FM2 Exam December 2019 Name: Maggie. 94%

1. The relation R is defined on $\mathbb{Z}$ by $x\ R\ y$ if 5 divides $x + y$. Prove that R is not an equivalence relation.

Proof. ~~Reflexivity:~~

R is not an equivalence relatron because it does not satisfy the reflexive axiom property

It's not true that $5 \mid 2x$ $\forall x \in \mathbb{Z}$.

A counter example : $x = 1$. $5 \nmid 2$.

Therefor, R is not an equivalence relation.

Very good!

2. Let S be the set of positive irrational numbers together with the number 1. Does $(S, \times)$ form a group?

Not. It is not closed.

A counterexample:

$\sqrt{2} \times \sqrt{2} = 2$,

$\sqrt{2}$ is irrational, but 2 is rational.

Therefore, $(S, \times)$ does not form a group.

3. Use the Maclaurin series for e^{-x} and $\sin 2x$ to evaluate the limit $\lim_{x\to 0} \dfrac{1 - e^{-x}}{\sin 2x}$.

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \cdots$$

$$\sin 2x = 2x - \frac{(2x)^3}{3!} + \cdots$$

$$\frac{1-e^{-x}}{\sin 2x} = \frac{1-(1-x+O(x^2))}{2x - O(x^3)} = \frac{x - O(x^2)}{2x - O(x^3)}$$

$$\lim_{x\to 0} \frac{1-e^{-x}}{\sin 2x} = \frac{1 - O(x)}{2 - O(x^2)} = \frac{1}{2}.$$

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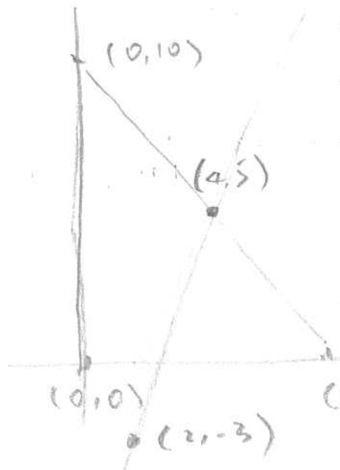
4. A circle intersects the axes at (0, 10), (0, 0) and (8, 0). A line through (2, −3) cuts the circle in half. Find the y-intercept of the line.



Since (0,10), (0,0), (8,0) form a right triangle and they are all on the circle, The ~~radius~~ center must be ~~at the midpoint of~~ at $\left(\frac{8}{2}, \frac{10}{2}\right) = (4,5)$.

Since a line through (2,-3) cuts the circle in half, it must also pass through the center at (4,5).

Therefore,

$$y = \frac{5-(-3)}{4-2}(x-4)+5$$
$$= 4(x-4)+5$$
$$= 4x-11$$

when $x=0$

$$\boxed{y = -11}$$

5. State the mean value theorem. If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be?

The MVT:

For a function $f(x)$ that is continuous on $[a,b]$ and is differentiable on $]a,b[$, there exist $c \in]a,b[$, such that

$$f'(c) = \cancel{\frac{f(a)-f(b)}{a-b}}\ \frac{f(b)-f(a)}{b-a}$$

$$\frac{f(4)-f(1)}{4-1} \geqslant 2.$$

$$f(4) - 10 \geqslant 6$$

$$f(4) \geqslant 16.$$

$f(4)$ ~~can be~~ is smallest at 16.

6. Let $y = f(x)$ be the particular solution to the differential equation $y' = x^2 + y^2$ for which $f(1) = 2$. Use Euler's method starting at $x = 1$ with a step size of 0.1 to approximate $f(1.2)$. Set out your work in a table.

n	x_n	y_n	h	$h*(x_n^2+y_n^2)$
0	1	2	0.1	0.5
1	1.1	2.5	0.1	0.746
2	1.2	3.246	0.1	

Therefore, $f(1.2) \approx 3.246$
≈ 3.25 (3 s.f.)

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7. Prove that a simple graph with more than one vertex contains two vertices of the same degree.

Proof: suppose G is a simple graph with v vertices.

The maximum degree of a vertex is $v-1$.

The minimum degree of a vertex is 1. why?

Since there are V vertices with $v-1$ possible degrees.

