Second lesson—Nim!

<u>Outline</u>

Objectives:

1. Learning the additions of games;
2. Applying the addition of games in Nim games;
3. Proving several theorems and construct an incomplete Nim table;
4. Learn MEX rule to construct the whole Nim table;
5. Using the Nim table to solve games.

Content:

2.1 What is the rule of a Nim game?
- Explain the rule;
- Play one round together.
- Exercise: prove that a Nim game is an impartial combinatorial game

2.2 Adding games
- Definition 2.1
- Addition of games and N/P positions
- Theorem 2.1 & 2.2

2.3 Nim table
- Theorem: 2.3
- Working out part of the Nim table;

2.4 Nim-Sum
- Definition 2.2
- Binary system
- Corollaries
- Theorem 2.4

2.5 MEX rule
- Definition 2.3
- Theorem 2.5
- Application of MEX rule to compute the whole Nim table.

2.6 Using the Nim table.
- Work out an exercise together hat uses Nim table to solve.

Exercise:

Focus on the practice of calculating Nim sums as well as applying MEX rules.

2.1 What is the rule of a Nim game?

Now it is time to introduce our new game called Nim game.
There are three piles of M&Ms. Two players take turns to select one of the piles and eat any number of M&Ms (at least one) in it. The one who eat the last M&M wins. You can practice playing this game on the web at Nim Game—http://www.dotsphinx.com/nim/.

It is easy yet boring when there is only one pile of M&Ms, and you can easily know that for a pile of size X:
It is an N-position if $X > 0$. (the player can simply take all the M&Ms!)
It is a P-position if $X=0$.

However, when 3 piles are played together, it is much more complicated. We will discuss that in Section 2.3.

2.2 Adding games together

Before we can add the three piles of Nim game together, we may first wonder: what does it mean by adding games together? And how can we do that?

Definition 2.1 Given two impartial combinatorial games, G_1 and G_2, we define their sum, $G_1 + G_2$, to be an impartial game that:
(1) For a position g_1 of the game G_1 and a position g_2 of the game G_2. We call it $g_1 + g_2$;
(2) A valid move from $g_1 + g_2$ is either a valid move in G_1 from g_1 to some other position g_1' or a valid move in G_2 from g_2 to some other position g_2' . i.e. we can only move to $g_1' + g_2$ or $g_1 + g_2'$;
(3) A player loses the game $G_1 + G_2$ if it is their turn and they are unable to move in either game. (Crash Course 6)

For example, suppose G_1 is a take-away game and G_2 is an Empty-and-Divid game. The initial position G_1+G_2 is 9 (with valid moves of 1, 2, 4) + (3, 3).
Now we need to first choose a game to play: if we start with G1, the next position can either be 8 + (3, 3), 7 + (3, 3), or 5 + (3, 3); if we choose G2, the next position can only be 9 + (1, 2).

Continuing is this way, the next player select one of the games to play, until the game reaches the terminal position 0 + (1, 1).

Some Notes:

1. Addition of games is **commutative**. i.e. G1+G2 is equivalent to G2+G1;
2. Addition of games is **associative**. i.e. (G1+G2)+G3 is equivalent to G1+ (G2+G3);

How do we know N/P positions of the games adding together?
There are two useful theorems (Crash Course 7):
Theorem 2.1 For any position g in an impartial game, g + g is always a P-position.
Theorem 2.2 If g is a P-position, then for any other position h in an impartial game, h + g has the same N/P position as h. i.e. N + P = N, P + P = P.

The proofs of these two theorems do not need much advanced math knowledge, but they does require some mathematical ways of thinking. So think it about that before moving on the see the proofs below.

