Introduction. We'll show the equivalent description of Geodesic

① Acceleration $\perp$ the manifold ($\in$ the normal bundle/sheaf), i.e. $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right)=0 \overset{\text{locally}}{\Longleftrightarrow} \frac{d^2x_k}{dt^2}+\sum_{i,j}\Gamma_{ij}^{k}\frac{dx_i}{dt}\frac{dx_j}{dt}=0$ (1)

② Translation of tangent vector is again tagent $\underset{\text{locally}}{\Longleftrightarrow} \begin{cases} y_k=\frac{dx_k}{dt} \\ \frac{dy_k}{dt}=-\sum_{i,j}\Gamma_{ij}^{k}y_iy_j \end{cases}$ (1')

③ Minimizing property (locally)

And for the hardest one ③ we develop 3 methods to prove it

ⓐ As our text, do step by step; for this we introduce a collection of notions

ⓑ Hopf–Rinow thm and it's proof

ⓒ Variation theory

Rk. 1. ①② is pure algebraic (in fact ① also), thus we have generalisation

(a) Given arbitary affine connection instead of Levi-Citiva

(b) for scheme instead of manifold

(c) When these generalization compatible with the "mertic" on other geometric objects such that ③ holds?

$\Downarrow$ Affine Geodesic

3. At last, we'll introduce some Exercise (not do!)

2. The Geodisc is the 1-dimension (5, 7, 9) minimal submanifold (Chap 6.)

Thm 1. The following is equivalence, and we call $\gamma: I\to M$ is Geodesic this case

① $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right)=0$ at all $t\in I$; $\left|\frac{d\gamma}{dt}\right|\equiv C$

② For $v\in T_{\gamma(t_0)}(\mathrm{Im}\,\gamma)$, the translation of v into $v(t)\in T_{\gamma(t)}(\mathrm{Im}\,\gamma)$ (by Levi-Citiva connection)

③ Locally minimize length: $\forall p\in M$, $\exists\, \varepsilon>0$, $\gamma|_{[0,1]}: [0,1]\to B_\varepsilon(p)$, for $\forall\, c:[0,1]\to M$ piecewise differentiable

$\gamma: \; I \longrightarrow M$ (with $[0,1]\subset I$, $B_\varepsilon(p)\subset M$)

and $c(\partial[0,1])=\gamma|_{[0,1]}(\partial[0,1])$

then $l(\gamma|_{[0,1]})\leq l(c|_{[0,1]})$

equality holds when $\gamma([0,1])=c([0,1])$

Pf. Here we prove ①$\Longleftrightarrow$② first

We canonical lift the map $I\xrightarrow{\gamma}M$ to $\tilde{\gamma}: I\longrightarrow TM$, $t\mapsto(\gamma(t),\gamma'(t))$ due to $TM\to M$ is fibration (bundle).

$$\begin{array}{ccc} & & TM \\ & \nearrow^{(\gamma,\gamma')} & \downarrow \\ I & \xrightarrow{\gamma} & M \end{array}$$

Now ①$\Longleftrightarrow \frac{d\gamma}{dt}(t)=\gamma'(t)$ is horizontal (Due to the connection, we can define the tangent/horizontal bundle/sheaf and normal bundle/sheaf along the γ

(orthogonal to $T_pM \Longleftrightarrow$ horizontal vector and horizontal vector field similiar) Ex 2

(this is a description of moving frame, we'll deal with it in Ex 1. for Geodesic frame)

We derive (1) by carculus of variation if δ(length functional) $=0 \Longleftrightarrow$ (1) then ①$\Longleftrightarrow$③ by approach ⓒ

(For more variation about Geodesic, see the first/second variation formula

Pf. (ⓒ) $\delta S=\delta\int_a^b g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\,dt$

$=\int_a^b \delta\left(g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right)dt=\int_a^b\left[(\delta g_{ij})\left(\frac{dx^i}{dt}\frac{dx^j}{dt}\right)+g_{ij}\frac{dx^j}{dt}\delta\left(\frac{dx^i}{dt}\right)+g_{ij}\frac{dx^i}{dt}\delta\left(\frac{dx^j}{dt}\right)\right]dt$

and $\delta(g_{ij})\frac{dx^i}{dt}\frac{dx^j}{dt}=\sum\frac{\partial g_{ij}}{\partial x^k}\delta x^k$

$\Rightarrow \delta S=\int_a^b\left[\sum_k\frac{\partial g_{ij}}{\partial x^k}\delta x^k\frac{dx^i}{dt}\frac{dx^j}{dt}+2g_{ij}\frac{dx^i}{dt}\frac{d}{dt}(\delta x^j)\right]dt$

integral by part

$\Rightarrow \delta S=\int_a^b\left(\sum_k\frac{\partial g_{ij}}{\partial x^k}\delta x^k\frac{dx^i}{dt}\frac{dx^j}{dt}\right)dt+2g_{ij}\frac{dx^j}{dt}(\delta x^j)\Big|_a^b-2\int_a^b\frac{d}{dt}\left(g_{ij}\frac{dx^j}{dt}\right)(\delta x^j)\,dt$

the variation of boundary $=0$

$\Rightarrow \delta(S)=\int_a^b\left[\sum\frac{\partial g_{ij}}{\partial x^k}\delta x^k\frac{dx^i}{dt}\frac{dx^j}{dt}-2\frac{d}{dt}\left(g_{ij}\frac{dx^j}{dt}\right)\right]$

$\sum_k\frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{dt}\frac{dx^j}{dt}-2\frac{d}{dt}\left(g_{ij}\frac{dx^j}{dx^k}\right)=0$

and the Christoffel symbol Γ_{ij}^{k} satisfy ($g^{kp}=g_{kp}^{-1}$)

$\Gamma_{ij}^{k}=\frac{1}{2}g^{kp}\left(\frac{\partial g_{ip}}{\partial x^j}+\frac{\partial g_{jp}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^p}\right)$

Let it into (1) to complete proof ☒)

$\Longleftrightarrow$ This canonical lift $\tilde{\gamma}=(\gamma,\gamma')$ is a horizontal lift

$\Longleftrightarrow$ The act $\gamma'(t_0)$ at $\gamma(t_0)$ to $\gamma(t)$ is horizontal act of translation $\Longleftrightarrow$ ② ☒

We have also a local charistization: (Due to the Thm 1 is local)

$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right)=0\Longleftrightarrow\sum_k\left(\frac{d\left(\frac{d\gamma}{dt}\right)^k}{dt}+\sum_{i,j}\Gamma_{ij}^{k}\left(\frac{d\gamma}{dt}\right)^j\frac{dx_i}{dt}\right)\frac{\partial}{\partial x^k}=0\Longleftrightarrow\sum_k\left(\frac{d^2x^k}{dt^2}+\sum_{i,j}\Gamma_{ij}^{k}\left(\frac{dx_i}{dt}\right)\left(\frac{dx_j}{dt}\right)\right)\frac{\partial}{\partial x^k}=0$

and $\frac{d\gamma}{dt}=\sum_i\frac{dx^i}{dt}\frac{\partial}{\partial x^i}$, i.e. $\left(\frac{d\gamma}{dt}\right)^k=\frac{dx^k}{dt}$ $\Longleftrightarrow \frac{d^2x^k}{dt^2}+\sum_{i,j}\Gamma_{ij}^{k}\left(\frac{dx_i}{dt}\right)\left(\frac{dx_j}{dt}\right)=0$, $\forall k$ ⋯ (1)

and after lift into TM, let local chart on TM inherited by $(x_1,\cdots,x_n,y_1,\cdots,y_n)$, $y_k=\frac{dx^k}{dt}$

$\Longleftrightarrow \begin{cases} y_k=\frac{dx^k}{dt} \\ \frac{dy_k}{dt}+\sum_{i,j}\Gamma_{ij}^{k}y_iy_j=0\end{cases}$ ⋯ (1'), we write the text: $\frac{dy_k}{dt}=-\sum_{i,j}\Gamma_{ij}^{k}y_iy_j$

This part shows that $y_i\frac{\partial}{\partial x_i}$ is the direction of v translation, and paralleled with the tangent of $\gamma(I)$

Rk. We Generalize to the definition

Totally geodesic submanifold is the following equivalence ($M\subset\bar{M}$ isometric embedding)

① Geodesic of M is Geodesic of $\bar{M}$

② T_pM translate to T_qM for $p,q\in M$

③ Locally M minimize the volume

For more study on Geometry of submanifold, see Chap 6.

E.g 1. $M=\mathbb{R}^n$ and $M_d\subset M$ to be the d-dimensional totally geodesic submanifold, then $M_d\cong\mathbb{R}^d$ by ③

Geodesic of M_d

$\Longleftrightarrow$ line $\Longleftrightarrow$ Geodesic of M

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To outline the structure of TM and then exponential map, we develop the following consequence

Prop 1. ① $\exists !$ vector field on TM : $\gamma : I \to TM$, $t \mapsto (\gamma(t), \dot\gamma(t))$ such that γ is a geodesic on M, called the Geodesic flow

(Rk. it's due to the fibration CHP (Covering homotopy property) directly, but here we haven't prove $TM \xrightarrow{\pi} M$ does a fibration)

② Give TM a Riemannian mertic by $\langle V, W\rangle_{(p,v)} = \langle d\pi(V), d\pi(W)\rangle_p + \langle \frac{D\lambda}{dt}(0), \frac{D\omega}{ds}(0)\rangle_p$ thaking TM a Riemannian manifold, then γ is a geodesic on TM

Pf. Due to ② is Ex2, we omit its proof

Only prove ①. The question isn't local but uniqueness is thus let $M = \mathbb{R}^n$, then unique is obvious by theory of ODE and equation (1), due to the uniqueness, the existence also local, then again by ODE ☐

Now on view of ODE, determine a geodesic is a initial value problem, we have

Prop 2. ① $\forall p \in M$, $\exists V \in N_p$, $\exists \delta > 0$, $\varepsilon > 0$ and map $\gamma : (-\delta, \delta) \times \underbrace{\{(q, v) \in TM \mid q \in V, v \in T_qM, |v| < \varepsilon\}}_{\text{denote as } \mathcal{U} \subset TM} \to M$, $(t, q, v) \mapsto \gamma(t, q, v)$ is the unique geodesic of M such that $\gamma'(0) = v$ for every $q \in V$ and every $|v| < \varepsilon$

② We can fix δ or ε to any $\mathbb{R} > 0$

Pf. ① is the analogue of ODE in Geometry, nothing to prove

② turns out to be prove $\gamma(t, q, kv) = \gamma(kt, q, v)$ (homogeneity) ☐

Lemma 1 (Homogeneity) $\gamma(t, q, kv)$ defined on $(-\frac{\delta}{k}, \frac{\delta}{k})$ induced by $\gamma(t, q, v)$ defined on $(-\delta, \delta)$ is same and $\frac{dh}{dt}(0) = a\frac{d\gamma}{dt}(0) = av$ ☐

Pf. Let $t \mapsto \gamma(at, q, v)$ =: h

if h is a Geodesic with $\frac{dh}{dt}(0) = av$, then by uniqueness: $\frac{D}{Dt}\left(\frac{dh}{dt}\right) \overset{\text{Prop 2.2 (1)}}{\underset{\text{chap 2}}{=}} \nabla_{h'(t)} h'(t) = \nabla_{a\gamma'(at, q, v)} a\gamma'(at, q, v) = a^2 \frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$

Due to the homogenouity of geodesic, we introduce various notions

Def. ① Exponential map $\exp : \mathcal{U} \to M$, $(q, v) \mapsto \gamma(1, q, v) = \gamma(|v|, q, \frac{v}{|v|})$ or write as $\exp_p : v \mapsto \gamma(1, p, v)$

② Normal ball/sphere / Geodesic ball/sphere

Let $B_r(p) = \exp_p B_r(0)$ and $S_r(p) = \partial B_r(p)$, where $B_r(0) \subset T_pM$ and the form $U = \exp_p V$ is called a normal neighborhood of p, $V \subset T_pM$

③ Geodesic complete if $\forall p \in M$, $\exp_p : B_r(0) \to M$ can be defined / extend to hole T_pM

④ Logarithm map for M Geodesic complete

$\exp_p : T_pM \to M$ induce diffeomorphism $\exp_p : \frac{T_pM}{\frac{T_pM}{Ke(\exp_p)}} \to Im(\exp_p)$

$\Rightarrow$ the inverse $\log_p : Im(\exp_p) \to \{v \in T_pM \mid \gamma(t) = \exp_p(tv) \text{ isn't Geodesic}\}$

$q \mapsto \log_p q$ the tangent vector $\in T_pM$

the Geometric picture will be shown point to q for Lemma 4.1.

E.g 2. $M = S^n \subset \mathbb{R}^{n+1}$

Due to Chap 2 EX4.(b), the great circles $\Leftrightarrow$ Geodesic

thus we can see the ball $B_r(p)$ in $M = S^n$ as

and exponential map send arc to arc length, that's why we call it exponential

Notice that both γ_1, γ_2 are Geodesic but only γ_1 minimize length

$\Rightarrow$ it must be local.

For exponential map, we have the following properties

Prop 3. ① $\exp_p$ is local diffeomorphism, i.e. $\exp_p : B_\varepsilon(0) \xrightarrow{\cong} M$ for $\exists \varepsilon > 0$, the we have log locally and $(Im(\exp_p), \log_p)$ is the normal corridnate chart

② (Gauss's Lemma) $\forall p \in M$, $\forall v \in T_pM : \exp_p v$ defined, $\forall w \in T_pM \cong T_v(T_pM)$

then $\langle (d\exp_p)_v(v), (d\exp_p)_v(w)\rangle = \langle v, w\rangle$

③ (Hopf-Rinow) If M connected, the Geodesic complete $\Leftrightarrow \exists$ d a mertic on M : (M, d) is a complete mertic space

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Pf: ① $(d\exp_p)_0(v) = \frac{d}{dt}(\exp_p)(tv)\Big|_{t=0} = \frac{d}{dt}\gamma(1, q, tv)\Big|_{t=0} = \frac{d}{dt}(\gamma(t, q, v))\Big|_{t=0} = v$

thus $(d\exp_p)_0 = Id_{T_pM} \Longrightarrow$ by inverse function thm, $\exists \varepsilon > 0$ ☒

② Similiar as parametrized curve,

we define the parametrized submanifold

to be $m: D \longrightarrow M$, where $D \subset \mathbb{R}^m \subsetneq \mathbb{R}^n$, and the vector field along m defined to be $V \in \Gamma(D, m^*(TM|_{Im(m)}))$

(D must need some smooth boundary properties..., we first not concern it)

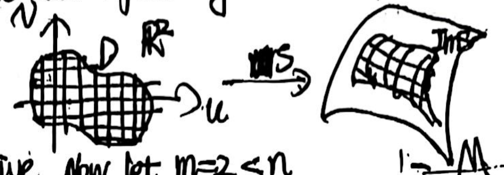
we think we weave the m-dimension parametrized submanifold by n-~~curves~~ parameterized curves

i.e. $m: D \longrightarrow M$ fix other and

$\quad D \subset \mathbb{R}^m$ change x^i only

$\quad (x^1 \cdots x^m)$ then we get a parametrized curve

and the covariant partial derivative



Now let $m = 2 < n$ and ∂D piece wise smooth, $\exists U \subset D \subset \bar{U}$, D connected

Inherit by the symmetry of Levi-Citiva connection

$\frac{D}{dt}\left(\frac{\partial f}{\partial s}\right) = \frac{D}{\partial t}\left(\sum_i \frac{\partial x^i}{\partial s}\frac{\partial}{\partial x^i}\right) = \sum_i \frac{\partial^2 x^i}{\partial t \partial s}\frac{\partial}{\partial x^i} + \sum_i \frac{\partial x^i}{\partial s}\nabla_{\sum_j \frac{\partial x^j}{\partial t}\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}$

$= \sum_i \frac{\partial^2 x^i}{\partial t \partial s}\frac{\partial}{\partial x^i} + \sum_{i,j} \frac{\partial x^i}{\partial s}\frac{\partial x^j}{\partial t}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}$ is symmetry on i, j, t, s

In particular, let

$f: [0, 1] \times (-\varepsilon, \varepsilon) \longrightarrow M$

$\quad (t, s) \longmapsto \exp_p tv(s)$

$v(s)$ is defined to be curve, $v(0) = v$

and $v'(0) = w_N$ the Normal part of w (due to the connection for without t), we ... $w = w_T + w_N$

thus for $s \in (-\varepsilon, \varepsilon)$ small, the curve is defined

Then $\langle (d\exp_p)_v(w_N), (d\exp_p)(w_N)\rangle = \langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle(1, 0)$ and $\frac{d}{dt}\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle = \langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle + \langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\rangle = \langle \frac{D}{\partial t}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle$

due to $t \mapsto f(t, s_0)$ is geodesic (radical), and $\frac{D}{dt}\frac{\partial f}{\partial s} = \frac{D}{ds}\frac{\partial f}{\partial t}$ by locally computation $\Rightarrow \frac{1}{2}\frac{\partial}{\partial s}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle = 0$

thus: $\langle (d\exp_p)_v(v), (d\exp_p)_v(w)\rangle = \langle (d\exp_p)_v(v), (d\exp_p)_v(w_T)\rangle + \langle (d\exp_p)_v(v), (d\exp_p)_v(w_N)\rangle$

$= \langle v, w_T\rangle + \langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle(1, 0) = \langle v, w_T\rangle + \langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle(0, 0) = \langle v, w_T\rangle + 0 = \langle v, w_T\rangle + \langle v, w_N\rangle = \langle v, w\rangle$ ☒

(due to it not depend on $t = 0$ or 1) $\quad (w_N \perp v)$

③ ($\Longrightarrow$) let $A \subset M$, A closed and bounded

i.e. $\exists M \geq 0$: $d(x, y) \leq M$ for $x, y \in A$, here let $d(x, y) = l(\gamma)$, γ is the Geodesic connecting x and y

