**1** What place does an "intuitive" rule such as Cavalieri's Principle have in mathematics?

**2** How can Cavalieri's Principle be used to find the volume of a sphere from the known volume of a circular cone? Can this be regarded as a valid proof of the formula?

In order to consider the volume and surface area we will look at a *finite* part of the trumpet from $x = 1$ to $x = N$ and then let $N \to \infty$.

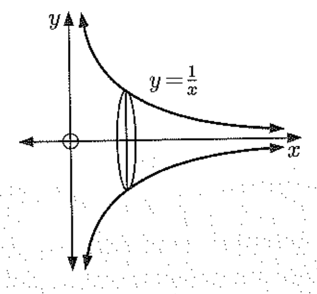
**VOLUME**

We imagine a plane perpendicular to the $x$-axis, cutting the axis at $x$.

The trumpet and the plane intersect at a circle of radius $\frac{1}{x}$.

The area of the circle is simply $\pi\frac{1}{x^2}$.

To calculate the volume we sum the infinitesimal areas of these discs as $\pi\int_1^N \frac{1}{x^2}\,dx = \pi\left[-\frac{1}{x}\right]_1^N = \pi\left(1-\frac{1}{N}\right)$.




As $N \to \infty$ we see that the volume converges to $\pi$. It is somewhat surprising that an infinite shape can have a finite volume. However, this is similar to the idea that an infinite series of positive terms can sum to a finite *length*.

**SURFACE AREA**

To calculate the surface area we use the formula $S = 2\pi\int_1^N y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$.

Since $y = \frac{1}{x}$ we have $\frac{dy}{dx} = -\frac{1}{x^2}$ and so the integral becomes $S = 2\pi\int_1^N \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$.

Noting that $\frac{1}{x}\sqrt{1+\frac{1}{x^4}} > \frac{1}{x}$, we find that

$$S = 2\pi\int_1^N \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx > 2\pi\int_1^N \frac{1}{x}\,dx = 2\pi\ln(N).$$

As $N \to \infty$, $\ln(N)$ diverges, and so the surface area $S$ also diverges.

Hence we have a finite volume with an infinite surface area. This is a strange and controversial result. Thomas Hobbes, the English Philosopher, said "*to understand this for sense, it is not required that a man should be a geometrician or logician, but that he should be mad*".

**3** Imagine filling the trumpet with a liquid. While the trumpet holds a finite volume, what happens when a thin layer coats the surface?

**4** What do we do when a physical problem yields a mathematical result with no physical meaning?



# APPENDIX A: METHODS OF PROOF

Greek mathematicians more than 2000 years ago realised that progress in mathematical thinking could be brought about by conscious formulation of the methods of **abstraction** and **proof**.

By considering a few examples, one might notice a certain common quality or pattern from which one could predict a rule or formula for the general case. In mathematics this prediction is known as a **conjecture**. Mathematicians love to find patterns, and try to understand why they occur.

Experiments and further examples might help to convince you that the conjecture is true. However, problems will often contain extra information which can sometimes obscure the essential detail, particularly in applied mathematics. Stripping this away is the process of **abstraction**.

For example, by considering the given table of values one may conjecture:

"If $a$ and $b$ are real numbers then $a < b$ implies that $a^2 < b^2$."

However, on observing that $-2 < 1$ but $(-2)^2 \not< 1^2$ we have a **counter-example**.

| $a$ | $b$ | $a^2$ | $b^2$ |
|---|---|---|---|
| 1 | 2 | 1 | 4 |
| 3 | 5 | 9 | 25 |
| 4 | 5 | 16 | 25 |
| 5 | 7 | 25 | 49 |
| 6 | 9 | 36 | 81 |

In the light of this we reformulate and refine our conjecture:

"If $a$ and $b$ are *positive* real numbers then $a < b$ implies $a^2 < b^2$."

The difficulty is that this process might continue with reformulations, counter-examples, and revised conjectures indefinitely. At what point are we certain that the conjecture is true? A **proof** is a flawless logical argument which leaves no doubt that the conjecture is indeed a truth. If we have a proof then the conjecture can be called a **theorem**.

