# Problem 1

**Definition 1.** Given an increasing sequence of positive integers $A = (a_n)$, a positive integer $m$ is **A-expressible** if it can be espressed as an alternating sum of a finite subsequence of A.

**Definition 2.** A sequence $A = (a_n)$ with increasing positive integers is called **alt-basis** if every positive integer is uniquely A-expressible.

**Lemma 1.** *Every positive integer can be uniquely expressed as a sum of distinct powers of 2.*

*Proof.* We use induction to prove this lemma.

**Base case:** $1 = 2^0$, which is the only possible sum of distinct powers of 2.

**Induction case:** Assume that every positive integer smaller than k can be expressed uniquely as sums of distinct powers of 2.
We want to show that this holds true for k also.

**Case 1:** $k$ is even.
Then $\frac{k}{2}$ is a number smaller than $k$, so exist distinct powers $p_1, p_2, \ldots p_n$, $0 \leq p_1 < \cdots < p_n$ and a unique sum of powers of 2:

$$\frac{k}{2} = 2^{p_1} + 2^{p_2} + \cdots + 2^{p_n}$$

Multiplying both sides by 2 gives

$$k = 2(2^{p_1} + 2^{p_2} + \cdots + 2^{p_n}) = 2^{p_1+1} + 2^{p_2+1} + \cdots + 2^{p_n+1},$$

which gives a unique sum of distinct powers of 2 for $k$.

**Case 2:** $k$ is odd.
Then $k - 1$ must be even and can be expressed uniquely as a sum of distinct powers of 2, with the least power at least 1:

$$k - 1 = 2^{p_1} + 2^{p_2} + \cdots + 2^{p_n}, \text{ where } 0 < p_1 < p_2 < \cdots < p_n.$$

$$\text{Thus, } k = 2^0 + 2^{p_1} + 2^{p_2} + \cdots + 2^{p_n},$$

which is a unique sum of distinct powers of 2.

Since the truth of the statement for all positive integers smaller than $k$ leads to the truth for $k$ in both cases, by the principle of mathematical induction, the statement holds true for all positive integers. □

**Proposition 1.** *Given a sequence $A = (a_n)$. Let $\Delta A$ be a sequence such that $\Delta A = (\Delta a_n) = (a_{n+1} - a_n)$. The sequence $A$ is an alt-basis only if*

$$\{\Delta a_n \mid n \in \mathbb{Z}^+\} \cup \{x \mid 2^x \in A, x \in \mathbb{N}\} = \{2^n \mid n \in \mathbb{N}\}.$$

*However, for any $x \in \mathbb{N}$, if $2^x \in A$, then $2^x \notin \Delta A$.*
*In other words, $A$ is an alt-basis if the joint of $\Delta A$ and $A$ must cover all the powers of 2 without repetition.*

*Proof.* We prove by the contrapositive, i.e. if

$$\{\Delta a_n \mid n \in \mathbb{Z}^+\} \cup \{x \mid 2^x \in A, x \in \mathbb{N}\} \neq \{2^n \mid n \in \mathbb{N}\},$$

$A$ is not an alt-basis. We proceed using mathematical induction.

**Base case:** $n = 0,\ 2^0 = 1$. Since 1 is a positive integer, it should be A-expressible. Therefore, either $1 \in A$ or $1 \in \Delta A$.

**Induction case:** Assume the truth of the statement for all non-negative integers smaller than $k$, i.e.

$$\{2^x \mid x \in \mathbb{N},\ x \leq k-1\} \subset \{\Delta a_n \mid n \in \mathbb{Z}^+\}$$

We want to show the statement holds for $k$.
First, we know that any positive number $m \leq 2^k - 1$ can be expressed using this subset, according to Lemma 1.

Therefore, if $m \in \{\Delta a_n \mid n \in \mathbb{Z}^+\}$, $m$ will not have an unique expression.
Hence, the next number $x$ in the set must satisfy that $x \geq 2^k$.

