Recall on J-holomorphic curves and Hamiltonian Floer theory

This note is helping to fill the gap between [MS, J-holo] [Audin] and [Siedel], from classical Morse theory toward Lagrangian Floer theory [FOOO] as a research monograph may be much harder.

Moduli space in differential & algebraic geometry have various use:

- Classification (Most in algebraic, esp. birational)
- Given $X$, moduli of some datum on $X =: \mathcal{M}$, the "$\mathcal{M}$ is closed to $X$"

E.g.: In differential side, Donaldson's gauge, $\mathcal{M}_\lambda \cong X \times (0,\lambda)$ local diffeomorphism

$X^4$ cpt.. $\mathcal{M}$ ASD moduli space, $\mathcal{M}_\lambda$ is close to boundary $\partial\mathcal{M}$

In algebraic side, $X$ is K3, with conditions pasted, $\mathcal{M} = \mathcal{M}^s$ is also K3

$\mathcal{M}$ semi-stable sheaves & $H(X;\mathbb{Z}) \cong H(\mathcal{M};\mathbb{Z})$

Rk. ① When one writing papers on moduli spaces expected to be used in such way, he should check $\mathcal{M}$ does "closed to" $X$, otherwise meaningless.

② From differential geometry, we can understand why this happens as the ASD equation is gained by variation principle, thus it recovers $X$ itself when it turns to boundary;

But I'm not sure why we can confirm this happens in algebraic side. I guess may be motivated by the dual Abelian variety = moduli of line bundles $\cong$ Abelian variety itself

- Detect $X$ and give invariants via counting / integral

GW, Donaldson, SW ... moduli invariant under isomorphism of underlying space (up to a reparametrization)

- Combination of upper two, for defining differential of a complex

cobordism counting $\rightsquigarrow$ homology theory

Rk. Usually, such homology theory recovers invariants easily.

Note focus on (SC).

(An excellent reference is [Bott, Morse Theory Indomitable] !)

PART II Morse $\rightsquigarrow$ Witten $\rightsquigarrow$ Floer (インドミタブル)

Morse theory is a far-reaching geometric method to detect topological data

$\boxed{\text{Morse} = \text{Singular}} \subset$ Sheaf

$\mathbb{C}$ — infinite-dim

Picard-Lefschetz — Floer — [RPS] — microlocal

Algebraic Geometry $\supset$ Weil conjecture — $\mathbb{F}_q$ — Lefschetz situation [S] $\rightsquigarrow$ Fukaya sector [RPS] — Symplectic Geometry

Start from 1928 Morse theory now still alive in Algebra, Geometry, Analysis and even applications. How other theory can be so powerful

We start at the story of Morse = Singular

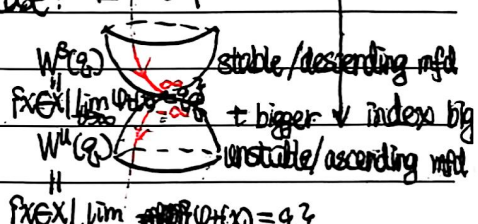
- Given $X^n$ smooth, closed, cpt (Riemannian) mfd and $f: X \to \mathbb{R}$ is Morse (nondegenerate critical pts) existence needs a proof) with $\mathrm{grad}(f) \leftrightarrow$ flow $\varphi_t$

Now, we have datum (A) Each critical pt (of $f$), we have a index (Morse)

(B) Any two critical pts with different indexes, we have flow lines connecting them, their moduli space is $M(p,q)$ (Subtle)

Let's explain (B) adapted to Morse case:

Step1




Our "stable" named due to $q$ is stable critical pt for such submfd

For flow lines connecting $p$ & $q$:

$\Rightarrow M(p,q) = (W^u(p) \cap W^s(q))/\sim$

$= (W^u(p) \cap W^s(q))/\mathbb{R}$

Step2

"Nontrivial transversality issue"

Subtle issue. $W^u(p) \pitchfork W^s(q)$ is assumed, we call as Morse-Smale condition

(Not Hard) Fact. $\dim W^s(q) = \mathrm{ind}(q)$ & $\dim W^u(q) = n - \mathrm{ind}(q)$ (negative eigenvalues of Hessian)

thus $\dim(M(p,q)) = \mathrm{ind}(p) - \mathrm{ind}(q) - 1$ when MS holds and they're both even disks

in particular, $p, q$ "nearby" $\Rightarrow \dim(M(p,q)) = 0$

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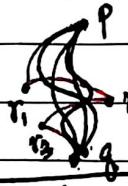
Compactness & Transversality both needed, latter one is assumed.

Compactification $\overline{M(p,q)}$ is mfd with corners

This corner structure is complicated, each k-dim corner can be given if critical pts are enough: it consists all broken flow lines k times:




$$\overline{M(p,q)}_k = \bigcup_{\substack{r_1\cdots r_k \in \mathrm{Crit}(f) \\ p\neq r_1\neq q,\ \forall i}} M(p,r_1)\times\cdots\times M(r_k,q)$$

only $\mathrm{ind}(p)<\mathrm{ind}(r_i)<\mathrm{ind}(q)$ contribute to this.

(1-corners are just boundary)

$\boxed{\text{Compactness + 0-dimensional = Able to count!}}$

Still needs a nontrivial pf, but idea is clear.

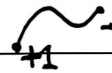
Morse cpx then: $CM_*(X,f)$ freely generated by critical pts

$$CM_*(X,f) \xrightarrow{d} CM_*(X,f)[-1]$$

$$[p] \longmapsto \sum_{\substack{\mathrm{ind}(q)=\mathrm{ind}(p)-1 \\ \text{all } q}} \#\overline{M(p,q)}\,[q] = \sum_{\substack{\mathrm{ind}(q)=\mathrm{ind}(p)-1 \\ \text{all } q}} \#M(p,q)[q]$$

$$d^2[p] = \sum_{\substack{\mathrm{ind}(q)=\mathrm{ind}(p)-2 \\ \text{all } q}} \Big(\#\bigcup M(p,r)\times M(r,q)\Big)[q] = \sum_q \#\partial M(p,q)[q] = 0$$

$M(p,q)$ is 1-mfd



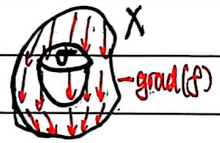

, boundary with sign

$\Rightarrow HM_*(X,f)$ defined

$HM_*(X,f) \cong H_*(X)$

critical $[p] \longmapsto W^s(p)$ a $\mathrm{ind}(p)$-dimensional cell

$\sum n_p[p] \longleftarrow \sigma$ a $m$-dim cell $\rightsquigarrow$




Done. (Need a proof, but not hard) ▨

$\rightsquigarrow$ Thus HM not depend on f and Riemann metric once MS holds and share all topological properties!

$t\to\infty$ in $\sigma \Rightarrow$ several pts $\sum n_p[p]$

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Witten's approach: de Rham instead of singular

Witten's deformation $d_t = e^{-tf} d e^{tf}$ for f Morse function given.

this gives deformed de Rham cpx

$\Omega^\bullet(t) = 0 \to \Omega^0 \xrightarrow{d_t} \Omega^1 \xrightarrow{d_t} \cdots \xrightarrow{d_t} \Omega^n \to 0$ called Witten's cpx $\xrightarrow{t\to 0}$ de Rham cpx

which gives same $H^\bullet_t(X) \cong H^\bullet_{dR}(X)$ as the conjugating not change ker / im

Then we link $H^\bullet_t(X;\mathbb{R})$ with Morse theory.

~~Witten's original paper stops at near critical points, so more Morse theory, Witten's paper is a good one.~~

Here we need Hodge a little, $\Delta = dd^* + d^*d$

and Hodge thm is: $\{\varphi\in\Omega^q \mid \Delta\varphi=0\} \cong H^q_{dR}(X)$ Harmonic forms

Rk, $d^*$ is the adjoint operator of $d$ given Riemann metric $g$

but when you introduce $d^* = (-1)^{q+1} * d *$, it seems we can do th without $g$ as we can give $T^*X$ a basis $\{dx_1,\cdots,dx_n\}$ and $*(dx_1\wedge\cdots\wedge dx_n) = (-1)^q (dx_{n+1}\wedge\cdots\wedge dx_n)$

Why we can define $*$ with "$g$ free"?

This's due to the basis $\{dx_1,\cdots,dx_n\}$ is choosen by dual $T^*X \cong TX$, give by $g$, thus $g$ is a must still.

We interpret Hodge thm as: we decompose $\Omega^\bullet = \bigoplus \Omega^q = \bigoplus_{q\in\mathbb{Z}}\bigoplus_{\lambda\in\mathbb{R}} \Omega^q_\lambda$

$\Omega^q_\lambda = \{\varphi\in\Omega^q \mid \Delta\varphi=\lambda\varphi\}$ eigensubspace, $\mathrm{Spec}(\Delta)\subset[0,\infty)$

$\Rightarrow \Omega^\bullet = (\bigoplus_q\bigoplus_{\lambda\neq 0}\Omega^q_\lambda) \oplus H^\bullet_{dR} \overset{①}{\Leftrightarrow} \Omega^q_0 = H^q_{dR}$

& ② $\forall\lambda>0$, $0\to\Omega^0_\lambda \xrightarrow{d^\lambda}\cdots\xrightarrow{d^\lambda}\Omega^n_\lambda\to 0$ with $d^\lambda$ restriction is exact

$\Rightarrow \forall\lambda_0>0$, $\Omega^*_{\lambda_0} = \bigoplus_{\lambda\le\lambda_0}(\bigoplus_q \Omega^q_\lambda)$ is a cpx and homology is $H^\bullet_{dR}(X)$.

$$= 0\to \bigoplus_{0\le\lambda\le\lambda_0}\Omega^0_\lambda \xrightarrow{d^\lambda} \bigoplus_{0\le\lambda\le\lambda_0}\Omega^1_\lambda \xrightarrow{d^\lambda}\cdots\xrightarrow{d^\lambda}\bigoplus_{0\le\lambda\le\lambda_0}\Omega^n_\lambda\to 0 \quad\cdots(*)$$

Why? contribute no homology, thus only left $\lambda=0$, by ①, it's $H^\bullet_{dR}$

By ②, exactness done.

Witten's deformation applied to $(*)$ $\Delta_t = d_t d_t^* + d_t^* d_t$ gives different decomposition $\Omega^\bullet(t) = \bigoplus \Omega^\bullet_\lambda(t) \Rightarrow \Omega^*_{\lambda_0}(t) = \bigoplus_{0\le\lambda\le\lambda_0}\Omega^\bullet_\lambda(t)$

here both ① & ② still holds! Thus cohomology still $H^\bullet_{dR}(X)$.

