FM2 Assignment #21 Name: Maggie. $\frac{9}{10}$

1. Must a matrix with an eigenvalue of 0 be a singular matrix? Make sure to justify your answer.

Let matrix A and λ be its eigenvalue.

$A\vec{v} = \lambda\vec{v}$.

$\det(A - \lambda I) = 0$

when $\lambda = 0$.

$\det(A) = 0$.

Thus matrix A is singular.

2. Find the image of the point $(1, 3)$ under a rotation of $30°$ clockwise about $(-1, 1)$ by first translating the centre of rotation to the origin, then rotating about the origin using a rotation matrix and lastly translating back.

① $\begin{pmatrix}1\\3\end{pmatrix} - \begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$

② $\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}$

$= \begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}$

$= \begin{pmatrix}\sqrt{3}+1\\ -1+\sqrt{3}\end{pmatrix}$

③ $\begin{pmatrix}\sqrt{3}+1\\ -1+\sqrt{3}\end{pmatrix} + \begin{pmatrix}-1\\1\end{pmatrix}$

$= \begin{pmatrix}\sqrt{3}\\ \sqrt{3}\end{pmatrix}$

3. Find the characteristic equation, eigenvalues and eigenvectors of the matrix $M = \begin{pmatrix}4 & -5\\ 1 & -2\end{pmatrix}$. $-8+5$

① Characteristic equation:

$\lambda^2 - trM\lambda + \det M$

$\Leftrightarrow \lambda^2 - 2\lambda - 3$. (*)

② Eigenvalues:

by solving (*), we get

$\lambda = 3$, or -1.

So the eigenvalues are 3 and -1.

③ Eigen vectors:

(i) when $\lambda = 3$.

$\begin{pmatrix}1 & -5\\ 1 & -5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 0$

we spot the solution $\begin{pmatrix}5\\1\end{pmatrix}$, which is an eigenvector corresponding to $\lambda = 3$.

(ii) When $\lambda = -1$.

$\begin{pmatrix}5 & -5\\ 1 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 0$.

~~the~~ We spot the solution $\begin{pmatrix}1\\1\end{pmatrix}$, which is an eigenvector corresponding to $\lambda = -1$.

$$(\arcsin x)^{(7)} = 7!\, C_7.$$

4. By integrating the binomial series for $\dfrac{1}{\sqrt{1-x^2}}$ find the seventh derivative of $\arcsin x$ at $x = 0$.

Recall that

$$(1-x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n. \quad w/ \quad -1 \le x \le 1.$$

So $$\int (1-x^2)^{\frac{1}{2}} = \arcsin x = c + \sum_{n=0}^{\infty} \frac{\binom{-\frac{1}{2}}{n} (-1)^{2n+1} x^{2n+1}}{2n+1}$$

Substituting $x=0$. $c=0$.

Hence, $$\arcsin x = \sum_{n=0}^{\infty} \frac{\binom{-\frac{1}{2}}{n} (-1)^{2n+1}}{2n+1}\, x^{2n+1}$$

So $$f^{(7)}(0) = -7! \cdot \binom{-\frac{1}{2}}{3} \cdot \frac{1}{7}$$

$$= -\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{4}\right)\left(\tfrac{1}{7}\right) \cdot \cancel{7} \cdot \cancel{6}^{3} \cdot 5 \cdot \cancel{4}$$

$$= 225$$

5. Write the matrix $M = \begin{pmatrix} -11 & 4 \\ -6 & 2 \end{pmatrix}$ as the product of elementary matrices and hence describe the transformation represented by M as a sequence of basic transformations.

$(M \mid I)$

$$= \left(\begin{array}{cc|cc} -11 & 4 & 1 & 0 \\ -6 & 2 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 1 & -\frac{4}{11} & -\frac{4}{11} & 0 \\ -6 & 2 & 0 & 1 \end{array}\right) \qquad E_1 = \begin{pmatrix} -\frac{4}{11} & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\sim \left(\begin{array}{cc|cc} 1 & -\frac{4}{11} & -\frac{4}{11} & 0 \\ 0 & -\frac{2}{11} & -\frac{24}{11} & 1 \end{array}\right) \qquad E_2 = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}$$

$$\sim \left(\begin{array}{cc|cc} 1 & -\frac{4}{11} & -\frac{4}{11} & 0 \\ 0 & 1 & 12 & -\frac{11}{2} \end{array}\right) \qquad E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{11}{2} \end{pmatrix}$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & 4 & -2 \\ 0 & 1 & 12 & -\frac{11}{2} \end{array}\right) \qquad E_4 = \begin{pmatrix} 1 & \frac{4}{11} \\ 0 & 1 \end{pmatrix}$$

