FM2 Assignment #11 Name: Maggie. $\frac{10}{10}$ Excellent!

1. Let $f(x) = \tan x$. Observe that $f(0) = f(\pi)$ but there is there no $c \in]0, \pi[$ such that $f'(c) = 0$. Explain why this does not contradict Rolle's theorem.

Because Rolle's thrm requires that $f(x)$ is continuous over the interval $[0, \pi]$.
However, for $f(x) = \tan x$, $f(x)$ is not continuous ~~of~~ at $f(\frac{\pi}{2})$.
Thus, Rolle's thrm doesn't apply.

2. Let $f(x) = x + |x|$. Prove that f is continuous but not differentiable at $x = 0$.

Proof: (a) f is continuous at $x=0$.

$$\lim_{x^+\to 0} f(x) = \lim_{x^- \to 0} f(x) = 0$$

$$\therefore \lim_{x\to 0} f(x) = 0 = f(0)$$

and thus conti.

(b) f is not differentiable at $x=0$.

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h+|h|}{h}$$

$$\lim_{h^+\to 0} \frac{h+|h|}{h} = 2.$$

$$\lim_{h^-\to 0} \frac{h+|h|}{h} = 0.$$

since $\lim_{h^+\to 0} \frac{h+|h|}{h} \neq \lim_{h^- \to 0} \frac{h+|h|}{h}$. the limit does not exist.

Therefore, f is not differentiable at $x=0$. $\square$

3. Use the mean value theorem to prove the inequality $|\sin a - \sin b| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

proof: Let $f(x) = \sin x$.
We know that $f(x)$ is conti. and differentiable on $\mathbb{R}$ by construction.
Then according to MVT,
$\exists\, c \in \mathbb{R}$. s.t

$$f'(c) = \frac{\sin a - \sin b}{a-b}, \quad \forall a,b \in \mathbb{R}.$$

Since for $f(x) = \sin x$,

$$f'(x) = (\sin x)' = \cos x.$$

and $-1 \leq \cos x \leq 1$. $\forall x \in \mathbb{R}$.

therefore,

$$-1 \leq f'(c) \leq 1.$$

which means

$$-1 \leq \frac{\sin a - \sin b}{a-b} \leq 1.$$

$$\frac{|\sin a - \sin b|}{|a-b|} \leq 1.$$

Thus, $|\sin a - \sin b| \leq |a-b|$ $\forall a,b \in \mathbb{R}$. $\square$.

4. In $\triangle ABC$, $a = 9$, $b = 6$ and $c = 12$. A circle with centre A and radius 4 meets sides $[AB]$ and $[AC]$ at E and F respectively. The secant (EF) meets (BC) at D. Use Menelaus's theorem to calculate the length CD.

According to Menelaus's thrm,

$$\frac{BC}{CA} \cdot \frac{AF}{FC} \cdot \frac{CD}{DB} = -1$$

$$\frac{8}{4} \cdot \frac{4}{2} \cdot \left(-\frac{CD}{CD+9}\right) = -1$$

$$\frac{CD}{CD+9} = \frac{1}{4}$$

$$4CD = CD+9$$

$$3CD = 9$$

$$\boxed{CD = 3}$$

/

5. Verify that $f(x) = 2x^4 - 3x^2 - x + 5$ satisfies the hypotheses of the mean value theorem on the interval $[0, 1]$ and find all numbers c that satisfy the conclusion of the mean value theorem.

$f'(x) = 8x^3 - 6x - 1$,

Which means $f(x)$ is differentiable on $]0,1[$.

and therefore continuous on $[0,1]$.

Thus, according to MVT,

$\exists\ c \in \mathbb{R}$, s.t.

$$f'(c) = \frac{f(1) - f(0)}{1-0} = \frac{3-5}{1} = -2.$$

$8c^3 - 6c - 1 = -2$.

$8c^3 - 6c + 1 = 0$.

using technology.

$c_1 \approx 0.174$

$c_2 \approx 0.766$.