Proof of Theorem 2.1:
Given two identical impartial games with same position g, a move in one game can always be copied by the next game. i.e. if the previous player moves to g′ + g, (it does not matter which one is chosen because the two games are identical), the next player can always find a move by moving to g′ + g′. Thus, the Previous player will eventually run out of move and loses. //

Proof of Theorem 2.2:
Case one: h is a P-position.
Then for both g and h, the second player has a winning strategy for both of the games individually. Now whatever game the first player move, either g′ or h′, the second player can find a winning strategy in the same game. Thus, the first player will finally run out of move and loses;
Case two: h is a N-position.
From the N-position, the first player can always move to a P-position, say h′(as proved in Lesson 1). So the game goes back to two P-position games g + h′ for the next player, which is the same as case one. Since case one is a P-position and the first player loses, the position that moves to such a P-position must be a N-position, in which the first player to move wins. //

2.3 Nim Table

Now that we know some basic rules of adding games now, let's look at 3-pile Nim game. In order to play it in a simpler way, we must need some theorems to simplify the problem.

We express the sum of three piles X, Y, Z by $X \oplus Y \oplus Z$. Still, order does not matter.

Theorem 2.3. For any two numbers X and Y , there exist at most one Z such that $X \oplus Y \oplus Z$ is a P-position. (Crash Course 9)

Proof by contradiction:

Suppose $X \oplus Y \oplus Z_1$ and $X \oplus Y \oplus Z_2$ are both P-positions with $Z_1 > Z_2$. Now the player can move from the Z_1 to Z_2 by taking $Z_1 - Z_2$ stones from Z_1. i.e. the player can move from a P-position to another P-position, which is impossible in am impartial game, contradiction.

Thus, the pile of Z, if exist, is unique. //

Because of the uniqueness of Z, given two numbers X and Y, we can always find a Z that makes $X \oplus Y \oplus Z$ a P-position. We can thus construct a table with Row x and Column Y.

	0	1	2	3	4	5	6	7	...
0	0	1	2	3	4	5	6	7	...
1	1	0							
2	2		0						
3	3			0					
4	4				0				
5	5					0			
6	6						0		
7	7							0	
...	...								...

Since we already know that $X \oplus X$ is a P-position, according to theorem 2.1, $X \oplus X \oplus 0$ must be a P-position, we get the incomplete Nim table as above.

Now in order to fill out the entire table, we may try to find the value of Z when $1 \oplus 2 \oplus Z$ is a P-position.

First, we know that Z cannot be either 0 or 1 or 2, according to the rule that X ⊕ X ⊕ 0 must is a P-position and Z is unique.
Thus, we may conjecture that Z = 3. You may want to try to write out all the possibilities and to see if this is true.

Indeed, we find that no matter what the first player do, he/she always loses. So we can verify that 1 ⊕ 2 ⊕ Z is a P-position.

However, as we can see, calculating the Nim table one by one is time-consuming, and it becomes even more difficult when we go to larger numbers. Thus, we need another technique to help us construct the Nim table, which is what we are going to discuss in the next section.

2.4 Nim-Sum

Definition 2.2 The Nim-sum of $(x_m x_{m-1} \cdots x_0)_2$ and $(y_m y_{m-1} \cdots y_0)_2$ is $(z_m z_{m-1} \cdots z_0)_2$, and we write $(x_m x_{m-1} \cdots x_0)_2 \oplus (y_m y_{m-1} \cdots y_0)_2 = (z_m z_{m-1} \cdots z_0)_2$, where for all k, $z_k = x_k + y_k \pmod 2$, i.e. $z_k = 1$ if and only if $x_k + y_k = 1$, otherwise, $z_k = 0$. (Ferguson 9)

Don't panic if you don't know what's going on here! Let's look at these notations one by one.
First, the $()_2$ is the binary representation. Every non-negative integer x can be expressed in the binary system in the form $x = x_m 2^m + x_{m-1} 2^{m-1} + \cdots + x_1 2 + x_0$ for some m, where each x_i is either zero or one. We use the notation $(x_m x_{m-1} \cdots x_1 x_0)_2$ to denote this representation of x to the base two.
For example, $27 = 16 + 8 + 2 + 1 = 1 * 2^4 + 1 * 2^3 + 0 * 2^2 + 1 * 2^1 + 1 * 2^0$, so m=4, $x_4 = 1$, $x_3=1$, $x_2=0$, $x_1=1$, $x_0=1$. Thus, $27 = (11011)_2$.

