Six functor formalism, Fourier theory and Sheaf theory

Our topics are two important issues in sheaf theory, to connect the gap between [Ha] and [KS90, Chap IV and later Chapters]:

One is using six functor formalism to derive Poincaré–Verdier duality (on mfd case and serveral other models)

One is serveral "Fourier transformation" on $D^b(X)$, Fourier–Deligne, Fourier–Sato, Fourier–Mukai & why they're "Fourier".

$\boxed{\text{PART I}}$ Six functor formalism, Recall.

Notation. To present a complete formalism, things like (Zariski) sheaf cohomology is not enough, our models are

- étale cohomology $H_{et}(X;\mathcal{F})$ of scheme X in $D_{et}(X)$
- topological $H(X;\mathbb{Z})$ of $\boxed{\text{locally cpt}}$ topological space X, (X, A_X) ringed space

$\rightsquigarrow H(X;M)$ for M is A_X-module given in $D(A_X) =: D(X)$

Zariski ✗ mfd ✓

• gld$(A_X) < \infty$ global homological

cpx case

- D-module on variety$/\Bbbk$, char $\Bbbk = 0$ (By [BR17], omitted here)

thus is a functor $\mathcal{C}_6 \to \mathrm{Cat}$ and six functor formalism is a structure of $D(X)$ (and their morphisms)

$$X \mapsto D(X)$$

I assume the familiarity of the first and focus on the second.

Abstract categorical framework: For "Geometric object" X

- Stage1. In AT, we need to define (co)homology groups $H(X;\mathbb{Z})$

now we know it's just $R\Gamma(X;\mathbb{Z}_X)$, $\mathbb{Z}_X$ is constant sheaf $\rightsquigarrow R\Gamma(X;\mathcal{F})$

Stage2. With Grothendieck's philosophy, the relative analogue for $X \xrightarrow{f} Y$ should "contain" datum of each fibre X_y. This's $Rf_*(\mathcal{F})$, now after derived cat, it's just induced by $X \xrightarrow{f} Y$ to $D(X) \xrightarrow{f_*} D(Y)$

$\{4 = 0\}$

Recall: When f is proper, $(Rf_*(\mathcal{F}))_y = R\Gamma(X_y;\mathcal{F})$

And the pullback $f^*: D(Y) \to D(X)$ is natural

$f^* \vdash f_* \vdash f^{-1}$

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Stage3. Generalise the additional algebraic structure of $H(X;\mathbb{Z})$

- $-\otimes- : D(X)\times D(X)\to D(X)$ and $\mathcal{H}om(-,-): D(X)^{op}\times D(X)\to D(X)$

$\otimes \dashv \mathcal{H}om$ is a partial right adjoint by UFC of $H(X;\mathbb{Z})$

- Poincáre duality $\mathcal{H}om(-;\mathbb{Z}) = \vee \Rightarrow H(X;\mathbb{Z})[\dim X] = H(X;\mathbb{Z})^\vee$ self-dual

Not only $\mathbb{Z}$ to $\mathcal{F}$ by relative ⇝ sheaf case

$H(X;\mathcal{F}^\vee)[\dim X] = H(X;\mathcal{F})^\vee \rightsquigarrow H(X;\mathcal{H}om(\mathcal{F},\mathcal{G}))[\dim X]$

dual to $\mathcal{G}$ $= \mathrm{Hom}(H(X;\mathcal{F}),\mathcal{G})$

⇝ Grothendieck duality $Rf_*(\mathcal{H}om(\mathcal{F},\text{"}\mathcal{G}\text{"}))[\text{rel.dim} f]$

−Verdier $= \mathcal{H}om(Rf_*(\mathcal{F}),\mathcal{G})$ not needed by def

Note the "$\mathcal{G}$" is subtle! $Rf_*: D(X)\to D(Y)\ni\mathcal{G}$, how back to $D(X)$?

thus should be $Rf_*(\mathcal{H}om(\mathcal{F}, f^{\square}\mathcal{G}))[\text{rel.dim} f] = \mathcal{H}om(Rf_*\mathcal{F},\mathcal{G})$

$f^{\square}: D(Y)\to D(X)$ is defined by Verdier $f^!: D(Y)\to D(X)$, thus

$Rf_*(\mathcal{H}om(\mathcal{F}, f^!\mathcal{G})) = \mathcal{H}om(Rf_*\mathcal{F},\mathcal{G})$

$f^!$ ← open

⇒ $f^!$, called the dualizing complex (functor)/exceptional inverse image

f^* ← closed

⇒ $f_!\dashv f^!$, $f_!$ is direct image of proper support

Stage4. Thus our six functor formalism is

$(f^*, f_*, \otimes, \mathcal{H}om, f^!, f_!)$, satisfy compatibility conditions

① Adjoint;

② $(f^*, f_*, \otimes)$ Projection formula I;

③ $(f^!, f_!, \otimes)$ Projection formula II;

④ $(f^*,\otimes)$ Symmetric monoidal: $f^*(\mathcal{F}\otimes\mathcal{G}) = f^*\mathcal{F}\otimes f^*\mathcal{G}$.

Hardest part of check six functor formalism is $f_!\dashv f^! \Leftrightarrow$ existence of $f^! \Leftrightarrow$ Poincáre-Verdier duality

Other properties, such as Künneth, can be deduced from ②+④...

Rk ① Understand dualizing complex $f^!$ via ω and $\omega^\bullet$ & orientation in Poincáre dual

$\omega_{X/Y} = \Lambda^{\text{rel}(X/Y)}\Omega_{X/Y}$ top differential dualizing

X/Y smooth f: A_Y complex smooth $\Lambda^{top}\Omega$ / Canonical

$X\xrightarrow{f} Y$, A_Y constant sheaf on Y, $U\mapsto \mathrm{Hom}(H_c(U;A_X),A)$

$\omega_{X/Y}[-\text{rel}(X/Y)] = or(X)$ called orientation sheaf; only f) term $\neq$ ...

By slogan: Poincare dual: Integral on top ⇔ Counting points, we have:

② Integral and de Rham cpx; $\dim X = n$, we have trace map/integral

$\int_X: \Gamma_c(X;\Omega^n_X\otimes or(X))/d\Gamma_c(X;\Omega^{n-1}_X\otimes or(X)) \to \mathbb{C}$ ---(a)

$or(X)\simeq[0\to\Omega^0_X\otimes or(X)\xrightarrow{d}\cdots\xrightarrow{d}\Omega^n_X\otimes or(X)\to 0]$ quasi-isomorphism

$\Rightarrow \int_X: H^n_c(X; or(X))\to\mathbb{C}$ ---(b) = (a) also analogous to char varieties

Our intersection/trace map on mfd case are characteristic cycle analogous to divisors/cycle classes in algebraic case.