Mathematics has evolved to accept certain types of arguments as valid proofs. They include a mixture of both logic and calculation. Generally mathematicians like elegant, efficient proofs. It is common not to write every minute detail. However, when you write a proof you should be prepared to expand and justify every step if asked to do so.

We have already examined in the HL Core text, proof by **the principle of mathematical induction**. Now we consider other methods.

1°

## DIRECT PROOF

In a **direct proof** we start with a known truth and by a succession of correct deductions finish with the required result.

**Example 1:** Prove that if $a, b \in \mathbb{R}$ then $a < b \Rightarrow a < \frac{a+b}{2}$

**Proof:** $a < b \Rightarrow \frac{a}{2} < \frac{b}{2}$ {as we are dividing by 2 which is $> 0$}

$\Rightarrow \frac{a}{2}+\frac{a}{2} < \frac{a}{2}+\frac{b}{2}$ {adding $\frac{a}{2}$ to both sides}

$\Rightarrow a < \frac{a+b}{2}$

Sometimes it is not possible to give a direct proof of the full result and so the different possible cases (called **exhaustive cases**) need to be considered and proved separately.

(分类讨论)

**Example 2:** Prove the **geometric progression**: For $n \in \mathbb{Z},\ n \geqslant 0,$

$$1 + r^1 + r^2 + .... + r^n = \begin{cases} \dfrac{r^{n+1} - 1}{r - 1}, & r \neq 1 \\ n + 1, & r = 1 \end{cases}$$

---

**Proof:** **Case $r = 1$:**

$$\begin{aligned} & 1 + r^1 + r^2 + .... + r^n \\ &= 1 + 1 + 1 + .... + 1 \qquad \{n + 1 \text{ times}\} \\ &= n + 1 \end{aligned}$$

**Case $r \neq 1$:**

$$\begin{aligned} \text{Let } \quad S_n &= 1 + r^1 + r^2 + .... + r^n. \\ \text{Then } \quad rS_n &= r^1 + r^2 + r^3 + .... + r^{n+1} \\ \therefore \quad rS_n - S_n &= r^{n+1} - 1 \qquad \{\text{after cancellation of terms}\} \\ \therefore \quad (r-1)S_n &= r^{n+1} - 1 \\ \therefore \quad S_n &= \frac{r^{n+1} - 1}{r - 1} \qquad \{\text{dividing by } r - 1 \text{ since } r \neq 1\} \end{aligned}$$

**Example 3:** Alice looks at Bob and Bob looks at Clare. Alice is married, but Clare is not. Prove that a married person looks at an unmarried person.

---

**Proof:** We do not know whether Bob is married or not, so we consider the different (exhaustive) cases:

**Case: Bob is married.** If Bob is married, then a married person (Bob) looks at an unmarried person (Clare).

**Case: Bob is unmarried.** If Bob is unmarried, then a married person (Alice) looks at an unmarried person (Bob).

Since we have considered all possible cases, the full result is proved.

## EXERCISE

1 Let $I = \sqrt{2}$, which is irrational. Consider $I^I$ and $I^{I^I}$, and hence prove that an irrational number to the power of an irrational number can be rational.

## PROOF BY CONTRADICTION (AN INDIRECT PROOF) 反证法

In **proof by contradiction** we deliberately assume the opposite to what we are trying to prove. By a series of correct steps we show that this is impossible, our assumption is false, and hence its opposite is true.

