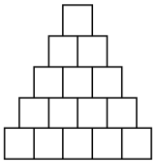
SUMaC 2020 Admission Exam

- Solve as many of the following problems as you can. Your work on these problems together with your grades in school, teacher recommendations, and answers to questions on the application form are all used to evaluate your qualifications for SUMaC. Although SUMaC is very selective with a competitive applicant pool, correct answers to every problem are not required for admission.
- There is no time limit for this exam other than the application deadline of March 11.
- Feel free to report partial progress toward a solution, in the event you are unable to solve a problem completely.
- Please include clear, detailed explanations with ***all of your answers***; numerical answers or formulas with no explanation are not useful for evaluating your application.
- None of these problems require a calculator or computer, and are designed to be done without these tools.
- You are expected to do your own work without the use of any outside source (books, internet search, etc). If you recognize one of the problems from another source, or if you receive any assistance, please indicate this on your solution.
- ***Please do not share these problems or your solutions with anyone.***

1. An n-pyramid is created with n blocks at the base, $n - 1$ blocks on the next level, until the peak which has 1 block. Here is a 5-pyramid:



Pyramids are colored according to the following rules: (1) Each block can be red, gold, or black, and (2) All 3-block units composed of one block on top of two others must either all be the same color or all be different colors. Answer the following:

a) How many ways are there to color a 7-pyramid?

b) What is the maximum number of black blocks in a 25-pyramid that has at least one red block and at least one gold block? Explain.

2. Find all positive integers x and y that satisfy $x^2 - y^2 = 2020$. Are there any integers x and y (positive or negative) such that $x^3 - y^3 = 2020$? Explain.

3. By definition, $p(x)$ is a polynomial with integer coefficients if

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

for some non-negative integer n, integers $a_0, a_1, a_2, \ldots, a_n$ and variable x. The number r is a root of the polynomial $p(x)$ if $p(r) = 0$.

(a) Suppose a and b are integers and $p(x)$ is a polynomial with integer coefficients. Prove that if $a + b\sqrt{2}$ is a root of $p(x)$, then $a - b\sqrt{2}$ is also a root of $p(x)$.

(b) Find a polynomial $q(x)$ with integer coefficients, and integers c and d such that c$+d\sqrt[3]{2}$ is a root of $q(x)$, but c $- d\sqrt[3]{2}$ is not a root of $q(x)$, where $\sqrt[3]{2}$ denotes the cube root of 2. Describe conditions on c, d, and $q(x)$ for this to hold.

4. Suppose $p(x)$ is a polynomial with integer coefficients. Show that if $p(a) = 17$ for some integer a, then $p(x)$ has at most three integer roots; that is, there are at most three distinct integers b, c, and d such that $p(b) = p(c) = p(d) = 0$.

5. In this problem we consider *sum-sequences* of numbers; that is, sequences that start with 1, and where each number that comes later in the sum-sequence is the sum of two previous numbers, including the possibility of adding a number to itself. For example, 1, 2, 4, 8, 16 is a sum-sequence, where each number is adding the previous number to itself. Also, 1, 2, 3, 5, 8, 13 is a sum-sequence, where after adding 1 to itself, each number is the sum of the two previous. The following is also a sum-sequence: 1, 2, 3, 4, 5, 6, obtained by adding 1 to the previous number to get the next.

More formally, let a k-sum-sequence be a sequence of integers $n_1, n_2, \ldots, n_k$, such that $n_1 = 1$ and for $r = 2, \ldots, k$, there are s, t such that. $s \le t < r$. and $n_r = n_s + n_t$. For each positive integer N we can look for the smallest k such that there is a k-sum-sequence ending in N.

For example, we can get $N = 10$, from the sequence 1, 2, 4, 8, 10, with $k = 5$.

a) Show that we cannot get $N = 10$, with a sequence where $k < 5$.

b) The method above used to get $N = 10$ with $k = 5$ was to add the previous number to itself until it was the largest power of 2 less-than-or-equal-to N, and then add the smaller powers of 2 together to get N. For example, we can get 13 from the sequence

$$1, 2, 4, 8, 12 = 8 + 4, 13 = 12 + 1.$$

We call this the *double first method.* Find an upper bound on the size of k in terms of N using the double-first method.

c) Show that for some N, the double-first method is *not* optimal. That is, find an N and an l-sum-sequence for N such that if the double first method is a k-sum-sequence, then $l < k$.

d) For $N = 100$, we can use the double-first method to get

$$1, 2, 4, 8, 16, 32, 64, 96 = 64 + 32, 100 = 96 + 4.$$

with $k = 9$. Is there a sum-sequence for 100 with $k < 9$?

6. Consider a with 3×3 grid where each cell contains a number of coins; for example, the following represents a possible configuration of coins on the grid (the integer in each cell is the number of coins in that cell):

12	3	1
1	8	4
2	13	0

This configuration is transformed in stages as follows: in each step, every cell sends a coin to all of its neighbors (horizontally or vertically, not diagonally), *but if there aren't enough coins in a cell to send one to each of its neighbors, it sends no coins at all*. For example, the above would result in the following after one step:

11	2	3
4	7	2
1	12	2

a) Show that every staring configuration results in stable configuration (one that no longer changes in this process), *or* repeatedly cycles through k configurations for some positive integer k (i.e., those same k configurations appear repeatedly in the sequence over and over as the transformation is applied).

b) In the case that the initial configuration eventually cycles through k configurations, what are the possible values of k?

c) Either prove that for some positive integer B, every configuration will reach a stable configuration or a repetition of a k-cycle in B or fewer steps, or prove there is no such B.

7. Consider the following game involving a deck of cards. We are only concerned about the color of the cards, red and black; we ignore the suits and the denominations. The rules for a given game are sequences of "R"s and "B"s corresponding to red and black cards. For example, one game has the three rules: (1) BB, (2) RR, and (3) BRBR.

   The object of the game is to see if you can use the rules to transform a starting sequence to a goal sequence. At stages along the way, you can apply the rules as follows, as many times as you wish: you can insert the sequence corresponding to the rule anywhere in your current sequence (and you always have enough cards to do this as many times as you would like), or you may remove the sequence corresponding to the rule from your current sequence.

   For example, we can transform the starting sequence RB to goal sequence BR as follows. From RB we can use Rule (1) to insert BB at the beginning and get BBRB, and then use Rule (2) to insert RR on the end, resulting in BBRBRR. Applying Rule (3) to this last sequence we can remove the four cards in the middle to get BR, so we have gone from RB to BR in three steps using each rule once.

   a) Suppose the rules are (1) BB, (2) RRR, and (3) RRBRB. Show that you *can* go from RB to BR, but you cannot go from R to B.

   b) Suppose the rules are (1) BB, (2) RRR, (3) RBRB. Show that you *cannot* go from RB to BR.

   c) For each set of rules in part a) and b), show that every sequence can be transformed into one of the following six, $\langle\,\rangle$, B, R, RR, BR, BRR, where $\langle\,\rangle$ is the "empty sequence" (an empty hand of cards).

   d) Show that for both the rules in part a), and for the rules in part b), for any sequence $x_1x_2x_3 \ldots x_m$ there is another sequence $y_1y_2y_3 \ldots y_n$ such that $x_1x_2x_3 \ldots x_m y_1y_2y_3 \ldots y_n$ can be eliminated completely by the rules, resulting in $\langle\,\rangle$.