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Here, we see $t \to 0$ is de Rham = Singular, taking classical limit, Witten's physical idea tells him $t \to \infty$ the ground state = Morse!

de Rham $\qquad$ Bott explained why in mathematical view.

$0 \longrightarrow t \longrightarrow \infty$

Hodge $\qquad$ Morse

first we make it more precise. Witten's idea is:

- $t$ large $\Omega^*_{S^1_0}(t)$ is stable, denoted $\Omega^*_{S^1_0}(\infty)$;
- $\dim \Omega^k_{S^1_0}(\infty) = \#\,CM_k(f)$ the critical pts of index $k$

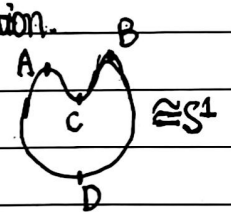
$d: \Omega^k_{S^1_0}(\infty) \to \Omega^{k+1}_{S^1_0}(\infty)$ = "Quantum tunnelling effect"

= "Proper instantons" from critical pt of index $k$ to index $(k+1)$

$\Rightarrow$ We can simply define Morse homology as $H^*(\Omega^*_{S^1_0}(\infty))$, Witten's version

Rk. The so-called physical terms essentially are using variation method (steepest descent) to compute flow lines connecting two critical pts. In physic, computing tunnelling effect classically in WKB, later is by instantons, both the two are variation.

* Bott's example (I don't understand!)




$\cong S^1$

$\mathrm{Spec}(\Delta|_{H^1})$:

$\mathrm{Spec}(\Delta|_{H^0})$:

$\lim_{t\to\infty} \mathrm{Spec}(\Delta_t|_{H^1})$, $\Delta_t$ local at max: $e^{\lambda t}$; at min $\lambda t$

$\lim_{t\to\infty} \mathrm{Spec}(\Delta_t|_{H^0})$, $\Delta_t$ local at min: $\lambda t$; at max $e^{\lambda t}$

thus distinct min & max ☑

- Floer's idea: infinite version applied to moduli space.

In original Witten's work, super symmetry is placed on YM moduli space, naturally infinite-dim, thus Floer's inspired then, he first develop Hamiltonian, then also Lagrangian, which is general, it's worth to mention that such a far-reaching mathematician killed himself in his 34 due to depression. ☹

(Upper discussion is after simplification!)

In this note, I'll not talk about other topological invariants.

Rk. It seems originally Hamiltonian story is related with the YM function and so on, except [B], there is nothing discussing about this.

Given $(M, \omega)$ symplectic & some properties, Arnold conjecture, as a generalisation of Poincaré's last Geometric conjecture, concerns about fixed pt of $f$.

We have learnt in fundamental symplectic geometry, we have,

fixed pt of $f \leftrightarrow$ critical pt of $\mathcal{A}_f : \mathcal{L}M \to \mathbb{R}$ ($\leftrightarrow$ some physics)

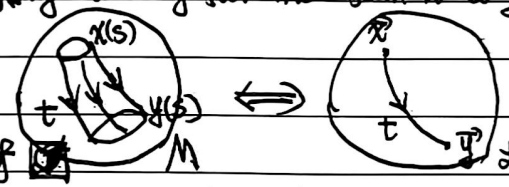
$\updownarrow$ $\qquad$ called action functional

1-period closed orbit $\vec{x} \in \mathcal{L}M$

thus we relate fixed pt with "Morse" successfully, but infinite-dimensional.

Modulo some technical results, we can build Floer homology $HF^*(M, f) = HM^*(\mathcal{L}M, \mathcal{A}_f)$ simply (B) by counting flow lines from $\vec{x}$ to $\vec{y} \in \mathcal{L}M$

$\Leftrightarrow x(t)$ to $y(t) \subset M$:




thus we have solution of ☑ is $u(s,t)$ 2-dimensional in $\mathcal{L}M$ with $L^2$-metric

Now by calculation of the flow lines simply, we have: $u$ satisfy Floer equation ☑, called

$$\underbrace{\frac{\partial u}{\partial t} + J(u)\frac{\partial u}{\partial s}}_{\text{CR equation of J-holomorphic curves}} = \underbrace{-\mathrm{grad} f(u)}_{\text{non-linear term/perturbation}} \quad ☑$$

inhomogenous J-holomorphic curve

(A') Each critical pts/closed orbit we need a index (Morse index)

Here our (A') & (B') are same as Morse's (A) & (B) case, and we repeat constructing moduli space as we done in Morse case, arguing $\boxed{\text{Opt \& transversality}}$ & finite-dim thus we done!

Gromov — energy finite — perturbation — Fredholm — [MS] and IA

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No.

**PART III** Lagrangian Floer theory.

Now we have Hamiltonian Floer, the key is counting $J$-holo curves under some perturbations, we expect the same thing happens in general case, also by counting $u: S \to M$, $S$ is a Riemann surface, cpt, with boundary (not only $\mathbb{R}\times[0,1]$ in Hamiltonian case!) & (i) some subtle boundary conditions (topological gluing); (ii) holds inhomogeneous pseudo-holo map equation (Here $J$ is changing, thus not fixed $J$-holo):

(*) $\partial_s u(z) + J(z,u)\partial_t u(z)\circ j = (u^*X(z,u) + J(z,u)\circ u^*X(z,u)\circ j$

($j$ is cpx structure on $S$, $f$ on $M$ determines vector field $X = -\mathrm{grad} f$ and pullback to $S$)

(analytic rigidity $\Rightarrow$ finiteness of counting).

We focus on (i) and forget (ii) as a finiteness condition.

Given two Lagrangians $L_1, L_2$, what $u$ (or $S$) we associate to them we need to count?

Back to Hamiltonian case, you'll hear that when you take $L_1 = \Gamma_f \subset (M\times M, \omega\oplus-\omega)$, then $\boxed{HF^*(L_1,L_2) = HF^*(M,f)}$ (or $L_2 = \Delta \subset (M\times M, \omega\oplus-\omega)$) because the left is $x \in L_1\cap L_2$

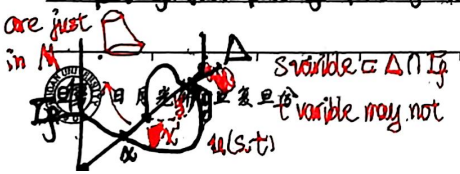
$= \Gamma_f \cap \Delta =$ fixed pt of $f =$ right."

But I'll point out it's nontrivial! The problem lies the "flowline" connecting two $x_1, x_2 \in L_1\cap L_2$:

① If $x_i$ fixed, so is the loop $\tilde{x}_i$ with $\tilde{x}_i|_{s=0} = x_i$, $i=1,2$.

thus $\tilde{x}_i(S) \subset L_1\cap L_2 = \Gamma_f\cap\Delta$ not transversal;

And the "flow line" connecting $x_1$ and $x_2$ (or $(x_1,x_2)$ & $(x_1',x_2') \in M\times M$) is just part of $u$, 1-dimensional:




Thus how we can extend to $u$ properly?

Ok. As an end word, it's still not enough for [S] as it's quite hard, but our motivation is enough, next step may read Auroux / Smith's survey on it to connect with [S]

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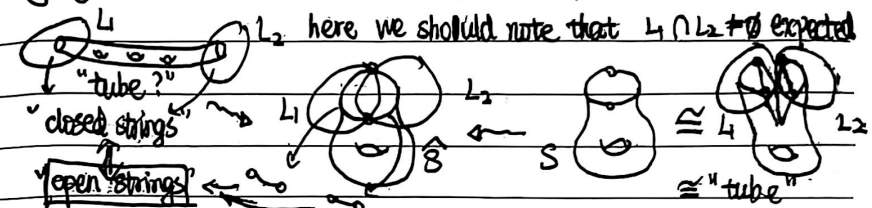
② Generally we expect $X$ not splittable into $M\times M$ and $L_1\cap L_2$, such extension is not possible to $L_1\cap L_2$ $\cup_i \Theta x_i$ locally as $x_i$ may isolated in $L_1\cap L_2$!

our expectation is modified to: one side boundary in $L_1$, another $L_2$, this will gives a symmetry between $HF^*(L_1,L_2)$ & $HF^*(L_2,L_1)$, such feature not occurs in Hamiltonian case. (at most, up to orientation of boundary of $S$)

Indeed, (*) can be done, but need some concrete computation, not obvious!

Who learnt Lagrangian Floer can take it as Exercise easily (Not Hard)

Anyway, ② had gives us a correct definition of what we need to cou

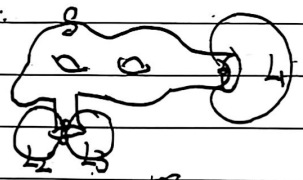



here we should note that $L_1\cap L_2 \neq \emptyset$ expected

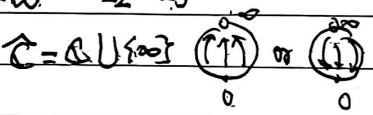
$S = \hat{S} - \Sigma$, $\Sigma \subset \partial\hat{S}$ is points $\subset$ 1-dim

Hence we have "Lagrangian label" in [S]: associate each connected component $C \subset \partial S = \partial\hat{S} - \Sigma$ a Lagrangian with a "orientation": $\Sigma \doteq \Sigma_+ \sqcup \Sigma_-$

means that $\Sigma_+$ outgoing to $\infty$, $\Sigma_-$ incoming from $\infty$.




same as compactify $\mathbb{C}$ to $\hat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$

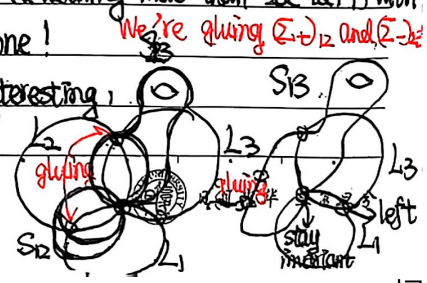



Gluing / Composition & Identity.

Note that gluing $\Sigma$, $\bigcirc$, $\Phi$ e. nothing more than 2d TQFT, with some pointed conditions, thus done!

But I still point out something interesting,

$\bigcirc$ = compactify upperplane $\subset \mathbb{C}$, delete $\infty$ is identity

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# Quantum Field Theory in Math

Motivated by the last one note, and due to I had learnt TQFT in Atiyah & Segal axioms, we need to motivate these axioms back to physical theories.

Our TQFT and FQFT's correspondence comes from the belief of physicists' two understandings of QFT: operator formalism (or canonical formalism, called by condensed) $\Leftrightarrow$ Wightman axioms & path-integral formalism. And if we forget dynamic properties and only record algebraic properties of path-integral, we have TQFT $\subseteq$ FQFT.

