First lesson—Go play some games

Outline

Objectives:

1.Learn about basic concepts;
2.Learn the presentation and notation used in game theory;
3.Identify different combinatorial games in daily life.

Content:

1.1 a simple take-away game as introduction.
1.2 What is a combinatorial game?
    Definition of combinatorial games;
    Difference between impartial and partizan games;
    Difference between normal and misere play rule.
1.3 The game graph
    Figure of a game graph;
    Definition of terminal position and height.
1.4 N-position and P-position
    Definition of N, P-positions;
    Theorem: every position in impartial combinatorial game is either P or N;
    Exercise: identifying N,P positions in simple take-away games.
1.5 Application—solving the game
    Provide an exact example of how to use what we learn in this lesson to solve a take-away game.

Exercise:

Identifying combinational games.
Solve a take-away game, and an Empty-and-Divide game.

1.1 Introduction—a simple take-away game.

Let's start the lesson by jumping right into the water and play a game.
The game rule is:

(1) There are two players, Alice and Bob;
(2) There is a pile of 13 M&Ms;
(3) Alice and Bob can eat one, two, or three M&Ms from the pile every time;
(4) Alternate play;
(5) The first one who cannot move loses.

This is a game called **take-away game**.
How can we win this game? Or which player would you rather be, going first or second?
Think about it, play a few rounds with your friends, before you read on.

Now let's analyze the game together from the end back to the beginning. This method is known as **backward induction** (Ferguson 3).

We know that if you are left with 0 M&M, you lose because you cannot move anymore;
Also, if there are 1, 2, 3 M&Ms left for you, you can just eat all of them and win the game;
Then suppose there are 4 M&Ms left, but no matter how many you take, you can only move to 1, 2, 3 M&Ms and you lose;
Now with 5, 6, 7 M&Ms, you can always move to 4 M&Ms, so the next player will absolutely lose the game, while you can win the game;
With 8 M&Ms left, you can only move to 5, 6, 7 M&Ms, and the next player can win, while you lose;
If this induction continues, we may conjecture that the positions with 0, 4, 8, 12, … M&Ms are positions to be avoided for you, but you want to move to them in order to win the game.

We may now analyze the game with 13 M&Ms. Since 13 is not divisible by 4, you want to move first and take one M&M to make it 12, which is a multiple of 4.

1.2 What is a combinatorial game?

The take-away game we played in Section 1.1 is a typical example of an impartial combinatorial game, but we need a more precise definition for impartial combinatorial games.

Definition 1.1 (Ferguson 4)

A combinatorial game is a game that satisfies all the following conditions:

(1) **Two players**. There should only be two players. Solitaire is not a combinatorial game because there is only one player.

(2) **Alternate.** The two players alternate their turns as we did in the take-away game.

(3) **No ties**. Chess can actually have a draw, so according to this, chess is not a combinatorial game.

(4) **Finite**. The game ends in a finite number of moves. We may not be able to say how many time that is, but we know it doesn't go on forever. Under the **normal play rule**, the last player to move wins. Under the **misere play rule** the last player to move loses.

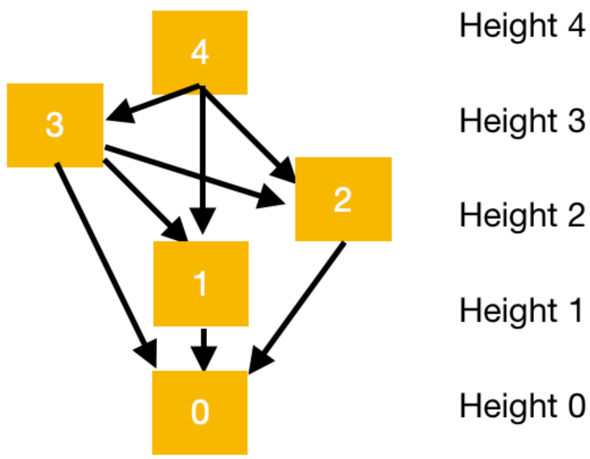
(5) **No distinction**. There should be no distinction between the players, that is if both players have the same options of moving from each position, the game is called **impartial**; otherwise, the game is called **partizan**.

A little summary: 2 players, alternate, no distinction, no ties, finite.

1.3 Games Played on Graphs

We can give an equivalent graph for every impartial combinatorial game that shows all the possibilities the game can proceed starting from that position. The **vertices** of the graph are positions of the game, and the **edges** are moves (Ferguson 14).

Here is an example showing another take-away game, but only with four M&Ms to begin with.

