Chapter 5 Topological Graph Theory

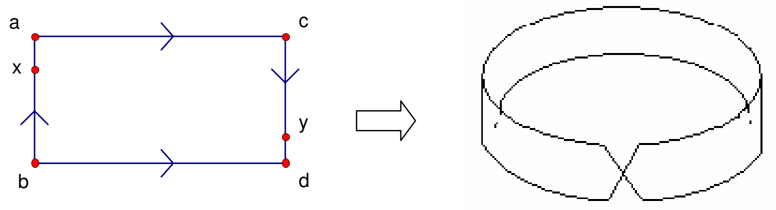
§ 5.1. Basic Notations

Topological graph theory studies the "drawing" of a graph on a surface. A proper drawing on a surface of a graph G with $|G| = p$ and $||G|| = q$ follows the rules :
(1). There are p points on the surface which corresponds to the set of vertices in G; and
(2). There are q curves joining points defined above which correspond to the set of edges and they are pairwise disjoint except possibly for the endpoints.

• The drawing is on a surface defined on $\mathbb{R}^3$.
• A **2-manifold** is a connected topological space in which every point has a neighborhood homeomorphic to the open unit disk defined on $\mathbb{R}^2$.
• An **n-manifold** is a connected topological space in which every point has a neighborhood homeomorphic to $B_n = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 < 1\}$.
• A subspace M of $\mathbb{R}^3$ is **bounded** if there exists a positive real number K such that $M \subseteq \{(x, y, z) \mid x^2 + y^2 + z^2 = K\}$.
• Let $M \subseteq \mathbb{R}^3$ be a 2-manifold. Then M is said to be **closed** if it is bounded and the boundary of M coincides with M.
• Let $M(\subseteq \mathbb{R}^3)$ be a 2-manifold; M is said to be **orientable** if for every simple closed C on M, a clockwise sense of rotation is preserved by traveling once around C. Otherwise, M is non-orientable.
• A 2-manifold M is orientable if and only if it is **two-sided**.

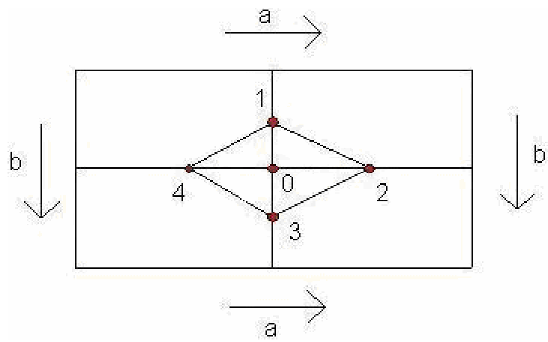
Definition 5.1.1. (Orientable Surface)
A surface is a compact orientable 2-manifold that may be thought of as a sphere on which has been placed(inserted) a number of "handles" (holes). A sphere, denoted by S_0, is the surface of a 3-dimension ball. More precisely, $S_0 = \{(x, y, z)|x^2 + y^2 + z^2 = r^2, r \in \mathbb{R}^+\}$. S_1 is known as a torus, S_2 a double torus, and S_h is a surface obtained by adding h handles to S_0.

Definition 5.1.2. (Non-Orientable Surface)
A surface obtained by adding k cross-caps to S_0 is known as the non-orientable surface N_k. (A cross cap is obtained from Möbius band described in what follows.)
Cross cap: Attach the boundary of a Möbius band to a cycle on S_0 to obtain a cross cap.

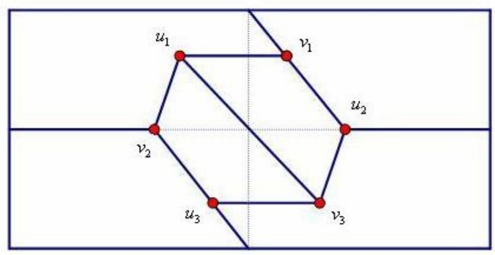


a→d, b→c, x→y

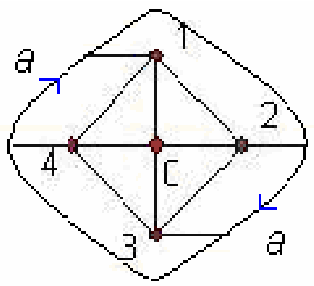
Definition 5.1.3. (Embedding or Imbedding, 2-cell embedding)
An embedding of a graph in a surface is a continuous 1-1 function from a topological representation of the graph into the **surface**. If every region of the embedding is homeomorphic to a 2-dim open disc, then the embedding is a **2-cell embedding**(圓盤嵌入).



Embedding of K_5 on S_1

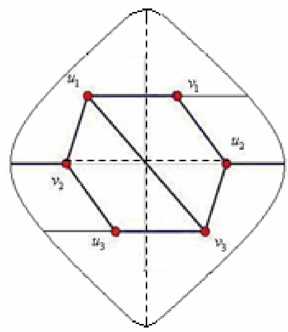


Embedding of $K_{3,3}$ on S_1



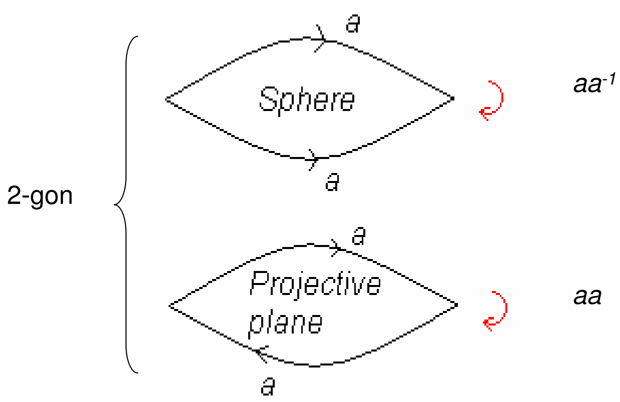
Embedding of K_5 on N_1

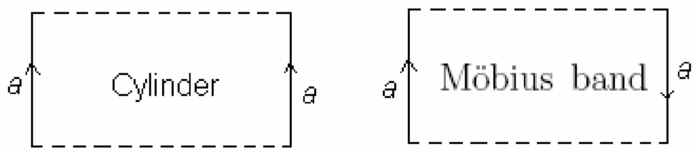
(N_1 is also known as a projective plane.)



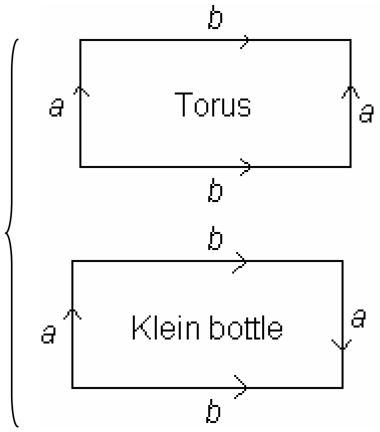
Embedding of $K_{3,3}$ on N_1

- Surfaces can be represented by a polygon (standard fundamental).





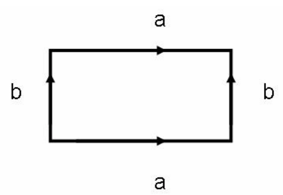
4-gon



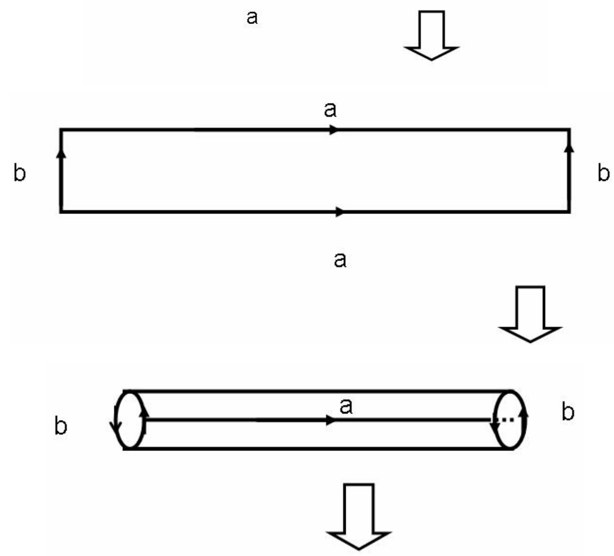
$$aba^{-1}b^{-1} = a_1 b_1 a_1^{-1} b_1^{-1}$$