Then, to calculate the Nim-sum of two integers, we first express the integers in binary system and use addition modulo 2 for every digit.
That is, for every digit,

$$0 + 0 = 0,$$
$$0 + 1 = 1,$$
$$1 + 0 = 1,$$
$$1 + 1 = 0.$$

For example, to compute $27 \oplus 13$, we first express them by

$$27 = (11011)_2$$

$$13 = (1101)_2$$

Since $(11011)_2 \oplus (1101)_2 = (10110)_2$,

we know that $27 \oplus 13 = 21$.

Below are some corollaries that may be useful:

Corollary 2.1 (Ferguson 10)

0 is an identity for addition, i.e. $0 \oplus x = x$;

Corollary 2.2 (Ferguson 10)

Every number is its own negative, i.e. $x \oplus x = 0$

Proof:

For every digit, either 1 or 0 in x, $0 + 0 = 0$, $1 + 1 = 0$.

Thus, every digit in x always adds up to zero, i.e. $x \oplus x = 0$. //

Corollary 2.3 (Ferguson 10)

$x \oplus y = x \oplus z$ implies $y=z$.

Proof:

For impartial games x, y and z,

If $x \oplus y = x \oplus z$,

then $x \oplus x \oplus y = x \oplus x \oplus z$.

Since $x \oplus x = 0$, $y=z$. //

Corollary 2.4

$X \oplus Y = Z$ if and only if $X \oplus Y \oplus Z = 0$

Proof:

If $X \oplus Y = Z$,

$X \oplus Y \oplus Z = Z \oplus Z = 0$

If $X \oplus Y \oplus Z = 0$,

$X \oplus Y \oplus Z = 0 = Z \oplus Z$,

$X \oplus Y = Z$. //

OK. Now we know how to calculate Nim-sums. But what has it to do with playing Nim games? The theorem below gives us the answer.

Theorem 2.4

A position, (X, Y, Z), in Nim is a P-position if and only if the Nim sum of its components is zero, i.e. $X \oplus Y \oplus Z = 0$. (Crash Course 14)

Proof by strong induction

Let P(n): be the sum of $X + Y + Z$, where n is a nonnegative integer.

First, when $n = 0$, $X = Y = Z = 0$.

(0, 0, 0) is the terminal position, and thus a P-position, so P(0) is true.

Now, we want to show that if P(0), P(1), P(2), …, P(k) are true, then P(k+1) is also true.

Assume P(0), P(1), P(2), …, P(k) are true.

Suppose first that $X \oplus Y \oplus Z = 0$. We need to show that in this case, (X, Y, Z) is a P-position.

A move from this position involves reducing the value of one of the numbers – say going from X to X', with $X' < X$. But any change will make $X' \oplus Y \oplus Z$ not equal to 0, according to Theorem 2.3.

Since $X' + Y + Z < k$, we can apply the inductive hypothesis to conclude that $X' \oplus Y \oplus Z$ is an N-position. This shows that all moves from $X \oplus Y \oplus Z$ are to N-positions, so $X \oplus Y \oplus Z$ itself is a P-position.

Now suppose $X \oplus Y \oplus Z$ do not equal 0. We want to show that this is an N-position by finding a move to a P-position.

Denote the binary digits of X by $X_1, X_2, \ldots X_n$. (Thus X_n is the 1's digit, X_{n-1} is the 2's digit, etc.) Find the smallest m such that $X_m+Y_m+Z_m = 1 \pmod 2$, so at least one of X_m, Y_m, and Z_m is not 0. Since X, Y and Z are identical, we assume $X_m = 1$.

Let X' be the integer with binary expansion

$$X' = \begin{cases} X_i \text{, if } i < m \\ 0\text{, if } i=m \\ Y_i + Z_i \pmod 2\text{, if } i > m \end{cases}$$

By construction, $X' \oplus Y \oplus Z = 0$ and $X' < X$. By the inductive hypothesis, $X' \oplus Y \oplus Z$ is a P-position, so $X \oplus Y \oplus Z$ is an N-position.

