SUMaC Admission Exam: Solution

Problem 1

Before answering the questions, We consider the general case of a n-pyramid.

Let B_j^i be the ith block at the jth level, and C_i be the number of ways to color all the blocks at the ith level.

Rule (1): Each block can be red, gold, or black.

Rule (2): All 3-block units composed of one block on top of two others must either all be the same color or all be different colors.

Lemma 1. *In a 3-block unit, if the colorings of 2 blocks are known, the color of the third block can be uniquely determined.*

Proof. In a 3-block unit with the colors of 2 blocks known, by rule (2),

i. If the two known blocks are of the same color, the third block is also that color.

ii. If the two know blocks are of distinct colors, then the third block is of the third color, different from the two known.

In either case, the color of the third block can be uniquely determined. □

Lemma 2. *Given a n-pyramid, with the coloring of all the blocks in the kth level known, if the coloring of B_{k+1}^1 is known, the coloring of all blocks at level $(k+1)$ can be uniquely determined.*

Proof. We prove by induction.

Base case: Consider the 3-unit block $B_{k+1}^1, B_k^1, B_{k+1}^1$, Since the colorings of B_{k+1}^1 and B_k^1 are known, by Lemma 1, the color of B_{k+1}^1 can also be uniquely determined.

Induction case: Assume that the coloring of B_{k+1}^i is known, we want to show that the coloring of B_{k+1}^{i+1} is also known.

Consider the 3-unit block $B_{k+1}^i, B_k^i, B_{k+1}^{i+1}$, Since the colorings of B_{k+1}^i and B_k^i are known, by Lemma 1, the color of B_{k+1}^{i+1} can also be uniquely determined.

Therefore, since the coloring of B_{k+1}^{i+1} can be uniquely deduced by the coloring of B_{k+1}^{i}, by mathematical induction, the colors of all blocks at the $(k+1)$th level are known given the coloring of the the kth level. $\square$

Proposition 1. *There are 3^n ways to color a n-pyramid.*

Proof. We proceed with induction.

Base case: there is 1 way to color a 1-pyramid.

Induction case: assume that the statement holds true when $n = k$. We want to show that it is also true for $n = k + 1$. i.e. there are 3^{k+1} ways to color a $(k+1)$-pyramid.

Consider B_{k+1}^1, the first block at the $(k+1)$th level.

There are 3 colors that B_{k+1}^1 can choose, and by Lemma 2, once the coloring for B_{k+1}^1 is determined, the coloring of all blocks can be uniquely determined. This means that based on each coloring of the kth level, there are 3 ways of coloring for the $(k+1)$th level, i.e.

$$C_{k+1} = 3C_k = 3 \times 3^k = 3^{k+1}.$$

Therefore, since the truth of the statement for k leads to the truth for $k+1$, by mathematical induction, There are 3^n ways to color a n-pyramid. $\square$

Proposition 2. *In a n-pyramid, there are maximum $\frac{n(n-1)}{2}$ block of the same color, given that at least one each of the other two colors are present.*

Proof. WLOG, let black be the color of blocks we want to maximize. In the n-pyramid, there are at least one red and at least one gold.

We claim that the maximum number is achieved when the blocks from Level 1 to Level $n-1$ are all black, and the blocks at the nth level are alternating colors of red and gold.

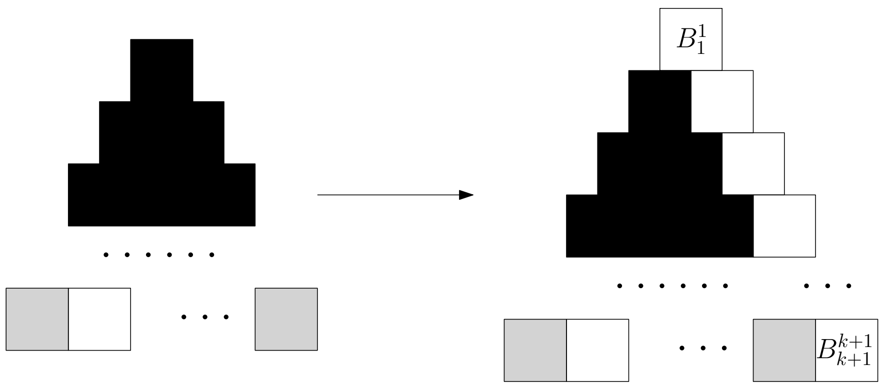
We use induction to show that the claim is true.

