Date

elected applications of graphic method in Geometry

Combinatrional model works well when

- Group action: this is a recall of your undergraduated course abstract algebra, prof tells you group = symmetry via action on polygons / graphs
- Geometric object it self as combinational construction e.g. triangluated mfd, toric varieties

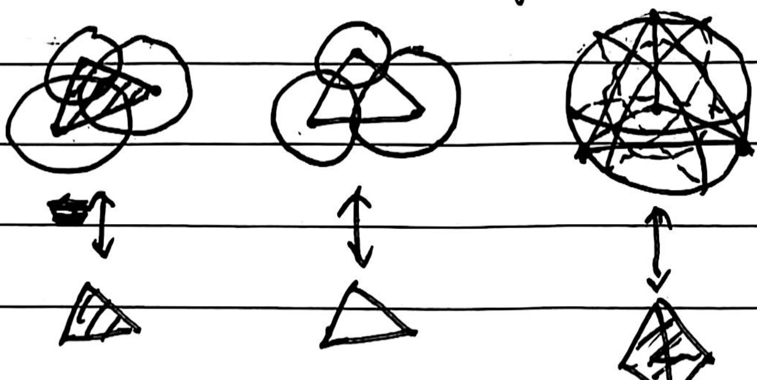
We'll give examples as many as I can, divided as: construct combinational model from covering; moduli of combinational objects; Galois group actions.

$\boxed{\text{PART I}}$ Nerve construction and toric variety

$\boxed{\text{PART II}}$ Moduli of quiver representations — Nakajima's quiver variety

$\boxed{\text{PART III}}$ Moduli of curves and Grothendieck's dream.

$\boxed{\text{I}}$. Recall the nerve of a covering (naive case) is given by



If there are intersection of two open cover, then use 1-simplex to connect them. each cover are 0-simpl[ex] if there's triple intersection then glue a 2-simplex...

Generally for a nerve of a category (simplicial set forms a), and take the category be the category of covering we recover the naive one

[think △ case above, there is

$$\begin{tikzcd}
 & U_{12} \arrow[r] \arrow[dr] & U_2 \arrow[dr] & \\
U_{123} \arrow[ur] \arrow[r] \arrow[dr] & U_{23} \arrow[ur] \arrow[dr] & & U_1 \\
 & U_{13} \arrow[r] \arrow[uur, bend right] & U_3 \arrow[ur] &
\end{tikzcd}$$

each morphism $\Leftrightarrow$ inclusion $\Leftrightarrow$ 1-simplex

i.e. U_{12} gives two 1-simplex connecting U_1 and U_2 ...

$\Rightarrow$ 

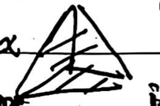
is the (barycentric) subdivision of above

- The nerve of topological cover may not homotopic to original space, hence we need good cover.
- Subdivision of complexes $\leftrightarrow$ Refinement of cover
- Nerve of cat have generality: almost all topological space can be constructed in this strategy: $\mathcal{C} \xrightarrow{\text{Nerve}} N(\mathcal{C}) \xrightarrow[\text{realization}]{\text{Geometric}} |N(\mathcal{C})|$, as we can view BG, or more generally, of groupoid as a special case of nerve construction: $B\mathcal{G} = N(\mathcal{G})$ ($\mathcal{G}$ viewed as single object category)

$$\mathrm{Cat} \to s\mathrm{Set} \to \mathrm{Top}$$

"Good cover" can't be compatible algebraic category. now for a variety X, it's better to talk about affine covering

E.g. • $\mathbb{P}^n$ covered by $(n+1)$ affine coverings, and their intersection gives the standard n-simplex

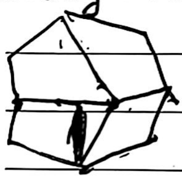


• $(\mathbb{CP}^n, \omega_{FS})$ has moment polytope is also this, but each edge has length standard 1 $\omega_{FS} = c_1(\mathcal{O}(1))$

• $(\mathbb{P}^n, \mathcal{O}(1))$ we associate this n-simplex with length 1, but other polarizations can give different volumes after scaling.