Thus, the proposition is also true for P(k+1), and by strong Mathematical induction, the proposition is true for all positive integer n. //

By using both Corollary 2.4 and Theorem 2.4, we know that in order to know the N/P position of any given Nim game with position (X, Y, Z), we can simply get the answer by calculating the Nim sum of any of the two numbers.
So here is what we can get by calculating the Nim sum, and if this continues, we will be able to get the complete Nim table.

	0	1	10	11	100	101	110	111	...
0	0	1	10	11	100	101	110	111	...
1	1	0	11	10	101	100	111	110	...
10	10	11	0	1	110	111	100	101	...
11	11	10	1	0	111	110	101	100	...
100	100	101	110	111	0	1	10	11	...
101	101	100	111	110	1	0	11	10	...
110	110	111	100	101	10	11	0	1	...
111	111	110	101	100	11	10	1	0	...
...	...	...	...	...	...	...	...	...	...

By converting into decimal system, we get:

	0	1	2	3	4	5	6	7	...
0	0	1	2	3	4	5	6	7	...
1	1	0	3	2	5	4	7	6	...
2	2	3	0	1	6	7	4	5	...
3	3	2	1	0	7	6	5	4	...
4	4	5	6	7	0	1	2	3	...
5	5	4	7	6	1	0	3	2	...
6	6	7	4	5	2	3	0	1	...
7	7	6	5	4	3	2	1	0	...
...	...	...	...	...	...	...	...	...	...

Try to find some other interesting patterns of Nim table before moving on the the next section.

2.5 MEX rule

Now when we look at the Nim table again, we may find some other interesting patterns.

Definition 2.3
Let S be a set of nonnegative integers. MEX(S) is defined to be the smallest nonnegative integer that does not appear in S. (MEX stands for "minimal excluded".) (Crash Course 11)

For example,
MEX(1,2,5,4) = 0;
MEX(2,1,6,0,3,5,8) = 4;
MEX(0,87,23455,35)=1.

Theorem 2.5 — "the MEX rule". (Crash Course 11)
For all integers X, Y ≥ 0,

$$[X,Y] = MEX\ (\{[X', Y] : X' < X\} \cup \{ [X, Y'] : Y' < Y\})$$

i.e. [X,Y] is the smallest integer that does not appear directly above or to the left of its position in the table.
Proof:
Given X, Y ≥ 0, let $M = MEX\ (\{[X', Y] : X' < X\} \cup \{ [X, Y'] : Y' < Y\})$.
We want to show that (X, Y, M) is a P-position by showing that it can only move to N-positions.
Case one: X/Y is moved to X'/Y
By definition of MEX, M != [X',Y], (or M != [X, Y']).
Thus it is not a P-position, so it is an N-position.
Case two: M is moved to M'
By definition of MEX, $M' \in \{[X', Y] : X' < X\} \cup \{ [X, Y'] : Y' < Y\}$.
Then there must be some X' that M' = [X',Y].
Because (X', Y, M') is a P-position, (X, Y, M') must be a N-position.
Since all possible moves from (X, Y, M) are N-positions, (X, Y, M) is a P-position. //

2.6 Using the Nim table.

Now that we construct the whole Nim table, what can we do to play the Nim game? Here is an example:

Suppose we are playing a Nim game with three piles of M&Ms, each of size 5, 7, and 15. If I am generous enough to let you first choose if you want to go first or second, which one should you choose?

Let's work this out together.

First, according to the Nim table,

	0	1	2	3	4	5	6	7	…
0	0	1	2	3	4	5	6	7	…
1	1	0	3	2	5	4	7	6	…
2	2	3	0	1	6	7	4	5	…
3	3	2	1	0	7	6	5	4	…
4	4	5	6	7	0	1	2	3	…
5	5	4	7	6	1	0	3	**2**	…
6	6	7	4	5	2	3	0	1	…
7	7	6	5	4	3	2	1	0	…
…	…	…	…	…	…	…	…	…	…

We know that (5, 7, 2) is a P-position, and according to theorem 2.3, which states the uniqueness of each one of these three numbers, we know that (5, 7, 15) must be a N-position. (you can also work out the Nom-Sum if you are not convinced).

Thus, you must want to be the first player and start the game by moving this N-position to a P-position.

