Neural Networks

Dr. Jameel Malik

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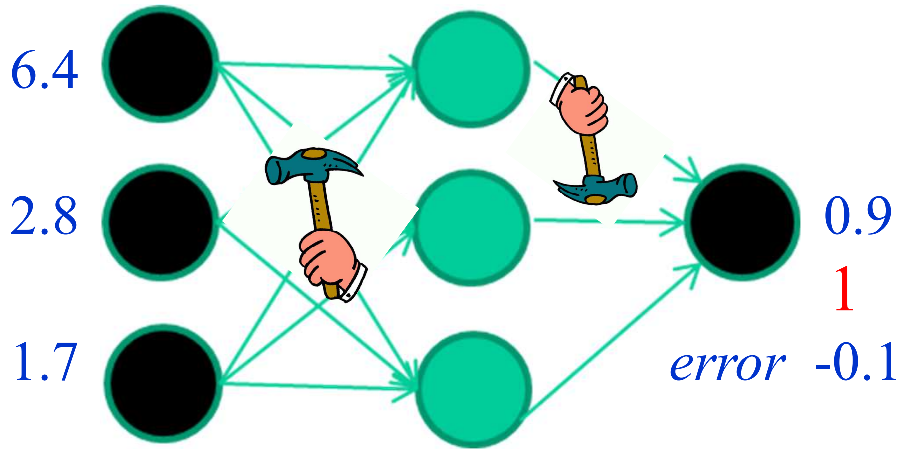
Backpropagation in Neural Networks

Training data

Features			***class***
1.4	2.7	1.9	0
3.8	3.4	3.2	0
6.4	2.8	1.7	1
4.1	0.1	0.2	0
etc …			

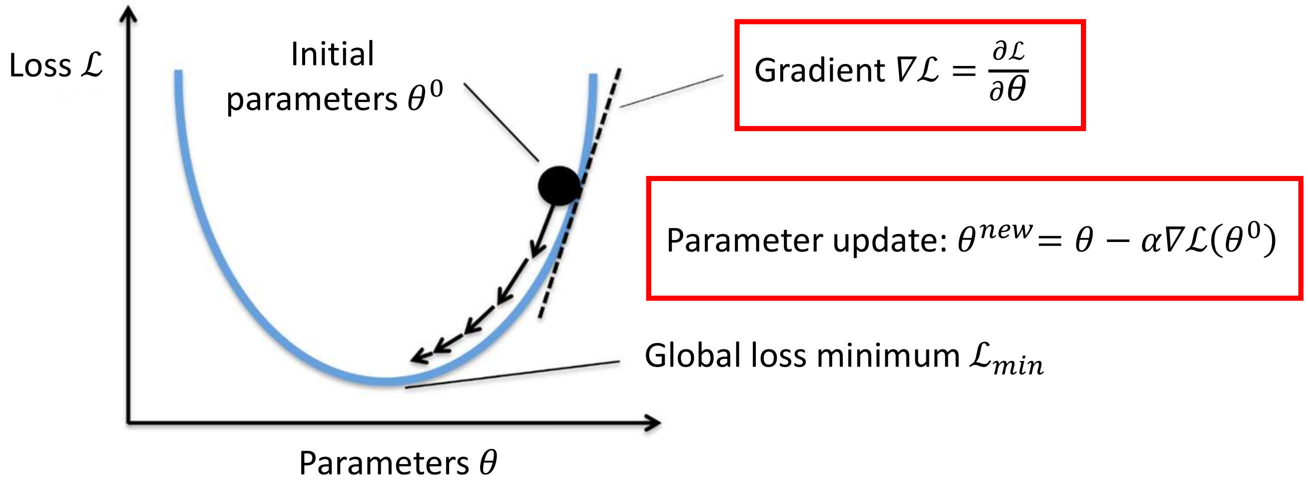
Adjust weights based on error



Neural Network

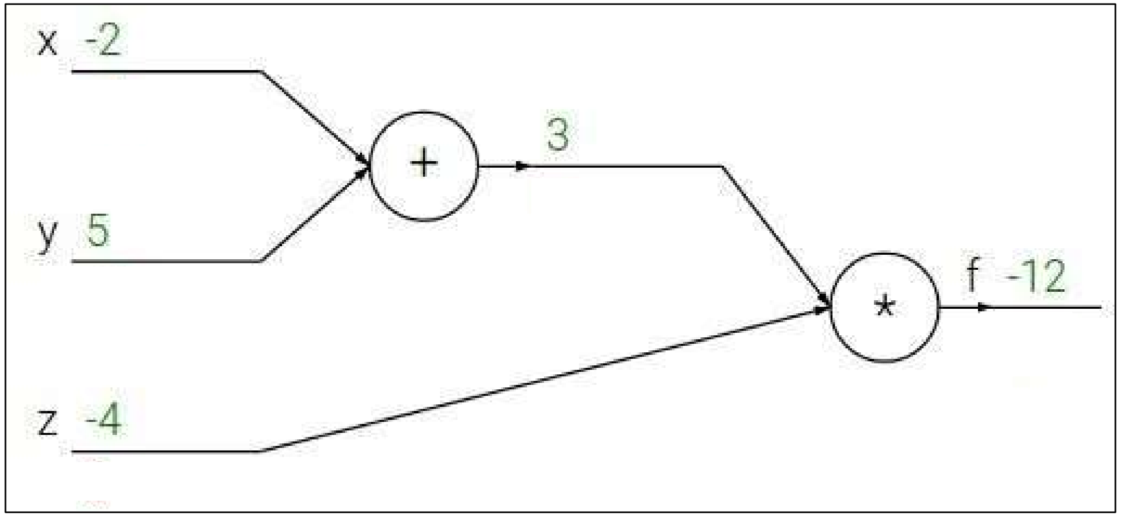
Gradient Descent Algorithm -- Recap

- Steps in the ***gradient descent algorithm***:
  1. Randomly initialize the model parameters, θ^0
  2. Compute the gradient of the loss function at the initial parameters θ^0: $\nabla\mathcal{L}(\theta^0)$
  3. Update the parameters as: $\theta^{new} = \theta^0 - \alpha\nabla\mathcal{L}(\theta^0)$
     - Where α is the learning rate
  4. Go to step 2 and repeat (until a terminating criterion is reached)



$$f(x, y, z) = (x + y)z$$

e.g. x = -2, y = 5, z = -4



A simple Computational Graph / Circuit

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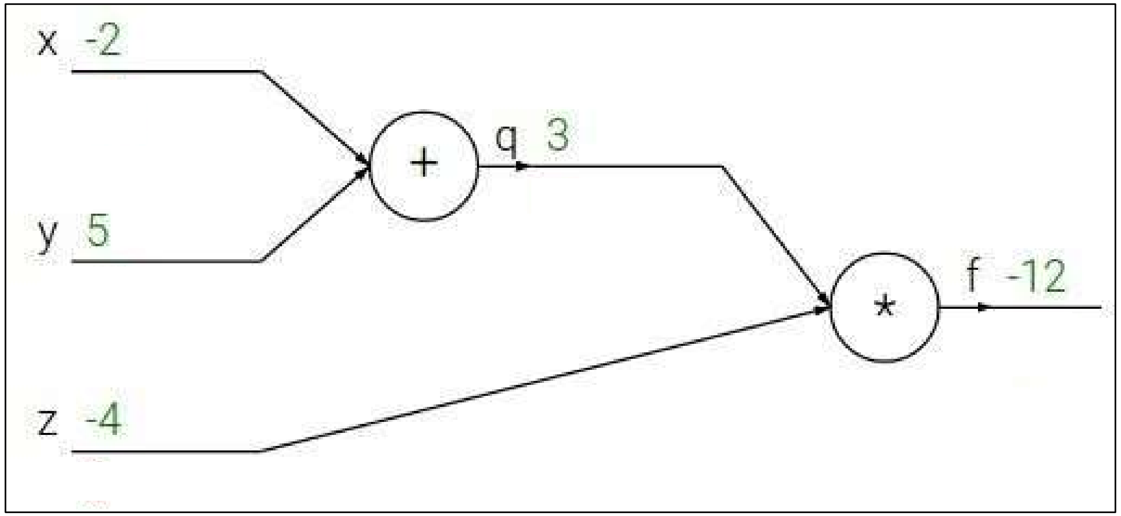
$$f = qz \qquad \frac{\partial f}{\partial q} = z, \frac{\partial f}{\partial z} = q$$

Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$



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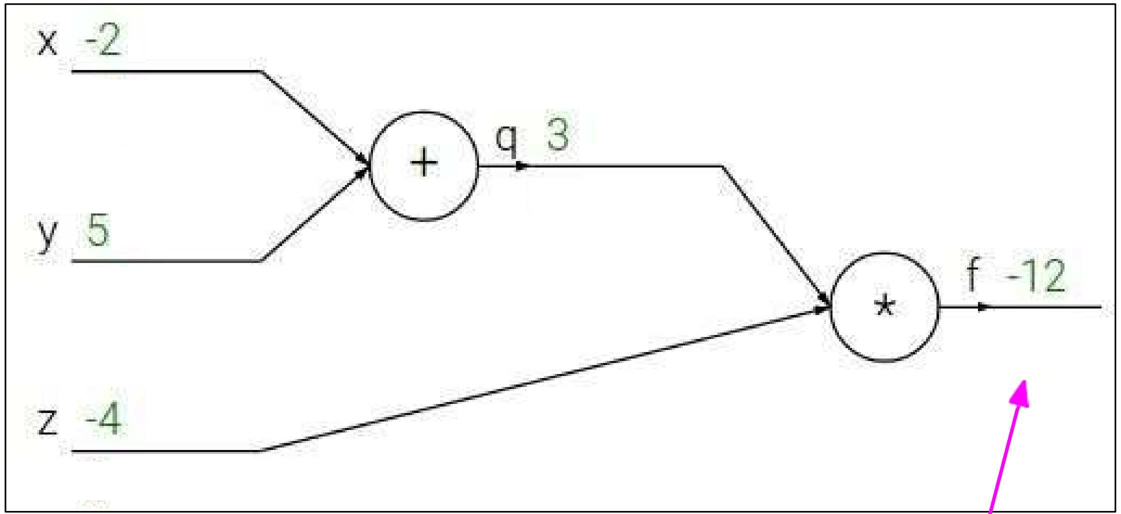

$$\boxed{\frac{\partial f}{\partial f}}$$

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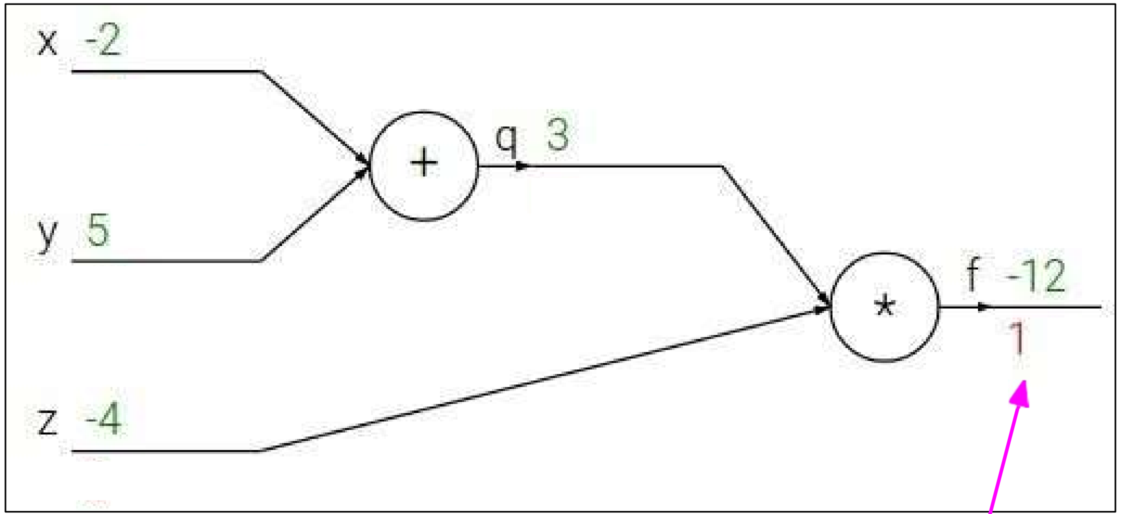

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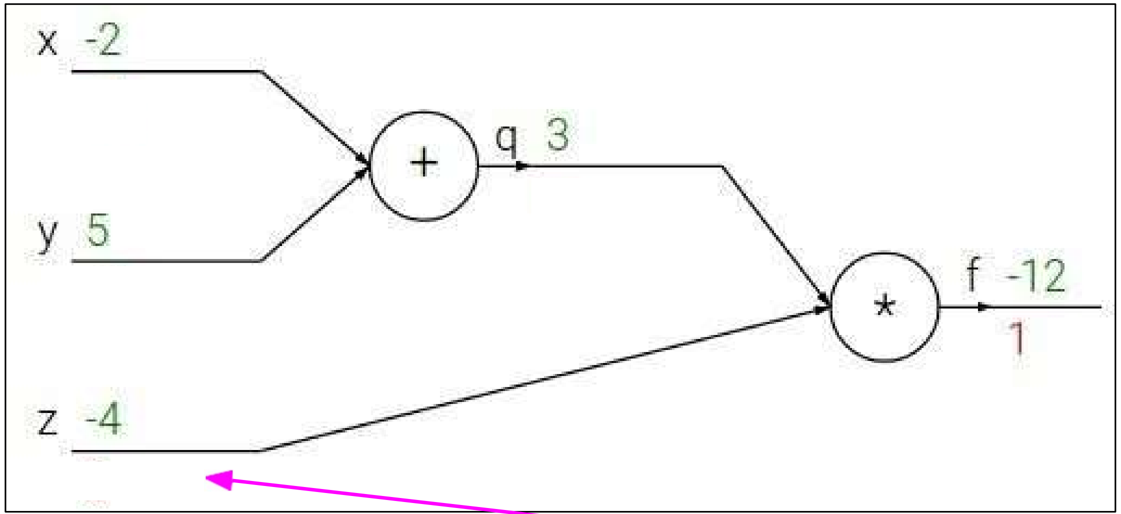

