FURTHER MATHEMATICS
HIGHER LEVEL
PAPER 1

Tuesday 18 February 2020

2 hour 30 minutes

Name in block letters

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INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- A graphic display calculator is required for this paper.
- A clean copy of the formula booklet is required for this paper.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

$\frac{136}{150}$ = 91%

$\boxed{7}$

Excellent!

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. Use L'Hôpital's rule to find $\lim_{x \to 0} \frac{\tan x - x}{1 - \cos x}$.

$\lim_{x \to 0} \frac{\tan x - x}{1 - \cos x} = \frac{0}{0}$ apply L'Hopital.

$\lim_{x \to 0} \frac{(\tan x - x)'}{(1 - \cos x)'}$

$= \lim_{x \to 0} \left(\frac{1}{\cos^2 x} - 1\right) \frac{1}{\sin x} = \lim_{x \to 0} \frac{1 - \cos^2 x}{\cos^2 x \sin x}$ ✓

$= \lim_{x \to 0} \frac{\sin^2 x}{\cos^2 x \sin x} = \lim_{x \to 0} \frac{\sin x}{\cos^2 x}$

$= \frac{0}{1} = 0$

10

2. Let $n \in \mathbb{N}$ and define $n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$.

   (a) Simplify

      i. $(3\mathbb{Z} \cap 6\mathbb{Z}) \cup 18\mathbb{Z}$

      ii. $6\mathbb{Z} \cap 15\mathbb{Z}$.

   (b) Let $n_1, n_2 \in \mathbb{N}$. Giving reasons, state whether the following assertions are true or false.

      i. $n_1\mathbb{Z} \cap n_2\mathbb{Z} = m\mathbb{Z}$ for some $m \in \mathbb{N}$.

      ii. $n_1\mathbb{Z} \cup n_2\mathbb{Z} = m\mathbb{Z}$ for some $m \in \mathbb{N}$.

(a) (i) $3\mathbb{Z} \cap 6\mathbb{Z} = 6\mathbb{Z}$.

$6\mathbb{Z} \cup 18\mathbb{Z} = 6\mathbb{Z}$.

So $(3\mathbb{Z} \cap 6\mathbb{Z}) \cup 18\mathbb{Z} = \boxed{6\mathbb{Z}} = \{\cdots, -6, 0, 6, 12, \cdots\}$

(ii) ~~$6\mathbb{Z} = \{ 6 \times 1 \times 2$~~

$6\mathbb{Z} = \{\cdots, -12, -6, 0, 6, 12, 18, \cdots\}$

$15\mathbb{Z} = \{\cdots, -15, 0, 15, 30, 45, \cdots\}$

$6\mathbb{Z} \cap 15\mathbb{Z} = \mathrm{lcm}(6, 15)\,\mathbb{Z}$

$= \boxed{30\mathbb{Z}} = \{ \cancel{30}, \cdots, -30, 0, 30, 60, \cdots\}$

(b) (i) $n_1\mathbb{Z} \cap n_2\mathbb{Z}$

$= \mathrm{lcm}(n_1, n_2)\,\mathbb{Z}$.

Since for $n_1, n_2 \in \mathbb{N}$, $\mathrm{lcm}(n_1, n_2) \in \mathbb{N}$,

let $m = \mathrm{lcm}(n, n_2)$, such $m \in \mathbb{N}$ exist,

the assertion's true.

(ii) It's false.

A counterexample: $n_1 = 3$, $n_2 = 5$.

$3\mathbb{Z} \cup 5\mathbb{Z} = \{\cdots, -6, -5, -3, 0, 3, 5, 6, 9, \cdots\}$.

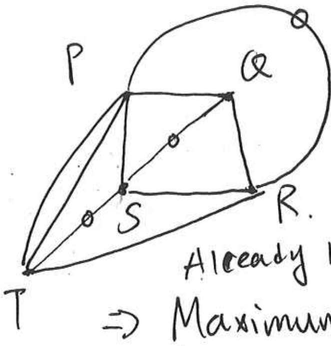
which can not be expressed as $m\mathbb{Z}$,

for any $m \in \mathbb{N}$.