Let's explain what the QFT need: $(n+1)$-QFT is datum:

① $(M^{n+1}, g)$ pseudo-Riemannian (not positive-definite, most case it's space-time like, thus $-\sum(dx_i)^2 + (dt)^2$) can have boundary $\partial M$

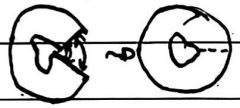
② For all mfd $X$ with $\dim \le n+1$, we have functor $\mathcal{F}(X)$ is "field" on $X$. If $\partial X \neq 0$, we need fix boundary condition $f \in \mathcal{F}(\partial X)$ and then denoted by $\mathcal{F}(X, f)$;

③ (Time involution), $\exists T: \mathcal{F}(M, f) \to \mathbb{R}/\mathbb{Z}$.

i.e. $T = e^{iS}$, $S: \mathcal{F}(M, f) \to \mathbb{R}$, s.t.

$T$ is multiplicative w.r.t. disjoint unions of different $M$

& commutative w.r.t. gluing of $M$



$\Leftrightarrow$ $T$ is topological invariant.

Rk. ① Such "field" always be some maps $\mathrm{Hom}(X, B)$, some $G$-bundles ... but $\mathcal{F}(X, f)$ can be infinite-dimensional, thus next path-integral may not rigorous in mathematics.

② 0d QFT is also not so trivial, but all are finite dim;

1d QFT is QM as $M$ is just $\mathbb{R}_t$, each time with Hilbert space.

2d QFT is the most important case we consider!

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To axiomize ③, now same as QM, we also have two (or more) picture: Schrödinger's & Heisenberg's:

- $\mathcal{F}(X, f)$ depend on time involution
- $\mathcal{F}(X, f)$ has unitary representations, strongly continuous family (operator depends on time)

We consider the former one, the path-integral $Z(X, f) = \int_{\alpha \in \mathcal{F}(X,f)} T \, d\alpha = \int_{\alpha \in \mathcal{F}(X,f)} e^{iS[\alpha]} d\alpha \in \mathbb{C}$.

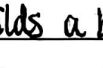
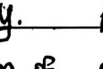
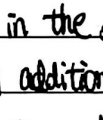
($\alpha$ is a field on $X$, satisfy boundary conditions)

Rk. The name "path" integral is easy to see from physics: in quantum case

each path has contribution proportional to $e^{-S[\alpha]/\hbar}$ (here we just not need constant $\hbar$, and so mathematic text also omit $i$, only $\int e^{S[\alpha]} d\alpha$), and $\int d\alpha$ is integral of this path, all paths together to $\mathcal{F}(X, f)$. (I guess...)

Then set $Z(X): \mathcal{F}(\partial X) \to \mathbb{C}$; $f \mapsto Z(X, f)$, it gives a functor $Z: \square_{n+1} \to \mathrm{Hilb}$, view $\mathcal{F}(\partial X) \to \mathbb{C}$ as dual of $\mathcal{F}(\partial X)$, we need a category "$\square_{n+1}$" concern's boundaries of all mfd with $\dim \le n+1$. This is the category of cobordism, we'll define it later.

Rk. ① In physics, $\square_{n+1}$ is $n$-dim general relativity, Hilb are quantum mechanics, thus it builds a bridge between them two, as a trial of quatization of gravity.

② TQFT lies in the simplification of cobordism to $(n+1)$-dim not depend on additional structure on (space-time) mfd mfds, such as pseudo-Riemannian structure $g$ (relativistic QFT, is most often discuss in physical QFT course) or conformal structure (CFT) ... due to forget $g$, there is no time involution. They take out algebraic property of path-integral $\Leftrightarrow$ Feynman diagram forget labels "  " as 1d TQFT. But "  "-like is singular in math, replace particle "by string" (closed) then 2d TQFT "  " by world sheet of such strings, 2d TQFT $\Leftrightarrow$ MS.

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**PART II** Category of cobordisms and TQFT, revisit.

Def1. (Symmetric monoidal structure)

Monoidal cat is just a cat with tensor $\otimes$ with $\mathbb{1}$, associative; symmetric means commutative of $\otimes$, we denote/upper $\otimes$

They satisfy compatibilities, omitted.

Axioms (such as $\mathbb{1}\otimes X \cong X$) with strict $=$ instead of isomorphism is called strict monoidal cat.

Def2. (Classical cobordism category)

We have (strict) symmetric monoidal cat $Cob^{\sqcup}_{n+1}$, with

- objects are $n$-mflds, oriented (, closed/open)
- $Hom(M,N) = \{B$ oriented $(n+1)$-mfld $\mid \partial B \xrightarrow{\cong} M^{\#} \sqcup N$ preserves orientation$\}$ ($M^{\#} \cong M$ with orientation reversed)

$B \cong B' \Leftrightarrow \partial B \xrightarrow{\cong} \partial B'$ preserves orientation

with identity $id_M = M \times [0,1]$

composition $Hom(M_1,M_2) \otimes Hom(M_2,M_3) \to Hom(M_1,M_3)$

$$B_1 \otimes B_2 \longmapsto B_1 \sqcup_{M_2} B_2$$

Rk. Many technical issues are omitted, such as orientation preserved under composition, and most time we consider smooth mfd, and the choice of smooth structure is subtle in gluing, depending on a collar neibourhood of $M_2$; most case we assume objects are closed without boundary, due to boundary case also important, closed isn't enough!

TQFT thus is symmetric monoidal functor $Z: Cob^{\sqcup}_{n+1} \to Vect(\mathbb{k})^{\otimes}$ (no distinction between $Vect(\mathbb{k})$ & $Hilb(\mathbb{k})$ in categorical level)

Rk. We lose many datum from the QFT in physic, but one should note that TQFT occurs without category theory: physicist that correlation function is topologically invariant $\Leftrightarrow$ not depend on Riemannian metric $g$ (In form, the correlation form seems do.)

Send boundary to correlation function $\in$ quantum state $\in$ Hilb

$\langle O_1 \cdots O_n \rangle = \langle \psi_0 \cdots O_n \psi_0 \rangle$ [illegible]

QFT origins at some spin equation such as Klein–Gordon... when we take the relastic form (special at first) into Schrödinger equation. Relastic/Spacetime + QM (and observe angular momentum not conserve $\Rightarrow \exists$ spin momentum $\Rightarrow$ QFT

A best function carries 50% datum, not 0% or 100%

Slogan.

"QFT = Representation of Space-time", take $Cob^{\sqcup}_{n+1}$ is something not enough to recover all QFT, but only a part of space-time. or only correlation function.

But not concerning too much physics, our biggest problem in geometry is the mflds with boundary, we have next observation:

- Let $M^{n+1}$ closed oriented, then $M$ determines $(\emptyset \to \emptyset) \in Cob^{\sqcup}_{n+1}$, view the empty set as $n$-mfld $\xmapsto{Z} Z(M): Z(\emptyset) \to Z(\emptyset) \in Vect(\mathbb{k})^{\otimes}$

By $Z$ preserves identity of tensor $\sqcup \Rightarrow Z: \emptyset \mapsto \mathbb{k}$

$\Rightarrow Z(M) \in End(\mathbb{k}) \cong \mathbb{k}$

This observation tells us: $Z: Cob^{\sqcup}_{n+1} \to Vect(\mathbb{k})^{\otimes}$ not only contains datum of $n$-mfld, also $(n+1)$-mfld; and when $(n+1)$-mfld is without boundary (closed), it's trivial datum.

Thus when $(n+1)$-mfld with $n$-dim boundary $\partial M \mapsto Z(\partial M)$, we should have some nontrivial datum "$Z(M)$" (formally now), associated to $Z(\partial M)$. This is the idea of $\boxed{\text{Lurie's Extending Down}}$, using higher cat to describe such t

Return back to the observation, $\partial M = \emptyset^{\#} \sqcup \emptyset \Rightarrow Z(\partial M) = Z(\emptyset^{\#}) \otimes Z(\emptyset)$

$= Hom(Z(\emptyset), Z(\emptyset)) = End(\mathbb{k}) = \mathbb{k} \ni Z(M)$ $\qquad = Z(\emptyset)^{\vee} \otimes Z(\emptyset)$

thus we find $Z(M) \in Z(\partial M)$ ! We need $n$-mfld with boundary (upper is $(n$-

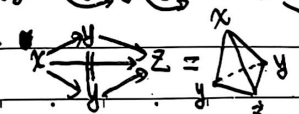
such thing only makes sense in 2-cat!

Our idea is adding it to the axiom of TQFT, to make it clear:

Def3. (2-category) 2-category is the morphism from $X$ to $Y$, $Map(X,Y)$ is an (1-) category, satisfying identity, composition, associativity...

$\Leftrightarrow$ Two morphisms in 2-category has a morphism $Hom(f,g)$ as (1-)category satisfy $x \underset{\Downarrow}{\rightrightarrows} y \underset{\Downarrow}{\rightrightarrows} z = x \underset{\Downarrow}{\rightrightarrows} z$ and .... (compatibility)

Write as



fill 2-morphism in each triangle $\Delta_i$, the polygon commutes surfaces.

thus $\Leftrightarrow$ each hom be filled to $\Delta_3$

3-dim

(For higher cat, we extend in such manner)

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Eg. We can build 2-category from 1-category, same thing we do for n-category, we take 2Vect(k) as example:

{objects of 2Vect(k) are cocomplete k-linear (1-) cats

morphisms $\mathrm{Hom}_{2\mathrm{Vect}(k)}(\mathcal{C},\mathcal{D}) = \{$cocontinuous k-linear functors $F: \mathcal{C}\to\mathcal{D}\}$

(Recall. cocomplete cat $\Leftrightarrow$ closed under colimit $\Leftrightarrow$ closed under $\coprod_{\text{finite}}$ and coker

cocontinuous functor $\Leftrightarrow$ commute with colimit $\Leftrightarrow$ commute with $\coprod_{\text{finite}}$ and coker)

$\Rightarrow$ Then $\mathrm{Hom}_{\mathcal{C}}(x,y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(Fx,Fy)$ are k-linear from an (1-) category.

n-cat is morphism is (n-1)-cat

Rk. Again we're speaking things in strict case, i.e. when we considering compatibilities, the result identity is strict, not isomorphism!