This should be M^{-1} but $M^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -4 \\ 4 & -11 \end{pmatrix}$

Also $E_4 E_3 E_2 E_1 M = I$

So $M = (E_4 E_3 E_2 E_1)^{-1} I$

$= E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$

You had $M = E_4 E_3 E_2 E_1$

$$\text{So } M = \begin{pmatrix} 1 & \frac{4}{11} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -\frac{11}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} -\frac{4}{11} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So the transformation is achieved by first expand ~~xxx~~ horizontally by a factor of $-\frac{4}{11}$, then vertical shear with shear factor 6, and then expand vertically by a factor of $-\frac{11}{2}$, and finally a horizontal shear w/ shear factor $\frac{11}{4}$

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/3

FM2 Assignment #22 Name: Maggie $\frac{10}{10}$ Excellent!

1. Let A be an invertible matrix with eigenvalue λ and corresponding eigenvector $\vec{v}$. Prove that A^{-1} must hawe the eigenvalue $1/\lambda$ together with the same corresponding eigenvector $\vec{v}$.

$A\vec{v} = \lambda\vec{v}$

$\vec{v} = A^{-1}\lambda\vec{v}$

as λ is a constant,

$\vec{v} = \lambda A^{-1}\vec{v}$

$\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$.

which means that A^{-1} have the eigenvalue $\frac{1}{\lambda}$ w/ correspondng eigenvector $\vec{v}$. □

✓

2. Let $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$. Show that $H_n \leq 1 + \ln n$. Hence show that $H_{1\,000\,000\,000} < 22$.

$H_n - 1$ is the lower Riemann Sum of the function $y = \frac{1}{x}$ on the interval $[1, n]$

since $Ln \leq \int_1^n f(x)dx \leq Un$.

$H_n - 1 \leq \int_1^n \frac{1}{x}dx$

since $\int_1^n \frac{1}{x}dx = \ln n$,

$H_n \leq 1 + \ln n$.

✓

when $n = 1\times10^9$

$H_{1\times10^9} \leq 1 + \ln(10^9)$

$\leq 21.7 < 22$ ✓

{using technology}.

3. Prove that similarity is an equivalence relation on the set of $n \times n$ matrices.

proof. Let $\sim$ be a relation s.t. $A \sim B$ if

$A = PBP^{-1}$, where A, B are $n\times n$ matrices, P is a non-sigular $n\times n$ matrix

① reflexive.

$AI = IA$, so $A = IAI^{-1}$. ✓ ✓

② symmetric ✓

if $A = PBP^{-1}$, then $AP = PB \Rightarrow P^{-1}AP = B$ ✓

③ transitive

if $A = P_1BP_1^{-1}$. $B = P_2CP_2^{-1}$. then ~~$AP = PB$~~ ~~$A = P$~~.

$A = P_1P_2CP_2^{-1}P_1^{-1} = (P_1P_2)C(P_1P_2)^{-1}$ ✓ $\Rightarrow A \sim C$. ✓ 6

therefore, similarity $\sim$ is an equivalence relation. □

4. Factorize the matrix $A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$ in the form PDP^{-1} where D is a diagonal matrix. Hence find A^8.

Let the eigenvalues of A be λ.

$\lambda^2 - tr(A)\lambda + det(A) = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$\lambda_1 = 1$. $\lambda_2 = 2$.

$1 \quad -1$
$1 \quad -2$

when $\lambda = 1$.

$\begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The corresponding eigenvector $\vec{v_1}$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

when $\lambda = 2$.

$\begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

the corresponding eigenvector $\vec{v_2}$ is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Let $P = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Thus,

$A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}^{-1}$

$A^8 = (PDP^{-1})^8$

$= PD^8P^{-1}$

$= \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 256 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 766 & -765 \\ 510 & -509 \end{pmatrix}$ ✓

5. Use matrix methods to solve the recurrence relation $u_n = 5u_{n-1} - 6u_{n-2}$ given $u_1 = 1$ and $u_2 = 2$.

$\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_{n-1} \\ u_{n-2} \end{pmatrix}$

$= \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}^{n-2}\begin{pmatrix} u_2 \\ u_1 \end{pmatrix}$

$= \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}^{n-2}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Let $\begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}$ w/ eigenvalues λ.