Base case: When $n = 2$, the maximum number of black blocks is 1, at the first level.

When $n = 3$, the maximum number is 3, the first and second level are all black, and the third level has alternating red and gold.

Induction case: Assume that the statement holds for $n = k$, we want to show that it is true for $n = k+1$ as well. i.e., when all blocks from Level 1 to $k+1$ are black, and the blocks at the $(k+1)$th level are alternating colors of red and gold, the number of black blocks is maximized to be $\frac{k(k+1)}{2}$.

We construct the $(k+1)$-pyramid based on the k-pyramid by adding one block at the end of each level, and one extra block at the top.



k-pyramid $(k+1)$-pyramid

Consider the color of B_1^1.

If is not black, then by Lemma 1, all the newly added blocks are alternating colors of red and gold;

If it is black, then all the newly added blocks have to be black, except for B_{k+1}^{k+1}. In this case, maximum black blocks is added; which equals to k.

Hence, the maximum number of black blocks for a $(k+1)$-pyramid is

$$\frac{k(k-1)}{2} + k = \frac{k(k+1)}{2}$$

Therefore, since the truth of the statement for $n = k$ leads to its truth for $n = k+1$, by mathematical induction, the statement holds for all $n \in \mathbb{Z}$. $\square$

Therefore,

a) By Proposition 1, when $n = 7$, there are 3^7 ways to color a 7-pyramid.

b) By Proposition 2, when $n = 25$, there are maximum $\frac{25(24)}{2} = 300$ black blocks.

Problem 2

a) Expand the equation $x^2 - y^2 = 2020$, we get

$$(x+y)(x-y) = 2020 = 2 \times 2 \times 5 \times 101$$

Thus, as $x, y \in \mathbb{Z}^+$,

$$\begin{cases} x+y = 2020 \\ x-y = 1 \end{cases} or \quad \begin{cases} 1010 \\ 2 \end{cases} or \quad \begin{cases} 505 \\ 4 \end{cases} or \quad \begin{cases} 202 \\ 10 \end{cases} or \quad \begin{cases} 101 \\ 20 \end{cases}$$

Solve for x and y, we get two possibilities:

$$\begin{cases} x = 506 \\ y = 504 \end{cases} or \quad \begin{cases} x = 106 \\ y = 102 \end{cases}$$

b) Such x, y do not exist.

Lemma 1. *$\forall x \in \mathbb{Z}$, one of the following holds:*

i. $x^3 \equiv 0 \mod 9$,

ii. $x^3 \equiv 1 \mod 9$,

iii. $x^3 \equiv -1 \mod 9$.

Proof. Let x be an integer. Consider $x \mod 3$, there are 3 cases:

i. $x \equiv 0 \mod 3$, $x = 3k$.

ii. $x \equiv 1 \mod 3$, $x = 3k+1$.

iii. $x \equiv 2 \mod 3$, $x = 3k+2$.

for some $k \in \mathbb{Z}$. Then,

i. $x^3 = (3k)^3 = 9(3k^3) \Rightarrow x^3 \equiv 0 \mod 9$.

ii. $x^3 = (3k+1)^3 = 9(3k^3 + 3k^2 + k) + 1 \Rightarrow x^3 \equiv 1 \mod 9$.

iii. $x^3 = (3k+2)^3 = 9(3k^3 + 6k^2 + 4k) + 8 \Rightarrow x^3 \equiv 8 \equiv -1 \mod 9$.

$\square$

Proposition 1. *There do not exist* $x, y, \in \mathbb{Z}$ *such that* $x^3 - y^3 = 2020$.

Proof. Since $\forall x \in \mathbb{Z}$, $x^3 \equiv 0, 1$, or $-1 \mod 9$.

By Lemma 1,

$$x^3 - y^3 \equiv 0, \pm 1, \text{ or} \pm 2 \mod 9$$

However,

$$2020 \equiv 4 \mod 9$$

Therefore, by lemma 1, such x, y do not exist. $\square$

Problem 3

a) If $a + b\sqrt{2}$ is a root of $p(x)$, $p(a + b\sqrt{2}) = 0$.

