Test III *[44 marks]*

1. Solve the simultaneous equations *[7 marks]*

$\log_2 6x = 1 + 2\log_2 y$

$1 + \log_6 x = \log_6 (15y - 25)$.

Markscheme

use of at least one "log rule" applied correctly for the first equation ***M1***

$\log_2 6x = \log_2 2 + 2\log_2 y$

$= \log_2 2 + \log_2 y^2$

$= \log_2 (2y^2)$

$\Rightarrow 6x = 2y^2$ ***A1***

use of at least one "log rule" applied correctly for the second equation ***M1***

$\log_6 (15y - 25) = 1 + \log_6 x$

$= \log_6 6 + \log_6 x$

$= \log_6 6x$

$\Rightarrow 15y - 25 = 6x$ ***A1***

attempt to eliminate x (or y) from their two equations ***M1***

$2y^2 = 15y - 25$

$2y^2 - 15y + 25 = 0$

$(2y - 5)(y - 5) = 0$

$x = \frac{25}{12}$, $y = \frac{5}{2}$, ***A1***

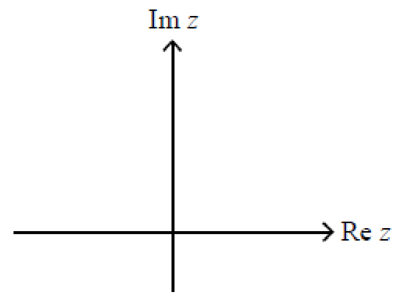
or $x = \frac{25}{3}$, $y = 5$ ***A1***

Note: x, y values do not have to be "paired" to gain either of the final two ***A*** marks.

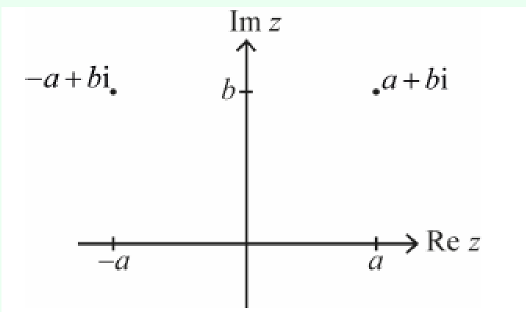
[7 marks]

Let $z = a + b\text{i}$, a, b $\in \mathbb{R}^+$ and let $\arg z = \theta$.

2a. Show the points represented by z and $z - 2a$ on the following Argand diagram. *[1 mark]*



Markscheme



A1

Note: Award ***A1*** for z in first quadrant and $z - 2a$ its reflection in the y-axis.

[1 mark]

2b. Find an expression in terms of θ for $\arg(z - 2a)$. *[1 mark]*

Markscheme

$\pi - \theta$ (or any equivalent) ***A1***

[1 mark]

2c. Find an expression in terms of θ for $\arg\left(\frac{z}{z-2a}\right)$. *[2 marks]*

Markscheme

$\arg\left(\frac{z}{z-2a}\right) = \arg(z) - \arg(z-2a)$ ***(M1)***

$= 2\theta - \pi$ (or any equivalent) ***A1***

[2 marks]

2d. Hence or otherwise find the value of θ for which $\mathrm{Re}\left(\frac{z}{z-2a}\right) = 0$. *[3 marks]*

Markscheme

METHOD 1

if $\mathrm{Re}\left(\frac{z}{z-2a}\right) = 0$ then $2\theta - \pi = \frac{n\pi}{2}$, ($n$ odd) ***(M1)***

$-\pi < 2\theta - \pi < 0 \Rightarrow n = -1$

$2\theta - \pi = -\frac{\pi}{2}$ ***(A1)***

$\theta = \frac{\pi}{4}$ ***A1***

METHOD 2

$\frac{a+b\mathrm{i}}{-a+b\mathrm{i}} = \frac{b^2-a^2-2ab\mathrm{i}}{a^2+b^2}$ ***M1***

$\mathrm{Re}\left(\frac{z}{z-2a}\right) = 0 \Rightarrow b^2 - a^2 = 0$

$b = a$ ***A1***

$\theta = \frac{\pi}{4}$ ***A1***

Note: Accept any equivalent, *eg* $\theta = -\frac{7\pi}{4}$.