It's sad for me to find that: both algebraic and symplectic, both J-holomorphic and microlocal, we cannot avoid counting, but mathematicians don't know how to do it...

- $\nu_X = \Omega^n_X\otimes or(X)$ called density sheaf due to:

integration $\int_X: \Gamma_c(X;\Omega^n_X)/d\Gamma_c(X,\Omega^{n-1}_X)\to\mathbb{C}$ defined ⇔ X is orientable

but for not oriented case not define. ⇔ $or(X)$ trivial

to fill this lack, we $\otimes\, or(X)$ and ν_X always holds integration.

- When X connected, $\int_X$ induces isomorphism $H^n_c(X;or(X))\cong\mathbb{C}$

by Poincare dual of cpt cohomology $\cong \mathrm{Hom}(H^0(X;\mathbb{C}_X),\mathbb{C})$

& de Rham cohomology in [BT]

- A direct way to (b) is considering X as $X\xrightarrow{\varphi_X}\{pt\}$ structure map

$\Rightarrow \varphi_{X!}\varphi_X^! A_{pt}\to A_{pt} \Leftrightarrow \varphi_{X!}\,\omega^0_{X/pt}\to A_{pt} \Leftrightarrow \varphi_{X!}\, or(X/pt)[n]\to A_{pt}$

$\Rightarrow H^0_c(\varphi_{X!}\, or(X/pt)[n]) \longrightarrow H^0(pt, A_{pt})$

$H^n_c(X; or(X/pt)) \longrightarrow A$ is also $\int_X$ ▨

f: • X orientable ⇔ $or(X/Y)\cong A$

- If $X\xrightarrow{f}Y$ is "mfd bundle" — fibres are étale balls / submersion of mfds

$\Rightarrow f^!\mathcal{G} = f^*\mathcal{G}\otimes\omega^\bullet_{X/Y}$

For singular case, $\omega_{X/Y}$ most useless and all are $\omega^\bullet_{X/Y}$!

..., due to $\in Ob(D^+(X))$, all $(-n)$-terms are cutted!

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PART II Fourier Theory (Grothendieck's functions-sheaves correspondence) (fonctions-faisceaux)

- We start at classical Fourier transformation/Fourier integral operator in Hörmander's school of microlocal analysis

$\mathcal{F}: S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$

$f(x) \mapsto \hat{f}(\xi) = \int e^{ix\xi} f(x)\,dx$

$\Leftrightarrow$

$$\begin{tikzcd} & S'(\mathbb{R}^n_x \times \mathbb{R}^n_\xi) \arrow[dl, "\int d\xi"'] \arrow[dr, "\int dx"] & \\ S'(\mathbb{R}^n_x) \arrow[ur, "e^{ix\xi} f(x)"] & & S'(\mathbb{R}^n_\xi) \\ f(x) \arrow[rr, "\mathcal{F}"] & & \hat{f}(\xi) \end{tikzcd}$$

or more generally

$$\begin{tikzcd} & S(T^*M) \arrow[dl] \arrow[dr] & \\ S'(M) \arrow[rr, "\mathcal{F}"] & & S'(\mathbb{R}^n_\xi) \end{tikzcd}$$

- Fourier–Pontryagin transformation

For G locally cpt Abelian grp with Haar measure

its Pontryagin dual $\hat{G} = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{T})$

Then $L^1(G) \longrightarrow L^1(\hat{G})$ $\Leftrightarrow$

$f(x) \longmapsto \hat{f}(\xi) = \int_G \xi(x) f(x)\, d\mu_x$

$$\begin{tikzcd} & L^1(G\times\hat{G}) \arrow[dl, "\int d\mu_\xi"'] \arrow[dr, "\int d\mu_x"] & \\ L^1(G) \arrow[ur, "\cdot\xi"] & & L^1(\hat{G}) \\ f(x) \arrow[rr] & & \hat{f} \end{tikzcd}$$

$\xi(x) \in \mathbb{T}$, for $G = \mathbb{R}^n = \hat{G}$, $\xi(x) = e^{ix\xi} \in \mathbb{T}$

Focus on this part more clearly than above

Thus by the red observation right, we know $f(x) \mapsto (\int d\mu_\xi)^* f \mapsto \int (\int d\mu_\xi)^* f(x)\, d\mu_x$

$= (\mathrm{pr}_2)_* (\int d\mu_\xi)^* f$ (pushforward = integral over fibres)

$\int \xi f\, d\mu_\xi = f \cdot (\int \xi\, d\mu_\xi)$

$= (\int \xi\, d\mu_\xi)^* f$

$= (\int d\mu_x)(\int d\mu_\xi)^* f$

$\boxed{\text{pullback}}$ of function

$(x, \xi) \mapsto x \cdot \int \xi\, d\mu_\xi$

$f \mapsto f(x) \cdot \int \xi\, d\mu_\xi$

- Fourier–Deligne transformation

For G Abelian group scheme & Unipotent

$$\begin{tikzcd} & D^b(G\times\hat{G}) \arrow[dr, "(\mathrm{pr}_2)_!"] & \\ D^b(G) \arrow[ur, "\mathrm{pr}_1^*"] & & D^b(\hat{G}) \end{tikzcd} \qquad \begin{tikzcd} & G\times\hat{G} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ G & & \hat{G} \end{tikzcd}$$

$\mathcal{F} \mapsto (\mathrm{pr}_2)_!(\mathrm{pr}_1^* \mathcal{F} \otimes \mathcal{L})$ for $\mathcal{L} \in D^b(G\times\hat{G})$ Poincaré line bundle

Take proper support! for "integral pr_2" well-defined. But Abelian $\Rightarrow$ projective, why not $(\mathrm{pr}_2)_*$?

- Fourier–Mukai transformation (1981)

$D^b(X\times Y)$ $M \in D^b(X\times Y)$ the FM kernel

$D^b(X) \xrightarrow{(\mathrm{pr}_2)_*(\mathrm{pr}_1^* \mathcal{F} \otimes M)} D^b(Y)$ generalizes FD case, X, Y smooth projective.

