KSBA moduli space ① codim 1 points of X are smooth / nodal), then K_X is defined by $\exists X^0$, s.t. $X^0 \xrightarrow{j} X$; $X - X^0$ has ① codim ≥ 2, j is open

and $(X^0) \subset \mathbb{P}^N$ l.c.i, locally closed $\Rightarrow \omega_{X^0} = \iota^*(\omega_{\mathbb{P}^N} \otimes \det(\mathcal{I}/\mathcal{I}^2))$ and $\omega_X = j_* \omega_{X^0}$

Normalization $\bar{X} \xrightarrow{\pi} X$, $\pi^* K_X = K_{\bar{X}} + \bar{D}$, we call pair (X, Δ) semi-lc if $(\bar{X}, \pi^*\Delta + \bar{D})$ is lc

Moduli functor of polarized schemes (Δ and $K_X + \Delta$ are $\mathbb{Q}$-Cartier to admit pullback).

$MP(Z) = \{f: X \to Z, \mathcal{L}\} | f$ flat & projective, ample$\} / \cong$ Z-isomorphism & $\mathcal{L}_1 - \mathcal{L}_2 = f^*\mathcal{L}$, $\exists \mathcal{L}$ line bundle on Z

It's not representable by scheme / fine, and it's too big, KSBA moduli functor is a subfunctor of it.

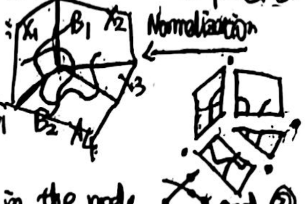
Pf. Otherwise, $\exists M$ represent it and $\exists$ universal family $\mathcal{U} \to M$, $\exists f: X \to Z$, s.t. smooth, proj, all fibres same, $X \ncong X_0 \times B$ nontrivial, but $\mathcal{U}$ ... $\Rightarrow MP(Z) = \mathcal{U}_{pt} \times Z$ product is trivial, contradiction ☒ $Z \to M$ is a point

$\boxed{\text{KSBA pair}}$ is (X, B)

X proj var, needn't to be irreducible, $B = \sum b_i B_i$, $b_i \in (0,1] \cap \mathbb{Q}$ and B_i Weil, s.t. ① (X, B) slc ; ② $K_X + B > 0$ is ample $\mathbb{Q}$-Car

• slc is generalization of SNC in irreducible case, it $\Rightarrow$ codim 1 SNC with boundary of X, B_i each ; Normalization

i.e. B doesn't contain any starata of $X = \bigcup X_i$ (NC $\neq$ SNC as it allows loops)



Thm. Fix $d = \dim X$, $\vec{b} = (b_1 \cdots b_n) \in \mathbb{Q}^n_{>0}$, $v = (K_X + B)^d$, $(n+2)$-numbers $\Rightarrow \exists$ (opt) projective moduli of KSBA pairs $M_{d, \vec{b}, v}$

• Notices KSBA pair is generalization of stable curves in $d = 1$: ① ensures that marked pts not in the node and ② is proven to be hold automatically, as a technical step in constructing projective moduli (coarse). Different with $\bar{M}_{g,n}$ ... here each marked pt may have multiplicity $\neq 1$. And in 1-dim, $g \leftrightarrow$ volume both descript birational properties holy.

Our main thm isn't proven totally now (2025). Toric / Abelian is done 25 years ago, and hyperplane arrangement $\mathbb{C}^k / \mathbb{C}$ are done 10 years ago, now we focus on K3 / CY case recently ; our idea: degenerate K3 to "almost toric".

Toric geometry recalled: (X, L) polarized $\leftrightarrow$ lattice polygon $Q_0 \subset M$, $Q \subset M_{\mathbb{R}}$ (Q_0 are vertices) given by symplectic $X(\mathbb{C})$'s moment map. with $\omega = c_1(L)$

Eg. • $\mathbb{P}^1$ case $\mathbb{P}^1_{\mathbb{C}} = S^2$ Lagrangian fibration $\to \mathbb{R}$ are just ⊷ "polarization" is useless but scaling.

• $\mathbb{P}^2$ case polarized by $\mathcal{O}(d)$ $\xleftrightarrow{\text{dual}}$ normal fan $\mathcal{F}$ polarization changes the scale and shape of the triangle, but the triangle is always triangle , , ... and scale is L^2 = area of Q

Now we have basic fact useful:

• General $X \in |-K_V|$ is K3 for V is Fano 3-fold

• Q reflective $\Leftrightarrow (X, -K_V)$ is Fano

($0 \in Q_0$, Q and Q^* are lattice / integral polygons)

Eg. $(\mathbb{P}^1)^2$, $\frac{\mathbb{P}^1 \times \mathbb{P}^1}{\mathbb{Z}/2 \times \mathbb{Z}/2}$, $(\mathbb{P}^1)^3$ take $(\mathbb{P}^1)^2$ as example, the toric MS works as correspondence of different normal fans A-side ; B-side

K3-surface and simple degenerations:

Eg. deg 2 K3-surfaces $X_2 = \{Y^2 = f_6(x_1, x_2, x_3)\} \xrightarrow{2:1} \mathbb{P}^2$; deg $2d$ K3-surfaces $\subset \mathbb{P}^3$ has $L^2 = 2d > 0$

Consider degeneration of $(\mathbb{P}^1)^3$ into 6-faces, i.e. $\bigcup^6 (\mathbb{P}^1)^2$ = special fibre and $(\mathbb{P}^1)^3$ = general fibre, each $(\mathbb{P}^1)^2$ has resolution $Bl(\mathbb{P}^1)^2$ and result 24 • singularities, we called the charge of K3 surfaces, charge $((\mathbb{P}^1)^3)$ = # internal blow up

"charge = obstruction to toric" is slogan

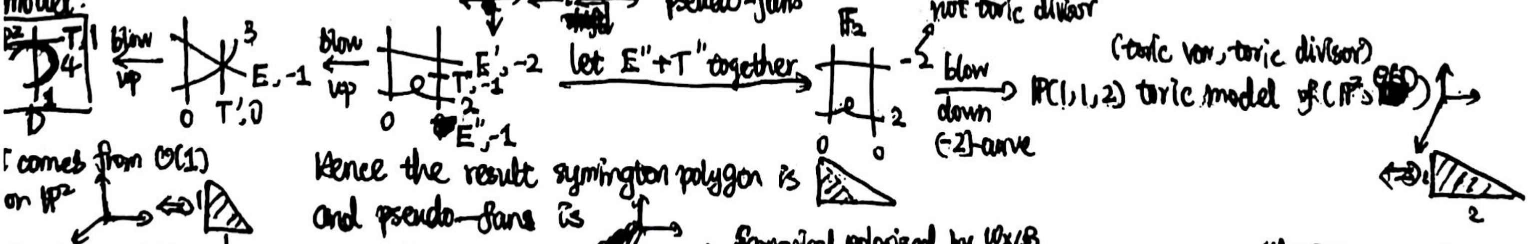
(25 years ago, Abelian has charge 0 and can degenerate to toric easily)

Generally, we have log CY surface is (X, D), D normal crossing, $K_X + D \sim 0$ and $-K_X > 0$, it have three types D (I) $D = 0$ (II) σ elliptic cl (III) rational loop

All type (III) has a toric model, however other cases we need to use generalizations: symington polygons and pseudo-fans.

So rational blow up of toric case, $\square \leftarrow$ symington polygon toric log CY

E.g. Blow-up & Blow-down on surfaces w.r.t. divisors to toric model: shear pseudo-fans



T comes from $\mathcal{O}(1)$ on $\mathbb{P}^2$

Hence the result symington polygon is and pseudo-fans is

Consider subfunctor $M^{sm}_n \subset MP$, $M^{sm}_n(B) = \{$canonical polarised by $\omega_{X/B}$, $f: X \to B \,|\, f$ sm. proj, ω_{X_b} ample and $\chi(X_b, \omega^{\otimes m}_{X_b}) = h(m)\} / \cong$

then the smaller functor admits a coarse moduli

If we view fine moduli is strict representable, then coarse is minimal weak representable, it don't admit universal family

Pf. $\exists m$ large, $X \hookrightarrow \mathbb{P}_S$ for $\omega^{\otimes m}_{X/B} \cong \mathbb{P}^N_S$, $N = h(\omega^{\otimes m}_{X_b}) - 1$ ... always as $\eta: \mathcal{M} \to \mathrm{Hom}(-, M)$ isn't iso, $\eta^{-1}(\mathrm{id}_M)$ not exist, we

However, $M_h^{sm} \subset MP$ is too small and not proper. Its compactification in MP (limit of smooth) is just KSBA compactification $M_h^{sm} \subset M^{KSBA} \subset MP$ and $M^{KSBA}(B) = \{f: X \to B$ canonically polarized as a stable pair$\}$. It's proper by semi-stable reduction, as higher dim analogue of $\overline{M}_g$, which highly depended on MMP.

| a base change $X' \xrightarrow{f'} B'$ is semi-stable family: X', B' are toric^{sm} and f' is toroidal, equidim with reduced fibres $\}$

Hence M^{KSBA} is proper, although our Main Thm only holds for some special case. But note that here we only consider the canonical polarized case (X, K_X), for bounded case (X, K_X+B) is true but much harder.

Moduli of polarized K_3-surfaces (X, L), $L^2 = 2d$ is denoted $F_{2d} \cong D/\Gamma$. D is the period domain $= \{[\omega] \in \mathbb{P}(L^\perp \otimes \mathbb{C}) \mid \omega^2 = 0, |\omega|^2 > 0\}$, $\Gamma = \{g \in O(L^\perp) \mid$ restriction from $H^2(X;\mathbb{Z})\}$ by Hodge theory.

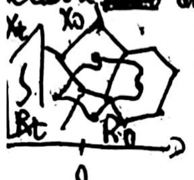
View L as $\langle L \rangle \cong \mathbb{Z} \subset \mathrm{Pic}(X)$, we generalize to $S \subset \mathrm{Pic}(X)$ and $\mathbb{Z}^r \cong S$, similarly $F_S \cong D_S/\Gamma_S$. And for pair (X, g) with action g, take $S = H^2(X;\mathbb{Z})^g$ is also done $\cong D_S/\Gamma_g$ ($\Gamma_g \supset \Gamma_S$)

Compactification of moduli of K_3-surfaces: Minimal one is Baily–Borel $\overline{F}_{2d}^{BB} = F_{2d} \cup \partial \overline{F}_{2d}^{BB} = F_{2d} \cup \{$rational one cusps$\} \cup \{pt\}$ model. $\overline{F}_{2d}^{T}$ toroidal and semi toroidal compactification & KSBA compactification (they former is for Abelian varieties, which is charge-free and reduce to toric model easily). For KSBA, we have other moduli problem $P_{2d} = \{(X, \varepsilon R) \mid R \in |L|$ is canonically chosen, $0 < \varepsilon \ll 1\}$ $((X, L) \xleftrightarrow{1:1} (X, \varepsilon R))$

$P_{2d}^{H} \to F_{2d}$ and $\overline{P}_{2d}^{KSBA}$ exists

E.g. Rational curves $\Sigma C = R_{rc}$ with $(d+1)$ nodes, has n_d counted by Yau–Zaslow on K_3 surfaces $\sum n_d t^d = \frac{1}{\prod (1-t^k)^{24}}$ and $n_1 = 324$, $n_2 = 3200 \ldots$

We call a divisor $R \in |L|$ rational recognizable if for degenerate family $\mathcal{X} \to C$, R on X only depend on (X_0, L_0), the special fibre (X_0 SNC, reduced, $K_{X_0} \cong 0$).



i.e. we have special fibre "special/complicated" enough in order to recover R from R_0

Thm. ① Given recognizable divisor R, $\overline{F}^{KSBA} = \overline{F}^{T}$; ② On F_{2d}, R_{rc} is recognizable divisor.

Degeneration of K_3-surfaces is known is semi-toroidal with several models: Kulikov, nef, divisor, KSBA model.

