• This series on QFT is base on the lecture in physic department, adding some mathematical comments and view (LN: lecture notes) LN

Lecture note on QFT I: introduction

QFT = QM + Relativistic, hence we recall the "relativistic QM" first: Schrödinger equation   Klein–Gordon equation

(first) quantization $E = \frac{p^2}{2m}$ → $\hat{E} = \frac{\hat{p}^2}{2m}$ (first) quantization $E^2 = p^2c^2 + m^2c^4$ → $\hat{E}^2 = \hat{p}^2c^2 + m^2c^4$

+ Lorenzian invariant

The problems of KG equations are two: $\boxed{\text{negative mass / energy}}$ (Dirac sea) & $\boxed{\text{negative possibility}}$ (Spin)

it can be resolved by some explainations, but here we have a key idea: (similiar to the philosophy of Yoneda Lemma)

If a particle / wave function can be determined by its observables, why can't we $\boxed{\text{consider the observable / operator instead of particle itself}}$?

In physical viewpoint, the mass changes induces the create & vanish of particles, thus the "background" of these particles live in is important.

Rk. For the "background" is 0-dim space & 1-dim time, it's just QM of one particle.

QM	QFT
$\hat{x}$	$\hat{\phi}(\vec{x})$
$\psi(x,t)$	$\psi(\phi(\vec{x}),t)$
$\hat{x}(t)$	$\hat{\phi}(\vec{x},t)$
$\psi(x(t))$	$\psi(\phi(\vec{x},t))$

Schrödinger picture & $\boxed{\text{Heisenberg picture}}$ better in QFT

日月光华 旦复旦兮

Q. Such equivalence can be seen in harmonic osscilator in [0, L]:

We have two views

• $x(t)$ has motion (QM) (0-dim)

• $\psi(x)$ describe osscilate on [0, L] (1-dim QFT)

Viewing higher energy as serval lower energy together: number / energy

Instead: particle from a to b

Another motivation (more physical): consider the many body system / lattice model in QM, the QFT can be viewed as the "continuous limit" of it:

$a^\dagger$ adds 1 particle, a minus 1 particle

lattice

$a \to 0$ (theory of RG flow)

Hence we can view QFT as a infinite many body system, interactions are given by relativisity on base space-time

TQFT means that we have simply no interactions.

Rk. It also explain the occurance of second quantization / "field" operator in many body system.

Rk. "Topological" in physic means that: no intristic interaction / no dynamics / trivial Lagrangian & Hamiltonian on time / ground state which is treated as a simplification of computation in physics. (the path integral formalism)

no special time-dependence operators

Three options:

• $H = 0$

• $V|_{\ker H}$ = all harmonic function (Liouville: all constant ⇒ trivial)

• Using supersymmetry

Those bundles without additional structure but just local, for interaction nontrivial, those connections but similiar allowed as additional structure under differential structure.

excited states always must have dynamics due it's not stable;

single one   interacted

reduce to $H \sim 0$ via chain-homotopy of de Rham theory

this's Witten's TQFT's idea, used most in math.

$\boxed{\text{But } \phi(x) \text{ have interaction at } x}$

QM is 1-particle (0-dim in space) effective QFT: $|x\rangle \Leftrightarrow \phi(x)|0\rangle$ (QM) (QFT)

Fock space

(do produce true particles than virtual ones: negativity)

$\mathcal{H} = \bigoplus \mathcal{H}_n$, $\mathcal{H}_n = \text{span}\{a_p \psi\}$, all $(p_1 \cdots p_n)$, $\psi \in \mathcal{H}_1$

$\mathcal{H}_0 = |0\rangle \cdot \mathbb{R}$ 1-dim, all $\mathcal{H}_n = \mathcal{H}_1^{\otimes n}$, each operator $a_p: \mathcal{H}_n \to \mathcal{H}_{n+1}$, $a_p: \mathcal{H}_1 \to \mathcal{H}_0$

vacuum state

$\mathcal{H}_1 = \{a_p|0\rangle \mid \text{all } p\}$, $p \Leftrightarrow$ particle only one

Instead change energy by ladder, now we change number

日月光华 旦复旦兮

掃描全能王 創建

Something interesting: The locality of fields makes one of the algebraic nature of QFT to be something able to be $\boxed{\text{sheafified}}$ (NOT rigorous!)

No.

Date

LN

Lectures on QFT II: algebraic taste

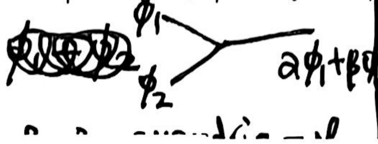
In this part I'd like to show the basic computation taste in (relativistic) QFT, ~~the~~ which creates different interesting phenomena in different cases: renormalization, BRST, anomalies..., based on correlation functions and Feynman rules.

Rk. In mathematical view-point, the QFT is axiomized by Weightman axioms and Weightman axioms $\Leftrightarrow$ Weightman functions (But without gauge symmetry!) i.e. correlation functions

Those different settings include different spin, different symmetry and free or interacted ... in physics.

Rk. Renormalization and gauge theory is planned to be extended next two lectures, but gauge theory need to be cited: those mathematicians study the principal bundles is something fake QFT but real classical ones, thus in the QFT series we'll focus on BRST, anomalies, symmetry broken and so on. This is due to historical reason: when physicists persued to unify QM and GR, Weyl first had his idea of gauge, where before QFT created. Gauge isn't symmetry in QFT, is a local symmetry in classical FT, but quantization breaks it, no conserved quantity in quantum theory, both local and global such as charge is global in classical, but not in quantum.

$\boxed{\text{Part I}}$ From Free fields ~~and~~ to interacted theory via scattering

Free fields	Fields with interaction
Linear $(\Box+m^2)\phi=0$	Potential term $(\Box+m^2)\phi=f$
solvable ϕ	generally unsolvable
	
	higher terms $=\mathcal{L}_{free}+\mathcal{L}_{int}$

掃描全能王 創建

Recall. Correlation function in QM (Heisenberg picture)

$\langle O_1 \cdots O_n \rangle := \langle \psi | U_{T-t_n} O_n U_{t_n - t_{n-1}} \cdots U_{t_2-t_1} O_1 U_{t_1} |\psi\rangle$ for time involution $[0,T]$, $U = e^{-iTH/\hbar}$ and observables $O_n \cdots O_1$

Acquiring some $\boxed{\text{measurement}}$, here for QFT, it's

Free field theory: measurement at different pts

The correlation function of two pts is

$D(x_1,x_2) = \langle 0|\psi(y)\psi(x)|0\rangle = \int \frac{dp}{(2\pi)^3}\frac{1}{2\omega_p} e^{-ip(x-y)}$

(expand $\psi(x)$ by a_p^+ and a_p^- by momentum)

We call the Green function as

$G(x_1-x_2) = D(x_1,x_2) \Rightarrow G(x) = \int \frac{dp}{(2\pi)^3}\frac{1}{2\omega_p} e^{-ipx}$ is solvable

For more pts, we reduce to 2 pts case by Wick's Thm:

(statement is about a^+, a^- operators, omitted but essentially it's about combination)

The n-pt correlation

$\langle 0|T\{\psi(x_1)\cdots\psi(x_n)\}|0\rangle$ is expressed as

(time-order product: $T\{\psi(x_1)\psi(x_2)\} = \begin{cases}\psi(x_1)\psi(x_2) \\ \psi(x_2)\psi(x_1)\end{cases}$ if $t(x_1) \geq t(x_2)$, $t(x_1) \leq t(x_2)$, t is time axis

which is a result of $\boxed{\text{causality}}$) contribute $a^-|a_p^+|0\rangle = 0$

$= \sum$ all contractions $\Rightarrow$ all odd are normal order $= 0$, even yet.