Q. Can every algebraic variety constructed by this manner?

A. No, when the singularity being very complicated, affine covering around it $\not\Rightarrow$ datum of this singularity as affine cover is to coarse.



non-normal cases

(this also explain why in AG some singularities' problem one dealt by transcendental method such as monodromy)

Hence at least, X should be normal.

On the otherhand, if we view it as symplectic polytope, then of course we have a torus action, i.e. $\mathbb{G}_m^n \subset X$ for $\dim X = n$ and this dense torus on $X \Rightarrow$ normal as codim 2 smooth (by Hartogs' Lemma).

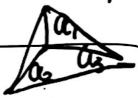
Hence this's toric variety in general: $\mathbb{G}_m^n \subset X$ dense orbit.

I don't want to give precise definition of a variety of toric conceptions but only figure out the correspondence of: Varieties $\xrightarrow[\text{affine opens}]{\text{Gluing}}$ Fans $\xleftarrow{\text{dual}}$ Polytopes (moment map), for more details, see Cox or Fulton's textbooks.

• Symplectic viewpoint gives us a Lagrangian fibration $(X(\mathbb{C}), \omega = c_1(L)) \to M$ to the moment polytope, for example, $\mathbb{CP}^1 = S^2$ and $M = \overset{1}{\longleftrightarrow} =: I \Rightarrow S^2 \to I$

Hence $\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} X(\mathbb{C}) = \frac{1}{2}\dim_{\mathbb{R}}(X(\mathbb{C}))$

Varieties	Fans (Dual Lattices)	Polytopes (Lattices)	
Polarization L	Length of vectors	Volume $= L^2$	
Toric affine coverings (Dense orbit of $\mathbb{G}_m^n$)	Chambers $\cong$ n-dim cones	Vertices	$\dim_{\mathbb{C}} \mathcal{O}_\sigma = \dim X - \dim_{\mathbb{R}} \sigma$ for cones σ in the fan
Toric divisors (Codim 1 orbit of $\mathbb{G}_m^n$)	1-dim cones i.e. vectors	Faces / edges	
Blow up	Add one vectors to chambers	Add one faces/edges	
Birational Properties			
Smooth	Chambers are $\mathbb{Z}$-basis of N	Faces are $\mathbb{Z}$-basis of M	
Resolution of singularity	Add vectors until $\mathbb{Z}$-basis	Add faces until $\mathbb{Z}$-basis	
L positive {g g b g / ample}	No $\geq 180°$ / No $\geq 180°$	convex / strictly convex	
Projective, Flat, Proper, Mirror	Subdivision, Reflexive	Subdivision, Reflexive	

E.g. • All weighted projective space are not smooth except usual (weight $(1,1,\dots,1)$) ones: they have polytope  and $a_i \geq 2$ not $\mathbb{Z}$-basis $\mathbb{P}(1,a_1,\dots,a_n)$

For example, $\mathbb{P}(1,2)$ = "football space"  $/\mathbb{R}$ have two singular points

• Hirzebruch surfaces $\mathbb{F}_n \leftrightarrow$ 

• $\mathbb{P}^1\times\mathbb{P}^1 \leftrightarrow$ $1\,\square\,1$, polarized by $O(m,n)$ is $n\,\square\,m$

hence the so called "polarized" can be viewed as rescaling in different directions

• Due to the dense orbit ⊂ toric, toric varieties are must rational, hence we can't expect any varieties are close to toric ones, but we can expect they have a toric model after some MMP process.

III Quiver rep ↔ Path algebra module

For a quiver $Q=(Q_0,Q_1)$ finite (oriented graph), its path algebra $\Bbbk Q$ is a graded algebra, spanned by all paths $Q_1/\Bbbk$, multiplication by composition of paths, graded by length of path.

• View a group G as one object groupoid $\mathcal{G}$. category ↔ quiver by taking Q_0 as object and Q_1 morphism

then group G rep ↔ $\Bbbk[G]$-module

free quiver $\mathcal{G}$ rep ↔ $\Bbbk\mathcal{G} = \Bbbk[g_1\cdots g_n]$

Hence group rep is a special case of quiver rep.

• $\Bbbk[x_1,x_2] = \dfrac{\Bbbk\langle x_1,x_2\rangle}{(x_1x_2-x_2x_1)}$ isn't path algebra! But quotient of path algebra, hence we need quotient of path algebra, i.e. quiver serves as a frame, but we need more relation on it.