If $x \geq 2^k$, then $2^k$ cannot be expressed from any sum of $\Delta a_n$.

Hence, $x = 2^k$, and

$$\{2^x \mid x \in \mathbb{N},\ x \leq k\} \subset \{\Delta a_n \mid n \in \mathbb{Z}^+\}.$$

Because the truth of the statement for all positive integers smaller than $k$ leads to the truth of the statement for $k$, by mathematical induction, the statement is true for all $n \in \mathbb{N}$, which means that all powers of 2 should be covered, and $\Delta A$ only contains powers of 2.

□

Now we proceed to the questions:

(a) If $D = (2^n - 1) = (1, 3, 7, 15, \dots)$, prove that $D$ is an alt-basis.

*Proof.* To prove that $D$ is an alt-basis, we want to show that every positive integer is uniquely D-expressible.

Let $\Delta D = (d_{n+1} - d_n) = (2, 4, 8, 16, \dots)$. Then

$$\Delta d_n = d_{n+1} - d_n = (2^{n+1} - 1) - (2^n - 1) = 2^n$$

**Expressible:** For any $m \in \mathbb{Z}^+$, exist positive integers $k_1 < k_2 < \dots, < k_t$ such that

$$\begin{aligned} m &= 2^{k_1} + 2^{k_2} + 2^{k_3} + \cdots + 2^{k_t} \\ &= \Delta d_{k_1} + \Delta d_{k_2} + \Delta d_{k_3} + \cdots + \Delta d_{k_t} \\ &= -d_{k_1} + d_{k_1+1} - d_{k_2} + d_{k_2+1} - d_{k_3} + d_{k_3+1} - + \cdots - d_{k_t} + d_{k_t+1} \end{aligned}$$

And since $d_{k_1}, d_{k_1+1}, d_{k_2+1} \dots$ are all terms in the sequence $D$, and the expression of $m$ is an alternating sequence, $m$ must be A-expressible.

Note that even if the adjacent terms might be equal, the terms cancel out. For example, if $k_1 + 1 = k_2$, then $d_{k_1+1} - d_{k_2} = 0$. This thus won't influence the alternating nature of the sequence.

**Uniqueness:** In order to show the uniqueness of the expression of each positive integer, we need to show the uniqueness of two things:

(a) the uniqueness of the expression of m as a sum of distinct powers of 2; and

(b) the uniqueness of $\Delta d_n$.

First of all, the uniqueness of m as sum of distinct powers of 2 is proved using Lemma 1.

Therefore, there is only one way such that $m = 2^{k_1} + 2^{k_2} + 2^{k_3} + \cdots + 2^{k_t}$, where $k_1 < k_2 < \cdots < k_t$ are positive integers.

Second, we use mathematical induction to show that $\Delta d_n$ is unique.

**Base case:** $\Delta d_1 = 3 - 1 = 2$. So the statement is true for $n = 1$.

**Induction case:** Assume the truth of $\Delta d_k = 2^{k-1}$, we want to show that $\Delta d_{k+1} = 2^k$.

Since $\Sigma \Delta d_k = \Sigma 2^{k-1} = 2^k - 1 < 2^k$, in order to make the number $2^k$ expressible, $\Delta d_{k+1} = 2^k$.

Therefore, since the truth of $\Delta d_k = 2^{k-1}$ leads to the truth of $\Delta d_{k+1} = 2^k$, by mathematical induction, we have shown that $\Delta d_n = 2^{n-1}$ is true $\forall n \in \mathbb{Z}^+$.

Now that the expression of each positive integer exists and is unique, sequence $D$ is alt-basis.

$\square$

(b) For $E$ to be an alt-basis, $E = (2, 3, 7, 15, 31, \dots)$, where $e_n = 2^n - 1, \forall\ n \geq 3$.

This is true following a similar reasoning as (a):

*Proof.* To prove that $E$ is an alt-basis, we want to show that every positive integer is uniquely E-expressible.