Rk ① It can be expressed as Moyal product, I know nothing

② $G(x)$ is Lorentz invariant, it acquires to check; then we can consider:

- space-like $\Rightarrow D(x-y) = D(y-x)$
- time-like $\Leftrightarrow x_t = y_t \Rightarrow$ the parity of Lorentz invariant gives D symmetry
- light-like (*hard)

$\Rightarrow [\psi(x), \psi(y)] = D(x-y) - D(y-x) \neq 0$

Interaction theory:

With potential, first thing is x_1 not symmetric with x_2.

Recall what we did in QM when wave function can't be solved: scattering perturbation, ...

On shell condition $\Leftrightarrow$ satisfy EOM $\Leftrightarrow$ Variation of Lagrangian works, Noether's thm holds

the mass $\Leftrightarrow$ $\boxed{\text{real}}$ particles

off shell $\Leftrightarrow \cdots \Leftrightarrow$ $\boxed{\text{virtual}}$ particles mass

It's a trick occurs in perturbation problem, e.g. the x,y (than 1,2)

For clarity, we denote vacuum state $|\Omega\rangle$ here

It conserves momentum and energy, massive but mass can changes $\Rightarrow$ not EOM

We show that $\boxed{\text{S-matrix} = \text{propagator}}$ by LSZ reduction

$\boxed{(\text{Functional formalism}) \quad (\text{Path integral formalism})}$

S-matrix is

$S = (S_{\alpha\beta}) = (\langle f_\beta | i_\alpha \rangle)$ for scattering problem:

its scattering (unitary) amplitudes $\langle f|S|i\rangle$

is a product of integral of initial final propagators, by Lehmann-Symanzik-Zimmermann: $|p_1 \cdots p_i\rangle$ $|f\rangle = |p_{i+1} \cdots\rangle$

$= \prod_{1\leq j \leq i} \left[i \int dx_j\, e^{-ip_j x_j}_{\text{in}(-1)} (\square_j + m^2)\right] \cdot \left[\prod_{i+1 \leq j \leq n} i \int dx_j\, e^{ip_j x_j}_{\text{out}(+1)} (\square_j + m^2)\right]$ $\boxed{\text{on-shell}}$

$\cdot \langle \Omega | T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle$

where the propagators are $\langle \Omega | T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle$

Pf: $\boxed{?}$ Assume it's true! (Schwinger-Dyson)

at the left vertex $\neq 0$ as $(D_{12}D_{34} + D_{14}D_{23} + D_{13}D_{24})$

Now we compute the $(\square_1 + m^2)\langle \Omega | T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle$, then:

- When $n=4$, after easy computation we have same result as Wick's thm: $\langle \Omega | T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle = D_{12}D_{34} + D_{14}D_{23} + D_{13}D_{24}$
- When $n \geq 3$, all these interaction theory obey Wick's thm too! Thus we reduce to $n=2$

It means

Causality $\Leftrightarrow$ Commutation

GR QM

(when time-like, existence of causality makes noncommutativity

$\boxed{\text{Connes's time} \Leftrightarrow \text{non commutativity}}$

- For $n=2$, we reduce to free case by Feynman rule / under perturbation

The first two are almost trivial computation:

$(\square_1 + m^2)\langle \Omega | T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle = \langle \Omega | T\{(\square_1 + m^2)\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle$

$- i\hbar \sum_{j \neq 1} \delta(x_1 - x_j) \langle \Omega | T\{\phi(x_2)\cdots\widehat{\phi(x_j)}\cdots\phi(x_n)\}|\Omega\rangle$

reduct on $n \geq 3$ $\boxed{4}$

No.

~~But~~ third assertion needs some careful computation on Green's function.

Part II <u>Path integral formalism and Feynman's rule</u>

$\langle \Omega | T\{\phi(x_1)\phi(x_2)\} | \Omega \rangle = D_{12} - g^2 \iint \left(\frac{1}{2} D_{1x} D_{xy}^2 D_{y2} + \frac{1}{4} D_{12} D_{xx} D_{yy} D_{y2} + \frac{1}{2} D_{1x} D_{2x} D_{xy} D_{yy} \right) dx\,dy \; + O(g^3)$

if the $\mathcal{L}_{int} = \frac{g}{3!}\phi^3 + O(g)$

(called ϕ^3-theory) in perturbation theory (*)

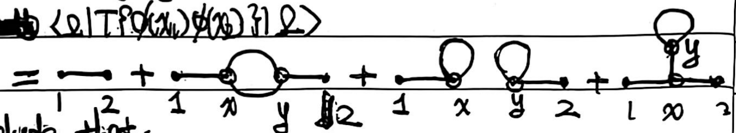
Rk ① The computation for $\phi^4 \cdots$ is similiar, but these computation are long integrals, omitted;

② Thus we reduce all propagator into free theories' Green function, theortically all S-amplitudes can be computed under a perturbation expansion;

③ Here we use more Lagrangian and propagators, indeed we had work with path integral formalism.

Omitted computations but look at the result, we rewrite (*) as graphical language: (Wick's perturbation)

$\langle \Omega | T\{\phi(x_1)\phi(x_2)\} | \Omega \rangle$



We conclude that:

① It equals to the free propagator term D_{12}, adding all possible diagrams, s.t.

② In perturbation order g^k, k additional $x_i \in$ space-time added to be integral, served as vertices;

③ Each vertex new contributes (ig) multiplied;

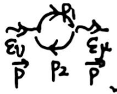
④ Each loop contribute $\frac{1}{2}$;

⑤ Each vertex new have l edges connect, when $\mathcal{L}_{int} = \frac{g}{l!}\phi^l$ (a loop to the vertex itself counted as 2 edges)

日月光华 旦复旦兮

The 1&2 $\Leftrightarrow$ in & out states in scattering
$\Leftrightarrow$ boundary of base mfd

E.g ② (Loop case)



has propagator $\in \mathbb{C}$ is a number

Key is (☆) term ~~of internal loop~~

$\underbrace{-}_{\substack{\text{Fermion}\\ \text{loop}(-1)}} \underbrace{(ie)^2 \int \frac{dp_1}{(2\pi)^4}\frac{dp_2}{(2\pi)^4}(2\pi)^4}_{\text{usual terms}} \underbrace{\delta(p - p_1 + p_2)}_{\substack{p = p_1 - p_2\\ \text{needed}}} \underbrace{\varepsilon_\mu(p)\varepsilon_\nu(p)}_{\substack{\text{external line}\\ \text{and , inner}\\ \text{product of 4-vectors}}} \boxed{☆}$