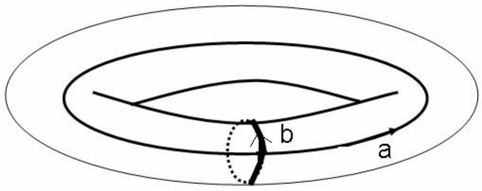
$$abab^{-1} = (ab)(ab)b^{-1}b^{-1} = a_1 a_1 a_2 a_2$$

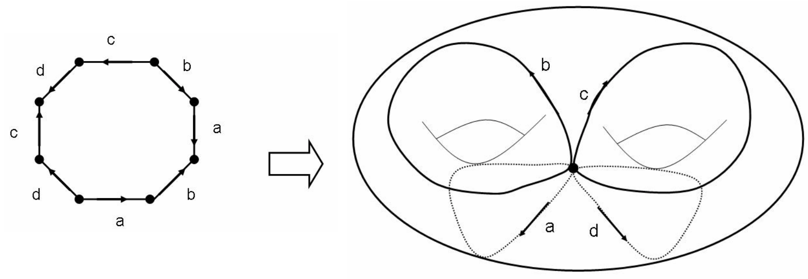
$$where\ a_1 = ab\ and\ a_2 = b^{-1}.$$



$$\boldsymbol{ab^{-1}a^{-1}b}$$







(*) The standard fundamental polygon for S_k is $a_1b_1a_1^{-1}b_1^{-1}...a_kb_ka_k^{-1}b_k^{-1}$.
(**) The standard fundamental polygon for N_h is $a_1a_1a_2a_2...a_hb_h$.

Definition 5.1.4. If a graph G can be embedded in S_0, then G is a planar graph.

Theorem 5.1.1.(Euler's Formula)
Let G be a **connected planar** graph which has p vertices, q edges and r regions. Then $p - q + r = 2$.
Proof. By induction on q. $\square$

Corollary 5.1.2.
Let G be a planar graph which has k components, p vertices, q edges and r regions. Then $p - q + r = 1 + k$.

Proof. By induction on k. $k = 1$ is true by Theorem 1. Assume the assertion is true for k. Let G be a graph with p vertices, q edges and r regions, and G have $k+1 (\geq 2)$ components. Now let $\tilde{G} = G + e$ (e connects two components). Then, $\tilde{G}$ has p vertices, $q + 1$ edges, r regions and k components. Hence, $p-(q+1)+r = 1+k$. This implies $p-q+r = 1 + (k + 1)$. The proof follows. $\square$

Definition 5.1.5. (Maximal planar graph)
A planar graph is maximal if $\forall\ u, v \in V(G)$, $uv \notin E(G)$, $G + uv$ is not planar.

Theorem 5.1.3. If G is a maximal planar (p, q)-graphs, then $q = 3p - 6$.
Proof. $3r = 2q$, whenever G is maximal. $\square$

Corollary 5.1.4. If G is a planar graph, then $q \leq 3p - 6$.

Theorem 5.1.5. If G is a planar graph with girth g, then G has at most $\frac{g}{g-2}(p - 2)$ edges, i.e., $q \leq \frac{g}{g-2}(p - 2)$.
Proof. $gr \leq 2q$; $p - q + r = 2$; $p - q + \frac{2q}{q} \geq 2$; $p - 2 \geq q(1 - \frac{2}{q})$; $p-2 \geq q\frac{g-2}{g}$. $\square$

Theorem 5.1.6. If G is a maximal planar graph (must be connected), then $3p_3 + 2p_4 + p_5 = p_7 + 2p_8 + \cdots + (n - 6)p_n + 12$ $(n = \Delta(G))$.
Proof. $p = \sum_{i=3}^{n} p_i$, $2q = \sum_{i=3}^{n} ip_i$, $q = 3p - 6$, $3p_3 + 2p_4 + p_5 \geq 12$ $\Longrightarrow \sum_{i=3}^{n} ip_i = 6\sum_{i=3}^{n} p_i - 12$. Now, the proof follows. $\square$

- Every planar graph has at least one vertex of degree less than 6.
- Not every graph((p, q)-graph) with $q = 3p - 6$ is planar.
- If G is planar, then every subgraph is planar.
- **Contracting an edge** of a planar graph gives a planar graph.
  G is planar$\Longrightarrow$ G/e is planar.
- Planar graphs and spherical graphs are planar graphs.

Definition 5.1.6. (Dual graph of planar graph)
The dual graph G^* of a planar graph G is a **plane graph** whose vertices correspond to the faces of G, whose edges xy correspond a common edge of two regions x and y (in G).

$$G : p,\ q,\ r$$
$$G^* : p^*,\ q^*,\ r^*$$
$$p^* = r,\ q^* = q,\ r^* = p.$$

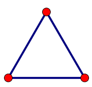
- The dual graph of a planar graph is also a planar graph.
- Some people like "faces" than "regions".

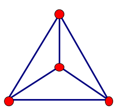
Second Proof of Euler's Formula (Pseudograph!!)

By induction on p. $p = 1$, G has q loops and $q+1$ regions. $p - q + r = 1 - q + (q+1) = 2$. done. Assume the assertion is true when $p = n$. Let G be a graph with $p = n+1$ vertices. Since G is connected and $p = n + 1 \geqslant 2$, G has an edge e. Now, contracting e of G, we obtain a graph G' such that G' has $p-1$ vertices, $q-1$ edges and r regions(?). By hypothesis $(p-1) - (q-1) + r = 2$. Hence $p - q + r = 2$. □

Corollary 5.1.7 Every planar graph is 5-degenerate.

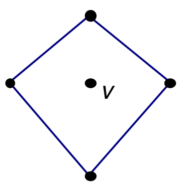
Theorem 5.1.8. Wagner[1936], **Fáry[1948]** and Stein[1951]
Every (finite) planar graph G has an embedding in "plane" where all edges are straight line segments. (Known as Fáry's Theorem)
Proof. Let $\widetilde{G}$ be the graph which is maximally planar and $\widetilde{G} \geq G$. Then, the proof follows by showing $\widetilde{G}$ can be embedded in plane such that all edges are straight line segments. By induction on $|\widetilde{G}|$.

$|\tilde{G}| = 3$, 

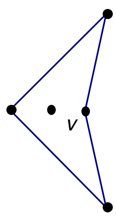
$|\tilde{G}| = 4$, 

$$\vdots$$

Assume $|\widetilde{G}| = n$ is true. Consider $|\widetilde{G}| = n+1$. By the fact that $\exists$ $v \in V(\widetilde{G})$, $deg_{\widetilde{G}}(v) \leq 5$, and $\widetilde{G} - v$ has a straight line segment drawing, we have five cases to consider depending on $deg_{\widetilde{G}}(v)$. Since $deg_{\widetilde{G}}(v) \leq 3$ are easy to see, we get $4 \leq deg_{\widetilde{G}}(v) \leq 5$. By hypothesis, $\widetilde{G} - v$ has a proper drawing. It suffices to add v back to $\widetilde{G} - v$. So, if $deg_{\widetilde{G}}(v) = 4$, we have



and



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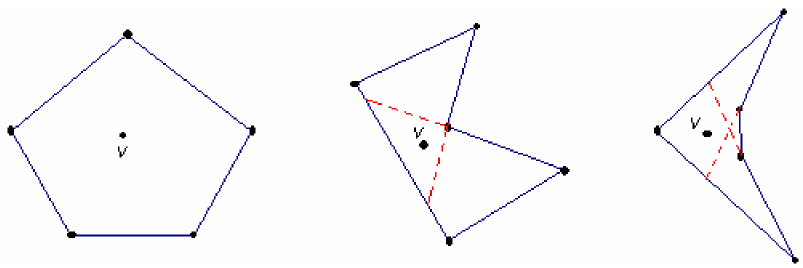
Finally, if $deg_{\widetilde{G}}(v) = 5$, we have



$\square$

Regular Polyhedra(正多面體)

Theorem 5.1.9. There are exactly five regular polyhedra.

Proof. Clearly, a platonic solid is a regular planar graph. Let it be G. Then G : k-regular, p vertices, q edges and r faces (have length l).

