FM2 Test #4 Name: Maggie.

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1. Use the Maclaurin series for $\cos x$ to evaluate the limit $\lim_{x\to 0} \frac{1-\cos x}{x^2}$.

$\cos x = 1 - \frac{x^2}{2!} + o(x^4)$

$\lim_{x\to 0} \frac{1-\cos x}{x^2}$

$= \lim_{x\to 0} \frac{1-(1-\frac{x^2}{2!}+o(x^4))}{x^2}$

$= \lim_{x\to 0} \left(\frac{1}{2!} + o(x^2)\right)$

$= \frac{1}{2!} + 0$

$= \frac{1}{2}$.

Very good

2. Find the images of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ under reflection in the line $y=-x$. Hence write down the matrix for the reflection.

$\begin{pmatrix}1\\0\end{pmatrix}$ after reflection is $\begin{pmatrix}0\\-1\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix}$ after reflection is $\begin{pmatrix}-1\\0\end{pmatrix}$.

The matrix for reflection is

$\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix} = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$

3. Find the values of a and b that make the given function f continuous everywhere.

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & x<2 \\ ax^2-bx+3 & 2\le x<3 \\ 2x-a+b & x\ge 3 \end{cases}$$

① $\lim_{x\to 2} f(x) = f(2)$

$\Rightarrow \frac{x^2-4}{x-2} = x+2 = 4$

so $4a-2b+3=4$

$4a-2b=1$!

② $\lim_{x\to 3} f(x) = f(3)$

$\Rightarrow 9a-3b+3 = 6-a+b$

$\Rightarrow 10a-4b=3$

so $\begin{cases}4a-2b=1\\10a-4b=3.\end{cases}$

$\Rightarrow \begin{cases}a=\frac{1}{2}\\ b=\frac{1}{2}.\end{cases}$

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4. Let G be a group containing the element a. We say a has a cube root in G if there is an $x \in G$ such that $x^3 = a$. Prove that if $a^2 = e$ then a has a cube root in G.

proof. if $a^2 = e$. $a \in G$.

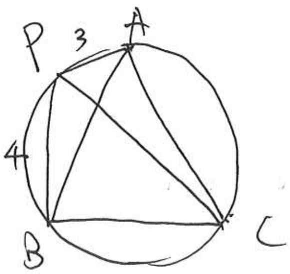
$a^2 \cdot a = e \cdot a = a$.

So $a^3 = a$.

~~exist~~.

so a has a cube root in G, which is a. $\square$.

5. Point P is on arc AB of the circumcircle of equilateral triangle ABC, $AP = 3$, and $BP = 4$. Find CP.



let ~~the~~ $AB = BC = CA = x$.

~~Then~~

Because $AB \cdot PC = AP \cdot BC + BP \cdot AC$

$x \cdot PC = 3x + 4x = 7x$.

~~So~~

So $CP = 7$.

6. Let $T\colon R^n \to \mathbb{R}^m$ be a linear transformation. Prove that $\operatorname{ran} T$ is a subspace of $\mathbb{R}^m$.

proof. a) as $T(\vec{0}) = \vec{0}$,

$\vec{0} \in \operatorname{ran} T$.

b). let $\vec{a}, \vec{b} \in \operatorname{ran} T$.

then exist $\vec{u}, \vec{v} \in \mathbb{R}^n$,

such that

$T(\vec{u}) = \vec{a}$

$T(\vec{v}) = \vec{b}$.

so $\vec{a} + \vec{b} = T(\vec{u}) + T(\vec{v})$

$= T(\vec{u} + \vec{v})$.

since $\vec{u} + \vec{v} \in \mathbb{R}^n$,

$T(\vec{u} + \vec{v}) \in \operatorname{ran} T$,

so $\vec{a} + \vec{b} \in \operatorname{ran} T$.

c) let $\vec{a} \in \operatorname{ran} T$.