According to PHP,

there must be at least two vertices that have the same degree. $\square$.

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8. Find a basis for the null space of the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 \end{pmatrix}$.

$A \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$

~~There are two free variables.~~

$Ax = 0$

There are two free variables.

let $x_3 = s$, $x_4 = t$, $s, t \in \mathbb{R}$.

$$\begin{cases} x_1 = -x_2 - s - t. \\ x_2 = -2s - 3t \end{cases}$$

So $x_1 = 2s + 3t - s - t$

$= s + 2t$

Therefore,

the null space is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} t,$$

which is ~~the space span~~

$$\left\langle \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

9. The circle $\mathscr{C}$ has centre O, the point X lies outside $\mathscr{C}$, the point Z lies on $\mathscr{C}$ and the secant $[XZ]$ cuts $\mathscr{C}$ at Y. If $XY = 6$, $YZ = 5$ and $XO = 9$, find the area of the circle.

usually as in my solutions.

let XO intersect $\odot O$ at P, Q.

let the radius be r.

$XZ = XY - YZ = 6 - 5 = 1$

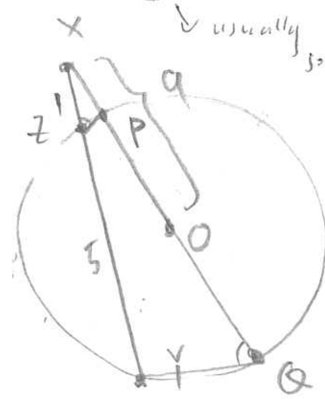
Then $XP \cdot XQ = XZ \cdot XY$.

$(9-r)(9+r) = 1 \times 6$

$81 - r^2 = 6$

$r^2 = 75$

The area is thus $\pi r^2 = \boxed{75\pi}$.



ok

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10. The function $f : \mathbb{Z}_{91} \to \mathbb{Z}_7 \times \mathbb{Z}_{13}$ with rule $f(x) = (x \bmod 7, x \bmod 13)$ is a bijection. Find $f^{-1}(1, 4)$.

Since it is a bijection,

$x \equiv 1 \mod 7$

$x \equiv 4 \mod 13$.

By inspection,

$x = 43$.

Therefore, $f^{-1}(1, 4) = 43$.

11. Prove that the order of a non-Abelian group cannot be prime.

Proof by contrapositive:

If the order of a group is prime, the group is abelian.

Let G be a group of order p, where p is a prime; $a \in G$. and $a \neq e$.

Since it is closed and finite, $\{a^0, a^1, a^2 \cdots a^{m-1}\}$ forms a subgroup of G, where m is the least positive number such that $a^m = e \Rightarrow$ The subgroup has order m.

According to Lagrange's theorem, $m \mid p$, and since p is prime, and $m \neq 1$, $m = p$, which means the group G is cyclic, and since all cyclic groups are abelian, G is abelian. //.

Therefore, the order of a non-Abelian group cannot be prime.

12. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove that $\ker T$ is a subspace of $\mathbb{R}^n$.

$\ker T = \{ \vec{v} \mid T(\vec{v}) = \vec{0}, x \in \mathbb{R}^n \}$.

① since $T(\vec{0}) = \vec{0}$, $\vec{0} \in \ker T$.

② Let $v, u \in \ker T$,

then $T(u+v) = T(u) + T(v)$

$= \vec{0} + \vec{0}$

$= \vec{0}$.

So it is closed under addition

③ $k \in \mathbb{R}$,

$T(ku) = k\,T(u)$

$= k\vec{0}$

$= \vec{0}$,

So $\ker T$ is closed under scalar multiplication

Therefore, we conclude that $\ker T$ is a subspace of $\mathbb{R}^n$. //

13. Let $f : G \to G'$ be a homomorphism of groups whose respective identity elements are e and e'. Prove

(a) $f(e) = e'$;

Let $a \in G$, due to homomorphism,

$f(e \cdot a) = f(e) \cdot f(a)$

on the other hand.

$f(e \cdot a) = f(a)$.

Therefore,

$f(e) \cdot f(a) = f(a)$. $\forall a \in G$.

So $f(e) = e'$. $\square$

(b) $f(a^{-1}) = [f(a)]^{-1}$.