Then $\lambda^2 - 5\lambda + 6 = 0$

$\lambda_1 = 2$, $\lambda_2 = 3$.

when $\lambda_1 = 2$

$\begin{pmatrix} 3 & -6 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

when $\lambda_2 = 3$.

$\begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\vec{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Thus,

$\begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}^{-1}$

$\begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}^{n-2} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{n-2}\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}^{-1}$

$= \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2^{n-2} & 0 \\ 0 & 3^{n-2} \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$

$= \begin{pmatrix} 2^{n-1} & 3^{n-1} \\ 2^{n-2} & 3^{n-2} \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$

$= \begin{pmatrix} -2^{n-1} + 3^{n-1} & 3\cdot 2^{n-1} - 2\cdot 3^{n-1} \\ -2^{n-2} + 3^{n-2} & 3\cdot 2^{n-2} - 2\cdot 3^{n-2} \end{pmatrix}$

Therefore,

$\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} -2^{n-1} + 3^{n-1} & 3\cdot 2^{n-1} - 2\cdot 3^{n-1} \\ -2^{n-2} + 3^{n-2} & 3\cdot 2^{n-2} - 2\cdot 3^{n-2} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$u_n = 2(-2^{n-1} + 3^{n-1}) + (3\cdot 2^{n-1} - 2\cdot 3^{n-1})$

$= \boxed{2^{n-1}}$ ✓

FM2 Assignment #23 Name: Maggie. $\frac{10}{10}$ Excellent!

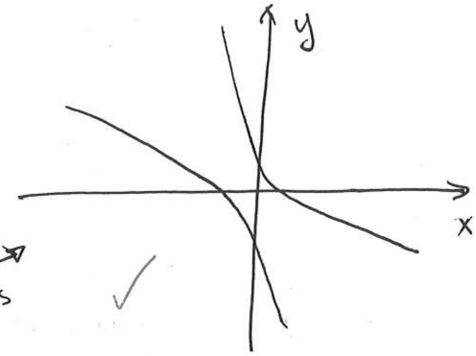
1. Calculate the discriminant of the conic $4x^2 + 10xy + y^2 = 1$. Hence determine if the conic is an ellipse, hyperbola or parabola. Confirm your result using desmos.

$\Delta = 10^2 - 4(4) = 84 > 0$

So it is hyperbola.

(confirmed w/ desmos!)

like this



✓

2. The ellipse $x^2 + xy + y^2 = 1$ is rotated 45° anticlockwise about the origin. Find the equation of the rotated ellipse.

The ellipse in matrix form is:

$(x\ \ y)\begin{pmatrix}1 & \frac12 \\ \frac12 & 1\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} = 1$

let the rotated ellipse be

$\begin{pmatrix}x' \\ y'\end{pmatrix} = \begin{pmatrix}\cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°)\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix}$

$\Leftrightarrow \begin{pmatrix}x \\ y\end{pmatrix} = \begin{pmatrix}\frac{\sqrt2}{2} & \frac{\sqrt2}{2} \\ -\frac{\sqrt2}{2} & \frac{\sqrt2}{2}\end{pmatrix}\begin{pmatrix}x' \\ y'\end{pmatrix}$

So $(x'\ y')\begin{pmatrix}\frac{\sqrt2}{2} & -\frac{\sqrt2}{2} \\ \frac{\sqrt2}{2} & \frac{\sqrt2}{2}\end{pmatrix}\begin{pmatrix}1 & \frac12 \\ \frac12 & 1\end{pmatrix}\begin{pmatrix}\frac{\sqrt2}{2} & \frac{\sqrt2}{2} \\ -\frac{\sqrt2}{2} & \frac{\sqrt2}{2}\end{pmatrix}\begin{pmatrix}x' \\ y'\end{pmatrix} = 1$

$(x'\ y')\begin{pmatrix}1 & -1 \\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & \frac12 \\ \frac12 & 1\end{pmatrix}\begin{pmatrix}1 & 1 \\ -1 & 1\end{pmatrix}\begin{pmatrix}x' \\ y'\end{pmatrix} = 2.$

$(x'\ y')\begin{pmatrix}1 & 0 \\ 0 & 3\end{pmatrix}\begin{pmatrix}x' \\ y'\end{pmatrix} = \cancel{2}.$

