Liebeck Chp 8. Math. Induction.

2. Let $P(n): \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

Note that $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \dots + n^2$.

Test $n=1$.

$$\begin{aligned} LHS &= 1^2 \\ &= \frac{6}{6} \\ &= \frac{1}{6}(1)(2)(3) \\ &= \frac{1}{6}(1)(1+1)(2(1)+1) \\ &= RHS. \end{aligned}$$

Thus, $P(n=1)$ is true.

Assume $P(n=k)$ is true for $k \geq 1$; i.e.

$\sum_{r=1}^{k} r^2 = \frac{1}{6}(k)(k+1)(2k+1)$.

We now wish to show that $P(n=k+1)$ is true; i.e. $\sum_{r=1}^{k+1} r^2 = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$

$= \frac{1}{6}(k+1)(k+2)(2k+3)$

Consider

$$\begin{aligned} LHS &= \sum_{r=1}^{k+1} r^2 \\ &= \sum_{r=1}^{k} r^2 + (k+1)^2 \\ &= \frac{k}{6}(k+1)(2k+1) + (k+1)^2 \\ &= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right] \\ &= (k+1)\left[\frac{2k^2+k}{6} + \frac{6k+6}{6}\right] \\ &= \frac{1}{6}(k+1)(2k^2+7k+6) \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) \\ &= RHS. \end{aligned}$$

Thus, $P(n=k+1)$ is true.

Since $P(n=1)$ is true and the truth of $P(n=k)$ implies the truth of $P(n=k+1)$ then by mathematical induction, $P(n): \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ is true for $n \in \mathbb{Z}^+$.

$$1(1)+2(3)+3(5)+4(7)+\cdots+n(2n-1) = \sum_{r=1}^{n} r(2r-1)$$

$$= \sum_{r=1}^{n}(2r^2-r)$$

$$= \sum_{r=1}^{n}2r^2 - \sum_{r=1}^{n} r$$

$$= 2\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r$$

$$= 2\left[\frac{n}{6}(n+1)(2n+1)\right] - \frac{n}{2}(n+1)$$

$$= \frac{2n}{6}(n+1)(2n+1) - \frac{3n}{6}(n+1)$$

$$= \frac{n(n+1)}{6}\left[2(2n+1) - 3\right]$$

$$= \frac{n(n+1)}{6}(4n+2-3)$$

$$= \frac{n(n+1)}{6}(4n-1)$$

$$1^2+3^2+5^2+\cdots+(2n-1)^2 = \sum_{r=1}^{n}(2r-1)^2$$

$$= \sum_{r=1}^{n}(4r^2-4r+1)$$

$$= \sum_{r=1}^{n}4r^2 - \sum_{r=1}^{n}4r + \sum_{r=1}^{n}1$$

$$= 4\sum_{r=1}^{n}r^2 - 4\sum_{r=1}^{n}r + n$$

$$= 4\left[\frac{n}{6}(n+1)(2n+1)\right] - 4\left(\frac{n}{2}\right)(n+1) + n$$

$$= \frac{n(n+1)}{6}\left[4(2n+1) - 12\right] + n$$

$$= \frac{n(n+1)}{6}\left[8n+4-12\right] + n$$

$$= \frac{n(n+1)}{6}\left[8n-8\right] + n$$

$$= \frac{8}{6}(n)(n+1)(n-1) + n$$

$$= n\left[\frac{4}{3}(n+1)(n-1) + 1\right]$$

5a) Let $P(n): 5^{2n} - 3^n = 11M$ for some $M \in \mathbb{Z}$ and $n \geq 0, n \in \mathbb{Z}$.

Test for $n=0$.

$$\begin{aligned} LHS &= 5^{2(0)} - 3^{(0)} \\ &= 5^0 - 3^0 \\ &= 1-1 \\ &= 0. \\ &= 11(0). \\ &= RHS. \end{aligned}$$

Zero is a multiple of 11 and $P(n=0)$ is true.

Assume $P(n=k)$ is true for $k \geq 0$; i.e

$$5^{2k} - 3^k = 11N \; ; \; N \in \mathbb{Z}.$$

We now wish to show that $5^{2(k+1)} - 3^{(k+1)}$ is a multiple of 11; i.e

$$5^{2(k+1)} - 3^{(k+1)} = 11A \; ; \text{ for some } A \in \mathbb{Z}.$$

Consider
$$\begin{aligned} 5^{2(k+1)} - 3^{k+1} &= 5^{2k+2} - 3^{k+1} \\ &= 5^{2k}(5^2) - (3^k)(3) \\ &= (25)(5^{2k}) - (3)(3^k) - (22)(3^k) + (22)(3^k) \\ &= 25[5^{2k} - 3^k] - (22)3^k \\ &= 25(11N) - 2(11)3^k \\ &= 11[25N - 2(3^k)] \end{aligned}$$

Since $N, K \in \mathbb{Z}$ then $25N \in \mathbb{Z}$, $3^k \in \mathbb{Z}$, $2(3^k) \in \mathbb{Z}$ and $[25N - 2(3^k)] \in \mathbb{Z}$.