Def4. $2\mathrm{Cob}^{\coprod}_{n+1}$ has (symmetric monoidal cat & 2-cat need compatibility)

{objects are closed oriented mfd of dim (n-1)

morphisms $\mathrm{Map}_{2\mathrm{Cob}^{\coprod}_{n+1}}(M,N)$ is a category, has:

{objects are cobordism B of M to N, $\partial B \cong M^{\#} \coprod N$

morphisms $\mathrm{Hom}_{\mathrm{Map}(\cdot,\cdot)}(B,B') = \{$cobordism X of B to B'$\}/\cong$ diff preserve orientation

Rk. ① Then it's not strict! (See [Lurie] for issues) due to up to

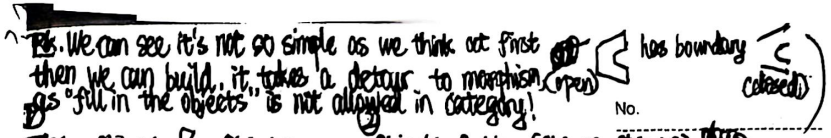
② B and B' must have boundary! How can we define their cobordism, and on $\partial B \cong M^{\#}\coprod N \cong \partial B'$ we need restrict to trivial cobordism? Here I think may be cyclic?

This means we need take X as (n+1)-mfd,

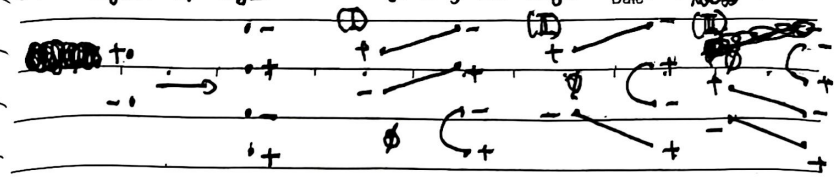
$$\partial X = B^{\#} \coprod_{(M^{\#}\coprod N)\times\{0\}} (M^{\#}\coprod N)\times[0,1] \coprod_{M^{\#}\coprod N\times\{1\}} B$$

Eg. (n=1) $2\mathrm{Cob}^{\coprod}_{2}$ denoted $2\mathrm{Cob}^{\coprod}$ easily for our later use of open strings' TQFT

Rk. We can see it's not so simple as we think at first then we can build, it takes a detour to morphism as "fill in the objects" is not allowed in category!



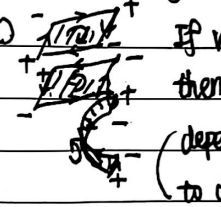

Fix Object1 & Object2 Objects of Hom(Object1, Object2)



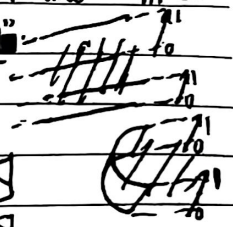

Here we understand why here we can't modulo diffeomorphism, otherwise all trivial

"Morphisms between objects of morphisms":

(II) → (I)



If we view all 0-dim "+" and "−" in ∞ then these surfaces "bound" (depend on orientation to define it rigorous!)



the collapse [0,1], we get it homotopical a desired cobordism of

the open string analogue of

$\in \mathrm{Mor}(2\mathrm{Cob}^{\coprod}_{2})$! One can consider other identity map is same, but (I) → (II) .. are strange.

Rk. (Not sure, it's too abstract for me!)

In mathematical word, the $\mathrm{Map}_{2\mathrm{Cob}^{\coprod}_{n+1}}(\mathbb{1},\mathbb{1}) \cong \mathrm{Cob}^{\coprod}_{n+1}$, we call the former or of a symmetric monoidal 2-cat the loop space $\Omega\mathcal{C}$ and we can extend it to $\Omega^k\mathcal{C}$ for $k\le n$, $\mathcal{C}$ is n-cat to recover its lower category analogue:

$\mathcal{C}_1$ is $k_1$-cat $\xleftarrow{\Omega^{k_2-k_1}}$ $\mathcal{C}_2$ is $k_2$-cat

$\mathrm{Mor}^{k_2-k_1}_{\mathcal{C}_2}(\cdot,\cdot) = \mathcal{C}_1$ construct

Thus we get our desired construction, and set $Z: 2\mathrm{Cob}^{\coprod} \to 2\mathrm{Vect}(k)^{\otimes}$ the symmetric monoidal 2-functor is called the 2-extended TQFT.

Similiarly we have n-extended TQFT, but for our use in sympletic geometric/mirror symmetry or T-duality, only 2d TQFT used, thus 2-extended, TQFT suffices. Higher dim TQFT if want to put conditions on boundary/corners, then need higher cats.

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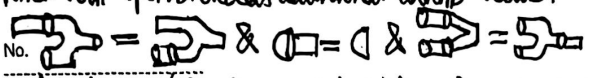
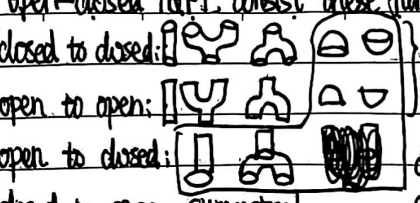
When both open & closed, additional axioms occurs:



And both open/closed seperately holds axioms same as closed.

Thm1. For 2d open-closed TQFT, consist these fundamental/elementary cobordisms:

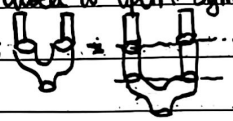


closed to closed: four not two due to orientation!

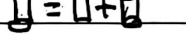
open to open:

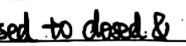
open to closed: due to $\phi$ is open and closed, "hots" are also open to closed

closed to open: symmetry!

and note that  not elementary

something like "open to closed + open" also elementary

If one think 

then only closed to closed & open to open are needed, but I think the gluing process of  need to be added artifically. Due to the lack of a standard reference of open-closed TQFT, I think both is acceptable.... (?)

Pf. Same as 2d (closed) TQFT, using Morse function $f$ to cut them into small tubes $f^{-1}([a-\varepsilon, a+\varepsilon]) = \coprod$ elementary ones (But here we have critical, we need modify Morse theory)

We know 2d closed TQFT $\Leftrightarrow$ Frobenius algebra;

here we can extend to "knowledgeable Frobenius algebra", see a paper by Lauda & Pfeiffer. I don't think it's important, thus we only recall closed case here:

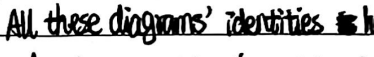
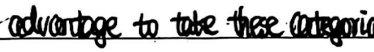
Def5. (Frobenius algebra) Frobenius algebra = monoid + comonoid + compatibility (Frobenius law), let's define them via pictures as in the textbook [Heunen, Vicary, Category for Quantum Theory], then we "see" why Frobenius algebra = 2d TQFT directly!

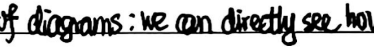
- A (co)monoid is $\Bbbk$-(associative) algebra $A$, with $: A \otimes A \to A$;

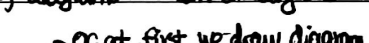
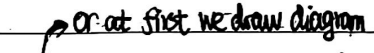
(or more generally, an object of monoidal cat with unit $I$)

$: I \to A$

(or "co": $: A \to A \otimes A$, $: A \to I$)

(co)associativity and (co)identity and (co)commutativity satisfy and  , if  we call it's commutative.

(or "co":  and  )

Frobenius if both monoid and comonoid and  ; 

(Ex. Drawing pictures to represent morphism between themselves)

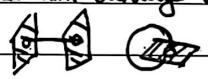
Observation. All these diagrams' identities holds under "topological transform" (isotopy), this is the advantage to take these categorical/algebraic structure into the viewpoint of diagrams: we can directly see how it's linked with topological quantum.

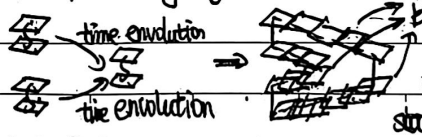
Now return back (or at first we draw diagram in "closed string way") 2D cobordism in string theory to Feynman diagram, then our categorical equivalence is obvious. Precisely, we have:

$\boxed{2\mathrm{TQFT}/\Bbbk \simeq \mathrm{CFrob}/\Bbbk \quad Z \mapsto Z(S^1)}$ is acceptable, and we can imagine that: when dimension goes up, there isn't such an easy story due to the cobordism is complex (also interesting: I. Bartlett, Douglas, Schommer-Pries, Ni)

But a question is: why 1d TQFT not Frobenius? It seems even more natural than 2d case! A. $Z(\bigcirc)$ has two choices: $Z(+)$ or $Z(-)$, and singular $\prec$ not allowed!

$$\begin{tikzcd}
\text{RK. D-branes} \arrow[rr, leftrightarrow, "\text{RR-fields...}"] \arrow[d, leftrightarrow] & & \text{K-theory} \arrow[d, leftrightarrow, "\text{Conner-Floyd (classical AT)}"] \\
\text{open strings} \arrow[r] & \text{open 2d TQFTs} & \text{(open) cobordisms} \arrow[l]
\end{tikzcd}$$

D-branes is cpx 1-dim (always counted in MS and AG), as the boundary of open strings  ; "D" means Dirichlet boundary condition for open string equations.




Boundary states in conformal field theory

An excellent paper explain the boundary states is [Moore, D-branes and K-theory in 2d TQFT]

Link of D-branes and (twisted) K-theory and RR-fields are various.

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usually when we talk about cohomology is a ring, it's equivalent to check its Frobenius structure (although our TQFT in Lagrangian Floer is open, here it's "closed"). Before Floer (co)homology groups, we first see some usual cohomologies with TQFT structure:

(small)

- Quantum cohomology (A-model) $QH^*(X;\mathbb{C}) \otimes QH^*(X;\mathbb{C}) \xrightarrow{*} QH^*(X;\mathbb{C}) \xrightarrow{\int} \mathbb{C}$

Not [illegible] $HH^*(X) = \bigoplus_{p+q=*} H^p(Y, \Lambda^q T_Y)$ (B-model) $\cong$ $H^p(Y, \Omega^q_Y) \otimes H^{n-p}(Y, \Lambda^{n-q} T_Y) \xrightarrow{\int} \mathbb{C}$

$H^p(Y, \Omega^q_Y) \cong H^q(Y, \Lambda^p T_Y)$, $H^{n-p}(Y, \Lambda^{n-q} T_Y) \cong H^{n-q}(Y, \Omega^p_Y)$

If $Y = (X, \omega)^\vee$ mirror pair $\Rightarrow$ isomorphic.

$\left( QH^*(X;\mathbb{C}) = H^*(X;\mathbb{C}) \overset{\text{Hodge}}{=} \bigoplus_{p+q=*} H^p(X, \Omega^q_X) = \bigoplus_{p+q=*} H^p(Y, \Omega^{n-q}_Y) = HH^*(Y) \text{ if Kähler} \right)$

What happens if we replace closed by open case? We have the open-closed string map:

| open string theory | = | closed string theory |
|---|---|---|
| $CX/dg/A_\infty$-cat | | Frobenius algebra |
| $HH^*$ (Hochschild cohomology) | open to closed $\cong$ closed to open | generic cohomology $QH^*$, SH*, HF* — |

this's one of the most important picture in symplectic topology.