$\Rightarrow A \subset \exp_x(\bar{B}_M(0))$ $\bar{B}_M(0)$ compact, $\exp_x$ continuous $\Rightarrow \exp_x(\bar{B}_M(0))$ compact, A closed $\Rightarrow A$ compact

($\Longleftarrow$) Otherwise, $\exists x \in M$ $\exp_x$ not defined at hole T_xM, i.e. $\exists v \in T_xM$

$\gamma(t) = \exp_x(tv)$ is only defined for $t \in [0, t_m)$, t_m not defined is a Geodesic

let $\{t_n\}$ Cauchy, tend to t_m, then Claim. $\{\gamma(t_n)\}$ Cauchy, thus by complete it converge to $\gamma(t_m)$ not exist $\Rightarrow$ contradiction

Pf the Claim $d(\gamma(t_i), \gamma(t_j)) \leq \left|\int_{t_i}^{t_j} \gamma'(t)\,dt\right|$

$\leq \int_{t_i}^{t_j} |\gamma'(t)|\,dt = |t_j - t_i|\left|\frac{d\gamma}{dt}\right|$ due to $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$

thus it follows that $\{\gamma(t_n)\}$ Cauchy ☒ $\quad \frac{d\gamma}{dt}$ not depend on t

Now we ~~use~~ use approach ⓐ ⓑ ⓒ to develop Thm 1. ②

ⓑ By Hopf-Rinow, ~~[illegible]~~ we'll show the minimizing property is global for Geodesic complete

For this, we prove that

Thm 2. (Hopf-Rinow) M geodesic-complete, $p, q \in M$, then $\exists$ a geodesic γ connecting p, q minimize length and the γ satisfy $d(p, q) = l(\gamma)$

thus we complete the ⓑ

Pf of Thm 2. (Non-trivial) Set $D: S_\delta(p) \longrightarrow \mathbb{R}_{\geq 0}$

$\quad y \longmapsto d(y, q)$

$S_\delta(p)$ is compact

$\Rightarrow \exists \delta$ is the minium of D, i.e. $\exists x \in S_\delta(p)$

$D(x) = \delta \leq D(y)$, $\forall y \in S_\delta(p)$

Let $v = \frac{\log_p x}{\|\log_p x\|}$ and $\gamma: t \mapsto \exp_p(tv)$ geodesic

and $\gamma(\delta) = \exp_p(\delta v) = \exp_p\left(\delta \frac{\log_p x}{\|\log_p x\|}\right) = \exp_p \log_p x = x$

by completeness, extend to $\gamma: \mathbb{R} \longrightarrow M$ (due $tv \in T_pM$ is defined for $\forall t$)

$\quad t \mapsto \exp_p(tv)$

$\gamma(d(p, q)) = q$ is our task

equivalently, $d(\gamma(d(p, q)), q) = 0$ or stronger $d(\gamma(t), q) = d(p, q) - t$

for $t \in [\delta, r]$ ... (*) ($r = d(p, q)$), $[t, t+\delta)$ holds

Now we prove the (*), we use the continuous induction

Step 1 Holds for δ Step 2 If hold for t, then $\exists \delta$ not depend on t

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Step 1 If $d(\gamma(\varepsilon), q) < r-\varepsilon$

$\Rightarrow d(p,q) \le d(p,x) + d(q,x) < \varepsilon + r - \varepsilon = r$

$\Rightarrow d(\gamma(\varepsilon), q) \ge r-\varepsilon$

and If $d(\gamma(\varepsilon), q) > r-\varepsilon$, take aribitary piecewise curve c joining p, q, and the first point $t_1 \in (0,1)$ s.t. $c(t_1) \in S_\varepsilon(p)$ (compare with prop3.6 in text!)

$\Rightarrow l(c) = l(c|_{[0,t_1]}) + l(c|_{[t_1,1]}) > \varepsilon + r - \varepsilon = r$

$\Rightarrow d(\gamma(\varepsilon), q) \le r-\varepsilon$

$\Rightarrow d(\gamma(\varepsilon), q) = r-\varepsilon$

Step 2 Now $d(\tilde\gamma(t_0), q) = r - t_0$

$\exists \delta > 0$: $B_\delta(\tilde\gamma(t_0))$ well-defined

We need show ① δ not depend on t_0.

② $d(\tilde\gamma(t_0+\delta), q) = r - (t_0+\delta)$ (due to for $t < t_0+\delta$)

① is due to Geodesic complete (we take $\delta' = t - t_0 < \delta$)

② Replace ε with δ, repeat the step 1, we have

Due to $S_\delta(\tilde\gamma(t_0))$ compact, $\exists x' \in S_\delta(\tilde\gamma(t_0))$: $d(x', q) = d(y, q)$

$\Rightarrow d(x', q) = r - (t_0+\delta)$

Let $w = \dfrac{\log_{\tilde\gamma(t_0)}(x')}{\|\log_{\tilde\gamma(t_0)}(x')\|}$ and $\beta: t \mapsto \exp_{\tilde\gamma(t_0)}((t-t_0)w)$ geodesic

then $x' = \beta(t_0+\delta)$, I claim $\beta \equiv \tilde\gamma$

This due to the uniqueness of Geodesic ☒

The approach ② in text (Prop3.6 and Coro3.9)

($\Rightarrow$) We can assume $c([0,1]) \subset B$ due to otherwise, let t_1 the first point $c(t_1) \in \partial B \Rightarrow l(c) \ge l_{[0,t_1]}(c) \ge$ radius of $B \ge l(\gamma)$

We write $c(t)$ normalised: $c(t) = \exp_p(r(t)v(t))$ determined by $r(t) = \|\log_p c(t)\|$ and $v(t) = \dfrac{\log_p c(t)}{r(t)}$

and denote as $c(t) = f(r,t)$

$\Rightarrow \dfrac{dc}{dt} = \dfrac{\partial f}{\partial r}\dfrac{dr}{dt} + \dfrac{\partial f}{\partial t}$ for almost all t

$\Rightarrow \left\langle \dfrac{dc}{dt}, \dfrac{dc}{dt} \right\rangle = \left\langle \dfrac{\partial f}{\partial r}\dfrac{dr}{dt}, \dfrac{\partial f}{\partial r}\dfrac{dr}{dt} \right\rangle + \left\langle \dfrac{\partial f}{\partial t}, \dfrac{\partial f}{\partial t} \right\rangle + 2\underbrace{\left\langle \dfrac{\partial f}{\partial r}\dfrac{dr}{dt}, \dfrac{\partial f}{\partial t} \right\rangle}_{=0}$ by acting $\exp_p$ then by Gauss Lemma

$\Rightarrow \left|\dfrac{dc}{dt}\right| \ge |r'(t)| \Rightarrow \int_\varepsilon^1 \left|\dfrac{dc}{dt}\right| dt \ge \int_\varepsilon^1 r'(t)\,dt = r(1) - r(\varepsilon)$

($\Leftarrow$) Now $[a,1] \subset [a,b]$ $l(c) \qquad l(\gamma) = l(\gamma) - 0$

and γ join $\gamma(0), \gamma(1) \in B_{\gamma(t_0)}(\varepsilon)$, $t_0 \in [0,1]$ is minimal the length

$\gamma|_{[0,1]}: [0,1] \to B_{\gamma(t_0)}(\varepsilon)$ and $l(\gamma|_{[0,1]})$ = length of geodesic connecting $\gamma(0)$ and $\gamma(1)$, denoted as $l(\tilde\gamma)$

$\Rightarrow \begin{cases} \left|\dfrac{\partial f}{\partial t}\right| = 0 \\ |r'(t)| = r'(t) > 0 \end{cases}$ in the proof of "$\Rightarrow$" $\Rightarrow$ c (now γ) is just change the velocity of $\tilde\gamma$ (now $\tilde\gamma$) $\Rightarrow c([0,1])$ (now $\gamma([0,1])$) $= \gamma([0,1])$ (now $\tilde\gamma([0,1])$)

$\Rightarrow \gamma$ is geodesic, but in the proof of "$\Rightarrow$", we need $B_p(\varepsilon)$ is a normal neiberhood of $p = \gamma(0) \ne \gamma(t_0)$

Thus we modify the following prop to complete the proposition

Prop4. $\forall p \in M, \exists W \in \mathcal{N}_p, \forall q$, $W \in \mathcal{N}_q$ is normal relative to q.

Pf. I.E, $\forall q \in W$, $\exp_q(B_\delta(0)) \cong B_\delta(0)$ and $\exp_q(B_\delta(0)) \supset W$, $\exists \delta > 0$

(Not trivial) Define $F: \mathcal{U}_{\text{?}} = \{(q,v) \mid q \in V, v \in T_qM, |v| < \varepsilon\} \to M \times M$
$(q,v) \mapsto (q, \exp_q(v))$

Then $\operatorname{Im}(F|_{v=0}) = \Delta_M$, given Δ_M the manifold structure by $(U \times U, (x,x)) \Rightarrow dF_{(p,0)}$ is $\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \Rightarrow$ local diffeomorphism by inverse function thm, i.e, $\exists \delta > 0$: $\mathcal{U}' = \{(q,v) \mid q \in V' \subseteq V, v \in T_qM, |v| < \delta\}$ and $F(\mathcal{U}') \cong \mathcal{U}'$

let $W \times W \subset F(\mathcal{U}') \Rightarrow W$ is the desired neiberhood (called totally normal NBS)

Verification left to reader due to content ☒

For higher dimensional minimize, we have (without proof)

Thm3. Minimalize the volume locally $\Leftrightarrow$ Minimal submanifold $\overset{\text{def}}{\Leftrightarrow}$ Main curvature vanish (Chap6. Def2.10)

Precisely, $M \subset \bar M$, $\forall p \in M$, $\exists \bar U \in \mathcal{N}_p$, $\exists \bar B \subset \bar U$: $B = \bar B \cap M$

such that $\forall \Omega \subset B$ with $\partial\Omega = \partial B \to V(\Omega) \ge V(B)$ // as Hahn-Banach, and results in PDE

Convex NBS (Sketch) (We can then generalize this in functional analysis)

Def (Convex) ① $S \subset M$ is (strongly) convex $\overset{\text{def}}{\Leftrightarrow}$ $\forall q_1, q_2 \in \bar S$, $\exists!$ Geodesic joining q_1 and q_2: $\operatorname{Im}\gamma \subset S$ ($\operatorname{Int}\operatorname{Im}\gamma$)

② $f \in \Gamma(M, \mathcal{O}_M)$ is Geodesic convex $\Leftrightarrow$ $\forall \gamma$ geodesic, $f \circ \gamma: I \to \mathbb{R}$ convex $\Leftrightarrow$ Hess $f \ge 0$ (Chap6. Ex11)

E.g3. (Complex Geodesic)

(Introduced by Vesentini, 1981)

For a complex manifold, or more easy, a domain D in a complex vector space V a complete LCH (locally convex Hausdorff)

Then is natural to have a geodesic is $f: \Delta$ (unit disk in $\mathbb{C}$) $\to D$ satisfy some property (P), which equivalent to given a restriction on mertic

That's where difficult, the mertic should contain the holomorphic structure

The Poincaré mertic is the most natural $ds^2 = \dfrac{|d\zeta|^2}{(1-|\zeta|^2)^2}$ (of Δ)

and the mertic of D has Kobayashi and Carathéodory

E.g. 4 (Convex not strong convex)

$p \ge 2$, $X = \sqrt{y^2 - x^3 + 3x^2 - 3x + 1}$ ... $\subset \mathbb{R}^3$ revolve X around x-axis get Y

$S = \{(x,y,z) \in Y \mid x \le p^2\}$ convex not strong convex in Y (Left as Ex)

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The last two thm in text shows ~~the existence of~~ the convexity of Geodesic ball when ε small

(Furthermore, strong convex)

Sketch) Compare the convex or not:

thus we use the Lemma 4.1. to show the Geodesic ball can stay out of for a neiberhood of outside point q, then prove the convexity by contradiction, and note that here we apply the $\log_p$, due to (Recall) the Geometric meaning of $\log_p$ just send q to the tangent to p, i.e. geodisc normal to $S_\varepsilon(p)$.

Then we skeletch the Exercises (5, 7, 9)

Ex5 (Killing field)

$X \in \mathcal{X}(M)$ is Killing when following equivalence condition hold

① $\mathcal{L}_X g \equiv 0$ ($\mathcal{L}_X$ is the Lie derivative)

② The Killing equation $\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0$ (Locally, $\nabla_{X_i} X_j + \nabla_{X_j} X_i = 0$)

③ X is an infinitesimal isometry: the flow $\varphi: (-\varepsilon,\varepsilon)\times U \to M$ has $\varphi_q(t)$, $\forall q$ the curve of X: $\varphi_q'(0) = X$ and $\forall t$, $\varphi_t: U \to M$ is isometry

Pf. ① $\Leftrightarrow$ ② (② $\Leftrightarrow$ ③ is Exercise 5)

$(\mathcal{L}_X g)(Y,Z) = 0 \Leftrightarrow X(g(Y,Z)) - g([X,Y],Z) - g([X,Z],Y) = 0$, write as inner product

$\Rightarrow \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle - \underbrace{(\langle [X,Y],Z\rangle + \langle [X,Z],Y\rangle)}_{0} = 0$

$\Rightarrow \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = 0$ ▨

We have the following property of Killing field (without proof), let X Killing on M

① X restrict to a Geodesic is Jacobi field (Chap 5)

② (Kostant) $\nabla_Y(\nabla X) = R(Y,X)$ (Curvature, Chap 4)

③ (Bochner) Let $f: M \to \mathbb{R}$, $x \mapsto \frac{\|X(x)\|}{2}$ then $\Delta f = -\mathrm{Ric}(X,X) + \|\nabla X\|$

(∇ is the gradient)

Ex7. (Geodesic frame)

$\dim M = n$, $\forall p \in M$, $\exists$ a local geodesic frame at p, i.e. a family $\{E_i\}_{i\le n} \subset \mathcal{X}(U)$, $\exists U \in N_p$ such that $\nabla_{E_i} E_j(p) = 0$, and orthgonal at every point of U

(Hint. Find orthgon basis of T_pM and translation them by geodesic.)

Ex9. (Beltrami-Laplace) $\Delta f = \operatorname{div} \nabla f$
$= \operatorname{tr}(\operatorname{Hess} f)$

(divergence and gradient is defined in Ex8)

It's the converse of the Hodge-Laplace $\tilde{\Delta}$: $\Delta f = -\tilde{\Delta} f$ ($\Delta f \ge 0 \Rightarrow f \equiv C$) (Hopf)

It's importance is Ex12, the basis of the Bochner technique, we can study f by study its Laplacian

Continue (Complex Geodesic)

$k_D(x,y) = \inf_{\{\zeta_i\} \text{ is an analytic chain}} \sum \omega(\zeta_i, \zeta_{i+1})$

$C_D(x,y) = \sup_{h \in \mathrm{Hol}(D,\Delta)} \omega(h(x), h(y))$

then $C_D(f(\zeta_1), f(\zeta_2)) \le k_D(f(\zeta_1), f(\zeta_2)) \le \omega(\zeta_1,\zeta_2)$, $\forall \zeta_1,\zeta_2 \in \Delta$

The analytic chain is a line partioned by connecting each ζ_i with a holomorphic funtion

Precisely $\exists \{f_i\} \subset \mathrm{Hol}(\Delta, D)$: $\begin{cases} f_0(\zeta_0) = x \\ f_i(\zeta_i) = f_{i-1}(\zeta_i) \\ f_l(\zeta_l) = y \end{cases}$

i.e. a line preserve the holomorphic information

Then

f is a complex Geodesic if following hold

① $\forall \zeta_1, \zeta_2 \in \Delta$, $C_D(f(\zeta_1), f(\zeta_2)) = k_D(f(\zeta_1), f(\zeta_2)) = \omega(\zeta_1,\zeta_2)$

② $\exists (\zeta_1,\zeta_2) \in \Delta^2$, $\cdots = \cdots = \cdots$

Pf. ① $\Rightarrow$ ② trivial (For $\exists \to \forall$, we always use Schwarz-Pick)

② $\Rightarrow$ ① (Sketch) $\exists \{h_n\} \subset \mathrm{Hol}(D,\Delta)$: $C_D(f(\zeta_1), f(\zeta_2)) = \lim \omega((h_n \circ f)(\zeta_1), (h_n \circ f)(\zeta_2))$

by Montel $\Rightarrow$ a subsequence converge compact uniformly in Δ

then we have $\{h_n \circ f\} \subset \mathrm{Hol}(\Delta)$, applying Schwarz-Pick ▨

The existence and uniqueness of complex Geodesic

① D convex $\Rightarrow \exists$ ② D strongly convex $\Rightarrow \exists !$

The convex (strong) neiberhood also holds for complex case

For complex manifold X, instead of D, the same way k_X and C_X is induced by ω instrintic mertic by Poincaré mertic (or other hyperbolic mertic) The study need the Fihner mertic.

Prob. For X Kähler, what about it for its Riemannian mertic compatible?