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RK. A thm by Yau, Zaslow and Leung says that it's closely related with T-duality, relating SYZ to YM theory.

Function $\rightsquigarrow$ Sheaf	Abstract our classical FT
multiplication $\rightsquigarrow \otimes$	into categorical level.
integral over fibres $\rightsquigarrow$ pushforward	(And Morse theory $\to$ Microlocal Morse/Sheaf)

- Fourier–Sato transformation (1982) can be seen as a special case of FM, but for fitting into mfd cases, we must do more modifications.

$$\begin{tikzcd} D^b(E\times_X E^*) \arrow[rr, "\otimes ?"] & & D^b(E\times_X E^*) \arrow[d, "(\mathrm{pr}_2)_!"] \\ D^b(E) \arrow[u, "(\mathrm{pr}_1)^{-1}"] \arrow[rr, "\Phi_?", "\text{adjoint}"'] & & D^b(E^*) \arrow[ll, bend left, "\Psi_{?'}"] \\ D^b(E\times_X E^*) \arrow[u, "(\mathrm{pr}_1)_*"] & & D^b(E\times_X E^*) \arrow[ll, "\otimes ?'"'] \arrow[u, "(\mathrm{pr}_2)^!"'] \end{tikzcd}$$

Commutativity gives the two FS are inverse to each other.

FM also have such a picture, but not admit full six functor, thus here FS or FD are more subtle! We can change $(-)_!$ and $(-)_*$.

$$\begin{tikzcd} D^b(E\times_X E^*) \arrow[rr, "\otimes ?"] & & D^b(E\times_X E^*) \arrow[d, "(\mathrm{pr}_2)_*"] \\ D^b(E) \arrow[u, "(\mathrm{pr}_1)^{-1}"] \arrow[rr, "\Phi_{?'}", "\text{adjoint}"'] & & D^b(E^*) \arrow[ll, bend left, "\Psi_{?'}"] \\ D^b(E\times_X E^*) \arrow[u, "(\mathrm{pr}_1)_!"] & & D^b(E\times_X E^*) \arrow[ll, "\otimes ?'"'] \arrow[u, "(\mathrm{pr}_2)^!"'] \end{tikzcd}$$

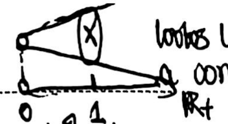
But it turns out $\boxed{\Phi_? = \Phi_{?'} \quad \Psi_? = \Psi_{?'}}$

Now we figure out what are the ? and ?' is:

A Convolution kernel $K = \ldots$ $F(x, y) = \langle \rho, y\rangle$, thus it's a special FM transform

Setting. $E \to X/\mathbb{R}$ vector bundle of rank r, E^* the dual bundle. We decorate $D^b(E)$ into $D^b_{\mathbb{R}^+}(E)$ after equipping a $\mathbb{R}^+ \curvearrowright X$ "equivariantly" defining sheaves, called conic sheaves/objects in derived cat.

Symmetrically $D^b(Y) \to D^b(X)$ for kernel given can be defined.



looks like a cone

this is dual to for $T^*X = E$

Def1. Conic objects $\in D^+(X)$ is all $\mathcal{H}^i(\mathcal{F})$ is conic sheaves, $\forall i$.
i.e. $\mathcal{H}^i(\mathcal{F})|_{O(x)}$ is locally constant for $O(x)$ orbit of $\mathbb{R}_+$-action of $\forall x \in X$

$\Leftrightarrow \mu^{-1}\mathcal{F} \simeq p^{-1}\mathcal{F}$ quasi-isomorphic $\quad X \times \mathbb{R}_+ \underset{pr_1 = p}{\overset{\mu}{\rightrightarrows}} X$

$\Leftrightarrow \mu^!\mathcal{F} \simeq p^!\mathcal{F}$ quasi-isomorphic

Latter two criterias allow us to check conic easily via some commutativity of six functors. full category

All derived categories in FS are $D^+_{\mathbb{R}_+}(X)$, consist all conic objects

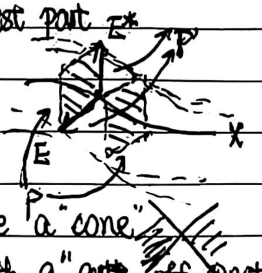
Determine ? and ?' is the strangest part

Set $P = \{(x,y) \in E \times_X E^* \mid \langle x,y \rangle \geq 0\}$

and $P' = \{(x,y) \in E \times_X E^* \mid \langle x,y \rangle \leq 0\}$

divide $E \times_X E^* = E \otimes E^*$ on X

into two parts, each is looks like a "cone"



Now we tensoring/multiplying with a "cut-off function"

$\Rightarrow$ the functors Γ_P and $(-)_P$, $i: P \hookrightarrow X$

sheaves with sections supported in P $\qquad \mathcal{F}_P|_{X-P} = 0$, $\mathcal{F}_P|_P = \mathcal{F}$

$\Rightarrow \Psi_{P'} = (pr_1)_* \circ \Gamma_{P'} \circ pr_2^!$ $\quad$ inverse $\quad$ $j_* j^{-1}$ when open

$\Phi_{P'} = (pr_2)_! \circ (-)_{P'} \circ pr_1^{-1}$ $\quad$ $j_! j^{-1}$ when closed [KS]

$\Psi_P = (pr_2)_* \circ \Gamma_P \circ pr_1^{-1}$ $\quad$ inverse

$\Phi_P = (pr_1)_! \circ (-)_P \circ pr_2^!$ $\quad$ is final result.

Note that we can understand it by observing that restrict to P or P' "orthogonal" to X not change $\mathbb{R}_+$-equivariance, and $(-)_*$, $(-)^!$, $(-)_!$ not change conic as they commute with either $(-)^!$ or $(-)^{-1}$ $\Rightarrow$ well-defined $\boxtimes$

$\langle x, \xi \rangle > 0$

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※ Multiply $e^{-\langle x,\xi \rangle}$ provide decay when $\langle x, \xi \rangle \geq 0$, thus distinct with $\langle x,\xi\rangle < 0$

meaningful decay $\Leftrightarrow$ restriction of support

Key point. Only X or T^*M not enough, must with a fibre scaling $(x,\xi) \mapsto (x, \lambda\xi)$

● used in changing of variable of estimate of integral in microlocal analysis $\Leftrightarrow$ reduce to sphere bundle and decay $\lambda^{-\infty}$, $\lambda \to \infty$

In algebraic case $\Leftrightarrow$ action $\mathbb{R}_+ \curvearrowright X$, and $\boxed{\text{conic}}$ = stable under dilation (this action)

Fourier–Sato is $(-)^\wedge = \Phi_{P'}$ and inverse $(-)^\vee = \Psi_{P'}$ defined

Understanding these Fourier transformations:

Eg.1: (Constant sheaf A_X)$^\wedge$ = Skyscraper sheaf A_{x}

• (Vector bundle E of rank r)$^\wedge = E^\wedge$ supported on r pts (with multi...)

for FM transform on Abelian variety, same holds with homogeneous conic sheaves for FS transform.