$\boxed{\mathrm{tr}\left(\gamma^\mu \frac{\not p_1 + m}{p_1^2 - m^2 + i\varepsilon} \gamma^\nu \frac{\not p_2 + m}{p_2^2 - m^2 + i\varepsilon}\right)}$ ① $\to$ trace of (4×4) matrix

Rk. It means that: higher order interactions comes from more edges ①, showing more interactions & higher order perturbation comes from more vertices, showing finer integrals on space-time to appromaxiate

from spinor sum $\sum u\bar{u}$ ② cancell the spinor indices

Our path integral formalism in relativistic case has propagator $K(x|y) = \langle \Omega | T\{\phi(x)\phi(y)\} | \Omega \rangle$, instead the time axis sepreately given $(K(x, t_1 | y, t_2))$. Why we have such form as a generalizati

A. In QM, the physical meaning characterize the propagator in non-relativistic case: $(i\hbar \frac{\partial}{\partial t} - H) K = \delta(x-y)\delta(t\cdots$ satisfying EOM up to a Dirac-δ; similiarly here $(\Box + m^2) G = \delta(x-y)$ $(-i\hbar\langle \Omega | \phi(x-y) | \Omega \rangle)$ also is Dirac-δ

(Many relativistic propagators/Green functions defined, $G_F, G_A, G_R \cdots$, here we not go deep on this)

Then the path integral is same as QM, with propagator changed, who write as $\int e^{iS[\phi]} d\phi$ can see it's a change of the Lagrangian above the expontential

Part III <u>Higher spin</u> as partition function (Write correlation/propagator in function)

All discussion above is about Klein-Gordon equation $(\Box + m^2)\phi = f$

as we known in QM, the classification of Boson and Fermion (Cluster decomposition) need to be proved in QFT, thus it's a must to

(The equivalence of $\begin{cases} \text{spin } \frac{1}{2} \Leftrightarrow \text{anti-symmetric} \\ \text{spin } \Box \Leftrightarrow \text{symmetric} \end{cases}$ can't be proven in QM!)

consider Dirac equations and $\cdots$

Here we call spin 0, the ϕ one component (of operator value!) is scalar field, the spin $\frac{1}{2}$, decomposed $\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$ two components

扫描全能王 創建

Rk. Here our ladder operator $a^\dagger$ is a notation $(a)^\dagger \neq a^+$ not conjugate as C-invariance not holds in general

C-invariant $\Leftrightarrow$ particles = anti-particles (thus higher spin)
$\Leftrightarrow$ real-valued particles $\Leftrightarrow$ non-charged

Date

the spinor field and more components vector fields, physists prefer spin-$\frac{1}{2}$ case, the quantum electrodynami[cs]

New "particles" are represented by various arrows in Feynmann diagram next:

The pf is via cluster decomposition: Fermion and Boson are describle by (anti-) symmetric relation on many particle states, taking the advantage of Fock space in QFT, we can generaliz[e] our computation to those particle by acting a and $a^\dagger$ properly and replace the spin 0 state ψ by some $a_{\square}\psi a^\dagger_{\square}$ [...]

Conclusion: Different spin $\nearrow$ Different $a, a^\dagger$ with $4s+2$ compon[ents]
$\searrow$ Different components

Take different $a, a^\dagger$ into propagators. all others not change in each component then write it into a vector.

For Fermion we write line as before, For Boson we write as dashed lines, e.g.  ;
(Fermion is reel, Boson is massless)

More notations of lines are used in different settings. e.g the photon is ;

E.g. ① Photon propagator of internal
Free

$$\langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle = \int \frac{dp}{(2\pi)^4}\underbrace{\frac{-i}{p^2+i\varepsilon}\left(g_{\mu\nu}+(\xi-1)\frac{p_\mu p_\nu}{p^2}\right)}\, e^{-ip(x-y)}$$

μ and ν are two components 4×4-matrix

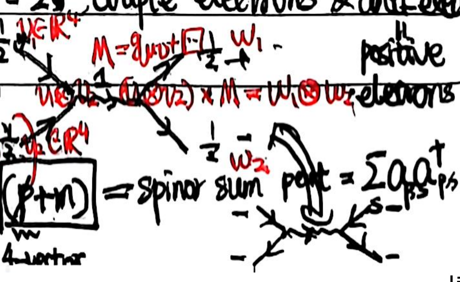
as QED is spin-$\frac{1}{2}$ $(2\times\frac{1}{2}+1=2)$, couple electrons & anti-elec[trons]
(although photon is spin-1)

Electrons (internal) are , e.g.

(essentially 2, $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

contribute a 4-vector $\frac{i}{p^2-m^2+i\varepsilon}\boxed{(\not p+m)}$ = spinor sum part $=\sum a_{ps}a^\dagger_{ps}$

we need sum over all $-2\le s\le 2$ scalar part.   4-vector



日月光华 旦复旦兮
FUDAN UNIVERSITY 1905

掃描全能王 創建

[N]

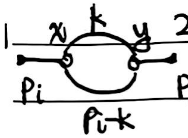
Lecture on QFT II': continue

Fourier $x \longleftrightarrow p$ transform

$\boxed{\text{PART I}}$ Momentum Feynmann Rule

Last time we have the Feymann rule of integral over position, but $\langle i|S|f\rangle$ is all momentums, using momentum Feymann rule is more useful, both physics and mathematics, as we'll show next; I only give result, the integral transform is easy.

Recall that, virtual particle satisfy the conservation of momentum



We assume the momentum is k on upper ~~lower~~ edge. $\Rightarrow$ the momentum on lower edge is $p_i - k$

(then after involution, momentum also conserved at y)

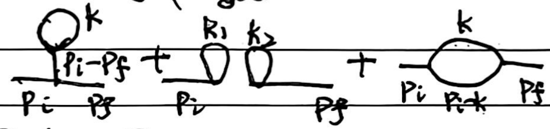
$\frac{1}{2}(ig)^2 \iint D_{1x} D_{xy}^2 D_{y2}\, dx\, dy$

$\xrightarrow[x \mapsto p]{\text{integral transform}} \frac{1}{2}(ig)^2 \int \frac{i}{k^2+m^2+i\varepsilon} \frac{i}{(p_i-k)^2+m^2+i\varepsilon} \frac{dk}{(2\pi)^4}$

thus it's easy to conclude momentum Feymann rule:

- Each internal edge of momentum k contribute $\frac{i}{k^2+m^2+i\varepsilon}$ in the integral
- Every loop $\Longleftrightarrow$ Every free momentum $\Longleftrightarrow$ Every integral (thus if the diagram is tree-level, no integral used and this is 0-order term) $\Longleftrightarrow$ contribute $\frac{1}{2}$
- Each new vertex contribute (ig)

Eg.