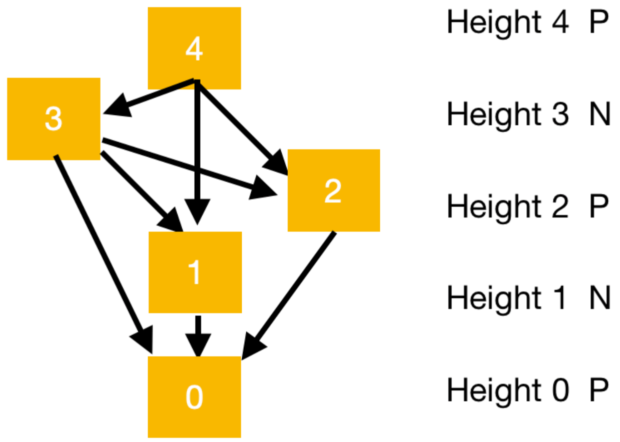


Since combinatorial games are finite, we are sure a game graph will always contain only a finite number of vertices and no loops. However, I do not draw the graph for 13 M&Ms because that will be too huge to draw, and we will have easier ways to deal with those with relatively large-number games in Section 1.5.

Here are some other definitions that may be useful later:
Definition 1.2 (Mathcamp 2)
Terminal position: A terminal position in a game is a position from which there are no valid moves.The terminal position in take-away games is 0
Height: The height of a position is the length of the longest path in its game graph. (Terminal positions have height 0.) 4, for example, 4 M&Ms have a height of 4, as shown in the graph.

1.4 N-position and P-position

Let's go back to the take-away game, we conjectured that 0,4,8,12,16,... are positions that are winning for your opponent, or what we call **P**revious player (the player who just moved) and that 1, 2, 3, 5, 6, 7, 9, 10, 11, . . . are winning for the **N**ext player to move, which is you. The former positions are called **P-positions**, and the latter ones are **called N-positions** (Ferguson 4-5).
Intuitively, N-positions are ones from which you want to go first; P-positions are ones from which you want to go second.
In impartial combinatorial games, one can find which positions are P- positions and which are N-positions by using the game graph as we introduced in Section 1.3.
We go back to the graph:



The terminal position must be P, so the former one, which goes to a P, should be N, and so on and so forth. So 4 M&Ms, where we start, is indeed a P-position.
Now you may wonder why this method works. Before proceeding to the proofs below, you can first think about it yourself.

In order to prove the method works, we need to prove the following statements (Mathcamp):

(1) A position from which you can move to a P-position is an N-position;
(2) A position from which the only moves are to N-positions is a P-position.
(3) Every position in an imperial combinatorial game is either N or P.

Proof for (1):
Think it backward: you don't know yet the position you are at right now, but the position you are moving to is a P-position, and you become the previous player. By definition of P, you have a winning strategy from that P-position.
Now we go back to your original position. Since you can move to a P-position, you have a winning strategy, so your current position is N.

Proof for (2):
Similarly, if you are now the next player and you can only move to N-positions, this means that whatever you do, the next player, you opponent, can win the game. This means your current position is a P-position.

Proof for (3);
<u>Proof by strong mathematical induction:</u>
First, consider Height 0. A position of height 0 is a terminal position, which means the Previous wins, so it's a P-position.

Now, we want to prove that if for all Height i, with 0<=i<=h, are all positions either N or P, we want to show that Height h+1 is also either N or P:
All moves from h+1 must be within the range of Height i, with 0<=i<=h, and since all Height i have positions either P or N, we can thus determine the position for h+1, either P or N according to the two statements we just proved (1) and (2).
Thus, because the truth of Height i, 0<=i<=h being either N or P can lead to the truth of Height h+1 being either N or P, by strong mathematical induction, we prove that every position in an imperial combinatorial game is either N or P. //

1.5 Application—solving a game

Let us now consider another take-away game.
This time, Alice and Bob are allowed to take 1, 2, or 5 M&Ms at a time.
So, let us analyze it.
This time, instead of drawing the full game graph for each position, which will be too big and too messy, let's just make a table for the different positions, which is easier to keep track of each position.
This is the table I constructed.

x	0	1	2	3	4	5	6	7	8	9	10	11	…
Position	P	N	N	P	N	N	P	N	N	P	N	N	…

Here is how it is constructed:
First, 0, the terminal position, must be P;
1, 2, 5 are N-positions, since they can be moved to 0, a P-position.
3 must be a P-position since it can only move to 1 or 2, which are all N-positions;
4 must then be N-positions as they can be moved to the P-position 3;
6 is a P-position because it can only move to 5, 4, or 1, which are all N-positions;
7, 8, and 11 must be N-positions because they can all move to the P-position 6;
9 is a P-position because it can only move to 8, 7, or 4, which are all N-positions;
10 is a N-position as it can move to the P-position 9;
…
We notice that the pattern PNN of length 3 repeats forever.

Now if there are 100 M&Ms in total, who will win the game, with Alice goes first and Bob second?

The P-positions are multiples of 3. Since 100 is not a multiple of 3, 100 is a N-position. Thus, Alice, if play optimally, can win the game by always moving to another N-position.
That's how you analyze a simple impartial combinatorial game.

