Second Year Further Mathematics

Term 1, 2019

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Tutoring: Tuesday 7–9pm, Wednesday 7–9pm, Friday 7–9pm

Exercises: The purpose of class and homework exercises is to deepen your understanding of mathematics and to improve your problem solving skills. Your exercise solutions should be clear, logical and complete. This practice will be very helpful when it comes to writing tests and examinations. You should ensure that you make a close study of the problems we solve on the board in class.

Evaluation: The end of term examination in the December block week is worth 50% of the total assessment. Four review tests will occur during the term, the best three will be counted, for a total of 30%. The review tests will occur on even numbered day 1's. Assignments make up the remaining 20% of the assessment. Late assignments, unless justified by good reason, and copied assignments will receive a score of zero.

Grades: The correspondence between percentage scores and IB grades for second year further mathematics is given in the following table.

Percentage	IB Grade
86–100	7
73–85	6
61–72	5
50–60	4
33–49	3
16–32	2
0–15	1

FL2 Assignment #1 Name: Maggie $\frac{9\frac{1}{2}}{10}$ Very Good

1. Let X be a set. How many solutions are there to $\{1, 2\} \subseteq X \subseteq \{1, 2, 3, 4, 5\}$?

Let $X = Y \cup \{1,2\}$.

$\because \{1,2\} \subseteq X \subseteq \{1,2,3,4,5\}$.

$\therefore \phi \subseteq Y \subseteq \{3,4,5\}$.

$\therefore$ There are $2^3 = 8$ solutions for Y.

$\therefore$ There are $\boxed{8}$ solutions for X also.

$28 \times 2 = 56$

2. The integers 5 and 15 are members of a set of 12 integers that form a group under multiplication modulo 28. List all 12 integers.

Since it is a group, there is an indentity elmt s.t.

$a \cdot e = e \cdot a = a \quad \forall a$ in the group.

By inspection, $e = 1$.

Now, $\forall a$ in the group, $\exists a^{-1}$ s.t. $a \cdot a^{-1} \equiv 1 \pmod{28}$.

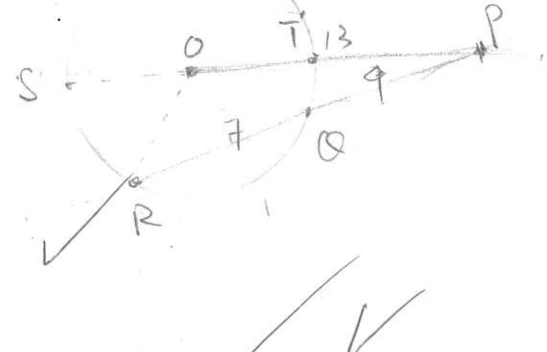
Thus, $\gcd(a, 28) = 1$. (by using Euclidean Algorithm).

$\therefore$ The 12 integers are 1, 3, 5, 9, 11, 13, 15, 17, 19, 23, 25, 27.

3. A point P is outside a circle and 13 cm from its centre. A secant from P cuts the circle at points Q and R so that the external segment $[PQ]$ of the secant is 9 cm and QR is 7 cm. Find the radius of the circle.

segment

Let radius be r cm. PO intersect $\odot O$ at point T.
Extend PO, s.t. it intersect $\odot O$ at point S.



According to secant-secant thrm,

$PT \cdot PS = PQ \cdot PR$.

$(13-r)(13+r) = 9 \times 16$.

$169 - r^2 = 144$

$r^2 = 25$

$\because r > 0$

$\therefore \boxed{r = 5}$

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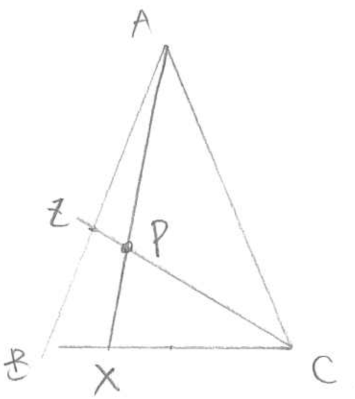
4. In $\triangle ABC$, cevians $[AX]$ and $[CZ]$ are drawn so that $CX : XB = 3 : 1$ and $AZ : ZB = 3 : 2$. Let $k = CP : PZ$, where P is the intersection point of $[CZ]$ and $[AX]$. Find k.