$\Rightarrow \boxed{x^2 + 3y^2 = 2.}$ ✓

3. Diagonalize the matrix of the hyperbola $x^2 + 2\sqrt{3}xy - y^2 = 2$. Hence determine the hyperbola's eccentricity.

Matrix of hyperbola:

$(x\ \ y)\underline{\begin{pmatrix}1 & \sqrt3 \\ \sqrt3 & -1\end{pmatrix}}\begin{pmatrix}x \\ y\end{pmatrix} = 2.$

Solving $\lambda^2 - 4 = 0$

$\lambda_1 = 2$ or $\lambda_2 = -2$.

when $\lambda_1 = 2$. $v_1 = \begin{pmatrix}-\sqrt3 \\ 1\end{pmatrix}$

when $\lambda_2 = -2$ $v_2 = \begin{pmatrix}1 \\ \sqrt3\end{pmatrix}$

So

$\begin{pmatrix}1 & \sqrt3 \\ \sqrt3 & -1\end{pmatrix} = \begin{pmatrix}1 & -\sqrt3 \\ \sqrt3 & 1\end{pmatrix}\begin{pmatrix}-2 & 0 \\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & -\sqrt3 \\ \sqrt3 & 1\end{pmatrix}^{-1}$

$= \begin{pmatrix}\frac12 & -\frac{\sqrt3}{2} \\ \frac{\sqrt3}{2} & \frac12\end{pmatrix}\begin{pmatrix}2 & 0 \\ 0 & -2\end{pmatrix}\begin{pmatrix}\frac12 & \frac{\sqrt3}{2} \\ -\frac{\sqrt3}{2} & \frac12\end{pmatrix}$

let $\begin{pmatrix}x' \\ y'\end{pmatrix} = \begin{pmatrix}\frac12 & \frac{\sqrt3}{2} \\ -\frac{\sqrt3}{2} & \frac12\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix}$

then $(x'\ y')\begin{pmatrix}2 & 0 \\ 0 & -2\end{pmatrix}\begin{pmatrix}x' \\ y'\end{pmatrix} = 2.$

So $2x^2 - 2y^2 = 2$.

$x^2 - y^2 = 1$. which has eccentricity $\sqrt2$,

As the ~~h~~ hyperbola is transformed through rotation, the eccentricity of the original matrix is still $\sqrt2$.

✓ 6

4. Diagonalize the matrix of the ellipse $5x^2 + 8xy + 11y^2 = 42$. Through what acute angle must the ellipse be rotated to align its major axis with the x-axis?

In matrix form:

$$(x \;\; y)\begin{pmatrix}5&4\\4&11\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=42.$$

$\begin{pmatrix}5&4\\4&11\end{pmatrix}$ has eigenvalues and eigenvectors:

$\lambda^2-16\lambda+39=0$

$\lambda_1 = 3.$ $\quad \vec{v}_1=\begin{pmatrix}2\\1\end{pmatrix}$

$\lambda_2 = 13$ $\quad \vec{v}_2=\begin{pmatrix}-1\\2\end{pmatrix}$

So
$$\begin{pmatrix}5&4\\4&11\end{pmatrix}=\begin{pmatrix}2&-1\\1&2\end{pmatrix}\begin{pmatrix}3&0\\0&13\end{pmatrix}\begin{pmatrix}2&-1\\1&2\end{pmatrix}^{-1}$$
$$=\begin{pmatrix}\frac{2}{\sqrt5}&-\frac{1}{\sqrt5}\\\frac{1}{\sqrt5}&\frac{2}{\sqrt5}\end{pmatrix}\begin{pmatrix}3&0\\0&13\end{pmatrix}\begin{pmatrix}\frac{2}{\sqrt5}&\frac{1}{\sqrt5}\\-\frac{1}{\sqrt5}&\frac{2}{\sqrt5}\end{pmatrix}$$

let $\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}\frac{2}{\sqrt5}&\frac{1}{\sqrt5}\\-\frac{1}{\sqrt5}&\frac{2}{\sqrt5}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$, so

$$(x' \;\; y')\begin{pmatrix}3&0\\0&13\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix}=42$$
$$\Leftrightarrow 3x'^2+13y'^2=42.$$

and since $\begin{pmatrix}\frac{2}{\sqrt5}&-\frac{1}{\sqrt5}\\\frac{1}{\sqrt5}&\frac{2}{\sqrt5}\end{pmatrix}$ is a rotation matrix w/ an angle of $\arctan(\frac{1}{2})$, anticlockwise.

