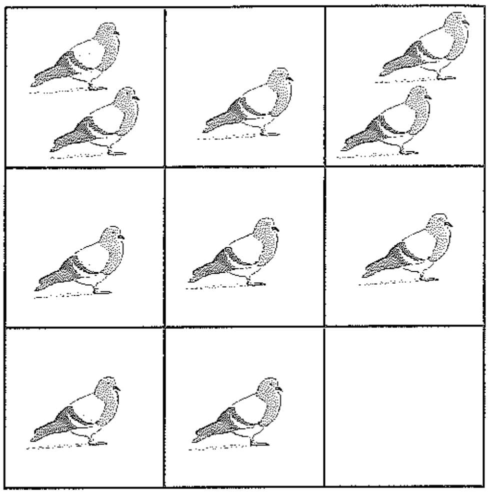
J THE PIGEONHOLE PRINCIPLE (DIRICHLET'S PRINCIPLE)

In the given picture there are 10 pigeons and 9 "pigeonholes".



In this case, two pigeonholes contain more than one pigeon.

Suppose the pigeons were taken out of their pigeonholes. Each pigeon is now systematically placed in a pigeonhole until they are all placed. Since the number of pigeons exceeds the number of pigeonholes, it is *guaranteed* that one of the holes will contain at least two birds.

Suppose now that 24 pigeons are placed into the 9 pigeonholes. They are placed systematically in separate pigeonholes, as far as is possible, until they are all placed.

Since $\frac{24}{9} \approx 2.33$, it is guaranteed that at least one pigeonhole will contain at least 3 pigeons.

The Pigeonhole Principle (PHP):

If n items are distributed amongst m pigeonholes with $n, m \in \mathbb{Z}^+$ and $n > m$, then:

(1) at least one pigeonhole will contain more than one item

(2) at least one pigeonhole will contain at least $\frac{n}{m}$ (or the smallest integer greater than $\frac{n}{m}$, if $\frac{n}{m}$ is not an integer) items.

Proof: (By contradiction)

(1) Suppose each pigeonhole contains 0 or 1 items.
$\therefore$ the total number of items in the pigeonholes is $\leqslant m < n$, a contradiction.
$\therefore$ at least one pigeonhole will contain more than one item.

(2) Suppose each pigeonhole contains less than $\frac{n}{m}$ items.

$\therefore$ the total number of items in the pigeonholes is $< m \times \frac{n}{m} = n$, a contradiction.

$\therefore$ at least one pigeonhole will contain at least $\frac{n}{m}$ (or the next integer greater than $\frac{n}{m}$, if $\frac{n}{m}$ is not an integer) items.

This simple and intuitive counting argument has many applications. Being able to determine when and how the pigeonhole principle can be applied is often the challenge.

Example 41

Consider a group of n people $(n > 1)$ meeting for the first time. Each person shakes hands with at least one other person. Prove there is a pair of people in the group who will shake hands the same number of times.

Each person shakes at least 1 and at most $n-1$ hands.

$\therefore$ each person shakes 1, 2, ..., or $(n-1)$ hands. Let these possibilities be the pigeonholes.

Since there are n people, and $n > n-1$, by the PHP at least two people are in the same pigeonhole.

$\therefore$ there is a pair of people in the group who shake hands the same number of times.

Example 42

Suppose five distinct points are arbitrarily drawn on the surface of a sphere.

Prove it is possible to cut the sphere in half so that four of the points will lie in one hemisphere. Assume that any point lying on the cut lies in both hemispheres.

Let O be the point at the centre of the sphere.

Any two of the points on the surface, together with O, define a plane which bisects the sphere.

The three remaining points lie in one or both of the two resulting hemispheres. Since $3 > 2$, by the PHP one hemisphere will contain at least two of these three remaining points. Together with the two original points chosen, this hemisphere contains at least four of the five original points.

Example 43

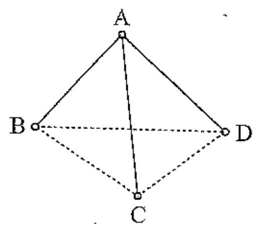
Six distinct points are arbitrarily drawn on a plane such that no three are collinear. Each pair of points is joined with a line segment called an *edge* which is coloured either red or blue.

Prove that in such a configuration it is always possible to find a triangle whose three edges have the same colour.

Choose one of the points and label it A.

Point A lies on five edges, each of which is either red or blue.

Since $\frac{5}{2} = 2.5$, by the PHP at least 3 edges through A will have the same colour. Call these edges AB, AC, and AD.



Consider the triangle formed by edges BD, BC, and CD.

If BD, BC, and CD all have the same colour, then $\triangle$BCD has all edges the same colour.

If BD, BC, and CD are not all the same colour, then both colours red and blue occur in $\triangle$BCD.

$\therefore$ one such edge will match the colour of edges AB, AC, and AD. Without loss of generality, we suppose this edge is BC.

$\therefore$ $\triangle$ABC is a triangle with all edges the same colour.

EXERCISE 1J

1. Show that in any group of 13 people there will be 2 or more people who are born in the same month.
2. Seven darts are thrown onto a circular dartboard of radius 10 cm. Assuming that all the darts land on the dartboard, show that there are two darts which are at most 10 cm apart.
3. 17 points are randomly placed in an equilateral triangle with side length 10 cm. Show that at least two of the points are at most 2.5 cm away from each other.
4. 10 children attended a party and each child received at least one of 50 party prizes. Show that there were at least two children who received the same number of prizes.
5. Show that if nine of the first twelve positive integers are selected at random, the selection contains at least three pairs whose sum is 13.
6. What is the minimum number of people needed to ensure that at least two of them have the same birthday (not including the year of birth)?
7. There are 8 black socks and 14 white socks in a drawer. Calculate the minimum number of socks needed to be selected from the drawer (without looking) to ensure that:
   a. a pair of the same colour is drawn
   b. two different coloured socks are drawn?
8. Prove that for every 27 word sequence in the US constitution, at least two words will start with the same letter.
9. The capacity of Wembley stadium in London is 90 000. Prove that in a full stadium there are at least 246 people with the same birthday (not including the year of birth).
10. Prove that if six distinct numbers from the integers 1 to 10 are chosen, then there will be two of them which sum to eleven.
11. Prove that if eleven integers are chosen at random, then at least two have the same units digit.
12. Prove that, at any cocktail party with two or more people, there must be at least two people who have the same number of acquaintances at the party.

    Hint: Consider the separate cases:

    (1) where everyone has at least one acquaintance at the party

    (2) where someone has no acquaintance at the party.
13. Draw a square of side length two units. Place five distinct points on the interior of the square. Prove that two of the points will be at most $\sqrt{2}$ units apart.

    Hint: Partition the square into four congruent squares.
14. There are 25 students in a class. Each student received a score of 7, 6, 5, or 4 for a test. What is the largest number of students which are guaranteed to have the same score?
15. Are there two powers of 2 which differ by a multiple of 2001?
16. A barrel contains 5 red, 8 blue, 10 green, and 7 yellow identically shaped balls. Balls are randomly selected one by one. Find the least number of balls which must be selected to guarantee choosing:
    a. at least 3 red balls
    b. at least 3 balls of the same colour
    c. at least 3 differently coloured balls.
17. Three dice are rolled and the sum total is recorded. Find the least number of rolls required to be guaranteed that a total will appear:
    a. twice
    b. three times.