International Baccalaureate
Baccalauréat International
Bachillerato Internacional

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 14]*

(a) The relation R is defined on $\mathbb{Z}^+$ by aRb if and only if ab is even. Show that only one of the conditions for R to be an equivalence relation is satisfied. *[5 marks]*

(b) The relation S is defined on $\mathbb{Z}^+$ by aSb if and only if $a^2 \equiv b^2 \pmod{6}$.

(i) Show that S is an equivalence relation.

(ii) For each equivalence class, give the four smallest members. *[9 marks]*

2. *[Maximum mark: 13]*

The binary operations $\odot$ and $*$ are defined on $\mathbb{R}^+$ by

$$a \odot b = \sqrt{ab} \text{ and } a * b = a^2 b^2 .$$

Determine whether or not

(a) $\odot$ is commutative; *[2 marks]*

(b) $*$ is associative; *[4 marks]*

(c) $*$ is distributive over $\odot$; *[4 marks]*

(d) $\odot$ has an identity element. *[3 marks]*

3. *[Maximum mark: 16]*

The group $\{G, \times_7\}$ is defined on the set $\{1, 2, 3, 4, 5, 6\}$ where $\times_7$ denotes multiplication modulo 7.

(a) (i) Write down the Cayley table for $\{G, \times_7\}$.

(ii) Determine whether or not $\{G, \times_7\}$ is cyclic.

(iii) Find the subgroup of G of order 3, denoting it by H.

(iv) Identify the element of order 2 in G and find its coset with respect to H. *[10 marks]*

(b) The group $\{K, \circ\}$ is defined on the six permutations of the integers 1, 2, 3 and $\circ$ denotes composition of permutations.

(i) Show that $\{K, \circ\}$ is non-Abelian.

(ii) Giving a reason, state whether or not $\{G, \times_7\}$ and $\{K, \circ\}$ are isomorphic. *[6 marks]*

4. *[Maximum mark: 9]*

The groups $\{G, *\}$ and $\{H, \odot\}$ are defined by the following Cayley tables.

G

$*$	**E**	**A**	**B**	**C**
E	E	A	B	C
A	A	E	C	B
B	B	C	A	E
C	C	B	E	A

$f(A \cdot A) = f(E) = e$.
$f(A) \cdot f(A) = e$.
$f(B) f(B) = f(A) = e$.

H

$\odot$	**e**	**a**
e	e	a
a	a	e

By considering a suitable function from G to H, show that a surjective homomorphism exists between these two groups. State the kernel of this homomorphism.

5. *[Maximum mark: 8]*

Let $\{G, *\}$ be a finite group and let H be a non-empty subset of G. Prove that $\{H, *\}$ is a group if H is closed under $*$.

1. (a) ① if aRb, ab is even, $a, b \in \mathbb{Z}^+$.
the $ba = ab$ must also be even,
and thus bRa.
So R is symmetric. ✓

② if a is even, then aRa;
if a is odd, then $a\not{R}a$. ✓
So ~~nf~~ R is not reflexive.

③ if aRb. bRc. $c \in \mathbb{Z}^+$.
ab is even, bc is even. ✓
ac is not necessarily even.
e.g. $a=1, b=2, c=3$ is an counterexample.
So R is not transitive.

Therefore, only one condition (symmetric) is satisfied.

(b) (i) <u>proof</u>:

① $a^2 \equiv a^2 \pmod 6$ $\forall a \in \mathbb{Z}^+$.
So aSa, S is reflexive.

② if $a^2 \equiv b^2 \pmod 6$ $a, b \in \mathbb{Z}^+$.
then $6 \mid (a^2 - b^2)$ according to defn.
$\Rightarrow 6 \mid (b^2 - a^2)$, i.e. $b^2 \equiv a^2 \pmod 6$
So $aSb \Rightarrow bSa$. S is symmetric.

③ if $a^2 \equiv b^2 \pmod 6$ $b^2 \equiv c^2 \pmod 6$ $a.b.c \in \mathbb{Z}^+$.
$6 \mid (a^2 - b^2)$. $6 \mid (b^2 - c^2)$
$\therefore 6 \mid ((a^2-b^2)+(b^2-c^2))$

→ Continued.

$\therefore 6 \mid (a^2-c^2)$ which means $a^2 \equiv c^2 \pmod 6$

So S is also transitive.

Therefore, S is an equivalence relation. $\square$.

(ii)

$[6] = \{6, 12, 18, 24\}$.