Where the categorial structure formed by boundary conditions

(Then apply the picture to A&B-model, we expect equivalence of cat: HMS)

**PART II** Floer theory adapted to picture

First, we can return back to the RHS via some famous isomorphism: $HF^*(M, \omega, H, J) \overset{\text{Floer}}{\cong} QH^*(M)$, thus Hamiltonian Floer theory done, with monotone case (By Floer): $[\omega] = \lambda c_1(M) := \lambda c_1(TM)$, $\lambda \in \mathbb{R}_{>0}$ ($J$ is $\omega$-tamed)

or general case with Novikov ring-coefficients (later)

Thus we can expect this, but a first problem is, what's the product of Morse cohomology (although we have cup product by $HM^* \cong H^*$, we need a direct construction to generalize to Floer / infinite-dim case), we'll see, the product of Morse-type naturally via Feynmann diagram / 2DTQFT with more product than two, $A_\infty$-property occurs naturally.

Our aim is: given Morse functions $f_0 - f_d$, we have

$\mu^d : C_*(f_0, f_1) \otimes C_*(f_1, f_2) \otimes \cdots \otimes C_*(f_{d-1}, f_d) \longrightarrow C_*(f_0, f_d)$

where $C_*(f, g) := CM_*(X, f-g)$, and satisfy the $A_\infty$-composition (omitted)

here it suffices to consider $\mu^2 : C_*(f_0, f_1) \otimes C_*(f_1, f_2) \to C_*(f_0, f_2)$

(we can see $\geq 3$ is similiar as 2-case, all counting flow lines)

$\mu^2 : C_*(f_0, f_1) \otimes C_*(f_1, f_2) \longrightarrow C_*(f_0, f_2)$ (Rk. Indroing Poincare dual $C_k(f) \cong C_{n-k}(-f)$)

$[p] \otimes [q] \longmapsto \sum_{\text{all } r \in \mathrm{Crit}(f_0 - f_2)} \# [?] [r]$ is a very natural idea.

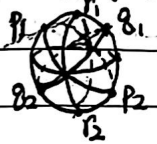
$[?]$ is 0-dimensional

thus we need to construct a moduli space $[?](p, q, r)$

consider the gradient flow is "linear" first and $X = S^2$

where $f_0 - f_1$, $f_1 - f_2$ are two height functions

then



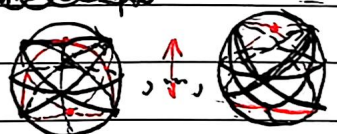

$-\mathrm{grad}(f_0, f_2)$
$= -\mathrm{grad}(f_0 - f_1) - (f_1 - f_2)$
$= -(\mathrm{grad}(f_1, f_2) + \mathrm{grad}(f_0$

For $[p_1] \otimes [q_1] \to [?][r_2]$, others all not exist!

the indices are [illegible] are $p_1$ and $q_1$ is [illegible]

the broken flow lines in this example

there are infinite choices:



Which is 2-dimensional

as we parametized these blue flow lines from $[p_1]$ & $[q_1]$ to $[r_2]$ by the intersection point of the three, which occurs in $\forall p \in S^2$, thus $\mathcal{M}(p_1, q_1, r_2) \cong S^2$, thus all are zero

(This means that the cup product on $H^*(S^2)$ is meaningless, as $H^4(S^2) = 0$, no intersection to 0-dim)

In fact, we have $\dim \mathcal{M}(p, q, r) = \mathrm{ind}(p) + \mathrm{ind}(q) - \mathrm{ind}(r)$

(I think the computation above is something for index, now I'm tired and not want to compute it)

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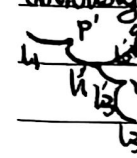
Write it strictly, we have $M(p,q,r) = \{(l_1,l_2,l_3) \mid l_1:(-\infty,0]\to X,\ l_2:(-\infty,0]\to X,\ l_3:[0,\infty)\to X,\ \frac{dl_i}{dt} = -\mathrm{grad}(f_{i-1}-f_i),\ f_3=f_0,\ l_1(0)=l_2(0)=l_3(0),\ l_1(-\infty)=p,\ l_2(-\infty)=q,\ l_3(\infty)=r\}$

$= \{l_i(0) \mid (l_1,l_2,l_3)\in M(p,q,r)\}$

only parametrize the intersection point,

obviously, the optification is broken $l_1, l_2, l_3$ seperately in form

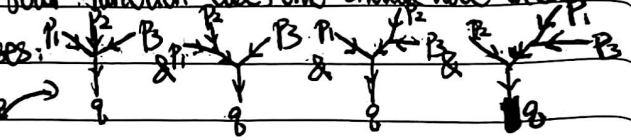


. We write object of the moduli as Feynman diagram

$\Leftrightarrow M(p,q,r) = M(p,q,r) \sqcup \partial M(p,q,r)$

$= M(p,q,r) \sqcup \bigcup_{\text{all } p'} (\overline{M(p,p')} \times M(p',q,r)) \cup \bigcup_{\text{all } q'} (\cdot,\cdot) \cup \bigcup_{\text{all } r'} (\cdot,\cdot)$

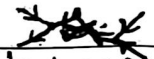
. Thus for four function case, one should note there are more possible cases:



degenerate

consists $M(p_1,p_2,p_3;q)$ desired, compactification same but note the case

$\dim M(p_1,p_2,p_3;q) = \mathrm{ind}(p_1)+\mathrm{ind}(p_2)+\mathrm{ind}(p_3)-\mathrm{ind}(q)\ \boxed{+1}$

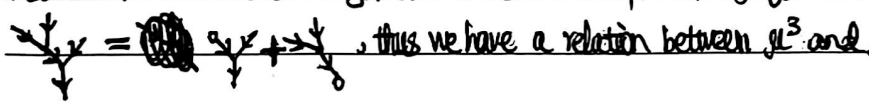


broken middle

Why "+1"? As above, $M(p,q,r) \Leftrightarrow$ parametrize $l_i(0)$ here is a line not a point! Thus add 1 dimension.

real line of

And we can see (nondegenerate case) as composition of $\mu^2$:



, thus we have a relation between $\mu^3$ and $\mu^2$

Generally, we conclude that:

① $\mu^d: C_{a_1}(f_0,f_1)\otimes\cdots\otimes C_{a_d}(f_{d-1},f_d) \to C_{a_1+\cdots+a_d+d-2}(f_0,f_d)$

$\Leftrightarrow \mu^d: C^\bullet(f_0,f_1)\otimes\cdots\otimes C^\bullet(f_{d-1},f_d) \to C^\bullet(f_0,f_d)[2-d]$ after we indexed by cohomology.

② $\mu^d$ can be written as a formula with $\mu^{d-m}$, $\forall m\geq 1$

$\rightsquigarrow$ These two are the definition of $A_\infty$-structure

Conclusion: Such Feynman diagram in Morse is "•", both open and closed. We know Hamiltonian is closed loop "O", but lift to $\mathcal{L}X$ is "•", what can we expect?

Rk. (Floer homotopy/Morse homotopy, classical version)

From Morse homology we back to CW-cpx by cellular decomposition, but it doesn't work for infinite-dim case & when we considering more complicated counting objects, CW-cpx isn't enough, our idea is CW-spectra, which had be introduced much earlier than my expectation in stable homotopy theory.

Furthermore, we have a $A_\infty$-category in Morse case with objects are Morse functions and $\mathrm{Hom}(f_1,f_2) = C^\bullet(f_1,f_2)$, we expect to recover $X$'s homotopy type from it. (Rational case is done, an older story is de Rham homotopy also done by homotopists)

After Morse, we take a look at TQFT or $A_\infty$-structure of Floer theories, due to their closed & open string-theoretic nature:

• Topological Floer theories.

We call them Floer theories due to two reasons: they're Morse-type; and they expected/proven as a Lagrangian Floer theory.

Our classical gauge theory of 4-mfd is coarsely classified into two approaches/equations: ① ASD moduli (Yang-Mill equations) & ② Monopole moduli (Seiberg-Witten)

instanton

Rk. ② is easier to compute than ① in practice, but until now (2025) mathematicians are still arguing which of them carries more datum than another one. In physics the first relationship is the solution of YM equation can also be described as monopoles, then in SW's paper they call SW equation is "dual" to Donaldson theory. Most of us believe these two are equivalent.

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Due to my limited knowledge of Gauge theory, I must recall the here:

① $X^4 \rightsquigarrow$ ASD YM equation $*F_A = -F_A$, A connection of $SU(2)$-bundle (critical pt of YM functional on all connections)

$\{A \mid *F_A = -F_A\}$/gauge is 5-dim with boundary $X \rightsquigarrow$ Donaldson's H

Counting points we get Donaldson's invariant $D(X)$

Our motivation of instanton homology is computing $D(X)$ by decompose $X$ to smaller ones, this'll give it TQFT structure, and it's thus on a 3-mfd $Y$. More precisely:

- The instanton homology $I_*(Y)$ is the Morse homology on the infinite-dimensional space $\{A\}$/gauge on the unique trivial $SU(2)$-bundle with Chern-Simons functional $CS(A) = \frac{1}{8\pi^2}\int_Y tr(A\wedge dA + A\wedge A\wedge A)$
- Decompose $X = X_1 \cup_Y X_2$ determines $D(X_1) \in I_*(\partial X_1) = I_*(Y)$ and pairing $I(Y) \otimes I_*(-Y) \to \mathbb{Z}$, $D(X_2) \in I_*(\partial X_2) = I_*(-Y)$, $D(X_1) \otimes D(X_2) \mapsto D(X)$

$\Leftrightarrow$ $D(W): I(Y) \to I(-Y)$ : $W$ can be seen as cobordism of $Y$ and $-Y$, then glue the ends $\rightsquigarrow D(X): I(Y_1) \to I_*(Y_2)$

thus we have $I_*: Corb^{\sqcup}_4 \to Vect^{\otimes}$ TQFT-structure.