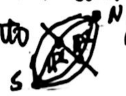
Kulikov model: (I) smooth; (II) $\mathbb{P}^1 \ldots \mathbb{P}^1$; (III) $\bigcup (V_i, D_i)$ union of log CY pairs and glued by dual graph is triangulation of S^2

Nef model: L_0 is nef; Divisor model: $R \in |L|$, s.t. R_0 big & nef, (X_0, R_0) is slc; definition

KSBA model: replace $R \in |L|$ by $R \in |mL|$ for some m, R_0 ample (X_0, R_0) slc, we have contraction from divisor model to KSBA model by

e.g. $X_t \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^1$ are general fibres

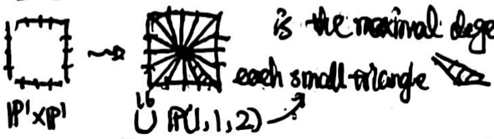
is divisor model, contract into



is KSBA model

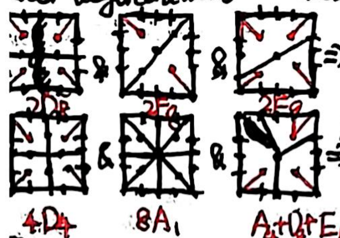
E.g. $F_{(2,20,0)} = \{X \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^1 \supset C \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4,4)|\}$, degeneration

$$\begin{array}{ccc} X & \rightsquigarrow & \bigcup^{16} \mathbb{P}^2 \\ \downarrow 2:1 & & \downarrow 2:1 \\ \mathbb{P}^1 \times \mathbb{P}^1 & \rightsquigarrow & \bigcup^{16} \mathbb{P}(1,1,2) \end{array}$$



is the maximal degeneration

Other degenerations are these types (all $\tilde{A}\tilde{D}\tilde{E}$ singularities):



$\Rightarrow$ Gluing along black line, with a double cover, it's elliptic curve

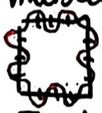
$\Rightarrow$ More complicated gluing

• Similarly, we denote maximal case as "0"

These coxeter diagrams generalizes Dynkin

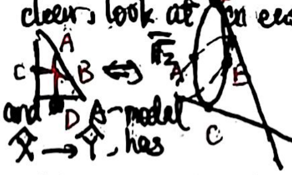
These black lines corresponds to $C \subset \mathbb{P}^1 \times \mathbb{P}^1$, or the red curve intersect with it as in right diagrams, serving as the ramification divisor of the ramified 2:1 covering.

Different types of degeneration comes from the divisor intersect with them, e.g. maximal case it's



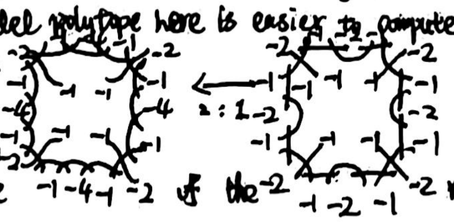
; $2\tilde{D}_8$ case is

To make this "intersection with face of polytope" more clear, look at an easier case when $\mathbb{F}_2$:



then this intersection achieves maximal and inducing maximal degeneration.

Due to toric MS, we know mirror pair are linked by reflexive polytopes and polytope are given by Lagrangians. here mirror of $X \to \mathbb{P}^1 \times \mathbb{P}^1$, denoted as $\check{X} \to \check{Y}$, has structure as (A-model polytope here is easier to compute, and then B-model can easily done by Batyrev's Construction) in maximal degeneration:



B-model originally. The symington polygon describes the shape of the moduli $\overline{F}_{(2,2,0)}^{KSBA}$, it vertices parametrizes KSBA model of $X \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^1$, are ADE types as above.

and symington polytopes are

, which is graphically same in (computed via lattices ..)

E.g. Enrique surface case can be deduced from above as

$$\begin{array}{ccc} X & \xrightarrow{2:1} & \mathbb{P}^1 \times \mathbb{P}^1 \\ \downarrow 2:1 & & \downarrow 2:1 \\ E & \xrightarrow{2:1} & \text{dPezzo} \end{array}$$

and a canonical divisor comes from pushforward of X's.

Wall-crossing for KSBA moduli ③ by Ziquan Zhuang

Hassett's weighted stable curves: $\overline{M}_{g,\vec{v}}$, $\vec{v}\in(0,1]^n$, s.t. $\omega_C+\sum v_iP_i$ are ample & $\forall p\in C$, $\sum v_i\le 1$, allowing point with multiplicities, and condition 1 still believes that when counting special pts with multiplicity, genus 0 ≥ 3 multiplicity and genus 1 ≥ 1, i.e. $2g-2+\sum v_i > 0$ (in integral case, they're ≥ 1, but here 2.0.1 is also stable)

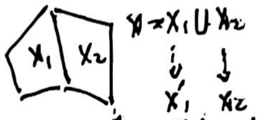
Thm (Hassett). $\forall\, 0<\vec{v}'\le\vec{v}$, $\overline{M}_{g,\vec{v}}\to\overline{M}_{g,\vec{v}'}$ $\exists$ reduction morphism $r_{\vec{v},\vec{v}'}$

• $\exists$ finite polyhedral decomposition of $\{\vec{v}\mid 2g-2+\sum v_i>0\}$, s.t. $\overline{M}_{g,\vec{v}}$ and $r_{\vec{v},\vec{v}'}$ only depend on the chamber containing $\vec{v}$ and $\vec{v}'$. hence finite (it means that they stay invariant in the chamber, and wall-crossing through the chamber)

Here $r_{\vec{v},\vec{v}'}$ is constructed via contract components C_i which $\deg(\omega_C+\sum v_i'p_i)|_{C_i}\le 0$

$v_1'+v_2'+v_3'\le 1$ $\quad v_{p'}'=v_1'+v_2'+v_3'\le 2$ (+1,)

$v_1+v_2+v_3>1$ $\quad$ (as the original node has 1)



Now we expect generalize these wall-crossing into higher dimensional KSBA moduli spaces: can't be glued

However, due to the contraction above generalizes to MMP in higher dimensional may not work for not normal cases, hence we should consider $\widetilde{M}_{\vec{v}}\to\overline{M}_{\vec{v},red}=(\bigcup_{\text{volume}}\overline{M}_{\vec{v},V})_{red}$, which such normalization just parametizes all normal varieties.

Thm (Meng, Zhang). $\forall\, 0<\vec{v}'\le\vec{v}$, $\exists$ reduction morphism $r_{\vec{v},\vec{v}'}:\widetilde{M}_{\vec{v}}\to\widetilde{M}_{\vec{v}'}$

• $\exists$ locally finite rational polyhedral decomposition of $\{\vec{v}\mid \omega_X+\sum v_iD_i \text{ is big}\}$, s.t. $\widetilde{M}_{\vec{v}}$ and $r_{\vec{v},\vec{v}'}$ only depend on the chamber containing $\vec{v}$ and $\vec{v}'$.

Irrationality of cubic 4-fold ① By Yu

Def. nc-Hodge structure is $(\mathcal{H},\mathcal{E}_B,\varphi)$, $\mathcal{H}$ is $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $/\mathbb{A}^1$, $\mathcal{E}_B$ is $\mathbb{Q}$-local system $/\mathbb{A}^1-\{0\}$, called Betti datum

$\varphi:\mathcal{E}_B\otimes\mathcal{O}_{\mathbb{A}^1-\{0\}}\xrightarrow[\text{flat}]{\cong}\mathcal{H}|_{\mathbb{A}^1-\{0\}}$ analytic isomorphism

• The last imply $\exists$ holomorphic connection ∇ on $\mathcal{H}|_{\mathbb{A}^1-\{0\}}$, called de Rham datum, by RH

Def. It's rational if ① ∇ extend to meromorphic connection of $\mathcal{H}$ with pole of order 2 at 0 and regular singularity at ∞, ② $\mathcal{E}$ compatible with Stokes's datum of $(\mathcal{H},\nabla)$ i.e. asymptotic of solutions on sectors; ③ oppositeness axiom ($H^{\pm}$-$H^{\mp}$)

Eg. Given pure rational Hodge structure $(V,F^\bullet V,V_{\mathbb{Q}})$ of weight w, Rees construction gives rational nc-HS via $\mathcal{H}=\sum u^{-i}F^iV[[u]]$,
$\nabla=d-\frac{w}{2}\frac{du}{u}$, $\mathcal{E}=V_{\mathbb{Q}}$. $\quad$ $\mathrm{Crit}(W)\xleftrightarrow{1:1}\mathrm{Eigenvalue}(E)$

A-model variation of nc-HS: now over base $B=\mathrm{Spf}\,\mathbb{C}[[Q=e^{\beta q},t=(t_i)\mid\beta\in NE]]$, we have "three directions":

u as $\mathbb{A}^1$-coordinate, $\tau\in H^*(X;\mathbb{Q})$ are $t=(t_i)$ directions, $\beta\in NE$ are Q directions:

$\nabla_{\partial_u}=\partial_u+u^{-2}E\star+u^{-1}G$; $\nabla_{\partial_\tau}=\partial_\tau+u^{-1}\tau\star$ $\quad$; $\nabla_{\xi q\partial_q}=\xi q\partial_q+u^{-1}\xi\star$, $\star$ are quantum products from A-model GW-theory

E Euler vector field and G is grading operator

Let $\mathcal{H}$ is trivial $H^*(X;\mathbb{Q})$-bundle and $\mathcal{E}\subset\mathcal{H}$ given by some convergence...

Conj. Then it's rational and exponential-type (latter is proven by symplectic methods)

B-model of LG model: $\quad$ $\mathcal{H}=\bigoplus\mathcal{E}^{c_i/u}\otimes R_i$ is exponential-like power series on u, dominate the order 2 pole of $(\mathcal{H},\nabla)$ at 0

Y smooth quasi-proj $/\mathbb{C}$ with potential $w:Y\to\mathbb{A}^1$

twisted de Rham $\mathcal{H}=\mathbb{H}^\bullet(Y,\Omega_Y^\bullet[[u]],d'=d-\frac{dw}{u}\wedge)$. Gauss-Manin connection $\nabla_{\partial_u}=\partial_u-\frac{w}{u^2}$, $\mathcal{E}=H^\bullet(Y,W^{-1}(\mathcal{H}_{\infty});\mathbb{Q})$ rapid decay cohomology

Rk. Hence given X, we have two nc-HS: A-model & B-model

Conj. It's rational and exponential type (except oppositeness, other proven) and classical HS is B-model, but after

de Rham datum: $(\mathcal{H},\nabla)$ $\longleftarrow\text{-----}\longrightarrow$ Betti datum / i.e. $\mathbb{Q}$-Stokes's structure $(\mathcal{L},\ldots)$
$\updownarrow$ Fourier transform $\qquad\qquad$ $\updownarrow$ topological Laplace transform
regular holonomic D-module $M/\mathbb{A}^1\xleftrightarrow{RH}$ constructible sheaf $\mathcal{F}$ of $\mathbb{Q}$-vector space $/\mathbb{C}$
$H^\bullet_{dR}(\mathbb{A},M)=0=R\Gamma(\mathcal{F})$

Rees out

construction vanishing cycle, it turns to be principle with A-model nc-HS

We have decomposition of constructible sheaves into (U_i) and $(T_{ij}:U_i\to U_j)$, $\mathbb{Q}$-vector space with linear maps (needn't iso)

Correspondingly, we have analytic decomposition of $(\mathcal{H},\mathcal{E},\varphi)$ too if exponential-type $(c_1\cdots c_n)\in\mathbb{C}^n$ lattice given: $(\mathcal{H}_i,\mathcal{E}_i,\varphi_i)$ all nc-HS and $T_{ij}:\mathcal{E}_i\to\mathcal{E}_j$ gluing datum.

Too much conjectures on nc-HS and depending on hard conjectures, e.g. Gamma conjecture, we can't use it directly, but replace $B=$ smooth non-Archimedean k-analytic space to ignore issues of complex analytic convergence

Def. F-bundle $(\mathcal{H},\nabla)/B$ is $\mathcal{H}/B\times D$ (D analytic disk $\in\mathbb{A}^1_u/k$) and meromorphic flat connection ∇ s.t. ...

Rk. It's same datum as Iritani's quantum D-module, except the non-Archimedean B.

