FM2 Test #2 Name: Maggie. 98%

Excellent!!

1. Find the order of $(3, 15)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{18}$.

~~(3,15)~~ $\mathbb{Z}_4 \times \mathbb{Z}_{18} = \{(x,y) \mid x \in \mathbb{Z}_4, y \in \mathbb{Z}_{18}\}$.

the identity e. is $(0,0)$.

the order of 3 in $\mathbb{Z}_4$ is 4,

the order of 15 in $\mathbb{Z}_{18}$ is 6.

and $\mathrm{lcm}(4,6) = 12$, so the order of $(3,15)$ is $\boxed{12.}$.

2. An equivalence relation on the set $\{1, 2, 3, 4, 5\}$ creates the partition $\{\{1, 2, 3\}, \{4\}, \{5\}\}$. Give the relation matrix.

Since every partition ~~[illegible]~~ creates ~~[illegible]~~ equivalent classes,

~~we know~~

we know that

$1R2$. $2R3$. $3R1$. $4R4$. $5R5$.

creating the equivalent classes.

$[1] = \{1,2,3\}$.

$[4] = \{4\}$

$[5] = \{5\}$

Let the relation denoted by R.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. For each of the following either explain why the graph cannot exist or draw a graph with the given property.

(a) A tree with seven vertices and seven edges.

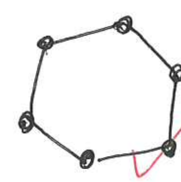
The tree ~~dos~~ does not exist.

Because for a tree ~~the~~ with e edges and v vertices,

$e = v - 1$.

yet $7 \neq 7 - 1 = 6$, so it can't exist.

(b) A simple bipartite graph on six vertices with an Eulerian circuit and a Hamiltonian cycle.



~~A graph has an Eulerian circuit iff all degrees are even.~~

~~~~So the only possible~~. bipartite graph,~~

~~However, for a ~~graph~~ to have a Hamiltonian cycle,~~

~~the bipartite graph should have a partition of vertices~~

~~s.t the number of vertices ~~are~~ is the same.~~

~~which means there should be 3 vertices on each side.~~
~~~~

4. Prove that every cyclic group is Abelian.

and thus a generates the whole group. i.e. any element can be written in the form a^n. $n \in \mathbb{Z}^+$.

$n \in \mathbb{Z}$ as the group may be infinite.

proof: let G be a cyclic group with generator a.

let ~~there be~~ a^i and a^j, $i, j \in \mathbb{Z}^{+}$ be any two elements in G.

$\mathbb{Z}$

$$a^i a^j = a^{i+j} = a^{j+i} = a^j a^i.$$

therefore, G is abelian. $\square$.

4

5. Prove that the function $f\colon \mathbb{N} \to \mathbb{N}$ with rule $f(n) = n + 1$ is injective but not surjective.

proof:

① If $n_1 \neq n_2$, $n_1, n_2 \in \mathbb{N}$.

~~① suppose $f(n_1) = f(n_2)$. where, $n_1, n_2 \in \mathbb{N}$ then $n_1+1 = n_2+1$, $n_1 = n_2$. thus, f is injective.~~

then $n_1+1 \neq n_2+1$.

$n_1+1, n_2+1 \in \mathbb{N}$.

which means $f(n_1) \neq f(n_2)$.

thus, f is injective.

② when $f(n) = 0 \in \mathbb{N}$

$f(n) = n+1 = 0$.

$n = -1$.

However, $-1 \notin \mathbb{N}$,

Thus, there is no preimage for $f(n) = 0$.

which means f is not surjective $\square$.

6. Determine the null space, nullity and rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$.

let $Ax = 0$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

which has AM:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right).$$

$$\sim R = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

{using technology}

let $x_2 = s$, $x_3 = t$.

$x_1 + 2x_2 + 3x_3 = 0$.

$x_1 + 2s + 3t = 0$

$x_1 = -2s - 3t$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} t, \quad s, t \in \mathbb{R}$$

therefore, $\boxed{\text{Null } A} = \text{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$

and $\boxed{\text{nullity} = 2}$

Since nullity + rank = number of columns,

$\boxed{\text{rank} = 3 - 2 = 1}$.

[the row space of A is $\{(1,2,3)\}$ which has rank $= 1$. which confirms that rank$(A) = 1$.]