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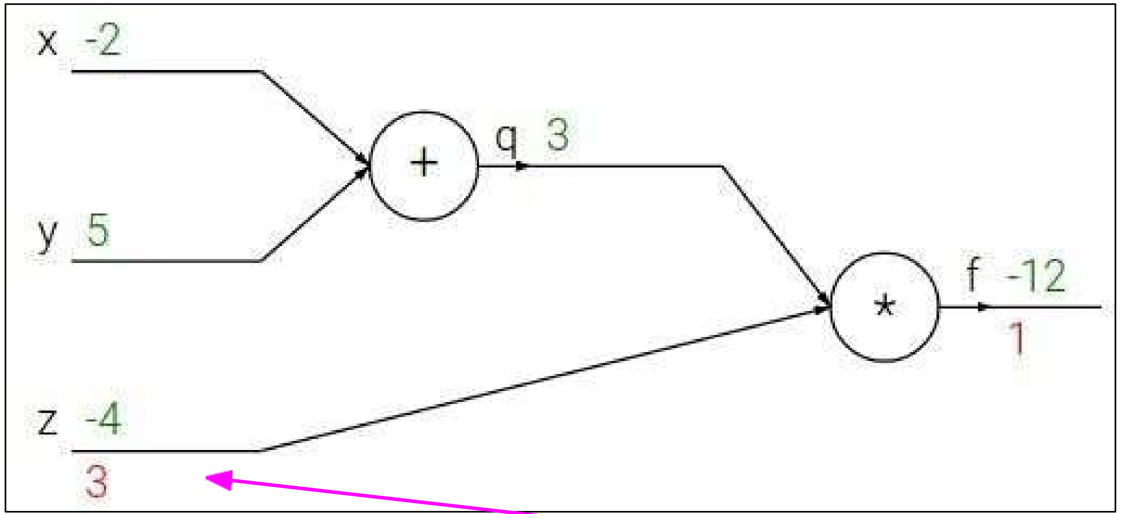

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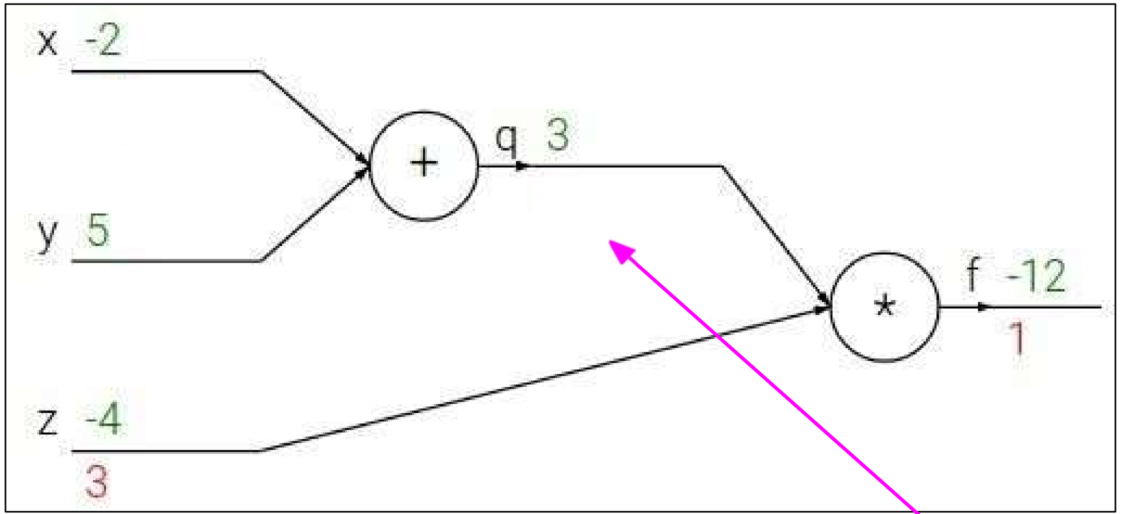

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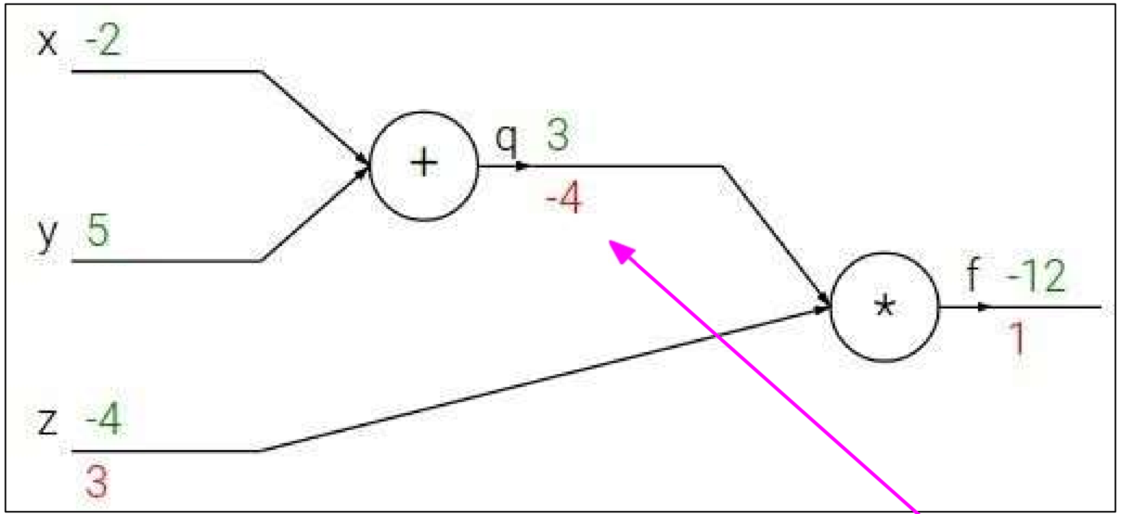

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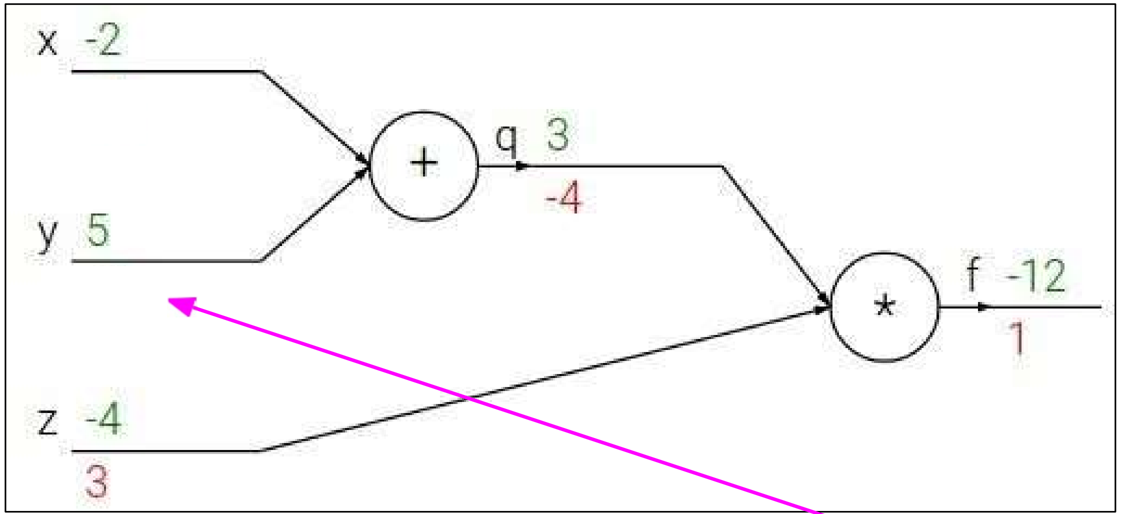

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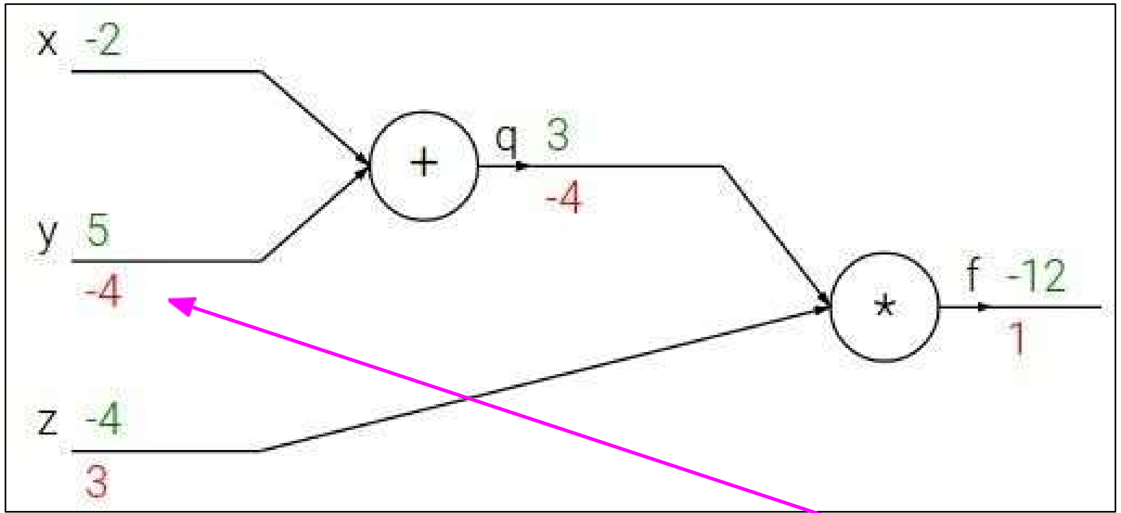

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$$\frac{\partial f}{\partial y}$$

Chain rule:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial q}\frac{\partial q}{\partial y}$$

Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

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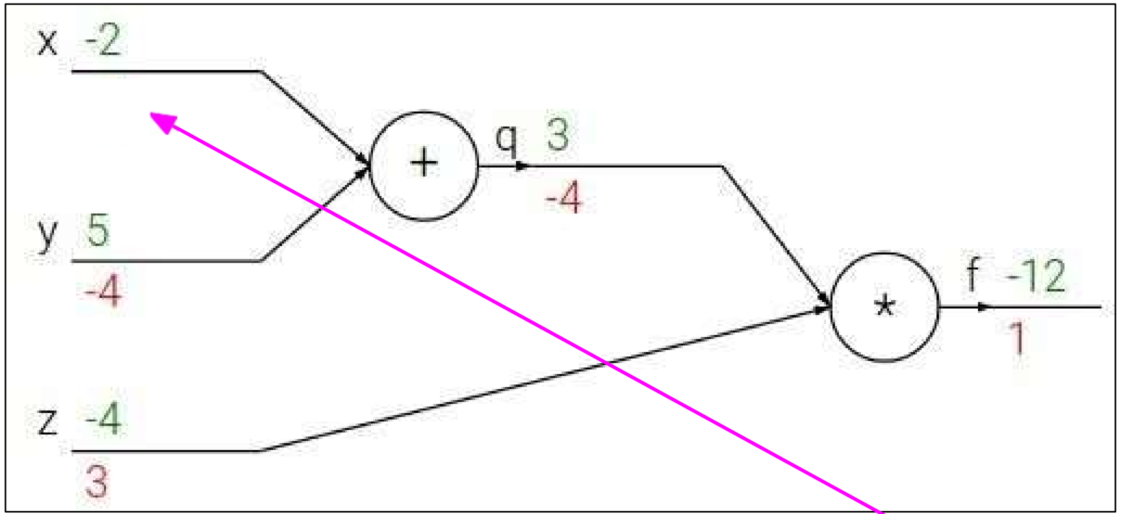

$\boxed{\frac{\partial f}{\partial x}}$

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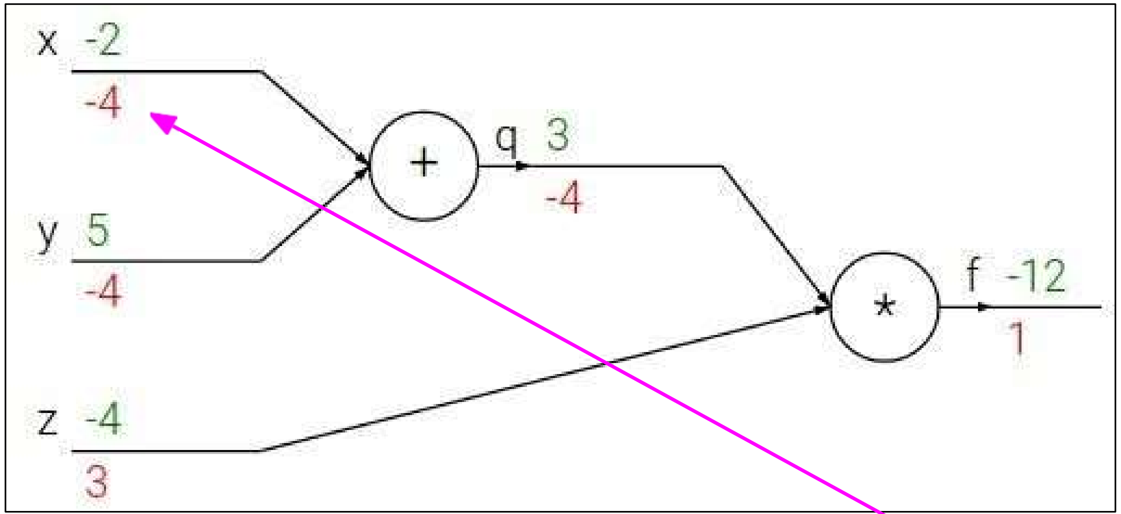