Called it Log F-bundle is $\mathcal{H}/B\times\mathrm{Spf}\,k[[u]]$ ($G=\nabla_{u\partial_u}$ by above)

The action of Euler vector field has eigenvalues are exponential order $c_1\sim c_n$ and induce spectral decomp $\mathcal{H}=\bigoplus\mathcal{H}_i$

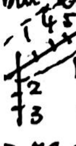
This only holds for non-Archimedean case, not also need to check convergence.
It coincides with vanishing cycle decomposition in some sense.
A framing of $(\mathcal{H},\nabla)$ means that $\nabla_{\partial u} = \partial_u + u^{-2}K + u^{-1}G$, $\nabla_{\partial_{\xi}} = \partial + u^{-1}A(\xi)$, A is connection 1-form valued in $End(\mathcal{H})$
It may not ~~to~~ exist, but in non-log case, we can extend it from single pt to global, log-case also proven recently.
Application: $H^{\cdot}(\mathbb{P}(E);\mathbb{Q}) = \bigoplus_{i\leq r-1} H^{\cdot}(X;\mathbb{Q})[-2i]$, $r=\mathrm{rank}(E)$, we can extend these into quantum cohomology

$$H^{\cdot}(Bl_Z X;\mathbb{Q}) = H^{\cdot}(X;\mathbb{Q}) \oplus \bigoplus_{i\leq r-1} H^{\cdot}(Z;\mathbb{Q})[-2i],\ r=\mathrm{codim}(Z;X)$$

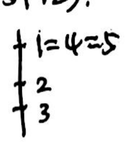
Via the isomorphism between two A-model F-bundle, est base pt first then extend, base pt is given by one $b\in B$ origin, another needs to be shifted & Two F-bundles are canonically given on $\mathbb{P}(E)$ and $X\times\mathbb{P}^{r-1}$ and this isomorphism of F-bundles gives decomposition of quantum cohomology (or $Bl_Z X$ and $X\times Z^{r-1}$) as the Euler vector field decomposition.
Small quantum cohomology $\approx$ Big one cancell all its except $\deg=2$ part. Konstevich–Manin's reconstruction says that if $H^{\cdot}$ is generated by $H^2 \Rightarrow$ big one can be reconstructed from small one.
Now in F-bundles, it's generalized into unfolding thm $\Rightarrow$ MS of G/P flag varieties
Atoms they defined via this is birational invariant and are used to prove cubic 4-fold irrational, ~~in~~ authors claimed that they'll post their paper in arxiv later (now 2025.7.9). It seems that KPY's proof originally have some failure but now filled.

Some computations of wall-crossing ~~[illegible]~~ ③ By Yang Zhou

Thm ① $\overline{M}_{0,5} \cong Bl_{3pts}(\mathbb{P}^1\times\mathbb{P}^1)$ is del Pezzo surface of degree 5;
② $\overline{H}_{2,5} \cong Bl_{4pts}(\mathbb{P}^2) \cong Bl_{3pts}(\mathbb{P}^1\times\mathbb{P}^1)$ also ... ; $\overline{H}_{n,m}$ is compactified moduli of hyperplane arrangements $(\mathbb{P}^n, H_1,\dots,H_m)$
We use wall-crossing to give a detail proof.
What's wall-crossing?
Making invariants (g,n) from discrete to continuous (Hassett's weighted moduli);
Stability condition gives subset $\subset\mathbb{R}^n$, namely $\{(v_i)\in\mathbb{R}^n \mid 0<v_i\leq 1, \sum v_i > 2-2g\} = S_{g,n}$
Walls are $S_{g,n}\cap H$, H some hyperplane $\in\mathbb{R}^n$, here $H = H_I = \{(v_i)\in\mathbb{R}^n \mid \sum_{i\in I} v_i = 1\}$, $I\subseteq\{1,\dots,n\}$
Chambers are components of $S_{g,n}$ minus these H, precisely: $\begin{cases}\text{coarse chambers here: components of } S_{g,n}-\bigcup_{2<|I|} H_I \\ \text{fine chambers here: } \dots\ S_{g,n}-\bigcup_{2\leq|I|}\end{cases}$ are chamber decompositions
[redacted] [A. Bayer & Y. I. Manin ...]
Wall-crossing says, each chamber has invariant constant/moduli isomorphic.
but when we pass these codim-1 walls, the invariant changes/moduli map isn't isomorphic
(later ~~is~~ is stronger than former; but former is more common)
Pf of ①. Consider $\overline{M}_{0,\vec{v}_0}$, $\vec{v}_0 = (1-\varepsilon, 1-\varepsilon, 1-\varepsilon, y, y)$, $0<y\ll\varepsilon\ll 1$, such coefficient $y\neq 1-\varepsilon$ ensures the former three & latter two don't hit $\Rightarrow$ determined by the last two coordinates $\in\mathbb{P}^1\times\mathbb{P}^1 \Rightarrow \overline{M}_{0,\vec{v}_0}\cong\mathbb{P}^1\times\mathbb{P}^1$
We have natural map $\overline{M}_{0,5}\to\overline{M}_{0,\vec{v}_0}$, we claim: it's blow-up of three pts, due to generally is 1-to-1
but exceptional case is $(145|23)$, $(245|13)$ & $(345|12)$:

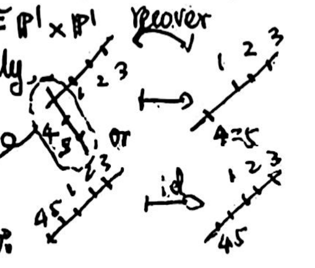


$1-\varepsilon+2y+1 = 2-\varepsilon+2y < 2$ due to $y\ll\varepsilon$ $\longmapsto$

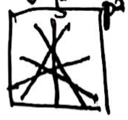


$\Rightarrow (145)$ isn't stable $\in\overline{M}_{0,\vec{v}_0}$
$2-2g+1$

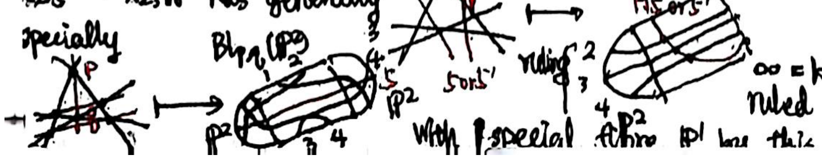
stable in $\overline{M}_{0,5}$, but not in $\overline{M}_{0,\vec{v}_0}$



What is the special fibre? It's $\overline{M}_{0,4}\cong\mathbb{P}^1$, hence the map is birational, exceptional at 3pts with fibre $\mathbb{P}^1$
$\Rightarrow \overline{M}_{0,5}\cong Bl_{3pts}(\mathbb{P}^1\times\mathbb{P}^1)$ $\boxtimes$
Pf of ②. Similarly consider $\overline{H}_{2,\vec{v}_0}$, $\vec{v}_0 = (1-\varepsilon, 1-\varepsilon, 1-\varepsilon, 1-\varepsilon, y)$, $0<y\ll\varepsilon\ll 1$. What is the stability here?
$\overline{H}_{2,\vec{v}_0}$ admits first four as $\overline{H}_{2,4}$, but the fifth placed arbitrary



No triple intersection $\Leftrightarrow$ No collinear three hyperplane in dual $(\mathbb{P}^2)^\vee$
$\Rightarrow \overline{H}_{2,\vec{v}_0}\cong(\mathbb{P}^2)^\vee\cong\mathbb{P}^2$
$\overline{H}_{2,5}\to\overline{H}_{2,\vec{v}_0}$ has generally is also 1-to-1
specially



$\overline{H}_{2,5} = Bl_{4pts}(\overline{H}_{2,\vec{v}_0}) = Bl_{4pts}(\mathbb{P}^2)$ $\boxtimes$
3 drawn planarly, left one in ∞
With special fibre $\mathbb{P}^1$ by this blow-up, here we have 4 cases of H has two triple pt

Moduli family of KSBA moduli space ② By Xu Chenyang

Notes: moduli ⟺ what we parametrize, geometric structure of moduli ⟺ what family we parametrize.

Scheme structure of KSBA moduli space:

Given flat family $X \to B$ with X_t has "good" singularities, then the key of the family is degeneration of stable family. here we just determine what KSBA-stable of a family above is: KSBA-stable = $\underbrace{\text{local KSBA-stable}}_{\text{local}}$ + $\underbrace{\omega_{X_t}\text{ ample}}_{\text{global}}$

(compare with curve case, X_t smooth & ω_{X_t} ample give, local stability is nodal)

we here only focus on local one, i.e. we're determining what X_t's shape changes in degeneration in KSBA.

Def. Local stable family $X \to S$ satisfy: ① $\omega_{X/S}^{[m_0]}$ Cartier, $\exists m_0$; ② (Kollár condition) $\omega_{X/S}^{[m]}|_{X_t} = \omega_{X_t}^{[m]}$, $\forall m$

②' differs ② by a nonreduced part in moduli and ② is more natural, hence we use ② instead of ②' (②' (Viehweg condition) $\omega_{X/S}^{[m_0']}|_{X_t} = \omega_{X_t}^{[m_0']}$, $\exists m_0'$)

Xu: "If one ask why we use ② instead ②', I think he just don't know what he is talking about."

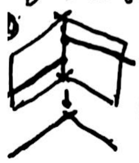
· Take $S = \operatorname{Spec} R$ in DVR case, ① implies ② as we deduce $\omega_{X/S}^{[m]}$ is CM via ① $\Rightarrow X_K$ is klt $\Rightarrow X$ is klt $\Rightarrow X$ CM $\Rightarrow \omega_{X/S}^{[m]}$ CM

hence we get the KSBA moduli $\subset (\mathcal{U} \to \text{Hilb})$ with scheme structure: $\exists Z_1 \subset \text{Hilb}$, s.t. $\omega^{[m]}_{U_{Z_1}/Z_1} \cong \mathcal{O}(1)$ (up to pullback of line bundle of Z_1), this we need a stratification of Z_1 to "approximate" the desired isomorphism, using Kollár's so-called husk-and-hull, then $\bigsqcup_{j \le m} Z_j =: Z$ is reduced part $\subset Z_1$, each Z_j makes $\omega^{[j]}_{U_{Z_j}/Z_j}$ is $\mathbb{Q}$-Cartier.

Thm (Numerical criterion) Over DVR here, ① L ample, X_t slc, i.e. $K_{X_t}^2 L_t^{n-2}$ not depend on t; ② K_{X_t} or $-K_{X_t}$ ample, i.e. $K_{X_t}^n$ or $(-K_{X_t})^n$ not depend on t; ③ L_t ample, i.e. L_t^n not depend on t. $\Big\}$ $X \to B = \operatorname{Spec} R$ is local stable $\Big\}$ L is $\mathbb{Q}$-Cartier

Now we lift the stability condition into log-case, i.e. $(X, D) \to B$, especially, we need to consider family of divisors.

Def. A relative family $(X, D) \to B$ is · $D_t \subset X_t$ Mumford divisor: $\operatorname{supp}(D_t)$ doesn't contain codim 1 singular pt of X_t & X_t is demi-normal; · $D \to B$ flat at generic pt. (Hence D not contain each fibre)



$D = D_1 + D_2$ isn't flat

When not flat, pullback at special fibre is impossible! Exclude these two cases are natural case?

① $D = D_1 + D_2$ contains some fibre generically pullback also not well-def.

Then we can pullback divisor to each fibre, what about non-DVR case?

$\underbrace{\text{Relative Mumford divisors}}_{\text{DVR case, weak}} \supset \underbrace{\text{K-flatness}}_{\text{Reduced base, by Kollár}} \supset$ Relative Cartier and for DVR, K-flatness = Relative Mumford;

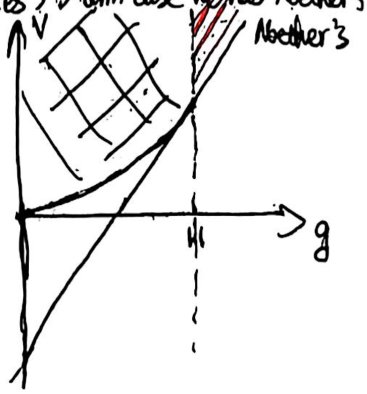
C-flat $\Rightarrow$ $\underbrace{\text{Stably C-flat} \supset \text{K-flat} \supset \text{locally K-flat}}_{\text{equal by Kollár}}$ $\Rightarrow$ formally K-flat; it's also expected that all these five conditions are equivalent.

Chow

With these "flat" families of $(X, D) \to B$, same local stability can be put into construct KSBA moduli.

Birational rigidity ② By Cheng Jiang and Z. Tian (Recalled version after lecture by me)

Geography result of 3-fold: Known 2-dim case dominated by Noether & Yau inequalities, 3-dim case we has Noether's inequality too $v \ge \frac{4}{3}g - \frac{10}{3}$, hence for general type X, we have the geograph:



when $g \in \mathbb{Z} \cap ([5, 10] \cup [11, \infty))$ and · $g \ge 11$ it's optimal, when $g < 11$ we have an arc-like bound expected

~~[illegible]~~ What impress me most is when $g \ge 11$, recently we find they're not continuously distributed:

· If $v > \frac{4}{3}g - \frac{10}{3}$ strictly $\Rightarrow v \ge \frac{4}{3}g - \frac{10}{3} + \frac{1}{6}$; (second Noether's inequality) ✓

· If $v > \frac{4}{3}g - \frac{10}{3} + \frac{1}{6}$ strictly $\Rightarrow v \ge \frac{4}{3} - \frac{10}{3} + \frac{2}{6}$; (third ..) ✓

(Conjecture) Then holds always (higher ..) ?