3. The graph G with vertices P, Q, R, S, T has the following adjacency table.

	P	Q	R	S	T
P	0	1	0	1	2
Q	1	0	1	0	0
R	0	1	0	1	1
S	1	0	1	0	0
T	2	0	1	0	0

(d)



Already has 6, ⇒ Maximum of 3 edges can be added.

(a) Make a drawing of G illustrating that G is planar.

(b) Giving reasons, state whether or not G is

i. simple;

ii. connected;

iii. bipartite.

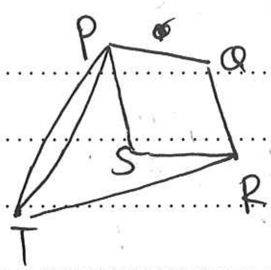
?

(c) Explain why G has an Eulerian trail but not an Eulerian circuit.

(d) Find the maximum number of edges that can be added to the graph G, not including loops or further multiple edges, whilst still keeping it planar.

Since planar, $e_{max} = 3V - 6 = 9$

(a)



⇒ planar.

(b) (i) not simple, because there are two edges connecting P, T.

(ii) it is connected, because every vertex is adjacent to ~~at~~ least one other vertex.

b) not connected

(iii) ~~it is not bipartite because it is not a simple graph~~. It is bipartite because the vertices can be divided into {R, P} and {Q, S, T}, such that there's no edge with in each set of vertices.

(c) It does not have an Eulerian circuit because not all vertices are of even degrees; However, it's an Eulerian trial because there are exactly 2 odd vertices (T and R).

9

4. (a) Consider the functions from $\mathbb{N}$ to $\mathbb{N}$ with rules $f(x) = \lfloor x/2 \rfloor$, $g(x) = x$, $h(x) = 1 + x$.

   i. Write down the function which is injective but not surjective.

   ii. Write down the function which is surjective but not injective.

   (b) Write down a function from $\mathbb{N}$ to $\mathbb{N}$ which is neither injective nor surjective.

   (c) If the function $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ with rule $f(x, y) = (2x + y, x + y)$ possesses an inverse find its rule, otherwise explain why no such inverse exists.

(a) injective : $g(x)$, $h(x)$

surjective : $f(x)$, $g(x)$,

(i) $h(x)$ (ii) $f(x)$

(b) $\ell(x) = \lfloor \frac{x}{2} + 2 \rfloor$

(c) ~~$f'(2x+y, x+y)$~~ let $f'(a,b)$ be the inverse of $f(a,b)$, $a, b \in \mathbb{Z}$.

$$\begin{cases} a = 2x + y \\ b = x + y \end{cases} \rightarrow \begin{cases} a - b = x \\ 2b - a = y \end{cases}$$

So $f'(a,b) = f(a-b, 2a-b)$

$f'(x,y) = f(x-y, 2x-y)$

$2y-x$

9

5. (a) Show that the vectors $\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\begin{pmatrix}5\\2\\5\end{pmatrix}$ form a basis for $\mathbb{R}^3$.

(b) Express the vector $\begin{pmatrix}12\\14\\16\end{pmatrix}$ as a linear combination of the above vectors.

(a) $\det\begin{pmatrix}1 & 2 & 5\\2 & 3 & 2\\3 & 1 & 5\end{pmatrix} = -30$, {using technology}

Since $\det \neq 0$, the vectors form a basis for $\mathbb{R}^3$.

(b) $\cancel{\begin{pmatrix}1 & 2 & 5\\2 & 3 & 2\\3 & 1 & 5\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = (}$

$$a\begin{pmatrix}1\\2\\3\end{pmatrix} + b\begin{pmatrix}2\\3\\1\end{pmatrix} + c\begin{pmatrix}5\\2\\5\end{pmatrix} = \begin{pmatrix}12\\14\\16\end{pmatrix}$$

$$\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}1 & 2 & 5\\2 & 3 & 2\\3 & 1 & 5\end{pmatrix}^{-1}\begin{pmatrix}12\\14\\16\end{pmatrix}$$

$$= \begin{pmatrix}3\\2\\1\end{pmatrix}$$

Therefore,

$$\begin{pmatrix}12\\14\\16\end{pmatrix} = 3\begin{pmatrix}1\\2\\3\end{pmatrix} + 2\begin{pmatrix}2\\3\\1\end{pmatrix} + \begin{pmatrix}5\\2\\5\end{pmatrix}.$$

10

6. Consider the two independent random variables X and Y where $X \sim \mathrm{Po}(3)$ and $Y \sim \mathrm{Po}(4)$.

   (a) Calculate $\mathrm{E}(2X + 7Y)$.

