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To be continues ... But not now...

Intersection theory and enumeration problems — Notes of reading problem I. Instructor is Yang Zhou.

Chap1. Chow rings

Intro: We construct an analogue of cohomology theory H^* in topology to algebraic setting. For the similarity, it's natural to consider: ① Construction of Chow group(ring) $A^*(X)$ using cycles: it's motivated by let X mfd, $Z\subset X$ submfd of $\dim = d = n-c \Rightarrow [Z]\in H_d(X)\cong H^{c}(X)\ni \eta_Z$, η_Z the volume form of normal bundle N_Z, $[Z]$ is the fundamental class. Thus $A^*(X)$ is something smaller than $H^*(X)$, consisting "classes from geometry".

Rk1. To outline "cohomology", one may denote it $CH^*(X)$; Rk2. Due to $A_*(X)=A^{n-*}(X)$ defined, one can view it naturally has a "Poincare duality" $\Leftrightarrow$ trace map $\int: A_0(X)\longrightarrow k \Leftrightarrow$ pair to top to give a number $\in k \Leftrightarrow$ intersection product [illegible]

② Universality of $A^*(X)$ among all cohomology theories: Classical Eilenberg–Steenrod axioms don't fit well in algebraic setting (and sometimes not enough): homotopy axiom isn't holding in algebraic setting & now we must need Poincaré duality. We call a Weil cohomology H^* is Poincare duality, Künneth formula and "relation with $A^*(X)$", i.e. closed cycle map $\gamma_X: A^i(X)\longrightarrow H^{2i}(X)$ as grp morphism naturally, s.t. $\gamma_X(\alpha\beta)=\gamma_X(\alpha)\cup\gamma_X(\beta)$ and $\int\gamma_P(P)=1$, $P=\operatorname{Spec}k$ pt.

Facts. i) γ_X preserves Künneth formula; ii) $Z\sim 0\Rightarrow [Z]=\gamma_X(Z)=0$ (iii) $A^*=H^*$ is a Weil cohomology, is one of our goal.

Rk. One adding Hard Lefschetz to Weil's cohomology for uses in proving Weil's conjecture (and Standard conjecture)

③ Categorification of $A^*(X)$ to Chow motive: to describe why topological cohomology not fit in algebraic category, consider the categorical embedding $\mathrm{Var}_k\subset \mathrm{Top}\xrightarrow{H^*}\mathrm{Comp}(Ab)=D_{ab}$ is dg-category, the category of (pure) motive is a category (dg) filling the diagram

$$\begin{tikzcd}
\mathrm{Var}_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & D\mathrm{motive} \arrow[d] \\
\mathrm{Top} \arrow[r, "H^*"'] & D_{ab}
\end{tikzcd}$$

One of Dmotive is constructed as Chow motive (although the category of Chow motive not Tannakian, a modification is) replace linear equivalence by numerical equivalence. Q. What is a "good" equivalence

Rk. Later we may describe the Bloch's higher Chow grps and to motivic cohomology.

§1.1 Chow grps & rings, as generalisation of degree

⇒ Generated by rational equi, thus Prop① should be modified as

Def1 (Chow group) $A_d(X)=A^c(X)=Z_d(X)/\sim$ rational/linear equivalence, where $d+c=n=\dim X$, $Z_d(X)$ consists all d-dim subvarieties' linear combination i.e. d-dim (X is only a scheme). $A_*(X)=\bigoplus A_d(X)$ also group graded by dim. This product structure on $A_*(X)$ is highly nontrivial (One also defining it as $A_*(X)=Z(X)/\sim$, they're equivalent)

$Z_0-Z_1=$ [illegible] integral closed + dim [illegible]

Generated by $[\operatorname{Div}(f)]=\sum\operatorname{ord}_V(f)[V]$, we can define on normal case by DVR; but when not normal case [illegible]

Prop1. X scheme, then ① Rational equivalence of two cycles Z_0, Z_1 ($Z_0-Z_1=[\operatorname{div}(f)]$) $\Leftrightarrow \exists\,\Phi\subset\mathbb{P}^1\times X$ subscheme intersect fibre X_0 and X_∞ properly ($\dim(\Phi\cap X_i)\le\dim\Phi-1$, $i=0,\infty$), and $X_0\cap\Phi=Z_0$, $X_\infty\cap\Phi=Z_1$ set-theoretically (construction) (consider as cycles) use the argument as Φ may not graph (but glued by several graphs)

② $Z_*(X)/\sim\;\cong\bigoplus Z_k(X)/\sim_k$ (Both Fulton and Stacks proj defined as right)

Pf. ① We only prove a useful case: $\Phi\xrightarrow{\pi}\mathbb{P}^1$ is dominant (avoid scheme-issue, to make it behave as X variety case and the fibres Φ_0 and Φ_∞ $=Z_0$ and Z_1 scheme-theoretically) Assume $Z_0=0$ ($\Leftarrow$) is easy, let $\Phi\longrightarrow\mathbb{P}^1$ dense image be nonvanishing

$\Rightarrow$) $Z_1=[\operatorname{div}(f)]$, where f has nonvanishing loci $W\subset X\Rightarrow f: W\longrightarrow\mathbb{P}^1$, then set $\Phi=\overline{\mathrm{Graph}(f)}\subset X\times\mathbb{P}^1$ locus of f_i

thus $\Phi\xrightarrow{p_1}W\xrightarrow{f}\mathbb{P}^1\Rightarrow Z_1=[\operatorname{div}(f)]$ (f rational) $=p_{1*}[\operatorname{div}(F)]$ (By Lemma1 next) U the domain of definition of f; $p_1|_\Phi$ birational, $W\subset X$, not domain of def and p_1 is proper $\Rightarrow p_1|_\Phi$ proper

The composite $p_1(\Phi)\subset U$, thus in fact the composite is morphism $F=[\Phi_0]-[\Phi_\infty]\sim 0$ ∎ $[F]-f$ ($\Leftarrow$) We accept that rational equi is preserved under proper pushforward

then we can choose pts $\in\mathbb{P}^1$, s.t. fibre is Z_1 and 0, Z_1 is always possible as its zero locus of f; and then change (t_0,t_1) to $(0,\infty)$ up to an automorphism of $\mathbb{P}^1$ ∎

$X\times\mathbb{P}^1\xrightarrow{p}X$ (it's proper as $\mathbb{P}^1$ proper)

② It suffices to show that $Z(X)/\sim$ is graded by dimension of cycles; we apply the description of ①: taking an affine chart for $\mathbb{P}^1$: then $\forall t_0\in\mathbb{A}^1\subset\mathbb{P}^1$, $\Phi_{t_0}=V(t-t_0)\subset\Phi$ codim 1 doesn't change, thus an affine cover connect 0 and ∞ ∎

Lemma1 (Push-forward of cycles) Recall $f_*[V]=\deg(f|_V)[f(V)]$ defined (same as divisor), then

① $f: X\to Y$ proper surjective, then $f_*[\operatorname{div}(r)]=\begin{cases}0; & \dim Y<\dim X\\ [\operatorname{div}(N(r))]; & \dim Y=\dim X\end{cases}$, where r rational and $N(r)=\det(t_r)$ the norm ($t_r: s\mapsto sr$)

② cycle $Z\sim 0\Rightarrow f_*Z\sim 0$, where f proper

$f|_V$ is proper: provided $f(V)$ does a variety. Otherwise $f_*[V]=\sum_i\deg(f: V\to f(V)_i)[f(V)_i]$
- $f(V)$ not closed (closure)
- $f(V)$ not irreducible (components)

Pf. ② follows ① obviously, surjectivity by replace Y by its image ∎

① [Fulton, Prop1.4] Case1 When $\dim Y=\dim X$, we reduce to f is a finite cover (finite map) $\tilde X\xrightarrow{\tilde f}\tilde Y$, $\tilde X,\tilde Y$ are normalization; thus we reduce to case X & Y are normal. Writing $[\operatorname{div}(r)]$, W is subvariety of Y. $A=\mathcal{O}_{W,Y}$ local ring + $\operatorname{codim}(W,Y)=1$ + Y normal (all local ring integrally closed) $\Rightarrow A$ is DVR. Thus we can use the valuation criteria of properness of f:

$$\begin{tikzcd}
\operatorname{Spec}K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec}A \arrow[r] \arrow[ur, dashed, "\exists!"] & Y
\end{tikzcd}$$

$K(X)/K(Y)\supset A$, the integral closure of A in $K(X)$ B

Now $V_i\subset X$ corresponds to $W\subset Y$ ($f(V_i)=W$) $\Leftrightarrow$ maximal ideals $m_i\trianglelefteq B$, $B_{m_i}=\mathcal{O}_{V_i,X}$ (the valuation criteria tells us B_{m_i} dominates $\mathcal{O}_{V_i,X}$ and $\mathcal{O}_{V_i,X}$ is normal one) this reduces claim $\Rightarrow B_{m_i}=\mathcal{O}_{V_i,X}$ in $\Leftarrow$ side

$f_*[\operatorname{div}(r)]=[\operatorname{div}(N(r))]$ to $\boxed{\sum_i\operatorname{ord}_{V_i}(r)[K(V_i):K(W)]=\operatorname{ord}_W(N(r))}$,

$\Leftarrow$ [illegible] $=l_A(\ldots)=\operatorname{ord}_A(\det(\ldots))$ [illegible]

Algebraic computation: $\operatorname{ord}_W(N(r))=\operatorname{ord}_W(\det(t_r^{K(X)}))$ and $\sum\operatorname{ord}_{V_i}(r)[K(V_i):K(W)]=\sum e_{B_{m_i}}(\ldots)$ [illegible]

Case II $\dim(Y) = \dim(X) - 1$ (In general $\dim(Y) < \dim(X)$ reduce to it by induction easily)

By base extension $X \leftarrow \operatorname{Spec} k \times X$ reduce to $X \times \operatorname{Spec} k \xrightarrow{f} \operatorname{Spec} k$ curve over affine line (assume X is this curve)

thus normalisation $X \xrightarrow{p} Y = \mathbb{P}^1$ In Fulton, he gives a computation

$\Rightarrow f_*[\operatorname{div}(r)] = p_* g_* [\operatorname{div}(r)]$ It's on $\operatorname{Ch}(X)/C$ order of pole "coefficient of Y in $f_*[\operatorname{div}(r)]$ is $\sum \operatorname{ord}_{y}(r)$ [illegible]"

Here f is given by $K(X) \simeq K(Y)$ (normalisation is birational) $\xrightarrow{g} \mathbb{P}^1$ finite cover as X is normal curve $[K(X):K(Y)]$, then $\sum \operatorname{ord}_y(r)[K(X):K(Y)]$ use base extension, I don't know what the

thus we reduce to p and g, g is done in Case I, and $p: \mathbb{P}^1 \to \mathbb{A}^1$ is a explicit computation (omitted) ▨ step is for

Now we construct intersection product to make $A_*(X)$ into ring. It's originally constructed by Chow using moving lemma under smooth & quasi-proj, motivated by Severi. Later Fulton's deformation to normal cone allows us construct it in a more general setting (only smooth). We show it next ("topological consequence" are admit in §1.1, they'll be proven in §1.2)

A intersection on Y is setting of the Cartesian diagram $W \hookrightarrow V$ $\operatorname{codim}(X,Y) = d$, $\dim V = k$ scheme pullback

we define $(X.V)$ first and then gives ring structure for $g \downarrow \quad \square \quad \downarrow f$ i is regular embedding with normal bundle N

$\in A_{k-d}(W)$ nonsingular variety. $X \xhookrightarrow{i} Y$ $W = f^{-1}(X)$ has natural subscheme structure $\subset V$

Prop 2 $(X.V) = s^*[C] = \{c(g^*N) \cap s(W,V)\}_{k-d}$, $s^*: A_k(C) \to A_{k-d}(W)$ Gysin map, C the normal cone, c Chern class and s Segre class. + diagonal embedding is Facts: ① without regular embedding gives I_X is regular sequence

Prop 3. (Chow ring) nonsingular Y intersection product makes $A_*(Y)$ a commutative graded ring. Here use cohomology for the via $A_k(Y) \otimes A_l(Y) \xrightarrow{\times} A_{k+l}(Y \times Y) \xrightarrow{\delta^*} A_{k+l-n}(Y)$, δ^* Gysin. the normal "=" not bundle, locally free (index of product is adding)

It's properties will showed later (When X is variety, it holds), as $N = \mathcal{I}/\mathcal{I}^2$, and $\mathcal{I}/\mathcal{I}^2$ generated by S_1, S_1 coherent,

Def 2 (Cone) A cone over scheme X is $C_X = \operatorname{Spec}(S^\bullet)$, $S^\bullet$ is a graded $\mathcal{O}_X$-algebra, $\mathcal{O}_X = S_0$ (or $\mathcal{O}_X \twoheadrightarrow S_0$ [illegible])

And projective cone is its projectivization $\mathbb{P}C_X = \operatorname{Proj}(S^\bullet)$, and its projective completion is $\mathbb{P}(C_X \oplus 1) = \operatorname{Proj}(S^\bullet[z])$ ($S^\bullet[z] = S^n \oplus S^{n-1} z \oplus \cdots \oplus S^0 z^n$) (One can visual their model as $C_X = \mathbb{A}^1$, $\mathbb{P}C_X = \{\infty\}$ (hyperplane at ∞) and $\mathbb{P}(C_X \oplus 1) = \mathbb{P}^1$), we denote them C simply

Def 3. (Segre class) $s(C) = q_*(\sum c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C \oplus 1)])$, $q: \mathbb{P}(C \oplus 1) \to X$ projection, applied to normal cone $C = \operatorname{Spec}(\oplus \mathcal{I}^n/\mathcal{I}^{n+1})$

$\Rightarrow s(X,Y) = s(C) = q_*(\sum c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C \oplus 1)]) = \sum p_*(c_1(\mathcal{O}(1))^i \cap [\mathbb{P}C])$, $p: \mathbb{P}C \to X$ projection as the normal cone is purely n-dim cone (we'll prove it next). In general, one can define Segre class of vector bundle similarly

Lemma 2. ① Normal cone is purely n-dim (for each component C_i of C, $\mathbb{P}C_i \neq \emptyset$) Only when dim differ 2, irreducible

② A cone satisfy these condition has $s(C) = \sum p_*(c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C)])$ is not change, when 1, $Y \leftarrow X$ changes

Pf ① $X \subseteq Y$ is pure dim $n \Rightarrow C_X$ also: $X \subset Y \xhookrightarrow{\text{zero section}} Y \times \mathbb{A}^1$, then the normal cone of $X \subseteq Y \times \mathbb{A}^1$ is $C \oplus 1$

Consider the blowing up $Bl_X(Y \times \mathbb{A}^1) \simeq Y \times \mathbb{A}^1$, ~~and X is nowhere dense in $Y \times \mathbb{A}^1$~~ $\Rightarrow Bl_X(Y \times \mathbb{A}^1)$ is pure dim $= n+1$ with exceptional divisor $\mathbb{P}(C \oplus 1) \Rightarrow$ pure dim $= n$ ▨ Thus tangent cone $\operatorname{Spec}(\oplus \mathfrak{m}^n/\mathfrak{m}^{n+1})$, normal cone C are singular

② Let $C = \bigcup C_i$, with geometric multiplicity $m_i(C_i, C) = l(\mathcal{O}_{C_i, C}) =: m_i$, then the divisor $\mathbb{P}(C \oplus 1)$ restrict to each C_i

$\Rightarrow [\mathbb{P}(C \oplus 1)] = \sum m_i [\mathbb{P}(C_i \oplus 1)] \Rightarrow s(C) = \sum m_i s(C_i)$ on each C_i, $c_1(\mathcal{O}(1)) \cap [\mathbb{P}(C \oplus 1)] = [\mathbb{P}C] \Rightarrow s(C) = \sum p_*(c_1(\mathcal{O}(1))^i \cap [\mathbb{P}C])$

Eg 1 (Bundles) ① Vector bundle $E = C(\operatorname{Sym}(\mathcal{E}^\vee))$ and $\mathbb{P}E = \mathbb{P}C(\operatorname{Sym}(\mathcal{E}^\vee))$, z of $S^\bullet[z]$ is a section of $\mathcal{O}(1)$

② (Blowing ups) Recall $Bl_X Y = \operatorname{Proj}(\oplus \mathcal{I}^n)$, compare with normal cone in $\mathbb{P}(C \oplus 1)$, zero locus of this section is C, nonvanishing is $\mathbb{P}(C)$ ▨

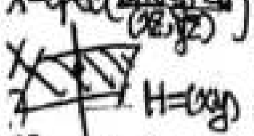
$\operatorname{Proj}(\oplus \mathcal{I}^n/\mathcal{I}^{n+1})$ (it's a bundle by ①), the exceptional projective divisor $E = \pi^{-1}(X)$, with $\mathcal{O}(1) = \mathcal{I}$, E is the projective cone as $(\oplus \mathcal{I}^n) \otimes \mathcal{O}_X = \oplus(\mathcal{I}^n/\mathcal{I}^{n+1}) \Rightarrow E = \mathbb{P}C$ ▨

Eg 2 (Cones) Consider the normal cone of affine variety $X \subset \mathbb{A}^2$; $X = \mathbb{A}^1 \cup \mathbb{A}^1$ One always compute

then $I = (xy)$, $I^2 = (x^2y^2) \Rightarrow C = \operatorname{Spec}(\oplus I^n/I^{n+1}) = \operatorname{Spec}(k[x,y]/(xy) [illegible])$ It's hard but finite

Eg 3 (Non regular embedding) $\operatorname{Spec}(\oplus I^n/I^{n+1})$, generators are 1 [illegible]

$X = \operatorname{Spec}\left(\frac{k[x,y,z]}{(xz,yz)}\right) \hookrightarrow \mathbb{A}^3$, $I^2 = (x^2z^2, xyz^2, y^2z^2) \Rightarrow C = \operatorname{Spec}\left(\frac{k[x,y,z][a,b]}{(xz, yz, ya-xb)}\right)$



$p \in H$ and $q \in L$ $C_p = \frac{k[a,b][z]}{(ya-xb)}$ $\mathbb{A}^1$ with slope

$C_q = \operatorname{Spec}[a,b] = \mathbb{A}^2$ is the affine cone

Rk. Passing to cone is a useful tool, as $\mathbb{P}^1 \times \mathbb{P}^1$ (by Segre embedding to $\mathbb{P}^3$) ▨

One study cone / proj cone by considering fibers at closed pts of X

$C_0 = \operatorname{Spec}(k[z]) = \mathbb{A}^1$

$C_p = \operatorname{Spec}(k[a,b][z]/(z)) = \mathbb{A}^2$

For replaced by Proj

$\mathbb{P}C_0 = \operatorname{Proj}(k[z]) = \{pt\}$

$(\operatorname{Proj}(k[x,y]) = \mathbb{P}^1$, not right

$= \mathbb{P}^0 \times \mathbb{P}^0$ (See [Fulton (4.1.3)])

Next we show the tool of deformation & specialisation by Fulton, (uses later)

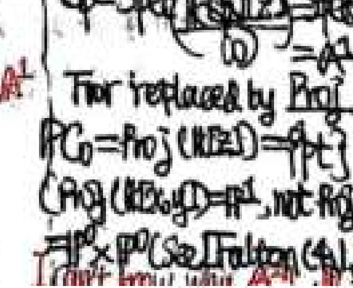
Prop 4. ① (Deformation) $X \subseteq Y$, we deform Y to "Y" and deform X to $\mathbb{P}C$:

$\exists M$, $X \times \mathbb{P}^1 \longrightarrow M$ with fibres on $\mathbb{P}^1$ flatly

(called deformation space) is $\mathbb{A}^1$ both effective Cartier

② (Specialisation) Pf: $M_\infty = \mathbb{P}(C \oplus 1) + \tilde{Y}$ as infinite divisor

We have specialisation morphism $\sigma: A_k(Y) \to A_k(C)$ well-defined $[V] \mapsto [C_{W}V]$



Pf ② is easier, ① contains more geometry, we prove ② first [illegible]

③ By Excision [illegible] (proved later), we have SES: $A_*C \xrightarrow{i_*} A_*M^\circ \xrightarrow{j^*} A_*(Y\times\mathbb{A}^1)$, where M° is deformation space $-\tilde{Y}$ [illegible]

We have ① Gysin map for divisor: $C \subset M^\circ$ is divisor, thus $\exists\, i^*$ [illegible] $\to A_*C$ one [illegible] effective Cartier

The left [illegible] by [illegible] [illegible] as [illegible] (A intersection class with divisor is always defined, [illegible] $\sum m_i [C\cdot V_i] = \sum m_i [i^*_i C]$ where [illegible] $C \subset \square$ $\downarrow_i$ (pullback of divisor is definable) $\hookrightarrow M$ In general, i^*C not a (well) divisor but a pseudo-divisor. However $[i^*C]$ can be defined)

The geometry of M° is $X\times\mathbb{A}^1 \to M^\circ$ where $0\in\mathbb{P}^1$; $p_0 \in \mathbb{P}^1$ is the [illegible] (as $\mathbb{P}(C\oplus 1)$ [illegible])

Flat pullback: $Y\times\mathbb{A}^1 \xrightarrow{p} Y$ flat by ① and flat map induces pullback [illegible] (and inclusion j^*) [illegible] $\mathbb{P}(C) \to$ delete $\tilde{Y}$

we need a technical lemma (next) to descend it to $A_*M^\circ \to A_*C$ [illegible]

Generally let $f: X \to \mathbb{A}^1$ flat with relative dim n, it descends to $A_*Y \xrightarrow{f^*} A_{*+n}X$ by [illegible] ; Using Prop 1①, if $a\sim 0 \Leftrightarrow a=[V_0]-[V_\infty]$, $\exists\, V \subset Y\times\mathbb{P}^1$ $\sum m_i[V_i] \mapsto \sum m_i [f^*V_i]$ $V \to \mathbb{P}^1$ dominant $\Rightarrow$ flat, let $W = (f\times 1)^{-1}(V) \subset X\times\mathbb{P}^1$

$$\begin{tikzcd} A_*C \arrow[r, "i_*"] \arrow[dr, "0"'] & A_*M^\circ \arrow[r, "j^*"] \arrow[d, "i^*"] & A_*(Y\times\mathbb{A}^1) \arrow[r] \arrow[dl, dashed] & 0 \\ & A_*C & A_*Y \arrow[u] \arrow[l, red] & \end{tikzcd}$$

inject also

We check the red arrow does send $[V]$ to $[C_{V\cap X}\, V]$: $[M^\circ_V] \mapsto [V\times\mathbb{A}^1]$ where M°_V is the deformation space of $V\cap X \subset V$, delete $\tilde{V}$ [illegible]; C_V is the normal cone of $V\cap X \subset V$ ▨

A computation then show $f^*a = [W_0]-[W_\infty]$ (taking components of W) $\Leftrightarrow f^*a \sim 0$ ▨

The lift is by surjective j^*, but not well-defined on A_*C $[C_V] \mapsto [V]$. But composite i^*, two pre-image $a_1, a_2 \in A_*M^\circ$ of $b \in A_*(Y\times\mathbb{A}^1)$ has same $i^*a_1 = i^*a_2 \in A_*C$, as $\boxed{i^*i_* = 0}$ (By $i^*i_*(a) = c_1(N)\cap a$ (proved next) N is the normal bundle of regular embedding $C \subset M$, [illegible] trivial)

① Construction1: $M = Bl_{X_\infty}(Y\times\mathbb{P}^1)$ (here $X_\infty = X$ means fibre at $\infty \in \mathbb{P}^1$) $X_\infty \subset Y\times\mathbb{P}^1$ is of codim 2

$\Rightarrow M$ $\xrightarrow{\text{blow down}} Y\times\mathbb{P}^1 \to \mathbb{P}^1$ is a morphism. Furthermore, ρ is flat due to [illegible] $\forall X \subset Y$, Y flat over T (universal as [illegible]) Bl_XY also (here is $Bl_X(X\times\mathbb{P}^1)$ flat over $\mathbb{P}^1$)

② Embedding $X_1\times\mathbb{P}^1 \hookrightarrow M$ by

$$\begin{tikzcd} Bl_{X_\infty}(X\times\mathbb{P}^1) \arrow[r, hook] \arrow[d, phantom, "\cong"] & Bl_{X_\infty}(Y\times\mathbb{P}^1) \arrow[d, phantom, "\parallel"] \\ X_1\times\mathbb{P}^1 \arrow[r, hook] & M \end{tikzcd}$$

(As $Bl_{X_\infty}(X\times\mathbb{P}^1)$ is smaller [illegible] Blow up not change

As X_∞ is Cartier (and effective)

This step ③ due to Bl_X [illegible] flat over T [illegible] effective [illegible] $\Leftrightarrow \oplus \mathcal{I}_X$ flat over $\mathcal{O}_T$, but all ideal sheaves $\mathcal{I}_X$ torsion-free + $\mathcal{O}_T$ is DVR $\Leftrightarrow \mathcal{I}_X$ flat over $\mathcal{O}_T$ ▨

Blow up along X_∞	Exceptional div
$Y\times\mathbb{P}^1$	$\mathbb{P}(C\oplus 1)+\tilde{Y}$
$Y\times\mathbb{A}^1$	$\mathbb{P}(C\oplus 1)$
Y	$\mathbb{P}C$

Thus we have M desired, except checking that $M_\infty = \mathbb{P}(C\oplus 1)+\tilde{Y} := \mathbb{P}(C\oplus 1) + Bl_X(Y)$ by algebraic computation locally on Y, first we assume $Y = \operatorname{Spec} A$ affine (why?) and $X \subset Y$ determined by $\operatorname{Spec}(A/I)$, and $\mathbb{P}^1$ [illegible] here over k [illegible]

(One can also not assume this and using Spec same, here [illegible] All the computation of exceptional [illegible] divisor, we simply write the sheaf [illegible] version of Spec/Proj and [illegible] replaced by Spec/Proj [illegible] easily. Arguments are all same)

$Bl_X(Y\times\mathbb{A}^1) = \operatorname{Proj}(\oplus (I,z)^n) = \operatorname{Proj}(\oplus (I^n + I^{n-1}z + \cdots + Iz^{n-1} + z^n)$ [illegible]) defined by $\mathcal{I}$

And then locally affine cover on $\operatorname{Proj}(\oplus (I,z)^n)$ are $\operatorname{Spec}(\oplus (I,z)^n)_{(a)}$ localise at all $a\in(I,z)$ is generators, then $\mathbb{P}(C\oplus 1)$ defined a/z by definition of projective cone [illegible] of $\operatorname{Proj}(\oplus (I,z)^n)$ [illegible] locally; $\tilde{Y}$ is due to its complement [illegible] $\tilde{Y}$ defined by z/a [illegible] $=\oplus$ [illegible]

$\operatorname{Spec}(S_{(z)}) = \operatorname{Spec}(\oplus I^n z^{-n} [illegible] \oplus I^{n-1} [illegible] \oplus A \oplus [illegible])$ $\oplus \{s/a^n \mid s\in S^n\}$ [illegible]
$= \operatorname{Spec}(\bigoplus_{-\infty}^{\infty} I^{-n}T^n)$ ($I^{-n} = A$ for $n \geq 0$) $=: \operatorname{Spec}(B)$

Consider this by compute example $Bl_0(\mathbb{A}^3)$ [illegible]

then $M^\circ = \operatorname{Spec}(B)$ [illegible] and when $z=0$ [illegible] B/zB, (corresponding to [illegible]) we have isomorphism $B/zB \cong \oplus I^n/I^{n+1} \Rightarrow$ [illegible] zero section ▨

$X = 0 \subset \mathbb{A}^3 - Y$ [illegible] $= P_x$ [illegible] $x_w = x_0 u$; $x_v w = x_0 v$ [illegible]

Construction2. It's a geometric version of Construction1 [illegible], i.e. MacPherson's graph construction

Set $M = \overline{\text{Graph}}$ [illegible] $\times\mathbb{A}^1 \to \mathbb{P}(E\oplus 1)\times\mathbb{P}^1$), where E vector bundle on Y, s.t. X is its zero section

$\mathbb{P}(E) \supseteq \mathbb{P}(E\oplus 1)$ compactification

s vanishing $X \Rightarrow \forall\, ts$, $t\in\mathbb{A}^1$, ts also

thus $Y\times\mathbb{A}^1 \to \mathbb{P}(E\oplus 1)\times\mathbb{P}^1$ defined
$(y,t) \mapsto (t\cdot s(y)) \times (1:t)$ [illegible] can be replaced by $\mathbb{A}^1$

Our geometric insight is when $t\to\infty \in \mathbb{P}^1 \supset \mathbb{A}^1$ then ts more closed to $\mathbb{P}(E)$ (not reach), thus our deformation of X into $\mathbb{P}(E\oplus 1)$, and $\subset \mathbb{P}E$ deform to $\mathbb{P}(C)$

i.e.

And section of $E \Rightarrow$ section of $\mathbb{P}(E\oplus 1)$; zero ① and C zero restrict to nearby X, thus done ▨

Affine version: let $I = (f_1, \ldots, f_n)$ ($\Leftrightarrow$ section s) then $Y\times\mathbb{A}^1 \to \mathbb{P}^n$ is desired map $(y,t) \mapsto (1 : tf_1 : \cdots : tf_n)$ [illegible] $\mathbb{P}(\mathbb{A}^n\oplus 1)$

$I = (x,y,z)$ [illegible] generated by [illegible] and $z = x_0$ a takes u or v or w [illegible] $Bl_0(\mathbb{A}^3)$ localisation on $\mathbb{P}^2$

Rk. (General Graph construction) $\mathbb{P}(E\oplus 1)$ here parameterizes lines $\subset E\oplus 1$ (i.e. lines with [illegible]). Generally $\mathbb{P}$ is replaced by Grassmannian, if there is no [illegible] $E \to$ [illegible] effective need lines to span a dim $=$ rank E subspace) [illegible] and $Y\times\mathbb{A}^1 \to$ [illegible] Here we assume [illegible] which exact out of X, it's used for [illegible] Chern [illegible] $G(E)$, [illegible] functions for kernel is subbundle on [illegible] this allow us giving an embedding. [illegible] $\mathbb{A}^1 \Rightarrow \mathbb{A}^1$ Generally in graph construction $Y \geq$ codim then its section [illegible]

Q. Where is the [illegible] of the ①

Rk. Why deformation to normal cone is key to intersection product $\Rightarrow$ both gives intersection product

① Intersection product $\Leftrightarrow$ Poincaré duality if we have a trace map

② Recall our Poincaré duality on mfd is given by $X \subset Y$ (both mfd), $[X] \leftrightarrow \alpha_X$, α_X is volume form of normal bundle's tubular neiberhood,

③ Deformation to normal cone is analogue of normal neiberhood $\subset \tilde{Y} + \mathbb{P}(C\oplus 1)$ [illegible] due to rigidity of algebraic (algebraic / complex analytic, usual a tubular neiberhood not exist in original space Y without deformation (even X, Y both smooth)

$\Rightarrow$ Such obstruction is due to $Y \subset X$ not regular. When regular, $C = N$ bundle, [illegible] bundle, zero section

Lemma3. (Flat pullback and Gysin map) All D effective Cartier, X scheme general (We need effective Cartier to [illegible]

① Gysin map of divisor $i^*: Z_k X \to A_{k-1} D$, [illegible] descends to Chow; its associated scheme
$\alpha \mapsto (D.\alpha)$

② $i^* i_*(\alpha) =$ [illegible] $c_1(N)\cap\alpha$, where $N = i^*\mathcal{O}_X(D)$ (when $D \subset X$ is regular, it's just normal bundle on D, as section of $i^*\mathcal{O}(D)$ is $i^*\mathcal{O}_X(-D)^\vee = (i^*\mathcal{O}_X(-D))^\vee$, and section of N is $(\mathcal{I}_D/\mathcal{I}_D^2)^\vee$ are same) $= (i^*\mathcal{I}_D)^\vee = (\mathcal{I}_D \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}_D)^\vee$ intersection formula

Here ∩ can't be viewed as cap product as intersection product haven't defined, but as intersection with divisor defined, here we set $c_1(L)\cap[V] = [C]$, where C is Cartier divisor s.t. $\mathcal{O}_V(C) \cong L|_V$ ⟹ $c_1(\mathcal{O}_X(D))\cap\alpha$ [illegible] $= (D.\alpha)$

② is easy by definition: $i^* i_*(\alpha) = (D.i_*\alpha)$ & $c_1(i^*\mathcal{O}_X(D))\cap\alpha = c_1(\mathcal{O}_X(D))\cap i_*\alpha = (D.i_*\alpha)$ ▨

① let $\alpha = [\mathrm{div}(r)]$, $r \in K(V)^*$, $V \subset X$ subvariety, replace X by V ⟹ $\mathcal{O}_X(\mathrm{div}(r))$ trivial (principal) ⟹ $(D.[\mathrm{div}(r)]) = (\mathrm{div}(r)\cdot[D]) = 0 \in A_{k-1}(D)$
($c_1(i^*\mathcal{O}_X(D))\cap[V] = [C]$, $i^*\mathcal{O}_X(D)|_V \cong \mathcal{O}_V(C)$, $j: W = V\cap D$, $\mathcal{O}_X(D)|_W \cong \mathcal{O}_{W}(C)$ ▨)

Communicativity: it's an algebraic computation on multiplicity $e_A(a, A/(bA)) = e_A(b, A/(aA))$ (omitted)

If $\mathcal{O}(D)$ trivial ⟹ $(D.[V]) = 0$; When Cartier nothing to prove as $\mathcal{O}(D)$ trivial ⟺ $D \sim 0$ ▨

Then we can start to define $(X.V)$ desired

Pf of Prop2.

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & V \arrow[d, "f"] \\ X \arrow[r, hook, "\mathrm{reg}"] & Y \end{tikzcd}$$

Deformation to normal cone gives

$$\begin{tikzcd} W & C_W V \arrow[l] \arrow[d] \\ X & C_X Y = N_X Y \arrow[l] \end{tikzcd}$$

Taking pullback (Cartesian diagram) of $N = N_X Y$ by regularity of embedding

Then $s^*: A_k(g^*N) \to A_{k-d}(W)$ determined by $s: W \to g^*N$ zero section, then $(X.V) := s^*[C]$ (denoted $D^!$ g^*N [C] also)

Then the red embedding is given by [illegible]
$g^*N = g^*\mathrm{Spec}(\oplus \mathcal{I}_X^n/\mathcal{I}_X^{n+1}) = \mathrm{Spec}(g^*(\oplus \mathcal{I}_X^n/\mathcal{I}_X^{n+1})) \hookleftarrow \mathrm{Spec}(\oplus \mathcal{I}_W^n/\mathcal{I}_W^{n+1})$
by $\mathcal{I}_W$ generated by $\mathcal{I}_X$ ($W = f^{-1}(X)$, the generation is obvious, i.e. $f^*\mathcal{I}_X \twoheadrightarrow \mathcal{I}_W$)
$g^*(\oplus \mathcal{I}_X^n/\mathcal{I}_X^{n+1}) \twoheadrightarrow \oplus \mathcal{I}_W^n/\mathcal{I}_W^{n+1}$
I say it's just Then isomorphic? why? See [Fu] for pf, but I think

Rk. Here $W \hookrightarrow V$ and $\boxed{C \subset E}$ ($E = g^*N$ on W) is a common setting of virtual fundamental cycle: $[V]^{vir}_{E_{W/V}} = s^*[C]$ (Here E any vector bundle) (call E the obstruction bundle of $W \hookrightarrow V$)

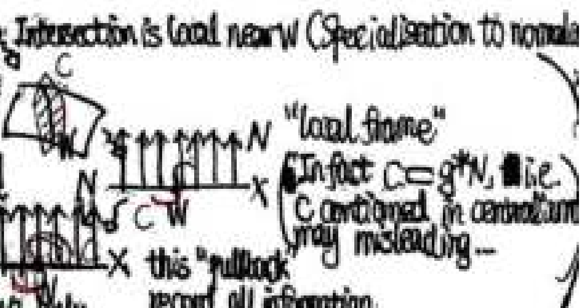
note $A_k(W) \xrightarrow[\cong]{\pi^*} A_{k+r}(E)$, $\xrightarrow{i_*} A_k(E) \xrightarrow{} A_k(W)$ pure dim. Thus $C \subset g^*N$ subscheme, then [C] on g^*N by taking geometric multiplicity

that's why it's called fundamental "virtual"

Here $W \to g^*N$ not inclusion of effective Cartier divisor, the Gysin map is defined as $(\pi^*)^{-1}$ (where $\pi: g^*N \to W$ bundle map flat): π^* isomorphism when bundle case (π^* is the flat pullback)

Understand of definition①:

Intersect X and V
⟺ (deformation) Intersect X and [C]
⟺ (regular) Intersect zero section of N and [C]
⟺ (Gysin) "Pullback" [C] to zero $= s^*[C]$



Intersection is local near W (Specialization to normal
In fact $C \subset g^*N$, i.e. C contained in central... may misleading...
this "pullback" record all information local at W

"It shows that in algebraic case, the "contact" case not equals but "coincide" case occurs

Then we show $s^*[C] = \{c(g^*N)\cap s(W,V)\}_{k-d}$ by computation of Chern classes... (By Lemma 2.0) we only explain why it's true here:

Understand of definition②: I think it's most direct: $c(g^*N)\cap s(W,V) =$ [illegible] ~~does record its multiplicity (algebra)~~ somewhat

Looking closely by components' intersection, we have (By dim $Z_i \le \dim C_i = \dim g^*N|_{Z_i}$, done ▨) $k-d \le$ (Severi, Lazarsfeld) $\le k$

Understand of definition③: Set $[C] = \sum m_i[C_i]$, Z_i are support of C_i on W ($Z_i = \pi(C_i)$) called distinguished varieties of V intersects X

They're varieties as: got rid of multiplicity m_i (& irreducible) ⟹ Intersect X and $[C_i]$ ⟺ Intersection zero section of N and $[C_i]$
⟺ "Pullback" $[C_i]$ to zero Z_i (When perfect field, smooth ⟺ nonsingular)

Pf of Prop3. We can modify conditions to ① Y nonsingular scheme + regular diagonal embedding, ② Y smooth scheme/k; ③ Y nonsingular scheme/k, char k = 0; ④ Y nonsingular variety ⟹ (give an example later to show when char k ≠ 0 non-perfect, nonsingular scheme/k not admit regular diagonal embedding (separated for diagonal closed) then can't define Chow ring.

Rk. We set in case ④ as [Fulton], ③ ⟺ ② by relation of nonsingular & smooth, ② ⟹ ① shown in [Fu]'s appendix and ① most di

Now we have $A^k(Y)\otimes A_l(Y) \to A_{k-d}(Y)$ (it descends as $(X.V) = \{c\cap s\}_{k-d}$ not depend on classes of [X] and [V])

Reduce to Diagonal: $[X]\otimes[V] \mapsto (X.V)$ due to V is pure dim k subscheme ⟹ [V] can be replaced by any cycle class α, but here X is regular embedding, we can't define it for all cycle classes! We should do this product by make use of our diagonal embedding $Y \xrightarrow{\delta} Y\times Y$ replace $X \hookrightarrow Y$, that is, $A_k Y \otimes A_l Y \to A_{k+l}(Y\times Y)$, and define $([V].[W]) := (\Delta.[V\times W])$

Write as form in Prop2, it's

$$\begin{tikzcd} \square \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V\times W \arrow[d] \\ Y \arrow[r, hook, "\delta"] & Y\times Y \end{tikzcd}$$

$[V]\otimes[W] \mapsto [V\times W]$ that is the idea of refined Gysin map

Thus it's enough to define $\delta^!: A_{k+l}(Y\times Y) \to A_{k+l-n}(Y)$ and we done our refined Gysin map and then composite to refined intersection product done ▨ (descending is easy by taking def (X.V) by $c\cap s$)

Defn (Refined Gysin ~~map~~ homomorphism) Usual Gysin is $A_k(Y) \xrightarrow{i^*} A_{k-d}(X)$ induced by $i: X \hookrightarrow Y$ (later we'll see it must regular) [illegible] which we had defined it in two special cases ① $d=1$ & ② Y is vector bundle/X, X as zero section. Here we generalise it into relative setting: $X' \xrightarrow{i'} Y'$ Cartesian diagram. Set $i^!: A_k Y' \to A_{k-d} X'$ as the composition $A_k Y' \xrightarrow[\text{specialisation}]{\sigma} A_k C' \xrightarrow{i^*} A_k(N) \to A_{k-d} X'$

$$\begin{tikzcd} X' \arrow[r, hook, "i'"] \arrow[d, "p"] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "f"] \\ X \arrow[r, hook] & Y \text{ regular} \end{tikzcd}$$

(just $i^![X]$ virtual fundamental cycle $[X]^{vir}_{g,N}$; $i^!$ called virtual pullback also) ($C' = C_{X'}Y'$) And we can write precisely $i^![V] = (\delta \cdot V)$ [illegible]

When $Y' = Y$, the usual Gysin homomorphism $i^*: A_k Y \to A_{k-d} X$. when i' regular, $i^! = (i')^*$

Refined/Generalised one can be used. With diagonal embedding replaced by graph embedding. And as relative analogue, in higher/stacky intersection its useful (We postpone some basic fact to stack's intersection theory)

Eg4. ($\delta: Y \to Y \times Y$ not regular) Set Y nonsingular $\Rightarrow$ Spec $\frac{\mathbb{F}_p(t)[x]}{(x^p - t)}$ / $\mathbb{F}_p(t)$ non-perfect (In fact it's not smooth [illegible])

Note that $\left(\frac{\mathbb{F}_p(t)[x]}{(x^p-t)}\right)^{\otimes 2} \cong \frac{\mathbb{F}_p(t)[x,y]}{(x^p-t, y^p-t)} \longrightarrow \frac{\mathbb{F}_p(t)[x]}{(x^p-t)}$ corresponds to diagonal embedding. But every closed pts [illegible] Spec [illegible] is non-reduced $\Rightarrow \delta$ not regular (Due to $\forall p \in \Delta$ diagonal, its prime $(x-y, x^p-t, y^p-t) \supsetneq (x^p-t)$ [illegible] height 2) (But $d(x^p-t) = px^{p-1} = 0$ not [illegible]) ② Tangent space at p has 1-dim greater as $(x-y)^p = 0$

Now we define Chow ring successfully either ①/② show not regular on smooth scheme/k.

Prop5. $A_*(X)$ is a communicative ring, moreover, $A^*: Sch^{sm}/_k \to$ GRing is well defined "functor" respect to proper push forward

Pf. We need show ① communitivity & associative; with unit. ② Functority $f_* g_* = (fg)_*$ (Trivial)

(Here we only consider functority of proper pushforward as it preserves degree, others also but omitted here. later with $\varphi: A_* X \to Gr K^0(X)$ also preserve w.r.t. to proper map, there may a better interpretation) ③ We take Unit is observe that $\delta_*[X]$: $\alpha \cdot [X] = \alpha$ ☑ $= \sum \deg(f: V \to f(V)) [f(V)]$ with when component dim($f(V)$) drops

Check both communitivity & associativity we back to $A_*(X \times X \times X)$

$$\begin{tikzcd} X \arrow[r, "\delta"] \arrow[d, "\delta"] \arrow[dr, phantom, "\circlearrowleft"] & X \times X \arrow[d, "\delta \times 1"] \\ X \times X \arrow[r, "1 \times \delta"'] & X \times X \times X \end{tikzcd}$$

$\alpha(\beta \cdot \gamma) = \delta^*(\alpha \times \delta^*(\beta \times \gamma))$ on $A_*(X \times X)$

$= \delta^*(1 \times \delta)^*(\alpha \times \beta \times \gamma) = \delta^*(\delta \times 1)^*(\alpha \times \beta \times \gamma)$

$= \delta^*(\delta^*(\alpha \times \beta) \times \gamma) = (\alpha \beta) \cdot \gamma$

And similiarly for communicative by $\delta^*(\alpha \times \beta) = \delta^*(\beta \times \alpha)$ ☑ (Here we use the communitivity of generalised [illegible] map, we omit it [illegible] function A^* done ☑)

Rk. Recall in AT, cup product of $H^*(X)$ isn't communicative but anti-communicative, why $A^*(X)$ is? It's due to in algebraic, over $\mathbb{R}$ is "$\frac{1}{2}$-dim" not exists, i.e. $A^*(X) \to H^{2*}(X)$, thus all coeffiencent of cup product is $(-1)^{2r \cdot 2s}$, $r, s \in \mathbb{Z} \Rightarrow$ commutes ☑

Eg5. (Chow rings) Finally, we gives several computation on Chow rings (almost trivial), more will be shown after our "AT tools" are built:

In general, a closed subscheme filtration $X \supset X_n \supset \cdots \supset X_0$, s.t. $X_j - X_{j-1} = \bigcup U_{ij}$, U_{ij} open quasi-affine

① ($\mathbb{P}^n$) We use tools called the affine stratification, to show $A^*(\mathbb{P}^n) = \mathbb{Z}[\zeta]/(\zeta^{n+1})$ generated by (equivalence class) of hyperplanes [illegible]

• Recall that $Pic(\mathbb{P}^n) = \mathbb{Z} = A^1(\mathbb{P}^n)$, in fact we first compute Chow ring by stratification with strata $\mathbb{P}^i - \mathbb{P}^{i-1}$ affine $\cong \mathbb{A}^i$

• The product of $A^*(\mathbb{P}^n)$ only consider the intersection of (k_1)-plane and (k_2)-plane, which is simple $(n-k_1-k_2)$-plane their set-intersection

Thus it suffices show $A^k(\mathbb{P}^n) = \mathbb{Z}[L]$. L is a general $(n-k)$-plane $\cong \mathbb{P}^{n-k}$ Thus $\mathbb{P}^1$ is kind of $\mathbb{A}^1$-cohomology (motivic) & H^*(topological) is S^1-cohomology

A stratification of space X, is finite set of strata (U_i), $\overline{U_i} = \bigcup U_j$, each U_i are irreducible & locally closed

One can consider it as take $U \subset X$, and $U \sqcup \partial U = X$, then repeat this for ∂U .. each $\overline{U_i}$ is compactification of U_i by adding lower strata

When X scheme, we need (U_i) subscheme. (Quasi-) affine stratification is each $U_i \cong$ (some subopen set of) $\mathbb{A}^{k_i}$, $\exists k_i$

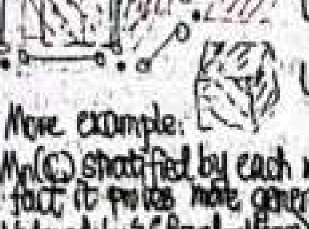
Claim. $A^*(X)$ generated by classes of closure of strata (closed strata called) for (U_i) is quasi-affine

Thus $A^k(\mathbb{P}^n) \cong \mathbb{Z}$ is obvious as $A^k(\mathbb{P}^k) \cong A^k(\mathbb{P}^k - \mathbb{P}^{k-1}) = A^k(\mathbb{A}^k) = \mathbb{Z}$

Pf of Claim. Induction on number of strata, U_0 is smallest, thus must closed and $(U_i)_{i \geq 1}$ form a stratification of $X - U_0$, inductive hypothesis shows that $A(X - U_0)$ is done, generated by $(U_i)_{i>0}$, thus it follows excision sequence and $A(U_0) = \mathbb{Z}[U_0] \cong \mathbb{Z}$ by U_0 quasi-affine ☑

Rk. Note here our excision sequence is only right exact (AT's case is exact) thus it's only generator, but we know nothing about relations of generator, thus finding basis is nontrivial.

A naive example of "stratification"



$X = $ cube $\subset \mathbb{R}^3$ admits a stratification. It's not stratification as not irreducible i.e. we can't admit boundary mfd case

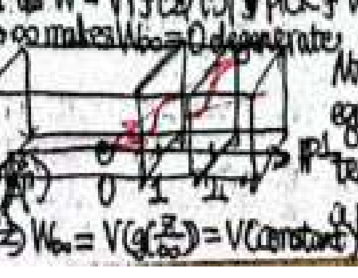
More example: $M_n(\mathbb{C})$ stratified by each rank

In Thm1.18 of [3264], refering to Totaro [2014] finding basis for affine case. In fact it proves more general case on $G_m^r \times \mathbb{A}^s$ strata. But using more AT tools on Bloch's formula/higher Chow grps and Voevodsky's (Borel-Moore) motivic cohomology (later)

②($\mathbb{A}^n$) First we show $A(\mathbb{A}^n) = A_n(\mathbb{A}^n) = \mathbb{Z}$, we deform arbitrary closed subvariety, by linearity, it suffices showing that $\exists W \subset \mathbb{P}^1 \times \mathbb{A}^n$, $W_1 = Z$ and $W_\infty = 0$ This $\mathbb{A}^n$ is S^n in topological, plays a role of "ball", fundamentally to 0 (as dim lower)

This is by set $W^0 \subset (\mathbb{A}^1 - \{0\}) \times \mathbb{A}^n$ first, s.t. $W^0_t = tY$ (it's closed as $W^0 = V\{f(z/t)\}$ pick f with vanishing locus is Y), thus $W^0 := \overline{W^0} \subset \mathbb{P}^1 \times \mathbb{A}^n$ is desired construction as the limit $t \to \infty$ makes $W_\infty = 0$ degenerate. $0 \notin Y \Rightarrow f(\tfrac{z}{\infty}) = f(0) \neq 0 \Rightarrow \phi$

(Rk. $t \to 0$ also degenerate: it degenerates to a cone singularly)



Note that when $t \to \infty$, the defining equation is $g(\frac{z}{t}) = 0$, g has constant term (otherwise we can't have such a picture as $0 \notin Y$, Impossible) $\Rightarrow W_\infty = V(g(\frac{z}{\infty})) = V(\text{constant}) = \emptyset$ ☑

Rk. After excision, it's easy to see $A^k(U) = \mathbb{Z}$, U quasi-affine [illegible] One can easy do this for graph & class Γ_f, $f: [illegible]$

③ (Diagonal class) $\Delta \subset \mathbb{P}^r \times \mathbb{P}^r$, we compute $\delta := [\Delta] \in A^r(\mathbb{P}^r \times \mathbb{P}^r)$ using the method of undetermined coefficients (as in high school)

First we need compute $A^*(\mathbb{P}^r\times\mathbb{P}^s)$ (or $A^*(\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}\times\cdots\times\mathbb{P}^{n_k})$ generally). But no Künneth formula holds for A^*
But when algebraic cellular decomposition, or in particular, affine stratification exists, the cycle map is isomorphism (later) then we can compute it using "Künneth". Here we compute directly the stratification (For any homogeneous projective varieties hold) (More general conclusion needs motivic $A(X)\cong H^{2k}(X;\mathbb{Z})$) Taking flag of $\mathbb{P}^r$ (Λ_i) and (Γ_j), their product to $(\Lambda_i\times\Gamma_j)$ forms the closed strata $\Rightarrow U_{ij}=(\Lambda_i\times\Gamma_j)-(\Lambda_i\times\Gamma_{j-1}\cup\Lambda_{i-1}\times\Gamma_j)$ is affine stratification (We not take (Λ_i) (Γ_j) complete flags $(\cong\mathbb{P}^r)$) thus $A^*(\mathbb{P}^r\times\mathbb{P}^s)$ generated by $[U_{ij}]$, $[U_{ij}]$ is intersection (transversally) of $[H]^{r-i}$ and $[L]^{s-j}$ (as $\mathbb{P}^r\times\mathbb{P}^s\not\cong\mathbb{P}^{r+s}$ is more complicated), H, L are hyperplane on $\mathbb{P}^r$ as generator of their Chow rings $\Rightarrow A^*(\mathbb{P}^r\times\mathbb{P}^s)$ generated by $[H], [L]$ $(H\cap L)$, the only nontrivial part is relation: Claim. Without $[H]^{r+1}=[L]^{s+1}=0$, there are no relation (In fact generated by pullback's here abuse) Pf of claim. By linearity, we show the monomials $[H]^{a_i}[L]^{b_i}$ and $[H]^{a_j}[L]^{b_j}$ are linearly independent using natural trace map in top $A_0(\mathbb{P}^r\times\mathbb{P}^s)\xrightarrow{\deg}\mathbb{Z}$, thus we have intersection pair $([H]^{a_i}[L]^{b_i}, [H]^{a_j}[L]^{b_j})=\begin{cases}1; & a_i+a_j=r, b_i+b_j=s\\ 0, & \text{otherwise}\end{cases}$
$\Rightarrow$ linear independent (otherwise $\sum m_i[H]^{a_i}[L]^{b_i}=0$, pair with $[H]^{a_{i_0}}[L]^{b_{i_0}}$ contradiction) ▨ and their [illegible]
Then set $A^*(\mathbb{P}^r\times\mathbb{P}^r)=\mathbb{Z}[\alpha,\beta]$, compute $\delta=c_0\alpha^r+c_1\alpha^{r-1}\beta+\cdots+c_r\beta^r$, product with $\alpha^{i}\beta^{r-i}$ $\Rightarrow c_i=\deg(\delta\cdot\alpha^{i}\beta^{r-i})$. By intersection with diagonal is $(\alpha^{r-i},\beta^{r-i})$ itself's intersection, i.e. $\deg(\delta\cdot\alpha^i\beta^{r-i})=(\alpha^i,\beta^{r-i})=1$ by transversality
$\Rightarrow\delta=\sum\alpha^{r-i}\beta^i$ ▨

§1.2. Intersection multiplicity and excess intersection

We study intersection multiplicity, and study not general case of intersection, called excess. On point of view of Fulton's deformation to normal cone, we need avoid the normal cone case, i.e. we'll take a quotient. Thus we all components proper, $(X\cdot V)=\sum i(Z,X\cdot V;Y)[Z]$

Def 5. (Intersection multiplicity on each components) A component $Z\subset W$ is called proper if it has expected dimension, then the intersection multiplicity is $i(Z,X\cdot V;Y)$, defined as: View Z as distinguished variety (with lowest dim), as $Z_i\subset W$ (support of cones)
we had known thus Z has lowest dimension, over it the obstruction bundle $g^*N|_Z\cong C|_Z$

$Z_i\subset W$
$\uparrow\quad\uparrow$
$C_i\subset C=C_WV$

Thus we can set i as the coefficient of $[Z]$ in $(X\cdot V)$'s $A_{k-d}(W)$ piece.

Prop 6. $i(Z,X\cdot V;Y)=e_Z(V)$ the Samuel's algebraic multiplicity := the coefficient of $[Z]$ in $s(W,V)$ (thus we can use Hilbert's polynomial's leading term) Geometric
Pf. By next lemma, when V is pure, we can restrict to each components with multiplicity so is $[Z]$'s coefficient $\Rightarrow e_Z(V)=\sum m_i(e_{W\cap V_i}V_i)_Z$. Note that here m_i geometric multiplicity of C on C_i = V on V_i

Thus $(X\cdot V)=[W]^{vir}_{g^*N}=\sum m_i s^*[C_i]$
Let $Z=Z_0=Z(C_0)$
$\Rightarrow [C_0]\in A_k(N_0)=A_k(C_0)$
$s^*[C_0]=[Z]\in A_{k-d}(Z)$ [illegible]
i.e. this case virtual fundamental class = fundamental class [illegible]
$\Rightarrow i(Z,X\cdot V;Y)=m_0$ [illegible]

Consider $V(z)\subset\mathbb{A}^2$ [illegible] $W\subset V$ V is irreducible $\to m_0=1$ [illegible] $W\cap V_0$, $V\cap Z_0$ [illegible] multiplicity 1
But $C_WV=\operatorname{Spec}(\oplus(x^2))=\operatorname{Spec}(\frac{k[x]}{(x^2)}[t])$ has multiplicity 2 [illegible] it's wrong!
We can compute m_0 on the left as $C_W V$ [illegible]

The pf is easy as $(X\cdot V)=\{c(g^*N)\cap s(W,V)\}_{k-d}$, thus coefficient of $[Z]$ on $(X\cdot V)\Leftrightarrow$ on $\{c(g^*N)\cap s(W,V)\}_{k-d}$
As Z is zero scheme of g^*N, $c(g^*N)$ not provides any multiplicity $\Leftrightarrow$ coefficient of $[Z]$ on $s(W,V)$ ▨

Lemma 4. $s(X,Y)=\sum m_i s(X_i,Y_i)$, $Y=\bigcup Y_i$ pure-dimensional (scheme)
$m_i=\operatorname{length}(\mathcal{O}_{Y_i,Y})$ geometric multiplicity, $X_i=Y_i\cap X$

Example 3.2.16 in [Fulton] says that let $Z\subset X\leftarrow E$, Z is zero section of S $\Rightarrow c_{top}(E)\cap[X]=[Z]$, this is same as topology

Pf. The Chern class part decomposes naturally, and the $P(C_{X}Y\oplus 1)$ decomposes by view it as exceptional divisor of blowing up (as the construction of deformation space: $Bl_{X\times 0}(Y\times\mathbb{A}^1)$) $\Rightarrow$ It's divisor and $[D]=\sum m_i[D_i]$ [illegible] ▨

Prop 7. ① (Commutativity) $i(Z,X\cdot V;Y)=i(Z,V\cdot X;Y)$ if $V\hookrightarrow Y$ regular
② (Associativity) Composite two (or more) regular embeddings $X\hookrightarrow Y\hookrightarrow Z$, then we have a Cartesian diagram
$W_i\subset g^{-1}(V)$ components containing S $\Rightarrow i(S,X\cdot V;Z)=\sum i(S,X\cdot W_i;Y)\cdot i(W_i,Y\cdot V;Z)$

$$\begin{tikzcd}
S\subset W \arrow[r] \arrow[d] & V \arrow[d, "g"] \\
X \arrow[r] & Z
\end{tikzcd}$$

③ (Projection formula) $h:V'\to V$ base change on V, then denote $h^{-1}(Z)=\bigcup Z_i\Rightarrow\deg(V'/V)\cdot i(Z,X\cdot V;Y)=\sum\deg(Z_j/Z)\cdot i(Z_j,X\cdot V';Y)$
Pf. They're directly sequence of commutativity, functoriality & proper-pushforward of refined Gysin map. May be seen much later on ▨ Rk. When $X\hookrightarrow Y$ not regular (but X pure), we modification is same as Chow ring:

Homological algebra and Serre's Tor formula, and we'll give a view on derived method in algebraic geometry so intersection theory
We'll not give a pf of Serre's formula (as [3264] said, it's too powerful thus useless; usually it much easier case [illegible]) but understand it
Set $A=\mathcal{O}_{Z,V}$, then $Z\subset W\subset V$, W corresponds to ideal $J\trianglelefteq A$ via $0\to\mathcal{I}_W\to\mathcal{O}_V\to\mathcal{O}_W\to 0$ localise to $0\to\mathcal{I}_{Z,W}\to\mathcal{O}_{Z,V}\to\mathcal{O}_{Z,W}\to 0$, $J:=\mathcal{I}_{Z,W}$. Consider $\mathcal{O}_{Z,W}\cong A/J$, $\operatorname{length}(\mathcal{O}_{Z,W})<\infty$: any chain of $\mathcal{O}_{Z,W}$ over $\mathcal{O}_{Z,W}$ has elements of form $\mathcal{O}_{Z',W}$ where $Z\subset Z'\subset W$, Z' consist more irreducible component, nearby two adding one component, thus their quotient corresponds to a prime ideal of $\mathcal{O}_{Z,V}$. And W has finite component always.
Set $\forall U\subset Y$ open affine, s.t. $X\cap U=\operatorname{Spec}B/I\hookrightarrow U=\operatorname{Spec}B$, $I=(b_1,\cdots,b_d)$ regular (by def), then $W\to X$ induces a algebra homomorphism, when $U\cap f(Z)\neq\emptyset$, it factors through local ring of $Z\Rightarrow B\to A$ maps $(b_1,\cdots,b_d)$ to $(a_1,\cdots,a_d)$ (not regular)
(When ~~[illegible]~~ f is flat, then regular sequence preserved)
Thus $A^d\to A$ determines a Koszul complex (denote $E=A^d$) $K_\bullet$ (used to tensoring $B/(b_1,\cdots,b_d)$)
$(a_1,\cdots,a_d)\mapsto\sum a_i e_i$ $\quad d_k:\Lambda^kE\to\Lambda^{k-1}E$
$e_1\wedge\cdots\wedge e_k\mapsto\sum(-1)^i\langle e_i,(a_1,\cdots,a_d)\rangle e_1\wedge\cdots\wedge\hat{e_i}\wedge\cdots\wedge e_k$

When Z not proper, $\chi_i=\infty$ otherwise $\chi_i>0$

Thm 1 (Serre's Tor formula)
$i(Z,X\cdot V;Y)=\chi_A(a_1,\cdots,a_d)$ ($i=$ [illegible] ~~[illegible]~~) [illegible] set $\chi(B/\cdots)$ Pf. Omitted ▨ See [Serre] ② is difficult than ①
Generally, we set $e_A(M)$ [illegible] to $\chi_A(a_1,\cdots,a_d;M)$ called $i(Z,X\cdot V;Y)=\chi(\mathcal{O}_{Z,X}\otimes\mathcal{O}_{Z,V})=\chi(\mathcal{O}_{Z,Y}/\cdots)$
$\cdots=\sum(-1)^i\operatorname{length}(H_i(K_\bullet))$
(algebraic) multiplicity of M (here $=A$) on $(a_1,\cdots,a_d)$ $=\sum(-1)^i\operatorname{length}(\operatorname{Tor}_i^{\mathcal{O}_{Z,Y}}(\mathcal{O}_{Z,X},\mathcal{O}_{Z,V}))=[\mathcal{O}_{Z,X}\otimes^{L}\mathcal{O}_{Z,V}]$ the derived tensor
[illegible]

ex. ① Look at the j=0 part of tor formula is $\text{length}_{\mathcal{O}_{Z,Y}}(\mathcal{O}_{Z,X}\otimes\mathcal{O}_{Z,V}) = \text{length}_{\mathcal{O}_{Z,Y}}(\mathcal{O}_{Z,Y}/\langle \mathcal{I}_{Z,X}, \mathcal{I}_{Z,V}\rangle)$ is the multiplicity
when X & V intersects naively/transversally ($\otimes$ means $\cap$), i.e. direct counting multiplicity $Z \sqsubset X \cap V$. All permutations of intersection was hided by higher terms, that is from $\mathcal{O}_{Z,X}\otimes\mathcal{O}_{Z,V}$ to $\mathcal{O}_{Z,X}\otimes^{L}\mathcal{O}_{Z,V}$ We have $\pi_0(X\times^R_k Y) = H_0(X\times^R_k Y) = X\times_k Y$ for X, Y

② We can reformulate it as [illegible] complicated information hided the higher schemes, and [illegible] the derived ones [illegible]

Denote $\mathcal{O}_{Z,X}\otimes^L_{\mathcal{O}_{Z,Y}}\mathcal{O}_{Z,V} =: M\otimes^L_R N \overset{\text{change}}{=} A\otimes^L_{A\otimes_k A}(M\otimes^L_k N)$, where R is taken to make $M\otimes^L_R N$ easier to compute $(M\otimes^L_R N)$ formal

$\Rightarrow \chi(M\otimes^L_R N) = \chi(A\otimes^L_{A\otimes_k A}(M\otimes^L_R N)) = \sum(-1)^i l_A(H_i(A\otimes^L_{A\otimes_k A}(M\otimes^L_R N))) = \underline{\sum(-1)^i l_A(HH^R_i(A, M\otimes^L_R N))}$ is same in derived case)

• Here we view $A\otimes^L_{A\otimes_k A}(M\otimes^L_R N)$ as simplicial module (i.e. "complex") $d_i: V_n \rightrightarrows V_{n-1}$ and degeneracy map $s_i: V_n \rightrightarrows V_{n+1}$ $(i\le n)$
then $\pi_i = H_i$ defined to be homology of it (= its normalisation to a complex $i \le n$)

③ A corollary of ① of Thm1 is: $i(Z, X.V; Y) \le l(A/I)$ as $i(Z, X.V; Y) = \boxed{e_W(V)_Z = e_A((a_1\cdots a_d), A)} \le l_A(\text{coker}(a_1\cdots a_d))$ (algebraic
And we are clear when it's equal (We postpone more DAG after excess) by written both of them as $= l_A(A/I)$

Cor1. TFAE: ① $i(Z, X.V; Y) = l(A/I)$; ② I regular; ③ $H_i(K_\bullet) = 0, i > 0$.   leading coefficient of Hilbert-Samuel polynomial
(When $V \subset Y$ flat, by base change of flat

Cor2. (Criteria of multiplicity 1) [illegible] When V is variety, then $i = 1 \iff A$ is regular local ring, $I = m$ maximal

Pf. ($\Leftarrow$) $l(A/I) = l(A/m)$, $\underline{\dim A} = d =$ height $\Rightarrow l(A/m) = 1 \ge i > 0 \Rightarrow i = 1$ (not need Y variety)

($\Rightarrow$) As $\underline{Y \text{ variety irreducible}}$ regular ($\dim = d$) $m = (a_1 \cdots a_d)$ regular (In case V not irreducible we can't)
after replace $X \hookrightarrow Y$ by $\text{Spec}(B/I) \hookrightarrow \text{Spec}\,B$, we can assume $\underline{Z = W \text{ is unique component}}$, thus we have diagram

$$\begin{tikzcd} \text{Spec}(A/I) \arrow[r, "\text{reg}"] \arrow[d] \arrow[dr, phantom, "\square"] & \text{Spec}\,\underline{\underline{A}} \arrow[d] \\ \text{Spec}(B/I) \arrow[r, "\text{reg}"] & \text{Spec}\,B \end{tikzcd}$$

local

$\Rightarrow (X.V) = i[W] = [W]$ and as coefficient of Z in [W] is $l(A/I) = 1$

We induction on $\dim A = d$: ① When $d = 1$, $l(A/I) = 1$ force $I = m$, done.

② After taking a normalisation s.t. $A \to \tilde{A}$, and show that $\tilde{A} = A + m\tilde{A}$, we can use Nakayama lemma to $A = \underline{\tilde{A} \text{ normal}}$ (For detail, see [Fulton]) Set $X' = \text{Spec}(B/(b_d))$

$$\begin{tikzcd} \text{Spec}(A/I) \arrow[r, "W"] \arrow[d] & \text{Spec}(A/(a_d)) \arrow[r] \arrow[d] & \text{Spec}\,A \arrow[d] \\ \text{Spec}(B/I) \arrow[r, "V"] & \text{Spec}(B/(b_d)) \arrow[r, "X'"] & \text{Spec}\,B \end{tikzcd}$$

$\Rightarrow$ Now by induction hypothesis, $(a_1 \cdots a_{d-1})$ is regular sequence of $A/(a_d)$, generating $m' \trianglelefteq A/(a_d)$, $m' = m/(a_d)$

Thus it suffices to prove $A/(a_d)$ local

[Recall our commutative algebra: when A local, then $m = (a_1 \cdots a_d)$ has d generators $\iff a_1, \cdots, a_d$ is regular sequence]

i.e. $\text{Spec}(A/(a_d))$ is irreducible (thus (a_d) is the only prime ideal containing itself), then $a_1 \cdots a_{d-1}$ regular in $A/(a_d)$
$\Rightarrow m/(a_d) = (a_1 \cdots a_{d-1}) \Rightarrow m = (a_1 \cdots a_d) = I$ regular, done.

$\text{Spec}(A/(a_d))$ has only component containing $\text{Spec}(A/(I))$, further $[\text{Spec}(A/(a_d))] = [\text{Spec}(A/(I))] + \cdots$ coefficient 1 due to
[illegible] $(X.W)$ (in X') $= (j')^![W] = i[V] = [Z] \Rightarrow (X.W) = [W] \Rightarrow [W] = [W] + \cdots \Rightarrow \underline{\text{length}_{A_p}(A_p/(a_d)A_p) = 1}$ and $\exists!\, p$
(One use $s^*[C]$ can also prove this, here $= [W]$ for simplicity we used [W] the functority of Refined Gysin) $\Downarrow$ minimal $\supset (a_d)$
By A normal $\Rightarrow (a_d)$ is p-primary, i.e. $\sqrt{(a_d)} = p$ $(a_d)A_p = pA_p$
Then $\sqrt{(a_d)} = p$ & $(a_d)A_p = pA_p \Rightarrow (a_d) = p$ (by definition) done ▨

Eg.6. (A classical computation of $i < l$) Set $Y = \mathbb{A}^4$, $X = V(x_1 - x_3, x_2 - x_4)$, $V = V(x_1x_3, x_1x_4, x_2x_3, x_2x_4)$

$l(A/I) = l\left(\frac{k[x_1, x_2, x_3, x_4]}{(x_1 - x_3, x_2 - x_4, x_1x_3, x_1x_4, x_2x_3, x_2x_4)}\right) = l\left(\frac{k[x_1, x_2]}{(x_1^2, x_1x_2, x_2^2)}\right) = 3$

$(X.V) \overset{\text{Serre's tor}}{=} \sum(-1)^i \text{length}\, \text{Tor}_i^{k[x_1 \cdots x_4]}\left(k[x_1, x_2], \frac{k[x_1, x_2, x_3, x_4]}{(x_1x_3, x_1x_4, x_2x_3, x_2x_4)}\right) = \sum(-1)^i \text{length}\, \text{Tor}_i^A(A/I_1, A/I_2)$ has length

① When $i = 0$, the naive intersection multiplicity is 2 ($V = V(x_1, x_2) \cup V(x_3, x_4)$), each component intersects X transversally
② When $i = 1$, $\text{Tor}_1^A(A/I_1, A/I_2) = \frac{I_1 \cap I_2}{I_1 I_2} =$ [illegible] Only it can't generated by [illegible]
③ When $i \ge 2$, $\text{Tor}_i^A(A/I_1, A/I_2) \cong \text{Tor}_{i-1}^A(A/I_1, I_2)$ by apply LES of $0 \to I_2 \to A \to A/I_2 \to 0$, and $\text{Tor}_j^A(A/I_1, A) = 0$ $(\forall j \ge 1)$
$\Rightarrow$ then take resolution of I_2 (free): $\cdots \to 0 \to 0 \to A^4 \to A^2 \to I_2 \Rightarrow$ when $i - 1 \ge 2$, $i \ge 3$, $\text{Tor}_i^A(A/I_1, A/I_2) = 0$, when $i = 2$, $\text{Tor}_2^A(A/I_1, I_2) = \ker(I_1 \otimes I_2 \to A \otimes I_2) = 0$
$\Rightarrow (X.V) = 2$ (only the 0-term makes sense)

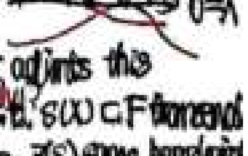
Excess [illegible] intersection

Eg.7. E vector bundle over X is pure $\dim = n$, $r := r(E)$. Fix a section $s: X \to E$, then the zero scheme of s is the intersection of s and zero section 0: $Z(s) \longrightarrow X$. When $s \pitchfork 0$ transversally, $Z(s)$ is of expected dimension $(n - r)$

What about when s intersect

$$\begin{tikzcd} Z(s) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "s"] \\ X \arrow[r, "0"] & E \end{tikzcd}$$

(s in a general position)
0 not proper (not of expected dim)   • A special case is $s = 0$, then it's the $\underline{\text{self-intersection}}$



$\underline{\text{Perturbation}}$ is used when topological setting, but rigidity in complex geometry/algebraic geometry not admits this
Quillen first uses excess intersection formula into topological, also set excess bundle as: $\boxed{\exists F \subset E}$ s.t. $s(X) \subset F$ transversal to 0_F, then E/F is obstruction of s_E to be transversal to 0_E, E/F is called excess bundle ① $e(E/F) \sim Z(s)$ some homological class of perturbation exists ② In particular, consider $Y \subset X$ submfd (X also mfd) [illegible] as zero mfd of s, then $E/F \cong N_{Y/X}$
Excess intersection formula computes intersection product (= refined Gysin) with the Chern class of excess bundle.

Def5. (Excess (normal) bundle) Consider a Cartisan diagram

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "f"] \\ X \arrow[r, hook, "reg"] & Y \end{tikzcd}$$

we set $E = g^*N/N'$ the excess normal bundle. Just as virtual fundamental class did, we can replace the outer space $X \hookrightarrow Y$ by obstruction bundle E' on X', then $E = E'/N'$ is called Excess bundle (As virtual fundamental class again, replace E by obstruction theory $E_\bullet$ (esp. on stacks) and its relation with virtual fundamental class showed next)

Prop1. (Excess intersection formula①) $i^!(\alpha) = c_r(g^*E) \cap i'^!(\alpha) \in A_{k-d}(X')$, $d = \text{codim}(X,Y)$, $r = r(E) = \text{codim}(X',Y')$ [illegible]

It says the E has the (intersection) [illegible] for $\alpha \in A_k(Y')$ (esp. the Euler class) obstruction datum between intersections on outer space $X' \hookrightarrow Y'$ and $X \hookrightarrow Y$

Pf. Postpone after Chern class (We need universal quotient bundle) ☑

Thus when $(X' \hookrightarrow Y') = (X \hookrightarrow Y)$ we have $i^!(\alpha) = c_d(g^*N) \cap \alpha$ [illegible]
$\Rightarrow$ when f, g also id, self-intersection is $i^* i_*(\alpha) = c_d(N) \cap \alpha$ (Fulton)

Prop2 (Excess intersection formula②) $(X_1, X_2, \dots, X_r, V)^Z = \{s(Z,V) \cap$ [illegible] $\prod c(N(X_i,Y)|_Z)\}_{\dim V - \sum \text{codim}(X_i,Y)}$

where $X_i \hookrightarrow Y$ all regular, V subvariety, $Z = \cap(X_i) \cap V$ component (connected component!)

The intersection product with supported on Z ($Z \subset V$ closed subset) is $(X.V)^Z = \sum_{\substack{\text{distinguished} \\ \text{variety } Z_i \subset S}} m_i \alpha_i$, where $\sum_i m_i \alpha_i$ is the canonical decomposition of $(X.V)$. Pf Postponed, also ☑

Rk. ① Residual intersection considers intersection information outside Z, i.e. residual class $(X_1, \dots, X_r, V) - (X_1, \dots, X_r, V)^Z = R$, omitted

② The name of "Excess" comes from 5-conic problems, it's done by excess intersection formula②.

※ [illegible]

Derived intersection theory: an overview (details build in derived stacks, will show finally)

We had known that $\mathcal{O}_X \otimes^L_{\mathcal{O}_Y} \mathcal{O}_{Z,V}$ { contains intersection datum: this gives "Geometry" to this algebraic itself (S_0 is the sheaf)
have simplicial communicative ring structure $\in$ sComm is model category $\simeq \text{Comp}_{\geq 0}(\text{Comm})$ (Dold-Kan correspondence), need Comm Abelian cat

Thus we have Sh(sComm), a derived scheme $= (X, \mathcal{O}_X)$, $\mathcal{O}_X \in$ Sh(sComm) and X topological scheme, s.t.
① $(X, \pi_0(\mathcal{O}_X))$ is scheme; ② $\pi_i(\mathcal{O}_X)$ on scheme $(X, \pi_0(\mathcal{O}_X))$ is quasi-coherent.

& (hidden smoothness principle) moduli space is always singular, ∃ Derived moduli, s.t. its truncation $(\pi_0(\mathcal{O}_X))$ is this (singular) moduli.

• Derived scheme is invented for a "smooth approximation" to scheme (singular) via simplicial resolution. This philosophy amazing rigorously holds by: "Any thing holds for smooth scheme holds for derived scheme" $\Rightarrow$ not simply derived category of stacks!)

Derived stack (is an étale sheaf of ∞-groupoids on affine derived schemes) has an abstract definition. We understand derived stack by the example of moduli stack Bun: taking "total space" of D(Bun) gives our moduli stack, is the gap between derived stack and "derived category of stacks": $D(\text{Stk}) \longleftrightarrow D(\text{Bun})$, it'll introduced precisely in obstruction theory) truncation [illegible]

Our core of this overview is take a look at the intersection theory total space behaves on derived stacks (derived scheme enough, but it's natural, one may accept intersection on stack ≈ scheme vaguely) ① Deformation to normal cone/stack ② Virtual fundamental class; ③ Excess intersection.

① Recall, our deformation to normal cone is this diagram:

$$\begin{tikzcd} X \arrow[r, hook, "1"] \arrow[d] \arrow[dr, phantom, "\square"] & X \times \mathbb{P}^1 \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[l, hook', "\infty"'] \arrow[d] \\ Y \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & M^0 \arrow[d] \arrow[dr, phantom, "\square"] & N_{X/Y} \doteq \mathbb{P}(C \oplus 1) \arrow[l, hook'] \arrow[d] \\ \{1\} \arrow[r, hook, "1"] & \mathbb{P}^1 & \{\infty\} \arrow[l, hook', "\infty"'] \end{tikzcd}$$

normal cone

With $(\mathbb{P}^1, 1, \infty)$ replace by $(\mathbb{A}^1, 1, 0)$, it's equivalent to

$$\begin{tikzcd} X \times \mathbb{G}_m \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & X \times \mathbb{A}^1 \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[l, hook', "0"'] \arrow[d] \\ Y \times \mathbb{G}_m \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & M^0 \arrow[d] \arrow[dr, phantom, "\square"] & M^0_0 = \mathbb{P}(C \oplus 1) \arrow[l, hook'] \arrow[d] \\ \mathbb{G}_m \arrow[r, hook] & \mathbb{A}^1 & \{0\} \arrow[l, hook', "0"'] \end{tikzcd}$$

normal cone

• Write $\mathbb{G}_m$ for lifting to ■. Contraction case.

