Rk. $\mathrm{Pic}(\circledcirc) = \mathbb{Z} \times \circledcirc = \text{degree} \times \mathrm{Pic}^0(\circledcirc)$ the excision of Chow ($A^0(\circledcirc) = \mathrm{Pic}(\circledcirc)$)

$$\begin{array}{ccccccccc} 0 \dashrightarrow & A_0(pt) & \to & A_0(\circledcirc) & \to & A_0(\circledcirc - p) & \to & 0 \\ & \| & & \| & & \| & & \\ & \mathbb{Z} & \to & \mathbb{Z} \times \circledcirc & \to & \boxed{\circledcirc} & \to & 0 \end{array}$$

Here injective, split

q ↦ $\mathcal{O}(q-p)$

deg

fix pt ⟺ minus pt ⟺ choose base pt ⟺ $\mathrm{Pic}^{\mathrm{deg}}(\circledcirc) = \circledcirc$ all fixed ⟺ ∃ section in upper SES by deg

digest on some geometry and arithmetic of curves

We'll consider C cpt Riemann surface, for $2g-2+n=0$ and $2g-2+n>0$ cases, i.e. toric and hyperbolic cases, we study the Abel-Jacobi theory and hyperbolic geometry, respectively. And see how these theories are well-generalized to other cases.

PART I Around Abel-Jacobi Theory

$H^0(\mathcal{C}, \mathbb{Z}) \cong \mathbb{Z}$, $H^1(\mathcal{C}, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, $H^2(C, \mathbb{Z}) \cong \mathbb{Z}$

$H^0(\mathcal{C}, \omega_C) \cong H^1(\mathcal{C}, \mathcal{O}_C)^\vee$ Serre duality

$\cong \mathbb{C}^g$ defining the geometric genus = algebraic genus

We can integral $\omega_1 \ldots \omega_g \in H^0(\mathcal{C}, \omega_C)$

g holomorphic 1-forms ($(1,0)$-forms)

over curves (due to holomorphic, it's 1-form instead 2-form, i.e. choice of cpx structure ⇒ give direction dz)

$\Rightarrow u: C \to J(C) \cong \mathbb{C}^g/\sim$

$p \mapsto (\int_{p_0}^{p} \omega_1 \ldots \int_{p_0}^{p} \omega_g)$ p_0 base pt chosen.

the module relation ensures it's well-defined ~~by modulo~~ (⟺ not depend on the path)

⇒ modulo all loops ⟺ $J(C) \cong H^1(\mathcal{C}, \mathcal{O}_C)/H^1(C, \mathbb{Z})$

$\Leftrightarrow \mathbb{C}^g/\Lambda_\gamma$, Λ_γ is lattice $(\int_\gamma \omega_1 \ldots \int_\gamma \omega_g)$, $\gamma \in H^1(\mathcal{C}, \mathbb{Z})$

Rk. ① $J(C)$ is Abelian variety, converse? integral class

② (Torelli) $J(C_1) \cong J(C_2) \Rightarrow C_1 \cong C_2$ (although topological is obviously same, here is the cpx structure)

③ (Abel-Jacobi thm) Jacobian ≅ Albanese

Albanese for X smooth proj is $\mathrm{Alb}(X)$ Abelian and it's universal: $\forall A$, $X \to A$, $\exists!$ $X \to \mathrm{Alb}(X) \to A$ (minimal)

④ (Period) Give $\mathcal{C}_g$ a ~~family~~ deformation family of cpx structure

then $\boxed{?}$ $\downarrow S$ $s \mapsto$

Here we not concern more Hodge theory and $\boxed{?}$ part is mystery. we call it period domain to parametrize deformation of Hodge structures, which is easier than Hodge structure itself: it can be Grassmannian / Flag in (or modulo some grp actions)

- Griffith's transversality tells us $\tilde{u}$ is holomorphic;
- We call $(\int_{\gamma_i} \omega_j)$ period matrix, (when $H(X,\mathbb{Z})$ has torsion, γ_i is torsion-free part)

named by period / integral on tori / elliptic curves

Rk. (Wilson loop operator & Hecke operator /C)

Consider $\pi_1(C)$, $\pi_1(C)^{ab} \cong H_1(C,\mathbb{Z}) \cong H^1(C,\mathbb{Z})^\vee \cong \pi_1(\mathrm{Jac}(C))^\vee$ canonically, as $\mathrm{Jac}(C)$ is torus (Geometric class field theory)