[3 marks]

3. Use the principle of mathematical induction to prove that *[7 marks]*

$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}$, where $n \in \mathbb{Z}^+$.

Markscheme

if $n = 1$

$\text{LHS} = 1\,;\ \text{RHS} = 4 - \frac{3}{2^0} = 4 - 3 = 1$ ***M1***

hence true for $n = 1$

assume true for $n = k$ ***M1***

Note: Assumption of truth must be present. Following marks are not dependent on the first two ***M1*** marks.

so $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ \ldots + k\left(\frac{1}{2}\right)^{k-1} = 4 - \frac{k+2}{2^{k-1}}$

if $n = k + 1$

$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ \ldots + k\left(\frac{1}{2}\right)^{k-1} + (k+1)\left(\frac{1}{2}\right)^k$

$= 4 - \frac{k+2}{2^{k-1}} + (k+1)\left(\frac{1}{2}\right)^k$ ***M1A1***

finding a common denominator for the two fractions ***M1***

$= 4 - \frac{2(k+2)}{2^k} + \frac{k+1}{2^k}$

$= 4 - \frac{2(k+2)-(k+1)}{2^k} = 4 - \frac{k+3}{2^k}\left(= 4 - \frac{(k+1)+2}{2^{(k+1)-1}}\right)$ ***A1***

hence if true for $n = k$ then also true for $n = k + 1$, as true for $n = 1$, so true (for all $n \in \mathbb{Z}^+$) ***R1***

Note: Award the final ***R1*** only if the first four marks have been awarded.

[7 marks]

The cubic equation $x^3 + px^2 + qx + c = 0$, has roots $\alpha,\ \beta,\ \gamma$. By expanding $(x-\alpha)(x-\beta)(x-\gamma)$ show that

4a. (i) $p = -(\alpha + \beta + \gamma)$; *[3 marks]*

(ii) $q = \alpha\beta + \beta\gamma + \gamma\alpha$;

(iii) $c = -\alpha\beta\gamma$.

Markscheme

(i)-(iii) given the three roots $\alpha,\ \beta,\ \gamma$, we have

$x^3 + px^2 + qx + c = (x-\alpha)(x-\beta)(x-\gamma)$ ***M1***

$= \left(x^2 - (\alpha+\beta)x + \alpha\beta\right)(x-\gamma)$ ***A1***

$= x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma$ ***A1***

comparing coefficients:

$p = -(\alpha+\beta+\gamma)$ ***AG***

$q = (\alpha\beta+\beta\gamma+\gamma\alpha)$ ***AG***

$c = -\alpha\beta\gamma$ ***AG***

[3 marks]

4b. It is now given that $p=-6$ and $q=18$ for parts (b) and (c) below. *[5 marks]*

(i) In the case that the three roots $\alpha,\ \beta,\ \gamma$ form an arithmetic sequence, show that one of the roots is 2.

(ii) Hence determine the value of c.

Markscheme

METHOD 1

(i) Given $-\alpha-\beta-\gamma=-6$

And $\alpha\beta+\beta\gamma+\gamma\alpha=18$

Let the three roots be $\alpha,\ \beta,\ \gamma$

So $\beta-\alpha=\gamma-\beta$ ***M1***

or $2\beta=\alpha+\gamma$

Attempt to solve simultaneous equations: ***M1***

$\beta+2\beta=6$ ***A1***

$\beta=2$ ***AG***

(ii) $\alpha+\gamma=4$

$2\alpha+2\gamma+\alpha\gamma=18$

$\Rightarrow \gamma^2-4\gamma+10=0$

$\Rightarrow \gamma=\frac{4\pm \mathrm{i}\sqrt{24}}{2}$ ***(A1)***

Therefore $c = -\alpha\beta\gamma = -\left(\frac{4+\mathrm{i}\sqrt{24}}{2}\right)\left(\frac{4-\mathrm{i}\sqrt{24}}{2}\right)2 = -20$ ***A1***