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Moving frame $M^n \subset \mathbb{R}^{n+m}$, the place vector $r = (x_1 \cdots x_{n+m})$ Topic1

taking orthogonal frame $\{e_i\}$ in M^n and $\{e_{\alpha+n}\}$ in NM

dual 1-form $\{\omega_i\}$ $\Rightarrow de_j = \sum_{k\le n+m} \omega_{jk} e_k$

$\Rightarrow dr = \sum \omega_i e_i$ $\Rightarrow d\langle e_i, e_j\rangle = \omega_{ij}+\omega_{ji}$, $d0 = 0$

and $I = \langle dr, dr\rangle = \sum \omega_i \omega_i$ (tensor) anti-symmertic

and $0 = d^2 r = d(\sum \omega_j e_j)$

$= \sum (d\omega_j e_j - \omega_j \wedge de_j)$

$= \sum (d\omega_j e_j - \omega_j \wedge \omega_{jk} e_k)$

$\Rightarrow \begin{cases} d\omega_j = \sum \omega_i \wedge \omega_{ij}, & j\le n \text{ intristic} \\ \boxed{0 = \sum \omega_i\wedge\omega_{ij}}, & j \ge n+1 \text{ extristic} \end{cases}$

Let $\omega_{ij} = \sum_k h^j_{ik}\omega_k$

$\Rightarrow 0 = \sum \omega_i \wedge h^j_{ik}\omega_k \Rightarrow \boxed{h^\alpha_{ij} = h^\alpha_{ji}}$ symmertic

Define $II_\alpha = -\langle dr, de_{n+\alpha}\rangle = -\langle \sum\omega_i e_i, \sum \omega_{n+\alpha,k} e_k\rangle$

$= \sum \omega_i \omega_{i,n+\alpha} = h^\alpha_{ij}\omega_i\omega_j$. In partiaular, consider $m=1$

hypersurface case $\Rightarrow II = h_{ij}\omega_i\omega_j$

$\Rightarrow K = \det(h_{ij})$ & $H = tr(h_{ij})$ defined.

and Codazzi equation: $0 = d^2 e_j = d(\sum \omega_{jk} e_k) \Rightarrow$

other equations $\Rightarrow \boxed{d\omega_{ik} = \sum \omega_{il}\wedge\omega_{lk}}$

(Ricci ...) all can be computed by this way.

We use moving frame to give these: Codazzi, Ricci, Simons..

Notation. Denote $f_{ijk} = \nabla^3 f(e_i,e_j,e_k)$ for $\{e_i\}$ frames

Similiar if one have $f_{ijkl} = \nabla^4 f(e_i,e_j,e_k,e_l) = \nabla f_{ijk}(e_l)$

When $M^n \subset \bar M^{n+m}$ has $\bar M$ constant curvature, Codazzi equation

turns out to be $\boxed{h_{ijk}=h_{kij}}$ ($h_{ij} = \langle B(e_i,e_j),\nu\rangle$ as usual)

Ricci equality $\boxed{\nabla^2 T(\cdots, e_i,e_j) - \nabla^2 T(\cdots, e_j, e_i) = R_{ij}T(\cdots)}$

$M^n \subset \mathbb{R}^{n+m}$ minimal $\Rightarrow H = 0 = \sum h_{ii}$, by Gauss equation $\boxed{R_{ijij} = \bar R_{ijij} - h_{ii}h_{jj} - h_{ij}^2}$

$\Rightarrow \text{scalar}_M = \sum R_{ijij} - \sum h_{ij}^2 = \sum \bar R_{ijij} - |B|^2$

$Ric(e_i,e_i) = \sum \bar R_{ijij} - \sum h_{ij}^2 \Rightarrow h_{ikjk} = h_{ikkj} + R_{kjim}h_{mk} + R_{kjkm}h_{mi}$

$= h_{ikkj} + (\bar R_{kjim} + h_{ki}h_{jm} - h_{km}h_{ij})h_{mk} + (\bar R_{kjkm} + h_{kk}h_{jm} - h_{km}h_{jk})h_{mi} = \bar R_{kjim}h_{mk} + \bar R_{kjkm}h_{mi} - h_{ij}|B|^2 + h_{ik}h_{jm}h_{mk} - h_{im}h_{jk}h_{km}$

The Simons equality $\frac12\Delta|B|^2 = |\nabla B|^2 - |B|^4 + n|B|^2$; $\bar M^{n+1} = S^{n+1}$

$\frac12\Delta|B|^2 = |\nabla B|^2 - |B|^4$; $\bar M^{n+1} = \mathbb{R}^{n+1}$

can be proven by upper computations;

$|\nabla B|^2 = \sum h_{ijk}^2$

$|\nabla|B||^2 = \left|\frac{\nabla|B|^2}{2|B|}\right|^2 = \frac{1}{|B|^2}|h_{ijk}h_{ij}e_k|^2$

$\le \frac{1}{|B|^2}(\sum h_{ijk}^2)(\sum h_{ij}^2) = |\nabla B|^2$ ∎

In M compact case.

$0 = \int|\nabla B|^2 + \int |B|^2(n-|B|^2)$ and $0\le|B|^2\le n$

$\Rightarrow |B|^2 \equiv 0$ or n.

e.g. ($|B|^2=n$, Clifford minimal hypersurface) $S^i\times S^j \subset S^{n+1}$, $i+j=n$ (Chern asked that:

precisely, $S^i(\sqrt{\frac{i}{n}})\times S^j(\sqrt{\frac{j}{n}})$ is minimal submanifold. $\Sigma = \{|B| \mid M^n \subset S^{n+1}$ minimal$\}$ is discrete? open)

Geometry of submanifold

Topic2

$M^n\subset\mathbb{R}^{n+m} \Rightarrow \bar\nabla_{e_i} x = e_i$ by $e_i = \sum a_{ij}E_j$

$\{E_i\}$

$\Rightarrow \bar\nabla_{e_i}x = \sum a_{ij}(\bar\nabla_{E_j}x) = \sum a_{ij}E_j = e_i$

$\Rightarrow \langle e_i,E_j\rangle = \langle\bar\nabla_{e_i}x, E_j\rangle$ ($x = (x_1\cdots x_j\cdots x_{n+m}) = \sum x_jE_j$) ∎

$\bar\nabla_{e_i}\langle x,E_j\rangle = \bar\nabla_{e_i}x_j = \nabla_{e_i}x_j$

$\Rightarrow \Delta x_j = \sum(\nabla_{e_i}\nabla_{e_i})x_j - (\nabla_{e_i}e_i)x_j = \sum(\nabla_{e_i}\langle e_i,E_j\rangle - \langle\nabla_{e_i}e_i, e_k\rangle\nabla_{e_k}x_j)$

$= \sum(\langle\bar\nabla_{e_i}e_i, E_j\rangle - \langle\nabla_{e_i}e_i,e_k\rangle\langle e_k,E_j\rangle) = \sum\langle\bar\nabla_{e_i}e_i - \nabla_{e_i}e_i, E_j\rangle = \langle H, E_j\rangle$

$\Rightarrow \Delta x = \sum\Delta(x_jE_j) = \sum\langle H,E_j\rangle E_j = \vec H$ ∎

$\Rightarrow \boxed{\Delta x = \vec H}$. In particular, $\vec H = 0 \Leftrightarrow x$ is harmonic map $\Leftrightarrow$ Minimal submanifold

In particular $m=1$, $\vec H = H\nu$, $\vec H = 0 \Leftrightarrow \nu = 0$

a vanishing of function

In particular graph case, $\Delta u = 0$ (height function, i.e. last corridnate) $\nu = \frac{-\nabla u + E_{n+1}}{\sqrt{1+|\nabla u|^2}}$

$\Rightarrow \frac{1}{\sqrt g}\partial_i(\sqrt g\, g^{ij}\partial_j u) = 0$

Conn. $H = \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$

$= \frac{1}{\sqrt{1+|\nabla u|^2}}\partial_i\left(\sqrt{1+|\nabla u|^2}\left(\delta_{ij} - \frac{u_iu_j}{1+|\nabla u|^2}\right)u_j\right)$ ∎

$= \frac{1}{\sqrt{1+|\nabla u|^2}}\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \Rightarrow \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$

Eg. When $M^n\subset\mathbb{R}^{n+1}$ is a graph of u: $\Omega\to\mathbb{R}$

$\Rightarrow r_j = \frac{\partial}{\partial x_j}(x,u) = (0\cdots \underset{j}{1} \cdots u_j)$ ($dr = r_j dx_j$)

$\Rightarrow \langle r_i,r_j\rangle = \delta_{ij}+u_iu_j$

and $g = I$: $g_{ij}dx_idx_j = \sum(dx_i^2) + du^2$

$= \sum dx_i^2 + (\sum u_idx_i)^2$

$= (\delta_{ij}+u_iu_j)dx_idx_j$

$\Rightarrow g_{ij} = \delta_{ij}+u_iu_j$

$\Rightarrow vol(M) = \int\sqrt{\det g_{ij}}$

$= \int\sqrt{1+|\nabla u|^2}$

denote $V = \sqrt{1+|\nabla u|^2}$ the volume form

Set $\nu = \frac{(-u_1\cdots-u_n,1)}{V} = \frac1V(-\nabla u + e_{n+1})$

$\Rightarrow d\nu = (-\nabla u + e_{n+1})d(\frac1V) - \frac1V u_{ij}dx_je_i$

$\Rightarrow II = -\langle dr, d\nu\rangle = -\langle r_jdx_j, -\frac1V u_{ij}dx_je_i\rangle$

$= \frac1V u_{ij}dx_idx_j$

Observation. Here $\{x_i\}$ not orthgonal frame!

$\Rightarrow K = \frac{\det II}{\det I} = \frac{\frac{1}{V^n}\det(\nabla^2 u)}{V^2} = \frac{\det\nabla^2 u}{V^{n+2}}$

(Gauss-Kronecker curvature)

$\Rightarrow H = tr(II\cdot I^{-1})$, For this we have

$I^{-1} = (g_{ij})^{-1} = (\delta_{ij}+u_iu_j)^{-1} = (I + uu^T)^{-1}$

$= \sum_j(-uu^T)^j = 1 - \frac{1}{1+|u|^2}uu^T$, i.e. $g^{ij} = \delta_{ij} - \frac{u_iu_j}{V^2}$

$\Rightarrow H = tr(I^{-1}\cdot II) = tr\left(\delta_{ij} - \frac{u_iu_j}{V^2}\right)\frac{u_{ij}}{V} = \frac1V\left(\Delta u - \frac{u_iu_ju_{ij}}{V^2}\right)$

$= \mathrm{div}\left(\frac{\nabla u}{V}\right)$ in $\mathbb{R}^{n+1}$

Another computation of H:

Moving ν in $\Omega\times\mathbb{R}$ parallel $\Rightarrow \tilde\nu = \frac1V(-\nabla u + E_{n+1})$

$\Rightarrow H = -\sum\langle e_j, \bar\nabla_{e_j}\nu\rangle = -\sum\langle e_j,\bar\nabla_{e_j}\nu\rangle = \underbrace{-\mathrm{div}(\tilde\nu)}_{\text{in } \mathbb{R}^{n+1}}$

$= \underbrace{-\mathrm{div}(\tilde\nu)}_{\text{in } \mathbb{R}^{n}} = \mathrm{div}\left(\frac{\nabla u}{V}\right)$ ∎

(E_{n+1} not provide any div)

This example relates moving frame with Caculus.

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$M^n \hookrightarrow S^{n+m} \subset \mathbb{R}^{n+m+1}$ $H = \sum_i (\bar\nabla_{e_i} e_i)^{NS^{n+m}/M} = \left(\sum(\bar\nabla_{e_i}e_i)^{TS^{n+m}}\right)^{NS^{n+m}/M} = \left(\sum(\bar\nabla_{e_i}e_i)^{N\mathbb{R}^{n+m+1}/M}\right)^{TS^{n+m}}$ (projection commutes)

$\nabla \quad \bar\nabla \quad \bar{\bar\nabla}$ (1) (2) $= \bar H^{TS^{n+m}}$

Thm. Embedding (1) Minimal $\Longleftrightarrow \Delta x = -nx$. Pf. $H=0 \Longleftrightarrow \bar H^{TS^n} = 0 \Longleftrightarrow \Delta x = \bar H$ normal to S^n

Embedding (1) has H, (2) has $\bar H \Longrightarrow \Delta x = \lambda x$, we need to determine λ (thus $\Longleftarrow$ is trivial)

(Yau) Is $-n$ the first eigenvalue of Laplacian? $0 = \frac{1}{2}\Delta|x|^2 = \langle \Delta x, x\rangle + |\nabla x|^2 = \langle \lambda x, x\rangle + \sum|\nabla_{e_i}x|^2 = \lambda|x|^2 + n = \lambda + n$ ☒

Thm. $\psi: M^n \to \mathbb{R}^{n+m+1}$ isometry & $\Delta\psi = -\lambda\psi \Longrightarrow$ (1) $\lambda > 0$ (2) $\psi(M) \subset S^{n+m}(\frac{n}{\lambda})$ (3) ψ is minimal.

Rk. Hyperbolic space is harder to deal in minimal theory:
(1) Models are various;
(2) Noncompact;
(3) Computation complex;
(4) Regularity fails.

Pf. Observation: $\Delta\psi = \bar H$ still holds (By $\bar\nabla_{e_i}\psi = e_i$, i.e. $\psi_* e_i = e_i$ ☒)

$\Longrightarrow$ Projection to normal: $\bar H = -\lambda\psi$, ψ is normal vector $\Longrightarrow \nabla_{e_i}\langle\psi,\psi\rangle = 2\langle\bar\nabla_{e_i}\psi,\psi\rangle = 0$

$\Longrightarrow \langle\psi,\psi\rangle = r^2$ constant $\Longrightarrow 0 = \frac{1}{2}\Delta|\psi|^2 =$ same argument as upper $= -\lambda r^2 + n$

$\Longrightarrow r^2 = \frac{n}{\lambda} > 0 \Longrightarrow \lambda > 0 \cdots (1)$, and $\psi(M) \subset S^{n+m}(r) = S^{n+m}(\frac{n}{\lambda}) \cdots (2)$, minimal $\cdots (3)$ ☒

Coarea formula. $\int_{\{x\in M \mid \psi(x)\le t\}} |\nabla\psi| = \int_{-\infty}^t \int_{\{x\in M \mid \psi(x) = s\}} ds$, $\psi \in C_c^\infty(M)$ or $\int_{\{\cdots\}} |\nabla\psi| f = \int_{-\infty}^t \int_{\{\cdots\}} f ds$

density in t $|\nabla\psi|$ $\{x\in M \mid \psi(x)\le t\}$

$\Longrightarrow$ Using this to:

Setting. $\Delta|x|^2 = 2\langle\Delta x, x\rangle + 2|\nabla x|^2 = 2n$ (minimal)

$B_t^M = B_t(0)\cap M$ the extrinsic ball, $\partial M \cap B_R(0) = \emptyset \Longrightarrow \partial B_t^M = \partial B_t(0) \cap M$

$\Longrightarrow 2n\,\mathrm{vol}(B_t^M) = \int_{B_t^M}\Delta|x|^2 = \int_{\partial B_t^M}\langle\nabla|x|^2, \frac{x^T}{|x^T|}\rangle$ (by Stokes) $= \int_{\partial B_t^M}\langle\nabla_{e_i}|x|^2 e_i, \frac{x^T}{|x^T|}\rangle = \int_{\partial B_t^M}\langle 2\langle\bar\nabla_{e_i}x, x\rangle e_i, \frac{x^T}{|x^T|}\rangle = \int_{\partial B_t^M}\langle 2\langle e_i, x\rangle e_i, \frac{x^T}{|x^T|}\rangle$

$= 2\int_{\partial B_t^M}\langle x^T, \frac{x^T}{|x^T|}\rangle = 2\int_{\partial B_t^M}|x^T|$ & $\int_{B_t^M} 1 = \int_{-\infty}^t \int_{\partial B_s^M}\frac{ds}{|\nabla|x||}$ (coarea formula) $\Longrightarrow \frac{d}{dt}\int_{B_t^M} 1 = \int_{\partial B_t^M}\frac{1}{|\nabla|x||}$, $\nabla|x| = \frac{\nabla|x|^2}{2|x|} = \frac{2x^T}{|x|}$

$\Longrightarrow \frac{d}{dt}\left(\frac{\mathrm{vol}(B_t^M)}{t^n}\right) = \frac{1}{t^n}\int_{\partial B_t^M}\frac{1}{|\nabla|x||} - \frac{n}{t^{n+1}}\int_{B_t^M} 1 = \frac{1}{2t^n}\int_{\partial B_t^M}\frac{|x|}{|x^T|} - \frac{n}{t^{n+1}}\int_{B_t^M} 1 = \frac{1}{2t^{n+1}}\int_{\partial B_t^M}\frac{|x|^2}{|x^T|} - \frac{n}{t^{n+1}}\int_{B_t^M} 1 = \frac{1}{2t^{n+1}}\int_{\partial B_t^M}\left(\frac{|x^N|^2}{|x^T|} + |x^T|\right) - \frac{n}{t^{n+1}}\int_{B_t^M} 1$

$\Longrightarrow = \frac{1}{2t^{n+1}}\int_{\partial B_t^M}\frac{|x^N|^2}{|x^T|} \ge 0 \Longrightarrow \frac{\mathrm{vol}(B_t^M)}{t^n} \nearrow$

Adding f: $2n\int_{B_t^M} f =$ same argument $= 2\int_{\partial B_t^M} f|x^T| - \underline{\int_{B_t^M}\mathrm{div}((|x|^2-t^2)\nabla f)} + \int_{B_t^M}(|x|^2 - t^2)\Delta f$

$\Longrightarrow \frac{1}{2t^{n+1}}\int_{\partial B_t^M}\frac{|x^N|^2}{|x^T|} f \ge 0$ when $f \ge 0$ $\quad = 2\int_{\partial B_t^M} f|x|^T + \int_{B_t^M}\underbrace{(|x|^2 - t^2)}_{\le 0}\Delta f$

Modify to $\Delta f \ge -Cf$ by Stokes

Setting $\Delta f \ge 0 \Longrightarrow$ Set $F(t) = \frac{1}{t^n}\int_{B_t^M} f$

i.e. $\frac{d}{dt}\left(\frac{1}{t^n}\int_{B_t^M} f\right) \ge 0 \Longrightarrow \boxed{\Delta f \ge 0 \Longrightarrow \frac{1}{t^n}\int_{B_t^M} f \le \frac{1}{r^n}\int_{B_r^M} f}$ $\quad 2n\int_{B_t^M} f \le 2\int_{\partial B_t^M} f|x|^T$

(Semi-haromonic.) $\frac{dF}{F} \ge -\frac{1}{2}Ct\,dt \Longrightarrow \boxed{F(t)e^{\frac{Ct^2}{4}} \le F(r)e^{\frac{Cr^2}{4}}}$

thus we have $t \to 0$ (assume $0 \in M$) $\Longrightarrow f(0) \le \frac{1}{\omega_n r^n}\int_{B_r^M} f$ the semi-mean value inequality. (ω_n the volume of standard ball) and $\mathrm{vol}(B_r^M) \ge \omega_n r^n$ $(r \to 0)$

Coro. When not minimal we need $\Delta|x|^2 \ge 2n$

$\Longleftrightarrow \Delta\rho^2 \ge 2n$, at least when need $\sec_M \le 0$

But we can weeken into $\mathrm{Ric}_M \ge -k^2$ case $\Longrightarrow$ (Scohen, Yau) $f(x) \le \frac{C(n,kr)}{r^n}\int_{B_r(x)} f$ (Proven by Heat Kernel) $B_r(x)$ (Geodesic ball)

Using Neumann-Poincaré inequality one can have Sobolev inequality and all things hold well.

