FM2 Test #1 Name: Maggie. 94 Very good!

1. The permutation $a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$. Find a^{10} giving your answer in cycle notation.

$a = (1\ 2\ 4\ 3)$ we found that $a^4 = e = \cancel{(1\ 2\ 3\ 4)}$.

$$a^{10} = \underbrace{(1\,2\,4\,3)(1\,2\,4\,3)\cdots(1\,2\,4\,3)}_{10\ (1243)}$$

$$= e \cdot e (1\,2\,4\,3)^2 = (1\,4)(2\,3)$$ ✓

2. Use the inverse matrix method without the aid of the calculator to solve the system $\begin{cases} x + 2y = 19 \\ 3x - y = 15 \end{cases}$.

~~The system has~~

~~we~~

~~The system has augumented matrix.~~

$\cancel{\left(\begin{array}{cc|c} 1 & 2 & 19 \\ 3 & -1 & 15 \end{array}\right)}$.

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 19 \\ 15 \end{pmatrix}.$$

let $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$.

$|A| = 1(-1) - (2)(3)$

$= -1 - 6$

$= -7 \neq 0$, So A^{-1} exists.

premultiple by A^{-1}, we get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 19 \\ 15 \end{pmatrix}.$$

$$= -\frac{1}{7}\begin{pmatrix} -1 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 19 \\ 15 \end{pmatrix}$$

$$= -\frac{1}{7}\begin{pmatrix} -49 \\ -42 \end{pmatrix}. = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$$

$\therefore$ the solution is $\begin{cases} x = 7 \\ y = 6 \end{cases}$ ✓

$-19 - 30$

$-49.$

$19 \times 3 = 57$

$57 - 15$

$-19(3) + 15^2$

3. The group isomorphism $f : \mathbb{Z}_4 \to G$ is defined by $f(0) = a$, $f(1) = b$, $f(2) = c$ and $f(3) = d$. Construct the operation table for G.

The operation table for $\mathbb{Z}_4$ is:

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2.

Be cause of isomorphism,

we know that the operation table for G is

	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c.

✓ 17

4. The three complex numbers 1, w and z form a cyclic group under multiplication. Find w and z.

Since it is a cyclic group. under multiplication.

and $1a = a \quad \forall a \in \mathbb{C}$, 1 must be the identity.

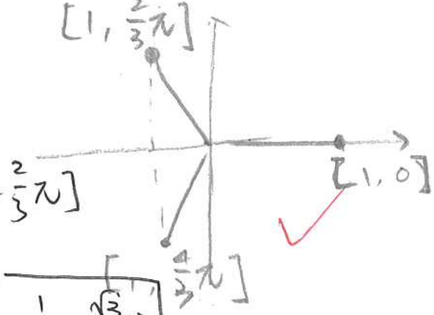
and thus. either w or z or both must be a generator

WLOG, w is the genrator and

$w^2 = z$, $w^3 = 1$.

by using the complex plane,

we know that $w = [-\frac{1}{2}, \frac{2}{3}\pi]$, $z = [-\frac{1}{2}, -\frac{2}{3}\pi]$



and thus w and z should be $\boxed{-\frac{1}{2} + \frac{\sqrt{3}}{2}i}$ and $\boxed{-\frac{1}{2} - \frac{\sqrt{3}}{2}i}$.

5. Calculate the values of x for which the determinant $\begin{vmatrix} x & 5 & -1 \\ 1 & 3 & x \\ 1 & 4 & 7 \end{vmatrix}$ is zero.

$$\begin{vmatrix} x & 5 & -1 \\ 1 & 3 & x \\ 1 & 4 & 7 \end{vmatrix} = x\begin{vmatrix} 3 & x \\ 4 & 7 \end{vmatrix} - 5\begin{vmatrix} 1 & x \\ 1 & 7 \end{vmatrix} + -1\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}$$

$$= x(21 - 4x) - 5(7 - x) - 1(4 - 3)$$

$$= 21x - 4x^2 - 35 + 5x - 1$$

$$= -4x^2 + 26x - 36 = 0$$

$2x^2 - 13x + 18 = 0$

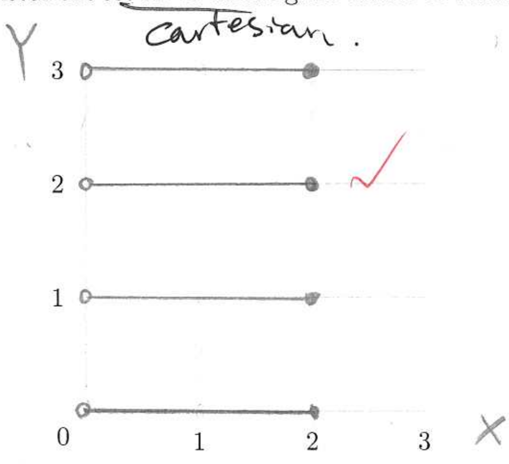
$(x - 2)(2x - 9) = 0$

thus, when $x = 2$ or $\frac{9}{2}$, the determinant is zero.

