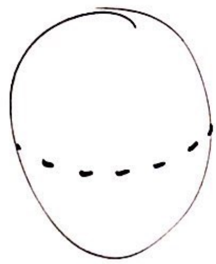
Euclide	Spherical geometry
Segment of lines	Segment of great circles
Circle	Circle on the sphere



Dfn: An a isometry on $\mathbb{S}^2$ is a map $\mathbb{S}^2 \to \mathbb{S}^2$ that preserves distance

Prp: Isometries on $\mathbb{S}^2$ are a group (for the composition).

Rotation:

In $\mathbb{R}^2$, $R(\theta): \mathbb{R}^2 \longrightarrow \mathbb{R}$

$x \longmapsto Ax$

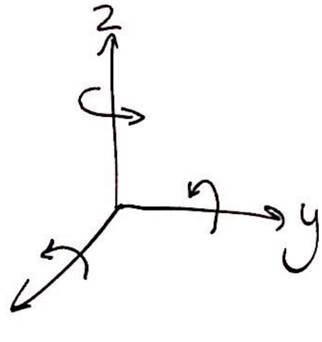
where $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

The rotation about the z-axis through an angle of θ is

$$R(z,\theta): S^2 \longrightarrow S^2$$
$$p \longmapsto A_{z,\theta}\, p$$

where $A_{z,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

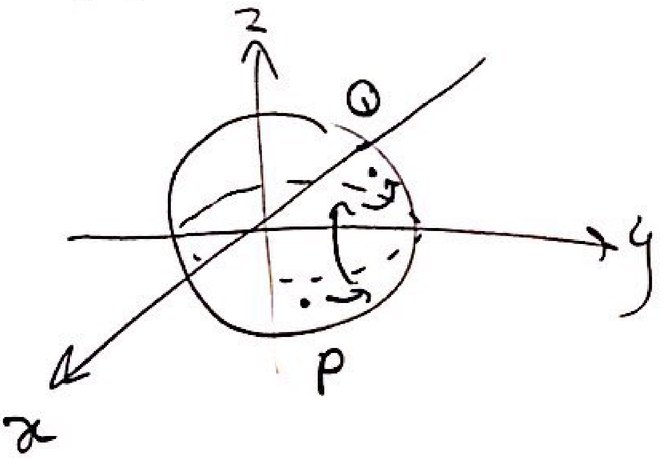
Similarly, $A_{x,\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$



xis Through

and $A_{y,\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

What about rotation in general?



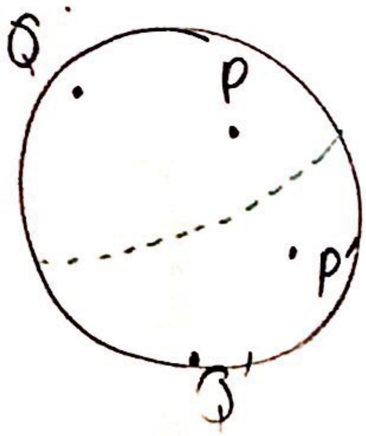
We compose many (at most 3) rotations about the main axes

What is the inverse of a rotation?

$$R(z,\theta)^{-1} = R(z,-\theta)$$

$$A_{y,-\theta} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Reflections in great circles



Let π be the plan cutting the great circle.

Let's write its equation as $ax + by + cz = 0$

We want to compute A_π s.t

$$S_\pi : \mathbb{S}^2 \longrightarrow \mathbb{S}^2 \quad \text{is the reflection in } \pi$$
$$p \longmapsto A_\pi p$$

WLOG, take a, b, c s.t $a^2+b^2+c^2=1$

Notation: For $u=(u_1, u_2, u_3)$ and $v=(v_1, v_2, v_3)$, write

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Facts: $(u_1+u_2)\cdot v = u_1 \cdot v + u_2 \cdot v$

$$u \cdot (v_1+v_2) = u \cdot v_1 + u \cdot v_2$$

$$(ku)\cdot v = k(u \cdot v) = u \cdot kv$$

Take $v=(a,b,c)$ and $p=(x,y,z)$

We know that $S(p) = p + kv$

$$S(p) \in S^2 \Rightarrow (p+kv)\cdot(p+kv) = 1$$
$$\Rightarrow p\cdot p + 2k\, v\cdot p + k^2 v\cdot v = 1$$
$$\Rightarrow \cancel{1} + 2k\, v\cdot p + k^2 = \cancel{1}$$
$$\Rightarrow 2k\, v\cdot p + k^2 = 0$$