5. Prove that a graph with no odd cycles is bipartite.

proof. Let G be a connected graph w/o odd cycles.
Let V denote the set of vertices in G. and $v_0 \in V$.
Let: $H_1 = \{v \mid v\in V$, the shortest path from v_0 to v is odd$\}$.
$H_2 = \{v \mid v \in V$, the shortest path from v_0 to v is even$\}$.
We claim that H_1, H_2 is a partition of G to make it bipartite.
To show this, we first know that $H_1 \cup H_2 = G$ and $H_1 \cap H_2 = \emptyset$ as G is connected, and the shortest path between two vertices can only either be even or odd.
Then, suppose the opposite: this partition cannot make G bipartite. i.e. exist adjacent vertices in either H_1 or H_2. and since in either H_1 or H_2, the shortest path between any two verities must be even, and if these two vertices are adjacent, there is an odd ~~cycle~~ cycle. which is a contradiction.
Therefore, H_1, H_2 is a partition of G and G is bipartite. $\square$.

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FM2 Assignment #24

Name: Maggie. $\frac{15}{10}$ Excellent!

1. Prove that the set $S = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid ax + by + cz = 0\}$ is a subspace of $\mathbb{R}^3$.

a) since $a(0)+b(0)+c(0)=0$, $\vec{0} \in S$

b) let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in S$.

$\vec{u} + \vec{v} = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix}$,

and $a(u_1+v_1)+b(u_2+v_2)+c(u_3+v_3)$

$= (au_1+bu_2+cu_3)+(av_1+bv_2+cv_3)$

$= 0+0=0$

so $\vec{u}+\vec{v} \in S$

c). Let $k \in \mathbb{R}$,

$k\vec{u} = \begin{pmatrix} ku_1 \\ ku_2 \\ ku_3 \end{pmatrix}$

$aku_1+bku_2+cku_3$

$= k(au_1+bu_2+cu_3) = k0 = 0$

so $k\vec{u} \in S$

therefore, S is a subspace of $\mathbb{R}^3$.

2. Let $T\colon R^n \to \mathbb{R}^m$ be a linear transformation. Prove that $\operatorname{ran} T$ is a subspace of $\mathbb{R}^m$.

proof. a) $T(\vec{0}) = \vec{0}$, so $\vec{0} \in \operatorname{ran} T$.

b) let $\vec{u}, \vec{v} \in R^n$. then

$T(\vec{u}), T(\vec{v}) \in \operatorname{ran} T$.

since $\vec{u}+\vec{v} \in R^n$,

$T(\vec{u}) + T(\vec{v}) = T(\vec{u}+\vec{v}) \in \operatorname{ran} T$.

c) since $k\vec{u} \in R^n$ for $\vec{u} \in R^n$, and $k \in R$,

$kT(\vec{u}) = T(k\vec{u}) \in \operatorname{ran} T$. for $T(\vec{u}) \in R^m$.

Therefore, $\operatorname{ran} T$ is a subspace of R^m.

3. The system below has a particular solution $x = -1.5, y = 1.5, z = 0$. Find the general solution.

$$\begin{aligned} x + y - 2z &= 0 \\ x - y &= -3 \\ 3x - y - 2z &= -6 \\ 2y - 2z &= 3 \end{aligned}$$

The system has Augumented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & -3 \\ 3 & -1 & -2 & -6 \\ 0 & 2 & -2 & 3 \end{array}\right)$$

Since we already know the particular solution

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1.5 \\ 1.5 \\ 0 \end{pmatrix}$ satisfies the system.

In order to find the general solution, we only need to know the homogenous solutions.

Solve for $A x = \vec{0}$:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & -1 & -2 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \{\text{using technology}\}.$$

So rank = 2 and nullity = 1.

null $= \langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rangle$ = kernel of transformation.

so the general solution is

$S = \{ \begin{pmatrix} -1.5 \\ 1.5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mid t \in R \}$.

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4. Let T be the linear transformation with matrix $\begin{pmatrix}2&-1&-1\\-1&0&3\\0&-1&5\end{pmatrix}$. Find Cartesian equations for $\ker T$ and $\operatorname{ran} T$.