**Example 2:** Prove the **geometric progression**: For $n \in \mathbb{Z}$, $n \geqslant 0$,

$$1 + r^1 + r^2 + \dots + r^n = \begin{cases} \dfrac{r^{n+1} - 1}{r - 1}, & r \neq 1 \\ n + 1, & r = 1 \end{cases}$$

**Proof:** **Case $r = 1$:**

$$\begin{aligned} &1 + r^1 + r^2 + \dots + r^n \\ &= 1 + 1 + 1 + \dots + 1 \qquad \{n+1 \text{ times}\} \\ &= n + 1 \end{aligned}$$

**Case $r \neq 1$:**

$$\begin{aligned} &\text{Let } S_n = 1 + r^1 + r^2 + \dots + r^n. \\ &\text{Then } rS_n = r^1 + r^2 + r^3 + \dots + r^{n+1} \\ \therefore\ & rS_n - S_n = r^{n+1} - 1 \qquad \{\text{after cancellation of terms}\} \\ \therefore\ & (r-1)S_n = r^{n+1} - 1 \\ \therefore\ & S_n = \frac{r^{n+1} - 1}{r - 1} \qquad \{\text{dividing by } r - 1 \text{ since } r \neq 1\} \end{aligned}$$

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## EXERCISE

1 Let $I = \sqrt{2}$, which is irrational. Consider $I^I$ and $I^{I^I}$, and hence prove that an irrational number to the power of an irrational number can be rational.

## PROOF BY CONTRADICTION (AN INDIRECT PROOF)

In **proof by contradiction** we deliberately assume the opposite to what we are trying to prove. By a series of correct steps we show that this is impossible, our assumption is false, and hence its opposite is true.



**Example 4:** Consider **Example 1** again but this time use proof by contradiction:

Prove that if $a, b \in \mathbb{R}$ then $a < b \Rightarrow a < \frac{a+b}{2}$.

**Proof (by contradiction):**

For $a < b$, suppose that $a \geqslant \frac{a+b}{2}$.

$$\begin{aligned} \Rightarrow\ & 2a \geqslant 2\left(\frac{a+b}{2}\right) \qquad \{\text{multiplying both sides by 2}\} \\ \Rightarrow\ & 2a \geqslant a + b \\ \Rightarrow\ & a \geqslant b \qquad \{\text{subtracting } a \text{ from both sides}\} \end{aligned}$$

which is false.

Since the steps of the argument are correct, the supposition must be false and the alternative, $a < \frac{a+b}{2}$ must be true.

**Example 5:** Prove that the solution of $3^x = 8$ is irrational.

**Proof (by contradiction):**

Suppose the solution of $3^x = 8$ is rational, or in other words, that $x$ is rational. Notice that $x > 0$.

$$\begin{aligned} \Rightarrow\ & x = \frac{p}{q} \quad \text{where } p, q \in \mathbb{Z},\ q \neq 0 \quad \{\text{and since } x > 0, \text{ integers } p, q > 0\} \\ \Rightarrow\ & 3^{\frac{p}{q}} = 8 \\ \Rightarrow\ & \left(3^{\frac{p}{q}}\right)^q = 8^q \\ \Rightarrow\ & 3^p = 8^q \end{aligned}$$

which is impossible since for the given possible values of $p$ and $q$, $3^p$ is always odd and $8^q$ is always even. Thus, the assumption is false and its opposite must be true. Hence $x$ is irrational.

**Example 6:** Prove that no positive integers $x$ and $y$ exist such that $x^2 - y^2 = 1$.

**Proof (by contradiction):**

Suppose $x, y \in \mathbb{Z}^+$ exist such that $x^2 - y^2 = 1$.

$$\begin{aligned} \Rightarrow\ & (x+y)(x-y) = 1 \\ \Rightarrow\ & \underbrace{x + y = 1 \text{ and } x - y = 1}_{\text{case 1}} \quad \textbf{or} \quad \underbrace{x + y = -1 \text{ and } x - y = -1}_{\text{case 2}} \\ \Rightarrow\ & x = 1,\ y = 0 \ \ (\text{from case 1}) \quad \textbf{or} \quad x = -1,\ y = 0 \ \ (\text{from case 2}) \end{aligned}$$

Both cases provide a contradiction to $x, y > 0$.
Thus, the supposition is false and its opposite is true.
There do *not* exist positive integers $x$ and $y$ such that $x^2 - y^2 = 1$.