Rk. ① Any Feymann Rule is expansion ordered by g^N, (or $g^N/N!$), N is the number of virtual particles, thus we call it Lagre N expansion: We approximate interaction of fields (infinite freedom) by interaction of virtual particles (finite freedom N);

日月光华 旦复旦兮

Date

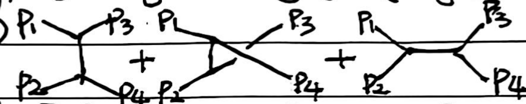
② (Tree level) The "stringification" of Feymann diagram relates the loops with the genus of surfaces, via replace particles/pts by S^1/closed string to give smoothness of involution, hence we have:

Tree level $\Longleftrightarrow g=0 \xLeftrightarrow[\text{cohomology}]{\text{Quantum}} \partial^2=0 \Longleftrightarrow$ no bubble

this means that loops gives the integral which can tend to infinity (renormalization needed), but tree-level terms always finite

$\Longleftrightarrow$ no BRST anomaly / Symmetry breaking (in gauge)

Eg. (2-to-2 tree level)



$= (ig)^2 \left[\frac{i}{(p_1-p_3)^2+m^2+i\varepsilon} + \frac{i}{(p_1-p_4)^2+m^2+i\varepsilon} + \frac{i}{(p_1+p_2)^2+m^2+i\varepsilon}\right]$

$\boxed{\text{PART II}}$ Equivalence of formalism

First we recall something fundamental of canonical quantization functional integral formalism:

Consider linear hyperbolic PDE with potential $\Box u + qu = 0 \cdots (*)$

$\Rightarrow$ the solution $u = \int e^{ipx}(a(p) + a^{+}(p))\, dp$

and we can compute $a(p)$ by expanding it to a_k homogeneous functions on cotangent bundle (classical mechanic system), called the Lax parametrix method.

Now let's forget how to compute a and a^{+}, but replace them by operators, then $\int e^{ipx}(a(p)+a^{+}(p))\, dp$ is a functional integral, it's an operator $\Rightarrow$ quantized field

- Then we need a and a^{+} satisfy CCR for Boson here (For Fermion, anti-CCR)

Now $(*)$ is the type of Klein-Gordon equation, for higher spin cases, there are $(2\sigma+1)$ freedom, we can factor into $(2\sigma+1)$ components of Klein-Gordon type

Q. How can we Spin $\rightsquigarrow$ EOM by quantization (Representation $\rightarrow$ PDE)?

$\Rightarrow$ there are operators $a(p,\sigma,n)$, $a^{+}(p,\sigma,n)$

日月光华 旦复旦兮

Rk. σ effects two things: number of components & the concrete equations

掃描全能王 創建

The canonical quantization is based on ladder operators instead of Hamiltonian operator, but same as QM, these operators are still related. we explain the equivalence of H and L:

($H \Rightarrow L$): H the Hamiltonian operator $\Rightarrow e^{itH/\hbar}$ is the involution operator $\Rightarrow$ its integral kernel is the propagator $= \int \boxed{\cdots} e^{\int L} Dp$

($L \Rightarrow H$): to represent the involution operator in path integral, consider the setting in analogue to scattering:

M manifold with boundary, ∂M has two components ∂M_{in} & ∂M_{out}

then the QFT functor $\partial M_{in} \xrightarrow{i} M \xleftarrow{j} \partial M_{out}$

$$\mapsto F(\partial M_{in}) \xleftarrow{i^*} F(M) \xrightarrow{j^*} F(\partial M_{out})$$

then the involution operator between vector spaces is defined

$$H_{in} := F(\partial M_{in})^\vee \xrightarrow{I} F(\partial M_{out})^\vee =: H_{out}$$

$$(\varphi : F(\partial M_{in}) \to \mathbb{C}) \longmapsto \left(\tilde{\varphi} : F(\partial M_{out}) \to \mathbb{C} \quad v \mapsto \int \varphi(i^* \tilde{v}) e^{-S} D\tilde{v} \right)$$

$$\{\tilde{v} \mid j^* \tilde{v} = v\} \subset F(M)$$

defined by path integral

$\Rightarrow I = e^{itH/\hbar}$ ▨

Rk. ① By Grothendieck's ~~six~~ six functor, $I(\varphi) = j_* i^* \varphi$

where i^* and j_* are in dual level $F(\partial M_{in})^\vee \xrightarrow{i^*} F(M)^\vee \xleftarrow{j_*} F(\partial M_{out})^\vee$

② $H \leftrightarrow I$ is a correspondence of ~~self~~ self-adjoint operators and unitary operators, the correlation function $\langle 0 | I | 0 \rangle$ = integral kernel of I = propagator;

③ (TQFT) The concrete functor F send to "fields" on M. what're they? They can be:

- σ-model $Map(M, X)$, $X = \mathbb{R}^1, \mathbb{C}^1, S^1, \cdots$

日月光华 旦复旦兮

- gauged σ-model $Map(M, X)^G$, $G \curvearrowright X$

- Section of bundles $\begin{cases} L^2\text{-functions} \\ L^2\text{-differential forms} \\ L^2\text{-cohomology classes} \\ G\text{-equivalent } \{\cdots\} \text{ in gauged model} \end{cases}$ $\boxed{?}$

The $\boxed{?}$ is given by supersymmetry, which allow a homotopy between operators, and we can define $H \sim 0$ to kill time involution. The SUSY is the basis for defining TQFT;

The SUSY is used to simplify path integral (hard) to a sum of zeros of S $\Leftarrow$ a topological intersection number.

~~Now it reduces to the question~~

~~Q. Given a particle, what's its EOM?~~

~~fundamental~~

Rk. $\boxed{\cdots}_\mu$ not enough, as it's varible on $x \in \mathbb{R}^{1,3}$, but

the interaction lost

$A_\mu(x_0)$ is just operator local at x_0, not spread like function on x (or over momentum space)

($\boxed{\cdots}_\mu(x_0)$ is eigenvector of $A_\mu(x_0)$)

Thus, the canonical quantization is two step

$EOM(x, p) \xrightarrow[\text{spin}]{①} EOM(x, \hat{p}_\mu) \xrightarrow{\text{solve}} \boxed{\cdots}$ (4s+2)-com

② $\Downarrow$ quantize CCR/AntiCCR A_μ

Fermion/B

A. Given spin $= s$, $s = 0, \frac{1}{2}, 1, \cdots$

field $\Downarrow$ Lagrangian $\Downarrow$ EOM $\Downarrow$ solve $\boxed{\cdots}$ (4s+2)-components $\Downarrow$ quantize A_μ at (4s+2)-components

particles $\Leftrightarrow$ rep

countable — many body system $\Downarrow$ second quantization of statistical $\psi(x_1, \cdots x_i \cdots) = \pm \psi(x_i \cdots x_1 \cdots)$ $\in$ Fock space

everywhere $\Leftrightarrow$ field

free $\Downarrow$ Klein-Gordon

with field A $\Downarrow$ Dirac, $\cdots, \cdots$ for each field we have EOM

their σ_μ are matrices, ~~eigen~~ generate $4s+2$ dim rep space

EOM is given by $p_\mu = i\frac{\partial}{\partial x_\mu}$, $\hat{p} = p_\mu \sigma_\mu$

日月光华 旦复旦兮

then replace classical EOM (x, p) by $\hat{p}_\mu$ ▨

掃描全能王 創建

Lecture on QFT III''': Summary of path integral

We had known that the correlation / partition function in QFT

$\langle \Psi' | \underbrace{\boxed{\cdots}}_{\text{composition of observables / field operators}} | \Psi \rangle$ = propagator, by written as Green functions (many kinds)

We carefully discuss it now.