The variation formula of minimal surfaces

Assume $M^n \subset \mathbb{R}^{n+1}$ orientible (due to variation is locally, and locally all manifold orientible) ν the unity vector field normal in M, $F: M \times [0,\varepsilon) \longrightarrow \mathbb{R}^{n+1}$ induced by $f \in C_c^\infty(M)$, $M_t := F(M,t)$ a family of ~~maps~~ manifolds, equipped the mertic

$(x, t) \longmapsto x + tf(x)\nu(x)$

$g_{ij}(t) := \langle F_i, F_j\rangle$, where $F: \mathbb{R}^{n+1} \longrightarrow M_t \subset \mathbb{R}^{n+1}$

$(x_1 \cdots x_n, t) \longmapsto x + tf(x)\nu(x)$

$\frac{\partial}{\partial t}(g_{ij}(t)) = \bar\nabla_{F_t}\langle F_i, F_j\rangle = \langle\bar\nabla_{F_t}F_i, F_j\rangle + \langle F_i, \bar\nabla_{F_t}F_j\rangle = \langle\bar\nabla_{F_i}F_t, F_j\rangle + \langle F_i, \bar\nabla_{F_j}F_t\rangle$

$\Longrightarrow g^{ij}(t)\frac{\partial}{\partial t}(g_{ij}(t)) = 2\mathrm{div}_{M_t}(F_t)$ $\quad$ (by $\bar\nabla_{F_i}F_t = \bar\nabla_{F_t}F_i$ as they're pullback of $\frac{\partial}{\partial x_i}$ & $\frac{\partial}{\partial t}$ communicative) $\quad x_i = \frac{\partial}{\partial x_i}$ the frame in M^n

$\Longrightarrow \frac{\partial}{\partial t}\log(\det g_{ij}(t)) = g^{ij}(t)\frac{\partial}{\partial t}(g_{ij}(t))$ by study of matrices.

Rk. $A(t)$ smooth $\not\Longrightarrow$ eigenvalue smooth it holds when $\lambda_1(t) > \lambda_2(t) > \cdots > \lambda_n(t)$ when $\lambda_i(t) = \lambda_j(t) \Longrightarrow$ not holds. for this we distribution to distinguish them.

$\Longrightarrow \frac{d}{dt}\mathrm{vol}(M_t) = \frac{d}{dt}\int_{M_t}\sqrt{\det(g_{ij}(t))}\,dx$

$\boxed{\text{Ex 5.15 (1)}}$ $\frac{\partial}{\partial t}\log(\det(A(t))) = A^{-1}(t)\frac{\partial}{\partial t}(A(t))$ for $A(t)$ positive defined and smooth

(Hint): For diagonalize first and diagonalise. Keep track the smoothness of $P(t)A(t)$ ...

$= \int_{M_t}\frac{1}{2}g^{ij}\frac{\partial}{\partial t}g_{ij}\sqrt{\det(g_{ij}(t))}\,dx$

$= \int_{M_t}\mathrm{div}_{M_t}(F_t)\sqrt{\det(g_{ij}(t))}\,dx(\mathrm{vol}\,M_t)$ & $\mathrm{div}_{M_t}(F_t)\big|_{t=0} = \mathrm{div}_M(f(t)\nu(t)) = f\,\mathrm{div}_M(\nu(t))$, $H = \langle\bar\nabla_{e_i}e_i, \nu\rangle$ ($H = 0$ by minimal)

the integral place M_t is a priori replaced by U locally we find it's independent of choice of local coordinate $\Longrightarrow$ holds globally in M_t

$\Longrightarrow \boxed{\frac{d}{dt}\Big|_{t=0}\mathrm{vol}(M_t) = -\int_M Hf\,d\mathrm{vol}_M}$

The first variation formula for M minimal $\Longrightarrow$ it's a critical point of functional.

Next we compute the second variation formula

Rk. For $M^n \subset \mathbb{R}^{n+m}$ it's similar but too hard

and the (1) $fh_{ij}g^{ik}g^{jl}\bar\nabla_{F_t}\big|_{t=0}\langle F_i, F_j\rangle =$

$2fh_{ij}g^{ik}g^{jl}\langle\bar\nabla_{F_k}F_t, F_l\rangle\big|_{t=0} = 2fh_{ij}g^{ik}g^{jl}\langle\bar\nabla_{\frac{\partial}{\partial x_k}}(f\nu), \frac{\partial}{\partial x_l}\rangle$

$= -2f^2 h_{ij}g^{ik}h_{kl}g^{jl} = -2f^2(\sum\lambda_j^2) = -2f^2|A|^2$

(2) $g^{ij}\langle\bar\nabla_{F_i}\bar\nabla_{F_t}F_t, F_j\rangle$ (diagonalize) $-\langle R(F_t,F_i)F_t, F_j\rangle + \langle\bar\nabla_{F_i}F_t, \bar\nabla_{F_j}F_t\rangle$

$= g^{ij}(\mathrm{div}_{M_t}(\bar\nabla_{F_t}F_t) - \mathrm{Ric}(F_t,F_t) + g^{ij}\langle\bar\nabla \cdots\rangle$

$\Longrightarrow \frac{\partial}{\partial t}(\mathrm{div}_{M_t}(F_t)) = \frac{\partial}{\partial t}(g^{ij}(t)\langle\bar\nabla_{F_i}F_t, F_j\rangle) = g^{ij}\frac{\partial}{\partial t}\langle\bar\nabla_{F_i}F_t, F_j\rangle$

$+\langle\bar\nabla_{\frac{\partial}{\partial x_i}}(f\nu), \frac{\partial}{\partial x_j}\rangle\frac{\partial}{\partial t}\Big|_{t=0}(g^{ij}(t)) = g^{ij}\bar\nabla_{F_t}\big|_{t=0}\langle\bar\nabla_{F_i}F_t, F_j\rangle - f\langle\bar\nabla_{\frac{\partial}{\partial x_i}}\nu, \frac{\partial}{\partial x_j}\rangle$

$g^{ik}\left(\frac{\partial}{\partial t}\Big|_{t=0}g_{kl}\right)g^{lj} = g^{ij}(\langle\bar\nabla_{F_i}\bar\nabla_{F_t}F_t, F_j\rangle - \langle R(F_t,F_i)F_t, F_j\rangle + \langle\bar\nabla_{F_i}F_t, \cdots$

$\cdots\bar\nabla_{F_j}F_t\rangle)\big|_{t=0} + fh_{ij}g^{ik}g^{jl}(\bar\nabla_{F_t}\big|_{t=0}\langle F_k, F_l\rangle)$, where we set $h_{ij} = -\langle\bar\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}, \nu\rangle$

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$\frac{g^{ik}f^{2}f_{k}}{\cdots}+g^{ij}f^{2}\langle\overline{\nabla}_{e_i}\nu,\overline{\nabla}_{e_j}\nu\rangle=|\nabla f|^{2}+f^{2}|A|^{2}\Rightarrow\frac{\partial}{\partial t}\Big|_{t=0}(\mathrm{div}_{M_t}(F_t))=\mathrm{div}_{M_t}(\overline{\nabla}_{F_t}F_t)-\overline{Ric}(f\nu,f\nu)+|\nabla f|^{2}-|A|^{2}f^{2}$ Page 8

$\Rightarrow\boxed{\left(\frac{d}{dt}\right)^{2}\Big|_{t=0}\mathrm{vol}(M_t)=\int_M|\nabla f|^{2}-(|A|^{2}+\overline{Ric}(\nu,\nu))f^{2}\,d\mathrm{vol}_M}$

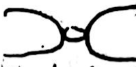
Compare geodesic: section curvature $\Leftrightarrow$ dim 1
Hypersurface: Ricci curvature $\Leftrightarrow$ codim 1

Here hold in $M^n\subset\overline{M}^{n+1}$, when $\mathbb{R}^{n+1}$, $\overline{Ric}\equiv0$. Stability $\Leftrightarrow\left(\frac{d}{dt}\right)^{2}\Big|_{t=0}\mathrm{vol}(M_t)\geq0\Leftrightarrow\int(|A|^{2}+\overline{Ric}(\nu,\nu))f^{2}\leq\int_M|\nabla f|^{2},\forall f\in C_c^\infty(M)$

Define. Jacobi operator $Lf=\Delta f+(|A|^{2}+\overline{Ric}(\nu,\nu))f$, stability $\Leftrightarrow-\int fLf\geq0,\forall f\in C_c^\infty(M)$

It's negative eigenvalue is the Morse index. Stability $\Rightarrow$ Morse index finite.

Ex 5.1582 (Catenoid) $x_1^2+x_2^2=\cosh^2 x_3$. Show $H=0$ minimal.

Hint.  Using $x_3=0$ cut it into two graphs and compute by the div $\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$ or by moving frame directly.

(2) Compute $\int K\,d\mathrm{vol}_M$ the integral of Gauss curvature.

Hint. Using graph or moving frame by $K=\frac{d\omega_{12}}{\omega_1\wedge\omega_2}$.

Rk. It's index finite but not stable.

Cohn-Vossen inequality. Using genus to dominate $\int K$. Here g...

E.g. $\Sigma^n\subset\mathbb{R}^{n+1}$ the graph $\Rightarrow\{E_i\}$ orthogonal basis in $\mathbb{R}^{n+1}$, Recall. $\nu=\frac{-\nabla u+E_{n+1}}{\sqrt{1+|\nabla u|^{2}}}$

Compute $\Delta\langle E_{n+1},\nu\rangle=\sum\nabla_{e_j}\nabla_{e_j}\langle E_{n+1},\nu\rangle=\sum\nabla_{e_j}\langle E_{n+1},\overline{\nabla}_{e_j}\nu\rangle=\nabla_{e_j}\langle E_{n+1},-h_{ij}e_j\rangle$

$=\langle E_{n+1},-h_{ij,i}e_j\rangle-\langle E_{n+1},h_{ij}h_{ij}\nu\rangle=-|A|^{2}\langle E_{n+1},\nu\rangle$

and $\Delta\langle E_{n+1},\nu\rangle=\Delta\frac{1}{\sqrt{1+|\nabla u|^{2}}}=-|A|^{2}\frac{1}{\sqrt{1+|\nabla u|^{2}}}$, i.e. $\boxed{\Delta\frac{1}{V}=-\frac{|A|^{2}}{V}}$ (Bochner's type)

Set Σ minimal. Claim. It's stable, i.e.

$\Rightarrow\int_\Sigma|A|^{2}f^{2}\leq\int_\Sigma|\nabla f|^{2}$: $\int_\Sigma|A|^{2}f^{2}=\int_\Sigma(-Vf^{2})(\Delta\frac{1}{V})=\int_\Sigma\nabla(Vf^{2})\cdot\nabla(\frac{1}{V})=-\int\frac{|\nabla V|^{2}}{V^{2}}f^{2}-2\int\frac{f\nabla f\cdot\nabla V}{V}\overset{\text{Cauchy}}{\leq}\int|\nabla f|^{2}$ ▨

Thus minimal graph is stable.

E.g. (Cone) $M^{n-1}\subset S^n$ closed, $C=\{tx\mid x\in M,t>0\}$ 

Prop. $M\subset S^n$ minimal $\Leftrightarrow CM\subset\mathbb{R}^{n+1}$ minimal

Pf. $\{e_j\}$ on M s.t. $e_n=\nu$ normal vector. $\{\xi_j=\frac{e_j}{r}\}$ is corresponding orthogonal basis in C (by the cone metric is dr^2+r^2g)

$\nabla_{\xi_i}\xi_j=-\frac{1}{r}\delta_{ij}\frac{\partial}{\partial r}+\frac{1}{r}h_{ij}\xi_n\Rightarrow H_C=\langle\nabla_{\xi_i}\xi_j,\xi_n\rangle_{dr^2+r^2g}$

$=\frac{1}{r}\langle h_{ii}\xi_n,\xi_n\rangle_{dr^2+r^2g}=\frac{1}{r}H\Rightarrow H=0\Leftrightarrow H_C=0$

It suffices to check the "~~" formula. $\langle\nabla_{\xi_i}\xi_j,\frac{\partial}{\partial r}\rangle=-\frac{1}{r}\langle\nabla_{\xi_i}\xi_j,x\rangle$

$=\frac{1}{r}\langle\xi_j,\overline{\nabla}_{\xi_i}x\rangle=-\frac{1}{r}\langle\xi_j,\xi_i\rangle=-\frac{\delta_{ij}}{r}$ & $\langle\nabla_{\xi_i}\xi_j,\xi_n\rangle_{dr^2+r^2g}=\langle\nabla_{\frac{e_i}{r}}\frac{e_j}{r},\frac{e_n}{r}\rangle_{dr^2+r^2g}$

$=\langle\nabla_{e_i}\frac{e_j}{r},e_n\rangle_g=\frac{1}{r}\langle\nabla_{e_i}e_j,e_n\rangle=\frac{1}{r}h_{ij}$ ▨

Other components all 0 by $\nabla_{\frac{\partial}{\partial r}}e_j=0$)

Additionally,

Ex. $M^n\subset\mathbb{R}^{n+1}$ closed $\Rightarrow\boxed{n\mathrm{vol}(M)\int_M H_r\geq\lambda_1\int_M H_{r-1}}$, and equal $\Leftrightarrow M^n=S^n$

(Recall ① $\int H_{r-1}+\langle x,\nu\rangle H_r=0$ the Minkowski formula)

② $\lambda_1=\inf\frac{\int|\nabla f|^{2}}{\int f^{2}}$ the first eigenvalue.

Pf. Assume $\int x=0$, $\int_M f=0$, $\forall a\in S^n$, $\int_M|\nabla\langle x,a\rangle|^{2}\geq\lambda_1\int\langle x,a\rangle^{2}$

$\Rightarrow\int_{S^n}\int_M|\nabla\langle x,a\rangle|^{2}\geq\lambda_1\int_{S^n}\int_M\langle x,a\rangle^{2}$

$\int_{S^n}\int_M\sum\langle e_i,a\rangle^{2}$ $=\lambda_1\int_{S^n}\int_M(\sum x_ia_i)^{2}=\lambda_1\int_{S^n}\int_M\sum x_ix_ja_ia_j$

$=\lambda_1\int\int x_ix_ja_ia_j$

By this we can write $\sum\int_M\frac{n}{n+1}\mathrm{vol}(S^n)$ (by $\int_{S^n}x_i^2=\frac{1}{n+1}\int(x_1^2+\cdots+x_{n+1}^2)=\frac{\mathrm{vol}(S^n)}{n+1}$)

$=\lambda_1\sum\int\int x_i^2a_i^2=\lambda_1\sum\int_M\int\ldots$

$\frac{n}{n+1}\mathrm{vol}(S^n)\mathrm{vol}(M)=\lambda_1\frac{|\ldots|}{n+2}\mathrm{vol}(S^n)\int|x|^{2}$

$\frac{n}{n+1}\mathrm{vol}(S^n)\mathrm{vol}(M)\geq\lambda_1\frac{1}{n+2}\mathrm{vol}(S^n)(\int_M|x|^{2})$

$\Leftarrow n\mathrm{vol}(M)\geq\lambda_1\int_M|x|^{2}$ and equality holds iff $\Delta\langle x,a\rangle+\lambda\langle x,a\rangle=0\Leftrightarrow\Delta x+\lambda x=\vec{H}+\lambda x=0\Leftrightarrow\vec{H}$ is normal vector $\Leftrightarrow\nabla_{e_i}\langle x,x\rangle=2\langle e_i,x\rangle=0\Leftrightarrow|x|=$ constant

At last to prove $n\mathrm{vol}(M)\geq\lambda_1\int|x|^{2}\Leftrightarrow S^n$ ▨

using $\int H_{r-1}+\langle x,\nu\rangle H_r=0\Rightarrow n\mathrm{vol}(M)\int H_r^2\geq\lambda_1\int|x|^{2}\int\langle x,\nu\rangle^{2}\int_M H_r^2\geq\lambda_1(\int_M\langle x,\nu\rangle H_r)^{2}=\lambda_1(\int_M H_{r-1})^{2}$ ▨ (also equal iff S^n)

② $0=X\langle e_A,e_B\rangle=\langle\nabla_X e_A,e_B\rangle+\langle e_A,\nabla_X e_B\rangle=\omega_{BA}(X)+\omega_{AB}(X)$ ▨

③ $R(X,Y)e_A=-\nabla_X\nabla_Y e_A+\nabla_Y\nabla_X e_A+\nabla_{[X,Y]}e_A$

$=\nabla_X(\omega_{BA}(Y)e_B)-\nabla_Y(\omega_{BA}(X)e_B)+\omega_{BA}([X,Y])e_B$ $\omega_{BA}(e_C)=\Gamma_{CA}^B$

$=X(\omega_{BA}(Y))-Y(\omega_{BA}(X))+\omega_{BA}([X,Y])]e_B$ $\Rightarrow\omega_{BA}=\Gamma_{CA}^B\omega_C$

$+[\omega_{BA}(Y)\omega_{CB}(X)-\omega_{BA}(X)\omega_{CB}(Y)]e_C=[-d\omega_{AB}(X,Y)+\omega_{BC}\wedge$

$\omega_{CA}(X,Y)]e_B$ ⊗ $\langle R(X,Y)e_A,e_B\rangle=(d\omega_{AB}+\omega_{AC}$

$\wedge\omega_{CB})(X,Y)\Rightarrow(d\omega_{AB}+\omega_{AC}\wedge\omega_{CB})(X,Y)=\frac{1}{2}K_{ABCD}\omega_C\wedge\omega_D(X,Y)$

$=\frac{1}{2}K_{ABCD}(\omega_C(X)\omega_D(Y)-\omega_C(Y)\omega_D(X))=\frac{1}{2}(K_{ABXY}-K_{ABYX})$

$=K_{ABXY}$ ▨

Estimate of curvature $\Sigma^2\subset M^n$ minimal surface [Topic 3 Geometric analysis]

Thm (Choi-Schoen) $\exists\varepsilon,R$ depending on M, s.t. $\forall r<R$, $\partial\Sigma\subset\partial B_r(x)$

(B_r the geodesic ball in M), $\forall\delta\in(0,1]$, $\boxed{\int_{B_r(x)\cap\Sigma}|A|^{2}<\delta\varepsilon}$ It's a natural condition due to it dominate $|A|$ (nonlinear) to locally linear $\sim(*)$

$\Rightarrow\forall\sigma\in(0,r)$, $\sigma^2\sup_{B_{r-\sigma}(x)\cap\Sigma}|A|^{2}<\delta$

Ex2 $M^n\subset\overline{M}^{n+1}$, $e_1\ldots e_{n+1}$ restrict to $e_1\ldots e_n$.

