Problem 2

Definition 1. A polynomial $f(x)$ is a **monic FSP polynomial** if $f(x)|f(x^2)$.

Definition 2. A monic FSP polynomial is **old** if it is a product of monic FSP polynomials of lower degrees; otherwise, it is **new**.

Proposition 1. *If $f(x)$ is a monic FSP polynomial of degree n, then*

$$\begin{aligned} f(x) = &(x-x_1)(x-x_1^2)(x-x_1^4)\dots(x-x_1^{2^{k_1}}) \\ &(x-x_2)(x-x_2^2)(x-x_2^4)\dots(x-x_2^{2^{k_2}}) \\ &(x-x_m)(x-x_m^2)(x-x_m^4)\dots(x-x_m^{2^{k_m}}) \end{aligned}$$

Where $\Sigma_{i=1}^{m}(k_i+1) = n$, $k_i \in \mathbb{Z}^+$.

Proof. Let f(x) be a monic FSP polynomial of degree n, $n \in \mathbb{Z}^+$. Then $f(x)|f(x^2)$, i.e. exists polynomial $g(x)$ such that $f(x)g(x) = f(x^2)$.

According to the fundamental theorem of algebra, there must exists a root x_1 such that $f(x_1) = 0$. Therefore, $f(x_1)g(x_1) = f(x_1^2) = 0$, so x_1^2 is also a root of $f(x)$.

And since $f(x_1^2)g(x_1^2) = f(x_1^{2^2}) = f(x_1^4) = 0,\ x_1^4$ is also a root of $f(x)$.

Using mathematical induction, if $f(x_1^{2^s}) = 0$, then $f(x_1^{2^s}) \cdot g(x_1^{2^s}) = f(x_1^{2^{s+1}}) = 0$, $s \in \mathbb{Z}^+$.

Since the truth of $f(x_1^{2^s}) = 0$ leads to the truth of $f(x_1^{2^{s+1}}) = 0$, we conclude that $f(x_1^{2^k}) = 0 \quad \forall\, k \in \mathbb{Z}^+$.

Since $f(x)$ is a polynomial of infinite degree, there must exist $t_i \leq k_i \leq n-1$, such that $f(x_i^{2^{k_i+1}}) = f(x_i^{2^{t_i}})$, $i = 1, 2, \dots, m$

Therefore, $f(x)$ can be expressed as:

$$\begin{aligned} f(x) = &(x-x_1)(x-x_1^2)(x-x_1^4)\dots(x-x_1^{2^{k_1}}) \\ &(x-x_2)(x-x_2^2)(x-x_2^4)\dots(x-x_2^{2^{k_2}}) \\ &(x-x_m)(x-x_m^2)(x-x_m^4)\dots(x-x_m^{2^{k_m}}) \end{aligned}$$

Where $\Sigma_{i=1}^{m}k_i + 1 = n$, $k_i \in \mathbb{Z}^+$, which proves the result. □

Now, using the result of Proposition 1, we solve the questions in specific cases:

(a) To show that x and $x-1$ are the only monic FSP polynomials when $n=1$, we know that $f(x)=(x-x_1)$.

Applying that $x_1^2=x_1$, we solve the equation and get $x_1=0$ or 1.

Therefore, x and $x-1$ are the only possibilities.

(b) When $n=2$, there are two cases

Case 1: $f(x)=(x-x_1)(x-x_2)$

Solve the system of equations $\begin{cases} x_1^2=x_1 \\ x_2^2=x_2 \end{cases}$, we get $\begin{cases} x_1=0 \text{ or } 1 \\ x_2=0 \text{ or } 1 \end{cases}$

Therefore, $x^2, x(x-1), (x-1)^2$ are all FSP polynomials of degree 2.

Case 2: $f(x)=(x-x_1)(x-x_1^2)$ We know that either $x_1^4=x_1$, or $x_1^4=x_1^2$.

(i) $x_1^4=x_1$.

$$x_1(x_1^3-1)=0$$
$$x_1(x_1-1)(x_1^2+x_1+1)=0$$

Hence, x^2+x+1 is also a monic FSP.

(ii) $x_1^4=x_1^2$

$$x_1^2(x_1^2-1)=0$$
$$x_1^2(x_1+1)(x_1-1)=0$$

Hence, $(x+1)(x-1)$ is also a monic FSP.

Therefore, all monic FSP polynomials of degree 2 include:

$$x^2$$
$$x(x-1)$$
$$(x-1)^2$$
$$x^2+x+1$$
$$(x+1)(x-1)$$

Among these, x^2+x+1 and $(x+1)(x-1)$ are new.