$A=\begin{pmatrix}2&-1&-1\\-1&0&3\\0&-1&5\end{pmatrix}\sim\begin{pmatrix}1&0&-3\\0&1&-5\\0&0&0\end{pmatrix}$,

which has

$\operatorname{rank} A = 2$. nullity$(A)=1$

and since $\ker T = \operatorname{null}(A)$,

$\ker T = \left\langle \begin{pmatrix}3\\5\\1\end{pmatrix}\right\rangle$, which is

the line in $\mathbb{R}^3$

$$\boxed{\frac{x}{3}=\frac{y}{5}=\frac{z}{1}}$$

✓

and since $\operatorname{ran} T = C(A)$,

and $\operatorname{rank}(A)=2$,

$\operatorname{ran} T = \left\langle \begin{pmatrix}2\\-1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\-1\end{pmatrix}\right\rangle$,

~~which has~~

$\begin{pmatrix}2&-1&0\\-1&0&-1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

we spot the solution

$\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}1\\2\\-1\end{pmatrix}$

So $\operatorname{ran} T$ is the plane in $\mathbb{R}^3$

$$\boxed{x+2y-z=0}$$

✓

5. Find a formula for M^n where $M=\begin{pmatrix}2b-a&a-b\\2b-2a&2a-b\end{pmatrix}$. Hence calculate M^{10} when $a=1$ and $b=2$.

To find M^n, we diagonalize M, by first finding the eigenvalues and eigenvectors;

$\lambda^2-(a+b)\lambda+ab=0$.

$\lambda_1=a$, $\vec{v}_1=\begin{pmatrix}1\\2\end{pmatrix}$ ✓

$\lambda_2=b$, $\vec{v}_2=\begin{pmatrix}1\\1\end{pmatrix}$

~~So $M=\begin{pmatrix}1&1\\-2&-1\end{pmatrix}\begin{pmatrix}\ &0\\0&b\end{pmatrix}($~~

So $M=\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}b&0\\0&a\end{pmatrix}\begin{pmatrix}1&1\\1&2\end{pmatrix}^{-1}$

$=\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}b&0\\0&a\end{pmatrix}\begin{pmatrix}2&-1\\-1&1\end{pmatrix}$

So $M^n=\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}b&0\\0&a\end{pmatrix}^n\begin{pmatrix}2&-1\\-1&1\end{pmatrix}$

$=\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}b^n&0\\0&a^n\end{pmatrix}\begin{pmatrix}2&-1\\-1&1\end{pmatrix}$

when $a=1$, $b=2$, $n=10$

$M^{10}=\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}2^{10}&0\\0&1^{10}\end{pmatrix}\begin{pmatrix}2&-1\\-1&1\end{pmatrix}$

$=\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}1024&0\\0&1\end{pmatrix}\begin{pmatrix}2&-1\\-1&1\end{pmatrix}$

$=\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}2048&-1024\\-1&1\end{pmatrix}$

$=$ ~~$\begin{pmatrix}2047&-1023\\-2046&-1022\end{pmatrix}$~~

$=\begin{pmatrix}2047&-1023\\2046&-1022\end{pmatrix}$ ✓.

4

FM2 Assignment #25 $L\begin{pmatrix}3\\3\end{pmatrix} = L\left(\begin{pmatrix}2\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}2\\4\end{pmatrix}\right)$ Name: Maggie. $\frac{9\frac{1}{2}}{10}$ Very good

1. The linear transformation L maps $(2,1)$ to $(4,1)$ and $(2,4)$ to $(0,6)$. Find $L(3,3)$.

let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the transformation matrix.

better would be $A\begin{pmatrix}2 & 2\\1 & 4\end{pmatrix} = \begin{pmatrix}4 & 0\\1 & 6\end{pmatrix}$ so $A = (\ )(\ )^{-1}$

$T\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}2a+b\\2c+d\end{pmatrix} = \begin{pmatrix}4\\1\end{pmatrix}$

$T\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}2a+4b\\2c+4d\end{pmatrix} = \begin{pmatrix}0\\6\end{pmatrix}$

By solving the system of equations

$$\begin{cases} 2a+b=4 \\ 2a+4b=0 \\ 2c+d=1 \\ 2c+4d=6 \end{cases}$$

we get

$$\begin{cases} a = \frac{8}{3} \\ b = -\frac{4}{3} \\ c = -\frac{1}{3} \\ d = \frac{5}{3} \end{cases}$$

So $L\begin{pmatrix}3\\3\end{pmatrix} = \begin{pmatrix}\frac{8}{3} & -\frac{4}{3}\\ -\frac{1}{3} & \frac{5}{3}\end{pmatrix}\begin{pmatrix}3\\3\end{pmatrix}$ ✓

