International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Maggie.

**MATHEMATICS**
**HIGHER LEVEL**
**PAPER 3 – CALCULUS**

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

*Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.*

**1.** *[Maximum mark: 14]*

The function $f$ is defined on the domain $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ by $f(x) = \ln(1+\sin x)$.

(a) Show that $f''(x) = -\frac{1}{(1+\sin x)}$. *[4 marks]*

(b) (i) Find the Maclaurin series for $f(x)$ up to and including the term in $x^4$.

(ii) Explain briefly why your result shows that $f$ is neither an even function nor an odd function. *[7 marks]*

(c) Determine the value of $\lim_{x\to 0}\frac{\ln(1+\sin x) - x}{x^2}$. *[3 marks]*

**2.** *[Maximum mark: 8]*

Consider the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y + \sqrt{x^2 - y^2}\,,\ x > 0,\ x^2 > y^2.$$

(a) Show that this is a homogeneous differential equation. *[1 mark]*

(b) Find the general solution, giving your answer in the form $y = f(x)$. *[7 marks]*

$\frac{d}{dx} vx$

$\frac{y+\sqrt{x^2-y^2}}{x}$

$\frac{d}{dx} vx$

$v$

Paper 3
Calculus

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50/60

V. good.

6/7

1. (a) $f'(x) = \frac{1}{1+\sin x} \cdot \cos x$

$$f''(x) = \frac{-\sin x\,(1+\sin x) - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - 1}{(1+\sin x)^2}$$

$$= -\frac{1}{1+\sin x}. \quad \boxed{}.$$

(b) (i) $P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(4)}(0)}{24}x^4.$

$$= 0 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{2}{24}x^4$$

$$= x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4.$$

(ii) $P_4(-x) = -x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{12}x^4$,

which is neither even nor odd. $\boxed{}$. more explanation is needed.

(c) $\lim_{x\to 0} \frac{\ln(1+\sin x) - x}{x^2}$

$$= \lim_{x\to 0} \frac{x - \frac{1}{2}x^2 + O(x^3) - x}{x^2}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{2}x^2 + O(x^3)}{x^2}$$

$$= \lim_{x\to 0} -\frac{1}{2} + O(x)$$

$$= \boxed{-\frac{1}{2}.}$$

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2. (a) $\frac{dy}{dx} = \frac{y + \sqrt{x^2 - y^2}}{x}$

$$= \frac{y}{x} + \sqrt{1 - \frac{y^2}{x^2}}$$, which is homogeneous.

→ continued.

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(b) let $y = vx$.

$$\frac{dy}{dx} = v + \sqrt{1-v^2}$$

$$\frac{dv}{dx}x + \cancel{v} = \cancel{v} + \sqrt{1-v^2}$$

$$\frac{dv}{dx}x = \sqrt{1-v^2}$$

$$\int \frac{1}{\sqrt{1-v^2}}\,dv = \int \frac{1}{x}\,dx.$$

$$\arcsin v = \ln x + c_1$$

$$\arcsin \frac{y}{x} = \ln x + c_1$$

$$\frac{y}{x} = \sin(\ln x + c_1)$$

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$$y = x\sin(\ln cx), \quad c = e^{c_1} \quad //.$$

3. (a) $\cancel{P_3(x) = 1 + 2x + \frac{d}{2dx}(2e^x + y\tan x)x^2 + \frac{d^2}{6dx^2}(2e^x + y\tan x)x^3}$

$\cancel{= 1 + 2x + \frac{1}{2}x^2 + \frac{1}{6}(\sec^2 x + 2\sec^2 x\tan x)x^3}$

$\cancel{= 1 + 2x + \frac{3}{2}x^2 + \frac{1}{6}x^3}$.

$f(0) = 1,$

$f'(0) = 2.$

$$f''(0) = 2e^x + \tan x\frac{dy}{dx} + y\cdot\sec^2 x.$$

$$= 2 + y. = 3$$

$$f'''(0) = 2e^x + \sec^2 x\cdot\frac{dy}{dx} + \frac{d^2y}{dx^2}\tan x + \sec^2 x\frac{dy}{dx} + 2\sec^2 x\tan x\cdot y$$

$$= 2 + \frac{dy}{dx} + \frac{dy}{dx}$$

$$= 2 + 4 = 6$$

So $P_3(x) = 1 + 2x + \frac{3}{2}x^2 + x^3$.

**3.** *[Maximum mark: 15]*

Consider the differential equation

$$\frac{dy}{dx} = 2e^x + y\tan x \text{, given that } y = 1 \text{ when } x = 0 .$$

$(2e^x + y\tan x)'$

The domain of the function $y$ is $\left[0, \frac{\pi}{2}\right[$.

