FURTHER MATHEMATICS
HIGHER LEVEL

98%

Excellent!

Tuesday 28 January 2020

Name in block letters

1 hour 10 minutes

M	A	G	G	I	E						

INSTRUCTIONS TO CANDIDATES

- Do not open this test until instructed to do so.
- Answer all 10 questions.
- A graphic display calculator is required for this test.
- A clean copy of the formula booklet is required for this test.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working or explanations. Where an answer is incorrect, some marks may be given for a correct method provided this is shown by written working. You are therefore advised to show all working. Working may be continued below the lines, if necessary.

1. Use L'Hôpital's rule to find $\lim_{x\to 0}(\csc x - \cot x)$.

$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{\tan x}\right)$

$= \lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right) = \frac{0}{0}$, so L'Hopital's rule applies.

So $= \lim_{x\to 0} \frac{(1-\cos x)'}{(\sin x)'}$

$= \lim_{x\to 0} \frac{\sin x}{\cos x}$

$= \lim_{x\to 0} \tan x$

$= 0$

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2. The relation R is defined on $\mathbb{R}$ by $x\,R\,y$ if $|x| + |y| = |x + y|$.

   (a) Show that R is reflexive.

   (b) Show that R is symmetric.

   (c) Show by means of a counterexample that R is not transitive.

$|0| + |2| = |2|$

$|0| + |-2| = |2|$.

(a) $\forall x \in \mathbb{R}$,

$$|x| + |x| = \begin{cases} 2x, & x \geq 0 \\ -2x, & x < 0. \end{cases}$$

$$|x + x| = |2x| = \begin{cases} 2x, & x \geq 0 \\ -2x, & x < 0. \end{cases}$$

Therefore, $|x| + |x| = |x+x|$, and xRx $\forall x \in \mathbb{R}$.

(b) If $|x| + |y| = |x+y|$. xRy, $x, y \in \mathbb{R}$,
then $|y| + |x| = |y+x|$, xRy,
therefore symmetric

(c) let $x = 2$, $y = 0$, $z = -2$,
$|x| + |y| = 2 = |x+y|$, so xRy,
$|y| + |z| = 2 = |y+z|$, so yRz.
However,
$|x| + |z| = |2| + |2| = 4$
$|x+z| = |2-2| = 0$.
So $|x| + |z| \neq |x+z|$. $x \not R z$.
Therefore, not transitive.

3. Consider the differential equation $dy/dx = y^3 - x^3$ with $y = 1$ when $x = 0$. Use Euler's method in table form with a step length of 0.1 to approximate the value of y when $x = 0.4$.

$\frac{dy}{dx} = y^3 - x^3$

n	x_n	y_n	h	$h \times f(x_n, y_n)$
0	0	1	0.1	0.1
1	0.1	1.1	0.1	0.133
2	0.2	1.233	0.1	0.18665
3	0.3	1.41965	0.1	0.28342
4	0.4	1.7031		

So when $x = 0.4$

$y \approx 1.70$ (3 s.f.)

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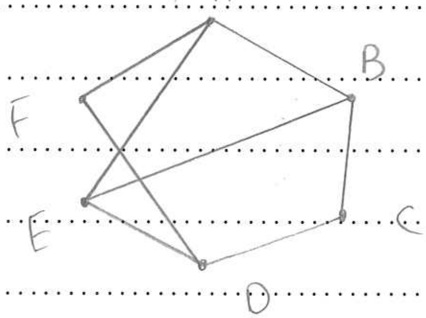
4. Consider the simple graph G with adjacency matrix $\begin{array}{c|c} & \begin{matrix} A & B & C & D & E & F \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$.

(a) Draw G and its complement G'.

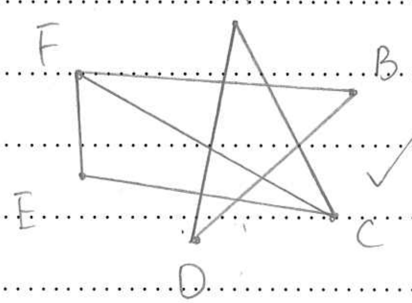
(b) State whether or not G is planar giving a reason for your answer.

(c) State whether or not G is bipartite giving a reason for your answer.