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FM2 Assignment #12 Name: Maggie. $\frac{10}{10}$ Excellent!

1. What value should be assigned to k to make the function $f(x) = \begin{cases} x^2 - 1, & x < 3, \\ 2kx, & x \geq 3, \end{cases}$ continuous at $x = 3$.

For $f(x)$ to be conti. at $x=3$

$\lim_{x\to 3} f(x) = f(3)$

which means

$\lim_{x^+\to 3} f(x) = \lim_{x^-\to 3} f(x)$

then $\lim_{x\to 3}(x^2-1) = \lim_{x\to 3}(2kx)$.

$3^2-1 = 2k(3)$

$8 = 6k$

$\boxed{k = \frac{4}{3}.}$

2. Construct a function that is continuous on $\mathbb{R}$ but fails to be differentiable at the four numbers $0, 1, 2, 3$.

$$f(x) = \begin{cases} x+1 & x<0 \\ \big|\,||x-2|-2|-1\,\big| & 0 \leq x \leq 3 \\ x-3 & x>3. \end{cases}$$

3. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f'(x) > 0$ for all $x \in \mathbb{R}$. Prove that if $a < b$ then $f(a) < f(b)$.

proof:

since f is differentiable on $\mathbb{R}$, it is also conti on $\mathbb{R}$.

then MVT applies.

If $a<b$.

the $\exists\, c \in]a,b[$ s.t.

$f'(c) = \frac{f(b)-f(a)}{b-a}$.

and since $f'(x) > 0$. $\forall x \in \mathbb{R}$,

$f'(c) > 0$

thus

$\frac{f(b)-f(a)}{b-a} > 0$.

and since $b-a > 0$.

$f(b) - f(a) > 0$

$f(b) > f(a)$.

$\square$.

6

4. The third degree Taylor polynomial of $\ln x$ about $x = 1$ is $a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3$. Find the values of a_0, a_1, a_2 and a_3 and hence estimate $\ln 1.2$.

$$P_3(x) = \frac{f^{(0)}(1)}{0!} + \frac{f^{(1)}(1)}{1!}(x-1) + \cdots + \frac{f^{(3)}(1)}{3!}(x-1)^3$$

$$= \ln(1) + \frac{1}{1}(x-1) + \frac{(-1)(1)^{-2}}{2!}(x-1)^2 + \frac{(2)(1)^{-3}}{3!}(x-1)^3$$

$$= 0 + x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

Thus, $a_0 = 0$, $a_1 = 1$, $a_2 = -\frac{1}{2}$, $a_3 = \frac{1}{3}$.

$$\ln(1.2) \approx P_3(1.2)$$

$$= 1.2 - 1 - \frac{1}{2}(1.2-1)^2 + \frac{1}{3}(1.2-1)^3$$

$$= 0.2 - 0.5(0.2)^2 + \frac{1}{3}(0.2)^3$$

$$\approx 0.183 \text{ (3s.f.)}.$$

5. In the trapezium $ABCD$, the midpoints of the parallel sides $[AB]$ and $[CD]$ are M and N respectively. The sides $[BC]$ and $[AD]$ are not parallel. Show that the diagonals and the line segment $[MN]$ are concurrent.

Lemma:

Prolong CA and DB,
let them intersect at point P.
then P, M, N are collinear.

proof:

Suppose not, ~~i.e.~~
then connect and prolong PM,
let PM intersect CD at N'.
since $AB \parallel CD$,
$\angle PAM = \angle PCN'$
$\angle APM = \angle CPN'$

So $\triangle APM \sim \triangle CPN'$.
similarly, $\triangle BPM \sim \triangle DPN'$.