$k \in \mathbb{R}$, $k \in \mathbb{R}$

exist $\vec{u} \in \mathbb{R}^n$, such that $T(\vec{u}) = \vec{a}$.

$k\vec{a} = kT(\vec{u})$

$= T(k\vec{u})$.

As $k\vec{u} \in \mathbb{R}^n$ also,

$T(k\vec{u}) \in \operatorname{ran} T$,

So $k\vec{a} \in \operatorname{ran} T$.

Therefore, as $\vec{0} \in \operatorname{ran} T$ and it is closed under addition and scalar multiplication $\operatorname{ran} T$ is a ~~subspa~~ subspace of $\mathbb{R}^m$. $\square$.

7. Find the radius of convergence and interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{3x^n}{2n}$.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{3x^{n+1}}{2(n+1)}}{\frac{3x^n}{2n}}\right| = \lim_{n\to\infty}\left|\frac{nx}{n+1}\right| = \lim_{n\to\infty}\left|\frac{x}{1+\frac{1}{n}}\right|$$

$$= |x| \le 1.$$

So the radius of convergence is 1;

when $x = 1$.

$\sum_{n=1}^{\infty}\frac{3}{2n} = \frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{n}$, which is a multiple of the natural series,

which is divergent.

when $x=-1$.

$\sum_{n=1}^{\infty}\frac{3(-1)^n}{2n} = \frac{3}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, which is ~~the~~ a multiple of the alternating natural series, which is convergent,

Therefore, the interval of convergence is

$[-1, 1[$.

8. Diagonalize the matrix of the ellipse $7x^2 - 8xy + 13y^2 = 150$. Hence determine the ellipse's eccentricity.

The matrix form of the ellipse is:

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 7 & -4 \\ -4 & 13 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 150.$$

Now we want to find the eigenvalues λ_1, λ_2 and eigenvectors $\vec{v}_1, \vec{v}_2$ for $\begin{pmatrix} 7 & -4 \\ -4 & 13 \end{pmatrix}$.

By solving the characteristic equation $\lambda^2 - 20\lambda + 75 = 0$.

we get

$\lambda_1 = 5, \quad \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda_2 = 15, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Therefore,

$$\begin{pmatrix} 7 & -4 \\ -4 & 13 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 15 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 15 \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix}$$

$$\cancel{= \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 15 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}}$$

let $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

then

$$\begin{pmatrix} x' & y' \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 15 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 150.$$

$$5x'^2 + 15y'^2 = 150.$$

So the ellipse is

$$\frac{x^2}{30} + \frac{y^2}{10} = 1$$

w/ ~~and the~~ eccentricity

$$= \sqrt{\frac{20}{30}} = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$$

Since the eccentricity remains the same after rotation, the eccentricity of the origional ellipse is also $\frac{\sqrt{6}}{3}$.

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9. Show that the series $\sum(-1)^{n+1}a_n$, where $a_n = 1/n$ if n is odd and $a_n = 1/n^2$ if n is even, is divergent. Why does the alternating series test not apply?

The series is convergent if it comes to a certain value.

harmonic

let ~~$b_n = \sum \frac{1}{2n-1}$, C_n~~ $b_n = \frac{1}{2n-1}$, $C_n = \left(\frac{1}{2n}\right)^2$

then $\sum (-1)^{n+1} a_n = \sum b_n - \sum C_n$.

~~as C_n~~

as ~~$\lim_{n\to\infty} \frac{b_n}{C_n} \frac{C_n}{b_n} = \frac{4n^2}{2n-1}$~~

~~$= \lim_{n\to\infty} \frac{4n^2}{2n-1} \frac{2n-1}{4n^2}$~~

~~$= \lim_{n\to\infty} \frac{2 - \frac{2}{n} - \frac{1}{n^2}}{4}$~~

~~$= 0$~~

As ~~①~~ $\sum C_n$ converges because ~~of~~ it is a multiple of p-series, and $p > 1$.

and b_n diverges ~~because~~ by company test with natural series,

so ~~$\sum$~~ $\sum C_n$ is bounded ~~by~~, say by β,

So $\sum (-1)^{n+1} a_n = \sum b_n - \beta$,

which is also divergent.