Indirect proof often seems cleverly contrived, especially if no direct proof is forthcoming. It is perhaps more natural to seek a direct proof for the first attempt to prove a conjecture.

## ERRORS IN PROOF

One must be careful not to make errors in algebra or reasoning. Examine carefully the following examples.

**Example 7:** Consider **Example 5** again: Prove that the solution of $3^x = 8$ is irrational.

Invalid argument:

$$
\begin{aligned}
& 3^x = 8 \\
\Rightarrow\quad & \log 3^x = \log 8 \\
\Rightarrow\quad & x \log 3 = \log 8 \\
\Rightarrow\quad & x = \frac{\log 8}{\log 3} \quad \text{where both } \log 8 \text{ and } \log 3 \text{ are irrational.} \\
\Rightarrow\quad & x \text{ is irrational.}
\end{aligned}
$$

The last step is not valid. The argument that an irrational divided by an irrational is always irrational is not correct. For example, $\frac{\sqrt{2}}{\sqrt{2}} = 1$, and 1 is rational.

Dividing by zero is *not* a valid operation. $\frac{a}{0}$ is not defined for any $a \in \mathbb{R}$, in particular $\frac{0}{0} \neq 1$.

**Example 8:** Invalid "proof" that $5 = 2$

$$
\begin{aligned}
& 0 = 0 \\
\Rightarrow\quad & 0 \times 5 = 0 \times 2 \\
\Rightarrow\quad & \frac{0 \times 5}{0} = \frac{0 \times 2}{0} \qquad \{\text{dividing through by } 0\} \\
\Rightarrow\quad & 5 = 2, \quad \text{which is clearly false.}
\end{aligned}
$$

This invalid step is not always obvious, as illustrated in the following example.

**Example 9:** Invalid "proof" that $0 = 1$:

$$
\begin{aligned}
& \text{Suppose } a = 1 \\
\Rightarrow\quad & a^2 = a \\
\Rightarrow\quad & a^2 - 1 = a - 1 \\
\Rightarrow\quad & (a+1)(a-1) = a - 1 \\
\Rightarrow\quad & a + 1 = 1 \quad .... \ (*) \\
\Rightarrow\quad & a = 0 \\
& \text{So, } 0 = 1
\end{aligned}
$$

The invalid step in the argument is $(*)$ where we divide both sides by $a - 1$. Since $a = 1$, $a - 1 = 0$, and so we are dividing both sides by zero.

Another trap to be avoided is to begin by assuming the result we wish to prove is true. This readily leads to invalid circular arguments.

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The invalid step in the argument is $(*)$ where we divide both sides by $a - 1$. Since $a = 1$, $a - 1 = 0$, and so we are dividing both sides by zero.

Another trap to be avoided is to begin by assuming the result we wish to prove is true. This readily leads to invalid circular arguments.



**Example 10:** Prove without decimalisation that $\sqrt{3} - 1 > \frac{1}{\sqrt{2}}$.

Invalid argument:

$$\begin{aligned} \sqrt{3} - 1 &> \tfrac{1}{\sqrt{2}} \\ \Rightarrow\quad (\sqrt{3}-1)^2 &> \left(\tfrac{1}{\sqrt{2}}\right)^2 \quad \{\text{both sides are } > 0 \text{, so we can square them}\} \\ \Rightarrow\quad 4 - 2\sqrt{3} &> \tfrac{1}{2} \\ \Rightarrow\quad \tfrac{7}{2} &> 2\sqrt{3} \\ \Rightarrow\quad 7 &> 4\sqrt{3} \\ \Rightarrow\quad 7^2 &> 48 \quad \{\text{squaring again}\} \\ \Rightarrow\quad 49 &> 48 \quad \text{which is true.} \end{aligned}$$

Hence $\sqrt{3} - 1 > \frac{1}{\sqrt{2}}$ is true.

Although $\sqrt{3} - 1 > \frac{1}{\sqrt{2}}$ is in fact true, the above argument is invalid because we began by assuming the result.