$\boxed{b_2^+ \geq 2}$ $\quad Y \longmapsto I_*(Y)$

$X \longmapsto D(X)$

② $X$ spin$^c$ structure $s$, and for 4-dim, the Seiberg-Witten equation of pair $(A, \varphi)$, $\varphi \in \Gamma(X, S^+)$

$A$ is Spin$^c$-connection and $S = S^+ \oplus S^-$ is spinor bundle, $\sigma$ is quadratic form

$F_A^+ = \sigma(\varphi)$, $D_A\varphi$ (covariant derivative) $= 0$

$\{(A,\varphi) \mid F_A^+ = \sigma(\varphi), D_A\varphi = 0\}$/gauge $\rightsquigarrow$ Counting points SW-invariant

Again we go to Spin$^c$-3-mfd$(Y, s)$, due to symmetry broken we consider only $U(1)$-bundle

$\{\varphi\}$/gauge with Chern-Simons-Dirac functional

$CSD(A,\varphi) = -\frac{1}{8}\int_Y (A^t - A_0^t)\wedge(F_A^+ + F_{A_0}^+) + \frac{1}{2}\int_Y \langle D_A\varphi, \varphi\rangle\, dvol$ critical

Rk. Both ① & ② not generally built due to the issue of reducible connections, for ①, it's still open! But ②, as we discussed, is easier to handle (for example, the SW moduli is automatically cpt), [KM] successfully dealt such issue and define $\widetilde{HM}_*$, $\widehat{HM}_*$, $\overline{HM}_*$ in general via a real blowup. $\widetilde{HM}_*$ (to the worst), $\widehat{HM}_*$ (from simplest), $\overline{HM}_*$ are called monopole homologies, with an exact triangle $\widehat{HM}_* \to \overline{HM}_* \to \widetilde{HM}_* \to \widehat{HM}_*$ (LES)

Each version we have $SW(X): HM_*(Y_1) \to HM_*(Y_2)$ close to TQFT and similiar as ①, only defined on $\boxed{b_2^+ \geq 2}$ case...

Realize as Lagrangian Floer on a proper symplectic mfd.

Either open or closed, we know that our expectation is 2d, down the dimension is choose a moduli with a point is the difference of dim decreases. (For example Hamiltonian case (2d) reduce to (1d))

critical $\Leftrightarrow$ flow lines are ASD instanton/solution to YM on $\mathbb{R}\times Y = X$

Hence from $Y^3$ we construct a symplectic moduli space with higher di

① $\{$Flat connection$\}$/gauge $\cong \{\pi_1(Y) \to SU(2)\}/SU(2)$-conjugation

is the generator of critical pts of CS

Given a Heegaard splitting $Y = Y_1 \cup_\Sigma Y_2 \overset{def}{\Leftrightarrow}$ handlebody decomposition: $Y$ into two handlebodies, intersecting in $\Sigma^2$.

$\{$Flat connection$\}$/gauge $\cong \{\pi_1(\Sigma) \to SU(2)\}/SU(2)$ namely $=: M(\Sigma)$

subspace $\{\rho \mid \rho$ extend to $\pi_1(Y_i)\}/SU(2) =: L_i \subset M(\Sigma)$ Lagrangian

(Atiyah-Floer conjecture) $I_*(Y) \cong HF_*(L_1, L_2)$

② $g = g(\Sigma)$, $Sym^g\Sigma = \underbrace{(\Sigma \times \cdots \times \Sigma)}_{g \text{ times}}/S_g$

A Heegaard diagram, roughly speaking, is $(\Sigma; \alpha_1 \cdots \alpha_g; \beta_1 \cdots \beta_g)$

$\alpha_1 \cdots \alpha_g \subset \Sigma$ are loops (simply closed curves), $[\alpha_1] \cdots [\alpha_g] \in H^1(\Sigma; \mathbb{Z}) = \mathbb{Z}^g$ is a basis (called (co)homologically linear independent), same for $\beta_1 \cdots \beta_g$.

$\Leftrightarrow$ flow lines are Seiberg-Witten monopoles/solution to SW on $\mathbb{R}\times Y = X$

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And $\alpha_1 \cdots \alpha_g$ bound $g$ disks $\subset Y_1$, $\beta_1 \cdots \beta_g$ bound $g$ disks $\subset Y_2$

then $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$, $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g \subset Sym^g(\Sigma)$ are Lagrangian tori

and the Heegard-Floer $:= HF^\circ(\mathbb{T}_\alpha, \mathbb{T}_\beta)$

But with some subtle things (I don't know) we also have $HF^+$, $HF^-$, $HF^\infty$ three types (distinct with upper slightly)

In [KM], they proved $HF^+ \cong \widetilde{HM}$, $HF^- \cong \widehat{HM}$, $HF^\infty \cong HM^\bullet$.

Conclusion: I draw a picture to depict it

| | | | | |
|---|---|---|---|---|
| Differential topology/Gauge theory | | Donaldson theory $\cong$ | SW theory | |
| (lower dimension) | 4-dim $X$ | Moduli of instantons/$Y_1$ | Moduli of mono | poles/SW functional |
| | 3-dim $Y$ | CS functional $\uparrow$ ASD | CSD | |
| (higher dimension) | | critical pts, flow lines $\leftarrow$ $X = Y \times \mathbb{R}^1$ | critical pts, flow lines | $X = Y \times \mathbb{R}^1$ |
| Symplectic topology/Floer theory | | $\updownarrow$ Morse type | $\updownarrow$ Morse | type |
| | | $I_*(Y) \leftarrow Y = Y_1 \cup Y \rightarrow HM_*(Y)$ | | |
| | | ? // Atiyah-Floer | ✓ // Kronheimer-Mrowka | |
| "All are Lagrangians" | | Lagrangian on $M(\Sigma)$ | Lagrangian on $Sym^g(\Sigma)$ | |

| | |
|---|---|
| | TQFT/$A_\infty$-structure (algebraic) |
| $X = Y \times \mathbb{R}^1$ | $\Rightarrow$ $X$ cobordism of $Y$ and $-Y$ $\Leftrightarrow$ 4-TQFT |
| | $\Vert$ closed |
| | $\Vert$ open string |
| | $\Longrightarrow$ $A_\infty$-structure |

Return back to (Hamiltonian and) Lagrangian case.

It's also worth to mention symplectic (co)homology, developed by Floer in order to deal with Arnold conjecture with boundary, but due to I know nothing about it, omit. contact

Why not read [8] directly? Nothing to copy here!

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# A last complement before Siedel

I'd like to point out several things refined:

① Technical condition for Fukaya category, and when we relax some of them, how we generalise;

② Homological algebra for HMS, noncommutative geometry and generation;

③ Some other properties, more classical applications;

Some of them are out of Siedel developed later (but not so new), including the Wrapped Fukaya category, Hochschild cohomology and closed-open string map, Symplectic cohomology these invariants ....

Our reference is three great surveys [A], [S], [Abbondandolo Schlenk]. The first one is the most fundamental and I like it most, but except (partially) wrapped Fukaya category there is nothing guidence of recent developments; the second one is the hardest, containing many algebraic aspects of $A_\infty$-categories, and giving the most numerical references for various Fukaya category in abstract levels related with HMS, but I think without enough knowledge, you can hardly understand his fantastic but oversimplified examples; the third underlines the contact boundary case and dynamical aspects, thus symplectic cohomology, but it's for Floer homology in dynamic and contact use without abstact algebraic structure, thus it not matches my main interests.

Restricted by my knowledge, here we do not discuss many interesting examples in [S], I'll read some of papers given by it, and wrapped and symplectic also be short as Reeb chord is not familiar object for me. Something not mentions by these three are microlocal method, [illegible] algebraic structure, topological Floer theory, moduli of objects in Fukaya category and quiver, extend Fukaya category to more...

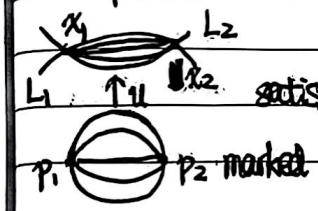
Date

① For a Morse type (co)homology (Here only Lagrangian Floer), what we read is only

- Differential (What we need to count) / (What moduli to construct)
- Transversal (Finite dim & smooth)
- Compactness (Counting finite), $d^2=0$) · Orientibility ("+" = "−")
- Product / Higher product
- Grading (Due to dim reason for count, it's natural in algebraic, but a geometric explaination is essential. Anyway it putted in last)

For these, we need put conditions on either M or L or both, in [S] we classify them into algebraic ones and geometric ones, not so artifical here!

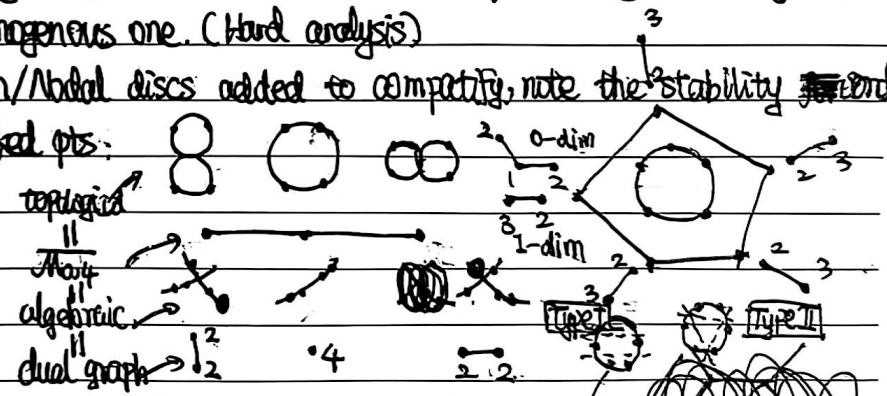
- (The picture I draw last time is correct, but too ugly)




satisfy
- topological: boundary condition
- analytic: Cauchy-Riemann equation
  - A "Rigidity" for counting
  - all compact ⇒ finiteness of intersection

Or relax opt by Nakikou coefficient to deal oops

- Adding perturbation datum to CR equation, say $(J, H)$ into nonhomogeneous one. (Hard analysis)
- Broken/Nodal discs added to compactify, note the stability condition in marked pts:




- Little complicated is $\Pi_{Mor}$ allowing broken twice.

Rk. (as loop ▽ is not stable!)