· When $g = 10$, we have counterexample of these second .. ones.

Why I think it's important is that it shows a change of rigidity in higher dim & genus: when one consider classification problem in AG, we can divide it into two parts: ① Choose proper birational invariants, draw a geograph; ② Ask that, each place of the geograph, the moduli < empty? / dimension/unique? / properties?

Hence here we can ask that: Is there's a "rigidity jump" ?

Birational superrigidity: consider hypersurface with type $(d, n) \subset \mathbb{P}^{n+1}$, we have $(d, n) = (3, 3)$ and $\operatorname{Pic}(X) \cong \mathbb{Z}$ generated by $-K_X$, or $(d, n) = (10, 10)$ and smooth $\Rightarrow \operatorname{Bir}(X) \cong \operatorname{Aut}(X)$

This result is surprising due to $Bir(X)$ is a so-large grp, but here it collapse into automorphism ☺

Birational invariants and rationality ①. By Lange (Recalled version after lecture by me)

Unramified cohomology and related cohomological invariants:

• Rationality problem always happens to be more interesting without $\Bbbk=\bar{\Bbbk}$ assumption, as in some special cases, X rational $\Leftrightarrow X(\Bbbk)\neq\emptyset$, hence H^* below can be chosen as H^*_{et} when $\Bbbk \subsetneq \bar{\Bbbk}$

Consider $X^{(i)}=\{x\in X \mid \operatorname{codim}\overline{\{x\}}=i\} \xrightarrow{i_x} X$, the residue field of each stratas give

• Gersten cpx: $0\to\bigoplus_{x\in X^{(0)}} j_* H^i(k(x)) \xrightarrow{\partial} \bigoplus_{x\in X^{(1)}} j_* H^{i-1}(k(x)) \xrightarrow{\partial} \cdots$ exact

• Bloch-Ogus thm: resolution of $\mathcal{H}^i: U\mapsto H^i(U)$ is $U\to\mathcal{H}^i \xrightarrow{\epsilon} \bigoplus_{x\in X^{(0)}} j_* H^i(k(x)) \xrightarrow{\partial} \bigoplus_{x\in X^{(1)}} j_* H^{i-1}(k(x)) \xrightarrow{\partial} \cdots$

The first two terms are $H^i(K(X))$ and $\bigoplus_{\text{Divisor}} j_* H^{i-1}(k(D))$, then $H^*_{nr}(X) := \operatorname{Ker}(H^*(K(X)) \to \bigoplus_D j_* H^{*-1}(k(D)))$ is birational invariant

• It's generalization of unramified cohomology in number theory for $X=\operatorname{Spec}k/\Bbbk$

• It's a special case of motivic cohomology, hence is homotopy/$\mathbb{A}^1$-invariant, we have Leray-Serre ss:

$H^p(X_{zar}, \mathcal{H}^q) \Rightarrow H^{p+q}_{sing}(X_{an}(\mathbb{C}))$, where $X(\mathbb{C}) \to X_{zar}$ can be view as a fibration.

"$\square$" parts are all birational invariants

$CH^q(X)=H^p(X;\mathcal{K}^p)$ by Bloch formula (where K-theory group holds. Bloch-Ogus too)

• $KH^p(X)=H^p(X;\mathcal{H}^d)$ for $p>d=\dim X$, Kato cohomology is also birational invariant

(Although some special cohomology is birational invariant, not general holds: CH isn't we know, blow-up changes the Chow grp).

Nicaise-Shinder's motivic volume: ...

These birational invariants provide a way to prove irrationality: Rational $\Rightarrow$ Stably rational

I just don't entre these definitions, but ask why intersection datum / cohomology can reflect on rationality: $\Rightarrow$ Rationally connected $\Rightarrow$ Diagonal factoried

Rationally connected means $\exists$ finite $\mathbb{P}^1$ connecting two pts;

A math Ph.D tells me cubic 3-fold (may some special kind?) rational $\Leftrightarrow$ each two $\mathbb{Q}$-pts (rational pts) can be connected by single $\mathbb{P}^1$;

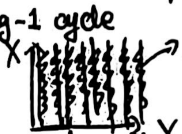
hence I'm convinced that cycles are important enough, to carry rational datum.

Factor the diagonal $[\Delta_X]=[X]\cdot \underset{\text{0-dim, deg-1 cycle}}{\underbrace{p}} + \underbrace{R}$ --- residue part (consider $\mathbb{P}^n$ and $\mathbb{A}^n$, both of them holds)

we need to pass to generic pt?

hence we consider $k(\xi)=K(X)/\Bbbk =: K/\Bbbk$

$CH_0(X_K)=\varinjlim_U CH_n(X\times U)$, $n=\dim X$



$\to X_K$ is "fibre-wise generic"

using degeneration/DVR, one can use these weaker-than-rationality properties to prove irrational via:

If X_R holds some property, we shall expect X also, then X_0 also; here, we take the diagonal decomposition

$\Rightarrow$ if special fibre isn't rational $\Rightarrow$ so is general fibre

Replace the "property" by "birational invariants", it also useful..

SCMS
Algebraic Geometry
Summer School
2025

Shanghai Center for Mathematical Sciences

Schedule &
Exercise Sheets

Week 1

Introduction to KSBA compactifications (J. Jiao)

Exercise

1. Let A, B be two $\mathbb{Q}$-divisors. Show that if $A \sim_{\mathbb{R}} B$, then $A \sim_{\mathbb{Q}} B$. In particular if K_X is $\mathbb{R}$-Cartier, then it is $\mathbb{Q}$-Cartier.

2. Show that if $f : Y \to X$ is a proper birational morphism of normal varieties and $D \in \mathrm{Div}(X)$ is a Cartier divisor on X and G is a Cartier divisor on Y, then

   (a) If D is big (resp. pseudo-effective) then so is f^*D;

   (b) If D is big (resp. pseudo-effective) and F is effective and its support contains all f-exceptional divisors, then $f_*^{-1}D + F$ is big (resp. pseudo-effective);

   (c) Show that if $G \sim G'$, then $f_*G \sim f_*G'$. Deduce from this that if G is big (resp. pseudo-effective) then so is f_*G;

   (d) Given an example where G is not big, but f_*G is big.

3. Suppose X is a projective variety and D is a divisor on X. Show that

   (a) $R(X, D) := \oplus_{m\geq 0}\mathcal{O}_X(mD)$ is a graded ring, and

   (b) if D is big and nef, then $R(X, D)$ is finitely generated if and only if D is semiample (that is, $|lD|$ is base point free for $l \in \mathbb{N}$ sufficiently divisible).

4. Show that if $(X, \Delta = \sum d_i\Delta_i)$ is a pair with $0 \leq d_i \leq 1$ and $\mathrm{Supp}(\Delta)$ has simple normal crossing, then $\mathrm{disp}(X, \Delta)$ is given by the minimum between 1, $1 - d_i$ and $1 - d_i - d_j$ where Δ_i intersects with Δ_j.

5. Use negativity lemma to show that if $(X, \Delta) \dashrightarrow (X', \Delta')$ is a step of $K_X + \Delta$-MMP, then $a(E, X, \Delta) \leq a(E, X', \Delta')$ for every prime divisor E over X. In particular, if (X, Δ) is lc (resp. plt, klt, canonical, terminal), then (X', Δ') is also lc (resp. plt, klt, canonical, terminal).

   Negativity Lemma. Let $X \to Y$ be a proper birational morphism of normal quasi-projective varieties. If $-B$ is an f-nef $\mathbb{R}$-Cartier divisor on Y, then

   (a) $B \geq 0$ if and only if $f_*B \geq 0$.

   (b) If $B \geq 0$ and $x \in X$, then either $f^{-1}(x) \subset \mathrm{Supp}(B)$ or $f^{-1}(x) \cap \mathrm{Supp}(B) = \emptyset$.

6. Let (X, Δ) be a projective klt pair such that $K_X + \Delta$ is big, (X', Δ') a minimal model of (X, Δ), and (X^c, Δ^c) the canonical model of (X, Δ). Use cone theorem to show that there exists a birational morphism $X' \to X^c$ and (X', Δ') is crepant birationally equivalent to (X^c, Δ^c) (that is, $a(E, X', \Delta') = a(E, X^c, \Delta^c)$ for every prime divisor E over X).

7. Let X/S be a projective scheme. Prove that $\mathrm{Hilb}_1(X/S) \cong X/S$.

8. Construct a projective morphism $f : X \to T$ such that

   - T is a smooth curve,
   - $X_a \cong X_b$ for every two closed points $a, b \in T$, and
   - K_{X_a} is ample.

9. Let X/S and Y/S be two projective schemes. Assume that X is flat over S. Let $\mathrm{Hom}_S(X, Y)$ be the functor

$$\mathrm{Hom}_S(X, Y)(T) := \{\text{morphisms over } T : X \times_S T \to Y \times_S T\}.$$

   Similarly we define $\mathrm{Isom}_S(X, Y)$ and $\mathrm{Aut}_S(X)$ that represent the functor of isomorphisms and automorphisms.

   Use the following lemma to show that $\mathrm{Hom}_S(X, Y)$ is represented by an open subscheme

$$\mathrm{Hom}_S(X, Y) \subset \mathrm{Hilb}(X \times_S Y/S).$$

   Lemma: Let $0 \in T$ be the spectrum of a local ring, U/T flat and proper and V/T arbitrary. Let $p : U \to V$ be a morphism over T. If $p_0 : U_0 \to V_0$ is an isomorphism, then p is an isomorphism.

10. Let $f_1 : X_1 \to S$ and $f_2 : X_2 \to S$ be two smooth morphisms over a variety S, such that $K_{X_1/S}$ and $K_{X_2/S}$ are relatively ample. Show that

   (a) for any $m \in \mathbb{N}$ sufficiently divisible, there exists a closed embedding

$$\mathrm{Isom}_S(X_1, X_2) \subset \mathrm{Isom}_S(\mathbb{P}_S(F_1), \mathbb{P}_S(F_2)),$$

   where $F_i := (f_i)_* \mathcal{O}_{X_i}(mK_{X_i/S})$;

   (b) $\mathrm{Isom}_S(X_1, X_2)$ is affine over S; } rel. dim 0 $\Rightarrow$ finite stablizer moduli

   (c) $\mathrm{Isom}_S(X_1, X_2)$ is proper over S.

11. Let X/S be projective and flat. Consider the functor $\mathrm{Cdiv}(X/S) : \{\text{Schemes over } S\} \to \{\text{sets}\}$ defined as

$$\mathrm{Cdiv}(X/S)(Z) := \{\text{Effective Cartier divisor of } X \times_S Z \text{ which is flat over } Z\}.$$

Show that $\mathrm{Cdiv}(X/S)$ is represented by an open subscheme

$$\mathrm{Cdiv}(X/S) \subset \mathrm{Hilb}(X/S).$$

12. Consider the following example:

Let $R \subset \mathbb{P}^4$ be a quartic rational normal curve, i.e., the image of the embedding of $\mathbb{P}^1$ into $\mathbb{P}^4$ by the global sections of $\mathcal{O}_{\mathbb{P}^1}(4)$. For example take

$$R = \{[u^4 : u^3v : u^2v^2 : uv^3 : v^4] \in \mathbb{P}^4 \mid [u : v] \in \mathbb{P}^1\}.$$

Let $T \subset \mathbb{P}^5$ be a quartic rational scroll, i.e., the image of the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^5$ by the global sections of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 2)$. For example take

$$T = \{[xz^2 : xzt : xt^2 : yz^2 : yzt : yt^2] \in \mathbb{P}^5 \mid ([x : y], [z : t]) \in \mathbb{P}^1 \times \mathbb{P}^1\}.$$

Let $C_R \subset \mathbb{P}^5$ and $C_T \subset \mathbb{P}^6$ be the projectivized cones:

$$\begin{aligned} C_R &= \{[u^4 : u^3v : u^2v^2 : uv^3 : v^4 : w^4] \in \mathbb{P}^5 \mid [u : v : w] \in \mathbb{P}^2\}, \text{ and} \\ C_T &= \{[xz^2 : xzt : xt^2 : yz^2 : yzt : yt^2 : pq^2] \in \mathbb{P}^6 \mid \\ &\qquad ([x : y : p], [z : t : q]) \in \mathbb{P}^2 \times \mathbb{P}^2 \setminus (l_1 \cup l_2)\} \end{aligned}$$

where $l_1 = (p = z = t = 0), l_2 = (x = y = q = 0) \subset \mathbb{P}^2 \times \mathbb{P}^2$ are two lines.