Our derived stacky analogue is
① Replace the red base by a general derived stack (relative)
② Using intristic normal cone replace $C_X Y$, thus the black part should be modified
③ Amazing, we have a much more precise description on the derived deformation space!
④ Note after lift to stack, all Cartisan diagram up to a homotopy

$$\begin{tikzcd} X \times \mathbb{G}_m \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & X \times \mathbb{A}^1 \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[l, hook', "0"'] \arrow[d, "0"] \\ Y \times \mathbb{G}_m \arrow[r, hook] \arrow[d, equal] \arrow[dr, phantom, "\square"] & D_{X/Y} \arrow[d] \arrow[dr, phantom, "\square"] & N_{X/Y} \arrow[l, hook'] \arrow[d] \\ Y \times \mathbb{G}_m \arrow[r, hook] & Y \times \mathbb{A}^1 & Y \arrow[l, hook', "0"'] \end{tikzcd}$$

$\Rightarrow$ For $X \to Y$ morphism between derived stacks [HKR] (here Y as a base, each fibre X_y has its intristic normal cone N_{X_y}, and $N_{X/Y}$ can be both understand as normal cone of $X \to Y$, "relative" gluing of each N_{X_y})

Here { $N_{X/Y}$ (or N_X) $= \mathbb{V}_X(\mathcal{N}_{X/Y}) = \mathbb{V}_X(\mathcal{L}_{X/Y}[-1])$ (or $\mathbb{V}_X(\mathcal{L}_X[-1])$), where $\mathbb{V}_X$ is the operation of "derived total space", $\mathcal{L}_X$ is the cotangent complex

$D_{X/Y}$ is called derived Weil restriction of $(X \to Y)$ along $(0: Y \hookrightarrow Y \times \mathbb{A}^1)$: it's the derived stack over $Y \times \mathbb{A}^1$, satisfy universal property: $\text{Hom}(S, D_{X/Y}) \cong \text{Hom}(S \times_{\mathbb{A}^1} \{0\}, X)$ (means ∃! descent from $N_{X/Y}$ to X, if it's global on $D_{X/Y}$) (its algebraicity is nontrivial)

② Recall our specialization to normal cone composites to virtual pullback: $X \hookrightarrow Y$ and $C_X Y \subset E$ obstruction bundle, $r = r(E)$

$A_k(Y) \xrightarrow[\sigma]{\text{specialize}} A_k(C_X Y) \xrightarrow{i_*} A_k(E) \xrightarrow[= (\pi^*)^{-1}]{0^!_E} A_{k-r}(X)$ & specialization is by the blue part of the upper diagram (the second one)

$[Y] \longmapsto^{s^!_E} [C]^{vir}_E$

$C \hookrightarrow M^0 \hookleftarrow$ ■ induce $A_{k+1}(C) \to A_{k+1}(M^0) \to A_{k+1}(Y \times \mathbb{G}_m)$ (*)

Observation, What we do $A_{k+1}(Y \times \mathbb{G}_m) \to A_k(C)$ is showing that $A_\bullet(C) \to A_\bullet(M^0) \to A_\bullet(Y \times \mathbb{G}_m) \xrightarrow{\text{connecting}} A_\bullet(C)[-1]$ is "exact" triangle; makes sense in derived case. Then quickly we composite $A_k(X) \to A_{k+1}(Y \times \mathbb{G}_m)$ to give σ

For derived stacky analogue, look at the blue part of the upper diagram (the third one) $f: X \to Y$ quasi-smooth, $N_{X/Y} \to D_{X/Y} \leftarrow Y \times \mathbb{G}_m$ induces the localization triangle of Borel-Moore (motivic) cohomology $C^{BM}_\bullet(N_{X/Y}) \to C^{BM}_\bullet(D_{X/Y}) \to C^{BM}_\bullet(Y \times \mathbb{G}_m) \to C^{BM}_\bullet(N_{X/Y})[-1]$, thus specialization is (It's definition will be showed in detail in §1.5) $\sigma: C^{BM}_\bullet(Y/S) \to C^{BM}_\bullet(N_{X/Y}/S)$, S is base derived algebraic stack

• Replace E by obstruction theory ($E = \mathbb{V}_X(\mathcal{E})$, $\mathcal{E}$ is perfect complex. For noninjective, r = virtual rank of $\mathcal{E}$)

$C^{BM}_\bullet(Y/S) \xrightarrow{\sigma} C^{BM}_\bullet(N_{X/Y}/S) \longrightarrow C^{BM}_\bullet(E/S) \longrightarrow C^{BM}_\bullet(X/S)[-2r]$ sends $[Y/Y]$ to $[X/Y]^{vir}_{\mathcal{E}}$ (set $S = Y$ [illegible])

③ Thus setting of excess intersection diagram is $X' \xrightarrow{\text{quasi-smooth}} Y'$ ⇒ excess sheaf $\mathcal{E}$ = [illegible] of $g^*N_{X/Y} \to N_{X'/Y'}$
And the excess intersection formula is a homotopy (replace equality):
$g^* f_*(a) \simeq f'_*(p^*(a) \cup c_r(\mathcal{E}))$
$a \in K(X)$, mapped into $K(Y')$ the space of algebraic K-theory of perfect complexes. (In particular, one can take $a \in C^{BM}_*(X)$)

$$\begin{tikzcd} X' \arrow[r, "\text{quasi-smooth}"] \arrow[d, "p"] \arrow[dr, phantom, "\triangle"] & Y' \arrow[d, "g"] \\ X \arrow[r, "\text{quasi-smooth}"] & Y \end{tikzcd}$$

We call such diagram is excess intersection square if ① It's Cartesian on cat of stacks (not derived) (denoted △) ② $E = V_X(\mathcal{E})$ is well-defined vector bundle of finite rank r

§1.3. Basic analogue from algebraic topology (Here is "analogue", next section is "technique", using K-theory reformulates it)
Gysin [illegible] morphism and Thom isomorphism also topological-analogous, had been introduced before. We'll show the ES/Well-axioms here (although we had seen some may not exist: Künneth formula...) and characteristic classes (esp. Chern classes) (Relative Chow grps) and using it to prove consequences we omitted before: (A) Prop 2: $s^*[C] = \{c(g^*N) \cap s(W,V)\}_{k-d}$ (B) Prop 4 & Prop 10: Excess intersection formula. At last we'll introduce Bivariant intersection & cycle maps

(With supp/Relative Chow grps are not usually used and can be reformulated later by Bloch's formula)

Prop 11. (Axioms) ① (Only dimension, functoriality, identity, excision holds for (absolute) Chow grps, only excision is nontrivial.) Excision pair (X,Y) with Y closed $X \hookleftarrow U = X - Y$ induces $A_*(Y) \xrightarrow{i_*} A_*(X) \xrightarrow{j^*} A_*(U) \to 0$ $X_1 \cap X_2$ isn't good in algebraic, we modify it by a Cartesian diagram
② (MV-sequence) $X_1, X_2 \sqsubset X$ closed subscheme ⇒ $A_*(X_1 \cap X_2) \to A_*(X_1) \oplus A_*(X_2) \to A_*(X_1 \cup X_2) \to 0$
③ (Trace map) (Only Künneth not holds for A^* among Weil's axioms) The top class $\in A^n(X)$ are 0-cycles $= \sum n_p [P]$; thus the counting map/degree is $\int : A^* X \to \mathbb{Z}$. Then $\int a = \int f_* a$ preserved under proper pushforward.
When X is complete/proper, this collection: $\sum n_p [P] \mapsto \sum n_p [k(p) : k]$ (Eilenberg–Moore...) [illegible] original (sheaf) ... residual field [illegible]

$$\begin{tikzcd} X_1 \arrow[r, hook, "\tau"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "p"] \\ {} & {} \end{tikzcd}$$

s.t. • this closed embedding
• p is proper
• p induce $X - X_1 \cong X_2 - X'$

Pf. ① is trivial, not 0-cycle ↦ 0. It's due to ... order of poles ... homology ...
① $j^* i_* = 0$ obvious, j^* surjective (a 0-cycle $[P]$ may consists several pts (with multiplicity), e.g. $[2]$ = two pts/pt with double multiplicity then $[k(p)/k] = 2$) ⇒ all not top class has deg 0 naturally
then $A_*(X') \to A_*(X_1) \oplus A_*(X_2) \to A_*(X) \to 0$
This universal/motive-like homology not satisfy any axioms! It holds ...
clear. Only check that, $\forall a \in A_k X$, $j^* a \sim 0 \Rightarrow a \sim i_* \beta$, $\exists \beta \in A_k Y$
by: $j^* a \sim 0 \Leftrightarrow j^* a = \sum [\operatorname{div}(r_i)]$, $\operatorname{div}(r_i) \subset U$ closed, extends to X by taking closure, and $K(\operatorname{div}(r_i)) = K(\operatorname{div}(\bar r_i)) \Rightarrow \exists!\ \bar r_i \in K(\bar W_i)$ s.t. $\operatorname{div}(\bar r_i) = j^* \operatorname{div}(r_i) \Rightarrow j^*(a - \sum \operatorname{div}(\bar r_i)) = 0 \Leftrightarrow$ [illegible] $a = \sum \operatorname{div}(\bar r_i) = i_* \beta$ as in chain-level $Z_* Y \to Z_* X \to Z_* U \to 0$ exact by linearity reduce to [illegible] $j^*[W] = $ [illegible] $[j^* W] = 0$, thus extend $j^*(W)$ to $\bar j^*(W) \subset X$, and $[\bar j^*(W)] = 0$ [illegible]
⇒ $[\bar j^*(W)] \in$ [illegible] $Z_* Y$ and then $i_* [\bar j^*(W)] = [W]$ done ▨
($\bar j^*(W)$ outside (up to deform) an open set ⇒ its closure ...)
② The proof is totally same as topological: set $A_k(X_1 \cap X_2) \to A_k(X_1) \oplus A_k(X_2) \to A_k(X_1 \cup X_2) \to 0$
Q. Why not left-exact? (Compare with the pf on Künneth's) Complexes $\alpha \mapsto (\alpha, -\alpha)$ ... replaced by ... some subspaces here we replace, $(\alpha, \beta) \mapsto \alpha + \beta$ ... subvariety ... α & i^* not exact ⇒ here we can see converse

Rk. (First Chern class) We have $c_1 = \det$ [illegible] $\approx \deg$ by $c_1(L) =$ the rational equivalence class of its associated Cartier divisor [illegible] L $=$ [illegible] $[L] \in A^1(X)$
thus $c_1 : \operatorname{Pic}(X) \to A^1(X)$
① $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \overset{\text{splitting}}{=}$ [illegible] $c(L_1 \otimes L_2) = \det(L_1$ [illegible] $\oplus L_2)$. Thus extending to all bundles by set $c_1(E) = c_1(\det E)$
② Comparison of $A^1(X)$ and cohomology grps: set M complex mfd ⇒ $0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to 0$ induces $c_1 : H^1(M, \mathcal{O}^\times) \to H^2(M, \mathbb{Z})$

$$\begin{tikzcd} H^1(M, \mathcal{O}^\times) \arrow[r] \arrow[d, equal] & H^2(M, \mathbb{Z}) \arrow[d, equal] \\ \operatorname{Pic}(M) \arrow[r] & A^1(M) \end{tikzcd}$$

③ Set X be projective curve ⇒ $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$, thus $\deg : \operatorname{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$
④ A view of cycle as differential form/currents gives why our notation of trace map is integral over X!
Eg 8. (Compute degree in both algebraic & analytic ways) Consider $S^2 \cong \mathbb{P}^1$ over $\mathbb{C}$ and $a = 2[pt] \in A^1(\mathbb{P}^1) \cong H^2(S^2, \mathbb{Z}) \ni 2\omega_{S^2}$
$\deg a = 2 \doteq 2 = \int 2\omega_{S^2}$ ▨ • For divisor class, $c_1([D]) = \frac{i}{2\pi} \int \partial\bar\partial \log h$, h is singular Hermitian metric of line bundle D

Chern class of vector bundle. ⑤ $c(L) = 1 + c_1(L)$, c_1 is same as before. Functoriality is ... ⑥ (Naturality) c_i respects flat pullback $c_i(f^* E) = f^* c_i(E)$
We had seen Segre class of a cone, we expect when the cone is good, s.t. degenerate to vector bundle (cone = singular vector bundle), Segre class ≃ Chern class [illegible]
Def 7. (Chern class) $\exists!\ c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + \cdots + c_r(\mathcal{E}) + \cdots \in A^*(X)$ for vector bundle $\mathcal{E}$ of rank r on X [illegible] (We ... need ...) ... splitting ... principle ... variety
s.t. [illegible] ② (Whitney) $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0 \Rightarrow c(\mathcal{F}) = c(\mathcal{E}) c(\mathcal{G})$ [illegible]
[illegible] Extend $c(\mathcal{E})$ [illegible] from vector bundles to $K_0(X) \to A^*(X)$ ring homomorphism. (Due to Grothendieck, he ... original topological one) to these axioms, equivalent to original ones for X mfd; here we explain why no ... no finiteness ... By ⑤ & ②, reduce to case of line bundle
Def 8. (K_0-grp and its ring structure) The $K_0(X) = \{[\mathcal{E}] \mid \mathcal{E}$ vector bundle, $[\mathcal{E}_1] = [\mathcal{E}_2]$ iff isomorphic • equivalent using ... $\}$ / ... $[\mathcal{E}_2] = [\mathcal{E}_1] + [\mathcal{E}_3]$ if $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$
has ring structure descended by tensor $\mathcal{E} \otimes \mathcal{E}'$ with ... no. $c_0 = 1$ [illegible] and $c_1(\mathcal{L}) = \ldots c_1$ defined as ...
identity the 1-dim trivial bundle and for line bundle it sends to $A^1(X)$ by ex ... (normalization condition)
(Thus $K_0(X) \xrightarrow{c} A^*(X)$ sends $X \mapsto [X]$ preserves identity ⇒ $c_0 = 1$), $c_1(L) = c_1(\mathcal{L})$ as Rk
Pf of Def 7. We had known topological construction of Chern classes by pullback from Grassmannian, here we can repeat this as [Bott], but here we give a pf of constructing by $c = s^{-1}$ via Segre classes, only algebraic as [Fulton] (obstruction classes/Chern–Weil)
Existence: Recall $s(E) = \sum p_*(c_1(\mathcal{O}(1))^i \cap [\mathbb{P}E])$ (E is vector bundle ⇒ pure-dimensional) and $c_1(\mathcal{L}) \cap [V] = [L|_V]$
and then we set $c(E) = s(E)^{-1} \Leftrightarrow c_0(E) = 1$, $c_1(E) = -s_1(E)$, $c_2(E) = -s_1(E) c_1(E) - s_2(E) c_0(E)$ ($L|_V = \mathcal{O}(D)$, $D = \amalg V_i$ up to ...)
..., $c_n(E) = -s_1(E) c_{n-1}(E) - \cdots - s_n(E) c_0(E)$ [illegible] ⇒ [illegible] we prove ② by the naturality of Segre class:
$s(f^* E) = \sum p_*(c_1(\mathcal{O}(1))^i \cap \mathbb{P}(f^* E)) = \sum p_*(c_1(\mathcal{O}(1))^i \cap \operatorname{Proj}(\operatorname{Sym}(f^* E^\vee))) = \sum p_*(c_1(\mathcal{O}(1))^i \cap f^* \operatorname{Proj}(\operatorname{Sym} E^\vee)) = \sum p_*(c_1(\mathcal{O}(1))^i \cap$
$f^* \mathbb{P}E) = f^*(\sum p_*(c_1(\mathcal{O}(1))^i \cap \mathbb{P}E)) = f^* s(E)$ ▨ ② By splitting [illegible] principle [illegible]

Splitting principle: we can reduce $c(\mathcal{E})$ to $c(\mathcal{L}_1\oplus\cdots\oplus\mathcal{L}_r)$, a precise description is the following

Splitting construction: $\exists f: X'\to X$ flat, s.t. $f^*: A^*X\to A^*X'$ injective and f^*E admits filtration $f^*E=E_r\supset\cdots\supset E_1\supset E_0=0$ with quotients are line bundles $E_i/E_{i-1}=L_i$

We can view this as Galois theory's view: [illegible] By proving

$\Rightarrow$ By naturality, we know splitting principle $\Leftrightarrow$ Whitney formula & Extension to K_0-ring. Using computation $c_t(\mathcal{E})=\prod(1+c_1(L_i)t)$

thus ($\Leftarrow$) allows us show splitting principle to complete existence, and ($\Rightarrow$) allows us to prove uniqueness as $c_1(\mathcal{L})$ is unique ▨

It suffices prove splitting construction [illegible] A^*X, thus we need ring extension to A^*X to $Gr K_0(X)\to A^*(X)$. I omitted. $c_1(L_i)$ called Chern roots of Chern polynomials.

Our idea of constructing filtration is easy. Solve the $c_t(\mathcal{E})=0$ equation. $\mathcal{E}\mapsto c(\mathcal{L}_1)\cdots c(\mathcal{L}_r)$, $c_1(\mathcal{L}_i)$ is Chern roots of $\mathcal{E}$

$$\begin{tikzcd}
\mathcal{Q}_{r-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{Q}_1 \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
X_r \arrow[r, "\pi_r"] \arrow[d, equal] & X_{r-1} \arrow[r, "\pi_{r-1}"] \arrow[d, equal] & \cdots X_1 \arrow[r, "\pi_1"] \arrow[d, equal] & X \arrow[d, equal] \\
\mathbb{P}\mathcal{Q}_{r-1} \arrow[r] & \mathbb{P}\mathcal{Q}_{r-2} \arrow[r] & \cdots \arrow[r] \mathbb{P}\mathcal{E} & X
\end{tikzcd}$$

each $\mathcal{Q}_i=\pi_i^*\mathcal{Q}_{i-1}/\mathcal{O}(-1)$; $\mathcal{O}(-1)$ on $\mathbb{P}\mathcal{Q}_{i-1}$ is tautological bundle, down one dimensional

$\Rightarrow f^*\mathcal{E}$ admits the filtration by $f^*\mathcal{E}\supset \mathcal{Q}_1\supset\cdots\supset\cdots\supset\pi^*\mathcal{O}(-1)$

and $c(\mathcal{L}_1)\cdots c(\mathcal{L}_r)=(1+c_1(\mathcal{L}_1))\cdots(1+c_1(\mathcal{L}_r))$; if $r>\dim X$, then $c_1(\mathcal{L}_1)\cap\cdots\cap c_1(\mathcal{L}_r)=0$, thus $c(\mathcal{E})=1+c_1+\cdots+c_r$, $r=\operatorname{rank}(\mathcal{E})$ holds vanishing.

$\Rightarrow c(E)=f^*c(E)=c(f^*E)=\prod(1+c_1(L_i))$

- Chern roots $c_1(L_i)$ may not written as Chern classes; we denote it $\alpha_i\in A^1(X)$ only

It allows us $f^*c(\mathcal{E})=c(f^*\mathcal{E})$ [illegible] to $A^*(X)$

And injectivity is by only check $\mathbb{P}\mathcal{E}\to X$, $\pi^*: A^*(X)\hookrightarrow A^*(\mathbb{P}\mathcal{E})$

this will be seen in later examples, we can compute $A^*(\mathbb{P}\mathcal{E})$ precisely, as generalisation of $A^*(\mathbb{P}^n)$ ▨

Rk. In this pf, the smoothness can be removed: although we used multiplicity of $A^*(X)$, but only involving multiply of Chern classes, they're defined by restriction to vector bundle, without smoothness. ⑦ $c_1(L^\vee)=-c_1(L)\Rightarrow c(E^\vee)=\prod(1-\alpha_i)$, α_i Chern roots;

Prop 12. ①(Chern character) $ch: K_0(X)\to A^*(X;\mathbb{Q})=A^*(X)\otimes\mathbb{Q}$ s.t. $ch([L])=e^{c_1(L)}$ [illegible] Chern roots

② (Grothendieck) When X is further projective (using Riemann–Roch) $K_0(X)\otimes\mathbb{Q}\to A^*(X;\mathbb{Q})$ is isomorphism. $c_1(\mathcal{E}\otimes\mathcal{F})=\alpha_i+\beta_j$ are their

③ $ch(E_1\otimes E_2)=ch(E_1)\cdot ch(E_2)$, thus $c_1(E_1\otimes E_2)=\operatorname{rank}(E_2)c_1(E_1)+\operatorname{rank}(E_1)c_1(E_2)$, further $c(E_1\otimes E_2)$ is symmetric polynomial

④ Set the todd class $Td(E)=\prod\frac{\alpha_i}{1-e^{-\alpha_i}}$, α_i are Chern roots of E, then we have Grothendieck–Riemann–Roch

(GRR): $ch(\pi_!\mathcal{F})=\pi_*(ch(\mathcal{F})\,Td(\mathcal{T}_\pi))$ where $\pi: X\to B$ is proper map between smooth (quasi-proj) varieties, with the relative tangent sheaf $\mathcal{T}_\pi$; $\pi_!\mathcal{F}=\sum(-1)^iR^i\pi_*\mathcal{F}$, $\mathcal{F}$ coherent on X; as a relative analogue of Hirzebruch–[illegible]

⑤ $c(\Lambda^p E)=\prod(1+\alpha_{i_1}+\cdots+\alpha_{i_p})$, $\Rightarrow ch(\pi_!\mathcal{F})=\sum(-1)^i ch(R^i\pi_*\mathcal{F})$ $\mathcal{F}\in K_0(X)$ [illegible] It's called exceptional direct image

α_i are Chern roots, $i_1<\cdots<i_p$; the core formula to compute invariants on moduli

⑥ (Adjunction formula) $X\subset Y$ closed codim d, both smooth variety with normal bundle $N_{X/Y}\Rightarrow 0\to\mathcal{T}_X\to\mathcal{T}_Y|_X\to N_{X/Y}\to 0$

we generalise the case of X is divisor to case of $X=D_1\cap\cdots\cap D_d$ (when $X=D$ we have adjunction formula as in [illegible] by $N_{D/Y}=\mathcal{O}(D)$ thus $c_1(\mathcal{T}_D)=c_1(\mathcal{T}_Y)/c_1(N_{D/Y})=c_1(\mathcal{T}_Y)\otimes\mathcal{O}(D)$ as determinant to canonical $\omega_D=\omega_Y\otimes\mathcal{O}(D)$) as $c(N_{X/Y})=\prod(1+c_1(\mathcal{O}_Y(D_i)))$ [redacted]

Pf. ① by splitting principle

② We have natural filtration on both $K_0(X)$ and $A^*(X)$: $F_kK_0(X)=\{[E]\in K_0(X)\mid \dim(\operatorname{supp}E)\le k\}$ and thus associated graded

Claim. $ch: F_kK_0X\to F_kA^*(X;\mathbb{Q})$ respects filtration. $F_kA^*(X)=\bigoplus A^i(X)\Rightarrow Gr A^*(X)=A^*(X)$, rings $Gr K_0(X)$ & $Gr A^*(X)$

By the claim $\Rightarrow Gr K_0X\xrightarrow{ch} Gr A^*(X;\mathbb{Q})=A^*(X;\mathbb{Q})$ and $ch(E)=e^{\alpha}+e^{\beta}$, α,β the Chern roots for rank $E=2$, then; called the topological filtration, communicates $\otimes\mathbb{Q}$

We define a map $\varphi: A_kX\twoheadrightarrow Gr_kK_0X$ with each $[E]\in Gr_kK_0X$ have preimage $\varphi^{-1}([E])=\sum m_V(E)[V]$; $m_V(E)=\operatorname{length}E_V$ (localise at generic) $\dim V=k$, V is component of $\operatorname{supp}E$

$\Rightarrow$ $[V]\mapsto[\mathcal{O}_V]$

$$\begin{tikzcd}
A^*(X) \arrow[r, two heads, "\varphi"] \arrow[rr, bend right, "\text{natural condition}"'] & Gr K_0(X) \arrow[r, "ch"] & Gr A^*(X;\mathbb{Q})=A^*(X;\mathbb{Q})
\end{tikzcd}$$

, tensoring $\mathbb{Q}$ respect to surjectivity of φ

Well-defined: ① $ch(E)\in A^*X'$ descends to $ch(E)\in A^*X$; ② Not depend on choice of f^*. From computation $=\sum(-1)^i\prod(1-\alpha_i)$

$$\begin{tikzcd}
Gr K_0(X;\mathbb{Q}) \arrow[r, "ch_{\mathbb{Q}}"] & Gr A^*(X;\mathbb{Q}) \\
A^*(X;\mathbb{Q}) \arrow[u, "\varphi_{\mathbb{Q}}"] \arrow[ur, phantom, "\circlearrowleft"] \arrow[ur, equal]
\end{tikzcd}$$

$\Rightarrow$ Both $\varphi_{\mathbb{Q}}$ & $ch_{\mathbb{Q}}$ isomorphism by communicative algebra, ch also isomorphism. Thus it suffices to prove the Claim:

this is trivial; higher direct image $=0$, thus χ [illegible] $ch(\mathcal{O}_D)=e^{[D]}=\sum\frac{[D]^i}{i!}$ [illegible]

Pf of Claim: ⑨ $ch: [\mathcal{O}_V]\mapsto i_*([V]+\alpha)$; $\dim V=k$ subvariety, $i: V\hookrightarrow X$, α is a $(k-1)$-cycle; then $ch([\mathcal{O}_V])\in F_kA^*(X;\mathbb{Q})$ respect filtration. ⑨ This is due to $ch[i_*\mathcal{O}_V]\overset{\text{GRR}}{=}i_*(ch\,Td(N)^{-1}\cap ch[\mathcal{O}_V])$, $ch[\mathcal{O}_V]\in A^*(V;\mathbb{Q})$, $ch[\mathcal{O}_V]=e^{c_1(\mathcal{O}_V)}=[V]$;

$\Rightarrow ch[i_*\mathcal{O}_V]=i_*(Td(N)^{-1}\cap[V])=i_*([V]+\alpha)$ done when $i: V\hookrightarrow X$ is regular $\Rightarrow$ the relative tangent sheaf is just dual of N. ⑩ We reduce to i regular by excision: $\exists S\subset V$ proper closed, $V-S\hookrightarrow X-S$ regular

$$\begin{tikzcd}
A^*S \arrow[r] \arrow[d, equal] & A^*X \arrow[r] & A^*(X-S) \arrow[r] & 0 \\
A^*S \arrow[r] & A^*V \arrow[r] \arrow[u, "i_*"] & A^*(V-S) \arrow[r] \arrow[u, "i_* \text{ regular}"] & 0
\end{tikzcd}$$

horizontal arrows all respect the filtration, thus for higher codim [illegible]

this all $ch_i(E)$ are symmetric polynomial of $c_1(E)$ & $c_i(E)$; holds: for ① all coefficients are $c_i(f^*E)=f^*c_i(E)$ in the base

- "regular embedding at a point" is an open condition as regular sequence, see [Mustrunyaroa] ▨

③ Set polynomial $P_k(S_1,\ldots,S_n)=\sum x_i^k$ ⟹ Newton's formula, $\forall n\ge k\ge 1$, $S_k=\sum x_{i_1}\cdots x_{i_k}$ symmetric polynomials on $(x_1,\ldots,x_n)$

(Existence is fundamental but nontrivial, omitted and acknowledged) Now set $x_i=c_i$ Chern-classes, we define $P_k(E)=P_k(\alpha_1,\ldots,\alpha_n)$ (the notation $\operatorname{com}(\alpha_i)$ is what here; Chern roots of E)

$\Rightarrow ch(E)=\sum_{k\ge 0}\frac{1}{k!}P_k(E)=\operatorname{rk}+c_1+\frac{1}{2}(c_1^2-2c_2)+\frac{1}{6}(c_1^3-3c_1c_2+3c_3)+\cdots$ by computation (omitted)

Thus it suffices to prove $P_k(E_1\oplus E_2)=P_k(E_1)+P_k(E_2)$ & $P_k(E_1\otimes E_2)=\sum P_k(E_1)\cdot P_k(E_2)\binom{k}{i}$, this is linear algebra of polynomial ▨

④(Pf of GRH) • Comments on putting condition on GRH: {proper f, nonsingular varieties / k [Fulton] ⇒ [redacted] $\mathcal{F}$ ... if it denotes τ_X
projective [redacted] nonsingular schemes / [BdA] ⇒ [redacted] $\mathcal{F}$ by τ_X [redacted]
proper f, nonsingular schemes [General case, 2017, Navarro & Navarro]
Here we adding quasi-projectivity to define K_0-ring, this is due to replace $\mathcal{F}$ by $\mathcal{F}^\cdot$, we need quasi-proj + smooth to give {coherent sheaf} / $\sim$ = {vector bundle}/$\sim$ original [Borel & Serre, 1958] S.f.d. proper f, ... quasi-proj) $\mathcal{F}$
In fact $\mathcal{F}^\cdot$ holds in all of these three cases ⇐ The last one used high K-grps via motivic cohomology
Remove proper in the first case is quite easy: Using $Ch: K_\cdot(X) \to H_{\bullet}(X;\mathbb{Q})$ higher Chern character. → $\pi_! \circ \mathcal{F}$
cohomology with compact support / Borel-Moore. • replace $Ch(\mathcal{F}^\cdot) = \sum (-1)^i Ch(\mathcal{F}_i)$ [redacted]
Thus we prove the baby case in [Fu] [redacted] (i.e. original one)

$$\begin{tikzcd} K_0(X) \arrow[r, "\pi_!"] \arrow[d, "\tau_X := Td(X)\cdot Ch"'] \arrow[dr, phantom, "\circlearrowleft"] & K_0(B) \arrow[d, "Td(B)\cdot Ch =: \tau_B"] \\ A^*(X;\mathbb{Q}) \arrow[r, "\pi_*"] & A^*(B;\mathbb{Q}) \end{tikzcd}$$

We reduce to the case of $X \xhookrightarrow{i} B\times\mathbb{P}^m \xrightarrow{p} B$ as it's functorial ▨
② For $B\times\mathbb{P}^m \to B$, we reduce to the case of $\mathbb{P}^m \xrightarrow{p} pt$ by:

$$\begin{tikzcd} K_0(B)\otimes K_0(\mathbb{P}^m) \arrow[r, "\tau_B\otimes\tau_{\mathbb{P}^m}"] \arrow[d, "\times"] \arrow[dd, bend right, "p_! \otimes p_!"'] & A^*(B;\mathbb{Q})\otimes A^*(\mathbb{P}^m;\mathbb{Q}) \arrow[d, "\times"] \arrow[dd, bend left, "1\otimes p_*"] \\ K_0(B\times\mathbb{P}^m) \arrow[r, "\tau_{B\times\mathbb{P}^m}"] \arrow[d, "p_!"] \arrow[u, phantom, "\circlearrowleft"] & A^*(B\times\mathbb{P}^m;\mathbb{Q}) \arrow[d, "p_*"] \\ K_0(B) \arrow[r, "\tau_B"] \arrow[d, equal] & A^*(B;\mathbb{Q}) \arrow[d, equal] \\ K_0(B)\otimes K_0(\operatorname{Spec}k) \arrow[r] & A^*(B;\mathbb{Q})\otimes A^*(\operatorname{Spec}k;\mathbb{Q}) \end{tikzcd}$$

⟺ The community is $Td(B\times\mathbb{P}^m) = Td(B)\,Td(\mathbb{P}^m)$, so is Chern character.
Then the big red square commutes + two small commutes
⇒ middle one also ▨
And big red is the case of $\mathbb{P}^m_k \xrightarrow{p} \operatorname{Spec}(k)$ (abuse notation also denoted p)

③ Compute $\mathbb{P}^m_k \xrightarrow{p} \operatorname{Spec}k$: By splitting principle, it suffices assume $\mathcal{F}$ line bundle, and $\operatorname{Pic}(\mathbb{P}^m) = \mathbb{Z}$ generated by $\mathcal{O}(1)$
⇒ assume $\mathcal{F} = [\mathcal{O}(n)]$, $0\le n\le m$ ⇒ [redacted] $Ch(p_!(\mathcal{O}(n))) = \sum(-1)^i Ch(R^ip_*(\mathcal{O}(n)))$, $R^ip_*(\mathcal{O}(n)) = H^i(\mathbb{P}^m, \mathcal{O}(n))$
$= \sum(-1)^i \dim H^i(\mathbb{P}^m,\mathcal{O}(n)) \overset{\text{[Ha]}}{=} \binom{n+m}{n}$; $Td(T_{\operatorname{Spec}k}) = 1$; $p_*(Ch(\mathcal{O}(n))\,Td(\mathbb{P}^m)) = \int_{\mathbb{P}^m} Ch(\mathcal{O}(n))\,Td(\mathbb{P}^m) = \int_{\mathbb{P}^m} e^{nc_1(\mathcal{O}(1))}\cdot$
this is HRR ($\chi(\mathbb{P}^m,\mathcal{O}(n))$ =)
$Td(1+c_1(\mathcal{O}(1)))^{m+1}$ $= \int_{\mathbb{P}^m} \frac{e^{nc_1}\,c_1^{m+1}}{(1-e^{-c_1})^{m+1}}$, by the Rk on c_1, we know deg = integral over $x = c_1(\mathcal{O}(1))$ ▨
④ For $X \hookrightarrow B\times\mathbb{P}^m$, ⇒ $p_*(Ch(\mathcal{O}(n))\,Td(\mathbb{P}^m)) = \int \frac{e^{nx}x^{m+1}}{(1-e^{-x})^{m+1}}dx = \binom{n+m}{n}$ by residue thm as complex analysis
we deform it into $X \hookrightarrow \mathbb{P}(N\oplus 1)$ by deformation to normal cone

Both questions {How to reduce to the "model of closed embedding" / Why GRH holds for the "model of closed embedding"} are the hardest part of our proof. We prove them as this order:
(**) Our deformation to normal cone diagram is $X \xrightarrow{i_0} X\times\mathbb{P}^1 \xleftarrow{i_\infty} X$ where we assume $X \xhookrightarrow{\pi_0} B$ closed embedding and $\mathcal{F} = E$ vector bundle

• We need reduce to $\bar{\pi}_0$, compute $Ch(\bar{\pi}_{0*}E)$
involving two parts: $\mathbb{P}(C\oplus 1)$ and $\tilde{B}$, we expect the part of $\tilde{B}$ contributes 0 to Ch.
Then using GRH of $\bar{\pi}_0$ and by $\pi_0 = g\circ k\circ\bar{\pi}_0$
Precisely, choose a resolution $G_\cdot$ of F_*p^*E, M flat ⇒ restrict (⊗) preserves exactness ⇒ $j_0^*G_\cdot$ is resolution of $\pi_{0*}E$
$\bar{f}(X) \cap \tilde{Y} = \emptyset$ by construct ⇒ $l^*G_\cdot$ is acyclic desired & $k^*G_\cdot$ is resolution of $\bar{\pi}_*E$; $j_\infty^* G_\cdot$ is resolution of $\bar{\pi}_{0*}E$
Split(G) into $j_\infty^* = l^* + k^*$ ⇒ $Ch(\pi_{0*}E) = g_*j_{0*}(Ch(\pi_{0*}E)) \overset{(a)}{=} g_*(j_{0*}(Ch(j_0^*G_\cdot)))$
$= g_*(Ch(G_\cdot)\cdot j_{0*}[B])$ by projection formula ($j_{0*}Ch(j_0^*G_\cdot) = j_{0*}j_0^*Ch(G_\cdot) = (j_{0*}[B])\cap Ch(G_\cdot)$) view Ch as cycle class instead of ...
$= g_*(Ch(G_\cdot)\cdot(k_*[\mathbb{P}(C\oplus 1)] + l_*[\tilde{B}]))$ by deformation $j_{0*}[B]$ to $j_{\infty*}[M_\infty] = l_*[\tilde{B}] + k_*[\mathbb{P}(C\oplus 1)]$ (rational equivalence)
$= g_*(k_*Ch(k^*G_\cdot) + l_*Ch(l^*G_\cdot))$ by projection back $\overset{(b)}{=} g_*(k_*(Ch(\bar{\pi}_{0*}E)) + 0) = g_*k_*Ch(\bar{\pi}_{0*}E) \overset{\text{GRH of (***)}}{=} g_*k_*\bar{\pi}_{0*}(Td(N)^{-1}$
$\cdot Ch(E)) = \pi_{0*}(Td(N)^{-1}\cdot Ch(E))$ ▨
(***) Abuse notation again and again $X \xrightarrow{\bar{\pi}_0} \mathbb{P}(N\oplus 1)$ [redacted] and $\mathcal{F} = E$ again vector bundle.
Here, $C =$ normal bundle $N_{X/Y}$ is due to when X, Y both smooth ⇒ $X \hookrightarrow Y$ regular
• Universal quotient bundle rank of $p: \mathbb{P}(N\oplus 1) \to X$ is Q s.t. ∃ natural $p^*(N\oplus 1) \to Q$ bundle and the red triangle also commutes (section level)
The universal quotient bundle gives Koszul complex $0 \to \Lambda^d Q^\vee \to \cdots \to \Lambda^2 Q^\vee \to Q^\vee$
and it's a resolution of $\bar{\pi}_{0*}\mathcal{O}_X$ via $0 \to \cdots \to Q^\vee \xrightarrow{s} \mathcal{O}_{\mathbb{P}(N\oplus 1)} \to \bar{\pi}_{0*}\mathcal{O}_X \to 0$
(By some algebraic facts on CM ring, I'm curious about why)
thus $\bar{\pi}_{0*}E$ can be computed by ⊗ ⇒ $0 \to \Lambda^r Q^\vee \otimes p^*E \to \cdots \to p^*E \to \bar{\pi}_{0*}E \to 0$
⇒ $Ch(\bar{\pi}_{0*}E) = (\sum(-1)^i Ch(\Lambda^i Q^\vee)) Ch(p^*E) = c_d(Q)\cdot Td(Q^{-1})\cdot Ch(p^*E) = [X]\cdot Td(Q^{-1})\cdot Ch(p^*E)$ as X is zero locus
$= \bar{\pi}_{0*}\bar{\pi}_0^*(Td(Q)^{-1}Ch(p^*E))$ (Compute the series precisely, omitted) of $s \in \mathbb{P}(N\oplus 1)$ ⇒ $c_{top}(Q) = c_d(Q)$
$= \bar{\pi}_{0*}(Td(N)^{-1}Ch(E))$ ▨ $= [X] \in A^*(\mathbb{P}(N\oplus 1))$
We ...

③ Bi-induction of pair (p,r), $r=r(E)$, by splitting principle, assume $E=E'\oplus L$, $c_1(L)=\alpha_r$ is last Chern root of E
$\Rightarrow 0\to \Lambda^{p-1}E'\otimes L\to \Lambda^pE\to \Lambda^pE'\to 0 \Rightarrow c(\Lambda^pE)=c(\Lambda^pE')\,c(\Lambda^{p-1}E'\otimes L)$, by ③ we compute $c(F\otimes L)=\sum_i \alpha_r^{\,r-i} c_i(F)$
$\Rightarrow c(\Lambda^pE)=\prod(1+\alpha_{i_1}+\cdots+\alpha_{i_p})\sum_{0\le i\le r}\alpha_r^{\,i}\Big\{\prod_{i_1<\cdots<i_{p-1}\le r-1}(1+\alpha_{i_1}+\cdots+\alpha_{i_p})\Big\}_i$ (take degree $=i$ part), then we can compute $\sum_{i\le r}$ ▨
④ Trivial ▨ $i_1<\cdots<i_p\le r-1$

Now we have fundamental tools to compute examples: (We not show Chern classes' ex as they're almost same as topological, but show general scheme how it's used to Chow ring)

Eg.9. (Projective bundles) It's a relative version of $A^*(\mathbb{P}^n)$, here set E over X, $\pi: \mathbb{P}E\to X$, rank $\mathbb{P}E = r=(\operatorname{rank}E)-1$, then $\pi^*: A^*(X)\to A^*(\mathbb{P}E)$ (flat pullback), then ① Each degree: $A_{k-r}X\xrightarrow{\pi^*}A_k(\mathbb{P}E)\to A_k(E)$ thus π^* is injective (as it's induced by $E\to \mathbb{P}E\to X$ naturally) ② $\forall\, \beta\in A_k(\mathbb{P}E)$ written uniquely $\beta=\sum c_1(\mathcal{O}(1))^i\cap \pi^*\alpha_i =: \sum(\pi^*\alpha_i)\zeta^i$, $\alpha_i\in A_{k-r+i}(X)$, thus $\bigoplus_i \zeta^i A_{k-r+i}(X)\cong A_k(\mathbb{P}E)$ Then isomorphism. Sometimes we identify $A^*(X)$ as $\pi^*A^*(X)$

• Both ①&② are level, for ring level we assume further X is smooth and projective scheme, then:

③ Ring isomorphism $A^*(\mathbb{P}E)\cong \dfrac{A^*(X)[\zeta]}{(\zeta^{r+1}+c_1(E)\zeta^r+\cdots+c_{r+1}(E))}$, $\zeta=c_1(\mathcal{O}(1))$. Next we prove ② & ③: $\zeta\le A^*(\mathbb{P}E)$ embedding, then ζ^i is absorbed into $A^*(X)$

$$\begin{tikzcd} \bigoplus_{i\le r} A^*(X) \arrow[r, shift left, "\varphi"] & A^*(\mathbb{P}E) \arrow[l, shift left, "\psi"] \end{tikzcd}$$

$(\alpha_0\cdots\alpha_r)\longmapsto \sum_{i\le r}(\pi^*\alpha_i)\zeta^i$ $\quad \sum \alpha_i\zeta^i$

$\sum_{i\le r}\pi_*(\zeta^{r-i}\beta)\zeta^i \longmapsfrom \beta$

• We check that $\pi_*(\zeta^{r-i}\beta)=\alpha_i$, where $\beta=\sum(\pi^*\alpha_i)\zeta^i$, it's due to left is graded over $A^*(X)$, right graded, thus it's induced by the left graded structure $\Rightarrow$ right, then ψ defined each $\beta_i\mapsto \pi_*(\zeta^{r-i}\beta_i)\cdot\zeta^i$, done, up to a constant ($\zeta^k\zeta^l$ and $\zeta^{r-i}\zeta^l$) ▨ (Note that similar with splitting principle, all multiplications here is $c_1(\mathcal{O}_{\mathbb{P}E}(1))^i\cap$, not real something)

③ Thus set $\beta=\zeta^{r+1}$ in particular, by pair, $\zeta^{r+1}=\sum_{i\le r}(\pi^*\alpha_i)\zeta^i \Rightarrow$ the $\varphi: A^*(X)[\zeta]\to A^*(\mathbb{P}E)$ fact through $A^*(X)[\zeta]/(f)\cong A^*(\mathbb{P}E)$ where f is monic polynomial on ζ, degree $=r+1$. We need to determine this f: $0\to \mathcal{O}_{\mathbb{P}E}(-1)\hookrightarrow \pi^*E\to Q\to 0$ (this we can't claim is a inverse) SES of sheaves on $\mathbb{P}E$, Q define to be quotient $\Rightarrow c(\mathcal{O}(-1))\,c(Q)=c(\pi^*E)\in \pi^*A^*(X)\cong A^*(X)\Rightarrow$ homotopy invariance

$\Rightarrow c(\mathcal{O}(-1))\,c(Q)=c(E)$, $c(\mathcal{O}(-1))=1-c_1(\mathcal{O}(1))=1-\zeta\to c(Q)=c(E)(1+\zeta+\zeta^2+\cdots)$, $r(Q)=r(E)-1=r\Rightarrow c_{r+1}(Q)=0$ i.e. $\zeta^{r+1}+c_1(E)\zeta^r+\cdots+c_{r+1}(E)=0$ ▨ • As $c(E\oplus\mathcal{O})=c(E)$, $A^*(\mathbb{P}(E\oplus\mathcal{O}))\cong A^*(\mathbb{P}E)$ By E affine bundle $\cong X\times\mathbb{A}$, $\mathbb{Z}[\zeta]/(\zeta^{n+1})$ $\cong A^*(\mathbb{P}^n)$

Geometric construction: Each $Z\subset\mathbb{P}E$ dim k subvariety can be viewed into two parts $[\pi(Z)]\in A^*(X)$ and fibre $[F_x]\in A^*(\mathbb{P}^r)=\mathbb{Z}[H]/(H^{r+1})$, Assume without degeneracy here $n=0$
$\Rightarrow [Z]=\zeta^{r+k-\dim}\cdot[\pi(Z)]$, we can see when degeneracy exists, there will be lower terms, but we don't care here. A rigorous argument is by dynamic projection in [3264] It's project $\mathbb{P}^n-A$ to B, $\mathbb{P}^n=\operatorname{span}(A,B)$ by retracting

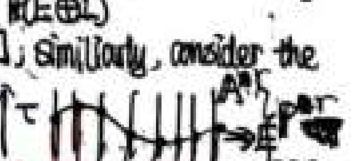
I don't know why [Fulton]/[3264] prove ② so long to prove inj + surj? Upper □ shows why constructive is not successful, for detail see [Fulton] at x ∃ sections $\tau_1\cdots\tau_r$ on U, $\{\tau_i(x)\}$ as here we don't is basis of this blue fibre, but in other fibres we have degeneracy locus $\tau_j=0$ section have zero

Red one is spanned by the blue fibre

Consider $F\subset E$ subbundle, we compute what $[\mathbb{P}F]\in A^*(\mathbb{P}E)$ is: ($s=\operatorname{rank}(F)$)
$[\mathbb{P}F]=\zeta^{r-s}+c_1(E/F)\zeta^{r-s-1}+\cdots+c_{r-s}(E/F)=c_{r-s}(\mathcal{O}(1)\otimes\pi^*(E/F))$
Note that $c_{r-s}(\mathcal{O}(1))\otimes\pi^*(E/F))=c_{top}(\mathcal{O}(1)\otimes\pi^*(E/F))=[Z]$, Z the zero of vector bundle $\mathcal{O}(-1)\otimes\pi^*(E/F)$. We show $Z=\mathbb{P}F$ as: $\forall\, p\in\mathbb{P}F\Leftrightarrow p\in\mathbb{P}E\cap(\mathbb{P}F)_{\pi(p)}=\mathbb{P}E\cap\mathbb{P}^s_{\pi(p)}$
$\Leftrightarrow$ p corresponds to the fibre of $\mathcal{O}(-1)$ at p (the fibre of tautological bundle at is (p,p)) & $\mathcal{O}(-1)_p\subset\mathbb{P}^s_{\pi(p)}$
$\Leftrightarrow \varphi: \mathcal{O}(-1)\hookrightarrow\pi^*E\to\pi^*(E/F)$ vanishes
$\Leftrightarrow$ zero locus of $\varphi\in\mathcal{H}om(\mathcal{O}(-1),\pi^*(E/F))\cong\mathcal{O}(1)\otimes\pi^*(E/F)$ section ▨

($\mathcal{O}_{\mathbb{P}E}(-1)\subset\pi^*E$, here we need this fibre $\subset\pi^*F$)

III Then we apply this to $E\subset E\oplus\mathbb{1}$, with is zero section in $\mathbb{P}(E\oplus\mathbb{1})$
$\Rightarrow[\mathbb{P}\mathcal{O}_X]=\zeta^r+c_1(E)\zeta^{r-1}+\cdots+c_r(E)$; we $\mathbb{P}\mathbb{1}=\mathbb{P}\mathcal{O}_X$ denote $\mathbb{P}\mathcal{O}_X$, the zero section as $[\Sigma_0]$; similarly, consider the section $(\tau,1)$ of $\mathbb{P}(E\oplus\mathbb{1})$, where τ is section of E, we denote it as $[\Sigma_\tau]$
this family $[\Sigma_{t\tau}]$ gives deformation from $[\Sigma_0]$ to $[\Sigma_\tau]\Rightarrow[\Sigma_\tau]=[\Sigma_0]$, now as most intersection problem can be formulated as intersection of section $\cap$ zero section, we reduce to $[\Sigma_0]^2$: $\pi_*([\Sigma_0]^2)=c_r(E)=c_{top}(E)$ (It's also a classical in topology) $[\mathcal{O}_X]=0$
It's very easy to prove $[\Sigma_0]^2=(\zeta^r+c_1(E)\zeta^{r-1}+\cdots+c_r(E))\cdot[\Sigma_0]$, note that Σ_0 is $\mathbb{P}(\mathbb{1})$ and ζ is c_1 of $\mathcal{O}(1)$ form a line bundle
$\pi_*([\Sigma_0]^2)=c_r(E)\,\pi_*([\Sigma_0])=c_r(E)[\Sigma_0]$, $\zeta[\Sigma_0]=0$ due disjoint to $[\Sigma_0]$ on X but $\mathbb{P}E$
$=c_r(E)$ ▨



$0\ni t$ connecting $\mathbb{P}\mathbb{1}\in\mathbb{P}^{r-1}$

Rk. $c_1(\mathcal{O}_{\mathbb{P}E}(1))$ may not effective Cartier divisor (different with $\mathbb{P}^n$ case!). The Hirzebruch surface is ruled surface $\pi: \mathbb{P}E\to\mathbb{P}^1$, $\pi_*\mathcal{O}(1)=\mathcal{O}\oplus\mathcal{O}(-n)$ thus not

Eg.10 (Ruled surface) As application of Eg.9, we prove that: a ruled surface contains at most 1 $(-n)$-curve $(n>0)$, here curve is in variety sense (Recall: ruled surface is total space of projective bundle of rank 1 on smooth curve)
Rk. Geometric meaning of intersection negative is it can't move, n more large, it's more hard. It depicts a rigidity out of algebraic sense, (self) (As when positive case, we need move it to intersect) this explain why $E^2=-1$ for exceptional divisor E:

With over $\mathbb{C}$, we can better explain it: in complex case all intersection >0 as orientation is given. (More large n, is the $(-n)$-curve is harder to move.)
But if we move E topologically, it's not $/\mathbb{C}$, but only real surface $\Rightarrow(-1)$ occurs as orientation reversed at this move!

E have no reason to move in this picture! (algebraic sense)

Pf. $\pi:\mathbb{P}E\to X$, X smooth curve, rank$(E)=2$. Otherwise ∃ two curves $C_1\neq C_2$, $[C_i]^2<0$; $i=1,2$.
First we reduce C_1 and C_2 to section Σ_1 and Σ_2 $=\mathbb{P}L_1$ and $\mathbb{P}L_2\subset\mathbb{P}E$, then by Eg.9, we can compute $[\Sigma_i]$!

observe that C_1 can't contained in fibre as $[C_1]^2 = [\pi^{-1}(x)]^2 = \pi^*[x]^2 = \pi^*0 = 0$ < 0 contradiction, thus $\pi|_{C_1} : C_1 \to X$ is finite map, set $\alpha_1 : \tilde{C}_1 \xrightarrow[\text{normalisation}]{\text{finite}} C_1 \hookrightarrow PE \xrightarrow{\pi} X$ and consider the pullback Cartesian diagram

$$\begin{tikzcd} \tilde{C}_1 \times_X PE \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & PE \arrow[d] \\ \tilde{C}_1 \arrow[r, "\alpha_1"] \arrow[u, bend left, "\sigma_1"] & X \end{tikzcd}$$

$\beta_{1*}[D_1] = 0$ (as the singularity contained in discrete points)

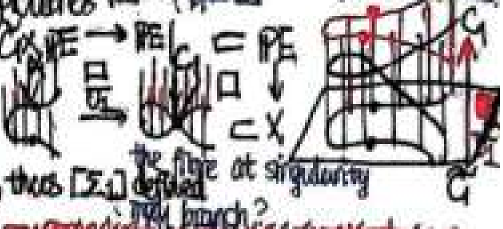
then [redacted] $A^1(\tilde{C}_1 \times_X PE) \ni [C_1] = \deg \beta_1 \cdot \beta_{1*}[\tilde{C}_1]$

$\tilde{C}_1 \times_X C_1 \subset \tilde{C}_1 \times_X PE$ induces a cycle $m_1[\Sigma_1] + [D_1] \in$ [redacted], Σ_1 is section of PE, by viewing $\tilde{C}_1 \times_X PE \supseteq \tilde{C}_1 \times_X C_1$ over $\tilde{C}_1$, the normalisation map induce the section as the following two pictures $\beta_{1*}[\Sigma_1] = \beta_{1*}[\Sigma_1]$

$\Rightarrow$ [redacted] $(m_1 > 0)$ $= [\tilde{C}_1 \times_X C_1] = \beta_1^*[C_1]$

$= \beta^*[C_1]$

$0 > \int_{PE} [C_1]^2 = \int_{PE} [\beta_{1*}\Sigma_1][C_1] = \int_{\tilde{C}_1 \times_X PE} [\Sigma_1]\beta_1^*[C_1]$ projection formula of degree $\Rightarrow \exists$ section $1 \times_X \sigma_1 : \tilde{C}_1 \to \tilde{C}_1 \times_X PE$ and its induced ~~$\tilde{C}_1 \times_X \sigma_1 \times_X \dots$~~, thus $[\Sigma_1]$ defined



$= \int_{\tilde{C}_1 \times_X PE} [\Sigma_1](m_1[\Sigma_1] + [D_1]) \geq m_1 \int [\Sigma_1]^2$ (Σ_1, D_1 are different)

Here we use $A(PE) = \dots$, i.e. $\xi^2 = -c_1(E)\xi$ & c_2 to

$\Rightarrow \int_{\tilde{C}_1 \times_X PE} [\Sigma_1]^2 < 0$, so $\int [\Sigma_2]^2 = \dots$ thus we noticed [redacted]

Σ_1 & Σ_2, then we give a contradiction; then $\Sigma_i = PL_i \subset PE$

$\Rightarrow [\Sigma_i] = \xi + c_1(E/L_i) = \xi + c_1(E) - c_1(L_i)$ [redacted]

$\xi^2 + c_1(E)\xi + c_1(L_i)\xi - 2c_1(E)c_1(L_i) = 0$

$\Rightarrow \int_{\tilde{C}_1 \times_X PE} [\Sigma_i]^2 = \int_{\tilde{C}_1 \times_X PE} \xi^2 + 2c_1(E) - 2c_1(L_i) < 0$ deg

$\Rightarrow \deg L_i > \deg c_1(E)$, and $\Sigma_1 \neq \Sigma_2$ as $C_1 \neq C_2 \Rightarrow$

the degree is same as deg of the bundle

$0 \to L_1 \oplus L_2 \to E \to Q \to 0$, rank $Q = 0$, thus has support on a finite set

$\Rightarrow \deg Q \geq 0 \Rightarrow \deg E \geq \deg(L_1 \oplus L_2) = \deg(L_1) + \deg(L_2) > \deg E$, ■

[redacted]

Can we replace by another $\beta_1 : \tilde{C}_1 \to C_1$ finite smooth

Q. ① Why we choose normalisation, it's normal, used?

② Where D_1 comes from? (D_1 has no component common with Σ_1)?

③ Where m_1 comes from, can we compute it? Prof Zhou claims that when $\tilde{C}_1 = C_1$ normal

↪ This comes from the singularity of C_1: when $\tilde{C}_1 = C_1$ normal $\Rightarrow [C_1 \times_X C_1] = [\Sigma_1]$ as $\Sigma_1 = C_1 \times_X C_1$ [redacted] $= 0 \Rightarrow m_1 = 1$ always

I can't give the picture of m_1 and D_1 ...

Eg 11. ((Projective) Grassmannian) Rk. After Eg 11 and Eg 9, we can combine them into Grassmannian bundle's Chow ring similarly, I omitted it ([3264, §9.6]) Our main task is ① $A^*(G(m,n)) = \mathbb{Z}[c_1 \cdots c_m]/((1+c_1+\cdots+c_m)^{-1})_{>n-m}$ $= (\{c^{-1}\}_{n-m+1}, \{c^{-1}\}_{n-m+2}, \cdots, \{c^{-1}\}_n)$ pick deg $= n-m+1$; ② using Schubert calculus compute each Chow groups (① is easy and ② is hard)

Rk. Our algebraic topology had shown that ① $H^*(G(m,n)) = A^*(G(m,n))$ and $H^*(G(m,\infty)) = \mathbb{Z}[c_1 \cdots c_m]$; $H(G(m,n); \mathbb{Z}/2)$ = replace Chern class by Stiefel-Whitney classes; ② using Schubert calculus compute the cellular decomposition the $1 - c_1 + c_1^2 - c_1^3 + \cdots$ n-th series

Thus we can see that Chow is easier to compute than H^*, as both using cellular decomposition, Chow grp can be computed out directly but topological it's just chain groups, and the ∂ are hard to compute.

① is reasonable as one of definition of Chern classes is pullback generators of $A^*(G(m,n))$ as G_m (as in [3264]), and easy! (but it's harder than topological case)

② acquires more detail description on the "boundary" of each "cells" intersects.

Here our Grassmannian $G(m,n) = \{m\text{-dim subspace } V \subset \mathbb{A}^n\} = \{(m-1)\text{-dim subspace } \mathbb{P}V \subset \mathbb{P}^{n-1}\} = \mathbb{G}(m-1, n-1)$, the first nontrivial one is $G(2,4) = \{$the projective lines $\subset \mathbb{P}^3\} = \mathbb{G}(1,3)$ (Recall by Plücker embedding, it's projective $G(m,n) \hookrightarrow \mathbb{P}(\wedge^m \mathbb{A}^n)$) and we have generalisation of tautological bundle $\mathcal{O}(-1)$ on projective spaces, says the universal subbundle S [redacted] $\Lambda \subset V \cong \mathbb{A}^n$ the fibre $S_{[\Lambda]} = \Lambda$. (Rk. Then the trivial bundle $(G(m,n) \times V)/S$ is just the universal quotient bundle) (Omitted check it does a vector bundle, it's linear algebra by affine covering showed next). [redacted] $T_G = Hom$

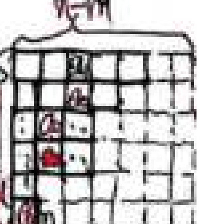
Pf of ①: Denote $A^* = A^*(G(m,n))$, $t_i = \{c^{-1}\}_i$, [redacted] $I = (t_{n-m+1}, \cdots, t_n)$, $R = \mathbb{Z}[c_1 \cdots c_m]/I$, we need prove $A \cong R$; for this we prove for any base extension $F \to \mathbb{Z}$, F is field, $R' = R \otimes_{\mathbb{Z}} F \xrightarrow{\varphi'} A' = A \otimes_{\mathbb{Z}} F$ is isomorphism to complete the proof, where we set $\varphi : R \longrightarrow A$, $t_i \longmapsto c_i(Q)$, where Q is universal quotient bundle: $0 \to S \to \mathcal{O}^n \to Q \to 0$, S universal subbundle.

The problem is why φ does ring homomorphism: $c(Q) = c(S)^{-1}$, thus [redacted] $\varphi : \{c^{-1}\}_i \mapsto \{c(S)^{-1}\}_i$, thus $c_i \mapsto$ [redacted] $c_i(S)$ We had known $A^*(Gr(m,n))$ generated by some special Schubert cycles via Giambelli's formula and these in such formula = Chern classes $c_i(S)$.

$c_i(S)$ is in fact a Schubert cycle, thus preserves the intersection product.

Then it's purely algebraic to show $R' \cong A'$: (i) Reduce φ to φ'; surj & inj; (ii) (t_i) is regular sequence; (iii) Using facts of $\dim R' = \dim A' = \binom{n}{m} \Rightarrow$ only injectivity; (iv) $\ker(\varphi') = 0$, only (iv) acquires more subtle analysis of structure of Schubert cycle, others are communicative algebra. Omitted ■ need consequence of ①

Schubert calculus: A Schubert cycle $\Sigma_a \subset G$, $a = (a_1 \cdots a_m)$ (Schubert index) s.t. $n-m \geq a_1 \geq \cdots \geq a_m \geq 0 \longleftrightarrow$ Young diagram



$\Sigma_a = \{\Lambda \in Gr(m,n) \mid \dim(V_{n-m+i-a_i} \cap \Lambda) \geq i, \forall i, \{0 \subset V_1 \subset \cdots \subset V_n = \mathbb{A}^n\}$ is a flag$\}$

not depend on choice of the flag after take class $\sigma_a = [\Sigma_a]$: as two flags up to a $G = GL_n(k)$-action

$\Rightarrow$ for G affine, $[gX] = [X]$ for any cycle by Kleiman's transversality thm (Omitted) ↪ sub or quotient universal bundle

- Thus $c_i(S) = (-1)^i \sigma_{1^i} = (-1)^i \sigma_{\underbrace{1,1,\cdots,1}_{i},0\cdots0}$

such form as $\boxed{}$ is called special Schubert class and = Chern class of universal subbundle σ_{1^i}; or σ_i = class of [pt]

$c(Q) = \frac{1}{c(S)} = \sum (-1)^i c_i(S)$

$= \sum (-1)^i \sigma_{1^i}$ as Q rank $(n-m)$

$\sigma_a \longleftrightarrow \sigma_{a^*}$ Using Young diagram arranging multi-index is easy to get proper

- Understand the definition of Σ_a. [redacted]

[redacted] It's equivalent to consider the restriction of flag $\mathcal{V}$: $0 = V_0 \cap \Lambda \subset \cdots \subset V_{n-1} \cap \Lambda = V_n \cap \Lambda = \Lambda$

$\Lambda \in \Sigma_a \Leftrightarrow$ [redacted] the i-th $0 \leq \dim(V_1 \cap \Lambda) \leq \cdots \leq \dim(V_{n-1} \cap \Lambda) \leq \dim \Lambda = m$ (*)

jump earlier [redacted] than [illegible]

Writing as Young diagram, it's

as here the meaning of this Young diagram differs from usual one, we use the matrix representation instead of it [Page 24]

Here our Young diagram is restricted s.t. not occur blocks like ⊞.

$$\begin{pmatrix} * & 0 & 0 & - & - & - & 0 \\ * & * & * & 0 & 0 & - & 0 \\ * & * & * & * & * & 0 & 0 & - & - & 0 \\ * & * & * & * & * & * & * & 0 & - & 0 \end{pmatrix} m \times n$$

$\leq a_i$

called the matrix representive of Λ after choosing basis $0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1 \cdots e_n \rangle = V = \mathbb{A}^n$ and each elements are determined by put the matrix [illegible] two basis.

(It's not easy, it [illegible])

When it chosen general, it's must thus [illegible] extra zeros ∈ the ith row (otherwise the special cycle [illegible] not exist)

$\left(\begin{smallmatrix} * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \\ * & * & * & * & * & * \end{smallmatrix} \right)$ First, right, but I not explain well.

Z_0 $\mathbb{V}_m$

where $\langle v_1 \rangle = V_{n-m+1-a_1}$ first dim 1 subspace $\subset \Lambda$
$\langle v_1, v_2 \rangle = V_{n-m+2-a_2}$ first dim 2 subspace $\subset \Lambda$
is basis of Λ

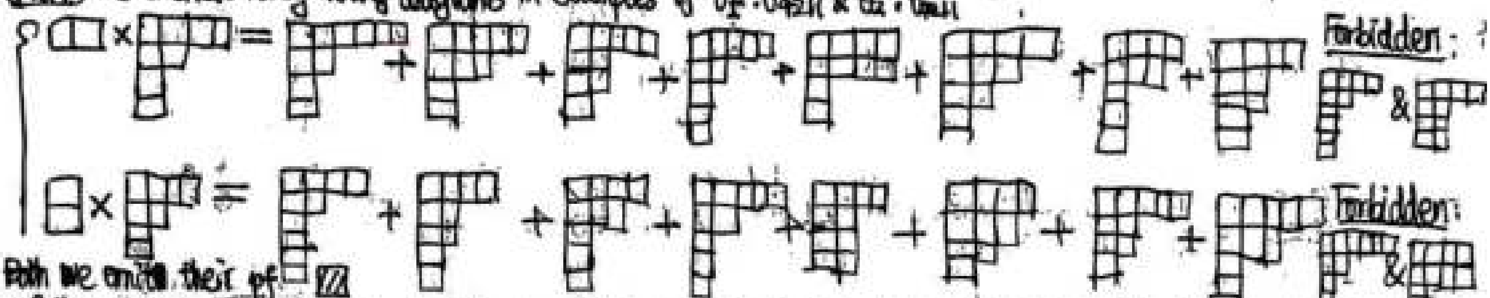
- Computation of intersection product

Step 1. All Schubert cycles are polynomial on special Schubert cycles.

(Giambelli) $\sigma_{a_1 \cdots a_m} = \begin{vmatrix} \sigma_{a_1} & \sigma_{a_1+1} & \cdots & \sigma_{a_1+m-1} \\ \sigma_{a_2-1} & \sigma_{a_2} & \cdots & \sigma_{a_2+m-2} \\ \vdots & & & \vdots \\ \sigma_{a_m-m+1} & - & - & \sigma_{a_m} \end{vmatrix}$ (It's too hard to compute when (m,n) large)

Step 2. Compute the intersection product of special × general

(Pieri) We describle it by Young diagrams in examples of $\sigma_2 \cdot \sigma_{4,2,1,1}$ & $\sigma_{1,1} \cdot \sigma_{4,1,1}$



Both we omit their pf ▨

- Cellular decomposition / Affine stratification: Schubert cells $\Sigma_a^0 = \Sigma_a - \bigcup_{b>a} \Sigma_b$ (It's trivial fact that $b > a \Leftrightarrow \Sigma_b \subsetneq \Sigma_a$)

Pf of ②. We start from the matrix representive and by fundamental operations to the form

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 0 \end{pmatrix} \cong \mathbb{A}^{m(n-m)-|a|}$$ Σ_a is locally closed, (Σ_a^0) forms a affine stratification such form

$A = \begin{pmatrix} 1 & 0 & 0 & - & - & - & - & 0 \\ 0 & * & 1 & 0 & - & - & - & 0 \\ 0 & * & 0 & * & 1 & - & - & 0 \\ 0 & * & 0 & * & 0 & * & 1 & 0 \end{pmatrix}$ Z_0. Taking its m-columns out form a m×m matrix A_b, $b = \{n-m+1-a_1, \cdots, n-a_m\} \Rightarrow A_b = I_{m\times m}$ is the standard affine covering of $Gr(m,n)$ $\dim = m(n-m)$ as except m^2 elements of $I_{m\times m}$ left all 0.

$\Rightarrow \Sigma_a^0 \subset Z_0^0 \cong \mathbb{A}^{m(n-m)}$, Z_0 open affine $\cong \mathbb{A}^{m(n-m)}$; and $\Sigma_a^0 \subset Z_0^0$ closed is cutted by vanishing locus of extra zero of each row $= \sum a_i = |a|$ ▨

This generators are all Schubert classes as free Abelian group $\Leftrightarrow \mathbb{Z}^r$, $r =$ number of Young diagram $\subset m(n-m)$ $\Rightarrow r = \binom{n}{m}$ by combination

- We know all things of $Gr(2,4)$: the first / trivialest nontrivial Grassmannian:

We write out all its Schubert cycles only, as multiplie structure is same as general $m=2$, thus we need compute $\Sigma_{0,0} \supset \Sigma_{1,0} \supset \Sigma_{2,0}, \Sigma_{1,1} \supset \Sigma_{2,1} \supset \Sigma_{2,2}$

$\Sigma_{0,0} = G(2,4)$, $\Sigma_{2,2} = \{\Lambda \mid \Lambda = V_2\} = \{pt\}$

Left four nontrivial cycles are: $\Sigma_{1,0} = \{\Lambda \mid \Lambda \cap V_2 \neq 0\}$
$\Sigma_{2,0} = \{\Lambda \mid V_1 \subset \Lambda\}$; $\Sigma_{1,1} = \{\Lambda \mid \Lambda \subset V_3\}$
$\Sigma_{2,1} = \{\Lambda \mid V_1 \subset \Lambda \subset V_3\}$,

By fixing a flag $0 = V_0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 = \mathbb{A}^4$

[3264] gives projective analogue {lines $\mathbb{P}^1 \subset \mathbb{P}^3$}, it can be describled in pictures; $\Sigma_{1,0} = \{\Lambda \mid \Lambda \cap L \neq \emptyset\}$; $\Sigma_{2,0} = \{\Lambda \mid p \in \Lambda\}$; $\Sigma_{1,1} = \{\Lambda \mid \Lambda \subset H\}$, $\Sigma_{2,1} = \{\Lambda \mid p \in \Lambda \subset H\}$; with flag $\{p\} = \overline{V_1} \subset L = \overline{V_2} \subset H = \overline{V_3} \subset \mathbb{P}^3$; then we can draw them as

$\Sigma_{1,0} \longleftarrow \Sigma_{1,1}$ $\Sigma_{2,0} \longleftarrow \Sigma_{2,1}$

thus we have desired inclusion ($\Sigma_{1,1} \subset \Sigma_{1,0}$ due to in $\mathbb{P}^2 = H$ all lines intersect and Schubert cells are easy to figure out)

- Degeneracy loci: It's a more geometric way to characterize the Chern classes as ~~[illegible]~~; when $\mathcal{F}$ is generated by global sections $s_1 \cdots s_r$ ($r = r(\mathcal{F})$, of vector bundle F) $\Rightarrow c_1(\mathcal{F}) = [D(s)]$, $D(s) = \{x \in X \mid \dim \operatorname{span}(s_1(x) \cdots s_r(x)) < r\}$ [illegible]

remove this, we call the class defined by following way is degeneracy class [illegible]

we generalise this into ① First for c_i, we pick $r-i+1$ sections let them linear dependent, it's then describle as the rank of the matrix: $D_i(s) = \{x \in X \mid \operatorname{rank}(s_\alpha(x)) \leq r-i+1\}$; ② Moreover, to $f: E \to F$ bundle map with rank $= e \to$ rank $= f$, the set $D_i(f) = \{x \in X \mid \operatorname{rank}(f(x)) \leq \min\{e,f\} - i + 1\}$. Then we finally have kth degeneracy class $D_k(f) = \{x \in X \mid \operatorname{rank}(f(x)) \leq k\}$

assume X is pure-dim $= k$

and the kth degeneracy class $D_k(f) \in A_{n-(e-k)(f-k)}(D_k(f))$ is then constructed by ① Grassmannian bundle $G = G(d, E)$, $k \leq \min\{e,f\}$, $d = e-k$

expected dim of $D_k(f)$

and $G(d,E) \longleftarrow \pi^* E \Rightarrow$ universal subbundle S of $\pi^* E$

② $S \subset \pi^* E \to \pi^* F$ ③ On $G(d,E)$, $Z(s_f)$ (zero locus) $\xrightarrow{\pi} D_k(f) \subset X$

$$\begin{tikzcd} D_k(f) \arrow[r, phantom, "\square"] & \\ X & E \arrow[l] \end{tikzcd}$$

④ $D_k(f) := \pi_*(Z(s_f), Z(s_f))$ is the localised top Chern class of $Z(s_f)$ $:= \zeta^! [G(d,E)]$, ζ is regular embedding of $G(d,E)$ to $S^\vee \otimes \pi^* F$

$G(d,E)$

$s_f \in \operatorname{Hom}(S, \pi^* F) \cong \operatorname{Hom}(G(d,E), S^\vee \otimes \pi^* F) \ni s_f$

And $E^!$ is refined Gysin map = virtual pullback. Thus $\begin{cases} Z(\sigma_{s_E}) = [Z(s_E)]^{vir} \\ D_k(\varphi) = [D(\varphi)]^{vir} \end{cases}$ and when regular embedding are chosen properly / X is Cohen–Macaulay, they're same as fundamental cycles, resp. Page 8

(Thom–Porteous) View $D_k(\varphi) \in A_{m-(e-k)(f-k)}(X)$ by natural pushforward $\Rightarrow D_k(\varphi) = \Delta^{(e-k)}_{f-k}(c(F)/c(E)) \cap [X]$, $\Delta^{(p)}_q(c) = \det\begin{pmatrix} c_q & c_{q+1} & \cdots & c_{q+p-1} \\ c_{q-1} & \cdots & \cdots & c_{q+p-2} \\ c_{q-p+1} & \cdots & \cdots & c_q \end{pmatrix}$. Omit it ▨

There are many classic researches before 2000 ...

Next several examples shows applications of GRR to (almost) every branches of AG: birational geometry, Derived category, Abelian variety's Arithmetic, and most important as Grothendieck's viewpoint of relative: moduli space.

Rk. So-called applications in [Ful] is all! applications of HRR, without any relative setting; intersection on moduli we accept now as same as scheme, later we'll make it rigorous.

Eg.12. (Covering) A double cover $f: X \to Y$ both smooth projective surfaces (finite degree 2 & surjective), $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus L^{-1}$, where $L^{\otimes 2} = \mathcal{O}_Y(E)$, E is the branch locus ($= f(R)$, R ramification divisor) (E or R determines this covering) (Hence we can view it a cyclic covering (E, L))

We can compute $c(f_*\mathcal{O}_X) = c(\mathcal{O}_Y \oplus L^{-1})$ by GRR, written in terms of L $\begin{cases} c_1(f_*\mathcal{O}_X) = [0] - [L] \\ c_2(f_*\mathcal{O}_X) = \frac{1}{2}(3L^2 - L) \end{cases}$ as rank $(f_*\mathcal{O}_X) = \mathrm{rank}(\mathcal{O}_Y \oplus L^{-1}) = 2$

- For $c(f_*\mathcal{O}_X(D))$ $\begin{cases} c_1(f_*\mathcal{O}_X(D)) = [f_*D] - L \\ c_2(f_*\mathcal{O}_X(D)) = \frac{1}{2}([f_*D]^2 - f_*D^2 - [f_*D] \cdot [L]) \end{cases}$
- In general, height of a cover $h(f) = \deg(c_1(f'_*\mathcal{O}_{Y'}))$, where $Y' \xrightarrow{\rho} Y \xrightarrow{f} X$, ρ is a resolution of singularity, then GRR is used to check it's independent with choice of ρ

Eg.13. (Compatibility of K-theoritic & cohomological Fourier–Mukai transform) $K(X) \xrightarrow{p_1^*} K(X \times Y) \xrightarrow{\otimes e} K(X\times Y) \xrightarrow{p_{2*}} K(Y)$, fix $\boxed{e \in K(X\times Y),\ a \in H^*(X\times Y)}$ and $H^*(X;\mathbb{Q}) \xrightarrow{p_1^*} H^*(X\times Y;\mathbb{Q}) \xrightarrow{\cup a} H^*(X\times Y;\mathbb{Q}) \xrightarrow{p_{2*}} H^*(Y;\mathbb{Q})$, then we have commutative diagram

$$\begin{tikzcd} K(X) \arrow[r, "\Phi_e"] \arrow[d, "v_X"] & K(Y) \arrow[d, "v_Y = ch(-)\sqrt{td(Y)}"] \\ H^*(X;\mathbb{Q}) \arrow[r, "\Phi^H_{v(e)}"] & H^*(Y;\mathbb{Q}) \end{tikzcd}$$

We'll see why radical taken in pf:

Φ^H_a: We use H^* instead of A^* after composite cycle map $cl: A^*(X) \xrightarrow{\cong} H^*(X)$, and here is clear.

By definition, we have

$$\begin{tikzcd} K_0(X) \arrow[r, "p_1^*"] \arrow[d, "v_X"] & K_0(X\times Y) \arrow[r, "\otimes e"] \arrow[d, "v_{X\times Y}(\sqrt{td(X)})"] & K_0(X\times Y) \arrow[r, "p_{2*}"] \arrow[d, "v_{X\times Y}(\sqrt{td(X)})"] & K_0(Y) \arrow[d, "v_Y"] \\ H^*(X;\mathbb{Q}) \arrow[r, "p_1^*"] & H^*(X\times Y;\mathbb{Q}) \arrow[r, "\cup v(e)"] & H^*(X\times Y;\mathbb{Q}) \arrow[r, "p_{2*}"] & H^*(Y;\mathbb{Q}) \end{tikzcd}$$

projection formula

projection formula

call e FM kernel

$\Rightarrow$ multiply both by $\sqrt{td(Y)}$

$$\Rightarrow \begin{tikzcd} K_0(X\times Y) \arrow[r, "p_{2*}"] \arrow[d, "Td_X"] & K_0(Y) \arrow[d, "Td_Y"] \\ H^*(X\times Y;\mathbb{Q}) \arrow[r, "p_{2*}"] & H^*(Y;\mathbb{Q}) \end{tikzcd}$$ by GRR ▨

Rk. (Different FM transformations) Both K & H are the lifting of chain-level (Derived cat) to cohomology-level of original FM transform: $D^b(X) \xrightarrow{p_1^*} D^b(X\times Y) \xrightarrow{\otimes^L \mathcal{P}} D^b(X\times Y) \xrightarrow{p_{2*}} D^b(Y)$, $\Phi_{\mathcal{P}}$, fix $\mathcal{P} \in D^b(X\times Y)$ FM kernel

Eg.14. (Abelian variety) We know that T_X is trivial on Abelian variety, thus all todd classes $= 1$, makes more easy to compute.

L ample line bundle on X Abelian, then $(\det \Phi^k_\wedge(L))^{-1}$ is also ample. $\wedge = \wedge(L) = m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$ the Mumford bundle (Abelian variety is one of the best place apply GRR (relative) and FM $(X\times X)$ as we always consider $X\times X \to X$) I don't know why

Eg.15. ($\overline{\mathcal{M}}_{g,n}$ and $\mathcal{M}_{g,n}$) First set $n=0$ (no marked point) for simplicity. And the coarse moduli $\overline{M}_g$ & M_g are proper scheme. We'll do things are ① quasi-projectivity of M_g follows computing Hodge bundle; ② DM compactification of $\overline{M}_g$ follows computing canonical bundle (Mumford's formula)

① quasi-projective $\Leftrightarrow$ open $\subset$ projective, and projective $\Leftarrow$ very ample $\Leftarrow$ ample $\overset{\text{Nakai–Moishezon numerical criteria}}{\Leftrightarrow} \int c_1(L)^{\dim X} > 0$ when proper scheme

$\mathbb{E} := \pi_*(\omega^\circ_{\mathcal{C}_g/\mathcal{M}_g})$ pushforward of relative dualising sheaf of the universal family $\mathcal{C}_g \xrightarrow{\pi} \mathcal{M}_g$. $\mathbb{E}$ is called Hodge bundle, and $c_1(\mathbb{E}) = \det \mathbb{E} =: \lambda$ called Hodge line bundle

$\mathrm{Proj}(\bigoplus H^0(M_g, \lambda^{\otimes n}))$, n large $= \overline{M}_g^{Sat}$ Arithmetic compactification, λ Hodge line bundle

$\overline{M}_g^{Sat} \supseteq \overline{M}_g \supseteq M_g$, $\overline{M}_g^{Sat}$ is bad bundle due to boundary $\overline{M}_g^{Sat} - M_g$ not

Fact. $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q} = \mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q} = \mathbb{Q}\{\lambda, \delta_0, \ldots, \delta_{\lfloor g/2 \rfloor}\}$ spanned as $\mathbb{Q}$-vector space. δ are boundary divisors (omitted)

- Using this we can reformulate Eg.14. by computing $(\det \Phi^k_\wedge(L))^{-1}) > 0$ as Abelian variety is for all projective $\Rightarrow$ proper. By GRR $\Rightarrow \int c_1(\Phi^k_\wedge(L))^{-1} = \int ch_1(p_{2*}(m^*L \otimes (p_1^*L)^{-1}))$ exists away locus of ... already

After DM compactification, GRR $= \int p_{2*} ch_1(m^*L \otimes (p_1^*L)^{-1}) = \int ch_1(m^*L \otimes (p_1^*L)^{-1}) = \int_{X\times X} c_1(p_1^*L) > 0$, Rk. is ample ▨

Thus λ and $\mathbb{E}$ abused into $\overline{M}_g$, M_g, $\overline{\mathcal{M}}_g$, $\mathcal{M}_g$ (As $\overline{M}_g^{Sat} \supseteq \overline{M}_g$, we restrict $\mathbb{E}$ and λ to $\overline{M}_g$), but $\mathbb{E}$ & λ on $\mathcal{M}_g$ stack only (for algebraic space we have also NM numerical criteria, thus for moduli stack also makes sense)

By definition $\mathbb{E}_{[C]} = H^0(C, K_C)$ for $[C] \in M_g$; and we now show that $\boxed{\lambda = \frac{\kappa}{12}} > 0$, κ the first tautological class κ_1 (Or Kappa class)

- Tautological classes κ_i: It's due to Mumford's standard conjecture that $H^*(\mathcal{M}_\infty) = \mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ (Later we can know all $\lambda_i = c_i(\mathbb{E})$ can be written as polynomials of κ_i in $\mathbb{Q}$ by GRR) $H^*(\mathcal{M}_\infty) = \lim H^*(\mathcal{M}_g)$ finally stable after $\forall k$, $H^k(\mathcal{M}_\infty) = H^k(\mathcal{M}_g)$ when $g \geq 3k$ [Harer]

$\kappa_i = \pi_*(c_1(\omega_{\mathcal{C}_g/\mathcal{M}_g})^{i+1})$ where $\mathcal{C}_g^{(l)} \xrightarrow{\pi} M_g^{(l)}$ are the rigidified moduli problem: due to M_g is not smooth, we modify it into smooth case $M_g^{(l)}$ via?

