Submission: FM2 Test # 5
Maggie Huang
Due: 03/27/20

1. Find unbiased estimates of the population parameters μ and σ^2 if a random sample of size 10 from this population gives $\sum x_i^2 = 180$ and $\sum x_i = 30$.

$$\mu = E(X) = \frac{\sum x_i}{10} = 3$$

$$\sigma^2 = \frac{n}{n-1} S_n^2 = \frac{10}{9}\left(\frac{\sum x_i^2}{10} - 3^2\right) = \frac{10}{9} \times 9 = 10$$

2. The lengths of king fish are normally distributed with mean μ metres and standard deviation 0.12 metres. If 20% of king fish are longer than 0.70 metres, find the value of μ.

Let X be the length of a randomly chosen king fish. Then $X \sim N(\mu, 0.12^2)$

$$P(X \geq 0.7) = 0.2$$

$$P(Z \geq \frac{0.7 - \mu}{0.12}) = 20,$$

$$\text{invNorm(0.2, 0, 1, RIGHT)} = 0.842$$

$$\Rightarrow \frac{0.7 - \mu}{0.12} = 0.842 \Rightarrow \mu = 0.599 \text{ (3 s.f.).}$$

3. Email messages arrive at an office at an average rate of four per hour. Find the probability that more than two messages are received between 11:00 am and 11:45 am on Monday morning.

Let X be the number of messages in a given hour. Then $X \sim Po(4)$.
Let $Y = 3/4X$, $Y \sim Po(3)$

$$P(Y > 2) = 1 - \text{poissoncdf}(3, 2) = 0.577(3s.f)$$

4. A random sample of 400 pet dogs shows a mean life expectancy of 10.3 years. Assuming that $\sigma = 3.2$ years, find a 90% confidence interval for the average life-span of all pet dogs.

Let X be a randomly chosen pet dog. Then $\bar{X} \sim N(10.3, \frac{3.2^2}{400})$

The 90% confidence interval is

$$\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}} \Rightarrow 10.3 \pm 1.645 \times \frac{3.2}{20}$$

Therefore, the confidence interval is $[10.0, 10.6]$.

5. The maximum load an elevator can carry is 550 kg. The weights of men are normally distributed with mean 70 kg and standard deviation 10 kg. The weights of women are normally distributed with mean 60 kg and standard deviation 5 kg. Find the probability that the elevator will be overloaded by five men and three women, if their weights are independent.

Let M be the weight of a randomly chosen man, W be the weight of a randomly chosen woman, and S be the total weight of 5 randomly chosen men and 3 randomly chosen women.

Then $M \sim N(70, 100)$, $W \sim N(60, 25)$. So $S = 5M + 3W \sim N(530, 575)$.

$$\mathrm{P}(\mathrm{S} \geq 550) = \mathrm{normalcdf}(550, \infty, 530, \sqrt{575}) = 0.202(3\text{ s.f.})$$

6. The sample mean of the random variable $X \sim NB(4, p)$ is $\bar{X}$.

$\bar{X} \sim N(4/p, 4q/p^2)$

(a) $\mathrm{E}(\bar{X}) = 4/p$.

(b) $\bar{X}/4$. $\mathrm{E}(\bar{X}/4) = \frac{4/p}{4} = 1/p$.

7. A large number of leaves of species A were measured and their lengths were found to have a mean of 64 mm and a standard deviation of 8 mm. A particular bush is discovered with leaves that are visually similar to species A. A random sample of 100 leaves from this bush were collected and they were shown to have a mean length of 61 mm. Do you think the bush is species A?

We use hypothesis testing at 5% level of significance to determine if the bush is species A.

Null hypothesis H_0: $\mu = 64$.
Alternative hypothesis H_1: $\mu \neq 64$.

Test statistic:

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

p-value $\approx 0 << 0.05$

Therefore, there is sufficient evidence for us to reject the null hypothesis at a 5% level of significance. Hence the bush is not species A.

8. The normal range of phosphorus in the blood is 26 to 48 mg/l. Alice is a patient who is thought to have kidney disease. Her blood is tested on six different occasions giving phosphorous readings in mg/l of

$$56, 53, 46, 48, 57, 61$$

If her phosphorus level varies normally, is there evidence at the 5% level of significance that Alice has a mean phosphorous level that exceeds 48 mg/l?

We apply hypothesis testing:

Null hypothesis H_0: $\mu = 48$
Alternative hypothesis H_1: $\mu > 48$.

Test statistic:

$$T = \frac{\bar{x} - \mu}{S/\sqrt{n}} \sim t(5)$$

p-value $= 0.0320 < 0.05$

Therefore, there is sufficient evidence to reject the null hypothesis at 5% level of significance. Hence Alice has a mean phosphorous level exceeding 48mg/l.

9. A student proposes a new method to produce random numbers from the interval $[0, 1]$. The method is implemented on a computer. His command to generate 100 random numbers gives a mean value $\bar{x} = 0.5784$. Is this result significant at the 1% level($\alpha = 0.01$)? What does this result tell you about his new method?

Let X be the number generated. The probability density function is

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & elsewhere \end{cases} \quad \text{, then}$$

$$Var(X) = \int_0^1 (x - 0.5)^2 f(x) dx = 1/12.$$

Null hypothesis H_0: $\mu = 0.5$
Alternative hypothesis H_1: $\mu \neq 0.5$.

Test statistic:

$$Z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1)$$

p-value $= 0.00661 < 0.01$

Therefore, there is sufficient evidence to reject the null hypothesis at 1% level of significance. Therefore, this mew method is not a good enough random number generator.

10. A random sample of size 100 is taken from a population with mean μ and variance 4 to test the null hypothesis that $\mu = 60$ against the alternative hypothesis that $\mu > 60$ at the 5% level of significance.

(a) $\alpha = 0.05$

(b) Test statistic:

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{x} - 60}{2/10} \sim N(0, 1)$$

The critical value is

$$\text{invNorm}(0.5, 0, 1, \text{RIGHT}) = 1.645$$

$$\Rightarrow \frac{\bar{x} - 60}{2/10} < 1.654 \Rightarrow x < 60.3$$

Since

$$\beta = P(X < 60.3 \mid \mu = \mu) = 0.25$$

$$P(Z < \frac{60.3 - \mu}{2}) = 0.25$$

$$\frac{60.3 - \mu}{2} = \text{invNorm(0.25, 0, 1, LEFT)} = -0.675$$

$$\Rightarrow \mu = 61.6 \text{ (3 s.f.)}$$