Then by the binomial theorem,

$$
\begin{aligned}
&p(a + b\sqrt{2}) + p(a - b\sqrt{2}) \\
&= \sum_{i=0}^{n} a_i (a + b\sqrt{2})^i + \sum_{i=0}^{n} a_i (a - b\sqrt{2})^i \\
&= \sum_{i=0}^{n} a_i [(a + b\sqrt{2})^i + (a - b\sqrt{2})^i] \\
&= \sum_{i=0}^{n} a_i \left[\sum_{j=0}^{i} \binom{i}{j} a^{i-j} (\sqrt{2}b)^j + \sum_{k=0}^{i} \binom{i}{k} a^{i-k} (-\sqrt{2}b)^k \right]
\end{aligned}
$$

The result is an integer, because when k is odd, the term $(-\sqrt{2}b)^k$ in the expansion of $p(a - b\sqrt{2})$ cancels out with $(\sqrt{2}b)^k$ in the expansion of $p(a + b\sqrt{2})$; when k is even, $\sqrt{2}$ is rationalized to 2.

Similarly,

$$
\begin{aligned}
&p(a + b\sqrt{2}) - p(a - b\sqrt{2}) \\
&= \sum_{i=0}^{n} a_i [(a + b\sqrt{2})^i - (a - b\sqrt{2})^i] \\
&= \sum_{i=0}^{n} a_i \left[\sum_{j=0}^{i} \binom{i}{j} a^{i-j} (\sqrt{2}b)^j - \sum_{k=0}^{i} \binom{i}{k} a^{i-k} (-\sqrt{2}b)^k \right]
\end{aligned}
$$

Since this time, the terms with $\sqrt{2}$ cannot cancel out, the result is some integer multiple of $\sqrt{2}$, which is an irrational number.

Now, since $p(a + b\sqrt{2}) = 0$, $p(a - b\sqrt{2})$ must be both an integer and some integer multiple of $\sqrt{2}$ at the same time. Hence, $p(a - b\sqrt{2}) = 0$. In other words, $a - b\sqrt{2}$ is also a root of $p(x)$.

b) We take a similar approach as part (a).

If $a+b\sqrt[3]{2}$ is a root of $q(x)$, $q(a+b\sqrt[3]{2})=0$. By the binomial theorem,

$$\begin{aligned}
&q(a+b\sqrt[3]{2})+q(a-b\sqrt[3]{2})\\
=&\sum_{i=0}^{n}a_i(a+b\sqrt[3]{2})^i+\sum_{i=0}^{n}a_i(a-b\sqrt[3]{2})^i\\
=&\sum_{i=0}^{n}a_i\left[\sum_{j=0}^{i}\binom{i}{j}a^{i-j}(\sqrt[3]{2}b)^j+\sum_{k=0}^{i}\binom{i}{k}a^{i-k}(-\sqrt[3]{2}b)^k\right]
\end{aligned}$$

However, even though in the expansion of $q(a-b\sqrt[3]{2})$, the terms with $-2^{1/3}$ are canceled out with terms with $2^{1/3}$ in the expansion of $q(a+b\sqrt[3]{2})$, and $(-\sqrt[3]{2})^3=-2$ is rationalized, the irrational terms $2^{2/3}$ cannot cancel out.

Hence, the sum is some integer multiple of $2^{2/3}$.

Similarly,

$$\begin{aligned}
&q(a+b\sqrt[3]{2})-q(a-b\sqrt[3]{2})\\
=&\sum_{i=0}^{n}a_i\left[\sum_{j=0}^{i}\binom{i}{j}a^{i-j}(\sqrt[3]{2}b)^j-\sum_{k=0}^{i}\binom{i}{k}a^{i-k}(-\sqrt[3]{2}b)^k\right]
\end{aligned}$$

This time, even though the terms with $2^{2/3}$ are canceled out and $(\sqrt[3]{2})^3=2$ is rationalized, the terms with $2^{1/3}$ cannot be canceled. The result is thus some integer multiple of $2^{1/3}$.