Let $B = 25N - 2(3^k)$, $B \in \mathbb{Z}$.

So $5^{2(k+1)} - 3^{k+1} = 11.B$, $B \in \mathbb{Z}$.

Thus, $P(n=k+1)$ is true.

Since $P(n=1)$ is true and the truth of $P(n=k)$ implies ~~the~~ truth of $P(n=k+1)$ then by mathematical induction, $P(n): 5^{2n} - 3^n = 11M$ is true for all integers $n \geq 0$.

5b) Let $P(n)$: 2^{4n-1} ends with an 8 for any integer $n \geq 1$.

Test for $n=1$.

$$2^{4(1)-1} = 2^{4-1}$$
$$= 2^3$$
$$= 8. \text{ which ends with an } 8$$

Thus, $P(n=1)$ is true.

Assume $P(n=k)$ is true for $k \geq 1$; i.e. the integer 2^{4k-1} ends with an 8. Let $2^{4k-1} = a_m \times 10^m + a_{m-1} \times 10^{m-1} + \ldots + a_1 \times 10 + 8$; for some $m \in \mathbb{Z}$.

We now wish to show that $P(n=k+1)$ is true, i.e. $2^{4(k+1)-1}$ ends with an 8.

$$\text{Consider } 2^{4(k+1)-1} = 2^{4k+4-1}$$
$$= 2^{4k-1+4}$$
$$= (2^{4k-1})(2^4)$$
$$= [a_m \times 10^m + a_{m-1} \times 10^{m-1} + \cdots + a_1 \times 10 + 8][16]$$
$$= [a_m \times 10^m + a_{m-1} \times 10^{m-1} + \cdots + a_1 \times 10 + 8][10 + 6]$$
$$= [a_m \times 10^{m+1} + a_{m-1} \times 10^m + \ldots + a_1 \times 10^2 + 8 \times 10] + [6a_m \times 10^m + 6a_{m-1} \times 10^{m-1} + \ldots + 6a_1 \times 10 + 48]$$
$$= [a_m \times 10^{m+1} + a_{m-1} \times 10^m + \ldots + a_1 \times 10^2 + 8 \times 10] + [6a_m \times 10^m + 6a_{m-1} \times 10^{m-1} + \ldots + (4 + 6a_1) \times 10 + 8]$$

So, $2^{4(k+1)-1}$ ends with an 8.

Thus, $P(n=k+1)$ is true.

Since $P(n=1)$ is true and the truth of $P(n=k)$ implies the truth of $P(n=k+1)$ then by mathematical induction $P(n)$: 2^{4n-1} ends with an 8 for any integer $n \geq 1$ is true.

5c) Let $P(n)$: $n^3+(n+1)^3+(n+2)^3 = 9M$; for some $M \in \mathbb{Z}$, $n \in \mathbb{Z}^+$.

Test for $n=1$.

$$LHS = 1^3+2^3+3^3$$
$$= 1+8+27$$
$$= 36$$
$$= 9(4)$$

36 is a multiple of 9 and $P(n=1)$ is true.

Assume $P(n=k)$ is true ; i.e. $k^3+(k+1)^3+(k+2)^3 = 9A$; $A \in \mathbb{Z}$, $k \in \mathbb{Z}^+$.

We now wish to show that $P(n=k+1)$ is true ; i.e. $(k+1)^3+(k+2)^3+(k+3)^3 = 9B$; for some $B \in \mathbb{Z}$.

Consider $(k+1)^3+(k+2)^3+(k+3)^3 = (k+1)^3+(k+2)^3+k^3+9k^2+27k+27$

$$= [(k+1)^3+(k+2)^3+k^3] + 9(k^2+3k+3)$$
$$= 9A + 9(k^2+3k+3)$$
$$= 9(A+k^2+3k+3)$$

Since $k \in \mathbb{Z}$ then $k^2 \in \mathbb{Z}$ and $3k \in \mathbb{Z}$.

Since $A, k^2, 3k, 3$ are all integers then $A+k^2+3k+3$ is an integer. So $(k+1)^3+(k+2)^3+(k+3)^3$ is a multiple of 9.

Thus, $P(n=k+1)$ is true.

Since $P(n=1)$ is true and the truth of $P(n=k)$ implies the truth of $P(n=k+1)$ then by mathematical induction, $P(n)$: $n^3+(n+1)^3+(n+2)^3 = 2M$ for some $M \in \mathbb{Z}$; $n \in \mathbb{Z}^+$ is true. That is, the sum of the cubes of three consecutive positive integers is always a multiple of 9.

5d) let $P(n): x^n \geq nx$ for $x \geq 2;\ x \in \mathbb{R}$.
$n \geq 1;\ n \in \mathbb{Z}$.

Test for $n=1$.

$$LHS = x^1 \geq 1 \cdot x \geq RHS.$$

Thus, $P(n=1)$ is true.