$kp = 2q = rl$, $p = \frac{2}{k}q$, $r = \frac{2}{l}q$

$p - q + r = 2$

$\Rightarrow \frac{2}{k}q - q + \frac{2}{l}q = 2$

$\Rightarrow q(\frac{2}{k} - 1 + \frac{2}{l}) = 2$, $\frac{2}{k} - 1 + \frac{2}{l} > 0$

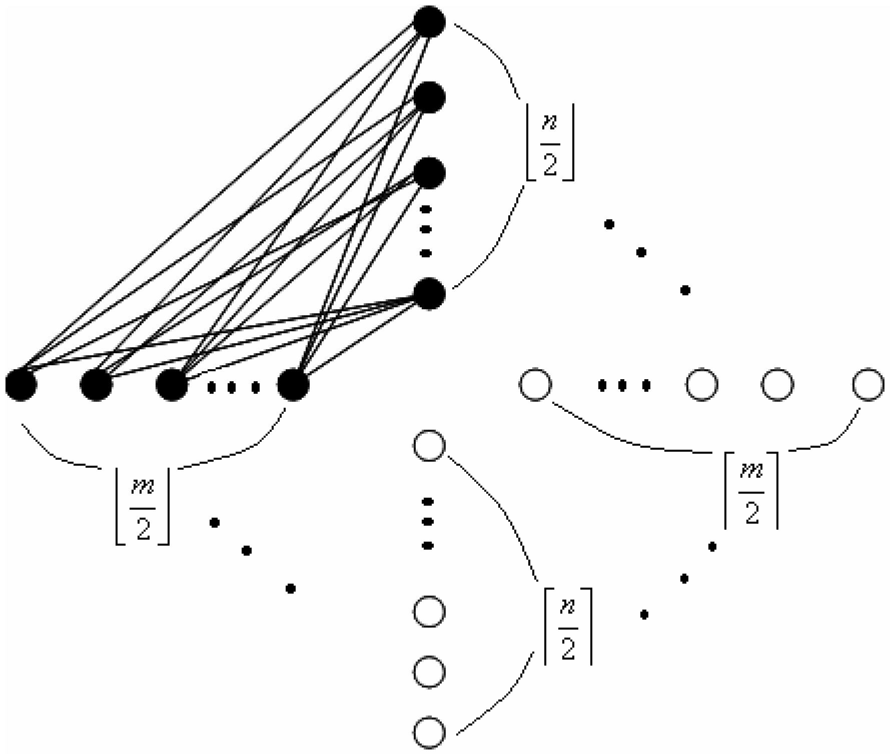
$\Rightarrow 2l - kl + 2k > 0 \Rightarrow kl - 2k - 2l < 0$
$\Rightarrow (k-2)(l-2) = kl - 2l - 2k + 4 < 4$
Now, $k \leq 5$ and thus $l \leq 5$ (?) (Dual graph is also planar!).
By direct checking the graph exists if and only if
$(k, l) = \{(3,3), (3,4), (3,5), (4,3), (5,3)\}$. This concludes the proof.

(k,l)	q	p	r	Name
(3,3)	6	4	4	Tetrahedron(正四面體)
(3,4)	12	8	6	Cube(正六面體)
(4,3)	12	6	8	Octahedron(正八面體)
(3,5)	30	20	12	Dodecahedron(正十二面體)
(5,3)	30	12	20	Icosahedron(正二十面體)

□

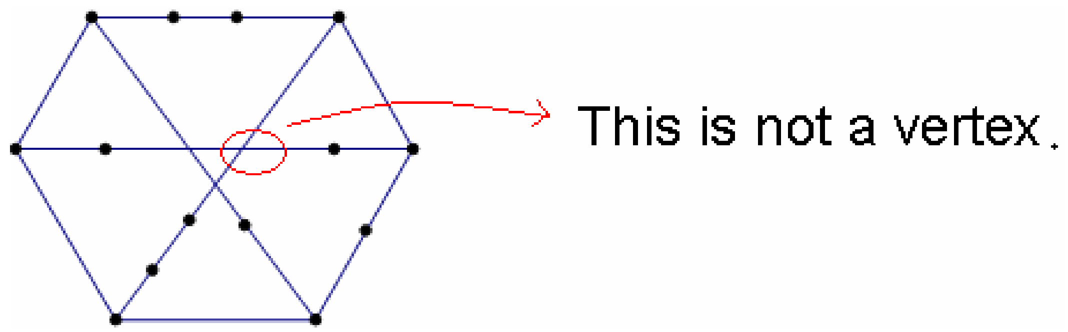
Definition 5.1.7. (Crossing Number)
The crossing number of G, $cr(G)$, is defined to the minimum number of crossings in a proper drawing of G on a plane.

• If G is a planar graph, then $cr(G) = 0$.
• If G is nonplanar, then $cr(G) > 0$.
• $cr(K_5) = 1$, $cr(K_6) = 3$.
• It is conjecture by Guy et al that $cr(K_p) = \frac{1}{4}\lfloor\frac{p}{2}\rfloor\lfloor\frac{p-1}{2}\rfloor\lfloor\frac{p-2}{2}\rfloor\lfloor\frac{p-3}{2}\rfloor$ and the conjecture has been verified for $p \leq 10$.
• (Conjecture) $cr(K_{m,n}) = \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$.
The conjecture has been verified for $1 \leq \min\{m, n\} \leq 6$. (Kleitman)
• The following drawing shows that $cr(K_{m,n}) \leq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$.


$\left\lfloor \frac{n}{2} \right\rfloor$
$\left\lfloor \frac{m}{2} \right\rfloor$
$\left\lceil \frac{m}{2} \right\rceil$
$\left\lceil \frac{n}{2} \right\rceil$

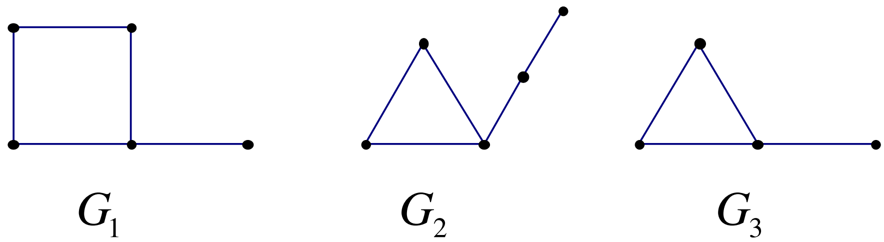
§ 5.2. **Characterization of Planar Graphs**

Definition 5.2.1. A subdivision of a graph is obtained from it by replacing edges with pairwise internally disjoint paths.



Definition 5.2.2. A graph H is said to be homeomorphic from G if either $H \cong G$ or H is isomorphic to a subdivision of G. A graph G_1 is homeomorphic with G_2 if there exists a graph G_3 such that G_1 and G_2 are both homeomorphic from G_3.

Both of G_1 and G_2 are homeomorphic from G_3.



Proposition 5.2.1. If a graph G has a subgraph that is homeomorphic from K_5 or $K_{3,3}$, then G is nonplanar.
Proof. Trivial.

In what follows, we shall prove that well-known theorem in characterizing planar graphs.

Theorem 5.2.2. (Kuratowski [1930])
A graph is planar if and only if it does not contain a subgraph which is homeomorphic from K_5 or $K_{3,3}$.

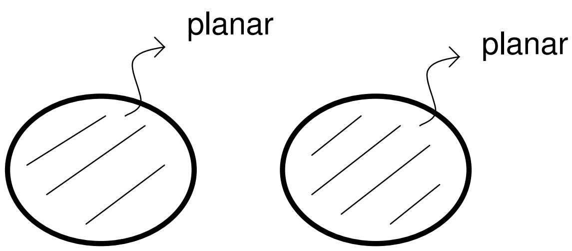
First, we need a couple of definitions.

Definition 5.2.3. A subdivision of K_5 or $K_{3,3}$ is called a Kuratowski subgraph.
Definition 5.2.4. A minimal nonplanar graph is a nonplanar graph such that every proper subgraph is planar.

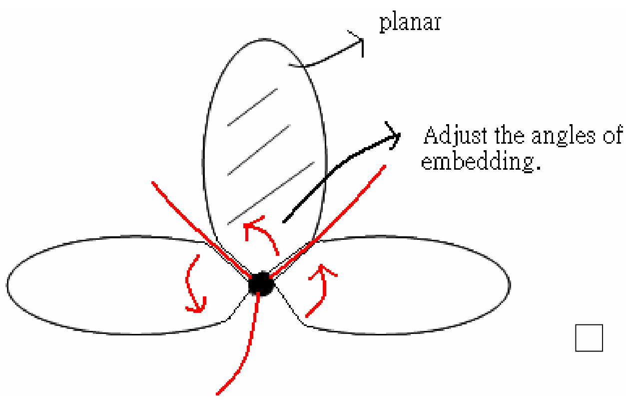
• If F is the edge set of a region in a planar embedding (in S) of G, then G has an embedding **(in plane)** with F being the edge set of the unbounded region.

