FM2 Assignment #36 Name: Maggie

1. In a corral there are cowboys and an odd number of horses. There are 20 legs in all: how many belong to horses?

Let there be x cowboys and y horses, $x, y \in \mathbb{Z}^+$, y is odd.

$2x + 4y = 20 \iff x + 2y = 10$

$\iff x \equiv 10 \pmod 2 \implies x = 2k + 10,\ k \in \mathbb{Z}$.
$y = \frac{10 - x}{2} = -k$.

Since $x, y \in \mathbb{Z}^+$, $-5 < k < 0$, and k is odd.

$\implies y = 1$ or 3. $\implies$ there are 4 or 12 legs belonging to horses.

2. Prove that consecutive Fibonacci numbers are relatively prime.

Defined as: $u_n = u_{n-1} + u_{n-2}$, $u_0 = u_1 = 1$.

Suppose $\exists\ u_n$ s.t. $\gcd(u_n, u_{n-1}) = d \neq 1$.

then $d \mid (u_{n-1} + u_{n-2}) \implies d \mid u_{n-2}$.

$d \mid (u_{n-2} + u_{n-3}) \implies d \mid u_{n-3}$

...

$d \mid (u_2 + u_1) \implies d \mid u_1$

However, $u_1 = 1 \implies d = 1$, a contradiction.

Therefore, consentive Fibonacci #s are relatively prime. $\square$

3. Find three consecutive integers such that the first is divisible by a square, the second by a cube and the third by a fourth power.

Let the three integers be x, $x+1$, $x+2$.

$x \equiv 0 \pmod{a^2}$

$x \equiv -1 \pmod{b^3}$

$x \equiv -2 \pmod{c^4}$, $a, b, c \in \mathbb{Z}^+$.

Since if $\exists$ such a, b, c, any factor of a, b, c will also satisfy this system, respectively, we assume that a, b, c are all distinct primes. By Chinese remainder thrm,

$\exists!\ x$ for any value of such a, b, c.

If we take $a = 5$, $b = 3$, $c = 2$,

$x = -25 \cdot 16 \cdot 16 - 2 \cdot 25 \cdot 27 \cdot 11 \equiv 350 \implies \boxed{350, 351, 352}$

4. Let a and b be elements of a group G. We say a is a *conjugate* of b if $a = xbx^{-1}$ for some $x \in G$. Define the relation $\sim$ on G by $a \sim b$ if a is a conjugate of b. Prove that $\sim$ is an equivalence relation on G. What are the equivalence classes when G is Abelian?

① reflexive:

$$a = eae^{-1} \Rightarrow a \sim a.$$

② symmetric:

if $a = xbx^{-1} \Rightarrow x^{-1}a(x^{-1})^{-1} = b \Rightarrow b \sim a.$

③ transitive:

if $a = xbx^{-1}$, $b = ycy^{-1}$,

$\Rightarrow a = xycy^{-1}x^{-1} = (xy)c(xy)^{-1} \Rightarrow a \sim c.$

Therefore, $\sim$ is an equiv. relation.

If G is abelian, the equiv. classes are each element in G.

5. Use induction on n to show that the Fibonacci numbers satsify $f_{m+n} = f_{m-1} \cdot f_n + f_m \cdot f_{n+1}$, $\quad m \geq 1, n \geq 0$.

Base case: $m=1$

LHS $= f_{n+1} = f_n + f_{n-1}$

RHS $= f_0 \cdot f_n + f_1 \cdot f_{n+1} = f_{n+1}$. $\Rightarrow$ RHS $=$ LHS.

Induction case:

Suppose statement holds for $m \leq k$,

we'd like to show $f_{k+1+n} = f_k \cdot f_n + f_{k+1} \cdot f_{n+1}$.

$$\text{LHS} = f_{k+n} + f_{k+n-1} = f_{k-1} \cdot f_n + f_k f_{n+1} + f_{k-2} \cdot f_n + f_{k-1} \cdot f_{n+1}$$

$$= (f_{k-1} + f_{k-2}) \cdot f_n + (f_k + f_{k-1}) \cdot f_{n+1}$$

$$= f_k f_n + f_{k+1} f_{n+1} = \text{RHS}$$

Since the truth of the statement for all $m \leq k$ leads to the truth of the statement for $m = k+1$, by strong mathematical induction, the statement is true for all $m \geq 1$. $\square$