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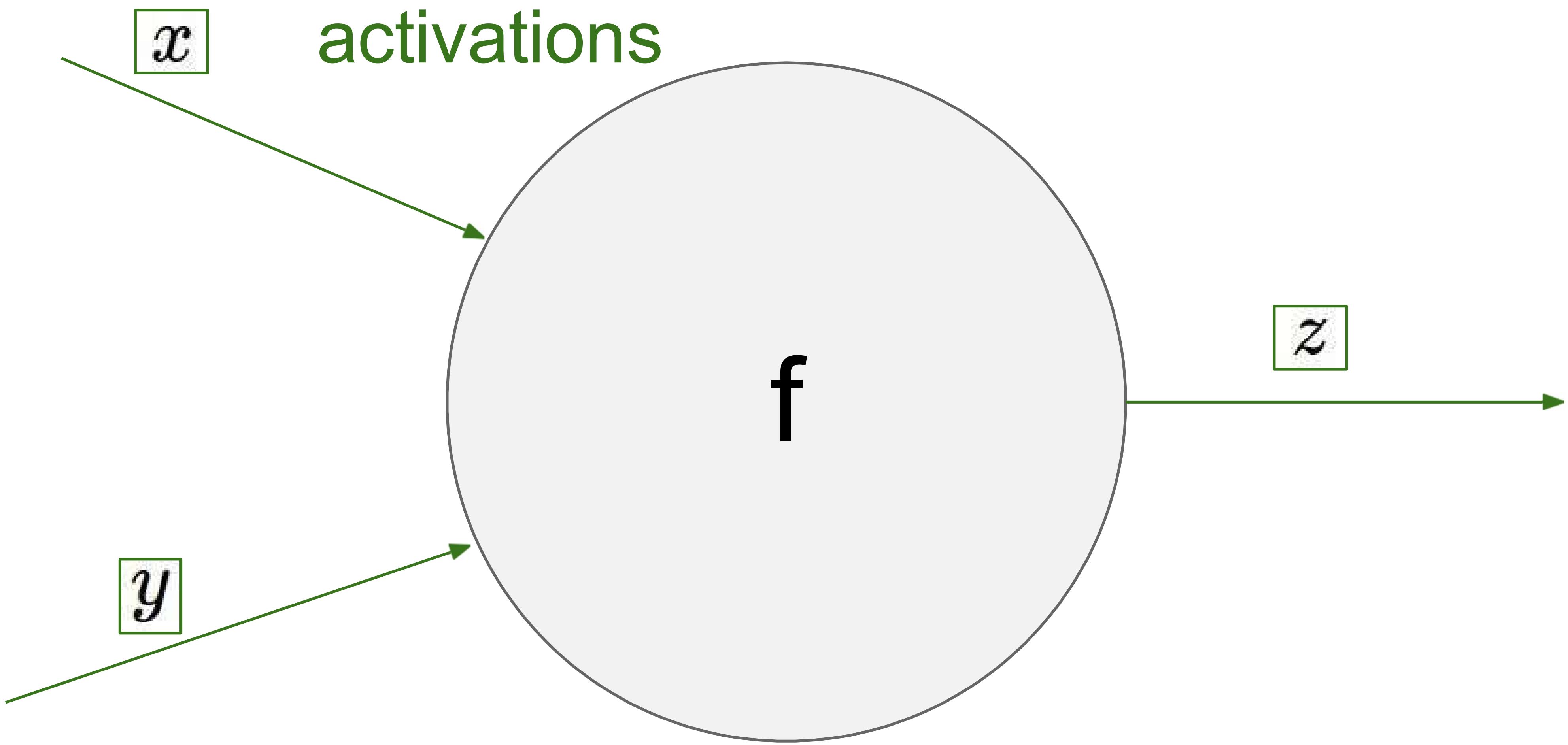

Any Function / Gate

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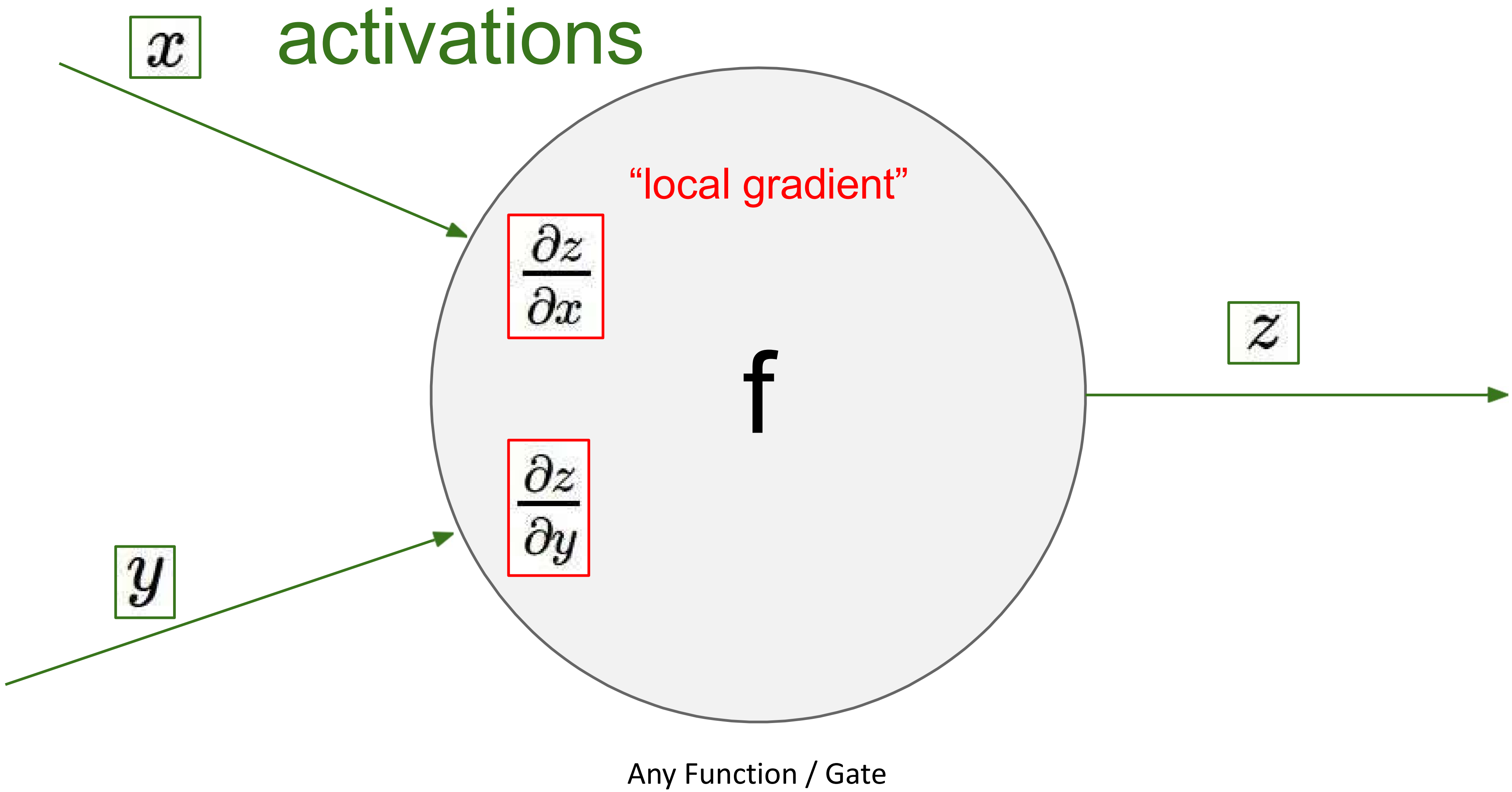

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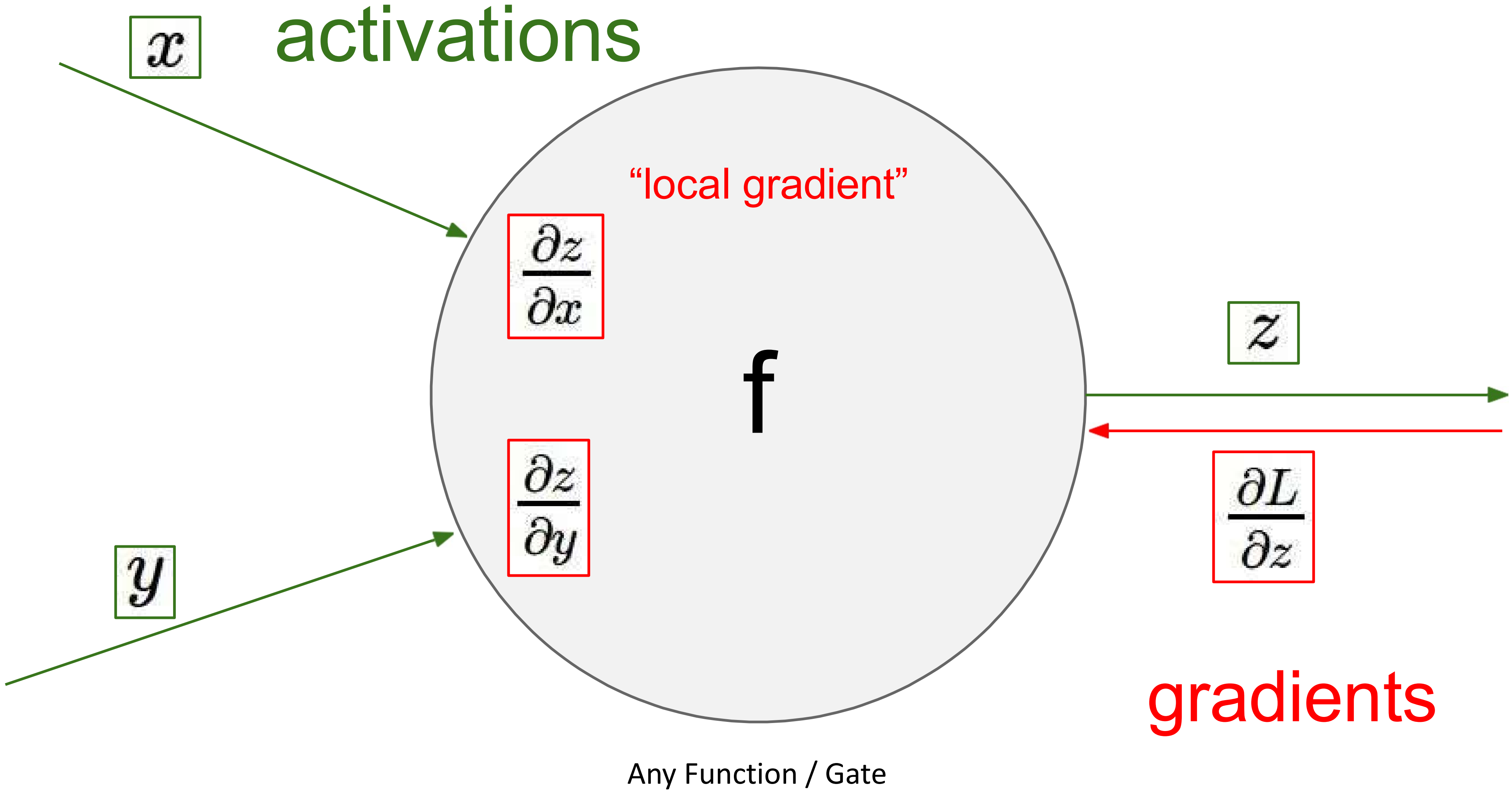

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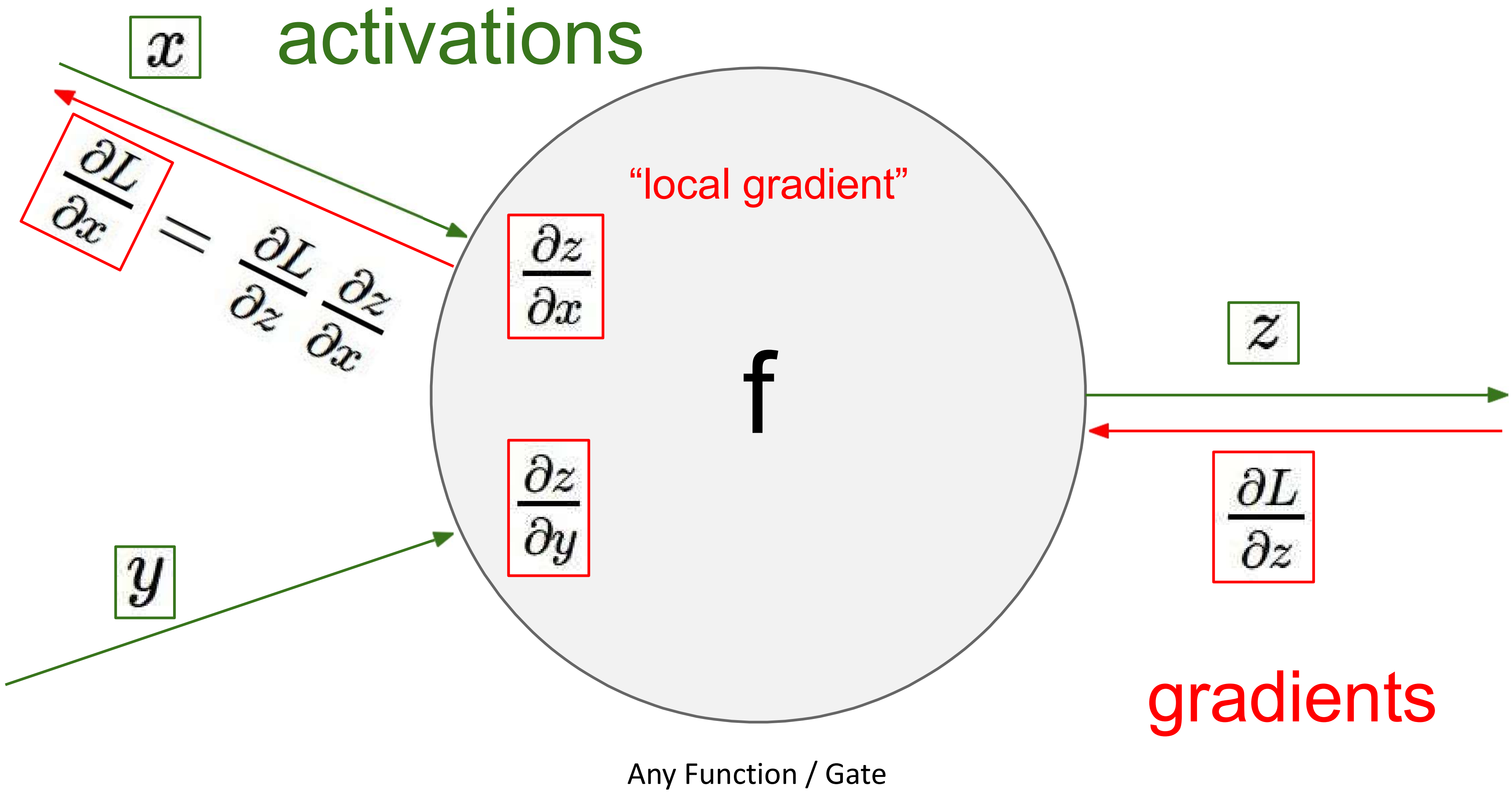