Lemma 5.2.3. Every minimal nonplanar graph G is 2-connected.
Proof. (By contraposition.)

(1). *G* is disconnected.

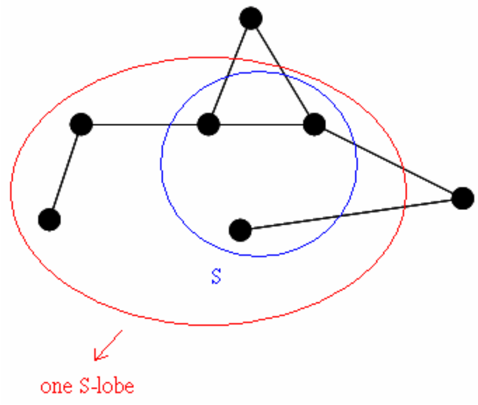


(2). *G* has a cut-vertex.



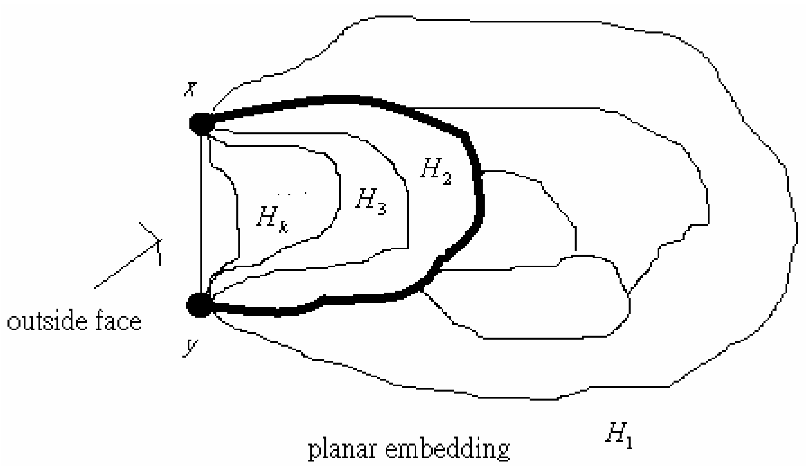
□

Definition 5.2.5. Let S be a set of vertices in a graph G. An S-lobe of G is an induced subgraph of G whose vertex set consists of S and the vertices of "a" component of $G - S$.



Lemma 5.2.4. Let $S = \{x, y\}$ be a separating 2-set of G. If G is nonplanar, then adding the edge xy to **some** S-lobe of G yields a non-planar graph.

Proof. Suppose not. Let $G_1, G_2, ..., G_k$ be the S-lobes of G and set $H_i = G_i + xy$, $i = 1, 2, ..., k$, where H_i is a planar graph. Let H_1 be embedded in a plane such that xy is in the unbounded face(region). Then $H_2, H_3, ..., H_k$ can be embedded one by one as shown in figure.

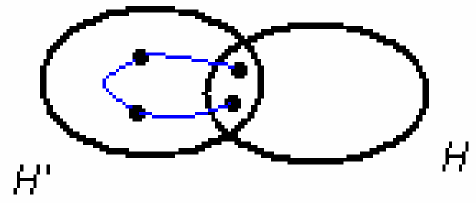


$\square$

Lemma 5.2.5. If G is a graph with fewest edges among all nonplanar graphs without Kuratowski subgraphs, then G is 3-connected.

Proof. Since G is a minimal nonplanar graph, G is 2-connected. Suppose that G is not 3-connected. Then G has a separating 2-set

$S = \{x, y\}$. Since G is nonplanar, there is an S-lobe H containing xy which is also nonplanar. By assumption, H has a Kuratowski subgraphs, F, since $\| H \| < \| G \|$. Now, if $xy \in E(F)$ then clearly F is a Kuratowski subgraphs of G, a contradiction. On the other hand, if $xy \in F$ but $xy \notin E(G)$, then a subdivision of F is contained in G (see the following figure).

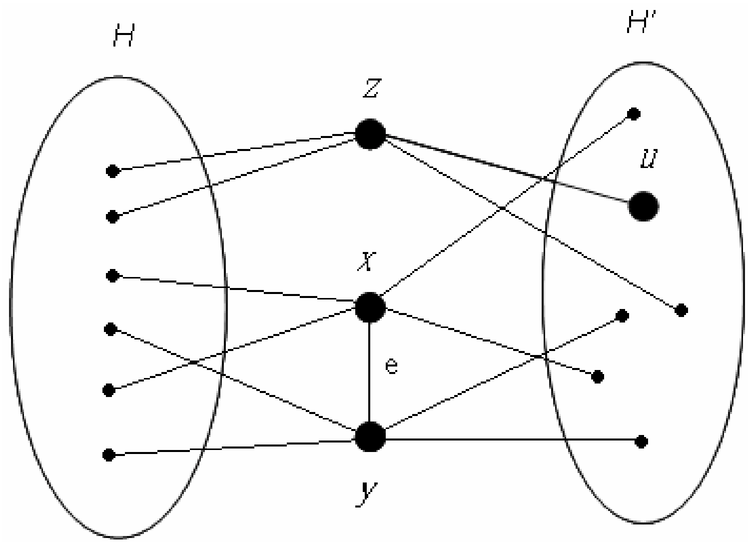


□

Definition 5.2.6. A convex embedding of a graph is a planar embedding in which each region boundary is a **convex polygon**.

Lemma 5.2.6. (Thomassen [1980])
Every 3-connected graph G with at least 5 vertices has an edge e such that G/e $(G \cdot e)$ is 3-connected.

Proof. Let $e = xy$ and the vertex obtained by shrinking e be $\overline{xy}$. Now, suppose that G/e is not 3-connected. Hence G/e has a separating 2-set S. Clearly, $\overline{xy} \in S$. Thus, let z be the other vertex in S. We call z, the mate of $\overline{xy}$. Observe that $\{x, y, z\}$ is a separating 3-set of G. Since every edge of G has a mate, (for otherwise the edge can be contracted and the new graph is also 3-connected), let xy and z be chosen so that the resulting disconnected graph $G - \{x, y, z\}$ has a largest component H. Let H' be another component in $G - \{x, y, z\}$ (See figure). Since $\{x, y, z\}$ is a minimal separating set, each vertex in $\{x, y, z\}$ has a neighbor in each H and H'. Let u be a neighbor of z in H' and v be a mate of uz. By the definition of a mate, $G - \{u, z, v\}$ is disconnected. However, $\langle V(H) \cup \{x, y\} \rangle_G$ is connected and for each vertex v' in H' is not a cut-vertex, for otherwise $\{z, v'\}$ is a 2-separating set of G. This implies that in $G - \{z, u, v\}$ we have a component which is of order at least $|H| + 1$, a contradiction.



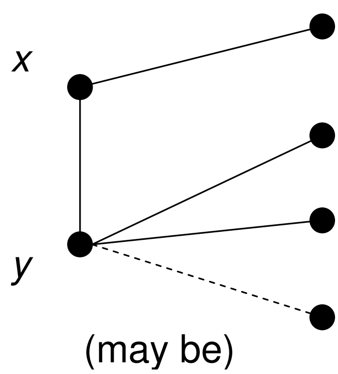
□

Lemma 5.2.7. If G has no Kuratowski subgraph, then for each edge e in G, G/e has no Kuratowski subgraph.

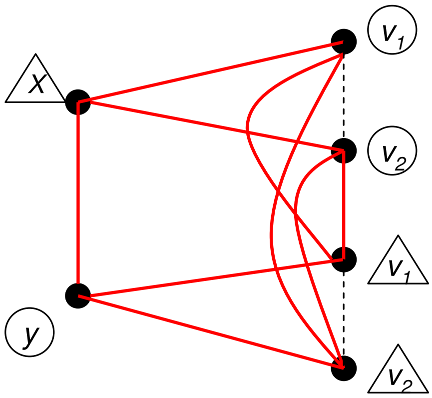
Proof. Suppose not. Let G/e contain a Kuratowski subgraph H. First, if $\bar{e} = \overline{xy}$ is not in H, then G contains H and the proof follows by the contraposition. Now, suppose that the vertex $\overline{xy} \in H$.

Case 1. $deg_H(\overline{xy}) = 2$.

Case 2. $deg_H(\overline{xy}) \geq 3$ and either x or y is incident to one edge of H in G. W.L.O.G. let x be incident to one edge of H in G. This implies G contains a subdivision of H by letting y be $\overline{xy}$ and x be a degree 2 vertex inserted to H (see figure).