Let $\Delta E = (e_{n+1} - e_n) = (1, 4, 8, 16, \dots)$. Then

$$\Delta e_n = e_{n+1} - e_n = (2^{n+1} - 1) - (2^n - 1) = 2^n, \text{ except that } e_1 = 1.$$

**Expressible:** the proof that every positive integer is E-expressible is exactly the same as the proof for D-expressible in (a), except that $2 \notin \Delta E,\ 2 \in E$.
However, the only case when there is another number preceding 2 is in the case of

$$\Delta e_1 + 2 = -e_1 + e_2 + 2 = -2 + 3 + 2 = 3.$$

Since $3 \in E$, all the numbers starting with $\Delta e_1 + 2$ are still E-expressible.
Therefore, all positive integers all E-expressible.

**Uniqueness:** In order to show the uniqueness of the expression of each positive integer, we need to show the uniqueness of two things:

i) the uniqueness of the expression of m as a sum of distinct powers of 2; and

ii) the uniqueness of $\Delta e_n$.

The truth of the first requirement follows from the same argument as in (a).

Now we wish to show that $\Delta e_n$ is unique.

**Base case:** Since $e_1 = 1$ is an exception to the formula, we start from $n = 2$, and

$$\Delta e_2 = 3 - 1 = 2.$$

So the statement is true for $n = 2$.

**Induction case:** Following the same reasoning as in (a), the truth of $\Delta e_k = 2^{k-1}$ leads to the truth of $\Delta e_{k+1} = 2^k$, by mathematical induction, know that $\Delta e_n = 2^{n-1}$ is true $\forall n \in \mathbb{Z}^+$.

Now that the expression of each positive integer exists and is unique, sequence $E$ is alt-basis.

$\square$

(c) $F = (1, 4, \dots)$ can not be an alt-basis. Because $\Delta f_1 = 4 - 1 = 3$, which is not a power of 2. This, according to Proposition 1, shows that $F$ cannot be an alt-basis.

Alternatively, for a more formal proof, we prove by contradiction.

*Proof.* Suppose exists such a sequence $F = (1, 4, \dots)$ that is an alt-basis.

First, 1 =1, and there must exist such $2 = -f_a + f_b - f_c + f_d - \dots$
Then, $3 = -1 + 4 = 1 + 2 = -f_a + f_b - f_c + f_d - \dots$.
Therefore, in this case, the expression of 3 is not unique, which is a contradiction.
Hence, $F$ is not an alt-basis. $\square$

(d) There are several tests that we could use to test if a sequence $G$ is an alt-basis.

**Proposition 2.** *If $G$ is an alt-basis, there is at most one power of 2 in $G$.*

*Proof.* We prove by contradiction. Assume that in increasing sequence $G$, there are two powers of 2, $2^m$, and $2^n$, where $m > n$.
If $n = 0$, $2^m - 1$ is odd, and since there cannot be 1 again in $\Delta G$, by Proposition 1, numbers in $\Delta G$ are all even, and cannot sum to an odd number. There is a contradiction.
If $n \neq 0$, by Proposition 1, $\Delta G$ must be powers of 2, which means exist $s < t$ such that

$$2^m - 2^n = \Sigma_{i=s}^{t} 2^i$$

$$\Rightarrow 2^n(2^{m-n} - 1) = 2^s(2^{t-s+1} - 1)$$

Therefore, $s = n$, $t + 1 = m$. Since the same power of 2 cannot be both in $G$ and $\Delta G$, there is a contradiction. Therefore, there cannot be more than one power of 2 in $G$. $\square$

Using Proposition 1 and 2, we have the following simple test for an increasing sequence of integers $H$:

(a) If more than one powers of 2 is in $H$, it is not an alt-basis;

(b) If $\Delta H$ has any number beside powers of 2, it is not an alt-basis;

(c) If $\Delta H$ and $H$ cannot cover all powers of 2, it is not an alt-basis;

(d) If $\Delta H$ and $H$ has any term in common, it is not an alt-basis.