$\lambda = c_1(\mathbb{E}) = ch_1(\mathbb{E})$, $ch(\mathbb{E}) = ch(\pi_!\omega^\circ_{\mathcal{C}_g/\mathcal{M}_g}) \overset{GRR}{=} \pi_*(ch(\omega_{\mathcal{C}_g/\mathcal{M}_g}) \cdot td(\omega^\vee_{\mathcal{C}_g/\mathcal{M}_g}))$, $ch(\omega_{\mathcal{C}_g/\mathcal{M}_g}) = (1 + c + \frac{c^2}{2} + \cdots)$, $td(\omega^\vee_{\mathcal{C}_g/\mathcal{M}_g}) = (1 - \frac{c}{2} + \frac{c^2}{12} - \cdots)$, $c = c_1(\omega_{\mathcal{C}_g/\mathcal{M}_g})$

$= \pi_*(1 + \frac{c}{2} + \frac{c^2}{12} + \cdots) = ch(\pi_*(\omega_{\mathcal{C}_g/\mathcal{M}_g})) + 1$

$\Rightarrow \lambda = \pi_*(\frac{c^2}{12}) = \frac{1}{12}\pi_*(c_1(\omega^\circ_{\mathcal{C}_g/\mathcal{M}_g})^2) = \frac{\kappa}{12}$, thus $M_g \subset \mathrm{Proj}(\cdots)$ quasi-projective ▨

② $\overline{M}_g = \mathrm{Proj}(\bigoplus H^0(\overline{M}_g, (\lambda-\mu\delta)^{\otimes m}))$ for $\mu < \frac{1}{11}$ is DM compactification (when $\mu \geq \frac{1}{11}$, $\overline{M}_g^{\mathrm{cusp}}$ ... other bigger compactifications)

The canonical line bundle $K = \det \Omega_{\overline{M}_g/k} = c_1(\Omega_{\overline{M}_g/k})$ on $\overline{M}_g = \pi_*(\Omega_{X/B} \otimes \omega_{X/B}^{\circ})$ by pushforward from $\overline{C}_g$ [illegible]

By deformation theory:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{C}_g \arrow[d, "\pi"] \\ B \arrow[r] & \overline{M}_g \end{tikzcd}$$

Universal property gives the $K = \pi_*\Omega_{\overline{C}_g/\overline{M}_g} \otimes \omega_{\overline{C}_g/\overline{M}_g}$

On each fibre C, $T_{[C]} = \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \overset{\text{Serre}}{=} H^1(C, \Omega_C^{\vee} \otimes \mathcal{O}_C) = H^1(C, T_C)$ (Ω locally free)

$\Rightarrow T^{\vee}_{[C]} \overset{\text{Serre}}{=} H^0(C, \Omega_C \otimes \omega_C)$

We show $K = 13\lambda - 2\delta$, and then by numerical criteria we can show it (or without it, we can know taking $\mu \ll 1$ can make it $\lambda - \mu\delta > 0$ ample also done)

$K = \mathrm{Ch}_1(\pi_*(\Omega_{X/B} \otimes \omega_{X/B})) \overset{\text{GRR}}{=} \pi_*(\mathrm{Ch}(\Omega_{X/B} \otimes \omega_{X/B})\,\mathrm{Td}(T_{\pi}))_1$

Now we set $\alpha = c_1(\Omega_{X/B})$, $\beta = c_2(\Omega_{X/B})$; we hope to compute. $\Omega_{X/B} = \omega_{X/B}$ smooth case, thus we minus the singular locus (complicated)

$[\mathrm{Sing}\,B] = [\text{all singularities of fibres of } X \xrightarrow{\pi} B]$ [illegible] desired

$= \pi_*(\mathrm{Ch}(\Omega_{X/B})(1 - \frac{\alpha}{2} + \frac{\alpha^2+\beta}{12}))$ [illegible] also $= \pi_*((1 + 2\alpha + \frac{(2\alpha)^2}{2} + 2\beta)(1 - \frac{\alpha}{2} + \frac{\alpha^2+\beta}{12}))$ [illegible]

$= \pi_*(2\alpha^2 - 2\beta - \alpha^2 + \frac{\alpha^2+\beta}{12}) = \pi_*(\frac{13}{12}\alpha^2 - \frac{11}{12}[\mathrm{Sing}\,B])$ $\Omega_{X/B} = \mathcal{I}_{[\mathrm{Sing}\,B]} \otimes \omega_{X/B}$ i.e. $\mathrm{Ch}_2(\Omega_{X/B}) = \alpha^2 - [\mathrm{Sing}\,B]$

$= \pi_*(13 \cdot \frac{\alpha^2 + [\mathrm{Sing}\,B]}{12} - 2[\mathrm{Sing}\,B]) = 13(\pi_*\frac{\alpha^2+[\mathrm{Sing}\,B]}{12}) - 2\delta$ thus here β is modified after singular part minused $\overline{\beta} = \frac{\beta}{2} - [\mathrm{Sing}\,B]$

$= 13\lambda - 2\delta$ (last step is a hard birational geometry) ∎

We prove several props we skipped before, all of their fundamental input is the universal bundle's description of them classes. This part includes complicated computation/technique in intersection theory, who not care can skip

Pf of Prop 2 ($s^*[C] = \{c(g^*N) \cap s(W, V)\}_{k-d}$) Recall setting is $W \hookrightarrow V$, $g: P(C\oplus 1) \to W$, we have SES on $P(C\oplus 1)$

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ X \arrow[r, hook, "\text{regular}"] & Y \supset N \end{tikzcd}$$

$0 \to \mathcal{O}(-1) \to g^*(g^*N \oplus 1) \to Q \to 0$
(universal subbundle) (universal quotient bundle)
Here all $\mathcal{O}(\pm 1)$, $g^*N \oplus 1$ is $\mathcal{O}_{P(C\oplus 1)}(\pm 1)$

$\Rightarrow c(Q)\,c(\mathcal{O}(-1)) = c(g^*g^*N) \cdot 1 = c(g^*g^*N)$ by Whitney

$\Rightarrow$ [illegible] $c(Q) = c(g^*g^*N) \cap s(\mathcal{O}(-1))$ (*)

[illegible] (Recall $s^*: A_* E \to A_{*-\mathrm{rank}\,E}$ Thom isomorphism, $E = g^*N$ here)

① [illegible] $s^*[C] = g_*(c_d(Q) \cap [P(C\oplus 1)])$ (Generally $s^*\beta = g_*(c_d(Q) \cap \overline{\beta})$, $\overline{\beta}$ closure in $P(\oplus 1)$) it means $\beta = \sum m_i [V_i] \Rightarrow \overline{\beta} = \sum m_i [\overline{V_i}]$

② $g_*(c_d(Q) \cap [P(C\oplus 1)]) = \{g_*(c(Q) \cap [P(C\oplus 1)])\}_{k-d} \overset{(\ast)}{=} \{g_*(c(g^*g^*N)\, s(\mathcal{O}(-1)) \cap [P(C\oplus 1)])\}_{k-d}$
$= \{c(g^*N)\, g_*(s(\mathcal{O}(-1)) \cap [P(C\oplus 1)])\}_{k-d} \overset{\text{def}}{=} \{c(g^*N)\, s(C)\}_{k-d}$ done

$\pi_* c(\pi^*E)$ [illegible] not true; $c(E) \cap \pi^* = \pi^*(c(E) \cap)$ is true (naturality) $\pi^*c(E) \neq c(E)$

For ①, recall Eg 1 on projective bundles, $[P(C\oplus 1)] \in A^*(P(g^*N\oplus 1))$ can be written as $[P(C\oplus 1)] = \sum c_1(\mathcal{O}(1))^i \cap \pi^*\alpha_i$ [illegible]
further written as $\pi^*(\alpha) + i_*(\delta)$: $j^*[P(C\oplus 1)] = \alpha$

Here our diagram is $P(g^*N) \xrightarrow{i} P(g^*N\oplus 1) \xleftarrow{j} g^*N$, $[P(C)]$

and by Exercise $i_*\delta + \pi^*\alpha \longmapsto \pi^*\alpha = [C]$

$$\begin{tikzcd} A_*(P(g^*N)) \arrow[r, "i_*"] & A^*(P(g^*N\oplus 1)) \arrow[r, "j^*"] & A^*(g^*N) \arrow[r] & 0 \\ & & A_*(X)[-d] \arrow[ull, "p^*"] \arrow[ul, "\pi^*"] \arrow[u, "\pi^*", "\cong"'] \end{tikzcd}$$

$\alpha = s^*[C]$

equivalent the blue triangle commutes.

$= \sum j^* c_1(\mathcal{O}(1))^i \cap j^*\pi^*\alpha_i = \sum c_1(\mathcal{O}(1))^i \cap \pi^*\alpha_i = \pi^*\alpha$ for some α

Assume $\alpha_i = [V_i]$, $c_1(\mathcal{O}(1)) \cap [V_i]$ is [illegible] from $g^{-1}(V)$ to $[g^{-1}(V)]$, then union all these $\pi^*(V_i)$ ($\alpha = s^*[C]$)

$\Rightarrow [P(C\oplus 1)] - \pi^*\alpha \in \mathrm{kernel}(j^*) = \mathrm{Im}(i_*)$

Then $g_*(c_d(Q) \cap [P(C\oplus 1)])$
$= g_*(c_d(Q) \cap \pi^*\alpha) + g_*(c_d(Q) \cap i_*\delta) = \alpha + 0 = s^*[C]$ done $\Rightarrow [P(C\oplus 1)] - \pi^*\alpha = i_*\delta$ done//

$g_*(c_d(Q) \cap \pi^*\alpha) \underset{e = c_1}{\overset{\text{Whitney}}{=}} g_*(c_1(\mathcal{O}(1))^d \cap \pi^*\alpha) = \alpha$ by $\Rightarrow c_d(Q) = \sum_{i \le d} c_1(\mathcal{O}(1))^i\, c_{d-i}(g^*g^*N\oplus 1)$ ($r = \mathrm{rank}\, g^*N$ = relative dim of $X \hookrightarrow Y$ $= d$)
$= \sum_{i \le d} c_1(\mathcal{O}(1))^i\, c_{d-i}(g^*g^*N\oplus 1)$

We don't use any property of α, thus it holds for all general class β

c_1 as $c_1(\mathcal{O}(1)) = H^{\cdot}$

and all lower order $i < d$ vanishes, i.e. $c_1(\mathcal{O}(1))^i\, c_{d-i}(\ldots) \cap g^*\alpha = 0$ for $\forall i < d$. Why? Due to dimension reason

$g_*(c_d(Q) \cap i_*\delta) = i_*(c_d(j^*Q) \cap \delta) = 0$ as i^*Q is trivial [illegible] bundle

Next are two excess intersection formulas, then we give some examples

Pf of Prop 9. ($i^!\alpha = c_e(g^*E) \cap i'^!\alpha$) Recall our setting

$$\begin{tikzcd} X'' \arrow[r] \arrow[d, "g"] & Y'' \arrow[d] \\ X' \arrow[r, "\text{regular}"] \arrow[d, "f"] & Y' \arrow[d] \\ X \arrow[r, "\text{regular}"] & Y \end{tikzcd}$$

$e := d - d'$, $g^*E = g^*N/N'$ excess normal bundle

Assume $\alpha = [V]$, $V \subset Y''$ closed subvariety $C_{II} = C_{V\cap X''}$ is the

$0 \to N' \to g^*N \to E \to 0$ on X' — obstruction of global section

$\Rightarrow 0 \to P(g^*N'\oplus 1) \to P(g^*g^*N\oplus 1) \to P(g^*E\oplus 1)$

The global section comes from $g^*g^*N\oplus 1$ due to $\mathcal{O}(-1)$ not kills the 1, i.e. $\mathcal{O}(-1) \subset g^*g^*N$ [illegible] but this admits a global section Why?

$0 \to \mathcal{O}(-1) \to \mathcal{O}(-1) \to$ [illegible]

$0 \to P(g^*N'\oplus 1) \to P(g^*g^*N\oplus 1) \to P(g^*E\oplus 1)$

$0 \to Q' \to Q \to$ [illegible] $\to 0$ on $P(g^*N\oplus 1)$

$C_{top}(F)$ = obstruction of k-dim trivial subbundle F, i.e. $\exists F \subset$ [illegible]

$\Rightarrow C_{top}(F) = 0$ (but not compose)

On $P(g^*N'\oplus 1)$ and $P(g^*g^*N\oplus 1)$, we have their universal quotient bundle; γ^*g^*E is their quotient as: (Consider embedding

$$\begin{tikzcd} 0 \arrow[r] & Q' \arrow[r, hook] & Q \arrow[r] & \gamma^*g^*E \arrow[r] & 0 \\ & g^*N' \arrow[u, "r^*"] \arrow[r, hook] & g^*g^*N \arrow[u, "r^*"] \end{tikzcd}$$

Same as ... compute its ...

Thus $i^![V] = (X, V) = \pi_*(c_d(Q) \cap [P(C_{V\cap X}\oplus 1)])$ by the definition of deformation to normal cone

$\overset{\text{Whitney}}{\underset{\text{by } (\ast)}{=}} \pi_*(c_{d'}(Q') \cap c_e(g^*E) \cap [P(C_{V\cap X''}\oplus 1)])$

$= c_e(g^*E) \cap \pi_*(c_{d'}(Q') \cap [P(C_{V\cap X''}\oplus 1)]) \overset{\text{def}}{=} c_e(g^*E) \cap i'^!(X', V)$

Pf of Prop 10. ($(X_1 \cdots X_r \cdot V)^Z = \{\prod c(N_i) \cap s(Z, V)\}_m$) Recall our setting $Z \subset \bigcap X_i \cap V$ a connected component (connected component) and $N_i = (N_{X_i}Y)|_Z$, each X_i are regular embedded. $m = \dim V - \sum \operatorname{codim}(X_i, Y)$ is just the top class (one take degree on right [illegible] be useful in many enumerative problems, such as 5 conic problem $\Rightarrow 3264$...) V is subvariety $\subset Y$ [illegible]

By Prop.[illegible] $(X \cdot V) = \{c(N|_{X \cap V}) \cap s(X \cap V, V)\}_{\dim V - \operatorname{codim}(X, Y)} = \sum m_i \alpha_i s_i^*[C_i]$

$\Rightarrow (X \cdot V)^Z = \sum_{\pi(C_i) \subset Z} m_i s_i^*[C_i] = \sum_{\pi(C_i) = Z} \{c(N|_{Z_i}) \cap s(Z_i, V)\}$ ... $\pi(C_i) = Z_i$ (Here if we take irreducible, there is geometric multiplicity $m_i \neq 1$ [illegible], thus not holds)

Now Z_i are irreducible components and $\pi(C_i) = Z_i \subset Z$ connected components is more fine: $= Z \Rightarrow \bigcup_{Z_i \subset Z} Z_i = Z$

$\Rightarrow (X \cdot V)^Z = \{\sum c(N|_{Z_i}) \cap \sum s(Z_i, V)\}_{\dim V - \operatorname{codim}(X,Y)} \overset{\text{Whitney}}{=} \{c(N|_Z) \cap s(Z, V)\}_{\dim V - \operatorname{codim}(X, Y)}$. Inductively and by Whitney, we can

Rk. When we splitting Z into irreducible components Z_i, it seems that one can refine our desired formula [illegible] statement into Z is irreducible component. It's of course true as Z irreducible $\Rightarrow$ connected; but our reason to stay at the level of connected component is due to then $(X_1 \cdots X_r \cdot V) = \sum_{\text{connected component}} \{\prod c(N_i) \cap s(Z, V)\}_m^Z$; this can't be refined into irreducible component as otherwise two irreducible components intersection is complicated, we'll show it by example next. (Due to the contribution [illegible] generally, we [illegible] irreducible one)

Eg. 16. (Connected v.s. Irreducible) $L_1, L_2, L_3, L_4 \subset \mathbb{P}^3$ four "parallel" distinct lines (in same $\mathbb{A}^2 \subset \mathbb{P}^3$ they parallel and intersect at ∞)

Two cycles $S = [L_1] + [L_2] + [L_3]$, $T = [L_2] + [L_3] + [L_4]$

$\Rightarrow S \cdot T = 9[L_2]$, Now our irreducible components $L_2 \cup L_3$ are L_2 & L_3 are even but 9 is odd. ▨

Due to excess intersection formula is one of the most enumerative formula for computing, we give some of its corollary

Prop 17 (Key formula) and Blow-up formula) if $S \cap T$ at least the right hand of excess is Z on-true and

① $X \xrightarrow{i} Y$ regular embedding of codim d; we taking $\tilde{Y} = Bl_X Y \xrightarrow{f} Y$ and $\tilde{X} = \mathbb{P}N \xrightarrow{g} X$, exceptional divisor $\Rightarrow \tilde{X} \xrightarrow{j} \tilde{Y}$

and $N_{\tilde{X}}\tilde{Y} = \mathcal{O}(-1)$ regularity, excess = universal quotient $E = g^*N/N' = g^*N/\mathcal{O}(-1)$ [illegible] symmetric $\Rightarrow$ right hand, even [illegible] $= \operatorname{codim}$

then $f^*i_*\alpha = j_*(c_{d-1}(E) \cap g^*\alpha)$; Further, we have $0 \to A_k X \xrightarrow{\Phi} A_k \tilde{X} \oplus A_k Y \xrightarrow{\Psi} A_k \tilde{Y} \to 0$ split: $\alpha \mapsto (c_{d-1}(E) \cap g^*\alpha, -i_*\alpha)$, $(\tilde{\alpha}, \beta) \mapsto j_*\tilde{\alpha} + f^*\beta$

② $V \subset Y$ subvariety, $\tilde{V}$ is the strict transform $\subset \tilde{Y} \Rightarrow f^*[V] = [\tilde{V}] + j_*\{c(E) \cap g^*s(V \cap X, V)\}_{\dim V}$

③ (Refined blow-up formula) $Y' \to Y$ [illegible] $\Rightarrow$ [illegible] $[V] = [V'] + j_*\{c(E) \cap g^*s(V \cap X', V)\}_{\dim}$, $X' = Bl_{X'} Y'$

Rk: For ①: When X & Y are smooth, it degenerates to [illegible] $c_d(N) \in A^d(X)$ (take $\alpha = [X] \in A_d(X)$)

Note that it's the special case of Steenrod's operation's construction: $Sq^i: A^d(X) \longrightarrow A^{d+i}(X)$ where $\beta = \sum [X_i]$; $X_i \xrightarrow{N_i} Y$ is

$\Rightarrow$ It's just Chow's Steenrod's operation. Motivic analogue exists: [illegible] $\mapsto \sum f_{i*}(c_i(N_i))$ smooth representatives

One idea is compatible with (motivic) suspension. [illegible] one is $\mathbb{Z}/2\mathbb{Z}$ (or $\mathbb{Z}/p\mathbb{Z}$) coefficient, $C_i / \mathbb{Z}/2 \mathbb{Z} N_i$

For ② & ③, later we'll see that a generalization/formalism by bi-intersection theory: a relative construction is optional SW Chow ring. We thus not prove ③, and after this, we'll introduce properties of refined Gysin and to bi-intersection class

Pf. ① the first claim is postponed after refined Gysin maps; we only prove the splitting SES: $\Psi \circ \Phi = 0$ by the first claim (admitted here)

this show the universality of Chow ring / motivic as Grothendieck predicted: $Sq^i: A^*(X)/p\mathbb{Z} \to A^{*+i}(X)/p\mathbb{Z}$

$\|$ $\|$ Brosch's formula

$Sq^i: H^{2*,*}(X, \mathbb{Z}(p)) \to H^{2*+2i, *+i}(X; \mathbb{Z}(p))$

is the motivic Steenrod operation

(i) Φ has left inverse Φ': $(\tilde{\alpha}, \beta) \mapsto g_*(\tilde{\alpha})$;

$\Phi' \circ \Phi(\alpha) = g_*(c_{d-1}(E) \cap g^*\alpha) = \alpha$ also by the first claim. [illegible]

thus injective and splitting shown; we had proved it for E = universal quotient bundle case in Prop.[illegible]

(ii) $\forall \beta \in A_k \tilde{Y}$, we can find its preimage $(\tilde{\alpha}, \beta)$ [illegible] indeed it holds for all quotient of [illegible] (why I don't know) [illegible] gives a reason via [illegible]

In fact we have: $\exists \tilde{\alpha} \in A_k \tilde{X}$, $\tilde{\beta} = f^*f_*\tilde{\beta} + j_*\tilde{\alpha}$ as $\tilde{\beta} - f^*f_*\tilde{\beta}$ is supported on $\tilde{X}$, then by excision on $\tilde{X} \hookrightarrow \tilde{Y} \longleftarrow \tilde{Y} - \tilde{X}$ iso [illegible]

$\Rightarrow \exists \tilde{\alpha} \in A_k \tilde{X}$, $\tilde{\beta} - f^*f_*\tilde{\beta} = j_*\tilde{\alpha}$ $\Rightarrow (\tilde{\alpha}, f_*\tilde{\beta}) \mapsto \tilde{\beta}$, done. for $f: \tilde{Y} - \tilde{X} \xrightarrow{\sim} Y - X$ due to $\tilde{X}$ is exceptional divisor of the blow up.

(iii) It left to check the exactness on middle: set $(\tilde{\alpha}, \beta) \in A_k \tilde{X} \oplus A_k Y$, $j_*\tilde{\alpha} + f^*\beta = 0$. We prove that $(\tilde{\alpha}, \beta) = \Phi(g_*\tilde{\alpha})$

$\tilde{\alpha}' := \tilde{\alpha} - c_{d-1}(E) \cap g^*g_*\tilde{\alpha} = 0$. (*)

$\beta = -i_*g_*\tilde{\alpha}$ (**)

(**) is by commutativity $-i_*g_*\tilde{\alpha} = -f_*j_*\tilde{\alpha}$, and apply f_* to $\Rightarrow f_*j_*\tilde{\alpha} + f_*f^*\beta = 0$ $\boxed{f_*f^*\beta = \beta}$, done

$\Downarrow$ g is neither flat (nor proper) in general, here we need assume l.c.i. due to $\exists E$ on Y, s.t. $Bl_X Y$ [illegible]

(*) $g_*\tilde{\alpha}' = g_*\tilde{\alpha} - g_*(c_{d-1}(E) \cap g^*g_*\tilde{\alpha})$

$= g_*\tilde{\alpha} - g_*\tilde{\alpha} = 0$

$\Rightarrow f^*i_*\tilde{\alpha}' = f^*0 = 0$

and $j_*\tilde{\alpha}' = j_*\tilde{\alpha} - j_*(c_{d-1}(E) \cap g^*g_*\tilde{\alpha}) = j_*\tilde{\alpha} - f^*i_*g_*\tilde{\alpha} = j_*\tilde{\alpha} + f^*\beta = 0$

Note that $\tilde{\alpha}' = 0$ implies $\tilde{\alpha}' = 0$ by the Chow ring of projective bundle admits unique representation of [illegible] but here set $\tilde{\alpha}' = \sum \zeta^i \cap g^*\alpha_i$ [illegible] due to $f^*i_*\alpha = c_d(N) \cap \alpha$ [illegible]

$\Rightarrow 0 = g_*\tilde{\alpha}' = \alpha_{d-1}$ [illegible] two representation [illegible]

$0 = f^*j_*\tilde{\alpha}' = -\sum \zeta^{i+1} \cap g^*\alpha_i$ $\Rightarrow$ all $\alpha_i = 0$

Left to check the [illegible] formula: assume $\beta = [V]$, V subvar $\subset Y$

[redacted]

When $V \subset X$, $f_*f^*[V] = f_*f^*i_*[V] = f_*j_*(c_{d-1}(E) \cap g^*[V])$ by first claim

$= i_*g_*(c_{d-1}(E) \cap g^*[V]) = i_*[V] = [V]$

By $f^*[V] - [\tilde{V}]$ is supported by $\tilde{X}$ $\Rightarrow$ $\exists \tilde{\alpha}$ and its due to [illegible] is exceptional divisor

Otherwise we can take the strict transform $\tilde{V} \Rightarrow f_*f^*[V] = f_*([\tilde{V}] + j_*\tilde{\alpha})$, $\exists \tilde{\alpha} \in A_k(\tilde{X})$ supported by $\tilde{V} \cap \tilde{X}$ $f^*[V]$ supported by $\tilde{X}$

$= f_*[\tilde{V}] + f_*j_*(\tilde{\alpha}) = [V] + i_*g_*\tilde{\alpha} \Rightarrow g_*\tilde{\alpha}$ supported on $V \cap X$, $\dim(V \cap X) < \dim X = k \Rightarrow g_*\tilde{\alpha} = 0 \in A_k(X)$ ▨. ①② are due to same reason, I both don't know [illegible] refers to Prop 6.2[illegible]

Prop 18. A immediate corollary of the first claim $f^*i_*\alpha = j_*(c_{d-1}(E) \cap g^*\alpha)$ is when both X, Y are smooth

$\Rightarrow f^*i_*\alpha = c_{d-1}(N_{\tilde{X}}\tilde{Y}) \cap \alpha$. One make use of this to give a useful intersection formula: for $\alpha_1, \alpha_2 \in A^*X$, then $i_*\alpha_1 \cdot i_*\alpha_2 = i_*(i^*i_*\alpha_1 \cdot \alpha_2) = i_*(c_{d-1}(N_X Y) \cap \alpha_1 \cdot \alpha_2)$, one can also call it key formulae.

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② As we proved the [illegible] formula, we consider two cases: $V \subset X \Rightarrow \tilde{V}=\emptyset \Rightarrow$ it's equivalent to key formula ①;
By (**) & (***)'s argument in ①, we know it suffices | $V \not\subset X$ is what we need to do, set $W = V \cap X \subsetneq V$
to check after applying f_* and j^*

- Applying f_* is easier: $f_* g^*[\tilde{V}] = [V]$; and $f_*[\tilde{V}] + f_* j_* \{c(E) \cap g^* s(V\cap X, V)\}_k = [V] + i_* g_* \{c(E) \cap g^* s(V\cap X, V)\}_k$
$= [V] + i_* \{c(E) \cap s(V\cap X, V)\}_k$, $s(V\cap X, V) \in A_*(V\cap X) \to A_*(V)$, $\dim V < k \Rightarrow$ latter term $= 0$, done;
- Applying j^*: $j^* f^* [V] = g^* i^* [V] = g^* \{c(N) \cap s(W,V)\}_{k-d} = \{c(g^*N) \cap g^* s(W,V)\}_{k-d}$, as $\tilde{X} \xrightarrow{g} X$ and $\tilde{X} = \mathbb{P}(N)$ is projective bundle $\Rightarrow s(W,V) = g_*(\sum \zeta^i [\tilde{W}])$ by the structure of Chow ring of $\tilde{X}$ (definition of Segre class); $\zeta = c_1(\mathcal{O}_N(1))$
$\Rightarrow j^* f^* [V] = \{c(g^*N) \cap g^* g_*(\sum \zeta^i [\tilde{W}])\}_{k-1}$

On the other hand, $j^*[\tilde{V}] + j^* j_* \{c(E) \cap g^* s(V\cap X, V)\}_k = [\tilde{W}] + j^* j_* \{c(E) \cap g^* s(W,V)\}_k \xrightarrow{\text{Whitney}} [\tilde{W}] + \{c(g^*N)\, c(\zeta - 1)^{-1} \cap g^* s(W,V)\}_k = [\tilde{W}] + \{c(g^*N)(\zeta)^{-1} \cap g^* s(W,V)\}_{k-1}$ [illegible]
$= [\tilde{W}] - \{c(g^*N) \sum \zeta^i) \cap g^* g_*(\sum \zeta^i [\tilde{W}])\}_{k-1}$ [illegible] and $(-\zeta)^{-1} = \sum_{i \geq 1} \zeta^i$ as formal power series

We check $[\tilde{W}] = \{c(g^*N) \sum \zeta^i \cap g^* g_*(\sum \zeta^i [\tilde{W}])\}_{k-1}$ (*)

By Eg. 9. projective bundle $[\tilde{W}] \in A_*\tilde{X}$, $[\tilde{W}] = \sum \zeta^i \cap g^* a_i$, assume that $[\tilde{W}] = \zeta^e \cap g^* a$, $e \leq d-1$
then (*) turns into $\zeta^e \cap g^* a = \{c(g^*N) \sum \zeta^i \cap g^* g_*(\sum \zeta^i \cap g^* a)\}_{k-1}$ ▨

Note that $g_*(\sum \zeta^i \cap g^* a)$ just $s(N) \cap a$ by definition $\Rightarrow$ RHS of (*) $= \{\sum \zeta^i \underline{c(g^*N)\frac{1}{s(g^*N)}} \cap g^* a\}_{k-1} = \{\sum \zeta^i \cap g^* a\}_{k-1} = [\tilde{W}]$

Eg. 11 (Chow ring & intersection multiplicity of blowing up)

$\tilde{X} \xrightarrow{j} \tilde{Y}$. Due to $\tilde{X} = \mathbb{P}N$ exceptional $\Rightarrow A^*(\mathbb{P}N)$ can be recovered   Set both X, Y smooth
$g\downarrow \quad \square \quad \downarrow f$ from $A^*(X)$ and $N_X Y$
$X \xrightarrow{i} Y$ It's natural to ask what the $A^*(\tilde{Y}) = A^*(\mathrm{Bl}_X Y)$ is via $A^*(\tilde{X})$, $A^*(Y)$, $A^*(X)$.

By Prop 10', it's almost done $\{A^*(\tilde{Y}) = \mathbb{Z}\{f^* A^*(Y), j_* A^*(\tilde{X})\}$ as abelian group (By $\beta = f^* f_* \tilde{\beta} + j_* \tilde{\alpha}$)
Thus we had clear about it. | Relations are given by $\Psi \circ \Phi(\alpha) = 0$ (By exactness)
We consider two concrete | Multiplication table is
examples

	$f^*\beta_1$	$j_*\tilde{\alpha}_1$
$f^*\beta_2$	$f^*(\beta_1\beta_2)$	$j_*(\tilde{\alpha}_1 \cdot g^* i^* \beta_2)$
$j_*\tilde{\alpha}_2$	$j_*(\tilde{\alpha}_2 \cdot g^* i^* \beta_1)$	$-j_*(\zeta \cap \tilde{\alpha}_1 \tilde{\alpha}_2)$

(By several computations in ①) ▨

① $\mathrm{Bl}_p(\mathbb{P}^n)$:

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Bl}_p(\mathbb{P}^n) \arrow[d, "\pi"] \\ \{p\} \arrow[r, "i"] & \mathbb{P}^n \end{tikzcd}$$

$A^*(\mathrm{Bl}_p(\mathbb{P}^n)) = \mathbb{Z}[[E],[H]]/(\cdots)$, denote $h = [H]$ hyperplane $\subset \mathbb{P}^n$, $e = [E]$ exceptional divisor
as $E \cong \{p\} \times \mathbb{P}^{n-1}$, thus $A^*(E)$ generated by $[E]$ (not view it as $\mathbb{P}^{n-1}$, but as the projective bundle!)
Figure out its relations: • Our moving H most away from p: $eh = 0$
• What the "away from" means? due to this restriction we can't easily write $h^n = 0$: here set $\mathrm{Bl}_p(\mathbb{P}^n) \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$
set $H = \tilde{H} \cap \Lambda$, $\Lambda \subset \mathrm{Bl}_p(\mathbb{P}^n)$ divisor isomorphic to $p_1(\Lambda)$ (away from E), then $h = [H]$
• e has no relation itself

$\Rightarrow A^*(\mathrm{Bl}_p(\mathbb{P}^n)) = \dfrac{\mathbb{Z}[e,h]}{(eh, h^n + (-e)^n)}$ done ▨

$e = [H] - [\tilde{H}]$
as $p_1(\tilde{H}) \cong \tilde{H}$, and $p_2(\tilde{H}) = \tilde{H}''$ is hyperplane on $\mathbb{P}^{n-1}$ $\Rightarrow [\tilde{H}]^n = 0$
$\Rightarrow (h-e)^n = 0 \overset{eh=0}{\Longleftrightarrow} h^n + (-e)^n = 0$

② $\mathrm{Bl}_C(\mathbb{P}^3)$, $C \subset \mathbb{P}^3$ is smooth curve, $\deg(C) = d$, $g(C) = g$.

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "\pi|_E"'] \arrow[dr, phantom, "\square"] & \mathrm{Bl}_C(\mathbb{P}^3) = B \arrow[d, "\pi"] \\ C \arrow[r, hook, "i"] & \mathbb{P}^3 \end{tikzcd}$$

Generators $\begin{cases} h \in A^1(\mathbb{P}^3) \text{ hyperplane (identified with } \in A^1(B)) \\ e = [E] \\ F_D = (\pi|_E)^* D,\ D \in Z^1(C) \text{ is divisor } \sum n_i [P_i],\ F_D = \sum n_i [F_{P_i}] \text{ union of fibres} \end{cases}$

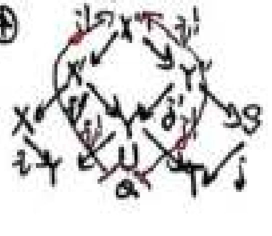
Note that here different with ① as $E = \mathbb{P}N$ not trivial, thus relation $e^2 = -j_* \zeta$ exists, but here over $A^*(\mathbb{P}^3)$, not needed
$\Rightarrow A^*(B) = \dfrac{\mathbb{Z}[e,h] \oplus \langle F_D \mid D \in A^1(C) \rangle}{(h^2 F_D = 0,\ h^2 e = 0)}$ Both relation as same as ① ▨

Revisit refined Gysin map / Virtual pullback:

Recall our setting $X' \xhookrightarrow{i} Y'$, $C_{X'}Y' \subseteq N$ obstruction bundle, $(i^!)_N : A_*Y' \xrightarrow[\text{specializing}]{\sigma} A_*(C_{X'}Y') \xrightarrow{\iota_*} A_*N \xrightarrow{(\pi^*)^{-1}} A_*X[-r]$
and when $N = N_X Y$, $X' \xhookrightarrow{i} Y'$ $\qquad \pi \downarrow$ rank $N = r$
we denote it $\qquad \downarrow \quad \square \quad \downarrow \qquad X$
$i^! = (i^!)_{N_X Y}$ Thus $\square^!$: $X \xhookrightarrow{i} Y$ regular $\Rightarrow$ virtual pullback
obstruction theory

All of this section focus on $N = N_X Y$ case, general (perfect) obstruction theory postpone to last.

Prop 13. (Refined Gysin map) ① (Proper pushforward) p proper, $i^! p_* \alpha = q_* i^! \alpha$ $\Rightarrow$ q as restriction also
② (Flat pullback) p flat, $i^! p^* \alpha = q^* i^! \alpha$ $\Rightarrow$ q also
③ (Compatibility) i' also regular with same codim, $i^! \alpha = (i')^! \alpha$
④ (Commutativity) $j^! i^! \alpha = i^! j^! \alpha$, i & j both regular
⑤ (Functoriality) $(ji)^! \alpha = i^! j^! \alpha$
Both i & j regular $\Rightarrow$ ji also

④ 

⑤
$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

①-③ 

Note that different $i^!$ has different meaning: left is by the black bottom square, right is by the red one.
Interchange the obstruction theory
Rk. All of our pf return back to the deformation space in Fulton's deformation to normal cone.

Pf. ① Assume $\alpha = [V'']$, $i^! p_* [V''] = i^!(\deg(V''/V)[V'])$ $(V' = p(V''))$ $= \deg(V''/V)(X \cdot V') = \deg(V''/V)\{c(g^*N) \cap s(X \cap V', V')\}_{k-d}$ Page 8
$= \{c(g^*N) \cap g_{*}s(X'' \cap V'', V'')\}_{k-d} = g_* \{c(g''^* g^* N) \cap s(X'' \cap V'', V'')\}_{k-d} = g_*(X \cdot V'') = g_* i^! [V'']$ (It's wrong, it should be $(g \circ g)$, here we abuse notation ~~[illegible]~~ here)

② Assume $\alpha = [V]$, $i^! \varphi^* [V] = i^! [V'']$ $(V'' = \varphi^{-1}(V)) = (X \cdot V'') = \{c(g^*N) \cap s(X \cap V'', V'')\}_{k-d} = \{c(g^*N) \cap g^* s(X \cap V, V)\}$
$= g^* \{c(g_* g^* N) \cap s(X \cap V', V')\}_{k-d} = g_*(X \cdot V') = g_* i^! [V']$

Thus for ①, ② respectively, we only to prove the red formula: the pushforward & pullback of Segre class

①′ $g_* s(W'', V'') = \deg(V''/V) \cdot s(W', V')$

②′ $g^* s(W', V') = s(W'', V'')$

Rk. Without properness ①′ also makes sense: $f_*[Y] = \sum \deg(Y/\overline{f(Y)})_i [\overline{f(Y)}_i]$, we set desired number $\deg(f(Y)/Y) := \sum m_i \deg(\overline{f(Y)}_i/Y)$, m_i is geometric multiplicity of $\overline{f(Y)}$ on each component $\overline{f(Y)}_i$

①′ $g_* s(W'', V'') = g_* g''_* (\sum c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C'' \oplus 1)])$ (By definition, $g'': \mathbb{P}(C'' \oplus 1) \to W''$, $C'' = C_{W''}V''$)

$= g'_* G_* (\sum c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C'' \oplus 1)])$ By the square

$$\begin{tikzcd} \mathbb{P}(C'' \oplus 1) \arrow[r, "G"] \arrow[d, "g''"] & \mathbb{P}(C' \oplus 1) \arrow[d, "g'"] \\ W'' \arrow[r, "g"] & X' \end{tikzcd}$$

$= g'_* (\sum c_1(G_* \mathcal{O}(1))^i \cap G_* [\mathbb{P}(C'' \oplus 1)])$

$= g'_* (\sum c_1(\mathcal{O}(1))^i \cap G_* [\mathbb{P}(C'' \oplus 1)])$ By G pushforward

$\overset{(*)}{=} g'_* (\sum c_1(\mathcal{O}(1))^i \cap \deg(V''/V')[\mathbb{P}(C' \oplus 1)])$ $\mathcal{O}_{\mathbb{P}(C'' \oplus 1)}(1)$ to $\mathcal{O}_{\mathbb{P}(C' \oplus 1)}(1)$

$= \deg(V''/V') g'_* (\sum c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C' \oplus 1)]) \overset{\text{def}}{=} \deg(V''/V') s(W', V')$

$(*)$ is due to $F_*[M''] = \deg(V''/V')[M']$ by definition of proper pushforward
$\Rightarrow G_* [\mathbb{P}(C'' \oplus 1)] = [\mathbb{P}(C' \oplus 1)] \deg(V''/V')$ ▨

$$\begin{tikzcd} W'' \arrow[r] \arrow[d] \arrow[dd, bend right] \arrow[dr, phantom, "\square"] & Y'' \arrow[d] \arrow[dd, bend left] \\ X'' \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & Y'' \arrow[d, "p"] \\ X' \arrow[r] & Y' \end{tikzcd}$$

Assume $W'' = X''$
$V'' = Y''$

by replace g, p by the composition red

Thus here

$$\begin{tikzcd} W'' \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & V'' \arrow[d, "p"] \arrow[r, Rightarrow, "\text{Deformation}"] & M'' \arrow[d, "F"] \\ X' \arrow[r] & Y' \arrow[r, Rightarrow, "\text{Deformation}"] & M' \end{tikzcd}$$

$F^* \mathbb{P}(C' \oplus 1) = \mathbb{P}(C'' \oplus 1)$

$F|_{\mathbb{P}(C'' \oplus 1)} = G: \mathbb{P}(C'' \oplus 1) \to \mathbb{P}(C' \oplus 1)$

②′ is similar, as $F^*[M'] = [M'']$ $\Rightarrow G^*[\mathbb{P}(C' \oplus 1)] = [\mathbb{P}(C'' \oplus 1)]$, then computation is same ▨

③ It says that ~~[illegible]~~ (composition of) morphism of regular embedding not change their destruction.
View it as elements ∈ optional Chow ring, this deduces that the "relative" rational equivalence

$G(i)^![V''] = (X' \cdot V'') = \{c(g^* N') \cap s(X'' \cap V'', V'')\}_{k-d}$
$= \{c(g^* g^* N) \cap s(X'' \cap V'', V'')\}_{k-d}$
$= \{c((g \circ g)^* N) \cap s(X'' \cap V'', V'')\}_{k-d} = i^! [V'']$ done
It's due to $N' = \mathcal{I}/\mathcal{I}^2 \twoheadleftarrow g^*(\mathcal{I}/\mathcal{I}^2)$ has same rank vector bundles $\Rightarrow$ iso ▨

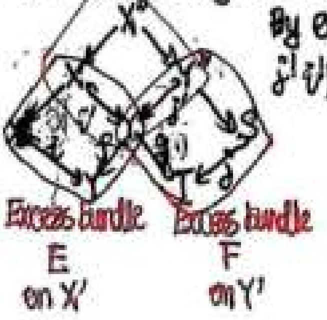
For ② case, our assumption is

$$\begin{tikzcd} X'' \arrow[r] \arrow[dr, phantom, "\square"] & Y'' \arrow[d] \\ W' \arrow[r] & V' \end{tikzcd}$$

and we abuse notation as in ①, $W'' \xrightarrow{g''} X$

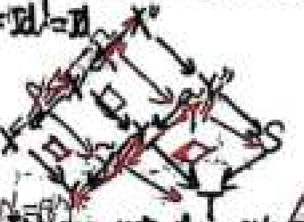
④ Assume $\alpha = [V']$, $V' \hookrightarrow Y'$ is proper and $Bl_{X'}Y' \to Y'$ two blow-downs are also proper
we reduce via these proper maps to $Bl_{Y''}Y' \to Y'$
$X', Y'' \subset Y'$ are Cartier divisors, $\alpha = [Y']$ the fundamental class

This is by given and proper map $\tilde{Y}' \to Y'$, we can use ① to pushforward: $\exists \tilde{\alpha}$, $h_* \tilde{\alpha} = \alpha$, then h_* commutes all $i^!$, $j^!$, done.
But note that a exceptional case is $X' = Y'$, then by excess intersection formula, as $G(i)^! = \mathrm{Id}^! = \mathrm{Id}$
then $i^!$ is just $c_d(g^*N) \cap \alpha$, then commutes done. Hence we done reduction.

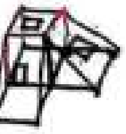


By excess intersection formula.
$j^! i^! [Y'] = c_{e}(F) \cap G(j)^! i^! [Y']$
$= c_e(F) \cap G(j)^! (c_{d-1}(E) \cap G(i)^! [Y'])$
$= c_e(F) \cap c_{d-1}(E) \cap G(j)^! G(i)^! [Y']$ (c_e & c_{d-1} are their top Chern class)
$= c_e(F) \cap c_{d-1}(E) \cap G(j)^! [X']$
$= c_e(F) \cap c_{d-1}(E) \cap \boxed{[Y''][X']}$
$= c_e(F) \cap c_{d-1}(E) \cap \boxed{[X'][Y'']}$
$= \cdots = c_e(F) \cap c_{d-1}(E) \cap G(i)^! G(j)^! [Y']$ ▨

Here $G(i)^!$ not pullback of Y'' to X'' but Y' to X' itself, i.e. $G(i)^! = \dots$ is refined in the top Cartesian diagram



It's may not clear, it's



It can't use commutivity of intersection product due to it need ④ conversely, it's cyclic
But in fundamental case we have a algebraic proof independent
is zero section

⑤ Assume $\alpha = [V']$, $V' \subset Z'$
we can assume $V' = Z'$, i.e. $\alpha = [Z']$ irreducible by pushforward ①, also by pushforward, assume Y' also irreducible then.
We prove it by two steps: (i) Reduce to $(Z' \to Z) = (g^*E \to E)$; (ii) Prove the bundle case.

(i)

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Z' \arrow[d] & N_{Y'}Z' =: M' \arrow[d] \\ Y \arrow[r] & Z & N_Y Z =: M \end{tikzcd}$$

deformation $N_{Y'}Z' =: M'$, and $X \hookrightarrow Y$ induce Cartesian diagram

$$\begin{tikzcd} X' \times \mathbb{P}^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & M' \arrow[d] \\ X \times \mathbb{P}^1 \arrow[r, hook, "\text{regular}"] & M \end{tikzcd}$$

Consider the fibre $t \in \mathbb{P}^1$
we take a look at two kinds of fibres $t \neq \infty$ and $t = \infty$

$$\begin{tikzcd} X' \times \{t\} \arrow[r] \arrow[d] & M'_t \arrow[d] \\ \boxed{X \times \{t\} \xrightarrow{(i \times \mathrm{Id})_t} M_t} \arrow[d] & \\ X \times \mathbb{P}^1 \arrow[r, "(i \times \mathrm{Id})"] & M \end{tikzcd}$$

regular
regular

$t = \infty$, it's $\tilde{j} i$, $\tilde{j}: Y \to N_Y Z$ zero section
$t \neq \infty$, it's $j i$

$\Updownarrow$ $\Updownarrow$
$(ji)^! [Z'] \underset{\text{of } t \in \mathbb{P}^1}{\overset{\text{invariance}}{=}} (\tilde{j}i)^! [Z']$
to complete our proof

$\Rightarrow (i \times \mathrm{Id})^! (\varphi_t)^! [M'] \overset{③}{=} (i \times \mathrm{Id})_t^! [M'_t] = \begin{cases} (ji)^! [Z'] \\ (\tilde{j}i)^! [C_{Y'}Z'] \end{cases} \underset{\text{case (ii)}}{\overset{\text{Special}}{=}} i^! \tilde{j}^! [C_{Y'}Z'] = i^! j^! [Z']$

$\varphi_t: \{t\} \to \mathbb{P}^1$ induces $(\varphi_t)^!$ by $M'_t \to \{t\}$
$\varphi_t^! [M'] = [M'_t] = \begin{cases} [Z'], & t \neq \infty \\ [C_{Y'}Z'], & t = \infty \end{cases}$ $M' \to \mathbb{P}^1$

Invariance of t:
By ④, $(i \times \mathrm{Id})^! \varphi_t^! [M'] = \varphi_t^! (i \times \mathrm{Id})^! [M']$
$= \varphi_t^! \beta$, $\beta \in A_*(X' \times \mathbb{P}^1)$ and $\varphi_t^! \beta \in A_*(X' \times \{t\})$
By definition of rational equivalence (deformation) $\Rightarrow \varphi_t^! \beta$ not depends on t ▨

(50)
$$\begin{tikzcd} Y' \arrow[r, hook] \arrow[d, "g"'] \arrow[rd, phantom, "\square"] & E' = g^*E \arrow[l, bend left, "\pi'"] \arrow[d] \\ Y \arrow[r, hook] & E \arrow[l, bend left, "\pi"] \end{tikzcd}$$

Recall our definition $i^![Y'] = s^*[C']$.
Here our setting is $h^*N_XE \xrightarrow{\pi} h^*N_XY \longrightarrow X'$ projections
and by Thom isomorphism $\Rightarrow$ π'^*, π^* are converse to s^*
$\Rightarrow$ $[C_{X'}Y'] = \pi^* i^![Y']$
$[C_{X'}E'] = (\pi')^*(g_0)^![E']$
$\Big\} \Rightarrow (g_0)^![E'] = i^![Y'] = i^!j^![E']$ done
① $\pi^*[C_{X'}Y'] = [C_{X'}E']$

It left to ask ①②
as $h^*N_XY \subset h^*N_XE$ is obvious
but we should still show that h^*N_XE splites: We prove $\boxed{C_{X'}E' \simeq C_{X'}Y' \times_{X'} \pi^*E'} \Rightarrow h^*N_XE = h^*N_XY \oplus (i')^*E'$ done.
By the definition of cone & $E' = \mathrm{Spec}(\mathrm{Sym}^\bullet \mathcal{E}^\vee)$. Set $\mathcal{I} = \mathcal{I}_{X'}Y'$, then $\mathcal{J} = \mathcal{J}_{X'}E' = \mathcal{O}_{E'}[\mathcal{I}, \mathrm{Sym}^1\mathcal{E}^\vee] \Longrightarrow \mathcal{J}^n = \mathcal{I}^n \oplus \mathcal{I}^{n-1}\mathrm{Sym}^1 \ldots$
$\Rightarrow \bigoplus_n (\mathcal{J}^n/\mathcal{J}^{n+1}) = \bigoplus_n [\mathcal{I}^n/\mathcal{I}^{n+1} \oplus [\mathcal{I}^{n-1}/\mathcal{I}^n \otimes \mathrm{Sym}^1\mathcal{E}^\vee] \oplus [\mathcal{I}^{n-2}/\mathcal{I}^{n-1}] \otimes (\mathrm{Sym}^2\mathcal{E}^\vee)] \oplus \ldots \oplus [\mathcal{O}_{Y'}/\mathcal{I} \otimes \mathrm{Sym}^n\mathcal{E}^\vee]$ as $\ldots \oplus \ldots \mathrm{Sym} \ldots \oplus \mathrm{Sym}^{n+1}\mathcal{E}^\vee \oplus \ldots$
$= (\bigoplus \mathcal{I}^n/\mathcal{I}^{n+1}) \otimes (\mathrm{Sym}^\bullet\mathcal{E}^\vee) \otimes \mathcal{O}_{X'}$ ▨

Refined Gysin map $\xrightarrow{\ \ \ }$ Virtual pullback $\Rightarrow$ equivalence
$\longleftarrow$ [illegible] by following

Note that for general $\mathfrak{C}_{X/Y}$, N obstruction bundle [illegible]
[illegible] then $X' \xrightarrow{i'} Y'$, π^*E' is vector bundle on Y'
$\Rightarrow (i')^!_{\mathfrak{C}} = i^!$ can be constructed, i.e. the obstruction bundle has $\boxed{\text{Geometrization}}$ to $X' \hookrightarrow E'$
It's wrong! We may have virtual pullback not refined Gysin map
zero locus $\longrightarrow E'$
when $\pi_* E'$ admits a global section, restrict to $X' \hookrightarrow E'$
$X' \xhookrightarrow{i'} Y'$ normal cone $C_{X'}Y'$
$\boxed{X' \hookrightarrow Y' \hookrightarrow E'}$ normal bundle E'

~~Remark~~

Bivariant intersection theory (We introduce this for stack $\Big\{$ intersection theory: by def. it provides singular: a formalism of various formulas before. It admits a singular GRR, see [Fu, Chap 18], we omit it.
It's a relative/singular generalisation of usual Chow ring
Def⁹: (Bivariant group) Any morphism $f: X \to Y$ of schemes and $g: Y' \to Y$ gives Cartesian diagram $X' \xrightarrow{f'} Y'$. Then $A^p(X \xrightarrow{f} Y) =: A(f)$ It can be done by descent result as [Fu, Thm 17.1] $X \xrightarrow{f} Y$ I omit it
$= \Big\{ c = (c_g^{(k)})_{(g,k)} \Big| \boxed{c_g^{(k)} : A_k Y' \longrightarrow A_{k-p} X'}$ all k and all (g, Y'), $c_g^{(k)}$ satisfy compatibility conditions $\Big\}$
(One sometimes write $c(\alpha)$ as $c \cap \alpha$ to distinguish with total Chern class)
Where compatibility conditions are: (C₁) $c_g^{(k)}(h_*\alpha) = h_* c_{gh}^{(k)}(\alpha)$, h proper pushforward as

$$\begin{tikzcd} X'' \arrow[r] \arrow[d, "h'"] \arrow[rd, phantom, "\square"] & Y'' \arrow[d, "h"] \\ X' \arrow[r] & Y' \end{tikzcd}$$

(C₂) $c_{gh}^{(k+d)}(h^*\alpha) = (h')^* c_g^{(k)}(\alpha)$, h flat pullback, relative dim $= d$
(C₃) $i^! c_g \alpha = c_{g'}(i^!\alpha)$, that is compatible with intersection product

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] \arrow[rd, phantom, "\square"] & Y'' \arrow[r] \arrow[d, "i'"] \arrow[rd, phantom, "\square"] & Z'' \arrow[d, "i"] \\ X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \\ X \arrow[r] & Y & \end{tikzcd}$$

i regular with codim $= c$

$\boxed{\text{(C₁)–(C₃) are axioms of bivariant theory; I emphasize they has for (co)homology due to Poincaré dual}}$

(• To avoid set-theoretic issues (as here all Y' is a large set), one should given fixed universe) with group structure by (again) freely generated over $\mathbb{Z}$
Ex. Thus it's easy to see $i^! \in A^c(X \xrightarrow{i} Y)$ the refined Gysin map is a bivariant class and $A^{-k}(X \longrightarrow \mathrm{Spec}\,k) \simeq A_k(X)$, X over k scheme by the map $c \overset{\varphi}{\mapsto} c([\mathrm{Spec}\,k])$ as c lift dimension k; thus $[\mathrm{Spec}\,k]$ as the fundamental class of a k-point (0-cycle) sends to k-cycle
We call $i^! = [i^!_\mathfrak{C}]$ the canonical class/orientation (Not all f has canonical class! Different with fundamental class)

as we checked in Prop 13 ①②④ corresponds to (C₁)–(C₃) ▨

$$\begin{tikzcd} X \arrow[r] & \mathrm{Spec}\,k \\ X \times_k Y \arrow[r] \arrow[u] \arrow[ru, phantom, "\square"] & Y \arrow[u, "g"] \end{tikzcd}$$

$A_{i+k}(X \times_k Y) \xleftarrow{c_g} A_i(Y)$
$x \times_k y \longleftarrow y$
$\psi: x \longmapsto \ldots$

One should check that
① $c_g^{(k)}$ is bivariant class ($x \times_k y = x \times y$ the product, thus obvious)
② They're converse to each others: $c[V] = c(p_V^*[\mathrm{Spec}\,k]) = (p|_V)^* c[\mathrm{Spec}\,k] = [V] \times c([\mathrm{Spec}\,k])$
(C₂) $= (\varphi \circ \psi)(c)[V]$
Here we can assume $\alpha = [V]$ by (C₁) ▨ $\Rightarrow \psi \circ \varphi = \mathrm{Id}$

That's why it's called "bivariant", it's $(\psi \circ \varphi)(x) = \psi(y \mapsto x \times y) = x \times [\mathrm{Spec}\,k] = x \Rightarrow \varphi \circ \psi = \mathrm{Id}$
mixes homology & cohomology / covariant & contravariant functor

And our tasks next are
① Poincaré dual after we set $\boxed{A^*X := A^*(X \xrightarrow{\mathrm{id}} X)}$ (when smooth)
② Ring structure; ③ Reformulation of previous conclusions; as we claimed (④ operational Chow ring almost omitted, as [Tabuada, 2014] says, it's less useful than motivic cohomological tools, I mention its definition here.
$\mathrm{op}A^*X := \mathrm{Im}\Big\{\varinjlim_{f: X \to Y} A^*Y \longrightarrow \prod_{(g: Z \to X)} \mathrm{End}(A_*Z) \Big| \text{induced by } A^*Y \times A_*Z \xrightarrow{f^*} A_{*-*}Z \Big\}$ as a singular generalisation of A^*X: when X smooth $A^*X \cong \mathrm{op}A^*X$
In fact, he proven that $\exists$ map: Motivic $\to \mathrm{op}A^*$
Thus $\mathrm{op}A^*X = \mathbb{Z}[(f, \alpha)] \mid f: X \to Y$, Y smooth quasi-projective, $\alpha \in A^*Y] \big/ \langle (f \circ g)^*\alpha \ne 0, \forall g: Z \to X \rangle$
or given by $ch: K(X) \twoheadrightarrow \mathrm{op}A^*X$ defined as $\mathrm{Im}\begin{pmatrix} K(X) \to A^*\tilde{X} \\ E \mapsto ch(\pi^*E) \end{pmatrix}$, $\tilde{X} \xrightarrow{\pi} X$ is resolution $\Rightarrow \tilde{X}$ smooth; later & all uses given by $\mathrm{op}A^*$
All of these make sense for motivic cohomology, can be proven by motivic, too.
Its relation with bivariant intersection lies that operational theory is a bivariant theory determined by a covariant theory (i.e. homology theory): here is from A_*X to $\mathrm{op}A^*X$. The procedure is similar with motives. For details, see [Fulton, Categorical Framework for the study of singular spaces])

① Def¹⁰. (Bivariant ring) Product $A^p(f) \otimes A^q(g) \xrightarrow{\cup} A^{p+q}(g \circ f)$ gives product on cohomology ring (just as topological setting; no direct way to put multiplication to homology) by $c \otimes d \mapsto c \cdot d$ as the following diagram:

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] \arrow[rd, phantom, "\square"] & Y' \arrow[r, "g'"] \arrow[d, "\psi'"] \arrow[rd, phantom, "\square"] & Z' \arrow[d, "\psi"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$$\begin{tikzcd} A_k Z' \arrow[r] & A_{k-q} Y' \arrow[r] & A_{k-q-p} X' \\ a \arrow[r, mapsto] \arrow[rr, bend right, "(c.d)"'] & da \arrow[r, mapsto] & c\circ d(a) \end{tikzcd}$$

One check that $(c\circ d)$ satisfy (c1)−(c5) easily as $(c\circ d)^{(k)}_{\psi} = (c^{(k-q)}_{\psi'}\circ d^{(k)}_{\psi})$

$\Longrightarrow c^{(k-q)}_{\psi'}\circ d^{(k)}_{\psi}(h^*_k a) = c^{(k-q)}_{\psi'}\, h^*_k\, d^{(k)}_{\psi h}(a) = h^*_k(c^{(k-q)}_{\psi' h}\circ d^{(k)}_{\psi h})(a)$
$= h^*_k(c\circ d)^{(k)}_{\psi h}(a)$

(c2)−(c5) also similar, are both trivial ▨

§5. Other operations: proper pushforward & flat pullback make sense as well:

$f_*: A^p(g\circ f)\to A^p(g)$; $c\mapsto f_*\circ c$ [redacted]

$g^*: A^p(f)\to A^p(f')$ via pullback Cartesian diagram, $c\mapsto c$

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] \arrow[rd, phantom, "\square"] & Y'' \arrow[d] \\ X' \arrow[r, "f'"] \arrow[d] \arrow[rd, phantom, "\square"] & Y' \arrow[d, "g"] \\ X \arrow[r] & Y \end{tikzcd}$$

Both g^*c and c sends $A_k Y''\to A_{k-p}X''$, $a\mapsto c(a)$

(No reason to say it's commutative although for refined Gysin it is)

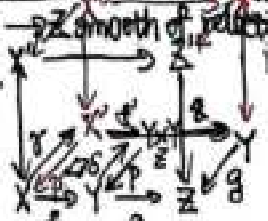
These operations satisfy compatibility conditions: associativity & functoriality, their commutativity and projection formula.

② Prop14. (Poincaré duality) ① $A^pY\cong A_{n-p}Y$; ② Ring structure of A^*Y coincides the older one.

Pf. ① First we need extend our orientation class of regular embedding to flat map (thus l.c.i as by [SGA], we can factor l.c.i to regular embedding & smooth map): for f is flat of relative dim $=n$, then $[f]\in A^{-n}(f)$ determined by $[f]=(f')^*$ as the base change of flat map f also flat, thus $(f')^*$ well-defined. (Thus for l.c.i, one define $[f]=[i].[p]$ where $f=i\circ p$). Then we have a general lemma for proving it:

Lemma A. $g: Y\to Z$ smooth of relative dim $=n$, then $\forall f: X\to Y$, $A^p(f)\xrightarrow{\cdot[g]} A^{p-n}(g\circ f)$ is isomorphism.

Pf of Lemma A.



[redacted] $A_k Z''\xrightarrow{c} A_{k-(p-n)}X''\in A^{p-n}(g\circ f)$

pullback to $g^*c: A_kZ''\to A_{k-(p-n)}X''\in A^{p-n}(X'\to Y)$

Then look at the $\boxed{\text{red Cartesian diagram}}$ induced by diagonal $\delta: Y\to Y\times_Z Y$ is regular [redacted] $\Rightarrow \gamma$ also $\Rightarrow [\gamma]\in A^n(X\to X')$ It's why we need smooth instead of only flat.

thus their product $[\gamma]\otimes g^*c\in A^n(X\to X')\otimes A^{p-n}(X'\to Y)\to A^p(X\to Y)=A^p(f)$

$([\gamma].g^*c)=:L(c)$ determines a converse to $\cdot[g]$

Verification: (all these square are Cartesian)

- $L(c)\cdot[g]$

$=[\gamma].(g^*c.[g])$ (associativity) $=[\gamma].[X'\xrightarrow{f'}X].c$ (pullback $Y\xrightarrow{g}Z$ by $X'\to Y$ [redacted] as $Y\xrightarrow{g}Z$ is smooth)

$=[\psi\circ\gamma]\cdot c=[id]\cdot c=c$

- $L(c\cdot[g])=[\gamma].g^*(c.[g])=[\gamma].p^*c.g^*[g]$ (By compatibility of pullback and $p: Y\times_Z Y\to Y$ naturally)

$=((f')^*[\delta].p^*c).g^*[g]=((p\circ\delta)^*c.[\delta])$ [redacted] $g^*[g]$

(when describe compatibility in relative setting, don't simply write but draw a diagram carefully!

$$\begin{tikzcd} A^p(X\to Y)\otimes A^q(Y\to Z) \arrow[r, "\cdot"] \arrow[d, "p^*\otimes g^*"] \arrow[rd, phantom, "\circlearrowleft"] & A^{p+q}(X\to Z) \arrow[d, "g^*"] \\ A^p(X'\to Y\times_Z Y)\otimes A^q(Y\times_Z Y\to Y) \arrow[r, "\cdot"] & A^{p+q}(X'\to Y) \end{tikzcd}$$

)

Here it's due to (a2):

[redacted]

$(f')^*[\delta].p^*c=(p\circ\delta)^*c.[\delta]$ due to $\forall a\in A_k(Y'')$ [redacted]

$((f')^*[\delta].p^*c)(a)=(f')^*[\delta](p^*c(a))=$ [redacted] $\delta^!$ [redacted] $c(a)\overset{(a2)}{=}(c\circ\delta^!)(a)$.

and $((p\circ\delta)^*c.[\delta])(a)=c([\delta](a))=c\circ\delta^! a$ done.

$=$ [redacted] $(p\circ\delta)^*c.[\delta]).[g_2]=(p\circ\delta)^*c.[\delta\circ g]=$ [redacted] $(id)^*c.$ [redacted] $=c$ [redacted] ▨

Then we apply it to the case $A^p(X\xrightarrow{id}X)\xrightarrow{\cdot[X\to\operatorname{Spec}k]}A^{p-n}(X\to\operatorname{Spec}k)$ is this isomorphism ▨

② We check that $L(\alpha.\beta)=L(\alpha).L(\beta)$

$L(\alpha.\beta)=L(\delta^*(\alpha\times\beta))=([\gamma].g^*\delta^*(\alpha\times\beta))=$ [redacted] $\gamma^!(\alpha\times\beta)$.

(here $\alpha\times\beta$ denotes the bivariant class $x\mapsto x\times(\alpha\times\beta)$, so is α,β resp.) Here we can abuse notation of operations on cycle & bivariant class as the isomorphism of homology is functorial.

$L(\alpha).L(\beta)=([\gamma].g^*\alpha).([\gamma].g^*\beta)=(\gamma^!\alpha,\gamma^!\beta)=\gamma^!(\alpha\times\beta)$ ▨

③ Prop15. (Reformulate) ① (Excess intersection formula)

$$\begin{tikzcd} X' \arrow[r, "f'", "\text{codim}=d'"'] \arrow[d] \arrow[rd, phantom, "\square"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f", "\text{l.c.i, codim}=d"'] & Y \end{tikzcd}$$

(To make it rigorous one may simply think f and f' are just regular embedding as before)

$\Longrightarrow g^*[f]=c_{top}(E).[f']$, E excess bundle

② (Thom isomorphism) $\pi^*: A^p(X\xrightarrow{f}Y)\xrightarrow{\sim}A^p(f^*E\to E)$ for $\forall f: X\to Y$; $\Longrightarrow$ (Projective bundle) $A^p(\mathbb{P}(f^*E)\to\mathbb{P}E)\cong\bigoplus A^{p-i}(X\to Y)$

③ (Key formula/Blowing up formula) We have split SES $0\to A^{p-d}(X'\to X)\to A^{p-1}(\tilde X'\to\tilde X)\oplus A^p(Y'\to Y)\to A^p(\tilde Y'\to\tilde Y)\to 0$

Pf, Rewrite ▨ Now we have completed $\boxed{\text{all}}$ fundamental intersection theories, who not concern abstract framework but concrete enumerative problems can skip to Chap2.

§1.4. Advanced topics from algebraic topology

We had known that Chow grp/rings not satisfy any of our known cohomology theory, in fact, it's part of Borel–Moore homology theory via cycle map (only even dimensional). For smooth case, Borel Moore = motivic Borel Moore = motivic cohom

But for not smooth case it's more complex.

$\boxed{\text{Bloch's formula}}$ ⇕ K-theory of sch $\overset{\text{smooth}}{=}$ K-theory of ... ⇕ $\boxed{\text{Poincaré dual}}$

We'll do all things smooth in this section, we'll first construct Chow motive, as the first pure motive in history, then mimic it (as Hodge done) to give motivic cohomology; finally we give the Borel–Moore homology and Bloch's higher Chow grps.

Construction of Chow motive ① Category of Correspondence $Corr(k)$ ② Effective Chow motives $Chow^{eff}(k)$; ③ $Chow(k)$

① **Correspondence** $X \vdash Y$ generalises morphism of varieties $X \to Y$. When $X = \sqcup X_i$

Def 11 (Correspondence) $\mathrm{Hom}_{\mathrm{Corr}(\mathbb{k})}(X,Y) := A^*(X\times Y)$. • One can also define $\mathrm{Hom}_{\mathrm{Corr}(\mathbb{k})}(X,Y) = Z^*(X\times Y)/\sim$, e.g. numerical, or homological
as dg-algebra$/\mathbb{k}$, we call $A^{\dim X + r}(X\times Y)$ is degree r correspondence.

We need $\sim$ is adequate to intersect, satisfy moving lemma; but in general, result we omit $\sim$ in general, but focus on these three.

Rk. ① For morphisms we have its correspondence as $[\Gamma_f] \in A^*(X\times Y)$, it gives a functor $\mathrm{Var}_{\mathbb{k}} \to \mathrm{Corr}(\mathbb{k})$ (Here all Var are smooth proj)
② Thus we have its horizontal categorification $\mathrm{Corr}^*(\mathbb{k})$ as a dg-category, with the problem is composition are given by
$A^*(X\times Y)\otimes A^*(Y\times Z) \to A^*(X\times Z)$ I omit its associativity... as they're really easy.
$\alpha\otimes\beta \longmapsto p_{XZ*}(p_{XY}^*\alpha \cdot p_{YZ}^*\beta)$ and we not concern all its classical applications, only view it as tool.
③ One sometimes define $\mathrm{Corr}^*(\mathbb{k})$ via cycle map $cl: A^r(X\times Y) \to H^{2r}(X\times Y)$ and it preserves product $\Rightarrow$ composition
$H^*(X\times Y)\otimes H^*(Y\times Z) \to H^*(X\times Z)$ where H^* are Weil cohomology theories, not matters desired motive.
$\alpha\otimes\beta \longmapsto p_{XZ*}(p_{XY}^*\alpha \cup p_{YZ}^*\beta)$ (later we'll see cl defined in BM homology more proper and study it in detail)

But when we need discuss its universal property we need Weil cohomology.

The red parts are categorical properties expected by Grothendieck's motive.

④ $\mathrm{Corr}^*(\mathbb{k})$ ~~[illegible]~~, when we set $X\oplus Y := X \sqcup Y$ ($X\otimes Y := X\times Y$), we expect gives "Abelian" category better.
this is   is thus additive$/\mathbb{k}$ our next step and will give a first universal property.

this expects the last step we need monoidal structure as one of Grm's motive for Weil cohomology acquires Künneth formula

② **Effective pure motives** (We denote $M^{\mathrm{eff}}(\mathbb{k}) = \mathrm{Chow}^{\mathrm{eff}}(\mathbb{k})$)
Set $M^{\mathrm{eff}}(\mathbb{k}) := \mathrm{Corr}^0(\mathbb{k})^\#$, where $\#$ is the Karoubian envelope (or idempotent completion, or pseudo-abelian hull)
and $\mathrm{Corr}^0(\mathbb{k})$ is the horizontal categorification of $\deg = 0$ correspondence

Def 12. (Karoubian category & Karoubian envelope)
An additive category is Karoubian if all its idempotent endomorphism p is split, i.e. $p = f\circ g$, where $g\circ f = \mathrm{Id}_A$
(or idempotent complete cat, or pseudo-Abelian cat) $\Longleftrightarrow A \cong \ker p \oplus \ker(1-p) \Leftrightarrow \ker(1-p)$, i.e. equaliser of (p, Id_A) exists.
For additive category $\mathscr{C}$, its Karoubian envelope $\mathscr{C}^\#$ is defined as
objects: $(A,p) \in \mathscr{C}\times \mathrm{End}\,A$, p is idempotent
morphisms $\{(A,p)\xrightarrow{f}(B,q)\} = \{f: A\to B \mid f\circ p = f\}$
$\mathscr{C} \hookrightarrow \mathscr{C}^\#$ naturally is fully faithful, and when $\mathscr{C}$ Karoubian they're equivalent.
$A \longmapsto (A, \mathrm{Id}_A)$

Thus $\mathscr{C}^\#$ is Karoubian as: $\forall r: (A,p)\to(A,p)$ idempotent, $p\circ r\circ p = r$ $\Rightarrow pr = rp = r$
~~[illegible]~~
$\Rightarrow (rp)\circ(rp) = (rp)$ & $(pr)\circ(pr) = pr$ idempotent
$\Rightarrow (A,p)\xrightarrow{pr}(A,r)\xrightarrow{rp}(A,p)$ splits r ▨

Thus we have

$$\begin{tikzcd} \mathrm{Var}(\mathbb{k}) \arrow[r] \arrow[rr, bend left, "h_\sim"] & \mathrm{Corr}^0(\mathbb{k}) \arrow[r] & M^{\mathrm{eff}}(\mathbb{k}) \end{tikzcd}$$

composed functor
$X \longmapsto (X, \Delta_X)$, where correspondence $\Delta_X = [\Gamma_{\mathrm{id}_X}]$ thus idempotent, we denote it $[X]$
$(f: X\to Y) \longmapsto (X,\Delta_X) \leftarrow (Y,\Delta_Y) : [\Gamma_f^t]$
And its direct sum $([X],\alpha)\oplus([Y],\beta) = ([X\sqcup Y], \alpha+\beta)$ and tensor product $([X],\alpha)\otimes([Y],\beta) = ([X\times Y], p_X^*\alpha\cdot p_Y^*\beta)$

Prop 16. (Universal property of $h_\sim$) For any Weil cohomology $H^*: \mathrm{Var}(\mathbb{k}) \to \mathrm{Gr}^{\geq 0}\mathrm{Vect}(\mathbb{k})$, we can descend to effective Chow motive $\mathrm{Var}(\mathbb{k}) \xrightarrow{h_\sim} M^{\mathrm{eff}}(\mathbb{k})$. Here $\mathrm{Var}(\mathbb{k})$ is smooth projective

$$\begin{tikzcd} \mathrm{Var}(\mathbb{k}) \arrow[r, "h_\sim"] \arrow[d, "H^*"'] & M^{\mathrm{eff}}(\mathbb{k}) \arrow[dl, "\tilde{H}^*\ \exists !"] \\ \mathrm{Gr}\,\mathrm{Vect}(\mathbb{k}) & \end{tikzcd}$$

as we need to define Weil cohomology.
($\mathrm{Top} \xrightarrow{H^*} \mathrm{Gr}\,\mathrm{Vect}(\mathbb{k})$ not! as in general Top: $\exists\ \mathrm{Gr}^{<0}$ part)

Rk. Thus our left to do is extend to $\mathrm{Gr}^*\mathrm{Vect}(\mathbb{k})$ hole, thus we need find a "inverse" of $\otimes$, thus we need to equip them symmetric monoidal structure of ~~[illegible]~~ finally from effective to all.

Pf. Set $H^*(X) = \bigoplus_k$; $x \mapsto p_{Y*}(p_X^*(x)\cup cl(\alpha))$, thus extend to $H^*: \mathrm{Corr}^0(\mathbb{k}) \to \mathrm{Gr}^{\geq 0}\mathrm{Vect}(\mathbb{k})$ as $\alpha \mapsto \alpha_*: H^*(X) \to H^*(Y) \in \mathrm{Gr}^{\geq 0}\mathrm{Vect}(\mathbb{k})$
We accept the fact that $\mathrm{Gr}^{\geq 0}\mathrm{Vect}(\mathbb{k})$ is Karoubian (It's linear algebra)
$\Rightarrow (H^*)^\#: \mathrm{Corr}^0(\mathbb{k})^\# \to (\mathrm{Gr}^{\geq 0}\mathrm{Vect}(\mathbb{k}))^\# = [\tilde{H}^*: M^{\mathrm{eff}}(\mathbb{k}) \to \mathrm{Gr}^{\geq 0}\mathrm{Vect}(\mathbb{k})]$ desired ▨

③ **Chow motive / Grothendieck motive $M(\mathbb{k})$**

Eg. 18 ($\mathbb{P}^1$ and variants) Consider $(\mathbb{P}^1, p)$, $p\in\mathbb{P}^1$ rational point $\leftrightarrow \mathrm{Spec}\,\mathbb{k}$, i.e. $i_p: \mathrm{Spec}\,\mathbb{k} \to \mathbb{P}^1$ & structure map $\varphi: \mathbb{P}^1 \to \mathrm{Spec}\,\mathbb{k}$
$\bullet \mapsto p$
$\mathrm{Corr}^0(\mathbb{P}^1,\mathbb{P}^1) = A^1(\mathbb{P}^1\times\mathbb{P}^1) = \mathbb{Z}^2$ generated by $[p\times\mathbb{P}^1] =: e_0$ and $[\mathbb{P}^1\times p] =: e_1$
Thus $h(\mathbb{P}^1) = (\mathbb{P}^1, [p\times\mathbb{P}^1]) \oplus (\mathbb{P}^1, [\mathbb{P}^1\times p])$ where $\mathbb{1} = (\mathrm{Spec}\,\mathbb{k}, [\mathrm{Spec}\,\mathbb{k}]) = h_\sim(\mathrm{Spec}\,\mathbb{k})$ is the identity of $M^{\mathrm{eff}}(\mathbb{k})$ w.r.t $\otimes$
$\cong \mathbb{1}$ $\cong \mathbb{L}$; $\mathbb{L}$ is a "formal inverse" to $\mathbb{1}$, called Lefschetz motive.

• The left isomorphism is given by: $\mathbb{1} \underset{\varphi_*}{\overset{\varphi^*}{\rightleftarrows}} (\mathbb{P}^1, [p\times\mathbb{P}^1])$ where Γ^t the transverse of graph, i.e. $[\Gamma_f^t]_* = f^*$, Γ_f is correspondence
• One generalise this into
$\mathbb{P}^n$: $h(\mathbb{P}^n) = \bigoplus_d (\mathbb{P}^n, \mathbb{P}^{n-d}\times\mathbb{P}^d) = \bigoplus_d \mathbb{L}^{\otimes d}$
C smooth proj curve: $\Delta_C = p\times C + C\times p + \alpha$

*We can see not in fact we not have of concrete inverse of $\mathbb{L}$. Later we'll see $\mathbb{L}$ is the ... $h(C) = \mathbb{1}\oplus(C,\alpha)\oplus\mathbb{L}$. $h(X)$ can be viewed as correspondence, and α_* is endomorphism of H^*, thus this action induces decomposition naturally

All of these decomposition of $h_\sim(X)$ gives a decomposition of cohomology grps

$h(\mathbb{P}^1) = \mathbb{1}\oplus\mathbb{L} \Leftrightarrow H^*(\mathbb{P}^1) = H^0(\mathbb{P}^1)\oplus H^2(\mathbb{P}^1)$
$h(\mathbb{P}^n) = \bigoplus_d \mathbb{L}^{\otimes d} \Leftrightarrow H^*(\mathbb{P}^n) = \bigoplus_d H^{2d}(\mathbb{P}^n)$
$h(C) = \mathbb{1}\oplus(C,\alpha)\oplus\mathbb{L} \Leftrightarrow H^*(C) = H^0(C)\oplus H^1(C)\oplus H^2(C)$

Thus α is obstruction of existence of H^1.
Set $M(\mathbb{k}) := (\mathrm{Corr}^*_\sim(\mathbb{k}))^\#$ is our desired motive (pure)
and then $\mathrm{Var}(\mathbb{k}) \xrightarrow{h_\sim} M(\mathbb{k})$ still sends X & $f: X\to Y$ to same as $M^{\mathrm{eff}}(\mathbb{k})$
Where now we set $\mathrm{Corr}^*(\mathbb{k})$'s objects by twisting $\mathrm{Corr}^0(\mathbb{k})$:

objects $h(X)(r)$, where $h(X) \in Corr^0(k)$, and then $Hom(h(X)(r), h(Y)(S)) = A^{\dim X + s - r}(X \times Y)$

This $h(X) = h(X)(0)$ defined. Now our problems are
- I. Inverting $\mathbb{L}$
- II. Why inverting $\mathbb{L}$ gives desired universal property

Lets denotes objects of $M(k)$ by (X, r, a), r is twist

$\Rightarrow M^{eff}(k) \hookrightarrow M(k)$, $(X, a) \mapsto (X, 0, a)$

First notices that $\mathbb{1}(-1) \cong \mathbb{L}$: $\boxed{f_0 : \mathbb{1}(-1) \to h(\mathbb{P}^1) \text{ induced by } 0 \in A^1(\mathbb{P}^1)}$ I don't know why.

thus $\mathbb{1}(-r) = \mathbb{L}^{\otimes r}$ and $\underline{h(X)(r) \cong h(X) \otimes (\mathbb{L}^{\otimes r})}$ $\Rightarrow$ when $r > 0$, $h(X)(r) \in M(k) - M^{eff}(k)$ that's why I holds.

Rk. We can return back A^* from $Chow(k)$ by $\underline{Hom_{Chow(k)}(\mathbb{1}, h(X)(r)) = A^r(X)$}

[illegible]

Prop I. ① $M(k) \cong M^{eff}(k)[(-\otimes \mathbb{L})^{-1}]$;

② (Universal property) $Var(k) \xrightarrow{h} M(k)$, $\searrow$ $\downarrow \tilde{H}^*$ $\exists!$ $Gr\,Vect(k)$

③ $M(k)$ is $\boxed{\text{rigid}}$ (Poincare Duality)

In fact, $M(k)$ is the rigidification of $M^{eff}(k)$

Pf. ① $M(k) \longleftrightarrow M^{eff}(k)[(-\otimes \mathbb{L}^{-1})]$

$(X, r, a) \longmapsto (X, 0, a) \otimes \mathbb{1}(r) = (X, 0, a) \otimes (\mathbb{L}^{-1})^{\otimes r}$

$(X, 1, a) \longleftarrow (X, 0, a) \otimes \mathbb{L}^{-1}$ ▨

We call the inverse $\mathbb{L}^{-1}$ the Tate motive & $\mathbb{1}$ the trivial Tate motive

② Extend $M^{eff}(k) \hookrightarrow M(k)$ only to extend $Corr^0(k) \hookrightarrow Corr^*(k)$

$$\begin{tikzcd}
M^{eff}(k) \arrow[r, hook] \arrow[d] \arrow[rd, phantom, "\circlearrowleft"] & M(k) \arrow[d] \\
Gr^{\geq 0}Vect(k) \arrow[r, hook] & Gr^*Vect(k)
\end{tikzcd}
\qquad
\begin{tikzcd}
Corr^0(k) \arrow[r, hook] \arrow[d, "H^*"'] \arrow[rd, phantom, "\circlearrowleft"] & Corr^*(k) \arrow[d, "\tilde{H}^*"] \\
Gr^{\geq 0}Vect(k) \arrow[r, hook] & Gr^*Vect(k)
\end{tikzcd}$$

Set $\tilde{H}^* : (X, r) \longmapsto (H^*(X), [illegible]$ then extend to their Karoubian envelope

③ Recall. Rigid (tensor) category is $\forall h(X)(r)$ (objects) $\exists$ another object $(h(X)(r)^\vee)$, s.t. $h(X)(r) \xrightarrow{\cong} h(X)(r) \otimes h(X)(r)^\vee \otimes h(X)(r) \xrightarrow{\cong} h(X)(r)$

Here it's given by $(h(X)(r))^\vee = (h(X) \otimes \mathbb{1}(-r))^\vee = h(X) \otimes \mathbb{1}(r)$

and $\begin{cases} \delta : \mathbb{1} \to h(X \times X)(\dim X) = h(X)(r) \otimes h(X)(r)^\vee \\ \varepsilon : h(X)(r)^\vee \otimes h(X)(r) = h(X \times X)(\dim X) \xrightarrow{} \mathbb{1} \end{cases}$ ▨

Rk. All things before holds for all adequate equivalence relation, not only Chow motive, all of them called Grothendieck motive.

For Chow & numerical motive we denote $Chow(k)$ and $NM(k)$ and we know $\forall H^*$, $\sim_{rat}$ stronger than $\sim_{H^*}$, stronger than $\sim_{num}$ (the numerical equivalence is $Z \sim_{num} 0$ ($Z \equiv 0$) $\Leftrightarrow \forall Z'$ complement dim, $(Z \cdot Z') = 0$; homologous $\Rightarrow$ degree same ▨)

Thus $\boxed{\begin{array}{l} Chow(k) \text{ is the most complex pure motive} \\ NM(k) \text{ is the most simple pure motive} \end{array}}$

we need more abstract categorical notaction to describe this in detail, omitted ($NM(k)$ is the only semi-simple Abelian cat when char $k = 0$ by Jannsen's semi-simplicity thm)

- (Another motif way (pre)sheaves with transfers is $F : \boxed{Corr^0(k)} \to Ab$, with a proper Grothendieck topology on $Corr^0(k)$. A important case for motivic is the zeroth homology $\underline{\mathbb{Z}_{tr}(X)}$ when we give Nisnevich topology; $U \mapsto Hom_{Corr^0(k)}(U, X)$ by set the equivalence relation $\sim$ to be homotopy, i.e. two correspondence $f, g : X \to Y$ $\underline{\mathbb{A}^1\text{-homotopy equi}}$ $\sim_{\mathbb{A}^1}$ iff $\exists h : X \times \mathbb{A}^1 \to Y$, $h|_{X \times \{0\}} = f$, $h|_{X \times \{1\}} = g$

<u>Voevodsky's category of mixed motives</u>

Same as mixed Hodge, mixed motive generalises smooth (projective) variety/$\mathbb{C}$ to all varieties; thus $H^* = H$ isn't split to each degree but a filtration as in Hodge.