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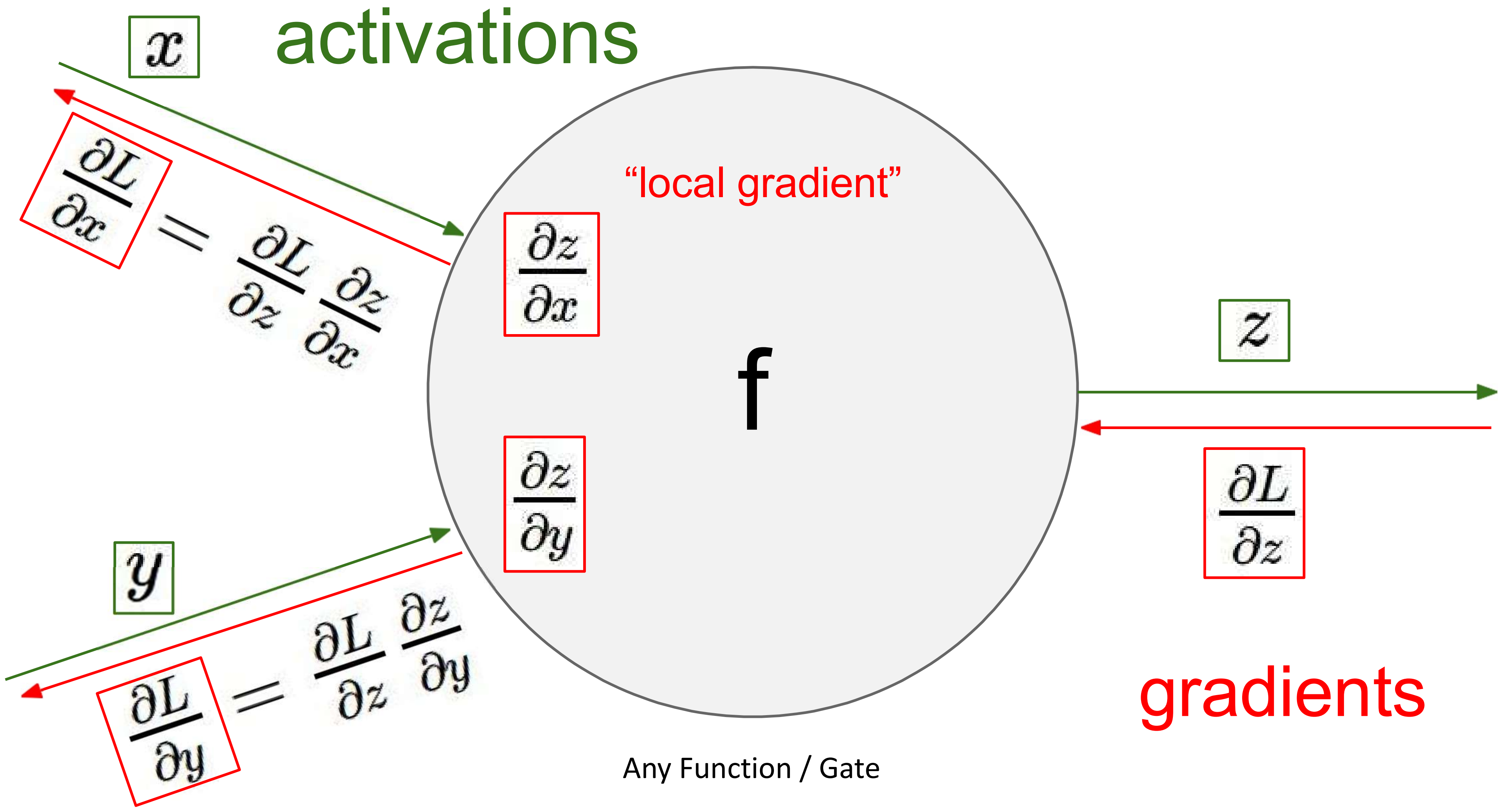

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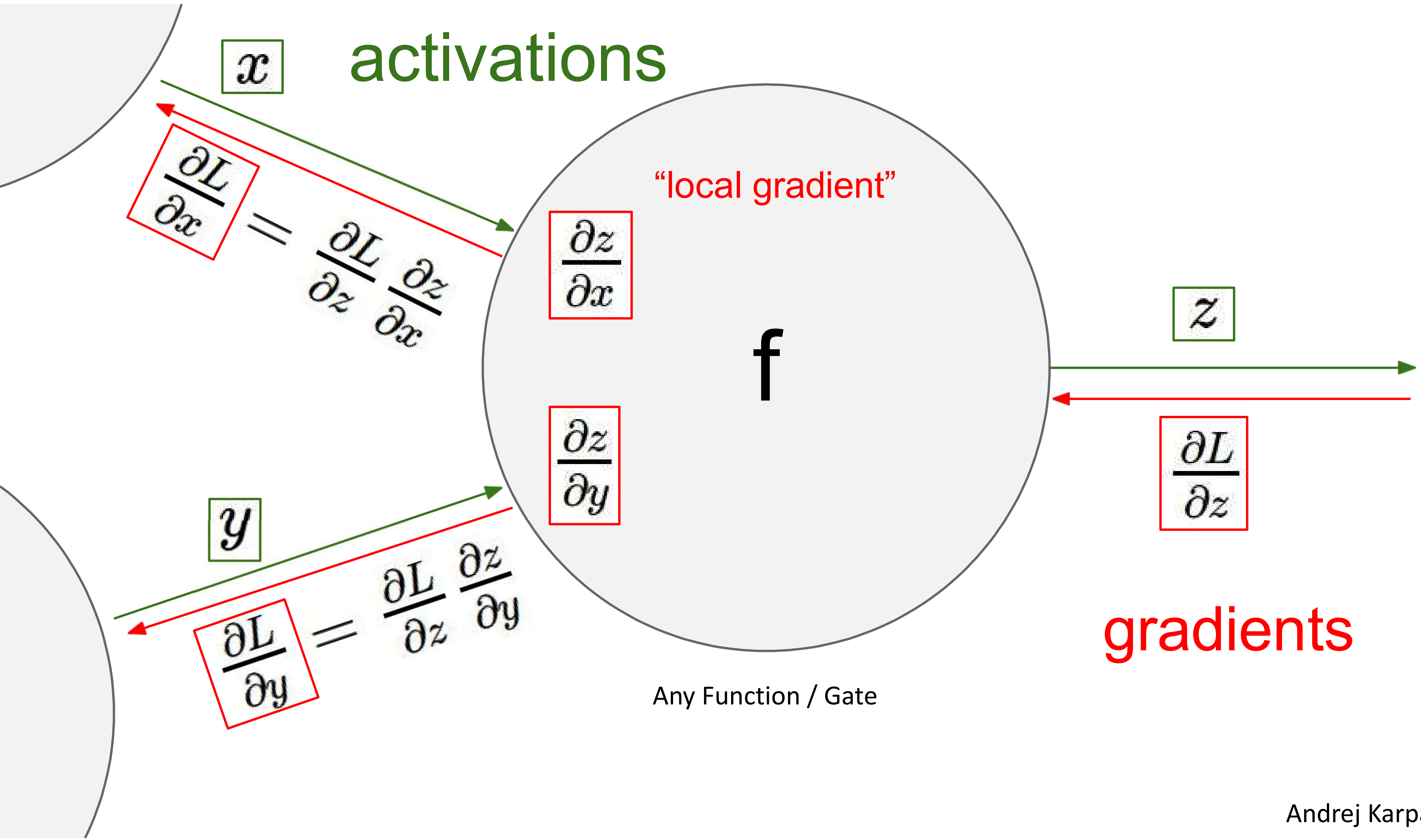

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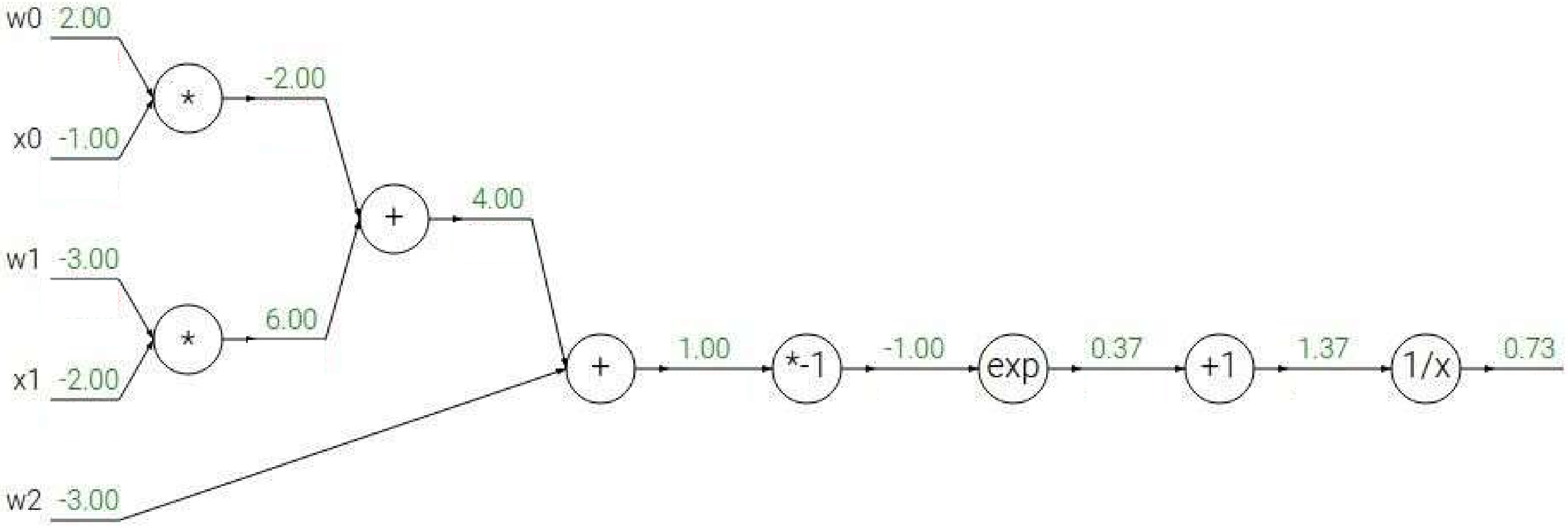
Another example: $$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$



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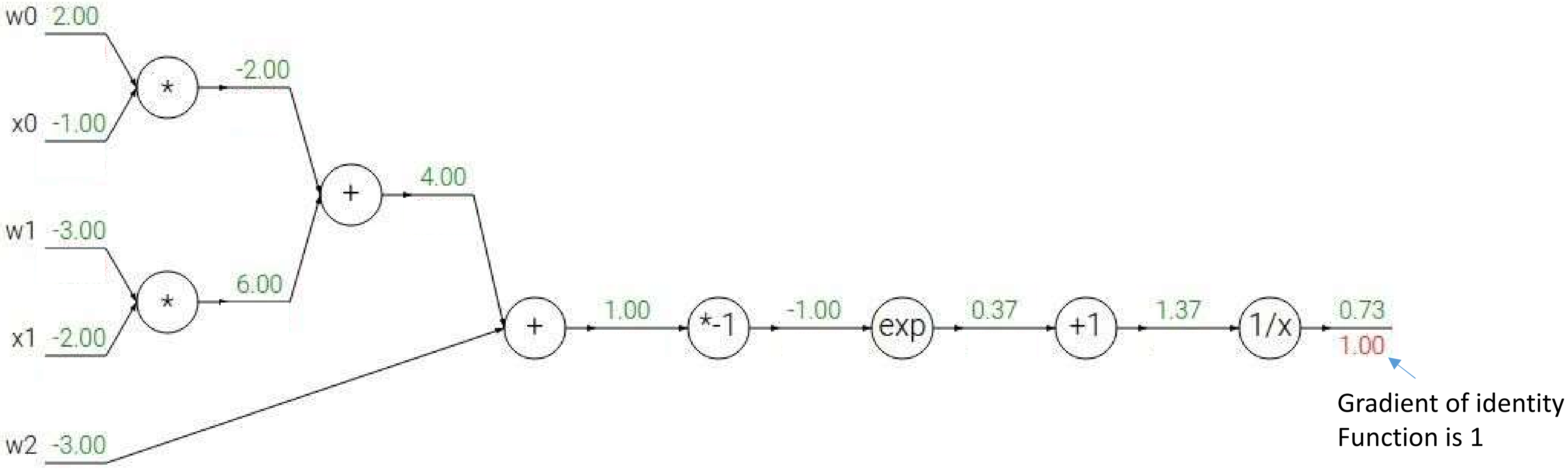

