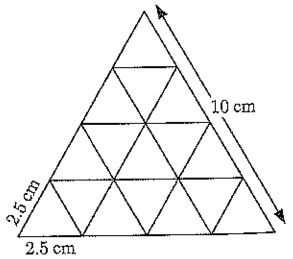
Solutions to Pigeonhole Principle

EXERCISE 1J

1. There are 12 months in a year, so by the Pigeonhole Principle there will be at least one month (pigeonhole) which is the birth month of two or more people (pigeons).

2. Divide the dartboard into 6 equal sectors. The maximum distance between any two points in a sector is 10 cm. Since there are 7 darts, at least two must be in the same sector (Pigeonhole Principle). Hence there are two darts which are at most 10 cm apart.

3. Divide the equilateral triangle into 16 identical triangles as shown. The length of each side of the small triangles is 2.5 cm.
If there are 17 points, then at least two must be in the same triangle (Pigeonhole Principle). Hence, there are at least two points which are at most 2.5 cm apart.



4. Suppose they each receive a different number of prizes. Since each child receives at least one prize, the smallest number of prizes there can be is
$1+2+3+4+5+6+7+8+9+10=55$.
But there are only 50 prizes. Hence, at least two children must receive the same number.

5. The pairs of numbers 1 & 12, 2 & 11, 3 & 10, 4 & 9, 5 & 8, 6 & 7 all add up to 13. Consider the three numbers which are *not* selected. These can come from at most 3 of the pairs. Hence, there are at least 3 pairs for which both numbers are selected.

6. The maximum number of days in a year is 366. So if 367 or more are present this will ensure that at least two people present have the same birthday.
$\therefore$ the minimum number of people needed $= 367$. {PHP}

7. a There are 2 different colours, so selecting 3 socks will ensure that 2 of the socks are the same colour.
b It is possible that if we select 14 socks all of them could be white.
$\therefore$ if we select 15 this will ensure that two different colours will be selected. {PHP}

8. There are 26 letters in the English alphabet and $27 > 26$. Therefore, at least two words will start with the same letter. {PHP}

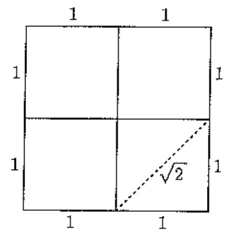
9. $\dfrac{90000}{366} \approx 2459$.
$\therefore$ by the PHP there will be a group of 246 people who have the same birthday.

10. The pairs with sum 11 are:
$\{1, 10\}, \{2, 9\}, \{3, 8\}, \{4, 7\}, \{5, 6\}$.
This set of subsets of $\{1, 2, 3, 4, ..., 10\}$ partition the integers 1, 2, 3, 4, ..., 10.
If the subsets are the pigeonholes and we select any 6 distinct numbers (pigeons) then there will be two such numbers with a sum of 11.

11. A units digit could be one of 10 possibilities, 0, 1, 2, 3, ..., 9. Let these possibilities be pigeonholes.
If we select 11 integers and place then into a pigeonhole corresponding to its units digit, then by the PHP at least one pigeonhole contains two of the integers and so at least two of them will have the same units digit.

12. Suppose there are $n \geqslant 2$ people at a cocktail party.
Case (1) (Each person has at least 1 acquaintance.)
Each person has $1, 2, 3, 4, ..., n-1$ acquaintances. If these values are the pigeonholes, we place each person in a pigeonhole corresponding to their number of acquaintances.
Since $n > n-1$, by the PHP, there will be two people in the same pigeonhole, that is, with the same number of acquaintances.
Case (2) (Someone has no acquaintances.)
Each other person can have at most $n-2$ acquaintances at the party.
Thus each of the other $n-1$ people have $1, 2, 3, ..., \text{or } n-2$ acquaintances. We let these $n-2$ values be the pigeonholes.
Then, by the PHP, since $n-1 > n-2$ there will be two people who have the same number of acquaintances.

13.



We divide the square into 4 squares which are 1 unit by 1 unit and let these smaller squares be the pigeonholes.
If 5 (> 4) points are arbitrarily placed inside the 2×2 square then by the PHP one smaller square will contain at least two points.
The distance between these points is at most the length of a diagonal of a small square, which is $\sqrt{2}$ units.
$\therefore$ the distance between these two points is at most $\sqrt{2}$ units.

14. Let their test scores 7, 6, 5, or 4 be the pigeonholes. Since there are 25 students and 4 pigeonholes, one pigeonhole contains at least $\frac{25}{4} = 6.25$ students. So, there exists one pigeonhole containing at least 7 students. Thus it is guaranteed that there will be 7 students having the same score.
(Although possible, no greater number can be guaranteed.)

15. There are infinitely many powers of 2 (the pigeons). The 2001 residue classes modulo 2001 are the pigeonholes.
By the PHP there will be two powers of 2 in the same residue class, and they will differ by a multiple of 2001.

16. a The 'worst case' is when the red balls are selected last.
$\therefore$ least number $= 8 + 10 + 7 + \underset{\uparrow \atop \text{red}}{3} = 28$.
b The 'worst case' is when two of each colour are selected first.
$\therefore$ least number $= 2+2+2+2+1 = 9$
c The 'worst case' is when all green and blue balls are selected first.
$\therefore$ least number $= 10 + 8 + 1$ other $= 19$.

17. a When 3 dice are rolled the possible totals are
$\underset{\uparrow \atop \text{three 1s}}{3}, 4, 5, 6, 7, ..., \underset{\uparrow \atop \text{three 6s}}{18}$.
So, there are 16 different totals.
$\therefore$ by the PHP, 17 rolls are needed to guarantee a repeated total.
b The 'worst case' is when each total appears twice first.
$\therefore$ least number $= 16 \times 2 + 1 = 33$ rolls.