<u>Analogue</u>: <u>Hodge</u> | <u>Motive</u>

- {polarized (pure) Hodge structure}$_{\mathbb{Z}}$ $\boxed{\text{all} \mid \text{semi-simple objects}}$ MHS Abelian rigid tensor cat.
- Filtration W, M pure HS
- $M(k)_{\mathbb{Q}}$ ($\mathbb{Q}$ is coeffiecent, usually it's $M(k)_F$ for $F \supseteq \mathbb{Q}$) $\boxed{\text{all semi-simple objects}}$ (Beilinson's conjecture) $MM(k)$
- Filtration W, M pure motive

- Note that, we used derived cat. thus some categorical tools on $\boxed{\text{triangulated cat}}$ & abelian cat. needed later.

In history, first Beilinson conjected the existence of $MM(k)$, but it's impossible to construct directly. Later Deligne had another idea: construct $D^b(MM(k))$ instead of $MM(k)$ itself; then Deligne first construct $DM(k)$, later Voevodsky constructed many $DM(k)$ vers. At now, $DM(k)$ is almost sufficent good, left conjecture is $DM(k)$ leads $MM(k)$ by motivic t-structure.

Here we construct Geometric motive $DM_{gm}(k)$, here we know all are expected to be $\boxed{\text{triangulated}}$

[illegible] first Chow (derived) [illegible] for (co)homology [illegible]

① Category of finite correspondence $Corr_{fin}(k)$; ② Effective geometric motives $DM^{eff}_{gm}(k)$; ③ Geometric motive $DM_{gm}(k)$; ④ Voevodsky's motivic cohomology.

Check when smooth: $[W] \in A^*(X \times Y)$, $[V] \in A^*(Y \times Z)$ [illegible] $(W \times Z) \cdot (X \times V)$ intersect properly

① Def 13. (Finite correspondence) A finite correspondence $X \vdash Y$ formal sum of $[W]$, $W \subseteq X \times Y$ closed subvariety, s.t. $p_X : W \to X$ is finite surj and $Corr_{fin}(k)$ is its categorification.

$T \to X$ is finite cover, $W \times V \to V$ base change to $W \times V$ [illegible] X ▨

One should check composition $X \vdash Y \vdash Z$ is still finite. Here X, Y even not smooth, we can't define intersection. It's natural due to Cone sequence/triangle

② (Balmer–Schlichting) $DM^{eff}_{gm}(k)^{\natural} := (DM^{eff}_{gm}(k))^{\#} := (K^b(Corr_{fin}(k))[W^{-1}])^{\#}$ where K^b is bounded homotopy [illegible]

and localization by inverting W, W is {(homotopy) $p_X : [X \times \mathbb{A}^1] \to [X]$; (MV sequence) $X = X_1 \cup X_2$, $Cone([X_1 \cap X_2] \to [X_1] \oplus [X_2]) \to [X]$ [illegible]}

③ Last step is <u>invert Tate motive $\mathbb{L}$</u>, we have $\mathbb{L} \to [\mathbb{P}^1] \xrightarrow[p_*]{} [\operatorname{Spec} k] \to \mathbb{L}[1]$, thus $\operatorname{Cone}([\mathbb{P}^1] \to [\operatorname{Spec} k])[-1] \to \mathbb{L}$ by $\boxed{\text{Page 24}}$ universal property of cone (induced by structure map p)

i.e. here, we define $\mathbb{L} = \operatorname{Cone}([\mathbb{P}^1] \to [\operatorname{Spec} k])$

(due to $\mathbb{L}$ is a priori before can't be used here)

$\Rightarrow$ replace $[\mathbb{P}^1]$ by $[X] \Rightarrow \widetilde{[X]} \to [X] \to [\operatorname{Spec} k] \to \widetilde{[X]}[1]$ where $\widetilde{[X]} \oplus \operatorname{Spec} k = [X]$ is splitted by $i: \operatorname{Spec} k \to \mathbb{P}^1$ a k-pt

we call $\widetilde{[X]}$ is <u>the reduced motive of X</u> (thus $\mathbb{L} = \widetilde{[\mathbb{P}^1]}$) i.e. $\widetilde{M_{gm}(X)} \oplus \operatorname{Spec} k = M_{gm}(X)$ similiar as cohomology functor $\widetilde{H}^0 \oplus \mathbb{Z} = H^0$

Set Tate motive / twist $\mathbb{L} =: \mathbb{Z}(1)[2]$ and $\mathbb{Z}(n) = \mathbb{Z}(1)^{\otimes n}$ ($\mathbb{Z}$ is coefficient, or $\mathbb{Q}$, $\mathbb{F}_q \cdots$)

$\Rightarrow DM_{gm}(k) \equiv DM^{\text{eff}}_{gm}(k)[(-\otimes \mathbb{Z}(1))^{-1}]$ desired. It's a rigid triangulated (tensor) category

i.e. set objects are $M(r)$, $M \in DM^{\text{eff}}_{gm}(k)$, and morphism $\operatorname{Hom}_{DM_{gm}(k)}(M(r), N(s)) = \varinjlim \operatorname{Hom}_{DM^{\text{eff}}_{gm}(k)}(M \otimes \mathbb{Z}(n+r), N \otimes \mathbb{Z}(n+s))$ ▨

④ Our motivic cohomology then just $\mathbb{R}$k, ét tools as ring spectra involves $\ldots$ Galois foundation M_{gm} type of $H^{p,q}$

$H^{p,q}(X;\mathbb{Z}) := H^p(X;\mathbb{Z}(q)) := \operatorname{Hom}_{DM_{gm}(k)}(M_{gm}(X), \mathbb{Z}(q)[p])$, where M_{gm} are functor: $\operatorname{Var}(k) \to \operatorname{Cor}_{\text{fin}}(k) \to DM^{\text{eff}}_{gm}(k)$

$\Rightarrow A^q(X) = \operatorname{Hom}_{\operatorname{Chow}(k)}(\mathbb{1}, h(X)(q)) = \boxed{\operatorname{Hom}_{\operatorname{Chow}(k)}(\mathbb{1}(-q), h(X)) = \operatorname{Hom}_{DM_{gm}(k)}(M_{gm}(X), \mathbb{Z}(q)[2q])}$

Thus our <u>Bloch's higher Chow grps</u> are $\operatorname{Chow}(k)^{op} \hookrightarrow DM_{gm}(k) = H^{2q,q}(X;\mathbb{Z})$ (Bloch's formula)

$A^q(X,p) := H^{2q-p,q}(X;\mathbb{Z})$ $\qquad h(X) \mapsto M_{gm}(X)$

$$\begin{tikzcd}
\operatorname{Cor}_{\text{fin}}(k) \arrow[r] \arrow[dr, "M_{gm} \text{ reduced}"'] & DM^{\text{eff}}_{gm}(k) \arrow[d] \\
& DM_{gm}(k)
\end{tikzcd}$$

A better index is $A^q(X, 2q-p) = H^{p,q}(X;\mathbb{Z})$ full embedding, we'll <u>prove it later</u>. It's <u>fully faithful embedding</u> but <u>nontrivial</u>: it's the

as then $A^q(X) = A^q(X, 0)$

Focus on motivic cohomology, we then need to discuss

I (Homological) Properties of $H^{p,q}$ examples

II Can we compute it by a "complex"? <u>Voevodsky's cancellation thm</u> & weight/mixed where?

III The role of Borel–Moore cohomology: motivic Poincaré dual

<u>Prop 18.</u> ① (Cup product / intersection product)

$H^{p,q}(X;\mathbb{Z}) \otimes H^{p',q'}(X;\mathbb{Z}) \to H^{p+p',q+q'}(X;\mathbb{Z})$

$a \otimes b \longmapsto (M_{gm}(X) \xrightarrow{(a \otimes b) \circ \delta} \mathbb{Z}(q)[p] \otimes \mathbb{Z}(q')[p'] = \mathbb{Z}(q+q')[p+p'])$, δ is diagonal embedding, thus $H^{*,*}(X;\mathbb{Z})$ is ring

② (Homotopical invariant) $p^*: H^{p,q}(X;\mathbb{Z}) \xrightarrow{\cong} H^{p,q}(X \times \mathbb{A}^1;\mathbb{Z})$ someone may write as vector bundle $E \to X$, $H^{p,q}(X) = H^{p,q}(E)$

③ (K-theory) $\operatorname{Ch}_{\mathbb{Q}}: K_n(X;\mathbb{Q}) \xrightarrow{\cong} \bigoplus A^i(X, n; \mathbb{Q})$, in particular initial $c_1: \operatorname{Pic}(X) \to H^{2,1}(X;\mathbb{Z}) = \operatorname{Hom}_{DM_{gm}(k)}(M_{gm}(X), \mathbb{Z}(1)[2])$

④ (MV sequence) $\cdots \to H^{p-1,q}(U \cap V;\mathbb{Z}) \to H^{p,q}(U \cup V;\mathbb{Z}) \to H^{p,q}(U;\mathbb{Z}) \oplus H^{p,q}(V;\mathbb{Z}) \to H^{p,q}(U \cap V;\mathbb{Z}) \to \cdots$

<u>Pf.</u> ① is trivial, ② & ④ we apply the functor $\operatorname{Hom}_{DM_{gm}}(-, \mathbb{Z}(q)[p])$ to the isomorphism determined by localisation ▨

(i.e. $H^{p,q}(X;\mathbb{Z}) \xrightarrow{\cong} H^{p,q}(X \times \mathbb{A}^1;\mathbb{Z})$ & $M_{gm}(U \cap V) \to M_{gm}(U) \oplus M_{gm}(V) \to M_{gm}(U \cup V) \to M_{gm}(U \cap V)[1]$)

③ we know nothing about higher K_n group, thus omitted ▨

$(\mathbb{A}^n - \mathbb{A}^{n-1}) \cap (\mathbb{A}^n - \mathbb{A}^1) = \mathbb{A}^{n-1} - \{0\} \times \ldots$

<u>Eg 19.</u> (Computing motive analogues to Chow grp / cohomology)

① Using MV to $\underline{\mathbb{A}^n - \{0\}}$: $M_{gm}(\mathbb{P}^1) = \mathbb{1} \oplus \mathbb{L} =: \mathbb{Z} \oplus \mathbb{Z}(1)[2]$ by definition of $\mathbb{Z}(1) = \mathbb{L}[-2]$. By $(\mathbb{A}^n - \{0\}) = (\mathbb{A}^n - \mathbb{A}^1) \cup (\mathbb{A}^n - \mathbb{A}^1)$,

Compute triangle is easier than SES, one only compute the cone and $\mathbb{A}^1 - \{0\} = \mathbb{A}^1 \cap \mathbb{A}^1$ in $\mathbb{P}^1$

$$\begin{tikzcd}
M_{gm}(\mathbb{A}^1 - \{0\}) \arrow[r] \arrow[d, equal] & M_{gm}(\mathbb{A}^1)^{\oplus 2} \arrow[r] \arrow[d, equal] & M_{gm}(\mathbb{P}^1) \arrow[r] \arrow[d, equal] & M_{gm}(\mathbb{A}^1 - \{0\})[1] \arrow[d, equal] \\
M_{gm}(\mathbb{A}^1 - \{0\}) \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z}(1)[2] \arrow[r] & M_{gm}(\mathbb{A}^1 - \{0\})[1]
\end{tikzcd}$$

$\Rightarrow M_{gm}(\mathbb{A}^1 - \{0\}) \simeq \mathbb{Z}(1)[1] \oplus \mathbb{Z}$ (often translation on)

and $M_{gm}((\mathbb{A}^{n-1} - \{0\}) \times \mathbb{A}^1) \to M_{gm}(\mathbb{A}^n - \mathbb{A}^{n-1}) \oplus M_{gm}(\mathbb{A}^n - \mathbb{A}^1) \to M_{gm}(\mathbb{A}^n - \{0\}) \to M_{gm}((\mathbb{A}^{n-1} - \{0\}) \times \mathbb{A}^1)[1]$ $\quad \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}(1)[2]$

$\cong$ MV, homotopy $\qquad \mathbb{Z}(\mathbb{A}^{n-1}) \cong$ homotopy $\qquad$ Induction

$M_{gm}(\mathbb{A}^{n-1} - \{0\})^{\oplus 2} \longrightarrow \mathbb{Z}^{\oplus 2} \oplus M_{gm}(\mathbb{A}^{n-1} - \{0\}) \to M_{gm}(\mathbb{A}^n - \{0\}) \to M_{gm}(\mathbb{A}^{n-1} - \{0\})^{\oplus 2}[1]$

$\Rightarrow M_{gm}(\mathbb{A}^n - \{0\}) = \mathbb{Z} \oplus \mathbb{Z}(n)[2n-1]$ ▨

② (Using projective bundle) to $\mathbb{P}E$ over X rank r, we prove $M_{gm}(\mathbb{P}E) \cong \bigoplus M_{gm}(X)(j)[2j]$

Consider covering (U_i) on X, inductively using MV to $X = \bigcup U_i$, we thus can assume $\mathbb{P}E$ is trivial $X \times \mathbb{P}^n$ as correspondence is ok

By $\ldots$, $M_{gm}(X \times \mathbb{P}^n) = M_{gm}(X) \otimes M_{gm}(\mathbb{P}^n)$, we only prove $M_{gm}(\mathbb{P}^n) \cong \bigoplus \mathbb{Z}(j)[2j]$

<u>M_{gm} is a tensor functor</u> The <u>Künneth formula</u> of motivic cohomology holds: $H^{p,q}(X;\mathbb{Z}) \cong \bigoplus H^{p-2i,q-i}(X;\mathbb{Z}) \otimes H^{2i,i}(\ldots;\mathbb{Z})$, $X = \ldots$

By $\ldots$ $\mathbb{P}^n = \mathbb{A}^n \cup (\mathbb{P}^n - \{0\})$, $(\mathbb{P}^n - \{0\}) \cap \mathbb{A}^n = \mathbb{A}^n - \{0\}$ (ball again): note that $\boxed{(\mathbb{P}^n - \{0\}) \text{ is } \mathbb{A}^1\text{-bundle over } \mathbb{P}^{n-1}}$

thus we can run induction $M_{gm}(\mathbb{P}^n - \{0\}) = \bigoplus \mathbb{Z}(j)[2j]$ • Why not true for Chow group $\ldots$ only when cellular decomposition are given as the sub occurs, not Chow grp part. In fact, it's the line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$'s total space

$$\begin{tikzcd}
M_{gm}(\mathbb{A}^n - \{0\}) \arrow[r] \arrow[d, "\cong"] & M_{gm}(\mathbb{A}^n) \oplus M_{gm}(\mathbb{P}^n - \{0\}) \arrow[r, "\oplus_{j=1}^{n-1}"] \arrow[d, "\cong"] & M_{gm}(\mathbb{P}^n) \arrow[r] \arrow[d, equal] & M_{gm}(\mathbb{A}^n - \{0\})[1] \\
\mathbb{Z} \oplus \mathbb{Z}(n)[2n-1] \arrow[r] & \mathbb{Z} \oplus \bigoplus \mathbb{Z}(j)[2j] \arrow[r] & M_{gm}(\mathbb{P}^n) \to (\mathbb{Z} \oplus \mathbb{Z}(n)[2n-1])[1] &
\end{tikzcd}$$

done ▨ i.e. $\mathbb{P}^n = \ldots$

Rk. Explicit isomorphism is $\bigoplus_{j=0}^{r-1} \alpha_j: M_{gm}(\mathbb{P}E) \to \bigoplus M_{gm}(X)(j)[2j]$, where $\alpha_j: M_{gm}(\mathbb{P}E) \xrightarrow{\delta} M_{gm}(\mathbb{P}E)^{\otimes 2} \xrightarrow{p \otimes c_1(\mathcal{Q})^j} M_{gm}(X)(j)[2j]$ fact

where $p: \mathbb{P}E \to X$, $\mathcal{Q}$ is universal quotient bundle $\mathcal{O}_{\mathbb{P}E}(1)$ on $\mathbb{P}E \Rightarrow$ <u>$\bigoplus \alpha_j$ is same as Chow one</u>

Then we sketch the <u>way</u> of Nisnevich sheaves with transfer to compute $DM^{\text{eff}}_{gm}(k)$, i.e. $M_{gm}(X)$

$DM^{\text{eff}}_{gm}(k) \xhookrightarrow{\text{full}} \boxed{DM^{\text{eff}}_{-}(k)} \xhookrightarrow{\text{full}} D^{-}(Sh^{Nis}(\operatorname{Cor}_{\text{fin}}(k)))$ the derived category with negative indices of Nisnevich sheaves

and $\boxed{RC_*(L(X)) \cong C_*(X)}$ $\ldots$ <u>RC_* localisation</u> functor induced by $C_*: Sh^{Nis}(\operatorname{Cor}_{\text{fin}}(k)) \to DM^{\text{eff}}_{-}(k)$ take Suslin complex

$L(X)$ is the sheaf (Nisnevich) associated with transfer $L(X): Y \mapsto \operatorname{Cor}_{\text{fin}}(X, Y)$ by homotopy $\ldots$ $(C_n(\mathcal{F}): X \mapsto \mathcal{F}(X \times \Delta^n))_n$

$RC_*(L(X))\cong C_*(X)$ is the key of computation: it ~~[illegible]~~ reduces Nisnevich topology to usual Zariski topology to compute: [redacted]

$\mathrm{Hom}_{DM^{eff}_{gm}(k)}(M_{gm}(Y), M_{gm}(X)[n]) \cong \mathbb{H}^n(Y_{Nis}, C_*X) \cong \mathbb{H}^n(Y_{Zar}, C_*(X))$, when X taken as Speck, $\tilde{M}_{gm}([redacted]) = \mathbb{Z}(1)$

$\Rightarrow H^{n-1}([redacted]\, Y;\mathbb{Z}) \cong H^n_{Nis}([redacted]\, Y;\mathbb{Z}(1)) \cong \mathbb{H}^n_{Zar}(Y;\mathbb{Z}(1))$ can be computed (Still nontrivial step!) $= H^{n-1}_{Zar}(Y;\mathcal{O}^*_Y) = \begin{cases} \mathcal{O}^*_{Zar}(X), & n=1 \\ \mathrm{Pic}(X), & n=2 \\ 0, & n\geq 3 \end{cases}$

I omitted almost all details.

Precisely, we have a "simplicial" description of Chow groups, and we'll define the bivariant (cycle) cohomology and give a dual between [redacted] $CH_{i+d}(X\times Y, j) \cong A_{i,j}(Y [redacted] X)$ (Here we'll write Chow groups $CH_j \neq CH^{n-j}$, they'll be defined by dual only in chain-level, but we still have $CH^i(X,\cdot)\cong A^i(X,\cdot)$; all of three are called Bloch's higher Chow groups.)

All these needn't smoothness!

This duality gives lots of conclusions we need:

$\{CH^i(X,j) \cong H^{2i-j,i}(X;\mathbb{Z})$ (and by def, $j=0$ case is A^i itself, thus we prove Bloch's formula)

DM_gm(k) is rigid

by uniqueness of cohomology theory, and we'll prove it another way later than dual

Def 14. (Bloch's higher Chow groups) Set the subgrp $Z_k(X,n) \subset Z_{k+n}(X\times\Delta^n)$ generated by $[W]$, $W\subset X\times\Delta^n$ closed subvariety s.t. $\dim(W\cap(X\times\Delta^m))\leq k+m$ for every faces $\Delta^m \xrightarrow{t_{i_1}=\cdots=t_{i_{n-m}}=0} \Delta^n \cong \mathrm{Spec}(k[T_0,T_1,\dots,T_n]/(\sum T_i-1))$ ($Z^k(X,n) := Z_{d-k}(X,n)$, $d=\dim X$.) They form a chain [redacted] by $Z_k(X,n+1)\rightrightarrows Z_k(X,n)$, where $\delta^n_i: \Delta^n \rightrightarrows \Delta^{n+1}$ are n face maps are obvious flat, thus we have pullback $(\delta^n_i)^*$ by $(\delta^n_i)^*: X\times\Delta^n \rightrightarrows X\times\Delta^{n+1}$

Thus set $d_n=\sum_{j=0}^{n+1}(-1)^j(\delta^n_j)^*: Z_k(X,n+1)\to Z_k(X,n)$ gives a chain [redacted] $\Rightarrow$ we set $\begin{cases} CH_k(X,n) := H_n(Z_k(X,\cdot),d_\cdot) \\ CH^k(X,n) := H_n(Z^k(X,\cdot),d_\cdot) \end{cases}$

By definition, $CH_k(X,0)=A_k(X)$ is obvious. In chain level we have some easy properties:

Prop 18 ① (Cup product/Intersection product) $Z^p(X,\cdot)\otimes Z^q(Y,\cdot)\xrightarrow{\times} Z^{p+q}(X\times Y,\cdot)$;

② (Homotopical invariant) $p^*: Z_k(X,\cdot)\xrightarrow{q.is} Z_{k+1}(X\times\mathbb{A}^1,\cdot)$ (i.e. same homology / Chow grps);

③ (Excision and MV sequence) We can extend our classical one to distinguished triangles with shifting n, and due to MV is called localization $n=0$ is maximal (extend in $D^-(Ab)$), classical one has the surjectivity $\to 0$ right side. (thus in LES, k not change as ...)

$Z_k(Y,\cdot)\xrightarrow{i_*} Z_k(X,\cdot)\xrightarrow{j^*} Z_k(U,\cdot)\longrightarrow Z_k(Y,\cdot)[1]$ for $Y\xrightarrow[\text{closed}]{i} X \xleftarrow[\text{open}]{j} U$, $X=Y\sqcup U$; $\Leftrightarrow$ LES $\cdots\to CH_k(U,n+1)\to CH_k(Y,n)\to CH_k(X,n)\to CH_k(U,n)\to CH_k(Y,n-1)\to\cdots\to CH_k(U)\to 0$ By this form,

$Z_k(U\cap V,\cdot)\longrightarrow Z_k(U,\cdot)\oplus Z_k(V,\cdot)\longrightarrow Z_k(U\cup V,\cdot)\longrightarrow Z_k(U\cap V,\cdot)[1]$ for $U,V\subset U\cup V$ open.

④ (K-theory) Chern: $K_n(X)\otimes\mathbb{Q}\longrightarrow\bigoplus CH^i(X,n;\mathbb{Q})$

Pf. ① Just the [redacted] product of cycles: $[W]\times[V]$, $W\subset X\times\Delta^n$, $V\subset Y\times\Delta^m \Rightarrow W\times V\subset X\times Y\times\Delta^{m+n}$ we know ②+Excision $\Rightarrow$ MV

and $\dim((W\times V)\cap(X\times Y\times\Delta^{p}))\leq [redacted]\,(t+s)+p$ as we can factor each face $\Delta^p\subset\Delta^{m+n}$ into Δ^n and Δ^m ▨

② and ③ are both highly nontrivial, both of them need triangulation for $X\times\Delta^n$ and using some simplicial arguments & some type of moving lemma.

due to our AT told, as ~~[illegible]~~ this two axioms in ES axiom formally gives MV (See AT books like [Munkres])

I'm confused about the form of moving lemma as an acyclicity of complex, as classical one says [redacted] we can move to transversal (i.e. in same rational equivalence class)

Pf see Bloch's works [Algebraic Cycles and Higher K-theory] & [The Moving Lemma for Higher Chow Groups] ▨ ④ omitted ▨

Rk. ① descend to cohomology level $CH^p(X,n)\otimes CH^q(Y,m)\to CH^{p+q}(X\times Y,m+n)$, but is also nontrivial and simplicial. We accept the ring structure on $CH^*(X,\cdot)$ and for $n=0$ $CH^*(X,0)\cong A^*(X)$ is obvious as $\times\Delta^0$

Def 15 (Equi-dimensional cycles) $Z_k^{equi}(X): \mathrm{Cor}^{op}_{Sm}(k)\longrightarrow Ab$ is a presheaf with transfers (i.e. such a functor) = PST, $Y\longmapsto\langle[W]\mid W\subset X\times Y,\ W\xrightarrow{flat} Y$ has equi-dimensional fibres, $\dim = k\rangle$ they're smooth

Thus we can take its Suslin complex

$Z_k^{equi}(X,\cdot) := C_*(Z_k^{equi}(X))(\mathrm{Spec}k)$ is the equi-dimensional cycles

$=\langle[W]\in Z_{k+\cdot}(X\times\Delta^\cdot)\mid W\xrightarrow{flat}\Delta^\cdot$ has equi-dimensional fibres, $\dim=k\rangle\subset Z_k(X,\cdot)$

A classical consequence is the generic equi-dimness says $\exists\, U\subset\Delta^\cdot$ has fibre equi-dim

Rk ① $C_*(F)$ is complex of ~~the~~ presheaf with transfers of F is, thus here our $C_*(Z_k^{equi}(X))(\mathrm{Spec}k)$ means each component of functor value at Speck.

② By Yoneda Lemma, we define the Suslin complex of X (not smooth needed) $C_*(X) := C_*(L(X))$ (Recall $L(X): Y\mapsto \mathrm{Cor}^0_k(Y,X)$)

This in fact gives a functor $SmVar(k)\xrightarrow{M_{gm}} DM^{eff}_{gm}(k)$. What is it mean? Here the upper Var(k) is smooth, but C_* don't need smooth to define, thus we can embedding them extend M_{gm} to real M_{gm} without smoothness (But cost is not effective)

This is a better understanding $Var(k)\xrightarrow{M_{gm}} DM^{eff}_{-}(k)$ of what $DM^{eff}_{-}(k)$ is (as we not define it as motivic complexes but); it's singular generalization of $DM^{eff}_{gm}(k)$.

③ $Z_k^{equi}(X)$ is in fact Nisnevich ST

This relates with resolution of singularities. Here we thus get k admits it, such as char $k=0$, and consists of blowing-ups. Our expection is: the blowing-up of singular varieties can cover itself $\Rightarrow\forall X\in Var(k)$, it can be covered by $SmVar(k)$ by RS, we reduce to DM_{gm} is motivation

Now giving a topology (Grothendieck) and we define a sheaf (hyper)cohomology of $C_*(Z_k^{equi}(X))(-)$ a sheaf, as desired bivariant cycle cohomology

Def 16. (cdh topology & bivariant cycle cohomology)

cdh topology on Var(k) (not smooth) is refinement of Nisnevich topology (omitted), adding blowing-up covering. Namely, $Cov(X)$ is generated by $Cov_{Nis}(X)$ and $p\sqcup i: Y\sqcup F\longrightarrow X$, where $\begin{cases} p: Y-p^{-1}F\xrightarrow{\cong} X-F. \\ i: F\hookrightarrow X \text{ closed}\end{cases}$ Then we set $A_{i,j}(Y [redacted] X)=\mathbb{H}^{-j}(Y_{cdh}, C_*(Z_i^{equi}(X))_{cdh})$

It's called bivariant as it's covariant on X & contravariant on Y

We need accept a property, as we know nothing about Nis-topology.

Prop19 ① (Acyclicity) $\mathcal{F}$ is PST, if $\mathcal{F}_{Nis}$ or $\mathcal{F}_{cdh}=0$ (i.e. their sheafification under Nisnevich or cdh topology) $\Rightarrow C_*(\mathcal{F})_{Zar}$ is acyclic (i.e. sheaf cohomology under Zariski topology $=0$)

② (Comparison) U smooth, $\mathcal{F}^* \in \mathrm{Comp}^-(PST)$ homotopy invariant (i.e. $H^i(\mathcal{F}^*)$, also PST, has same value on X and $X\times\mathbb{A}^1$; our $DM^{eff}_{-}(k) \subset D^-(Sh^{Nis}(Cor_{fin}(k)))$ also define $\in \mathrm{Comp}^-_{fin}(k)$ as this manner)

$\Rightarrow \mathbb{H}^n(U_{Zar}, \mathcal{F}_{Zar}) = \mathbb{H}^n(U_{Nis}, \mathcal{F}_{Nis})$ has cohomology is $= \mathbb{H}^n(U_{cdh}, \mathcal{F}_{cdh})$ (cdh-Nis-Zar comparison)

③ (Global PST thm)

Same condition as ②, we have $Sm/Var(k) \to \mathrm{Comp}^-(Ab)$, $U \longmapsto \mathbb{H}^*(U, \mathcal{F}^*)$ is homotopy invariant presheaf.

④ (Suslin's moving lemma/Geometric comparison)

$z^{equi}_k(X,\bullet) \hookrightarrow z_k(X,\bullet)$ is qis

⑤ (Friedlander-Lawson moving lemma/Geometric duality) Set $z^{equi}_k(Z,X):$ $Sm/Var(k) \to Ab$ (it's the inner-Hom)

When Z is smooth, quasi-proj, $\dim Z = d$, we have the natural inclusion $U \longmapsto z^{equi}(X, Z\times U)$

$z^{equi}_k(Z,X) \hookrightarrow z^{equi}_{k+d}(X\times Z)$ inducing $C_*(z^{equi}_k(Z,X))_{Zar} \to C_*(z^{equi}_{k+d}(X\times Z))_{Zar}$ is qis.

Using them (their pf needs comparison between Nis, cdh, Zar via spectral sequence & simplicial argument for moving lemma), we can prove properties of $A_{i,j}$:

Prop20 ① (Cdh comparison) U smooth, quasi-projective, $H_{i}(z^{equi}_k(X))(U) \to A_{i,j}(U,X)$ is isomorphism

② (Cdh duality) U smooth, $d=\dim U$, $A_{i,j}(Y\times U, X) \to A_{i+d,j}(Y, X\times U)$

③ (MV excision & blowing-up sequence) (due to cdh topology)

$\to A_{i,j}(Y, U\cap V) \to A_{i,j}(Y,U)\oplus A_{i,j}(Y,V) \to A_{i,j}(Y, U\cup V) \to A_{i,j-1}(Y, U\cap V) \to \cdots$

$\to A_{i,j}(Y,Z) \to A_{i,j}(Y,X) \to A_{i,j}(Y,U) \to A_{i,j-1}(Y,Z) \to \cdots$ for $Z \xrightarrow[closed]{i} X \xleftarrow[open]{j} U$

$\to A_{i,j}(Y, p^{-1}(F)) \to A_{i,j}(Y, X')\oplus A_{i,j}(Y, F) \to A_{i,j}(Y,X) \to A_{i,j-1}(X, p^{-1}(F))$ for $p \amalg i: X' \amalg F \to X$ described in Def. (It's the associated LES of Prop15③'s SES)

Pf. ① First by Prop19②, we reduce to Zariski topology $\mathbb{H}^{-j}_{Zar}(U, C_*(z^{equi}_i(X)))$ $= A_{i,j}(U,X)$, now to compute it in Zariski topology, we need to consider its MV sequence $V_1\cup V_2 = U$ (induction): $C_*(z^{equi}_i(X))(V_1\cup V_2) \to C_*(z^{equi}_i(X))(V_1)\oplus C_*(\ldots)(V_2) \to C_*(z^{equi}_i(X))(V_1\cap V_2)$ (*) (it comes from embedding this, send $[V_1\cap V_2] \to [V_1]\oplus[V_2] \to [V_1\cup V_2]$ into $DM^{eff}_{-}(k)$, thus $C_*(V_1\cap V_2)_{Nis} \to C_*(V_1)_{Nis}\oplus C_*(V_2)_{Nis} \to C_*(V_1\cup V_2)_{Nis}$ distinguish triangle, under Zariski topology the morphism of chain complex exists still, but it's may not triangle (in $DM^{eff}_{-}(k)$, $[V] \cong M_{gm}(V)$ denoted, it's distinguish triangle) (in fact it is)) And we need prove it's distinguish triangle $\Rightarrow H_i(C_*(z^{equi}_i(X))(U)) = \mathbb{H}^{-i}_{Zar}(U, C_*(z^{equi}_i(X)))$ so we reduces to $U=\mathbb{A}^n$, thus $U=\mathrm{Spec}\,k$ by homotopy invariance, and this case $H_i(C_*(z^{equi}_i(X))(\mathrm{Spec}\,k)) = H_i(z^{equi}_i(X,\bullet))$ & $\mathbb{H}^{-i}_{Zar}(\mathrm{Spec}\,k, C_*(z^{equi}_i(X))) = ?$ We compute by spectral sequence $E_2^{pq} = R^p\Gamma(\mathrm{Spec}\,k, \mathcal{H}^q(C_*(z^{equi}_i(X))))$ $\xRightarrow{Prop19④}$ $H_i(z_i(X,\bullet))$

Now we reduce to prove (*) is distinguish triangle $\Rightarrow H^{-i}(\mathrm{Spec}\,k$, (ker sheaf/im sheaf also sheaf)$) = CH_{k,i}(X)$

consider $0 \to z^{equi}_{i+d}(X\times(V_1\cup V_2)) \to z^{equi}_{i+d}(X\times V_1)\oplus z^{equi}_{i+d}(X\times V_2)$

$\xrightarrow{\Sigma} z^{equi}_{i+d}(X\times(V_1\cap V_2))$ the left exact presheaf sequence ($\mathcal{H}^q(C_*(z^{equi}(X)))$) only $i=0$ not $0 \Rightarrow \Gamma(\mathrm{Spec}\,k, \mathcal{H}^q(C_*(z^{equi}(X)))) = \mathbb{H}^{-i}(\mathrm{Spec}\,k, C_*)$

and under cdh-sheafification it has $(\mathrm{coker}\,\Sigma)_{cdh} = 0$ ⓐ $\xrightarrow[SS]{another}$ $H_i(\Gamma(\mathrm{Spec}\,k, C_*))$ ⓑ It's obvious that ② is more useful: ▨

ⓐ is due to our classical MV sequence $\xrightarrow{cdh}$ (and when smooth, it descends to Chow groups)

$\mathbb{H}^{-i}_{Zar}(\mathrm{Spec}\,k, C_*(z^{equi}(X))) = H^{-i}(z^{equi}_k(X,\bullet)) = H_i(z^{equi}_k(X,\bullet)) = CH_{k,i}(X)$

ⓑ Why?

Then by Prop19① $\Rightarrow C_*(\mathrm{coker}\,\Sigma)_{Zar}$ acyclic $\Rightarrow$ we have a distinguished triangle $C_*(z^{equi}_{i+d}(X\times(V_1\cup V_2)))_{Zar} \to C_*(z^{equi}_{i+d}(X\times V_1))_{Zar}\oplus C_*(\ldots)$ $\to C_*(z^{equi}_{i+d}(X\times(V_1\cap V_2)))_{Zar} \xrightarrow{+1} \cdots$, take geometric duality thm Prop19⑤ $\Rightarrow C_*(z^{equi}(V_1\cup V_2, X))_{Zar} \to C_*(z^{equi}(V_1, X))_{Zar}\oplus C_*(\ldots)$ $\to C_*(z^{equi}(V_1\cap V_2, X))_{Zar} \xrightarrow{+1}$ $\cdots$ Zariski sheaves; Take value at $\mathrm{Spec}\,k$ $\Rightarrow$ $C_*(z^{equi}(V,X))(\mathrm{Spec}\,k)$ $= z^{equi}_k(V,X)(\mathrm{Spec}\,k\times\Delta^\bullet) = z^{equi}_k(V,X)(\Delta^\bullet)$ $= z^{equi}(X)(V\times\Delta^\bullet) = C_*(z^{equi}(X))(V)$

$\Rightarrow$ distinguish triangle $C_*(z^{equi}(X))(V_1\cup V_2) \to C_*(z^{equi}(X))(V_1)\oplus C_*(z^{equi}(X))(V_2)$ $\to C_*(z^{equi}(X))(V_1\cap V_2) \xrightarrow{+1} \cdots$ is just (*) ▨

② $A_{i,j}(Y\times U, X) = \mathbb{H}^{-j}_{cdh}(Y\times U, C_*(z^{equi}_i(X))) \cong \mathbb{H}^{-j}_{cdh}(Y, C_*(z^{equi}_{i+d}(X\times U))) = A_{i+d,j}(Y, X\times U)$ (It seems that we don't need use comparison to Zariski topology, as we not use MV for Zariski open)

$\Uparrow p+q=-j$ $\qquad \Uparrow p+q=-j$

$E_2^{pq} = H^p(R^q\Gamma(Y\times U, C_*(z^{equi}_i(X))))$ $\qquad H^p(R^q\Gamma(Y, C_*(z^{equi}_{i+d}(X\times U)))) = E_2^{pq}$

$\Gamma(Y\times U, C_*(z^{equi}_i(X)))$ $\cong$ Geometric dual $= \Gamma(Y, C_*(z^{equi}_i(U,X)))$ $H^p(R^q\Gamma(Y, C_*(z^{equi}_i(U,X))))$ ▨

③ By the properties of hyperchomology using cdh dual, take $U\cap V = (U\cap V)\times\mathrm{Spec}\,k \Rightarrow A_{i,j}(Y, U\cap V) \cong A_{i-d,j}(Y\times(U\cap V), \mathrm{Spec}\,k)$ and ..., note that for the first variable Y in hyperchomology holds these LES ▨ (I'm confused why excision needs an independent pf in text book / why can we use cdh dual: $U\cap V$... may not smooth!)

Now we can prove desired duality thm (duality) $CH_{i}(X\times Y, j) = \ldots(Y, X)$; $d=\dim Y$, char $k=0$ (or admitting resolution + k perfect) pf. We reduce to Y smooth & quasi-projective by ∃ a filtration when k perfect: $\emptyset = Y_{-1} \subset Y_0 \subset \cdots \subset Y_m = Y$, s.t. $Y_i - Y_{i-1}$ [illegible] is smooth and quasi-projective. Thus $Y = \bigcup (Y_i - Y_{i-1})$ reduced by excision sequence inductively.

We have q.i.s by cdh comparison $C_*(\underline{Z}^{equi}_{trd}(X\times Y))(\mathrm{Spec}\,k) = Z^{equi}_{trd}(X\times Y, \bullet) \overset{qis}{\simeq} Z_{trd}(X\times Y, \bullet)$, taking cohomology of these two chain complexes of Ab (Geometric) $\Rightarrow H_j(C_*(\underline{Z}^{equi}_{trd}(X\times Y))(\mathrm{Spec}\,k)) \cong CH_{i+d}(X\times Y, j)$ (Here both cdh comparison & dual used smooth + quasi-projectivity).

$$\begin{tikzcd} H_j(C_*(\underline{Z}^{equi}_{trd}(X\times Y))(\mathrm{Spec}\,k)) \arrow[r, "\cong"] \arrow[d, equal, "\text{cdh comparison}"] & CH_{i+d}(X\times Y, j) \arrow[d, "\cong"] \\ A^{equi}_{i,j}(\mathrm{Spec}\,k, X\times Y) \arrow[r, "\text{cdh dual}"] & A_{i,j}(X,Y) \end{tikzcd}$$

return back to $DM^{eff}_{gm}(k)$ & $DM_{gm}(k)$ ▨

we need a localisation property to ensure descendent of $DM^{eff}_{gm}(k) \subset DM^{eff}_{-}(k)$. And we can relate this to pure Chow motive [illegible] prove Bloch's formula.

First recall $\mathrm{Sm}\,\mathrm{Var}(k) \xrightarrow{M_{gm}} DM^{eff}_{gm}(k)$ one have $\mathrm{Var}(k) \xrightarrow{M^c_{gm}} DM^{eff}_{-}(k)$, $X \longmapsto C_*(\underline{Z}^{equi}(X))$

$$\begin{tikzcd} \mathrm{Sm}\,\mathrm{Var}(k) \arrow[r, "M_{gm}"] \arrow[d] & DM^{eff}_{gm}(k) \arrow[d] \\ \mathrm{Var}(k) \arrow[r, "M_{gm}"] \arrow[ur, phantom, "\cong"] & DM^{eff}_{-}(k) \end{tikzcd}$$

$X \longmapsto C_*(L(X))$

compare that $C_*(L(X))(U) \subsetneq C_*(\underline{Z}^{equi}(X))(U)$ note that they're equal if $\underline{X \text{ proper}}$

$\text{Cor}_{fin}(U,X) \subsetneq \text{Cor}_{quasi\text{-}fin}(U,X)$ (equi-dimensional 0 = quasi-finite strictly weaker than finite) (both "$\overset{def}{=}$")

$\Rightarrow M^c_{gm} = M_{gm}$ ⇒ the projection is closed ⇒ each fibres closed in Zariski meaning ⇒ 0-dim fibre must finite ▨

thus we have $\mathrm{Var}(k) \xrightarrow{M^c_{gm}} DM^{eff}_{-}(k)$ (not extend M_{gm})

$$\begin{tikzcd} \mathrm{Var}(k) \arrow[r, "M^c_{gm}"] \arrow[dr, "M_{gm}"'] & DM^{eff}_{-}(k) \\ & DM^{eff}_{gm}(k) \arrow[u] \end{tikzcd}$$

Our localisation result says $M^c_{gm}(X) \in DM^{eff}_{gm}(k)$

Prop 21. (Localisation) $\forall X \in \mathrm{Var}(k)$, $M^c_{gm}(X) \in DM^{eff}_{gm}(k)$

pf. We reduce to case of $\underline{X \text{ smooth \& proper}}$ by $\underline{\text{excision}}$, then $M^c_{gm}(X) \overset{proper}{=} M_{gm}(X) \overset{smooth}{\in} DM^{eff}_{gm}(k)$ done.

- How the $\underline{\text{excision}}$: $M^c_{gm}(X) = C_*(\underline{Z}^{equi}(X))$, recall or $\underline{(*) \text{ in Prop 19's pf}}$ is MV sequence from embedding thm, so is excision, and again, by $(\text{Lake})_{cdh} = 0$, using comparison & acyclicity again: We have $C_*(\underline{Z}^{equi}(Z))_{Nis} \to C_*(\underline{Z}^{equi}(X))_{Nis} \to C_*(\underline{Z}^{equi}(U))_{Nis}$ is distinguish triangle in $DM^{eff}_{-}(k)$ as $DM^{eff}_{-}(k)$ is given Nisnevich topology
- Reduce to $\underline{\text{smooth}}$: Again taking filtration of X, $X_i - X_{i-1}$ is smooth (Here we don't need (quasi-) projectivity, not need k perf
- Reduce to $\underline{\text{proper}}$ (in fact projective): X smooth, we blow-up it through $Z \subset X$ also smooth $\Rightarrow \tilde{X} = Bl_Z X$ also smooth, further it's projective (By definition $\mathrm{Proj}(\oplus I^n)$), and $\tilde{X} \underset{i}{\overset{\pi}{\rightleftarrows}} X$ splittes to $X \subset \tilde{X}$, and $\tilde{X} = X \sqcup E = X \sqcup \pi^{-1}(Z)$, apply excisi this ⇒ pf $M^c_{gm}(\tilde{X}) \in DM^{eff}_{gm}(k) \Rightarrow M^c_{gm}(X) \in DM^{eff}_{gm}(k)$

This is due to $M^c_{gm}(E) \in DM^{eff}_{gm}(k)$ always as $\dim E < \dim X$: one only induction on $\dim X$ to complete our pf ▨

Eg 21. We know $M^c_{gm}(\mathbb{P}^n) = M_{gm}(\mathbb{P}^n) = \bigoplus \mathbb{Z}(j)[2j]$ as $\mathbb{P}^n$ is proper, we can compute $\mathbb{A}^n$ by excision $\mathbb{P}^n - \mathbb{P}^{n-1}$:

$$\begin{tikzcd} M^c_{gm}(\mathbb{P}^{n-1}) \arrow[r] \arrow[d, "\cong"] & M^c_{gm}(\mathbb{P}^n) \arrow[r] \arrow[d, "\cong"] & M^c_{gm}(\mathbb{A}^n) \arrow[r, "+1"] \arrow[d, equal] & \cdots \arrow[d, "\cong"] \\ \bigoplus_{j\le n-1} \mathbb{Z}(j)[2j] \arrow[r] & \bigoplus_{j\le n} \mathbb{Z}(j)[2j] \arrow[r] & M^c_{gm}(\mathbb{A}^n) \arrow[r] & \bigoplus_{j\le n-1} \mathbb{Z}(j)[2j+1] \end{tikzcd}$$

(Note that for M_{gm} we don't have such excision sequence, only MV sequence)

$\overset{?}{\Longrightarrow} M^c_{gm}(\mathbb{A}^n) = \mathbb{Z}(n)[2n]$

We can return back to Chow motive & Geometric motive's relations

Prop 22 ① We have a full embedding $i: \mathrm{Chow}^{eff}(k)^{op} \longrightarrow DM^{eff}_{gm}(k)$, satisfying $i(h(X)(-r)) = M_{gm}(X)(r)$, i.e.

$$\begin{tikzcd} \mathrm{Sm}\,\mathrm{Var}(k)^{op} \arrow[r, "h"] \arrow[dr, "M_{gm}"'] & \mathrm{Chow}^{eff}(k) \arrow[d, "i"] \\ & DM^{eff}_{gm}(k) \end{tikzcd}$$

commutes up to a twist.

(for X smooth projective as we set in pure motive case) $X \longmapsto M_{gm}(X)$ (In fact $(X, 0, \Delta)$)

② (Bloch's formula) $CH^i(X, j) \cong H^{2i-j}(X, \mathbb{Z})$, $i \ge 0$, $j \in \mathbb{Z}$

③ The full embedding extends to $\hat{i}: \mathrm{Chow}(k)^{op} \longrightarrow DM_{gm}(k)$

④ $DM_{gm}(k)$ is rigid.

pf. ① Only check morphism level, $\mathrm{Hom}_{\mathrm{Chow}^{eff}(k)}(X,Y) = CH^{\dim X}(X\times Y) \overset{\text{Duality}}{=} A_{\dim Y, 0}(\mathrm{Spec}\,k, Y\times X) \underset{\text{duality}}{\overset{\text{cdh}}{=}} \boxed{A_{0,0}(Y,X) = \mathrm{Hom}_{DM^{eff}_{gm}(k)}(M_{gm}(Y), M_{gm}(X))}$ when $i \ge 0$

Only to prove the last red formula showing $DM^{eff}_{gm}(k)$ is horizontal thm categorification of bivariant cycle cohomology.

We prove the red formula: $\mathrm{Hom}_{DM^{eff}_{-}(k)}(M_{gm}(Y)(i)[2i+j], M^c_{gm}(X)) \cong A_{i,j}(Y,X)$, then by $\underline{\text{localisation Prop 21}}$ we complete the pf

~~it holds for general scheme X & Y, but here for us we prove the case of Y smooth (as here X, Y all smooth proper) [illegible] For i=0 case: $A_{0,j}(Y,X)$ [illegible] embedding thm $DM^{eff}_{gm}(k) \subset DM^{eff}_{-}(k)$ [illegible] smooth ⇒ $M_{gm}(X) = C_*(L(X)) \in DM^{eff}_{gm}(k)$ by embedding thm [illegible] full done by $DM^{eff}_{-}(k)$ [illegible] is homotopy invariance [illegible] $\mathrm{Hom}_{DM^{eff}_{gm}(k)}(M_{gm}(Y)[j], M^c_{gm}(X))$ [illegible] $(M_{gm}(Y)[j], M^c_{gm}(X))$~~

I'm lost in the pf as it uses a lot localisation & embedding thm we omitted. We accept it only ▨

② $H^{2i-j}(X;\mathbb{Z}) = \mathrm{Hom}_{DM^{eff}_{gm}(k)}(M_{gm}(X), \mathbb{Z}(i)[2i-j]) \overset{?}{=} \mathrm{Hom}_{DM^{eff}_{-}(k)}(M_{gm}(X), \mathbb{Z}(i)[2i-j]) \overset{\text{cancellation}}{=} \mathrm{Hom}_{DM^{eff}_{-}(k)}(M_{gm}(X), \mathbb{Z}(i)[2i-j])$

$\cong \mathrm{Hom}_{DM^{eff}_{-}(k)}(M_{gm}(X), M^c_{gm}(\mathbb{A}^i)[-j]) = \mathrm{Hom}_{DM^{eff}_{-}(k)}(M_{gm}(X)[j], M^c_{gm}(\mathbb{A}^i)) \cong A_{0,j}(X, \mathbb{A}^i) \overset{\text{cdh dual}}{=} A_{\dim X, j}(\mathrm{Spec}\,k, X\times\mathbb{A}^i) \overset{\text{Duality thm}}{=} CH^i(X\times\mathbb{A}^i, j) \overset{\cong}{\longrightarrow} CH^i(X, j)$ ▨

③
$$\begin{tikzcd} DM^{eff}_{gm}(k) \arrow[d] & \mathrm{Chow}^{eff}(k) \arrow[l, "\text{homotopy}"'] \arrow[d, hook] \\ DM_{gm}(k) & \mathrm{Chow}(k)[(-\otimes\mathbb{L})^{-1}] \arrow[l, dashed] \end{tikzcd}$$

Due to $DM_{gm}(k)$'s inverting $[(-\otimes\mathbb{Z}(1))^{-1}]$, $\mathbb{Z}(1)$ and $\mathbb{L}$ up to a 2-shifting, thus embed exists, and it's full by writing out Hom-sets:

$\mathrm{Hom}_{\mathrm{Chow}(k)}(X(i), Y(j)) = CH^{\dim X + j - i}(X\times Y) = CH_{\dim Y - j + i}(X\times Y) = A_{i-j,0}(Y,X) = \mathrm{Hom}_{DM^{eff}_{gm}(k)}(M_{gm}(Y)(i-j)[2(i-j)], M^c_{gm}(X))$

$= \mathrm{Hom}_{DM^{eff}_{gm}(k)}(M_{gm}(Y)(i-j)[2(i-j)], M_{gm}(X))$ if $i-j\geq 0$, thus our concern the case of $i-j<0$, assume $i<0$, we need to prove the vanishing in this case $H^{2i-2j,i-j}(Y,\mathbb{Z})=0$ (In fact to invert original $i\geq 0$)

i.e. our extension is zero extension for negative indices. (it compatible our feeling of Chow grp for $i<0$ orders)

Thus we have two things to prove (a) (Voevodsky's cancellation thm) $(-\otimes \mathrm{id})$: $\mathrm{Hom}(A,B) \xrightarrow{\cong} \mathrm{Hom}(A(1),B(1))$, thus we have $DM^{eff}_{gm}(k) \hookrightarrow DM_{gm}(k)$ is full; (b) $H^{2i-2j,i-j}(Y;\mathbb{Z})=0$, $i<0$ vanishing.

It's due to $DM_{gm}(k)$ invert $\otimes\mathbb{Z}(1)$, i.e. $\otimes\mathbb{Z}(1)$ to $A\otimes\mathbb{Z}(1)^{-1}$, the isom gives $\mathrm{Hom}(A(1),B(1)) \to \mathrm{Hom}(A,B)$, and $A = A(1)\otimes\mathbb{Z}(1)^{-1}$ desired, thus full ▨

We need preparation work on Gysin map/Gysin distinguish triangle, will shown next example: it says that $DM^{eff}_{gm}(k) = \langle M_{gm}(X) \mid X\in Var(k)\rangle = \langle M_{gm}(X) \mid X\in SmProjVar(k)\rangle$ (In fact it's similiar to excision triangle, but twisted)

(We know Mgm has no excision triangle) and using to argument before: Operation 1: filtration by smooth, we can have this generation relation ($M_{gm}(X)$ as $DM^{eff}_{gm}(k) = K^b(\mathrm{Cor}_{fin}(k))$ generated by each degree) Operation 2: Blow-up to proj

(a) By upper argument, we assume $A=M_{gm}(Y)[i]$, $B=M_{gm}(X)$, X, Y smooth & proj

$\Rightarrow \mathrm{Hom}(A(1),B(1)) \cong \mathrm{Hom}(M_{gm}(Y)(1)[i], M_{gm}(X)(1)) \overset{\text{red}}{\cong} A_{1,i}(Y, X\times\mathbb{A}^1) \overset{\text{homotopy}}{=} A_{0,i}(Y,X) \overset{\text{red}}{\cong} \mathrm{Hom}(A,B)$ ▨

(b) $H^{2i-2j,i-j}(Y;\mathbb{Z}) \overset{\text{def}}{=} \mathrm{Hom}_{DM^{eff}_{gm}(k)}(M_{gm}(Y)(-i)[-2i], \mathbb{Z}) = \mathrm{Hom}_{DM^{eff}_{gm}(k)}(M_{gm}(Y)(-i)[-2i], M^c_{gm}(\mathrm{Spec}\,k))$

$\overset{\text{red formula}}{\cong} A_{-i,-2i}(Y, \mathrm{Spec}\,k) \overset{\text{duality}}{=} A_{\dim Y - i, -2i}(\mathrm{Spec}\,k, Y)$ (Both Speck & Y smooth) $=$ We know the SS of H is E_2-degenerate $H^{-2i}(\Gamma(\mathrm{Spec}\,k, C_*(z_{equi}(Y, \dim Y - i))))$, but here $\dim Y - i > \dim Y$ $= 0$ ▨

⊕ A categorical fact:

> If $S\subset D$ subset of objects, D tensor triangulated cat (must), if $\forall M\in S$ has dual M^* & (*) $\exists D' \subsetneq D$ full triangulated subcat, $S\subset D'$, closed under ⊗
> $\Rightarrow$ D is rigid

By this fact, we only need to find dual of objects which is smooth & projective: it's obvious as in this case $M^c_{gm} \overset{(n.2)}{=} M_{gm}$ and $\mathrm{Chow}(k)^{op} \overset{L}{\hookrightarrow} DM_{gm}(k)$, Chow(k) rigid $\Rightarrow \exists$ dual$(M_{gm}(X))^* = L(h(X)(-r))$

$h(X)(-r) \longmapsto M_{gm}(X)(r) = M^c_{gm}(X)(r)$

Rk. In general X, the dual is $(M_{gm}(X))^* = M^c_{gm}(X)(-\dim X)[-2\dim X]$

It suffices to check the condition (*): again by $\left|\begin{array}{l}\text{Op1: Filtration}\\ \text{Op2: Blow-up}\end{array}\right.$ to reduce, and Gysin ($\in DM_{gm}(k)$ localization) triangle in place of excision inducing

Eg.21 (Generalize construction of Chow grps to Mgm)

① (Deformation to normal bundle) Here our discussion in $DM^{eff}_{gm}(k)$, thus all smooth $\Rightarrow$ normal cone is normal bundle. $i: Z\hookrightarrow X$ closed, we have deformation space $M_i := Bl_{Z\times\{0\}}(X\times\mathbb{A}^1) - \tilde{X}$ has fibre at 0 is $s: Z\to N$, N is normal bundle ($M_i^0 = N$)

We have isomorphisms $M_{gm}(N/N\setminus s(Z)) \xrightarrow{\cong} M_{gm}(M_i/M_i\setminus(Z\times\mathbb{A}^1)) \xleftarrow{\cong} M_{gm}(X/X\setminus Z)$,

where $M_{gm}(X/X\setminus Z) = M_{gm}([X\setminus Z]\to[X])$ is image of complex

We can't prove it, as it uses Nisnevich excision, this also answers why Mgm doesn't have a (Zariski) excision sequence for $DM^{eff}_{gm}(k)$ no Zar-Nis comparison exists.

② (Gysin distinguished triangle) By definition of $X/X\setminus Z \Rightarrow M_{gm}(X\setminus Z)\to M_{gm}(X)\to M_{gm}(X/X\setminus Z)\xrightarrow{+1}$ (A) is distinguished triangle

Our desired triangle is by Gysin isomorphism $M_{gm}(X/X\setminus Z) \cong M_{gm}(Z)(n)[2n] \Rightarrow M_{gm}(X\setminus Z)\to M_{gm}(X)\to M_{gm}(Z)(n)[2n]\xrightarrow{+1}$ (B)

Pf. By ①, we reduces to case X = E over Z, and $i: Z\to E$ is zero section, rank(E) = n

By triangle (A), if $M_{gm}(E\setminus s(Z)) \cong M_{gm}(Z)\oplus M_{gm}(Z)(n)[2n-1]$ (*), then by $M_{gm}(E) = M_{gm}(Z)$ (homotopy invariance)

$\Rightarrow M_{gm}(E/E\setminus s(Z)) = M_{gm}(Z)(n)[2n]$ done, thus we prove (*) using MV sequence: $\mathbb{P}(E\oplus 1) = E\cup(\mathbb{P}(E\oplus 1)-s(Z))$ and $E\cap(\mathbb{P}(E\oplus1)-s(Z)) = E\setminus s(Z)$

$\Rightarrow M_{gm}(E\setminus s(Z)) \to M_{gm}(E)\oplus M_{gm}(\mathbb{P}(E\oplus1)-s(Z)) \to M_{gm}(\mathbb{P}(E\oplus1)) \xrightarrow{+1}$

$$\begin{tikzcd}
M_{gm}(E\setminus s(Z)) \arrow[r] \arrow[d, equal] & M_{gm}(Z)\oplus M_{gm}(\mathbb{P}(E)) \arrow[r] \arrow[d, equal, "\text{Eg.19②}"] & M_{gm}(\mathbb{P}(E\oplus1)) \arrow[d, equal, "\text{Eg.19②}"] \\
M_{gm}(Z)\oplus M_{gm}(Z)(n)[2n-1] \arrow[r, phantom, "\overset{+1}{\leftarrow}"] & M_{gm}(Z)\oplus \bigoplus_{j\leq n-1} M_{gm}(Z)(j)[2j] \arrow[r] & \bigoplus_{j\leq n} M_{gm}(Z)(j)[2j]
\end{tikzcd}$$

homotopy ‖ $\mathbb{P}(E\oplus1)-s(Z)\to\mathbb{P}(E)$ is $\mathbb{A}^1$-bundle

(It shift $[2n-1]$ as $\bigoplus_{j\leq n} M_{gm}(Z)(j)[2j] \xrightarrow{+1} \cdots [2n]$) ▨

[illegible]

Borel-Moore motivic homology

Our classical Borel-Moore homology (as in [Fulton]) is homology with compact support $H^{BM}_i(X;\mathbb{Z}) := \varprojlim_{K\subset X \text{ compact}} H_i(X, X-K;\mathbb{Z})$ (Poincaré dual for noncpt mfds) as generalization of Poincaré dual for noncpt mfds in topology. In [Fulton] he focus on the case $X/\mathbb{C}$ to apply topological results directly (more generally, $\exists\, k\hookrightarrow\mathbb{C}$ and take $X(\mathbb{C})$, $\mathbb{C}$-pts), and gives cycle map in homology level $A_*(X) \xrightarrow{cl} H^{BM}_*(X)$ ($\Leftrightarrow A^*(X) \xrightarrow{cl} H^*(X)$ by Poincaré dual of both)

Borel-Moore motivic homology is dual to motivic cohomology: set $H^{BM}_{i,j}(X;\mathbb{Z}) := \mathrm{Hom}_{DM_{gm}(k)}(\mathbb{Z}(j)[i], M^c_{gm}(X))$, here $M^c_{gm}(X)$ is the compact supported (geometric) motive of X, $M_{gm,c} = M_{gm}$ when smooth X

By our red formula again: $H^{BM}_{i,j}(X;\mathbb{Z}) = CH^{d-j}(X, i-2j)$, $d=\dim X$, $= CH_j(X, i-2j) = H^{2d-i,d-j}(X;\mathbb{Z})$ is the motivic Poincaré dual.

Note that for singular case Poincaré dual not holds; in smooth case, all the properties of LES holds for BM homology and in chain level, we denote $H^{BM}_* = R\Gamma(X, D(X))$, $D(X)$ is the dualizing complex similar with the bivariant cycle cohomology, one set $H^{BM}_n(X) := H^{-n}(C^{BM}_\bullet(X)) \simeq H^{-n}(X, D_\bullet(X))$, $D_\bullet(X)$ is dualizing complexes on X. From Borel-Moore motivic cohomology ⇒ Borel-Moore homology is by $C^{BM}_\bullet(X)(-i)[-2i] \cong Z^{d-i}(X)$ $\Rightarrow H^{BM}_n(X)(-i)[-2i] \cong CH^{d-i}(X, n) \cong H^{BM}_{n+2i, i}(X)$ up to a reindex (twist & shift) when smooth (accept)

Thus we reduce our computation of $H^{BM}_{n+2i,i}(X)$ to $H^{BM}_n(X)$, and we can focus on the chain-level properties (compatible with Poincaré

We accept facts of dualizing complexes: $C^{BM}_\bullet(X)$ is dual to $C^\bullet(X) = R\Gamma(X, \mathbb{Z})$, $\mathbb{Z}$ the constant sheaf, and $D_\bullet(X) = p^!(\mathbb{Z}_{\operatorname{Spec} k})$ pull back via the structural map $p: X \to \operatorname{Spec} k$ (Here $p^!$ is defined as next)

This fact is for the fundamental classes taking value in chain-level (compatible with our feeling of cycle map)

Prop 23. ① (Functoriality) $f: X \to Y$ proper, we have $f_*: C^{BM}_\bullet(X) \to C^{BM}_\bullet(Y)$ proper pushforward

$f: X \to Y$ smooth, we have $f^!: C^{BM}_\bullet(Y) \to C^{BM}_\bullet(X)(-d)[-2d]$ Gysin map
relative dim $= d$
$f^! \simeq f^*(d)[2d]$ flat pullback

② $C^{BM}_\bullet(\operatorname{Spec} k)$ has unit 1
we set

$[X] = p^!(1) \in C^{BM}_\bullet(X)(-d)[-2d]$ the fundamental class, and Poincaré dual $\cap [X]: C^\bullet(X) \to C^{BM}_\bullet(X)(-d)[-2d]$

③ (Excision) $i: Z \underset{\text{closed}}{\hookrightarrow} X \underset{\text{open}}{\hookleftarrow} U : j$, $X = U \amalg Z$ ⇒ distinguished triangle $C^{BM}_\bullet(Z) \to C^{BM}_\bullet(X) \to C^{BM}_\bullet(U) \xrightarrow{+1}$

④ (Descent) $C^{BM}_\bullet(X) \simeq \lim C^{BM}_\bullet(S)$, where $s: S \to X$ smooth, S affine

Pf. ② & ③ needn't to prove, ④ ~~($s: S \to X$)~~ omitted. We point out: if one extend $C^{BM}_\bullet(\mathfrak{X})$ to $\mathfrak{X}$ algebraic stack (Artinian) we define by this manner with $s: S \to \mathfrak{X}$, S schemes, s is smooth stack morphisms.

① It's Grothendieck's six functors formalism and $C^{BM}_\bullet(X)(-i)[-2i] \cong Z^{d-i}(X)$ gives f_* and f^*; for $f^!$, consider the $f_* = f_!$ as f proper ⇒ $f_* f^! = f_! f^! \to \mathrm{id}$ in monoidal category ⇒ its right transpose when f smooth (not proper needed) is $\mathrm{id} \to f_* f^*(-d)[-2d]$ gives $f^!$. ☒ (I don't understand it)

This $C^{BM}_\bullet(X)$ is a good replacement in derived intersection theory

Deligne's realisation functors description of pure/mixed motive:

There're many construction of mixed motives, our Voevodsky's Geometric motive, his stable Geometric motive $SH_{gm}(k)$, and Nori's mixed motive.... Deligne gives an abstract description, saying that a mixed motive/$\mathbb{Q}$ is a compatible system of realisation and axioms:

Def 17. (Deligne, [Le Groupe Fondamental de la Droite projective Moins Trois Points])
A (mixed) motive over $\mathbb{Q}$, $MM(\mathbb{Q})$ consists objects $(M_B, M_{DR}, M_{A^f}, M_{cris,p}, \mathrm{comp}_{DR,B}, \mathrm{comp}_{A^f,B}, \mathrm{comp}_{cris,p,DR}, W^\bullet, F^\bullet, F_\infty, \varphi, \phi)$
(M1) $\mathbb{Q}$-vector space M_B, called Betti's realisation; (M2) $\mathbb{Q}$-vector space M_{DR}, called de Rham's realisation
(M3) A^f-module M_{A^f}, called Étale cohomology's realisation
(M4) $\mathbb{Q}_p$-vector space $M_{cris,p}$, called cristalline's realisation (mod p)

(M5) Comparison isomorphisms (preserve filtrations later)
$\mathrm{comp}_{DR,B}: M_B \otimes \mathbb{C} \xrightarrow{\simeq} M_{DR} \otimes \mathbb{C}$
$\mathrm{comp}_{A^f,B}: M_B \otimes A^f \xrightarrow{\simeq} M_{A^f}$
$\mathrm{comp}_{cris,p,DR}: M_{DR} \otimes \mathbb{Q}_p \xrightarrow{\simeq} M_{cris,p}$

(M6) $(M_B, M_{DR}, M_{A^f}, M_{cris,p})$ are all filtered by $W^\bullet$
(M7) (Frobenius at infinity) $\exists F_\infty: M_B \to M_B$ involution
(M8) (Hodge) M_{DR} admits a Hodge filtration $F^\bullet$
(M9) $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \overset{\varphi}{\curvearrowright} M_{A^f}$, respect filtration
(M10) (Cristalline Frobenius) $\exists \phi: M_{cris,p} \to M_{cris,p}$ auto

(A1) via $\mathrm{comp}_{DR,B}$ the Hodge Filtration pass to M_B, this makes $(M_B, W^\bullet, F^\bullet)$ has mixed $\mathbb{Q}$-Hodge structure
(A2) $M_B \otimes \mathbb{C} \xrightarrow{\simeq} M_{DR} \otimes \mathbb{C} \xrightarrow{C_{DR}} M_{DR} \otimes \mathbb{C} \xrightarrow{\simeq} M_B \otimes \mathbb{C} \xrightarrow{C_B} M_B \otimes \mathbb{C}$ composite and restrict to M_B gives F_∞
Where C_B and C_{DR} are involution by conjugate of $\mathbb{C}$ over $\mathbb{R}$
(A3) $A^f = (\prod \mathbb{Z}_l) \otimes \mathbb{Q}$ (restricted product) is adèle ⇒ $A^f \hookrightarrow (\prod \mathbb{Z}_l) \otimes \mathbb{Q} \twoheadrightarrow \mathbb{Z}_l$ ⇒ ∀ l prime, $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \curvearrowright M_l$ is unramified except at l itself and finite primes

$$\begin{tikzcd} M_{A^f} \arrow[r, dashed] & M_l \end{tikzcd}$$

(A4) ... I don't understand this condition on cristalline Frobenius.
(A5) $c \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is complex conjugation
then $\varphi(c): M_{A^f} \to M_{A^f}$ preserves $M_B \hookrightarrow M_B \otimes A^f \xrightarrow{\simeq} M_{A^f}$ and $\varphi(c)|_{M_B}: M_B \to M_B$ just F_∞

And morphisms are morphism of these vector spaces/modules, respect filtrations and Frobenius, actions. /Gal

Def 18. (Realisation functors of $DM_{gm}(k)$) A realisation functor is $\mathrm{Sm/Var}(k) \xrightarrow{\tilde{R}} \mathrm{Comp}(\mathcal{A})$ abelian

Where $\mathcal{A}$ is Abelian cat with enough injectives. little in next

$$\begin{tikzcd} \mathrm{Sm/Var}(k) \arrow[r, "\tilde{R}"] \arrow[d] & \mathrm{Comp}(\mathcal{A}) \arrow[d] \\ DM_{gm}(k) \arrow[r, "R"] & D^b(\mathcal{A}) \end{tikzcd}$$
triangulated

We discuss the properties of R abstractly, only for "next" "examples" (In fact these are not difficult, properties of DM^{eff}_{gm} & DM_{gm} before is enough; more detail: Realisation of Voevodsky's motives, Huber.

Eg 22. ① (Betti realisation) $\mathrm{Sm/Var}(k) \xrightarrow{R_{Betti}} \mathrm{Comp}^{\geq 0}(\mathrm{Vect}(\mathbb{Q}))$ and extend to R_{Betti} gives Betti cohomology
$X \longmapsto (\mathrm{Hom}(\Delta^\bullet, X(\mathbb{C})) \otimes \mathbb{Q})^\vee$ $H^\bullet_{B,\sigma}(X; \mathbb{Q}) = H^\bullet(X(\mathbb{C}), \mathbb{Q})$, fix $\sigma: k \to \mathbb{C}$
(dual of $\mathbb{Q}$-vector space)

② (Étale realisation)
$\mathrm{Comp}_{Sm}(k) \to \mathrm{Comp}^{\geq 0}(\mathrm{Ab})$
$X \longmapsto (\mathrm{Hom}^\bullet_{et}(M_{gm}(X) \otimes (\mathbb{Z}/l^n\mathbb{Z}), \mathcal{I}^\bullet))$ (*) where $\mathcal{I}^\bullet$ is injective resolution of $\mathbb{Z}/l^n\mathbb{Z}$ constant (pre) sheaf

$$\begin{tikzcd} \mathrm{Comp}_{Sm}(k) \arrow[r, "\tilde{R}"] \arrow[d] & \mathrm{Comp}(\mathcal{A}) \arrow[d] \\ DM^{eff}_{gm}(k) \arrow[r, "R"] & D^b(\mathcal{A}) \end{tikzcd}$$

all these sheaf are Étale sheafification of original Nisnevich sheaf

extends to $R_{et}: DM_{gm}(k) \to D^b((\operatorname{Spec} k)_{et}, \mathbb{Z}/l^n\mathbb{Z})$ as (*) computes Étale motivic cohomology

. I have a poor understand on cristalline cohomology, omitted. And de Rham is easy as it's $H^{*}_{dR}(X,\Omega^{\bullet}_{X/k})$ is just a [illegible] $\boxed{\text{[illegible]}}$ — Moore homology.

Prop2.4, [redacted] We can extend $\mathrm{Comp}^{b}_{fin}(k)\xrightarrow{\tilde{R}}\mathrm{Comp}^{\geq 0}(A)$ if $\tilde{R}$ has descent for open coverings MV(sequence) and it's homotopy invariant ($R(X\times\mathbb{A}^1)\simeq R(X)$). Further if $R(\mathbb{Q}(1))$: $D^b(A)\to D^b(A)$ equivalence we extend to $DM_{gm}(k)\to D^b(A)$

$$\begin{tikzcd}
\mathrm{Comp}^{b}_{fin}(k) \arrow[r, "\tilde{R}"] \arrow[d] \arrow[dr, phantom, "R"] & \mathrm{Comp}^{\geq 0}(A) \arrow[d] \\
DM_{gm}(k) \arrow[r, "R"] & D^b(A)
\end{tikzcd}$$

Pf, Recall $DM_{gm}(k) =$ [redacted] $(K^b(\mathrm{Comp}^{b}_{fin}(k))[\mathbb{L}^{-1}])^{\#}$

As $\tilde{R}$ respects MV sequence & homotopy

$\Rightarrow \tilde{R}$ extend to $K(\mathrm{Comp}^{b}_{fin}(k))[\mathbb{L}^{-1}]$ if we extend to $K^b(\mathrm{Comp}^{b}_{fin}(k))$

and by $\mathrm{Comp}^{\geq 0}(A)$ is Karoubian $\Rightarrow$ induce $(R)^{\#}$ done.

Thus only to extend to and Diff $\mathrm{Comp}(\mathrm{Comp}^{b}_{fin}(k))$ (the morphisms are same extension) as $K^b(\mathrm{Comp}^{b}_{fin}(k))$ is quotient cat of $\mathrm{Comp}^b(\mathrm{Comp}^{b}_{fin}(k))$

R: $X^{\bullet}\mapsto R^{\bullet}X^{\bullet}$ is a bicomplex, taking its total complex is desired ■ $\Rightarrow K^b(\mathrm{Comp}^{b}_{fin}(k))\to K^b(A)\to D^b(A)$ ☒ (Naturally as total complex not loss information of (co)homology)

Chap2, Enumeration

Intro: We had built fundamental formulas in Chap1, now we'll use it to count basically via many examples, especially in lower dimensional. Naturally {counting some types geometric objects} = {counting points in moduli space}, thus our second things is to show how parameter spaces used into enumeration. Enumeration is more old than algebraic geometry itself and may concern about non-algebraic cases, we focus on algebraic settings, but tools from topological, esp. after Donaldson we using enumerative invariants on a moduli space to count, this leads to two questions; ① Intersection on moduli stack, as these invariants take value $\in$ (co)homology ② Using these invariants back to counting. We may point out some relation with other fields as symplectic topology, representation theory, physics I familiar with. If time permitting, at last we'll sketch derived intersection theory.

§2.1. Counting curves

Q. What are enumerative problems?

We need to describe the set $\Phi=\{C\mid C \text{ satisfy conditions } (C_1),(C_2),\dots,(C_n)\}=\bigcap\Phi_i$, each $\Phi_i=\{C\mid C \text{ satisfy } (C_i)\}$: that's why intersection. Expect that $\#(\Phi)<\infty$ and compute $\#(\Phi)=\int_{\mathcal{M}}[\Phi]$, where $\Phi\subset\mathcal{M}$ is zero-dimensional, $\mathcal{M}$ is a proper & smooth parameter space/moduli space. (Seek for compactification of $\mathcal{M}$ otherwise & obstruction theory → smoothness; In a primary stage, our examples are Grassmannian & projective bundles..)

Aiming to find a way to solve it, more questions arose naturally

Q1, How we can return $\prod[\Phi_i]=[\Phi]^{vir}_{E}$, where E is normal bundle of $\underline{\Delta\subset\mathcal{M}\times\mathcal{M}\times\dots\times\mathcal{M}\text{ (n times)}}$ (that's why we need $\mathcal{M}$ smooth) back to the geometry of Φ itself? As the class of Φ loses information of Φ itself.

Q2, What's more additional conditions should be put into the intersection of Φ_i, makes Φ does have expected 0-dimension? Then by the properness/compactness $\Rightarrow \#\Phi<\infty$ and our counting is good.

This method is most natural to understand, and it's called "Naive counting" in [$1\frac{1}{2}$ ways counting curves] as the first "$\frac{1}{2}$" way.

A1, We have following facts: [redacted] If $\boxed{[\Phi]^{vir}_{E}=0}$ ~~we loss all informations of Φ~~

$[\Phi]^{vir}_{E}\neq0\Rightarrow\Phi$ non-empty (obvious)

$\Rightarrow$ we can discuss $\int_{\mathcal{M}}[\Phi]^{vir}_{E}=:x<\infty$ $\begin{cases} x>0\Rightarrow\#\Phi\leq x \text{ as some point of } \Phi \text{ has multiplicity (obvious)}\\ \boxed{x=0}\Rightarrow\Phi=\emptyset \text{ or } \underline{\Phi \text{ infinite}} \text{ (Why may infinite?)}\\ x<0\Rightarrow\Phi \text{ not expected dim, must infinite! (Why?)}\end{cases}$

x is called the virtual number

Let Φ_i intersect in expected dimension [redacted]

A2. To exclude exceptional cases in A1, we put condition on intersection: $\boxed{\text{Generically transversal}}$: $A\cap B=C$ iff local deg of each generic pts of component C_i of C is transversal. and investigate the geometry of Φ itself to determine Φ is of expected dimension $\Rightarrow x=\#\Phi$ is ensured.

We show However, it's hardest as Φ_i are much complexer than cycle $[\Phi]$, need more (local) information.

[redacted] many examples on this "naive" method [redacted] next;

Rk. Here our method is called "naive" due to $\mathcal{M}$ assume to be scheme. For more complex problems, $\mathcal{M}$ only DM stack and we need describe its Chow ring. Analytic story? Compactness ⟺ Stable

[redacted]

Conclusion. Parameter space $\xrightarrow{\text{compactification}}$ virtual number $\xrightarrow{\text{transversality}}$ Answer ①&② is two fundamental steps in both algebraic & topology (by Donaldson)

We'll proceed from easy, classical (such as counting polynomial) into modern, complicated case (how we use them ...?)

Rk. ② sometimes useless, it's up to concrete problem ② is too hard, a general method is local deformation + DT invariants ... and Def = tangent space to compute dimension; but in concrete problem, it's almost impossible to do, and various way developed.

construction of Morse/Floer is by counting trajectories also need $\boxed{\text{compactness}}$ + $\boxed{\text{transversality}}$

Eg. (Counting lines in $\mathbb{P}^1$, $\mathbb{P}^2$, $\mathbb{P}^3$) ⟺ Fredholm

① ($\mathbb{P}^1$) Our "—" for lines is due to in $\mathbb{P}^1$ we not have any lines. Our problem is: counting roots of a polynomial $f(x)=a_0x^d+\dots+a_d$ on $\mathbb{A}^1=\mathbb{C}^1$. ($a_0=0$ is allowed as degenerated case)

The degenerate case makes it differs algebraic fundamental thm; let's forget it now.

First consider $d=1$ and 2, $\begin{cases} d=1: x=-\frac{a_1}{a_0} \text{ is root } \begin{cases} a_0\neq0, \text{ done}\\ a_0=0, \begin{cases} a_1=0, \text{ infinite solution}\\ a_1\neq0, \boxed{\text{no}} \text{ solution let } a_1\to0,\ x\to\infty \text{ is a solution "in fact"}\end{cases}\end{cases}\\ d=2: x=\frac{-a_1\pm\sqrt{a_1^2-4a_0a_2}}{2a_0} \text{ two roots}\end{cases}$

Thus it's easy to imagine higher cases.

Except infinite solution case we handle degenerate case via red words, in fact, i.e. solution on $\mathbb{A}^1$.

$\begin{cases} a_0\neq0, \text{ done, in high school, two or one (double multiplicity) roots / general two roots}\\ a_0=0 \begin{cases} a_1\neq0, \boxed{\text{one}} \text{ solution } x=-\frac{a_2}{a_1} \text{ let } a_0\to0, \text{ another solution } x\to\infty \text{ is a "solution" in fact}\\ a_1=0 \begin{cases} a_2\neq0, \text{ no solution. let } a_1\to0,\ x_1,x_2\to\infty \text{ is a solution "in fact" with multiplicity 2}\\ a_2=0, \text{ infinite solution}\end{cases}\end{cases}\end{cases}$

This is (may) the easiest motivation of compactifying moduli space: here our parameter space $\mathbb{A}^d$ parameterizes roots of equations, but not closed under limits $a_0\to 0, a_1\to 0\cdots$, i.e. not compact/proper, can't admit $\{\infty\}$.