   (b) Calculate $\mathrm{Var}(4X - 3Y)$.

   (c) Calculate $\mathrm{E}(X^2 - Y^2)$.

(a) $E(2X + 7Y)$

$= 2E(X) + 7E(Y)$

$= 2 \times 3 + 7 \times 4 = \underline{34}$

(b) $Var(4X - 3Y)$

$= 16\,Var(X) + 9\,Var(Y)$

$= 16(3) + 9(4) = \underline{84}$

(c) $E(X^2 - Y^2)$

$= E(X^2) - E(Y^2)$

$= \{Var(X) + [E(X)]^2\} - \{Var(Y) + [E(Y)]^2\}$

$= (3 + 9) - (4 + 16)$

$= -8$

10

7. Consider the system of equations $\begin{pmatrix} 1 & -1 & 2 \\ 2 & 2 & -1 \\ 3 & 5 & -4 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \\ k \end{pmatrix}$.

(a) Find the rank of the coefficient matrix.

(b) Find the value of k for which the system has a solution.

(c) For this value of k determine the solution.

(a) $\Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 2 & 2 & -1 & 3 \\ 3 & 5 & -4 & 1 \\ 3 & 1 & 1 & k \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 4 & -5 & -7 \\ 0 & 4 & -5 & 1-k \\ 0 & 4 & -5 & k-15 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 1 & -\frac{5}{4} & -\frac{7}{4} \\ 0 & 0 & 0 & 8-k \\ 0 & 0 & 0 & k-8 \end{array}\right)$

(a) Since there are 2 leading 1s,
rank = 2.

(b) To have a solution, $k-8 = 8-k = 0$,
so $k = 8$.

(c) Let $z = s$,

$y - \frac{5}{4}s = -\frac{7}{4}$

$y = \frac{5}{4}s - \frac{7}{4}$,

$x - y + 2s = 5$

$x = \frac{5}{4}s - \frac{7}{4} - 2s + 5$

$= -\frac{3}{4}s + \frac{13}{4}$

So the solutions are $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s\begin{pmatrix} -\frac{3}{4} \\ \frac{5}{4} \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{13}{4} \\ -\frac{7}{4} \\ 0 \end{pmatrix}$. $s \in \mathbb{R}$.

10

8. The function f is defined by $f(x) = e^x \cos x$.

   (a) Show that $f''(x) = -2e^x \sin x$.

   (b) Determine the fourth degree Maclaurin polynomial for $f(x)$.

   (c) By differentiating your polynomial, determine the cubic Maclaurin polynomial for $e^x \sin x$.

(a) $f'(x) = e^x \cos x - e^x \sin x$

$f''(x) = e^x \cancel{\cos x} - e^x \sin x - e^x \sin x - \cancel{e^x \cos x}$

$= -2e^x \sin x$

(b) $f^{(3)}(x) = -2e^x \sin x - 2e^x \cos x$

~~So~~ $\cancel{P_4(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1}x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3}$

$f^{(4)}(x) = -2(e^x \sin x + e^x \cos x) - 2(e^x \cos x - e^x \sin x)$

$= -2(2e^x \cos x) = -4e^x \cos x)$

$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4$

$= 1 + x + 0x^2 + (-2)\frac{x^3}{6} + (-4)\frac{x^4}{24}$

$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6}$

(c) $P_4'(x) = 1 - x^2 - \frac{2}{3}x^3$, leg $g(x) = e^x \sin x$

since $f'(x) = f(x) - e^x \sin x$

$= f(x) - g(x)$

$g(x) = f(x) - f'(x)$

$P_3(x) = \left(1 + x - \frac{x^3}{3} \cancel{- \frac{x^4}{6}}\right) - \left(1 - x^2 - \frac{2}{3}x^3\right)$

$= x + x^2 + \frac{1}{3}x^3$

10

9. (a) The point $T(at^2, 2at)$ lies on the parabola $y^2 = 4ax$. Show that the tangent to the parabola at T has equation $x - ty + at^2 = 0$.

(b) The distinct points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, where $p, q \neq 0$, also lie on the parabola $y^2 = 4ax$. If the line (PQ) passes through the focus show that the tangents to the parabola at P and Q intersect at the directrix.