Since we know that (2, 5, 7) is a P-position, what you can do next is to eat 13 M&Ms from the 17 M&Ms pile, making it to be (2, 5, 7).

That's how you play a Nim game as a mathematician. :)

Do some exercise to make sure you have mastered this trick and go play with your friends (Don't tell them the strategy and impress them!!)

2.7 Exercises.

1. Calculate the Nim-sums below:

   (1) $2 \oplus 4 = ?$

   (2) $9 \oplus 8 = ?$

   (3) $100 \oplus 324 = ?$

2. Suppose you are playing a Nim game with three piles of M&Ms of size 2019, 215, and 9102, and still, I am so generous to let you choose whether you want to be the first or second, which one will you choose, and why?

3. Suppose you are playing a Nim game (again!) with three piles of M&Ms of size 9, 10 and 8, determine **all** possible move if you are the first player.

4. Now, still a Nim game, but you only know there is a nimheap of size 1, but you don't know the size of the other two piles yet. Without knowing the specific size of the other two piles, how can you win this game?

   * Not knowing specific size does not necessarily mean that you don't know anything about these two piles.

   * You may want to discuss this under different cases.

Answer key:

1. (1) $2 \oplus 4 = 6$

$$\begin{array}{rr} & 10\ (2) \\ + & 100\ (4) \\ \hline & 110\ (6) \end{array}$$

(2) $9 \oplus 8 = 1$

$$\begin{array}{rr} & 1001\ (9) \\ + & 1000\ (8) \\ \hline & 0001\ (1) \end{array}$$

(3) $100 \oplus 324 = 288$

$$\begin{array}{rr} & 1100100\ (100) \\ + & 101000100\ (324) \\ \hline & 100100000\ (288) \end{array}$$

2. If I were you, I would choose to go first. Here is why:
The Nim sum of 2019 and 215 is

$$\begin{array}{rr} & 11111100011\ (2019) \\ + & 11010111\ (215) \\ \hline & 11100110100\ (1844) \end{array}$$

Since $2019 \oplus 215 = 1844 \mathrel{!=} 9102$, (2019, 215, 9102) is a N-position.
In order to win, you should go first.

3. Before solving this game, we can fill in more of the Nim table below:

From the table, we can easily get that

$$9 \oplus 8 = 1;$$
$$9 \oplus 10 = 3;$$
$$8 \oplus 10 = 2.$$

Thus, we can either eat 9 M&Ms from 10 to make it 1, eat 5 M&Ms from 8 to make it 3, or eat 7 M&Ms from 9 to make it 2.

	0	1	2	3	4	5	6	7	8	9	10	...
0	0	1	2	3	4	5	6	7	8	9	10	...
1	1	0	3	2	5	4	7	6	9	8	11	...
2	2	3	0	1	6	7	4	5	10	11	8	...
3	3	2	1	0	7	6	5	4	11	10	9	...
4	4	5	6	7	0	1	2	3	12	13	14	...
5	5	4	7	6	1	0	3	2	13	12	15	...
6	6	7	4	5	2	3	0	1	14	15	12	...
7	7	6	5	4	3	2	1	0	15	14	13	...
8	8	9	10	11	12	13	14	15	0	**1**	2	...
9	9	8	11	10	13	12	15	14	1	0	**3**	...
10	10	11	8	9	14	15	12	13	**2**	3	0	...
...	...	...	...	...	...	...	...	...	...	...	...	...

4. Suppose we have the position of (X, Y, 1), where X and Y are nonnegative integers.

Case one: X is even, i.e the binary number of X ends in 0.
Then $X \oplus 1$ must end in 1, because 0+1=1.
Thus, $X \oplus 1 = X + 1$.
If $Y = X + 1$, then (X, Y, 1) is a P-position and you should go second; otherwise, it is a N-position and you should go first.

Case two: X is odd, i.e. the binary number of X ends in 1.
Then $X \oplus 1$ must end in 0, because $1 + 1 = 0$.
Thus, $X \oplus 1 = X - 1$.
If $Y = X - 1$, then (X, Y, 1) is a P-position and you should go second; otherwise, it is a N-position and you should go first.

Reference:

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