6. Let $X =]0, 2]$ and $Y = \{0, 1, 2, 3\}$. Sketch the set $X \times Y$ in the grid. Hence or otherwise determine $|(X \times Y) \cap (Y \times X)|$.

cartesian.

~~]0,0]~~
~~]0,2]~~
~~]0,4]~~
~~]0,6]~~



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and $(Y \times X)$ is flipping the graph along $y = x$.

thus. $(X \times Y) \cap (Y \times X) = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$.

Therefore $|(X \times Y) \cap (Y \times X)| = 6$ × 4

7. The set $S = \{261x + 126y \mid x, y \in \mathbb{Z}\}$ forms a group under addition. Explain why this group must be cyclic and hence explain why 333 must be in S.

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First, we want to show that $S \leq \mathbb{Z}$.

~~Since every subgroup of a~~

we use the 3-step subgroup test:

(i) let $(x_1, y_1), (x_2, y_2) \in S$.

$261x_1 + 126y_1 + 261x_2 + 126y_2$

$= 261(x_1 + x_2) + 126(y_1 + y_2)$

Since $x_1, x_2, y_1, y_2 \in \mathbb{Z}$,

$x_1 + x_2,\ y_1 + y_2 \in \mathbb{Z}$.

So S is closed.

(ii) let $x = 0$ $y = 0$.

the $261x + 126y = 0$

identity preserved.

(iii) The inverses of $x, y \in S$ are $-x$ and $-y$ as $261x + 126y + 261(-x) + 126(-y) = 0$. $\forall x, y \in S$.

Thus, inverse exists.

Therefore, $S \leq \mathbb{Z}$.

Since every subgroup of a cyclic group is also cyclic, and $\mathbb{Z}$ is cyclic, S is also cyclic. $\square$

not needed as the question states S is a group

Since $261x + 126y$

$= 9(29x + 14y)$

and $\gcd(29, 14) = 1$.

$\exists s, t$ s.t $29s + 14t = 1$. $\therefore 9 \mid 9(29x + 14y)$

more simply $S = \langle 9 \rangle$ as $\gcd = 9$.

and since $9 \mid 333$. 333 is in S.

8. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 5 & 5 & 5 & 0 \end{pmatrix}$.

(a) Find a basis for the null space of A.

let $Ax = 0$. which has AM.

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 4 & 3 & 2 & 1 & 0 \\ 5 & 5 & 5 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

x_3 is the free varible. let $x_3 = s$.

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 0 \\ x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = s \\ x_2 = -2s \\ x_3 = s \\ x_4 = 0 \end{cases}$$

Thus, Every matrix of form $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ can be expressed as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}. \quad s \in \mathbb{R}.$$

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Thus, $\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$ is a basis for the null space of A.

(b) Find all vectors $\vec{v} \in \mathbb{R}^4$ such that $A\vec{v} = \begin{pmatrix} 10 \\ 10 \\ 15 \end{pmatrix}$.

Since $\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$ ~~is a basis for the null space of A,~~

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 5 & 5 & 5 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \\ 15 \end{pmatrix}$

$\begin{pmatrix} 10 \\ 10 \\ 15 \end{pmatrix}$ is in the column space of A

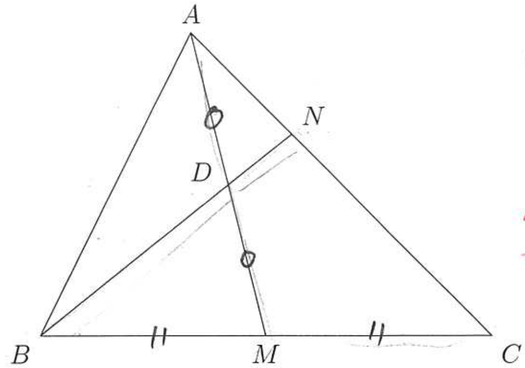
This does not work!

Alternatively you can just solve the AM

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 4 & 3 & 2 & 1 & 10 \\ 5 & 5 & 5 & 0 & 15 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right). \Rightarrow \begin{pmatrix} s \\ 3-2s \\ s \\ 1 \end{pmatrix}.$$

9. In $\triangle ABC$, median $[AM]$ has midpoint D. Prove that the cevian $[BN]$ trisects side $[AC]$.



In $\triangle AMC$

$$\frac{AD}{DM} \cdot \frac{MB}{BC} \cdot \frac{CN}{NA} = -1$$

$$\frac{1}{1}\left(-\frac{1}{2}\right) \cdot \frac{CN}{NA} = -1$$

$$\frac{CN}{NA} = 2$$

$\therefore AN = \frac{1}{3}AC$

and thus $[BN]$ trisects $[AC]$.