A valid method of proof for $\sqrt{3} - 1 > \frac{1}{\sqrt{2}}$ can be found by either:

- reversing the steps of the above argument, or by
- using proof by contradiction (supposing $\sqrt{3} - 1 \leqslant \frac{1}{\sqrt{2}}$).

It is important to distinguish **errors in proof** from a **false conjecture.**

Consider the table alongside, which shows values of $n^2 - n + 41$ for various values of $n \in \mathbb{N}$.

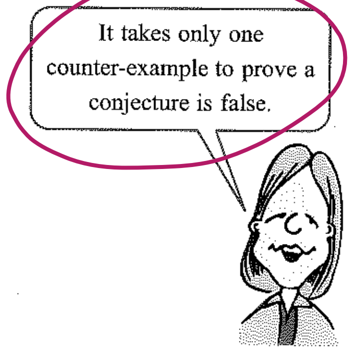
| $n$ | $n^2 - n + 41$ |
|---|---|
| 1 | 41 |
| 2 | 43 |
| 3 | 47 |
| 4 | 53 |
| 5 | 61 |
| 6 | 71 |
| 7 | 83 |
| 8 | 97 |
| 9 | 113 |
| 10 | 131 |
| 11 | 151 |
| 12 | 173 |
| 13 | 197 |
| ... | ... |
| 30 | 911 |
| ... | ... |
| 99 | 9743 |
| ... | ... |

From the many examples given, one might conjecture:

"For all natural numbers $n$, $n^2 - n + 41$ is prime."

This conjecture is in fact false.

For example, for $n = 41$, $n^2 - n + 41 = 41^2$ is clearly not prime.



## IMPLICATIONS AND THEIR CONVERSE

**If .... then ....**

Many statements in mathematics take the form of an **implication** "If $A$ then $B$", where $A$ and $B$ are themselves statements. The statement $A$ is known as the **hypothesis**. The statement $B$ is known as the **conclusion**.

Implications can be written in many forms in addition to "If $A$ then $B$". For example, the following all have the same meaning:

$$A \left\{ \begin{array}{c} \text{implies} \\ \text{so} \\ \text{hence} \\ \text{thus} \\ \text{therefore} \end{array} \right\} B.$$

Given a statement of the form "If $A$ then $B$", we can write a **converse** statement "If $B$ then $A$".

If we know the truth, or otherwise, of a given statement, we can say nothing about the truth of the converse. It could be true or false.

A statement and its converse are said to be (logically) *independent*.

For example, suppose $x$ is an integer.

- The statement "If $x$ is odd, then $2x$ is even" is *true*, but its converse "If $2x$ is even, then $x$ is odd" is *false*.
- The statement "If $2x$ is even, then $x$ is odd" is *false*, but its converse "If $x$ is odd, then $2x$ is even" is *true*.
- The statement "If $x > 1$, then $\ln x > 0$" is *true*, and its converse "If $\ln x > 0$, then $x > 1$" is also *true*.
- The statement "If $x = 5$, then $x^2 = 16$" is *false*, and its converse "If $x^2 = 16$, then $x = 5$" is also *false*.

### EXERCISE

Prove or disprove:

1 If $x$ is rational then $2^x \neq 3$. $2^{\frac{p}{q}} = 3$ $2^p = 3^q$ (x) ⇒ prove.

2 If $2^x \neq 3$ then $x$ is rational.

## EQUIVALENCE

Some conjectures with two statements $A$ and $B$ involve **logical equivalence** or simply **equivalence**.

We say *$A$ is equivalent to $B$*, or $A$ is true *if and only if* $B$ is true.

The phrase "if and only if" is often written as "iff" or $\Leftrightarrow$.

$$A \Leftrightarrow B \text{ means } A \Rightarrow B \textit{ and } B \Rightarrow A$$

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### EXERCISE

Prove or disprove:

1 If $x$ is rational then $2^x \neq 3$.

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$$A \Leftrightarrow B \text{ means } A \Rightarrow B \text{ and } B \Rightarrow A$$



In order to prove an equivalence, we need to prove both implications: $A \Rightarrow B$ *and* $B \Rightarrow A$.

