample of 9 grade B apples sold on the market stall has a total mass exceeding 1.5 kg,
   (iii) $P(X - Y > 37)$. (NEAB)

3. In the game of snooker, a match is decided by playing a series of 'frames'. The length of time taken to play one frame is modelled by a normal distribution with mean 28 minutes and standard deviation 5 minutes, and the time for any frame is assumed to be independent of the time for other frames. In the first session of a championship final, four frames are scheduled, with no interval between frames.
   (i) Define fully the distribution of the total time to play four frames of snooker.
   (ii) Calculate the probability that play in the first four frames of the championship will last for more than $2\frac{1}{4}$ hours. (C)

4. The time, W minutes, for a certain commuter to travel from home to work each morning may be modelled by a normal random variable with mean 45 and variance 10. The time, H minutes, for the commuter's journey home each evening may be modelled by a normal random variable with mean 60 and variance 35. All journeys may be considered to be independent.
   (a) Calculate the probability that, on a particular day, the total time for the commuter's two journeys is more than two hours.
   (b) The commuter makes each journey five times per week. Calculate the probability that, in a particular week, the commuter's total travelling time to and from work is less than 8 hours.
   (c) Calculate the probability that, on a particular evening, the commuter's journey home takes more than half as long again as the time taken to travel to work that morning. (NEAB)

5. Students have to sit examinations in English and Science. The marks obtained in these examinations are independent and may be approximated by normal distributions having means and standard deviations as shown in the table below.

	Mean	Standard deviation
English	100	18
Science	80	22

Find the probability that the sum of the marks obtained in the two examinations by a student chosen at random is greater than 210.
The marks obtained in Science are increased by 25% for all students. Show that the probability that the sum of the marks obtained in the two examinations by a student chosen at random is greater than 210 is now approximately 0.38. (C)

6. An engineering company buys steel rods and steel tubes. Without heating the tubes so that they expand, an insufficient proportion of the rods will fit inside the tubes. Measured in centimetres, the internal diameter at air temperature of a randomly chosen tube is denoted by T, and the diameter at air temperature or a randomly chosen rod by R. It is given that $T \sim N(4.000, 0.10^2)$, $R \sim N(4.020, 0.10^2)$, and that T and R are independent. By considering the distribution of $T - R$, find the probability that a randomly chosen rod would fit inside a randomly chosen tube, without heating the tube. The tubes are heated so that the internal diameter of each tube increases by 5%. Find the probability that a randomly chosen rod fits inside a randomly chosen tube, after the tube has been heated. (C)

7. The distribution of the masses of adult husky dogs may be modelled by the normal distribution with mean 37 kg and standard deviation 5 kg.
   (a) Calculate the probability that an adult husky has a mass greater than 30 kg.
   (b) Calculate the probability that a randomly chosen team of six huskies has a total mass lying between 198 kg and 240 kg, giving your answer to 3 decimal places. (NEAB)

1. Let M be the weight, in kg, of a man. $M \sim N(60, 10^2)$
Let W be the weight, in kg, of a woman. $W \sim N(55, 5^2)$.
Let $X = M_1 + M_2 + M_3 + M_4 + M_5 + W_1 + W_2$
$E(X) = 5 \times 60 + 2 \times 55 = 410$
$\text{Var}(X) = 5 \times 10^2 + 2 \times 5^2 = 550$
So $X \sim N(410, 550)$

$$P(X > 450) = P\left(Z > \frac{450 - 410}{\sqrt{550}}\right)$$
$$= P(Z > 1.706) = 0.044 \text{ (3 d.p.)}$$

2. $X \sim N(212, 12^2)$, $Y \sim N(150, 20^2)$

(i) $$P(X < 230) = P\left(Z < \frac{230 - 212}{12}\right)$$
$$= P(Z < 1.5) = 0.993 \text{ (3 d.p.)}$$

(ii) Let $B = Y_1 + Y_2 + \ldots + Y_9$
$E(B) = 9 \times 150 = 1350$
$\text{Var}(B) = 9 \times 20^2 = 3600 = 60^2$
So $B \sim N(1350, 60^2)$
$$P(B > 1500) = P\left(Z > \frac{1500 - 1350}{60}\right)$$
$$= P(Z > 2.5) = 0.006 \text{ (3 d.p.)}$$

(iii) $X - Y \sim N(212 - 150, 12^2 + 20^2)$
i.e. $X - Y \sim N(62, 544)$
$$P(X - Y > 37) = P\left(Z > \frac{37 - 62}{\sqrt{544}}\right)$$
$$= P(Z > -1.072) = 0.858 \text{ (3 d.p.)}$$