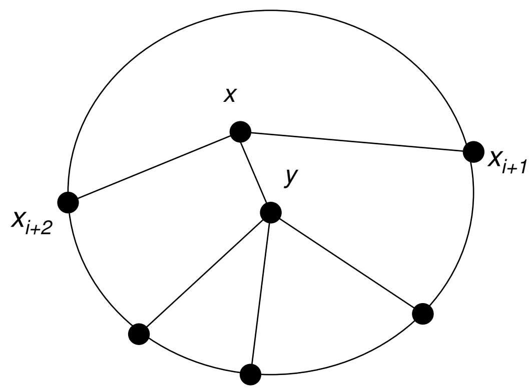


Case 3. $deg_H(\overline{xy}) = 4$ and both x and y are incident to two edges of H (homeomorphic to K_5). Then, by keeping the red paths, we have a graph which is homeomorphic to $K_{3,3}$.

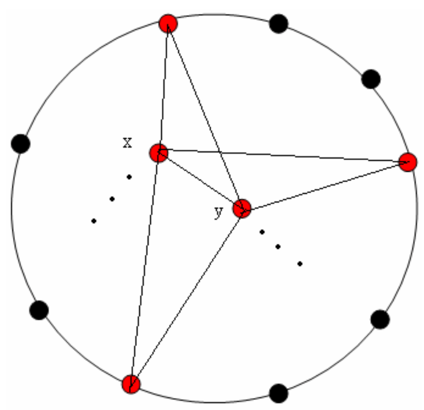


$\square$

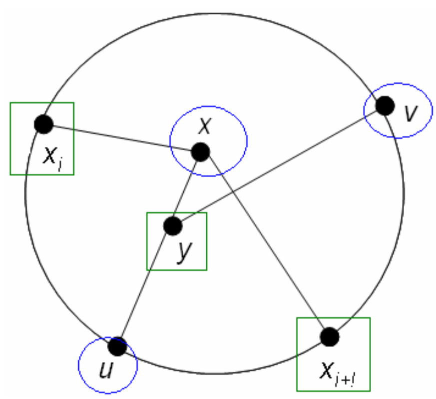
Theorem 5.2.8. (Tutte [1960,1963])
If G is 3-connected and G contains no Kuratowski subgraph, then G has a convex embedding in the plane with no three vertices on a line.
Proof. (Thomassen [1980,1981]) By induction on $|G|$.
Clearly, it is true for $|G| = 4$. Assume that the assertion is true for $|G| = n(\geq 5)$. Consider a graph G of order $n+1$. Let $e = xy$ be the edge such that G/e is 3-connected. By above lemma G/e contains no Kuratowski subgraph and by induction hypothesis, G/e has a convex embedding in the plane with no three vertices on a line.
Since $G/e - \overline{xy}$ is 2-connected, there exists a region whose boundary (a cycle C) contains all the neighbors of $\overline{xy}$. Note that the neighbors of $\overline{xy}$ may be incident to x or y or both (in G) and these neighbors can be joined to x or y by a straight line segments. (convex).
Now, let the neighbors of x (in G) be $x_1, x_2, ..., x_k$ on cycle C in counterclockwise order.
Case 1. The neighbors of y lie in between x_i and x_{i+1} for some i.



Case 2. x and y have three common neighbors.
There exists a subdivision of K_5.



Case 3. y has neighbors u, v that are alternate on C with neighbors x_i, x_{i+1} of x. Then G has a subdivision of $K_{3,3}$.



□

Theorem 5.2.9. (Kuratowski [1930])
A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.
Proof. ($\Longrightarrow$) If a graph does contain a subdivision of K_5 or $K_{3,3}$, then the graph is not planar by using Euler's Formula for K_5 and a revised version for $K_{3,3}$.
($\Longleftarrow$) If the graph does not contain a subdivision of K_5 or $K_{3,3}$ and it is nonplanar, then we can delete some of its edges to obtain a 3-connected "nonplanar" graph which contains no Kuratowski subgraphs. Then, by Tutte's Theorem for convex planar embedding, this graph must be planar, a contradiction. $\square$

Theorem 5.2.10. (Schnyder [1990])
Every n-vertex planar graph has a straight-line embedding in which the vertices are located at integer grid points in $[0, n-1] \times [0, n-1]$.
Proof. Omitted.

Definition 5.2.7. A graph H is a minor of a graph G if a copy of H can be obtained from G by deleting "or" contracting edges of G.

e.g. K_5 is a minor of the Petersen graph but Petersen graph does not contain a subdivision of K_5.

- If G contains a subdivision of H, say H', then H is a minor of G.

We can also prove the following theorem.

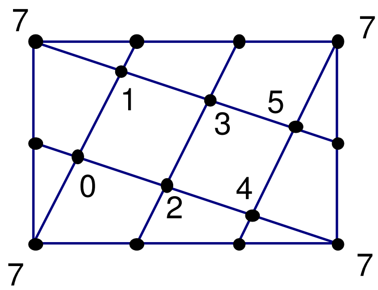
Theorem 5.2.11. (Wagner [1937])
G is planar if and only if neither K_5 nor $K_{3,3}$ is a "minor" of G.

e.g. Petersen graph is nonplanar.

§ 5.3. Graph Embeddings

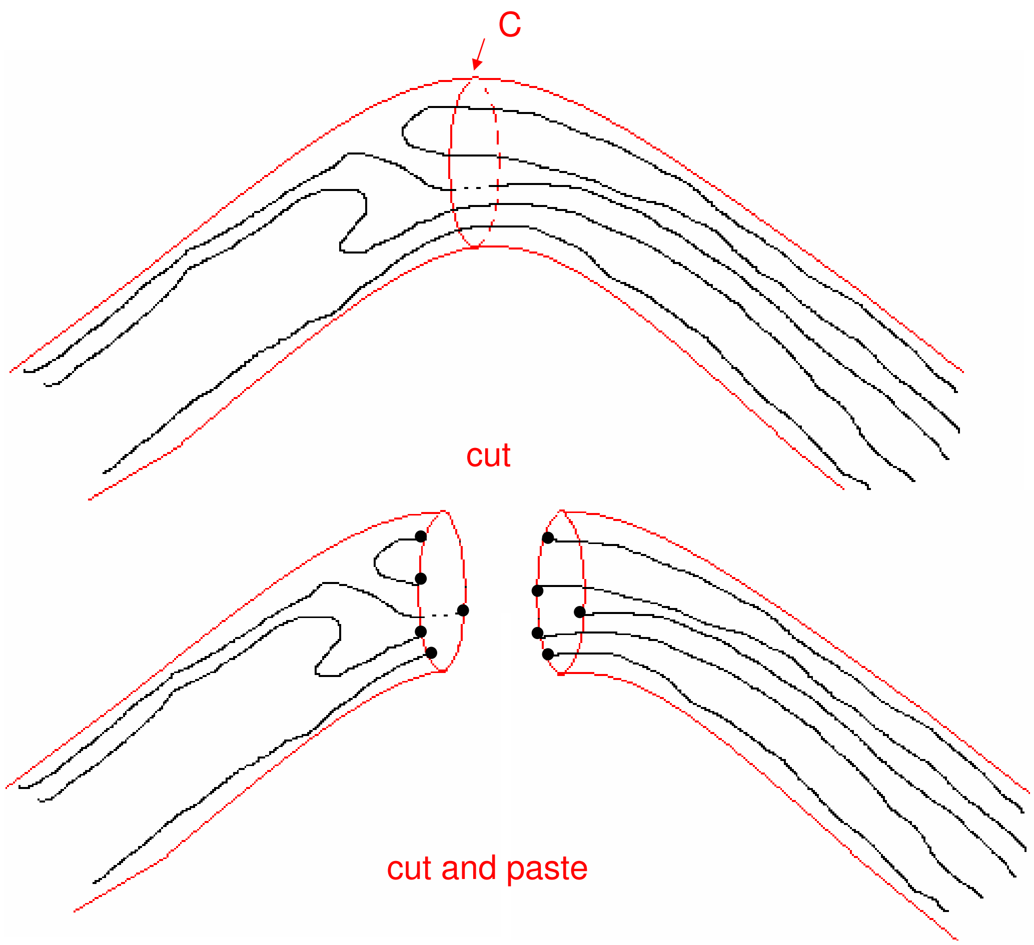
Definition 5.3.1. The orientable genus of a graph G, $\gamma(G)$, is the minimum genus of a surface in which G can be embedded. The non-orientable genus of a graph G, $\overline{\gamma}(G)$, is the minimum non-orientable genus of a surface (crosscaps) in which G can be embedded.

e.g. $\gamma(K_5) = \gamma(K_6) = \gamma(K_7) = 1$.
$\overline{\gamma}(K_5) = 1 = \overline{\gamma}(K_{3,3})$.