$$f(x) = e^x \quad \rightarrow \quad \frac{df}{dx} = e^x$$

$$f_a(x) = ax \quad \rightarrow \quad \frac{df}{dx} = a$$

$$f(x) = \frac{1}{x} \quad \rightarrow \quad \frac{df}{dx} = -1/x^2$$

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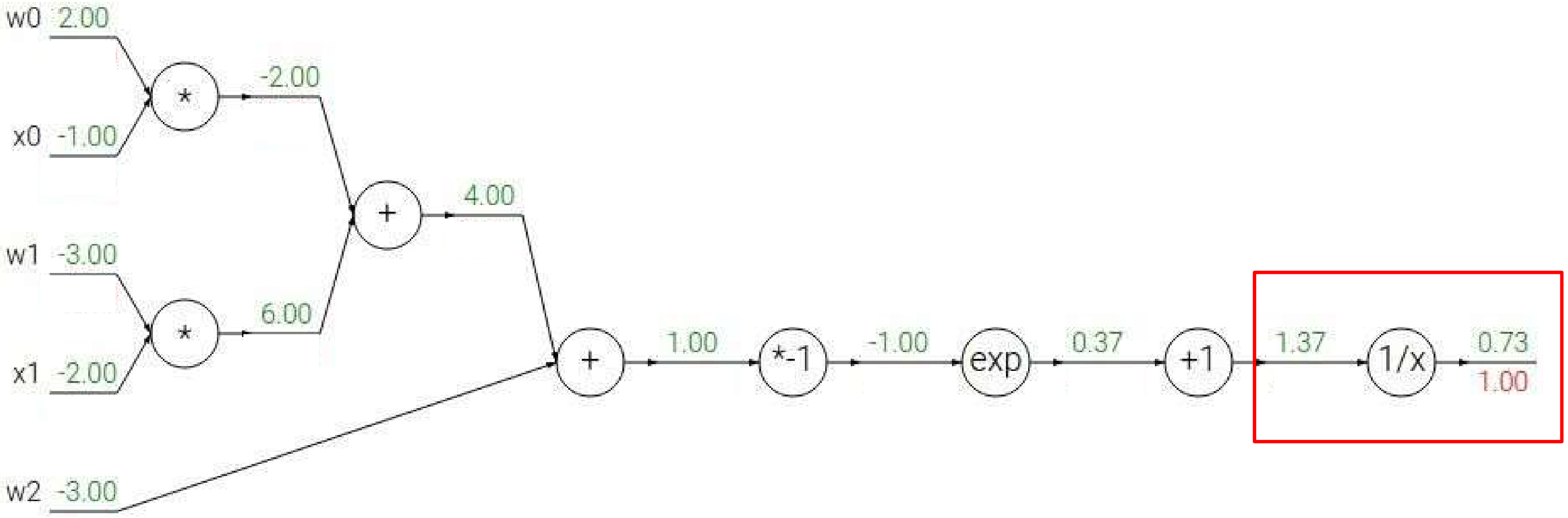

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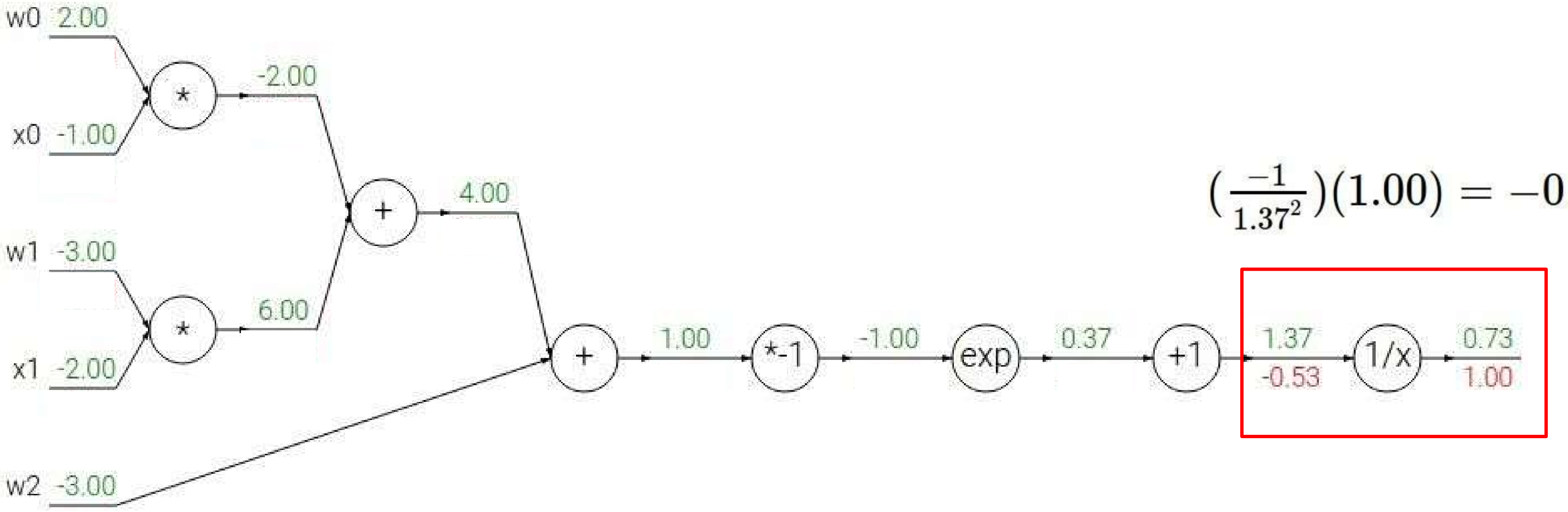

$$\left(\frac{-1}{1.37^2}\right)(1.00) = -0.53$$

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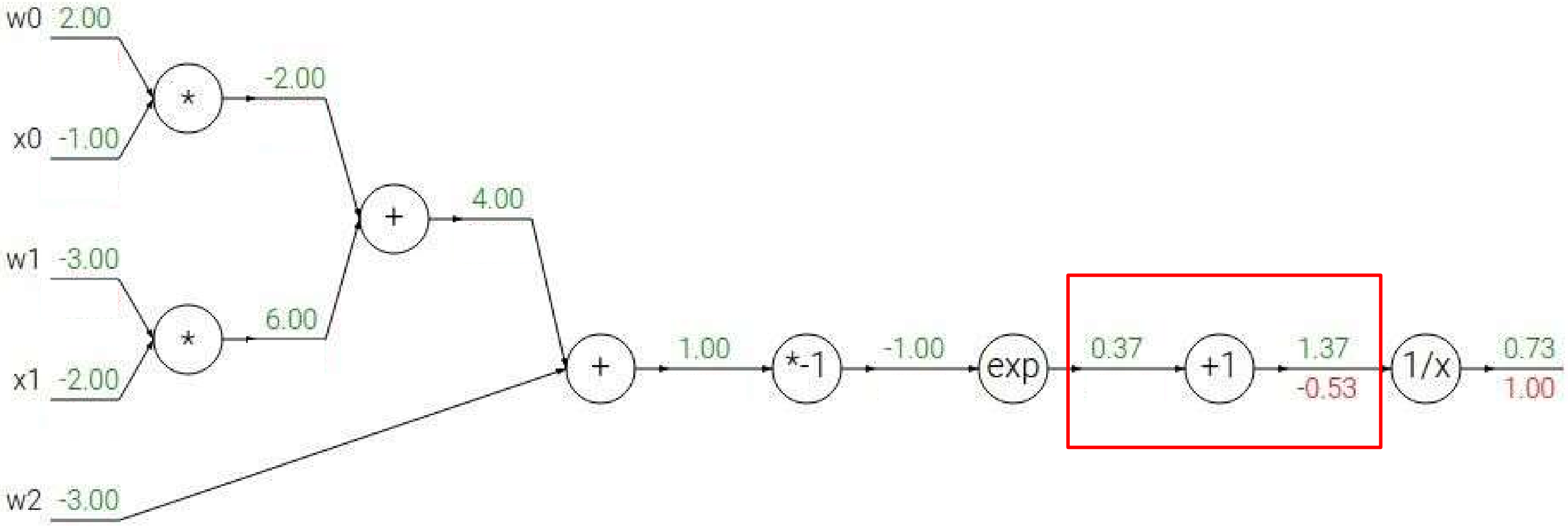

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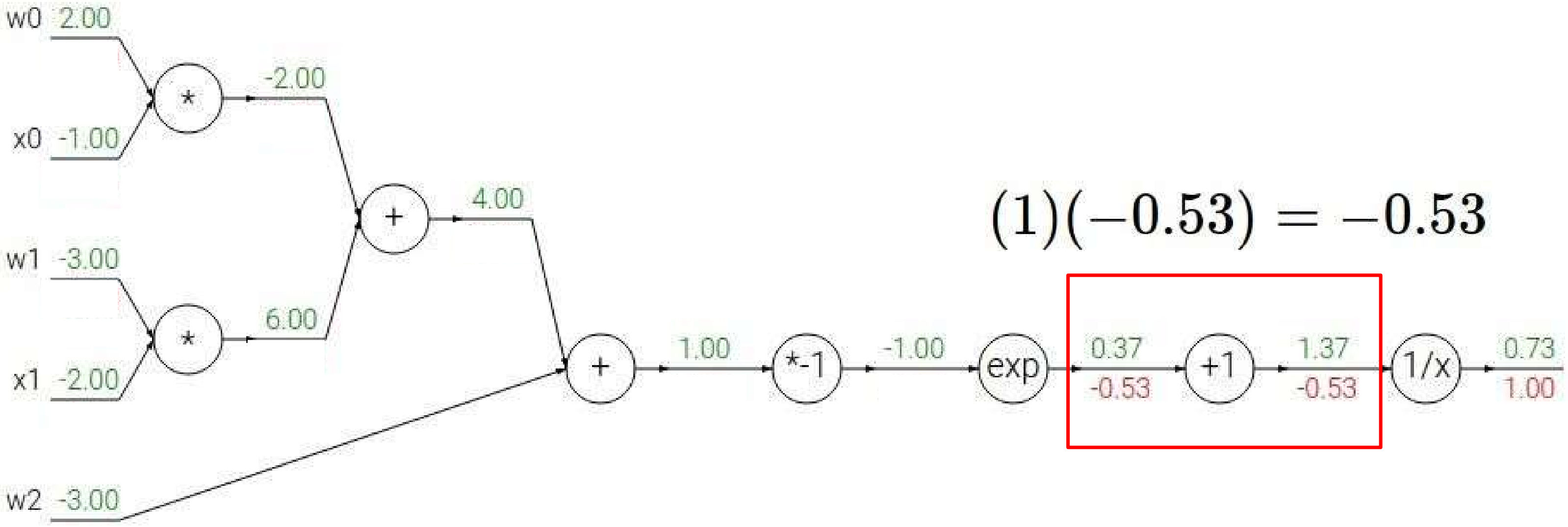

$$(1)(-0.53) = -0.53$$

$f(x) = e^x$	$\rightarrow$	$\frac{df}{dx} = e^x$	$f(x) = \frac{1}{x}$	$\rightarrow$	$\frac{df}{dx} = -1/x^2$
$f_a(x) = ax$	$\rightarrow$	$\frac{df}{dx} = a$	$\boxed{f_c(x) = c + x}$	$\rightarrow$	$\boxed{\frac{df}{dx} = 1}$

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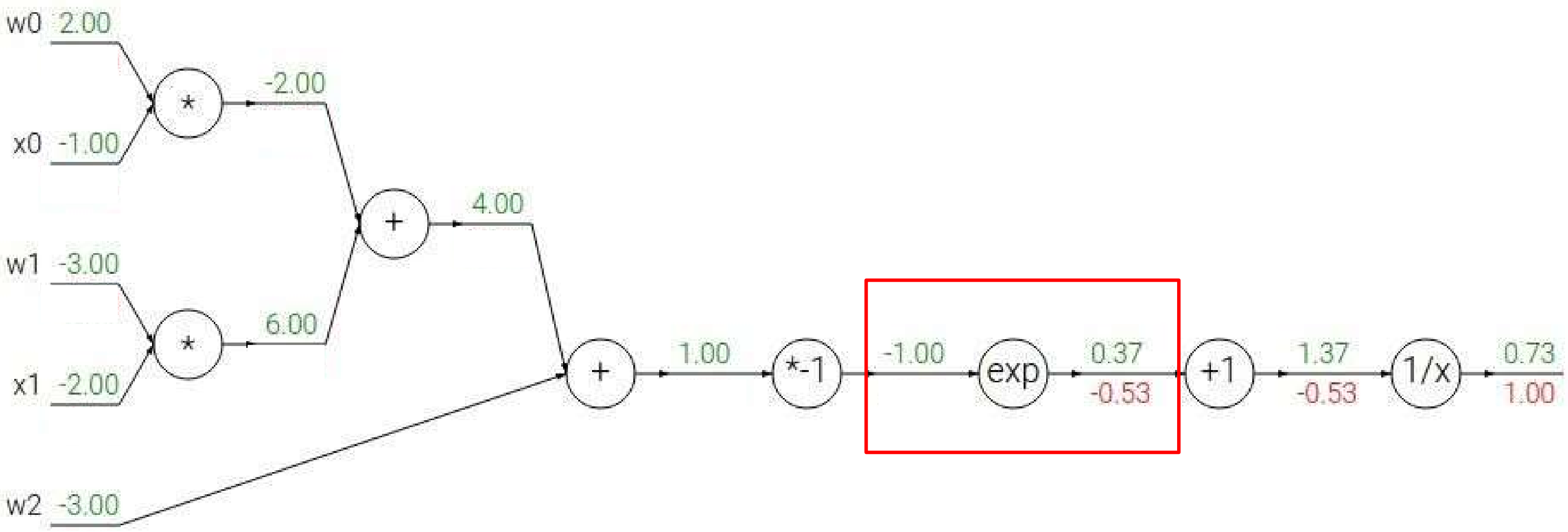