$\Rightarrow \mathrm{Loc}_G(C) \cong \mathrm{Loc}_G(\mathrm{Jac}(C))$ if G Abelian: $\pi_1(C) \to G$ for G abelian factor through $\pi_1(C)^{ab}$ by universal property

The Abel-Jacobi map then becomes

$\pi^*: \mathrm{Loc}_G(\mathrm{Jac}(C)) \to \mathrm{Loc}_G(C)$ isomorphism

$(\pi_1(\mathrm{Jac}(C)) \to G) \mapsto (\pi_1(C) \to \pi_1(\mathrm{Jac}(C)) \to G)$

$\pi_1(C)^{ab} \cong \pi_1(\mathrm{Jac}(C))$, determined by u original AJ map

i.e. induced by the non-canonical dual given by u

Hecke operator $H_x: \mathrm{Pic}(C) \to \mathrm{Pic}(C)$

$$L \mapsto L(x) = L \otimes \mathcal{O}(x)$$

$x \in C$, $\mathcal{O}(x)$ defined via viewing x as a divisor.

an Hecke eigenfunction $f \in \Gamma(\mathrm{Pic}(C), \boxed{/\!/})$ → some coefficient

eigenvalue $\in \boxed{/\!/}$

is $f(H_x(L)) = \Box f(L)$ (Write as $H_x^* f = f$, the it doesn't "eigenvalue")

The Wilson loop operator $\rho: \pi_1(C) \to \boxed{/\!/}$

$\Rightarrow \rho \in \mathrm{Loc}_{\boxed{/\!/}}(C)$ $\qquad$ $\mathrm{Frob}_x \mapsto$ eigenvalue of H_x

For example $\boxed{/\!/} = \mathbb{G}_m = \mathbb{C}^\times$ in QFT and balabala..

We can rewrite this into (tensor) product and pullback (like what we did for Abelian Vars, their duals and Poincaré bundles) or more abstract nonsense (?), into Tannakian formalism in categorical level. (omitted?)

Structure of $\mathrm{Pic}(C) \Rightarrow C$ only see initial component of $\mathrm{Pic}(C)$!

No prefered base point is given for $\mathrm{Pic}(C) - \mathrm{Pic}^0(C)$, hence those $\mathrm{Pic}^d(C)$ are principal $\mathrm{Pic}^0(C)$-homogenous spaces, once a base pt of C given, say p_0 as we did in defining $\Rightarrow \mathcal{O}(p_0)^{\otimes d} \in \mathrm{Pic}^d(C)$ is a chosen base pt and we have non-canonical iso $\mathrm{Pic}^0(C) \cong \mathrm{Pic}^d(C)$.

$\mathrm{Pic}^0(C) \cong \mathrm{Jac}(C)$, as the Λ lattice determined by degrees.

Then the extension of u to $\tilde{u}$ in all higher level > 0 is

$\tilde{u}: \mathrm{Sym}^\infty(C) \to \mathrm{Pic}(C)$, each degree d $u_d: \mathrm{Sym}^d(C) \to \mathrm{Pic}^d(C)$

$$\Rightarrow u_0: C \to \mathrm{Pic}^0(C) = \mathrm{Jac}(C)$$

Rk. (Stable homotopy theory) We have Dold-Thom thm:

$\pi_n(\mathrm{Sym}^\infty(X)) \cong H_n(X)$, one can understand it as a higher dim generalization of $\pi_1(X)^{ab} \cong H_1(X)$, as Sym^∞ on spaces is some "Abelianization" in analogue as Abelianization of groups.

In stable homotopy theory, Sym^∞ is E_∞-space

$H_n(X) \cong \pi_n(\mathbb{Z} \wedge \Sigma^\infty X) \cong \pi_n(\mathbb{Z}[X]) \cong \pi_n(\mathrm{Sym}^\infty(X))$

$\mathrm{Top} \xrightarrow{\Sigma^\infty} \mathrm{Sp} \dashrightarrow \mathrm{Mod}(\mathbb{Z})$

$\mathrm{Sp} \cong \mathrm{Mod}(\mathbb{S}) \xrightarrow{\mathbb{Z} \wedge -} \mathrm{Mod}(\mathbb{Z})$