Theorem 5.3.1. (Euler-Poincaré) (for pseudo-graphs)
Let G be a (p, q)-graph which has a 2-cell embedding in an orientable surface of genus n. Then $p - q + r = 2 - 2n$ where r is the number of regions.

Proof. By induction on n and it is true for $n = 0$ which is a direct consequence of Euler's formula. Now, assume the assertion is true for the surface with genus less than n and G is a graph which has an orientable 2-cell embedding on S_n.
First, draw a cycle C along a handle which does not meet any vertex of $V(G)$, see figure below. Let C intersect m edges of G, see the following figures.



Moreover, let the total number of "intersections" be k. Now, by cutting off the handle from C and paste two regions to the plaves they form from cut we obtain a surface with one less handles. Also, let the points of intersection be new vertices, then the new graph has $p' = p + 2k$ vertices, $q' = q + 3k$ edges and $r' = r + k + 2$ regions. Since the new graph is embedded on a surface of genus $n-1$, $p' - q' + r' = 2 - 2(n-1)$. This implies that $(p + 2k) - (q + 3k) + (r + k + 2) = 2 - 2(n - 1)$ and thus $p_q + r = 2 - 2n$. $\square$

Observe that the above theorem does extend the Euler's formula when the graph is planar, in that case, $n = 0$. Moreover, we also have a version for non-orientable.

Theorem 5.3.2. Let the non-orientable genus of G be h. Then

$p-q+r=2-h$.

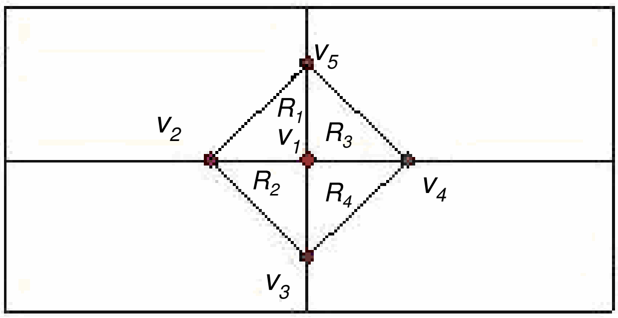
Proof. Omitted.

Theorem 5.3.3. (The Rotational Embedding Scheme)
Let G be a nontrivial connected graph with $V(G)=\{v_1, v_2, ..., v_p\}$. For each 2-cell embedding of G on a surface, there exists a unique p-tuple $(\pi_1, \pi_2, ..., \pi_p)$, where for $i=1,2,...,p$, $\pi_i : V(i) \longrightarrow V(i)$ is a cyclic permutation that describes the subscripts of the vertices adjacent to v_i in counterclockwise order about v_i. Conversely, for each such p-tuple $(\pi_1, \pi_2, ..., \pi_p)$, there exists a 2-cell embedding of G on some surface such that for $i=1,2,...,p$, the subscripts of the vertices adjacent to v_i and in counterclockwise order about v_i are given by π_i.

Key idea: $\pi((v_i, v_j)) = \pi(v_i, v_j) = (v_j, v_{\pi_j(i)})$.

Proof. Omitted.

e.g. (1) One embedding of K_5 on S_1.



$\pi_1 =(2\ 3\ 4\ 5)$	$R_1 = v_1 - v_2 - v_5 - v_1$
$\pi_2 =(3\ 1\ 5\ 4)$	$R_2 = v_1 - v_3 - v_2 - v_1$
$\pi_3 =(4\ 1\ 2\ 5)$	$R_3 = v_1 - v_4 - v_3 - v_1$
$\pi_4 =(3\ 2\ 5\ 1)$	$R_4 = v_1 - v_5 - v_4 - v_1$
$\pi_5 =(1\ 4\ 3\ 2)$	$R_5 = v_2 - v_3 - v_5 - v_2 - v_4 - v_5 - v_3 - v_4 - v_2$

e.g. (2) One embedding of K_5 on S_2.

$\pi_1 =$(3 2 4 5)
$\pi_2 =$(3 1 5 4)
$\pi_3 =$(4 1 2 5)
$\pi_4 =$(3 2 5 1)
$\pi_5 =$(1 4 3 2)
$R_1 = v_1 - v_2 - v_5 - v_1 - v_3 - v_2 - v_1 - v_4 - v_3 - v_1 - v_2$
$R_2 = v_1 - v_5 - v_4 - v_1 - v_5$
$R_3 = v_2 - v_3 - v_5 - v_2 - v_4 - v_5 - v_3 - v_4 - v_2 - v_3$

e.g. (3) One embedding of $K_{10,24}$ on S_{24}.

Let $A = \{v_1, v_3, ..., v_{19}\}$ and $B = \{v_2, v_4, ..., v_{28}\}$.
Let $\pi_1 = \pi_5 = \pi_9 = \pi_{13} = \pi_{17} =_{def} (2, 4, 6, ..., 28)$ (cycle)
$\pi_3 = \pi_7 = \pi_{11} = \pi_{15} = \pi_{19} =_{def} (28, 26, 24, ..., 2)$
$\pi_2 = \pi_6 = ... = \pi_{26} =_{def} (1, 3, 5, ..., 19)$
$\pi_4 = \pi_8 = ... = \pi_{28} =_{def} (19, 17, 15, ..., 1)$.
$\Longrightarrow$ Every region has 4 sides! Let's see an example.
$\pi(v_7, v_{12}) = (v_{12}, v_{\pi_{12}(7)}) = (v_{12}, v_5)$
$\pi(v_7, v_{12}) = v_7 - v_{12} - v_5 - v_{14} - v_7 - v_{12} = (v_{12}, v_5)$
$\pi(v_{12}, v_5) = (v_5, v_{\pi_5(12)}) = (v_5, v_{14})$
$\pi(v_5, v_{14}) = (v_{14}, v_{\pi_{14}(5)}) = (v_{14}, v_7)$
$\pi(v_{14}, v_7) = (v_7, v_{\pi_7(14)}) = (v_7, v_{12})$

We give the following result without a proof.

Theorem 5.3.4. (Interpolation Theorem, Duke [1966])
If there exist 2-cell embeddings of a connected graph G on surfaces S_m and S_n where $m < n$ and k is any integer $m \leq k \leq n$, then there exists a 2-cell embedding of G on S_k.

So, it is interesting to know the maximum n such that G has a

2-cell embedding in S_n. Clearly, the largest possible value occurs when the embedding has either one region or two regions depending on the parity of $p-q$ or equivalently $q-p+1$, this is also known as the Betti number of a $(p,q)-$graph.

We remark here that "Interpolation Theorem" also holds for non-orientable 2-cell embeddings.

Definition 5.3.2. (Maximum Genus)
The maximum genus $\gamma_{_M}(G)$ of G is the maximum among the genera of all surfaces on which G can be 2-cell embedded.

• There are $\prod_{i=1}^{p}(deg_{_G}(v_i)-1)!$ p-tuples of a graph G.
$\Rightarrow \gamma_{_M}(G)$ exists !
$\Rightarrow$ A connected graph G has a 2-cell embedding on S_k if and only if $\gamma(G) \leq K \leq \gamma_{_M}(G)$.
• Finding $\gamma(G)$ for a graph is very difficult, but finding $\gamma_{_M}(G)$ is comparatively easier.

Definition 5.3.3. (Betti Number of a Graph)
Let $c(G) = k$ denote the number of components in G. Then, the Betti number is defined to be $\beta(G) = q - p + k$. Thus, if G is connected, then $\beta(G) = q - p + 1$.

• A tree has Betti number "0". So, if G is connected, $\beta(G)$ "measures" how far is G from a tree.