Using moving frame $\omega_1\ldots\omega_{n+1}$ also to $\omega_1\ldots\omega_n$

$\Rightarrow$ the structure equations ① $d\omega_A=-\omega_{AB}\wedge\omega_B$, ② $\omega_{AB}+\omega_{BA}=0$

③ $d\omega_{AB}=-\omega_{AC}\wedge\omega_{CB}+\frac{1}{2}K_{ABCD}\omega_C\wedge\omega_D$

Using correspondence $X=\sum\omega_A(X)e_A\Rightarrow\nabla e_A=\omega_{BA}e_B$

thus ① is the no torsion (Levi-Civita), ② is connection compatible with metric ③ is def of curvature, precisely

Pf ① $\nabla_XY-\nabla_YX=[X,Y]$ & $\nabla_XY=\nabla_X(\omega_A(Y)e_A)=X(\omega_A(Y))e_A+\omega_A(Y)\omega_{BA}(X)e_B\Rightarrow\nabla_XY-\nabla_YX=(X(\omega_A(Y))-Y(\omega_A(X))-\omega_A([X,Y]))e_A+(\omega_{BA}(X)\omega_A(Y)-\omega_{BA}(Y)\omega_A(X))e_B$

$=(d\omega_A(X,Y)+\omega_{AB}\wedge\omega_B(X,Y))e_A=0$ ▨

Pf. By Gaussian formula, $R_{ijkl}-\overline{R}_{ijkl}=\langle A_{ik},A_{jl}\rangle-\langle A_{il},A_{jk}\rangle$

the curvature increasing as $|A|^3$

$\frac{1}{2}\Delta|A|^{2}=\Delta(\sum(h_{ij})^2)=h_{ijk}h_{ijk}+h_{ijkk}h_{ij}=|\nabla A|^{2}+(h_{kijk}+\text{codazzi})h_{ij}$

at least we can have the Codazzi part $\geq|\nabla A|^{2}+h_{kikj}h_{ij}-C_1(|A|^N+|A|^2)$

$\geq h_{kikj}h_{ij}-C_2(|A|^N+|A|^2)\geq-C_3(|A|^N+|A|^2)$, taking $N=4$ is possible

$\Rightarrow\Delta|A|^{2}\geq-C_4(|A|^4+1)$, however, such order of estimate 4 is too high

that's why we need $(*)$ to dominate it to linear;

Otherwise $\exists\sigma\in(0,r)$, $\sigma^2\sup|A|^{2}\geq\delta$, $\exists z\in B_r(x)$, s.t. $(r-d(z,x))^2|A(z)|^2$

$=\sup(r-d(y,x))^2|A(y)|^2\geq\delta$ (by $\partial B_r(x)$ takes 0 $\Rightarrow$ sup achieve in interior)

By expand the ball $B_{r-\sigma}(x)$ to $B_r(x)$. Set $\sigma>0$, s.t. $4\sigma^2|A(z)|^2=\delta$

$\Rightarrow 2\sigma\leq r-d(z,x)\Rightarrow\sigma^2\sup|A|^2\leq\sigma^2(r-\sigma-d(z,x))^{-2}\ldots$

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By triangle inequality $\Rightarrow \sigma^2 \sup_{B_\sigma(\xi)} |A|^2 \le \frac{\sigma^2(r-d(\xi,x_0))^2}{(r-\sigma-d(\xi,x_0))^2}|A(\xi)|^2 = \frac{(r-d(\xi,x_0))^2}{4(r-\sigma-d(\xi,x_0))^2}\delta \le \frac{(2\sigma)^2}{4\sigma^2}\delta = \delta$ (by $2\sigma \le r-\rho(\xi)$)

$\boxed{\text{Scaling}}$ $B_\sigma(\xi)$ to $B_1(\xi)$ with new metric $\tilde g = \frac{g}{\sigma^2}$, it also gives scaling of Σ to $\tilde\Sigma \Rightarrow |\tilde A(\xi)|^2 = \frac{\delta}{4} \le \frac{1}{4}$

$\Rightarrow \sup_{B_1(\xi)\cap\tilde\Sigma} |\tilde A|^2 \le \delta \le 1 \Rightarrow |\tilde A|^2 \le \delta|\tilde A|$, set $f=|\tilde A|^2 \Rightarrow \Delta f \ge -C(|\tilde A|^4+|\tilde A|^2) \ge -Cf \Rightarrow \Delta f \ge -Cf \Rightarrow f(\xi) \le C\int_{B_1(\xi)\cap\tilde\Sigma} f$

$\le C\int_{B_\sigma(\xi)\cap\Sigma}|A|^2$ (By $\sigma \tilde A = |A|$ & $\tilde\nabla = \frac{1}{\sigma}\nabla$ we use $\dim\Sigma = 2$) $< C\delta\varepsilon \xRightarrow{\text{Smooth}} \frac{\delta}{4} < C\delta\varepsilon \Rightarrow \varepsilon > \frac{1}{4C}$, contradiction ▨

(conformal invariant)

Thm. (E. Heinz) Σ = graph of $u \subset \mathbb{R}^3$ and $\Sigma \subset B_r$ the domain $\Rightarrow \sigma^2 \sup_{B_{r\sigma}} |A|^2 \le C,\ \forall \sigma\in(0,r)$. Pf. By stability $\Rightarrow \int_\Sigma |A|^2\eta^2 \le \int_\Sigma |\nabla\eta|^2, \forall \eta\in C_c^\infty(\Sigma)$

$\Sigma \cap B_r$, $0 = \int_{\Sigma^+} \operatorname{div}\frac{-\nabla u + E_3}{\sqrt{1+|\nabla u|^2}} = \int_{\Sigma\cap B_r}\langle\frac{-\nabla u+E_3}{\sqrt{1+|\nabla u|^2}}, -\nu\rangle$

$+\int_{S^+}\langle\frac{-\nabla u+E_3}{\sqrt{1+|\nabla u|^2}}, \frac{x}{|x|}\rangle = -\operatorname{vol}(\Sigma\cap B_r) + \int_{S^+}\langle\frac{-\nabla u+E_3}{\sqrt{1+|\nabla u|^2}}, \frac{x}{|x|}\rangle$

$\Rightarrow \operatorname{vol}(\Sigma\cap B_r) \le \int_{S^+} 1 = \operatorname{vol}(S^+)$

$\Rightarrow \operatorname{vol}(\Sigma\cap B_r) \le \min\{\operatorname{vol}(S^+), \operatorname{vol}(S^-)\} \le 2\pi r^2$

$\boxed{\text{Lec 20}}$ The positive harmonic function over catenoid $\subset \mathbb{R}^3$ = constant

(One can generalise it the upper & below harmonic)

(Hint. Using conformal coordinate $\Delta f = \frac{1}{\cosh^2 u}\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right)$)

Positive harmonic function on $\mathbb{R}^n$ = constant by the Harnack inequality. However in general it's open as Harnack inequality not exists.

Thm. If $\lim_{r\to\infty}\frac{\operatorname{Area}(\Sigma\cap B_r)}{r^2} < \infty$, then all positive subharmonic function = constant (In general, open) (Or $\lim \frac{\operatorname{Vol}(\Sigma\cap B_r)}{r^n} < \infty$ in higher dim) False, consider the higher catenoid

Pf. $\Delta u \le 0$, set $f = \log u \Rightarrow \Delta f = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} \le -|\nabla f|^2$

$\Rightarrow \forall \eta \in C_c^\infty(\Sigma), -\int_\Sigma \eta^2\Delta f \ge \int_\Sigma \eta^2|\nabla f|^2$

$\int_\Sigma 2\eta\nabla\eta\nabla f \le \frac{1}{2}\int_\Sigma \eta^2|\nabla f|^2 + 2\int_\Sigma|\nabla\eta|^2 \le \frac{1}{2}\left(4\int_\Sigma|\nabla\eta|^2\right)$ (Cauchy)

$\Rightarrow \int_\Sigma \eta^2|\nabla f|^2 \le 4\int_\Sigma|\nabla\eta|^2$, set η the sectional function with logrithm in Pf of Heinz thm (Precisely later) $+2\int_\Sigma|\nabla\eta|^2$

$\int|\nabla\eta|^2 \to 0 \Rightarrow f \equiv C \Rightarrow u \equiv C$ ▨

Taking $\eta(x) = \begin{cases} 1; & x\in B_{\sqrt R} \\ 2\left(1-\frac{\log|x|}{\log R}\right); & x\in B_R - B_{\sqrt R} \\ 0; & x \in B_R \end{cases}$

$\Rightarrow \int_\Sigma|\nabla\eta|^2 = \int_{\Sigma\cap(B_R - B_{\sqrt R})}\frac{4}{(\log R)^2}\frac{1}{|x|^2}$

$\le \frac{4}{(\log R)^2}\sum_{\frac{1}{2}\log R\le j\le\log R}\int_{\Sigma\cap(B_{e^j}-B_{e^{j-1}})}\frac{1}{|x|^2} \le \frac{4}{(\log R)^2}\sum e^{2-2j}\operatorname{Vol}(\Sigma\cap(B_{e^j}-B_{e^{j-1}}))$

$\le \frac{4}{(\log R)^2}\sum e^{2-2j}2\pi e^{2j} = \frac{4}{(\log R)^2}\left(\frac{\log R}{2}\right)(2\pi e^2)$

$\le \frac{C}{\log R}\xrightarrow{R\to\infty}0 \Rightarrow \int_{B_r\cap\Sigma}|A|^2 \to 0$, then use the thm of Choi-Schoen ▨

Set $\Sigma \subset M$, the intristinic distance and ball $B_r^\Sigma(x) = \{y\in\Sigma \mid \operatorname{dist}_\Sigma(x,y) < r\}$, $B_r^\Sigma(x)\cap\partial\Sigma = \emptyset$, the cut locus $\mathcal{C}(\Sigma)\cap B_r^\Sigma(x) = \emptyset$

$\rho_\Sigma : y \mapsto \operatorname{dist}_\Sigma(x,y)$ the distance function, $\rho_\Sigma \in C_c^\infty(B_r^\Sigma(x)\setminus\{x\})$ (Geodesic balls)

$\frac{d}{dt}\ell(\partial B_t^\Sigma) = \frac{d}{dt}\int_{\partial B_t^\Sigma} 1 = \int_{\partial B_t^\Sigma}\nabla\rho_\Sigma$ (unit out normal vector) $= \int_{\partial B_t^\Sigma}k_g$ (Geodesic curvature) $\overset{\text{Gauss Bonnet}}{=} 2\pi - \int_{B_t^\Sigma}K_\Sigma$ $\xRightarrow{\text{integral}} \int \ell(\partial B_r^\Sigma) - 2\pi r = -\int_0^r\int_{B_t^\Sigma}K_\Sigma \Rightarrow \operatorname{Area}(B_R^\Sigma) - \pi R^2$

(By coarea formula, $|\nabla\rho_\Sigma| = 1$) $= -\int_0^R\int_0^r\int_{B_t^\Sigma}K_\Sigma\,dt\,dr$

Dealing the integrand $-\int_0^R\int_0^r\int_{B_t^\Sigma}K_\Sigma\,dt\,dr$, set $M=\mathbb{R}^n$, $K_\Sigma = \lambda_1\lambda_2$,

$|A|^2 = \lambda_1^2+\lambda_2^2 = -2\lambda_1\lambda_2 = -2K_\Sigma$ (by minimal $\Leftrightarrow \lambda_1+\lambda_2=0$)

$\Rightarrow \frac{1}{2}\int_0^R\int_0^r\int_{B_t^\Sigma}|A|^2\,dt\,dr \underset{\text{Fubini}}{=} \frac{1}{4}\int_{B_R^\Sigma}|A|^2(R-\rho_\Sigma)^2$

$\ge t^2\int_{B_{R-t}^\Sigma}|A|^2$; $\Rightarrow t^2\int_{B_{R-t}^\Sigma}|A|^2 \le \operatorname{Area}(B_R^\Sigma) - \pi R^2 \le R\,\ell(\partial B_R^\Sigma) - \pi R^2$

Thm. (Colding-Minicozzi) $\Sigma^2 \subset \mathbb{R}^3$ oriented stable $B_r^\Sigma(x)$ simply connected $\Rightarrow \operatorname{Area}(B_r^\Sigma(x)) \le \frac{4\pi}{3}r^2$

If Σ^2 bounded, i.e. $\Sigma^2 \subset \Omega \subset \mathbb{R}^3$, we need adding $B_r^\Sigma \cap\partial\Sigma = \emptyset$ to hold the inequality.

If Σ^2 unbounded $\Rightarrow \Sigma^2 = \mathbb{R}^2$ by Bernstein

Coro. (Bernstein thm). $\Sigma \subset \mathbb{R}^3$ complete oriented stable $\Rightarrow \Sigma$ flat. Pf. By covering ▨

Pf. $4(\operatorname{Area}(B_R^\Sigma) - \pi R^2) = \int_{B_R^\Sigma}|A|^2(R-\rho_\Sigma)^2 \underset{\text{stable}}{\le} \int_{B_R^\Sigma}|\nabla(R-\rho_\Sigma)|^2 = \operatorname{Area}(B_R^\Sigma)$ ▨ (Here stable minimal $\Rightarrow K\le 0 \Rightarrow$ not cut locus)

Define. Excess = density -1

$= \Theta_x(s) - 1 = \frac{1}{\pi s^2}\operatorname{Area}(B_s(x)\cap\Sigma) - 1 \xrightarrow{s\to 0} 0$

$\Rightarrow \left(\int_{B_{\frac{7}{8}}^\Sigma}|A|^2\right)\frac{2}{16} \ge \frac{2}{16}\int_{B_1^\Sigma}|A|^2 \ge C|A|^2(0) = \frac{C}{4}$

$\Rightarrow \varepsilon \ge \frac{2}{\pi}C > 0$, contradiction ▨ ~~The mean value inequality~~

Idea. Using exterior metric dominate intristinic metric and write as the domination of balls in integrand.

Only Colding-Minicozzi use $\dim = 2$, using $\operatorname{Area}(\text{Ball}) \ge \int|A|^2$ isn't trivial in higher dimension

We complete our study of minimal surfaces.

$\to$ one can use elliptic estimate to give $\int|\nabla u|^2 \le C\sup|u|^2$ excess $\to 0$ then can dominate ▨

$\boxed{\text{Metric Geometry}}$ (Z, d) metric space [Topic 4]

$2_c^Z = \{$compact subset$\}$, define Hausdorff metric $2_c^Z \times 2_c^Z \xrightarrow{d_H} \mathbb{R}$

$(A_1, A_2) \longmapsto d_H(A_1, A_2)$

(i.e. $A_1 \subset B_\varepsilon(A_2)$ & $A_2 \subset B_\varepsilon(A_1)$) $\inf\{\varepsilon > 0 \mid A_1 \subset B_\varepsilon(A_2)\ \&\ A_2\subset B_\varepsilon(A_1)\}$

Lemma. $(2_c^Z, d_H)$ also metric space

Pf. Only check triangle inequality:

$\forall x\in B_{1-s}\cap\Sigma$, $s\in(0,1)$ and $\langle\nu(x),\nu(0)\rangle^2 = \cos^2(dx)$

Here $dx = \operatorname{dist}(\nu(x),\nu(0))$

$\Rightarrow 1+|\nabla u|^2 \le 2 \Rightarrow |\nabla u| \le 1$

$|A|^2 = g^{ij}h_{jk}g^{kl}h_{li}$ ~①

$g^{ij} = \delta_{ij} - \frac{u_iu_j}{1+|\nabla u|^2}$ ~②

$h_{ij} = \frac{u_{ij}}{\sqrt{1+|\nabla u|^2}}$ ~③ $\Rightarrow |A|^2 = \operatorname{trace}(g^{-1}\mathrm{II}\,g^{-1}\mathrm{II})$

$\Rightarrow |A|^2 \ge (1+|\nabla u|^2)^{-3}|\nabla^2 u|^2$

$\Rightarrow |\nabla^2 u|^2 \le 8|A|^2 \le 8$

$\Rightarrow B_{\frac{7}{8}}^\Sigma \supset B_{\frac{7}{8}}\cap\Sigma$

i.e. $\boxed{|x-y| \le \operatorname{dist}_\Sigma(x,y) \le \sqrt{2}|x-y|}$

$\left(\int_{B_{\frac{7}{8}}^\Sigma}|A|^2\right)\frac{2}{16} \le 2(\operatorname{Area}(B_{\frac{7}{8}}^\Sigma) - \frac{2}{16}\pi) \le \frac{2\pi}{8}$

$\left(\frac{7}{4} - 2\cdot\frac{1}{16} = \frac{1}{8}\right)$ and by Choi-Schoen, using extra

Thm. $\exists\,\varepsilon>0$ s.t. $0\in\Sigma^2\subset B_{2r}(0)$ smooth compact minimal surface, $\partial\Sigma\subset\partial B_{2r}(0)$

$\Theta_y(r) - 1 < \varepsilon$, $\forall y\in B_r\cap\Sigma$

$\Rightarrow \sigma^2\sup_{B_{r-\sigma}}|A|^2 \le 1$

It holds for Σ^n still, called Allard regularity thm in GMT

Pf. Otherwise, assume $|A|^2(0) = \frac{1}{4}$ & $\sup_{B_1\cap\Sigma}|A|^2 \le 1$ by scalaring Σ.