Thus we consider bounded quiver (Q,I) to describe the relations.

But we need I is admissible, i.e. $\exists m$, s.t. $R_Q^m \subset I \subset R_Q^2$, where R_Q is arrow ideal consists all nontrivial arrows (trivial means it's constant at a vertex) $=\sqrt{\Bbbk Q}$, $\cdots \subset R_Q^2 \subset R_Q^1 \subset \Bbbk Q$ filtration with $R_Q^m/R_Q^{m+1} = \Bbbk Q_m$ arrows of length m

Hence admissible ↔ length ≥ 2 and $\exists m$, all length $\geq m$ paths all 0 modulo I

• Then $(x_1x_2-x_2x_1)$ isn't admissible for $\circlearrowleft \cdot \circlearrowright$ (x_1, x_2), hence $\Bbbk[x_1,x_2]$ isn't bounded path algebra! Then one may ask where is the generality: consider $\dfrac{\Bbbk[x_1,x_2]}{(x_1^2,x_2^2)}$ is bounded path algebra with $I=(x_1x_2-x_2x_1, x_1^2, x_2^2)$ admissible, we have all Artinian algebra are bounded quiver algebra (This lead to Ext^1/deformation later)

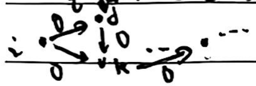
• On the other hand, the above example explains the obstruction occurs in the loops, if the quiver is acyclic (doesn't contain any loops, including $\circlearrowleft$), then $\Bbbk Q$ is f.d./$\Bbbk$, then we can expect all f.d./$\Bbbk$ given by a quiver, hence we have following correspondence:

Quiver (Q,I)	f.d. algebra $A/\Bbbk$
$Q_0 = n$ pts, $1\cdots n$	$e_1\cdots e_n$ idempotent basis
Q_1	$\sqrt{A}/(\sqrt{A})^2 = R_Q/R_Q^2$
$i \rightrightarrows j$	$e_i \frac{\sqrt{A}}{(\sqrt{A})^2} e_j$

and lift in representation/module cat level, we have:

(Q, I)

Quiver rep M	Algebra A-mod M
dirac δ at i	simple module $S(i)$ $\dim M_i$
	semi-simple module $\bigoplus_i S(i)$ $\dim M_i$
$\bigcap$ Ker of arrows	socle of M, i.e. generated by all $S(i)$ (maximal semi-simple)
$\bigoplus$ Im of arrows	$\sqrt{M}$
$\bigoplus$ Coker of arrows	$\sqrt{M}/(\sqrt{M})^2$
	$P(i) = e_i \Bbbk Q$ projective module
	$\updownarrow$ dual
	injective module



Rk. ① Simple and (basic) projective module have the difference is $S(i) = e_i A = e_i \Bbbk Q/I$

$$P(i) = e_i \Bbbk Q;$$

② $\mathrm{Ext}^1_A(S(i), S(j)) = e_i \dfrac{\sqrt{A}}{(\sqrt{A})^2} e_j$

$\mathrm{Ext}^1_{\Bbbk Q}(P(i), P(j)) = e_i \Bbbk Q e_j$

are easy ways to compute extension, for example, $\mathrm{Ext}^1_{\Bbbk[x]}(\Bbbk, \Bbbk) = \dfrac{\Bbbk[x]}{(x^2)}$ is well-known, now we can see it directly from quiver $\bullet\circlearrowleft$; for deformation theory

③ We can view quiver rep as a category $\subset \mathrm{Vect}_{\Bbbk}$ with objects are each M_i and $\mathrm{Hom}(M_i, M_j) = e_i M e_j$

this category is A-module if we view $\Bbbk Q/I$ also a category, then these simples/projectives ... are functors and when we talk about category $\mathrm{Mod}(A) \simeq \mathrm{Rep}(Q, I)$, we can view the object of this category is also category;

~~Tilting theory~~

Taking $\mathrm{Rep}(Q, I)$ as an entirety, we can describe it via Auslander-Reiten quiver.