Theorem 5.3.5. If G is a connected graph, then $\gamma_{_M}(G) \leq \lfloor \frac{\beta(G)}{2} \rfloor$. Furthermore, equality holds if and only if there exists a 2-cell embedding of G with $1+\delta_{_{\beta(G)}}$ regions where $\delta_{_{\beta(G)}} = \begin{cases} 0 & \text{if } \beta(G) \text{ is even ,} \\ 1 & \text{if } \beta(G) \text{ is odd.} \end{cases}$
Proof. By Theorem 5.3.1., $p - q + r = 2 - 2\gamma_{_M}(G)$.
$2\gamma_{_M}(G) + r - 1 = q - p + 1 = \beta(G)$

$\gamma_M(G) = \frac{\beta(G)+1-r}{2} \leq \frac{\beta(G)}{2}$
$\Rightarrow \gamma_M(G) \leq \lfloor \frac{\beta(G)}{2} \rfloor$.
$\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$ if and only if $\begin{cases} r = 1 & \text{if } \beta(G) \text{ is even ,} \\ r = 2 & \text{if } \beta(G) \text{ is odd.} \end{cases}$ □

Definition 5.3.4. A (connected) graph G is upper embeddable if $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$.

Definition 5.3.5. A spanning tree of a connected graph G is a splitting tree of G if at most one component of $G - E(T)$ has odd "**size**"(邊數).

Theorem 5.3.6. Let T be a splitting tree of a (p, q)-graph G. Then every component of $G - E(T)$ has even size if and only if $\beta(G)$ is even.

Proof. ($\Rightarrow$) $G - E(T)$ is of size $q - p + 1 = \beta(G)$.
($\Leftarrow$) Since at most one component of $G - E(T)$ is of odd size, the possible one must be even too. □

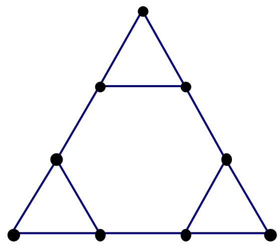
Theorem 5.3.7. (Jungerman, Xuong, [1978,1979])
A graph G is upper embeddable if and only if G has a splitting tree.
Proof. A direct consequence of Theorem 5.3.9. □

Corollary 5.3.8. Let G be a graph which contains two edge-disjoint spanning trees. Then G is upper embeddable.

Problem Which graph contains two edge-disjoint trees? 2 edge-connected? 3-connected? 4-edge-connected?

- $\gamma_M(K_p) = \lfloor \frac{(p-1)(p-2)}{4} \rfloor$.
- $\gamma_M(K_{m,n}) = \lfloor \frac{(m-1)(n-1)}{2} \rfloor$.
- $\gamma_M(Q_n) = (n-2)2^{n-2}$.
- $\gamma_M(K_{m(n)}) =?$.

e.g. This graph is not upper emmeddable! (Contains no splitting trees!)



How about the maximum genus of a graph G which is not upper embeddable.

Definition 5.3.6. For a graph H, defined $\xi_0(H)$ as the number of odd components in H. Define $\xi(G) = \min \xi_0(G - E(T))$.

Theorem 5.3.9. (Xuong,[1979])
$\gamma_M(G) = \frac{1}{2}(\beta(G) - \xi(G))$.
We shall prove this theorem later.

Theorem 5.3.10. (Neboskŷ)
Let $c(H)$ be the number of components in H and $b(H)$ be the number of components with odd Betti number.
Then $\xi(G) = \max \{v(G, A) = c(G-A)+b(G-A)-|A|-1 : A \subseteq E(G)\}$.
Proof. Omitted.

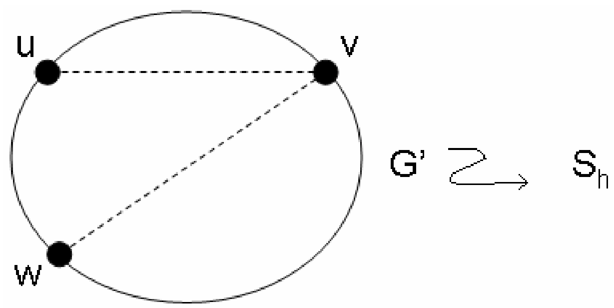
Theorem 5.3.11. (Ringeisen-White Edge-Adding Lemma)
Let G be a connected graph with $V(G) = \{v_1, v_2, ..., v_p\}$ s.t. $v_i \nsim_G v_j$. Suppose there exists a 2-cell embedding of G on an S_h with r regions s.t. $v_i \in bdd(R_i)$ and $v_j \in bdd(R_j)$. Let $H = G + v_i v_j$. Then
(a). if $R_i \neq R_j$, then there exists a 2-cell embedding of H on S_{h+1} with $r-1$ regions in which v_i and v_j are on the boundary of the same region, while
(b). if $R_i = R_j$, then there exists a 2-cell embedding of H on S_h with $r+1$ regions in which v_i and v_j belongs to the boundaries of (the same)

two distinct regions.

Proof. By direct checking. □

Corollary 5.3.12. Let e and f be adjacent edges of a connected graph G. If there exists a 2-cell embedding of $G' = G - e - f$ with one region, then there exists a 2-cell embedding of G with one region.

Proof. Let $e = uv$ and $f = vw$. Consider $G' + uv$. By (b), u and v are in two regions of S_h. By (a), $G' + uv + vw$ lies in one region of S_{h+1}.



□

Proof of Xuong's Theorem (Theorem 5.3.9.)

Proof. Let G be a (p, q)-graph which is embedded on $S_{\gamma_M(G)}$ with r regions.

Claim 1. $r = 1 + \xi(G)$.

If $\xi(G) = 0$, then G has a splitting tree with no odd size components, hence $\beta(G)$ is even. This implies that $r = 1$ and $r = 1 + \xi(G)$. On the other hand, if $r = 1$, then $\beta(G)$ is even and G is upper embeddable, we have $\xi(G) = 0$, so $r = 1 + \xi(G)$ if $r = 1$. (If $r = 1$ or $\xi(G) = 0$, then done.) Assume $r \geq 2$ and $\xi(G) > 0$.

Let T_1 be a spanning tree of G s.t. $\xi_0(G - E(T_1)) = \xi(G)$. Moreover, let $G_1, G_2, ...G_{\xi(G)}$ be the components of odd size in $G - E(T_1)$. For $i = 1, 2, ..., \xi(G)$, let $e_i \in (G_i)$ where e_i is a leaf if G_i is a tree and e_i is a cycle-edge if G_i is not a tree. Now, define $H = G - \{e_1, e_2, ..., e_{\xi(G)}\}$. Clearly, H is connected and T_1 is a splitting tree of H, moreover $\beta(H)$

is even. Hence, H can be embedded on $S_{\gamma_M(H)}$ with one region. Adding the edges $e_1, e_2, ..., e_{\xi(G)}$ back to H to obtain G. So, there exists a 2-cell embedding of G on some surofaces with at most $1+\xi(G)$ regions. Since $G \hookrightarrow S_{\gamma_M(G)}$ produces r regions, $r \leq 1 + \xi(G)$.
Now, assume that G is 2-cell embedded on $S_{\gamma_M(G)}$ with $r(\geq 2)$ regions. Let f_1 be an edge belonging to the boundary of two regions of G. Then $G - f_1 \hookrightarrow S_{\gamma_M(G)}$ with $r - 1$ regions. If $r - 1 \geq 2$, continuing the above process to obtain a graph $G' = G - \{f_1, f_2, ..., f_{r-1}\} \hookrightarrow S_{\gamma_M(G)}$ with one region. Therefore, G' has a splitting tree T'. This implies that $G' - E(T')$ contains only even size components and $\xi(G) \leq \xi_0(G - E(T')) \leq r - 1$. Thus, $r = 1 + \xi(G)$.
Now, by Euler-Poincaŕe's Theorem
$p - q + r = 2 - 2\gamma_M(G)$
$2\gamma_M(G) = q - p + 1 - \xi(G) = \beta(G) - \xi(G)$
$\gamma_M(G) = \frac{1}{2}(\beta(G) - \xi(G))$. □

Note. $\beta(G) - \xi(G)$ is always an even integer.

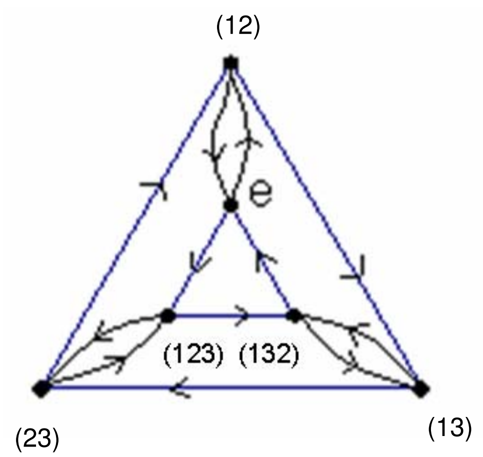
§ 5.4. Genus of Groups

We may use graph notion to give a picture of a group. It is known that a group G can be obtained by using generators, i.e., every element of G can be represented by a sequence (word) of $g_1, g_2, ..., g_1^{-1}, g_2^{-1}, ...$ where $g_1, g_2, ...$ are generators. For example, D_4 can be generated by $\{r = (13)$ and $s = (1234)\}$.