(a)

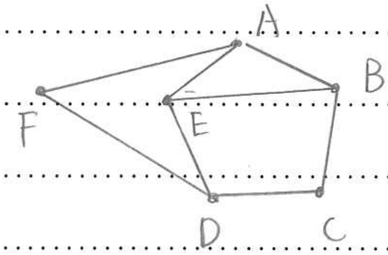


G



G'

(b) YES. It can be drawn in a way such that no ~~edg~~ edges intersect.



(c) G ~~A~~ is not bipartite since it contains the triangle (3-cycle) AEB.

If A and B are in different partitions, then E should be different from both A and B, which is impossible.

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$-(1-a-d+ad)$

$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$ $\quad -(a+d-1)$

-1

5. Consider the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) If $a + b = c + d = 1$ show that 1 is an eigenvalue of M.

(b) Find the eigenvectors for M when $a = 2$, $b = -1$, $c = 3$ and $d = -2$.

(a) $\lambda^2 - (a+d)\lambda + ad - bc = 0$

$b = 1-a$, $\cancel{d = 1-c}$. $c = 1-d$.

$\lambda^2 - (a+d)\lambda + ad - (1-a)(1-d) = 0$

$\lambda^2 - (a+d)\lambda + \overset{+ad}{a+d} - 1 - ad = 0$.

~~when $\lambda = 1$.~~

$\cancel{1 - (a+d) + a+d - 1 \; ad = 0}$

So ~~$\lambda = 1$ is a solution and thus an eigenvalue~~.

$[\lambda - (a+d-1)](\lambda - 1) = 0$.

So $\lambda_1 = a+d-1$, $\lambda_2 = 1$.

So 1 is an eigenvalue of M.

(b) $\lambda_1 = a+d-1 = 2-2-1 = -1$

$\lambda_2 = 1$.

when $\lambda = -1$.

$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

when $\lambda = 1$

$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

6. Consider the permutation $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 2 & 1 \end{pmatrix}$.

(a) Find the order of p justifying your answer.

(b) Find p^2.

(c) The permutation group G is generated by p. Find the element of G that is of order 2 giving your answer in cycle notation.

(a) $p = (1\ 3\ 5\ 2\ 4\ 6)$

So ~~the order of~~ $p^6 = e$,

so the order is 6.

(b) $p^2 = (1\ 3\ 5\ 2\ 4\ 6)(1\ 3\ 5\ 2\ 4\ 6)$

$= (1\ 5\ 4)(3\ 2\ 6)$

(c) Since p is of order 6,

p^3 is of order 2.

$p^3 = (1\ 5\ 4)(3\ 2\ 6)(1\ 3\ 5\ 2\ 4\ 6)$

$= (1\ 2)(3\ 4)(5\ 6)$

7. A linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^4$ is represented by the matrix $M = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 7 & 5 \\ -3 & 1 & 4 \\ 1 & 5 & 4 \end{pmatrix}$.

(a) Find the rank of M.

(b) Find a basis for the range of T.

(c) Find the kernel of T.

(a) $M \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & 7 & 7 \\ 0 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = R$

Therefore, there are 2 rows of leading 1s, so rank = 2.

(b) ~~From~~ Take the two columns with leading 1s in R, we find in M the basis

$\begin{pmatrix} 1 \\ 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 1 \\ 5 \end{pmatrix}$, Hence the range $\langle \begin{pmatrix} 1 \\ 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 1 \\ 5 \end{pmatrix} \rangle$ ✓

(c) The kernel of T is the null space of M.

let $x_3 = s$,

$x_1 = s$, $x_2 = -s$,

So $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} s$, $s \in \mathbb{R}$.

So the kernel of T is

$\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \rangle$. ✓

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8. Determine whether each of the following infinite series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{2n^2+5}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{5^n (n!)^2}$

(a) Diverges.

$\cancel{\lim_{n\to\infty} \frac{3n}{2n^2+5}}$

$\cancel{\lim_{n\to\infty} \frac{\frac{3}{n}}{2+\frac{5}{n^2}}}$

Using the comparison test

since $3n^2 > 2n^2+5 \quad \forall\, n \geq 3$

$$\frac{3n}{3n^2} < \frac{3n}{2\ 3n^2+5} \quad \forall\, n \geq 3.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{3n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges, it follows that $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{2n^2+5}$ diverges.