In $\triangle BCZ$. consider the transversal APX.

According to Menelaus' thm,

$$\frac{CP}{PZ} \cdot \frac{ZA}{AB} \cdot \frac{BX}{XC} = -1.$$

$$\because \frac{CP}{PZ} = k \quad \frac{ZA}{AB} = -\frac{3}{5}. \quad \frac{BX}{XC} = \frac{1}{3}.$$

$$k \times \left(-\frac{3}{5}\right) \times \frac{1}{3} = -\frac{1}{5}k = -1$$

$$\therefore \boxed{k = 5}.$$

5. If H and K are subgroups of G, show that $H \cap K$ is also a subgroup of G.

proof. We use the 3-step subgroup test.

(i) Let $x, y \in H \cap K$.
then $x, y \in H$ and $x, y \in K$.
Because H and K are both subgroups, of G,
$xy \in H$, $xy \in K$.
$\therefore xy \in H \cap K$.

(ii) Since $H \leq G$, $e \in H$.
Since $K \leq G$, $e \in K$.
$\therefore e \in H \cap K$.

(iii) $\forall x \in H \cap K$.
$x \in H$, and $x \in K$. ?
Thus, $\exists x^{-1} \in H$, $(x^{-1})' \in K$.
$x^{-1}x = e = (x^{-1})' x$?
$x^{-1} = (x^{-1})'$?

$\therefore x^{-1} \in H \cap K$.

Therefore, according to the 3-step subgroup test,
$H \cap K \leq G$. $\square$.

$3\frac{1}{2}$

FM2 Assignment #2 Name: Maggie.

$9\frac{1}{2}/10$ Very good.

1. List the subgroups of $\mathbb{Z}_{18}$.

$\{0\}$.

$\mathbb{Z}_{18}$.

$\{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$.

$\{0, 3, 6, 9, 12, 15, 18\}$.

$\{0, 6, 12, 18\}$.

$\{0, 9, 18\}$.

2. Suppose a cyclic group's only proper subgroup has order 7. What is the order of the group?

Since the order of a subgroup is always a factor of the group order. The order of the group should be a multiple of 7 and has 7 as its only factor, as it only has one proper subgroup of order 7.
Thus, the order of the group can only be $7 \times 7 = \boxed{49}$.

3. Suppose groups G and G' are isomorphic. Show that if G is Abelian then G' must also be Abelian.

Pf: let $(G, *) \cong (G', \circ)$.

i.e. $\exists\ f: G \to G'$ s.t. $f(a*b) = f(a) \circ f(b)\ \forall a, b \in G$.
$f(a), f(b) \in G'$

Since G is Abelian,

$f(a*b) = f(b*a) = f(b) \circ f(a) = f(a) \circ f(b)$.

Since $f(a), f(b) \in G'$ and are commutative,

G' is also Abelian $\square$.

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$(-1)^n \tan(\frac{1}{n})$

4. Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\tan n}$ converges conditionally.

pf. ① We want to show that $\sum_{n=1}^{\infty} \left|(-1)^n \tan(\frac{1}{n})\right| = \sum_{n=1}^{\infty} \tan(\frac{1}{n})$ diverges.

We use the limit comparison test.

Since $\tan(\frac{1}{n}) > 0$ and $\frac{1}{n} > 0$. $\forall n \in \mathbb{Z}^+$.

$$\lim_{x\to\infty} \frac{\tan(\frac{1}{x})}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\sin(\frac{1}{x})}{\cos(\frac{1}{x})} \cdot x = \lim_{x\to\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}}.$$

Recall that $\lim_{x\to 0} \frac{\sin x}{x} = 1.$

$$\lim_{x\to\infty} \frac{\tan(\frac{1}{x})}{\frac{1}{x}} = \lim_{\frac{1}{x}\to 0} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} = 1.$$

Thus, $\sum_{n=1}^{\infty} \tan(\frac{1}{n})$, like $\sum_{n=1}^{\infty} \frac{1}{n}$, diverges.