Broken once is 6-polygon, twice has [illegible] cases,

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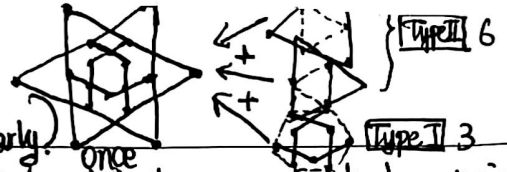
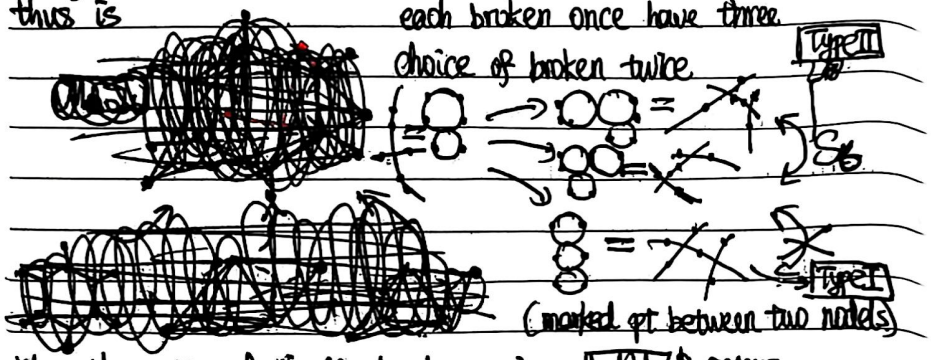
$\mathcal{M}_{0,6}$ is determined by

(due to $\dim \mathcal{M}_{0,6}=2$ we can't draw it clearly)




and nearby two broken once have no same broken twice, thus is

each broken once have three choice of broken twice

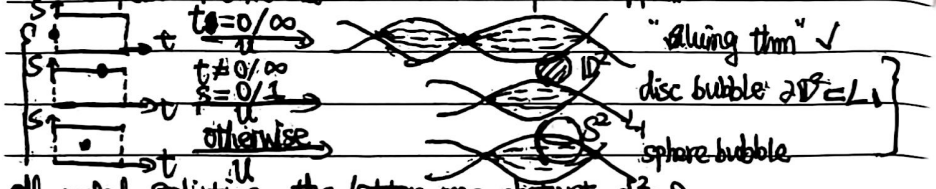



(marked pt between two nodes)

When these mapped via $u$ to Lagrangians, bubble occurs

this's determined by the concentration of energy $\int|\nabla u|^2$

$\Leftrightarrow$ the place the modal split happens (How up)




all modal splitting, the latter one obstruct $d^2=0$

bubble $\Leftrightarrow$ obstruction $\Leftrightarrow$ BRST symmetry broken/Anoma $\Leftrightarrow d^2\neq 0$

See example given in [A]:

$d^2x = d(1-Iy)=0$

$L_1$ same is $y$

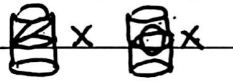
intersection

$d^2x = 2dy$

$= 4x \neq 0$

same is $y$

(The Lagrangian of $T^*S^1=\mathbb{R}^1\times S^1$ is nontrivial $\Leftrightarrow$ two areas equal

otherwise $\exists\, \phi_H^t$ separate them  )

Some ways to exclude bubbles:

(a) $I[u]\,\pi_2(L,M)=0$ for all $L$ $\Leftrightarrow$ out Lagrangian, $\pi_2(L,M)=0$ (a) Exact for all

(b) Monotone (I don't understand Maslov index now, thus it's vague)

(c) Obstruction theory to deal with bubbles directly, not exclude dim reg in [FOOO], but not recognized by all!

Date Why?

The orientation choice of moduli space $\Leftrightarrow$ All $L$ are oriented and spin

One can avoid orientation by let $\mathbb{Z}_2$-coefficient with a coherent choice

bw we had given all our conditions include of orientation (if for more $L$:

cpt/Nakikov + $\int[u]\,\pi_2(L,M)=0$/exact/curved + oriented & Spin $L$

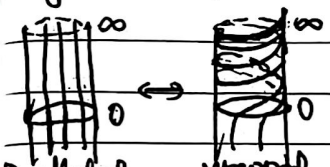
monotone

Strategy of controling noncompact $L$ toward $\infty$ have several choices:

Asymptotic well toward $\infty$ (a natural idea)

Large perturbation at $\infty$ (seems strange!)

look at our fundamental model of $T^*S^1=\mathbb{R}^1\times S^1$, indeed due to we're considering so-called "$\infty$", thus their object at first also things like tangent bundle/Liouville mfd/fibration.




Parallelled $\Leftrightarrow$ Wrapped

The first idea include Sedel's $\mathcal{F}(\pi)$ for Picard–Lefschetz fibration, latter to Liouville sectors & Nadler–Zaslow's sheaf

The second idea due to Abouzaid & [GPS] sheaf cat ... (See next)

perturbation small $\leftrightarrow$ Perturbation Large

(Partially wrapped) $\hookrightarrow$ such wrap $\Leftrightarrow$ rep of $\pi_1(S^1)$/Loop space

$A_\infty$-structure/Product structure is then natural for such Morse-type.

Only give several trivial observation by myself:

(1) We can easily get Hamiltonian case from Morse case:

on $M\times M$ with Lagrangian restrict to Hamiltonian graphs. $\Gamma_{\phi_H}$, then

$HF(\Gamma_{\phi_H},\Gamma_{\phi_G})=HF(\Delta,\Gamma_{\phi_G-\phi_H})=HF(\Delta,\Gamma_{\phi_{G-H}})=HM(\mathcal{A}_{G-H})=HM(\mathcal{A}_G,\mathcal{A}_H)$ ☒

relax (a1)/(a2) to curved $A_\infty$-category by a enhancement of algebraic structure. $\hookrightarrow$ curvature, $d_\nabla^2\neq 0$ case

how can we do homological algebra?

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We can draw Feynman diagram same as Morse case: for example

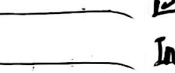
- this's just a clear notation.

Another $A_\infty$-structure (most classical one!) is the singular chain cpx of loop space $C(\mathcal{L}M)$, the $A_\infty$-structure just given by homotopy and higher homotopy:

If given homotopy between $x$ and $y$, $z$ and $y$ given, then the homotopy between $x$ and $z$ $\Leftrightarrow$ 2-homotopy between the two homotopy.

In $\mathcal{L}M$, it's equivalent to say that:

connecting $x$ and $z$ $\Leftrightarrow$ filling the hom



I find such thing is closed related with definition of $\infty$-cat via quasi-cat model. (I still think I don't fully understand it. May be critical to undergraduate students in Chicago who learn $\infty$-cat too early, he said: "you have to learn to walk first before you fly!" I still don't have enough motivation, although I had seen many examples/models)

It turns out $A_\infty$-cat is a model of stable $(\infty,1)$-cat. "All homological algebra built in stable $\infty$-cat".

Wrapped cpx, $\mathcal{L}_q = T^*_q N$ Loop space base $q$

Thm:

A Beautiful thm by Abouzaid shows that: $CW^*(\mathcal{L}_q, \mathcal{L}_q) \cong C_*(\Omega_q N)$.

and there is $\mathcal{W}(T^*N) \longrightarrow \mathrm{Mod}(\Omega_*(\Omega_q N))$

fully faithful quasi-embedding of $A_\infty$-categories.

- Maslov index for grading Lagrangians, Conley-Zendler index for grading symplectics (used for contact boundary case), both of them I'm not familiar! Omitted.

② The deeper I walk through the world of math, the hole I know I know, especially in homological algebra. Indeed the hole (pure) mathematics after 1900s are all doing homological algebra: arithmetic, geometry and topology, representation theory, physics, dynamic and category theory... For a category (with additional structure), when and how to "make" it into able to do homological algebra?

What we need is something "short" to induce Long exact sequence.

Short exact sequence $\in$ Abelian category $\mathcal{A}$

exact/distinguish triangle $\in$ Triangulated category $\mathcal{T}$ $\mathcal{T} = D(\mathcal{A})$ deri

$\boxed{\text{Slogan}}$ Triangulated category is more general & easier to construct.

Indeed, reconstruct $\mathcal{A}$ from $\mathcal{T}$ is hard problem!

A "possible" approach: $\mathcal{T} \to$ heart $\mathcal{T}^\heartsuit = \mathcal{A}$, however the existence of t-structure & its heart is harder than the standard conjecture.

Luckily, $\mathcal{T}$ is enough for LES and homological algebra.

For dg-category we have a good enough theory to get triangulated

dg-category $\xrightarrow{\text{dg enhancement}}$ triangulated category with dg-structure equivalent,

(pre)triangulation $\leftarrow$ (construction concretely) + papers later...

(pretriangulated + Homotopy category = triangulated)

But for $A_\infty$-category? Bondal & Kapranov, Enhanced triangulated category

($A_\infty$ is dg, but we have more datum to collect)

using twisted complex to have a triangulated category, here we also use twist $Tw(\mathcal{A})$. Further for (splitting) generation property, we take its Karoubian envelope $Tw^\pi(\mathcal{A}) = \mathrm{Perf}(\mathcal{A})$ called the category of perfect modules (usually, Karoubian = splitting, later I'll explain this), lastly we take the derived category $D^\pi(\mathcal{A}) = H(Tw^\pi(\mathcal{A}))$ the homology category (For HMS, of course)

(take localization simply) For dg/cpx case, it's simply

Before return back to $\mathcal{A} = Fuk$ generically, the perfect cpx. Perfect module we need understand the splitting generation is special to $A_\infty$-case only, and Karoubian: defined in [Higher Algebra, Lurie]

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No.

Date

(Adding idempotent $\Leftrightarrow$ splitting due to idempotents can be viewed as projection)

Due to I had known what the Karoubian envelope is when I study Grothendieck's pure motive, I not give definition here, but examples:

E.g.1 Set $\mathcal{O} = \{$open sets $\subset \mathbb{R}^n \mid$ all $n\}$

then $\mathcal{O}^\pi = $ Mfd, thus we can say the generators of $\mathcal{O}^\pi$ are unit disks...

and note that we then have a coordinate-free approach to mfd.

To see this only recall the object $\in \mathcal{O}^\pi$ are retraction of $\mathcal{O}$, in topological spaces, these retraction just same. Thus by tubular neighborhood thm & Whitney embedding thm any mfd is retraction of open set $\blacksquare$

E.g.2 Set $\mathcal{T}(X) = \{$trivial bundles $X \times \mathbb{R}^n \to X \mid$ all $n\}$

then $\mathcal{T}(X)^\pi = \mathrm{Vect}(X)$ for same reason, also coordinate-free.

By Serre-Swan, the Karoubian envelope of free modules are projective mod.

Due to algebraic/topological K-theory, the $K_0$-grp are free splitted generator

E.g.3 We know $\mathrm{Cor}(\mathbb{K}) \to \mathrm{Cor}^\pi(\mathbb{K}) = \mathrm{Mot}(\mathbb{K}) \to \mathcal{M}(\mathbb{K})$ is the procedure of Grothendieck's pure motive.

$\mathrm{Cor}(\mathbb{K})$ is something "geometric", thus can't have "ker" or "coker", thus different with sheaves... born with algebraic, it's impossible to directly give a "good" homological method without acted by a functor toward an "algebraic" category — it must lose some datum.

That's why Karoubian envelope! Instead acting a functor ($H^*$), we direct make the category itself to allow "ker", but Abelian is too strong, thus only (but enough) pseudo-Abelian/Karoubian.