In QM, $K(x',t'|x,t) =: \underbrace{\langle x',t' | x,t \rangle}_{\text{Schrödinger picture}} = \underbrace{\langle x' | e^{i(t'-t)H/\hbar} | x \rangle}_{\text{Heisenberg picture}}$ is propagator.

and we known $K(x',t'|x,t) = \int_{\{\gamma\}} e^{iS[\gamma]/\hbar} \, d\mu[\gamma]$

that's, $\langle x' | e^{i(t'-t)H/\hbar} | x \rangle = \int_{\{\gamma\}} e^{iS[\gamma]/\hbar} \, d\mu[\gamma]$ — $(*)$

We have dictionary comparing QM v.s. QFT:

QM	QFT
$\hat{X}\|x\rangle = x\|x\rangle$ $\|x\rangle$ complete basis	$\hat{X}_x\|X\rangle = X(x)\|X\rangle$ $\|X\rangle$ complete basis
$\hat{P}\|p\rangle = p\|p\rangle$ $\|p\rangle$ complete basis	$\hat{P}_x\|\Pi\rangle = \Pi(x)\|\Pi\rangle$ $\|\Pi\rangle$ complete basis
$\langle p\|x\rangle = e^{-ipx}$	$\langle \Pi \| X\rangle = e^{-i\int \Pi(x)X(x)dx}$
Phase space is a fixed Hilbert space	Fock space

$\Rightarrow$ we enhance $(*)$ to QFT version

$\langle \Psi' | e^{i(t'-t)H/\hbar} | \Psi \rangle = \int_{\{\Phi\}} e^{iS[\Phi]/\hbar} \, d\mu[\Phi]$, and $\langle \Psi' | \boxed{\cdots} | \Psi \rangle = \int_{\{\Phi\}} e^{iS[\Phi]/\hbar} \boxed{\cdots} \, d\mu$

$\Phi(x,t) = \Psi(x,t), \forall x$; $\Phi(x,t') = \Psi(x,t'), \forall x$

are all field configuration with these boundary conditions, using some computation:

Pf. Insert $|\Phi_j\rangle\langle\Phi_j|$, $|\Phi_j\rangle$ eigenstates of $e^{it_jH(t_j)/\hbar}$, cutting $[t,t']$ into pieces $t = t_0 < t_1 < \cdots < t_n = t'$, $t_j - t_{j-1} = \Delta t$

$\Rightarrow \sum \langle \Psi' | e^{it_nH(t_n)\Delta t} | \Phi_n \rangle \langle \Phi_n | e^{it_{n-1}H(t_{n-1})\Delta t} | \Phi_{n-1} \rangle \langle \Phi_{n-1} | \cdots$

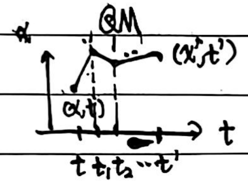
sum over: each $|\Phi_k\rangle$ eigenbasis of $e^{it_kH(t_k)/\hbar}$ is $\{|\Phi_k^{(i)}\rangle\}_{i \in \mathbb{N}}$.

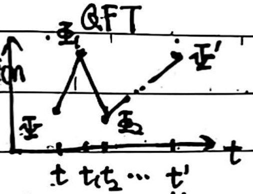
sum over all basis i for each k, product over all k

FUDAN UNIVERSITY 1905 日月光华 旦复旦兮

$\Rightarrow$ take $\lim_{\Delta t \to 0}$, done (Detail same as QM) $\square$

掃描全能王 創建





QM: sum over all eigenstates at $t_1, t_2 \cdots$ (real eigenvalue is position)

$\Rightarrow$ result is a real path

QFT: sum over all eigen-field-configuration at t_1, t_2

$\Rightarrow$ result is not real path

$\boxed{\text{last path in field configuration space}}$

Rk.

① For example, in σ-model, the "path" in QFT can be $\gamma: [t, t'] \to \mathrm{Map}(M, X)$;

When $d=0$ in space, field configuration is $\Phi(x,t) = \Phi(t)$ only has position X, not depend on x

② Although QM has Wiener measure makes path integral well, QFT has various field configuration spaces, have no known measure!

Computation method:

Solution to nonlinear (interacted) PDE: $\Box u = f$

Canonical Quantisation — Path integral (Physical word)

Anstatz — Time ordered exponential (Mathematical word)

$u = \int \Box(a + a^\dagger) f(p)\, dp$ — $P\exp\left(\int_t^{t'} f\right) = u$

$\boxed{\text{Asyptotic expansion}}$ (Mathematical computation)

Our asyptotic expansion for path integral, i.e. perturbative method, is expected to get same result as CQ way, such as Wick theorem and balabala, express interacted propagator as free ones:

Trick: Generating function

- This is the original form of many generating series in enumerative geometry e.g. Gromov–Witten potential; in physical setting here, we also reduce it into combinatorial problems.

We're computing $\langle \Omega | T\{\phi(x_1) \cdots \phi(x_n)\} | \Omega \rangle$

$$= \int \phi(x_1) \cdots \phi(x_n) e^{iS[\phi]/\hbar} d\mu[\phi]$$

Set generating functional

$$Z(J) = \int e^{i\int \mathcal{L}[\phi] + J(x)\phi(x)\, dx/\hbar} d\mu[\phi]$$

$$\Rightarrow -i \frac{\delta Z(J)}{\delta J(x)}\Big|_{J=0} = \langle \Omega | \phi(x) | \Omega \rangle$$

$$\rightsquigarrow (-i)^n \frac{\delta^n Z(J)}{\delta J(x_1) \cdots \delta J(x_n)}\Big|_{J=0} = \langle \Omega | \phi(x_1) \cdots \phi(x_n) | \Omega \rangle \quad (☆)$$

Compute path integral $\Leftrightarrow$ compute $Z(J)$

- For free theory, $Z_0(J) = e^{-\frac{1}{2\hbar}\iint J(x)\, \boxed{G}\, J(y)\, dx dy}$

$\boxed{G}$ the Green function $\Rightarrow$ recover free propagator as before;

- For perturbation $Z(J) = \int e^{i\int \mathcal{L}[\phi] + J\phi + g\Box\, dx/\hbar} d\mu[\phi]$

$$= \int e^{i\int \mathcal{L}[\phi] + J\phi\, dx/\hbar} \cdot e^{i\int g\Box\, dx/\hbar} d\mu[\phi]$$

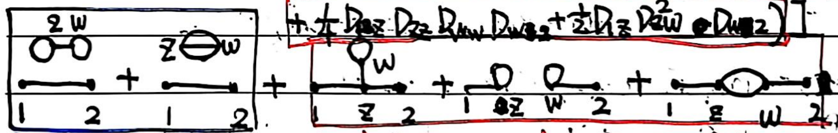
due to g is small: $= \int e^{i\int \mathcal{L}[\phi] + J\phi/\hbar} \left(1 + \frac{ig}{3!}\int \Box\, dx + \left(\frac{ig}{3!}\right)^2 \iint \Box\Box + \cdots\right) d\mu[\phi]$

$$= Z_0(J) + \frac{ig}{3!}\int e^{i\int \mathcal{L}[\phi] + J\phi\, dx/\hbar}\left(\int \Box\, dx\right) d\mu[\phi] + \cdots$$

$\rightsquigarrow$ we get perturbation expansion!