Thus FM/FS behaves similar to classical Fourier transform, sending a continuous function to some discrete data.

To make it more general, we consider cones $\subset E$, they form the γ-topology, but it's another story in microlocal theory.

Def2. We define cones $\gamma \subset E$ by restrict on each fibres:

• Subset $\gamma \subset E$ is convex, conic if $\gamma_x \subset E_x = \mathbb{R}^n$ is convex, conic (cone $\subset \mathbb{R}^n$ are just [cone], cannot convex)

Cone is proper if it contains no lines (of each fibre)

• Antipodal map $a: E \to E$, $x \mapsto -x$, $a^{-1}: D^+(E) \to D^+(E)$ and reduce to $D^+_{\mathbb{R}_+}(E) \to D^+_{\mathbb{R}_+}(E)$. We denote $a^{-1}\mathcal{F} = \mathcal{F}^a$

Thm1. (Properties of FS transform) All quasi-isomorphisms of equivalence category

① $\mathcal{F}^{\vee} \simeq (\mathcal{F}^\wedge)^a \otimes \omega_{E^*/X}$; ② $\mathrm{Hom}_{D^+_{\mathbb{R}_+}(E)}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{D^+_{\mathbb{R}_+}(E^*)}(\mathcal{F}^\wedge, \mathcal{G}^\wedge)$

③ • γ proper convex cone of E, closed, containing zero section $X \subset E$

$\Rightarrow (A_\gamma)^\wedge \simeq A_{\mathrm{Int}\,\gamma^\circ}$ (constant sheaves); $\gamma^\circ = \gamma - X$

• convex cone of E, open, containing zero section, $U^\circ = U - X$

$\Rightarrow (A_U)^\wedge \simeq A_{U^{\circ a}} \otimes or_{E^*/X}[-n] \simeq A_{U^{\circ a}} \otimes \omega_{E^*/X}[-2n]$;

④ $\mathcal{F}^{\wedge\wedge} \simeq \mathcal{F}^a \otimes \omega_{E/X}[-2n]$;

⑤ Dual $D\mathcal{F} = \mathcal{H}om(\mathcal{F}, \omega_X)$ (related with cohomologically constructible sheaves)

$D'\mathcal{F} = \mathcal{H}om(\mathcal{F}, A_X)$

$\Rightarrow (D'\mathcal{F})^\vee = D'(\mathcal{F}^\wedge)$ & $(D\mathcal{F})^\vee \simeq D(\mathcal{F}^\wedge)$

Pf. See [KS] (specialize to normal cone)

Last word:

$\boxed{\text{Microlocalization} = \text{Specialization} + \text{FS}}$

Fundamental microlocal sheaf theory

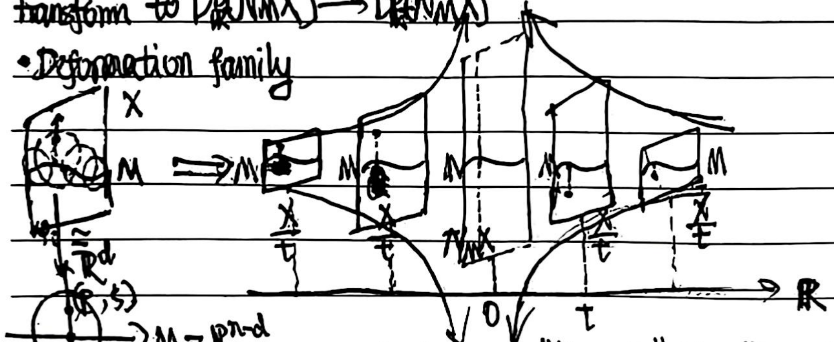
Our task is building the involutivity thm and understand it, and by the way we motivate notations and properties from analytic view in propagation codirection $\subset T^*X$.

PART I

Microlocalization vs. Deformation to normal cone:

We define a functor $\mu_M: D^b(X) \to D^b_{\mathbb{R}^+}(N^*_M X)$, $M \hookrightarrow X$ closed submfd same as Fulton's deformation trick in intersection theory, we construct the deformation family to $M \overset{c}{\hookrightarrow} N_M X$ and the composite the Fourier-Sato transform to $D^b_{\mathbb{R}^+}(N_M X) \to D^b_{\mathbb{R}^+}(N^*_M X)$

• Deformation family



at O, pumps "blow-up" into fibre of normal bundle.

push away in ξ-direction to $\frac{\xi}{t}$, $t \to 0$

Our construction is base on this method by taking Eucildean neibourhood

$\Rightarrow$ We have
$$\begin{tikzcd} X \arrow[r, equal] & X & M \arrow[l, hook'] \\ X\times(\mathbb{R}-\{0\}) \arrow[u, "p_1"] \arrow[r, hook, "j"] \arrow[d, "p_2"] & \widetilde{X}_M \arrow[u, "\widetilde{p}_1"] \arrow[d, "p_2"] & N_M X \arrow[l, hook', "i"] \arrow[u] \arrow[d] \\ \mathbb{R}-\{0\} \arrow[r, hook] & \mathbb{R} & \{0\} \arrow[l, hook'] \end{tikzcd}$$

It's a limit process. $\widetilde{p}_1^!$ is nothing more than copy to each "X/t"
$i^! j_!$ just taking limit $t \to 0$ and take value at O out

$\widetilde{X}_M$ is desired $\mathbb{R}$-family

Specialization

and $\nu_M = i^! j_! \widetilde{p}_1^! : D^b(X) \to D^b_{\mathbb{R}^+}(N_M X)$, symmetrically, we can replace $\mathbb{R}-\{0\}$ by $\mathbb{R}_{>0}$ or $\mathbb{R}_{<0}$ $\Rightarrow \mu_M(F) = \nu_M(F)^\wedge$ done ▨