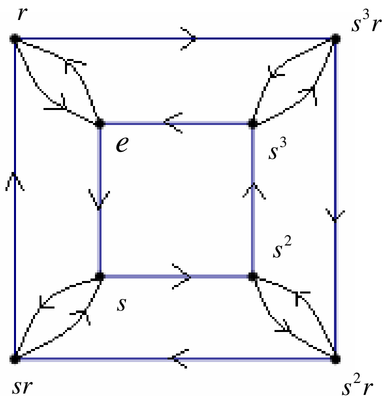
Definition 5.4.1. (Cayley Color Graph)
A Cayley color graph of a group "G" with generating set S :
(1) vertices are the elements of the group G, let $v_i \longleftrightarrow g_i \in G$.
(2) generators of G, i.e., elements of S are colors.
(3) (v_i, v_j) is an arc with color h if and only if $g_i h = g_j$.

For convenience, we use $C_S(G)$ to denote the Cayley color graph of G with generating set S. If $h^2 = e$ for each $h \in S$, then we have (v_i, v_j) and (v_j, v_i) at the same time. In this case, we have a Cayley graph $C_S(G)$ which is a simple graph by letting $(v_i, v_j) \cup (v_j, v_i) = v_i v_j$.

Examples:

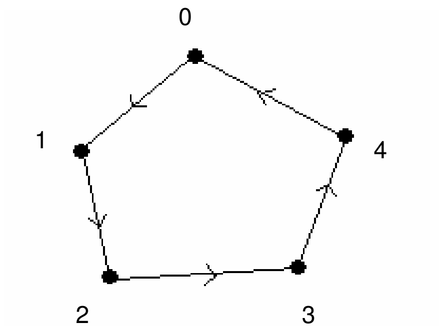


$G = S_3, \quad S = \{(12), (123)\}$



$r = (13)$
$s = (1234)$
$S = \{r, s\}$
$G = D_4$

Multiply from the right!



$G = \mathbb{Z}_5$
$S = \{1\}$

Definition 5.4.2. (Genus of a group)
Let G be a group with generating set S and $\gamma(C_S(G))$ denote the genus of the underlying graph of $C_S(G)$. Then the genus of G denoted by $\gamma(G) = \min\{\gamma(C_S(G)) \mid S \text{ is a generating set of } G\}$.
Definition 5.4.3. A group is said to be planar if $\gamma(G) = 0$.

Theorem 5.4.1. Let $\Gamma_n = \mathbb{Z}_2^n$. Then $\gamma(\Gamma_n) = 1 + 2^{n-3}(n-4)$, $n \geq 2$.
Proof. Omitted.

Theorem 5.4.2. $\gamma(S_n) \leq 1 + \frac{(n-2)!}{4}(n^2 - 5n + 2)$, $n \geq 2$.
Proof. Omitted.

Problem Find $\gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$.
(Note: So far, it is known that $5 \leq \gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) \leq 10$.)

§ 5.5. Selected Problem

Topological graph theory has been one of the most important topics in the study of graph structures. Starting from the 4-color problem of planar graphs, the progress of graph theory shows the importance of researches in this topic. Almost all beautiful problems or conjectures in graph theory are closely related to it. Clearly, we are not able to include all the problems here. Nevertheless, we shall list some of them in this section. For convenience, the begining of each problem is marked by "$\bullet$".

• Thickness Problem

The **thickness** of a graph G is defined as the minimum number of planar graphs in which G is decomposed, i.e., $\theta_1(G) = \min\{n|G$ can be decomposed into n planar graphs$\}$. Clearly, the problem intends to find $\theta_1(G)$ for any given graph G. Some of the known results are:
(1). If G is a (p, q)-graph, then $\theta_1(G) \geq \frac{q}{(3p-6)}$.
(2). $\theta_1(K_p) = \begin{cases} \lfloor \frac{p+7}{6} \rfloor & \text{if } p \neq 9, 10; \text{ and },\\ 3 & \text{if } p = 9, 10. \text{ (Beineke et al.)} \end{cases}$
(3). $\theta_1(Q_n) = \lfloor \frac{n+1}{4} \rfloor$.
(4). You may try to find $\theta_1(K_{m(n)})$.

• Crossing Number Problem (Review)

The **crossing number** of a graph G, $v(G)$, is defined to be the minimum number of crossings in all drawings of G on a plane or correspondingly a sphere. Clearly, we are interested in knowing $v(G)$ for all graphs of G. But, it turns out that this is also a very difficult problem. Not much is known in general, people are working on special graphs. Three of the most important ones are:
(1). Determine if $v(G) = \frac{1}{4}\lfloor \frac{p}{2} \rfloor \lfloor \frac{p-1}{2} \rfloor \lfloor \frac{p-2}{2} \rfloor \lfloor \frac{p-3}{2} \rfloor$ or not. (For $p \leq 10$, it

is true.)

(2). Determine if $v(K_{m,n}) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ or not. (Some results on small m and n have been obtained.)

(3). A good problem to try is finding $v(C_m \times C_n)$. (It was conjectured that $v(C_m \times C_n) = (m-2)n$ when $m \leq n$.)

- **Genus Problem** (Review)

The orientable (resp. non-orientable) genus of a graph G, $\gamma(G)$ (resp. $\overline{\gamma}(G)$), is defined as the minimum number of handles of an orientable (resp. non-orientable) surface in which G has an embedding on the surface. Clearly, if the surface is S_k (resp. N_h), then $\gamma(G) = k$ (resp. $\overline{\gamma}(G) = h$).

For example, $\gamma(K_5) = 1$ and $\overline{\gamma}(K_5) = 1$. One of the most celebrating theorems in graph genus in the following.

(1). $\gamma(K_p) = \lceil \frac{(p-3)(p-4)}{12} \rceil$, $p \geq 3$. (Ringel and Youngs)

The result on $K_{m,n}$ is also outstanding.

(2). $\gamma(K_{m,n}) = \lfloor \frac{(m-3)(m-4)}{4} \rfloor$, $m, n \geq 2$. (Ringel)

(Note: Rotational scheme can be applied to prove (2).)

(3). Good problem to try : Find $\gamma(Q_n)$ and $\overline{\gamma}(Q_n)$.

- **Chromatic number of a surface**

The chromatic number of a surface S_n, denoted by $\chi(S_n)$ is the maximum chromatic number among all graphs that can be embedded on S_n. So, the Four Color Theorem states that $\chi(S_0) = 4$. Heawood proved $\chi(S_1) = 7$.

Theorem (The Heawood Map Coloring Theorem)
For every positive integer n, $\chi(S_n) = \lfloor \frac{7+\sqrt{1+48n}}{2} \rfloor$.

Proof. By the fact that $\gamma(K_p) = \lceil \frac{(p-3)(p-4)}{12} \rceil$ and letting $p = \lfloor \frac{7+\sqrt{1+48n}}{2} \rfloor$, we conclude that $\chi(S_n) \geq \lfloor \frac{7+\sqrt{1+48n}}{2} \rfloor$. Then, the proof follows by showing $\chi(S_n) \leq \lfloor \frac{7+\sqrt{1+48n}}{2} \rfloor$. This result has been done by Heawood long time ago. We omit the detail here. $\square$

(1). Why $n = 0$ is not working?

(2). Can you prove the second inequality yourself?

• **Book Embedding Problem**

A page is a closed half-plane. A book is a collection of pages identified along the boundary of the half-planes. This common boundary is called the spine. A book embedding is a drawing such that all vertices lie on the spine and no edge contains a vertex on the spine other that its ends. The page number of a graph G, $pn(G)$, is the fewest number of pages in a book embedding of G.

(1). For each positive integer n, $pn(C_n) = 1$.

(2). $pn(K_{2,3}) = 2$. (?)

(3). Find $pn(K_{m,n})$.

To conclude this chapter, we make the following comments:

(1). Find the genus (oriedtable or non-orientable) of a graph is going to be very difficult, so far, no polynomial time algorithms have been found. On the other hand, there exists a polynomial time algorithm to find the maximum genus of a graph.

(2). Topological graph theory is an interesting topic but almost all problems posed are very difficult to solve in general. So, special graphs are the main ones which are of more attention.