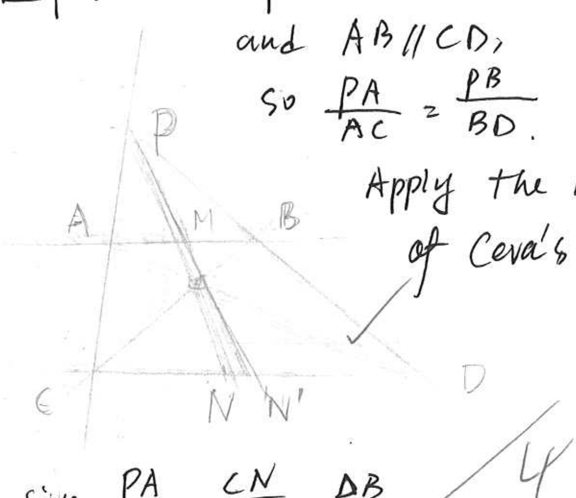
So $\frac{AM}{CN'} = \frac{PM}{PN'} = \frac{BM}{DN'}$

Since M is the midpoint of AB,
$\frac{AM}{BM} = 1$.

then $\frac{CN'}{DN'} = \frac{AM}{BM} = 1$.

i.e. $CN' = DN'$, N' is the midpoint of CD, which means
N and N' coincide.
Since P, M, N' are collinear,
P, M, N are collinear. $\square$.

proof: Since P, M, N are collinear.
and $AB \parallel CD$,
So $\frac{PA}{AC} = \frac{PB}{BD}$.
Apply the inverse of Ceva's thrm,



Since $\frac{PA}{AC} \cdot \frac{CN}{ND} \cdot \frac{DB}{BP}$
$= \frac{PB}{BD} \cdot \frac{CN}{ND} \cdot \frac{DB}{BP} = 1$

PN, CB, DA are concurrent. $\square$.

4

FM2 Assignment #13 Name: Maggie.

10/10 Excellent!

1. Use l'Hôpital's rule to evaluate $\lim_{x\to 1} \dfrac{\arctan x - \pi/4}{x-1}$.

$$\lim_{x\to 1} \frac{\arctan x - \frac{\pi}{4}}{x-1} = \frac{0}{0},$$

Using L'Hopital's Rule,

$$\lim_{x\to 1} \frac{\arctan x - \frac{\pi}{4}}{x-1} = \lim_{x\to 1} \frac{(\arctan x - \frac{\pi}{4})'}{(x-1)'}$$

$$= \lim_{x\to 1} \frac{1}{1+x^2} = \frac{1}{2}.$$

2. Let $A = (-1, 0)$ and $B = (1, 0)$. Find the locus of a point P that moves so that $PA^2 + PB^2 = 10$.

$$\cancel{\frac{PA^2}{10} + \frac{PB^2}{10} = 1}.$$

Let $P = (x, y)$. ~~PA =~~

then $(x+1)^2 + y^2 + (x-1)^2 + y^2 = 10$.

$$2x^2 + 2y^2 = 8$$

$$\cancel{\frac{x^2}{4} + \frac{y^2}{4}} \quad x^2 + y^2 = 4 = 2^2.$$

So the locus of P is a circle at the origin with radius = 2.

3. The third degree Taylor polynomial for the function f centred at 1 is $4 - (x-1) + 3(x-1)^2 - 5(x-1)^3$.

(a) Write down the value of $f''(1)$.

$$\frac{f''(1)}{2!} = 3. \qquad f''(1) = 6.$$

(b) Approximate $f'(1.2)$.

$$f'(x) = -1 + 6(x-1) - 15(x-1)^2.$$

$$f'(1.2) = -1 + 6(0.2) - 15(0.2)^2$$

$$= -1 + 1.2 - 0.6$$

$$= -0.4.$$

6

4. The sequence $\{u_n\}$ is defined recursively by $u_1 = 2$ and $u_{n+1} = \frac{1}{2}(u_n + 4)$. Use mathematical induction to show that $\{u_n\}$ is an increasing sequence bounded above by 4. What is the limit of the sequence?

proof by induction:

① $u_1 = 2$. $u_2 = \frac{1}{2}(u_1+4) = 3$.

~~$u_2 > u_1$~~. $u_1 < u_2 < 4$.