2 $= 2e^x + y\tan x$

(a) By finding the values of successive derivatives when $x = 0$, find the Maclaurin series for $y$ as far as the term in $x^3$. *[6 marks]*

(b) (i) Differentiate the function $e^x(\sin x + \cos x)$ and hence show that

$$\int e^x \cos x \, dx = \frac{1}{2}e^x(\sin x + \cos x) + c .$$

(ii) Find an integrating factor for the differential equation and hence find the solution in the form $y = f(x)$. *[9 marks]*

$\frac{d}{dx}(y \tan x)$

**4.** *[Maximum mark: 10]*

$\frac{d}{dx}\left(\tan x \frac{d}{dx} y + y \cdot \sec^2 x.\right)$

Let $f(x) = 2x + |x|$, $x \in \mathbb{R}$.

(a) Prove that $f$ is continuous but not differentiable at the point $(0, 0)$. *[7 marks]*

$\tan x + \sec^2 x.$

(b) Determine the value of $\int_{-a}^{a} f(x)\,dx$ where $a > 0$. *[3 marks]*

$\sec^2 x + 1 \cdot \frac{1}{\cos 0}$

**5.** *[Maximum mark: 13]*

Consider the infinite series $\sum_{n=1}^{\infty} \frac{(n-1)x^n}{n^2 \times 2^n}$.

$\frac{d}{dx} y \cdot \sec^2 x + \frac{d^2}{dx^2} y$ $\qquad \frac{d}{dx} \sec^2 x.$

(a) Find the radius of convergence. *[4 marks]*

(b) Find the interval of convergence. *[9 marks]*

$\frac{d}{dx} \tan x$

$\frac{d}{dx}$ $2\sec x \cdot \sec x \tan x$

$2e^x$

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(b) (i) $\frac{d}{dx} e^x (\sin x + \cos x)$

$= e^x (\sin x + \cos x) + e^x \frac{d}{dx} (\sin x + \cos x)$

$= e^x (\sin x + \cos x + \cos x - \sin x)$

$= 2e^x \cos x$

Therefore, $\int 2e^x \cos x \, dx = e^x (\sin x + \cos x) + C_1$

$\Leftrightarrow \int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + c,$

where $c = \frac{1}{2} C_1$. //. ✓

(ii) $\frac{dy}{dx} - \tan x \cdot y = 2e^x$.

Since $(\ln \cos x)' = -\tan x$,

$\cos x$ is an integrating factor. ✓

Hence, $(\cos x \cdot y)' = 2e^x \cos x$.

$y = 1$ when $x = 0$

$y \cos x = 2 \int e^x \cos x \, dx$

So $y = \tan x \, e^x + e^x$.

$= e^x (\sin x + \cos x) + c$ , $c \in \mathbb{R}$.

$\Rightarrow y = c \cdot \sec x + \tan x \cdot e^x + e^x$.

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4. (a) $\lim_{x \to 0} f(x) = \lim_{x \to 0} (2x + |x|) = 0$.

$f(0) = 2(0) + |0| = 0$. ,

So $\lim_{x \to 0} f(x) = f(0)$, therefore $f$ continuous. ✓

→ continued.

However, $f$ is not differentiable be cauce

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{2h+|h|}{h}.$$

$$= 2 + \lim_{h\to 0}\frac{|h|}{h}.$$

since $\lim_{h^+\to 0}\frac{|h|}{h} = 1 \neq \lim_{h^-\to 0}\frac{|h|}{h} = -1$,

limit does not exist, and thus $f$ is not differentiable.//

(b) $\int_{-a}^{a} 2x + |x| \, dx$

$$= \int_{-a}^{a} 2x\,dx + \int_{-a}^{a} |x| \, dx$$

$$= \left[x^2\right]_{-a}^{a} + \left[\tfrac{1}{2}x^2\right]_0^a + \left[-\tfrac{1}{2}x^2\right]_{-a}^{0}.$$

$$= a^2 - a^2 + \tfrac{1}{2}a^2 + \tfrac{1}{2}a^2$$

$= a^2$. where $a > 0$.

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5. (a) when $x = -1$, we get.

$\sum_{n=1}^{\infty} (-1)^n \frac{n-1}{2^n n^2}$, apply alternating series test,

$\lim_{n\to\infty} \frac{n-1}{n^2} = 0$, and since ~~$n>0$,~~

missing.

$$\frac{n-1}{n^2} - \frac{n}{(n+1)^2} = \frac{n^2-n-1}{(n+1)^2 \cdot n^2} > 0 \quad \forall\, n \geq 2.$$

$\frac{n-1}{n^2} > \frac{n}{(n+1)^2}$, therefore the series converges.

when $x = 1$, we get

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$\sum_{n=1}^{\infty} \frac{n-1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n} - \sum_{n=1}^{\infty}\frac{1}{n^2}$. since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges yet

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is bounded, by $\beta$ say.

Then $\sum_{n=1}^{\infty}\frac{n-1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n} - \beta$, is divergent.

Therefore, $R = 1$, interval of convergence is $[-1, 1[$.