$\mathbb{Z}$ as E_∞-ring spectrum $/\mathbb{S}$

$\boxed{\text{$\mathbb{Z}$-module spectrum on X}}$

Both $\mathbb{Z} \wedge \Sigma^\infty(X)$ (usual singular homology) and $\mathrm{Sym}^\infty(X)$ are concrete model of $\mathbb{Z}[X]$

$\chi_{L_1\otimes L_2} = \chi_{L_1} \otimes \chi_{L_2}$

Thm (GCFT, unramified)

$\boxed{Loc_1(Pic) \cong Loc_1(C)}$ $Loc_1(C) := Loc_{\mathbb{G}_m}(C)$ character pro

$\{(\chi, \varphi) \in Loc_1(Pic) \mid \varphi: \mu^*\chi \to \chi \boxtimes \chi$ is isomorphism$\}$

μ is multiplication map of Abelian variety $Pic(C)$

Pf (Due to Deligne)

We have $L: Loc_1(Pic(C)) \to Loc_1(C)$ the Abel-Jacobi map

$\chi \mapsto u^*\chi$

$Pic \times Pic \xrightarrow{\mu} Pic$ $(\mu\circ(u\times id))^*\chi = (u\times id)^*\mu^*\chi$

$\uparrow u\times id$ Hecke operator $= (u\times id)^*(\chi\boxtimes\chi)$ character property

$C\times Pic \xrightarrow{\mu\circ(u\times id)}$ $= u^*\chi \boxtimes \chi = \underline{L(\chi)} \boxtimes \chi$

eigenvalue Hecke eigen property

~~Denote~~

Here we can see the Abel-Jacobi pullback is generalization of Wilson loop operator

We construct the inverse, given $L \in Loc_1(C)$, we define $\chi_L \in Loc_1(Pic)$

Our startgy:

① Define $\chi_L^d \in Loc_1(Pic^d(C))$ for d is large;

② Extend to lower degree by multiplicity / character property.

Namely, ①: $Sym^d \to Pic^d$ has fibres of D is the linear system $|D|$ of $\dim = d-(2g-2)$ is linear $\Rightarrow$ simply connected for $d > 2g-2$ $\Rightarrow$ we descend map from Sym^d to map from Pic^d;

Let $L^d: Sym^d \to \mathbb{G}_m$ $\in Loc_1(Sym^d)$

$\sum x_i \mapsto \otimes L(x_i)$ rank 1

$\Rightarrow$ descend to $\chi_L^d \in Loc_1(Pic^d)$

② : any divisor is difference of two large degree divisor

$d = d_1 - d_2$, $D = D_1 - D_2$, $d_1, d_2 > 2g-2$

$\Rightarrow$ set $\chi_L^d(D) = \chi_L^{d_1}(D_1) \otimes \chi_L^{d_2}(D_2)^{-1}$, done

$\Rightarrow$ $\chi_L \in Loc_1(Pic(C))$ is desired inverse $\square$

Rk. ① We check nothing, as it's an equivalence of groupoid / stack, hence it's meaningless without defining them;

② Ramified case we consider line bundles trivialized near S;

③ Such fibration of linear system over Pic is also used in the proof of Weil conjecture in curve case.

PART II Hyperbolic curves and modular forms

For something interesting, I shall write something above the words "modular"; a modular form is holomorphic function on a hyperbolic locally symmetric space

satisfy growth condition

$SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}) / SO(2)$ maximal cpt Lie gp

$\cong SL_2(\mathbb{Z}) \backslash \mathbb{H}^2$; $\mathbb{H}^2$ the upper plane model of 2-dim hyperbolic space (explain next)

$=: \Gamma \backslash \mathbb{H}^2$, the fundamental domain

modular group

But why "modular" and why "form"?

- modular comes from the modular group Γ, named as it's a quotient $\Gamma' \backslash \mathbb{H}^2$, forming moduli space of elliptic curves, $\Gamma' \leq \Gamma$;
- old times, functions are called form, esp. cpx analysis; they're historical reason $X(\Gamma(N)) \Leftarrow$ N-torsion on elliptic curves
- Mordenly, modular form $\in \Gamma(\ , \omega^{\otimes \frac{k}{2}})$ a form

I is first studied in Erlangen program, they noticed it contains ample symmetry datum.