E.g. Let $\Box = \frac{g}{3!}\phi^3$, and the $\frac{ig}{3!}$ term $= 0$ by Wick (both in $Z(J)$ and propagat[or]) we compute $\left(\frac{ig}{3!}\right)^2$ term by counting, the result is:

$$\frac{1}{(\cdots)}\Big[D_{12} - g^2 \int dz\, dw \Big(\boxed{\tfrac{1}{8} D_{1z} D_{zz} D_{zw} D_{ww} + \tfrac{1}{12} D_{12} D_{zw}^3} + \boxed{\tfrac{1}{2} D_{1z} D_{2z} D_{zw} D_{ww} + \tfrac{1}{4} D_{1z} D_{2z} D_{1w} D_{w2} + \tfrac{1}{2} D_{1z} D_{zw}^2 D_{w2}}\Big)\Big]$$



Bad term, strange coefficients/automorphism $\left(\frac{1}{8}, \frac{1}{12}\right)$

good term, normal coefficients $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\right)$

Q. How to correct bad terms to coincide original results?

A. Before we forget normalization factor, i.e. $\langle \Omega | T\{\phi(x) \cdots\} | \Omega \rangle = \int e^{iS[\phi]} \cdots d\mu[\phi]$

$\Rightarrow$ there is a factor $\boxed{\frac{Z_0(0)}{Z(0)}}$ to multiply, which really cancel bad terms!

掃描全能王 創建

Lecture on QFT III, Symmetry

In physics, the "symmetry"/invariance occurs everywhere as an additional structure given by Lie group/Lie algebra, Lie group is global and Lie algebra is local:

$G \curvearrowright M \Leftrightarrow \mathfrak{g} \curvearrowright T_pM$ action is same in $\forall p \in M \Leftrightarrow$ global;

$\mathfrak{g}_p \curvearrowright T_pM$, $\mathfrak{g}_p$ varies by points $\Leftrightarrow$ local $\Leftrightarrow$ representation

replace T_pM by $\mathcal{H}_p$ space of states & maps by G-equivariant ones, the quantum case is similiar;

For example

- Gauge invariance : local
- Lorentz invariance : global
- Conformal invariance : global or local ~

I'll give an informal explenation to some "words" about symmetry but we start at the most fundamental Lorentz invariance.

$\boxed{\text{PART I}}$ Spin and Representation

$SO(1,3) \curvearrowright \mathbb{R}^{1,3}$ the Lorentz grp generated by rotation and boost, 3+3 generators:

the Poincáre grp is the inhomogenous Lorentz group $\mathbb{R}^{1,3} \rtimes_\rho SO(1,3)$ the semi-direct product given by ρ the upper action, adding the generators called space-time transform/shift;

$\Rightarrow$ $so(1,3)$ has infinitsimal generators J_i, K_i, $i=1,2,3$, from the rotation and boost respectively;

$\Rightarrow$ Let $J_i^{\pm} = \frac{1}{2}(J_i \pm iK_i)$, then $J_i^{\pm}$ is also a set of generators and $J_i^{\pm}$ satisfy CCR of $su(2)$/Pauli matrices

lck station

(By write out these matrices precisely)

$\Rightarrow so(1,3) \cong \underbrace{su(2)}_{J_i^+} \oplus \underbrace{su(2)}_{J_i^-}$ (Rigorously, up to a cpxification)

$\cong spin(3) \oplus spin(3)$

$\cong su(2) \oplus su(2)$

(irreducible unitary) we'll always assume this!

Thus the rep of $SO(1,3)$ is given by $\underbrace{(s_1, s_2)}_{\text{two weight}} \in \frac{1}{2}\mathbb{Z} \oplus \frac{1}{2}\mathbb{Z}$ and dimension $(2s_1+1)(2s_2+1)$, as we know what's the rep of $su(2)$;

$\Rightarrow$ the rep of $\mathbb{R}^{1,3} \rtimes SO(1,3)$ is given by (m, s), $\mathbb{R}^{1,3}$ contribute m and $SO(1,3)$ contribute $s = s_1 + s_2$, thus note that the irreducible rep of Poincáre grp not restrict to irreducible rep of Lorentz grp, there is no way to distinct $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, we define $s = s_1 + s_2$ the spin number of a particle, but what's the physical meaning of $s_1, s_2, J_i, K_i, J_i^{\pm}$? And why the sum concide the spin of non-realistic particles?

$J_i^+ + J_i^- = J_i$ comes from rotation generator

$\Leftrightarrow$ the $s = s_1 + s_2$ comes from intrinstic rotation of particles

$\Leftrightarrow$ $SO(3) \leq SO(1,3)$, it's $SO(3)$-invariant

$\Leftrightarrow$ it's nothing than spin, called Dirac Spinor

K_i comes from Boost generator is the relactivistic effect, is the outer effect of motion

$\rightarrow$ spin, $\rightarrow$ motion	$\rightarrow$ spin, motion $\leftarrow$
J_i^+	J_i^-
right-handed	left-handed
Weyl spiner	Weyl spinor

Roughly, the $J_i^{\pm}$ are like this, but one have notice that: $J_i^{\pm}$ is obvious Lorentz invariant, but upper picture not: if observer moves faster than particles' motion $\Rightarrow$ it's converse!

The upper is hellcity right-hand (or left-hand), but the lower is chairality right-hand (or left-hand)

Upper not depend on propagation while lower can change after propagation; upper not Lorentz invariant by lower;

(When particle is massless $\Leftrightarrow$ light-like, they're same: no observer can move faster than light.)