One is able to pass $\mathrm{Mod}(A)$ to $\mathrm{Mod}(\mathrm{End}(T_A))$ via tilting theory

We don't continue to go deeper now but refer to [ASS], drawing AR quiver of given (Q, I) is like a game and tilting theoretic conceptions are widely used in studying $\mathrm{Mod}(\mathcal{O}_X)$ and $D^b(X)$

Quiver varieties

I'd like to show the so-called category–moduli correspondence by my self (maybe) at first:

$$\text{Category} \xrightarrow[\text{+ fix invariants}]{\text{+ geometric structure}} \text{moduli}/\text{geometric object}$$

moduli of objects

Groupoid

stability $\longleftrightarrow$ properness/degeneration

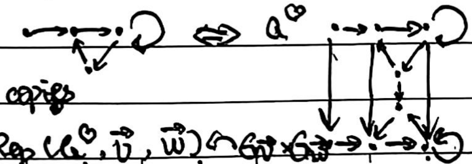
Now we start at the category $\mathrm{Rep}(Q, \vec{v})$ ← fix dimension vector $\vec{v}$, we'll construct a symplectic (indeed cotangent bundle) variety, called Nakajima's quiver variety then.

First abuse notation the vector space parametrized objects also $\mathrm{Rep}(Q, \vec{v})$

then let $G_{\vec{v}} = \prod GL(V_i) \curvearrowright \mathrm{Rep}(Q, \vec{v})$

- Then we need King's θ-stability for $\mathrm{Rep}(Q, \vec{v}) \Rightarrow$ we have geometric quotient $\mathrm{Rep}^{ss}(Q, \vec{v}) /\!/_{\chi_\theta} G_{\vec{v}} =: \mathcal{R}_\theta(Q, \vec{v})$
- But for fixed Q, may all $\theta \in \mathbb{Z}^I$ have $\mathcal{R}_\theta(Q, \vec{v}) = \emptyset$

hence for each Q we associate $Q^\heartsuit$ the framed quiver of Q



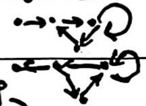
Q: $\Leftrightarrow$ $Q^\heartsuit$ is double of $Q^\heartsuit$ connected two copies

$\Rightarrow \mathrm{Rep}(Q^\heartsuit, \vec{v}, \vec{w}) \curvearrowleft G_{\vec{v}} \times G_{\vec{w}}$, but we still consider the subgrp $G_{\vec{v}} \subset G_{\vec{v}} \times G_{\vec{w}}$ - action

then we have $\mathcal{R}_\theta(Q, \vec{v}, \vec{w}) = \mathrm{Rep}^{ss}(Q^\heartsuit, \vec{v}, \vec{w}) /\!/_{\chi_\theta} G_{\vec{v}}$

• But it's still not enough (why?).

We consider $\bar{Q}$ double of Q and $\mathrm{Rep}(\bar{Q}, \vec{v})$ means

Q:  $\Leftrightarrow$ $\bar{Q}$:  same dimension vector

in the reversed copy Q^{op} $\Rightarrow$ $\mathrm{Rep}(\bar{Q}, \vec{v}) = T^*\mathrm{Rep}(Q, \vec{v})$

as vector space, hence $= \mathrm{Rep}(Q, \vec{v}) \times \mathrm{Rep}(Q^{op}, \vec{v})$

Then we can use symplectic reduction, equivalent the GIT quotient.

• Lastly, $\mathrm{Rep}(Q^{\heartsuit}, \vec{v}, \vec{w}) = \mathrm{Rep}(Q, \vec{v}) \times \mathrm{Rep}(Q^{op}, \vec{v}) \times \mathrm{Hom}(V, W) \times \mathrm{Hom}(W, V)$

and $\mathcal{M}_{\lambda,\theta}(\vec{v}, \vec{w}, Q) := \mu^{-1}(\lambda)^{ss}_{\theta} / G_{\vec{v}}$ the symplectic reduction

Rk. AR quiver describe different Q-rep in different dimension vector, here the moduli focus on fixed dim vector

III Here we all talking about $/\mathbb{C}$, $S^2 = \mathbb{P}^1$

We have following isomorphism of sets:

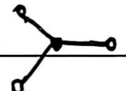
$\{$all cpt oriented surfaces $X\}$ $\overset{\text{Bieyli pairs}}{\cong}$ $\{X \xrightarrow{\beta} S^2 \mid \beta$ ramified at $0, 1, \infty\}$

After Grothendieck saw $\cong \{\beta^{-1}[0,1] \mid [0,1]$ is geodesic$\}$

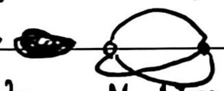
Bieyli's thm, he find $= \{$Some specific graphs$\}$

these graphs called Dessin d'enfant, and by taking look at local of $[0,1]$, it's a planar graph

$\mathbb{P}^1 \to \mathbb{P}^1$ is $3:1$ cover by x^3 (extend to homogenous one)

$\Leftrightarrow$  $(x, y) \mapsto y$

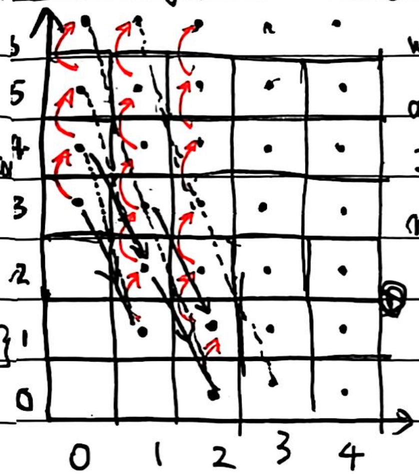
elliptic curve $\{y^2 - y = x^3\} \to \mathbb{P}^1$ (extend to homogenous one)

$\Leftrightarrow$ 

They're all bipartite graphs (not containing odd loops).

But Grothendieck's main idea is, the Bielyi pair (X, β) has β is $\bar{\mathbb{Q}}(z)$ ~~polynomial~~, hence equipped a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action naturally, so is the set of all Dessins, and writing out the paths of Dessin to study its action, we'll not go deep in these computations, but consider how $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ act on so-called Teichmüller tower.

For $\overline{\mathcal{M}}_{g,n}$ we have operations $\overline{\mathcal{M}}_{g_1,n_1} \times \overline{\mathcal{M}}_{g_2,n_2} \xrightarrow{\text{gluing}} \overline{\mathcal{M}}_{g_1+g_2, n_1+n_2-2}$ and $\overline{\mathcal{M}}_{g,n} \xrightarrow{\text{reduction}} \overline{\mathcal{M}}_{g+1, n-2}$, start at $\overline{\mathcal{M}}_{0,3}$, we can generate all $\overline{\mathcal{M}}_{g,n}$ with generators $\overline{\mathcal{M}}_{0,3}$, $\overline{\mathcal{M}}_{1,1}$ and relations $\overline{\mathcal{M}}_{0,4}$, $\overline{\mathcal{M}}_{0,5}$, $\overline{\mathcal{M}}_{1,2}$



when $\searrow$ is reduction and $\curvearrowleft$ is gluing with $\overline{\mathcal{M}}_{0,3}$

I use --- to connect same dimensional moduli spaces, making of

With the above discussion keep in mind, now $\overline{\mathcal{M}}_{g,n}/\mathbb{Q}$ and $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \curvearrowright \overline{\mathcal{M}}_{g,n}$ naturally, due to some reason (explained next), Grothendieck says it's not enough to consider single moduli stacks (he called "multiplicities") but all of the tower, but let recall the definition of fundamental grp on algebraic "varieties" first / Grothendieck's Galois theory:

We use the Deck transformation grp = fundamental grp, then given X/k, $k \neq \bar{k}$, what's the covering?

Date [?]

• Any field extension $\mathbb{K}/\mathbb{k} \subset \overline{\mathbb{k}}/\mathbb{k}$ gives a base change

$$\begin{matrix} X_{\mathbb{K}} & - & X_{\mathbb{k}} \\ | & & | \\ \mathbb{K} & - & \mathbb{k} \end{matrix}$$

[?] should be closed to covering in topological meaning, hence $\mathbb{K}/\mathbb{k}$ is separable/unramified

• Geometric $\widehat{X} \to X_{\overline{\mathbb{k}}}$

Couple these two versions of covering, which corresponds to $Gal(\overline{\mathbb{k}}/\mathbb{k})$ and $\pi_1(X) = Aut(\widehat{X}^{\text{universal}})$, we expect a combination and this's étale fundamental grp.