(a) Show that both T and C_R are hyperplane sections of C_T (intersection with a hyperplane in $\mathbb{P}^6$). In particular, there exists a flat morphism $X \to R$ to an irreducible smooth curve such that both T and C_R are fibers of $X \to R$.

(b) Let $V \subset \mathbb{P}^5$ be the embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$ by the composition of global sections of $\mathcal{O}_{\mathbb{P}^2}(2)$ and the 4-to-1 endomorphism of $\mathbb{P}^2, [u : v : w] \mapsto [u^2 : v^2 : w^2]$:

$$V = \{[u^4 : v^2w^2 : u^2v^2 : u^2w^2 : v^4 : w^4] \mid [u : v : w] \in \mathbb{P}^2\}.$$

Show that there exists a flat morphism $Y \to S$ to an irreducible smooth curve such that both V and C_R are fibers of $Y \to S$.

(c) Show that the canonical divisor K_X and K_Y cannot be both $\mathbb{Q}$-Cartier.

KSBA compactifications of moduli of K3 surfaces (V. Alexeev)

Lecture 1

- Basics of KSBA theory and known concretely described cases.
- K3 surfaces from reflexive polytopes. The $\sum \ell(e_i)\ell(e_i^*) = 24$ identity.
- First degenerations of K3 surfaces from toric geometry.
- Kulikov degenerations.
- Anticanonical pairs and their charge.
- Symington polytopes and pseudo-fans.
- Integral affine spheres IAS^2 from the A-side (symplectic geometry).
- Integral affine spheres IAS^2 from the B-side (algebraic geometry).

Lecture 2

- Polarized K3 surfaces and their moduli spaces F_{2d}.
- Lattice-polarized K3 surfaces and their moduli spaces F_S.
- Moduli spaces F_ρ of K3 surfaces with a nonsymplectic automorphism.
- Compactification of arithmetic quotients $\mathbb{D}/\Gamma$: Baily-Borel, toroidal, semitoroidal, KSBA.
- Two moduli problems: $\overline{P}_{2d}$ and $\overline{F}_{2d}$; resp. $\overline{P}_S$ and $\overline{F}_S$.
- Degenerations of K3 surfaces: Kulikov, nef, divisor models, KSBA.
- Canonical polarizing divisor R.
- Recognizable divisors R. Main theorems for $\overline{F}^R$.
- Mirror symmetry moves.
- The mirror theorem of Engel–Friedman.

Lecture 3

- KSBA compactification of the moduli space $F_{(2,2,0)}$ of K3 surfaces which are double covers of $\mathbb{P}^1 \times \mathbb{P}^1$.
- Kulikov degenerations and KSBA compactifications of moduli spaces of K3 surfaces with a nonsymplectic involution (75/50 cases).
- KSBA compactification of the moduli space E_2 of Enriques surfaces of degree 2.
- KSBA compactifications of moduli spaces of K3 surfaces with a nonsymplectic automorphism of order 3.
- KSBA compactifications of the moduli spaces of anticanonical pairs $(X, D + \epsilon B)$.

References

AAB24 Valery Alexeev, Hulia Arguz, and Pierrick Bousseau, *The KSBA moduli space of stable log Calabi-Yau surfaces*, Preprint (2024), arXiv:2402.15117.

ADH24 Valery Alexeev, Anand Deopurkar, and Changho Han, *Compactifications of moduli spaces of K3 surfaces with a higher-order nonsymplectic automorphism*, Preprint (2024), arXiv:2412.11256.

AE22 Valery Alexeev and Philip Engel, *Compactifications of moduli spaces of K3 surfaces with a nonsymplectic involution*, arXiv:2208.10383 (2022).

AE23 by same author, *Compact moduli of K3 surfaces*, Ann. of Math. (2) **198** (2023), no. 2, 727–789.

AE25 by same author, *On lattice-polarized K3 surfaces*, Preprint (2025), arxiv:2505.22557.

AEGS25 Valery Alexeev, Philip Engel, D. Zack Garza, and Luca Schaffler, *Compact moduli of Enriques surfaces with a numerical polarization of degree 2*, Nagoya Math. J., to appear (2025), arXiv:2312.03638.

AET23 Valery Alexeev, Philip Engel, and Alan Thompson, *Stable pair compactification of moduli of K3 surfaces of degree 2*, J. Reine Angew. Math. **799** (2023), 1–56.

Ale02 Valery Alexeev, *Complete moduli in the presence of semiabelian group action*, Ann. of Math. (2) **155** (2002), no. 3, 611–708.

Ale15 by same author, *Moduli of weighted hyperplane arrangements*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser/Springer, Basel, 2015, Edited by Gilberto Bini, Martí Lahoz, Emanuele Macrì and Paolo Stellari.

EF21 Philip Engel and Robert Friedman, *Smoothings and rational double point adjacencies for cusp singularities*, J. Differential Geom. **118** (2021), no. 1, 23–100.

Eng18 Philip Engel, *Looijenga's conjecture via integral-affine geometry*, J. Differential Geom. **109** (2018), no. 3, 467–495.

Sea87 Francesco Scattone, *On the compactification of moduli spaces for algebraic $K3$ surfaces*, Mem. Amer. Math. Soc. **70** (1987), no. 374, x+86.

Sym03 Margaret Symington, *Four dimensions from two in symplectic topology*, Topology and geometry of manifolds (Athens, GA, 2001), Proc. Sympos. Pure Math., vol. 71, Amer. Math. Soc., Providence, RI, 2003, pp. 153–208.

Exercises

- **Toric varieties from lattice polytopes and their fans**

Exercise 1. Understand that the following lattice polytopes Q in M correspond to the given toric varieties V with ample line bundle L.

1. Simplex with vertices $(0,0,0)$, $(4,0,0)$, $(0,4,0)$, $(0,0,4)$ gives $(\mathbb{P}^3, \mathcal{O}(4))$.
2. The 6-6-6 triangle, with vertices $(0,0,0)$, $(6,0,0)$, $(0,6,0)$ gives $(V,L) = (\mathbb{P}^2, \mathcal{O}(6))$.
3. The 4×4-square, with vertices $(0,0,0)$, $(4,0,0)$, $(0,4,0)$, $(4,4,0)$ gives $(V,L) = (\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}(4,4))$.
4. The 12-3-3 triangle, with vertices $(-6,0,0)$, $(6,0,0)$, $(0,3,0)$ gives $V = \mathbb{P}(1,1,4) = \overline{\mathbb{F}}_4$.
5. The 8-4-4 triangle, with vertices $(-4,0,0)$, $(4,0,0)$, $(0,4,0)$ gives the quadratic cone $V = \mathbb{P}(1,1,2) = \overline{\mathbb{F}}_2$.

- **K3 surfaces from reflexive polytopes**

Exercise 2. Prove that the following polytopes are reflexive (shift them appropriately, so that the central point is at the origin $(0,0,0)$).

1. Simplex with vertices $(0,0,0)$, $(4,0,0)$, $(0,4,0)$, $(0,0,4)$; shifted by $(1,1,1)$.
2. Pyramid of height 2 over the 6-6-6 triangle, with vertices $(0,0,0)$, $(6,0,0)$, $(0,6,0)$, $(2,2,2)$; shifted by $(2,2,1)$.
3. Pyramid of height 2 over the 4×4-square, with vertices $(0,0,0)$, $(4,0,0)$, $(0,4,0)$, $(4,4,0)$, $(2,2,2)$; shifted by $(2,2,1)$.
4. Pyramid of height 2 over the 12-3-3 triangle, with vertices $(-6,0,0)$, $(6,0,0)$, $(0,3,0)$, $(0,2,2)$; shifted by $(0,2,1)$.
5. Pyramid of height 2 over the 8-4-4 triangle, with vertices $(-4,0,0)$, $(4,0,0)$, $(0,4,0)$, $(0,2,2)$; shifted by $(0,2,1)$.

Exercise 3. Prove that the generic K3 surfaces $X \in |-K_V|$ corresponding to the above polytopes are:

1. Quartic surfaces in $\mathbb{P}^3$.

2. Double covers over $\mathbb{P}^2$ ramified in a curve of degree 6.

3. Double covers over $\mathbb{P}^1 \times \mathbb{P}^1$ ramified in a curve of bidegree $(4,4)$.

4. Double covers over the cone $\mathbb{P}(1,1,4) = \overline{\mathbb{F}}_4$ ramified in a 3-section of this cone and at the vertex.

5. Double covers over the quadratic cone $\mathbb{P}(1,1,2) = \overline{\mathbb{F}}_2$ ramified in a 2-section of this cone.

- **Symington polytopes and pseudo-fans**

Exercise 4. Describe the pseudofans for the following anticanonical pairs (X,D).

1. $X = \mathbb{P}^2$, $D = D_1 + D_2 + D_3 =$ three lines, intersecting normally.

2. $X = \mathbb{P}^2$, $D = D_1 + D_2 =$ a line and a conic, intersecting normally.

3. $X = \mathbb{P}^2$, $D =$ a nodal cubic.

4. $X = \mathbb{P}^1 \times \mathbb{P}^1$, $D = D_1 + D_2$, with both D_i smooth curves in $|\mathcal{O}(1,1)|$, intersecting normally.

Exercise 5. Describe the Symington polytopes for the above anticanonical pairs, with the given ample line bundles L.

1. $X = \mathbb{P}^2$, $D = D_1 + D_2 + D_3 =$ three lines, intersecting normally, $L = \mathcal{O}(1)$.

2. $X = \mathbb{P}^2$, $D = D_1 + D_2 =$ a line and a conic, intersecting normally, $L = \mathcal{O}(1)$.

3. $X = \mathbb{P}^2$, $D =$ a nodal cubic, $L = \mathcal{O}(1)$.

4. $X = \mathbb{P}^2$, $D =$ a smooth cubic, $L = \mathcal{O}(1)$.

5. $X = \mathbb{P}^1 \times \mathbb{P}^1$, $D = D_1 + D_2$, with both D_i smooth curves in $|\mathcal{O}(1,1)|$, intersecting normally, $L = \mathcal{O}(1,1)$.

- **Canonical polarizing divisors for moduli of K3 surfaces**

Exercise 6. Can you think of a canonical choice of a polarizing divisor $R \in |L^N|$ in some other cases, for example for the moduli space F_4 of K3 surfaces of degree 4, which generically are quartics $X_4 \subset \mathbb{P}^3$? Other examples?

Exercise 7. Can you think of a canonical choice of a polarizing divisor $R \in |L^N|$ for some moduli spaces of anticanonical pairs $(X, D + \epsilon R)$, $R \in |L|$?

- **Explicit KSBA compactifications of moduli of K3 surfaces**

Exercise 8. Repeat and understand the construction of the Symington polytope for the mirrors to the K3 double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified in $B \in \mathcal{O}(4,4)$ given in the lecture.

Exercise 9. Repeat and understand the construction of the corresponding Kulikov models given in the lecture.

Exercise 10. Repeat and understand the construction of the corresponding KSBA stable models given in the lecture.

Exercise 11. Repeat and understand the construction of the KSBA stable models of Enriques surfaces of degree 2 given in the lecture.

Irrationality of Hypersurfaces (J. Lange)

Lecture 1

$k(X) = k(\xi)$ residue field of generic pt
Hence base change of K/k is closely related with $X \times X$.
$X_K \subset X \times X$ as a X with each pt has a generic fibre of X =
$CH_0(X_K) = \varinjlim_{U} CH_n(X \times U)$

1. **Problem 1.** Let K/k be a finitely generated field extension and let m be a positive integer that is invertible in k. Consider the structure morphism $\mathbb{A}^1_K \to \operatorname{Spec} K$. Show that the canonical morphism

$$f^*\colon H^i_{nr}(K/k, \mu_m^{\otimes j}) \longrightarrow H^i_{nr}(K(\mathbb{A}^1_K)/k, \mu_m^{\otimes j})$$

is an isomorphism.