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10. The centre of a group G, denoted $Z(G)$, is the set of elements in G that commute with every element of G. That is, $Z(G) = \{a \in G \mid ax = xa \text{ for all } x \in G\}$. Prove that $Z(G)$ is a subgroup of G.

Proof: we use the 3-step subgroup test to show $Z(G) \le G$.

(i) let $m, n \in Z(G)$.

$(mn)x$
$= m(nx)$
$= (mx)n$
$= x(mn) \quad \forall x \in G.$

Thus, $mn \in Z(G)$. and closure is verified.

(ii) Since

$ex = xe = x \quad \forall x \in G,$
$e \in Z(G)$

and identity axiom is verified.

(iii) Let $m \in Z(G)$.

$mx = xm \quad \forall x \in G.$

Since $m \in G$,
$\exists\, m^{-1} \in G$ as G is a group.

premultiply by m^{-1} we get

$m^{-1}mx = m^{-1}xm$
$x = m^{-1}xm.$

postmultiply by m^{-1} we get

$xm^{-1} = m^{-1}xmm^{-1}$
$= m^{-1}x. \quad \forall x \in G.$

and thus $m^{-1} \in Z(G)$.
Thus, inverse exists. 5

Therefore, by 3-step subgroup test, we conclude that

$Z(G) \le G$. $\square$

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Solutions to FM1 Test #1

1. Since a is a 4-cycle, $a^4 = e$. So $a^{10} = a^2 = (14)(23)$.

2. We have $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 19 \\ 15 \end{pmatrix} = -\frac{1}{7} \begin{pmatrix} -1 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 19 \\ 15 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$. Hence $x = 7$, $y = 6$.

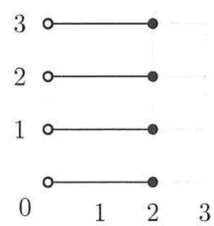
3. The operation table for G is

	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

4. The three roots of unity form the required cyclic group. These roots are 1, $[1, 120°]$ and $[1, 240°]$. So $w = [1, 120°]$ and $z = [1, 240°]$ will do.

5. Expanding the determinant across the first row gives $x(21-4x)-5(7-x)-(4-3)$. Hence we solve $2x^2-13x+18 = 0$, whence $x = 2, \frac{9}{2}$.

6. The diagram illustrates $X \times Y$. The diagram for $Y \times X$ will be the reflection of the given diagram in the line $y = x$. We conclude $|(X \times Y) \cap (Y \times X)| = 4$.



7. We are given that $(S, +)$ is a group and clearly S is a proper subset of $\mathbb{Z}$. So $(S, +) \leq (\mathbb{Z}, +)$. Since $(\mathbb{Z}, +)$ is cyclic we conclude $(S, +)$ is cyclic since every subgroup of a cyclic group is also cyclic. A generator for $(S, +)$ is $\gcd(261, 126) = 9$. So $S = \langle 9 \rangle$. Since $9 \mid 333$, we conclude $333 \in S$.

8. (a) Using the calculator $\text{rref}(A) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. So a basis for $\text{null}(A)$ is $\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \}$.

   (b) We spot $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ as a particular solution. Hence the full solution is $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$, $t \in \mathbb{R}$.

9. Menelaus's theorem with unsigned lengths gives

$$\frac{AN}{NC} \times \frac{CB}{BM} \times \frac{MD}{DA} = 1.$$

$$\begin{pmatrix} t+1 \\ 1-2t \\ t+1 \\ 1 \end{pmatrix}$$

Solving for $AN : NC$, gives $AN : NC = 1 : 2$, which is to say cevian $[BN]$ trisects side $[AC]$.

10. We use the 3-step subgroup test.

    i. Suppose $a, b \in Z(G)$ and $x \in G$. Then $(ab)x = a(bx) = a(xb) = (ax)b = (xa)b = x(ab)$. Hence $ab \in Z(G)$. So $Z(G)$ is closed under the group operation.

    ii. Since $ex = xe$ for all $x \in G$, we have $e \in Z(G)$.

    iii. Suppose $a \in Z(G)$ and $x \in G$. Then $ax = xa$. So $axa^{-1} = x$, from which it follows that $xa^{-1} = a^{-1}x$. Thus $a^{-1} \in Z(G)$.

    Hence $Z(G)$ is a subgroup of G.