1.6 Exercises.

1. Identify which of the following games are impartial combinatorial games and which ones are not, and explain your decision.
   (1) Aeroplane chess
   (2) Notakto
   (3) Dots-and-boxes

   Note: Notakto is played on a finite number of empty 3-by-3 boards. Then, each player takes turns placing an X on the board(s) in a vacant space (a space not occupied by a X already on the board). If a board has a three-in-a-row, the board is dead and it cannot be played on anymore. When one player creates a three-in-a-row and there are no more boards to play on, that player loses (Notakto).

   The game of dots-and-boxes starts with an empty grid of dots. Usually two players take turns adding a single horizontal or vertical line between two unjoined adjacent dots. A player who completes the fourth side of a 1×1 box earns one point and takes another turn. (A point is typically recorded by placing a mark that identifies the player in the box, such as an initial.) The game ends when no more lines can be placed. The winner is the player with the most points (Dots-and-Boxes).

2. Another Take-away game: analyze the take away game with 203 M&Ms in total and each player can take either 1, 2, 3, or 4 M&Ms; construct a position table, and tell whether you should go first or second if you want to win the game.

3. We have a new game called Empty and Divide: There are two piles of M&Ms. Initially, one pile has m M&Ms and the other has n M&Ms. Such a position is denoted by (m,n), where $m > 0$ and $n > 0$. The two players alternate moving. A move consists of eating all the M&Ms in one pile, and dividing the other into two new piles with at least one M&M in each pile. Whoever cannot move loses, i.e. the terminal position is (1,1).

For example, if m=3 and n=4, you can either eat up all the 3 M&Ms in the first pile and divide the 4 M&Ms into two piles of either (1, 3) or (2, 2); or you can eat all the 4 M&Ms in the second pile and divide the 3 M&Ms in the first pile into (1, 2).

(1) Show that this is indeed an impartial combinatorial game;
(2) If m=3 and n=7, is it better to go first or second?
(3) What about m=2019 and n=9021, should you go first or second?

Answer key:

1. (1) Aeroplane chess is not an impartial game because it involves chances. According to definition, an impartial game should have no distinction between two players, and they should have the same options. Thun, aeroplane chess is not an impartial combinatorial game.

   Note: any game involves dies is NOT a combinatorial game.

   (2) Notakto is an impartial combinatorial game because it satisfies all the requirements to be an impartial game, with two players alternate playing, no ties, with finite game. Lastly, the two players have no distinction in the moving, as all moving all available for each player.

   (3) Dots-and-boxes is an impartial combinatorial game because it satisfies all the requirements to be an impartial game, with two players alternate playing, no ties, with finite game. Lastly, the two players have no distinction in the moving, as all moving all available for each player.

   However, dots-and-boxes is a little bit different from other impartial games given its different way of determining who wins the game. (counting the boxes instead of just who moves last)

2. The table:

x	0	1	2	3	4	5	6	7	8	9	10	11	…
Position	P	N	N	N	N	P	N	N	N	N	P	N	…

   the pattern of PNNNN repeats forever.

   203 has a remainder of 3 when divided by 5, which is N, so 203 is a N-position, and we wish to go first.

3. (1) Empty-and-Devide is indeed an impartial combinatorial game because it satisfies all the requirements to be an impartial game, with two players alternate playing, no ties, no distinction, finite game.

(2) In order to solve the problem, I constructed the game table. However, I only show half of it because (m, n) and (n, m) do not make a difference:

(1,1) P					
(1,2) N	(2,2) N				
(1,3) P	(2,3) N	(3,3) P			
(1,4) N	(2,4) N	(3,4) N	(4,4) N		
(1,5) P	(2,5) N	(3,5) P	(4,5) N	(5,5) P	
(1,6) N	(2,6) N	(3,6) N	(4,6) N	(5,6) N	(6,6)N
(1,7) P	(2,7) N	(3,7) P	(4,7) N	(5,7) P	…
…					

Thus, (3,7) will be a P-position, and it will be better to go second.

Keep constructing the table. You may find the pattern that whenever there is an even number present, it will be a N-position.
Here is why:
If you are at a position of (n, m):
Case one: at least one of n, m is an even number.
An even number can always be divided into a sum of two odd numbers. Your opponent will then have two odd numbers in front. The only choice for him/her is to split one of the two odd numbers into two piles with one even and one odd. Now it comes to your turn again, and you can keep dividing the even number into a sum of two odd numbers. If this continues, you will eventually get a pile with 2 M&Ms, and you can eat the other pile, and split the 2 M&Ms into (1,1), and your opponent has no moves available and loses.

Case two; n, m are both odd numbers.
n, m can only be divided into a pile with a sum of one odd and one even number. Then your opponent, as what you do in Case one, will keep splitting her even-number pile into two odd-number piles for you until you get (1,1) and loses.

(3) this question is relatively easy now as we have figure out the rules behind the game. Since both 2019 and 9021 are odd number, it is a P-position, and thus it will be better to go second.

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