Q. Why we're asking for K/k unramified part?

Hint: Use the Faddeev exact sequence:

DVR case $\overset{\operatorname{Spec}(R)}{\longleftrightarrow}$ k K ; $k = \bar{k}$ closed, generic

$$0 \longrightarrow H^i_{nr}(K/k, \mu_m^{\otimes j}) \xrightarrow{f^*} H^i_{nr}(K(\mathbb{A}^1_K)/k, \mu_m^{\otimes j}) \xrightarrow{\sum \partial_x} \bigoplus_{x \in \mathbb{P}^1_K} H^{i-1}(\kappa(x), \mu_m^{\otimes(j-1)}),$$

where x runs through all closed points of $\mathbb{P}^1_K$ and $\partial_x = \partial_{\mathcal{O}_{\mathbb{P}^1_K, x}}$.

If $\bar{X} = X \times_k \bar{k}$ holds (P), then X also holds (P)
then special fibre Y/k also holds (P)

2. **Problem 2.** Recall the Hassett-Pirutka-Tschinkel quadric surface bundle example

$$X = \{yzs^2 + xzt^2 + xyu^2 + (x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)v^2 = 0\} \subset \mathbb{P}^2_k \times_k \mathbb{P}^3_k$$

where x, y, z are the coordinates of $\mathbb{P}^2$ and s, t, u, v are the coordinates of $\mathbb{P}^3_k$. Let $\alpha = (x, y) \in H^2(k(\mathbb{P}^2), \mathbb{Z}/2)$ and let $f\colon X \to \mathbb{P}^2$ be the projection onto the first factor. Then $f^*\alpha \in H^2_{nr}(k(X), \mathbb{Z}/2)$ and $f^*\alpha \neq 0$. Prove that the restriction of $f^*\alpha$ to the generic point of the closed subschemes D_x, D_s, D_t, D_u, and D_v in X vanishes, where

Here our (P) is used as $\mathbb{Z}$-decomposition of diagonal, (or other conditions related with rationality)

$$D_w := \{w = 0\} \subset X$$

for each $w \in \{x, s, t, u, v\}$.

If we need prove something general, it suffices to prove for something special via this type limit process

3. **Problem 3.** Let k be a field and $n \in \mathbb{Z}_{\geq 0}$ a non-negative integer. Show that

$$\mathrm{CH}_i(\mathbb{A}^n_k) = \begin{cases} \mathbb{Z} \cdot [\mathbb{A}^n_k] & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Deduce via the localization exact sequence that $\mathrm{CH}_i(\mathbb{P}^n_k) = \mathbb{Z} \cdot [\mathbb{P}^i_k]$ for all $0 \leq i \leq n$.

4. **Problem 4.** Show directly that $\mathbb{P}^n_k$ admits a decomposition of the diagonal for every integer $n \geq 1$ and for every field k.

Hint: You might want to show first

$$\mathrm{CH}_n(\mathbb{P}^n_k \times_k \mathbb{P}^n_k) = \bigoplus_{i=0}^{n} \mathbb{Z} \cdot [\mathbb{P}^i_k \times_k \mathbb{P}^{n-i}_k].$$

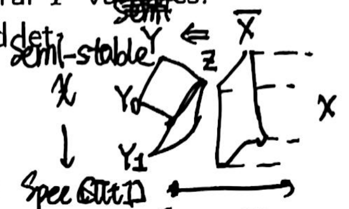
Q. If X admits $\mathbb{Z}$-decomposition of diagonal, $\delta_X = [X] \cdot p_0 + Z$
then all $H^i(X(\mathbb{C}); \mathbb{Z})$ are torsion-free ($\mathbb{Z}$-module)
0-dim residue part (degree 1)

5. **Problem 5.** Let F be a field of characteristic 0 and let $\mathbb{Z}[\mathrm{SB}_F]$ be the free abelian group generated by stable birational equivalence of integral F-varieties. Let $K = \mathbb{C}\{\{t\}\} = \bigcup_{n>0} \mathbb{C}((t^{\frac{1}{n}}))$ be the field of Puiseux series and let

$= \overline{\mathbb{C}((t))}$

$= \mathrm{Frac}(\overline{\mathbb{C}[[t]]})$

$$\mathrm{Vol}\colon \mathbb{Z}[\mathrm{SB}_K] \longrightarrow \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$$



be the motivic volume morphism from Nicaise-Shinder. Show that

(a) $\mathrm{Vol}([\mathrm{Spec}\,K]_{sb}) = [\mathrm{Spec}\,\mathbb{C}]_{sb}$;

(b) If $\mathcal{X} \to \mathrm{Spec}\left(\bigcup_{n>0} \mathbb{C}[[t^{\frac{1}{n}}]]\right)$ is smooth and proper, then $\mathrm{Vol}([\mathcal{X}_K]) = [\mathcal{X}_k]$.

Deduce that stable rationality specializes in smooth and proper families.

Volume is also a limit to the special fibre, and Nicaise-Shinder shows that,
$\mathrm{Vol}([\overline{X}]_{sb}) = \sum_{J \subset I} (-1)^{|J|+1} [Y_J]_{sb}$, $Y_J = \bigcap_{j \in J} Y_j$

e.g.  has $\mathrm{Vol}([\overline{X}]_{sb}) = [Y_0]_{sb} + [Y_1]_{sb} - [Z]_{sb}$ as $Z = Y_0 \cap Y_1$

6. **Problem 6.** Consider the family

$$\mathcal{X} := \{x_0^3 + x_1^3 + x_2^3 + tx_3^3 = 0\} \subset \mathbb{P}^3_{k[t]_{(t)}}.$$

$\mathbb{P}^2$ blow up six pts $\Rightarrow$ rational

The generic fibre is a cubic surface and the special fibre is a projective cone over an elliptic curve. Why is this not a contradiction to Problem 5?

$P = (0,0,0,1), t=0$ is the singularity/vertex of cone.

Blow up $\tilde{\mathcal{X}} \to \mathcal{X} \to \mathrm{Spec}(k[t]_{(t)}) \Rightarrow \mathrm{Vol}([\mathcal{X}_K]_{sb}) = [E_0]_{sb} + [\tilde{X}_0]_{sb} - [E \cap \tilde{X}_0]_{sb} \overset{E \cong \mathbb{P}^2}{=} [\mathrm{Spec}\,k]_{sb} + [C]_{sb} - [C]_{sb}$

$E \cup \tilde{X}_0$

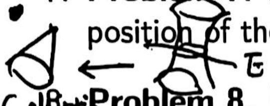
~~$= [\mathrm{Spec}\,K]_{sb}\ [\mathrm{Spec}\,k]_{sb}$ (By ex 5)~~

$[\tilde{X}_0]_{sb} = [C]_{sb}$

$[E \cap \tilde{X}_0]_{sb} = [C]_{sb}$

(By ex 6) Hence not contradi[ction]

7. **Problem 7.** Let X be a proper k-variety. Show that X does not admit a decomposition of the diagonal relative to the empty set.



Problem 8. Let X be a variety over a field k, let $W \subset X$ be a closed subset and let Λ be a commutative ring with 1. Prove the following properties of the relative decomposition of the diagonal.

(a) If X admits a $\mathbb{Z}$-decomposition of the diagonal relative to W, then X admits also a $\mathbb{Z}/m$-decomposition of the diagonal relative to W for every $m \in \mathbb{Z}$;

(b) If X admits a Λ-decomposition of the diagonal relative to W, then X admits a Λ-decomposition of the diagonal relative to W' for all $W \subset W' \subset X$ closed subset.

(c) If $\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$ is an isomorphism, then X admits a decomposition of the diagonal if and only if X admits a $\mathbb{Z}$-decomposition of the diagonal relative to every zero-dimensional closed subscheme containing a zero-cycle of degree 1.

9. **Problem 9.** We aim to show that a very general hypersurface of bidegree $(2,3)$ in $\mathbb{P}^1_k \times_k \mathbb{P}^4_k$ over a field of characteristic different from 2 does not admit a decomposition of the diagonal.

   (a) Write down a special bidegree $(2,3)$ hypersurface Z in $\mathbb{P}^1_k \times_k \mathbb{P}^4_k$, which is birational to the Hassett-Pirutka-Tschinkel example X in Problem 2.

   (b) Show that Z does not admit a $\mathbb{Z}/2$-decomposition of the diagonal using the Merkurjev pairing.

   (c) Conclude the statement.

10. **Problem 10.** We reprove the result that a very general quartic fivefold over a field k of characteristic different from 2 does not admit a decomposition of the diagonal.

   (a) Show that the hypersurface $Z = \{f = 0\} \subset \mathbb{P}^5_k$ given by

   $$f = x_0^2 y_1^2 + x_1 x_2 y_2^2 + x_0 x_2 y_3^2 + x_0 x_1 (x_0^2 + x_1^2 + x_2^2 - 2x_0x_1 - 2x_0x_2 - 2x_1x_2)$$

   in $k[x_0, x_1, x_2, y_1, y_2, y_3]$ is birational to X from Problem 2. In particular, Z has a nontrivial unramified cohomology class.

   (b) Consider the degeneration

   $$\mathcal{X} := \{tx_0^2 + zw = f + x_0^3 z + x_0^2 y_1 w = 0\} \subset \mathbb{P}^7_{k[t]_{(t)}} \longrightarrow \operatorname{Spec} k[t]_{(t)}.$$

   Let $W_{\mathcal{X}} \subset \mathcal{X}$ be the closed subscheme given by $x_0 x_1 x_2 y_1 = 0$. Check that $\mathcal{X} \setminus W_{\mathcal{X}}$ is a strictly semi-stable $k[t]_{(t)}$-scheme.

   (c) Describe the special fibre of $\mathcal{X}$ and prove that the assumption of the criterion are satisfied.

   (d) Conclude that the very general quartic fivefold over k does not admit a $\mathbb{Z}/2$-decomposition of the diagonal.

SCMS
Algebraic Geometry
Summer School
2025

Shanghai Center for Mathematical Sciences

Exercise Sheets
Week 2

Week 2

Wall crossing for moduli of stable pairs (Z. Zhuang)

Exercise

Question 1. In this problem, we show that $\overline{M}_{0,5}$, the (coarse) moduli space of stable curves of genus 0 with 5 (labeled) marked points, is a del Pezzo surface of degree 5.

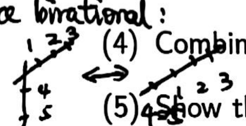
(1) Show that stable pointed curves of genus 0 have trivial automorphism groups. Conclude that $\overline{M}_{0,5}$ is smooth. $\overline{M}_{0,5} = Bl_{3pts}(\mathbb{P}^1\times\mathbb{P}^1) = Bl_{4pts}(\mathbb{P}^2)$ ▨

(2) Let $\overline{M}_{0,\mathbf{v}}$ be the moduli space of weighted pointed stable curves of genus 0 with coefficient vector $\mathbf{v}$. Show that $\overline{M}_{0,\mathbf{v}_0} \cong \mathbb{P}^1\times\mathbb{P}^1$ when $\mathbf{v}_0 = (1-\varepsilon, 1-\varepsilon, 1-\varepsilon, \eta, \eta)$ and $0 < \eta \ll \varepsilon \ll 1$. (2) Coefficient ensures that the second two (λ,μ) not hit $(0,1,\infty)$ first three: hence not ~~blow-up of~~ diagonal pts, but just determined by (λ,μ) $\in \mathbb{P}^1\times\mathbb{P}^1 \Rightarrow \overline{M}_{0,\vec{v}_0} = \mathbb{P}^1\times\mathbb{P}^1$ ▨

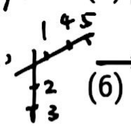
(3) Describe the reduction morphism $\overline{M}_{0,5} \to \overline{M}_{0,\mathbf{v}_0}$. Is it birational? What are the exceptional divisors?

(3) $\overline{M}_{0,5} \to \overline{M}_{0,\vec{v}_0}$
$\;\;\shortparallel \qquad\qquad \shortparallel$
$Bl_{3pt}(\mathbb{P}^1\times\mathbb{P}^1) \to \mathbb{P}^1\times\mathbb{P}^1$ is just the blow up, and hence birational:

• Generally,



• Exceptionally,



1=4=5 here we use $-2+2g<0$
we can contract (1,4,5) component with fibre is $\cong \overline{M}_{0,4} = \mathbb{P}^1$
and (1,4,5) (2,4,5) (3,4,5) has three disjoint exceptional divisor $\mathbb{P}^1 \Rightarrow$ it's blow-up $\overline{M}_{0,\vec{v}_0} \cong \mathbb{P}^1\times\mathbb{P}^1$ with three pts ▨

(4) Combine (2) and (3) to show that $\overline{M}_{0,5}$ is a del Pezzo surface of degree 5.