(a) $\frac{dy}{dt} = 2a$. $\frac{dx}{dt} = 2at$.

So $\frac{dy}{dx} = \frac{1}{t}$.

So $y - 2at = \frac{1}{t}(x - at^2)$

$\Rightarrow ty - 2at^2 = x - at^2$

$\Rightarrow x - ty + at^2 = 0$.

(b) focus : $(a, 0)$ directrix : $x = -a$

(PQ) : $y = \frac{2\cancel{a}\cancel{(p-q)}}{\cancel{a}\cancel{(p-q)}(p+q)}(x-a)$

$= \frac{2}{p+q}(x-a)$.

Since P on (PQ),

$2ap = \frac{2}{p+q}(ap^2 - a)$

$\Leftrightarrow \cancel{p^2+q}\ p^2 + pq = p^2 - 1 \Leftrightarrow pq = -1$

so the tangents are

$$\begin{cases} x - py + ap^2 = 0 \\ x - qy + aq^2 = 0 \end{cases}$$

So $y = \frac{x + ap^2}{p} = \frac{x + aq^2}{q}$, $pq = -1$.

$qx - ap = px - aq$.

$\underline{x = -a}$, which means the two tangents intersect at the directrix. //

10

10. (a) Write down the values of p for which the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

(b) Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{3\sqrt{n}}{2n^2+1}$.

(c) Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

n^n

$(n+1)^n \cdot (n+1)$.

(a) P-series $\Rightarrow$ $p > 1$

$\frac{n^n}{(n+1)^n}$

(b) $\frac{3\sqrt{n}}{2n^2+1} < \frac{3\sqrt{n}}{3n^2}$, $\forall n > 1$

$\left(1+\frac{1}{n}\right)^n$

and since $\sum_{n=1}^{\infty} \frac{3\sqrt{n}}{3n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$.

because $p = \frac{3}{2} > 1$, it converges,

using comparison test.

$\sum_{n=1}^{\infty} \frac{3\sqrt{n}}{2n^2+1}$ also converges.

(c) ~~$\lim_{n\to\infty} \left| \frac{(n+1)!}{n^{n+1}} \cdot \frac{n^n}{n!} \right|$~~

~~$= \lim_{n\to\infty} \left| \frac{n+1}{n} \right| = 1$~~

Since $n(n-1)(n-2)(\cdots) < n \cdot n \cdot n \cdots$

$n! < n^n$.

~~diverges~~ : So

~~$\frac{n!}{n^n}$~~

$\sum_{n=1}^{\infty} \frac{n!}{n^n} > \sum_{n=1}^{\infty} \frac{k!}{n^n} > \sum_{n=1}^{\infty} \frac{k!}{n^k} > \sum_{n=1}^{\infty} \frac{1}{n}$.

$\Rightarrow$ diverges.

6

11. The weights of peaches are normally distributed with mean 98 grams and standard deviation 16 grams.

  (a) A shopkeeper places 100 randomly chosen peaches on a weighing machine. Find the probability that their total weight exceeds 10 kilograms.

  (b) Find the minimum number of randomly selected peaches needed to ensure their total weight exceeds 10 kilograms with probability greater than 0.95.

Let X denote the weights of peaches.

$X \sim N(98, 16^2)$

(a) let $S = \sum_{i=1}^{100} X_i$.

$S \sim N(98 \times 100, 100 \times 16^2)$

$P(S > 10^4) = \text{normalcdf}(10^4, \infty, 9800, 160)$

$= 0.106$ (3 s.f.)

(b) ~~let Y~~

let $Y = \sum_{i=1}^{m} X_i$

~~and~~ $Y \sim N(98m, 16^2 m)$

$P(Y > 10^4) = 0.95$

$P\left(Z > \frac{10^4 - 98m}{\sqrt{16^2 m}}\right) = 0.95$

$\Rightarrow \frac{10^4 - 98m}{\sqrt{16^2 m}} = -1.645$

$98m - \cancel{65.94}\, 26.318\sqrt{m} - 10^4 = 0$

using technology:

~~m = 3.184 (3 s.f.)~~ $m = 10.23669^2 > 104$

~~m = 107.4 → 107~~

So the minimum number is ~~108~~ 105

10

12. The points A, B have coordinates $(-3,0)$, $(5,0)$ respectively. Consider the circle $\mathscr{C}$ with centre $(13,0)$ which is the locus of the point P where $PA : PB = k : 1$ for $k \neq 1$.