(b) Apply the ratio test.

$$\lim_{n\to\infty} \left| \frac{(2n+2)!}{5^{n+1}[(n+1)!]^2} \cdot \frac{5^n (n!)^2}{(2n)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)}{5(n+1)^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2}{5} \cdot \frac{2n+1}{n+1} \right| = \frac{4}{5} < 1.$$

Therefore, it converges.

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9. In $\triangle ABC$, the angles are $A = \frac{5\pi}{8}$, $B = \frac{\pi}{8}$ and $C = \frac{\pi}{4}$. Point P is the foot of the perpendicular from A to side $[BC]$, point Q is the midpoint of side $[AC]$, and point R is the intersection of the angle bisector of angle C with side $[AB]$.

(a) Show $\dfrac{AR}{BR} = \tan\frac{\pi}{8}$.

(b) Show $\dfrac{BP}{CP} = \tan\frac{3\pi}{8}$.

(c) Hence show that the lines (AP), (BQ) and (CR) are concurrent.

(a) $\dfrac{AR}{BR} = \dfrac{AC}{BC} = \dfrac{\sin B}{\sin A}$.

$= \dfrac{\sin\left(\frac{\pi}{8}\right)}{\sin\left(\frac{5\pi}{8}\right)}$ $\cancel{= 0.414 \ (3 \text{ s.f.})}$

$= \dfrac{\sin\left(\frac{\pi}{8}\right)}{\cos\left(\frac{\pi}{8}\right)} = \tan\left(\frac{\pi}{8}\right)$

(b) $\dfrac{BP}{CP} = \dfrac{BP}{AP}$, since $\angle ACP = \frac{\pi}{4}$, $\angle APC = 90°$.

$\dfrac{BP}{AP} = \dfrac{\sin\left(\frac{3\pi}{8}\right)}{\sin\left(\frac{\pi}{8}\right)} = \dfrac{\sin\left(\frac{3\pi}{8}\right)}{\cos\left(\frac{3\pi}{8}\right)} = \tan\left(\frac{3\pi}{8}\right)$.

(c). since $\dfrac{CQ}{QA} = 1$.

$\dfrac{AR}{RB} \cdot \dfrac{BP}{PC} \cdot \dfrac{CQ}{QA} = \tan\left(\frac{\pi}{8}\right) \cdot \tan\left(\frac{3\pi}{8}\right)$

$= 1$.

Therefor, accordng to Ceva's theorem, (AP), (BQ), (CR) are concurrent.

10. The random variable $X \sim \mathrm{NB}(r, p)$ has probability generating function $G_X(t) = \dfrac{p^r t^r}{(1-qt)^r}$.

   (a) Use this probability generating function to find $\mathrm{E}(X)$.

Consider another independent random variable $Y \sim \mathrm{NB}(s, p)$ and let $W = X + Y$.

   (b) i. Find the probability generating function for W.

   ii. Hence identify the distribution that W follows and state its parameters.

   iii. Given that $r = 2$ and $s = 3$, calculate $P(X = 3 \mid W = 7)$.

(a) $G_X'(t) = [p^r t^r (1-qt)^{-r}]'$

$= p^r t^r \cdot rq(1-qt)^{-r-1} + rp^r t^{r-1}(1-qt)^{-r}$.

$E(X) = G_X'(1)$

$= p^r \cdot rq(1-q)^{-r-1} + rp^r(1-q)^{-r}$

$= p^r \cdot rq \cdot p^{-r-1} + rp^r \cdot p^{-r}$

$= p^{-1} rq + r$

$= r(p^{-1}q + 1) = \dfrac{r}{p}$ ✓

(b) (i) $W = G_X(t) \times G_Y(t)$

$= \dfrac{p^r t^r}{(1-qt)^r} \cdot \dfrac{p^s t^s}{(1-qt)^s}$

$= \dfrac{p^{r+s} t^{r+s}}{(1-qt)^{s+r}}$. ✓

(ii) So $W \sim NB(s+r, p)$. ✓

(iii) $P(X=3 \mid W=7) = \dfrac{P(Y=4), P(X=3)}{P(W=7)}$

$= \dfrac{\binom{3}{2} p^3 q^1}{\binom{6}{4} p^5 q^2} = \dfrac{3p^3 q}{15 p^5 q^2}$

$= \dfrac{1}{5p^2 q}$ ✗ $\dfrac{2}{5}$

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