If $k=0$, then p lies on π, and then $v\cdot p = 0$

Else, $2v\cdot p + k = 0$ (always true)

$\Rightarrow k = -2v\cdot p$

$\Rightarrow S(p) = p - 2(v\cdot p)v$

$$\Rightarrow A_\pi = \begin{pmatrix} 1+2a^2 & -2ab & -2ac \\ -2ab & 1+2b^2 & -2bc \\ -2ac & -2bc & 1+2c^2 \end{pmatrix}$$

Rmk: $S_\pi^{-1} = S_\pi$

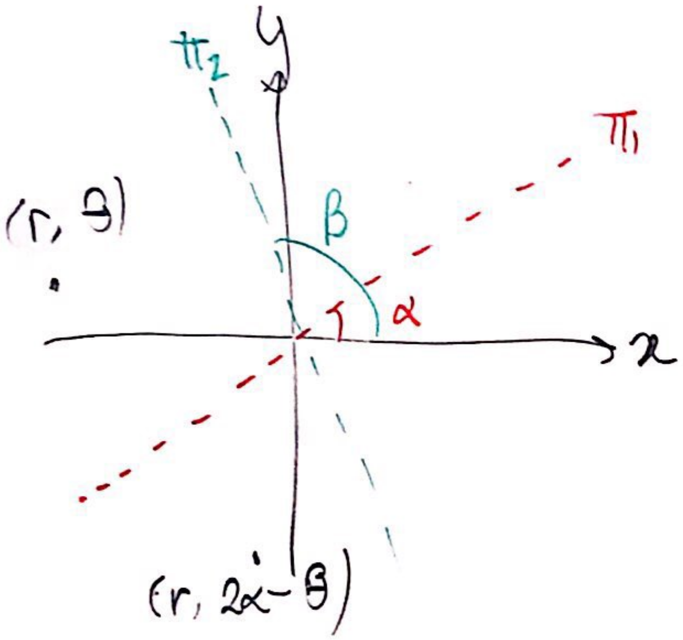
Prp: The product of any $\overbrace{\text{two reflections}}^{(S_{\pi_1},\ S_{\pi_2})}$ of $\mathbb{S}^2$ is a rotation around the line at the intersection of π_1 and π_2

Idea of the proof

By choosing coordinates axes suitably, we can arrange the the intersection line is the z-axis.

$\Rightarrow$ The z-coordinate is unaltered by both S_{π_1} and S_{π_2}

Look at the effect on π_1 and π_2 only on the $(x-y)$-plane



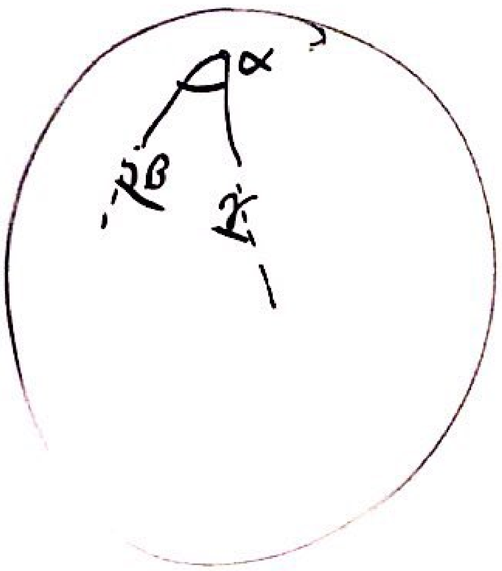
$$\pi_1 : (r, \theta) \mapsto (r, 2\alpha - \theta)$$

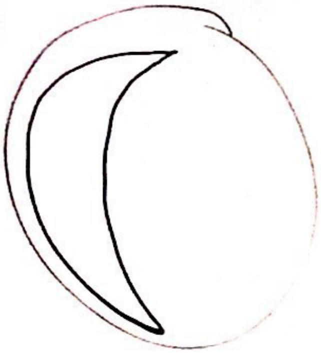
$$\pi_2 : (r, 2\alpha - \theta) \mapsto (r, 2\beta - (2\alpha - \theta))$$

$$= (r, \theta + 2\beta - 2\alpha)$$

Lemma: A triangle in S^2 is determined by its angles

Idea of the proof





The area of a lune of angle θ is 2θ because the sphere is a lune of angle 2π and of aera 4π.

Girard's theorem: The area of a triangle of angles α, β, and γ is $\alpha+\beta+\gamma-\pi$.