② Now we'd like to show that $\sum_{n=1}^{\infty} (-1)^n \tan(\frac{1}{n})$ converges.

We apply the alternating series test.

(i) check that $\lim_{n\to\infty} \tan(\frac{1}{n}) = 0$.

and (ii) Since $\tan x \nearrow$ for $x \in [0, \frac{\pi}{2})$, $\frac{1}{x} \searrow$ for $x \in \mathbb{Z}^+$.

$\frac{1}{n+1} < \frac{1}{n}$, $\forall n \geq 1$

$\therefore \tan(\frac{1}{n+1}) < \tan(\frac{1}{n})$ $\forall n \geq 1$.

Thus, according to the alternating series test, $\sum_{n=1}^{\infty} (-1)^n \tan(\frac{1}{n})$ converges. □

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5. Let G be a simple graph with p vertices and q edges. Show that if $q > \frac{1}{2}(p-1)(p-2)$ then G is connected.

proof by contrapositive:

Suppose that G is disconnected.

i.e. at least one vertix is not connected to G.

consider the extreme case when all other vertices are all connected to each other except one disconnected one,

there are. $\sum_{i=1}^{p-2} i = \frac{(p-2+1)(p-2)}{2} = \frac{1}{2}(p-1)(p-2).$

Does this extreme case require the most edges?

Thus, if G is disconnected, the maximum # of ~~the~~ edges is $\frac{1}{2}(p-1)(p-2)$

i.e. $q \leq \frac{1}{2}(p-1)(p-2)$.

Which is equivalent to saying that if $q > \frac{1}{2}(p-1)(p-2)$ G is connected. □

a, $p-a$

$q_{max}: \frac{a(a-1)}{2}$, $\frac{(p-a)(p-a-1)}{2}$

$q \leq \frac{a(a-1)}{2} + \frac{(p-a)(p-a-1)}{2}$

$q \leq \binom{a}{2} + \binom{p-a}{2}$

$= \frac{a^2 - a + p^2 - 2pa + 1 - p + a}{2}$

$= \frac{a^2 + p^2 - 2pa - p + 1}{2}$

$= \frac{a^2 - (2p)a + p^2 - p + 1}{2}$.

3 ?

FM2 Assignment #3 Name: Maggie $\frac{10}{10}$ Excellent!

1. The minimum degree and maximum degree of a graph are denoted δ and Δ respectively.

   (a) What are δ and Δ for a 3-regular graph?

   $\delta = \Delta = 3$

   (b) What is δ for any tree with more than one vertex?

   $\delta = 1$. (both at leaves)

2. Find $\lim\limits_{x\to 1} \dfrac{\ln x}{\sin \pi x}$.

$\because \lim\limits_{x\to 1} \ln x = 0$

$= \lim\limits_{x\to 1} \sin \pi x = 0$

We apply L'Hôpital's Rule,

$$\lim_{x\to 1} \frac{\ln x}{\sin \pi x} = \lim_{x\to 1} \frac{(\ln x)'}{(\sin \pi x)'} = \frac{\frac{1}{x}}{\pi \cos \pi x} = \frac{1}{\pi x \cos \pi x} = -\frac{1}{\pi}$$

3. Prove that $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

pf. $V_4 = \{e, v, h, vh\}$

where v is fliping vertically.
h is fliping horizontally.

$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0)\ (0,1)\ (1,0)\ (1,1)\}$.

~~By send~~ Define $f: V_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ by

$f(e) = (0,0)$

$f(v) = (0,1)$

$f(h) = (1,0)$

$f(vh) = (1,1)$.

we get a bijection.

Thus, $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

To see operation preservity, we use the cayley tables:

V_4:

	e	v	h	vh
e	e	v	h	vh
v	v	e	vh	h
h	h	hv	e	v
vh	vh	h	v	e

$\mathbb{Z}_2 \times \mathbb{Z}_2$:

	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,0)	(0,1)	(0,0)

And there is a correspondence btw the two tables.

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4. Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges or diverges.

Since $f(n) = \frac{1}{n \ln n}$ is continuous, positive, and decreasing, we apply the Integral test.