Here we have all "shrink" not "star" due to all things not, cpt due to our parameters are $\mathbb{R}^1$ not $\mathbb{P}^1$

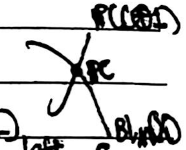
Check: ν_M is well-defined & target is $\mathbb{R}^+$-conic

Comparision: • Algebraic case do not have a clean way to reduce to Euclidean case and glue, but fortunately, satisfying our "blow-up" geometric intuition, blow-up works here, but take care of exceptional part:

•
$$\begin{tikzcd} X \arrow[r, equal] & X & M \arrow[l] \\ X\times\mathbb{A}^1 \arrow[u] \arrow[r, hook] \arrow[d] & \widetilde{X}_M \arrow[u] \arrow[d] & \mathrm{exc}(\widetilde{X}_M) \arrow[l, hook'] \arrow[u] \arrow[d] \\ \mathbb{A}^1 \arrow[r, hook] & \mathbb{P}^1 & \mathrm{pt}\infty \arrow[l, hook'] \end{tikzcd}$$

$\widetilde{X}_M = Bl_M(X\times\mathbb{P}^1)$.

$\mathrm{exc}(Bl_M(X\times\mathbb{P}^1)) = Bl_M(X) + \mathbb{P}(C\oplus 1)$



Replace $\widetilde{X}_M$ by $\widetilde{X}_M^\circ = \widetilde{X}_M - Bl_M(X)$, then left $C \to M$

$M \hookrightarrow X$ needn't regular (replace $N_M X$ by $C_M X =: C$)

the specialization $\sigma: A(X) \to A(C)$ induced by

$$\begin{tikzcd} AC \arrow[r] & A\widetilde{X}_M^\circ \arrow[r] & A(X\times\mathbb{A}^1) \arrow[r] & 0 \\ 0 \arrow[r] & AC[-1] \arrow[u, "\text{Gysin map} \atop \text{Cof divisor}"] & A(X)[-1] \arrow[u, equal, "\text{Thom} \atop \text{isomorphism}"'] \end{tikzcd}$$

MV sequence of flat pullback + proper pushforward

the Gysin map descend to $A(X\times\mathbb{A}^1) \to AC[-1]$, thus $\sigma: A(X) \to A(C)$

$\boxed{\text{same path}}$ of specialization! (But here our functor is not so general.)

• Cones. similiar with normal cone in algebraic case above. If we further acquire intersect with zero section, then it's the intersection product

$C(M,S) = N_M X \cap \overline{\widetilde{p}_1^{-1}(S)}$, $S \subset X$ subset only

and $C(S_1,S_2) = C(\Delta, S_1\times S_2) \subset N_\Delta(X\times X) \cong TX$ canonically

thus we can talk about its fibres $C_x(S_1,S_2)$

Then the normal cone of $S\subset X$ (as generalization of normal bundle of embedded closed submfd) is $N_S X = \bigcup_{x\in X} N_{x,S} X = \bigcup_{x\in X} (T_x X - C_x(X-S, S))$

and conormal cone $N^*_S X = (N_S X)^\circ = \{\xi \in T^*X \mid \langle v, \xi\rangle \geq 0, \forall v \in N_S X\}$

Properties: • $N_{X-S}(X) = (N_S X)^a$ • When S is regular, they're bundles

μhom functor: Set $\mu hom: D^b(X)\otimes D^b(X) \to D^b_{\mathbb{R}^+}(N^*_\Delta(X\times X)) \cong D^b_{\mathbb{R}^+}(T^*X)$

$F \otimes G \longmapsto \mu_\Delta \mathcal{H}om(p_1^! F, p_2^! G)$

singular datum concentrated on the diagonal. We always need cpt supported, but here we only need target suffices, thus $p_1^! F \to p_2^! G$.

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PART I

Microlocal support.

Def 1 Given $F \in Ob(D^b(X))$, $SS(F) \subset T^*X$ is the subset defined by:

$SS(F) = \{(x_0,\xi_0) \in T^*X \mid \exists U \ni (x_0,\xi_0), \forall \varphi \in C^2(X), \boxed{\varphi(x)=0, d\varphi(x) \in U}$ but $(R\Gamma_{\{x;\varphi(x)\geq 0\}}(F))_x = 0\}^c$

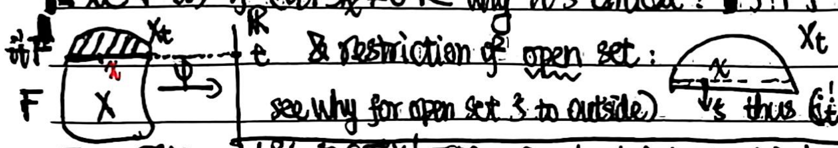
micro-local ... Morse

$= \bigcup_{\varphi \in C^2(X)} \{(x,\xi) \in T^*X \mid \exists U \ni (x,\xi), \varphi(x)=0, d\varphi(x) \in U, (R\Gamma_{\{\varphi \geq 0\}}(F))_x \neq 0\}$

move critical point to (x,ξ) by a small perturbation

$\bigcup_{\varphi \in C^2(X)} \{(x,\xi) \in T^*X \mid \xi = d\varphi(x), \varphi(x)=0, (R\Gamma_{\{\varphi \geq 0\}}(F))_x \neq 0\}$

• We can write it in a Morse-theoretic form once one note that "$d\varphi \in U$" is a generalized condition of critical point. For this we set: $X_t = \{\varphi \geq t\} \overset{i_t}{\hookrightarrow} X$, we say x is a cohomological F-critical pt of φ if $x \in \varphi^{-1}(t)$ if $(i_t^! F)_x \neq 0$ ← why it's critical? $f_! \dashv f^!$ is for extension & restriction of open set:



(later we can see why for open set ξ to outside) thus $(i_t^! F)_x = 0$ usual

Thus $SS(F) = \bigcup_{\varphi \in C^2(X)} \{(x,\xi) \in T^*X \mid x$ is cohomological F-critical pt of φ, $\xi = d\varphi\}$ is a micro-lift of the singular support (which in X only) $\exists \gamma$ satisfy (P)

• Equivalently, $SS(F) = \{(x,\xi) \in T^*X \mid \exists U \ni (x,\xi), \exists \varepsilon > 0, \forall x' \in U,$ $R\Gamma(L_\varepsilon \cap (x'+\gamma); F) \cong R\Gamma(H_\varepsilon \cap (x'+\gamma); F)\}^c$

Here our condition (P) for cone γ is: (a) $\gamma - \{0\} \subset \{v \mid \langle v, \xi \rangle < 0\}$ ⟷ $\xi \in \mathrm{Int}(\gamma^{oa})$

(b) γ is proper closed convex cone; (c) $0 \in \gamma$.