Since $q(a+b\sqrt[3]{2})=0$, $q(a-b\sqrt[3]{2})$ must be some integer multiple of both $2^{2/3}$ and $2^{1/3}$.

The only case when this could happen is for the coefficient of both $2^{2/3}$ and $2^{1/3}$ to be 0. This makes $b=0$.

Therefore, $a-b\sqrt[3]{2}$ is not a root of $q(x)$, unless $b=0$.

Problem 4

Proof by contradiction. Suppose there are four distinct integer roots for $p(a)$, say b, c, d, e. Then

$$p(x) = (x - b)(x - c)(x - d)(x - e)q(x),$$

where $q(x)$ is another polynomial without integer roots. Since $p(a) = 17$,

$$|p(a)| = |a - b||a - c||a - d||a - e||q(a)| = 17$$

since b, c, d, e are distinct integers, so will $a - b, a - c, a - d, a - e$ be distinct integers.

However, since $17 = 1 \times 17$, at least 3 absolute values will be 1, say, WLOG,

$$|a - b| = |a - c| = |a - d| = 1.$$

Then, $(a - b), (a - c), (a - d)$ takes the values of 1 or -1.

By Pigeonhole Principle, two of them must take the same value, which is a contradiction.

Therefore, there can be at most 3 distinct integer roots b, c, d for $p(x)$.

For example, $a - b = -1, a - c = 1$, and $a - d = 17$ will do. $\square$

Problem 5

a) *Proof by contradiction.* Suppose that it is possible to get $N = 10$ with $k = 4$.

As $n_r = n_s + n_t$, $s \leq t < r$, $n_1 = 1$, we have $n_r \geq r$.

Hence, this is a monotonic increasing sequence, and it increases fastest when $s = t = r - 1$, and $n_r = 2n_{r-1}$ for $r = 2, 3, 4$.

This gives that

$$a_2 \leq 2, \ a_3 \leq 4, \ a_4 \leq 8 < 10.$$

Therefore, when $k = 4$, the maximum value of $a_4 = 8$, and it is impossible to get $N = 10$, which is a contradiction.

So we cannot get $N = 10$ with a sequence where $k < 5$. □

b) An upper bound of k is $2\lfloor log_2 N \rfloor + 1$.

Proof. Double first method gets to a number N by expressing N as a sum of powers of 2.

We thus write N in binary,

$$N = k_0 1 + k_1 2 + k_2 2^2 + k_3 2^3 + \cdots + k_n 2^n$$

Where k_i is either 1 or 0, $i = 0, 1, 2, \ldots, n$.

First, in order to get to 2^n,

$$n = \lfloor log_2 N \rfloor + 1.$$

However, in order to get N, we still need to add all powers of 2 that sums to N, and the values of k_i change as N changes.

The upper bound is reached when $k_i = 1, \ i = 0, 1, 2, \ldots, n$, in which case every power of 2 is added up to n, and that give a total number of $\lfloor log_2 N \rfloor$.

Therefore, the upper bound for k is

$$k \leq \lfloor log_2 N \rfloor + 1 + \lfloor log_2 N \rfloor = 2\lfloor log_2 N \rfloor + 1$$

□

c) The double-first method is not optimal. An example is when $N = 15$.

When using double-first method, we get the sequence

$$1, 2, 4, 8, 12, 14, 15, \text{ with } k = 7.$$

However, there exists a 6-sum-sequence:

$$1, 2, 3, 5, 10, 15, \text{ with } l = 6 < 7.$$

d) A sum sequence for 100 with $k < 9$ does not exist.

Proof by contradiction. Suppose such sequence exists, $k = 8$.

We use backward induction, and $n_8 = 100$.

Hence,

$$n_7 \geq 50 \Rightarrow n_6 \geq 25 \Rightarrow n_5 \geq 13 \Rightarrow n_4 \geq 7 \Rightarrow n_3 \geq 4.$$

On the other hand, by double-first method, we also know that

$$n_2 \leq 2,\ n_3 \leq 4,\ n_4 \leq 8,\ n_5 \leq 16,\ n_6 \leq 32,\ n_7 \leq 64.$$

Using these inequalities, we first get $n_2 = 2,\ n_3 = 4$.