Let $x = \ln n$. $dx = \frac{1}{n} dn$.

$$\int_2^{\infty} \frac{1}{n \ln n} dn = \lim_{b \to \infty} \int_2^b \frac{1}{x} dx.$$

$$= \lim_{b \to \infty} [\ln |x|]_2^b.$$

$$= \lim_{b \to \infty} [\ln(\ln(n))]_2^b.$$

$$= \lim_{b \to \infty} [\ln(\ln b) - \ln(\ln 2)].$$

$= \infty$, which diverges.

Thus by the integral test the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ must diverge.

5. For what values of x is the following subset of $\mathbb{R}^3$ independent?

$$\{ \begin{pmatrix} x \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ x \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ x \end{pmatrix} \}$$

Consider $c_1 \begin{pmatrix} x \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ x \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 2 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$$\begin{pmatrix} x & 1 & 2 \\ 1 & x & 2 \\ 1 & 2 & x \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad c_1 . c_2 . c_3 \in \mathbb{R}.$$

The system has AM.

$$\left(\begin{array}{ccc|c} x & 1 & 2 & 0 \\ 1 & x & 2 & 0 \\ 1 & 2 & x & 0 \end{array} \right).$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & x & 0 \\ 1 & x & 2 & 0 \\ x & 1 & 2 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 2 & x & 0 \\ 0 & x-2 & 2-x & 0 \\ 0 & 1-2x & 2-x^2 & 0 \end{array} \right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & x & 0 \\ 0 & x-2 & 2x. & 0 \\ 0 & 0 & -x^2-2x+3 & 0 \end{array} \right). \quad R_3 + \left(\frac{2x-1}{x-2}\right) R_2 \to R_3$$

The vectors are linearly independent if the system only has the trivial solution $c_1 = c_2 = c_3 = 0$.

from Row 2 & 3, this is when

$-x^2 - 2x + 3 \neq 0$, and $x - 2 \neq 0$.

$\therefore \; x \neq 1$, or 2 or -3.

4

FM2 Assignment #4 Name: Maggie. $\frac{10}{10}$ Excellent!

1. (a) Is $(3\mathbb{Z} \cap 4\mathbb{Z}, +)$ a group? If so describe it, if not explain why.

$3\mathbb{Z} = \{3k \mid k \in \mathbb{Z}\}$.
$4\mathbb{Z} = \{4k \mid k \in \mathbb{Z}\}$.
$3\mathbb{Z} \cap 4\mathbb{Z} = \{12k \mid k \in \mathbb{Z}\}$.

So $(3\mathbb{Z} \cap 4\mathbb{Z}, +)$ is a group. that contains all multiples of 12, and is equivalent to the group $(12\mathbb{Z}, +)$.

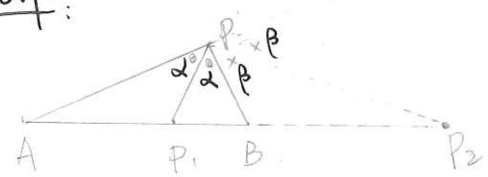
(b) Is $(3\mathbb{Z} \cup 4\mathbb{Z}, +)$ a group? If so describe it, if not explain why.

No. Because it is not closed.
To see this, we know $3 \in 3\mathbb{Z}$, $4 \in 4\mathbb{Z}$.
and thus $3, 4 \in (3\mathbb{Z} \cup 4\mathbb{Z})$.
However, $3+4=7$ is neither in $3\mathbb{Z}$ nor in $4\mathbb{Z}$.
which do not obey the closure of a group.
Thus we conclude $(3\mathbb{Z} \cup 4\mathbb{Z}, +)$ is not a group.

2. Let $A = (0, -1)$ and $B = (0, 2)$. Describe the locus of a point P that moves so that $PA = 2PB$.

The Locus of P is the circle. with the equation $x^2 + (y-3)^2 = 4$.

~~proof~~:



let P be a point that is not collinear w/ A and B. and $PA = PB$.

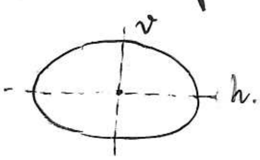
Draw the internal and external of $\angle APB$, which meet $[AB]$ at P_1 and P_2.