(c) When n = 3, there are three cases:

Case 1: $f(x) = (x - x_1)(x - x_2)(x - x_3)$

Solve the system of equations $\begin{cases} x_1^2 = x_1 \\ x_2^2 = x_2 \\ x_3^2 = x_2 \end{cases}$, we get $\begin{cases} x_1 = 0 \text{ or } 1 \\ x_2 = 0 \text{ or } 1 \\ x_3 = 0 \text{ or } 1 \end{cases}$

Therefore, $x^3, x(x-1)^2, x^2(x-1), (x-1)^3$ are all FSP polynomials of degree 3.

Case 2: $f(x) = (x - x_1)(x - x_1^2)(x - x_2)$

We know that either $x_1^4 = x_1$, or $x_1^4 = x_1^2$, which are the same cases as (b), and $x_2 = 0$ or 1.

Therefore, we know $(x-1)(x^2+x+1), x(x^2+x+1), x(x-1)(x+1), (x+1)(x-1)^2$, are all monic FSP polynomials.

Case 3: $f(x) = (x - x_1)(x - x_1^2)(x - x_1^4)$

In this case, we solve for the equations

$$\begin{gathered} x_1^8 = x_1 \\ x_1^8 = x_1^2 \\ x_1^8 = x_1^4, \end{gathered}$$

respectively. We get:

$$\begin{aligned} &x_1 = 0 \\ &x_1 = cis\frac{2k\pi}{7}, \quad k = 0,1,2\ldots,6 \\ &x_1 = cis\frac{k\pi}{3}, \quad k = 0,1,2\ldots,5 \\ &x_1 = cis\frac{k\pi}{2}, \quad k = 0,1,2,3 \end{aligned}$$

Therefore, summarizing all cases, old monic FSP polynomials of degree 3 include:

$$\begin{gathered} x^3, \\ x(x-1)^2, \\ x^2(x-1), \\ (x-1)^3 \end{gathered}$$

$$(x-1)(x^2+x+1),$$
$$x(x^2+x+1),$$
$$x(x-1)(x+1),$$
$$(x+1)(x-1)^2$$

Those new ones are of the form $(x-x_1)(x-x_1^2)(x-x_1^4)$, where

$$x_1 = cis\frac{2k\pi}{7}, \quad k = 0,1,2\ldots,6$$
$$x_1 = cis\frac{k\pi}{3}, \quad k = 0,1,2\ldots,5$$
$$x_1 = cis\frac{k\pi}{2}, \quad k = 0,1,2,3$$

When $n = 4$, there are five cases:

Case 1: $f(x) = (x-x_1)(x-x_2)(x-x_3)(x-x_4)$

Following the same pattern as the former cases, we get $x_i = 1$ or 0, $i = 1,2,3,4$.

Therefore, $x^4, x^3(x-1), x^2(x-1)^2, x(x-1)^3, (x-1)^4$ are all FSP polynomials of degree 4.

Case 2: $f(x) = (x-x_1)(x-x_1^2)(x-x_2)(x-x_3)$

Following the same pattern, we get $x_2, x_3 = 1$ or 0, and $x_1 = \pm 1$ or 0, or $x_1^2 + x_1 + 1 = 0$,

Therefore, $x^2(x^2+x+1), x(x-1)(x^2+x+1), (x-1)^2(x^2+x+1)$,
$x^2(x-1)(x+1), x(x+1)(x-1)^2, (x+1)(x-1)^3$, are all monic FSP polynomials of degree 4.

Case 3: $f(x) = (x-x_1)(x-x_1^2)(x-x_2)(x-x_2^2)$

Following the same pattern, we know that $x = \pm 1$ or 0, or $x^2+x+1 = 0$, for both x_1 and x_2.

Hence, $(x^2+x+1)^2, x^2(x+1)^2, x(x-1)(x+1)^2, (x-1)^2(x+1)^2$ are also monic FSP polynomials of degree 4.

Case 4: $f(x) = (x-x_1)(x-x_1^2)(x-x_1^4)(x-x_2)$

Similarly, we know that for $(x-x_1)(x-x_1^2)(x-x_1^4)$,

$$
\begin{aligned}
&x_1 = 0 \\
&x_1 = cis\frac{2k\pi}{7}, \quad k = 0,1,2\ldots,6 \\
&x_1 = cis\frac{k\pi}{3}, \quad k = 0,1,2\ldots,5 \\
&x_1 = cis\frac{k\pi}{2}, \quad k = 0,1,2,3
\end{aligned}
$$

and $x_2 = 0$ or 1.