Let $a \in G$.

$f(a \cdot a^{-1}) = f(e) = e'$

On the other hand,

because of homomorphism,

$f(a\, a^{-1}) = f(a) \cdot f(a^{-1})$.

Therefore,

$f(a) \cdot f(a^{-1}) = e'$

and since $f(a) \cdot [f(a)]^{-1} = e'$,

$f(a^{-1}) = [f(a)]^{-1}$. $\square$.

14. Find the general solution of the differential equation $\frac{dy}{dx} + y \cot x = x$, $0 < x < \pi$. Give your answer in the form $y = f(x)$.

$e^{\int \cot x \, dx}$

$= e^{\int \frac{(\sin x)'}{\sin x} dx}$

$= e^{\ln \sin x}$

$= \sin x$.

So $(\sin x \cdot y)' = x \sin x$

$\sin x \cdot y = \int x \, dx$

$= \frac{1}{2} x^2 + C_1$

~~$y = \frac{1}{2} x^2 + C_1$~~

so $y = \frac{x^2 + C}{2 \sin x}$,

where $C = 2C_1 \in \mathbb{R}$.

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15. Prove that the intersection of two subgroups of a group is also a subgroup of that group.

Proof: let H and K be two subgroups of G.

since $e \in H$, and $e \in K$, $H \cap K \neq \emptyset$,

① therefore, $e \in H \cap K$;

② let $a \in H \cap K$,

so $a \in H$ and $a \in K$.

since $H, K \leq G$,

$a^{-1} \in H$ and $a^{-1} \in K$,

so $a^{-1} \in H \cap K$,

③ let $a, b \in H \cap K$.

$a \in H$, $b \in H$,

$a \in K$, $b \in K$,

so $ab \in H$, and

$ab \in K$.

Thus $ab \in H \cap K$. and it is closed.

Therefor, by using the 3-step subgroup test, $H \cap K$ is also a subgroup of G.

16. Determine the interval of convergence for the power series $1 + \frac{x+2}{3 \times 1} + \frac{(x+2)^2}{3^2 \times 2} + \frac{(x+2)^3}{3^3 \times 3} + \ldots$

Let a_n denote the $(n+1)^{th}$ term of the series.

$$a_n = \frac{(x+2)^n}{3^n \times n}, \quad a_0 = 1$$

Apply the ratio test,

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{(x+2)^{n+1}}{3^{n+1} \times (n+1)} \cdot \frac{3^n \times n}{(x+2)^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{(x+2)}{3} \frac{(n)}{(n+1)} \right|$$

$$= \left| \frac{x+2}{3} \right| < 1$$

So $-5 < x < 1$.

① when $x = 1$.

$$\sum_{n=0}^{\infty} a_n = 1 + \sum_{n=1}^{\infty} \frac{1}{n}.$$

which is the harmonious series and thus diverges.

② when $x = -5$

$$\sum_{n=0}^{\infty} a_n = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the alternating series test.

Therefore, the interval of convergence is $[-5, 1[$.

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17. Find the equation of the line containing the major axis of the ellipse $2x^2 - 4xy + 5y^2 = 6$.

$$(x \quad y)\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 6.$$

We want to diagonalize $\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$

Solving $\lambda^2 - 7\lambda + 6 = 0$,

$\lambda_1 = 1$, $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda_2 = 6$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

So $\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$

$$= \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} \\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \\ -\frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{pmatrix}$$

Let $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \\ -\frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

then

$$(X \quad Y)\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 6.$$

$$X^2 + 6Y^2 = 6.$$

$$\frac{X^2}{(\sqrt6)^2} + \frac{Y^2}{1^2} = 1.$$

The major axis is $Y = 0$.

After the transformation with $\begin{pmatrix} \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} \\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{pmatrix}$, it is rotated $\arctan(\frac{1}{2})$ anticlockwise, so the major axis originally is $\boxed{y = \frac{1}{2}x}$.

18. Solve the differential equation $x^2 \dfrac{dy}{dx} = y^2 + xy + 4x^2$ given $y = 2$ when $x = 1$. Give your answer in the form $y = f(x)$.