Fact. Any nonzero homogenous polynomial $F(x_0,x_1)$, $\deg F=d$ defined on $\mathbb{P}^1=\{[x_0,x_1]\cdots\}$ has d roots, counting multiplicity.
The proof is easy argument on polynomial: Induction on d, $F(x_0,x_1)=\sum_{i\le d}a_i x_0^i x_1^{d-i}$, $a_0\neq 0$ (otherwise $F=x_0G$ and reduce to $\deg G=d-1$ case, adding one root $(0,1)$, complete proof)
Then the dehomogenization $f(x)=\sum_{i\le d}a_i x_1^{d-i}$ (i.e. restrict on $\mathbb{A}^1$ with $\{x_0=0\}$) is nondegenerate $\Rightarrow$ also d roots ☒

② $(\mathbb{P}^2)^\vee\{\text{Lines}\subset\mathbb{A}^2\}\supsetneq\mathbb{A}^2$, $\mathbb{A}^2$ can't parameterize lines of $y=c$, i.e. $a\to\infty$, again we compactify to $\mathbb{P}^2$, and fundamental fact of AG tells us $(\mathbb{P}^2)^\vee\cong\mathbb{P}^2$ desired via $\{a_0x_0+a_1x_1+a_2x_2=0\}\leftrightarrow\{(a_0,a_1,a_2)\}$
$\{y=ax+b\}\longleftrightarrow(a,b)$

Two things on $\mathbb{P}^2$ are easy to understand
For example, consider fix $p\in\mathbb{P}^2$, {lines pass p} $=\{L\subset\mathbb{P}^2\mid p\in L\}=\{[a_0x_0+a_1x_1+a_2x_2=0]\mid a_0p_0+a_1p_1+a_2p_2=0\}$
$=\{(a_0,a_1,a_2)\in\mathbb{P}^2\mid a_0p_0+a_1p_1+a_2p_2=0\}\subset\mathbb{P}^2$ is line of dual $\mathbb{P}^2$
Thus by Bézout thm #{lines passing two points $p\neq q$}
$=\#\{$two general lines intersects$\}=1$

③ $(\mathbb{P}^3)\{\text{lines}\subset\mathbb{P}^3\}\cong G(2,4)$ introduced. Let's consider the enumerative question: $\#\{L\subset\mathbb{P}^3\mid L\cap L_i\neq\emptyset, i=1,2,3,4\}$ with L_i are general lines.
Recall structure of $A^*(G(2,4))$: Schubert cycles are $\Sigma_{2,2}\subset\Sigma_{2,1}\subset\begin{smallmatrix}\Sigma_{2,0}\\ \Sigma_{1,1}\end{smallmatrix}\subset\Sigma_{1,0}\subset G(2,4)$, and $\sigma_{i,j}=[\Sigma_{i,j}]$ the class $\in A^*(G(2,4))$
By Pieri, we have its multiplication table by

$\sigma_{1,0}^2=\square\times\square=\ydiagram{1,1}+\ydiagram{2}=\sigma_{1,1}+\sigma_{2,0}$ $\quad A^1\otimes A^1\to A^2$
$\sigma_{1,0}\sigma_{1,1}=\square\times\ydiagram{1,1}=\ydiagram{2,1}=\sigma_{2,1}=\square\times\ydiagram{2}=\sigma_{1,0}\sigma_{2,0}$ $\quad A^1\otimes A^2\to A^3$
$\sigma_{1,0}\sigma_{2,1}=\square\times\ydiagram{2,1}=\ydiagram{2,2}=\sigma_{2,2}$ $\quad A^1\otimes A^3\to A^4$
$\sigma_{1,1}^2=\ydiagram{1,1}\times\ydiagram{1,1}=\ydiagram{2,2}=\sigma_{2,2}=\ydiagram{2}\times\ydiagram{2}=\sigma_{2,0}^2$
$\sigma_{1,1}\sigma_{2,0}=\ydiagram{1,1}\times\ydiagram{2}=0$ $\quad A^2\otimes A^2\to A^4$

Thus $\int_{G(2,4)}\sigma_{1,0}^4=\int_{G(2,4)}(\sigma_{1,1}+\sigma_{2,0})^2=\int_{G(2,4)}(\sigma_{1,1}^2+\sigma_{2,0}^2)=\int_{G(2,4)}2\overset{\text{one pt}}{\boxed{\sigma_{2,2}}}=2$ (we'll omit "$G(2,4)$" later without confusion)
where $\sigma_{1,0}=[\Sigma_{1,0}]=[\{\Lambda\mid\Lambda\cap L\neq\emptyset\}]$ desired.
Thus our expectation of $\#\{\cdots\}=2$ is solution
Similarly we can "invert" enumerative question for a top cycle class
e.g. $\int\sigma_{1,0}^2\sigma_{1,1}=1\Rightarrow\#\{L\subset H\cong\mathbb{P}^2\subset\mathbb{P}^3\mid L\cap L_1\neq\emptyset\neq L\cap L_2\}$ is the number of lines intersect two general lines is:
(all are special cycle in lower dimension, otherwise it's more complex to compute determinant)
⊂ fixed general plane
When not general:

How can we check the solution does 2, compatible with our expectation? We make use the "general" of L_1-L_4 each
Step1 General of $L_2\&L_3$ makes them looks like
Step2 General of $L_1\Rightarrow\forall x\in L_1$ is general
$\Rightarrow\exists!\,M_x$ intersect both L_2,L_3 transversally:

Step3 $\coprod_{x\in L_1}M_x=:Q$ is a quadratic surface

Step4 General of $L_4\Rightarrow L_4\cap Q$ is two points by Bézout thm.
For higher $G(m,n)$, it's obvious $\int\sigma_1^k=$ [illegible] generalized [illegible]

Thus it suffices to prove Step3:
We prove $\exists$ quadric surface $Q\supset L_1\cup L_2\cup L_3$
then by Bézout again, $M_x\cap L_i$ [illegible] $=0$
for $i=1,2,3\Rightarrow M_x\subset Q\Rightarrow Q\supset\bigcup M_x$
then as $\bigcup M_x$ is surface too $\Rightarrow Q$ is unique $=\bigcup M_x$
(otherwise $\bigcup M_x\subset Q_1\cap Q_2$ can't be a surface) done
Now {quadric surface $Q\supset L_1\cup L_2\cup L_3$}
$=$ {quadric homogenous polynomial vanishing at L_i, $i=1,2,3$}

$$\begin{array}{ccc}H^0(\mathcal{O}_{\mathbb{P}^3}(2)) & \longrightarrow & H^0(\mathcal{O}_{\coprod L_i}(2))\\ & & \|\\ & & \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(2))\end{array}$$

has dimension
$\geq h^0(\mathcal{O}_{\mathbb{P}^3}(2))-3(h^0(\mathcal{O}_{\mathbb{P}^1}(2))-h^0(\mathcal{O}_{[illegible]}(2)))$
$=\binom{5}{2}-3(\binom{[illegible]}{}-\binom{3}{2})=10-9=1>0$
(Recall $h^0(\mathcal{O}_{\mathbb{P}^n}(m))=\binom{m+n}{m}$) $\Rightarrow\exists Q$ ☒

Eg.2. (Rational curve) Rational curve $C\cong\mathbb{P}^1$, i.e. an embedding $f:\mathbb{P}^1\hookrightarrow X$, usually we can embedding $X\hookrightarrow\mathbb{P}^n$, as X as parameter space is hard to handle, thus $f:\mathbb{P}^1\hookrightarrow\mathbb{P}^n$ of degree d means $\deg P_0=\cdots=\deg P_n=d$
It's natural to generalize $(x,y)\mapsto(P_0(x,y),\cdots,P_n(x,y))$, $\gcd\{P_0\cdots P_n\}=1$ into $\mathrm{Map}_d(\mathbb{P}^1,\mathbb{P}^n)$ parameterized all rational maps of degree d, and {all rational curves} $\subset\mathrm{Map}_d(\mathbb{P}^1,\mathbb{P}^n)$, we have known former is projective, one asks is $\mathrm{Map}_d(\mathbb{P}^1,\mathbb{P}^n)$ good? $\mathrm{Map}_d(\mathbb{P}^1,\mathbb{P}^n)\hookrightarrow\mathbb{P}(\mathrm{Hom}(\mathrm{Sym}^d(\mathbb{A}^2),\mathbb{A}^{n+1}))$ is the projectized of a linear space, but $\mathrm{Map}_d(\mathbb{P}^1,\mathbb{P}^n)$ isn't "closed"
What here "closed" means? $PGL(2,k)\curvearrowright\mathbb{P}^1$ induces $PGL(2,k)\curvearrowright\mathrm{Map}_d(\mathbb{P}^1,\mathbb{P}^n)$ not free, the result "quotient space" (how to define) has automorphism of some pts $\Rightarrow$ stacky issue. Our conclusion is $\mathrm{Map}_d(\mathbb{P}^1,\mathbb{P}^n)/PGL(2,k)$ not proper. It's later story to compactify it to moduli of stable maps.

IIB side | mirror symmetry | IIA side = Symplectic. Mirror symmetry is useful tool to compute GW and quantum ring.

Rk. Over $\mathbb{C}$, $\mathbb{P}^1_{\mathbb{C}}=S^2$, $f:S^2\to X$ algebraically $\Longleftrightarrow$ $u:(S^2,j)\to(X,J)$ analytically
(or higher genus Riemannian surfaces) algebraic
$du\circ j=J\circ du$, ⋯ Cauchy–Riemann equation $\Leftrightarrow$ J-holomorphic curves

The number result is same GW-invariant proved by [Jun Li and Gang Tian] [illegible]
Later we'll introduce the left side algebraically/physically, two different AG

$\{f:S^2\to X\}$ $\Longleftrightarrow$ $\{u:S^2\to X\}$
rational maps $\Downarrow$ compactification — J-holomorphic curves $\Downarrow$ compactification
stable maps $\Longleftrightarrow$ stable curves
moduli is dim 0 + pt
GW invariants

Same as GW invariant, closely related is the quantum cohomology rings, are same via the two approaches

Return back to rational curves counting, even this easy case we can relate big pictures like mirror symmetry
First consider an analogous question as Eg.1③: replace L_1-L_4 general by C_1-C_4 four curves of $\deg=d_1,\cdots,d_4$ as general

The answer is similiar ⓐ $\Sigma_C=\{L\subset\mathbb{P}^3 \mid L\cap C\neq\emptyset\}$ has $[\Sigma_C]=\deg(C)\cdot\sigma_{1,0}$ $\Rightarrow \int[\Sigma_{C_1}]\cdot[\Sigma_{C_2}]\cdot[\Sigma_{C_3}]\cdot[\Sigma_{C_4}]=2d_1d_2d_3d_4$ Page 2

ⓑ Check its geometry is more complex: one study the local geometry by tangent space of $G(2,4)$

① First we show $[\Gamma_C]\in A^1(G(2,4))$, consider the incidence correspondence $\Sigma=\{(p,L)\in C\times G(2,4)\mid p\in L\}$ note $p_2(\Sigma)=\Gamma_C$ by definition

$$\begin{tikzcd} & \Sigma \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ C & & G(2,4)\supset\Gamma_C \end{tikzcd}$$

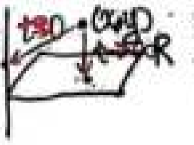
$\Rightarrow \dim\Gamma_C=\dim\Sigma=\dim C+\dim p_1^{-1}(c)$
$= 1+2=3$ all these dimension is taken generic

Here $\dim p^{-1}(c)=2$ for $c\in C$ general as $p^{-1}(c)=\{L\in G(2,4)\mid c\in L\}=\Sigma_{2,0}$.
Thus $[\Gamma_C]\in A^1(G(2,4))=\mathbb{Z}[\sigma_{1,0}] \Rightarrow [\Gamma_C]=a\sigma_{1,0}$

● To determined coeffiecent $a\in\mathbb{Z}$, we pair into top class and integral $\int[\Gamma_C]\cdot\sigma_{2,1}=\int a(\sigma_{1,0}\cdot\sigma_{2,1})=a\cdot 1=a$

Now we need to compute $\int([\Gamma_C]\cdot\sigma_{2,1})$ by look at its geometry: we'll use the specialization/degeneration arguments (differ [Fulton]'s specialize to normal cone). What it means? Now our parameter space ($\mathbb{P}^n$, $G(m,n)$, $\mathbb{P}E$) all have a natural PGL-action (Furthermore, general parameter space always have one), then the orbits has limit $t\to 0$ to the closure of orbit, usually have a special fibre/object.

Here we construct the orbit by acting C by a one-parametered family $\subset PGL(4,k)$, this family given by trick of dynamic projection: On $\mathbb{P}^n$ spanned by two projective planes: A (attractor) & R (repellor)

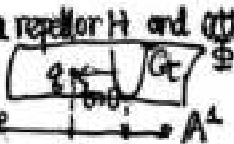


We focus on $t\to 0$ that's why A is called attractor

$p\in\mathbb{P}^n$ has $p=(x,y)$, A is $\{x=0\}$, then $\varphi_t:\mathbb{P}^n\to\mathbb{P}^n$
and R is $\{y=0\}$ $\quad (x,y)\mapsto\varphi_t(x,y)=(x,\frac{y}{t})$

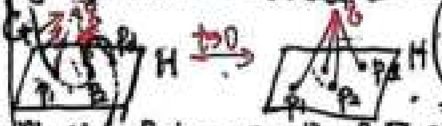
and for $X\subset\mathbb{P}^n$, we have $\{\varphi_t(X)\}\subset$ parameter space desired $\mathbb{G}_m$-orbit.

Now for $C\subset\mathbb{P}^3$, $\Sigma_{2,1}=\{L\mid p\in L\subset H\}$ with p, H generally fixed, take a repellor H and attractor $q\notin H$ gives $(\varphi_t)\subset PGL(4,k)$
then $C_t:=\varphi_t(C)$, we get $\Phi=\{(t,p)\in\mathbb{A}^1\times\mathbb{P}^3\mid t\neq 0,\ p\in C_t\}\to\mathbb{A}^1$



$\varphi_t=\begin{pmatrix}1&&\\&t&\\&&t\\&&&t\end{pmatrix}$ a projectized matrix

Then By Φ is deformation space, it gives a rational equivalence to the fibre at 0
$[\Phi_0]\sim$ By degeneration to Φ_0, $\sum[\Sigma_{2,1}(p_i,q)]$, where showed in the following picture:



(Rk. The $[\Phi_0]\neq\sum[\Sigma_{2,1}(p_i,q)]$ may have embedded pts. but rational equivalence not consider what Φ_0 is.)
$\Rightarrow d$ points $\Rightarrow \int[\Gamma_C]\cdot\sigma_{2,1}=d$ ▨

② Verification of transversality of $\Gamma_{C_i}\cap\Gamma_{C_j}$; $i,j=1,2,3,4$ (It's Exercises 3.30–3.33 in [3264])

We need to prove: (a) $[L]\in\Gamma_C$ is smooth point we tangent space $T_{[L]}\Gamma_C$ [redacted] (here L as 2-dim linear space)
(b) Γ_{C_i} intersects transversal at $[L]\Leftrightarrow$ Four projective lines $\mathbb{T}_{p_i}C_i$ are skew, $p_i=C_i\cap L$ (projectized of $T_{p_i}C_i$)
thus when C_1, C_2, C_3, C_4 are taken to be general, Γ_{C_i} intersect transversal

$(p=C\cap L)=\{\alpha\in\operatorname{Hom}(L,k^4/L)\mid p\subset L$ as 1-dim linear space, $\alpha(p)\subset(\mathbb{T}_pC+L)/L\}$

(Rk. Here all projective points, lines, p, L, ... are taking their affine cone to vector space with dim higher 1.)

Both (a) & (b) are easy to prove next, I mention that the only hard part is how to deal with the case when general $\not\Rightarrow \mathbb{T}_{p_i}C_i$ skew.

[redacted] The projectized bundle is $\mathbb{P}(E\oplus 1)$, [redacted] projectized space [redacted] also $\mathbb{P}(\mathrm{Vect})$ $\Rightarrow \mathbb{T}_{p_i}C_i$ is projective line

Here $T_{p_i}C_i$ is (expected to be) one dimensional

I omit this hardest step's detail, its idea is by: otherwise $\mathbb{T}_{p_i}C_i$ not skew, i.e. two of them are [redacted] $\subset$ some plane H
We take four general automorphism of $\mathbb{P}^3\in PGL(4,k)$, then we have an incidence correspondence
$\Phi=\{(\varphi_1,\varphi_2,\varphi_3,\varphi_4,p_1,p_2,p_3,p_4,H,$ [redacted] $q_1,q_2,q_3,q_4, L)\mid p_i\in C_i,\ q_i\in L\subset H,$ [redacted] $q_i=\varphi_i(p_i)$, $\mathbb{T}_{q_1}\varphi_1(C_1)_{q_1}, \mathbb{T}_{q_2}\varphi_2(C_2)_{q_2}\subset H\}$
$\subset(PGL(4,k))^4\times(\mathbb{P}^3)^4\times(\mathbb{P}^3)^4\times\mathbb{P}^{3*}\times G(2,4)$

$$\begin{tikzcd} & \Phi \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ (PGL(4,k))^4 & & \square \end{tikzcd}$$

after general automorphism holds

We can counting dimension of Φ (59-dimensional) to give a contradiction ▨ (Why we must take general auto?)

Pf of (a) and (b): First note (a) $\Rightarrow$ (b) directly, only prove (a)
(a) It's all linear algebra, we take local neiborhood $U_\Gamma=\{\Lambda\in G(2,4)\mid\Lambda\cap\Gamma=0\}\ni L$ [redacted]
And take coordinates (x_0,x_1,x_2,x_3) on $G(2,4)$ $\subset G(2,4)$
s.t. $\Lambda=V(x_2,x_3)$, $\mathbb{T}_pC=V(x_1,x_3)$, $\Gamma=V(x_0,x_1)$ I don't know why.

● Two much more modern enumerative problems:
Q1. How many rational quartic curves $C\subseteq\mathbb{P}^2$ contains fixed 11 general pts $\in\mathbb{P}^2$?
Q2. (Clemens' conjecture) How many rational deg $=d$ curves $C\subset$ a general quintic threefold $X^3\subset\mathbb{P}^4$ hypersurface

Quadric rational curve called conic (distinct!)

Recall our notation
quadric : degree $=2$
cubic : degree $=3$
quartic : degree $=4$
quintic : degree $=5$

The first shows a case of parameter space noncompact & compactification to Kontsevich's space of stable maps, the second has no primary AG proof but using physics to give some proceed.
Clemens' conjecture conjected that it does finite (later we'll show why we expect this), and proven by [Katz] in $d\leq 9$, but still open now (2024.10.16) in general. Counting it we find the number is large and hard (although fundamental) (see the diagram next pages), when degree is low. But in 1991, physicists first used mirror symmetry to count all $d>0$.

d	number	time
1	2875	19th
2	609250	1980
3	317206375	1991
4	...	1991
5	...	1991

We'll explain how so-called Mirror theory used to counting Clemens conjecture, and it gives a motivation toward stable map (as $\textcircled{2}$ is former than $\textcircled{3}$) which start as the famous [CDGP], i.e. P. Candelas, X. de la Ossa, P. Green, L. Parkes, a pair of CY-mfds as an exactly soluble superconformal field theory.

let $X^3 = V(f)$, $\deg f = 5$ is homogenous & general$/\mathbb{C}$.

Fact 1 X^3 is Calabi–Yau 3-fold, i.e. $K_X \cong \mathcal{O}_X \Rightarrow X$ has mirror mfds.

Fact 2 [CDGP] $F_A \xlongequal[\text{symmetry}]{\text{Mirror}} F_B = \int_\alpha \Omega_{Y_\psi}$ [illegible], where $\alpha \in H_3(Y_\psi)$ (top as complex $\mathbb{C}$) and Y is mirror quintic, $\int_\alpha$ means period integral

Rk. These are virtual numbers. Or we can check the local geometry (or virtual, i.e. we only count its virtual counting)

F_A is our desired counting of genus$=0$ = rational holomorphic curves defined in type IIA theory (Also not algebraic, but by complex geometry, we know this does what we want) (We not explain this physic terms here)

Fact 3 (Yukawa couplings) We need to explain why the limit of period integral is computed

$$\begin{tikzcd}
SCFT_A(X) \arrow[r, phantom, "A(X)=B(Y)"] \arrow[drr] & & SCFT_A(Y) \arrow[dll] \\
SCFT_B(X) \arrow[r, phantom, "B(X)=A(Y)"] & \arrow[l, "\text{mirror}"'] & SCFT_B(Y)
\end{tikzcd}$$

If the red equality holds, then the black equality holds and then $A(X)$, $B(X)$ computes certain limits for $SCFT_B(X)$, $SCFT_A(X)$ as the purple arrow shows in the diagram.

(They can be realised as functors in math as TQFT) Where SCFT is super(symmetric) conformal field theory of supersymmetric σ-model, and A, B are Witten's topologically twisted σ-model two theories

Now we can only prove **Fact 1**, and we explain why we explain the Clemens' conjecture following, indeed (i), (ii); if time permits we'll (much) later compute the 317206375 curves.

(i). Let $\mathcal{I}_X = (f)$ the ideal sheaf of $X \subset \mathbb{P}^4 \Rightarrow 0 \to \mathcal{I}_X/\mathcal{I}_X^2 \to \Omega_{\mathbb{P}^4}|_X \to \Omega_X \to 0$ on X as X is smooth (general $\Rightarrow$ smooth)

By smoothness again $\mathcal{N}_{X/\mathbb{P}^4} \cong \mathcal{O}_{\mathbb{P}^4}(5)|_X$ as a section of normal bundle on $\mathbb{P}^4$ with $\deg = 5$ the normal bundle $\mathcal{N}_{X/\mathbb{P}^4} = \mathcal{I}_X/\mathcal{I}_X^2$

Rk. An exercise of [Ha] say's that for a smooth variety $X \subset \mathbb{P}^n$, $X = \cap V(f_i)$, $d_i = \deg f_i$ $\Rightarrow \omega_X \cong \mathcal{O}_X(d_1 + \cdots + d_r - n - 1)$ take $d = n+1 = 5$ $\Rightarrow$ Calabi–Yau ▨

By adjunction formula $\mathcal{T}_{\mathbb{P}^4}|_X = \mathcal{T}_X \oplus \mathcal{N}_{X/\mathbb{P}^4} = \mathcal{T}_X \oplus \mathcal{O}_{\mathbb{P}^4}(5)|_X$, taking determinant of its dual

$K_{\mathbb{P}^4}|_X = K_X \otimes \mathcal{O}_{\mathbb{P}^4}(-5)|_X$

Recall the fact that $K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-(n+1)) \Rightarrow$ Left hand is $\mathcal{O}_{\mathbb{P}^4}(-5)|_X \Rightarrow K_X \cong \mathcal{O}_X$ ▨

(ii) $u: \mathbb{P}^1 \to X^3 \subset \mathbb{P}^4$ with $\boxed{f(f_0(x,y), \ldots, f_4(x,y)) \equiv 0, \forall (x,y) \in \mathbb{P}^1}$ $(*)$

$(x,y) \longmapsto (f_0(x,y), \ldots, f_4(x,y))$, each $f_j(x,y) = \sum_{i \le d} a_{ij} x^i y^{d-i}$ parameterised by $(d+1)$-dim vector space thus $5(d+1)$-dim as 5 components

We write $(*)$ to $\sum h_k(\{a_{ij}\}) x^k y^{5d-k} = 0$, $\{a_{ij}\}$ is a $(d+1) \times 5$ matrix $\Rightarrow h_k(\{a_{ij}\}) = 0$, $0 \le k \le 5d$ has $(5d+1)$ equations

$\Rightarrow$ we have a $5(d+1) - (5d+1) = 4$-dimensional parameter space. Modulo choices of parameter isation, i.e. choices of (x,y) has 4-dimensional $\Longrightarrow$ we expect a 0-dim parameter space, adding a compactness we can have such expectation ▨

But left to compute $\textcircled{3}$: here we only use coarse moduli to restrict as not scheme (of course, compactness is hardest and lead to Kontsevich's stable maps) we point out that general Kontsevich space is not enough good: it needn't to be smooth or irreducible. But here rational ($\Leftrightarrow g=0 \Leftrightarrow$ "tree", the "tree" is called by physicists) & marked 0 pts $\overline{M}_{0,0}(\mathbb{P}^n, d)$ is quite good: $\overline{M}_{0,0}(\mathbb{P}^n, d)$ is the coarse moduli of smooth DM stack & irreducible (Although it is smooth, it shouldn't but it's (finite) quotient singularity most)

Here 11 generic pts are expected to pass as $11 = 3 \times 4 - 1$, for general, consider $(3d-1)$ pts $\in \mathbb{P}^2$ general and d-degree curve passing them, we denote our answer $N(d) \in \mathbb{Z}_{>0}$, but stop at virtual number level (otherwise its too hard and open)

We define $\overline{M}_g(\mathbb{P}^N, d) = \overline{M}_{g,0}(\mathbb{P}^N, d) := \{f: C \to \mathbb{P}^N$ | C connect curve, $p_a(C) = g$; C has worst nodal singularity; All $\varphi: C \to C$ automorphism s.t. $f \circ \varphi = f$ is finite over on C; $\deg = d$ to its target $\}$

$\boxed{f_*[C] = d[H], [H] \text{ is generator of } H_2}$

stable curve + stable map; two stable maps are same $\Leftrightarrow$ $C \xrightarrow{f_1} \mathbb{P}^N$, $\exists\, \varphi$, $C \xrightarrow{f_2}$

Marked pt case, naively is easy to define, for moduli stack, it's no section maps from base. proper

We not prove $\overline{M}_{g,n}(\mathbb{P}^N, d)$ proper as by Keel–Mori & $\overline{M}_{g,n}(\mathbb{P}^N, d)$ proper gives it directly: $\overline{M}_{g,n}(\mathbb{P}^N, d) \to M_{g,n}(\mathbb{P}^N, d)$

Rk. ① Our computing $N(d) \Longleftrightarrow$ compute the quantum cohomology ring of $\mathbb{P}^2$;

② It holds for all homogenous varieties, this approach in [FP] may be shown later.

[redacted]

Recall. Marked pts satisfy stable, then it's proper and stable map need preserve them

$g=0$, ≥ 3 pts (include intersection)

$g=1$, ≥ 1 pt (include intersection)

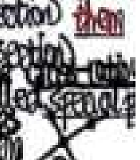
$g \ge 2$, ≥ 0 pt (include intersection)

Our proof have several steps:

Step 1 Induction on d, we need to express $N(d) = f(N(d_1), N(d_2), \ldots)$, $d_i < d$.

Step 2 We need to compute $f(-,-,\ldots)$ by counting $\boxed{\text{pole-zero numbers}}$ of a proper function on $\mathbb{P}^1$, called the cross-ratio

Step 3 $(3d-1)$-points is separated into three parts: $(3d-4)$-points $+ q$ and r two points + two pts on two lines L_1, L_2 marked pts, keep them in stability; marked pt + intersection called special pt; to define our $B \subset \overline{M}_{0,4}(\mathbb{P}^2, d)$ a curve parameterise them: as more 1 point marked but our move keep transversal



Step 4 $[C] \in B$ (as point of B), C has two cases: irreducible: putted (as L_1, L_2 fixed) line to move $\Rightarrow (d+1)$-dim is 1-dim; reducible: finite C and $C = C_1 \cup C_2$, must two components

Computing both of them

Rk. Denote $M := \overline{M}_{0,4}(\mathbb{P}^2, d)$ and $B = \{f: (C, p_1, p_2, p_3, p_4) \to \mathbb{P}^2$ stable map | $f(p_1) = q$, $f(p_2) = r$, $f(p_3) \in L_1$, $f(p_4) \in L_2$, $\Gamma \subset f(C)$, Γ consists $(3d-4)$ general pts$\}$ $\subset M$ a curve (we explained why its dim $= 1$)

Recall $C \cong \mathbb{P}^1$ is rational, the four pts $(p_1, \ldots, p_4)$'s cross-ratio is same as complex analysis: view $\mathbb{P}^1$ covered by $\mathbb{A}^1 = \mathbb{C}$, then on the covering $\mathbb{A}^1$, [illegible]

For C_1 not irreducible, $\succ\!\!\prec_{2}^{1}$, for each node $\times$, we reduce it into (flat) limit of $(C_t, p_1(t), \dots, p_4(t))$

This we have a blow+down map

$C_1 \cong \mathbb{P}^1$, $\mathbb{P}^1 \cong C_2$

$C = C_1 \cup C_2$ and p_1, p_2, p_3, p_4

other cases are always possible; always $\exists$ two pts $\in C_i$ by Pigeonhole principle

▨

(The singularity is more complex, in fact, the inverse is one step of stable reduction)

Thus we reduce to Case I to define the cross-ratio for all stable maps $\in \overline{M}_{0,4}(\mathbb{P}^2)$

Now we prove **Fact 1** C has no more than 2 components

Fact 2 $\#$ of C is B [$C = C_1 \cup C_2$] [redacted] $= \binom{3d-2}{3d_1-1} N(d_1)N(d_2)$, $d_i = \deg(f(C_i))$

Fact 1 We use stability condition: when f is constant on one of the component, then as rational $\Rightarrow$ it has ≥ 3 special pts.

But [redacted] each component of $f(C)_{\mathrm{red}} = D$, say D_i, $\deg D_i = d_i$ has $\leq 3d_i - 1$ marked pts by induction $\Rightarrow \sum(3d_i - 1) = 3d - k \leq 3d - 2$ [redacted] $(3d-2)$ is by def of $B \subset M$, k is the number of components of $D \Rightarrow k \geq 2 \Rightarrow f(C)$ has at least 2 component $\Rightarrow$ so is C as f finite cover. Assume $k \geq 3$

We know a stable curve has no "loop" ($\triangle$), thus have at least ① 2 rational tails and 1 rational bridge

② For 3 rational tails $\pitchfork$ $\to$ $\#$

Thus only consider ①

thus at least [redacted] marked $2\times2+1 = 5$ pts > 4 pts we have by def of B

$\Rightarrow$ we can't have ~~all~~ f constant on all components, this contradicts to stable $\Rightarrow k = 2$ ▨

Why?

Fact 2 not used, omitted.

Now we start our [redacted] computation

Case I C irreducible,

To mark $(3d-1)$ pts, as $f(p_1), f(p_2)$ totally fixed, only the case $f(p_3) = f(p_4) = L_1 \cap L_2 = \{p\}$

It's equivalent to the number of zeros of ~~a~~ cross-ratio φ

Case II $C = C_1 \cup C_2$ not irreducible: $\deg C_1 = d_1$, $\deg C_2 = d_2$, $d_1 + d_2 = d$

① Zeros $\Leftrightarrow p_1 = p_2$ or $p_3 = p_4$, thus we must let $p_1, p_2 \in C_1$, $p_3, p_4 \in C_2$ (or converse) in same component

$f(C_1)$: q, r, $(3d_1 - 1) - 2$ other pts by induction $\Longrightarrow$ we take $(3d_1 - 3)$ pts from $(3d-4)$ pts

$f(C_2)$: remaining $(3d-4) - (3d_1 - 3) = (3d_2 - 1)$ desired $\Rightarrow$ multiply $\binom{3d-4}{3d_1-3}$

And for p_3 & p_4, by Bézout's thm, $p_3 \in C_2 \cap f^{-1}(L_1)$, $p_4 \in C_2 \cap f^{-1}(L_2)$ has d_2 & d_2 choice and the nodal pt $f(C_1 \cap C_2) \in f(C_1) \cap f(C_2)$ $d_1 d_2$ choice

$\Rightarrow \sum d_1 d_2^3 \binom{3d-4}{3d_1-3} N(d_1)N(d_2)$ is zeros

② Poles $\Leftrightarrow p_1 = p_3$ or $p_2 = p_4$, let $p_1, p_3 \in C_1$, $p_2, p_4 \in C_2$

$f(C_1)$: q, $(3d_1 - 1) - 1$ other pts

$f(C_2)$: r, $(3d-4) - (3d_1 - 2) = (3d_2 - 2)$ desired

Again we pick $(3d_1 - 2)$ pts $\Rightarrow \binom{3d-4}{3d_1-2}$

and $\underset{d_2}{C_2 \cap f^{-1}(L_2)}$, $\underset{d_1}{C_1 \cap f^{-1}(L_1)}$, $\underset{d_1 d_2}{f(C_1) \cap f(C_2)}$

$\Rightarrow \sum d_1^2 d_2^2 \binom{3d-4}{3d_1-2} N(d_1)N(d_2)$ is poles

At last, zeros $-$ poles $= \boxed{\sum_{d_1 + d_2 = d} d_1 d_2 \left[d_1 d_2 \binom{3d-4}{3d_1-2} - d_2^2 \binom{3d-4}{3d_1-3} \right] N(d_1)N(d_2)}$ desired answer & $N(4) = 620$ is our answer to original question ▨

I think it's due to ~~minus poles for give~~ "pure" number of zero, i.e. under permutation to poles $= 0$, zeros minus same on $\mathbb{P}^1$.

Rk (GIT's view) [Harris, Example 6.7.9] Cross-ratio is powerful trick to use, but restricted in the $M_{0,4}$, we can't extend it to marked more or less pts.

① 4-tuple (p_1, p_2, p_3, p_4) of distinct points $\longleftrightarrow p = \varphi(p_1, p_2, p_3, p_4)$ bounded ② Two pts $p, q \in X^s$ have same $SL_2(k)$-orbit iff

$\|_s$

Stable locus $X^s \subset X$

Why we need consider linear family in parameter space?

When our element contains not expected, ③ $X^{ss} = \{$at most two pts equal tuple$\}$

Where $X = (\mathbb{P}^1)^4$ equipped the diagonal action $SL_2(k) \curvearrowright (\mathbb{P}^1)^4$ but dim k, we use a ~~codim k~~ ~~linear~~ to cut it into dim 0 ▨ then we can classify the orbits of $p \in X^{ss} \setminus X^s$

Eg 3. (Special elements $\in$ family via pencils/nets

The meaning of special can have various realisations ① containing line

② [redacted] singular/non reduced

③ contact pts as flex/hyperflex

closed: $O_{(0,0,\infty,\infty)}$, $O_{(0,\infty,0,\infty)}$, $O_{(0,\infty,\infty,0)}$ ①

not-closed: same closure $O_{(0,0,\infty,\infty)}$

For ①, we consider the hypersurface $X \subset \mathbb{P}^n$ of degree, this is parameterized by Fano scheme, we provide a way by Hilbert scheme/representable functors, but omit ~~all~~ constructions in detail.

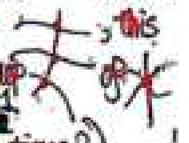
Our notation is $\boxed{\begin{array}{ll}\text{tangent} & \geq 2 \\ \text{flex} & = 3 \\ \text{hyperflex} & \geq 4\end{array}}$

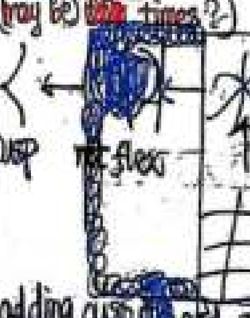
We'll define the order of contact later. $F_m(\mathbb{P}^n)$

G for projective k in $\mathbb{P}^n$ plane

Recall the Fano scheme is defined by $F_m(X) \subseteq G(m, n)$ for $X \subset \mathbb{P}^n$, parameterizes all m-planes $\subset X$

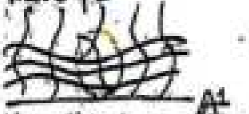
Look at the local has singular locus, this is the blow-up of (may be) times?



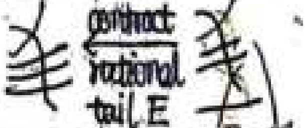


adding cusp and flex its the one step of stable reduction.

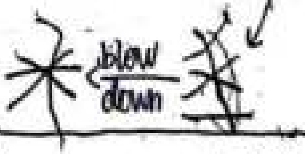
- More generally, marked more pts



Here this local is not able to blow down to desired we need additional operation called the contraction:



$E \cong \mathbb{P}^1$ isn't stable $E \cong \mathbb{P}^1$ that's why we contract it in stable reduction blow down



$\Phi(n,d,m)=\{(X,L)\in\mathbb{P}^N\times G(m,n)\mid L\subset X\}\Rightarrow \forall [X]\in\mathbb{P}^N, X\subset\mathbb{P}^n$

$$\begin{tikzcd} & \Phi(n,d,m) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}^N & & G(m,n) \end{tikzcd}$$

$\boxed{p_1^{-1}(X)=F_m(X)}\subset G(m,n)$

$N=\binom{n+d}{d}-1$ is relative analogue of Fano scheme.

parametrized all hypersurface $\subset\mathbb{P}^n$ of degree d.

Rk. ① The subscheme structure of $F_m(X)$ and $\Phi(n,d,m)$ are similar as Grassmannian by finding an open covering, omitted.

② The fibre of $p_1\Leftrightarrow$ find $L\subset X$ fixed has a complicated locus $F_m(X)$ in $G(m,n)$, thus our study usually through p_2 ($\Leftrightarrow$ find X containing fixed L), such $p_2^{-1}(L)\subset\mathbb{P}^N$ is [redacted] a projectived linear space, this is one step of counting dim of $\Phi(n,d,m)$: $p_2^{-1}(L)=\{X\in\mathbb{P}^N\mid X\supset L\}=\mathbb{P}(\mathrm{Ker}(H^0(\mathcal{O}_{\mathbb{P}^n}(d))\to H^0(\mathcal{O}_L(d))))$ as $\mathbb{P}^N=\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ and $X\supset L\Leftrightarrow X$ restrict to L is L itself $\Leftrightarrow X\cap L$ is the polynomial $0\Leftrightarrow$ in the kernel ▨

Then we can count dimension $\dim\Phi(n,d,m)=\dim G(m,n)+\dim p_2^{-1}(L)=(m+1)(n-m)+$ [redacted] $N-\binom{m+d}{m}$ ▨ minus N, $(m+1)(n-m)-\binom{m+d}{d}$ is $\dim(F_m(X))$, we expect it=

Recall we can rewrite Grassmannian & Hilbert scheme into representable functors, so is Fano scheme (as a special case of Hilbert

[redacted]	Universal family	Universal property	Functorial (2-Yoneda lemma)
Grassmannian $G(m,n)$	$\Phi(m,n)=\{(\Lambda,p)\in G(m,n)\times\mathbb{P}^n\mid p\in\Lambda\}$	$\mathcal{L}=B\times_{G(m,n)}\Phi\to\Phi$ $G(m,n)$ replaced by $F_m(X)$ or $\mathcal{H}_P(X)$ $\exists!$ $G(m,n)$	$F_{m,n}: B\mapsto\{$flat families $\mathcal{L}\to B$ of m-planes in $\mathbb{P}^n\}\cong\mathrm{Mor}(-,G$
Fano $F_m(X)$	$\Phi(n,d,m)$ defined		$F_{m,X}: B\mapsto\{$flat families $\mathcal{L}\to B$ of m-planes in $X\}$ [redacted] $\cong\mathrm{Mor}(-,F$
Hilbert $\mathcal{H}_P(X)$	The universal family of $Y\subset X$ with Hilbert polynomial p		$F_{P,X}: B\mapsto\{$flat families $\mathcal{L}\to B$ of Hilbert polynomial $P\}\cong\mathrm{Mor}(-,\mathcal{H}$
Quot ----	Omitted, Quot schemes more general, used usually for moduli of sheaves...		

For more general moduli stack $\mathcal{M}$, also $\exists$ $\mathcal{U}$ universal family $\mathcal{U}\xrightarrow{p}\mathcal{M}$: universal family is defined as: $p^{-1}([X])=X$, and have some universal properties upper $\Rightarrow\mathcal{M}$ is a functor.

The local geometry of them such as tangent spaces help us to count, and when counting lines $\subset$ hypersurface case, we determine the virtual number first by computing Chern class: we first consider two baby cases ① 27 lines $\subset$ general cubic $X\subset\mathbb{P}^3$

[redacted] The local geometry of $F_m(X)$ can be complex: not smooth & not [redacted] reduced; but generically they hold

Global Geometry of $F_m(X)\subset G(m,n)$: Now our virtual number is $\int_{G(m,n)}[F_m(X)]$ for $(m+1)(n-m)=\binom{m+d}{m}$ ⇒ 27 lines as Eg.2 predicted

For computation of $[F_m(X)]\in A^*(G(m,n))$ general we need Chern class: $\boxed{[F_m(X)]=c_{top}(\mathrm{Sym}^d S^*)}=c_{\binom{m+d}{m}}(\mathrm{Sym}^d S^*)\in A^*(G(m,n))$, S is universal subbundle/ tautological bundle, $S\subset$ [redacted] $\mathbb{A}^{n+1}\otimes\mathcal{O}_{G(m,n)}$, where $\mathbb{A}^{n+1}$ thinked as rank (n+1)-bundle on $G(m,n)$ and $\mathbb{P}^n=\mathbb{P}\mathbb{A}^{n+1}$ is in the def of $G(m,n)$, rank $S=m+1$

We'll not check that local geometry, but later check another one.

Pf. Let X defined by $f\in H^0(\mathcal{O}_{\mathbb{P}^n}(d))$, we'll use it to determine $\sigma_f\in H^0(\mathrm{Sym}^d S^*)$ with zero locus is just $F_m(X)$, thus done. First we consider what [redacted] $\mathrm{Sym}^d S^*$ is: S has fibre $L\cong\mathbb{P}^m$ over $[L]\in G(m,n)$, thus its dual has the fibres to be linear forms $\in H^0(\mathcal{O}_{\mathbb{P}^m}(1))$, and taking d symmetric powers comes into $H^0(\mathcal{O}_{\mathbb{P}^m}(d))\Rightarrow f$ restrict to $f|_{\mathbb{P}^m}\in H^0(\mathcal{O}_{\mathbb{P}^m}(d))=(\mathrm{Sym}^d S)^*$, it determines dual element to $\sigma_f\in H^0(\mathrm{Sym}^d S^*)$: the value of $\sigma_f([L])=f|_L=f|_{\mathbb{P}^m}$, thus $\{\sigma_f=0\}\Leftrightarrow\{L\subset X\}$ [redacted] $=F_m(X)$ ▨

(ii) We compute $c(\mathrm{Sym}^3 S^*)$, as rank $S=2$ and rank $(\mathrm{Sym}^3 S)^*=4$, Recall $\begin{cases} c(Q)=1+\sigma_1+\sigma_2+\cdots+\sigma_{n-m} & \text{rank }(n-m)\\ c(S)=1-\sigma_1+\sigma_{1,1}-\cdots+(-1)^m\sigma_{\underbrace{1,\ldots,1}_{m\text{ "1"}}} & \text{rank } m\end{cases}$

$\Rightarrow c(S^*)=1+\sigma_1+\sigma_{1,1}$

By splitting principle, we have two Chern roots α,β, s.t. $(1+\alpha)(1+\beta)=1+\sigma_1+\sigma_{1,1}$

$\Rightarrow\alpha+\beta=\sigma_1$ & $\alpha\cdot\beta=\sigma_{1,1}$

$\mathrm{Sym}^3 S^*=\mathrm{Sym}^3(S^*)=\oplus(L_1^i\otimes L_2^{3-i})$ (here $L^2=L\otimes L$) $\Rightarrow c(\mathrm{Sym}^3 S^*)=(1+3\alpha)(1+2\alpha+\beta)(1+\alpha+2\beta)(1+3\beta)$ by Whitney sum and splitting

where L_1 and L_2 corresponds to chern roots $c_1(L_1)=\alpha$, $c_1(L_2)=\beta$

$\Rightarrow c_{top}(\mathrm{Sym}^3 S^*)=3\alpha(2\alpha+\beta)(\alpha+2\beta)3\beta=9\alpha\beta(2\alpha^2+5\alpha\beta+2\beta^2)=18\alpha\beta(\alpha+\beta)^2+9(\alpha\beta)^2=18\sigma_{1,1}\sigma_1^2+9\sigma_{1,1}^2\overset{\text{Pieri}}{=}18\sigma_{2,2}+9\sigma_{2,2}=27\sigma_{2,2}$

$\Rightarrow\int[F_1(X)]=\int c_{top}(\mathrm{Sym}^3 S^*)=\int 27\sigma_{2,2}=27$ ▨

Rk. ③ The virtual number is true for general cubic surface, are two special (not contradict to general) elements famous and classical:

Clebsch surface has 27 distinct lines

smooth

Hesse surface has 3 distinct lines, but each multiplicity 9.

not smooth

609250 < 698005

rational curves > lines why?

(iii) $c(S^*)=1+\sigma_1+\sigma_{1,1}+\sigma_{1,1,1}=(1+\alpha)(1+\beta)(1+\gamma)\Rightarrow\begin{cases}\alpha+\beta+\gamma=\sigma_1\\ \alpha\beta+\beta\gamma+\gamma\alpha=\sigma_{1,1}\\ \alpha\beta\gamma=\sigma_{1,1,1}\end{cases}$

$\mathrm{Sym}^5 S^*=\oplus(L_1^i\otimes L_2^j\otimes L_3^k)$, $i+j+k=5$

$\Rightarrow c(\mathrm{Sym}^5 S^*)=(1+5\alpha)(1+4\alpha+\beta)(1+4\alpha+\gamma)(1+3\alpha+\beta+\gamma)(1+3\alpha+2\beta)(1+3\alpha+2\gamma)(1+2\alpha+2\beta+\gamma)(1+2\alpha+\beta+2\gamma)(1+2\alpha+3\beta)(1+2\alpha+3\gamma)(1+\alpha+2\beta+2\gamma)(1+\alpha+3\beta+\gamma)(1+\alpha+\beta+3\gamma)(1+\alpha+4\beta)(1+\alpha+4\gamma)(1+5\beta)(1+5\gamma)$

$\Rightarrow c_{top}(\mathrm{Sym}^5 S^*)=5^3\alpha\beta\gamma(4\alpha+\beta)(4\alpha+\gamma)(3\alpha+\beta+\gamma)(3\alpha+2\beta)(3\alpha+2\gamma)(2\alpha+2\beta+\gamma)(2\alpha+\beta+2\gamma)(2\alpha+3\beta)(2\alpha+3\gamma)(\alpha+2\beta+2\gamma)(\alpha+3\beta+\gamma)(\alpha+\beta+3\gamma)(\alpha+4\beta)(\alpha+4\gamma)=(5^3\times 23)\sigma_{2,2,1}=2875\sigma_{2,2,1}$ (Without compute, I know it's $c_{2,2,1}$ for dim, as $\alpha\beta\gamma=\sigma_{1,1,1}$, by Pieri, finally we obtain [illegible] the degree of hypersurface)

It's of course more computation, thus we left it to computers

Using computers as [Dek96.5], we founded 698005 & 305093061: they're different with rational curve's 609250 & 317206375 predicted by mirror symmetry Q. Why can we expect 698005 > 609250? 609250 is (virtual) number of rational curves should more for but for 2875 them coincides: in this special cases, these lines are homologous all about not linear equivalent)

After these two baby case, we focus on the family case, our first main question asks that: (virtual) number of lines in our degree

Q1: Consider a general pencil $\{X_t \subset \mathbb{P}^3\}$ of quartic surfaces $\#\{X_t \mid X_t$ containing a line$\} = ?$ A pencil / net / - is called Lefschetz. (pencil)

Correspondingly, we need use relative analogue of $F_1(X_t)$; i.e. universal Fano scheme $\Phi(n,d,1)$ general means that a general

We need compute $[\Phi(3,4,1)] \in A^*(\mathbb{P}^{34} \times G(1,3))$ & using argument on tangent & normal bundles on $\mathbb{P}^{34} \times G(1,3)$ to give concrete

But our best thing is we can skip the later step! It's due to $\boxed{\text{Bertini's thm}}$, in the family (projective line $\subset$ the complete linear system $\mathbb{P} \cong \mathbb{P}^N$)

case, the $\Phi(n,d,m) \cap$ the family $M \cong \mathbb{P}^k$ in $\mathbb{P}^k \times G(m,n)$ is reduced & expected dimension.

It suffices to show: let $M \cong \mathbb{P}^m$ is a linear family $\{X_t \subset \mathbb{P}^n\}$, then our universal Fano scheme $\Phi(n,d,1)|_M \subset \mathbb{P}^m \times G(1,n)$

is the zero locus of $E = p_2^*(\mathrm{Sym}^d S^*) \otimes p_1^*(\mathcal{O}_{\mathbb{P}^m}(1))$ for $\mathbb{P}^m \times G(1,n) \Rightarrow [\Phi(n,d,1)|_M] = c_{top}(E) = c_{d+1}(E)$

$$\begin{tikzcd} & \mathbb{P}^m \times G(1,n) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}^m & & G(1,n) \end{tikzcd}$$

Pf: $M = \mathbb{P}^m \subset \mathbb{P}^N = \mathbb{P}(\mathrm{Sym}^d V^*)$, where $\mathbb{P}^n = \mathbb{P}V$

thus we can assume $M = \mathbb{P}^N$, $m = N \Rightarrow \Phi(n,d,1)|_M \subset \mathbb{P}^N \times G(1,n)$

as then the general case is m-plane of $\mathbb{P}^N$, adding a restriction as we done in absolute case. Thus we reduce to $m = N$

$p_1^*(\mathcal{O}_{\mathbb{P}^N}(-1)) \xrightarrow[\varphi]{} p_1^*(\mathrm{Sym}^d V^*) \xrightarrow{\cong} p_2^*(\mathrm{Sym}^d V^*) \xrightarrow[\psi]{} p_2^*(\mathrm{Sym}^d S^*)$

$(W = \mathrm{Sym}^d V^*)$ Why?

φ is induced by the tautological bundle ~~[illegible]~~ $\mathcal{O}_{\mathbb{P}^N}(-1) \to W$ on $\mathbb{P}^N$, where we view W as vector bundle on $\mathbb{P}^N$?

ψ is induced by the same as absolute case $\mathrm{Sym}^d V^* \to \mathrm{Sym}^d S^*$ on $G(1,n)$ via degree d ~~[illegible]~~ function V (left) $\mapsto \sigma_F$

Thus restrict to $[F] \in \mathbb{P}^N$ (i.e. the hypersurface $X = V(F)$), ~~[illegible]~~ $p_1^*(\mathcal{O}_{\mathbb{P}^N}(-1))|_{[F]} \xrightarrow[\Phi]{\text{composite map}} p_2^*(\mathrm{Sym}^d S^*) \Rightarrow \{$the zero locus of $\Phi\} = \Phi(n,d,1)$

as the fibre ~~[illegible]~~ at $[F]$ to $\sigma_F = 0 \Leftrightarrow F_1(X)$ with $X = V(F)$

generator $\longmapsto \sigma_F$

all these fibres' union $\Leftrightarrow \Phi(n,d,1)$

$\Updownarrow$

zero locus of Φ

and $\Phi \in$ ~~[illegible]~~ $\mathcal{H}om(p_1^*\mathcal{O}_{\mathbb{P}^N}(-1), p_2^*\mathrm{Sym}^d S^*) = p_2^*(\mathrm{Sym}^d S^*) \otimes p_1^*(\mathcal{O}_{\mathbb{P}^N}(1))$ ▨

Here $n = 3$, $d = 4$, $m = N = \binom{3+4}{4} - 1 = 35 - 1 = 34$

Due to $d = 4$, by our experience before we know it's almost impossible to compute it by ~~[illegible]~~ hands. I give the answer by computer $[\Phi(3,4,1)] = 320\sigma_{2,2}\zeta + 220\sigma_{2,1}\zeta^2 + (80\sigma_2 + 55\sigma_{1,1})\zeta^3 + 10\sigma_1\zeta^4 + \zeta^5$, here our ζ comes from $\mathcal{O}_{\mathbb{P}^{34}}(1)$'s pullback

$\boxed{\begin{array}{l}\text{Fact. ① No element can contain} \geq 2 \text{ lines} \\ \text{② No line can contained in} \geq 2 \text{ elements}\end{array}}$ $\Rightarrow \#\{X_t \in M \mid \exists L \subset X_t\} = \#\{L \subset \mathbb{P}^3 \mid \exists X_t \in M, X_t \supset L\}$ i.e. generator of $A^1(\mathbb{P}^{34})$

Pf of Fact: ①② are symmetry, I prove ① (3264, Exercise 6.64)

Two cases $\begin{cases}\text{Two lines are skew} \\ \text{Two lines are incident}\end{cases}$

Both has the locus of quartic containing them

$L_1 = \{X \supset$ skew lines$\} \subset \mathbb{P}^{34}$

$L_2 = \{X \supset$ incident lines$\} \subset \mathbb{P}^{34}$ $\boxed{\dim L_1 = \dim L_2 = 32}$

Thus the general pencil $\mathbb{P}^1$ not intersect them as its codim 2, only codim 1 can intersect finite pts by Bézout thm.

dimensions are easy to expect: $H^0(\mathcal{O}_{\mathbb{P}^3}(4)) = \mathbb{P}^{34}$ $\nearrow H^0(\mathcal{O}_{M_1}(4))$ $\searrow H^0(\mathcal{O}_{M_2}(4))$ $\to 0$ or $H^0(\mathcal{O}_p(4)) \Rightarrow 34 - 2 = 32$-dimensional. ▨

dim 34 (dim 1) $\times$ 2 dim 0

Rk. Here our computation $= \int [\Phi(3,4,1)] \cdot [\mathbb{P}^1]$ (We know it's no more than virtual as Bertini gives $\Phi(3,4,1) \cap \mathbb{P}^1$ is genuine)

only use the leading term $= \int [\Phi(3,4,1)] \cdot \zeta^{33}$ (codim 33)

as on $\mathbb{P}^{34} \times G(1,3)$, only $= \int 320 \boxed{\sigma_{2,2}\zeta^{34}} + \cdots$ top

$\sigma_{2,2} \cdot \zeta^{34}$ is top class.

Later terms can also $= 320$ ▨

Reflect information: they can solve following type of problem:

$C \subset G(1,3)$, $C = \{L \mid \exists X_t \in M, X_t \supset L\}$ a curve of $G(1,3)$, we can compute $\deg C$ & $g(C)$ by other terms. (See 3264, Coro 6.37 ①)

③ Now we consider singular elements of linear systems.

Rk: Its singularity can be described without algebraic structure, the another method via "topological Hurwitz formula" as in 3264 §7.1 can describe this enumerative problem by $\chi_{top}(X_t)$. we omit this.

② Due to we need to use bundles of principal part / Jets bundle to study universal singular locus; for one singular pt, we study its order of the local equation: this needs record higher derivatives on the vector space T_pX, i.e. $T_pX \oplus S^2_pX \oplus \cdots \to$ for a ~~[illegible]~~ vanishing, we need a bundle construction. Thus we require our base field must char $= 0$ (& Bertini thm)

I'm not sure whether they're totally same, but they're quite "same" concepts. [illegible] "for P and" "for J"

Our bundle of kth-order principal parts $\Rightarrow$ kth-Jet bundle has fibre $\mathcal{P}^k$ or J^k = $\dfrac{\{\text{germs of sections of } \mathcal{O}_X \text{ at } p\}}{\{\text{germs vanishing to order} \geq k \text{ at } p\}}$

and for general sheaf $\mathcal{F}$, $\mathcal{P}^k(\mathcal{F})$ (or $J^k(\mathcal{F})$) $= p_{2*}(p_1^*\mathcal{F} \otimes \mathcal{O}_{X\times X}/\mathcal{I}^{k+1})$

Fibrewise its sum $\cong \cdots / \sim$ (Two germs equivalent $\Leftrightarrow$ their kth-Taylor expansion same)

Rk. $k = \infty$ is possible to take, by limit of $J^k \hookrightarrow J^{k+1}$ containing most rich datum $= \mathcal{F} \otimes (1 \oplus \Omega_X \oplus \cdots \oplus \wedge^k \Omega_X)$

~~[illegible]~~ We can describe it by inductively $\mathcal{P}^0(\mathcal{F}) = \mathcal{F}$ $\Rightarrow \mathcal{P}^k(\mathcal{O}_X) = 1 \oplus \Omega_X \oplus \wedge^2 \Omega_X \oplus \wedge^3 \Omega_X \oplus \cdots \oplus \wedge^k \Omega_X$

$\mathcal{F} \otimes \mathrm{Sym}^m(\Omega_X) \to \mathcal{P}^m(\mathcal{F}) \to \mathcal{P}^{m-1}(\mathcal{F}) \to 0$ (*) and when $\mathcal{F} = F$ vector bundle, so is $\mathcal{P}^m(F)$ $\doteq p_{2*}(\mathcal{O}_{X\times X}/\mathcal{I}^{k+1})$, where $\mathcal{I}$ is ideal sheaf of $X \xrightarrow{\Delta} X \times X$

We accept these facts, and (*) is left exact.

Return back to our problem: Still focus on family of hypersurfaces $X_t \subset \mathbb{P}^n$ degree d, the universal singular loci is $\Sigma_{n,d} = \{(X,p) \in \mathbb{P}^N \times \mathbb{P}^n \mid p \in \mathrm{Sing}(X)\}$ as an incidence correspondence. $p_1(\Sigma_{n,d}) = \{$singular hypersurfaces$\} =: D_{n,d}$ the discriminant defined by Jacobian $= 0$ $p_1^{-1}(X) = \mathrm{Sing}(X)$

$p_1 \swarrow \mathbb{P}^N$ $\searrow p_2\ \mathbb{P}^n$

Consider the Geometry of $\Sigma_{n,d}$ and $D_{n,d}$. ① first we have generalisation to $\Sigma_{n,d,m}$ and $D_{n,d,m}$ as note here $\Sigma_{n,d} = \Sigma_{n,d,2}$ as $p \in \mathrm{Sing}(X) \Leftrightarrow \mathrm{mult}_p(X) \geq 2$, thus we can have m-multiplicity points $\Rightarrow$ $\begin{array}{l}\Sigma_{n,d,m} \subset \Sigma_{n,d,m-1} \subset \cdots \subset \Sigma_{n,d,2} = \Sigma_{n,d} \\ D_{n,d,m} \subset D_{n,d,m-1} \subset \cdots \subset D_{n,d,2} = D_{n,d}\end{array}$

and thus if we want to consider a family's element containing m-mult pts

turns out to be a general [?]-plane $\subset \mathbb{P}^N$ intersects $\mathcal{D}$ & computing its intersection number. [?] determined by n. (page 2)
If $\mathcal{D}$ is smooth, by Bertini again, we only to compute virtual number, but $\mathcal{D}$ not smooth. However, we'll show $\mathcal{D}$ is irreducible ✓ generically smooth to give also we can apply Bertini in ② next.

④ Focus on $\pi_2 : \Sigma \to \mathcal{D}$, it turns out it's a resolution of singularity: ($m=2$ case)
(i) $\Sigma \simeq \mathcal{D}$ birational; (ii) Σ smooth, $\mathcal{D}$ hypersurface $\subset \mathbb{P}^N$; (iii) Σ & $\mathcal{D}$ both irreducible; (iv) general $[X] \in \mathcal{D}$ has unique ordinary double pt (more special & singular complicated, singularity of course dimension lower)

We prove ②: (ii) & (iii) $\pi_2^{-1}(p) = \{$ hypersurface singular at $p\} \subset \mathbb{P}^N$ defined by $(n+1)$-linear equation $\Rightarrow \pi_2^{-1}(p) = \mathbb{P}^{N-n-1} \subset \mathbb{P}^N$
$\Rightarrow \Sigma$ irreducible & smooth $\Rightarrow \pi_1(\Sigma)$ also irreducible.

To write out the $(n+1)$-linear equations: assume $p = (1, 0, \dots, 0) \in \mathbb{P}^n$ and a general polynomial on $\mathbb{P}^n$, $F(x_0 \dots x_n) = \sum a_I x^I$, I multi
$\Rightarrow \Sigma \subset \mathbb{P}^N \times \mathbb{P}^n$ defined by $V(F, \partial_i F)$. Take affine covering $p = 0 \in \mathbb{A}^n \Rightarrow f(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}) = x_0^{-d} F(x_0 \dots x_n)$ dehomogenousification, then
$\Sigma \cap (\mathbb{P}^N \times \mathbb{A}^n) = V(f, \partial_i f) \Rightarrow$ defined by vanishing of constant & linear terms, constant $f_0 = 0$, $\partial_i f(0) = A x_0 = 0$ for matrix $\}$ $(n+1)$-linear equation ▨

(iv) $\pi_2^{-1}(p) \subset \mathbb{P}^N$ linear $\ni [X]$, where X is union of $(d-2)$ hyperplanes away from p $\cup$ cone over $X' \subset \mathbb{P}^{n-1}$ quadric hypersurface with vertex p
This X' does a singular hypersurface $\Rightarrow p$ singular
such type of hypersurfaces only ordinary double pts p & generically reduced
moving all d hyperplanes away from p, it forms a linear system Φ
with only base point is p
By Bertini $\Rightarrow$ general member Y of Φ is smooth if Y
away from p.

Now our left to move X' the quadric form on $\mathbb{P}^{n-1}$: $\dim \pi_2^{-1}(p) = N-n-1 = \dim \Phi + \dim\{$quadric form on $\mathbb{P}^{n-1}\}$
and we need X' smooth $\Rightarrow p$ double, smoothness is open condition $\Rightarrow$ finally we have a open set $\subset \mathbb{P}^{N-n-1}$ with single double pt.
(i) It easily follows (iv) as then the general fibre $\pi_1^{-1}(D)$ for $D \in \mathcal{D}$ general is single pt (i.e. the single double pt in (iv)) ▨

Return back to our counting questions, we asks $\{$ Q$_1$. The singular members $\in$ A general pencil of hypersurface deg $d \subset \mathbb{P}^n$
Q$_2$. The triple members $\in$ A general 4-dim linear family of plane curve deg d.

Q$_1$. $\#\{X_t \mid X_t$ singular$\}$
$\overset{\text{general}}{\underset{\text{1 singular pt}}{=}} \#\{p \in \mathbb{P}^n \mid \exists X_t, p \in X_t$ singular$\}$

write $\{X_t\}_{t \in \mathbb{P}^1} = V(t_0 F + t_1 G)_{t = [t_0 : t_1]} \subset \mathbb{P}^1 \times \mathbb{P}^n$, then singular $p \in X_t \Leftrightarrow [dF]_p, [dG]_p$ linear independent; thus we can write it as the degeneracy loci of the bundle $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$, that is, view X_t as section of $\mathcal{O}_{\mathbb{P}^n}(d)$ and, then
$t_0 F + t_1 G$ vanish order $2 \Leftrightarrow F$ & G ...

$\#\{p \in \mathbb{P}^n \mid \exists X_t, p \in X_t$ singular$\} = \#\{p \in \mathbb{P}^n \mid$ section $(\tau_F, \tau_G) : \mathbb{P}^n \to \mathcal{O}_{\mathbb{P}^n}(d)$ degenerate at $p\} = \#\{p \in \mathbb{P}^n \mid \tau_F(p), \tau_G(p)$ are linear dependent in the fibre $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))|_p\} = \#\{p \in \mathbb{P}^n \mid p \in$ degeneracy loci of $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))\} \overset{\text{below}}{=} \int_{\mathbb{P}^n} c_n(\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d)))$
the last step is due to linear dependent of two down 1-dim $\Rightarrow c_{r-1}$ by definition of Chern class (rank$(\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))) = n+1$)

Thus it left to do two things $\{$ (a) Degeneracy loci D is reduced $\Rightarrow [D] = c_n$
(b) Compute c_n

(a) It's equivalent to the loci of $\{p \in \mathbb{P}^n \mid \begin{pmatrix} f, & \frac{\partial f}{\partial x_1}, & \dots, & \frac{\partial f}{\partial x_n} \\ g, & \frac{\partial g}{\partial x_1}, & \dots, & \frac{\partial g}{\partial x_n} \end{pmatrix}_p$ rank $1\}$ in affine covering f, g dehomogenous polynomial, by general of pencil, we can assume $G(p) \neq 0 \Rightarrow f = f_1 + \dots, g = 1 + g_1 + g_2 + \dots$
Reduced $\Leftrightarrow$ it consists n function with independent linear terms at p vanishing.
it's $\left\{\begin{pmatrix} 0 & \frac{\partial f}{\partial x_i} \\ 1 & 0 \end{pmatrix}\right\}_{i \in [1, n]}$ all (2×2)-minors of upper $2 \times (n+1)$ matrix $\Rightarrow$ linear independent ▨ (When char $k \neq 2$)

(b) More generally, we compute $\boxed{c(\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))) = (1 + (d-m)\zeta)^{\binom{n+m}{n}}}$ (*) (ζ is generator of $A^1(\mathbb{P}^n)$) for higher order of multiplicity
Then $c_n(\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))) = (n+1)(d-1)^n \zeta^n \Rightarrow \int c_n = (n+1)(d-1)^n$ done ▨

Pf of (*). Due to $\mathcal{O}_{\mathbb{P}^n}(d)$ is vector bundle, so is $\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))$, and we have $0 \to (\mathrm{Sym}^m \Omega_{\mathbb{P}^n})(d) \to \mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d)) \to \mathcal{P}^{m-1}(\mathcal{O}_{\mathbb{P}^n}(d))$
inductively $\Rightarrow c(\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))) = \prod_{j \leq m} c((\mathrm{Sym}^j \Omega_{\mathbb{P}^n})(d))$. Recall our Euler sequence $0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0$ and apply next Lemma A: (Whitney)

Lemma A. $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ vector bundles on proj variety X, $\mathcal{C}$ line bundle $\Rightarrow$ for any line bundle $\mathcal{L}_0$, any $j \geq 1$, we have $c(\mathrm{Sym}^j(\mathcal{A}) \otimes \mathcal{L}_0) = c(\mathrm{Sym}^j(\mathcal{B}) \otimes \mathcal{L}_0)\, c(\mathrm{Sym}^{j-1}(\mathcal{B}) \otimes \mathcal{C} \otimes \mathcal{L}_0)^{-1}$

Pf of Lemma A. By $\mathcal{A}, \mathcal{B}, \mathcal{C}$ vector bundle $\Rightarrow 0 \to \mathcal{C}^* \to \mathcal{B}^* \to \mathcal{A}^* \to 0$ dual sequence $\Rightarrow \mathcal{C}^* \otimes \mathrm{Sym}^{j-1}(\mathcal{B}^*) \to \mathrm{Sym}^j(\mathcal{B}^*) \to \mathrm{Sym}^j(\mathcal{A}^*)$
and by rank $\mathcal{C}^* = $ rank $\mathcal{C} = 1$, it's left exact by dimension reason $\Rightarrow 0 \to \mathcal{C}^* \otimes \mathrm{Sym}^{j-1}(\mathcal{B}^*) \to \mathrm{Sym}^j(\mathcal{B}^*) \to \mathrm{Sym}^j(\mathcal{A}^*) \to 0$, three terms are bundles again $\Rightarrow 0 \to \mathrm{Sym}^j \mathcal{A} \to \mathrm{Sym}^j \mathcal{B} \to \mathrm{Sym}^{j-1} \mathcal{B} \otimes \mathcal{C} \to 0$ as bidual of bundle is itself, then Whitney done ▨

Thus let $\mathcal{L}_0 = \mathcal{O}_{\mathbb{P}^n}(d)$ twist d times $\Rightarrow c((\mathrm{Sym}^j \Omega_{\mathbb{P}^n})(d)) = c(\mathrm{Sym}^j(\mathcal{O}_{\mathbb{P}^n}(-1)^{n+1})(d))\, c(\mathrm{Sym}^{j-1}(\mathcal{O}_{\mathbb{P}^n}(-1)^{n+1})(d))^{-1}$
multiply righthand by indice j, then all terms reduced $\Rightarrow c(\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))) = c(\mathrm{Sym}^m(\mathcal{O}_{\mathbb{P}^n}(-1)^{n+1})(d)) = c(\mathcal{O}_{\mathbb{P}^n}(-m)^{\binom{n+m}{n}}(d))$
($c(\mathcal{O}(1)) = (1+\zeta)$, and $c(\mathcal{O}(n)) = (1+n\zeta)$ all line bundles)
Whitney $= c(\mathcal{O}_{\mathbb{P}^n}(d-m))^{\binom{n+m}{n}}$
$= (1 + (d-m)\zeta)^{\binom{n+m}{n}}$ ▨

Q$_2$ Quickly we can give $c(\mathcal{P}^2(\mathcal{O}_{\mathbb{P}^2}(d))) = (1 + (d-2)\zeta)^6 = 15(d^2 - 4d + 4)\zeta^2 + 6(d-2)\zeta + 1 + \cdots$
$\Rightarrow \int c_2 = 15(d^2 - 4d + 4)$ done ▨

It left to analyse why the triple singular hypersurface can be computed via degeneracy loci & why degeneracy loci reduced
The reason of second is similar to double pt case as here our equation is : for f, we have $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \dots$ ... $= 0$ six linear equations

$\Sigma_{2,d,3} = \{(C,p)\in\mathbb{P}^N\times\mathbb{P}^2 \mid \mathrm{mult}_p C \geq 3\}$ By a similiar argument for $\Sigma_{n,d,2} = \Sigma_{n,d}$

$$\begin{tikzcd} & \Sigma_{2,d,3} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}^N & & \mathbb{P}^2 \end{tikzcd}$$

We have $p_1(\Sigma_{2,d,3}) = D_{2,d,3}$ is irreducible variety of dim $N-4$. We need let d large to make such number

(By six equations, $\dim p_1(\Sigma_{2,d,3}) = \dim(\Sigma_{2,d,3}) - \dim F_{p_1} = \dim\Sigma_{2,d,3} - 0$ (generically only 1 triple pt)

$= \dim(\mathbb{P}^N\times\mathbb{P}^2) - 6 = N+2-6 = N-4$ Here $d\geq 3$

Thus we need to consider linear family of dim4 to count:

$\{t_0F_0+t_1F_1+t_2F_2+t_3F_3+t_4F_4\}_{t\in\mathbb{P}^4}$, then the degeneracy loci $V(\tau_{F_0}\wedge\tau_{F_1}\wedge\cdots\wedge\tau_{F_4})$ ▨

⑤ A closely related concept of singular pt is contact pt; it's related due to the singularities occur as cusp/tacnode they can't be describled as vanishing of linear terms, i.e. their definiting equations of Σ not linear $\Rightarrow$ methods in ② not holds! To describle them, it's easy to consider a line $L \ni p$ cusp/tacnode, s.t. $m_p(C,L)\geq 3/4$

thus we reduce it into contact problem: $\Sigma = \{(C,L,p)\in\mathbb{P}^N\times\Phi \mid m_p(C,L)\geq 3\}$, $N=\binom{d+2}{2}$ $(d=2)$

Due to we adding a "varible" of line L, we need to develop as relative analogue of bundle of principal part, and char k = 0.

For simplicity, we only consider linear family of plane curves of degree d, then we have the space $\Phi = \{(L,p)\in\mathbb{P}^{2*}\times\mathbb{P}^2 \mid p\in L\}$ singularity

then our bundle of relative principal part associates to π relatively $m_p^3 + I_{\tilde{p}} = I_{L,p}$ in replacement of local equations (as it's singular & not linear / homogenous)

$$\begin{tikzcd} \Phi \arrow[d, "\pi"] \\ \mathbb{P}^2 = \{L\} \cong \mathbb{P}^{2*} \end{tikzcd}$$

y^2-x^3 locally

Generally, $\pi: Y\to X$ proper smooth map, X, Y schemes, E vector bundle on Y

(Recall in absolute case we consider $\Delta\subset Y\times Y$ and push-pull via two projections $Y \xleftarrow{p_1} (\ ) \xrightarrow{p_2} Y$)

Here we use fibre product: $\Delta \longrightarrow Y\times_X Y \xrightarrow{p_1} Y$ Then $P^m_{Y/X}(E) = p_{2*}(p_1^*E\otimes\mathcal{O}_{Y\times_X Y}/\mathcal{I}_\Delta^{m+1})$,

$$\begin{tikzcd} Y\times_X Y \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\pi"] \\ Y \arrow[r, "\pi"] & X \end{tikzcd}$$

Again we can describle it by SES

$0\to E\otimes\mathrm{Sym}^m(\Omega_{Y/X})\to P^m_{Y/X}(E)\to P^{m-1}_{Y/X}(E)\to 0$

and $P^0_{Y/X}E = E$

by taking $I = I_{L,p}$ is a quadric plane curve s.t. $T_pI_{L,p} = L$: Using this way reduce to ordinary double pt

Consider the incidence correspondence $\Sigma_1 = \{(L,p,C)\in\Phi\times\mathbb{P}^N \mid m_p(L,C)\geq m\}$. From this one can asks $\#\{\text{flex on } C\}$

$$\begin{tikzcd} & \Sigma_1 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \Phi & & \mathbb{P}^N \end{tikzcd}$$

We only consider the hyperflex case

$\#\{\text{hyperflex on } C_t \mid t\in\mathbb{P}^1\}$ (here our hyperflex is m=4)

Q1. $\#\{C_t \text{ has } (L,p)\in\Phi \text{ hyperflex}\}$?

Then $\dim\Sigma_1 = \dim p_1^{-1}(L,p) + \dim\Phi = (4-1) + \dim(p_1^{-1}(L,p)) = 3+(N-4) = N-1$, here $p_1^{-1}(L,p)\subset\mathbb{P}^N$ defined by four equations of $m_p(L,C)\geq 4$ (linear), thus general fibre is dimension $(N-4)$

And $p_2^{-1}(C)$ is generically one pt for $C\in p_2(\Sigma_1)$, i.e. general hyperflex curve contains unique by same argument as in ② $\Rightarrow \dim p_2(\Sigma_1) = \dim\Sigma_1 = N-1$ hypersurface. Our computation is just its degree $\int[p_2(\Sigma_1)]$ (By reduced)

Again we do two things ① Realise it as degeneracy loci $V(\tau_F\wedge\tau_G)$; ② Show it's reduced

But here we need to add one thing ③ Compute $\int c_2(E)$ as here relative case the Chern class is slightly difficulter than before

① Write the pencil by $\{t_0F+t_1G\}_{t=(t_0,t_1)\in\mathbb{P}^1}$ again, then τ_F, τ_G gives section of $E = P^3_{\Phi/\mathbb{P}^{2*}}(g_2^*\mathcal{O}_{\mathbb{P}^2}(d))$; our difference with absolute cases in just adding g_2^*, where g_1 & g_2 is defined $\Phi = \{(L,p)\in\mathbb{P}^{2*}\times\mathbb{P}^2 \mid p\in L\}\subset\mathbb{P}^2\times\mathbb{P}^2$, apply our construction to $\Phi/\mathbb{P}^{2*}\cong\Phi/\mathbb{P}^2 \Rightarrow E|_{(L,p)} = H^0(\mathcal{O}_L(d)/\mathcal{I}_p^4(d))$

$$\begin{tikzcd} & \Phi \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ \mathbb{P}^{2*} & & \mathbb{P}^2 \end{tikzcd}$$

Thus by same reason as absolute case, $p_2(\Sigma_1) = \{\text{hyperflex plane curves}\}$

$= V(\tau_F\wedge\tau_G) = c_3(E)$, where E is rank 4.

② (Check is always hardest)

Locally around $(L,p)\in\mathbb{P}^2\times\mathbb{P}^2$, write as $p = (x,y)$, $L = \{(x+t, y+bt)\}_{t\in\mathbb{A}^1}$

let $f(x,y) = \sum a_{ij}x^iy^j$, $g(x,y) = \sum b_{ij}x^iy^j$. Trivialise the E near (p,L), then τ_F & τ_G is given by four terms in Taylor expansion of coefficient of $f(x+t, y+bt)$ & $g(x+t,y+bt)$ near $t=0$, w.r.t varible t

$f(x+t,y+bt) = \sum a_{ij}(x+t)^i(y+bt)^j = (a_{00}+a_{10}x+a_{01}y+a_{20}x^2+a_{11}xy+a_{02}y^2+\cdots) + (a_{10}+a_{01}b+a_{20}(2x)+a_{11}(y+bx)+a_{02}(2by)\cdots)t + (a_{20}+a_{02}b^2+\cdots)t^2 + (a_{30}+a_{03}b^3+\cdots)t^3+\cdots$

$\Rightarrow \tau_F: (L,p) = \{(x+t, y+bt)\} \longmapsto (a_{00}+\cdots, a_{10}+a_{01}b+\cdots, a_{20}+a_{02}b^2+a_{11}b+\cdots, a_{30}+a_{03}b^3+a_{21}b+a_{12}b^2+\cdots)$

$\Rightarrow \tau_F\wedge\tau_G$ is given by $\begin{pmatrix} a_{00}+\cdots & a_{10}+a_{01}b+\cdots & a_{20}+a_{02}b^2+a_{11}b+\cdots & a_{30}+a_{03}b^3+a_{21}b^2+a_{12}b^2+\cdots \\ b_{00}+\cdots & b_{10}+b_{01}b+\cdots & b_{20}+b_{02}b^2+b_{11}b+\cdots & b_{30}+b_{03}b^3+b_{21}b+b_{12}b^2+\cdots \end{pmatrix}$'s minors

▨

Due to these coeffiecents (a_{ij}) & (b_{ij}) taken to be general, these minors are linear independent $\Rightarrow$ only vanishing at p reduced

I'm confused that in 3264, §11.3.1 I, before compute locally at (L,p), he adding an "irreducibility argument", I don't know what it's for.

③ Again by SES (*), we have $c(P^3_{\Phi/G}(g_2^*\mathcal{O}_{\mathbb{P}^n}(d)))$, where $\Phi = \{(L,p)\in G(1,n)\times\mathbb{P}^n \mid p\in L\}$.

$$\begin{tikzcd} & \Phi \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ G(1,n) & & \mathbb{P}^n \end{tikzcd}$$

But we ~~don't~~ does have a relative analogue of Euler sequence, ~~but~~ them can't use directly

$\prod c(\mathrm{Sym}^i(\Omega_{\Phi/G(1,n)})\otimes g_2^*\mathcal{O}_{\mathbb{P}^n}(d))$

thus one need to compute $c(\Omega_{\Phi/G(1,n)})$ carefully. Here our $n=2$ and $G(1,n)=\mathbb{P}^2$.

First we need to compute $A^*(\Phi)$ by realising it as projective bundle: consider the tautological bundle $\mathcal{O}(-1)$ on $\mathbb{P}^2\cong\mathbb{P}^{2*}$

$\Rightarrow \Phi = \mathbb{P}(\mathcal{O}(-1)^{\oplus 2}) \to \mathbb{P}^2$, thus $A(\Phi) = A^*(\mathbb{P}^2)[\zeta]/(\zeta^2-\sigma\zeta+\sigma^2)$ where $\sigma\in A^1(\mathbb{P}^2)$ generating hyperplane & $\zeta = c_1(\mathcal{O}_\Phi(1))$

$\Rightarrow c_1(T_{\Phi/\mathbb{P}^2}) = [?] \Rightarrow c_1(\Omega_{\Phi/\mathbb{P}^2}) = -[?] \Rightarrow c(\mathrm{Sym}^i\Omega_{\Phi/\mathbb{P}^2}) = 1+i(-[?]) = 1-i[?]$ $\Rightarrow c(\mathrm{Sym}^i(\Omega_{\Phi/\mathbb{P}^2})\otimes g_2^*\mathcal{O}_{\mathbb{P}^2}(d))$

(as Φ is 3-dim $\Rightarrow T_{\Phi/\mathbb{P}^2}$ is line bundle)

It turns out how to compute Chern class of relative tangent bundle via a proper SES relative Euler sequence $1-i[?]+d\zeta$

We prove the general fact $c_k(T_{\mathbb{P}E/X}) = \sum_j \binom{r+1-j}{k-j} c_j(E) \xi^{k-j}$, where $r+1 = \operatorname{rank}(E)$, $\xi = c_1(\mathcal{O}_{\mathbb{P}E}(1))$, $\pi: \mathbb{P}E \to X$

Our key is finding relative Euler sequence $0 \to \mathcal{O}_{\mathbb{P}E} \to \pi^*E \otimes \mathcal{O}_{\mathbb{P}E}(1) \to T_{\mathbb{P}E/X} \to 0$, then $c(T_{\mathbb{P}E/X}) = c(\pi^*E \otimes \underbrace{\mathcal{O}_{\mathbb{P}E}(1)}_{\text{line bundle}})$

Then c(rank $(r+1)$-bundle $\otimes$ line bundle), we can compute each c_k precisely.

The SES comes from $0 \to S \to \pi^*E \to Q \to 0$ on $\mathbb{P}E$, tensoring S^* $\Rightarrow 0 \to \mathcal{O}_{\mathbb{P}E} \to \pi^*E \otimes \mathcal{O}_{\mathbb{P}E}(1) \to Q \otimes S^* \to 0$

$\operatorname{Hom}(S, Q) = T_{\mathbb{P}E/X}$ due to we had known $T_{Gr(m,E)/X} = \operatorname{Hom}(S,Q)$ $\to \mathcal{O}_{\mathbb{P}E} \to \pi^*E \otimes \mathcal{O}_{\mathbb{P}E}(1) \to \boxed{\operatorname{Hom}(S,Q)} \to 0$

and here is $m=1$ case ▨

Eg 4. (Excess intersection) Our start point of excess intersection is Chap 1. Prop 10, the second excess intersection formula. We'll look at a baby case to see where it comes from / why it's called "excess". Then we can use it as very powerful tool, to compute generally. Last we'll use it to give 3264's five conic problem, another way will be shown too.

Recall excess intersection formula $(X_1 \cdots X_r \cdot V)^Z = \{\prod c(N_{X_i}Y|_Z) s(Z,V)\}_{\dim V - \sum \operatorname{codim}(X_i,Y)}$ for each connected component $Z \subset \bigcap X_i$, all $X_i \hookrightarrow Y$ regular & $V \hookrightarrow Y$ $\Rightarrow$ $\boxed{(X_1 \cdots X_r \cdot V) = \sum_Z (i_Z)_* \{\prod c(N_{X_i}Y|_Z) s(Z,V)\}_{\dim V - \sum \operatorname{codim}(X_i,Y)}}$ (✱), $i_Z: Z \hookrightarrow Y$ natural inclusion

Our excess means that the connected component may have not expected dimension (even not pure-dimensional), but it can be expressed by Chern class & Segre class in expected dimension, thus it's highly nontrivial.

Baby case: Eg. Three (hyper)surfaces $S_1, S_2, S_3 \subset \mathbb{P}^3$, $\deg S_i =: s_i$ & $\bigcap S_i = \underset{\text{residue part}}{\boxed{L}} \sqcup \underset{\text{excess part}}{\boxed{\Gamma}}$, L is a reduced line $\cong \mathbb{P}^1$, Γ is zero-dimensional scheme. We ask $\int [\Gamma]$ as $[\Gamma] \in A(\mathbb{P}^3)$ or L is a smooth curve of degree d & genus g

We'll give an excess intersection formula $F(\deg(\Gamma), s_i, d, g) = 0$ in this setting.