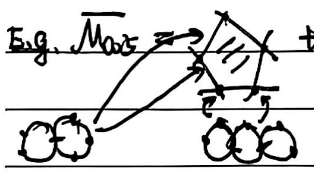
• Some subtle things need to be kept in mind: $Gal(\overline{\mathbb{k}}/\mathbb{k})$ needn't to be finite extension, hence profinite, thus we prefer the profinite completion $\widehat{\pi}_1(X)$

"⇒" $1 \to \widehat{\pi}_1(X) \to \pi_1^{et}(X) \to Gal(\overline{\mathbb{k}}/\mathbb{k}) \to 1$ SES of profinite grps, or equivalently, $\pi_1^{et}(X) = \widehat{\pi}_1(X) \times_{\rho} Gal(\overline{\mathbb{k}}/\mathbb{k})$ the twisted product with rep $\rho: Gal(\overline{\mathbb{k}}/\mathbb{k}) \to Out(\widehat{\pi}_1(X))$

Rk. Due to the process of profinite completion, we can hardly recover $\pi_1(X)$ from $\pi_1^{et}(X)$

Then we explain the reason of using tower totally: $\pi_1(X) = \pi_1^{\infty}(X) := \varinjlim_{K\ \text{cpt}} \pi_1(X-K)$, hence the topological/étale fundamental grps of $\overline{M}_{g,n}$ can be reduced to $\partial \overline{M}_{g,n} = \coprod \overline{M}_{g',n'}$ severed lower (g', n') forming boundary divisors. unioned together via combinational rules.

E.g. $\overline{M}_{0,5}$ two level of degenerations



• In analytic story, such boundary given by length of geodesic $\to 0$

first degeneration $\cong \overline{M}_{0,4} \cong \mathbb{P}^1$

maximal degeneration pt

It's the key conjecture that, the fundamental grp of Teichmüller tower GT, has $\widehat{GT} \cong Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ (injective $Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{GT}$ is shown) (Then determine $\widehat{GT}$ precisely by Dessin — also nontrivial

• Fundamental group of single $M_{g,n}$, $\pi_1(M_{g,n}) = \pi_1(\overline{M}_{g,n}) = \pi_1(\partial M_{g,n})$ is easy to determine, as $T_{g,n} \to M_{g,n} = T_{g,n}/\Gamma_{g,n}$, $\Gamma_{g,n}$ is the modular group / Teichmüller group / mapping class group as although hyperbolic structure $\Leftrightarrow$ cpx structure $\Leftrightarrow$ metric structure $\Leftrightarrow$ symplectic structure $\Leftrightarrow$ algebraic structure, this doesn't means that any structure isotopic to identity one, thus $T_{g,n} \ncong M_{g,n}$, but up to $\Gamma_{g,n}$

It's known that $T_{g,n}$ contractible ($T_g \cong \mathbb{R}^{6g-6}$), hence $\pi_1(M_{g,n}) = \Gamma_{g,n}$! (But $\widehat{\Gamma_{g,n}}$ not so easy to depict completely still)

What's "Anabelian"?

Lastly, we'd like to describe that, Grothendieck expect such a space X, $\pi_1^{et}(X)$ is "sufficiently away from" abelian to dominant the geometry of X" ($\pi_1(X)$ topological?) / $Gal(\overline{\mathbb{k}}/\mathbb{k}) \hookrightarrow \pi_1^{et}(X)$ can be used to study (absolute) Galois grp('s actions)

But what "Anabelian" precisely means? (a class known by fibration

Why we need $\pi_1^{et}(X)$ away from Abelian?

I think he got this point in M_g: $g \geq 2$ are hyperbolic $\Leftrightarrow$ anabelian and $\pi_1(\Sigma_g)$ is $\langle \alpha_1, \beta_1, \cdots \alpha_g, \beta_g \mid [\alpha_i; \beta_j] = \begin{cases} 0; & i \neq j \\ 1; & i = j \end{cases} \rangle$ are becoming away from abelian when $g \nearrow$, but why Σ_g anabelian $\Rightarrow M_g$ also?

Even the basic idea itself stays mysterious!