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{y}{x} + 4.$$

let $y = vx$. $v = \frac{y}{x}$

$$x\frac{dv}{dx} + v = v^2 + v + 4$$

$$x\frac{dv}{dx} = v^2 + 4$$

$$\int \frac{1}{v^2+4}\,dv = \int \frac{1}{x}\,dx$$

$$\frac{1}{2}\arctan\left(\frac{v}{2}\right) = \ln|x| + c_1$$

$$\arctan\left(\frac{v}{2}\right) = 2\ln|x| + 2c_1$$

$$v = 2\tan(2\ln|x| + 2c_1).$$

$$y = 2x\tan(2\ln|x| + c). \quad c = 2c_1.$$

since $x = 1$, $y = 2$.

$$2\tan(2\ln 1 + c) = 2.$$

$$\tan(c) = 1$$

~~So $c = \frac{\pi}{4}$ is a solution.~~

So ~~$y = 2x\tan(2\ln|x| + \frac{\pi}{4})$~~

$c = \frac{\pi}{4} + \frac{n\pi}{2}$, $n \in \mathbb{R}$

So $y = 2x\tan(2\ln|x| + \frac{\pi}{4} + \frac{n\pi}{2})$

which is equivalent to

$$y = 2x\tan\left(2\ln|x| + \frac{\pi}{4}\right).$$

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$G(t) = t\left(\frac{1}{1-(t-1)}\right) = t + t(t-1) + t(t-1)^2 + \cdots$

19. The random variable X has probability generating function $G(t) = \dfrac{t}{2-t}$, mean μ and variance σ^2. Find $P(|X-\mu| < \sigma)$.

$P(|X-\mu|<\sigma)$

$= P(X=\mu+\sigma) - P(X=\mu-\sigma)$.

$G'(t) = \dfrac{(2-t) - \cancel{t}(-t)}{(2-t)^2}$

$= \dfrac{t^2-t+2}{(t-2)^2}$

$G''(t) = \dfrac{(t-2)^2(2t-1) - 2(t^2-t+2)(t-2)}{(t-2)^4}$

$G'(1) = 2$ ✓

$G''(1) = 5$ ✗

So $\mu = G'(1) = 2$

$\sigma = \sqrt{G''(1) + G'(1) - (G'(1))^2}$ ✓

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$= \sqrt{5+2-2^2}$

$= \sqrt{3}$. $= \sqrt{p(1-p)n}$. ✗

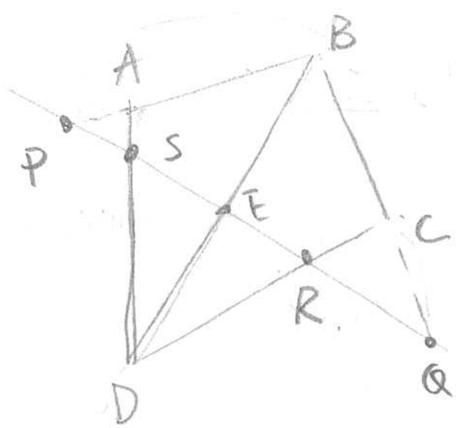
So $P(|X-\mu|<\sigma)$

$= P(X = 2+\sqrt{3}) - P(X = 2-\sqrt{3})$

$= 0.68$ ✗

20. The sides $[AB]$, $[BC]$, $[CD]$, $[AD]$ of quadrilateral $ABCD$ (produced if necessary) are cut by a transversal in the points P, Q, R and S, respectively. Prove that

$$\frac{AP}{PB} \times \frac{BQ}{QC} \times \frac{CR}{RD} \times \frac{DS}{SA} = 1.$$



Connect BD.

WLOG, let the transversal intersect BD at E

According to Menelaus' theorem,

$$\frac{AP}{PB} \cdot \frac{BE}{ED} \cdot \frac{DS}{SA} = -1$$

and $$\frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DE}{EB} = -1$$

$$\frac{AP}{PB} \cdot \frac{DS}{SA} = -\frac{BE}{ED}$$

$$\frac{BQ}{QC} \cdot \frac{CR}{RD} = -\frac{DE}{EB}.$$

So $$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = \left(-\frac{BE}{ED}\right)\left(-\frac{DE}{EB}\right) = 1 \quad \square.$$

✓

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