② Now we suppose that $\{u_n\}$ is increasing and $u_n < 4$.

then $u_{n+1} = \frac{1}{2}(u_n+4)$

~~Since $u_{n+1} - u_n$~~.

Since $\frac{1}{2}(u_n+4) < \frac{1}{2}(8) = 4$

$u_{n+1} < 4$, ~~which~~

and $u_{n+1} - u_n$

$= 4 - \frac{1}{2}u_n > 0$.

So $u_{n+1} \geq u_n$,

which means the sequence is increasing.

Therefore, since the truth of $\{u_n\}$ implies the truth of $\{u_{n+1}\}$. by induction, $\{u_n\}$ is an increasing sequence bounded by 4. ✓

Then according to Monotonic Sequence thrm for convergence, the sequence has a limit, say L.

so $\lim_{n\to\infty} u_{n+1} = L$.

on the other hand, $\lim_{n\to\infty} u_{n+1} = \frac{1}{2}(u_n+4)$
$= \frac{1}{2}(L+4)$ ✓

so $L = \frac{1}{2}(L+4)$

$L = 4$.

So the limit of the sequence is 4. □

5. The function f has derivatives of all orders for all real numbers. The third degree Taylor polynomial for f centred at 2 is $7 - 9(x-2)^2 - 3(x-2)^3$. If $|f^{(4)}(x)| \leq 6$ for all x in the open interval $]0, 2[$, show that $f(0)$ must be negative.

proof: According to Taylor's thrm,

$f(x) = 7 - 9(x-2)^2 - 3(x-2)^3 + R_3$

where $R_3 = \frac{f^{(4)}(c)}{4!}(x-2)^4$

for some ~~$c \in]$~~

c between 2 and x.

let $x = 0$.

then $f(0) = 7 - 9(-2)^2 - 3(-2)^3 + \frac{f^{(4)}(c)}{4!}(-2)^4$ for some $c \in]0, 2[$.

and since $|f^{(4)}(x)| \leq 6$. $\forall x \in]0, 2[$.

$-6 \leq f^{(4)}(c) \leq 6$.

thus $-5 - 4 \leq f(0) \leq -5 + 4 < 0$.

Therefore, $f(0)$ must be negative. □

4

FM2 Assignment #14 Name: Maggie. $\frac{9}{10}$ v. good

1. Write the Maclaurin series for e^x, $\cos x$ and $\sin x$ in sigma notation.

$$e^x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad k \in \mathbb{N}$$

unnecessary as k is indicated by the index.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} (-1)^{\frac{k}{2}} \frac{x^k}{k!}. \quad k \in \mathbb{N}.$$

$1\frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} (-1)^{\frac{k-1}{2}} \frac{x^k}{k!}. \quad k \in \mathbb{N}$$

what if $k = 1$?

2. Find the n-th degree Taylor polynomial for $\frac{1}{x}$ about $x = 1$ and write the polynomial in sigma notation.

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!} (x-1)^k$$

$$= \sum_{k=0}^{n} \frac{(-1)^k (k!) (1)^{-k-1}}{k!} (x-1)^k$$

$$= \sum_{k=0}^{n} (1-x)^k.$$

3. Use the alternating series estimation theorem to find an interval centre 0 throughout which $\cos x$ can be approximated by $1 - \frac{1}{2}x^2$ to three decimal places.

The nth Maclaurin poly for $\cos x$ is

$$P_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^{\frac{n}{2}} \frac{x^n}{n!}.$$

~~So the absolute difference~~

$$\cancel{\left|\cos x - (1 - \tfrac{1}{2}x^2)\right| < \left|\tfrac{x^4}{4!}\right| < 0.5 \times 10^{-3}}.$$

By the Alternating series estimation thrm,

$$\left|\cos x - (1 - \tfrac{1}{2}x^2)\right| < \frac{x^4}{4!} < 0.5 \times 10^{-3}$$

$$x^4 < 0.012.$$

so $-0.331 < x < 0.331$.

the interval is $]-0.331, 0.331[$.