For example: $x^2 = 9 \Leftrightarrow x = 3$ is a false statement.
$x = 3 \Rightarrow x^2 = 9$ is true
but $x^2 = 9 \not\Rightarrow x = 3$ as $x$ may be $-3$.

**Example 11:** Prove that $(n+2)^2 - n^2$ is a multiple of 8 $\Leftrightarrow$ $n$ is odd.

**Proof:** $(\Rightarrow)$ $(n+2)^2 - n^2$ is a multiple of 8
$\Rightarrow n^2 + 4n + 4 - n^2 = 8a$ for some integer $a$
$\Rightarrow 4n + 4 = 8a$
$\Rightarrow n + 1 = 2a$
$\Rightarrow n = 2a - 1$
$\Rightarrow n$ is odd.

$(\Leftarrow)$ $n$ is odd
$\Rightarrow n = 2a - 1$ for some integer $a$
$\Rightarrow n + 1 = 2a$
$\Rightarrow 4n + 4 = 8a$
$\Rightarrow (n^2 + 4n + 4) - n^2 = 8a$
$\Rightarrow (n+2)^2 - n^2$ is a multiple of 8.

In the above example the $(\Rightarrow)$ argument is clearly reversible to give the $(\Leftarrow)$ argument. However, this is not always the case.

**Example 12:** Prove that for all $x \in \mathbb{Z}^+$, $x$ is not divisible by 3 $\Leftrightarrow$ $x^2 - 1$ is divisible by 3.

**Proof:** $(\Rightarrow)$ $x$ is not divisible by 3
$\Rightarrow$ either $x = 3k + 1$ or $x = 3k + 2$ for some $k \in \mathbb{Z}^+ \cup \{0\}$
$\Rightarrow x^2 - 1 = 9k^2 + 6k$ or $9k^2 + 12k + 3$
$= 3(3k^2 + 2)$ or $3(3k^2 + 4k + 1)$
$\Rightarrow x^2 - 1$ is divisible by 3.

$(\Leftarrow)$ $x^2 - 1$ is divisible by 3
$\Rightarrow 3 \mid x^2 - 1$
$\Rightarrow 3 \mid (x+1)(x-1)$
$\Rightarrow 3 \mid (x+1)$ or $3 \mid (x-1)$ {as 3 is a prime number}
$\Rightarrow 3 \nmid x$
or in other words, $x$ is not divisible by 3.

## NEGATION

For any given statement $A$, we write not $A$ or $\neg A$ to represent the negation of the statement $A$.

For example:

| | $A$ | $\neg A$ |
|---|---|---|
| | $x > 0$ | $x \leqslant 0$ |
| | $x$ is prime | $x$ is not prime |
| | $x$ is an integer | $x$ is not an integer |
| For $x \in \mathbb{R}$: | $x$ is rational | $x$ is irrational |
| For $z \in \mathbb{C}$: | $z$ is real | $z = a + bi,\ a, b \in \mathbb{R},\ b \neq 0$ |
| For $x \in \mathbb{Z}^+ \cup \{0\}$: | $x$ is a multiple of 3 | $x$ is not a multiple of 3 *or* $x = 3k+1$ or $3k+2$ for $k \in \mathbb{Z}^+ \cup \{0\}$ |

## PROOF OF THE CONTRAPOSITIVE

To prove the statement "If $A$ then $B$", we can provide a direct proof, or we can prove the logically equivalent **contrapositive** statement "If not $B$, then not $A$" which we can also write as "If $\neg B$, then $\neg A$".

For example, the statement "If it is Jon's bicycle, then it is blue"
is logically equivalent to "If that bicycle is not blue, then it is not Jon's".

**Example 13:** Prove that for $a, b \in \mathbb{R}$, "$ab$ is irrational $\Rightarrow$ either $a$ or $b$ is irrational".