$L_\varepsilon = \{x' \in X \mid \langle x'-x, \xi \rangle = -\varepsilon\} \subset \{x' \in X \mid \langle x - x', \xi \rangle \geq -\varepsilon\} = H_\varepsilon$

Why this holds?

• Philosophy level: it describe a $\boxed{\text{propagation}}$ from L_ε to H_ε thus $SS(F)$ describe the point, codirection that can't propagation $\in T^*X$, we can see this clearly in the example of constant sheaf.

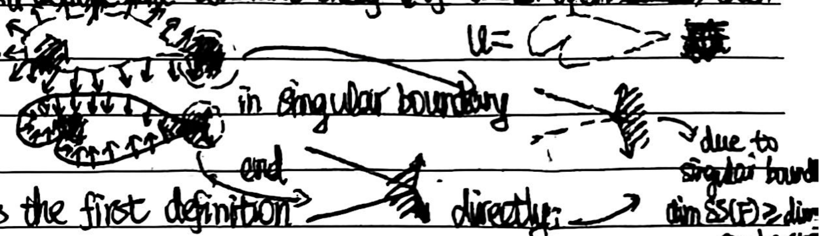
• Technical level: The proof of equivalence is Prop 5.1.1 in [KS]; we can replace $L_\varepsilon \subset H_\varepsilon$ by any two γ-open sets $\Omega_0 \subset \Omega$ with $\Omega - \Omega_0 \subset U$ the idea of proving this is propagate $R\Gamma(\Omega_0 \cap X; F)$ by several small balls $B(\varepsilon, x_i)$, thus (Prop 5.2.1 in [KS])

$SS(F) = \{(x,\xi) \in T^*X \mid \exists U \ni (x,\xi), \exists \Omega_0 \subset \Omega$ above $R\Gamma(\Omega_0 \cap X; F) \cong R\Gamma(\Omega \cap X; F)\}^c$

Eg. 1 ① k_U denote the constant sheaf of $U \subset X$ open, then

$SS(k_U) =$ ... $U =$

$SS(k_{\bar U}) =$ ... in singular boundary and



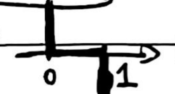
due to singular bound $\dim SS(F) \geq \dim$ can be str thus not Lagrangian but we can expect isotropic.

this follows the first definition directly:

② $M \subset X$ embedding submfd, then $SS(k_M) = N_M^* X \subset T^*X$

in particular, $SS(k) = N_X^* X = X$ (zero section)

$\boxed{\text{as two direction reversed}}$ ③ Let $X = \mathbb{R}$, $F = \mathbb{C}_{[0,1]}$

then $SS(F) =$  $\subset \mathbb{R}^2 = T^*\mathbb{R}$

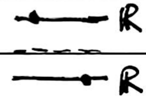
(Due to F the direction at 0 is positive, 1 negative)

Similarly, we have $SS(\mathbb{C}_{\{0\}}) =$  $\subset \mathbb{R}$

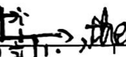
$SS(\mathbb{C}_{[0,\infty)}) =$ , $SS(\mathbb{C}_{(0,\infty)}) =$ 

this advise an exact sequence $0 \to \mathbb{C}_{(0,\infty)} \to \mathbb{C}_{[0,\infty)} \to \mathbb{C}_{\{0\}} \to 0$ induce relation between their SS (Omitted here and later, refer [KS])

④ Now view $T^*\mathbb{R}$ as symplectic mfd, then $S^*T^*\mathbb{R} = \mathbb{R} \sqcup \mathbb{R}$ is contact

$SS(F) \subset T^*\mathbb{R}$ $\quad \Lambda := SS(F) \cap S T^*\mathbb{R} =$  two pts

$\cup \qquad \cup$ Here our isotopy is

$\boxed{SS(F) \cap S T^*\mathbb{R}} \subset S^*T^*\mathbb{R}$ by moving "closer" , then

Lagrangian ⊂ Symplectic ⟲ Homogenous Hamiltonian isotopy

Legendrian ⊂ Contact ⟲ Legendrian isotopy

this is the story of [GKS]

Rk. $W \supset \dot{SS}(F) \cap K$ conic open is a must for $\dot{SS}(F) \cap K$ itself, there not exists such cut-off

No.

Date

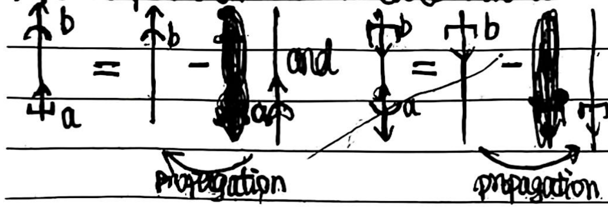
Date

⑤ (Microlocal Morse theory) We generalise discussion before and write precisely: $\varphi: X \to \mathbb{R}$ restrict to $\mathrm{supp}(F) \to \mathbb{R}$ proper

- If $d\varphi \notin SS(F)$ for level set $a \le \varphi(x) < b \Rightarrow R\Gamma(\varphi^{-1}(-\infty, b); F) \cong R\Gamma(\varphi^{-1}((-\infty, b)); F)$
- If $-d\varphi \notin SS(F)$ for level set $a < \varphi(x) \le b \Rightarrow R\Gamma_{\varphi^{-1}([a,b))}(X; F) \cong R\Gamma_{\varphi^{-1}((-\infty, b])}(X; F)$

(Recall that $\Gamma_{\square}$ is the section with supported on a closed set, and replaced by $\Gamma(-; F)$ is for usually open sets)

Here we partition into two cases due to



thus the first case propagation is $d\varphi$ but second case propagation is $-d\varphi$

This is a refined version of classical Morse theory: replace F by local system as coefficient, then it shows the (co)homology stays invariant away critical pts.