(a) Find the radius of $\mathscr{C}$.

(b) If M is a point on $\mathscr{C}$ and N is the x-intercept of $\mathscr{C}$ between A and B, prove $\angle AMN = \angle NMB$.

(a) let the radius be r.

So $\frac{13-r+3}{r-8} = \frac{13+3+r}{r+8}$

$(16-r)(r+8) = (16+r)(r-8)$

$r = 8\sqrt{2}$

(b) $k = \frac{16-r}{r-8} = \sqrt{2}$.

So $\frac{MA}{MB} = \frac{NA}{NB} = k = \sqrt{2}$.

$\frac{MA}{NA} = \frac{MB}{NB}$, ~~angle~~

So $\angle AMN = \angle NMB$

By what theorem?

Important to state this

9

13. (a) State Lagrange's theorem.

(b) Prove that every group of prime order is cyclic.

(c) The Abelian group G is generated by the distinct group elements a and b where $|a| = |b| = 3$.

i. What is the order of G?

ii. How many proper subgroups does G have?

(a) In a finite group, the order of a subgroup always divides the order of the group.

(b) ~~proof.~~ ~~First~~,

let G be a group of prime order

G $\neq \{e\}$. otherwise $|G| = 1$ ~~cannot be empty~~; So there exists $a \in G$, $a \neq e$.

consider $a^0, a^1, a^2, a^3, \ldots$

Since G is of prime order, it has no proper subgroup. So a must generate the whole group.

siuece ~~Ga~~ $G = \langle a \rangle$, the group is cyclic. //.

(c) (i) $|G| = |\langle a \rangle| = 3$.

(ii) Sina $|G| = 3$, 3 is a prime, only $1|3$ and $3|3$,

So there is no proper subgroups

$a^2b \cdot a^2b$

$ab^2 \cdot a^2b$

Not what I had in mind, but perhaps the question is ambiguous.

e

ab^2 a^2b

10

14. Consider the matrix $A = \begin{pmatrix} 11 & \sqrt{3} \\ \sqrt{3} & 9 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of A.

(b) The ellipse $\mathscr{E}$ has equation $X^T A X = 24$ where $X^T = (x \quad y)$.

i. Show that $\mathscr{E}$ can be rotated about the origin onto the ellipse $\mathscr{E}'$ having equation $2x^2+3y^2 = 6$.

ii. Find the acute angle through which $\mathscr{E}$ must be rotated to coincide with $\mathscr{E}'$.

(a) $\lambda^2 - 20\lambda + 99 - 3 = 0$

$\lambda_1 = 12 \qquad \lambda_2 = 8$

① when $\lambda = 12$

$$\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$

② when $\lambda = 8$

$$\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$$

(b) (i) $(x \quad y)\begin{pmatrix} 11 & \sqrt{3} \\ \sqrt{3} & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 24$

$$(x \quad y)\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 12 & 0 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = 24$$

let $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$12x'^2 + 8y'^2 = 24 \Leftrightarrow 3x^2 + 2y^2 = 6$

(ii) The rotation matrix $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$.

$\Rightarrow \theta = 30^\circ$, clockwise.

8

$\frac{1+\sin x}{\cos x}$ $\frac{1-\sin^2 x}{\cos^2 x}$ $\sec x + \tan x$ $\sec^2 x + \sec x \cdot \tan x$.

15. Let $y = f(x)$ be the solution to the differential equation $y' + y\sec x = x(\sec x - \tan x)$ with $f(0) = 3$.

(a) Use Euler's method in table form with a step length of 0.1 to approximate $f(0.2)$.

(b) i. Determine the second degree Maclaurin polynomial for $f(x)$.

ii. Use this polynomial to approximate $f(0.2)$.

(c) i. Solve this differential equation to find $y = f(x)$.

ii. Hence determine which of the above two approximations for $f(0.2)$ is closer to the true value.

(a)

n	x_n	y_n	h	$h \times f(x, y)$
0	0	3	0.1	-0.3
1	0.1	2.7	0.1	-0.26
2	0.2	2.44		

So $f(0.2) \approx 2.44$. (3 s.f.)