**Proof using contrapositive:**

$a$ and $b$ are both rational $\Rightarrow$ $a = \frac{p}{q}$ and $b = \frac{r}{s}$ where $p, q, r, s \in \mathbb{Z}$, $q \neq 0$, $s \neq 0$

$\Rightarrow$ $ab = \left(\frac{p}{q}\right)\left(\frac{r}{s}\right) = \frac{pr}{qs}$ {where $qs \neq 0$, since $q, s \neq 0$}

$\Rightarrow$ $ab$ is rational {since $pr, qs \in \mathbb{Z}$}

Thus $ab$ is irrational $\Rightarrow$ either $a$ or $b$ is irrational.

**Example 14:** Prove that if $n$ is a positive integer of the form $3k+2$, $k \geqslant 0$, $k \in \mathbb{Z}$, then $n$ is not a square.

**Proof using contrapositive:**

If $n$ is a square then
$n$ has one of the forms $(3a)^2$, $(3a+1)^2$ or $(3a+2)^2$, where $a \in \mathbb{Z}^+ \cup \{0\}$.

$\Rightarrow$ $n = 9a^2$, $9a^2 + 6a + 1$ or $9a^2 + 12a + 4$

$\Rightarrow$ $n = 3(3a^2)$, $3(3a^2 + 2a) + 1$ or $3(3a^2 + 4a + 1) + 1$

$\Rightarrow$ $n$ has the form $3k$ or $3k+1$ only, where $k \in \mathbb{Z}^+ \cup \{0\}$

$\Rightarrow$ $n$ does not have form $3k+2$.

Thus if $n$ is a positive integer of the form $3k+2$, $k \geqslant 0$, $k \in \mathbb{Z}$, then $n$ is not a square.

## NEGATION

For any given statement $A$, we write not $A$ or $\neg A$ to represent the negation of the statement $A$.

For example:

| | $A$ | $\neg A$ |
|---|---|---|
| | $x > 0$ | $x \leqslant 0$ |
| | $x$ is prime | $x$ is not prime |
| | $x$ is an integer | $x$ is not an integer |
| For $x \in \mathbb{R}$: | $x$ is rational | $x$ is irrational |
| For $z \in \mathbb{C}$: | $z$ is real | $z = a + bi,\ a, b \in \mathbb{R},\ b \neq 0$ |
| For $x \in \mathbb{Z}^+ \cup \{0\}$: | $x$ is a multiple of 3 | $x$ is not a multiple of 3 *or* $x = 3k+1$ or $3k+2$ for $k \in \mathbb{Z}^+ \cup \{0\}$ |

## PROOF OF THE CONTRAPOSITIVE

To prove the statement "If $A$ then $B$", we can provide a direct proof, or we can prove the logically equivalent **contrapositive** statement "If not $B$, then not $A$" which we can also write as "If $\neg B$, then $\neg A$".

For example, the statement "If it is Jon's bicycle, then it is blue"
is logically equivalent to "If that bicycle is not blue, then it is not Jon's".

**Example 13:** Prove that for $a, b \in \mathbb{R}$, "$ab$ is irrational $\Rightarrow$ either $a$ or $b$ is irrational".

**Proof using contrapositive:**

$a$ and $b$ are both rational $\Rightarrow$ $a = \frac{p}{q}$ and $b = \frac{r}{s}$ where $p, q, r, s \in \mathbb{Z},\ q \neq 0,\ s \neq 0$

$\Rightarrow ab = \left(\frac{p}{q}\right)\left(\frac{r}{s}\right) = \frac{pr}{qs}$ {where $qs \neq 0$, since $q, s \neq 0$}

$\Rightarrow ab$ is rational {since $pr, qs \in \mathbb{Z}$}

Thus $ab$ is irrational $\Rightarrow$ either $a$ or $b$ is irrational.

**Example 14:** Prove that if $n$ is a positive integer of the form $3k+2$, $k \geqslant 0$, $k \in \mathbb{Z}$, then $n$ is not a square.