Let's look at several examples where these tests apply:

**Example 1.** $H = (2, 5, 7, 8, 12, 28, \dots)$
This sequence is apparently not alt-basis because both 2 and 8 are powers of 2 by test (a). Moreover, since $5 - 2 = 3$ is not a power of 2, it does not satisfy test (b). Therefore, H is not an alt-basis.

**Example 2.** $I = (2, 6, 7, 9, 17, \dots)$
In this example, $\Delta I = (4, 1, 2, 8, \dots)$ However, it does not satisfy test (d) because $I$ and $\Delta I$ has 2 in common, so the expression of 2 is not unique. Therefore, $I$ is not an alt-basis.

**Example 3.** $J = (2^n - 3) = (1, 5, 13, 29, 61, \dots)$
$\Delta J = (4, 8, 16, 32, \dots)$. However there is not 2 in either $J$ or $\Delta J$, which does not past test (d) and is thus not an alt-basis.

**Example 4.** $K = (3, 4, 6, 14, 30, \dots)$
$\Delta K = (1, 2, 8, 16, \dots)$, and since 4 is in $K$, it passes all the tests. However, we cannot determine whether it is an alt-basis or not, because all the tests above test for necessity rather than sufficiency.
In fact, $K$ is not an alt-basis, which we can check using the two methods below.

**Definition 3.** Two numbers are a **couple** if, given a increasing sequence of integers $M$, the expressions of these two numbers are not attainable at the same time, i.e. they either share certain number(s) with the same sign, or they are both in $M$.

Hence I propose some not-as-simple checks that could help us better determine whether a sequence is an alt-basis:

(a) **Repetition check:** for each even number $2n$ in $L$, apply a check: if $n$ can also be expressed as $-n + 2n$ when substituted as terms in $L$, the expression is repetitive, and $L$ is thus not an alt-basis.

**Example 5.** In the case of $L = (4, 6, 7, 15, 31, \dots)$, even though it satisfies all the requirements in the former tests, when we double check for 6, we find that $3 = -3 + 6 = 4 - 7 + 6$, which means non-unique representation, so $L$ in this case is not an alt-basis.

(b) **Recursive check:** for each positive integer $m$ that is not in $K$ or $\Delta I$, we subtract the largest power of 2 smaller then $m$ to get, say, $n$. Then we subtract the largest power of 2 smaller than $n$ ... We stop either if we get a power of 2 or if we get a term in the sequence. If these two final numbers we get are a couple, then the number is unattainable.

**Example 6.** Consider again $K = (3, 4, 6, 14, 30, \dots)$.
$K$ passes all the simple tests, but when we apply the recursive check, we find $7 = 2^2 + 3 = 4 + 3$. However, both 4 and 3 are in $K$, so they are a couple, which means 7 is not K-expressible, and thus $K$ is not an alt-basis.

**Example 7.** Consider $N = (6, 8, 9, 13, 29, 61, \dots)$
$N$ also passes all simple tests, as well as the repetition check. However, we find that $14 = 8 + 6 = 8 + 4 + 2$. However, 8 and 2 are a couple as 2 = -6+8, so they cannot be obtained at the same time. Therefore, $N$ fails the recursive check and is not an alt-basis.

**Example 8.** Consider $P = (1, 5, 7, 23, 31, \dots)$ and $\Delta P = (4, 2, 16, 8, \dots)$.
$P$ passes every test and check I proposed, but we still cannot determine whether it is an alt-basis.
In fact, it is, and can be proved using a similar method as in Question (a). To avoid redundancy, I will not write the proof again here.

Therefore, I do recognize the deficiency in the tests and checks I proposed. They only provide the necessary conditions but not the sufficient ones. We still have to prove a sequence is an alt-basis depending on different cases, as shown in Example 8. Therefore, future investigation will be continued in search for a simpler result.