$5\frac{1}{2}$

4. Show that the power series $\sum_{n=1}^{\infty} \frac{n^2}{n!}(x-1)^n$ converges for all $x \in \mathbb{R}$.

proof: We apply the ratio test. Let $a_n = \frac{n^2}{n!}(x-1)^n$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{(n+1)!}(x-1)^{n+1} \cdot \frac{n!}{n^2(x-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| (x-1)\left(\frac{n+1}{n^2}\right) \right|$$

$$= \lim_{n\to\infty} \left| (x-1)\left(\frac{1}{n}+\frac{1}{n^2}\right) \right|$$

$$= \lim_{n\to\infty} |(x-1)\cdot 0| = 0 \quad \forall x \in \mathbb{R}.$$

and since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$. the series $\sum_{n=1}^{\infty} \frac{n^2}{n!}(x-1)^n$ converges. $\square$.

5. Let $f(x) = e^x \sin x$. Show that $f''(x) = 2(f'(x) - f(x))$. Hence find the fifth degree Maclaurin polynomial for f.

$$f'(x) = e^x \sin x + e^x \cos x.$$

$$f''(x) = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$$

$$= 2e^x \cos x.$$

$$2(f'(x) - f(x)) = (e^x \sin x + e^x \cos x - e^x \sin x)\times 2$$

$$= 2e^x \cos x.$$

So $f''(x) = 2(f'(x) - f(x))$.

$f'''(x) = 2(f''(x) - f'(x))$

$u_n = f^{(n)}(0)$.

So $u_n = 2(u_{n-1} - u_{n-2})$; $u_0 = 0$, $u_1 = 1$.

$\Rightarrow u_5 = -4$.

$1\frac{1}{2}$

Thus, the fifth deg. Mac. poly for f is

$$P_5(x) = \sum_{k=0}^{5} \frac{f^{(k)}(0)}{k!} x^k.$$

$$= 0 + x + \frac{2x^2}{2!} + \frac{2x^3}{3!} + \frac{2x^4}{4!} + \frac{2x^5}{5!}$$

$$= x + x^2 + \frac{2}{3!}x^3 + \cancel{\frac{2}{4!}x^4} + \frac{2}{5!}x^5.$$

0

$-\frac{1}{30}x^5$

$3\frac{1}{2}$

FM2 Assignment #15 Name: Maggie $9\frac{1}{2}$ / 12 Very good

1. Write down the coefficient of x^6 in the Maclaurin series for $\cos 2x$. Hence determine the coefficient of x^6 in the Maclaurin series for $\cos^2 x$ giving your answer as a fraction in lowest terms.

coefficient of x^6 for $\cos 2x$:

$$\frac{f^{(6)}(0)}{6!} = \frac{-64 \cdot \cos(0)}{6!} = \boxed{-\frac{4}{45}}$$

since $\cos 2x = 2\cos^2 x - 1$

$$\cos^2 x = \frac{\cos 2x + 1}{2}$$

and thus the coefficient of x^6 for $\cos^2 x$ is

$$-\frac{4}{45} \cdot \frac{1}{2} = \boxed{-\frac{2}{45}}$$

2. Find the radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n2^{n-1}}$.

Observe that when $x = 1$,

the series becomes $\sum_{n=1}^{\infty} \frac{(-2)^n}{n\,2^{n-1}} = \sum_{n=1}^{\infty} (-1)^n \frac{2}{n}$,

which converges by applying alternating series test; $R \geq 2$.

Also, when $x = 5$, the series becomes

$$\sum_{n=1}^{\infty} \frac{2^n}{n\,2^{n-1}} = \sum_{n=1}^{\infty} \frac{2}{n},$$ which diverges, so $R \leq 2$.

Therefore, $\boxed{R = 2}$.

3. Use the substitution $y = 1/x$ and L'Hôpital's rule to evaluate $\lim_{y\to\infty} y - \sqrt{1+y^2}$. Confirm you answer using a series approach.