Case 5: $f(x) = (x-x_1)(x-x_1^2)(x-x_1^4)(x-x_1^8)$

In this case, we solve for the equations

$$
\begin{aligned}
x_1^{16} &= x_1 \\
x_1^{16} &= x_1^2 \\
x_1^{16} &= x_1^4 \\
x_1^{16} &= x_1^8
\end{aligned}
$$

respectively. We get:

$$
\begin{aligned}
&x_1 = 0 \\
&x_1 = cis\frac{2k\pi}{15}, \quad k = 0,1,2\ldots,14 \\
&x_1 = cis\frac{k\pi}{7}, \quad k = 0,1,2\ldots,13 \\
&x_1 = cis\frac{k\pi}{6}, \quad k = 0,1,2,11 \\
&x_1 = cis\frac{k\pi}{4}, \quad k = 0,1,2,7
\end{aligned}
$$

Thus, the list of old monic FSP polynomials of degree 4 includes:

$$
\begin{gathered}
x^4, \\
x^3(x-1), \\
x^2(x-1)^2, \\
x(x-1)^3, \\
(x-1)^4,
\end{gathered}
$$

$$x^2(x^2+x+1),$$
$$x(x-1)(x^2+x+1),$$
$$(x-1)^2(x^2+x+1),$$
$$x^2(x-1)(x+1),$$
$$x(x+1)(x-1)^2,$$
$$(x+1)(x-1)^3$$
$$(x^2+x+1)^2,$$
$$x^2(x+1)^2,$$
$$x(x-1)(x+1)^2,$$
$$(x-1)^2(x+1)^2$$

The list of new monic FSP polynomials of degree 4 includes:

Polynomials of the form $x(x-x_1)(x-x_1^2)(x-x_1^4)$ and $(x-1)(x-x_1)(x-x_1^2)(x-x_1^4)$, where

$$x_1 = cis\frac{2k\pi}{7}, \quad k = 0,1,2\ldots,6$$
$$x_1 = cis\frac{k\pi}{3}, \quad k = 0,1,2\ldots,5$$
$$x_1 = cis\frac{k\pi}{2}, \quad k = 0,1,2,3$$

As well as polynomials of the form $(x-x_1)(x-x_1^2)(x-x_1^4)(x-x_1^8)$, where

$$x_1 = cis\frac{2k\pi}{15}, \quad k = 0,1,2\ldots,14$$
$$x_1 = cis\frac{k\pi}{7}, \quad k = 0,1,2\ldots,13$$
$$x_1 = cis\frac{k\pi}{6}, \quad k = 0,1,2,11$$
$$x_1 = cis\frac{k\pi}{4}, \quad k = 0,1,2,7$$

(d) Monic FSP polynomials of degree 3 with integer coefficients are the same as the list of old polynomials of degree 3 in answer(c).

Monic FSP polynomials of degree 3 with complex nnumber coefficients are the same as the new polynomials of degree 3 in answer (c).

The list of monic FSP polynomials of degree 4 with integer coefficients are the same as those old ones of degree 4, as listed in answer(c).

The list of monic FSP polynomials of degree 4 with complex number coefficients are the same as those new ones of degree 4 in answer(c).

There are no example of monic FSP polynomials with real number coefficients that are not all integers. The reason is that all the roots of the polynomials are solutions to the equation $x^{2^n} = x^{2^m}, m, n \in \mathbb{Z}^+$.

That means, the possible values of x can either be 0, or roots of unity. Therefore, there can never be real yet non-integer coefficients.

In general, using this property, all functions with solutions as roots of unity are monic FSP polynomials. For example, $x^k - 1$ is a monic FSP polynomial $\forall k \in \mathbb{Z}^+$;

Moreover, all functions that are products of powers of $x, x-1, x+1$, x^2+x+1 are monic FSP polynomials. For example, x^k is a monic FSP polynomial $\forall k \in \mathbb{Z}^+$. This can also be justified by the fact that if $f(x)$ and $g(x)$ are monic FSP polynomials, there exist polynomials $h(x), l(x)$ such that

$$\begin{aligned} f(x)h(x) &= f(x^2) \\ g(x)l(x) &= g(x^2). \end{aligned}$$

Hence,

$$\begin{aligned} f(x^m)h(x^m) &= f(x^{2m}), \\ g(x^n)l(x^n) &= g(x^{2n}). \end{aligned}$$

Multiplying these two equations, we get

$$[f(x^m)g(x^n)][h(x^m)l(x^n)] = f(x^{2m})g(x^{2n})$$

Let $F(x) = f(x^m)g(x^n)$, $H(x) = h(x^m)l(x^n)$, The equation becomes

$$F(x)H(x) = F(x^2),$$

which again shows that $F(x) = f(x^m)g(x^n)$ is also a monic FSP polynomial, $\forall m, n \in \mathbb{Z}^+$.