(4) (2)+(3) proves $\overline{M}_{0,5} = Bl_{3pts}(\mathbb{P}^1\times\mathbb{P}^1)$

(5) Show that the symmetric group S_5 has a faithful action on the del Pezzo surface of degree 5.

Permute marked pts may makes not survive as stable curve $\Rightarrow$ faithful ▨

(6) The del Pezzo surface of degree 5 has 10 (-1)-curves. Describe these curves on $\overline{M}_{0,5}$?

(6) have $\binom{5}{2}$ choices, each corresponds to $\cong \mathbb{P}^1$ of (3|2) form $\Rightarrow 10 = \binom{5}{2}$ exceptional (-1)-curves ▨

(7) Let E be a (-1)-curve on $\overline{M}_{0,5}$. Can you realize the blow down of E as a reduction morphism $\overline{M}_{0,5} \to \overline{M}_{0,\mathbf{v}}$ for some coefficient vector $\mathbf{v}$?

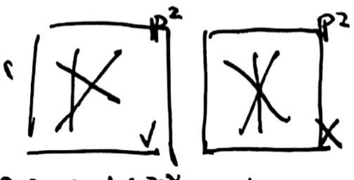
(7) Compare (3) and (6), this is just a subtle deal with coefficients ▨

Question 2. A hyperplane arrangement is an ordered collection $H_1, \ldots, H_d \subset \mathbb{P}^n$ of hyperplanes in the projective space, viewed as a stable pair $(\mathbb{P}^n, H_1 + \cdots + H_d)$. Let $\overline{H}_{n,d}$ be the irreducible component of the KSBA moduli space that generically parametrizes hyperplane arrangement $(\mathbb{P}^n, H_1 + \cdots + H_d)$, and let $\overline{H}_{n,\mathbf{v}}$ be the component that generically parametrizes weighted hyperplane arrangement $(\mathbb{P}^n, a_1H_1 + \cdots + a_dH_d)$ with coefficient vector $\mathbf{v} = (a_1, \cdots, a_d)$.

In this problem, we show that $\overline{H}_{2,5}$ is also a del Pezzo surface of degree 5.

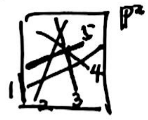
(1) Show that $\overline{H}_{2,\mathbf{v}_0} \cong \mathbb{P}^2$ when $\mathbf{v}_0 = (1-\varepsilon, 1-\varepsilon, 1-\varepsilon, 1-\varepsilon, \eta)$ and $0 < \eta \ll \varepsilon \ll 1$.

• Stability in hyperplane arrangement is no triple intersections (when no coefficients)

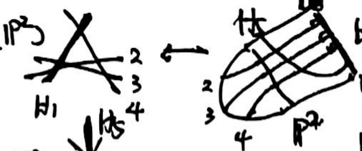


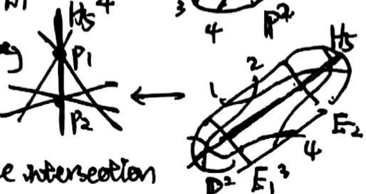
$\Leftrightarrow$ in dual $(\mathbb{P}^2)^\vee$, no three pts colinear.

Here $\overline{H}_{2,\vec{v}_0}$ is



Here 1, 2, 3, 4 determines coordinates of $\mathbb{P}^2 \Rightarrow$ determined by $H_5 \in (\mathbb{P}^2)^\vee \cong \mathbb{P}^2$
$\Rightarrow \overline{H}_{2,\vec{v}_0} \cong \mathbb{P}^2$ ▨

(2). Generally, $Bl_{H_1}(\mathbb{P}^2)$  ruling surface F_1

• Exceptionally, $Bl_{2pts}(\mathbb{P}^2)$ 

Such H_5 pass two triple intersection pts has 4 choices

 and one at ∞

(2) Show that the reduction morphism $\overline{H}_{2,5} \to \overline{H}_{2,v_0}$ is a blowup of ~~4~~ 4 points on $\overline{H}_{2,v_0}$. (Which 4 points?) Conclude that $\overline{H}_{2,5}$ is a del Pezzo surface of degree 5.

they're four disjoint exceptional

$\Rightarrow \overline{H}_{2,5} = Bl_{4pts}(\overline{H}_{2,v_0}) = Bl_{4pts}(\mathbb{P}^2)$ ☒

(3) From the previous problems we see that $\overline{H}_{2,5} \cong \overline{M}_{0,5}$. Can you give a modular interpretation of this isomorphism? Hint: Lines in $\mathbb{P}^2$ are parametrized by points in the dual plane $\tilde{\mathbb{P}}^2$, and through 5 general points in $\tilde{\mathbb{P}}^2$ there is a unique conic. Show that this induces a birational map $\overline{H}_{2,5} \dashrightarrow \overline{M}_{0,5}$, and that it extends to an isomorphism.

The next two problems are more open-ended (the answers may or may not be known).

Question 3. Are there other nontrivial isomorphisms between the moduli spaces $\overline{H}_{n,d}$? (Notice that $\overline{H}_{1,d} = \overline{M}_{0,d}$, so $\overline{H}_{1,5} \cong \overline{H}_{2,5}$ by the previous problems.)

Question 4 (*cf.* [1]). Let $\mathcal{A}_{n,\varepsilon}$ be the irreducible component of the KSBA moduli stack that generically parametrizes pairs $(A, \varepsilon\Theta)$ where (A, Θ) are polarized abelian varieties of dimension n. Prove (or disprove) that $\mathcal{A}_{n,\varepsilon} \cong \mathcal{A}_{n,\varepsilon'}$ for any $\varepsilon, \varepsilon' \in (0, 1]$.

Hint: when Θ is principal, this was proved by Tenini.

References

[Ale02] Valery Alexeev, *Complete moduli in the presence of semiabelian group action*, Ann. of Math. (2) **155** (2002), no. 3, 611–708.

Introduction to Gromov-Witten invariants and quantum cohomology (Y. Zhou)

Exercise

Question 1. We say a set of points in $\mathbb{CP}^2$ are in general position if any three of them do not lie on a line. Given 5 points on $\mathbb{CP}^2$ in general position, show that there is exactly 1 smooth conic through them.

Question 2. Show that there are exactly 2875 lines on a generic quintic hypersurface in $\mathbb{CP}^4$. Give an example where there are indeed infinitely many lines on a specific smooth quintic.

Question 3. Consider the family of stable maps $f_t : \mathbb{P}^1 \to \mathbb{P}^2$,

$D_1 = (u)$

$$f_t([u, v]) = [2tu^2, 2tuv, u^2 + v^2].$$

$D_2 = (v)$

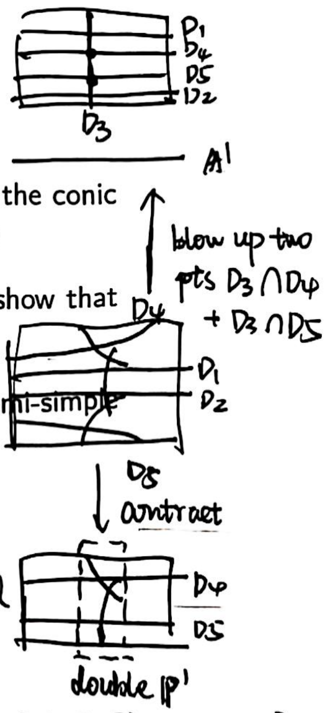
$2D_1 + D_3 \quad D_1 + D_2 + D_3 \quad D_4 + D_5$

$D_3 = (t)$

Describe its stable map limit as $t \to 0$. Note that f_t is a parameterization of the conic $x^2 + (y - t)^2 = t^2$.

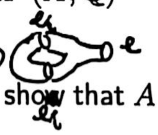
$D_4 = (u + \sqrt{-1}\, v)$

$D_5 = (u - \sqrt{-1}\, v)$



Question 4. Let X be a Calabi-Yau manifold (we assume $H^1(X, \mathbb{Q}) = 0$), show that its (big) quantum cohomology is nilpotent.

$e = \sum e_i e_i^*$ given a basis $\Leftrightarrow$



Question 5. Let A be a finite dimensional Frobenius algebra, show that A is semi-simple as an algebra if and only if its Euler class is invertible.

Semi-simple $\Leftrightarrow$ $A = \oplus A_i \Rightarrow \exists\ a * b = 0$

$\Rightarrow (a, b) = (a * b, 1) = 0 \Rightarrow$ Euler class trivial

Conversely, otherwise not semi-simple $\Leftrightarrow \exists$ component $A' \subset A$ not field $\Leftrightarrow$ nontrivial nilpotent $\Rightarrow$ e also has nilpotent component $\Rightarrow e$ not invertible $\blacksquare$

(c.f. [Lowell Abrams, The quantum Euler class and the quantum cohomology of the Grassmannians])

扫描全能王 創建

Birational (super) rigidity in higher dim

Hypersurfaces $(d,n)=(3,3)$ $\mathrm{Bir}(X)\cong \mathrm{Aut}(X)$ when $\mathrm{Pic}(X)$ generated by $-K_X$

$(d,n)=(4,4)$ $\mathrm{Bir}(X)\cong\mathrm{Aut}(X)$ when smooth

Clemens–Griffith Type $(d,n)=(3,4)$ is irrational. Via Hodge theory/$\mathbb{C}$ to prove.

Q. Cubic 4-fold is unknown but Konsterich.. claimed that they proved it's irrational

Cubic threefold. $\Rightarrow$ Due to $/\mathbb{C}$ is irrational, $/\mathbb{k}$ also irrational

Birational invariants of rationally connected varieties (Z. Tian)

Lecture 1

$\mathbb{k}$ needn't to be algebraically closed.

Problem 1. *Let $f : X \dashrightarrow Y$ be a rational map between two k-varieties. Assume that X is smooth and Y is proper. Prove: if $X(k) \neq \emptyset$, then $Y(k) \neq \emptyset$.*

Problem 2. *Let $X/\mathbb{R}$ be a smooth projective variety defined over the real numbers $\mathbb{R}$. Suppose that $X(\mathbb{R}) \neq \emptyset$ and consider it with the Euclidean topology. Prove that $\pi_0(X(\mathbb{R}))$ is a birational invariant of X.*

Problem 3. *Prove: let $F(X_0,\dots,X_n)$ be a homogeneous polynomial of $n+1$ variables and degree $d \le n$ defined over a field K. Then $F=0$ has a non-trivial solution if K is finite or $\mathbb{C}(t)$.*

Problem 4. *Prove: a smooth hypersurface of bidegree $(2,n)$ in $\mathbb{P}^m \times \mathbb{P}^1$ is rational.*

Problem 5. *Let X/k be a Brauer-Severi variety. Prove that the following are equivalent:*

$X_{\bar{\mathbb{k}}}\cong\mathbb{P}^n_{\bar{\mathbb{k}}}$

(1) $X \cong \mathbb{P}^n_k$.

(2) $X(k) \neq \emptyset$.

(3) $Pic(X)$ is generated by $\mathcal{O}(1)$.

$X^{(i)}=\{x\in X \mid \mathrm{codim}\,\bar{x}=i\} \xrightarrow{i} X$

$X_{(i)}=\{x\in X \mid \dim\bar{x}=i\}$ both gives a "stratification" of X

Lecture 2

Convention 1. We use the following conventions for cohomology and homology.

Gerstem cpx: $0\to \bigoplus_{x\in X^{(0)}} i_{x*}H^i(\mathbb{k}(x)) \xrightarrow{\partial} \bigoplus_{x\in X^{(1)}} i_{x*}H^{i-1}(\mathbb{k}(x)) \xrightarrow{\partial}\cdots$

- Let X be a smooth projective variety of dimension d defined over an algebraically closed field k.

Bloch–Ogus: $0\to \mathcal{H}^i \to \bigoplus_{x\in X^{(0)}} i_*H^i(\mathbb{k}(x)) \xrightarrow{\partial}$ ... when $\mathcal{H}^i: U\mapsto H^i(U)$ is "good"

- Given an abelian group A that is one of $\mathbb{Z}/m, \mathbb{Z}_\ell, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}_\ell$, where m is a positive integer, ℓ a prime number, both relatively prime to the characteristic of k, we simply write $H^k(X,A)$ as the étale cohomology with coefficient A (resp. the singular cohomology with coefficient A, if X is a complex variety), and $H_i(*,A)$ as the étale Borel-Moore homology (resp. singular Borel-Moore homology).