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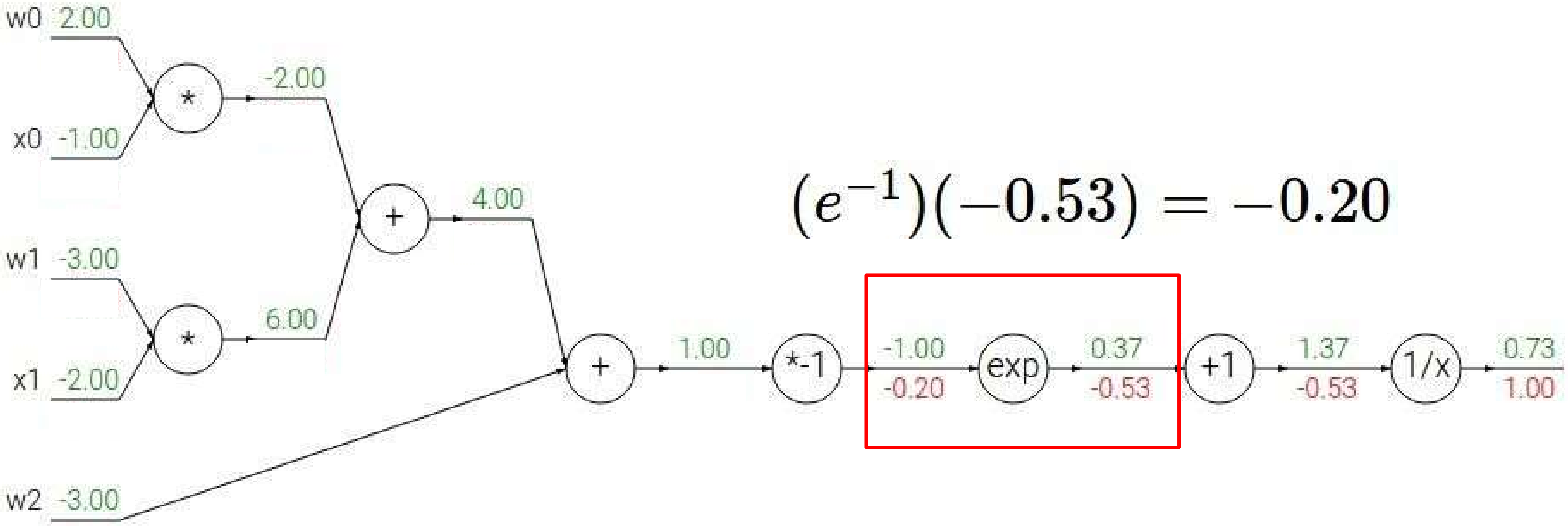

$$(e^{-1})(-0.53) = -0.20$$

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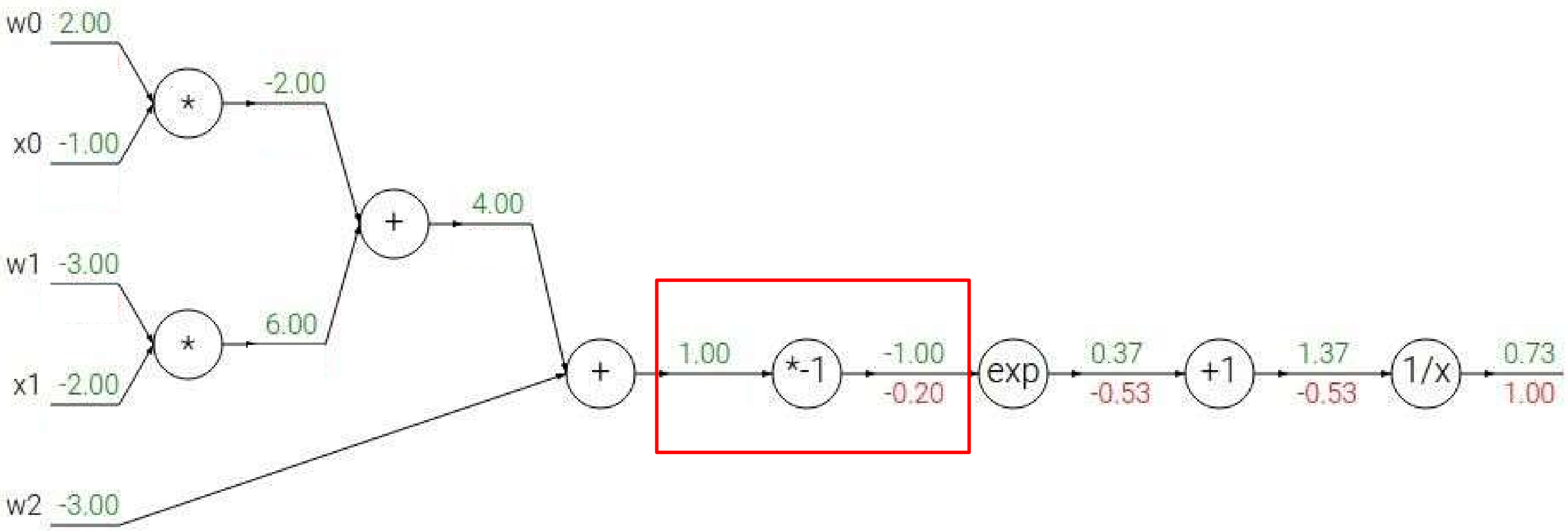

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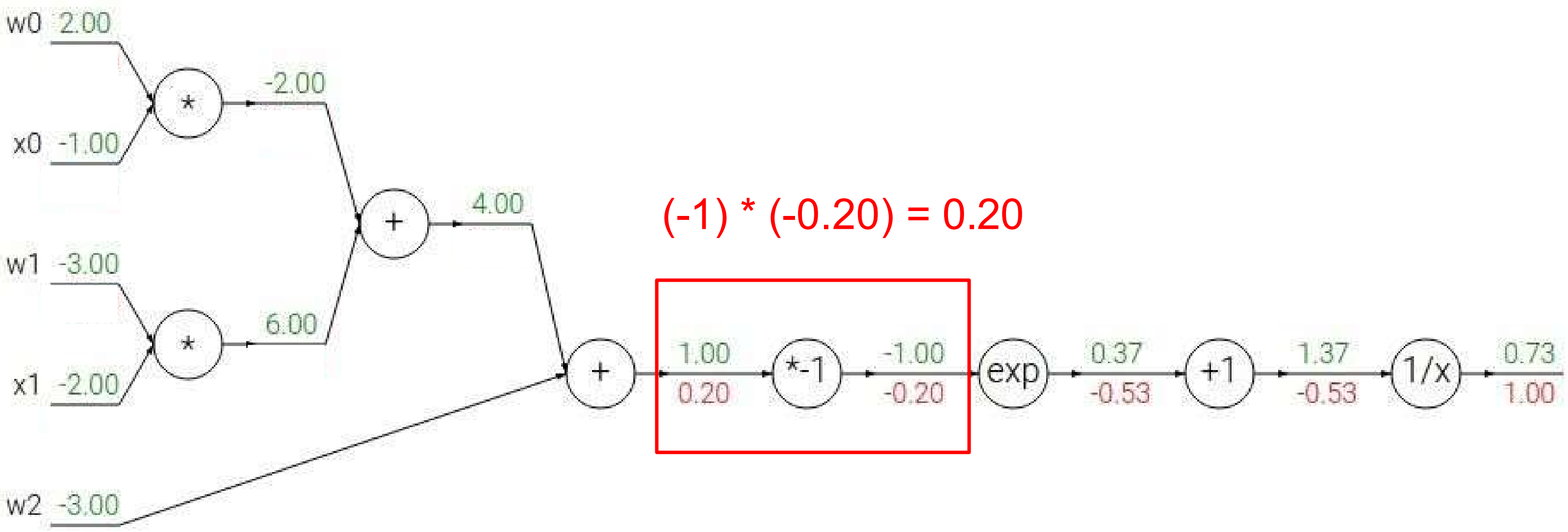

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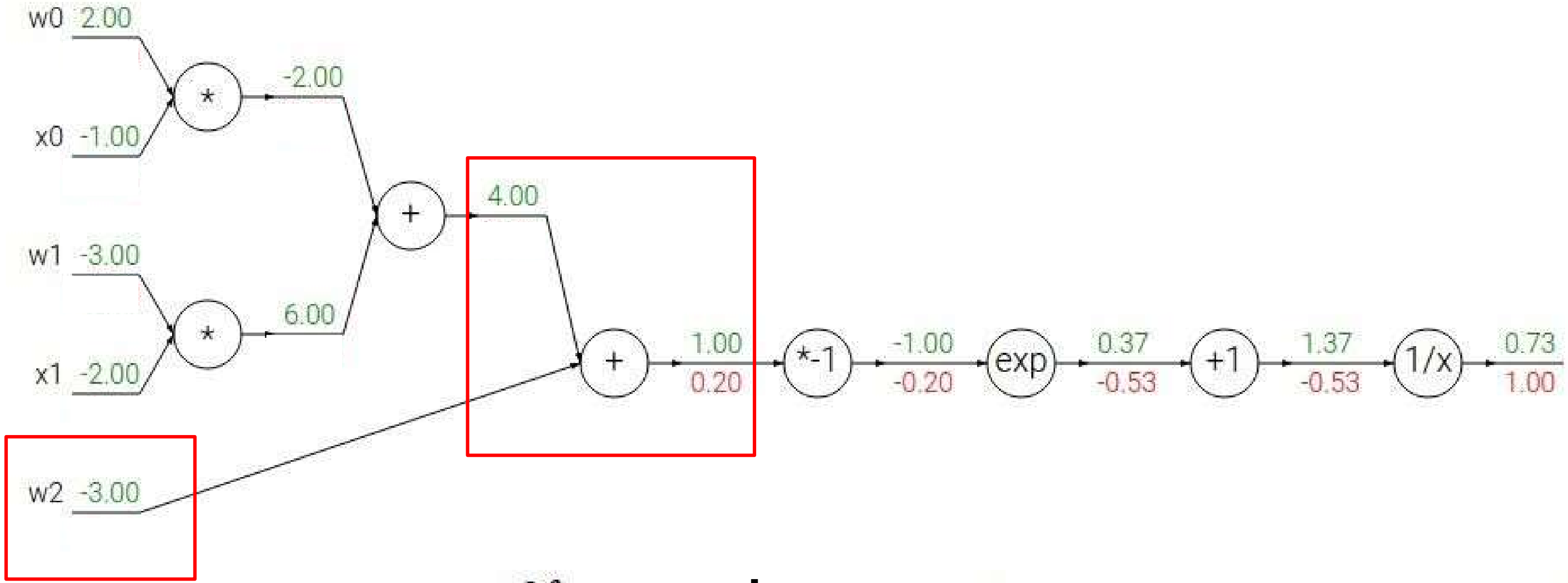

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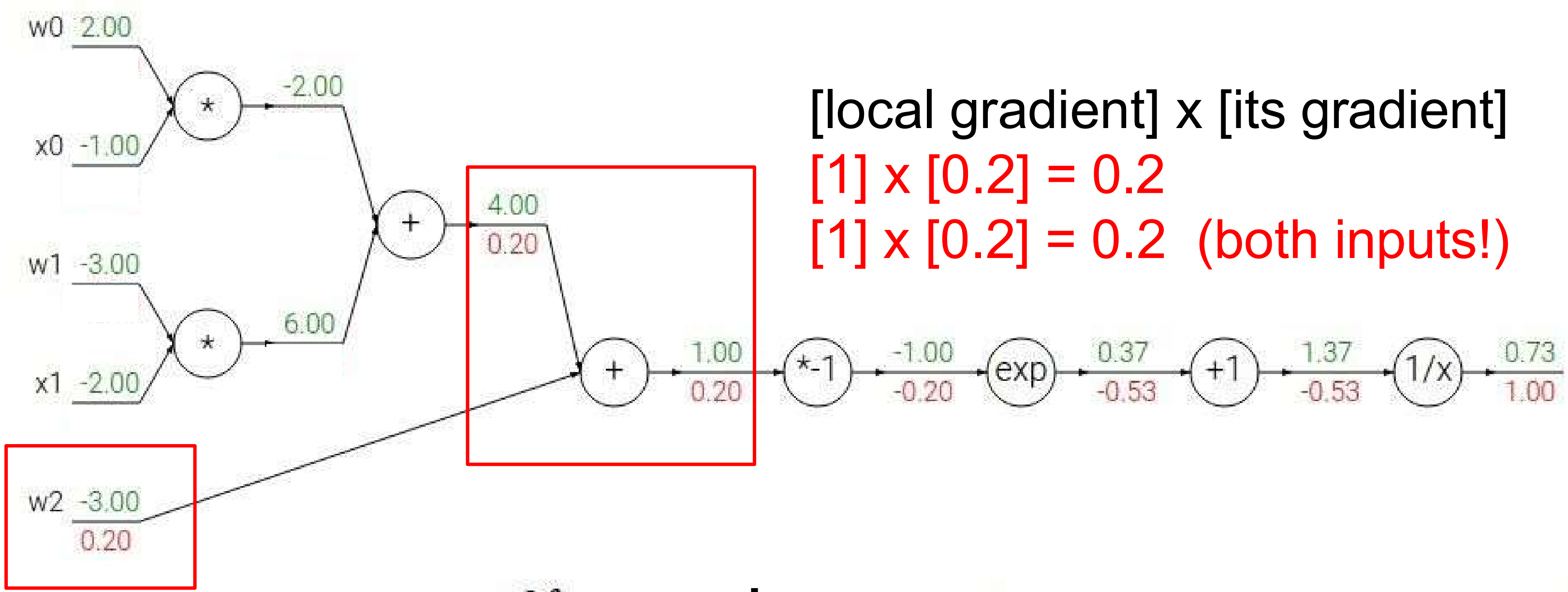

[local gradient] x [its gradient]

[1] x [0.2] = 0.2

[1] x [0.2] = 0.2 (both inputs!)

$$f(x) = e^x \quad \rightarrow \quad \frac{df}{dx} = e^x \qquad\qquad f(x) = \frac{1}{x} \quad \rightarrow \quad \frac{df}{dx} = -1/x^2$$