4

7. The relation $\sim$ is defined on $\mathbb{Z}^+$ by $x \sim y$ if $x + y$ is even. Prove that $\sim$ is an equivalence relation and give the equivalence classes.

proof. Let $x, y, z \in \mathbb{Z}^+$.

① Since $x + x = 2x$ which is always even, $x \sim x$, and $\sim$ is reflexive. ✓

② If $x \sim y$, $x+y$ is even and thus $y + x = x + y$ is even and $y \sim x$, which means $\sim$ is symmetric. ✓

③ If $x \sim y$, $y \sim z$, $x+y$, $y+z$ are both even. Then $(x+y)+(y+z) = x+z+2y$ is even. ~~we can~~ rewrite as $x+z = k-2y$ for some even number $k \in \mathbb{Z}$. So the RHS is even and thus LHS must also be even, which means $x \sim z$, and $\sim$ is transitive.

As $\sim$ is reflexive, symmetric and transitive, it is an equiv. relation □

The equiv. classes are:

$[1] = \{$all odd positive integers$\}$. ✓

$[2] = \{$all even positive integers$\}$

8. Use the mean value theorem to find $a, b \in \mathbb{Q}$ so that $a < \sqrt[3]{10} < b$.

Define function $f : \mathbb{Q} \to \mathbb{Q}$ by
$$f(x) = \sqrt[3]{x}.$$
which is both continuous and differentiable on $\mathbb{Q}^*$, and thus MVT applies:
$$\exists c \in \mathbb{Q}^* \text{ s.t. } f'(c) = \frac{f(b)-f(a)}{b-a}.$$
let $b = 10$, $a = 8$.
$$f(b) = f(a) + f'(c)(b-a).$$
$$f(10) = f(8) + f'(c)(10-8).$$
$$\sqrt[3]{10} = 2 + 2f'(c), \quad c \in\,]8, 10[.$$
and $f'(x) = (x^{\frac{1}{3}})' = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$.
$$\frac{1}{3\sqrt[3]{27^2}} < \frac{1}{3\sqrt[3]{10^2}} < f'(c) < \frac{1}{3\sqrt[3]{8^2}}$$
$$\frac{1}{27} < f'(c) < \frac{1}{12}.$$
and since $f'(c) = \frac{\sqrt[3]{10}-2}{2}$.

So that
$$\frac{1}{27} < \frac{\sqrt[3]{10}-2}{2} < \frac{1}{12}$$
$$2\tfrac{2}{27} < \sqrt[3]{10} < 2\tfrac{1}{6}.$$
so
$$\boxed{a = 2\tfrac{2}{27} \quad b = 2\tfrac{1}{6}}$$

10

9. Let G be the group of 2×2 matrices under addition and $H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + d = 0 \right\}$. Prove that $H < G$.

proof: we first use the 3-step subgroup test to prove $H \leq G$.

① let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in H$. $a+d=0$. $e+h=0$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$

since $a+e+d+h = (a+d)+(e+h) = 0$.

$\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \in H$, and H is closed.

② since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

and $0+0=0$. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in H$

and it is the identity.

③. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and since

$-a-d = -(a+d) = -0 = 0$.

$\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \in H$.

and thus inverse exists $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$. with $a+d=0$.

Therefore. by the 3-step subgroup test. $H \leq G$.

and since $H \neq G$, $H < G$. □

because $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a+d \neq 0 \right\} \notin H$.

10. Suppose $f\colon G \to G'$ is a group homomorphism with $a \in G$. If a has finite order prove $|f(a)|$ divides $|a|$.

proof: Suppose a has order n. i.e. $|a| = n$.

$f(a^n) = f(e) = e'$. where e' is the identity in G'.

on the ~~orther~~ other hand,

$f(a^n) = [f(a)]^n$

So $[f(a)]^n = e'$.

suppose. $|f(a)| = m$. i.e. $[f(a)]^m = e'$.

and $n = m \cdot q + r$.

$q, r \in \mathbb{Z}$, $0 \leq r < m$.

then $[f(a)]^n$

$= [f(a)]^{m \cdot q + r} = [f(a)]^{mq} \cdot [f(a)]^r$

$= e' \cdot [f(a)]^r = e'$.

and since $|f(a)| = m$ and $0 \leq r < m$.

r has to be 0.

which means.