①

$$\lim_{y\to\infty} (y - \sqrt{1+y^2})$$

$$= \lim_{x\to 0^+} \left(\frac{1}{x} - \sqrt{1+\frac{1}{x^2}}\right)$$

$$= \lim_{x\to 0^+} \left(\frac{1}{x} - \frac{\sqrt{x^2+1}}{|x|}\right)$$

$$= \lim_{x\to 0^+} \frac{1-\sqrt{1+x^2}}{x} = \frac{0}{0}.$$

By L'Hopital's rule,

$$\lim_{x\to 0^+} \frac{(1-\sqrt{1+x^2})'}{x'}$$

$$= \lim_{x\to 0^+} \frac{2x \cdot \frac{1}{2}(1+x^2)^{-\frac{1}{2}}}{1}$$

$$= 0$$

② Series Approach.

the nth degree polynomial for $f(x) = (1+x^2)^{\frac{1}{2}}$.

is $P_n(x) = \binom{\frac{1}{2}}{0} + \binom{\frac{1}{2}}{1} x^2 + \cdots + \binom{\frac{1}{2}}{n} x^{2n}$

Now

$$\frac{1-\sqrt{x^2+1}}{x} = \frac{1 - (1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - + \cdots)}{x}$$

$$= \frac{1 - (1 + O(x^2))}{x}$$

$$= O(x).$$

So

$$\lim_{y\to\infty} (y - \sqrt{1+y^2})$$

$$= \lim_{x\to 0} \left(\frac{1-\sqrt{x^2+1}}{x}\right)$$

$$= \lim_{x\to 0} O(x) = 0.$$

6

4. Find the interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

Observe that when $x = \frac{1}{3}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-3)^n (\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n+1}},$$ which converges by alternating series test, and thus radius $R \geq \frac{1}{3}$;

Also, when $x = -\frac{1}{3}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-3)^n (-\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1^n}{\sqrt{n+1}},$$ which diverges.

$1\frac{1}{2}$ ✓

and thus $R \leq \frac{1}{3}$

Therefore, $R = \frac{1}{3}$, and the interval of convergence is $]-\frac{1}{3}, \frac{1}{3}]$. ✓

5. Prove in a conditionally convergent series both the series of positive terms and the series of negative terms diverge.

Proof: Suppose $\sum_{n=1}^{\infty} a_n$ is a series w/ positive and negative terms that is conditionally convergent.

i.e. $\sum_{n=1}^{\infty} a_n$ is convergent, yet $\sum_{n=1}^{\infty} |a_n|$ is divergent.

the series of positive terms is $\sum_{n=1}^{\infty} \frac{a_n + |a_n|}{2}$;

and the series of negative terms is $\sum_{n=1}^{\infty} \frac{a_n - |a_n|}{2}$.

Now, since $\sum_{n=1}^{\infty} a_n$ converges, $|a_n|$ is bonded $\forall n$, by β say.

then $\sum_{n=1}^{\infty} \frac{a_n + |a_n|}{2} \leq \sum_{n=1}^{\infty} \left(\frac{\beta}{2} + \frac{|a_n|}{2}\right) ~~= \frac{n\beta}{2} + \sum_{n=1}^{\infty} \frac{|a_n|}{2}~~$. ✓

and $\sum_{n=1}^{\infty} \frac{a_n - |a_n|}{2} \leq \sum_{n=1}^{\infty} \left(\frac{\beta}{2} - \frac{|a_n|}{2}\right)$.

but since $\sum_{n=1}^{\infty} |a_n|$ diverges, and $\frac{\beta}{2}$ is a constant,

$3\frac{1}{2}$ ✓

both $\sum_{n=1}^{\infty} \left(\frac{\beta}{2} + \frac{|a_n|}{2}\right)$ and $\sum_{n=1}^{\infty} \left(\frac{\beta}{2} - \frac{|a_n|}{2}\right)$ diverge,

Therefore, both the series of positive and negative terms diverge. ∎