Now. $2\alpha + 2\beta = 180°$.
$\alpha + \beta = 180° = \angle P_1PP_2$.
and since PP_1, PP_2 are bisectors,
$\frac{P_1A}{P_1B} = \frac{PA}{PB} = 2$. $\frac{P_2A}{P_2B} = \frac{PA}{PB} = 2$.
$\because AB = 3$. $\therefore$ $P_1(0,1)$.
$P_2(0,5)$.
and since P_1. P_2 are fixed points and $\angle P_1PP_2 = 90°$, the locus of P is the circle w/ diameter P_1P_2.
which is $x^2 + (y-3)^2 = 4$.

3. Describe the symmetry group of the graph of $x^2 + 4y^2 = 1$.

$$x^2 + \frac{y^2}{(\frac{1}{2})^2} = 1.$$

which is an ellipse. w/ center O,
x-intercepts $(\pm 1, 0)$. y-intercepts $(0, \pm\frac{1}{2})$



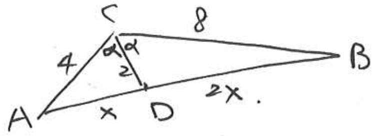
consider the operations $\{e, v, h, vh\}$.
where h is flipping along the horizontal line
and v is flipping along the vertical line.

and the cayley table for it is:

	e	v	h	vh
e	e	v	h	vh
v	v	e	vh	h
h	h	vh	e	v
vh	vh	h	v	e.

and thus is isomorphic to V_4, or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

4. A triangle has sides of length 4 and 8. If the bisector of the angle between the sides has length 2, find the length of the third side, giving your answer in the form $\sqrt{a}$ where $a \in \mathbb{Z}^+$.



Let $AC=4$. $CB=8$. $CD=2$.
CP ~~biset~~ bisects $\angle ACB$.
and intersect AB at D.

Let $AD=x$.
$\because \angle ACD = \angle DCB$
$$\frac{AC}{CB} = \frac{AD}{BD} = 2.$$
$\therefore BD = 2x$

In $\triangle ACD$:

$$x^2 = (2)^2 + (4)^2 - 2(4)(2)\cos\alpha.$$

In $\triangle CDB$:

$$(2x)^2 = (2)^2 + (8)^2 - 2(2)(8)\cos\alpha.$$

$$\therefore \begin{cases} x^2 = 20 - 16\cos\alpha \\ 4x^2 = 68 - 32\cos\alpha \end{cases} \Rightarrow \cos\alpha = \frac{20-x^2}{16}$$

$$\therefore 4x^2 = 68 - \cancel{32}\left(\frac{20-x^2}{\cancel{16}}\right)$$
$$= 68 - 40 + 2x^2$$
$$x^2 = 14$$
$\because x > 0$
$\therefore x = \sqrt{14}$
$\therefore AD = 3\sqrt{14} = \sqrt{126}$.

✓

5. Let G be a group and H a non-empty subset of G. Show that $H \leq G$ if H is closed under division; by this we mean xy^{-1} is in H whenever x and y are in H.

pf. We use the 3-step subgroup test. ✓

(1) Let $x = y = a \in H$.
$\therefore aa^{-1} = e \in H$.
So H has the identity. ✓

(2) Now let $x = e$, $y = a \in H$.
$\therefore ea^{-1} \in H$.
$\therefore ea^{-1} = a^{-1} \in H$ ✓
So ~~the~~ inverse exists.

(3) Let $x, y \in H$.
$\therefore y^{-1} \in H$. according to (2). ✓
$\therefore x(y^{-1})^{-1} = xy \in H$.
So ~~the~~ closure is verified.

Thus, by the three-step subgroup test, $H \leq G$.