We'll see a family of consequences called cut-off lemmas:

PART III

Localization of category and cut-off type thms

We set $D^b(X; \Omega) = D^b(X)[W^{-1}]$ localization by weak equivalence W:
$F \overset{W}{\cong} G \Leftrightarrow SS(F) \cap \Omega = SS(G) \cap \Omega$, we call them isomorphic on Ω
($\Omega \subset T^*X$ only a subset)

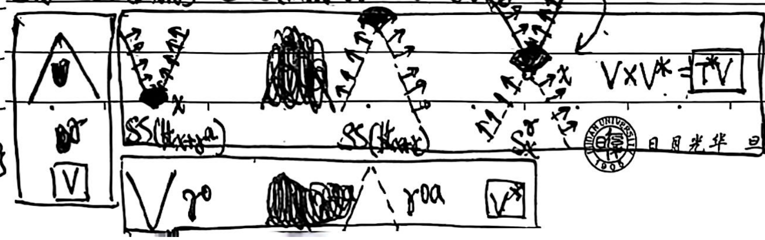
Lemma 1 (Cut-off Lemmas, [KS, Prop 6.1.4, Prop 6.1.6], local version)
$K \subset T^*_x X$ a proper closed convex cone, $U \subset K$ open cone,
$W \supset K \cap \dot{SS}(F)$ — (or we also denote as homogeneous one: $\dot{SS}(F)$) is a conic neighbourhood, then:

① $\exists u: F' \to F$, s.t. $(F' \cong F) \in D^b(X; U)$;
② $\exists u: F \to F'$, s.t. $(F' \cong F) \in D^b(X; U)$; $\}$ and $\cap\, SS(F') \subset W \cup \{0\}$
(Dual version)

This is: we cut the F into F' on two open (conic) sets U & W, s.t. $F \cong F'$ on U and the $SS(F)_x$ is cut into W. Such a version is local at x.
One can view this in analogue of the usual cut-off function (locally bounded by K)
Thus we can have a global version and a partition of sheaf:
Lemma 2. (Global Cut-off lemma, [Guillermou, Ch III.1])
Rk. See [Guillermou, Ch III.2-3] for local version which only ② above, which ever cut-off functor $D^b(k_V) \to D^b(k_W)$, target not hole V, below.
Our base space setted to be vector space $V = T_pV$ and $V^* = T_p^*V^*$
we have Fourier–Mukai functors, $\gamma \subset V$ closed convex cone.

$P_\gamma: F \mapsto Rp_{2*}(k_{\gamma} \otimes p_1^{-1} F)$ $\quad \gamma = V \times V$ defined as $\gamma = \{(x,y) \in V \times V \mid x - y \in \gamma\}$
$Q_\gamma: F \mapsto Rp_{2!}(\mathcal{H}om(k_{\gamma a}, p_1^! F))$
$P'_\gamma: F \mapsto Rp_{2!}(\mathcal{H}om(k_{\gamma - \Delta}[1], p_1^! F))$
$Q'_\gamma: F \mapsto Rp_{2*}(k_{\gamma - \Delta}[1] \otimes p_1^{-1} F)$, then using them as cut-off functor $Q_\gamma, P_\gamma, P'_\gamma, Q'_\gamma: D^b(k_V) \to D^b(k_V)$

① $u: P_\gamma(F) \to F$ is isomorphism on $V \times \mathrm{Int}(\gamma^{\circ a})$, and it's isomorphism globally on $V \times V^*$ iff $SS(F) \subset V \times \mathrm{Int}(\gamma^{\circ a})$;
② We have distinguished triangles:
$P_\gamma(F) \to F \to Q'_\gamma(F) \xrightarrow{+1}$; $P'_\gamma(F) \to F \to Q_\gamma(F) \xrightarrow{+1} \in D(k_V)$
and restrict $\overset{\text{open}}{W} \subset V$, s.t. $S^\gamma_x \cap \dot{SS}(F) = \emptyset, \forall x \in W$
then the adjoint pair is equivalence functors, thus $\exists L$ locally
$P_\gamma(F)|_W \oplus P'_\gamma(F)|_W \to F|_W \to L \xrightarrow{+1}$
$Q_\gamma(F)|_W \oplus Q'_\gamma(F)|_W \to F|_W \to L \xrightarrow{+1}$
and $\dot{SS}(P_\gamma(F)|_W) = \dot{SS}(F) \cap (W \times \mathrm{Int}(\gamma^{\circ a}))$; $\dot{SS}(P'_\gamma(F)|_W) = \dot{SS}(F) \cap (W \times (V - \gamma$
$S^\gamma_x := (\dot{SS}(k_{x+\gamma})^a \cup \dot{SS}(k_{x+\gamma^a})) \cap (V \times \partial\gamma^{\circ a})$



Rk. ① All are correspondence: assume supp(F) cpt, then $P_\gamma = - \circ \mathbb{k}_\gamma$ the composition of sheaves defined in a correspondence manner: (Same for Fourier-Mukai/Sato transformation with different kernels)
Given $K_{12} \in D^b(\mathbb{k}_{X_1 \times X_2})$, $K_{23} \in D^b(\mathbb{k}_{X_2 \times X_3})$, then $K_{13} = K_{12} \circ K_{23} = Rp_{13!}(p_{12}^{-1}K_1 \otimes p_{23}^{-1}K_2)$
It gives raise a Lagrangian correspondence (expected case) of their microlocal support: $SS(K_{12} \circ K_{23}) \subset SS(K_{12}) \overset{a}{\circ} SS(K_{23})$
② Why we denote P_γ is $= p_{13}(p_{12}^{-1}a^{-1}SS(K_{12}) \cap p_{23}^{-1}SS(K_{23}))$
due to it's idempotent elements, thus it's a "projection", we call it the Tamarkin projectors, we set $D^b_{V \times \gamma^{\circ a}}(\mathbb{k}_V)$ the subcategory with complexes are SSed on $V \times \gamma^{\circ a} \subset T^*V$, and denote its right orthogonal complement as $D^{b,\perp}_{V \times \gamma^{\circ a}}(\mathbb{k}_V)$, we have:
$P_\gamma \circ P_\gamma \cong P_\gamma$, $P_\gamma \circ Q_\gamma' \simeq 0$ (if left orthogonal complement, use $Q_\gamma' \circ P_\gamma \simeq 0$)
($P_\gamma' \circ P_\gamma' \simeq P_\gamma'$, $P_\gamma' \circ Q_\gamma' \simeq 0$ similiar, omitted, they're related by adjoint properties: $\lambda^{l} \searrow$ & $\lambda^{l^\perp} \searrow$, not same!
$Q_\gamma \dashv P_\gamma$ into $D^b_{V \times \gamma^{\circ a}}(\mathbb{k}_V)$ & $Q_\gamma' \vdash P_\gamma'$ into $D^{b,\perp}_{V \times \gamma^{\circ a}}(\mathbb{k}_V)$)
thus $P_\gamma : D^b(\mathbb{k}_V) \longrightarrow D^b_{V \times \gamma^{\circ a}}(\mathbb{k}_V)$ and $Q_\gamma : D^b(\mathbb{k}_V) \longrightarrow D^{b,\perp}_{V \times \gamma^{\circ a}}(\mathbb{k}_V)$