For L is line case, no d, g $\Rightarrow$ we can compute $\int [\Gamma]$ via only s_i. Here we restrict to L line case, curve case is similar:

First we do reductions (i) Reduce to S_1 smooth
(ii) $\int_{\mathbb{P}^3} [\Gamma] = \int_{S_1} [\Gamma]$ due to S_1 smooth.

(i) The intersection $\bigcap S_i$ ⟺ solution to system of equations: $S_i = V(F_i) \Rightarrow \bigcap S_i = V(F_1; F_2, F_3) = V(F_1 + gF_2 + hF_3, F_2, F_3)$ where we can assume $s_1 \geq s_2 \geq s_3$ and take g, h two general homogeneous polynomial of deg $(s_1 - s_2)$ and $(s_1 - s_3)$. Thus we modify $S_1 = V(F_1)$ to $V(F_1 + gF_2 + hF_3)$

Base locus's $\dim < \frac{\dim X}{2}$, X big space, then the linear system's general element is smooth.

$\Rightarrow S_1 \in$ general pencil, the locus has $\dim 1 < \frac{3}{2}$, by $\boxed{\text{Strong Bertini}}$, the general element S_1 can be chosen smooth ▨

(ii) $S_1 \cap S_2 \cap S_3 = L \sqcup \Gamma$ means that $S_1 \cap S_2 \cap S_3 = (S_1 \cap S_2) \cap (S_1 \cap S_3) = \boxed{(L + D_1) \cap (L + D_2) = L + (D_1 \cap D_2)} = L \sqcup \Gamma$

where D_1, D_2 are divisors (Weil) on S_1 (may 0)

Here can we let $L \cap D_i = \emptyset$? Or not, consider $p \in L \cap D_2$
$\Rightarrow p$ singular pt in $S_1 \cap S_2 = L + D_2$
$\Rightarrow p$ also singular pt in $S_1 \cap S_2 \cap S_3$; it's impossible as we know $S_1 \cap S_2 \cap S_3 = L \sqcup \Gamma$ smooth.
Thus although $L \cap D_i \neq \emptyset$, $L \cap D_1 \cap D_2 = \emptyset$, thus upper equality holds ▨

$\Rightarrow$ We only compute $[D_1][D_2]$. Simply take the hyperplane class $[H] \in A(S_1)$

$\Rightarrow [S_1 \cap S_2] = s_2[H]$ & $[S_1 \cap S_3] = s_3[H]$ (i.e. the class determined by intersects with a general hyperplane $\subset \mathbb{P}^3$)

(it's due to $H = S_1 \cap \tilde{H}$, $\tilde{H}$ hyperplane deg $= 1$, here S_i has degree s_i

$\Rightarrow [D_1][D_2] = (s_2[H] - [L])(s_3[H] - [L]) = s_2 s_3 [H]^2 - (s_2 + s_3)[H][L] + [L]^2$

$\Rightarrow \int [\Gamma] = \int [D_1][D_2] = s_2 s_3 \int [H]^2 - (s_2 + s_3) \int [H][L] + \int [L]^2 = s_1 s_2 s_3 - 1(s_2 + s_3) + (2 - s_1)$ why?

First term: $[H]^2$ is $[S_1 \cap \tilde{H}]^2$; two general hyperplane section S_1 ⟺ general line section S_1, $\deg S_1 = s_1$;

Second term: 1 is obvious as $\tilde{H} \cap L$ single pt;

· For an independent proof without Prop 10, see [3264, §13.1.1]

Third term: We have a baby baby case of excess intersection formula: $\boxed{(X \cdot D) = (i_D)_* c_1(N_{D/X}|_D)}$ for D effective Cartier divisor

$\Rightarrow [L]^2 = (i_L)_* c_1(N_{L/S_1})$ and $\int [L]^2 = \int c_1(N_{L/S_1})$, then its the only difference between L is line/general curve. For L is line:

$\int c_1(N_{L/S_1}) = \int ([K_L] - [L][K_{S_1}]) = (2g - 2) - \int [L][\mathcal{O}_{S_1}(s_1 - 4)] = (2 \cdot 0 - 2) - (s_1 - 4) = 2 - s_1$ ▨ Details of using adjunction.

$K_D = (K_X + D)|_D$, D smooth divisor.

Here $K_L = [K_{S_1}]|_L + [L]$

Generally [illegible], due to $\int c_1(N_{L/S_1}) = \int [L]^2 = 2g - 2 - d(s_1 - 4)$

$\Rightarrow \boxed{\int [\Gamma] = \prod s_i - d(s_2 + s_3) + (4d + 2g - 2)}$ is our excess intersection formula ▨ (✱✱)

$\int [\Gamma]$ is residual part, we denote as

We interpret (✱✱) into the form of (✱): $\prod s_i - (4d + 2g - 2) = \sum \int c_1(\mathcal{O}_{\mathbb{P}^3}(s_i)) \cdots$

$0 \to I/I^2 \to \Omega_{X} \to \Omega_{Y} \to 0$

$\det(I/I^2)^* = \cdots$

Apply (✱), we can deal with more complex questions.

$= \int \{\sum c_1(N_{S_i}\mathbb{P}^3|_L) - c_1(N_{L}\mathbb{P}^3)\} = \int \{\prod c(N_{S_i}\mathbb{P}^3|_L) \cdot s(L, \mathbb{P}^3)\}_1$

Eg. $Q_1, \cdots, Q_n \subset \mathbb{P}^n$ quadric hypersurfaces (they're "general" in meaning that general choice) and $\bigcap Q_j = \Lambda \cap \Gamma$, $\Lambda \cong \mathbb{P}^{n-r}$ linear subspace, $\dim \Gamma = 0$

$= \int \left\{\frac{\prod c(N_{S_i}\mathbb{P}^3|_L)}{c(N_L\mathbb{P}^3)}\right\}_1 = \int \{s(N_L\mathbb{P}^3) \cdot \prod c(N_{S_i}\mathbb{P}^3|_L)\}_1$

then as generalisation of Eg., $\int [\Gamma]$ also can be computed by (✱)

$= \int \{s(L, \mathbb{P}^3) \cdot \prod c(N_{S_i}\mathbb{P}^3|_L)\}_1$ ▨

Furthermore Γ is reduced pts' union thus $\#\Gamma = \int [\Gamma]$ also can be counted.

Rk. Look at $r = 2$ case, assume $\Lambda = V(X_0, X_1)$, (X_i) are projective coordinates of $\mathbb{P}^n$, $Q_i = V(X_0 A_i + X_1 B_i)$

thus $\int [\Gamma] = 2^n - \int [\Lambda] = 2^n - \sum \binom{n}{i}$

then $\Gamma = \left\{ X \in \mathbb{P}^n \,\middle|\, \operatorname{rank}\begin{pmatrix} X_0 & B_1 & B_2 & \cdots & A_n \\ X_1 & A_1 & A_2 & \cdots & A_n \end{pmatrix} \leq 1 \right\}$ the degeneracy locus, $\int [\Gamma]$ can be computed by Porteous' formula as determinant or

· Reduced: First we show $\Lambda \cap \Gamma = \emptyset$, it suffices to prove Λ is reduced (otherwise $\int$ Λ not reduced) $= \boxed{\sum \binom{n}{i}}$ desired number

Then as the linear system $\{t_1 F_1 + \cdots + t_n F_n\}$ spanned by (Q_i) has base loci just $\Lambda \to \Gamma$ not base pts $\Rightarrow$ reduced done

To show Λ reduced, we have $\mathcal{O}(-2) \xrightarrow{\Phi} \mathcal{O}(-1)$, where section of $\mathcal{O}_\Lambda(-1)$ linear forms l_i defining $\Lambda \mapsto \sum x_{ij} l_j$ quadrics

Φ is a $(n \times n)$-matrix, thus generically surjection as all minors vanishing in codim $= n - r + 1$ [illegible]

than expectation $\Rightarrow \Lambda$ is scheme-theoretic cut by Q_i, reduced ▨

· Computation: $\int [\Gamma] = \int \{\cdots\}_n$ (as all Q_i are same general) $= \int \{\cdots\}_0 = \sum \cdots$ [illegible]

The last step is combinatorial trick; omitted (but nontrivial!), thus it suffices to compute $c(N_{\Lambda/\mathbb{P}^m}|_\Lambda)$ & $c(N_{Y/\mathbb{P}^m})$; $\Lambda\cong\mathbb{P}^{n+r}$, thus let $\zeta\in A^1(\mathbb{P}^{n+r})$ the generator hyperplane class. $N_{\mathbb{P}^{n+r}/\mathbb{P}^m}=(\mathcal{O}_{\mathbb{P}^{n+r}}(1))^{\oplus r}\Rightarrow c(N_{\Lambda/\mathbb{P}^m})=(1+\zeta)^r$ by Whitney; $N_{Y/\mathbb{P}^m}=\mathcal{O}_{\mathbb{P}^m}(2)|_Y$ as it's defined by quadric polynomial ⟺ section of $\mathcal{O}_{\mathbb{P}^m}(2)\Rightarrow N_{Y/\mathbb{P}^m}|_\Lambda=\mathcal{O}_\Lambda(2)\Rightarrow c(N_{Y/\mathbb{P}^m}|_\Lambda)=1+2\zeta$ ▨

Last, it's <u>the five conic problem</u>. Eg. The general plane conics $C_1,\dots,C_5\subset\mathbb{P}^2$, $\#\{C\subset\mathbb{P}^2 \text{ conic} \mid C \text{ tangent to all the five}\}=$?

We carry out the problem via two approaches: compactification & excess; second is easier and later by Fulton, but I think first is more important.

Our original parameterize space is: $\{(C,p)\in\mathbb{P}^5\times C_i \mid m_p(C\cdot C_i)\geq 2\}=\Phi_i$, $p_1(\Phi_i)=Z_i\subset\mathbb{P}^5$ parameterize C tangent to C_i, then our desired computation is $\int_{\mathbb{P}^5}[Z_1][Z_2][Z_3][Z_4][Z_5]$ = virtual number. (Z_i are hypersurfaces $\subset\mathbb{P}^5$)

$$\mathbb{P}^5 \xleftarrow{p_1} \Phi_i \xrightarrow{p_2} C_i$$

Thus we note that our virtual number is quite easy to compute: $\deg Z_i=\int(Z_i\cdot L)$ for $L\subset\mathbb{P}^5$ general line

$$\int_{\mathbb{P}^5}[Z_1]\cdots[Z_5]=\prod\deg Z_i=(\deg Z_i)^5$$

it's the question: $\mathbb{P}^5$ how many element of a general pencil contact C? By same (and easier) method as <u>Eg 2②③</u> hyperflex, we can compute it's 6

$\Rightarrow$ virtual number $=6^5=7776$. But we known the correct answer is 3264, thus our virtual number is totally wrong! Why?

Ex. This shows that to make virtual number = desired number, we need carefully defining the virtual number: the intersection of classes; the domain of integral/parameter space effects virtual number; to make generically transversal.

Study local geometry of Z_i hypersurfaces: can we assume $Z_1\cdots Z_5$ intersect transversally?

We find (admit): <u>Fact1</u>. let $U\subset\mathbb{P}^5$ the loci of smooth conics, $U_i=Z_i\cap U$ intersect generically transversally.

<u>Fact2</u>. Generally, $\bigcap Z_i$ has dimensional ≥ 1! Conic degeneration is × (via))(or $\boxed{=2}$

Thus we can see $\bigcap Z_i=\boxed{\Lambda}\sqcup(\bigcap(Z_i\cap U))$, thus our approach of excess intersection formula can be carried out successfully now. One can see excess intersection (residual + excess part) formula as improvement of blowing-up bad boundary (Griffith & Harris I) to better boundary: Here view $U\subset\mathbb{P}^5$ is compactification of U, with boundary divisor $\mathbb{P}^5-U$. $Bl_\Lambda(\mathbb{P}^5-U)$ makes Z_i <u>not intersect</u> after blow up in exceptional divisor ⇒ only left U's intersection. (Blow-up can <u>separate Z_i</u>). That's why in <u>Chap1</u>, we relates excess intersection formula① with blow-up formula.

Λ = surface of $\boxed{\text{double lines}}\cong\mathbb{P}^2 \xrightarrow{\text{Veronese}}\mathbb{P}^5$

Another approach is: focus on U, we find a better (smaller than $\mathbb{P}^5$ & ...) compactification. (As for how to find, [3264] says it's $\boxed{\text{ART}}$) (a basic examples is smooth curves ⇝ stable curves = nodal curves ⊂ (singular) all curves; stable is better & smaller than all)

Our answer is <u>the space of complete conics</u> (one also can use Kontsevich's stable maps, [illegible] we point out all these turns out to be equal to $Bl_S(\mathbb{P}^5)$). (Here we are lucky to they're equal, but not always: for example we had seen $\overline{M}_g$, $\overline{M}_g$)

<u>①Excess intersection formula.</u> $\bigcap Z_i=\Gamma\sqcup T$, S is Veronese surface $\subset\bigcap Z_i$, T is component of $\bigcap Z_i$ supported on S

$$\int_{\mathbb{P}^5}[T]=\int_{\mathbb{P}^5}\left\{\frac{c(N_{Z/\mathbb{P}^5}|_T)^5}{c(N_{T/\mathbb{P}^5})}\right\}_0=\int_{\mathbb{P}^5}\left\{\frac{c(N_{Z/\mathbb{P}^5}|_S)^5}{c(N_{T/\mathbb{P}^5})}\right\}_0=\int_{\mathbb{P}^5}(8-144\zeta+1632\zeta^2)(1+12\zeta)^5$$

(T has nonreduced structure, thus we can't write $\bigcap Z_i=S\sqcup\Gamma$ simply. (Restrict to T ⟺ Restrict to $T_{red}=S$ is same; but $N_{T/\mathbb{P}^5}\neq N_{S/\mathbb{P}^5}$))

$\Rightarrow\int[\Gamma]=7776-\int[T]=7776-4512=3264$ ▨ $=1632-60\cdot 144+1440\cdot 8=4512$

To compute $s(T,\mathbb{P}^5)$ and $c(N_{Z/\mathbb{P}^5}|_S)$, we have: let $\zeta\in A^1(S)=A^1(\mathbb{P}^2)$ the hyperplane generator. S is double line means that our $\xi\in A^1(\mathbb{P}^5)$ hyperplane generator has $\xi|_S=2\zeta$ (We admit this local geometry result) $\Rightarrow c(N_{Z/\mathbb{P}^5}|_S)=c(\mathcal{O}_{\mathbb{P}^5}(6)|_S)=1+6\xi=1+12\zeta$.

- For $s(T,\mathbb{P}^5)$, we need to point out what T's nonreduced structure is: $\mathcal{I}_{T/\mathbb{P}^5}=\mathcal{I}_{S/\mathbb{P}^5}^2\Rightarrow Bl_T\mathbb{P}^5$ is $Bl_S\mathbb{P}^5$ with exceptional divisor doubled, recall Segre class is a power of the class of exceptional divisor: $s(C,X)=\pi_*\left(\sum(-[E])^{\operatorname{codim}C-1+k}\right)$ (π is map of blowing up) (by definition $s(C,X)=\pi_*(\sum c_1(\mathcal{O}_E(1))^k)$, $c_1(\mathcal{O}_E(1))^k=(-[E])^{\operatorname{codim}C-1+k}$ ▨) $\Rightarrow s(T,\mathbb{P}^5)$ can be computed by $s(S,\mathbb{P}^5)$.

$s(S,\mathbb{P}^5)=c^{-1}(N_{S/\mathbb{P}^5})$. Compute Chern class we use SES & Whitney $0\to T_S\to T_{\mathbb{P}^5}|_S\to N_{S/\mathbb{P}^5}\to 0$

$c(N_{S/\mathbb{P}^5})=\frac{c(T_{\mathbb{P}^5}|_S)}{c(T_S)}=\frac{(1+\xi)^6}{(1+\zeta)^3}=\frac{(1+2\zeta)^6}{(1+\zeta)^3}=1+9\zeta+30\zeta^2$ (only order ≤ 2 as our final is top 7, minus $(1+12\zeta)^5$, left degree 2)

$\Rightarrow s(N_{S/\mathbb{P}^5})=1-9\zeta+51\zeta^2\Rightarrow s(T,\mathbb{P}^5)=1\times 2^3-9\times 2^4\zeta+51\times 2^5\zeta^2=8-144\zeta+1632\zeta^2$ ▨

②<u>Complete conics/Kontsevich's stable maps</u>. I omit all details, as after we have a "<u>good</u>" compactification (no contains too much excess parts), then we <u>compute its Chow ring</u> and carry out computation all tricks we had known. Thus I only gives a description of these compactification, and understand why they're good.

(C^* may not smooth, thus we degenerate them)

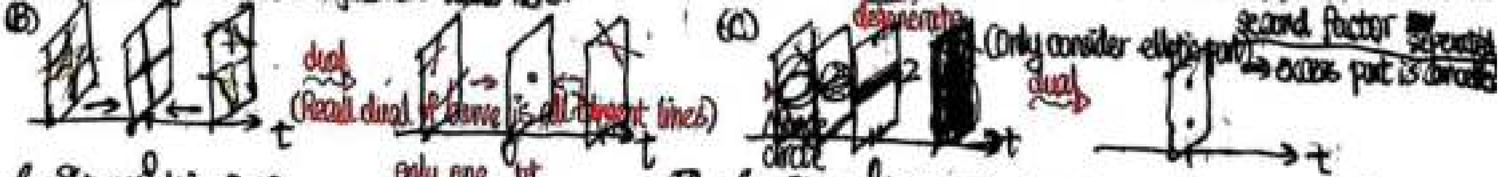
- <u>Complete conics</u>. Set $V=\{(C,C^*)\in U\times U^*\}$ the dual of smooth conics, then our desired construction is simply $X=\overline{V}\subset\mathbb{P}^5\times\mathbb{P}^{5*}$

X consists <u>four types</u> of complete conics: (A) $(C,C^*)\in V$; (B) $C=L_1\cup L_2$, $C^*=2p^*$ ($p=L_1\cap L_2$); (C) $C=2L$, $C^*=p^*\cup q^*$ ($p,q\in L$); (D) $C=2L$, $C^*=2p^*$ ($p\in L$)

(degenerated conics × & $=2$) degenerate; dual (switch C & C^* as $C^{**}=C$); degenerate

Thus here when C singular conic degenerated, the Z_i induce on X has the <u>second factor</u> separated → excess part is

We explain why (B) & (C) degenerates occurs here:



$\mathcal{C}=\{(t,x,y)\mid y^2=x^2-t\}$ ▨ $\mathcal{C}=\{(t,x,y)\mid y^2=t(x^2-2y)\}$ ▨

- <u>Kontsevich's space</u> $\overline{M}_{0,0}(\mathbb{P}^2,2)$ (deg 0, marked 0 pt, degree 2 = conics) $t\to 0$, the <u>Monge circle</u> $\to\infty$

as X upper. The proof uses the Hilbert scheme $Hilb(\mathbb{P}^2)=\mathbb{P}^5$, show that $\overline{M}_{0,0}(\mathbb{P}^2,2)\cong X$

W is open set, defined as <u>except double lines</u> ⇒ extend i to I via "i" both ... projection to $\mathbb{P}^5$

Then upper triangle also commutes ... $Hilb(\mathbb{P}^2)$ ▨

... I to ... then show I isomorphism

§2.2. Recall on stacks & intersection theory on stacks

Our main objects are DM stacks ⇔ the diagonal $\delta: \mathcal{X} \to \mathcal{X}\times\mathcal{X}$ is unramified not smooth closed embedding in smooth scheme, how we can define the intersection product? We'll follow the paper [Vistoli] our | X smooth X separated X scheme | Slogan is Intersection theory on smooth DM stack = Intersection theory on smooth scheme in $\mathbb{Q}$-coefficient & /$\Bbbk$ field only.

Before Vistoli's paper, some proceeding is taken for quotient scheme with $\mathbb{Q}$-coefficient. Vistoli's result is much more general. Then we can see our next step (not in this section) is to weaken the smooth condition, as our mentioned before. Our first idea is "derived approximate smooth". (Some way for singularity is $\mathbb{Q}$-coefficient too)

Precisely, our main things to do in this section is { Chow group, proper pushforward, flat pullback, Gysin map, Chow ring; but use another Chow group, see two papers [Joshua I, II] or [redacted] [Kresch]

Except intersection: { (not in [Vistoli]) We'll take some arguments from

Chern class, GRR — [Vistoli] said he can't prove, it's later proven in [Edidin] after more than 20 years, using equivariant K-theory (A much earlier is by [Toen], but it's French and it's complexer, I not take use of this (in fact [E] simplify it))

Bivariant intersection

Rk. ① For Artinian stack, we must use higher Chow [redacted]

② GRR for Artinian must not exist! (even DM is, under some situation) A paper of Joshua gives even an explicit counterexample (the broken issue is the representability), In our GRR next, we need to add representable condition (another way is revise the topology). If we have time to deeper than stack-level, we'll focus on more derived case instead of motivic uses.

Recall on stacks. First: stack $\mathcal{X}$ is algebraic & over $\Bbbk$ & Noetherian as schemes. In this section, also Deligne–Mumford. After we recall the smoothness, all $\mathcal{X}$ smooth assumed in this section only.

(A interesting thing is that: it's later comes to be one part of his thesis)

Def 1. (Properties of stacks) ① A stack $\mathcal{X}$ is DM iff $\mathcal{X}$ over étale site $Sch_{\Bbbk}$ and the presentation $U \to \mathcal{X}$ surj + étale + representable (étale groupoid)

Here our site condition over $\Bbbk$, thus $Sch/\Bbbk$

$\Downarrow$ ⇔ δ unramified ⇔ every pt $\operatorname{Spec} k \to \mathcal{X}$ (closed) it's called has finite & reduced stabilizer

Stabilizer = automorphism, the most essential property for stacky-issue.

② A substack is $\mathcal{X}' \hookrightarrow \mathcal{X}$ representable and it's open/closed if their presentations are,

$$\begin{tikzcd} & \mathcal{X}' \arrow[r, hook] & \mathcal{X} \\ U'' \arrow[ur, "pre"] \arrow[r, dashed, "\exists!"] & U' \arrow[u] \arrow[r, hook] & U \arrow[u, "pre"] \end{tikzcd}$$

s.t. composite $U' \to U$ embedding ($U'' \to U'$ due to universal property of $U' \to \mathcal{X}'$ presentation)

$U' = U\times_{\mathcal{X}}\mathcal{X}'$ scheme due to $\mathcal{X}' \to \mathcal{X}$ representable

③ All étale local properties P of $\mathcal{X}' \to \mathcal{X}$ (representable) are defined same as ② via lifting to their presentation scheme (ensure it not depend on choice of presentation, as $U_i \to \mathcal{X} \Rightarrow U_2\times_{\mathcal{X}} U_1$ covering U_1 & U_2) P can be [redacted] iso/closed/open/quasi-affine dense/dominant

(Topologically property) ④ The underlying topological space $|\mathcal{X}| = \{\operatorname{Spec} k \to \mathcal{X}\}/\langle \operatorname{Spec} k_1 \to \mathcal{X} \Leftrightarrow \operatorname{Spec} K \rightrightarrows \mathcal{X} \rangle$ with topology is $\mathcal{U} = \{|U| \subset |\mathcal{X}| \mid U \hookrightarrow \mathcal{X}$ open substack$\}$, then $\mathcal{X}$ is α iff $|\mathcal{X}|$ is α, for α is topological property: quasi-cpt, irreducible, connected;

Variety ↓ nonreduced Scheme ↓ automorphism / stabilizer Stack

Our main details are dealing this condition!

(Valuative criteria) ⑤ $f: \mathcal{X} \to \mathcal{Y}$ is universal closed/separated/proper iff

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "\exists"] & \mathcal{Y} \end{tikzcd}$$

is exist/unique/exist & unique

(finiteness) ⑥ $f: \mathcal{X} \to \mathcal{Y}$ is locally quasi-finite/finite type (representable) iff ∀ $T \to \mathcal{Y}$, T scheme, then $T\times_{\mathcal{Y}}\mathcal{X} \to T$ is locally quasi-finite/finite type

$$\begin{tikzcd} T\times_{\mathcal{Y}}\mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ T \arrow[r] & \mathcal{Y} \end{tikzcd}$$

(Recall for scheme, locally quasi-finite = locally finite type + fibre discrete) quasi-finite = locally quasi-finite + quasi-cpt; finite = proper + quasi-finite; generically finite iff $\exists\, \mathcal{Y}' \subset \mathcal{Y}$ open dense s.t. $\mathcal{X}\times_{\mathcal{Y}}\mathcal{Y}' \to \mathcal{Y}'$ finite.

(smoothness / unramified) ⑦ $f: \mathcal{X} \to \mathcal{Y}$ is locally finite type, f is smooth/étale/unramified iff

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists"] & \mathcal{Y} \end{tikzcd}$$

exist / unique & exist / unique ; [redacted] $0 \to \Bbbk \to A \to A_0 \to 0$ (Artinian).

Here it's on 2-diagram and the meaning of unique should be awared.

$\mathcal{X}$ is smooth iff $\exists$ U presentation smooth.

(Structure) Sheaves on $\mathcal{X}$ is $\mathcal{O}_{\mathcal{X}}(U) = \Gamma(U, \mathcal{O}_U)$ for $U \to \mathcal{X}$ étale. $\mathcal{O}_{\mathcal{X}}$-module is then defined & quasi-coherence also by

(Inertia group stack) ⑧

$$\begin{tikzcd} I_{\mathcal{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\delta"] \\ \mathcal{X} \arrow[r, "\delta"] & \mathcal{X}\times\mathcal{X} \end{tikzcd}$$

$I_{\mathcal{X}}$ has grp structure via $I_{\mathcal{X}}\times_{\mathcal{X}} I_{\mathcal{X}} \to I_{\mathcal{X}}$ on T-values $I_{\mathcal{X}}(T)\times_{\mathcal{X}(T)} I_{\mathcal{X}}(T) \to I_{\mathcal{X}}(T)$

$\Leftrightarrow I_{\mathcal{X}}(T) = \{(x,\alpha) \in \mathcal{X}(T)\times \mathrm{Aut}_{\mathcal{X}(T)}(x)\}$

$((x,\alpha),(y,\beta);\gamma) \mapsto (x, \alpha\circ\gamma^{-1}\circ\beta\circ\gamma)$

(γ is isomorphism $x \to y$ as $\mathcal{X}(T)$ is groupoid)

Properties of them we'll not list here, but when we use, we'll mention it. Now we start to consider geometry of stacks in detail.

[redacted] Def 2. (Degree) $f: \mathcal{X} \to \mathcal{Y}$ is [redacted] separated, finite type between integral stacks (= reduced + irreducible, reduced ⇔ $\exists$ U presentation) K(X) is a field; general nonreduced scheme/stack K(X) not field.

Then $\deg(f) = \deg(\mathcal{X}\times_{\mathcal{Y}} V/V)$ where $V \to \mathcal{Y}$ étale presentation. again by

When f not representable | f representable, $\mathcal{X}\times_{\mathcal{Y}} V$ is scheme, $\in \mathbb{Z}$

as $U \to \mathcal{X} \to \mathcal{Y}$ is map from scheme ⇒ must representable, define $\deg(f) = \deg(U/\mathcal{Y})/\deg(U/\mathcal{X})$, where $U \to \mathcal{X}$ is presentation $= \deg(U\times_{\mathcal{Y}} V/V)/\deg(U\times_{\mathcal{X}} U/U) \in \underset{\sim}{\mathbb{Q}}$

Thus $\mathbb{Q}$-coefficient seems can't be avoided! Two approaches if we need $\mathbb{Z}$: ① Adding representable; (obvious) ② Not use the Naive Chow group $A^* = Z^*/R^*$, but define another functor A'

[redacted] It's well-defined: [redacted] Take two presentation of $\mathcal{Y}$

$$\begin{tikzcd} & V \arrow[dr, "pre"] & \\ V'' \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\lozenge"] & & \mathcal{Y} \\ & V' \arrow[ur, "pre"] & \end{tikzcd}$$

⇒ $\deg(\mathcal{X}\times_{\mathcal{Y}} V/V) = \deg((\mathcal{X}\times_{\mathcal{Y}} V)\times V''/V'')$ using the definition again as the two red maps are also étale & representable & dominant, and preserved under base change to $\mathcal{X}$

$= \deg(\mathcal{X}\times_{\mathcal{Y}} V''/V'')$ as $\mathcal{X}\times_{\mathcal{Y}} V'' \xrightarrow{\cong} (\mathcal{X}\times_{\mathcal{Y}} V)\times_{V} V''$ same $= \deg(\mathcal{X}\times_{\mathcal{Y}} V'/V')$ ∎

Here due to $\mathcal{Y}$ is integral, so is V & V' ⇒ $V\times_{\mathcal{Y}} V'$ but only reduced, thus we take a irreducible component V'' integral (open dense in the component)

② Not representable case:

$$\begin{tikzcd} U\times_{\mathcal X}U' \arrow[r, shift left] \arrow[r, shift right] & U'' \arrow[dd] \\ U \arrow[d] & U' \arrow[dl] \\ \mathcal X \arrow[r] & \mathcal Y \end{tikzcd}$$

again take the irreducible (integral) open [redacted] subset, then

Prop3. ① $\deg(\mathcal X/\mathcal Z) = \deg(\mathcal X/\mathcal Y)\deg(\mathcal Y/\mathcal Z)$;

$\deg(\mathcal X/\mathcal Y) = \frac{\deg(U/\mathcal Y)}{\deg(U/\mathcal X)} = \frac{\boxed{\deg(U''/U)}\deg(U/\mathcal Y)}{\boxed{\deg(U''/U)}\deg(U/\mathcal X)} = \frac{\deg(U''/\mathcal Y)}{\deg(U''/\mathcal X)} \overset{\text{same argument}}{=} \frac{\deg(U'/\mathcal Y)}{\deg(U'/\mathcal X)}$ ▨

② $\deg(\mathcal X/\mathcal Y) = \frac{\delta(\mathcal Y)}{\delta(\mathcal X)}$ [redacted] $[K(\mathcal X):K(\mathcal Y)]$, $\delta(\mathcal X) = \deg(I_{\mathcal X}/\mathcal X)$, rational function field is def. $\mathcal X' \to$ [redacted] $\mathcal X' \subset \mathcal X$ open subst

It's same as in ANT ramification theory & inertia group ... when it integral

Rk. ① The $I_{\mathcal X}$ measures the stabilizer: when $\mathcal X$ is scheme, $\delta(\mathcal X)=1$, [redacted] degenerates to usual case; thus for our degree of stacks, the degree includes the datum of stabilizer. A easy example is classifying stack $B\mathfrak S_m$, each pt has automorphism $\mathfrak S_m$ group, the $I_{B\mathfrak S_m} = B\mathfrak S_m\times\mathfrak S_m$. [redacted] $I_{B\mathfrak S_m}\to B\mathfrak S_m$ forget (automorphisms) functor

② $I_{\mathcal X}\to\mathcal X$ forgetful functor is [redacted] quasi-finite ⇒ finite type ⇒ we can define the $\delta(\mathcal X)$ above.

representable

separated

All these properties comes quickly from $I_{\mathcal X} = \mathcal X\times_{\mathcal X\times\mathcal X}\mathcal X$

$I_{\mathcal X}$ is integral if $\mathcal X$ is by $I_{\mathcal X} = \mathcal X\times_{\mathcal X\times\mathcal X}\mathcal X$ (i.e. a locally free $\mathcal O_V$-module)

Pf. ① Representable case:

Observation: $\deg(\mathcal X/\mathcal Y) = \operatorname{rank}(\mathcal O_{\mathcal X\times_{\mathcal Y}V})$ [redacted], view $\mathcal O_{\mathcal X\times_{\mathcal Y}V}$ as vector bundle on [redacted] V, $V\to\mathcal Y$ étale presentation

Then $\deg(\mathcal X/\mathcal Z) == \deg(\mathcal X/\mathcal Y)\deg(\mathcal Y/\mathcal Z)$

$$\underbrace{\operatorname{rank}(\mathcal O_{\mathcal X\times_{\mathcal Z}W})}_{\text{on } W} = \underbrace{\operatorname{rank}(\mathcal O_{\mathcal X\times_{\mathcal Y}V})}_{\text{on } V}\underbrace{\operatorname{rank}(\mathcal O_{\mathcal Y\times_{\mathcal Z}W})}_{\text{on } W}$$

The rank formula due to $\mathcal O_{\mathcal X\times_{\mathcal Z}W}$ is vector bundle on $\mathcal Y\times_{\mathcal Z}W$, $\operatorname{rank}(\mathcal O_{\mathcal X\times_{\mathcal Z}W})$ [redacted] has $\operatorname{rank}(\mathcal O_{\mathcal Y\times_{\mathcal Z}W})$ and here we take V by: $V\subset \underbrace{V' \subset [redacted]\ \mathcal Y\times_{\mathcal Z}W}_{\text{connect component}} \Rightarrow \mathcal O_{\mathcal X\times_{\mathcal Z}W}|_V = \mathcal O_{\mathcal X\times_{\mathcal Y}V}$ (as it is presen

It's étale ⇐ étale open subset presentation of $\mathcal Y$ (For rank constant on it, connected needed) ⇒ upper rank is just $\operatorname{rank}(\mathcal O_{\mathcal X\times_{\mathcal Y}V})$

● Not representable case: Denote U, W, V presentations of $\mathcal X$, $\mathcal Y$, $\mathcal Z$, using definition of degree in not representable case repeatedly we have $\deg(\mathcal X/\mathcal Y)\deg(\mathcal Y/\mathcal Z) = \frac{\deg(U/\mathcal Y)}{\deg(U/\mathcal X)}\cdot\frac{\deg(V/\mathcal Z)}{\deg(V/\mathcal Y)} \overset{(*)}{=} \frac{\deg(U/V)\deg(V/\mathcal Y)}{\deg(U/\mathcal X)}\cdot\frac{\deg(V/\mathcal Z)}{\deg([redacted]V/\mathcal Y)} = \frac{\deg(U/V)\deg(V/\mathcal Z)}{\deg(U/\mathcal X)}$

$= \frac{\deg(U/\mathcal Z)}{\deg(U/\mathcal X)} = \deg(\mathcal X/\mathcal Z)$

(*) is due to

$$\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[dr] & V \arrow[d] \\ \mathcal X \arrow[r] & \mathcal Y \end{tikzcd}$$

$\Rightarrow \deg(V/\mathcal Y) = \frac{\deg(U/\mathcal Y)}{\deg(U/V)}$, another term is same.

U→V defined by $U \xrightarrow{?} U\times_{\mathcal Y}V \xrightarrow{pr_2} V$, composite of étale also étale and due to the projection it's surj?

The U, V both integral scheme and U→V étale surj (representable for scheme is tautological.) ⇒ U→V is étale presentation ▨

③ We'll use ① to prove it: note both left & right satisfy multiplicative property in ①, then

, we reduce to case of morphisms of type $U\to\mathcal X$, U scheme: $\deg(U/\mathcal X) = \delta(\mathcal X)[K(U):K(\mathcal X)]$

Then we need to use the equivalent description of DM stack by an étale groupoid to $\mathcal X$:

An DM stack $U\to\mathcal X \Leftrightarrow$ an étale groupoid in (Sch/k) $R \overset{s}{\underset{t}{\rightrightarrows}} U$, diagonal $R \xrightarrow{(s,t)} U\times U$ is quasi-cpt and seperated.

$(U\to\mathcal X) \longmapsto (U\times_{\mathcal X}U \overset{p_1}{\underset{p_2}{\rightrightarrows}} U)$

$(U\to[U/R]) \longleftarrow\!\shortmid (R \overset{s}{\underset{t}{\rightrightarrows}} U)$

We omit the detail construction of groupoid quotient [U/R] (I assume one learnt stack & moduli before as texts [Ols])

A standard example is group action $G\times U \overset{pr}{\underset{a}{\rightrightarrows}} U$, from projection (start) to action (target), then [U/R] = [U/G] the quotient stack. [redacted] scheme ⇔ injection ⟹ trivial equivalence

Algebraic space ⇔ diagonal [redacted] monomorphism ⇔ the étale groupoid is étale equivalence ⇔ trivial stabilizer

Our idea is reduce $K(\mathcal X) = K(X)$, X is scheme, it's by the étale groupoid has local trivialisation: we restrict to the target to make it locally has groupoid quotient is scheme.

$R = U\times_{\mathcal X}U \xrightarrow{h} U\times U \rightrightarrows U$, $R' = \overline{h(R)} \subset U\times U$

$R \xrightarrow{h} R' \rightrightarrows U$; $R' \rightrightarrows U$ assumed to be trivial equivalence $\Rightarrow X := [U/R']$ is scheme

Then $\deg(U/\mathcal X) = \deg(R/U) \overset{\text{scheme}}{=} \deg(R/R')\deg(R'/U) \overset{[redacted]}{=} \delta(\mathcal X)[K(U):K([redacted])]$ [redacted] $= \delta(\mathcal X)[K(U):K(\mathcal X)]$ done.

We check: (i) $\deg(U/\mathcal X) = \deg(R/U)$; [redacted] $\deg(R/R') = \delta(\mathcal X)$; [redacted] $K(\mathcal X) = K(X)$

(i) R→U by definition, as $U\to\mathcal X$ is representable & U is scheme

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ U \arrow[r] & \mathcal X \end{tikzcd}$$

$\deg(U/\mathcal X) = \deg(U\times_{\mathcal X}U/U) = \deg(R/U)$ Why $U\to\mathcal X$ is presentation? I don't assume this, only étale & representable but not surj.

(ii) Study the fibre on h is generically finite, with same number as $\mathcal X \to \mathcal X\times_k\mathcal X$ why? (h is unramified)

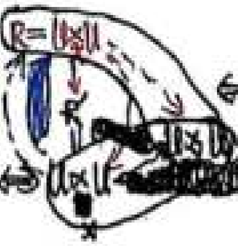
(iii)

three cartesian diagram / [redacted] étale groupoids

$\Rightarrow \mathcal X = [U/U\times_{\mathcal X}U]$

$X = [U/R']$ as groupoid quotient

thus $\{\mathcal X$ [redacted] $\xrightarrow{f} \mathbb A^1_k\} \cong \{X \xrightarrow{F} \mathbb A^1_k\}$ ▨



⇔ $\mathcal X \to Y$ done.

I don't think it works?

[redacted] due to $F: X \dashrightarrow Y \Leftrightarrow U\times_{\mathcal X}U$ [redacted]

Then we define $\mathbb A_k$ fundamental objects: normal cones, [redacted] regular embeddings, Chow groups...

Def3. ① (Cones) It's not easy to define a relative spectrum on stacks, but we have a more efficient way via presentation. Let $\mathcal S^\bullet$ is graded $\mathcal O_{\mathcal X}$-algebra ($\mathcal S^0 = \mathcal O_{\mathcal X}$, generated by $\mathcal S^1$ not needed), then $R\rightrightarrows U$ presentation of F induces another groupoid $\operatorname{Spec}(\mathcal S_R) \rightrightarrows \operatorname{Spec}(\mathcal S_U)$, then this groupoid quotient gives desired $C(\mathcal S)$ on F.

Spec$(\mathcal{F}_R^\bullet) \rightrightarrows$ Spec$(\mathcal{F}_U^\bullet) \to C(\mathcal{F}^\bullet)$, where $\mathcal{F}_R^\bullet$ (or $\mathcal{F}_U^\bullet$) are pullback of $\mathcal{O}$-modules (same as scheme case)

$$\begin{tikzcd} \mathcal{F}_R^\bullet & \mathcal{F}_U^\bullet \arrow[l, shift left, "p_1^*"'] \arrow[l, shift right, "p_2^*"] & \mathcal{F}^\bullet \arrow[l] \\ R \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & U \arrow[r] & \mathcal{X} \end{tikzcd}$$

Q. How we pullback $\mathcal{F}_R^\bullet$? A. Descent condition $p_1^*\mathcal{F}_U^\bullet = p_2^*\mathcal{F}_U^\bullet =: \mathcal{F}_R^\bullet$ (satisfy cocycle condition)

② Defining normal cone [illegible] $f:\mathcal{X}\to\mathcal{Y}$ local embedding: representable & finite type & unramified (our problem occurs here: this case we can't blow up, some [illegible])

this is for descent, as we have fact: for representable finite type map, unramified $\iff \exists$ [illegible] étale presentation [illegible] dimension [illegible]

thus we descend $C_{U/V}$ to $C_{\mathcal{X}/\mathcal{Y}}$ normal cone ⟶ these two descends are nontrivial but not hard to check, I omit [illegible]

③ (Regular embedding) $f:\mathcal{X}\to\mathcal{Y}$ local embedding, it's regular of codim d if its $U\to V$ is regular embedding of codim d $\to \mathcal{Y}$

Thus $C_{U/V}$ is normal bundle, so is $C_{\mathcal{X}/\mathcal{Y}} = N_{\mathcal{X}/\mathcal{Y}}$ over $\mathcal{O}_{\mathcal{X}}$

④ (Chow group) Same as scheme case, as the cokernel $R_*(\mathcal{X}) \xrightarrow{\partial_*} Z_*(\mathcal{X}) \to A_*(\mathcal{X}) \to 0$ (By here may is not left exact) in the category of graded Abelian group.

and due to degree $\in \mathbb{Q}$, we focus on $A_*(\mathcal{X})_{\mathbb{Q}}$, for simplicity we write as $A_*(\mathcal{X})$ stands for it.

$$\bigoplus_{\mathcal{X}'\subset\mathcal{X} \text{ integral substacks}} (k(\mathcal{X}')^*) \to \bigoplus_d \langle \mathcal{X}'\subset\mathcal{X} \text{ integral closed dim } d \mid [\mathcal{X}'] \rangle$$

naturally

$\bigoplus\bigoplus\langle$ principal cycle $\partial\beta\rangle$ on $\mathcal{X}' \mid \langle\beta\rangle\rangle$ thus it's same as scheme case. $\mathcal{X}'\subset\mathcal{X}$, dim$\mathcal{X}'=d$

The cycle $[\mathcal{X}']$ is also by descent from $R \rightrightarrows U \to \mathcal{X}$

$R\times_{\mathcal{X}}\mathcal{X}' \rightrightarrows U\times_{\mathcal{X}}\mathcal{X}' \to \mathcal{X}'$, [illegible] $\in Z_*(U)$ and [illegible] $= [\,\ldots R] = p_2^*[\ldots]$

Rk. (Beyond Naive Chow group) For many reasons we mentioned before: generalise to Artinian stack; generalise to $\mathbb{Z}$-coeffecent...

We call the upper as naive Chow groups and denote it $A^\circ_*(\mathcal{X})$

Some improvements of Chow groups: (They're same in DM [illegible] $\mathbb{Q}$-coeffecent cases): $A^\circ_*\mathcal{X} \to \hat{A}_*\mathcal{X} \to A_*\mathcal{X}$ is iso after $\otimes\mathbb{Q}$ [illegible]

① Edidin-Graham-Totaro Chow group $\hat{A}_*(\mathcal{X}) = \bigoplus_{\mathcal{X}_i\subset\mathcal{X} \text{ connected components}} \hat{A}_*(\mathcal{X}_i) = \bigoplus_i \left(\bigoplus_d \varprojlim_{E\in \text{Bundle}} A^\circ_{d+\text{rank}(E)} E \right)$, the limit takes between isomorphism class of all vector bundles, and partially ordered by $E_1\preceq E_2 \iff \exists\, p: E_2\to E_1$ projection

(Here is clear why we need decompose by connected component: we need the rank of vector bundle constant)

for $\mathcal{X}\to\mathcal{Y}$, $\hat{A}_*(\mathcal{X}) = \bigoplus\bigoplus \varprojlim A^\circ_* f^*E$ relative analogue

② Kresch's Chow group. In his Ph.d Thesis, he generalise a well-behaved intersection theory of $\mathbb{Z}$-coeffecent Artinian stack via this Chow group $A_*\mathcal{X} = \varinjlim (\hat{A}_*\mathcal{Y}/B_*\mathcal{Y})$, $B_*\mathcal{Y} = \bigoplus_{\mathcal{Z}\xrightarrow{i}\mathcal{Y}} \langle p_{2*}\beta_2 - p_{1*}\beta_1 \mid (\beta_1,\beta_2)\in \hat{A}_*\mathcal{Z}\oplus\hat{A}_*\mathcal{Z},\ i(\beta_1)=i(\beta_2)\rangle$

the limit takes via $\mathcal{Y}\to\mathcal{X}$ projective

$(\mathcal{Y}_1,f_1)\preceq(\mathcal{Y}_2,f_2) \iff \mathcal{Y}_1\xrightarrow{g}\mathcal{Y}_2$ (proj $f_1 \searrow \mathcal{X} \swarrow f_2$ proj), s.t. g induce isomorphism between $\mathcal{Y}_1$ and some union of connected components of $\mathcal{Y}_2$

with stratification on $\mathcal{X}$ by $[U/G]$, U algebraic space with G-action. to define intersection product. (G also algebraic)

Def4. (Operations)

① (Proper pushforward) $f:\mathcal{X}\to\mathcal{Y}$ proper (separated) finite type I'm confused why [Vistoli] not have, as [illegible] degree not defined

then $f_*: A_*(\mathcal{X})\to A_*(\mathcal{Y})$

$[\mathcal{X}'] \mapsto \deg(\mathcal{X}'/\mathcal{Y}')[\mathcal{Y}']$ (Here degree defined due to $\mathbb{Q}$. [illegible] only consider integral substacks)

$\mathcal{Y}' = f(\mathcal{X}') = \overline{f(\mathcal{X}')}$ (When f representable, it holds for $\mathbb{Z}$-coeffecent)

② (Flat pullback) $f:\mathcal{X}\to\mathcal{Y}$ flat (only!), then $f^*: A_*(\mathcal{Y}) \to A_*(\mathcal{X})[\text{rel.dim} f]$

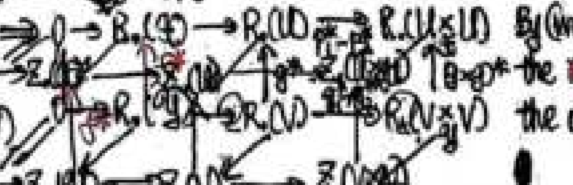
Pf. We should prove it's well-defined to descend to $[\mathcal{Y}'] \mapsto [\mathcal{Y}'\times_{\mathcal{Y}}\mathcal{X}]$ cycle class-level i.e. $\partial_{\mathcal{X}}\circ f^* = f^*\circ\partial_{\mathcal{Y}}$ & $\partial_{\mathcal{Y}}\circ f_* = f_*\circ\partial_{\mathcal{X}}$

Rk. If we want to modify below (all) arguments (except bivariant), we consider pair $(f,\mathcal{Z})\in A_*\mathcal{X}$ and

$E \to T \xrightarrow{f} \mathcal{X}$ projective, $\mathcal{Z}\in A^\circ_*(E)$, $E\to T$ vector bundle

③ [illegible] apply the functor R_* and Z_*, and due to for scheme $U\to V$ (take limit always exist) $U\times_{\mathcal{X}}U \to V\times_{\mathcal{Y}}V$ we have descent.

$$\begin{tikzcd} U\times_{\mathcal{X}}U \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, "g\times g"'] & U \arrow[r] \arrow[d, "g"] & \mathcal{X} \arrow[d, "f"] \\ V\times_{\mathcal{Y}}V \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & V \arrow[r] & \mathcal{Y} \end{tikzcd}$$



By (weak) four Lemma the red f^* fill into the diagram communicatively ☒ as the blue showed

(g induced by $U\to\mathcal{X}\times_{\mathcal{Y}}V\to V$)

④ • Representable case

It's local on $U\to\mathcal{X}$, thus it direct follows scheme case, same as arguments in ①;

• General case $U\xrightarrow{p}\mathcal{X}\xrightarrow{f}\mathcal{Y}$, for $\zeta\in Z_*(\mathcal{X})$, $\zeta\sim0$, we expect to have $f_*\zeta\sim0$

For this consider the proper surjective map $U\xrightarrow{p}\mathcal{X}$ as its étale presentation, if $\exists\,\eta\in Z_*(U)$, $\eta\sim0$, $p_*\eta=\zeta\sim0$ (*)

then $f_*\zeta = f_*p_*\eta = (f\circ p)_*\eta\sim0$ due to $f\circ p: U\to\mathcal{Y}$ is representable, by upper, $\eta\sim0 \Rightarrow (f\circ p)_*\eta\sim0$

It left to prove the fact (*) $\iff W_*(U)\xrightarrow{p_*}W_*(\mathcal{X})$ is surjective, take $(\mathcal{X}',\varphi)\in W_*(\mathcal{X})$ (i.e. $\varphi\in k(\mathcal{X}')$, $\mathcal{X}'\subset\mathcal{X}$ integral)

then as $\boxed{f \text{ proper surjective} \Rightarrow \exists\, U'\subset U \text{ integral s.t. } f|_{U'}: U'\to\mathcal{X}' \text{ generically finite of degree } n}$ $\Rightarrow p_*(\frac{1}{n}(U', p^*\varphi)) = (\mathcal{X}',\varphi)$ ☒

This fact holds for all proper surj representable $\mathcal{X}\to\mathcal{Y}$ between stacks, $\mathcal{Y}$ irreducible

Def5. (Refined Gysin map & intersection product) ⟶ We can define for substack $\mathcal{T}\subset\mathcal{Y}$, $(\mathcal{X}\cdot\mathcal{T}) := (\mathcal{X}, T)$ $U\to T$

① (Baby case) $U\to T$, T is purely k-dim scheme, thus $\varphi^*N_{\mathcal{X}/\mathcal{Y}} \supset C_{U/T}$ defines the Gysin map $s^*: A_k(\varphi^*N_{\mathcal{X}/\mathcal{Y}}) \to A_{k-d}(U)$ is only occurs scheme-level, thus well-defined.

$$\begin{tikzcd} U \arrow[r] \arrow[d, "p"'] & T \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "\text{regular local embedding}"'] & \mathcal{Y} \end{tikzcd}$$

codim d $N_{\mathcal{X}/\mathcal{Y}}$ vectorbundle $\Rightarrow (\mathcal{X}\cdot T) = s^*[C_{U/T}]$

But to define $A^*(\mathcal{X})$ as ring, we need allow T to be integral substacks

In [Vistoli] he stopped here and prove the refined Gysin map in this setting [illegible]

$A_*(T)\xrightarrow{\sigma} A_*(C_{U/T}) \to A_*(\varphi^*N_{\mathcal{X}/\mathcal{Y}}) \xrightarrow{s^*} A_*(U)[-d]$

especially for $[T] \longmapsto f^![T]$

the specializing σ as $U = \mathcal{X}\times_{\mathcal{Y}}T$ we can't use scheme case (as [illegible] didn't use Fulton's method)

② $\boxed{\text{Fact. } \mathcal{Y} \text{ DM} \Rightarrow \mathcal{Y}\xrightarrow{\Delta}\mathcal{Y}\times\mathcal{Y} \text{ regular local embedding}}$ [Admit]

And we have refined Gysin map (smooth)

$f^!: A_*\mathcal{Y}' \to A_*\mathcal{X}'[-d]$ for $\mathcal{X}'\to\mathcal{Y}'$, regular [illegible] embedding codim d

Thus $A_*(\mathcal{Y})\otimes A_*(\mathcal{Y}) \to A_*(\mathcal{Y}\times\mathcal{Y}) \xrightarrow{\Delta^!} A_*(\mathcal{Y}\times_{\mathcal{Y}\times\mathcal{Y}}\mathcal{Y}) = A_*(\mathcal{Y})$ has well-defined intersection product.

Before state the well-definedness of desired refined Gysin map, we admit basic fact coincides with scheme case.

Prop. ① $g'_* p^! = f^! g_* : A_*(\mathcal{X}) \to A_*(\mathcal{Y}')$; ② When g proper : $f'^! g_* = p_* f'^! : A_*(\mathcal{Y}'') \to A_*(\mathcal{X}')$; When g flat : $f'^! g^* = p^* f^! : A_*(\mathcal{Y}') \to A_*(\mathcal{X}'')$

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f'"] \arrow[d, "p"] \arrow[rd, phantom, "\square"] & \mathcal{Y}' \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{X}'' \arrow[r] \arrow[d, "p"] \arrow[rd, phantom, "\square"] & \mathcal{Y}'' \arrow[d, "g"] \\ \mathcal{X}' \arrow[r] \arrow[d] \arrow[rd, phantom, "\square"] & \mathcal{Y}' \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

③ $f_2^! f_1^! = f_1^! f_2^! : A_*(\mathcal{Y}) \to A_*(\mathcal{X})$

④ $\mathcal{X}' \subset \mathcal{X}$ closed substack $\Rightarrow \mathcal{X}' \xhookrightarrow{i} \mathcal{X} \xhookleftarrow{j} \mathcal{U} = \mathcal{X}\setminus\mathcal{X}' \Rightarrow A_*\mathcal{X}' \xrightarrow{i_*} A_*\mathcal{X} \xrightarrow{j^*} A_*\mathcal{U} \to 0$

⑤ $\pi : \mathcal{X}\times\mathbb{A}^1 \to \mathcal{X} \Rightarrow \pi^* : A_*\mathcal{X} \to A_*(\mathcal{X}\times\mathbb{A}^1)[1]$ is isomorphism.

Pf. All these can be proven by take presentation, but we can image that the diagram chasing is too complicated, omitted ▨ (See [Vistoli] & [Kresch] for details) then argue same as scheme-case.

Construction of refined Gysin map (In next section, we'll generalise $f^*N_{\mathcal{X}/\mathcal{Y}}$ obstruction bundle to a perfect obstruction theory)

$$\begin{tikzcd} A_*(\mathcal{Y}') \arrow[r, "\sigma"] \arrow[d, "\cong"] & A_*(C_{\mathcal{X}'}\mathcal{Y}') \arrow[r] & \boxed{A_*(C_{\mathcal{X}'}\mathcal{Y}' \times_{\mathcal{X}} \mathcal{X}') \arrow[r] & A_*(\mathcal{X}')[-d]} \\ A_*(\mathcal{Y}'\times\mathbb{A}^1) \arrow[d, equal] & & \left(A_*(p^*N_{\mathcal{X}/\mathcal{Y}}) = A_*(p^*C_{\mathcal{X}/\mathcal{Y}})\right) \arrow[u, equal] & \\ A_*(\mathcal{Y}'\times(\mathbb{P}^1-0)) \arrow[r, "\cong"] & A_*(M^0_{\mathcal{X}'}\mathcal{Y}')[1] \arrow[uu, "\partial"] & & \\ & A_*(C_{\mathcal{X}'}\mathcal{Y}')[1] & & \end{tikzcd}$$

Our left two things to do

① Construction of $M^0_{\mathcal{X}'}\mathcal{Y}'$

② The Thom isomorphism in the red □ gives.

the construction of specialising map is same as scheme case as we know this construction is pure topological

② Just right we didn't prove the Thom isomorphism is scheme reason due to topological reason, luckily we can prove it detail here, as one way to construct Chern classes

① As we mentioned, here local embedding $\mathcal{X}' \hookrightarrow \mathcal{Y}'$ [redacted] (regular not preserved under base change) can't blow-up easily, but Kresch's another paper gives alternative approach (idea is still by descent from scheme cases). We need to prove (same as scheme case)

• For $\mathcal{X} \to \mathcal{Y}$ local embedding of stacks, $\exists$ deformation space $M^0_{\mathcal{X}}\mathcal{Y}$ flat $/\mathbb{P}^1$ with general fibre $\mathcal{Y}$ and special fibre $C_{\mathcal{X}}\mathcal{Y}$

Pf. Unramified gives diagram

$$\begin{tikzcd} R = U\times_{\mathcal{X}} U \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & U \arrow[r, "f"] \arrow[d, hook, "\text{embedding}"] & \mathcal{X} \arrow[d, hook, "i", "\text{local embedding}"'] \\ S = V\times_{\mathcal{Y}} V \arrow[r, shift left] \arrow[r, shift right] & V \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

Then the étale groupoid quotient stack is desired $M^0_{\mathcal{X}}\mathcal{Y} = [M^0_U V / M^0_R S]$ $\Rightarrow M^0_R S \rightrightarrows M^0_U V$ is an étale groupoid induced by $M^0_R S \to M^0_{U\times_{\mathcal{X}} U} V\times_{\mathcal{Y}} V \underset{\text{pr}_2}{\overset{\text{pr}_1}{\rightrightarrows}} M^0_U V$ (first by $R = U\times U \hookrightarrow U\times_{\mathcal{Y}} V$, second by pullback) via write them as Proj of blow-up, then it follows the descent of sheaf-level (Recall $M^0_X Y = Bl_X(Y\times\mathbb{P}^1) - Bl_X Y$)

Then it left to show the projection to $\mathbb{P}^1$ and study its fibres: Flatness directly follows a descent result (I forgot but I believe it holds), so is its fibres ▨

Chern class

It's the way of [Milnor] takes: first construct Euler class $e = c_{top}$, then to lower term, our only use top here. Namely, $s_i(E)\cap - : \alpha \mapsto p_*(c_{top}(\mathcal{O}(1))^{r-1+i}\cap p^*\alpha)$, $p : \mathbb{P}E \to \mathcal{X}$, and $c = s^{-1}$ (as $s_1 = -c_{top}$, it's coincide original one)

Prop 2. (Top Chern class operations)

① E is rank r vector bundle over stack $\mathcal{X}$, $c_{top}(E)\cap - : A_*\mathcal{X} \to A_*\mathcal{X}[-r]$ is well-defined;

② (Properties of c_{top}) • $c_{top}(E)(c_{top}(F)\cap\alpha) = c_{top}(F)(c_{top}(E)\cap\alpha)$ • $f_*(c_{top}(f^*E)\cap\alpha) = c_{top}(E)\cap f_*\alpha$ • $c_{top}(f^*E)\cap f^*\alpha = f^*(c_{top}(E)\cap\alpha)$

• If E admits a nowhere-vanishing global section $\Rightarrow c_{top}(E) = 0$

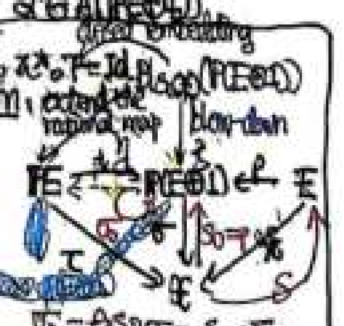
③ Let R is universal quotient bundle of τ^*E on $\mathbb{P}E$ $\xRightarrow{(a)} \pi^*\tau_*(c_{top}(R)\cap\alpha) = p^*s_0^* i^*\alpha$ [redacted], $\alpha\in A_*(\mathbb{P}E)$; $\tau_*(c_{top}(R)\cap\pi^*s_0^*\alpha) = \alpha$, $\alpha\in A_*\mathcal{X}$

Let Q is universal quotient bundle of $\sigma^*(E\oplus 1)$ on $\mathbb{P}(E\oplus 1)$ $\xRightarrow{(b)} c_{top}(Q)\cap i_*\alpha = 0$, $\alpha\in A_*(\mathbb{P}E)$; $\pi^*\sigma_*(c_{top}(Q)\cap\alpha') = p^*s_0^*$ [redacted], $\alpha'\in A_*(\mathbb{P}(E\oplus 1))$

④ $A_*E \xrightarrow{T} A_*\mathbb{P}E$, $\bar\alpha\in A_*(\mathbb{P}(E\oplus 1))$ restricts to α (via excision)

$\alpha \mapsto \sigma_*(c_{top}(Q)\cap\bar\alpha)$. T not depend on choice of $\bar\alpha$, and T is isomorphism, $T\circ\pi^* = Id$, $\pi^*\circ T = Id$

Pf. ①② are same as scheme case, omitted, we prove ③-④, first our notations from the diagram, extend the



$\sigma_0 = \rho s_0 \sigma = s_0\circ\sigma$,
$\mathbb{P}(E\oplus 1) \to s_0(\mathcal{X}) \subset \mathbb{P}(E\oplus 1)$
collapsing $\mathbb{P}(E\oplus 1)$ to zero section $s_0(\mathcal{X})$

③ we proceed by (a)(b)(c)(d)

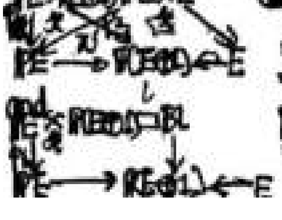
(a) $0 \to \mathcal{O}_E(-1) \to \tau^*E \to R \to 0$ on $\mathbb{P}E$ $\Rightarrow \alpha = [Z]$, $Z \subset \mathbb{P}E$, then

$c_{top}(R)\cap[Z] = [Z\cap \mathcal{O}_E(-1)]$ here $\mathcal{O}_E(-1)$ represents its cycle class $\in A^1$ ($\mathrm{Pic} \to A^1$)

$\pi^*\tau_* c_{top}(R)\cap[Z] = (\tau)_*(Z)^* c_{top}(R)\cap[Z]$

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] \arrow[rd, phantom, "\square"] & \tau^*E = \mathbb{P}E\times_{\mathcal{X}} E \arrow[l, "\tau'"'] \arrow[d, "\pi'"] \arrow[r, hook] & \mathbb{P}E\times_{\mathcal{X}}\mathbb{P}(E\oplus 1) \\ \mathcal{X} & \mathbb{P}E \arrow[l, "\tau"] & \end{tikzcd}$$

By commutativity of [redacted] two "ways"



[redacted] factor through [redacted] $A_*(\mathbb{P}E\times_{\mathcal{X}}\mathbb{P}(E\oplus 1))$, the by right red Cartesian square

$\Rightarrow$ factor through $\mathbb{P}E \leftarrow \mathbb{P}E\times\mathbb{P}(E\oplus 1) \to \mathbb{P}(E\oplus 1) \hookleftarrow E$ ($\pi^*\tau_* c_{top}(R)\cap\alpha$ side)

Factor through $\mathbb{P}E \leftarrow \mathbb{P}E\times\mathbb{P}(E\oplus 1) \leftarrow Bl \to \mathbb{P}(E\oplus 1) \hookleftarrow E$ ($p^*s_0 i^*\alpha$ side)

We focus on $\mathbb{P}E\times_{\mathcal{X}}\mathbb{P}(E\oplus 1) \leftarrow Bl$, blow-up is blow-up the $s_0(\mathcal{X}) \subset \mathbb{P}(E)$ disjoint from $\mathbb{P}E$

We reduce to show that $p_{2*}(C_{top}(p^*R)\cap[Z\times_S \mathbb{P}(E\oplus 1)]) = s_* i^*[Z\times_S \mathbb{P}(E\oplus 1)]$: this is due to the above non-commutative [illegible] diagram (✱) in level of Chow group as the [redacted] s_* sends cycles to the zero section ⇒ equivalent to $C_{top}(p^*R)$ operation as [redacted] collapse to zero section of $\mathbb{P}E$'s pullback ▨ (Conjure the last argument of $\cap\, \mathbb{P}E\times_S \mathbb{P}$)

① (2) $0\to \mathcal{O}_{E\oplus 1}(-1)\to \pi^*(E\oplus 1)\to Q\to 0$ on $\mathbb{P}(E\oplus 1)$

then we have trivial $\mathbb{A}^1$-bundle $\mathbb{P}(E\oplus 1)\ominus \pi^*E\oplus 1 \longrightarrow \mathbb{P}(E\oplus 1)\ominus \pi^*E$

thus we can identify $C_{top}(Q)\cap i_{E*}\alpha = C_{top}(Q)\cap [i(Z)]$ ← zero section $= [i(Z)]\cap \mathcal{O}_{E\oplus 1}(-1)$ $= [i(Z)\times \mathcal{O}_{E\oplus 1}(-1)]$, $i(Z)\not\subset \mathbb{P}E$ $= 0$ ▨

(a) $\pi^*\pi_* C_{top}(Q)\cap[Z] \overset{(?)}{=} \pi^*\pi_* C_{top}(Q)\cap(s_* j^*[Z\times_{\mathbb{P}(E\oplus 1)} \mathbb{P}E] + \pi_*[Z]) = p^* s_* j^*[Z\times_{\mathbb{P}(E\oplus 1)}\mathbb{P}E] + s_*\pi_*[Z]$

Here we use two formula(✱) as $\overset{(b)}{=} p^*[Z]$

$[Z] = s_* j^*[Z\times_{\mathbb{P}(E\oplus 1)} \mathbb{P}E] + \pi_*[Z]$ (✱) Pf of (✱) is very hard ▨

(b) $C_{top}(R)\cap \pi^*\alpha =$ [redacted] $[Z\times_S \mathcal{O}_E(-1)] \Rightarrow \pi_* C_{top}(R)\cap \pi^*\alpha = \pi_*[Z\times_S \mathcal{O}_E(-1)] = [Z\times_S E] \overset{\text{affine bundle}}{\underset{\text{homotopical}}{=}} [Z]$ ▨

© Then by ⓐ we can prove Thom isomorphism: $\pi_*\alpha = \pi_* C_{top}(Q)\cap\bar{\alpha} \overset{(a)}{=} \pi_*(p^*\pi_* C_{top}(Q)\cap\bar{\alpha})$
$= \pi_* p^*\bar{\alpha} = \pi_*\alpha$ well-defined & surjective.

By (b), if $\alpha = 0$; i.e. $\bar{\alpha}\in A^*(\mathbb{P}(E\oplus 1))$
can restrict to $A^*(\mathbb{P}E)\ni \tilde{\alpha}$

$\Rightarrow \pi_* C_{top}(R)\cap\bar{\alpha} = \pi_*(C_{top}(Q)\cap i_*\tilde{\alpha}) \overset{(c)}{=} \pi_* 0 = 0 \Rightarrow$ injective ▨ [redacted]

Our excess intersection formula is totally follows these properties:
$$\begin{tikzcd} \mathfrak{Z}'' \arrow[r] \arrow[d, "g"] & \mathcal{Y}'' \arrow[d, "p"] \\ \mathfrak{Z}' \arrow[r] \arrow[d, "g"] \arrow[rd, phantom, "\square"] & \mathcal{Y}' \arrow[d, "f"] \\ \mathfrak{Z} \arrow[r] & \mathcal{Y} \end{tikzcd}$$
$i^!\alpha = C_{top}(g^* N_{\mathfrak{Z}/\mathcal{Y}}/N_{\mathfrak{Z}'/\mathcal{Y}'})\cap (i')^!\alpha$

as both $i^!$ and C_{top}-operations are well-defined

- Projection formula of C_{top} holds ▨

Before Keel–Mori; all facts holds for Artinian stacks with Kresch's Chow group

Bivariant intersection theory & application to coarse moduli space

We have famous Keel–Mori thm: Given $\mathfrak{X}$ stack (with properties we assumed before), ∃ a coarse moduli space $\mathfrak{X}\xrightarrow{\pi} X$, s.t.

(1) $\forall\, k = \bar{k}$, $\mathfrak{X}(k)/_\sim \xrightarrow{\cong} X(k)$, $\sim$ is the isomorphism class of category $\mathfrak{X}(k)$;
(2) (Universal)
$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\forall f"] \arrow[rd, "\pi"'] & X' \\ & X \arrow[u, dashed, "\exists !"'] \end{tikzcd}$$

X algebraic space, here for simplicity we assume X is a scheme ⟺ definition of coarse moduli space

(KM1) $\mathcal{O}_X = \pi_*\mathcal{O}_{\mathfrak{X}}$; (KM2) X also separated & finite type; (KM3) π proper; (KM4) π is universal homeomorphism (topological)
(KM5) $\forall\, X'\to X$ flat, $\mathfrak{X}\times_X X' \longrightarrow X'$ also coarse moduli space ▨

Our task of this part is proof/sketch $A^*(\mathfrak{X})\cong A^*(X)$; or stronger, $A^*(Y\to X)\cong A^*(Y\times_X\mathfrak{X}\to\mathfrak{X})$.

Definition of $A^*(\mathcal{Y}\xrightarrow{f}\mathfrak{X})$ is same as scheme case, but need $\mathcal{Y}\xrightarrow{f}\mathfrak{X}$ is additionally representable

$A^*(\mathcal{Y}\xrightarrow{f}\mathfrak{X}) = \left\{ C = (C_g^{(p)}) \,\middle|\, C_g^{(p)} : A_* \mathfrak{X}' \to A_{*-p}\mathcal{Y}' \right\}$ satisfy axioms (1)–(3) same as scheme-case (all push-pull properties & refined Gysin is defined)
$$\begin{tikzcd} \mathcal{Y}' \arrow[r] \arrow[d] \arrow[rd, phantom, "\square"] & \mathfrak{X}' \arrow[d, "g"] \\ \mathcal{Y} \arrow[r] & \mathfrak{X} \end{tikzcd}$$

Thm. $\mathfrak{X}\xrightarrow{\pi} X$ is coarse moduli (satisfy (1)+(2) only), then ① $\pi_*: A^*(\mathfrak{X})\xrightarrow{\cong} A^*(X)$;
② Pullback $\pi^*: A^*(Y\to X)\xrightarrow{\cong} A^*(Y\times_X\mathfrak{X}\to\mathfrak{X})$

Accept this fact
$\pi(\mathfrak{Z})\times_X \mathfrak{X})_{red} = \mathfrak{Z} \Rightarrow [\mathfrak{Z}] = [(\pi(\mathfrak{Z})\times_X\mathfrak{X})_{red}]$
$\pi((Z\times_X\mathfrak{X})_{red}) = Z \Rightarrow [Z] = \pi_*[(Z\times_X\mathfrak{X})_{red}]$

Pf. ① Surjectivity. let $\mathfrak{Z}\subset\mathfrak{X}$ closed integral substack, $[\mathfrak{Z}]\in Z^*(\mathfrak{X}) \Rightarrow$ [redacted]
let $Z\subset X$ closed integral subscheme, $[Z]\in Z^*(X)\Rightarrow$ [redacted]

Thus $Z^*(\mathfrak{X})\xrightarrow{\pi_*} Z^*(X)$
$[\mathfrak{Z}]\longmapsto[\pi(\mathfrak{Z})]$ is isomorphism ⇒ descend to (at least) surjective.
$[Z\times_X\mathfrak{X}_{red}] \longleftarrow [Z]$

Injectivity. If $\pi_*\alpha \sim 0$, then by Def 4.①(✱) ⇒ $\alpha\sim 0$ done ▨

② Injectivity. $\pi^*\alpha = 0$, $\alpha\in A^*(Y\to X)$, $\forall\, x\in A^*(X)$. we need to prove $\alpha\cap x = 0$
By $\alpha\cap x = \alpha\cap\pi_* y \overset{}{=\!=} (\pi')_*(\pi^*\alpha\cap y) = (\pi')_*(0\cap y) = 0$
$Z^*(X)\cong Z^*(\mathfrak{X})$ | axiom (C) compatible with proper pushforward

Here π' is base change by
$$\begin{tikzcd} \mathfrak{X}\times_X Y \arrow[r, "\pi'"] \arrow[d] \arrow[rd, phantom, "\square"] & Y \arrow[d] \\ \mathfrak{X} \arrow[r, "\pi"] & X \end{tikzcd}$$

[Here the bivariant class action is by α_g : $A_* X' \to A_{*-p} Y'$ sends $\alpha_g : x\mapsto \alpha_g(x) \in A^{\vee}_{*-p}(X'\times_X Y)$ and it's denoted as $\alpha_g\cap x$, for all g, $\alpha\cap x = \alpha_g(x) = \alpha_g\cap x$)
Recall!

Surjectivity.
$$\begin{tikzcd} \mathfrak{X}\times_X Y \arrow[rr, "\pi'"] \arrow[dd] & & Y \arrow[dd] \\ & \mathfrak{X}\times_X Y' \arrow[lu] \arrow[rr, "\pi'"] \arrow[dd] & & Y' \arrow[lu] \arrow[dd] \\ \mathfrak{X} \arrow[rr, "\pi"] & & X \\ & \mathfrak{X}' \arrow[lu] \arrow[rr, "\pi"] & & X' \arrow[lu] \end{tikzcd}$$
$\forall\,\beta\in A^*(Y\times_X\mathfrak{X}\to\mathfrak{X})$
then its preimage $\alpha\in A^*(Y\to X)$ is determined by
$\alpha : x'\longmapsto \alpha\cap x' = (\pi')_*(\beta\cap (\pi_*^{-1})^{\text{(1)}} x')$
$A_*(X')\longrightarrow A_{*-p}(Y')$
Here we use ① ▨

§23. Normal cone, deformation and virtual [redacted] theories

After [Kontsevich, Manin]'s restricted construction of GW-classes in terms of numbers; [LT] develop a general method from Kuranishi theory in symplectic geometry (via analytics), in terms of virtual class in replacement of numbers; Here we focus on the famous algebraic approach constructing virtual class in [BF]

One year later [BF], Donaldson pushed his PhD student Thomas construct invariants ~~of~~ of CY 3-folds, this is DT-invariant (closely related) with GW (Earlier [D-T] also expect to construct this, but not general), in Thomas' PhD thesis he used (also) symplectic method (also not general) DT is for moduli of semistable sheaves, GW is for moduli of stable curves)

In this section, we mainly follow [BF], and next section use it to construct GW invariants. Idea of [BF] is DM not smooth, how can we do? smooth DM stack (but its coarse not smooth) gluing the quotient cone stack $[C_{U/M}/T_M|_U]$ to intrinsic normal cone (can be described as a Picard stack), note that the cone $\mathfrak{C}_X$ is Artinian, thus we must use [Kresch] (Kresch's is after [BF], in [BF] he not use it, but deal a restricted case)

Then
$$\begin{tikzcd} ▨ \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "0"'] \\ \mathfrak{C}_X \arrow[r] & \mathfrak{C}_{E^\bullet} \arrow[u, bend right, "\text{zero section } 0"'] \end{tikzcd}$$
$\mathfrak{C}_{E^\bullet}$ is determined by perfect obstruction theory $E^\bullet$ (Both these $\mathfrak{C}$ over $\mathfrak{X}$) called vector bundle stack, then $[X]^{vir}_{E^\bullet} = 0^![\mathfrak{C}_X] = (\mathfrak{C}_X \cdot \mathfrak{X})$ desired

Glued by $[T_M|_U \to C_{U/M}]$ locally on U

It's related with baby case as before

$$\begin{tikzcd} ▨ \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "0"] & \Leftarrow & X' \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d] \\ C_{X/Y} \arrow[r] & g^*N_{X/Y} & & X \arrow[r] & Y \end{tikzcd}$$
normal cone, obstruction bundle.
↓ ↓
Intrinsic normal cone perfect obstruction theory

Later we'll give a relative analogue of intrinsic normal cone + virtual fundamental class via intersection theory on Artinian stacks

Preparations: Cones and cotangent complexes All $\mathfrak{X}$ is at least DM here

Def. (More cones) ① A cone $\mathrm{Spec}(\mathrm{Sym}\,\mathcal{F})$ is called Abelian (named due to symmetric), and the abelianization (/associated abelian cone/abelian hull) is $A: \mathrm{Spec}(\oplus \mathcal{S}^i) \mapsto \mathrm{Spec}(\mathrm{Sym}\,\mathcal{S}^1)$ is functorial (Thus $C \subset A(C)$, that's why "hull")

② E vector bundle, $E \xrightarrow{d} C$ as map between cones over $\mathfrak{X}$, then $E \curvearrowright C$ via $E \times C \xrightarrow{\varphi} C$ naturally, so is it induced on $A(C)$, C is called E-cone if $(e, c) \mapsto d(e)+c$ (it's natural. Often then the cone quotient $[C/E]$ groupoid ... By the symmetric product we can extend φ) $E \curvearrowright A(C)$ preserves $C \subset A(C)$ invariant $E \xrightarrow{d} C \leadsto [C/E]$ vector bundle-cones

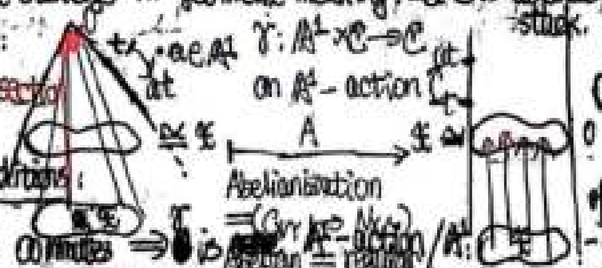
* Here all our cones needs $\mathcal{S}^0 \cong \mathcal{O}_{\mathfrak{X}}$ & generated by $\mathcal{S}^1$ étale locally

③ Functorially, morphism & homotopy (2-morphism) between [redacted] is by $E \xrightarrow{d} C$, $\phi \simeq \psi$ iff $\exists\, k: C \to F$ s.t.
$$\begin{tikzcd} E \arrow[r, "d"] \arrow[d, "\phi"'] & C \arrow[d, "\phi"] \arrow[dl, "k"'] \\ F \arrow[r, "d"] & D \end{tikzcd}$$
the two triangles commutes with $\psi - \phi$ in replacement of ϕ ($kd = \psi - \phi$, $dk = \psi - \phi$)

④ What's a cone (naively) in geometric meaning? Let $\mathfrak{C}$ is algebraic stack. $(\mathfrak{C}, 0, \gamma)$ over $\mathfrak{X}$: $\gamma: \mathbb{A}^1 \times \mathfrak{C} \to \mathfrak{C}$ on $\mathbb{A}^1$-action

$0: \mathfrak{X} \to \mathfrak{C}$ zero section

γ satisfy some compatibility conditions:



$$\begin{tikzcd} \mathfrak{C} \arrow[r, "(1{,}id)"] \arrow[dr, "id"'] & \mathbb{A}^1 \times \mathfrak{C} \arrow[d, "\gamma"] \\ & \mathfrak{C} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{C} \arrow[r, "(0{,}id)"] \arrow[dr, "0"'] & \mathbb{A}^1 \times \mathfrak{C} \arrow[d, "\gamma"] \\ & \mathfrak{C} \end{tikzcd}$$
• (1) • (2)

$$\begin{tikzcd} \mathbb{A}^1 \times \mathbb{A}^1 \times \mathfrak{C} \arrow[r, "id\times\gamma"] \arrow[d, "m\times id"'] & \mathbb{A}^1 \times \mathfrak{C} \arrow[d, "\gamma"] \\ \mathbb{A}^1 \times \mathfrak{C} \arrow[r, "\gamma"] & \mathfrak{C} \end{tikzcd}$$
• (3)

If (1)-(3) are commutes ⇒ γ is $\mathbb{A}^1$-action / $\mathbb{A}^1$-equivariant theory of cones

If (1)-(3) are $\boxed{\text{2-commutes}}$ ⇒ γ is $\mathbb{A}^1$-action / $\mathbb{A}^1$-equivariant theory of cone stacks (i.e. ∃ 2-morphisms • 2(1) $\theta_1: id \Rightarrow \gamma\circ(1, id)$ • 2(2) $\theta_0: 0 \Rightarrow \gamma\circ(0, id)$ • 2(3) $\theta_\sigma: \gamma\circ(m\times id) \Rightarrow \gamma\circ(id\times\gamma)$, and the three 2-maps also need to be compatible)

A cone stack $\mathfrak{C}$ is then $(\mathfrak{C}, 0, \gamma)$ (γ is $\mathbb{A}^1$-action of cone stacks) if it admits an $\boxed{\text{étale local}}$ $\boxed{\text{smooth presentation}}$ $C \xrightarrow{\pi} \mathfrak{C}$ (i.e. π smooth & surjective & $\mathbb{A}^1$-equivariant between two cones), it's abelian/vector bundle if C is. (i.e. étale local [redacted] w.r.t étale topology on $\mathfrak{X}$)

Morphism between cone stacks is $\mathbb{A}^1$-equivariant stack morphism.