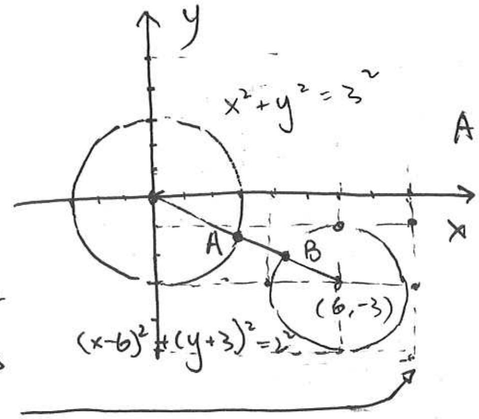
$= \begin{pmatrix}4\\4\end{pmatrix}$

2. Find the shortest distance between a point on the circle $x^2+y^2=9$ and a point on the circle $x^2+y^2-12x+6y+41=0$.

$x^2+y^2-12x+6y+41=0$

$\Leftrightarrow (x-6)^2+(y+3)^2=4$,

which is →



and the shortest distance is the line segment AB. which is

$AB = \sqrt{6^2+3^2} - 3 - 2$

$= 3\sqrt{5} - 5$ ✓

3. Sketch the slope field of the differential equation $\frac{dy}{dx} = y - x$ using a window of $[-3,3]\times[-3,3]$ and a rectangular grid of lattice points. Identify the isoclines and write down a particular solution to the differential equation.

- isoclines:

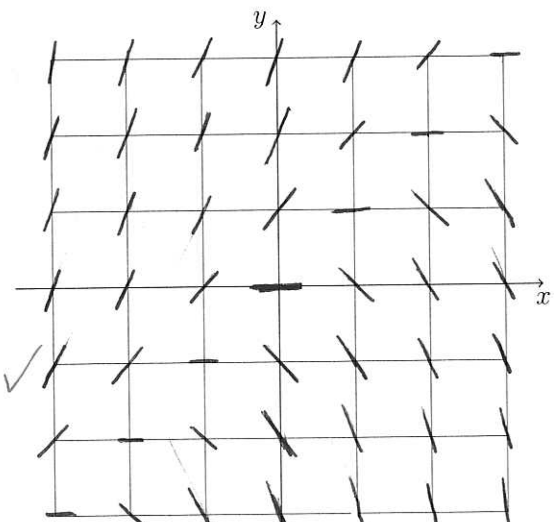
  the lines $y = x + k$, $k \in \mathbb{R}$. ✓

- one solution:

  $y = x + 1$ ✓

- all solutions:

  $y = x + 1 + ke^x$, $k \in \mathbb{R}$.

✓


4. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = y - x$ with $f(0) = 2$. Give the recurrence relation found by applying Euler's method with a step size of 0.1. Hence approximate $f(1)$ aided by the GDC.

By entering into the calculator:

$$\begin{cases} x_n = x_{n-1} + H \\ y_n = y_{n-1} + H(y_{n-1} - x_{n-1}) \end{cases}$$

where $H = 0.1$

$x_0 = 0$

$y_0 = 2.$

we get the approximation, ✓

$f(1) \approx \boxed{4.59}$ (3.s.f.)

5. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = \frac{y}{8}(6 - y)$ with $f(0) = 8$.

(a) Use Euler's method in tabular form with a step size of 0.5 to approximate $f(1)$.

n	x_n	y_n	h	$h \times f(x_n, y_n)$
0	0	8	0.5	-1
1	0.5	7	0.5	$-\frac{7}{16}$
2	1	$\frac{105}{16}$ ✓	0.5	

✓

$\Rightarrow f(1) \approx \boxed{6.56}$ (3 s.f.)

(b) Find the second degree Maclaurin polynomial for f and use it to approximate $f(1)$.

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2.$$

$$= 8 + \frac{8}{8}(6-8)x + \frac{f''(0)}{2}x^2 \checkmark$$

$$= 8 - 2x + \frac{-2}{8}(6+2)\cdot\frac{1}{2}x^2$$

✗

$$= 8 - 2x - x^2$$

$x = 1$, $P_2(1) = 8 - 2 - 1 = \boxed{5}$ ✗

$1\frac{1}{2}$

$3\frac{1}{2}$

Should be $P_2(x) = 8 - 2x + 1.25x^2$