日月光华 旦复旦兮

掃描全能王 創建

Pf. EOM is $\begin{cases} \sigma^{\mu} p_{\mu} \psi_R = (E - \vec{\sigma}\cdot\vec{p})\psi_R = m\psi_L \xRightarrow{m\ll|\vec{p}|} \text{let } m=0 \\ \bar{\sigma}^{\mu} p_{\mu} \psi_L = (E + \vec{\sigma}\cdot\vec{p})\psi_L = m\psi_R \end{cases} \Rightarrow \left.\begin{aligned} \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\psi_R = \psi_R \\ \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\psi_L = -\psi_L \end{aligned}\right\}$ helicity

$\left(\begin{pmatrix} -m & i\sigma^{\mu}\partial_{\mu} \\ i\bar{\sigma}^{\mu}\partial_{\mu} & -m \end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} = 0\right)$

is chirality left & right Weyl spinor

($J_i^+ \Leftrightarrow J_i^-$ is given by parity symmetry)

Now we know what is spin:

- mathematically from SO(1,3)-rep, or collapse to SO(3)-rep;
- historically, they found orbital angular momentum operator not commute with $\hat{H}$, they add additional term called spin angular momentum operator by relativistic QM;

The last question is spin half $\Leftrightarrow$ Fermion

spin integral $\Leftrightarrow$ Boson

it's the spin-statistic thm ("statistic" comes from Boson & Fermion is property of many particles, but for field it's more precise), I omit it's pf.

SS thm $\Rightarrow$ CPT thm, C is central charge conjugate. (exchange particles & anti-particles)

P is Parity & T is time reversal

are something you check for many fields in QFT (although in some really complicated cases I don't know, it holds)

$(CPT)\phi(x)(CPT)^{\dagger} = \overline{\phi(-x)}$

the scattering has Feynman diagram rotation 180°

$x \to y \iff y \leftarrow x$

- beginning state x, final state y reversed $\Leftrightarrow$ T
- Arrow reversed $\Leftrightarrow$ C and P

Rk. Particles $\Leftrightarrow$ Fields, and due to our fields are always Lorentz invariant, particles $\Leftrightarrow$ Lorentz/Poincare grp rep, we can abuse the word "particle" freely.

PART II Symmetry breaking

(No. May not subgrp, the subgrp is called dynamical breaking)

Symmetry breaking means the invariance of G collapse to $H \leq G$, and we collapse the H-action directly; if quantization makes symmetry breaking, we call it anomaly (many kinds)

The symmetry breaking has two kinds:

$|0\rangle \curvearrowleft G$ ①

$\Downarrow$ equation of motion / involution / dynamic $\curvearrowleft G$ ②

$|n\rangle$

① is spontaneous symmetry breaking, ② is explicit symmetry breaking

Most cases we concerned are ①, $\Leftrightarrow$ Hamiltonian G-equi under topological / gauge theory such as Higgs Mechanism ...

Rk. Occurance of symmetry breaking is equivalent to the blow-up of energy / renomalization

E.g. In QED, i.e. U(1)-gauged theory quantized, there is ABJ anomaly (Adler-Bell-Jackiw)

$U(1): \psi \mapsto e^{-ia}\psi$ for vectors (classical)

("G-gauged" is classical notation for EOM of particle classical / quantization not respect it)

$\frac{\psi}{2}$

but the quantized theory is spin$(\frac{1}{2}, \frac{1}{2})$, i.e. $U(1)\times U(1)$ action

$U(1)_V \times U(1)_A : \psi \mapsto e^{ia\gamma^5}\psi, \ \bar{\psi} \mapsto \bar{\psi}e^{ia\gamma^5}$, ψ_L eigenvalue

[$\gamma^5 := i\gamma^0 \cdots \gamma^3$ the total spin $= \begin{pmatrix} -1_{2\times2} & 0 \\ 0 & 1_{2\times2} \end{pmatrix}$ under Weyl basis, ψ_R eigenvalue]

as we decompose it into ψ_L and ψ_R (left & right action, from 2 dim (only U(1)) to 4 dim ($U(1)\times U(1)$) quantise)

the additional U(1) is called axial

① It breaks the conservation of chiral currents and thus it's also called chiral / axial currents

axial anomaly;

② The chiral anomaly is "left handed ≠ right handed", more general notation, it's easy to see, here the anomaly comes from the difference between left & right-handed!

日月光华 旦复旦兮

日月光华 旦复旦兮

掃描全能王 創建

Lecture on QFT IV. Renormalization and effectivity

Date

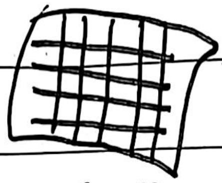
Recall when we compute $\boxed{\text{interaction propagator}}$, we have to deal the integral $\int \frac{\square}{k^4} dk$

but we write them as $\int \frac{\square}{k^4 + i\varepsilon} dk$ regularized integral.

Calculus tell us these integral have two ways to diverge, when $k \to 0$ and $|k| \to \infty$, i.e. the momentum turn to $\boxed{\text{infrared (IR)}}$ & $\boxed{\text{ultraviolet (UV)}}$

and our regularization is by a cut-off

$\varepsilon < |p| < \Lambda$

Rk. In lattice model  , the cut-off / regularization restrict the lattice into finite points

p-space

then we take $\boxed{\text{continuous limit}}$ $\leftrightarrow$ renormalization, but how to do the limit?

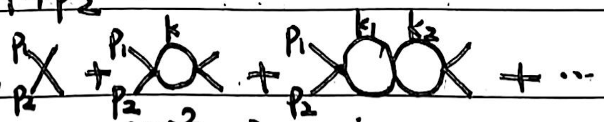
- ϕ^4 - theory $\Leftarrow$ Higgs Boson in quantized Yang-Mills theory

If no loop, then no such integral, only $|x| \to \infty$ diverge $\Rightarrow$ only IR diver

If $\exists$ loop, then such integral in i.e. $|p| \to 0$ (by Fourier transform)

p-space exists, both IR and UV divergence exist.

Consider 2-to-2 scattering $p = p_1 + p_2$

The scattering amplitude $iA(p) =$  $+ \cdots$

$$= -i\lambda + \frac{(-i\lambda)^2}{2} \int \frac{i}{k^2} \frac{i}{(p-k)^2} dk + O(\lambda^3)(p)$$

λ is coupling "constant", i.e. $L = L_{free} - \frac{\lambda}{4!}\phi^4$,

After regularization λ is small, as λ absorb $\hbar$

Fact ① $\frac{\partial}{\partial p}\left(\frac{(-i\lambda)^2}{2}\int \frac{i}{k^2+i\varepsilon}\frac{i}{(p-k)^2+i\varepsilon} dk\right) = -i\frac{\lambda^2}{16\pi^2}\frac{1}{p}$ (by contour integral)

$\Rightarrow \frac{(-i\lambda)^2}{2}\int \frac{i}{k^2+i\varepsilon}\frac{i}{(p-k)^2+i\varepsilon} dk = -i\frac{\lambda^2}{16\pi^2}\log\left(\frac{p}{\Lambda}\right)$. Λ is constant

but due to its diverge behavior, we know "$\Lambda = \infty$" & "$\varepsilon = 0$", we need to cancell it (mathematically)

② $A(p)$ is an amplitude, it's observable / experimental, but λ is mysterious (physically)

掃描全能王 創建

By the two reason above, we're doing a methematical trick to cancell both λ and $\Lambda+\varepsilon$: $\underbrace{}_{\text{by experiment} < \infty}$

Regularization ① Assume $\lambda_R = -A(p_0)$ is known, called renormalized coupling constant

$i\lambda_R = -i\lambda + \frac{(-i\lambda)^2}{2}\int \frac{i}{k^2}\frac{i}{(p_0+k)^2}dk + O(\lambda^3)$.