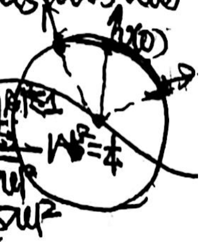
$\Rightarrow \operatorname{Area}(B_s\cap\Sigma) < \pi(1+\varepsilon)s^2$, $\forall s\in(0,1)$

$|\nabla\nu| \le |A| \le 1$, ν the unit outside normal vector $\Rightarrow \sup_{B_{\frac{1}{2}}^\Sigma}\operatorname{dist}(\nu(x),\nu(0))$

$\le \frac{1}{2} < \frac{\pi}{4}$ $\Rightarrow$ choose u function on $T_0\Sigma$

$\Rightarrow B_{\frac{1}{2}}^\Sigma \subset \Gamma_u \Rightarrow \nu = \frac{-\nabla u + E_3}{\sqrt{1+|\nabla u|^2}}$

$\nu(0) = E_3$ $\Rightarrow \langle\nu(x),\nu(0)\rangle^{-2} = 1+|\nabla u|^2$



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$d_H(A,B)\le d_H(B,C)+d_H(A,C)$ by $\forall\varepsilon>0$ $A\subset B_{r_1+\varepsilon}(C)\subset B_{r_1+\varepsilon}(B_{r_2+\varepsilon}(B))=B_{r_1+r_2+2\varepsilon}(B)$ $=: r_2+r_1$

converse also $B\subseteq B_{r_1+r_2+2\varepsilon}(A)\ \underset{\varepsilon\to0}{\Longrightarrow}\ d_H(A,B)\le r_1+r_2$ ▨

Lemma. (Blaschke) (Z,d) complete (or compact) $\Longleftrightarrow$ $(\mathcal{C}_c^Z, d_H)$ complete (or compact)

$\Leftarrow$) Trivial $\Rightarrow$) $\{A_j\}$ Cauchy $\subset 2_c^Z$, $d_H(A_j,A_{j+k})<2^{-j}$, $\forall k>0$; A_j compact $\Rightarrow$ it has finite ε-net $A_j(\varepsilon)\subset A_j$

$A_j(\varepsilon_i)=\{a_j^1\cdots a_j^m\}$ by taking m large (it's possible by $\sum 2^{-j}<\infty$) $\Rightarrow$ $a_j^{ik}\xrightarrow[\text{By Cauchy}]{} a^{ik}$, set $A(\varepsilon_i)=\{a^{ik}\}_{k\le m}$

$\Rightarrow A(\varepsilon_{i+1})\supset A(\varepsilon_i)\supset\cdots$, set $A'=\bigcup_{i\ge1}A(\varepsilon_i)$ is desired limit ▨ (Using ε-net replace compact set)

(By $d_H(A_j,A)\le d_H(A_j,A_j(\varepsilon_i))+d_H(A_j(\varepsilon_i),A(\varepsilon_i))+d_H(A(\varepsilon_i),A)$)

For compactness $\{A_j\}\subset 2_c^Z$

$\exists Z(i,\varepsilon)\supset A_i(\varepsilon)$ ε-net of Z, also finite $\Rightarrow \#A_i(\varepsilon)\le\#Z(i,\varepsilon)\le\#Z(\frac{\varepsilon}{2})$; repeat the argument above by taking

$\{j_{k}\}\subset\{j_{k-1}\}\subset\cdots\subset\{j\}\subset\{j\}\Rightarrow A_{j_k}(\varepsilon_{j_k})$ a subsequence, s.t. $A_{j_k}(\varepsilon_{j_k})\to A(\varepsilon_k)$, $A(\varepsilon_k)\supset A(\varepsilon_{k-1})\supset\cdots$

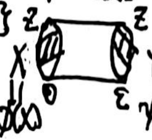
$\Rightarrow (A_{j_k})_k$ is Cauchy subsequence, by completeness converge $\Rightarrow$ sequentially compact $\underset{\text{Metric}}{\Longrightarrow}$ compact ▨

Gromov-Hausdorff metric $d_{GH}(X,Y)=\inf\{d_H(\varphi(X),\psi(Y))\mid$ All Z and all isometry embedding $\varphi: X\to Z$, $\psi: Y\to Z\}$ for $\mathrm{diam}(X)<\infty$, $\mathrm{diam}(Y)<\infty$

For the second equality: $=\inf\{d_H(X,Y)\mid$ All d the admissible metric on $X\sqcup Y\}$

$\forall\varepsilon>0$, $\exists\varphi: X\to Z$, $\psi: Y\to Z$, $d_H(\varphi(X),\psi(X))<d_{GH}(X,Y)+\varepsilon$ ($d|_X=d_X$, $d|_Y=d_Y$)

Consider $X\sqcup Y\subset Z\times[0,\varepsilon]$ by $X\subset Z\times\{0\}$ & $Y\subset Z\times\{\varepsilon\}$ (equipped product (flat) metric)



$\Rightarrow$ RHS $\le d_H^{X\sqcup Y}(\varphi(X)\times\{0\},\psi(Y)\times\{\varepsilon\})$

$=d_H^{Z\times[0,\varepsilon]}(\varphi(X)\times\{0\},\psi(Y)\times\{\varepsilon\})$

$\le d_H^{Z\times[0,\varepsilon]}(\varphi(X)\times\{0\},\psi(Y)\times\{0\})+d_H^{Z\times[0,\varepsilon]}(\psi(Y)\times\{0\},\psi(Y)\times\{\varepsilon\})$ ▨

$\le d_H(\varphi(X),\psi(Y))+\varepsilon<d_{GH}(X,Y)+2\varepsilon$

Lemma

d_{GH}: Comp $\times$ Comp $\to\mathbb{R}$ defines a ~~pseudo-metric~~ in the set of all compact metric space.

$d_{GH}(X,Y)=0\Leftrightarrow X$ isometry Y (i.e. metric on Comp/$\sim$isometry)

Pf. Suffices check $d_{GH}(X,Z)\le d_{GH}(X,Y)+d_{GH}(Y,Z)$

Given admissible metric d_{XY} & d_{YZ}, s.t. $\begin{cases} d_{XY}(X,Y)<d_{GH}(X,Y)+\frac{\varepsilon}{2}\\ d_{YZ}(Y,Z)<d_{GH}(Y,Z)+\frac{\varepsilon}{2}\end{cases}$ $\Rightarrow$ Choosing compatible metric in $X\sqcup Y\sqcup Z$ s.t. $d_H(X,Y)=d_{XY}(X,Y)$, $d_H(Y,Z)=d_{YZ}(Y,Z)$

$\Rightarrow d_{GH}(X,Z)\le d_H(X,Z)\le d_H(X,Y)+d_H(Y,Z)$

$=d_{XY}(X,Y)+d_{YZ}(Y,Z)<d_{GH}(X,Y)+d_{GH}(Y,Z)+\varepsilon$

And $\Leftarrow$) Trivial

$\Rightarrow$) Taking $\{d_i\}$ admissible metric on $X\sqcup Y$

s.t. $d_i(X,Y)<\frac{1}{i}\Rightarrow$ countable dense subset $\{x_k\}\subset X$, $\exists y_{1,i}\in Y$

s.t. $d_i(x_1,y_{1,i})<\frac{1}{i}$, Y compact $\Rightarrow y_{1,i}\to y_1\in Y$ repeat for $x_2\cdots x_k\cdots\Rightarrow$ get $y_2\in Y\sim y_k\in Y\cdots$

s.t. $d_{i_k}(x_k,y_k)\le d_{i_k}(x_k,y_{k,i_k})+d_{i_k}(y_{k,i_k},y_k)\to0\Rightarrow$ taking subsequence $\{d_{i_k}\}\subset\{d_i\}$, define $f:\{x_k\}\to Y$ extending to $f: X\to Y$

$x_k\mapsto y_k$

then f is desired isometry ▨

(By $d_Y(f(x_j),f(x_i))=d_{i_k}(f(x_j),f(x_i))=\lim_{k\to\infty}d_{i_k}(y_j,y_i)\le\lim_{k\to\infty}(d_{i_k}(y_j,x_j)+d_{i_k}(x_j,x_i)+d_{i_k}(x_i,y_i))=d(x_j,x_i)$ ▨

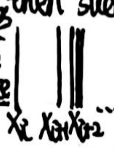
$M^{all}=\text{Comp}/\simeq$ isometry, then M^{all} is a complete metric space

Pf. (X_i) Cauchy, take $Y=\sqcup X_i$ with admissible metric ① and $Z=\hat{Y}$ completion, then taking ε-net and diagonal sequence converge ② to desired X ▨ (This is a standard argument; ① & ② are only subtle)

① First in $X_i\sqcup X_{i+1}$, $d_{i,i+1}(X_i,X_{i+1})\le d_{GH}(X_i,X_{i+1})+2^{-i}<2^{1-i}$ and $d_{GH}(X_i,X_{i+1})<2^{-i}$,

② Standard argument, omitted ▨

then such $d_{i,i+1}$ is admissible in $X_i\sqcup X_{i+1}$



seperated

then taking $d_Y(x_i,x_{j+1})=\inf\{\sum d_{l,l+1}(x_l,x_{l+1})\mid x_{l+1}\in X_{l+1}\}$, $l\le j+1$

For compactness, or at least paracompactness of this Cauchy metric space

is desired construction: d_Y is admissible in Y

Check. ① Triangle inequality; ② Admissible (easy)

③ (X_i) still Cauchy under d_Y

Lemma. (Gromov) $M\subset M^{all}$ is paracompact $\Leftrightarrow$ (i) $\sup_{X\in M}\mathrm{diam}(X)<\infty$

(ii) $\sup_{X\in M}|X(\varepsilon)|<l(\varepsilon)$ (equi-continuous)

here $|X(\varepsilon)|$ (A function of ε dominate $|X(\varepsilon)|$)

$=\mathrm{Cap}_X(\varepsilon)=\{$the maximal number of ε-balls intersect empty $\subset X\}$

$\Rightarrow$) Otherwise, either $\mathrm{diam}(X_j)\to\infty$ or $|X_j(\varepsilon)|\to\infty$. Taking Cauchy subsequence $\{X_{i_k}\}$ Cauchy $\Rightarrow\mathrm{diam}(X_{i_k})<\mathrm{diam}(X_{i_N})$

for $\forall k\ge N$, and $|X_{i_k}(\varepsilon)|\le|X_{i_N}(\frac{\varepsilon}{2})|$, contradiction ▨

$\Leftarrow$). $d(X_{j_k},X_{j_l})\le d(X_{j_k},X_{j_k}(\varepsilon))+d(X_{j_k}(\varepsilon),X_{j_l}(\varepsilon))+d(X_{j_l}(\varepsilon),X_{j_l})<2\varepsilon+d(X_{j_k}(\varepsilon),X_{j_l}(\varepsilon))<2\varepsilon+\underset{0}{\underline{\lim\varepsilon_i}}\ \underset{\text{by (i) bounded}}{\underline{\mathrm{diam}(X_i)}}$

$<3\varepsilon\to0\Rightarrow$ paracompact ▨

taking ε_i and diagonal subsequence

$\hookrightarrow$ it converges by (ii) $|X_j(\varepsilon_i)|<l(\varepsilon_i)$ bounded.

Thm. (Gromov's paracompact thm) $M_{n,H,D}\subset M^{all}$, $Ric\ge(n-1)H$, $\dim X=n$, $\mathrm{diam}(X)\le D$'s all Riemannian manifold ($/\simeq$ isometry) $\Rightarrow M_{n,H,D}$ is paracompact.

Pf. Suffices give $\sup_{X\in M_{n,H,D}}|X(\varepsilon)|$ estimate. $\mathrm{Vol}(M)\ge\sum\mathrm{vol}(B_\varepsilon(x_i))$ and assume $B_\varepsilon(x_i)$ increasing when i increasing

$\Rightarrow\mathrm{vol}(B_D(x_1))=\mathrm{Vol}(M)\ge k\,\mathrm{vol}(B_\varepsilon(x_1))$ $x_i\in M(\varepsilon)$, $1\le i\le k$

$\Rightarrow k\le\frac{\mathrm{vol}(B_D(x_1))}{\mathrm{vol}(B_\varepsilon(x_1))}\underset{\text{Bishop-Gromov}}{\le}C(\varepsilon,n,H,D)$ ▨

Def. (ε-Gromov-Hausdorff approximation) ε-GHA $f: X\to Y$, X,Y compact metric space s.t. (1) ε-surjective) $B_\varepsilon(f(X))\supseteq Y$ (Even not continuous)

(2) $|d(x_1,x_2)-d(f(x_1),f(x_2))|<\varepsilon$ (ε-isometry)

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Set $d_{GH}(X,Y) = \inf\{\varepsilon \mid \exists\ \varepsilon\text{-GHAs}, f: X\to Y\ \&\ g: Y\to X\}$. Lemma. $\frac{2}{3} d_{GH} \le d_{GH}' \le 2 d_{GH}$ thus we considering Gromov-Hausdorff convergence, these two are same. Pf. Omitted ▨

For noncompact case, marked point (pointwise Gromov-Hausdorff convergence). It has good properties: makes sense in noncompact case.

Def. ε-pointed GHA is $f: (X,p) \to (Y,q)$ if (1) $B_{\frac{1}{\varepsilon}}(q) \subset B_{\varepsilon}(f(B_{\frac{1}{\varepsilon}}(p)))$ (ε-surjective); (2) $|d(x_1,x_2) - d(f(x_1), f(x_2))| < \varepsilon$ (ε-isometry)

Def. $d_{GH}((X,p),(Y,q)) = \inf\{\varepsilon \mid \exists\ \varepsilon\text{-PGHAs}: f:(X,p)\to(Y,q),\ g:(Y,q)\to(X,p)\}$

Then we proceed as before to construct the moduli space $\mathcal{M}_1^{\text{all}}$, with topology $(X_i,p_i) \to (X,p)$ iff $\exists\ \varepsilon$-PHGA $f_i: (X_i,p_i)\to(X,p)$ and $\varepsilon_i \to 0$

and completeness, compactness ... same as before.

Tangent cone $\exists\ \{r_i\} \to 0$, s.t. $(X, p, r_i^{-1} d) \xrightarrow[\text{converge}]{PGH} (T_pX, p^*)$ the tangent cone at p. (infinitesimal metric)

Setting. M^n, $Ric \ge (n-1)\Lambda$ **Topic 5. Splitting thm by Cheeger-Colding**

M^n_Λ is the space form with $sec M^n_\Lambda = \Lambda$, $g = dr^2 + sn_\Lambda^2 g$, with $sn_\Lambda = \begin{cases} \frac{\sin(\sqrt{\Lambda} r)}{\sqrt{\Lambda}} & ; \Lambda > 0 \\ r & ; \Lambda = 0 \\ \frac{\sinh(\sqrt{-\Lambda} r)}{\sqrt{-\Lambda}} & ; \Lambda < 0 \end{cases}$

set $\underline{G}(r) = \frac{1}{(n-2)\mathrm{vol}(S^{n-1})}\int_r^\infty sn_\Lambda^{1-n}(s)\,ds$ Geodesic polar coordinates

the Green function on M^n_Λ & Laplacian $\Delta = sn_\Lambda^{1-n}\partial_r(sn_\Lambda^{n-1}\partial_r)$

$\Rightarrow \Delta \underline{G} = -c\, sn_\Lambda^{1-n}\partial_r(sn_\Lambda^{n-1} sn_\Lambda^{1-n}) = 0$

$\Rightarrow \underline{G}$ is harmonic, with pole at 0

set $\underline{U}(r) = \int_0^r sn_\Lambda^{1-n}(s)\left(\int_0^s sn_\Lambda^{n-1}(t)\,dt\right)ds \Rightarrow \Delta \underline{U} = sn_\Lambda^{1-n}\partial_r\left(\int_0^r sn_\Lambda^{n-1}(t)\,dt\right)$

and $\underline{U}(0) = \lim_{r\to 0}\underline{U}(r) = 0$ and its differentials $= sn_\Lambda^{1-n}(r)\, sn_\Lambda^{n-1}(r) = 1$

set $\underline{A}(r) = sn_\Lambda^{n-1}(r)$. $\Rightarrow \underline{H}(\partial B_r(0)) = \frac{\underline{A}'(r)}{\underline{A}(r)}$ by Gromov-Bishop's proof

thus the mean curvature decreasing

set $\underline{G}_R = \underline{G} - \underline{G}(R)$, $\underline{U}_R = \underline{U} - \underline{U}(R)$ boundary. And without "_", $A(r)$, $H(\partial B_r(0))$ denote M^n's $\Rightarrow \frac{A(r)}{\underline{A}(r)} \searrow \Rightarrow \frac{H(\partial B_r(0))}{\underline{H}(\partial B_r(0))} \searrow$

The most notorious theas by Colding

$M_t \subset \mathbb{R}^{n+1}$ flowed by mean curvature (MCF)

if $\partial_t x = -H\vec{n}$ ($H = \mathrm{div}\,\vec{n}$) **Topic 6. Entropy, self-shrinker and F-stable minimal surface**

(1) $\partial_t \vec{n} = \nabla H$ & $\partial_t H = \Delta H + |A|^2 H$

(2) Define $\Phi(x,t) = (-4\pi t)^{-\frac{n}{2}} e^{\frac{|x|^2}{4t}}$, $\Phi_{(x_0,t_0)}(x,t)$

$= \Phi(x - x_0, t - t_0) \Rightarrow \frac{d}{dt}\int_{M_t}\Phi_{(x_0,t_0)} \le 0$ decreasing

F-functional: $F_{x_0,t_0}(\Sigma) = (4\pi t_0)^{-\frac{n}{2}}\int_\Sigma e^{-\frac{|x-x_0|^2}{4t_0}}$

Entropy of Σ: $\lambda(\Sigma) = \sup_{x_0,t_0} F_{x_0,t_0}(\Sigma)$ is decreasing: $t > s$

by $F_{x_0,t_0}(M_t) = (4\pi(t_0 - t))^{-\frac{n}{2}}\int_{M_t} e^{-\frac{|x-x_0|^2}{4(t_0-t)}}$

$= \int_{M_t}\Phi_{(x_0,t_0)}(x,t) \le \int_{M_s}\Phi_{(x_0,t_0)}(x,s) = F_{x_0,t_0}(M_s)$ ▨

(3) Critical point of F-functional

① First variation $= 0$ ② Second variation < 0

$\Rightarrow$ Local maximal

③ Global maximal: $\forall$ other $q \neq p$ $\exists$ path $q \xrightarrow{\gamma} p$, F_{x_0,t_0} decrease in γ.