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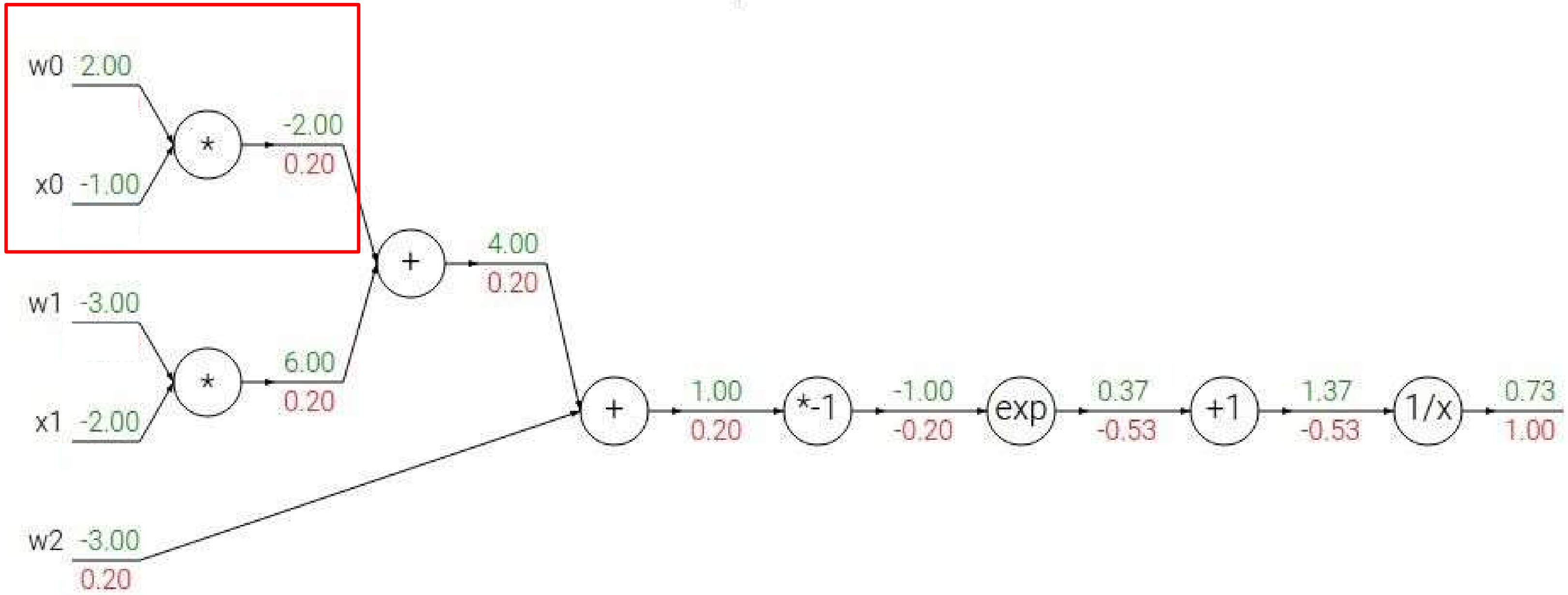

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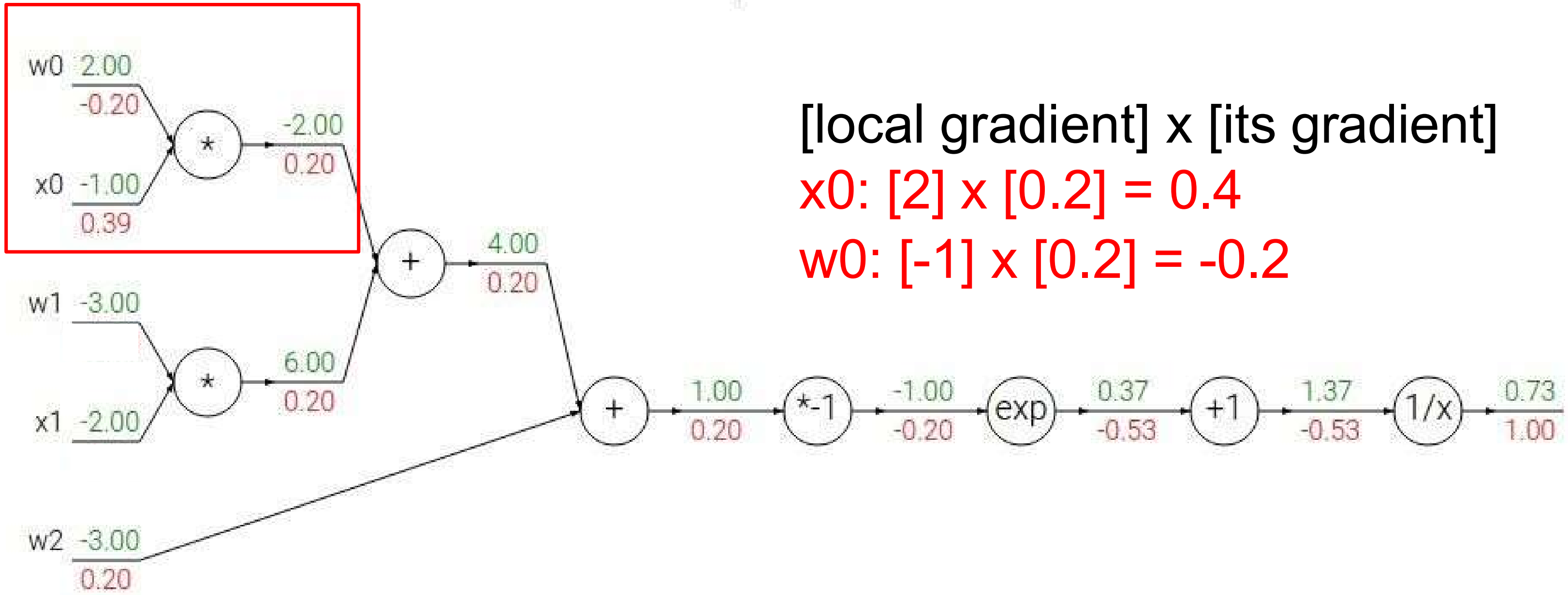

[local gradient] x [its gradient]

x0: [2] x [0.2] = 0.4

w0: [-1] x [0.2] = -0.2

$$f(x) = e^x \quad \rightarrow \quad \frac{df}{dx} = e^x \qquad\qquad f(x) = \frac{1}{x} \quad \rightarrow \quad \frac{df}{dx} = -1/x^2$$

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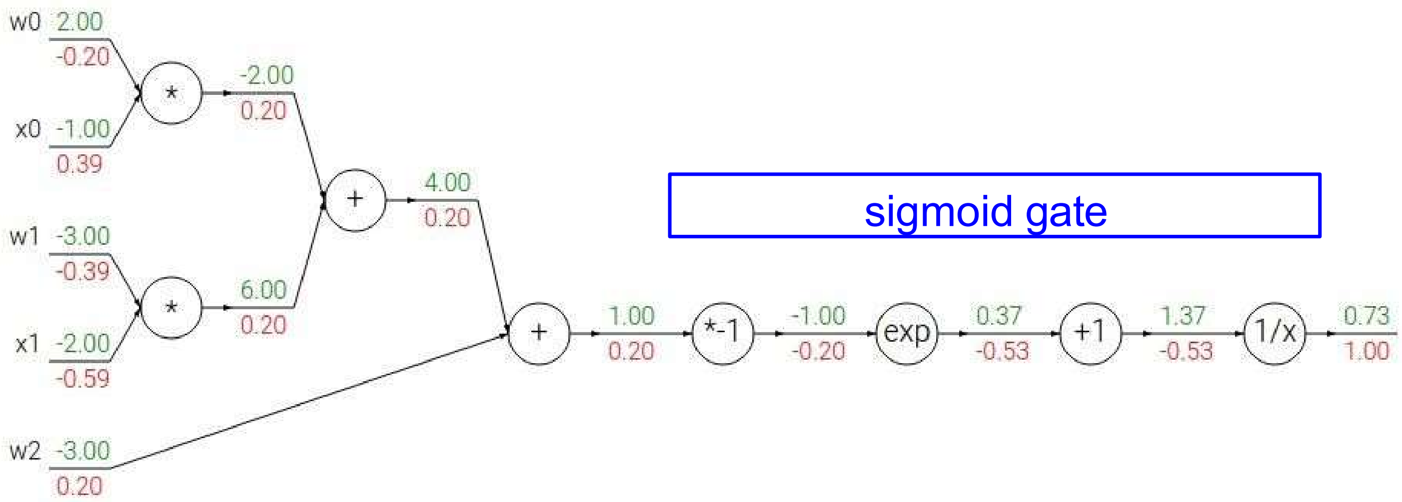
$$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$

$$\boxed{\sigma(x) = \frac{1}{1+e^{-x}}}$$ sigmoid function

$$\frac{d\sigma(x)}{dx} = \frac{e^{-x}}{(1+e^{-x})^2} = \left(\frac{1+e^{-x}-1}{1+e^{-x}}\right)\left(\frac{1}{1+e^{-x}}\right) = (1-\sigma(x))\sigma(x)$$



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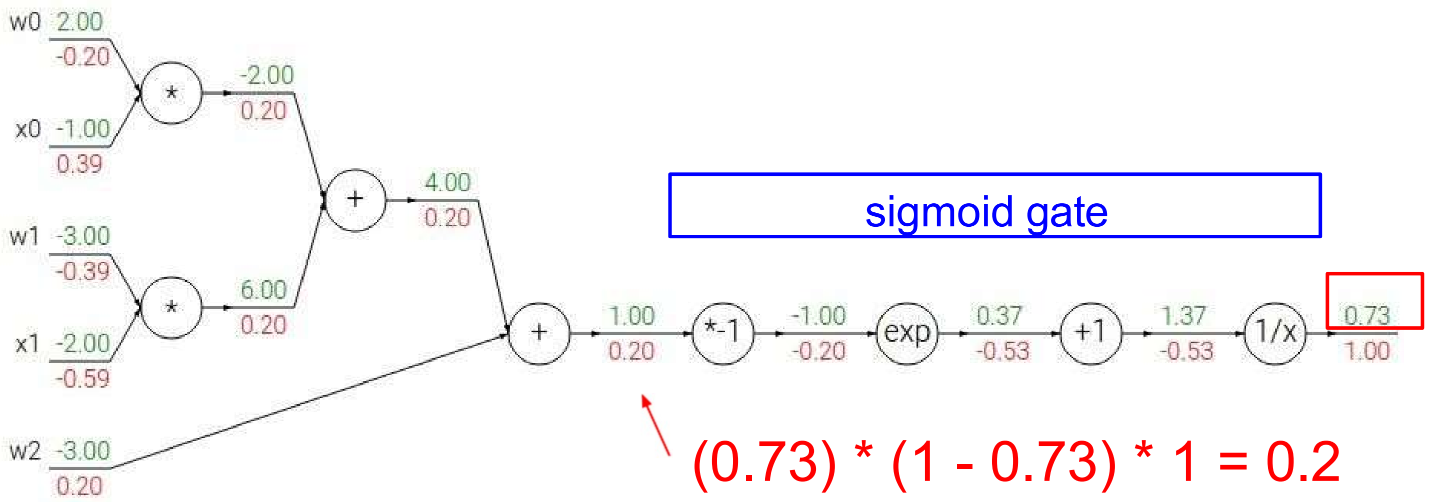

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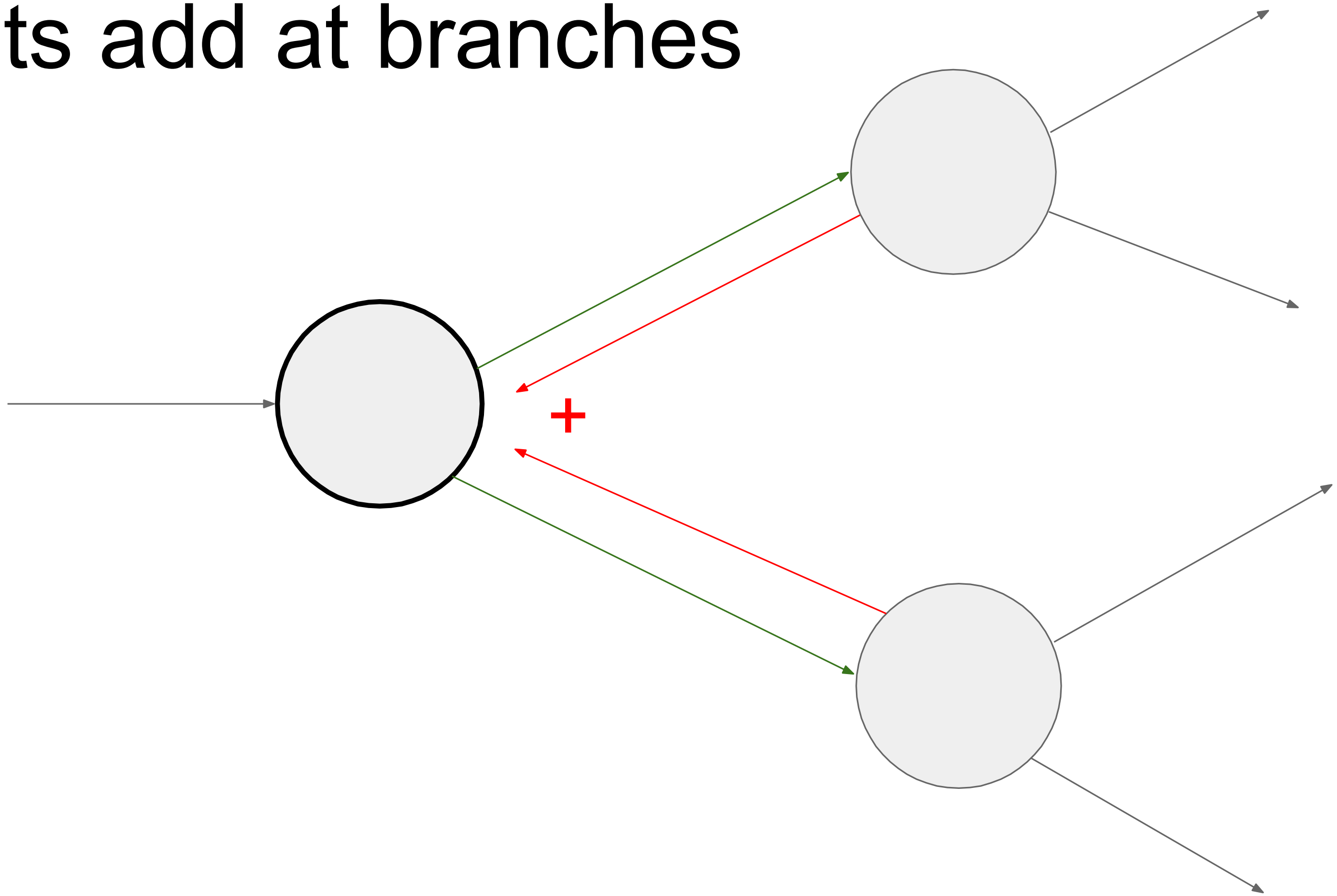
Gradients add at branches