Then we notice $n_4 = 7$ or 8. However, it is impossible to get to 7 by adding two numbers among 1, 2, 4. Hence $n_4 = 8$.

Now $n_5 = 13, 14, 15$ or 16. However, as $4 + 8 = 12 < 13$, $8 + 8 = 16$, it is impossible to get to 13, 14, or 15 as a sum of two numbers among 1, 2, 4, 8. Hence $n_5 = 16$.

Similarly, as $8 + 16 = 24 < 25$, yet $16 + 16 = 32$, $n_6 = 32$.

As $16 + 32 = 48 < 50$, $32 + 32 = 64$, $n_7 = 64$.

However, n_8 =100 can not be expressed as a sum of two powers of 2, which leads to a contradiction.

Since 100 can not be expressed with $k = 8$.

When $k < 8$, the maximum number can be reached by n_{k-1} is $32 < 50$, and thus not possible to get to 100 at n_k. Hence, $k \not< 8$.

Therefore, a sum-sequence for 100 with $k < 9$ is impossible. $\square$

Problem 6

Let A_{ij} be the value of entry at the ith row, jth column in the 3×3 grid, $A_{ij} \in \mathbb{N}$, and let $S = \sum_{i,j=1}^{3} A_{i,j}$.

Lemma 1. *For $n, k \in \mathbb{N}$, and $x_1 + x_2 + \cdots + x_k = n$, where x_i are non-negative integers, there are $\binom{n+k-1}{k-1}$ possibilities for the values of $x_1, x_, \ldots, x_k$.*

Proof. Since the sum of integers $x_1, x_2, \ldots x_k$ is n, we want to distribute n to these integers.

We imagine that process as dividing n objects in a line using $k-1$ bars.

Hence, there are a total of $n+k-1$ spots to place either an object or a bar. In order to place the bars, there are

$$\binom{n+k-1}{k-1}.$$

ways to do so, which is also the number of possible values of $x_1, x_2, \ldots x_k$. □

Proposition 1. *Every configuration will reach a stable configuration or a repetition of a k-cycle in B or fewer steps, where*

$$B = \binom{S+8}{8}$$

Proof. Given a fixed sum S of 9 integers, by Lemma 1, the number of values the entries can take are

$$\binom{n+k-1}{k-1} = \binom{S+8}{8}$$

By Pigeonhole Principle, any transformation beyond this value is a repetition of some former configuration, and will continues on from that configuration. This will result in a repeated cycle through, say, k configurations. Specifically, when $k = 1$, it doesn't change and is a stable configuration.

Now we consider the specific conditions for those to happen:

This is the least number required for each cell to discharge:

2	3	2
3	4	3
2	3	2

The sum of numbers in all cells is 24.

This is the most number that each cell could have so that none discharges:

1	2	1
2	3	2
1	2	1

The sum of numbers in all cells is 15.

By this two thresholds, we consider 3 cases:

1. $S \geq 24$:

   In this case, the initial configuration transforms until there are enough coins in each cell to be sent. When every cell is sending adjacent cells coins, it also receives an equal number of coins from its neighbors, so this will result in stable configurations.

2. $15 < S < 24$:

   In this case, there is always some cell(s) that can not discharge. However, the rest of the cells keep discharging, resulting in a cyclic pattern.

3. $S \leq 15$:

   In this case, the initial configuration transforms until none of the cells can discharge anymore, resulting in a stable configuration.

Therefore, considering these 3 cases, we conclude that every starting configuration either leads to a stable configuration or a repetition, within $\binom{S+8}{8}$ steps.

$\square$

Conjecture 1. *An initial configuration can only cycle through 2 or 4 configurations. In other words, the possible k values are 2, and 4.*

An example of a 2-cycle:

2	0	2
0	4	0
2	0	2

$\Rightarrow$

0	3	0
3	0	3
0	3	0

$\Rightarrow$

2	0	2
0	4	0
2	0	2

An example of a 4-cycle:

0	1	1
3	3	3
1	2	0

$\Rightarrow$

1	1	2
0	5	0
2	2	1

$\Rightarrow$

1	3	0
2	1	2
0	4	1

$\Rightarrow$

2	0	1
2	3	2
1	1	2

$\Rightarrow$

0	1	1
3	3	3
1	2	0

Problem 7

Let S be the set of all sequences of "R"s and "B"s. Let $\sim$ be a relation on S. For $X, Y \in S$, $X \sim Y$ iff X can be transformed to Y using the rules of the game.