$[1] = \{1, 5, 7, 11\}$.

$[2] = \{2, 4, 8, 10\}$.

$[3] = \{3, 9, 15, 21\}$.

2. (a) let $a, b \in \mathbb{R}^+$.

$a \odot b = \sqrt{ab} = \sqrt{ba} = b \odot a$.

so $\odot$ is commutative.

(b) let $a, b, c \in \mathbb{R}^+$.

$(a * b) * c = (a^2b^2) * c = (a^2b^2)^2 c^2$

while $a*(b*c) = a*(b^2c^2) = a^2(b^2c^2)^2$.

since $(a^2b^2)^2c^2 \neq a^2(b^2c^2)^2$.

$(a*b)*c \neq a*(b*c)$ and $*$ is not associative.

(c) $a*(b \odot c) = a*(\sqrt{bc}) = a^2(\sqrt{bc})^2 = a^2bc$.

and $a*b \odot a*c = a^2b^2 \odot a^2c^2$

$= \sqrt{a^2b^2a^2c^2} = a^2bc$

So $a*(b \odot c) = a*b \odot a*c$.

and thus $*$ is distributive over $\odot$.

(d) Suppose $\odot$ has an identity x,

$a \odot x = \sqrt{ax} = a.\ \forall a \in \mathbb{R}^+$.

$a^2 = ax. \Rightarrow x = a$. However, in this case

x can not be the identity as identity is one elmt.

→ Continued.

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Therefore, there is no identity elmt for ⊙.

3. (a) (i)

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

Excellent!!

(ii) G is cyclic. It can be generated by 3 or 5.

$\langle 3\rangle = \{3, 2, 6, 4, 5, 1\}$; $\langle 5\rangle = \{5, 4, 6, 2, 3, 1\}$.

(iii) The subset $\{1, 2, 4\}$ is a subgroup as

	1	2	4
1	1	2	4
2	2	4	1
4	4	1	2

so $H = \{1, 2, 4\}$.

(iv) The elmt of order 2 is 6.

the coset of 6 w.r.t. H is

$6H = H6 = \{6, 5, 3\}$.

(b) (i) We claim that K is not Abelian.

Consider $(12)(123)$ and $(123)(12)$.

$(12)(123) = (23)$.

$(123)(12) = (13)$

Since $(23) \neq (13)$, $(12)(123) \neq (123)(12)$

which is an counterexample and shows that K is not Abelian.

(ii) Since G is cyclic from (a)(ii).
and every cyclic group is Abelian.
So G is Abelian.
yet K is not Abelian.
So there cannot be an isomorphism between G and K. and thus they are not isomorphic.

4. Define fcn. $f: G \to H$ by

$$\begin{cases} f(E) = f(A) = e. \\ f(B) = f(C) = a. \end{cases}$$

By checking the two cayley table, we can establish a corresponden btw G and H.
and $f(xy) = f(x)f(y) \quad \forall . x, y \in G$.
Also, since f exhausted all elmts in H (both a & e)
f is a surjection.
and $\ker(f) = \{x \mid f(x) = e, x \in G\}$.
$= \{E, A\}$.

5. proof. We use the 3-step subgroup test to show $H \leq G$.
① closure is assumed
② since H is non-empty . $\exists a \in H$.
Consider the sequence $a, a^2, a^3, a^4 \ldots$
Since G is finite and $H \subseteq G$, H is finite.
and thus $\exists\, i, j > 0$ s.t. $a^i = a^j$.
and since $a^i \in G$, $(a^i)^{-1}$ exists.
WLOG, assume $j > i$.
premultiply by $(a^i)^{-1}$. we get

→ Continued.

$a^{-i} \cdot a^{i} = a^{-i} a^{j}$

$e = a^{j-i}$. $j > i$. $j - i > 0$

So a^{j-i} must be in the sequence.

i.e. $a^{j-i} \in H$. and thus $e \in H$.

③ i> if $a = e$, then $a^{-1} = e$. and thus a^{-1} exists.

ii> if $a \neq e$. we claim that $a^{j-i-1} = a^{-1}$.

To show that,

since $a \neq e$, $j-i-1 > 0$.

$a^{j-i-1} \cdot a = a^{j-i-1+1}$

$= a^{j-i} = e$.

and since $j-i-1 > 0$, a^{j-i-1} is in the sequence.

i.e. $a^{j-i-1} \in H$.

and thus $a^{-1} \in H$.

Hence by the 3-step subgroup test, we prove that $H \leq G$. □.