Now we'll show these hyperbolic geometry used:

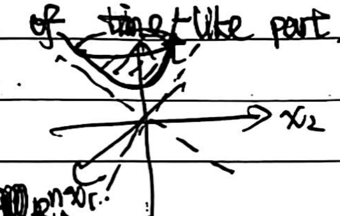
- Models of $\mathbb{H}^n$
- Geodesics of $\mathbb{H}^n$
- Hyperbolic group action and all hyperbolic mfds comes from quotient

* Some interesting proofs about hyperbolic curves (real dim 2) / cover

Finally we'll see why hyperbolic world indicates complicated symmetry

- Three models for $\mathbb{H}^n$:

① Lorentz mfd has space-like, light-like, time-like parts.
we take a component of time-like part, i.e.
$\{x \in \mathbb{R}^n_x \times \mathbb{R}^n_t \mid \|x\|^2 = -1, t > 0\}$



② Poincare disk model $\mathbb{D}^n$

③ Upper half plane model $\mathbb{R}^n_+$

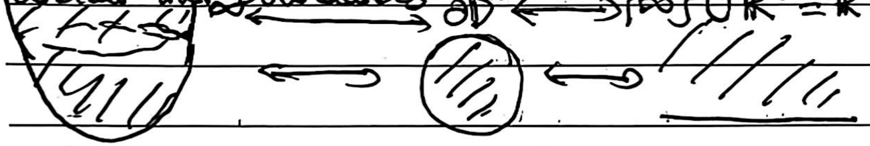
$\Longrightarrow$ ① gives the metric structure to $\mathbb{H}^n$, then we can write

- diffeomorphisms between ①②③, the metric on ②③ then induced

(Another model is by mapping to $\mathbb{RP}^n$, I don't think this model can give any benifit to deal something)

Their isometry is a simple excrise of complex analysis in $n=2$ (conformal mapping), and we can imagine it easily:

Consider their boundaries: $\longleftrightarrow \partial\mathbb{D}^n \longleftrightarrow \{\infty\} \cup \mathbb{R}^{n-1} = \overline{\mathbb{R}^{n-1}}$



and those mapping are easy to contact.

Rk. We have isomorphism of groups: $\mathrm{Iso}(\mathbb{H}^n) \cong \mathrm{Conf}(\partial\mathbb{H}^n)$

the isometry of hyperbolic space $\longleftrightarrow$ conformal map of boundary.

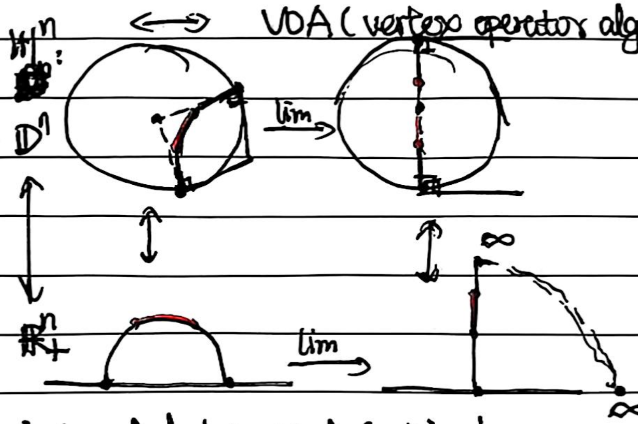
This correspondence is deep and interesting, having further generalizations, which might be big projects: (holography principle in physics)

AdS $\longleftrightarrow$ CFT

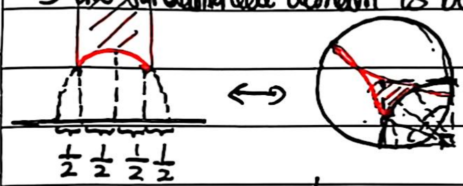
moduler forms $\longleftrightarrow$ VOA (vertex operator algebra)

- We look at geodesic of $\mathbb{H}^n$:

Usually, the model ① isn't easy one to draw pictures



$\Longrightarrow$ the fundamental domain is bounded by geodesic triangle:



A hyperbolic isometry ϕ of $\mathbb{H}^n$ is an isometry with no fix pt $\in \mathbb{H}^n$, 2 fixed pts $\in \partial\mathbb{H}^n \Longleftrightarrow$ determined by a complete geodesic (which is ϕ-invariant)

For hyperbolic mfd than $\mathbb{H}^n$ itself, we define it by gluing $\mathbb{H}^n$ via translation functions respect the hyperbolic metric.