① Set Σ_s normal variation of Σ, variation field Σ

$\frac{\partial}{\partial s}\Big|_{s=0} F_{x_0,t_0}(\Sigma_s) = (4\pi t_0)^{-\frac{n}{2}}\int_{\Sigma_0} e^{-\frac{|x-x_0|^2}{4t_0}}\Big(f H$

$+ \left(\frac{|x-x_0|^2}{4t_0} - \frac{n}{2t_0}\right)h + \frac{\langle y, x-x_0\rangle}{2t_0}\Big) = \int f\left(H - \frac{\langle x - x_0, \vec{n}\rangle}{2t_0}\right)$ is Critical point ▨

$+ \left(\frac{|x-x_0|^2}{4t_0} - \frac{n}{2t_0}\right)h + \frac{\langle y, x-x_0\rangle}{2t_0}\Big] = 0 \Leftrightarrow \left[\frac{|x_0 - x|^2}{t_0} - 2n\right]_0 = 0 \Leftrightarrow [x_0 - x_0]_0 = 0$

(ii) Σ is parabolic, i.e.

$\Delta_\Sigma u \le 0$, $u > 0 \Rightarrow$ constant

$\Delta_\Sigma = \frac{1}{\cosh^2 u}\left(\left(\frac{\partial}{\partial u}\right)^2 + \left(\frac{\partial}{\partial v}\right)^2\right)$

prove $\boxed{\mathrm{Area}(B_r \cap \Sigma) \le Cr^2}$

(*) Euclidean developing.

Lemma. (*) $\Rightarrow$ parabolic

Proof of Lemma.

(*) For graph Γ_u, $w = \log u$

$\Delta w = \mathrm{div}\left(\frac{\nabla u}{u}\right) = \frac{\Delta u}{u} - |\nabla w|^2$

$\le -|\nabla w|^2$

$\Rightarrow \int_\Sigma |\nabla w|^2 \le -\int(\Delta w)\eta^2$

$= \int 2\eta\langle\nabla w, \nabla\eta\rangle \le \frac{1}{2}\int|\nabla w|^2\eta^2$

$+ 2\int|\nabla\eta|^2 \Rightarrow \int|\nabla w|^2\eta^2 \le 4\int|\nabla\eta|^2$

$\eta = \begin{cases} 1 & ; x \in B_R \\ 2 - \log r/\log R & ; x \in B_{R^2} - B_R \\ 0 & ; x \in B_{R^2}^c \end{cases}$

$\Rightarrow \int|\nabla w|^2 \le \frac{4}{(\log R)^2}\int_{\Sigma\cap(B_{R^2} - B_R)}\frac{1}{r^2}$

$\le \frac{4}{(\log R)^2}\sum_{k=\log R}^{2\log R}\int_{\Sigma\cap B_{e^{k+1}} - B_{e^k}}\frac{1}{r^2}$

$\le \frac{C}{\log R} \to 0$ ▨

We do Exercises before about catenoid here

(i) Σ is minimal, taking $f(u) = \cosh u$, $g(u) = u$

$E = f'^2 + (g')^2 = \sinh^2 u + 1 = \cosh^2 u$, $F = 0$

$G = \cosh^2 u \Rightarrow I = \cosh^2 u(du^2 + dv^2)$

(harmonic coordinates) $\Rightarrow \begin{cases}\Delta f(u)\cos v = 0 \\ \Delta f(u)\sin v = 0 \\ \Delta u = 0\end{cases}$

$\Rightarrow$ minimal ▨

(ii) Any rotation surface minimal is catenoid or plane.

Now $x = f(s)$, $z = g(s)$, $f'^2 + g'^2 = 1$

$\Rightarrow E = 1$, $F = 0$, $G = f^2$ and also harmonic coordinates $(g_{ij}) = \begin{pmatrix}1 & 0 \\ 0 & f^2\end{pmatrix}$, $\det(g_{ij}) = f^2$, $f > 0$

$\Delta = \frac{1}{f}\frac{\partial}{\partial u}\left(f\frac{\partial}{\partial u}\right) + \frac{1}{f}\frac{\partial}{\partial v}\left(f\left(\frac{1}{f^2}\frac{\partial}{\partial v}\right)\right)$

$= \left(\frac{\partial}{\partial u}\right)^2 + \frac{f'}{f}\frac{\partial}{\partial u} + \frac{1}{f^2}\left(\frac{\partial}{\partial v}\right)^2$

$\Rightarrow \begin{cases} f''f + f'^2 = 1 \\ f''g' + f'g'' = 0 \\ f'^2 + g'^2 = 0 \end{cases}$ by minimal $\Delta x = 0$ and $\Delta z = 0$

$\Rightarrow$ Solve it out to give catenoid ▨

(nontrivial: $\frac{dx}{dz} = \frac{f'g' - f'g'}{g'^3}$ & $\frac{dx}{dz} = \frac{f'}{g'}$

$\Rightarrow x\left(\frac{d^2x}{dz^2}\right) - \left(\frac{dx}{dz}\right)^2 = 1 \Rightarrow x = A\cosh(\ldots)$

(iii) $\int_\Sigma K =$ total curvature compute:

$\int_\Sigma K = -2\pi\int\det(I^{-1}II) \xrightarrow[II = 1]{\text{minimal}} -2\pi\int\frac{\cosh^2 u}{\cosh^4 u}$

$= -2\pi\cdot 2 = -4\pi$ ▨

Using such way, we can prove $M^n \subset \mathbb{R}^{n+1} \Rightarrow \int|H|^n \ge \mathrm{vol}(S^n)$ ($H = \frac{k_1 + \cdots + k_n}{n}$)

Pf. $\int_M\left(\frac{k_1 + \cdots + k_n}{n}\right)^n \ge \int_M k_1\cdots k_n \ge \boxed{\int_M K \ge \mathrm{vol}(S^n)}$ when all $k_i \ge 0$ by moving frame the unit volume $\mathrm{vol}_{N^*g_2}$

it holds generally for k_i arbitrary ▨

Coro. $M^n \subset S^{n+1} \Rightarrow \mathrm{vol}(M) \ge \mathrm{vol}(S^n)$

Pf. $M^n \subset S^{n+1}$ minimal $\Rightarrow \mathrm{vol}(M) \ge \int_M|H|^n \ge \mathrm{vol}(S^n)$ ▨

Another method (Moving frame & Gaussian map)

$N: (\Sigma^2, g_1) \to (S^2, g_2)$, $N^*g_2 \neq g_1$

$\Rightarrow \mathrm{vol}_{g_1} = \omega^1\wedge\omega^2$ & $\langle N_*(e_i), N_*(e_j)\rangle$

$= \langle\nabla_{e_i}N, \nabla_{e_j}N\rangle = h_{ik}h_{jm} \Rightarrow N^*g_2 = (h_{ik}h_{jk})$

$\Rightarrow \mathrm{vol}_{N^*g_2} = |\det(h_{ik})|\,\omega^1\wedge\omega^2 = |K|\,\omega^1\wedge\omega^2$

$\Rightarrow \int K = -\int|K| = -\mathrm{vol}(N(\Sigma)) = -4\pi$ ▨

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② For compute second variation

Σ is self-shrinker $\overset{\text{Def}}{\Longleftrightarrow}$ $H=\frac{\langle x,n\rangle}{2}$

$\Downarrow$

$\int[|x|^2-2n]=0$ & $\int[x|x|^2]=0$ & $\int[|x|^4-2n(2n+4)+|\sigma|^2]=0$ & $\int[\langle x_0,y\rangle^2-2|y^T|^2]=0$ & $\int[\frac{|x|^2}{4}-\frac{n}{2}\ldots-\frac{H}{2}+H^2]=0$

By scaling $\tilde\Sigma=\frac{\Sigma-x_0}{\sqrt{t_0}}$, $y=\frac{x-x_0}{\sqrt{t_0}}$

$\Rightarrow H_{\tilde\Sigma}(y)=\sqrt{t_0}H_\Sigma(x)=\sqrt{t_0}\frac{\langle x-x_0,n\rangle}{2t_0}=\frac12\langle\frac{x-x_0}{\sqrt{t_0}},n\rangle=\frac12\langle y,n\rangle$

$x_0=0$, $t_0=-1$ ... $H=\frac12\langle x,n\rangle \Rightarrow \int[|x|^2-2n]=0$ & $\int[x]=0$ done

$\int[f]=(4\pi)^{-\frac n2}\int f e^{-\frac{|x|^2}{4}}$

Drift Laplace operator $\mathcal{L}$: $\mathcal{L}\cdot:=e^{\frac{|x|^2}{4}}\operatorname{div}(e^{-\frac{|x|^2}{4}}\nabla(\cdot))=\Delta(\cdot)-\frac12\langle x,\nabla(\cdot)\rangle$

$\Rightarrow \int[u\mathcal{L}v]=(4\pi)^{-\frac n2}\int u\operatorname{div}(e^{-\frac{|x|^2}{4}}\nabla v)=-(4\pi)^{-\frac n2}\int\langle\nabla u,\nabla v\rangle e^{-\frac{|x|^2}{4}}$

$=-\int[\langle\nabla u,\nabla v\rangle] \Rightarrow \int[\phi u\mathcal{L}v]=-\int[\langle\nabla(\phi u),\nabla v\rangle]=-\int[\phi\langle\nabla u,\nabla v\rangle]$ $+\int[\langle\nabla\phi,\ldots\rangle]$ ... (*)

$\Rightarrow \mathcal{L}|x|^2=\Delta|x|^2-\frac12\langle x,\nabla|x|^2\rangle=2\langle\Delta x,x\rangle+2|\nabla x|^2-\frac12\langle x,2x^T\rangle$

$=-|x^\perp|^2+2n-|x^T|^2=2n-|x|^2$

$\mathcal{L}x_i=\Delta x_i-\frac12\langle x,\nabla x_i\rangle=\langle\Delta x,\partial_i\rangle-\frac12\langle x,\nabla\langle x,\partial_i\rangle\rangle=-\frac12\langle x^\perp,\partial_i\rangle$ $-\frac12\langle x^T,\partial_i\rangle=-\frac12\langle x,\partial_i\rangle=-\frac12 x_i$ ... (**)

$\Rightarrow \int[u\mathcal{L}v]\overset{(**)}{=}\int[-\frac12 x_i]$; $\int[|x|^2\mathcal{L}x_i]=-\frac12\int[|x|^2x_i]=\int[x_i\mathcal{L}|x|^2]$ $=\int[x_i(2n-|x|^2)]=\ldots-\int[x_i|x|^2]$ done

$\overset{(*)}{\Downarrow}$ $\int[\langle\nabla u,\nabla v\rangle]=0$ done

The last three formulas ...

Splitting theorem

Given $\Delta f\le 0$ and $f\ge 0$ on M^n, $Ric\ge 0$ and the polar coordinates $g=dr^2+r^2\sigma_{ij}d\theta_i d\theta_j$

Denote $G=\det\sigma_{ij}$, by Bishop–Gromov

$\Rightarrow \frac{n-1}{r}\ge\Delta r=\frac{\mathcal{A}'}{\mathcal{A}}$ with $\mathcal{A}=r^{n-1}\sqrt{\det\sigma}=r^{n-1}\sqrt G$

$=\frac{n-1}{r}+(\log\sqrt G)' \Rightarrow G'\le 0$ ...(1)

and $\int_{B_r(p)}\Delta f=\int_{\partial B_r(p)}\nabla f\cdot\nabla r\overset{\text{polar}}{=}\int_{\partial B_r(p)}\frac{\partial f}{\partial r}=\int_{S^{n-1}}\frac{\partial f}{\partial r}r^{n-1}\sqrt G$ ...(2)

as the topology of $\partial B_r(p)$ is a ball when $r\to 0$

$\frac{1}{r^{n-1}}\int_{B_r(p)}\Delta f\le 0$

$\int_{S^{n-1}}\frac{\partial f}{\partial t}\sqrt G\underset{f\ge0}{\overset{(1)}{\ge}}\int_{S^{n-1}}\frac{\partial f}{\partial t}\sqrt G+f\frac{\partial\sqrt G}{\partial t}=\frac{d}{dt}\int_{S^{n-1}}f\sqrt G$ decreasing

$\Rightarrow \int_{S^{n-1}}f\sqrt G\le f(p)|S^{n-1}| \Rightarrow r^{n-1}f(p)|S^{n-1}|\ge\int_{S^{n-1}}ft^{n-1}\sqrt G\overset{(2)}{=}\int_{\partial B_t(p)}f \Rightarrow \int_{B_r(p)}f\le\frac{r^n}{n}f(p)|S^{n-1}|=f(p)|B^n|r^n=\omega_n \Rightarrow f(p)\ge\frac{1}{\omega_n r^n}\int_{B_r}f$

minimal

Cheeger–Gromoll) M^n with $Ric\ge0$ complete, if M contains a line (i.e. $\gamma:(-\infty,+\infty)\to M$, geodesic) $\Rightarrow M$ isometry splitting (i.e. $M\cong N\times\mathbb{R}$)

Pf. Construct Busemann function $b_t^\pm=d(x,\gamma(\pm t))-t$ and $b^\pm=\lim_{t\to\infty}b_t^\pm$, $b_t=b_t^++b_t^-$

Observation ① $|b_t^\pm|$ is bounded uniformly by triangle equality; ② $b_t^\pm$ is decreasing; ③ $b_t\ge0$ and vanishing at γ;

then $\Delta b_t\le\frac{n-1}{d(x,\gamma(t))}+\frac{n-1}{d(x,\gamma(-t))}$, by ② it converges into $\Delta b\le0$, by $0=b(\gamma(t))\ge\frac{1}{\omega_n r^n}\int b \Rightarrow b\equiv0$

$\Rightarrow 0\le-\Delta b^-=\Delta b^+\le0\Rightarrow\Delta b^+=0$, and distance function $|\nabla b^+|=1\Rightarrow$ Bochner $\frac12\Delta|\nabla b^+|^2=|Hess\,b^+|^2+Ric(\nabla b^+,\nabla b^+)=0$

$\Rightarrow Hess\,b^+=\nabla^2b^+\equiv0$, then by de Rham decomposition thm $M\cong(b^+)^{-1}(0)\times\mathbb{R}$ through the gradient flow ∇b^+ ▨ (it's paralleled and killing).

Cheeger–Colding) M^n complete, $Ric\ge-(n-1)\delta$ in $B_{2R}(p)$, $p_\pm\in M$, $d(p_\pm,p)\ge L\gg R$

the excess $E(p)<\varepsilon$ ($E(x)=d(x,p_+)+d(x,p_-)-d(p_+,p_-)$), recall it's also defined as density−1.)