METHOD 2

(i) let the three roots be $\alpha,\ \alpha - d,\ \alpha + d$ ***M1***

adding roots ***M1***

to give $3\alpha = 6$ ***A1***

$\alpha = 2$ ***AG***

(ii) α is a root, so $2^3 - 6 \times 2^2 + 18 \times 2 + c = 0$ ***M1***

$8 - 24 + 36 + c = 0$

$c = -20$ ***A1***

METHOD 3

(i) let the three roots be $\alpha,\ \alpha - d,\ \alpha + d$ ***M1***

adding roots ***M1***

to give $3\alpha = 6$ ***A1***

$\alpha = 2$ ***AG***

(ii) $q = 18 = 2(2-d) + (2-d)(2+d) + 2(2+d)$ ***M1***

$d^2 = -6 \Rightarrow d = \sqrt{6}\mathrm{i}$

$\Rightarrow c = -20$ ***A1***

[5 marks]

4c. In another case the three roots $\alpha,\ \beta,\ \gamma$ form a geometric sequence. *[6 marks]*
Determine the value of c.

Markscheme

METHOD 1

Given $-\alpha-\beta-\gamma=-6$

And $\alpha\beta+\beta\gamma+\gamma\alpha=18$

Let the three roots be $\alpha,\ \beta,\ \gamma$.

So $\frac{\beta}{\alpha}=\frac{\gamma}{\beta}$ ***M1***

or $\beta^2=\alpha\gamma$

Attempt to solve simultaneous equations: ***M1***

$\alpha\beta+\gamma\beta+\beta^2=18$

$\beta(\alpha+\beta+\gamma)=18$

$6\beta=18$

$\beta=3$ ***A1***

$\alpha+\gamma=3,\ \alpha=\frac{9}{\gamma}$

$\Rightarrow\gamma^2-3\gamma+9=0$

$\Rightarrow\gamma=\frac{3\pm\mathrm{i}\sqrt{27}}{2}$ ***(A1)(A1)***

Therefore $c=-\alpha\beta\gamma=-\left(\frac{3+\mathrm{i}\sqrt{27}}{2}\right)\left(\frac{3-\mathrm{i}\sqrt{27}}{2}\right)3=-27$ ***A1***

METHOD 2

let the three roots be $a,\ ar,\ ar^2$ ***M1***

attempt at substitution of $a,\ ar,\ ar^2$ and p and q into equations from (a) ***M1***

$6=a+ar+ar^2\left(=a(1+r+r^2)\right)$ ***A1***

$18=a^2r+a^2r^3+a^2r^2\left(=a^2r(1+r+r^2)\right)$ ***A1***

therefore $3=ar$ ***A1***

therefore $c=-a^3r^3=-3^3=-27$ ***A1***

[6 marks]

Total [14 marks]

Consider the following system of equations

$$2x + y + 6z = 0$$

$$4x + 3y + 14z = 4$$

$$2x - 2y + (\alpha - 2)z = \beta - 12.$$

5a. Find conditions on α and β for which *[6 marks]*

(i) the system has no solutions;

(ii) the system has only one solution;

(iii) the system has an infinite number of solutions.

Markscheme

$2x + y + 6z = 0$

$4x + 3y + 14z = 4$

$2x - 2y + (\alpha - 2)z = \beta - 12$

attempt at row reduction ***M1***

eg $R_2 - 2R_1$ and $R_3 - R_1$

$2x + y + 6z = 0$

$y + 2z = 4$

$-3y + (\alpha - 8)z = \beta - 12$ ***A1***

eg $R_3 + 3R_2$

$2x + y + 6z = 0$

$y + 2z = 4$ ***A1***

$(\alpha - 2)z = \beta$

(i) no solutions if $\alpha = 2,\ \beta \neq 0$ ***A1***

(ii) one solution if $\alpha \neq 2$ ***A1***

(iii) infinite solutions if $\alpha = 2,\ \beta = 0$ ***A1***

Note: Accept alternative methods e.g. determinant of a matrix

Note: Award ***A1A1A0*** if all three consistent with their reduced form, ***A1A0A0*** if two or one answer consistent with their reduced form.

[6 marks]

5b. In the case where the number of solutions is infinite, find the general solution of the system of equations in Cartesian form. *[3 marks]*

Markscheme

$y + 2z = 4 \Rightarrow y = 4 - 2z$

$2x = -y - 6z = 2z - 4 - 6z = -4z - 4 \Rightarrow x = -2z - 2$ ***A1***

therefore Cartesian equation is $\frac{x+2}{-2} = \frac{y-4}{-2} = \frac{z}{1}$ or equivalent ***A1***

[3 marks]

Total [9 marks]