But by its curvature $= -1$ and space forms in Riemannian geometry, all orientible simply connected hyperbolic mfd are quotient of $\mathbb{H}^n$ by discrete group / covering space.

• Consider real-dimension 2 case now.

We ~~[illegible]~~ roughly give the idea of proof of these very easy statements via a very interesting way

① $g \geq 2 \Leftrightarrow$ hyperbolic

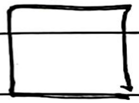
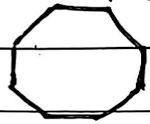
② ~~[illegible]~~

The Teichmüller space T_g has real dim $6(g-1)$

, puntrued n pts has real dim $6(g-1)+2n$

(Then what's its cpx/algebraic structure?)

Pf ① C_g is quotient of a polygon, $4g$ edges, minus a pt

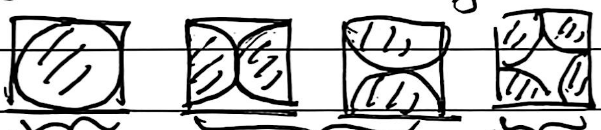
$g=1$  $g \geq 2$  ...

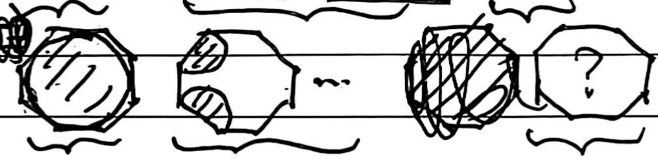
We find covering for them

(i) $g=1$ covered by $\mathbb{R}^n \supset U$;

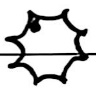
(ii) $g \geq 2$ can't be covered by $U \subset \mathbb{R}^n$;

(iii) $g \geq 2$ can be covered by $U \subset \mathbb{H}^n$.

(i)  all balls

(ii) 

it's unable to define a chart at angles as sum of all iner angles $\geq 4\pi \geq 2\pi$! $\Rightarrow$ unable to cover the vertex

iii) Replace  by , sum of inner ~~[illegible]~~ angles $\searrow$ ▨

② We have fibre sequence

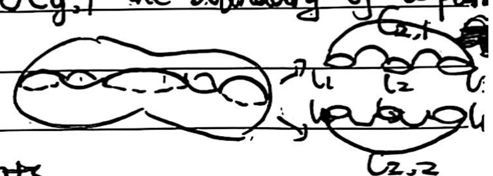
$$\begin{array}{ccc} \mathbb{R}^{3g-3} \hookrightarrow & T_g & \text{hyperbolic metric} \\ \text{metric} & \downarrow P & \downarrow \\ \downarrow P & \mathbb{R}^{3g-3}_{>0} & (3g-3) \text{ geodesics} \\ & (l(C_1), \dots, l(C_{3g-3})) & \end{array}$$

under this metric; we use these geodesic to cut C_g into pair of pants.

• Decomposition to pair of pants by $(3g-3)$ curves are easy to done in topology, given metric, we perturb them into geodesics

• Each fibre of $(l_1 \cdots l_{3g-3})$ is what?

Fact (admitted): each l_1, l_2, l_3, $l_1 \cup l_2 \cup l_3 = \partial C_{g,1}$ the boundary of a pant, their length determine the hyperbolic metric on the pant.



$\Rightarrow$

The fibre of $P \Leftrightarrow$ gluing of pants

$\Leftrightarrow$ Dehn twist at each C_i, $i=1,\dots,3g-3$ by rotating $\theta \in \mathbb{R}$

$\cong \mathbb{R}^{3g-3}$

• When puntured n pts, adding n loops

$\Rightarrow \mathbb{R}^{3g-3+n} \hookrightarrow T_g \twoheadrightarrow \mathbb{R}^{3g-3+n}_{>0}$ ▨

Rk. Thus we have $0 \to Map(C_g) \to T_g \to M_g \to 0$ with kernel is mapping class group of topological C_g, as although algebraic iso $\Leftrightarrow$ isometry, they needn't isotopical to identity.

As T_g simply connected $\Rightarrow$ universal cover $\Rightarrow \pi_1(M_g) = Map(C_g)$ ▨