Assume $P(n=k)$ is true for $k \geq 1$; i.e.

$$x^k \geq kx.$$

We now wish to show that $P(n=k+1)$ is true; i.e.

$$x^{k+1} \geq (k+1)x.$$

Consider
$$\begin{aligned} x^{k+1} &= (x^k)(x) \\ &= x \cdot x^k \\ &\geq x \cdot kx \\ &\geq (xk)x \end{aligned}$$

Since $k \geq 1$ and $x \geq 2$ then $xk \geq 2$.
$xk \geq (k+1)$.

So $x^{k+1} \geq (k+1)x$

Thus, $P(n=k+1)$ is true.

Since $P(n=1)$ is true and the truth of $P(n=k)$ implies the truth of $P(n=k+1)$ then by mathematical induction, $P(n): x^n \geq nx$ is true for all integer $n \geq 1$ and real $x \geq 2$.

5e) $P(n)$: $5^n > 4^n + 3^n + 2^n$; $n \geq 3$; $n \in \mathbb{Z}$

Test for $n=3$.

$LHS = 5^3$

$LHS = 125$

$LHS > 99$

$LHS > 64 + 27 + 8$

$LHS > 4^3 + 3^3 + 2^3$.

Thus, $P(n=1)$ is true.

Assume $P(n=k)$ is true for $k \geq 3$; i.e.

$5^k > 4^k + 3^k + 2^k$

We now wish to show that $P(n=k+1)$ is true; i.e.

$5^{k+1} > 4^{k+1} + 3^{k+1} + 2^{k+1}$

Consider $5^{k+1} = 5(5^k)$

$> 5(4^k + 3^k + 2^k)$

$> 5(4^k) + 5(3^k) + 5(2^k)$

Since $5(4^k) > 4(4^k)$, $5(3^k) > 3(3^k)$ and $5(2^k) > 2(2^k)$

then $5^{k+1} > 4^{k+1} + 3^{k+1} + 2^{k+1}$.

Thus, $P(n=k+1)$ is true.

Since $P(n=1)$ is true and the truth of $P(n=k)$ implies the truth of $P(n=k+1)$ then by mathematical induction, $P(n): 5^n > 4^n + 3^n + 2^n$ is true for all integers $n \geq 3$.

Strong Induction.

7: Let $P(n)$: $f_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$ where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ and $n \geq 1$, $n \in \mathbb{Z}$.

Test for $n=1$.

$$\begin{aligned} LHS &= f_1 \\ &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{2}\right) \\ &= 1. \end{aligned}$$

Thus, $P(n=1)$ is true.

Assume $P(n=1), P(n=2), \ldots, P(n=k)$ are all true for $k \geq 1$.

We now wish to show that $P(n=k+1)$ is true.

Consider
$$\begin{aligned} f_{k+1} &= f_k + f_{k-1} \\ &= \frac{1}{\sqrt{5}}(\alpha^k - \beta^k) + \frac{1}{\sqrt{5}}(\alpha^{k-1} - \beta^{k-1}) \\ &= \frac{1}{\sqrt{5}}\alpha^k - \frac{1}{\sqrt{5}}\beta^k + \frac{1}{\sqrt{5}}\alpha^{k-1} - \frac{1}{\sqrt{5}}\beta^{k-1} \\ &= \frac{1}{\sqrt{5}}\alpha^{k-1}(1+\alpha) - \frac{1}{\sqrt{5}}\beta^{k-1}(1+\beta) \end{aligned}$$

Since $\alpha + \beta = \frac{1}{2} + \frac{\sqrt{5}}{2} + \frac{1}{2} - \frac{\sqrt{5}}{2}$
$= 1$

and $\alpha\beta = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)$
$= \frac{1-5}{4}$
$= -1$.

then α and β are the roots of

$$0 = x^2 + (\alpha\beta)x - (\alpha+\beta)$$
$$0 = x^2 - x - 1.$$

so $\alpha^2 - \alpha - 1 = 0 \Rightarrow \alpha^2 = \alpha + 1$

and $\beta^2 - \beta - 1 = 0 \Rightarrow \beta^2 = \beta + 1$.

so
$$\begin{aligned} f_{k+1} &= \frac{1}{\sqrt{5}}\alpha^{k-1}(\alpha^2) - \frac{1}{\sqrt{5}}\beta^{k-1}(\beta^2) \\ &= \frac{1}{\sqrt{5}}\alpha^{k+1} - \frac{1}{\sqrt{5}}\beta^{k+1} \end{aligned}$$

Thus, $P(n=k+1)$ is true.

Since $P(n=1)$ is true and the truth of $P(n=1), P(n=2), \ldots, P(n=k)$ implies the truth of $P(n=k+1)$ then by mathematical induction.

$P(n)$: $f_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$ where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $n \geq 1$, $n \in \mathbb{Z}$, $f_1 = 1$ and $f_2 = 1$.