② We do cut-off in $\varepsilon<|p|<\Lambda$, denote all cut-off by adding ~

Renormalization ③ $i\tilde{A}(p) = -i\lambda - i\frac{\lambda^2}{16\pi^2}\log\left(\frac{p}{\Lambda}\right) + O(\lambda^3)(p)$ ← (b) take $\lambda = \square(\tilde\lambda_R)$ in it

$i\tilde\lambda_R = -i\lambda - i\frac{\lambda^2}{16\pi^2}\log\left(\frac{p_0}{\Lambda}\right) + O(\lambda^3)$ ⇝ (a) express λ by $\tilde\lambda_R$

$\Longrightarrow \tilde A(p) = \tilde\lambda_R - \frac{\tilde\lambda_R^2}{16\pi^2}\log\left(\frac{p}{p_0}\right) + O(\lambda^3)(p)$

④ Take limit to cancell $\varepsilon+\Lambda$, as here doesn't have $\varepsilon+\Lambda$

$\Longrightarrow A(p) = \lambda_R - \frac{\lambda_R^2}{16\pi^2}\log\left(\frac{p}{p_0}\right) + O(\lambda^3)(p)$. is our final result

This progress is methematical trick only? No, it has suprising physical meaning:

Observation: Λ is chosen arbitrarily, when we do ③(a) above, we see $\lambda = \lambda(\Lambda)$ depend on the cut-off, it's not constant;

Physically, it says that, the coupling changes when the energy scale ⇔ momentum scale changes, the scale effects (contrast to length scale) what the physic theory is;

• $A(p)$ is well-define function after renormalization in all scales, we call ϕ^4-theory / Yang-Mills theory renormaliable.

Rk. A strange notation is that, we call $|k|>\varepsilon$ the UV cut-off as they named from the saved part; $|k|<\Lambda$ the IR cut-off

• QED

The 1-to-1 photon scattering gives amplitudue is the Columb potential $V(p) = \sim\sim + \sim\bigcirc\sim + \cdots$

After regularization

Fact ① $V(p) = \frac{e^2}{p}\left(1 - e^2\Pi_2(p) + O(e^4)\right)$, e is coupling "constant" and $\Pi_2(p)$ is some complicated special function (known);

② The Columb potential is observable

Rk. In both ϕ^4 theory and QED, we use regularization to compute integral first, but in ϕ^4, we use cut-off $\varepsilon<|p|<\Lambda$ i.e. $\boxed{\text{scale regularization}}$ it has clear physical meaning

in QED, we use $\boxed{\text{dimension regularization}}$

Roughly, dimension regulazation totally methematic trick, no known physical explanation, is by analytical extend $d=4$ to all $d\in\mathbb{C}$, then take $\lim_{\substack{d\in\mathbb{C}\\ d\to 4}}$, this is the same trick as the fractional integral/derivative;

Then we can proceed as ϕ^4-theory

① $e_R = (p_0 V(p_0))^{\frac{1}{2}}$;

② Dimension regularization;

③ Cancel e by two power series;

④ Take limit $d\to 4$.

Our result is $V(p) = -\frac{1}{p}\frac{e_R^2}{1-\frac{e_R^2}{12\pi^2}\log\frac{p}{m^2}}$

Rk. Surprisingly, here we can collect all higher terms together and give a precise result, all strange special functions disapper ($\Pi_2(p)-\Pi_2(p_0)$ is good)

Mathematicians: It's good!!

Physists: No, it's not: when $p = m^2 e^{\frac{12\pi^2}{e_R^2}}$ (finite number) $V(p)$ is infinity, ill behavior even occurs in $|p|<\infty$, this is called Landau pole

⇝ QED is not renormaliable

QED is not a complete theory

QED is only $\boxed{\text{effective}}$ in low energy scale

I'll give more remark later.

日月光华 旦复旦兮

掃描全能王 創建

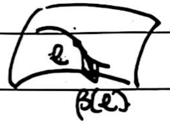
Set $e_{eff}^2 = \frac{e_R^2}{1-\frac{e_R^2}{12\pi^2}\log\frac{p}{m^2}}$, e_{eff} depend on p.

$\Leftrightarrow \frac{1}{e_{eff}^2(p)} - \frac{1}{e_{eff}^2(q)} = \frac{1}{12\pi^2}\log\left(\frac{q}{p}\right)$, fix p, take derivative $\frac{d}{dp}$

$\Rightarrow q\, e_{eff}'(q) = e_{eff}^3(q)/12\pi^2 =: \beta(e_{eff})$ called β function of QED

β describles the coupling e's behavior when energy scale changes, roughly, β is its derivative.

We call β is the RG flow (Renormalization Group Flow) in this meaning.



space of running coupling (here is $\frac{1}{e_{eff}^2(q)}$)

$\cong$ space of Lagrangian

β act on Lagrangian $\leftrightarrow$ coupling e

$\rightsquigarrow$ running coupling

In QED, the β is uniformly positive, it means that in general

β	
$+$	: $e \nearrow$ when energy scale $\nearrow$ $\Rightarrow$ Unrenormalizable theories
$-$	: $e \searrow$ when energy scale $\nearrow$ $\Leftrightarrow$ asymptotic free
0	: scale invariance $\Leftrightarrow$ CFT $\Leftrightarrow$ Fixed points under RG flow

Rk. ① Classical YM theory is conformal invariant, but quantization breaks it;

② Renormalizable $\Leftrightarrow$ β is well-defined in hole space

$\Leftrightarrow$ the induced vector field has no singularity;

③ QCD is example of asymptotic free, with asymptotic freedom $n_f \le 12$, $\beta(g) = -(11-\frac{2}{3}n_f)\frac{g^3}{16\pi^2}$

④ "Quantum Gravity" is hard because it not only unrenomalizable but also β's definition domain $= \phi$;

⑤ AdS – CFT can be understood as a correspondence between different scales: high energy CFT in boundary (UV)

$\Leftrightarrow$ low energy gravity interior (IR)

Mathematical application: Most application to math is about CFT, here our example is Kontsevich's formality thm:

- Indeed, it's conformal string theory (no Minkowski space-time, thus), but the renomalization techniques are shared.

We needed to prove a sum (very complicated form) "converges", the intitution is:

- Turn combinational data into Feymann diagrams;
- These Feymann diagram lives in a base space of some physical theory (open string)
- Let the theory has conformal boundary $\Leftrightarrow$ hyperbolic interior; ($\overline{\mathbb{H}}$)
- Scale inv $\Leftrightarrow$ Renormalizable $\Leftrightarrow$ Converge! (Not Rigorous!)

Convergence's proof needs some geometry of $\overline{\mathbb{H}}$, but physical idea tells us why it will be true.

- From Mathematical perspective, why QFT is some complicated?

Someone says it's due to we didn't understand the "interaction at a point" correctly: as here each point is equipped with operator

how these operators interact locally are mysterious

日月光华 旦复旦兮

日月光华 旦复旦兮

掃描全能王 創建