Gerstem–Quillen: Holds for algebraic K-theory, too

Milnor–Kerz: Holds for Milnor K-theory, too, if $\#\mathbb{k}=\infty$

H above can be étale with torsion coefficient $\leftarrow$ due to $\mathbb{k}\neq\bar{\mathbb{k}}$

Definition 1. With conventions as in Convention 2.1, we define the following filtrations on the cohomology $H^*(X,A)$.

Bloch: $H^p(X;\mathcal{K}^p)=CH^p(X)$

Given filtration $X(B)\to$

gives $H^p(X_{Zar},\mathcal{K}^p)\Rightarrow H^p$

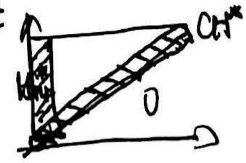
Unramified cohomology

$H^*_{nr}(X;A):=H^0(X;\mathcal{H}^*_A)$, $\mathcal{H}^*_A: U\mapsto H^*(U;A)$

Bloch–Ogus $\mathrm{Ker}\left(H^*(K(X);A)\to \bigoplus_{D\in X^{(1)}} H^{*-1}(\mathbb{k}(D))\right)$ is birational invariant & homotopy/$\mathbb{A}^1$-invariant

divisor classes

it tells us why it's called "unramified":



as spe... of

in dim 0 case it's unramified Galois cohomology

$= \mathrm{Ker}(H^1(\mathbb{k};G(\bar{\mathbb{k}}))\to H^1(I_v;G(\bar{\mathbb{k}})))$ $\subset H^1(\mathbb{k};G(\bar{\mathbb{k}}))$ ($G(\bar{\mathbb{k}})$ is $\bar{\mathbb{k}}$-pt. of algebraic grp)

the inertia at unramified place v is divisor class group

$q>\dim X$, H $=d$

4

(1) The **coniveau filtration** $\{N^*H^k(X,A)\}$ with

$$N^cH^k(X,A) := \sum_{\substack{f:Y\to X\\ \dim Y\le d-c}} f_*(H_{2d-k}(Y,A)) \subset H_{2d-k}(X,A) \cong H^k(X,A),$$

where the sum is taken over all morphisms from projective algebraic sets $f: Y \to X$ with $\dim Y \le d-c$.

(2) The **strong coniveau filtration** $\{\widetilde{N}^*H^k(X,A)\}$ with

$$\widetilde{N}^cH^k(X,A) := \sum_{\substack{f:Y\to X\\ \dim Y\le d-c}} f_*(H_{2d-k}(Y,A)) \subset H_{2d-k}(X,A) \cong H^k(X,A),$$

where the sum is taken over all morphisms from *smooth* projective varieties $f: Y \to X$ with $\dim Y \le d-c$.

(3) The **strong cylindrical filtration** $\{\widetilde{N}_{*,\mathrm{cyl}}H^k(X,A)\}$ with

$$\widetilde{N}_{c,\mathrm{cyl}}H^k(X,A) := \sum_{\substack{Z,\Gamma\\ p:\Gamma\to Z \text{ dom.}\\ \dim_{\mathrm{rel}} p=c}} \Gamma_*(H_{2d-k-2c}(Z,A)) \subset H_{2d-k}(X,A) \cong H^k(X,A)$$

where the sum is taken over all *smooth* projective varieties Z and subvarieties $\Gamma \subset Z \times X$ such that the projection $\Gamma \to Z$ is dominant of relative dimension c.

(4) We use the notations N^cH_k etc. to denote the filtrations on Borel-Moore or singular homology H_k. Since X is smooth, this is the same as the filtrations N^cH^{2d-k}.

Problem 6. *For $A = \mathbb{Z}, \mathbb{Z}_\ell$ and X of dimension d, prove that we have the following equality:*

$$\widetilde{N}_{d-c,\mathrm{cyl}}H^{2c-1}(X,A) = \widetilde{N}^{c-1}H^{2c-1}(X,A), \quad c \ge 1.$$

Problem 7. *Let X be a smooth projective variety of dimension d defined over an algebraically closed field. Fix a prime number ℓ different from the field characteristic. All cohomology groups considered are étale cohomology groups. Prove that the following invariants are stable birational invariants of X.*

(1) The kernel and cokernel of the cycle class map

$$A_1(X) \otimes \mathbb{Z}_\ell \to H^{2d-2}(X, \mathbb{Z}_\ell(d-1)).$$

Here $A_1(X)$ is the Chow group of one-cycles on X modulo algebraic equivalence.

/C
zar, the LS SS

(X)an

d case

otwic

(X, $\mathbb{Z}^d$) = KH^q(X) Kato cohomology

birational invariant

(2) *The kernel and cokernel of the cycle class map*

$$\varprojlim_n \mathrm{CH}_1(X, r, \mathbb{Z}/\ell^n\mathbb{Z}) \to H^{2d-2-r}(X, \mathbb{Z}_\ell(d-1)).$$

(3) *The quotients*

$$H_i(X,\mathbb{Z}_\ell)/\widetilde{N}_{1,cyl}H_i(X,\mathbb{Z}_\ell) = H^{2d-i}(X,\mathbb{Z}_\ell)/\widetilde{N}_{1,cyl}H^{2d-i}(X,\mathbb{Z}_\ell),$$
$$H^{2d-3}(X,\mathbb{Z}_\ell)/\widetilde{N}^{d-2}H^{2d-3}(X,\mathbb{Z}_\ell), \quad \text{and}$$
$$H^{2d-3}(X,\mathbb{Z}_\ell)/N^{d-2}H^{2d-3}(X,\mathbb{Z}_\ell).$$

(4) *The cokernel*

$$\mathrm{cl}^2 : \mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(d-2)).$$

Problem 8. *Let X be a smooth variety defined over an algebraically closed field k. Fix ℓ a prime number distinct from the characteristic of k. Using the Beilinson-Lichtenbaum conjecture proved by Voevodsky, prove that all the torsion classes in the étale cohomology $H^{i+1}(X,\mathbb{Z}_\ell(i))$ ($i \geq 0$) are supported in divisors, and $H^{i+1}(X,\mathbb{Z}_\ell(i))/N^1H^{i+1}(X,\mathbb{Z}_\ell(i))$ is torsion free.*

The same conclusion holds if X is a complex variety and we use singular cohomology with $\mathbb{Z}$-coefficients.

Problem 9. *Let X be a smooth projective variety defined over an algebraically closed field. How is the cokernel of cl^2 related to $H^3_{\mathrm{nr}}(X, \mu_n^{\otimes i})$?*

Problem 10. *For Kato homology, prove that the projective bundle isomorphism and birational invariance are equivalent at least in characteristic* 0.

Lecture 3

Problem 11. *Study degeneration of rationally connected surfaces in positive and mixed characteristic.*

Problem 12. *Let $f : X \to S$ be a smooth projective morphism between smooth quasi-projective varieties. Assume that the fibers are rationally connected and satisfy the integral Hodge conjecture for one-cycles. Prove that $KH_2(X) \cong KH_2(S)$.*

Local Stability: a notion of family in higher dimension (C. Xu)

Deformation Rigidity of Canonical Singularities

In this exercise, we work over an algebraically closed field k of characteristic 0.

Problem 13. *Construct a family over a smooth pointed curve that degenerates* $\mathbb{P}^1 \times \mathbb{P}^1$ *into* $\mathbb{P}(1,1,2)$.

Problem 14. *For the above degeneration, how does a line* L *in a ruling degenerate?*

Problem 15. *Denote the degeneration family by* $(\mathcal{X}, \mathcal{L}) \to C$ *in (1) and (2). Show that the family* $(\mathcal{X}, a\mathcal{L}) \to C$ *with a coefficient* $a \in [0,1)$ *is a family of log Fano pairs, i.e., every fiber is log Fano (with klt singularities).*

Problem 16. *Compute the volume of the anti-canonical bundle* $-K_{\mathcal{X}_t} - a\mathcal{L}_t$ *for each fiber. Are they constant?*

Problem 17. *For every* $a \in (0,1)$, *show that* $(\mathcal{X}, a\mathcal{L})$ *is not klt, though every fiber* $(\mathcal{X}_t, a\mathcal{L}_t)$ *is klt.*

Problem 18. *Now consider another family of projective normal varieties over a smooth curve:* $\mathcal{Y} \to C$. *Suppose every fiber* $\mathcal{Y}_t$ *is canonical. Show that* $\mathcal{Y}$ *is* $\mathbb{Q}$*-Gorenstein (which implies that* $\mathcal{Y}$ *is canonical).*

C-flatness

In this exercise, we learn the concept of C-flatness.

Relative Cartier Divisor

Let $f : X \to S$ be a flat morphism with S_2 fibers, $x \in X$ a point, and $s := f(x)$. Let $D \subset X$ be a subscheme. Show that the following three statements are equivalent:

(1) D is flat over S at x, and $D_s := D|_{X_s}$ is a Cartier divisor on X_s at x.

(2) D is a Cartier divisor on X at x, and a local equation $g_x \in \mathcal{O}_{X,x}$ of D restricts to a non-zero divisor on the fiber X_s.

(3) D is a Cartier divisor on X at x, and it does not contain any irreducible component of X_s that passes through x.

If one of the above conditions is satisfied, we say D is *relatively Cartier* at x. If D is relatively Cartier at any $x \in X$, we say D is a *relative Cartier divisor.*

Projection

Problem 19. *Given a closed point $(a_0 : a_1 : \cdots : a_n) \in \mathbb{P}^n$, how do you perform the projection $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1} := \{x_n = 0\}$ from $(a_0 : \cdots : a_n)$?*

Problem 20. *Suppose $Y \subset \mathbb{P}^n$ is a hypersurface that does not pass through $(a_0 : \cdots : a_n)$. Show that $\pi|_Y : Y \to \mathbb{P}^{n-1}$ is a finite morphism.*

Problem 21. *Now consider the relative version. Let S be an affine scheme and $a_i \in \mathcal{O}_S$ for $0 \leq i \leq n$. How do you perform the projection $\pi : \mathbb{P}^n_S \dashrightarrow \mathbb{P}^{n-1}_S := \{x_n = 0\}$ from $(a_0 : \cdots : a_n)$?*

Problem 22. *Notation as in (3). How do you project from $\mathbb{P}^n_S$ to $\mathbb{P}^r_S := \{x_n = \cdots = x_{r+1} = 0\}$ from $(a_0 : \cdots : a_n)$?*

C-flatness

Let (s, S) be a local scheme with infinite residue field, and let $Y \to S$ be a generically flat family of pure schemes of relative dimension d with an embedding $Y \to \mathbb{P}^n_S$ over S. We say Y is *C-flat* over S if for any $\mathcal{O}_S$-projection $\pi : \mathbb{P}^n_S \dashrightarrow \mathbb{P}^{d+1}_S$ that is finite on Y, the divisorial support of $\pi_* \mathcal{O}_Y$ is relatively Cartier over S. (Note that the definition depends on the embedding $Y \to \mathbb{P}^n_S/S$.)

Problem 23. *Under the above definition, show that a flat family of d-dimensional hypersurfaces $Y \to \mathbb{P}^{d+1}_S/S$ is C-flat.*

K-flatness vs. Formally K-flatness

Let (s, S) be a local scheme with infinite residue field, and let $Y \to S$ be a generically flat family of pure schemes of relative dimension d.

- We say Y is *K-flat* over S if the divisorial support of $\rho_* \mathcal{O}_Y$ is relatively Cartier over S for any finite morphism $\rho : Y \to \mathbb{P}^{d+1}_S$.
- We say Y is *formally K-flat* over S at $y \in Y$ (mapping to $s \in S$) if the divisorial support of $\rho_* \widehat{\mathcal{O}_{Y,y}}$ is relatively Cartier over $\widehat{\mathcal{O}_{S,s}}$ for any finite morphism $\rho : \widehat{Y} \to \mathbb{A}^{d+1}_{\widehat{S}}$, where $\widehat{Y}$ (resp. $\widehat{S}$) denotes the completion of Y at y (resp. S at s).

Problem 24. *Define formal K-flatness in a natural way. (We only defined it locally above.)*

Problem 25. *Show that K-flatness implies C-flatness.*

Problem 26. *Show that formal K-flatness implies K-flatness.*

Problem 27. *Does K-flatness imply formal K-flatness?* (Open)