$\Rightarrow\exists$ metric space (X,x), s.t. $d_{GH}(B_R(p),B_R((0,x)))<\Psi(\delta,L^{-1},\varepsilon|n,R)$. (It's also called the quantitive splitting thm)

lemma (Abresch–Gromoll) Locally form we have $E\le\Psi(\delta,L^{-1},\varepsilon|n,R)$ when they $\to0$ ($\Psi\to0$) $(0,x)\in\mathbb{R}\times X$

Pf. Set $L_R=c_1G_R+U_R$, $c_1=-\frac{U'(R)}{G'(R)}>0$

$\Rightarrow\Delta L_R=1$ and $L_R'=-\frac{1}{G'(R)}(U'(R)G'-U'G'(R))\le0$ decreasing to 0 in boundary, set $\rho_x=d(x,-)$

Using maximal principle, we have if $\Delta f\ge C\Rightarrow f(x)\le\max_{\partial\Omega}(f-CU(\rho_x))$ (as $\Delta(f-CU(\rho_x))\ge C-C\Delta U(\rho_x)\ge C-C\Delta U=$

and when $\Delta f\le-C\Rightarrow f(x)\ge\min_{\partial\Omega}(f+CU(\rho_x))$ similarly

$\Rightarrow\Omega\subset A_{R_1,R_2}(p)=B_{R_2}(p)-B_{R_1}(p)$, $\Delta f\le C>0$, $t\ge0\Rightarrow$ $f(x)\ge(L_{R_2}+tG_{R_2})(\rho_p)\ge\min_{\partial\Omega}(f-C(L_{R_2}+tG_{R_2}))(\rho_p)$

$\Rightarrow$ either ① $\min_{\partial B_{R_1}(p)}f<(L_{R_2}+tG_{R_2})(R_1)$ or ② $f(x)\ge(L_{R_2}+tG_{R_2})(d(x,p))$ for $\forall x\in A_{R_1,R_2}(p)$

It's by $\Delta f=\Delta\ldots\le\delta\Delta L_R\ldots\to0$

Now in the setting of Abresch–Gromoll ① holds $\Rightarrow$ ② not, i.e. $f(y)<(L_{R_2}+tG_{R_2})(R_1)+kR_1$ for $x\in A_{R_1,R_2}(p)$

and $\Delta E\le\Psi_1(\delta,L^{-1},\varepsilon|n,R)$ by definition of E as distance function $\Rightarrow E(x)\le\varepsilon\le\Psi_1L_{R+1}(R)\le\Psi_1L_{R+1}(y)$ ($\forall y<R$)

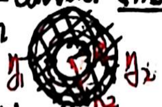
$\le\Psi L_{R+1}(y)+2y$ ($y=R_1$, $R+1=R_2$) $\le2y+2y=4y\underset{\Psi\to0}{\to}0\Rightarrow$ It can be dominated by some modification of Ψ ▨

due to $L_R\le1$ this not release

Lemma (Cheeger–Colding) $Ric\ge-(n-1)$, $\forall R_2>R_1>0$, $\exists\varphi:M\to[0,1]$ test function (i.e. $\varphi|_{B_{R_1}(p)}=1$, $\operatorname{supp}\varphi\subset B_{R_2}(p)$)

s.t. $|\nabla\varphi|+|\Delta\varphi|\le C(n,R_1,R_2)$ (Idea: closed to boundary $|\nabla\varphi|$ can't be estimated (By Cheng–Yau), thus we need η_1,η_2 distinct bound)

Pf. Consider Dirichlet problem on $\begin{cases}\Delta f=1 \text{ in } B_{R_2}(p)-B_{R_1}(p)\\ f|_{\partial B_{R_2}(p)}=0;\ f|_{\partial B_{R_1}(p)}=L_{R_2}(R_1)\\ (L_{R_2}(R_2)=0)\end{cases}$ Given not enough $\exists\eta_1,\eta_2>0$, $L_{R_2}(R_1)>\underbrace{L_{R_2}(R_1+\eta_1)}_{a}\ge\underbrace{L_{R_2}(R)-U(R_2-R_1-\eta_2)}_{b}>0$



and $1=\Delta f\le\Delta L_{R_2}\Rightarrow f\ge L_{R_2}(d(x,p))$ (Rightside is well-defined, as at least its order is 2)

② When $R_2-\eta_2\le d(x,p)\le R_2$, $f(x)\le a$

① When $R_1\le d(x,p)\le R_1+\eta_1$, $f(x)\ge b$

$\Delta(f-U)\le0\Rightarrow$ subharmonic $\Rightarrow f(x)\le\max_{\partial B_{R_2}(p)-B_{R_1}(p)}(f-U)$ $\le L_{R_2}(R_1)-U(d(x,p)-R_1)$

Then we can construct $\psi:[0,L_{R_2}(R_1)]\to[0,1]\in C^\infty$ and $\psi=1$; $x\in[b,L_{R_2}(R_1)]\Rightarrow\varphi=\psi(f)$, done ▨

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Consider $\mathbb{b}_t$ a harmonic function, $\Delta \mathbb{b}_t = 0$, $\mathbb{b}_t|_{\partial B_{(p)}} = b_t$

Lemma0. $\|\mathbb{b}_t - b_t\| \le \psi$ (ψ is the same as the Abresch–Gromoll, up to constant) [Page 8]

Pf. Observe that $b_+(x) + b_-(x) - \underline{E}(x) - \underline{E}(p) \le \psi$ by Abresch–Gromoll and $\Delta(\mathbb{b}_\pm - b_\pm) \le \psi$, $\mathbb{b}_\pm - b_\pm \ge -\psi \Rightarrow -\varepsilon \le b_+ - \mathbb{b}_- \Rightarrow -\varepsilon \le \mathbb{b}_+ - \mathbb{b}_-$ by maximal principal $\Rightarrow \mathbb{b}_+ - \psi \le b_+ \le -b_- + \psi \le -\mathbb{b}_- + 2\psi \le \mathbb{b}_+ + 2\psi + \varepsilon \Rightarrow \mathbb{b}_+ \le 3\psi + \varepsilon$ ▨

this is the $\boxed{C^0\text{-estimate}}$, next is C^1-estimate by meanvalue inequality.

Lemma1. $⨍_{B(p)} |\nabla \mathbb{b}_+ - \nabla b_+|^2 \le \psi$ Pf. $|\Delta b_+| \le (\Delta b_+)_- + \psi$

$\Rightarrow ⨍_{B(p)} |\nabla \mathbb{b}_+ - \nabla b_+|^2 \le ⨍_{B(p)} \mathrm{div}(\mathbb{b}_+ - b_+)\nabla(\mathbb{b}_+ - b_+)|$

$= ⨍_{B(p)} |\Delta b_+|\,|\mathbb{b}_+ - b_+|$

$\le \psi ⨍_{B(p)} |(\Delta b_+)_-| + \psi$

$\le \psi ⨍_{B(p)} \Delta b_+| + \psi = \psi ⨍ 1 + \psi$

$\frac{\partial B(p)}{\psi \, B(p)}$

$= \psi(1 + \square) \le C\psi$ ▨

Lemma. $\forall x, y \in B_{\frac{1}{2}}(p)$, $s = b_+(x)$, $\exists z \in b_+^{-1}(s)$, s.t. $d(z, w) = d(w, b_+^{-1}(s))$

$\Rightarrow |d(x,z) + d^2(z, w) - d^2(x, w)| \le \psi$

Pf. $\forall z_* \in B_\psi(z)$, $w_* \in B_\psi(w)$

the minimal geodesic $\sigma_{z_* w_*}$, $x_* \in B_\psi(x)$

$\Rightarrow \exists x_*, z_*, w_*$ s.t. with length $K_{z_* w_*}$

$\int_0^{K_{z_* w_*}} |\nabla \mathbb{b}_+(\sigma_{z_* w_*}) - \sigma'_{z_* w_*}(s)| \le \psi$ $=: K$

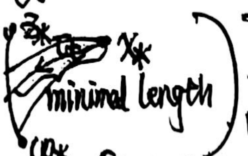
(We can't directly estimate as "every direct" is unknown due to sectional curvature unknown.)

$\int_0^{K_{z_* w_*}} \int_0^{l(s)} |\mathrm{Hess}\,\mathbb{b}_+(\tau_s'(t))| \,dt\, ds < \psi$

Lemma2. (Hessian estimate) (Same reason, must $⨍_{B(p)}$)

$⨍_{B_{\frac{1}{2}}(p)} |\mathrm{Hess}\,\mathbb{b}_+|^2 \le \psi$ (boundary $\nearrow \infty$)

τ_t is



This is by Poincare inequality we first derivative

Pf. $\frac{1}{2}\Delta|\nabla \mathbb{b}_+|^2 = |\mathrm{Hess}\,\mathbb{b}_+|^2 + \mathrm{Ric}(\nabla \mathbb{b}_+, \nabla \mathbb{b}_+)$ (Bochner)

$\ge |\mathrm{Hess}\,\mathbb{b}_+|^2 - (n-1)\delta|\nabla \mathbb{b}_+|^2$

then by Lemma2, $⨍ \mathrm{Hess} \le \psi$, and $\sigma'' = 0$, done

$⨍_{B_{\frac{1}{2}}(p)} |\mathrm{Hess}\,\mathbb{b}_+|^2 \le C ⨍_{B(p)} |\mathrm{Hess}\,\mathbb{b}_+|^2 \varphi$ (φ is the test function of $R_1 = \frac{1}{2} < R_2 = 1$)

$\Rightarrow \forall t \in [0, l(s)]$, $K\langle \nabla \mathbb{b}_+(\tau_s(t)), \tau_s'(t)\rangle$

$\le C\left[(n-1)\delta ⨍_{B(p)} |\nabla \mathbb{b}_+|^2 \varphi + \frac{1}{2} ⨍_{B(p)} \varphi \Delta(|\nabla \mathbb{b}_+|^2 - 1)\right] \le C\psi + \frac{1}{2} ⨍_{B(p)} \Delta\varphi(|\nabla \mathbb{b}_+|^2 - 1) \le C\psi + C ⨍_{B(p)} ||\nabla \mathbb{b}_+|^2 - 1|$

$-\langle \nabla \mathbb{b}_+(\tau_s(l(s)), \tau_s'(l(s))\rangle|$

$\le \int_t^{l(s)} |\mathrm{Hess}\,\mathbb{b}_+(\tau_s'(r), \tau_s'(r))\, dr|$

$\le C(\psi + ⨍ ||\nabla \mathbb{b}_+| - 1|) \le C\psi$ (C is different constants) ▨

$C_1\psi + C_2\sqrt{\psi} =: \psi$

$\Rightarrow \frac{1}{2} d^2(z, w) = \int_0^K s\,ds \pm \psi = \int_0^K (\mathbb{b}_+(\sigma(s)) - \mathbb{b}_+(\sigma(0)))\,ds \pm \psi$ (Cauchy)

$= \int_0^K (\mathbb{b}_+(\tau_s(l(s)) - \mathbb{b}_+(\tau_s(0))) \, ds \pm \psi = \int_0^K \int_0^{l(s)} \langle \nabla \mathbb{b}_+(\tau_s(t)), \tau_s'(t)\rangle \,dt\, ds \pm \psi = \int_0^K l(s) \langle \nabla \mathbb{b}_+(\tau_s(l(s))), \tau_s'(l(s))\rangle ds$

$= \int_0^K l(s) \langle \nabla \mathbb{b}_+(\sigma(s)), \tau_s'(l(s))\rangle ds \pm \psi = \int_0^K l(s) \langle \sigma'(s), \tau_s'(l(s))\rangle ds \pm \psi = \int_0^K l(s) l'(s)\, ds \pm \psi = \frac{1}{2}(l^2(K) - l^2(0)) \pm \psi \pm \psi$

$= \frac{1}{2}(d^2(x, w) - d^2(x, z)) \pm \psi$ ▨

Finally, we can prove the Cheeger–Colding's quanitive splitting thm

Pf. We need prove $d_{GH}(B_{\frac{1}{2}}(p), B_{\frac{1}{2}}(0, x)) < \psi$. Define $f: B_{\frac{1}{2}}(p) \to b_+^{-1}(b_+(p))$

The desired metric space $X := b_+^{-1}(b_+(p)) \cap B_{\frac{1}{2}}(p)$ (Ricci limit space) $\Rightarrow (y, b_+(y) - b_+(p))$

taking limit in X, s.t. the Lemma holds $d^2(x, w) + d^2(z, w) - d^2(x, w)$, by this $X \times \mathbb{R}$ coincides the given one at first

(Gromov–Hausdorff limit) $\Rightarrow d_{GH}(B_{\frac{1}{2}}(p), B_{\frac{1}{2}}(0, x)) < \psi$ is then tautological ▨

The second variation formula of self-shrinker: $\left(\frac{\partial}{\partial s}\right)^2\Big|_{s=0} F_{x_s, t_s}(\Sigma_s) = \left[-f L f + 2fhH - h^2 H^2 + f\langle y, \vec{n}\rangle - \frac{1}{2}\langle y, \vec{n}\rangle^2\right]$ [Topic6 (Continue)]

Pf. Recall the first variation formula is $\frac{\partial}{\partial s}\Big|_{s=0} F_{x_s, t_s}(\Sigma_s) = \left[f\left(H - \frac{\langle x - x_0, \vec{n}\rangle}{2t_0}\right) + h\left(\frac{|x - x_0|^2}{4t_0} - \frac{n}{2t_0}\right) + \frac{\langle x - x_0, y\rangle}{2t_0}\right]$

$\Rightarrow \delta^2 F = \left[f\left(H' - \frac{\langle f\vec{n} - y, \vec{n}\rangle + \langle x, \vec{n}'\rangle}{2}\right) + \langle \vec{n}, \ldots\rangle h\right] + h\left(\frac{\langle f\vec{n} - y, x\rangle}{2} - \frac{x^2}{4}t_0 + \frac{n}{2}h\right)$

$+ \frac{\langle f\vec{n} - y, y\rangle}{2} - \frac{\langle x, y\rangle}{2}h + h\left(\frac{|x|^2}{2} - \frac{n}{2}\right) - \frac{\langle x, y\rangle}{2}\Big]$

(vanishing at $s = 0$, this only derivative in $\vec{n}$, the inner)

$= \Big[-f\Delta f - |A|^2 f^2 - \frac{1}{2}f^2 + \boxed{\frac{\langle y, \vec{n}\rangle}{2} f} + \frac{1}{2}\langle x, \nabla f\rangle f + \frac{1}{2}\langle x, \vec{n}\rangle fh + \frac{1}{2}\langle x, \vec{n}\rangle fh - \frac{1}{2}\langle x, y\rangle h - \frac{1}{2}|x|^2 h^2 + \frac{n}{2}h^2 + \boxed{\frac{1}{2}\langle y, \vec{n}\rangle f} - \frac{1}{2}|y|^2$

$- \frac{1}{2}\langle x, y\rangle h + h^2\left(\frac{n}{2} - \frac{|x|^2}{4}\right) + \frac{1}{2}|y|^2\Big] = \left[-fLf + 2fhH - h^2 H^2 + f\langle y, \vec{n}\rangle - \frac{1}{2}\langle y, \vec{n}\rangle^2\right]$ ▨

Variation: $\delta F = 0$, $\delta^2 F < 0$ maximal

Locally, Globally, a path needed

A self-shrinker Σ is F-stable if $\forall f\vec{n}$, $\exists x_s, t_s$, s.t. $\left(\frac{\partial}{\partial s}\right)^2\Big|_{s=0} F_{x_s, t_s}(\Sigma_s) \ge 0$

E.g. (Spheres) When $S^n(r)$ is self-shrinker? Compute $\frac{n}{r} = \frac{1}{2}\langle x, H\rangle = \frac{1}{2}\langle x, \frac{1}{2}\langle x, \vec{n}\rangle\rangle = \frac{r}{2} \Rightarrow r = \sqrt{2n}$. Here take $g(s)$

Claim. $S^n(\sqrt{2n})$ not only self-shrinkers but also F-stable!

$F_{s,y, 1+s^2}(\Sigma)$ and $g'(s) \le 0$

Pf. The eigenvalue of Δ in S^n is $k(n+k-1) \Rightarrow$ in $S^n(\sqrt{2n})$ is $\frac{k(n+k-1)}{2n}$; $\lambda_0 = 0$: constant, $\lambda_1 = \frac{1}{2}$: linear, $\lambda_2 = \frac{n+1}{n}$: ...

Computation of δF, using $f_{\varepsilon, h}$ as here has's

$\Rightarrow L = \Delta + |A|^2 + \frac{1}{2} - \frac{1}{2}\langle x, \nabla(\cdot)\rangle = \Delta + 1$

$\Rightarrow \delta^2 F = \left[-f(\Delta f + f) + \sqrt{2n}fh - \frac{n}{2}h^2 - \frac{1}{2}\langle y, \frac{x}{\sqrt{2n}}\rangle^2\right]$ Using spectral decomposition $f = a + \langle x, z\rangle + f_0 \Rightarrow \Delta f = -\frac{\langle x, z\rangle}{2} + \Delta f_0$

$= \left[-(a + \langle x, z\rangle + f_0)\left(-\frac{\langle x, z\rangle}{2} + \Delta f_0 + a + \frac{\langle x, z\rangle}{2} + f_0\right) + \sqrt{2n}\,ah - \frac{n}{2}h^2 + \langle x, z\rangle\langle y, \vec{n}\rangle - \frac{1}{4n}\langle y, x\rangle^2\right]$ and $\frac{⨍|\nabla f_0|^2}{⨍ f_0^2} \ge \frac{n+1}{n}$

$= \left[-a^2 - \frac{1}{2}\langle x, z\rangle^2 + \frac{1}{n}f_0^2 + \sqrt{2n}\,ah - \frac{n}{2}h^2 + \langle x, z\rangle\langle y, \vec{n}\rangle - \frac{1}{2}\langle y, \vec{n}\rangle^2\right] = \left[\frac{1}{n}f_0^2 - \left(a - \sqrt{\frac{n}{2}}h\right)^2 - \frac{1}{2}(\langle x, z\rangle - \langle y, \vec{n}\rangle)^2\right] \ge 0$ ▨

The entropy λ_k of $S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ (It's a $\square$ thus using δ and δ^2) If Σ can't split line $\Rightarrow \lambda(\Sigma) = F_{0,1}(\Sigma)$

When $\delta^2 F = 0 \Rightarrow h = 0$ or $H^2 = 0 \Rightarrow H = 0$ minimal $\Rightarrow$ cone $\xrightarrow{\text{smooth}}$ plane X. By Σ invariant, $f = 0 \Rightarrow$ ① $\delta^2 F < 0$ done!

② $F_{x_0, t_0}(\Sigma^n \times \mathbb{R})$ If can split as form $\square$ ruled surface X $\Rightarrow \delta^2 F < 0$ ▨ and $\delta F|_{(0,1)} = 0$ is easy. By ① & ②, $\lambda(\Sigma) = F_{0,1}(\Sigma)$

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