Proposition 1. *The relation $\sim$ on S is an equivalence relation.*

Proof. We want to show that $\sim$ is reflexive, symmetric, and transitive:

Reflexive: Since all sequences can be transformed into itself, $X \sim X \quad \forall X \in S$.

Symmetric: If $X \sim Y$, then X can be transformed into Y. Since the rules can be added as well as subtracted, the transformation is reversible by exchanging all the addition and subtraction. So $Y \sim X$.

Transitive: If $X \sim Y$, $Y \sim Z$, $X, Y, Z \in S$, then it is possible for X to transform to Z by first transforming to Y and then from Y to Z. i.e. $X \sim Z$.

Therefore, $\sim$ is an equivalence relation on S. $\square$

Proposition 2. *If the rules are (1) BB, (2) RRR, and (3) RRBRB, RB $\sim$ BR, but R$\nsim$ B.*

Proof. By the rules, $\forall$ R, B $\in S$,

$$\text{RB} \sim \text{BBRB} \sim \text{BRRRBRB} \sim \text{BR}.$$

Which also means that all the elements of the sequences in S are commutative.

However, R $\nsim$ B. Because the elements are commutative, the rules are equivalent to B^2, R^3, and $\text{B}^2{\cdot}\text{R}^3$, respectively. Therefore, there must be some multiple of 2 of "B"s, and some multiple of 3 of "R"s.

It is impossible to have $\text{R}{\cdot}\text{B}^{2s}{\cdot}\,\text{R}^{3t} = \text{B}$, for some $s, t \in \mathbb{Z}$. Therefore, it is impossible to go from R to B, i.e. R$\nsim$B. $\square$

Proposition 3. *If the rules are (1) BB, (2) RRR, and (3) RRBRB, there are six equivalence classes: [$\langle\rangle$], [B], [R], [RR], [BR], [BRR].*

Proof. Since by Proposition 2, the rules imply all elements are commutative, we can move all "B"s to the left and all "R"s to the right. Therefore, any sequence in S can be transformed into the form B^sR^t, $s, t \in \mathbb{Z}$.

Since for any $s \geq 2$, we can eliminate B^2 , and for any $t \geq 3$, we can eliminate R^3, any sequence can thus be transformed into $\mathrm{B}^w\mathrm{R}^z$, where $0 \leq w \leq 1,\ 0 \leq z \leq 2$:

When $w = z = 0$, $\mathrm{B}^w\mathrm{R}^z \sim \langle\rangle$;
when $w = 0, z = 1$, $\mathrm{B}^w\mathrm{R}^z \sim$ R;
when $w = 0, z = 2$, $\mathrm{B}^w\mathrm{R}^z \sim$ RR;
when $w = 1, z = 0$, $\mathrm{B}^w\mathrm{R}^z \sim$ B;
when $w = 1, z = 1$, $\mathrm{B}^w\mathrm{R}^z \sim$ BR;
when $w = 1, z = 2$, $\mathrm{B}^w\mathrm{R}^z \sim$ BRR; □

Proposition 4. *Let $G = \{\langle\rangle$, B, R, RR, BR, BRR$\}$, under the rules (1) BB, (2) RRR, and (3) RRBRB. Then $G \in S$ forms an abelian group.*

Proof. We construct the Cayley table for G:

	$\langle\rangle$	B	R	RR	BR	BRR
$\langle\rangle$	$\langle\rangle$	B	R	RR	BR	BRR
B	B	$\langle\rangle$	BR	BRR	R	RR
R	R	BR	RR	$\langle\rangle$	BRR	B
RR	RR	BRR	$\langle\rangle$	R	B	BR
BR	BR	R	BRR	B	RR	$\langle\rangle$
BRR	BRR	RR	B	BR	$\langle\rangle$	R

Closure: from the Cayley table above, we can see G is closed.