**Proof using contrapositive:**

If $n$ is a square then
$n$ has one of the forms $(3a)^2$, $(3a+1)^2$ or $(3a+2)^2$, where $a \in \mathbb{Z}^+ \cup \{0\}$.

$\Rightarrow n = 9a^2,\ 9a^2 + 6a + 1$ or $9a^2 + 12a + 4$

$\Rightarrow n = 3(3a^2),\ 3(3a^2 + 2a) + 1$ or $3(3a^2 + 4a + 1) + 1$

$\Rightarrow n$ has the form $3k$ or $3k+1$ only, where $k \in \mathbb{Z}^+ \cup \{0\}$

$\Rightarrow n$ does not have form $3k+2$.

Thus if $n$ is a positive integer of the form $3k+2$, $k \geqslant 0$, $k \in \mathbb{Z}$, then $n$ is not a square.



## USING PREVIOUS RESULTS

In mathematics we build up collections of important and useful results, each depending on previously proven statements.

**Example 15:** Prove the conjecture:

"The recurring decimal $0.\overline{9} = 0.999\,999\,99....$ is exactly equal to 1".

**Proof (by contradiction):**

Suppose $0.\overline{9} < 1$

$\Rightarrow 0.\overline{9} < \frac{0.\overline{9}+1}{2}$ {We proved earlier that $a < b \Rightarrow a < \frac{a+b}{2}$}

$\Rightarrow 0.\overline{9} < \frac{1.\overline{9}}{2}$ {Ordinary division: $2\,\big\lfloor\, 1.999\,999\,99....$ gives $0.999\,999\,99....$ }

$\Rightarrow 0.\overline{9} < 0.\overline{9}$ clearly a contradiction

Therefore the supposition is false, and so $0.\overline{9} \geqslant 1$ is true.

Since, $0.\overline{9} > 1$ is absurd, $0.\overline{9} = 1$.

**Proof (Direct Proof):**

$$\begin{aligned}0.\overline{9} &= 0.999\,999\,99....\\ &= 0.9 + 0.09 + 0.009 + 0.0009 + ....\\ &= 0.9\left(1 + \tfrac{1}{10} + \tfrac{1}{100} + \tfrac{1}{1000} + ....\right)\\ &= \tfrac{9}{10}\left(\sum_{i=0}^{\infty}\left(\tfrac{1}{10}\right)^i\right)\\ &= \tfrac{9}{10}\left(\frac{1}{1-\frac{1}{10}}\right) \quad \text{\{Using the previously proved Geometric Series with } r = \tfrac{1}{10} \text{ and } \left|\tfrac{1}{10}\right| < 1\text{\}}\\ &= \tfrac{9}{10} \times \tfrac{10}{9}\\ &= 1\end{aligned}$$

## THEORY OF KNOWLEDGE — AXIOMS AND OCCAM'S RAZOR

In order to understand complicated concepts, we often try to break them down into simpler components. But when mathematicians try to understand the foundations of a particular branch of the subject, they consider the question "What is the minimal set of assumptions from which all other results can be deduced or proved?" The assumptions they make are called **axioms.** Whether the axioms accurately reflect properties observed in the physical world is less important to pure mathematicians than the theory which can be developed and deduced from the axioms.

**Occam's razor** is a principle of economy that among competing hypotheses, the one that makes the fewest assumptions should be selected.

**1** What value does Occam's razor have in understanding the long-held belief that the world was flat?

**2** Is the simplest explanation to something always true?

**3** Is it reasonable to construct a set of mathematical axioms under Occam's razor?

One of the most famous examples of a set of axioms is given by Euclid in his set of 13 books called *Elements*. He gives five axioms, which he calls "postulates", as the basis for his study of Geometry:

1. Any two points can be joined by a straight line.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as centre.
4. All right angles are congruent.
5. **Parallel postulate:** If two lines intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

**4** Is the parallel postulate genuinely an axiom, or can it be proved from the others?

**5** What happens if you change the list of axioms or do not include the parallel postulate?

**6** What other areas of mathematics can we reduce to a concise list of axioms?