/4

FM2 Assignment #5 Name: Maggie. $9\frac{1}{2}$ / 10 Very good

1. The adjacency matrix of graph G is $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$. What information do the diagonal elements of A^2 give?

$$A^2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

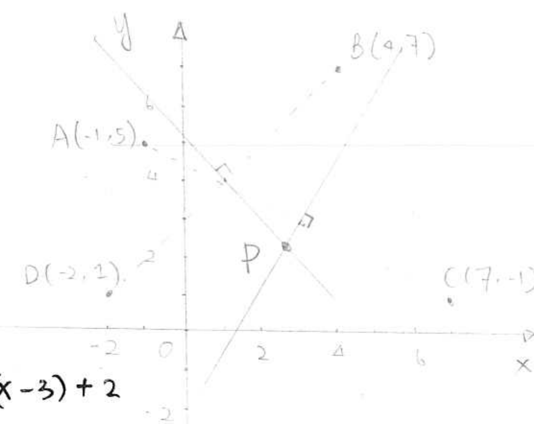
which means there are two vertices in graph G with 3 walks of length 2 back to themselves, and two vertices with 2 walks of length 2 back to themselves $\frac{1}{2}$

2. The circle group $T = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ is a subgroup of $\mathbb{C}^*$. Give a geometric description of the coset $(3+4i)T$.

Since T is the circle w/ center at origin and radius 1.
$(3+4i)$ has modulus 5, so by multiplying $3+4i$ to T,
the circle is enlarged to a radius of 5.
Thus, the coset $(3+4i)T$ is the circle w/ center at origin
~~at~~ and radius 5.

3. A quadrilateral has vertices $A(-1,5)$, $B(4,7)$, $C(7,-1)$ and $D(-2,1)$. Find the coordinates of the point P such that $PA = PC$ and $PB = PD$.

Since $PA=PC$.
P must be on the perpendicular bisector of AC.
and since $PB=PD$
P must be on the perpendicular bisector of BD.



AC: $y = \frac{5-(-1)}{-1-7}(x-7)-1$
$= -\frac{3}{4}x + \frac{17}{4}$.

BD: $y = \frac{7-1}{4-(-2)}(x-4)+7$.
$= x+3$.

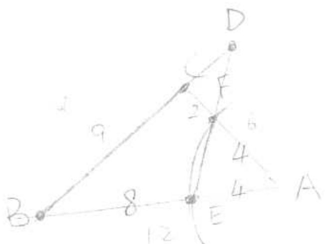
so p is the intersection of the two lines:

$$\begin{cases} y = \frac{4}{3}(x-3)+2 \\ y = -(x-1)+4 \end{cases}$$

$\Rightarrow \frac{4}{3}(x-3)+2 = -(x-1)+4$
$x = 3$.
$\therefore P(3,2)$.

$\frac{1}{2}$

4. In $\triangle ABC$, $a = 9$, $b = 6$ and $c = 12$. A circle with centre A and radius 4 meets sides $[AB]$ and $[AC]$ at E and F respectively. The secant (EF) meets (BC) at D. Use Menelaus's theorem to calculate the length CD.



$4CD = CD + 9$

$\boxed{CD = 3}$

$$\frac{AF}{FC} \cdot \frac{CD}{DB} \cdot \frac{BE}{EA} = -1$$

$$\frac{\cancel{6}4}{2} \cdot -\frac{CD}{CD+9} \cdot \frac{8}{4} = -1$$

$$-\frac{CD}{CD+9} = -\frac{1}{4}$$

5. Let $G = \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \text{ and } a^2 + b^2 \neq 0\}$. Show that the groups $(G, \times)$ and $(\mathbb{C}^*, \times)$ are isomorphic.

Pf. Define $f : G \to \mathbb{C}^* : f(A) = a_{11} + a_{21} i$.

we want to show f is an isomorphism.

① To show f is a bijection.

Let $f(A) = f(B)$. $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. $B = \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \in G$.

then $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$

$\Rightarrow \begin{cases} a = c \\ b = d \\ -b = -d \end{cases} \Rightarrow \begin{cases} a = c \\ b = d \end{cases} \Rightarrow a + bi = c + di$.

Thus, f is a bijection.

② $f(A \cdot B) = f\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & -c \end{pmatrix}\right)$

$= f\left(\begin{pmatrix} ac - bd & -ad - bc \\ ad + bc & ca - bd \end{pmatrix}\right)$

$= (ac - bd) + (ad + bc)i$

$= (a + bi)(c + di) = f(A) \cdot f(B)$

Therefore, we proved that $G \cong \mathbb{C}^*$. □

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