(b)(i) $\frac{d^2y}{dx^2} + \sec x \frac{dy}{dx} + \sec x \tan x \cdot y = \sec x - \tan x + x(\sec x \tan x - \sec^2 x)$

~~$\int \sec x \, dx = \sec x \tan x$~~

$P_2(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2}$

$= 3 + -3x + 2x^2$

(ii) $f(0.2) = 3 + -3(0.2) + 2(0.2)^2$

$= 3 - 0.6 + 0.08 = \underline{2.48}$

(c) (i) $\int \sec x \, dx =$

(ii) the Maclaurin polynomial method is closer.

Reason?

$-$

5

Block Week Exam
Correction
Maggie Feb.18

Excellent! — (Mistake in 13c)

3 (b). G is connected because there exists a path joining each pair of vertices in G. Wrong definition, and When I tried define it myself, I didn't take all circumstances into consideration. ✓

4. (c). let $f^{-1}(a,b)$ be the inverse of $f(a,b)$. $a, b \in \mathbb{Z}$.

$$\begin{cases} a = 2x+y \\ b = x+y \end{cases} \Rightarrow \begin{cases} x = a-b \\ y = 2b-a \end{cases}$$ ✓

So $f^{-1}(a,b) = f(a-b, 2b-a)$ I copied it wrongly from above...

$f^{-1}(x,y) = f(x-y, 2y-x)$

10. (c).

$$\lim_{n\to\infty} \left| \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{(n+1)^{n+1}} \cdot n^n \right|$$

$$= \lim_{n\to\infty} \left| \left(\frac{n}{n+1}\right)^n \right|$$

$= \frac{1}{e}$. therefore, it converges.

✓ I didn't calculate the limit correctly at first, and got $\lim_{n\to\infty} = \left|\frac{n}{n+1}\right| = 1$. So I continued trying different methods instead of calculating it again. ":

12 (b). since N, M are both points on $\mathscr{C}$,

$$\frac{MA}{MB} = \frac{NA}{NB} = k$$ ✓

$\Rightarrow \frac{MA}{NA} = \frac{MB}{NB}$. by Angle Bisector thrm, I didn't state the thrm explicitly

$\angle AMB = \angle NMB$. □.

13 (c) (i) since $|a| = |b| = 3$, $\langle a\rangle \neq \langle b\rangle$, I mistood the question. the group elmts are $e, a, a^2, b, b^2, ab, a^2b, ab^2, a^2b^2$; So $|G| = 9$.

(ii) Since only $3|9$ and $|a| = |b| = |ab| = |a^2b|$ generate different subgroups. there are 4 proper subgroups ✓

14 (b). (i)

$$\mathcal{E}: \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 11 & \sqrt{3} \\ \sqrt{3} & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 24.$$

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 8 & 0 \\ 0 & 12 \end{pmatrix} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = 24$$

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 8 & 0 \\ 0 & 12 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 24$$

let $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

then

$$\begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} 8 & 0 \\ 0 & 12 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 24.$$

$\Leftrightarrow \quad 2x^2 + 3y^2 = 6.$ → I didn't copy the coeffients right from the question, so I got $3x^2+2y^2=6$

(ii) The rotation matrix $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

So $\mathcal{E}$ must be rotated $60°$ anticlockwise.

15 (c)(i)

$$(\sec x + \tan x) y' + y(\sec^2 x + \sec x \tan x) = x(\sec^2 x - \tan^2 x)$$

I didn't think of how to do $\int \sec x \, dx$, and I couldn't find a clever way like this either :(

$$y(\sec x + \tan x) = \int x(\sec^2 x - \tan^2 x)\, dx$$

$$y(\sec x + \tan x) = \int x \, dx$$

$$y(\sec x + \tan x) = \tfrac{1}{2}x^2 + c_1$$

$$3(\sec(0) + \tan(0)) = 0 + c_1$$

$$\Rightarrow c_1 = 3$$

So $y = \dfrac{\frac{1}{2}x^2 + 3}{\sec x + \tan x}.$

(ii) $y = \dfrac{0.2 + 3}{\sec(0.2) + \tan(0.2)} = 2.62 \ (3\text{s.f.})$

therefore, the Maclaurin polynomial method is closer.