Identity: for any $X \in G$, $X \cdot \langle\rangle = X$, so $\langle\rangle$ is the identity element.

Inverse: from the Cayley table, for each element $X \in G$, exists one other element $Y \in G$ such that $XY = \langle\rangle$.

Associativity: this is assumed in this sequence.

Commutativity: this is proved in Proposition 2 that RB $\sim$ BR.

Therefore, $G \in S$ is an abelian group. □

Proposition 5. *If the rules are (1) BB, (2) RRR, (3) RBRB, RB $\not\sim$ BR.*

Proof by contradiction. Assume that RB $\sim$ BR.

Since

$$\text{RB} \sim \text{RBRRR} \sim \text{RBRBBRR} \sim \text{BRR},$$

$$\text{BR} \sim \text{RRRBR} \sim \text{RRRBRBB} \sim \text{RRB},$$

we know that RB $\sim$ BRR, and BR $\sim$ RRB. And because RB $\sim$ BR,

$$\begin{aligned} & \text{RB} \sim \text{RRB} \\ \Rightarrow\quad & \text{RBBRR} \sim \text{RRBBRR} \\ \Rightarrow\quad & \text{RRR} \sim \text{RRRR} \\ \Rightarrow\quad & \langle\rangle \sim \text{R}. \end{aligned}$$

Which is absurd.

Therefore, it is impossible for BR to be transformed to RB. $\square$

Proposition 6. *If the rules are (1) BB, (2) RRR, (3) RBRB, there are six equivalence classes: [$\langle\rangle$], [B], [R], [RR], [BR], [BRR].*

Proof. By Proposition 5, since RB $\sim$ BRR, and BR $\sim$ RRB, every time B and R exchanges their places, an extra R is added.

However, this does not affect us when we move all the ”B”s to the left and all ”R”s to the right.

Therefore, the reasoning in the proof for Proposition 3 still holds, and all sequences can be transformed into the form B^wR^z, where $0 \leq w \leq 1,\ 0 \leq z \leq 2$.

Following the same construction as in Proposition 3, [$\langle\rangle$], [B], [R], [RR], [BR], and [BRR] are the six equivalence classes in S. $\square$

Proposition 7. *Let $H = \{\langle\rangle$, B, R, RR, BR, BRR$\}$, under the rules (1) BB, (2) RRR, and (3) RBRB. Then $H \in S$ forms a group.*

Proof. We construct the Cayley table for H:

	$\langle\rangle$	B	R	RR	BR	BRR
$\langle\rangle$	$\langle\rangle$	B	R	RR	BR	BRR
B	B	$\langle\rangle$	BR	BRR	R	RR
R	R	BRR	RR	$\langle\rangle$	B	BR
RR	RR	BR	$\langle\rangle$	R	BRR	B
BR	BR	RR	BRR	B	$\langle\rangle$	R
BRR	BRR	R	B	BR	RR	$\langle\rangle$

Closure: from the Cayley table above, we can see H is closed.

Identity: for any $X \in H$, $X \cdot \langle\rangle = X$, so $\langle\rangle$ is the identity element for H.

Inverse: from the Cayley table, for each element $X \in H$, exists one other element $Y \in H$ such that $XY = \langle\rangle$.

Associativity: this is assumed in this sequence.

Therefore, $H \in S$ is a group. $\square$

Now we proceed to answer the questions:

a) See Proposition 2.

b) See Proposition 5.

c) Under the rules in part a), by Proposition 3, every sequence is in one of the six equivalence classes, and thus every sequence can be transformed into one of these six sequences;

   Under the rules in part b), by Proposition 6, similarly, every sequence can be transformed into one of these six sequences.

d) Under the rules in part a), by Proposition 3, $x_1 x_2 x_3 \ldots x_m$ must belong to one to the six equivalence classes in G. Therefore, by Proposition 4, G is an abelian group and inverse always exists, so $y_1 y_2 \ldots y_n$ is that inverse.

   Under the rules in part b), by Proposition 6, $x_1 x_2 x_3 \ldots x_m$ must belong to one to the six equivalence classes in H. Therefore, by Proposition 7, H is a group and inverse always exists, so $y_1 y_2 \ldots y_n$ is that inverse.