Eg5. (More cones) ① As the green above, we show that $A(\mathfrak{C}_{X/Y}) = \mathfrak{N}_{X/Y}$ for $X \xrightarrow{f} Y$ local immersion between DM stacks. it's due to: $\mathfrak{N}_{X/Y} = $ [redacted] $(\mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y})^\vee \Rightarrow V(\mathfrak{N}_{X/Y}) = \mathrm{Spec}(\mathrm{Sym}\,\mathfrak{N}^\vee_{X/Y}) = \mathrm{Spec}(\mathrm{Sym}(\mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y})) = A(\mathrm{Spec} \oplus \mathcal{I}^n/\mathcal{I}^{n+1})$ ▨

② [redacted] ($\mathfrak{E}$-cone stack as generalisation of E-cone) $\mathfrak{E}$ is a vector bundle stack, $\mathfrak{E} \to \mathfrak{C}$ morphism between cone stacks, $\mathfrak{C}$ is a $\mathfrak{E}$-cone stack if étale locally isomorphic to $[E_1/E_0] \to [C/F]$ via $E_0 \to F$ where C is both E_1-cone & F-cone (thus we have $E_1 \to C$ and action $F \curvearrowright C$), and E_1 is E_0-cone (thus $E_0 \curvearrowright E_1$) gives this descent commute diagram.
$$\begin{tikzcd} E_1 \arrow[r] \arrow[d] \arrow[loop above, "\gamma\,id"] & C \arrow[d] \arrow[loop above, "\gamma\,id"] \\ \mathfrak{E} \arrow[r] & \mathfrak{C} \end{tikzcd}$$
"smooth" presentation

⇒ Due to C is E_1-cone, we descend $E_1 \curvearrowright C$ to $\mathfrak{E} \curvearrowright \mathfrak{C}$ action

• Due to our Abelian hull, $A(\mathfrak{C})$ also constructed by gluing étale locally the abelian hull of smooth presentation, i.e. C above ⇒ we can also describe $\mathfrak{E}$-cone stack by invariance of $\mathfrak{C} \subset A(\mathfrak{C}) \curvearrowleft \mathfrak{E}$ ▨

③ [redacted] Why cone can be descend to quotient stack in ②? We here gives cone structure of $[C/E]$, C is E-cone.

$0: \mathfrak{X} \to [C/E]$, [redacted] $\bigcup_{\substack{U \to \mathfrak{X} \\ U \text{ scheme}}} [C/E](U) = \bigcup_{\substack{U \to \mathfrak{X} \\ U \text{ scheme}}} \left\{ \begin{matrix} P \xrightarrow{f} C \\ \downarrow E\text{-torsor} \\ U \end{matrix} \right\}$ ("—" means it's determined by all $U \to \mathfrak{X}$ via Yoneda Lemma)

$= \{(E_U, 0)\}_{\substack{U \to \mathfrak{X} \\ U \text{ scheme}}}$ [redacted] $=: \{(P, f)\}_{\substack{U \to \mathfrak{X} \\ U \text{ scheme}}}$

[redacted] $:= \{(U \times E,\ 0: U\times E \to C$ vertex map is composite $U \times E \to$ [redacted] $\xrightarrow{0\times id} C \times E \xrightarrow{\varphi} C)\}$ determined.

$\mathbb{A}^1$-action is due to $\mathbb{A}^1(U) \times [C/E](U) \to [C/E](U)$

$$\left(a, \begin{matrix} P \xrightarrow{f} C \\ \downarrow \\ U \end{matrix}\right) \longmapsto \left(\begin{matrix} P \times_{a,E} E \xrightarrow{af} C \\ \downarrow \\ U \end{matrix}\right)$$ given by
$$\begin{tikzcd} P \times_{a,E} E \arrow[r] \arrow[d] & P \arrow[d] \\ E \arrow[r, "a"] & E \end{tikzcd}$$
and $af: P \times_{a,E} E \to C$, $(p, e) \mapsto af(p) + de$. ▨

④ (SES of cones) $0 \to E \underset{\rho}{\rightleftarrows} C \to D \to 0$ of cones is exact ⟺ E is vector bundle & étale locally over $\mathfrak{X}$, ∃ρ splittes it (thus induce $C \cong E \times D$) ⟺ C is E-cone & $E \times C \xrightarrow{\varphi} C$ Cartesian

$0 \to \mathfrak{E} \to \mathfrak{C} \to \mathfrak{D} \to 0$ of cone stacks is exact ⟺ $\mathfrak{E}$ is vector bundle cone stack & étale locally over $\mathfrak{X}$, $\mathfrak{C} \cong \mathfrak{E} \times \mathfrak{D}$ ⟺ $\mathfrak{C}$ is $\mathfrak{E}$-cone stack & $\mathfrak{C} \to \mathfrak{D}$ smooth & $\mathfrak{E} \times \mathfrak{C} \xrightarrow{\varphi} \mathfrak{C}$ Cartesian ⟺ We have a descending diagram in next, étale locally over $\mathfrak{X}$:

$$\begin{tikzcd}
0 \arrow[r] & E_0 \arrow[r] \arrow[d, Rightarrow] & F \arrow[r] \arrow[d, Rightarrow] & G \arrow[r] \arrow[d, Rightarrow] & 0 & \text{vector bundles} \arrow[d, Rightarrow] \\
0 \arrow[r] & E_1 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 & \text{Cones} \arrow[d] \\
0 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{C} \arrow[r] & \mathcal{D} \arrow[r] & 0 & \text{Cone stacks}
\end{tikzcd}$$

i.e.
$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d, equal] & \mathcal{C} \arrow[r] \arrow[d, equal] & \mathcal{D} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & {[E_1/E_0]} \arrow[r] & {[C/F]} \arrow[r] & {[D/G]} \arrow[r] & 0
\end{tikzcd}$$

Rk. Our cone stack is a special case of Picard stack: a stack + an additive operation. We write the Picard stack $(h^1/h^0)(E^\bullet) = [E^1/E^0]$ via $E^0 \xrightarrow{d} E^1$ between sheaves of Abelian groups if $E^\bullet = [0 \to E^0 \to E^1 \to 0]$, generally $\cdots \to E^{-1} \xrightarrow{d^{-1}} E^0 \to E^1 \xrightarrow{d^1} E^2 \to \cdots \Rightarrow (h^1/h^0)(E^\bullet) = [\mathrm{coker}(d^{-1}) / \ker(d^1)]$ ← $\boxed{\mathrm{coker}(d^{-1}) \xrightarrow{d^0} \ker(d^1)}$

Thus we have functor $(h^1/h^0): D(\mathfrak{X}) \to \mathrm{AbStk}/\mathfrak{X}$

Here a cone stack is taken the $E^\bullet$ to be simply $[E^0 \to E^1]$ with E^0 is locally free.

Next we define the Cotangent complex $\mathbb{L}^\bullet_{\mathfrak{X}}$ for a finite type $/k$ DM-stack $\mathfrak{X}$ ($\mathbb{L}^\bullet$ is relative, i.e. here we define $\mathbb{L}_{\mathfrak{X}/\mathrm{Spec}\,k}$) $\in D(\mathcal{O}_{\mathfrak{X}})$ — thus we know our real $\mathbb{L}_{\mathfrak{X}}$ is supported on $[-1,0]$

then we have $\mathfrak{N}_{\mathfrak{X}} := (h^1/h^0)((\mathbb{L}^{\geq -1}_{\mathfrak{X}})^\vee)$ the intrinsic normal sheaf

and our task later is construct $\mathfrak{C}_{\mathfrak{X}} \xrightarrow[\text{closed}]{} \mathfrak{N}_{\mathfrak{X}} = A(\mathcal{C}_{\mathfrak{X}})$ the intrinsic normal cone

Further, to $E^\bullet \to \mathbb{L}^{\geq -1}_{\mathfrak{X}}$ perfect obstruction theory $\mathfrak{C}_{\mathfrak{X}} \xrightarrow{\text{closed}} (h^1/h^0)((E^\bullet)^\vee) = [\ldots]$

A useful special case:

Let
$$\begin{tikzcd}
E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d, "\text{smooth}"] \\
\mathfrak{X} \arrow[r, "i"] & D
\end{tikzcd}$$
, then $0 \to E \to C \to D \to 0$ is SES of cones

Pf. It suffices to prove $0 \to E \xrightarrow{i} A(C) \to A(D) \to 0$ exact as the composite with inclusion $E \hookrightarrow U$, $U \to C$ gives desired splitting;

Now we prove it algebraically:

> Recall an exercise in [Ha]
> $0 \to \mathcal{G}' \to \mathcal{F} \to \mathcal{E} \to 0$ induces $0 \to C(\mathcal{E}) \to C(\mathcal{F}) \to C(\mathcal{G}') \to 0$
> i.e. $\underline{\mathrm{Spec}}\,\mathrm{Sym}(-)$ is exact when $\mathcal{E}$ locally free

Write $E = C(\ldots)$, $A(C) = C(S^1)$, $A(D) = C(S'^1)$ by Zariski locally (It's possible due to Spec is defined Zariski locally)
$D = \mathrm{Spec}(\oplus S'^i)$

> By base change, we can assume $\ldots = \mathrm{Spec}(\ldots \otimes S'^i)$

How can we choose such a base change? I don't know.

$\Rightarrow E = C \times_D \mathfrak{X} = \mathrm{Spec}(\oplus S^i \otimes_{S'^i} \mathcal{O}_{\mathfrak{X}}) = \mathrm{Spec}(\oplus (S^i / S^1 S'^i)) = \mathrm{Spec}(\mathrm{Sym}(S^1/S'^1))$

(i.e. $\mathcal{O}_{\mathfrak{X}} \otimes S'^i \to \mathcal{O}_{\mathfrak{X}}$)

$\Rightarrow \mathcal{E} \cong S^1/S'^1 \Rightarrow 0 \to S'^1 \to S^1 \to \mathcal{E} \to 0$ ☒

Def 1. (Cotangent complexes) It's the left derived functor of Ω (Kähler differential) (by Quillen) It's first Illusie's thesis formally give a basis for cotangent cpx coming into AG.

$\Omega_{-/R}: R\text{-Alg} \to \mathrm{Gr}(R\text{-Alg}) (\cong {}_R\mathrm{Mod}$ simplicial R-module$)$

$A \longmapsto \Omega_{A/R}$

$\mathbb{L}^\bullet_{A/R} := (L\Omega_{-/R})(A) \in D(\mathrm{Gr}(R\text{-Alg}))$ is called cotangent complex of A over R

One can also refer to [Jard]

For $\mathfrak{X} \to \ldots$ $\mathbb{L}_{\mathfrak{X}} = (L\Omega_{-/\ldots})(\mathfrak{X}) \in D(\mathfrak{X})$ as $\Omega_{\mathfrak{X}/\mathcal{Y}}$ had defined same as [Ha]

Due to the properties need model category to prove, we list basic facts here: (only ☐ part is needed later)

① $\mathbb{L}^\bullet_{A/R} \cong A \otimes \Omega_{A_\bullet/R}$ which $A_\bullet \to A$ is simplicial resolution (for example, we can take the bar resolution)

② $\mathrm{Hom}(\mathbb{L}^\bullet_{A/R}, N) \cong \mathrm{Hom}(A \otimes_{A_\bullet} \Omega_{A_\bullet/R}, N) \cong \mathrm{Hom}_{A_\bullet}(\Omega_{A_\bullet/R}, N) \cong \mathrm{Der}(A_\bullet, N)$ (we have André–Quillen cohomology $H^n(R, A, N)$ can be computed via $\mathrm{Der}(A_\bullet, N)_i = H_n(\mathrm{Hom}(\mathbb{L}^\bullet_{A/R}, N))$)

③ As derived functor of Ω, $\mathbb{L}^\bullet$ has same properties as Ω: $\mathfrak{X} \xrightarrow{f} \mathcal{Y} \to \mathcal{Z}\,(=\mathrm{Spec}\,k)$

$\boxed{f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}} \to \mathbb{L}_{\mathfrak{X}/\mathcal{Z}} \to \mathbb{L}_{\mathfrak{X}/\mathcal{Y}} \to f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}[1]}$ is distinguished triangle $\in D(\mathcal{O}_{\mathfrak{X}})$ (others like base change is also similiar) useful later

④ $\mathfrak{X} \to \mathcal{Y}$ is l.c.i $\Rightarrow$ $\boxed{\mathbb{L}^\bullet_{\mathfrak{X}/\mathcal{Y}} \overset{\text{q.is}}{=} [I/I^2 \to \Omega_{\mathcal{Y}}|_{\mathfrak{X}}]}$, $\mathfrak{X} \hookrightarrow \mathcal{Y}$ closed embedding $\Rightarrow$ $\boxed{\mathbb{L}^\bullet_{\mathfrak{X}/\mathcal{Y}} \overset{\text{q.is}}{\cong} C_{\mathfrak{X}/\mathcal{Y}}}$ & $\mathfrak{X} \to \mathcal{Y}$ smooth $\Rightarrow$ $\boxed{\mathbb{L}^\bullet_{\mathfrak{X}/\mathcal{Y}} \overset{\text{q.is}}{\cong} \Omega_{\mathfrak{X}/\mathcal{Y}}}$

(Grothendieck) $\mathfrak{X} \to \mathcal{Y}$ smoothable (closed embedding + smooth) $[-1, 0]$ is generalization of conormal sequence; $\Omega_{\mathcal{Y}}|_{\mathfrak{X}}$ = pullback = restriction

perfect in amplitude $[-1,0]$: q.is to sequence of vector bundle supported on $[-1,0]$

(b) & (c) gives us motivation to construct intrisic normal cone, we need glue $[I/I^2 \to \Omega_{\mathcal{Y}}|_{\mathfrak{X}}]$ locally for $U \subset V$, then locally they're cones

$\Rightarrow \mathbb{L}^\bullet_{\mathfrak{X}/\mathcal{Y}} = [I/I^2 \to \Omega_{V/\mathcal{Y}}|_{\mathfrak{X}}]$ where $\mathfrak{X} \to \mathcal{Y}$ generalised l.c.i case (l.c.i = closed embedding ∘ projection), $\mathfrak{X} \hookrightarrow V \to \mathcal{Y}$, $I = I_{\mathfrak{X}/V}$ (not depend on V)

$[-1, 0]$ perfect

Thus although $\mathbb{L}^\bullet_{\mathfrak{X}/\mathcal{Y}}$ seems generalise cotangent bundle to singular case, but it's almost impossible to compute for not l.c.i case (smoothable is used less), one always makes it easy when $\mathfrak{X}/\mathcal{Y}$ l.c.i or weaker.

Gluing intrinsic normal cone

We define the intrinsic local embedding of $\mathfrak{X}$ as $U \xrightarrow{f} M$, i étale, U, M affine and finite type scheme, M smooth
$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "i"', "\text{étale}"] & M \\
\mathfrak{X} \arrow[r] & \mathrm{Spec}\,k
\end{tikzcd}$$
f is local immersion (i.e. immersion étale local subset of $\mathfrak{X}$ to another affine) — It's used in counting dimension

the morphism between $(U, M) \to (U', M')$ is
$$\begin{tikzcd}
U' \arrow[r] \arrow[d, "\text{étale}"'] \arrow[dr, phantom, "\circlearrowleft"] & M' \arrow[d, "\text{smooth}"] \\
U \arrow[r] & M
\end{tikzcd}$$

Thm 2. $\exists!$ closed subcone $\mathfrak{C}_{\mathfrak{X}} \hookrightarrow \mathfrak{N}_{\mathfrak{X}}$, s.t. $\forall$ intrinsic local embedding, $\mathfrak{C}_{\mathfrak{X}}|_U = [C_{U/M} / f^* T_M]$

i.e. Cartesian diagram
$$\begin{tikzcd}
C_{U/M} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & N_{U/M} \arrow[d] \\
\mathfrak{C}_{\mathfrak{X}} \arrow[r] & \mathfrak{N}_{\mathfrak{X}}
\end{tikzcd}$$

(Due to this local resolution, we can glue to $\mathfrak{C}_{\mathfrak{X}}$ & $\mathfrak{N}_{\mathfrak{X}}$, but for general $E^\bullet$, we can only use $(h^1/h^0)((E^\bullet)^\vee)$, can't glue locally)

and then such $\mathfrak{C}_{\mathfrak{X}}$ is 0-dimensional & $A(\mathcal{C}_{\mathfrak{X}}) = \mathfrak{N}_{\mathfrak{X}}$

Pf. By (b) $\exists\, \varphi: \mathbb{L}^\bullet_{\mathfrak{X}}|_U \overset{\text{q.is}}{\to} [I/I^2 \to f^*\Omega_M]$ (i.e. truncation to $[-1,0]$ in $D^-(\mathcal{O}_{U_{\text{ét}}})$)

it induces $\varphi^\vee: (h^1/h^0)([I/I^2 \to f^*\Omega_M]^\vee) \to (h^1/h^0)((\mathbb{L}^\bullet_{\mathfrak{X}}|_U)^\vee)$ (Recall dual is $R\mathcal{H}om(-, \mathcal{O}_{U_{\text{ét}}})$) by functorial of h^1/h^0, the q.is = isomorphism in $D^-(\mathcal{O}_{U_{\text{ét}}}) \to$ isomorphism between stacks

(*) due to the conormal sequence is dual to the tangent sequence ☒

$(h^1/h^0)([f^* T_M \to N_{U/M}]) \xrightarrow{\cong} \mathfrak{N}_{\mathfrak{X}}|_U$

$\boxed{[N_{U/M} / f^* T_M] \xrightarrow{\cong} \mathfrak{N}_{\mathfrak{X}}|_U}$

$\Rightarrow \{N_{U/M} \to \mathfrak{N}_{\mathfrak{X}}|_U\}_{U \to \mathfrak{X}\ \text{étale}}$ is an étale local presentation

Our problem then lies on the gluing condition: take another intrinsic local embedding $(U', M') \to (U, M)$ ($f': U' \to M'$) then on U', we have $(I/I^2)|_{U'} \to f^*\Omega_M|_{U'}$ (restrict = pullback through $U' \xrightarrow{\text{étale}} U$) $\Rightarrow$ now we need check compatibility of $[N_{U'/M'} / f'^* T_{M'}]$ and $[N_{U/M} / f^* T_M]|_{U'}$ as cone stacks

$I'/I'^2 \to f'^*\Omega_{M'}$

and $C_{U/M} \hookrightarrow N_{U/M}$ is invariant under the action of f^*T_M $\Rightarrow$ same isomorphism pass to $[C_{U/M}/f^*T_M] \cong [C_{U/M}/f^*T_M]|_{U'}$ together (i.e. $C_{U/M}$ is f^*T_M-cone)
thus we have desired $\mathfrak{C}_{\mathfrak{X}}$ by gluing them together.

For the first assertion, return back to $\mathbb{L}^{\geq -1}_{\mathfrak{X}}|_U \to f^*\Omega_M|_U$, φ in derived category $D^-(\mathcal{O}_{U_{et}})$ have $\varphi \circ (\varphi|_{U'}) = \varphi'$ by naturality

for the second assertion, induces $h^1/h^0(\varphi^\vee)$ isomorphism (i.e. $(\mathbb{L}_{U/M})|_{U'} = \mathbb{L}_{\mathfrak{X}}|_{U'}$)

$U \xrightarrow{f} M$ induces splitting SES $\mathbb{L}^{\geq -1} \to f^*\Omega_M \to$ As both φ and φ' are qis

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "\Delta_f"'] & M \\ & M \times M \end{tikzcd}$$

$$\begin{cases} 0 \to f^*T_M \to N_{U/M\times M} \rightleftarrows N_{U/M} \to 0 \\ 0 \to f^*T_M \to C_{U/M\times M} \rightleftarrows C_{U/M} \to 0 \end{cases} \Rightarrow \begin{cases} N_{U/M\times M} = f^*T_M \times N_{U/M} \xrightarrow{pr_2} N_{U/M} \\ C_{U/M\times M} = f^*T_M \times C_{U/M} \xrightarrow{pr_2'} C_{U/M} \end{cases}$$

$\Rightarrow$ they have same $h^1/h^0(-)$ ▨

Here pr_2 (& pr_2') are just the action of f^*T_M to $N_{U/M}$ by the SES given. ▨

It left to prove properties of $\mathfrak{C}_{\mathfrak{X}}$: $A(\mathfrak{C}_{\mathfrak{X}}) = \mathfrak{N}_{\mathfrak{X}}$ and $\dim \mathfrak{C}_{\mathfrak{X}} = 0$. The first is obvious as the cone spec is constructed locally and locally $A(C_{U/M}) = N_{U/M}$ ▨

Counting dimensional also local: $f^*T_M \times C_{U/M} \xrightarrow[pr]{\sigma} C_{U/M} \to [C_{U/M}/f^*T_M] \Leftrightarrow$ Cartesian diagram

$$\begin{tikzcd} f^*T_M \times C_{U/M} \arrow[r, "\sigma"] \arrow[d] \arrow[dr, phantom, "\square"] & C_{U/M} \arrow[d] \\ C_{U/M} \arrow[r, "q"] & {[C_{U/M}/f^*T_M]} \end{tikzcd}$$

$\Rightarrow \dim \mathfrak{C}_{\mathfrak{X}} \overset{\text{local}}{=} \dim [C_{U/M}/f^*T_M] = \dim C_{U/M} - \dim q \overset{\text{Cartesian}}{=} \dim C_{U/M} - \dim q' = \dim M - \dim f^*T_M \overset{\text{local immersion}}{=} \dim M - \dim T_M \overset{\text{smooth}}{=} 0$ ▨

Prop2. ① $\mathfrak{X}(\to \operatorname{Spec} k)$ l.c.i $\Leftrightarrow \mathfrak{C}_{\mathfrak{X}} = \mathfrak{N}_{\mathfrak{X}} \Leftrightarrow \mathfrak{C}_{\mathfrak{X}}$ is vector bundle stack;
② $\mathfrak{C}_{\mathfrak{X}\times\mathfrak{Y}} = \mathfrak{C}_{\mathfrak{X}} \times \mathfrak{C}_{\mathfrak{Y}} \subset \mathfrak{N}_{\mathfrak{X}} \times \mathfrak{N}_{\mathfrak{Y}} = \mathfrak{N}_{\mathfrak{X}\times\mathfrak{Y}}$; ③ $f: \mathfrak{X} \to \mathfrak{Y}$ l.c.i $\Rightarrow$ SES $\mathfrak{N}_{\mathfrak{X}/\mathfrak{Y}} \to \mathfrak{C}_{\mathfrak{X}} \to f^*\mathfrak{C}_{\mathfrak{Y}}$.

Pf. ① (i) $\Rightarrow$ (iii) l.c.i makes local immersion $U \to M$ is regular $\Leftrightarrow C_{U/M} = N_{U/M}$
(iii) $\Rightarrow$ (ii) Trivial (ii) $\Rightarrow$ (i) Vector bundle stack $\Rightarrow \mathfrak{C}_{\mathfrak{X}} = A(\mathfrak{C}_{\mathfrak{X}}) = \mathfrak{N}_{\mathfrak{X}} \Leftrightarrow C_{U/M} = N_{U/M} \Leftrightarrow$ regular $U \to M \Leftrightarrow$ l.c.i ▨
② It's local ▨
③ Apply the functor $h^1/h^0((-)^\vee_{\geq -1})$ to (*) $f^*\mathbb{L}_{\mathfrak{Y}} \to \mathbb{L}_{\mathfrak{X}} \to \mathbb{L}_{\mathfrak{X}/\mathfrak{Y}} \to f^*\mathbb{L}_{\mathfrak{Y}}[1]$ to give SES $\mathfrak{N}_{\mathfrak{X}/\mathfrak{Y}} \to \mathfrak{N}_{\mathfrak{X}} \to f^*\mathfrak{N}_{\mathfrak{Y}}$ (**)
How we pass to cone-level? We return back to local structure, the glue respects the exactness due to (**) is exact $\Rightarrow$ (**) locally exact $\supset$ cone-sequence $\Rightarrow$ if cone-sequence exact, it has same gluing as normal sheave $\Rightarrow$ also exact.

Locally, we need a local presentation of $\mathfrak{X} \xrightarrow{f} \mathfrak{Y}$ of relative setting (local immersion $U \to M$ induce) local embedding induce map $M \to V$ thus $[C_{U/P}/T_P|_U]$, $[C_{V/N}/T_N|_U]$, $[N_{U/M}/T_{M/V}|_U]$ is desired locally

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\text{local immersion}"'] \arrow[dr, "\text{smooth}"] & M \arrow[dl, "\text{smooth}"] \\ V \arrow[r, "\text{local immersion}"'] & N \end{tikzcd}$$

then composite $U \to P$ also an intrinsic local embedding of $\mathfrak{X}$

By cone exact sequence & pullback sequence of tangent sheaf (I'm confused on where the P comes from) (or differential sheaf)

$$\begin{tikzcd} T_{M/V}|_U \arrow[r] \arrow[d] & T_P|_U \arrow[r] \arrow[d] & T_V|_U \arrow[d] \\ N_{U/M} \arrow[r] & C_{U/P} \arrow[r] & C_{V/N} \end{tikzcd}$$

$\Rightarrow$ descend to quotient ▨

It left to check the functor $(h^1/h^0)((-)^\vee_{\geq -1})$ changes distinguished triangle to SES (in the cat of Abelian cone stacks): set $E^\bullet \to F^\bullet \to G^\bullet \to E^\bullet[1] \in D(\mathcal{O}_{\mathfrak{X}_{et}})$
Assume $G^\bullet = \operatorname{Cone}(E^\bullet \to F^\bullet)$ (up to an isomorphism between distinguished triangles?), and assume all of them supported on $[-1, 0]$ (This is due to the definition of (h^1/h^0) is reduce to ker → coker in $[0,1]$)
Thus we have $[E^{-1} \to E^0] \to [F^{-1} \to F^0] \to [F^{-1} \oplus E^0 \to F^0] \xrightarrow{+1}$.
I'm confused the argument in [BF]
using $(E^{-1})^\vee \to \ldots \leftarrow [(F^{-1})^\vee/(F^0)^\vee] \leftarrow [(F^{-1} \oplus E^0)^\vee/(F^0)^\vee]$ exact ▨
i.e. the heart/middle is adding E^0
and seems that the black not exact

$$\begin{tikzcd} (E^{-1})^\vee \arrow[d, equal] & (F^{-1})^\vee \arrow[l] \arrow[d, equal] & (F^{-1} \oplus E^0)^\vee \arrow[l] \arrow[d, equal] \\ (E^0)^\vee & (F^0)^\vee \arrow[l] & (F^0)^\vee \arrow[l, equal] \end{tikzcd}$$

exact + exact

after $F^{-1} \xrightarrow{0} F^{-1} \oplus E^0$, $E^{-1} \oplus E^0$ exact is true.

Obstruction theory

Def8. An obstruction theory for $\mathfrak{X}$ is $\varphi: E^\bullet \to \mathbb{L}_{\mathfrak{X}} \in D(\mathcal{O}_{\mathfrak{X}_{et}})$, s.t. $h^0(\varphi)$ is isomorphism & $h^{-1}(\varphi)$ surjective
$\Leftrightarrow \varphi^\vee: \mathfrak{N}_{\mathfrak{X}} \to \mathfrak{C}(E^\bullet) = (h^1/h^0)(E^\bullet_{\geq -1}{}^\vee)$ is closed immersion between cone stacks (the equivalence by functoriality of h^1/h^0)
It's perfect if $E^\bullet$ is of perfect amplitude $\subset [-1, 0]$ (i.e. $E^\bullet \overset{\text{qis}}{=} E'^\bullet$, $E'^\bullet$ supported on $[-1, 0]$ & locally free)

Why it's called obstruction theory?

I answer this via two ways: ① Similar with AT, we can identify it as obstruction of lift/extending morphism, (Deformation theory)
② What our baby case before behaves with obstruction $\mathcal{N}_{X/Y}$ (by example we motivate a definition of E for smooth Y and $T_Y|_X \to E_{X/Y}$, then for singular, replace T by $\mathbb{L}^\bullet$)
① from [BF], ② from [BCC. Virtual Classes for working M]
① We don't give details (not used later). We had known as a higher order generalisation of $\operatorname{Def}(X) = H^1(T_X)$ to $\operatorname{Obs}(X) - H^2(T_X)$ in deformation theory. To make this vague description precisely, consider extension problem $T \xrightarrow{g} \mathfrak{X}$, $\bar{T}$, $J = \mathcal{I}_{T/\bar{T}}$ (square-zero extension, $J^2 = 0$)
$g^*\mathbb{L}_{\mathfrak{X}} \to \mathbb{L}_T \to \mathbb{L}_{T/\bar{T}} \in D^-(\mathcal{O}_{T_{et}})$, and $\mathbb{L}^{\geq -1}_{T/\bar{T}} = J[1] \Rightarrow (g^*\mathbb{L}_{\mathfrak{X}} \to \mathbb{L}_{T/\bar{T}}) \in \operatorname{Hom}(g^*\mathbb{L}_{\mathfrak{X}}, J[1])$ closed immersion $= \operatorname{Ext}^1(g^*\mathbb{L}_{\mathfrak{X}}, J)$
thus we have $\omega: \operatorname{Hom}_{\mathfrak{X}}(T, \mathfrak{X}) \to \operatorname{Ext}^1(g^*\mathbb{L}_{\mathfrak{X}}, J)$

Fact (1) Extension $\bar{g}$ exists $\Leftrightarrow \omega(g)$ vanishes (2) $E^\bullet$ is obstruction theory $\Leftrightarrow$ (i) $\varphi^*(\omega(g)) \in \operatorname{Ext}^1(g^*E^\bullet, J) = 0 \Leftrightarrow$ extension

③ then $g: T \to \mathfrak{X}$ is a $\mathcal{E}xt^1(g^*E^\bullet, I) = \mathcal{H}om(g^* h^1(E^\bullet)_\bullet, I)$ — torsor

④ Eg.b. $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ Segre embedding / twisted cubic, $I_{\mathbb{P}^1/\mathbb{P}^3} = \langle xz - y^2, yw - z^2, xw - yz \rangle$ projective ideal

$(s:t) \mapsto (s^3 : s^2t : st^2 : t^3)$ is degree

To compute $E_{\mathbb{P}^1}\mathbb{P}^3$, we have a description by $I_{\mathbb{P}^1/\mathbb{P}^3}$'s generators via: $\bigoplus_{j \in T} \mathcal{O}_Y(-(f_j)) \xrightarrow{(f_1 \cdots f_r)} \mathcal{I}_{X/Y} \to 0$ on Y, pullback/restrict to X

$\Rightarrow 0 \to \mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y} \to \bigoplus \mathcal{O}_Y(-(f_j))|_X$ and apply the functor

$\mathrm{Spec}(\mathrm{Sym}(-))$ to it (i.e. the total space of locally free sheaf $\mapsto$ vector bundle) for $\mathcal{I}_{X/Y} = \langle f_1 \cdots f_r \rangle$, (f_j) is divisor generated by f_j

$\Rightarrow C_{X/Y} \hookrightarrow \mathrm{Spec}(\mathrm{Sym}(\mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y})) \hookrightarrow \mathrm{Spec}(\mathrm{Sym} \bigoplus_{j \in T} \mathcal{O}_Y(-(f_j))|_X) =: E_X Y$, an obstruction bundle

$\overset{\shortparallel}{A(C_{X/Y})}$ is the Abelian hull of $C_{X/Y}$

we call such construction gives the naive obstruction bundle, this makes sense when $X \hookrightarrow Y$ not regular

Here by this construction of obstruction bundle (distinct from the normal bundle as obstruction bundle)

$\Rightarrow \mathcal{O}_{\mathbb{P}^1}(-(f_i))$ [illegible] $\in \mathrm{Pic}(\mathbb{P}^1) = \mathbb{Z}$ determined by $\deg f_i$, i.e. $\mathcal{O}_{\mathbb{P}^1}(-(f_i)) = \mathcal{O}_{\mathbb{P}^1}(\deg f_i)$ [illegible] $\deg f_i) = \mathcal{O}_{\mathbb{P}^1}(3\deg f_i)$

$\Rightarrow E_{\mathbb{P}^1}\mathbb{P}^3 = \mathrm{Spec}(\mathrm{Sym}\, \mathcal{O}_{\mathbb{P}^1}(6)^{\oplus 3}) = \mathbb{V}(\mathcal{O}_{\mathbb{P}^1}(6)^{\oplus 3})$ desired obstruction bundle, we simply write it as $\mathcal{O}_{\mathbb{P}^1}(6)$

Our expectation is $\mathcal{O}_{\mathbb{P}^1}(6)^{\oplus 3} = E_{\mathbb{P}^1}\mathbb{P}^3 \longrightarrow N_{\mathbb{P}^1}\mathbb{P}^3$ i.e. $N_{\mathbb{P}^1}\mathbb{P}^3 \subseteq T_{\mathbb{P}^3}|_{\mathbb{P}^1} \twoheadrightarrow E_{\mathbb{P}^1}\mathbb{P}^3$ as $(f) \in \mathbb{P}^3$, we pullback f to $(s^3 \cdots)$ $\in \mathbb{P}^1$ to compute $\Rightarrow$ multiply $(\deg = 3)$ ▨

We compute $N_{\mathbb{P}^1}\mathbb{P}^3$ by $0 \to N_{\mathbb{P}^1/\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^1}(6)^{\oplus 3} \xrightarrow{(\varphi)} \mathcal{O}_{\mathbb{P}^1}(9)$ given by the relations between the generators $(xz - y^2), (yw - z^2), (xw - yz)$ by linear algebra,

$((xz-y^2)(yw-z^2)(xw-yz)) \mapsto 0$

[This sequence comes from excess intersection formula; $0 \to N_{X/Y} \to E_{X/Y} \to E \to 0$]

(co)ker (φ) computable

$(A,B,C) \longmapsto wA + yB - zC$ (I'm confused on why it's so defined)

$\Rightarrow 0 \to \mathcal{O}_{\mathbb{P}^1}(6)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1}(6)^{\oplus 3} \to \boxed{\mathcal{O}_{\mathbb{P}^1}(9) \to \mathcal{O}_{\mathbb{P}^1}(1)} \to 0$ exact, i.e. $N_{\mathbb{P}^1/\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^1}(5)^{\oplus 2} \hookrightarrow E_{\mathbb{P}^1}\mathbb{P}^3 = \mathcal{O}_{\mathbb{P}^1}(6)^{\oplus 3}$ excess

obstruction (Here is $\mathcal{O}_{\mathbb{P}^1}(8)$)

A corollary is $[X]^{vir}_{E_{\mathbb{P}^1}\mathbb{P}^3} = \mathrm{Gysin}(\mathcal{O}_{\mathbb{P}^1}(8)) \cap [X] = (C \cdot X) = 8[pt]$, where $X = \mathcal{S}(\mathbb{P}^1) \subset \mathbb{P}^3$, $\deg = 8$, Bezout

Generalise to X not smooth, by blowing-up formula $[X]^{vir}_{E} = \pi^* \mathrm{Gysin}(\ldots \cap [\tilde{X}])$, $\tilde{X} \to X = \mathbb{P}^1$, $\tilde{Y} = \mathrm{Bl}_X Y$ ▨, $Q = \pi^* E_{X/Y}/\mathcal{O}(E)$

This it motivates a baby case

Here due to $C_{X/Y} \subset E_X Y$ universal factor through $N_{X/Y}$ (by $A = A(C)$)

Def 8' Obstruction theory $E_{X/Y}$ on X is given by datum: $X \hookrightarrow Y$ (global) embedding, Y smooth

② $X \hookrightarrow Y$, (V_i) open cover of Y and gluing condition (Here locally we can also take each E_i naive obstruction bundle)

$\boxed{C_{U_i}V_i \hookrightarrow E_i \text{ locally}}$ $\varphi_{ij}: E_i|_{V_{ij}} \to E_j|_{V_{ij}}$

$$\begin{tikzcd}
X \arrow[r, hookrightarrow] & Y \\
U_i \arrow[u] \arrow[r, hookrightarrow] \arrow[ur, phantom, "\square"] & V_i \arrow[u]
\end{tikzcd}$$

$\Rightarrow$ gluing to $E_{X/Y}$ globally

$C_{U_{ij}}V_{ij} \to C_{U_j}V_j|_{U_{ij}}$

Again due to $N_{X/Y} = A(C_{X/Y})$ is smallest Abelian cone $\supset C_{X/Y}$

$\Rightarrow N_{X/Y} \subset E_{X/Y}$ ▨

These are just what our done when gluing $\mathfrak{N}_{\mathfrak{X}}$ and $\mathfrak{C}_{\mathfrak{X}}$ (But we mentioned $(h^1/h^0)(E^\bullet_{\mathrm{trunc}}) =: \mathfrak{C}(E^\bullet)$ can't glued so) for $E^\bullet$ concentrated on 0, things work well and it's $\mathfrak{C}(E^\bullet)$ and this gives $\mathfrak{N}_{\mathfrak{X}} \to \mathfrak{C}(E^\bullet)$ motivated ▨

Return back to Def 8, forget upper baby case now. Rk (Virtual & Derived Geometry; by Pardon)

[illegible] Virtual Geometry

• Why "Geometry"? X topological space enough to $[X] \in H^{BM}_*(X)$

We motivate this via excess intersection theory: $E \to Y$ cut out, $\dim Y - \mathrm{rank}(E)$ (assume $E = L$, locally determines f cut on Y) but nothing about virtual fundamental class

We call $d = \dim Y - \mathrm{rank}\, E$ the virtual dimension of X. When section transversal, virtual dimension $= \dim X$

Why derived geometry? $[X]$ is harder than $[X] \in$ in [illegible] geometry

Thus our "virtuality" of $[X]^{vir} = \mathrm{Gysin}(E/E^\bullet) \cap [X]$ lies on the not transversal cases / excess cases

otherwise: degenerates to less local equations determined by $E \in E^\bullet$

this in symplectic, E [illegible] intersection, transversal: more global geometry is included as J-holomorphic curves, to define virtual fundamental class.

[illegible] excess bundle is E/E

That's why we mentioned that: in some modern enumerative problems, transversality not important if we care at virtual level sometimes is enough (e.g. for uses in physics) $\Rightarrow$ thus we expect a derived information carries these J-hol curves.

Def 9 (Virtual concepts) ① Now we have $\mathfrak{C}_{\mathfrak{X}} \hookrightarrow \mathfrak{C}(E^\bullet)$, the virtual dimension of $\mathfrak{X}$: $\dim^{vir}\mathfrak{X} := \mathrm{rank}(E^\bullet) = \dim(E^0) - \dim E^{-1}$ (as $\dim \mathfrak{C}_{\mathfrak{X}} = 0$, the definition is natural)

② [illegible]

$$\begin{tikzcd}
\mathfrak{C}_{\mathfrak{X}} \times_{\mathfrak{C}(E^\bullet)} \mathfrak{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "0"] \\
\mathfrak{C}_{\mathfrak{X}} \arrow[r] & \mathfrak{C}(E^\bullet)
\end{tikzcd}$$

Here $\dim(\mathfrak{C}_{\mathfrak{X}} \times_{\mathfrak{C}(E^\bullet)} \mathfrak{X}) = \dim \mathfrak{C}_{\mathfrak{X}} - \mathrm{codim}(X; \mathfrak{C}(E^\bullet))$
$:= 0 - (-n) = n$

$(\dim (h^1/h^0)(E^\bullet_{\mathrm{trunc}}) = -n$ over X, we admittes it$)$

then by Prop 2's Thom isomorphism between Chow grps of Artin stacks

$\Rightarrow 0^!_{\mathfrak{C}(E^\bullet)}: A^*(\mathfrak{C}(E^\bullet)) \to A^*(\mathfrak{X})[\mathrm{rank}\, E^\bullet]$

$\mathfrak{C}_{\mathfrak{X}} \longmapsto 0^!_{\mathfrak{C}(E^\bullet)}[\mathfrak{C}_{\mathfrak{X}}] =: [\mathfrak{X}]^{vir}_{\mathfrak{C}(E^\bullet)}$ or $[\mathfrak{X}]^{vir}_{E^\bullet}$ simply

Rk. Without it in [BF], he takes a global resolution of $E^\bullet$ to reduce to quotient stack, then prove independence.

③ For $C \hookrightarrow \mathfrak{E}$ closed subcone stack $\subset$ vector bundle stack

the $\mathcal{O}_{C/\mathfrak{E}}$ is defined via gluing locally $E_0 \rightrightarrows E_1 \to \mathfrak{E}$ presentation, assume $\mathfrak{E} \cong [E_1/E_0]$

Why we can glue them?

(i.e. let the local φ's in perfect amplitude to be global $\exists F^\bullet \xrightarrow{\sim} E^\bullet$, $F = [F^{-1} \to F^0]$ locally free)

$$\begin{tikzcd}
E_1 \arrow[r] & \mathfrak{E} \\
C \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \mathfrak{C} \arrow[u]
\end{tikzcd}$$

then take $\mathfrak{C}_{\mathfrak{X}} \hookrightarrow \mathfrak{C}(E^\bullet)$

set $\mathcal{O}_{C/\mathfrak{E}} = \bigoplus \mathcal{O}^i_{C/\mathfrak{E}} = \bigoplus \mathrm{Tor}_i^{\mathcal{O}_{\mathfrak{E}}}(\mathcal{O}_C, \mathcal{O}_{\mathfrak{E}})$

$\Rightarrow \mathcal{O}^{vir}_{\mathfrak{X}, E^\bullet} := \mathcal{O}_{\mathfrak{C}_{\mathfrak{X}}/\mathfrak{C}(E^\bullet)}$ [illegible] called virtual structure sheaf predicted by Kontsevich (Such form of graded structure sheaf is useful in derived scheme.)

He expect that such sheaf satisfy $[\mathfrak{X}]^{vir}_{E^\bullet} = \mathrm{td}(E^\bullet) \cap \mathrm{ch}(\mathcal{O}^{vir}_{\mathfrak{X}, E^\bullet})$ a type of GRR then

[In Addition], he had describe how we can lift td and ch to derived level (virtual GRR)

Prop 4. ① $\mathfrak{X}$ smooth, $h^1(E^\bullet)$ locally free $\Rightarrow [\mathfrak{X}]^{vir}_{E^\bullet} = c_{top}(h^1(E^\bullet)^\vee) \cap [\mathfrak{X}]$ ($h^1(E^\bullet)$ locally free $\Leftrightarrow h^1(E^\bullet)^\vee$ locally free)

in particular, if obstruction vanishes, i.e. $h^1(E^\bullet) = 0$, $[\mathfrak{X}]^{vir}_{E^\bullet} = [\mathfrak{X}]$

② $[\mathfrak{X} \times \mathfrak{Y}]^{vir}_{E^\bullet \boxplus F^\bullet} = [\mathfrak{X}]^{vir}_{E^\bullet} \times [\mathfrak{Y}]^{vir}_{F^\bullet}$ if $E^\bullet$ & $F^\bullet$ admit global resolution ($E^\bullet \boxplus F^\bullet = p_1^* E^\bullet \oplus p_2^* F^\bullet$) $\Rightarrow \mathbb{L}_{\mathfrak{X} \times \mathfrak{Y}} = \mathbb{L}_{\mathfrak{X}} \boxplus \mathbb{L}_{\mathfrak{Y}}$)

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f'"] \arrow[d, "g'"] \arrow[dr, phantom, "\square"] & \mathcal{Y}' \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$ $E^\bullet \to \mathbb{L}_{\mathcal{X}}$, $F^\bullet \to \mathbb{L}_{\mathcal{Y}}$ [redacted] $\Rightarrow g^![\mathcal{Y}']^{vir}_{F^\bullet} = [\mathcal{X}']^{vir}_{E^\bullet}$

two perfect obstruction theory & admits global resolutions. In particular take $\mathcal{Y}' = \mathcal{Y}$, $\mathcal{X}' = \mathcal{X}$, then $f^*[\mathcal{Y}]^{vir}_{F^\bullet} = [\mathcal{X}]^{vir}_{E^\bullet}$

For two cases (A) f l.c.i. & $\mathcal{X}, \mathcal{Y}$ are smooth; (B) f smooth. & compatible: $f^*F^\bullet \xrightarrow{} E^\bullet \longrightarrow \mathbb{L}_{\mathcal{X}/\mathcal{Y}} \xrightarrow{+1} \cdots$

Pf. ①②③(E) all proven for scheme before: ① is excess intersection formula; ② is product of tangent bundle (here is cotangent complex); ③(B) is a property of refined Gysin map. For (A), use another square and imitate the proof of (B), see [BF]

$f^*\mathbb{L}_{\mathcal{Y}} \longrightarrow \mathbb{L}_{\mathcal{X}} \longrightarrow \mathbb{L}_{\mathcal{X}/\mathcal{Y}} \xrightarrow{+1} \cdots$ is morphism between distinguished triangles.

Note that replace previous obstruction by perfect obstruction, scheme by DM stack not essentially effect our proof: via the global resolution, replace $E^\bullet$ & $(F^\bullet)$ by simply $[E_0 \to E_1]$ & $[F_0 \to F_1]$, then $[E_1/F_0]$ & $[E_1/F_0]$ gives the underlying DM stacks, we can descent our result before via this quotient ▨

Relative theory. Replace our base Spec k by smooth Artinian stack $\mathcal{S}$ over k.

Then replace $(h^1/h^0)(\mathbb{L}_{\mathcal{X}}^\vee)$ by $(h^1/h^0)(T_{\mathcal{X}/\mathcal{S}})$, replace $E^\bullet \to \mathbb{L}_{\mathcal{X}}$ by $E^\bullet \to \mathbb{L}_{\mathcal{X}/\mathcal{S}}$, and we can give $\mathfrak{C}_{\mathcal{X}/\mathcal{S}} \subset \mathfrak{C}(E^\bullet)$, relative intrinsic normal cone.

One then may think we'll define $[\mathcal{X}]^{vir}_{E/\mathcal{S}} \overset{?}{=}$ [redacted] $[\mathfrak{C}_{\mathcal{X}/\mathcal{S}}]$ It's wrong! One can notice that then when $E^\bullet$ trivial

Our correct definition is still $0^!_E[\mathfrak{C}_{\mathcal{X}/\mathcal{S}}]$ (Here $\mathfrak{C}(E^\bullet)$ contains enough datum of relative to $\mathcal{S}$) $\Rightarrow [\mathcal{X}]^{vir}_{E/\mathcal{S}} \neq [\mathcal{X}]$ (even dimension not coincide)

Now consider diagram $\begin{tikzcd} \mathcal{X}' \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$ (Cartesian is not natural holds, but an additional condition!)

$\mathcal{X}(\mathcal{X}')$ over $\mathcal{Y}(\mathcal{Y}')$ relatively, then similar properties as before: just a refined statement, proof is same as refined Gysin maps' arguments

Props. ① $g^![\mathcal{X}]^{vir}_{E/\mathcal{Y}} = [\mathcal{X}']^{vir}_{g^*E/\mathcal{Y}'}$ if $E^\bullet$ admits global resolution (then so is $g^*E^\bullet$) and g is flat/g regular local immersion

② $[\mathcal{X}]^{vir}_{E/\mathcal{Y}} = \mathrm{Ctop}(h^1(E^{\bullet\vee})) \cap [\mathcal{X}]$ if $E^\bullet$ admits global resolution, $h^1(E^\bullet)$ locally free, $\mathcal{X} \to \mathcal{Y}$ smooth;

③ $[\mathcal{X} \times \mathcal{X}']^{vir}_{E \boxplus F/\mathcal{Y} \times \mathcal{Y}'} = [\mathcal{X}]^{vir}_{E/\mathcal{Y}} \times [\mathcal{X}']^{vir}_{F/\mathcal{Y}'}$ if $E^\bullet$ & $F^\bullet$ admits global resolution.

④ $g^![\mathcal{X}]^{vir}_{E/\mathcal{Y}} = [\mathcal{X}']^{vir}_{F/\mathcal{Y}'}$ if $E^\bullet$ & $F^\bullet$ admits global resolution ([BF] don't have this condition, I think it's not true) & compatible & g smooth / g' & g smooth.

Pf. Same ▨

[redacted]

Understand virtual class: ① It has good invariance: deformation invariance;

② We know the intersection (X_t, Y_t) can jump as $t \to 0$ if $(X_t), (Y_t)$ has deformation to $t = 0$ set of naive obstruction theories, we compute $[X]^{vir}_{X/\mathbb{P}^3} = [X]^{vir}_{\cdots}$

$\boxed{\text{A useful tool: } [X]^{vir}_{X/Z} = [X]^{vir}_{(E_{X/Y} \oplus E_{Y/Z}|_X)}, \text{ for } X \hookrightarrow Y \hookrightarrow Z \text{ embedding}}$

(For a proof via bi-deformation, see [BCC])

③ virtual class (resp. to. deformation theory $E^\bullet$) of (X, Y) is the (X_t, Y_t) should be for $t \neq 0$, ($(X_t), (Y_t)$ reconstructed from $E^\bullet$ via deformation theory)

i.e. $\lim_{t \to 0}[(X_t, Y_t)]^{vir} \neq [(X, Y)] = (X_0, Y_0)$

Eg.7. (Several examples of virtual fundamental classes)

① $X \subset \mathbb{P}^3$, $X = V(xz, yz)$, we compute the virtual fundamental class (VFC) of X respect to the obstruction theory given next:

$X \subset Y \subset \mathbb{P}^3 \Rightarrow$
$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_X(2) \arrow[r] \arrow[d, equal] & \mathcal{O}_X(2) \oplus \mathcal{O}_X(2) \arrow[r] \arrow[d, equal] & \mathcal{O}_X(2) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & E_{X/Y} \arrow[r] & E_{X/\mathbb{P}^3} \arrow[r] & E_{Y/\mathbb{P}^3}|_X \arrow[r] & 0 \end{tikzcd}$$
$V(xz, yz) \subset V(z)$

Thus $C_{X/C_{Y/\mathbb{P}^3}|_X} \xrightarrow{i} E_{X/Y} \oplus E_{Y/\mathbb{P}^3}|_X$

$\cong C_{X/Y} \times_X (C_{Y/\mathbb{P}^3}|_X)$
$\cong C_{X/Y} \oplus \mathcal{O}_X(2)$

Now $X = V(x, y) \cup V(z) \subset V(z) \cup V(z) = Y$

$\Rightarrow$ [redacted]

$C_{X/Y} = C_{X \times_{V(z)} V(z)} \oplus C_{X \times_{V(z)} V(z)/V(z)} = C_{V(x,y)/V(x,z)/V(z)} \oplus C_{L_2/V(z)} = C_{V(x,y)/V(z)} \oplus C_{V(z)/V(z)} \oplus 1$
$= \mathcal{O}_{L_1}(2) \oplus \mathcal{O}_{L_2}(2) \oplus 1 = \mathcal{O}_{L_1 \cup L_2}(2) \oplus 1$

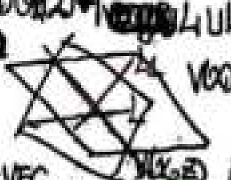
let's denote $V(x, y) = L_1$, $V(x, z) = L_2 \subset V(x)$

$\Rightarrow$ The first component 1.1 $\oplus \mathcal{O}_X(2)|_{V(z)} = \mathcal{O}_{V(z)}(2) \hookrightarrow \mathcal{O}_{V(z)}(2) \oplus \mathcal{O}_{V(z)}(2)$ is $2[L]$ for $L \subset V(z)$ a line

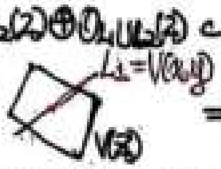
The second component $C_{L_1 \cup L_2/V(x)} \oplus \mathcal{O}_X(2)|_{L_1 \cup L_2} = \mathcal{O}_{L_1 \cup L_2}(2) \oplus \mathcal{O}_{L_1 \cup L_2}(2) \hookrightarrow \mathcal{O}_{L_1 \cup L_2}(2) \oplus \mathcal{O}_{L_1 \cup L_2}(2)$ is $[L_1] + [L_2]$

$\Rightarrow [X]^{vir}_{X/\mathbb{P}^3} = [X]^{vir}_{i} = 2[L] + [L_1] + [L_2]$,
$= [L_1] + 3[L_2]$

as L, $L_2 \subset V(z)$ both ▨



resp.



$= X \hookrightarrow \mathbb{P}^3$ thus VFC for non-regular embedding can not equi-dim can also have a equi-dim cycle

② We had known that the pullback of VFC behaves well if the compatibility in the triangulated level holds. Here we see pushforward: not good enough!

Set $Bl_0\mathbb{P}^2 \longrightarrow \mathbb{P}^2$ $(0 = (0:\cdots:1))$ and obstruction theories $E_1^\bullet = [\mathcal{O}(-H_1) \xrightarrow{0} \Omega_{Bl_0\mathbb{P}^2}]$, $E_2^\bullet = [\mathcal{O}(-H_2) \xrightarrow{0} \Omega_{Bl_0\mathbb{P}^2}]$ on $Bl_0\mathbb{P}^2$ with H_1 and H_2 are two different lines on $Bl_0\mathbb{P}^2$, $[H_1] \neq [H_2]$ (Here we blow-up to avoid that $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}$, all are same)

By the locally free obstruction, $[Bl_0\mathbb{P}^2]^{vir}_{E_i} = \mathrm{Ctop}(\mathcal{O}(-H_i)) \cap [Bl_0\mathbb{P}^2] = [H_i]$ different

$\Rightarrow id_{*}[Bl_0\mathbb{P}^2]^{vir}_{E_1} \neq [Bl_0\mathbb{P}^2]^{vir}_{E_2}$

Thus we call $f: \mathcal{X} \to \mathcal{Y}$ satisfy the virtual push-forward property if $f_*[\mathcal{X}]^{vir}_{E^\bullet} = \sum c_i [\mathcal{Y}_i]^{vir}_{F^\bullet}$, where $[\mathcal{Y}]^{vir}_{F^\bullet} = \sum [\mathcal{Y}_i]^{vir}_{F^\bullet}$

it's strong if $c_i = c_j$ all equal.

Fact. f proper & DM type (or $\mathcal{X}$, $\mathcal{Y}$ DM) & $\mathcal{Y}$ connected & compatible $f^*F^\bullet \longrightarrow E^\bullet$ is morphism of perfect obstruction theories & $f^*F^\bullet \xrightarrow{\phi} E^\bullet \to \mathrm{Cone}(f^*F^\bullet \to E^\bullet) \xrightarrow{+1}$

$$\begin{tikzcd} f^*F^\bullet \arrow[r] \arrow[d] & E^\bullet \arrow[d] \\ f^*\mathbb{L}_{\mathcal{Y}} \arrow[r] & \mathbb{L}_{\mathcal{X}} \end{tikzcd}$$

Eg.4. Gromov-Witten (supported in [LOT]) invariants, Quantum cohomology & its application mirror symmetry. $N(d) = \sum_{d_1 + d_2 = d} N(d_1)N(d_2)d_1^2 d_2 \cdots$

Recall in Eg.2, we use counting zeros & poles of cross ratios, to get Kontsevich's plane curve formula (d>3)

Start from this, we mainly do three things:

① The moduli stack of stable maps; ② Gromov–Witten invariants, WDVV equation and Quantum product via intersection theory on the VFC of the moduli stack. Standard ref is [Fulton & Pandharipande]; ③ Why these invariants useful in counting [illegible]. The cross ratio tells us why ① need marking pts & the Kontsevich's formula tells us why Quantum product is so defined (as an associativity). Main references of this sections are [FP], [$\frac{?}{2}$], [Bertram's notes], [Abramovich's notes] & [Vakil's notes].

In history, the construction of GW-invariants was first in symplectic geometry via counting J-holomorphic curves. [KM] first gives an algebraic description for rational curves ($g=0$), then [LT] uses Kuranishi theory (also idea from symplectic geometry), gives a general algebraic approach. Later due to the Kuranishi structure is too technical for AGists, after [BF] as replacement of it to define VFC, [Behrend, GW-invariants in AG, 1996] finally gives our well-known construction for AGists moderly. Further developments introduced later. Q. How we reduce the problem of defining GW-invariants to defining VFC?

Construction of moduli of stable maps: Kontsevich's space

I think C not irreducible, $p_i \neq p_j$

We need to define the moduli stack $\overline{M}_{g,n}(X;\beta)$ of stable maps $\{f: C \to X \mid f_*[C] = \beta \in H_2(X;\mathbb{Z})$, $g(C) = g$, $(C, p_1 \cdots p_n)$ stable$\}$. Stable map $(C; p_1 \cdots p_n) \xrightarrow{f} X$ is a connected projective nodal curve s.t. $(p_i) \in C$ are smooth pts Rk. ① is more essential than ② [illegible] and $|\mathrm{Aut}(f)| < \infty \Leftrightarrow$ ① If f constant on irreducible component C_i, $g(C_i) = 0 \Rightarrow C_i$ has ≥ 3 special pts due to ② automatical except that $(g, n, \beta) = (1, 0, 0)$

Kontsevich

② If f constant on irreducible component C_i, $g(C_i) = 1 \Rightarrow C_i$ has ≥ 1 special pts

Thm 2 (Genus 0 case) X projective, $\mathbb{C}$-scheme/variety

① Our functor $\overline{M}_{0,n}(X;\beta)(S) = \{ [illegible] \mathcal{C} \xrightarrow{f} X$ is stable, family of stable curves $\}$ is smooth, proper, DM stack $(\mathbb{P}^1 \xrightarrow{f} X) \cong (\mathbb{P}^1 \xrightarrow{f\circ\varphi} X)$ and φ preserve marked pts but [illegible] morphism is base change of $S \to T$ via construct it from $\overline{M}_{0,n}(\mathbb{P}^m; d) = \coprod \overline{M}_{0,n}(\mathbb{P}^m; \beta)$ [illegible]

② $\dim \overline{M}_{0,n}(X;\beta) = \dim X + \int_\beta c_1(T_X) + n - 3$. Here $\int_\beta \alpha_{top} = \sum_i b_i \int_{B_i} j_i^* \alpha_{top}$ for $j_i: B_i \hookrightarrow X$, α_{top} is top class, $\deg \beta = d$; all ③④⑤ assume $\deg = d \geq 0$

All $\mathcal{C} \to S$ is flat in this section

③ We have evaluation map

Sometimes I omit the

$ev: \overline{M}_{0,n}(X;\beta) \to \overline{M}_{0,n} \times X^n$

$(C, p_1 \cdots p_n, f) \mapsto (st(C, p_1 \cdots p_n), f(p_1) \cdots f(p_n))$ Where note that the domain of stable map may not stable curves, as nonconstant irreducible components are arbitrary, via contract unstable components we have the stable curve. Then $I^X_{0,n,\beta}: A^*(X;\mathbb{Q})^{\otimes n} \to A^*(\overline{M}_{0,n}(X;\beta); \mathbb{Q})$, we can extend it into $H^*(X;\mathbb{Q})^{\otimes n}$

$\gamma_1 \otimes \cdots \otimes \gamma_n \mapsto (ev_1^* \gamma_1) \cdot (ev_2^* \gamma_2) \cdots (ev_n^* \gamma_n)$ the GW class $= ev^*([\overline{M}_{0,n}] \times \gamma_1 \times \cdots \times \gamma_n)$ ($\gamma_1 \sim \gamma_n$ satisfy a [illegible] condition)

and $\langle I^X_{0,n,\beta} \rangle(\gamma_1 \otimes \cdots \otimes \gamma_n) = \int_{\overline{M}_{0,n}(X;\beta)} (I_{0,n,\beta})(\gamma_1 \otimes \cdots \otimes \gamma_n) = \int_{\overline{M}_{0,n}(X;\beta)} (ev_1^* \gamma_1) \cdots (ev_n^* \gamma_n)$ the GW invariant $\in \mathbb{Q} =: \langle \gamma_1 \cdots \gamma_n \rangle_\beta$ for

Even we take $A^*(X)^{\otimes n}$ domain, simplicity if no doubt ($\mathbb{Z}$-coefficient), GW is $\mathbb{Q}$!

[illegible] $\langle I_{0,n,\beta} \rangle([V_1] \otimes \cdots \otimes [V_n])$ = virtual number of [illegible].

Q. How many rational curve $(C, p_1 \cdots p_n) \subset X$ has $p_i \in V_i$. (One can compare it with Rk.1 and find it's quite strong!)

④ Our WDVV equations are

$$\sum_{\beta_1+\beta_2=\beta} \sum_{I_1 \sqcup I_2 = I} \sum_{i,j} \langle \gamma_1, \gamma_2, S_{I_1}, e_i \rangle_{\beta_1} \left(\int_X e_i \cup e_j\right)^{-1} \langle e_j, \gamma_3, S_{I_2}, \gamma_4 \rangle_{\beta_2}$$

$$= \sum_{\beta_1+\beta_2=\beta} \sum_{I_1 \sqcup I_2 = I} \sum_{i,j} \langle \gamma_1, \gamma_3, S_{I_1}, e_i \rangle_{\beta_1} \left(\int_X e_i \cup e_j\right)^{-1} \langle e_j, \gamma_2, S_{I_2}, \gamma_4 \rangle_{\beta_2}$$

With I = set of marked pts $\{p_1 \cdots p_n\} \equiv X^n$

It's given marked pts via section (codim 1)

S_I is the boundary divisor of [illegible]

$(e_i) \subset H^*(X;\mathbb{Q})$ taken to be a basis

the $\int e_i \cup e_j$ intersection form is quadratic, this gives a inverse matrix. They're written. Divisors, [illegible] Verlinde's

⑤ We have small quantum cohomology & big quantum cohomology What are [illegible]'s coefficients? They're Novikov rings.

$SQH^*(X;\mathbb{Q}) = H^*(X)[[e^{t_1} \cdots e^{t_m}]]$, $t_1 \cdots t_m \in H_2(X;\mathbb{Z})$ is span the space of all curve classes $\{\beta = [C]\}$ and independent over $\mathbb{Q}$. It's well-defined (not depend on choice of (t_i)) & have a small quantum product;

Explain it later in ⑥.

$BQH^*(X;\mathbb{Q}) = H^*(X)[[e_1^\vee \cdots e_m^\vee]]$, $e_1^\vee \cdots e_m^\vee$ is dual basis of $e_1 \cdots e_m$ in ④, it has a big quantum product.

Rk. Due to ②, we know $\dim \overline{M}_{0,n}(X;\beta)$ is still some number "easy" to be positive, things goes worse when $\dim \overline{M}_{g,n}(X;\beta) = \cdots + (1-g)$ [illegible] is always highly negative, and our VFC comes into use. I'll explain what's its obstruction theory later.

C is stable curve without the marked pts

Def (f) infinite $\Rightarrow$ Aut(C) not reduced

Issues exist when [illegible] consider stable map [illegible]

④ & ⑤ are special for only tree-level (i.e. $g=0$). Before proving Thm 2, we state ① [illegible] for higher genus case

Thm 2' (Higher Genus case) ①' $\overline{M}_{g,n}(X;\beta)$ as functor (similar with $g=0$ case) admits a coarse moduli space $\overline{M}_{g,n}(X;\beta)$ [illegible]

②' $\dim \overline{M}_{g,n}(X;\beta) = (1-g)(\dim X - 3) + \int_\beta c_1(T_X) + n$; $(= \dim \overline{M}_{g,n}(X;\beta))$

Here $\overline{M}_{g,n}(X;\beta)$ is still DM due to X/k, for X/S [illegible]

③' We have evaluation map $ev: \overline{M}_{g,n}(X;\beta) \to \overline{M}_{g,n}(X) \times X^n$, $I_{g,n,\beta}: A^*(X;\mathbb{Q})^{\otimes n} \to A^*(\overline{M}_{g,n}(X;\beta);\mathbb{Q})$, $\langle I^X_{g,n,\beta} \rangle(\gamma_1 \otimes \cdots \otimes \gamma_n)$ [illegible]

$\overline{M}_{g,n}(X;\beta)$ [illegible] the [illegible] curves [illegible]

Rk. It's not surprising to replace moduli stack by coarse moduli: $A^*(\mathcal{X}) = A^*(X)$ has same intersection-theoretic datum, and all our definition maps can easily changes from relative $= \int_{[\overline{M}_{g,n}(X;\beta)]^{vir}} (I_{g,n,\beta})(\gamma_1 \otimes \cdots \otimes \gamma_n)$ to correct the naive cases. I'm sad to say I find nothing useful about $\overline{M}_{g,n}(X;\beta)$ itself

deficiency in [Abramovich & Vistoli]

Pf of Thm 2 & Thm 2'. We'll proceed by ①①'②③④⑤ [illegible] $\overline{M}_{0,n}(X;\beta)$ [illegible] need separated to define in [BF], [illegible] why

① First we reduce to $X = \mathbb{P}^m$ and $\beta = d[\text{line}]$ generates $A^1(\mathbb{P}^m)$ [illegible] $X \subset \mathbb{P}^m$ has $I(X) = (F_1 \cdots F_r)$ homogeneous ideal, $\deg F_i = d_i$ [illegible] smoothness

$\Rightarrow S_i \in H^0(\overline{M}_{0,n}(\mathbb{P}^m; d), \pi_* ev^* \mathcal{O}_{\mathbb{P}^m}(d_i))$, where π [illegible] functor is same as on the base S; one can understand [illegible] these sections $V(S_1 \cdots S_r) \subset \overline{M}_{0,n}(\mathbb{P}^m; d)$ is desired $\overline{M}_{0,n}(X; d)$ due to we restrict $\mathcal{C}$ into zero locus of (F_i);

The $\overline{M}_{0,n}(\mathbb{P}^m; d) = \overline{M}_{0,n}(\mathbb{P}^m; d[H])$, but for $\overline{M}_{0,n}(X; d)$ [illegible] decomposition $\overline{M}_{0,n}(X;d) = \coprod \overline{M}_{0,n}(X;\beta)$ due to $\beta = ev_*[C]$ is constant on connected components of $S \to \mathrm{Aut}(X)$, thus disjoint [illegible]

Now we reduce $\overline{M}_{0,n}(\mathbb{P}^m; d)$ as a [illegible] $\mapsto \beta = e_*[C]$ component of Zariski closed subset of $\overline{M}_{0,n}(\mathbb{P}^m; d)$, thus all prop

$\mathcal{M}_{0,n}(\mathbb{P}^N;d)$ inherited by $\mathcal{M}_{0,n}(X;\beta)$. Now the second step is check (ii)—(iii) of $\mathcal{M}_{0,n}(\mathbb{P}^N;d)$

(ii) To show $\mathcal{M}_{0,n}(\mathbb{P}^N;d)$ is DM-type, we construct an étale groupoid:

First, we reduce to $\mathcal{M}_{0,0}(\mathbb{P}^N;d)$ via the forgetful natural transformations $\mathcal{M}_{0,n+1}(\mathbb{P}^N;d)(S) \to \mathcal{M}_{0,n}(\mathbb{P}^N;d)(S)$, $\forall S \in \mathrm{Sch}/k$, this identify $\mathcal{M}_{0,n+1}(\mathbb{P}^N;d) \to \mathcal{M}_{0,n}(\mathbb{P}^N;d)$

There is no problem for DM type & proper, but why affected by $\tilde{S}$, the smoothness is preserved? Thus only (ii)-(iii) ~~not~~.

$\{ \mathcal{C} \xrightarrow{\varphi_S} \mathbb{P}^N, \pi \downarrow, S \}$ forget the last marked $\{ \tilde{\mathcal{C}} \xrightarrow{\varphi_S} \mathbb{P}^N, \tilde{\pi} \}$ the universal family over the moduli stack $\mathcal{M}_{0,n}(\mathbb{P}^N;d)$ that the DM-type passes via preserved by universal family but for proper

stabilization by contract unstable components (preserve the isomorphism of stable curve is obvious)

{ Smoothness ✓ Properness ✓ } Why? both need

We admit the fact that it's universal family, as we know that $\mathcal{M}_{0,n+1} \to \mathcal{M}_{0,n}$ is an universal family similarly. the same argue & unique

Then $\mathcal{M}_{0,0}(\mathbb{P}^N;d)(S) = \{ \mathcal{C} \xrightarrow{\varphi_S} \mathbb{P}^N \} \not\cong \{ \mathbb{P}^1 \times S \xrightarrow{\deg 1} \mathcal{C} \xrightarrow{\varphi_S} \mathbb{P}^N \}$ (does not holds) With a natural $\mathcal{C}_S \to \mathbb{P}^1$ into the stable[illegible] also same, as each fibre are rational. one step is stable reduction why degree = 1?

But we can recover left from right that properties by simply forget φ_S, then taking all irreducible components to $\mathbb{P}^1$ (equivalence is not change)

Our identification via making $\mathcal{M}_{0,0}(\mathbb{P}^N;d)$ as quotient of $U \subset \mathbb{P}^1[N]$ (stack morphism) ∴ DM-type. Then we denote $\mathcal{M}_{0,0}(\mathbb{P}^N;d)$ is (#)

$\mathbb{P}^1[N]$ the Fulton–MacPherson's configuration space (of $\mathbb{P}^1$)

$\mathbb{P}^1[N](S) = \{ \mathbb{P}^1 \times S \to \mathcal{C} \to S \}$, degree 1, is smooth projective N-dimensional $\cong \mathrm{Bl}_\Delta(\mathbb{P}^N)$; here the Δ is chosen subtle, $\Delta = \bigcup \Delta_{ij}$ all diagonal

$(\mathcal{C}_S, p_i(S))$ is Auto-free $\to \mathbb{P}^1[N]$ yes. we prove it next: replaced by general X, X^N $(\mathbb{P}^1)^N - \Delta$, Δ the big diagonal, i.e. $\exists x_i = x_j \Leftrightarrow x \in \Delta$

It's a compactification of the configuration space of N pts $\in \mathbb{P}^1$, i.e. motivated by the five conic problem, we can due to the excess part is when these pts same as diagonal. suggest that some blow-up to be its compactification

We construct $X[N]$ inductively:

We call diagonal of X^n is { small: $\Delta_{12\cdots n}$ all distinct; big/fat: Δ_{ij} / $\bigcup \Delta$ }

this not compact/proper due to $((\mathbb{P}^1_k)^N - \Delta)/\mathrm{PGL}_2(k) \cong ((\mathbb{A}^1_k)^{N-1} - \Delta)/\mathrm{Aff}(k)$ the affine group $= ((\mathbb{A}^1_k)^{N-2} - \Delta)/(k \times k^*)$ no reason to be compact ▨ Why?

$X[1] = X$

$\uparrow p_1$

$X^+[1] = X \times X$ universal family over $X[1]$: parameterize one pt $\in X$

$\uparrow p_{\mathrm{Bl}}$

$X[2] = \mathrm{Bl}_\Delta(X \times X)$ blow-up along the only diagonal $\Delta = \Delta_{12}$

$\uparrow p_2$

$X^+[2] = \boxed{?}$ universal family over $X[2]$: parameterize two distinct pts $\in X$

How to construct $\boxed{?} = X^+[2]$ concretely? (We need this due to we need determine which subvariety we need blow-up next step)

$E = g_2^{-1}(\Delta_{12}) \subset X^+[2]$ exceptional divisor

then $E \subset X^+[2] \subset X[2] \times X$ as the graph $E \xrightarrow{g_2} \Delta_{12} \cong X$ $\Rightarrow X[2] \times X = \mathrm{Bl}_\Delta(X \times X) \times X \to X \times X \times X$

$(e \longmapsto (e, x_e))$ $(e \mapsto g_2(e) \mapsto x_e)$ $E \longmapsto \Delta_{123}$ small diagonal

(i.e. $x \in \Delta_{123} \Leftrightarrow x_1 = x_2 = x_3$)

as $(e, x_e) \mapsto (g_2(e)_1, g_2(e)_2, x_e)$, and $x_e = (g_2(e))_1 = (g_2(e))_2 \in \Delta_{12}$

then $\mathrm{Bl}_E(X[2] \times X) = X^+[2]$ desired. $X^+[2] \to X[2] \times X \xrightarrow{pr_1} X[2]$.

Then we blow-up $X^+[2]$ to give $X[3]$: here we have two "diagonals" comes from $X^+[2] = \mathrm{Bl}_E(X[2] \times X) \to X[2] \times X$

denote the strict transforms $\tilde{\Gamma}_{pr_2 \circ g_2} = D_2$, $\tilde{\Gamma}_{pr_1 \circ g_2} = D_1$, then $X[3] = \mathrm{Bl}_{D_1 \cup D_2}(X^+[2])$

$$\begin{tikzcd}
 & X[2] \times X \arrow[d] \arrow[ld, "\Gamma_{pr_1 \circ g_2}"'] \\
\Delta_{13} \arrow[r] & X \times X \times X
\end{tikzcd}$$

strict transform

This (2 ⇒ 3) construction gives our construction in general:

- From $X[m]$ to $X^+[m]$, we embedding $E = g_m^{-1}(\Delta_{12\cdots m}) \subsetneq X[m] \subset X[m] \times X$ then $X^+[m] = \mathrm{Bl}_E(X[m] \times X) \to X[m] \times X \to X[m]$

$(\mathrm{Bl}_\Delta(X \times X) = X[2] \xrightarrow{g_2} X \times X \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} X)$

- From $X^+[m]$ to $X[m+1]$, all [illegible]-subset $S \subset \{1, \ldots, m\}$, has a pullback of $\Delta_S \subset X \times \cdots \times X$ ($(m+1)$ times) to $X[m] \times X$, then strict transform to $X^+[m]$, denoted as D_S

$\Rightarrow X[m+1] = \mathrm{Bl}_{\bigcup_{S \subset \{1 \ldots m\}} D_S}(X^+[m])$ ▨ (Thus we call rewrite $X[N] = \mathrm{Bl}_\Delta(X^N)$ by pullback all the diagonal of X^n, $m < N$ together by the sequence of blow-up, then the union of them is Δ, the fat diagonal)

Now we check the universal property of $X[N]$ to rewrite it as functor: $\mathrm{Hom}(S, X[N]) = \{ X \times S \leftarrow \mathcal{C}, pr \searrow, \circlearrowleft, \downarrow \pi, S \} p_i$

via $\begin{pmatrix} g: S \to X[N] \\ s \mapsto g(s) \end{pmatrix} \longmapsto S \xrightarrow{f} X[N] \to X^N \rightrightarrows X$ gives (p_i) by set $\mathcal{C} \to S$ has each fibres

$s \longmapsto p_i(s)_i \in \mathcal{C}_s$ [illegible] $X \Rightarrow \mathcal{C} \to X \times S$

$\bigcup \{p_i(s)\} \hookrightarrow \mathcal{C}_s$ degree 1

Conversely, each $s \in S$ gives $p_i(s) \in \mathcal{C}_s \xrightarrow{g_s} X$ distinct N pts $\in X$, this configuration determines $\in X[N] \ni \mathcal{C}_s$ a pointed curve forgetting all of $\{p_i(s)\} \in X$ (distinct) and $(\mathcal{C}_s, p_1(s), \cdots, p_N(s))$ is Automorphism-free.

$\varphi(s) \in X[N] \Rightarrow \varphi: S \to X[N]$ defined. $\exists!\, \varphi$

Is the φ algebraic? $S \to \mathcal{C}^N \longleftrightarrow X^N \times S \xrightarrow{pr_1} X^N$ (I'm not sure, but at least for $X = \mathbb{P}^1$, take $\mathcal{C} = S \times \mathbb{P}^1$ it works)

$s \mapsto (p_1(s), \ldots, p_N(s), s) \mapsto (g_s p_1(s), \ldots, g_s p_N(s), s) \mapsto (g_s p_1(s), \ldots, g_s p_N(s))$ ▨ But then it seems like that all blow-up of X^N can have such correspondence? I think there is something additional need to be checked subtlely

Thus now we equip $\mathbb{P}^1[N]$ a variety structure.

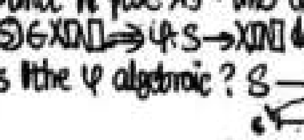
Rk. (Geometry of $X[N]$) One can understand the construction by:

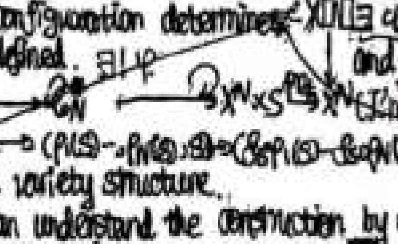
the boundary $X[N] - ((X^N - \Delta)/\mathrm{Aut}(N))$ parameterises N pts $\in X$ has the form not distinct.

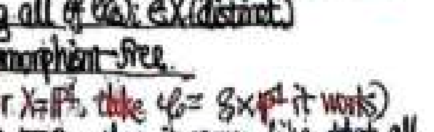
Usual case

Boundary case

Thus the boundary has more complex "funnel structure", different level of "closed" corresponds to different level of blow-up



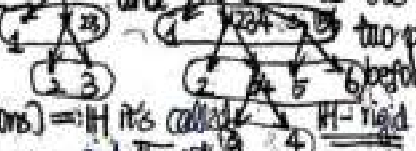




When only three pts, the funnel only have 1 level ⟺ only 1 diagonal to blow-up (the correspondence due to $p_2, p_3 \in \Delta$

When four pts, the funnel have two levels: the red one & the orange one ⟺ two big diagonals to blow-up ⟺ p_2 & p_3 closed ⟺ p_2 & p_3 in the same funnel

Note that such structure gives a tree to describe elements of boundary and as the these tree gives a rigorous way to study the boundary datum, see the two pictures before. paper of [Kontsevich, Deformation Quantization of poisson manifolds. I]



Mark more pts rigidify curves

For $\mathbb{P}^N$, we take a basis of all hyperplanes $\{H_0, \dots, H_N\}$ (as linear equations) $=: \mathbb{H}$ it's called H-rigid stable family

then we have subsets $\{\mathbb{P}^1 \times S \leftarrow \mathcal{C}_S \boxed{\xrightarrow{f} \mathbb{P}^N}$ | $f^*(H_i) \subset \mathcal{C}_S$ is union of d sections and $\overline{M}_{0,0}(\mathbb{P}^N; d)_{\mathbb{H}} \subset \overline{M}_{0,0}(\mathbb{P}^N; d)(S)$ is called H-rigidification (why it's so named?)

denoted as $\overline{M}_{0,0}(\mathbb{P}^N; d)_{\mathbb{H}}$ ... $p_{i1}(S) + \dots + p_{id}(S)$ distinguish each $p_{ij}(S) \in \mathcal{C}_S$ avoid nodes

and call the family meets $\mathbb{H}$ transversally

It's acceptable: $f(\mathcal{C}_S) \subset \mathbb{P}^N$ is deg d curves, cutted by H_i gives d pts

Observation $\overline{M}_{0,0}(\mathbb{P}^N; d)_{\mathbb{H}}(S) \longrightarrow \mathbb{P}^d[d(N+1)](S)$ we need $\forall i$, H_i not meet the node of $\mathcal{C}_S$, $\forall s \in S$ idea of rigidification to cover

by forget the right "□" part, and due to $/(S_d)^{N+1}$ (p_{ij}) unordered, quotient symmetric grps naturally determine f stable locally

But it's only a morphism between familys, now take $U \subset \mathbb{P}^1[d(N+1)]$ called (H-)-balanced open subset by ... is determined

$U(S) = \mathbb{P}^1 \times S \leftarrow \mathcal{C}_S$ | $\mathcal{O}_{\mathcal{C}_S}(p_{i1}(S) + \dots + p_{id}(S))$ (where $p_{ij}(S) \in \mathcal{C}_S$ 0-dimension pts ∈ curve are divisors addition) by its up to time action

$f^*\mathcal{O}(1)$ ... restricts to same line bundle on every irreducible components of $\mathcal{C}_S$

Then • $U \subset \mathbb{P}^1[d(N+1)]$ open Why? [I think $\mathcal{O}_C(p)$ locally near p is the log-form $\frac{dz}{z}$ ... (we used it in later gluing)

• $(S_d)^{N+1} \curvearrowright \mathbb{P}^1[d(N+1)]$ respect U keep invariant. It's obvious; switching p_{ij} not change $\mathcal{O}_{\mathcal{C}_S}(p_{i1}(S) + \dots + p_{id}(S))$

Then $(S_d)^{N+1} \curvearrowright U$ invariant induces a $(\mathbb{C}^*)^N$-torsor $(\mathbb{C}^*)^N \xrightarrow{\text{action}} U_{\mathbb{H}}$, we'll prove it later in ① via a direct construction of $\Rightarrow (U_{\mathbb{H}}/(S_d)^{N+1}) \times U_{\mathbb{H}} \cong \overline{M}_{0,0}(\mathbb{P}^N; d)_{\mathbb{H}}$ $(\mathbb{C}^*)^N$-torsor for $X/\mathbb{R}$ ↓ $U_{\mathbb{H}}$ (Bertram's note refers to [P1], but I don't find it)

Finally we cover $\overline{M}_{0,0}(\mathbb{P}^N; d)$ by all its rigidifications; it's due to [$\forall$ $C \subset \mathbb{P}^N$ curve degree $= d$

thus $\overline{M}_{0,0}(\mathbb{P}^N; d)$ is DM stack ▨

$\exists$ $H \cap C =:$ marked pts of C smooth $\in C$ by Bertini

and adding these pts makes $(C, p_1, \dots, p_d)$ stable

H generic and we can take $H_0, \dots, H_N$

I think there is still last one thing to do: patch together, we also two ways { verify that $U_{\mathbb{H}}$ does a open set $\subset \overline{M}_{0,0}(\mathbb{P}^N; d)$ (or $\overline{M}_{0,0}(\mathbb{P}^N; d)_{\mathbb{H}}$) prove it locally

Find an etale equivalence relation $R \Rightarrow \coprod U_{\mathbb{H}} \longrightarrow \overline{M}_{0,0}(\mathbb{P}^N; d)$, R is two parts < $(S_d)^{N+1}$-action; two $\mathbb{H}$ & $\mathbb{H}'$ gives gluing datum

③ Smoothness is a standard argument of deformation theory at $(C, f) \in \overline{M}_{0,0}(X; \beta)(\operatorname{Spec} \mathbb{C})$ closed pts extend to its

$0 \to H^0(C, T_C) \to H^0(C, f^*T_X) \to T_{(C,f)}(\overline{M}_{0,0}(X;\beta)) = \operatorname{Def}_{(C,f)}(\overline{M}_{0,0}(X;\beta))$ infinitesimal neiborhood $\overline{M}_{0,0}(X;\beta)(\operatorname{Spec} \mathbb{C}[\varepsilon])$

$\to H^1(C, T_C) \to H^1(C, f^*T_X) \to \operatorname{Ob}_{(C,f)}(\overline{M}_{0,0}(X;\beta)) \to H^1(C, T_C) = 0$

$\operatorname{Ob}_{(C,f)}(\overline{M}_{0,0}(X;\beta)) = 0 \Leftrightarrow \overline{M}_{0,0}(X;\beta)$ smooth at $(C, f) \Leftarrow H^1(C, f^*T_X) = 0$ (Such argument holds for $\forall g \geq 0$, but only when $g(C) = 0$, $H^1(C, f^*T_X) = 0$ holds)

(By Grothendieck's infinitesimal lifting critia)

Fact. ① $H^1(C, f^*T_X) = 0$ for $C \to X$ projective connected reduced nodal $g = 0$ morphism (all conditions needed!?) [Kollár]

② Thus $\dim \overline{M}_{0,0}(X;\beta) = \dim T_{(C,f)}(\overline{M}_{0,0}(X;\beta)) = \chi(C, f^*T_X) - \chi(C, T_C) = \dim X + \int c_1(T_X) - 3$ (**)

(and general $g > 0$, except the first equality fails, others holds as $\dim T_{(C,f)}(\overline{M}_{g,0}(X;\beta)) = (1-g)\dim X + \int c_1(T_X) - (3 - 3g)$)

• We'll prove (**) in ① Why we can reduce smoothness to $n = 0$ case?