The alternating series test does not apply because there is no uniform formula for the whole series and the amount adding or substracting ~~(an)~~ each time (a_n) is different.

10. Suppose the group G has subgroup H. Define the relation $\sim$ on G by $a \sim b$ if $ab^{-1} \in H$. Prove that $\sim$ is an equivalence relation and describe the equivalence classes.

proof. a) as $H \leq G$, $e \in H$,

so $aa^{-1} = e \in H$,

so $a \sim a$, and

$\sim$ is reflexive.

b) if $a \sim b$, ~~ab~~

then $ab^{-1} \in H$.

as each element has an inverse in H,

$(ab^{-1})^{-1} = ba^{-1} \in H$,

So $b \sim a$, and thus

$\sim$ is symmetric

c) if $a \sim b$, $b \sim c$,

then $ab^{-1} \in H$,

$bc^{-1} \in H$

since H is closed,

$(ab^{-1})(bc^{-1})$

$= ab^{-1}bc^{-1}$

$= ac^{-1} \in H$, so $a \sim c$

and $\sim$ is transitive.

Therefore, as $\sim$ is reflexive, symmetric and transitive, $\sim$ is an equivalence relation.

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the equivalence relation creates a partition of H

and ~~G/H~~. G/H.

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What are the subsets of the partition?

Solutions to FM2 Test #4

1. $(1-\cos x)/x^2 = [1-(1-x^2/2+O(x^4))]/x^2 = 1/2+O(x^2)$. So the limit is $1/2$.

2. $M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. The required images are the first and second columns of the matrix.

3. We have $4a-2b+3=4$ and $9a-3b+3=6-a+b$. Solving gives $a=b=\frac{1}{2}$.

4. If $a^2=e$ then $a^3=a$. So a has a cube root in G, namely itself.

5. Let the side length of the equilateral triangle be x. Using Ptolemy's theorem, we find $3x+4x=xCP$. So $CP=7$.

6. See assignment #24.

7. The radius of convergence is $R=1$ and the interval of convergence is $[-1.1[$.

8. $\begin{pmatrix} 7 & -4 \\ -4 & 13 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 \\ 0 & 15 \end{pmatrix} \cdot \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$. So the canonical form of the ellipse is $x^2/30+y^2/10=1$. Hence $e=\sqrt{2/3}$.

9. Suppose $\sum(-1)^{n+1}a_n$ is convergent. Now we know $\sum 1/(2n)^2$ is convergent by comparison with the known convergent series $\sum 1/n^2$. Hence $\sum[(-1)^{n+1}a_n + 1/(2n)^2] = \sum 1/(2n-1)]$ must also converge. But this is a contradiction as $\sum 1/(2n-1)$ diverges, by for example the integral test. Hence $\sum(-1)^{n+1}a_n$ must diverge.

   The alternating series test does not apply as the terms do not decrease in absolute value.

10. To show that $\sim$ is an equivalence relation, we must show that $\sim$ is reflexive, symmetric and transitive.

    i. Since $aa^{-1}=e$ and $e\in H$, it follows that $\sim$ is reflexive.
    ii. If ab^{-1} is in H then $(ab^{-1})^{-1}=ba^{-1}$ is also in H since subgroups contain their inverses. It follows that $\sim$ is symmetric.
    iii. Suppose ab^{-1} and bc^{-1} are in H. Then their product $ab^{-1}bc^{-1}=ac^{-1}$ is also in H since subgroups are closed under the group operation. It follows that $\sim$ is transitive.

    Now $[g]=\{x\in G \mid xg^{-1}\in H\}=\{x\in G \mid x\in Hg\}=Hg$. So the equivalence classes are the right cosets of H in G.