3. (i) Let S be the time, in minutes, to play a frame, then $S \sim N(28, 5^2)$.
Let $X = S_1 + S_2 + S_3 + S_4$
$E(X) = 4 \times 28 = 112$
$\text{Var}(X) = 4 \times 5^2 = 100 = 10^2$

(ii) So $X \sim N(112, 10^2)$
$$P(X > 135) = P\left(Z > \frac{135 - 112}{10}\right)$$
$$= P(Z > 2.3) = 0.0107$$

4. $W \sim N(45, 10)$, $H \sim N(60, 35)$

(a) $W + H \sim N(45 + 60, 10 + 35)$
$\sim N(105, 45)$
$$P(W + H > 120) = P\left(Z > \frac{120 - 105}{\sqrt{45}}\right)$$
$$= P(Z > 2.236) = 0.0127$$

(b) Let $X = W_1 + W_2 + W_3 + W_4 + W_5 + H_1 + H_2 + H_3 + H_4 + H_5$
$E(X) = 5 \times 45 + 5 \times 60 = 525$
$\text{Var}(X) = 5 \times 10 + 5 \times 35 = 225 = 15^2$
So $X \sim N(525, 15^2)$
$$P(X < 480) = P\left(Z < \frac{480 - 525}{15}\right)$$
$$= P(Z < -3) = 0.001\,35$$

(c) $P(H > 1\tfrac{1}{2} W) = P(2H > 3W)$
$= P(2H - 3W > 0)$
$E(2H - 3W) = 2E(H) - 3E(W) = -15$
$\text{Var}(2H - 3W) = 4\,\text{Var}(H) + 9\text{Var}(W)$
$= 230$
So $2H - 3W \sim N(-15, 230)$
$$P(2H - 3W > 0) = P\left(Z > \frac{0 - (-15)}{\sqrt{230}}\right)$$
$$= P(Z > 0.989) = 0.1612$$

5. $E \sim N(100, 18^2)$, $S \sim N(80, 22^2)$
Let $T = E + S$
$T \sim N(100 + 80, 18^2 + 22^2)$
i.e. $T \sim N(180, 808)$
$$P(T > 210) = P\left(Z > \frac{210 - 180}{\sqrt{808}}\right)$$
$$= P(Z > 1.055) = 0.1457$$
Science marks are multiplied by 1.25, so
$S \sim N(1.25 \times 80, 1.25^2 \times 22^2)$
$S \sim N(100, 27.5^2)$
$\therefore\ T \sim N(100 + 100, 18^2 + 27.5^2)$
$T \sim N(200, 1080.25)$
$$P(T > 210) = P\left(Z > \frac{210 - 200}{\sqrt{1080.25}}\right)$$
$$= P(Z > 0.304) = 0.381 \text{ (3 d.p.)}$$

6. $T \sim N(4.000, 0.10^2)$, $R \sim N(4.020, 0.10^2)$
$T - R \sim N(4.000 - 4.020, 0.10^2 + 0.10^2)$
$T - R \sim N(-0.020, 0.20)$
$$P(T - R > 0) = P\left(Z > \frac{0 - (-0.020)}{\sqrt{0.02}}\right)$$
$$= P(Z > 0.1414) = 0.4439$$
After heating,
$T \sim N(1.05 \times 4.000, 1.05^2 \times 0.10^2)$
i.e. $T \sim N(4.2, 0.105^2)$
$T - R \sim (4.2 - 4.020, 0.105^2 + 0.10^2)$
so $T - R \sim N(0.18, 0.145^2)$
$$P(T - R > 0) = P\left(Z > \frac{0 - 0.18}{0.145}\right)$$
$$= P(Z > -1.241) = 0.8927$$

7. $H \sim N(37, 5^2)$

(a) $$P(H > 30) = P\left(Z > \frac{30 - 37}{5}\right)$$
$$= P(Z > -1.4) = 0.9192$$

(b) Let $X = H_1 + H_2 + H_3 + H_4 + H_5 + H_6$
$E(X) = 6 \times 37 = 222$
$\text{Var}(X) = 6 \times 5^2 = 150$
So $X \sim N(222, 150)$
$$P(198 < X < 240) = P\left(\frac{198 - 222}{\sqrt{150}} < Z < \frac{240 - 222}{\sqrt{150}}\right)$$
$$= P(-1.96 < Z < 1.47) = 0.904 \text{ (3 d.p.)}$$