In $\mathrm{Cor}^\pi(\mathbb{K})$, $[\mathbb{P}^1] = \ker(p) \oplus \ker(1-p)$, $p: \mathbb{P}^1 \to \mathrm{Spec}\,\mathbb{K}$

$= [\mathrm{Spec}\,\mathbb{K}] + \mathbb{L}$, the Lefschetz motive

$= [pt] + [\mathbb{A}^1] = 1 + \mathbb{L}$

The follow step rigidification $\Leftrightarrow$ give a "Poincare dual", but indeed I think is for arithmetic use: $\mathbb{L}^{-1} = \mathbb{Q}(1)$ the Tate twist of $l$-adic Galois representation, called Tate motive so.

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Date

Now apply these algebra to the $A_\infty$-category (compact) Fukaya category $\mathrm{Fuk}(X)$, objects are Lagrangians (with technical restriction given in ①), morphism $\mathrm{Hom}_{\mathrm{Fuk}(X)}(L_1, L_2) = CF^*(L_1, L_2)$ (who taking $HF^*(L_1, L_2)$ directly is called Donaldson-Fukaya category, but it only records datum & differential, losing all $A_\infty$ structure! This's similiar (but different) as the Grothendieck & Verdier's idea of derived cat: not (co)homology but cpx.

This is also a "geometric" category, thus we do things including Karoubian envelope as the $A_\infty$-cat case. (Similiar for $\mathcal{W}\mathcal{F}uk(X)$...)

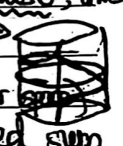
We enhance Thm1 to next, which show the power of split generation

Thm2 (Abouzaid, 2010) $X$ cpt, connected, $(T^*X, \omega_{std})$

$\Rightarrow D^\pi \mathcal{W}\mathcal{F}uk(T^*X) = \langle T^*_x X \rangle$ generated by only one fibre.

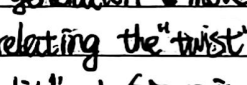
and then $D^\pi \mathcal{W}\mathcal{F}uk(T^*X) \simeq H^*(\mathcal{L}X)$ derived-equivalence.

Another important generation more concrete is Seidel's Dehn twist, this is interestingly relating the "twist" in algebra and geometry.



and show why "split" works (tensoring geometric object by graded vector space)

geometrically:

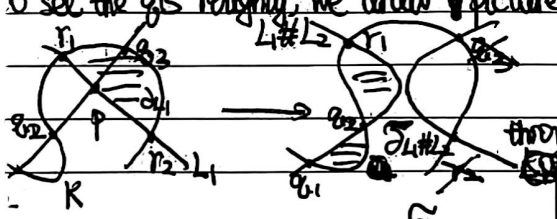


$L_1 \# L_2$ Lagrangian connected sum

then $CF^*(K, L_1 \# L_2) \simeq (CF^*(K, L_1) \oplus CF^*(K, L_2), \begin{pmatrix} \partial_{L_1} & 0 \\ \mu^2(p, -) & \partial_{L_2} \end{pmatrix})$ is qis

by definition of mapping cone $\Leftrightarrow$ exact triangle $L_1 \xrightarrow{p} L_2 \to L_1 \# L_2 \xrightarrow{+1} L_1$ some $c$ is Dehn

to see the qis roughly, we draw pictures:



And $CF(K, L_1 \# L_2) \circlearrowleft \partial_{L_1 \# L_2}$ is just $r_1 \leftrightarrows q_2 \leftrightarrows q_1$

throw away

$(r_1, q_2) \mapsto (r_1, 0)$

$(r_1, 0) \mapsto 0$

$(0, q_1) \mapsto (0, q_2)$

$CF(K, L_1) \oplus CF(K, L_2) \xrightarrow{\tilde{\delta}}$

$(r_i, q_j) \mapsto (\partial_{L_1} r_i + \mu^2(p q_j), \partial_{L_2} q_j)$

$= r_{3-i} + \boxed{?} \begin{cases} r_i, & j = 1, 2 \\ r_1 - r_2, & j = 3 \end{cases}, q_{i+1})$

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Rk. Things like $A_\infty$-module algebraically not mention due to fully treatment in [Siedel]. Only to mention that, the category of cpxs/dg-algebras, Ch, is the "base" of all $A_\infty$-cat, just like Set is the "base" of all cat and A is the "base" of all A-module. Thus we have a similiar $A_\infty$-Yoneda embedding of Hom(-, Ch). That's why we define $A_\infty$-module simply such a functor, then due to the good properties inherited from Ch, cat of $A_\infty$-mod behaves well.

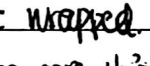
③ Only treat two examples in [S]: Nearby Lagrangian conjecture & closed-open string map revisited.

• Given $(M,\omega)\supset L$, we call another K is nearby of L if: due to Weinstein's Lagrangian neiberhood, $\exists\, U\supset L$, $(U,\omega|_U)\cong(T^*L,\omega_{can})$ (and fix L), if $K\subset U$, then we reduce to Lagrangian intersection of K and L to $K\in Fuk(T^*L)$ and the zero section.

(Arnold's conjecture) Nearby K if closed and exact, then it's Hamiltonian isotopic to L / zero section (S)

Rk. Exactness is used in $\omega=d\theta$, $\theta|_{L_i}=df_i$, $L_0=L$, $L_1=K$

$\Rightarrow \int u^*\omega=(f_1(q_1)-f_0(q_1))-(f_1(p)-f_0(p))$ :




$\Rightarrow$ the Novikov coefficient reduces to $p\mapsto T^{f_1(p)-f_0(p)}p$, thus we use the cpt Fukaya category is enough, not wrapped.

Of course we're using Floer homology doing something weaker than (S), but we expect coming back. The obstructions I think:

(i) $L\simeq K=\phi(L)$, $\phi$ Hamiltonian symplectomorphism $\Leftarrow$ Hamiltonian isotopic

Converse says $\phi$ is same connected component in $Ham(T^*L)$;

(ii) Extend to hole M from U;

(iii) $HF(L,K)=H(L)=H(K)\Leftarrow K=\phi(L)$

Converse holds?

(I think all of the three not hard, but restrict to my knowledge I don't know)

Thus our aim is weaker: $Fuk(T^*L)$ only one object $\Leftrightarrow HF(L,K)=H(L)$ the trivial $H(L)$-module.

Rk. We can give a quick proof of $HF(L,L)=H(L)$ here, as a Floer-Morse correspondence: L is nearby L itself $\Rightarrow$ reduce to $M=T^*L$

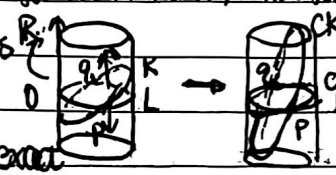
then a Morse function $L\xrightarrow{f}\mathbb{R}$ gives 1-form $\varepsilon df$, and $L'=\Gamma_{\varepsilon df}$ is a small ($\varepsilon>0$) perturbation of zero section L, where $L\cap L'$

$\Leftrightarrow df=0\Leftrightarrow f$ is critical

and Morse's gradient flow equation $\Leftrightarrow$ Floer equation

$\Rightarrow HF(L,L)=HF(L,L')=HM(L,f)=H(L)$ ▨

Our idea is for $K\neq L$ and $K\neq T^*_xL$ for some x, we have a rescaling action of each fibres




due to K exact

(thus (iii) is ok!)

$\Rightarrow cK$ exact $\Rightarrow$ K Hamiltonian isotopic to $\forall cK$, same as $c\to 0$ or $c\to\infty$

$H(L)\overset{c\to 0}{=}HF(cK,L)=HF(K,L)=HF(cK,L)\overset{c\to\infty}{=}HF(T^*_pL,L)\oplus HF(T^*_qL,L)$

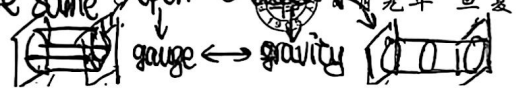
The right hand has a Floer to Morse spectral (or generally $\bigoplus_{x\in L\cap K}HF(T^*_xL,L)$) sequence $\bigoplus_{x\in L\cap K}HF(T^*_xL,L)\Rightarrow H(K)$ is induced by a filtration of CF

• We have $\forall L$, $\exists$ closed-open string map $QH(M)\to HF(L,L)$, given by sending marked pts to deleted pts and bring to boundary $\subset L$

Thus it's obvious to say open string contains more data than closed string case. (Thus the open-closed string map is even harder) But in these not extended to higher operations $\mu^d$ case, closed-open string duality always occurs, including the PSS isomorphism give Hamiltonian Floer $\simeq$ Morse and in physics the division of string between branes & states their string scattering amplitude same

open $\leftrightarrow$ closed

gauge $\leftrightarrow$ gravity



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Now we have an extension to $A_\infty$-category chain-level not only in Donaldson category / $HF^\bullet$-level by Hochschild cohomology:

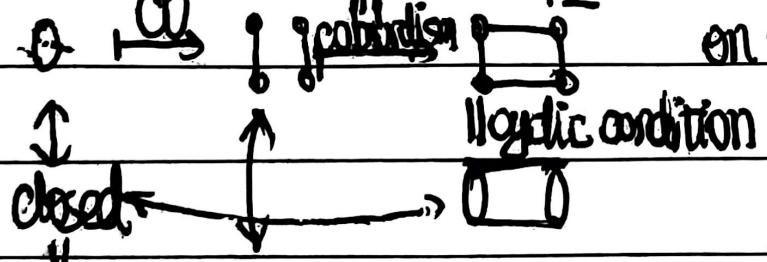
What's Hochschild cohomology? We had known the Hochschild cohomology (and cyclic cohomology) of associative algebra is given by the chain cpx with each $CH^d(A;A) = \{ \underbrace{A \otimes \cdots \otimes A}_{d+1 \text{ times}} \to A \}$, now for $A_\infty$-algebra we have a natural one: the $\mu^d$ in it, induce not only differential (shift 1) but shift all $(2-d)$.

Lift to category level we have Hochschild cohomology of dg / $A_\infty$ / derived cats as their self-Ext algebra (why so define is computation)

Then $CO: QH(X) \to HH(\mathcal{F}uk(X), \mathcal{F}uk(X))$ extend original one as the order 0 truncation.

Rk. A closed-open view point to Connes's IBS LES (NC Gysin sequence)

$$\xrightarrow[+2]{S} HC \xrightarrow[\uparrow]{I} HH \xrightarrow[\downarrow]{B} HC \xrightarrow[+2]{S}$$




Here our idea based on the HC and HR's relationship.

not related here! But at least HH is open side

Noncommutative geometry (but not Connes's spectral triple...) comes here.

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