$n = mq + 0 = mq$.

and $m \mid n$.

i.e. $|f(a)| \mid |a|$.

□.

Excellent!

10

Solutions to FM2 Test #2

1. The order of the element in the direct product is the least common multiple of the component orders. Now the order of 3 in $\mathbb{Z}_4$ is 4 and the order of 15 in $\mathbb{Z}_{18}$ is 6. We conclude that the order of $(3, 15)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{18}$ is $\operatorname{lcm}(4, 6) = 12$.

2. The relation matrix is $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A is the 3×3 matrix of ones, B is the 3×2 zero matrix, C is the 2×3 zero matrix and D is the 2×2 identity matrix.

3. (a) In a tree $|E| = |V| - 1$. Since $7 \neq 7 - 1$, there is no such tree. (b) The cycle graph C_6 satisfies the criteria.

4. See your notes.

5. Suppose $f(n_1) = f(n_2)$. Then $n_1 + 1 = n_2 + 1$ or equivalently $n_1 = n_2$. Hence f is not injective. There is no $n \in \mathbb{N}$ satisfying $f(n) = 0$. Hence f is not surjective.

6. Observe matrix A has only one independent row vector, so $\operatorname{rank}(A) = 1$, and so $\operatorname{nullity}(A) = 3 - 1 = 2$. The null space is the span of the vectors $-2\vec{\imath} + \vec{\jmath}$ and $-3\vec{\jmath} + \vec{k}$.

7. We wish to show that $\sim$ is reflexive, symmetric and transitive.

   i For any $x \in \mathbb{Z}^+$, $x + x = 2x$, which is even. So $\sim$ is reflexive.

   ii If $x \sim y$ then $x + y$ is even but then $y + x$ is even, which implies $y \sim x$. So $\sim$ is symmetric.

   iii Suppose $x \sim y$ and $y \sim z$. Then $x + y$ is even and $y + z$ is even. So their sum $x + 2y + z$ is also even, from which it follows that $x + z$ is even. So $\sim$ is transitive.

   We conclude that $\sim$ is an equivalence relation. There are two equivalence classes [1] and [2], namely the odd and even positive integers.

8. Using MVT in the form $f(b) = f(a) + f'(c)(b - a)$ with $a = 8$, $b = 10$ and $f(x) = \sqrt[3]{10}$ gives $\sqrt[3]{10} = 2 + 2f'(c)$ where $c \in]8, 10[$. Next $f'(x) = \frac{1}{3}x^{-2/3}$. Now observe

$$\frac{1}{3} \cdot 27^{-2/3} < \frac{1}{3} \cdot 10^{-2/3} < f'(c) < \frac{1}{3} \cdot 8^{-2/3},$$

   whence $1/27 < f'(c) < 1/12$. Hence $56/27 < \sqrt[3]{10} < 13/6$.

9. The *trace* of a matrix is the sum of the elements on the main diagonal. So H is the set of 2×2 matrices with zero trace.

   We use the 3-step subgroup test.

   i Since $\operatorname{trace}(A + B) = \operatorname{trace}(A) + \operatorname{trace}(B)$, we conclude H is closed under addition.

   ii The trace of the zero matrix is zero. So H contains the zero matrix, which is the additive identity.

   iii The additive inverse of A is $-A$. Since $\operatorname{trace}(-A) = -\operatorname{trace}(A)$, we conclude $-A \in H$.

   Now note that $I \notin H$ since $\operatorname{trace}(I) = 2$. So H is a proper subset of G and $H < G$ as required.

10. Let $f(a) = a'$, $f(e) = e'$ and let the order of a be n. By homomorphism $f(a^n) = (a')^n$. But $f(a^n) = f(e) = e'$ as the image of the identity in G is the identity in G' by a standard homomorphism result. So $f(a')^n = e'$. This means the order of a' in G' is finite, say m. Now by the division algorithm $n = mq + r$ where $0 \le r < m$. So $(a')^n = [(a')^m]^q\,(a')^r$, whence $(a')^r = e'$. Since m was the order of a', it is the least positive integer m for which $(a')^m = e'$, so $r = 0$. Hence m divides n, and we are done.