E.g. 2. $V = \mathbb{R}$, $\gamma = (-\infty, 0]$, then (using Lemma 2, ②)

F	$P_\gamma(F)$	$Q_\gamma(F)$	$P_\gamma'(F)$	$Q_\gamma'(F)$
$\mathbb{k}_{[0,1]}$	$\mathbb{k}_{[0,\infty)}$	$\mathbb{k}_{(-\infty,0)}[1]$	$\mathbb{k}_{(-\infty,1]}$	$\mathbb{k}_{(1,\infty)}[1]$
$\mathbb{k}_{[0,1)}$	$\mathbb{k}_{[0,1)}$	$\mathbb{k}_{[0,1)}$	0	0
$\mathbb{k}_{(0,1]}$	0	0	$\mathbb{k}_{(0,1]}$	$\mathbb{k}_{(0,1]}$
$\mathbb{k}_{(0,1)}$	$\mathbb{k}_{[1,\infty)}[-1]$	$\mathbb{k}_{(-\infty,1)}$	$\mathbb{k}_{(-\infty,0]}[-1]$	$\mathbb{k}_{(0,\infty)}$

Rk. Compare our Fukaya category v.s. (some) sheaf category, what can we find the advantage of sheaf category?
- Avoid hard PDEs, obviously
- Easier to localize via cut-off
- Algebraic category has better decomposition property
- Even (locally) constant sheaves carrys many datum

- I don't add things about constructibility due to I'm fully understand it)

However, although it's expected that sheaf category can provide detour of hard analysis and provide technical advantage, and Guillermou... did prove a lot of consequence on classical symplectic geometry such as Gromov's non-squeezing thm..., there is still no evidence that one can totally replace another: they're just the difference between Morse $\longleftrightarrow$ singular cohomology.

PART IV (Floer $\longleftrightarrow$ sheaf) $(C_p(S,S) \supset C_p(S,S)^\omega, \forall p \in S)$

Involutivity thm (Involutivity is the generalization of coisotropic)
Thm 1. $SS(F) \subset T^*X$ is involutive subset, i.e. $\forall p \in S$ $\forall \theta \in T_p^*T^*X$, s.t. $C_p(S,S) \subset \{v \in T_pT^*X \mid \langle v, \theta \rangle = 0\}$, $\overline{\omega}(\theta) \in C_p(S,S)$
($\overline{\omega}$ iso induced by ω: $\omega(v, \overline{\omega}(\theta)) = \langle v, \theta \rangle$)

- We do not prepare enough techniques (mainly functorial properties in microlocal level) at before parts, these parts are occupied by the understanding of definitions and properties.

The proof of Thm 1 depends on the key microlocalization thm
Lemma 3. (Microlocalization) $SS(\nu_M(F)) = SS(\mu_M(F)) \subset C(T_M^*X, SS(F))$ and $SS(\mu Hom(G,F)) \subset C(SS(F), SS(G))$

Pf. • View Fourier-Sato transformation $\wedge$ and its inverse $\vee$ as composition with convolution kernel, then by $SS(\mu_M(F)) = SS(\nu_M(F) \circ K) \subset SS(\nu_M(F)) \overset{a}{\circ} C$ $= SS(\nu_M(F))$
and conversely same; (this differs proof in [KS]) ("bideformation trick")
- $SS(\nu_M(F)) \subset C(T_M^*X, SS(F))$: check locally, via definition of ν_M;
- $SS(\mu Hom(G,F)) = SS(\mu_\Delta Hom(q_1^{-1}G, q_2^! F)) \subset C(N_\Delta^*(X \times X), SS(Hom(q_1^{-1}G, q_2^! F)))$ $\subset C(N_\Delta^*(X \times X), SS(F) \times SS(G)^a)$ ([KS, Prop 4.2]) $=: C(SS(F), SS(G)^a)$ $= C(SS(F), SS(G))$ ▨

for example, DT^*M is symplectic mfd with boundary of contact type, then the Legendrian $\Lambda \subset ST^*M$ can be easily dealt by restrict micro-supp

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Pf of Thm1. Otherwise, $\exists p \in SS(F)$, $\exists \theta \in T_p^*T^*X$, s.t. ① $C_p(SS(F), SS(F)) \subseteq \{v \in T_pT^*X \mid \langle v, \theta\rangle = 0\}$ and ② $\varpi(\theta) \notin C(p, S)$. We then show that $p \notin SS(F)$ to derive a contradiction:

① $\Rightarrow SS_p(\mu Hom(F,F)) \overset{\text{Lemma3}}{\subset} C_p(SS(F), SS(F)) \subset \{\theta = 0\} \subset \{\varpi(\theta) = 0\}$

$\Rightarrow SS(\mu Hom(F,F)) \cap N_p^*Z =$ if $\exists Z \supset SS(F) \ni p$ and $\forall \lambda \in N_p^*Z$, $\langle \varpi(\theta), \lambda \rangle < 0$

then $\mu Hom(F,F)_p = (R\Gamma_Z(\mu Hom(F,F)))_p = 0$ ~ We need take the stalk as limit under γ-topology, see KS book

$\Leftrightarrow p \notin SS(F)$ done

② $\Leftarrow$ the existence of such Z ☑

Rk. In these sets above, we note they're "different" subset of cotangent bundle, thus we should indentify them first.

Coro1 $L \subset (T^*M, \omega_{std})$ conic Lagrangian, connected; for $F \in D^b(k_X)$ with $\dot{SS}(F) \subset L$ and nonempty $\Rightarrow \dot{SS}(F) = L$.

This is a consequence of propagation of singularity, thus we also connected condition to extend whole L.