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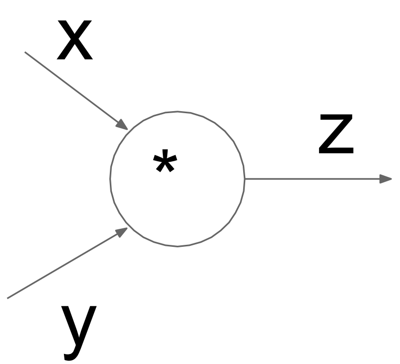
Implementation: forward/backward API




```
class MultiplyGate(object):
    def forward(x,y):
        z = x*y
        self.x = x # must keep these around!
        self.y = y
        return z
    def backward(dz):
        dx = self.y * dz # [dz/dx * dL/dz]
        dy = self.x * dz # [dz/dy * dL/dz]
        return [dx, dy]
```

(x,y,z are scalars)

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Implementation: forward/backward

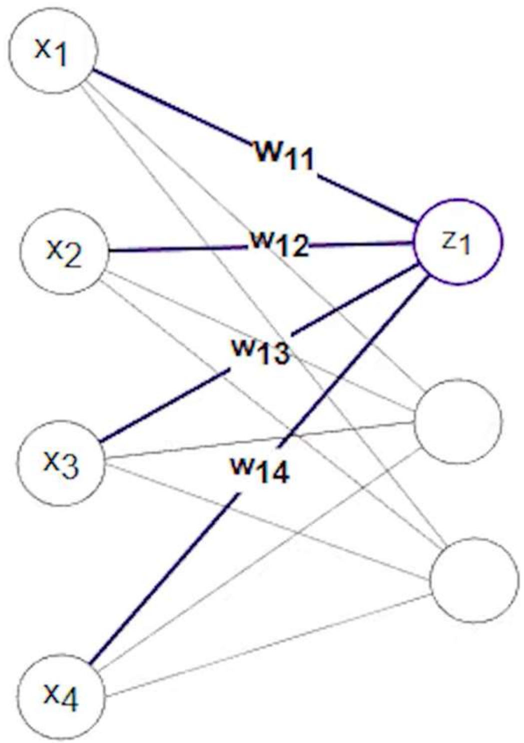
1- Find generic equations for forward and backward propagation

2- Find gradient values for the following weights

weights = numpy.array([0.80, 0.87, 0.16, 0.96, 0.89, 0.87, 0.31, 0.08, 0.09, 0.69, 0.03, 0.42])

inputs = numpy.array([0.75,0.98, 0.74, 0.28])

biases = numpy.array([0.68, 0.83, 0.01])

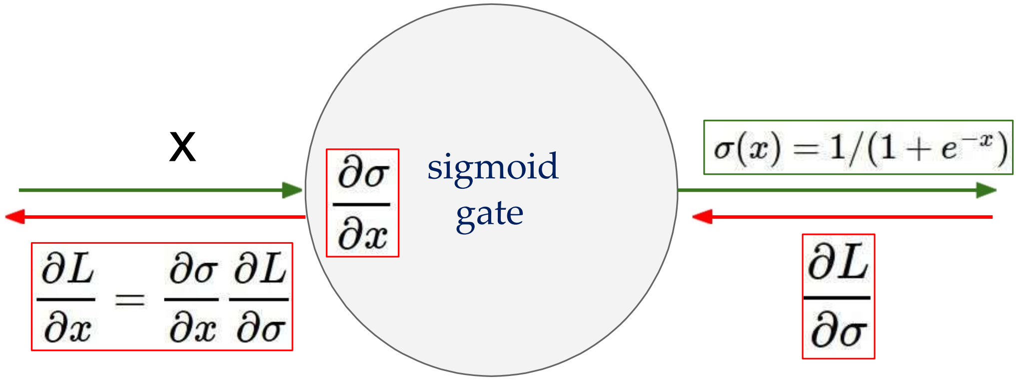


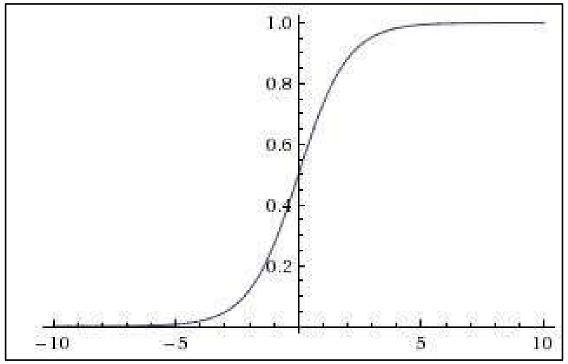
Backpropagation Summary

- Modern NNs employ the ***backpropagation*** method for calculating the gradients of the loss function $\nabla\mathcal{L}(\theta) = \partial\mathcal{L}/\partial\theta_i$
  - Backpropagation is short for "backward propagation"
- For training NNs, forward propagation (forward pass) refers to passing the inputs x through the hidden layers to obtain the model outputs (predictions) y
  - The loss $\mathcal{L}(y, \hat{y})$ function is then calculated
  - Backpropagation traverses the network in reverse order, from the outputs y backward toward the inputs x to calculate the gradients of the loss $\nabla\mathcal{L}(\theta)$
  - The chain rule is used for calculating the partial derivatives of the loss function with respect to the parameters θ in the different layers in the network
- Each update of the model parameters θ during training takes one forward and one backward pass (e.g., of a batch of inputs)
- Automatic calculation of the gradients (automatic differentiation) is available in all current deep learning libraries
  - It significantly simplifies the implementation of deep learning algorithms, since it avoids deriving the partial derivatives of the loss function by hand

Vanishing Gradient Problem

- In some cases, during training, the gradients can become either very small (vanishing gradients) of very large (exploding gradients)
  - They result in very small or very large update of the parameters
  - Solutions: change learning rate, ReLU activations, regularization





What happens when x = -10?
What happens when x = 0?
What happens when x = 10?

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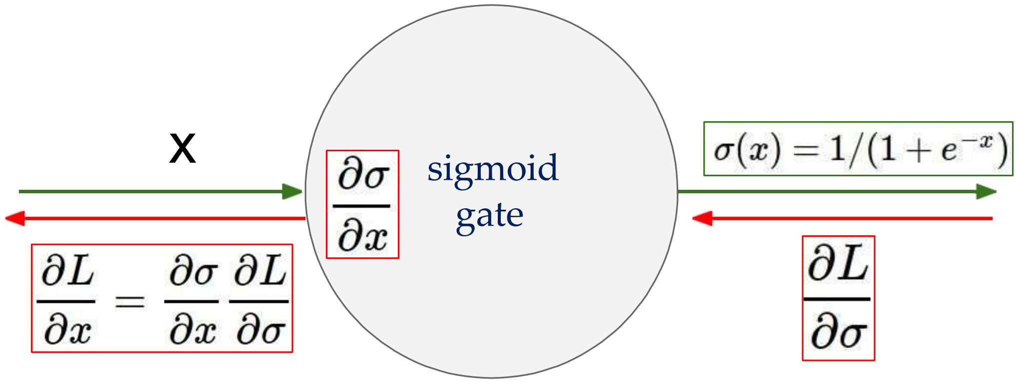
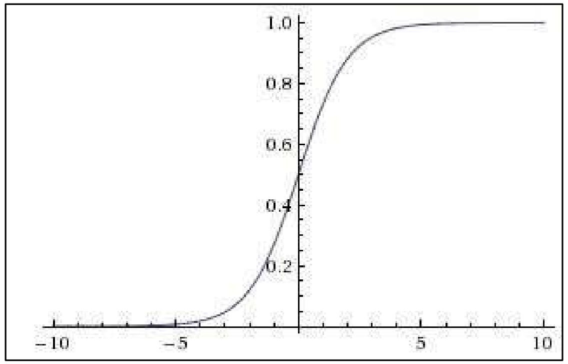
Vanishing Gradient Problem





What happens when x = -10? (almost zero)
What happens when x = 0? (small value)
What happens when x = 10? (almost zero)

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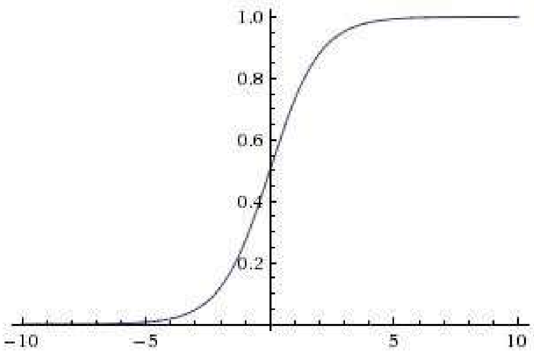
Vanishing Gradient Problem

Activation Functions



Sigmoid

$$\sigma(x) = 1/(1 + e^{-x})$$

- Squashes numbers to range [0,1]

- 3 problems:
  1. Saturated neurons "kill" the gradients
  2. exp() is a bit compute expensive

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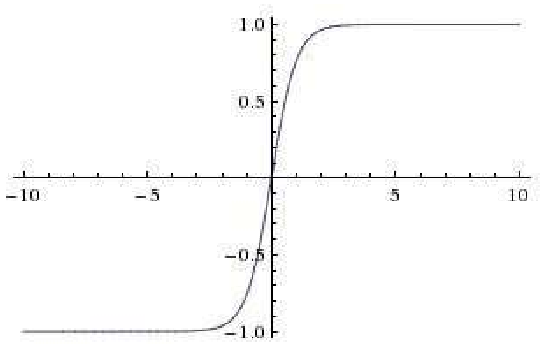
Vanishing Gradient Problem

Activation Functions



tanh(x)

- Squashes numbers to range [-1,1]
- zero centered (nice)
- still kills gradients when saturated :(

[LeCun et al., 1991]

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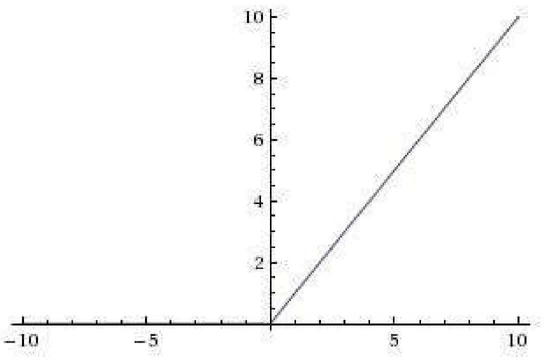
Vanishing Gradient Problem

Activation Functions



ReLU
(Rectified Linear Unit)

- Computes **f(x) = max(0,x)**

- Does not saturate (in +region)
- Very computationally efficient
- Converges much faster than sigmoid/tanh in practice (e.g. 6x)

- What happens when input is less than zero ?

[Krizhevsky et al., 2012]

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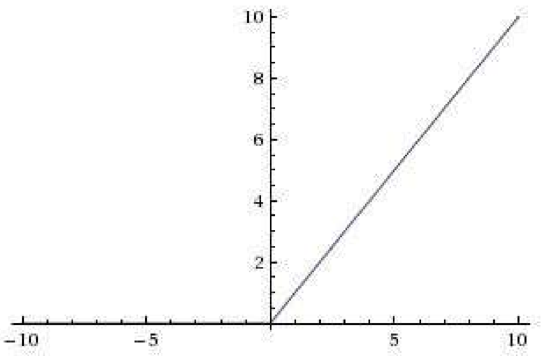
Vanishing Gradient Problem

Activation Functions



ReLU
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- Computes **f(x) = max(0,x)**

- Does not saturate (in +region)
- Very computationally efficient
- Converges much faster than sigmoid/tanh in practice (e.g. 6x)

- What happens when input is less than zero ?
  - Gradients are zero

[Krizhevsky et al., 2012]

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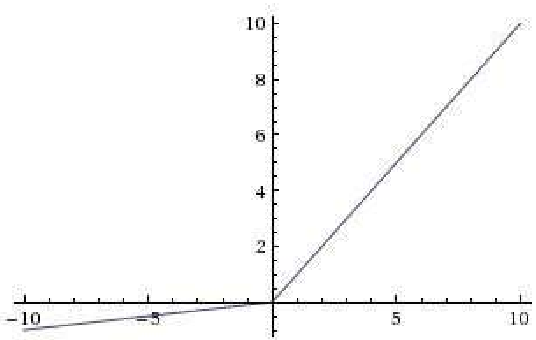
Vanishing Gradient Problem

Activation Functions

[Mass et al., 2013]
[He et al., 2015]



Leaky ReLU

$$f(x) = \max(0.01x, x)$$

- Does not saturate
- Computationally efficient
- Converges much faster than sigmoid/tanh in practice! (e.g. 6x)
- **will not "die".**

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Vanishing Gradient Problem

In Practice,

- Use ReLU. Be careful with your learning rates
- Try out Leaky ReLU
- Try out tanh but don't expect much
- Don't use sigmoid

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