④ Properness recall in $\overline{M}_{g,n}$, we do stable reduction and use valuative critia (and this allow us to prove the coarse $\overline{M}_{g,n}$ projective by positivity later, as we'll do in ①': both $\overline{M}_{g,n}$ & $\overline{M}_{g,n}(X;\beta)$'s projectivity is due to Kollár), such procedure have two facts: (A) Do for $\overline{M}_{g,n}(X;\beta)$ fine ⟺ Do for $\overline{M}_{g,n}(X;\beta)$ coarse; (B) $g > 0$ also holds.

Thus we also postpone it to ①' ▨ (We used this name is derived scheme, in fact this is an abstract idea: scheme → stack; derived)

Rk. Comparing coarse and fine, only the hidden smoothness is the only good property for stack higher structure

all constructions & properties are similar: in next ①', we'll first show that locally $\overline{M}_{g,n}(\mathbb{P}^N; d)$ (strengthen the smoothness) admittes a coarse moduli $\overline{M}_{g,n}(\mathbb{P}^N; d)_{\mathbb{H}}$, then gluing them together to $\overline{M}_{g,n}(\mathbb{P}^N; d)$, the check its properties: separatedness ⟶ properness ⟶ projectivity.

①' First due to we don't need check smooth, all reduction to $\overline{M}_{g,n}(\mathbb{P}^N; d)$ makes sense as ①. The rigidification also makes sense due to the $H_i \subset \mathbb{P}^N$ intersects with $f(C)$ not depend on the genus (Bézout)

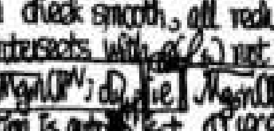
• $\boxed{\overline{M}_{g,n}(\mathbb{P}^N; d) \text{ admits coarse } M_{g,n}(\mathbb{P}^N; d)}$ i.e. $\overline{M}_{g,n}(\mathbb{P}^N; d)_{\mathbb{H}} \xrightarrow{\varphi} \operatorname{Hom}(-, M_{g,n}(\mathbb{P}^N; d)_{\mathbb{H}})$ natural transformation s.t. ① $\varphi(\operatorname{Spec} \mathbb{C})$ set bijection; ② (Universal) $\overline{M}_{g,n}(\mathbb{P}^N; d)_{\mathbb{H}} \xrightarrow{\varphi} \operatorname{Hom}(-, M_{g,n}(\mathbb{P}^N; d))$ Or $\exists! \; \overline{M}_{g,n}(\mathbb{P}^N; d)_{\mathbb{H}} \to Y$ ... $\exists!$ $\operatorname{Hom}(-, Y)$

thus after gluing, verification is auto for general $M_{g,n}(\mathbb{P}^N; d)$, so is $M_{g,n}(X;\beta)$

Def of coarse as before

$$\begin{tikzcd}
X \times \overline{M}_{g,n} \arrow[r, "pr_2"] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{g,n} \arrow[d, "\text{universal}"] \\
X \arrow[r, "f"] & M_{g,n}
\end{tikzcd}$$

$\mathbb{P}^N(\mathbb{C})$ ... $\mathcal{H}(p_{ij})$ ...



$(N+1)(N+1)$ sections ⟺ marked pts

We call $f: X \to \overline{M}_{g,n}$ is H-balanced if ① $\mathcal{O}(p_{i1}) \otimes \dots \otimes (p_{id})^N (\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$ ... ② ... $(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$ ... ($g = 0$ case ... in Abelian ... but when $g > 0$ ...)

Set $\mathcal{H}_i$ Idea, mark more pts makes curve all stable

$= \mathcal{O}_{\overline{M}_{g,n}}(p_{i1}(\overline{M}_{g,n}) + \dots + p_{id}(\overline{M}_{g,n}))$; sections are ... $\exists$ $B_{g,n} \subset \overline{M}_{g,n}$ universal, locally closed substack ... s.t. the inclusion is H-balanced, (ii) $\forall X \xrightarrow{f} \overline{M}_{g,n}$ H-balanced $\exists!$ $B_{g,n}$

and $\exists$ $\mathcal{L} \in H^0(\overline{M}_{g,n}, \dots)$ ... $\mathcal{L}$ is such the bundle well-defined irreducible component need $\overline{M}_{g,n}$ smooth / reducible ... [Both not true] Even $\overline{M}_{g,n}$ smooth

(The existence is ... by taking ...

$$\begin{tikzcd} \mathcal{G}_{\tau} \arrow[r] \arrow[d] & \overline{U}_{g,m} \arrow[d] \\ \mathcal{B}_{g,m} \arrow[r, "\iota"] & \overline{M}_{g,m} \end{tikzcd}$$

Set $\mathcal{G}_i = \mathbb{P}_{\mathcal{B}}\pi_*(\iota)^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$ locally free by (a) as ι is $\mathcal{H}$-balanced. [FP] says that for $g>0$, $\exists$ universal family [illegible] but this is the [illegible] below [illegible] for $g>0$.

$\Rightarrow$ set $P_i : \mathcal{G}_i \to \mathcal{B}_{g,m}$ this $(\mathbb{C}^*)$-bundle, i.e. the $V(\mathcal{G}_i)$ [illegible]

$\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_N$ [illegible] $\mathcal{G}_i$

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\overline{\tau}"] \arrow[d] & \overline{U}_{g,m} \arrow[d] \\ \mathcal{G} \arrow[r, "\tau"] & \mathcal{B}_{g,m} \arrow[r, hook] & \overline{M}_{g,m} \end{tikzcd}$$

$\Rightarrow (\overline{\tau}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}))$ [illegible] $= P_i^*\tau^*\mathcal{G}_i = P_i^* [illegible]$

The first equality [illegible]

But $P_i^*\mathcal{G}_i$ is trivial as $\mathcal{G}_i = V(\mathcal{G}_i) - \{0\}$, $P_i^*\mathcal{G}_i$ is the tautological $\mathcal{O}_{\mathcal{G}_i}$ trivial

Claim: $\overline{M}_{g,m}(\mathbb{P}^N;d)_{\mathcal{H}} = \mathcal{G}$ [illegible]

To prove this, we need to show $\mathcal{U} \to \mathcal{G}$ is a Universal family of $\mathcal{H}$-rigid (induced by universal property of fibre product/pullback) [illegible] to $\mathcal{H}_i$; ② Stable [illegible] ③ Universal.

$\Rightarrow \overline{\tau}^*\mathcal{H}_i = \overline{\tau}^*\mathcal{H}_0 =: \mathcal{L}$, all same [illegible] $\mathcal{U}$ and $\mathcal{G}$ is $\mathcal{H}$-rigid

then we call pullback sections (P_j) to $\mathcal{U} \to \mathcal{G}$ [illegible]

recall $s_i \in H^0(\overline{U}_{g,m}, \mathcal{H}_i = \mathcal{O}_{\overline{U}_{g,m}}([P_{i1}] + \cdots + [P_{id}]))$ canonical section representing the Cartier divisor $[P_{i1}] + \cdots + [P_{id}]$,

then $\overline{\tau}^* s_i \in H^0(\mathcal{U}, \mathcal{L})$ are $(N+1)$ sections will determines the pullbacks of (P_j) to $\mathcal{U} \to \mathcal{G}$ by:

$\overline{\tau}^* s_i \Leftrightarrow \mathcal{L} = \overline{\tau}^*\mathcal{H}_i = \overline{\tau}^*\mathcal{O}_{\overline{U}_{g,m}}(\sum [P_{ij}(\overline{M}_{g,m})]) = \mathcal{O}_{\mathcal{U}}(\sum D_{ij})$ for some $D_i = \sum_j D_{ij}$, here each D_{ij} is recovered from D_i via [redacted] $D_i|_s = (\sum D_{ij})|_s = \sum D_{ij}|_s$ is the intersection [illegible] of $H_i \cap \mathcal{U}_s$, given the sum $\sum D_{ij}|_s$ [illegible] $\Leftrightarrow D_i|_s \Leftrightarrow$ each $D_{ij}|_s$ as they on same plane H_i (But I'm not clear how to write a precise argument in relative case)

[redacted] not vanish at same fibre, this $\mathcal{U} \xrightarrow{f} \mathbb{P}^N$ [redacted] is defined (We don't compute [illegible] directly!)

[redacted]

$H_i \mapsto \overline{\tau}^* s_i$ on section level $(\mathbb{P}^N)^\vee = H^0(\mathbb{P}^N, \mathcal{O}(1)) \to H^0(\mathcal{U}, \mathcal{L})$ induces f desired ($(\mathbb{P}^N)^\vee$ denoted all hyperplanes)

② We need check the stability of [redacted] family $\mathcal{U} \xrightarrow{f} \mathbb{P}^N$; $\mathcal{H}$-rigidity follows from the construction of f upper. Is the $g>0$ case falls here?

Consider $C = \mathcal{U}_g$ the fibre of the family is connected curve and let $E \subset C$ irreducible component $f(E) = pt$ constant, then $f(E) \cap H_i$

$\Rightarrow f(E) \cap H_i = \emptyset$, i.e. E has no marking from (P_{ij}), only (P_i) marked on E (and intersection with other irreducible components), thus due to P_i pullback from $\overline{U}_{g,m} \to \overline{M}_{g,m}$ stable, so is $C = \mathcal{U}_g$ fibre $\Rightarrow$ done. [FP] say $\int \omega_C(P_1 + \cdots + P_n) > 0 \Rightarrow$ stable, I don't know the meaning?

③ The universal property of family $\mathcal{U}$ [illegible]. It suffices find the map in base $S \to \mathcal{G}$, then pullback $\mathcal{U}$ to $\mathcal{E}$ is, for $\forall$ another family $\mathcal{E} \to S$, and this operation preserves sections/marked pts, due to both base S and $\mathcal{G}$ has pullback from $\overline{U}_{g,m} \to \overline{M}_{g,m}$, i.e.

$$\begin{tikzcd} \mathcal{U} \arrow[d] & \mathcal{E} \arrow[l] \arrow[d] \\ \mathcal{G} & S \arrow[l] \end{tikzcd}$$

To construct $S \to \mathcal{G}$,

consider
$$\begin{tikzcd} S \arrow[r, dashed, "\exists!"] \arrow[d, "\lambda"] \arrow[dr] & \mathcal{G} \arrow[d, "\tau"] \\ \overline{M}_{g,m} & \mathcal{B}_{g,m} \arrow[l, "\iota"] \end{tikzcd}$$

By universal property of $\mathcal{B}_{g,m} \subset \overline{M}_{g,m}$, $\exists! S \to \mathcal{B}_{g,m}$

If $\boxed{\lambda : S \to \overline{M}_{g,m} \text{ is } \mathcal{H}\text{-balanced}}$, then we can descend to $\mathcal{G}$ due to the construction of τ: τ is fibre products of $(\mathbb{C}^*)$-bundle over $\mathcal{B}_{g,m}$, thus we need to give a canonical nowhere vanishing section of $\lambda^*\mathcal{G}_i$, $\forall\, 1 \le i \le N$

$\lambda^*\mathcal{G}_i = \lambda^*\mathbb{P}_{\mathcal{B}}\pi_*(\iota)^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$
$= \lambda^*\iota_*(\mathbb{P}_{\mathcal{B}}\pi_*(\iota)^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}))$
$= \pi_*(\overline{\lambda})^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\overline{\lambda}"] \arrow[d, "\pi"] & \overline{U}_{g,m} \arrow[d] & \square \arrow[l, "\overline{\iota}"] \arrow[d, "p_{\mathcal{B}}"] \\ S \arrow[r, "\lambda"] & \overline{M}_{g,m} & \mathcal{B}_{g,m} \arrow[l, hook] \end{tikzcd}$$

$H^0(S, \lambda^*\mathcal{G}_i) = H^0(S, \pi_*(\overline{\lambda})^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}))$

$\Updownarrow$

$H^0(S, \pi_*(\overline{\lambda})^*\mathcal{H}_i)) \underset{\lambda \text{ is } \mathcal{H}\text{-balanced}}{=} H^0(S, \pi_* [illegible] \mu^*(\mathcal{O}_{\mathbb{P}^N}(1)))$

then $H^0(S, \pi_*\mu^*(\mathcal{O}_{\mathbb{P}^N}(1)))$ has nowhere vanishing section $\Rightarrow$ so is $H^0(S, \lambda^*\mathcal{G}_i)$, done!

Now we check λ is $\mathcal{H}$-balanced:

again as in ② done, we take $(\mathbb{P}^N)^\vee \to H^0(\mathcal{E}, \mu^*(\mathcal{O}_{\mathbb{P}^N}(1)))$, we need to prove $\mu^*(\mathcal{O}_{\mathbb{P}^N}(1)) \cong (\overline{\lambda})^*\mathcal{H}_i$

(induced by $\mathcal{E} \xrightarrow{\mu} \mathbb{P}^N$) $H_i \mapsto s_i$, then the Cartier divisor $\sum [P_{ij}(\mathcal{E})]$ on $\mathcal{E} \Leftrightarrow ((\overline{\lambda})^*\mathcal{H}_i, (\overline{\lambda})^* s_i)$ as the construction in ②

By $\mathcal{E} \to S$ is also $\mathcal{H}$-rigid, $\mu^*[H_i] = \sum [P_{ij}(\mathcal{E})]$ also; $\Rightarrow (\mu^*(\mathcal{O}_{\mathbb{P}^N}(1)), s_i)$ also $\Leftrightarrow \sum [P_{ij}(\mathcal{E})]$, thus $\mu^*(\mathcal{O}_{\mathbb{P}^N}(1)) \cong (\overline{\lambda})^*\mathcal{H}_i$ ▨

• **Gluing $\overline{M}_{g,n}(\mathbb{P}^N;d)_{\mathcal{H}}$ together**

We need do things: ① modulo permutation between $(P_{ij})_j$s; ② Check the intersection $\overline{M}_{g,n}(\mathbb{P}^N;d)_{\mathcal{H}_1,\mathcal{H}_2}/\mathcal{G} \cong \overline{M}_{g,n}(\mathbb{P}^N;d)_{\mathcal{H}_2,\mathcal{H}_1}/\mathcal{G}$

③ Cocycle condition; ④ $\overline{M}_{g,n}(\mathbb{P}^N;d)$ is covered by finite.

① let $\mathcal{G} = \mathcal{G}_{d,N+1} = \prod_{i=0}^{N} S_d$, permutes $\{P_{i1}, \ldots, P_{id}\}$ for $0 \le i \le N$ (It's obvious that we don't need permutes different i in the picture, the basis $H = \{H_i\}$ is fixed.)

② There is no stacky issue to quotient by $\mathcal{G}$ as $\#\mathcal{G} < \infty$. ($\mathcal{E}$) also needn't as in $\overline{M}_{g,n}$ we had done

and we define $\overline{M}_{g,n}(\mathbb{P}^N;d)_{\mathcal{H}_1,\mathcal{H}_2} \subset \overline{M}_{g,n}(\mathbb{P}^N;d)_{\mathcal{H}_1}$ as the open loci such that $f^*(H_{2,0}), \ldots, f^*(H_{2,N})$ also disjoint & disjoint away $(P_i)_{i\le n}$ & they're étale divisors

($\mathcal{H}_1 = (H_{1,j})$ and $\mathcal{H}_2 = (H_{2,j})$ and $f : \mathcal{U} \to \mathbb{P}^N$) the action $\mathcal{G} \curvearrowright \overline{M}_{g,n}(\mathbb{P}^N;d)_{\mathcal{H}_1,\mathcal{H}_2}$ is naturally invariant, thus also well-defined (permutes $f^*(H_{1,j})$ not effect $f^*(H_{2,j})$)

$\{f^*(H_{2,j})\}_{j\le N}$ étale divisors gives an étale Galois covering $E \xrightarrow{g} \overline{M}_{g,n}(\mathbb{P}^N;d)_{\mathcal{H}_1,\mathcal{H}_2}$ with Galois group just $\mathcal{G}$

Precisely, the fibre of g at $([C], \{P_i(C)\}, \{P_{ij}(C)\}, f)$ is simply the ordering of $\{P_{2j}(C)\}$, and the Galois group $\mathcal{G}$-action is permuting them (different permutation!) (i.e. $\mathrm{Gal}(E/M) = \mathcal{G}$)

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[dr, "g_2"'] & \overline{M}_{g,n}(\mathbb{P}^N;d)_{\mathcal{H}_1,\mathcal{H}_2} \arrow[r] & \overline{M}_{g,n}(\mathbb{P}^N;d)_{\mathcal{H}_1} \\ & \overline{M}_{g,n}(\mathbb{P}^N;d)_{\mathcal{H}_2,\mathcal{H}_1} \arrow[ur] & \end{tikzcd}$$

Via this action, we have g_2, so factor through $\overline{M}_{g,n}(\mathbb{P}^N;d)_{\mathcal{H}_1,\mathcal{H}_2}$ due to this action respects $\{P_{2j}(C)\}$

$\Rightarrow \overline{M}_{g,n}(\mathbb{P}^N;d)_{\mathcal{H}_1,\mathcal{H}_2}/\mathcal{G} \to \overline{M}_{g,n}(\mathbb{P}^N;d)_{\mathcal{H}_2,\mathcal{H}_1}/\mathcal{G}$, by symmetric, we have a inverse of it ▨ ③ omitted.

• **Properness & Properties** ④ [FP] said one [illegible] of [illegible] valuative criteria, [illegible] given two family $(\mathcal{C} \to S, \ldots)$, $(\mathcal{C}' \to S, \{P_i'\}, f')$ isomorphic over $U = S - \{0\}$

then the isomorphism extend from $U = S - \{0\}$ to S (uniqueness)

Properness ⟺ After base change, family $(\pi: \mathcal{C} \to U, \{p_i\}, f)$ extend to hole base; i.e. $\mathcal{C} \leftarrow \mathcal{C}'$, s.t. $\mathcal{C}' \to U'$ extends to stable family $\mathcal{C}' \to S'$ (Existence)

The pullback of family of stable curves has sections and $f: \mathcal{C}' \to \mathbb{P}^N$ same as we constructed before

$$\begin{tikzcd}
\mathcal{C} \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}' \arrow[l] \arrow[d] \\
U \arrow[d, phantom, "\cap"] & U' = S' - \{0\} \arrow[l] \\
S & S' \arrow[l, "\exists \pi"']
\end{tikzcd}$$

$0, 0'$

Separatedness is always easier, extend to 0 is (étale) local, thus take H on $\mathbb{P}^N$, this makes $\mathcal{C}$ and $\mathcal{C}'$ are stable curves (family) with marked pt $d(N+1)$, by $\overline{M}_{g,n+d(N+1)}$ separated ⇒ $\mathcal{C} \cong \mathcal{C}'$ as stable curves. It left to show f and f' compatible: it's obvious that f & f' coincide on $S - \{0\} \subseteq S$ Zariski open set will extend to same ▨

This argument also makes sense for stack level

(Thus this argument can't used into properness: both not étale local and the extend of f to 0 is nontrivial)

Properness & Projectivity I omitted them!

Again our trick to prove properness is standard semi-stable reduction & projectivity by Kollár is by find an ample line bundle $\mathcal{L}$ on a proper scheme X then $\mathcal{L}^{\otimes m}$ very ample $\Rightarrow X \hookrightarrow \mathbb{P}(H^0(X, \mathcal{L}^{\otimes m})) \cong \mathbb{P}^N$

Originally, they're both dealt in Kollár's paper [Projectivity of complete moduli], do this for $\overline{M}_{g,n}$ and $\overline{M}_{g,n}(X;\beta)$ once time, alternatively, [FP] also deal it in detail. ▨

Here I omit it due to in [BF], the VFC's definition suffices separatedness (Although I assumed a much stronger condition)

② We use deformation theory to compute dimension of $\overline{M}_{g,n}(X;\beta)$ (We had mentioned that we can't reduce the ...) and show it's equal to $\overline{M}_{g,n}(X;\beta)$, thus smooth. Here our task is explain smoothness to $\overline{M}_{0,n}(\mathbb{P}^r; d)$ (※) in detail. Here in [FP] he assumed convexity to prove ① in (※) ($H^1(\mathbb{P}^1, f^*T_X) = 0$ for $\forall f: \mathbb{P}^1 \to X \Leftrightarrow X$ convex)

※. In fact we don't need any smoothness, only counting the virtual dim of $[\overline{M}_{g,n}(X;\beta)]^{vir}$ / "expected dimension" of $\overline{M}_{g,n}(X;\beta)$

① is $H^1(C, f^*T_X)$ for not irreducible case (1-dimensional)

$T_{[C,f]}\overline{M}_{g,n}(X;\beta) = \mathrm{Def}(C,f)$

$\mathrm{Ob}(C,f)$

The Deformation LES of $[C, \{p_i(C)\}, f] \in \overline{M}_{g,n}(X;\beta)$ is $0 \to \mathrm{Aut}(C, f) \to \mathrm{Aut}(C) \to \mathrm{Def}(f) \to \mathrm{Def}(C, f) \to \mathrm{Def}(C) \to \mathrm{Ob}(C,f)$ via associated with the SES $0 \to T_C(-p_1(C) - \dots - p_n(C)) \to f^*T_X \to N_{f/X} \to 0$ (simply denoted C later)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Aut}(C,f) \arrow[r] \arrow[d, phantom, "\parallel"] & \mathrm{Aut}(C) \arrow[r] \arrow[d, phantom, "\parallel"] & \mathrm{Def}(f) \arrow[r] \arrow[d, phantom, "\parallel"] & \mathrm{Def}(C,f) \arrow[r] \arrow[d, phantom, "\parallel"] & \mathrm{Def}(C) \arrow[r] \arrow[d, phantom, "\parallel"] & \mathrm{Ob}(C,f) \arrow[d, phantom, "\parallel"] \\
& 0 \arrow[r] & \boxed{1} \arrow[r] & H^0(C, f^*T_X) \arrow[r] & \boxed{2} \arrow[r] & \boxed{3} \arrow[r] & H^1(C, f^*T_X)
\end{tikzcd}$$

stable map

The blue sequence directly gives our desired information: • $\mathrm{Ob}(f) = 0$ unobstructed ⇒ $\overline{M}_{g,n}(X;\beta)$ is nonsingular (convex)

• $\dim^{vir}\overline{M}_{g,n}(X;\beta) = \dim\boxed{2} = \dim\boxed{3} - \dim H^0(C, f^*T_X) - \dim\boxed{1} = (\dim\boxed{3} - \dim\boxed{1}) + h^0(C, f^*T_X) = (3g-3) - (-n) + \chi(C, f^*T_X) = (3g-3+n) + \chi(C, f^*T_X)$

(Here we had smooth, the "vir" only refers to the result can be negative) (By facts about $\overline{M}_{g,n}$)

By Riemann-Roch. $\chi(C, f^*T_X) = \deg(f^*T_X) - \mathrm{rank}(f^*T_X)(g-1) = \deg(f^*T_X) - \boxed{(\dim X)(g-1)}$

$\overset{\text{def}}{=} \int_C c_1(f^*T_X) + \dim X = \int_C f^* \underbrace{c_1(T_X)}_{= c_1(X)} + \dim X = \int_{f_*[C]} c_1(X) + \dim X = \int_\beta c_1(X) + \dim X$ $\boxed{\text{Done}}$! this proves ②'

Thus we have obviously: ① The Green exactly holds for general $g \Longrightarrow \dim^{vir}\overline{M}_{g,n}(X;\beta) = 3(g-1) - (\dim X)(g-1) + n + \int_\beta c_1(X)$

② Without convexity, $\overline{M}_{g,n}(X;\beta)$ not smooth, but dimension not change: we only add $-H^1(C, f^*T_X)$, then also $\chi(C, f^*T_X)$!

Now we return back the last thing to prove: X convex ⇒ for $f: C \to X$ proj & connected & reduced & nodal & $g=0$ ▨

Pf. $C = \bigcup E_i$, induction on the number of irreducible components $\boxed{H^1(C, f^*T_X) = 0}$

$E_i \cong \mathbb{P}^1$ are irreducible components, now $C = C' \cup E \xrightarrow{f} X$ ~~$H^1(C', f^*T_X|_{C'}) = H^1(C', f^*T_X|_{C'}) = 0$~~ It's wrong as C' can be not connected!

$E \cap C = E \cap (\bigcup_{i \neq 1} E_i) = \bigcup_{i \neq 1} (E \cap E_i)$

$$\begin{tikzcd}
& & X \\
E \arrow[urr, "f|_E"] & C' \arrow[ur, "f|_{C'}"'] & \\
& E \cap C' \arrow[ul] \arrow[u] &
\end{tikzcd}$$

$H^1(E, f^*T_X|_E) = 0$

By inductive hypothesis they're done ($E \cong \mathbb{P}^1$, by convexity, it's 0)

this we need study $E \cap C'$ left.

Due to we known a tree can't happen "loops" i.e. "⋏" or "⤫", thus $E \cap C' = \{p_1 \cdots p_m\}$, for each p_j $(j \le m)$ each connected components $C_j' \cap E$ of C'

Thus we modify ①' $H^1(C_j', f^*T_X|_{C_j'}) = 0$

$\Rightarrow 0 \to f^*T_X|_{C'} \otimes \mathcal{O}_{C'}(-\sum p_j) \to f^*T_X \to f^*T_X|_E \to 0$ associates LES $\to \boxed{H^1(C', f^*T_X|_{C'} \otimes \mathcal{O}_{C'}(-\sum p_j))} \to H^1(C, f^*T_X) \to 0 \to$

$\overset{\text{By ①'}}{=} 0$

③ Nothing to prove but we only to explain two things: Obstruction bundle of VFC

How to evaluate/integral a class over another one (I'm confused on the pf of [FP] about this) ▨

• Construction of VFC

Here we have a relative obstruction theory of the forgetful map $\overline{M}_{g,n}(X;\beta) \to \mathfrak{M}_{g,n}^{all}$ (forget $f: \mathcal{C} \to X$, not stabilize the curve), then by nodal, we factor through $\mathfrak{M}_{g,n} \subseteq \mathfrak{M}_{g,n}^{all}$ the prestable (or quasi-stable; I think prestable is better by GIT, stable = semi-stable + prestable = auto-finite (rational tails ×) (worst nodal) + stability/limit class respectively by GIT) Then $\overline{M}_{g,n}(X;\beta) \to \mathfrak{M}_{g,n}$ gives a relative obstruction theory $\varphi: (R\pi_* f^*T_X)^\vee \to \mathbb{L}_{\overline{M}_{g,n}(X;\beta)/\mathfrak{M}_{g,n}}$, for $\mathcal{C} \xrightarrow{f} X$, $\downarrow \pi$, due to locally free, it's naturally perfect

i.e. $h^0(\varphi)$ isomorphism & $h^{-1}(\varphi)$ surjective

$\Rightarrow \mathfrak{C}_{\overline{M}_{g,n}(X;\beta)/\mathfrak{M}_{g,n}} \hookrightarrow \mathfrak{N}_{\overline{M}_{g,n}(X;\beta)/\mathfrak{M}_{g,n}} \xrightarrow{\varphi^\vee} h^1/h^0((R\pi_* f^*T_X)^\vee)$ $\overline{M}_{g,n}(X;\beta)$

Thus we set $=: \mathcal{E}(R\pi_* f^*T_X)$ vector bundle stack $\overset{0}{\leftrightarrows} \overline{M}_{g,n}(X;\beta)$

$[\overline{M}_{g,n}(X;\beta)]^{vir} = 0^![\mathfrak{C}_{\overline{M}_{g,n}(X;\beta)/\mathfrak{M}_{g,n}}]$, done ▨ (Thus when defining GW, we'll need $\sum \mathrm{codim}\, \gamma_i = (3 - \dim X)(g-1) + n + \int_\beta c_1(X)$)

• Evaluation/Integral $\int \alpha \overset{\text{def}}{=} \deg(\alpha.\beta) = \int \alpha.\beta$ for $\mathrm{codim}\,\alpha + \mathrm{codim}\,\beta = \dim X$, and $\int_\beta \alpha = \int_{\sum a_i [W_i]} = \sum_i a_i \int_{W_i} \alpha$ (※※)

this formula $\int \alpha = \int \alpha$

(※※) gives us the computation in ③: $\int f^*\alpha = \int_{f_*} \alpha$ the projection formula ▨

※. Via Poincaré dual of α, view it as differential form, it's just an integral over each W_i. It's called period integral / Feynman integral, and $H_k(X) \times H^k_{dR}(X) \to \mathbb{R}$ induces the period map (in Hodge theory, we learnt it, here is a baby case) $H_k(X) \cong H^k_{dR}(X)^*$ explains why it's called "period": take X elliptic curve, the integral is integral of elliptic function, known to be periodic.

Pf of WDVV needs a detail description of boundary of $\overline{\mathcal{M}_{g,n}}(X;\beta)$, thus we let X to be convex. Smoothness/Convexity is for [illegible]
(or weaker, satisfy $ev_i : \overline{\mathcal{M}_{0,n}}(X;\beta) \xrightarrow{\text{(1)}} \overline{\mathcal{M}_{0,n}} \times X^n \xrightarrow{pr_i} X$ smooth; (2) contraction $\overline{\mathcal{M}_{0,n}}(X;\beta) \to \overline{\mathcal{M}_{0,n}}$ smooth and each fiber are [illegible] smooth are independently have the same equation. What's "smooth are independently" means?
Why these two conditions hold by convex variety?

Rk: WDVV ⟺ associativity of small quantum cohomology ring; this gives a different model of GW-invariant in IIA (symplectic side); Frobenius mfd equip such multiplication naturally.

Study of boundary of moduli (See [FP] for more details)
$n = n_1 + n_2$ & $\beta = \beta_1 + \beta_2$, then

$$\begin{tikzcd}
\overline{\mathcal{M}_{0,n_1+1}}(X;\beta_1) \times_X \overline{\mathcal{M}_{0,n_2+1}}(X;\beta_2) \arrow[r, "I"] \arrow[d, "pr_1"] \arrow[dr, phantom, "\circlearrowleft"] & \overline{\mathcal{M}_{g,n}}(X;\beta) \arrow[d, "ev_i"] \\
\overline{\mathcal{M}_{0,n_1+1}}(X;\beta_1) \arrow[r, "ev_i"] & X
\end{tikzcd}$$
commutes if $i \le n_1$.

Why the commutativity holds? We need consider the definition of F

$$\begin{tikzcd}
\mathcal{C}_1 \arrow[r, "f_1"] \arrow[d, "\pi_{n+1}"'] & X & \mathcal{C}_2 \arrow[r, "f_2"] \arrow[d, "\pi_{n+1}"'] & X \\
S_1 \arrow[u, bend right] & X & S_2 \arrow[u, bend right] &
\end{tikzcd}$$

Likewise we construct $\mathcal{C}_0$ by $(\mathcal{C}_1, S, p_1, \cdots, \boxed{p_{n_1+1}})$, $(\mathcal{C}_2, S, q_1, \cdots, \boxed{q_{n_2+1}})$ Gluing $\square$ together to a new node.

this not change the stability and $S_0 : \mathcal{C}_0 \to X$ constructed via isomorphism: $C \cong C_1 \cup C_2$
$\Rightarrow \mathrm{Hom}(C,X) = \mathrm{Hom}(C_1 \cup C_2, X)$
$\cong \mathrm{Hom}(C_1,X) \times_X \mathrm{Hom}(C_2,X)$
$\cong \mathrm{Hom}(C_1,X) \times \mathrm{Hom}(C_2,X)$
and then $(f_1 \cup f_2)_*[C]$
$= (f_1)_*[C_1] + (f_2)_*[C_2]$
$= \beta_1 + \beta_2 = \beta$ ▨

Thus the ev_i ($i \le n_1$) is "unused", of course commutes!

How boundary information is given from this picture?
Such gluing via node to  is called degenerate curve
and for stable curves
$\overline{\mathcal{M}_{0,n_1+1}} \times \overline{\mathcal{M}_{0,n_2+1}} \to \overline{\mathcal{M}_{0,n}}$ is the boundary divisor, so is stable map case.

Due to our order important in next, we set $\overline{\mathcal{M}_{0,n}}(X;\beta)$'s boundary divisor to be $D(A,B;\beta_1,\beta_2)$ ($A \amalg B = I$, $\beta_1+\beta_2=\beta$) and $D(A,B) = \delta_A$ of $\overline{\mathcal{M}_{0,n}}$.

Here we order the n pts result by $(\underbrace{p_1 \cdots p_{n_1}}_{\text{first } n_1}, p_{n_1+1} = q_{n_2+1} \cdots)$ first n_1 [illegible] n

(Mention again, order important! $\delta_A = D(A,B) \neq D(B,A) = \delta_B$, a reason is that if we want forget last some marked pts, δ_A not change by δ_B changes)

Pf of WDVV. Recall WDVV:
$$\boxed{\begin{aligned}&\sum_{\beta_1+\beta_2=\beta}\sum_{A\amalg B=I}\sum_{i,j}\langle\gamma_1,\gamma_2,\delta_A,e_i\rangle_{\beta_1} g^{ij} \langle e_j,\gamma_3,\gamma_4,\delta_B\rangle_{\beta_2}\\ =&\sum_{\beta_1+\beta_2=\beta}\sum_{A\amalg B=I}\sum_{i,j}\langle\gamma_1,\gamma_3,\delta_A,e_i\rangle_{\beta_1} g^{ij}\langle e_j,\gamma_2,\delta_B,\gamma_4\rangle_{\beta_2}\end{aligned}} \quad (*)$$

later introduce the GW potential we rewrite it as a PDE

First, consider fix β_1, β_2, fix A, B, we need given a construction, then take fibre product $\coprod_{\beta_1+\beta_2=\beta}\coprod_{A\amalg B=I}$ over all moduli we construct over X.

• Second, we left γ_4 to be $*$ (i.e. not do evaluate/integral) and stay at cycle class level, then $(*)$ is written as
$$\boxed{\sum_{\beta_1+\beta_2=\beta}\sum_{A\amalg B=I}\langle\langle\gamma_1,\gamma_2,\delta_A,*\rangle_{\beta_1},\gamma_3,\delta_B,*\rangle_{\beta_2}=\sum_{\beta_1+\beta_2=\beta}\sum_{A\amalg B=I}\langle\langle\gamma_1,\gamma_3,\delta_A,*\rangle_{\beta_1},\gamma_2,\delta_B,*\rangle_{\beta_2}} \quad (**)$$

Here our notation of $\langle\gamma_1 \cdots \gamma_{n-1}, *\rangle_\beta$ is a variant of GW class of $\overline{\mathcal{M}_{0,n}}(X;\beta)$ s.t. $\langle\gamma_1 \cdots \gamma_n\rangle_\beta = \int_X \langle\gamma_1 \cdots \gamma_{n-1}, *\rangle_\beta \cup \gamma_n$
it's defined via $(ev_n)_*((ev_1 \times \cdots \times ev_{n-1})^*(\gamma_1 \times \cdots \times \gamma_{n-1}) \cap [\overline{\mathcal{M}_{0,n}}(X;\beta)])$.

Our $(*) \Leftrightarrow (**)$ is an easy linear algebra on the linear space — (Why we need this part?)
$H^*(X;\mathbb{Q})/\mathbb{Q}$ as $g^{ij} = g_{ij}^{-1} = (\int e_i \cup e_j)^{-1} \Rightarrow$ [illegible] our $g_{ij} = \int e_i \cup e_j$ is number, $(g^{ij}) = (g_{ij})^{-1}$
thus it holds g^{ij} is the inverse matrix's element.

I'm not sure how this linear algebra works: $\langle\gamma_1, \gamma_3, e_i\rangle g^{ij} \langle e_j, \gamma_2\rangle_{\beta_2} = \langle\gamma_1, \gamma_3, *\rangle_{\beta_1}, \gamma_2\rangle_{\beta_2}$
~~as g^{ij} is [illegible] vague for me as a intersection pairing~~

• Third, combining the first second points, we mainly reduce the two-sides of $(**)$ into a symmetric term w.r.t index 2 & 3

Construction. Set $\overline{\mathcal{M}_1} := \overline{\mathcal{M}_{0,A \amalg ▸}}(X;\beta_1)$, the marked pts are: (1.1) A consists first two are pullback of γ_1, γ_2 (left) from $\overline{\mathcal{M}_{0,n}}(X;\beta)$ to $\overline{\mathcal{M}_1}$ and γ_1, γ_3 right
then are pullback of δ_A [illegible] ($A = \{A_1, \cdots, A_a\}$)
(1.2) ▸ consists the pushforward of $a = |A|$ pts.
$\overline{\mathcal{M}_2} := \overline{\mathcal{M}_{0,◂ \amalg B \amalg •}}(X;\beta_2)$, the marked pts are:
(2.1) ◂ consists only one, pullback of $\langle\gamma_1, \gamma_2, \delta_A, *\rangle_{\beta_1}$ left (from $\overline{\mathcal{M}_{0,n}}(X;\beta)$ to $\overline{\mathcal{M}_2}$)
and $\langle\gamma_1, \gamma_3, \delta_A, *\rangle_{\beta_1}$ right
(2.2) B consists first one is pullback of γ_3 left, γ_2 right,
then are pullback of δ_B ($b = |B|$ pts)
(2.3) • consists the pushforward (only one)

(One may confuse about what it means for be pullback/push to/from a mark pt from/to divisor. Here just a "feeling", i.e. we divide these marked $3+|A|$ pts into two/three parts)

For understand we can write it by marked in WDVV $(**)$: $\langle\langle\boxed{\gamma_1,\gamma_2}\boxed{\delta_A}\boxed{*}\rangle_{\beta_1}\boxed{\gamma_3}\boxed{\delta_B}\boxed{*}\rangle_{\beta_2} \leftrightarrow \langle\langle\boxed{\gamma_1,\gamma_3}\boxed{\delta_A}\boxed{*}\rangle_{\beta_1}\boxed{\gamma_2}\boxed{\delta_B}\boxed{*}\rangle_{\beta_2}$ (A, ◂, ▸, B, • over the boxes; (1.1) (1.2) (2.1) (2.2) (2.3) under them)

Thus $\overline{\mathcal{M}_1} \times_X \overline{\mathcal{M}_2} \xrightarrow{F} \overline{\mathcal{M}} := \overline{\mathcal{M}_{0,A \amalg B \amalg •}}(X;\beta)$ gluing ▸◂ by node
(both ◂ & ▸ are single pt) (The gluing is natural as the pushforward of ▸ = pullback of ◂, they corresponds to the $\overline{\mathcal{M}_1}$ part and $\overline{\mathcal{M}_2}$ part of the GW-class $\langle\gamma_1, \gamma_2, \delta_A, *\rangle_{\beta_1}$) again I don't know why.

$$\begin{tikzcd}
\overline{\mathcal{M}_1} \times_X \overline{\mathcal{M}_2} \arrow[r, "pr_2"] \arrow[d, "pr_1"] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}_2} \arrow[r, "ev_•"] \arrow[d, "ev_◂"] & X \\
\overline{\mathcal{M}_1} \arrow[r, "ev_▸"] & X &
\end{tikzcd} \quad (A)$$

Then $\langle\langle\gamma_1, \gamma_2, \delta_A, *\rangle_{\beta_1}, \gamma_3, \delta_B, *\rangle_{\beta_2} = (ev_•)_*(\boxed{ev_◂^*(ev_▸)_*}(ev_{1,2,A}^*(\gamma_1 \times \gamma_2 \times \delta_A) \cap [\overline{\mathcal{M}_1}]) \cdot ev_{3,B}^*(\gamma_3 \times \delta_B) \cap [\overline{\mathcal{M}_2}])$
(here all spaces are proper (proj), thus pushforward makes sense) By definition of GW-class where (A) is used is the pullback of evaluation!
(smooth ⇒ flat)
projection formula via left diagram: $= (ev_•)_*(pr_2)_*(pr_1^* ev_{1,2,A}^*(\gamma_1 \times \gamma_2 \times \delta_A) \cup pr_2^* ev_{3,B}^*(\gamma_3 \times \delta_B) \cap [\overline{\mathcal{M}_1} \times \overline{\mathcal{M}_2}])$
(B) $= (ev_•)_*(pr_2)_*(F^* ev_{1,2,A,3,B}^*(\gamma_1 \times \gamma_2 \times \delta_A \times \gamma_3 \times \delta_B) \cap [\overline{\mathcal{M}_1} \times_X \overline{\mathcal{M}_2}])$
$= (ev_•)_*(ev_{1,2,A,3,B}^*(\gamma_1 \times \gamma_2 \times \delta_A \times \gamma_3 \times \delta_B) \cap C^*[F(12|3•)])$
is symmetric! ▨

Next,
$$\begin{tikzcd}
\overline{\mathcal{M}_1} \times \overline{\mathcal{M}_2} \arrow[r] \arrow[rr, bend left, "F"] & \Delta(12|3•) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}} \arrow[d, "C \text{ contraction map}"] \\
& \{(12|3•)\} \arrow[r, hook] & \overline{\mathcal{M}_{0,4}}
\end{tikzcd} \quad (B)$$
[illegible] used to pullback C^*

Where $(12|3•)$ is the curves  of this form & $\Delta(12|3•) \subset \overline{\mathcal{M}}$.
Rigorously, one can write it as Chow class defined by pullback.

⑤ We'll describe the ring structure/quantum product of $QH^*(X)$ and $BQH^*(X)$. They're deformation of the ordinary cohomology ring (Slogan: Quantization ⟺ Deformation), originally inspired by Chiral ring of CY 3-mfd in physics.

Small case: $QH^*(X) = H^*(X)[[e^{t_1},\dots,e^{t_m}]]$, $t_1 \dots t_m \in H_2(X;\mathbb{Z})/$torsion independent and span all curve class/$\mathbb{Z}$

The first explanation is what $H^*(X)$ is, someone write $H^*(X) = H^*(X;\mathbb{Z})/\{$torsion part$\}$ (the torsion of $H^*(X;\mathbb{Z})$ as ring)

Second explanation is the $QH^*(X;\Lambda)$'s coefficient ring Λ, we have then $QH^*(X;\Lambda) \cong H^*(X)\otimes\Lambda$ as ring isomorphism, thus only check the well-defined of the Novikov ring (if we assume $X/\mathbb{C}$ smooth projective ⇒ cpt Kähler with Kähler form $\omega \in H^2(X;$ [redacted] $\mathbb{R})$) $\Lambda = \left\{ \sum_{A\in H_2(X)} \lambda_A e^A \;\middle|\; \lambda_A \in \mathbb{Z},\ \forall c \in \mathbb{R}, \exists \text{ finite } A \in H_2(X) \text{ s.t. } \omega(A) \le c \text{ and } \lambda_A \neq 0 \right\}$ $\cong \mathbb{Z}[[e^{t_1},\dots,e^{t_m}]]$ obviously ($\omega(A)$ is Poincaré pair $H^2 \times H_2 \to \mathbb{R}$)

Lastly we need to depict the multiplication precisely (We had known that a formal multiplication $(\sum \lambda_A e^A)(\sum \lambda_B e^B) = \sum \lambda_A \lambda_B e^{A+B}$)

$c_1, c_2 \in QH^*(X;\Lambda)$, we define $c_1 * c_2$ uniquely determined by (✱):

(✱) $\int_X (c_1 * c_2) \cup c = \sum_{\beta} \langle c_1, c_2, c\rangle^X_\beta \cdot e^\beta$, $\forall c \in QH^*(X;\Lambda)$ with expected dim ($\Lambda = \mathbb{Z}[H_2(X)]$)

$\beta \in H_2(X)$ all curve class

the integral of $c \in QH^*(X;\Lambda)$ makes sense due to it makes sense of $H^*(X)$ and e^β is formal multiplication

[redacted] for a basis $e^{t_1} \dots e^{t_m}$ over $H^*(X)$ [redacted]

We know $\beta = 0$ case $\langle c_1, c_2, c_3\rangle_0 = \int_X c_1 \cup c_2 \cup c_3$, thus $c_1 * c_2 = c_1 \cup c_2$ recovers usual intersection product; $\langle c_1, c_2, c_3\rangle = \int c_1 \cup c_2 \cup c_3 + \sum_{\beta\neq 0} \langle c_1, c_2, c_3\rangle_\beta e^\beta$ called **Yukawa coupling** by physicists (n=3)

Then we can write the small quantum product [redacted]

$c_1 * c_2 = \sum_{\substack{1\le i\le m\\ 1\le j\le m}} \sum_{\beta} \langle c_1, c_2$ [redacted] $e^\beta t_j$ where g^{ij} = [redacted] $(t_i \cup t_j)^{-1}$ as before we done in WDVV

(✱') [redacted]

Further it's super commutative, i.e. $c_1 * c_2 = (-1)^{\deg c_1 \deg c_2} c_2 * c_1$, this is similar and easier than associativity, omitted.

[redacted] Thus by (✱') we can write the associativity of $QH^*(X;\Lambda)$

$(c_1 * c_2) * c_3 = \left(\sum_{i,j} \sum_\beta \langle c_1, c_2, t_i\rangle_\beta g^{ij} e^\beta t_j\right) * c_3 = \sum_{i,j} \sum_{\beta_1} g^{ij} e^{\beta} t_j\, (\langle c_1, c_2, t_i\rangle_\beta * c_3) = \sum_{i,j} \sum_{\beta_1} g^{ij} e^{\beta_1} t_j \sum_{k,l} \sum_{\beta_2} \langle\langle c_1, c_2, t_i\rangle_{\beta_1} c_3, t_k\rangle g^{kl} e^{\beta_2}$

$= \sum_{i,j,k,l} \sum_{\beta_1+\beta_2} \langle\langle c_1, c_2, \boxed{t_i}\rangle_{\beta_1} c_3, \boxed{t_k}\rangle_{\beta_2} g^{kl} e^{\beta_2} \boxed{t_l} g^{ij} e^{\beta} t_j = \sum_{i,j,k,l} \sum_{\beta_1+\beta_2} \boxed{\langle c_1, c_2, t_k\rangle_{\beta_1} \langle t_l, c_3, t_i\rangle_{\beta_2}} g^{kl} g^{ij} e^\beta t_j$ ($e^\beta = e^{\beta_1} e^{\beta_2}$, here sum ⋯ (1) over both (β_1, β_2) and β)

Similarly, $c_1 * (c_2 * c_3) = \sum_{i,j,k,l} \sum_{\beta_1+\beta_2} \boxed{\langle c_2, c_3, t_k\rangle_{\beta_1} \langle c_1, t_l, t_i\rangle_{\beta_2}} g^{kl} g^{ij} e^\beta t_j$

(Later we'll prove this via axiom characterization of GW by [KM] Here it's equivalence axiom) $\overset{\text{super-commutativity of GW invariants}}{=} \sum_{i,j,k,l} \sum_{\beta_1+\beta_2} \langle c_2, c_3, t_k\rangle_{\beta_1} \langle t_l, c_1, t_i\rangle_{\beta_2} g^{kl} g^{ij} e^\beta t_j (-1)^{\deg c_1 \deg t_l} \cdots (2)$ [redacted]

Our key to identify (1)=(2) is **the splitting axiom**: (up to a pullback, we write a more marked pts GW into multiplication of two/more marked less pts' GW, we'll describe it precisely later via $\overline{M}_{0,n_1+1}(X;\beta_1) \times \overline{M}_{0,n_2+1}(X;\beta_2) \to \overline{M}_{0,n}(X;\beta)$)

$F: \overline{M}_{0,3}(X;\beta_1) \times \overline{M}_{0,3}(X;\beta_2) \to \overline{M}_{0,4}(X;\beta)$

$= (-1)^{\deg c_1 \deg c_2} (-1)^{\deg c_1 \deg c_3} F^*\langle c_1, c_2, c_3, t_i\rangle_\beta$

$F^*\langle c_1, c_2, c_3, t_i\rangle_\beta = \sum_{\beta_1+\beta_2} \sum_{k,l} \langle c_1, c_2, t_k\rangle_{\beta_1} \langle t_l, c_3, t_i\rangle_{\beta_2}$ & $F^*\langle c_2, c_3, c_1, t_i\rangle_\beta = \sum_{\beta_1+\beta_2} \sum_{k,l} \langle c_2, c_3, t_k\rangle_{\beta_1} \langle t_l, c_1, t_i\rangle_{\beta_2}$

thus it left to check the $(-1)^{\deg c_1 \deg t_l} (-1)^{\deg c_1 \deg c_2} (-1)^{\deg c_1 \deg c_3} = 1$, i.e. $\deg c_1 (\deg t_l + \deg c_2 + \deg c_3) \equiv 0 \pmod 2$

~~Return back to definition~~ we know $\deg c_1 + \deg c_2 + \deg t_k = 2\dim X - 2\int_{\beta_1} c_1(X)$ to make the $\langle c_1, c_2, t_k\rangle_{\beta_1}$ & $\langle c_2, c_3, t_i\rangle_{\beta_2}$ well-defined ▨

Degree Axiom

$\deg c_2 + \deg c_3 + \deg t_l = 2\dim X - 2\int_{\beta_2} c_1(X)$

Big case: What we done in small case is make $H^*(X;\mathbb{C})$ into a supercommutative ring ($QH^*(X;\Lambda) \subset H^*(X;\mathbb{C})$ due to e^a is allowed in $\mathbb{C}$-coefficient, and $c_1, c_2 \in H^*(X;\mathbb{C})$ is able to take) (Further $H^*(X;\mathbb{C})$ is Frobenius algebra) Big case says that $H^*(X;\mathbb{C})$ is a supermanifold (what's [illegible]) quantum product by definition

First we need GW-potential, and rewrite our ④ WDVV into PDE via GW potential, it's a "generating function" collects all enumerative information $\Phi(\gamma) = \sum_n \sum_\beta \frac{1}{n!} \langle \gamma, \dots, \gamma\rangle_\beta \cdot e^\beta$ ([FP] need $n\ge 3$ for not virtual, but here we only need $n\ge 0$) (named from combinatorics), Here $\langle \gamma, \dots, \gamma\rangle_\beta$ in $\overline{M}_{0,n}(X;\beta)$

for $\gamma \in H^*(X;\mathbb{C}) \Rightarrow \gamma = \sum_{j=0}^{m} t_j e_j$, "$e_j$ is basis" chosen in ④ (Note that it makes sense in genus 0 GW, the example is $\langle [pt], \dots, [pt]\rangle_\beta$ ⟵ $3d-1$ the number of pass $3d-1$ pts of deg d plane curves)

(distinct $\vec{e}$ with formal e^β, somewhat without [illegible])

$\Rightarrow \langle \gamma^n\rangle_\beta = \langle \gamma, \dots, \gamma\rangle_\beta$

$\frac{1}{n!}\langle \gamma^n\rangle_\beta = \sum_{|a|=n}$ [redacted] $\langle \vec{e}^{\,a}\rangle_\beta \frac{\vec{t}^{\,a}}{\vec{a}!}$, Here a is multiindex and $\vec{e} = (e_0 \cdots e_m) \Rightarrow \Phi(\gamma) = \sum \sum$ [redacted] $\langle \vec{e}^{\,a}\rangle_\beta \frac{\vec{t}^{\,a}}{\vec{a}!}$

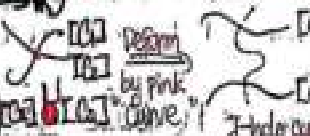
Rk. A subtle thing is the non/supercommunicativity, one may have a $\delta(a) = \pm 1$ determined by [redacted] $(t_0 e_0)^{a_0} \cdots (t_m e_m)^{a_m} = \delta(a) e_0^{a_0} \cdots e_m^{a_m} t_0^{a_0} \cdots t_m^{a_m}$, anyway I omit a ± 1 here not effect our final result for simplicity

Then our $BQH^*(X) = H^*(X)[[e_0, \dots, e_m]]$ with multiplication $e_i * e_j = \sum_{k,l} \Phi_{ijk}\, g^{kl} e_l$, $\Phi_{ijk} = \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_k}$ makes sense due to $\Phi(\gamma)$ is a formal power series on $\vec{t}$ $= \sum_k \Phi_{ij}{}^k e_k$ (raising index in physics/GR) $= \sum_k \Phi_{ijk} e_k^\vee$ called **big quantum product**

[redacted] Without loss, similar with QH's case, we can describe it by pairing: $c_1 \cdot c_2$ defined by (Why here we take a dual basis? I suppose it not affect. We take $H^*(X)[[e_0 \dots e_m]]$ (here is answer))

$\int (c_1 \cdot c_2) \cup c = \sum_a \sum_\beta \langle c_1, c_2, c, \vec{e}^{\,a}\rangle_\beta$ [redacted]

Associativity is why? We'll translate it into WDVV④:



It's easy to understand by def of $*$ as

$[c_1] \cup [c_2]$ ⟷ "number" of pts $\in c_1 \cap c_2$

$[c_1] * [c_2]$ ⟷ "number" of J-holo curves connecting c_1 & c_2 ⟷ $\langle c_1, c_2, *\rangle_\beta^X$ the GW-invariant

We have a symplectic characterization of such quantum deformation (but as [FP] says, it's not algebraic as the J-holo curves are not allowed in algebraic)

Ordinary $H^*(X;\mathbb{Z})$'s multiplication $\cup$ is intersection

$QH^*(X;\mathbb{Z})$'s multiplication $*$ is "intersection by connected with J-holo curves":

$\Phi_{ijk} = \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_k} = \frac{\partial^3}{\partial t_i \partial t_j \partial t_k} \sum_{n} \sum_{\beta} \frac{1}{n!} \langle \gamma^n \rangle_\beta \, e^{\beta} = \sum_{|\alpha|} \sum_{\beta} \frac{\partial^3}{\partial t_i \partial t_j \partial t_k} \langle \gamma^n \rangle_\beta$ [illegible] (for denote residue indices)

$= \sum_{|\alpha|} \sum_{\beta} \langle e_i, e_j, e_k, \gamma \rangle$ [illegible] $\frac{1}{(a_i-1)!(a_j-1)!(a_k-1)!}$ [illegible] $\cdot e^{\beta}$. Let's assume that $a_i = a_j = a_k = 1$ due to we can assume it's only a simple root. Supermanifold approach / rewrite:

$= \sum_{|\alpha|} \sum_{\beta} \langle e_i, e_j, e_k, \gamma \rangle_\beta$ [illegible] $\cdot e^{\beta}$ Again assume $\cdot$ is supercommutative, then

View $H^*(X; \mathbb{C})$: each even graded part $\mathbb{S}\mathbb{H}^*(X; \mathbb{C})$ is complex [illegible], adding the Frobenius [illegible] algebra since [illegible]

$= \sum_{|\alpha|} \sum_{\beta} \langle e_i, e_j, e_k, \gamma \rangle_\beta$ [illegible] $\cdot e^{\beta}$. Thus $(e_i \cdot e_j) \cdot e_k = \sum \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_a} g^{ab} \frac{\partial^3 \Phi}{\partial t_b \partial t_k \partial t_c} g^{cd} e_d$

$n+3$

Rk. Compute SQH^* & BQH^* are [illegible] then [illegible] / [illegible] the

$= \sum_{|\alpha|} \sum_{\beta} \langle e_i, e_j, e_k, \gamma \rangle_\beta$ [illegible] $A^*(\overline{M}_{g,n}(X,\beta))$ Thus we have simplify to $\sum$ [illegible] Σ to [illegible] $\langle e_0, e_i, e_k \rangle$ [illegible] $\langle e_0, e_k, e_i \rangle$ [illegible] computing

as more form-like ▨ $\Rightarrow (e_i \cdot e_j) \cdot e_k =$ [illegible] [illegible] determined [illegible] and target [illegible] $+3 =$ [illegible] $= (-1)$ [illegible]

(If one have $\epsilon(\alpha)$ considered, things are subtle when compute Φ_{ijk}, but then the [illegible] [illegible]

intersection [illegible] GW-class $\longmapsto$ GW-invariant [illegible] $H^*(X; \mathbb{Q})$ can be consided to simplify [illegible] is complicated called Dubrovin connection, so is big [illegible]

Axioms for GW-classes

Thm 4. GW-class $\Leftrightarrow I^X_{g,n,\beta} : H^*(X;\mathbb{Q})^{\otimes n} \to H^*(\overline{M}_{g,n}; \mathbb{Q})$ [illegible] ($\overline{M}_{g,n}$ or $\overline{M}_{g,n}(X,\beta)$ is not [illegible] [illegible] evaluation product)

satisfy following axioms: A tree-level system, A cohomological field theory (CohFT) is such a system $\overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n} \times X^n$

A0) Linearity. $I^X_{g,n,\beta}$ linear on each variables $\in H^*(X;\mathbb{Q})$; **A1) Effectivity.** β not effective $\Rightarrow I^X_{g,n,\beta} = 0$ satisfy D1') and C1) mainly, see also [KM]

(We denote A0) due to in [KM] originally without it)

A2) Degree axiom $\deg I^X_{g,n,\beta}(\gamma_1, \dots, \gamma_n) = 2(g-1)\dim X + 2\int_\beta c_1(X) + \sum \deg \gamma_i$ (Due to $\langle \gamma_1 \cdots \gamma_n \rangle = \int I^X_{g,n,\beta} = 0$ iff not top degree, thus $\langle \gamma_1 \cdots \gamma_n \rangle \neq 0 \Rightarrow \sum \deg \gamma_i = 2(1-g)\dim X + 2\int_\beta c_1(X) + (3g-3+n)2$)

A3) S_n-equivariance. $I^X_{g,n,\beta}$ is S_n-equivariant.

($\overline{M}_{g,n}$ has S_n-action by permuting marked pts & $H^*(X;\mathbb{Q})^{\otimes n}$ is obvious)

B4) Fundamental class axiom. $\pi : \overline{M}_{g,n} \to \overline{M}_{g,n-1}$ forget last pt, then $I^X_{g,n,\beta}(\gamma_1 \cdots \gamma_{n-1}, [X]) = \pi^* I^X_{g,n-1,\beta}(\gamma_1 \cdots \gamma_{n-1})$ (thus $\langle \gamma_1 \cdots \gamma_{n-1}, [X] \rangle_{\beta} = 0$) $\beta \neq 0$

B5) Divisor axiom. $\gamma_n \in H^2(X;\mathbb{Q})$ (i.e. divisor class [illegible] $n+2g \geq 4$ [illegible]), then $\pi_* I^X_{g,n,\beta}(\gamma_1 \cdots \gamma_n) = (\int_\beta \gamma_n) I^X_{g,n-1,\beta}(\gamma_1 \cdots \gamma_{n-1})$ without $n+2g \geq 4$, even π is not defined, we can use virtual tricks to prove

(thus $\langle \gamma_1 \cdots \gamma_n \rangle_\beta = (\int_\beta \gamma_n) \langle \gamma_1 \cdots \gamma_{n-1} \rangle$, and holds without π / $n+2g \geq 4$ same reason as B4)

B6) Point mapping axiom. When $\beta = 0$, $I^X_{g,n,0}$ is given precisely [illegible]

Here one should write a formula, but I can't find it ; in fact $g > 0$ case is quite complicated by [KM] due to $[\overline{M}_{g,n}(X,\beta)]^{vir} = \pi^* [\overline{M}_{g,n-1}(X,\beta)]^{vir}$ π forget the first pt (See [LT])

But $g = 0$ case $I^X_{0,n,0}(\gamma_1 \cdots \gamma_n) = \begin{cases} \int_X (\gamma_1 \cup \cdots \cup \gamma_n) [\overline{M}_{0,n}] ; \sum \deg \gamma_i = 2\dim X \\ 0 ; \text{otherwise.} \end{cases}$ (not only replace $[\overline{M}_{g,n}]$ but $[\overline{M}_{g,n}]^{vir}$!)

C1) Splitting axiom.

$\varphi^* I^X_{g,n,\beta}(\gamma_1 \cdots \gamma_n) = \sum_{\beta = \beta_1 + \beta_2} \sum_{i,j} g^{ij} I^X_{g_1, n_1+1, \beta_1}(\gamma_1 \cdots \gamma_{n_1}, e_i) \otimes I^X_{g_2, n_2+1, \beta_2}(e_j, \gamma_{n_1+1} \cdots, \gamma_n)$

C2) Genus reduction.

$\psi : \overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ gluing last two marked pts (How this glue works?)

Then $\psi^* I^X_{g,n,\beta}(\gamma_1 \cdots \gamma_n) = \sum g^{ij} (\gamma_1 \cdots \gamma_n, e_i, e_j)$

D1) Motivic axiom. $I^X_{g,n,\beta} : \gamma_1 \cdots \gamma_n \longmapsto \mathrm{pr}_{2*}\, \mathrm{pr}_1^*(\gamma) \cap [C^X_{g,n,\beta}]$, $[C^X_{g,n,\beta}] \in Z^*(X^n \times \overline{M}_{g,n})$, $X^n \times \overline{M}_{g,n} \xrightarrow{\mathrm{pr}_1} X^n$, pr_2 to $\overline{M}_{g,n}$

D1') Deformation axiom. For a family of $\mathfrak{X} \to T$ with fibre X_t, GW-class is deformation invariant: β_t locally constant section of $H_2(X_t, \mathbb{Z})$ and γ is locally constant sections of $H^*(X_t;\mathbb{Q})^{\otimes n} \Rightarrow I^{X_t}_{g,n,\beta_t}(\gamma_1 \cdots \gamma_n)$ constant.

Pf/Rk. I'll not give a precise pf: Both [LT] & [Behrend] pays most of their pages to check them via their different construction of virtual fundamental class.

It's not hard to guess (by myself) what [KM] thought in their way: B4)–B6) gives enough start condition and C1)–C2) (most important) gives enough inductive data, and they done well in $g = 0$ cases inductively.

The issue of D1) & D1') is: original [KM] uses D1) instead of D1'), then after [LT] & [B], VFC directly $\Rightarrow$ D1') and provides that D1) is contained in other axioms. Same is B4), which not noticed in [KM]. Both of D1) can be excluded.

[Cox & Katz] says that A0)–D1) all easy in symplectic setting but complicated in algebraic setting (In fact, I know to construction of $\overline{M}_{g,n}(X,\beta)$ also this manner

Eg 8 ① (Kontsevich's plane rational curve formula) Thus we have two ways { $\overline{M}_{0,4}(\mathbb{P}^2, d)$ & cross ratio [illegible] ; WDVV equation & physics

$N_d = \langle [pt]^{3d-1} \rangle_d$, $\Phi(pt) = \sum_{n} \sum_{\beta} \frac{1}{n!} \langle [pt]^n \rangle_\beta \, e^{\beta}$, due to $\beta \in H_2(X;\mathbb{Z}) = H_2(\mathbb{P}^2;\mathbb{Z})$ [illegible] $= \mathbb{Z}$ generated by $[H]$

Take it into WDVV equation

$\sum_{a,b} \Phi_{ija} g^{ab} \Phi_{bkl} (-1)^{[illegible]} = \sum_{a,b} \Phi_{jka} g^{ab} \Phi_{bil}$ (※※) [illegible] $\sum \frac{1}{n!} \langle [pt]^n \rangle_d \, e^{d[H]} = \sum_{d \geq 1} \frac{1}{(3d-1)!} \langle [pt]^{3d-1} \rangle_d \, e^{d[H]}$ due to the degree axiom, we known that $\langle [pt]^n \rangle_d \neq 0 \Leftrightarrow n = 3d-1$; $= \sum_d \frac{1}{(3d-1)!} N_d \, e^{d[H]}$ (※)

A physical story is the duality diagram in TFT

Let $(i,j,k,l) = (1,1,2,2) \Rightarrow \boxed{\Phi_{112} + \Phi_{11a}\Phi_{b22} = \Phi_{12}}$ (※※) with the homogenous basis is $e_0 = [X], e_1 = [H], e_2 = [pt]$

Thus only for $a+b = 2$ case the term is not vanishing $\Rightarrow g_{00} = 0, g_{11} = 1, g_{22} = 0$, matrix (left) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$\Phi_{112}\Phi_{2\,\cdot} + \Phi_{111}\Phi_{\cdot\,22}$ [illegible] $\Rightarrow \Phi_{111}\Phi_{122} + \Phi_{\cdot}\,0 + \cdots \Phi_{222} = \Phi_{112}^2$ $g_{01} = 0$, $g_{02} = 1$

[illegible] Φ_{ij0} [illegible]

we have (※※) desired. $g_{12} = 0$ only $i+j = 2$ is 1 $\Rightarrow$ inverse is itself

(may need a subtle care on the sign, omitted) otherwise all vanish

We compute Φ_{0ij} (note that t_i can be too) to $\Phi_{0ij} = g_{ij} = \begin{cases} 1 ; i+j = 2 \\ 0 ; \text{otherwise.} \end{cases}$ (always $e_0 = [X]$ fundamental class)

$\Phi_{0ij} = \sum_n \sum_\beta \frac{1}{n!} \langle e_0, e_i, e_j, \gamma^n \rangle_\beta \, e^{\beta}$

☆ $\langle e_0, e_i, e_j \rangle_{0} = \int [X] \cup e_i \cup e_j = \int e_i \cup e_j = g_{ij}$ ▨

☆ is due to fundamental class axiom (latter is point map axiom)

$\Rightarrow$ Due to this, it's reasonable by degree axiom, GW-potential into { $\beta = 0$ and $\beta \neq 0$; $e^{\beta} = 1$ and $e^{\beta} \neq 1$; e_0 and without e_0 } (By left argument) $\Rightarrow$ classical [illegible] quantum [illegible] $= \frac{1}{2} t_0 t_1^2 + \frac{1}{2} t_0^2 t_2 + \sum \sum \frac{1}{n!} \langle \gamma^n \rangle_\beta e^{\beta}$ classical intersection number

(Here we can also see why we need multiply e^{β} last formally: e^{β} is the canonical quantization method that's also why it's a [illegible] deformation)

We determine N_d from (**) via first consider the only term in Φ_{222} contains it, then compare the coefficient in two sides: [redacted]

First besides (*), we need write Φ by basis $\{e_0, e_1, e_2\}$ to compute derivatives: (Why Φ_{222}? Due to it's the only term with degree ...)

$\Phi = \Phi_{classical} + \Phi_{quantum} = \sum \langle e^{a}\rangle_0 \frac{t^a}{a!} + \sum_{|a|}\sum_{\beta\neq 0} \langle e^a\rangle_\beta \frac{t^a}{a!} e^\beta = \boxed{\sum_{a=(2,0,1),\ a=(1,2,0)} \langle e^a\rangle_0 \frac{t^a}{a!}} + \sum_{|a|}\sum_{\beta\neq 0}\langle e^a\rangle_\beta \frac{t^a}{a!} e^\beta = \frac{1}{2}(t_0^2 t_2 + t_0 t_1^2) + \sum \langle e^a\rangle \frac{t^a}{a!} e^d$

$= \frac{1}{2}(t_0^2 t_2 + t_0 t_1^2) + \sum \langle e_1^{a}, e_2^{b}\rangle_d \frac{t_1^a}{a!}\frac{t_2^b}{b!} e^d$

(Due to $\beta=0$, by degree axiom, only $e_0^2 e_2$ and $e_0 e_1^2$ makes sense) thus $|a|=3$

Quantum part is much harder due to $|a|$ can be large.

due to our pick deformation had shown that t_0 only occurs in $\Phi_{classical}$

By degree axiom again, we know the a, b, d should satisfy a equation for nonvanishing

If you is clever enough, it's $\boxed{b=3d-1}$, our excepted number of pts passed by $d[H]=de_1$

$\Rightarrow \Phi = \frac{1}{2}(t_0^2 t_2 + t_0 t_1^2) + \sum_{a,d}\langle e_1^a, e_2^{3d-1}\rangle_d \frac{t_1^a}{a!}\frac{t_2^{3d-1}}{(3d-1)!} e^d \overset{(**)}{=} \frac{1}{2}(t_0^2 t_2 + t_0 t_1^2) + \sum N_d \frac{(dt_1)^a}{a!}\frac{t_2^{3d-1}}{(3d-1)!} e^d$

$\boxed{\text{divisor axiom}}$

$\underbrace{\langle e_1^a, e_2^{3d-1}\rangle_d}_{a+3d-1} \overset{e_1=[H]}{\underset{\text{is divisor of } \mathbb{P}^2}{=}} \left(\int_d [H]\right)^a \underbrace{\langle e_2^{3d-1}\rangle_d}_{3d-1} = d^a \langle e_2^{3d-1}\rangle_d$

$$\boxed{\Phi = \frac{1}{2}(t_0^2 t_2 + t_0 t_1^2) + \sum_{d\geq 1} N_d \frac{t_2^{3d-1}}{(3d-1)!} e^{dt_1} e^d \quad (*)}$$

(It's not adviced to write $e^{d t_1}$ due to the quantization e^β always have addition coefficient, thus sometimes it's denoted as q^β)

$\boxed{q^\beta = e^{2\pi i \int_\beta (B + i\omega)}}$ by physics

Combining (*) & (**) we can finally have: $\Phi_{222} = \sum_{d\geq 2} N_d \frac{t_2^{3d-4}}{(3d-4)!} e^{dt_1} e^d$; $\Phi_{111} = \sum_{d\geq 1} N_d \frac{t_2^{3d-1}}{(3d-1)!} d^3 e^{dt_1}\cdot e^d$

$\Phi_{112}^2 - \Phi_{111}\Phi_{122} = \sum_d \left[\sum_{d_1+d_2=d} N_{d_1} N_{d_2} d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - \sum N_{d_1} N_{d_2} d_1^3 d_2 \binom{3d-4}{3d_1-1}\right] \frac{t_2^{3d-4}}{(3d-4)!} e^{dt_1} e^d$ ▨

$\Phi_{122} = \sum_{d\geq 1} N_d \frac{t_2^{3d-3}}{(3d-3)!} d e^{dt_1}\cdot e^d$; $\Phi_{112} = \sum_{d\geq 1} N_d \frac{t_2^{3d-2}}{(3d-2)!} d^2 e^{dt_1}\cdot e^d$

[PP] says: quantum formalism has removed any necessity to be understood

Lastly, if you is not clever enough, we can have $\boxed{b=3d-1}$ precisely by degree axiom (It's more general method to compute expected ...

Φ is a homogenous polynomial of deg -2 [illegible] t_0, t_1, t_2 have degree $-2, 0, 2$ but $e^d = e^{d[H]}$, $e^{[H]}$ has degree -6, thus in $t_1^a t_2^b e^d$, we have $a\cdot 0 + b\cdot(-2) + d\cdot(-6) = -2 \Rightarrow b = 3d-1$ done. How to compute these degrees? In general we have

$\deg t_i = \deg(e_i) - 2$ & $\deg e^\beta = 2\int_\beta c_1(X)$ by: $\langle e_0^{a_0} \cdots e_n^{a_n}\rangle_\beta \frac{t_0^{a_0}}{a_0!} \cdots \frac{t_n^{a_n}}{a_n!} e^\beta \Rightarrow \sum a_i \deg e_i = 2\dim X - 2\int_\beta c_1(X) + 2(3g-3+n)$ $(g=0)$

(Write the left deg is the degree in polynomial Φ, not same meaning as right) What is degree of monomial? i.e. $\sum a_i \deg e_i = 2\dim X - 2\int_\beta c_1(X) - 6 + 2(\sum a_i)$

$\underset{t}{\underline{\sum a_i(\deg e_i - 2)}} + \underset{e^\beta}{2\int_\beta c_1(X)} = \underset{\Phi}{\underline{2\dim X - 6}}$

A corollary of these computation is the $QH^*(\mathbb{P}^2)$ and $BQH^*(\mathbb{P}^2)$ can be determined:

We write the multiplication table

$*$	e_0	e_1	e_2
e_0	e_1	e_2	1
e_1	e_2	e_2	0
e_2	1	0	0

$\bullet$	e_0	e_1	e_2
e_0	e_0	e_1	e_2
e_1	e_1	▨	▨
e_2	e_2	▨	▨

By dual basis: $e^0 = \sum g_{i0} e_i = e_2$

$e^1 = \sum g_{i1} e_i = e_1$

$e^2 = \sum g_{i2} e_i = e_0$

thus we can see $\bullet > * > \cup$ contains more intersection datum. (I omit e^d in the table)

$\Rightarrow e_i \cdot e_j = \sum_k \Phi_{ijk} e_{2-k}$ & $e_i * e_j = \sum_k \sum_d \langle e_i, e_j, e_k\rangle_d e_{2-k} \cdot e^d = \sum_d \left(\langle e_i, e_j, e_0\rangle_d e_2 \cdot e^d + \underset{\text{divisor}}{\underline{d\langle e_i, e_j\rangle_d e_1 \cdot e^d}}\right)$

$+ \underset{\text{fundamental}}{\underline{\langle e_i, e_j, e_2\rangle_0 e_0}}$

($\deg e_i + \deg e_j + \deg e_2 \leq 4+4 = 8 < 10 = 4+6 = 2\dim X + 2n$)

$= \sum_d (\langle e_i, e_j, e_0\rangle_d e_2 \cdot e^d + d\langle e_i, e_j\rangle_d e_1 e^d)$

$e_0 * e_0 = \sum_d 0 + \sum_{d=1} \langle e_0, e_0\rangle_d d e_1 = N_1 e_1 = e_1$;

$e_0 * e_1 = \sum_{d=1} \langle e_0, e_0\rangle_d d e_2 + 0 = N_1 e_2 = e_2$;

$e_1 * e_1 = 0 +$ [redacted] $= 0$; $e_2 * \square = \langle \square, e_0\rangle_0 + 0$

due to $\deg \square + \deg e_0 \leq 4+4 = 8$ [illegible] $= 2\dim X + 2n \Rightarrow \square = e_0$ is only case not vanishes

$e_0 * e_0 = N_1 + 0 = 1$. The $\bullet$ can't write out precisely, for example $e_1 \cdot e_1 = \underset{e_1 * e_1}{\underline{e_2}} + \underset{\text{additional datum}}{\underline{\Phi_{111} e_1 + \Phi_{110} e_0}}$ due to except additional datum $\Phi_{222}, \Phi_{112}, \Phi_{122}$, all other Φ_{ijk} is easy to compute by (*): $\Phi_{000} = 0$, $\Phi_{001} = 0$, $\Phi_{002} = 1$, $\Phi_{011} = 1$, $\Phi_{012} = 0$, $\Phi_{022} = 0$ all comes from classical part ▨ [redacted]

ⓐ (Elliptic curve case) [redacted] Due to no nontrivial (nonconstant) map $\mathbb{P}^1 \to E \Rightarrow \beta = 0$ is especially easy:

$\Phi = \sum_{|a|}\sum_{\beta=0} \langle e^a\rangle_\beta \frac{t^a}{a!} e^\beta = \sum_{|a|} \langle e^a\rangle_0 \frac{t^a}{a!}$ classical

(Call quantum part vanishing, what's its physical meaning?)

and computed easily by point-mapping axiom.

let $e_0 = 1, e_1, e_2, e_3 = [pt]$, $\int_E e_1 \cup e_2 = 1$

$\Rightarrow \Phi = \sum_{a=(2,0,0,1),\ a=(1,1,1,0)} \langle e^a\rangle_0 \frac{t^a}{a!} = \frac{1}{2} t_0^2 t_3 - t_0 t_1 t_2$ ▨

[redacted]

Thus we can compute $BQH^*(E)$:

$\bullet$	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	0	e_3	0
e_2	e_2	e_3	0	0
e_3	e_3	0	0	0

$\Phi_{ijk} = \begin{cases} 1; & (i,j,k) = (0,0,3) \text{ or } (0,1,2) \\ 0; & \text{otherwise} \end{cases}$ is easy to determine;

Dual basis $g_{ij} = \int_E e_i \cup e_j = \begin{cases} 1; & i+j=3 \\ 0; & \text{otherwise} \end{cases}$ is $\begin{pmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{pmatrix}$ converse itself

$\Rightarrow e^i = e_{3-i}$ also easy to determine ▨

Rk. Both QH^* & BQH^* is computable by hand, if we know well for the $A^*(X)$, then $QH^*(X)$ & $BQH^*(X)$ follows. But due to the GW-invariants / potential are not so trivial to compute, I don't think them recovers $A^*(X)$ or $H^*(X;\mathbb{Z})$ well (Due to they're even easier to compute).

But the additional intersection/topological datum will tell us more: Can we read more geometric datum from these?

[Ruan] Conjecture. Two minimal model X', X'' of X has $QH^*(X') \cong QH^*(X'')$

(I view it as a generalisation of Castelnuovo's contractibility thm: both of them build property of minimal model with $\Leftrightarrow$ a intersection datum (no (-1)-curve))

Another totally different case is the $(I^X_{1,n,\beta})$, i.e. stable map $E \to X$ called elliptic GW-invariants

I not carry out its computation here but point out some interesting stories:

ⓐ It's first study by physicists in [Bershadsky, Cecotti, Ooguri, Vafa, Holomorphic anomalies in TFT], they introduce a generalisation of $I_{1,n,\beta}$ called Gravitational correlators ("GS" is physical and "correlator" always a correct of symmetry broken. Recently the [illegible] flow theory by Abouzaid is a model make GS without correlations)

ⓑ Return back to enumerative problem: let $N_d^1 = \langle [pt]^{3d}\rangle_{1,d}$ is number of all plane elliptic curves deg$=d$ pass $3d$ pts, the behavior is still good and computation is similar; problem comes when $E \subset \mathbb{P}^3$: let $N_{a,b} = \langle [L]^a, [pt]^b\rangle$ the number of elliptic curves of degree $\frac{a+2b}{4} \in \mathbb{Z}$ (Same number $N_{a,b} = \langle [l]^a, [pt]^b \rangle_{a,b}$ for $g=0$ rational curves $C \subset \mathbb{P}^3$, $N_{a,b}$ behaves well too) passing a lines & b pts. However, it does virtual and far away to be a true enumerative number! All $N^1_{a,b} \in \mathbb{Q}_{<0}$ and even integral is few. Finally is found that $E^1_{a,b} = N^1_{a,b} + \frac{2d-1}{12} N_{a,b}$ is the true number for this problem. Next I'll give some examples of GW not enumerative

③ (Not enumerative GW-invariants) • $I_{1,\emptyset,1}$ the genus 1 GW-invariant, degree 1 map from tori to $\mathbb{P}^1$ (elliptic curve/$\mathbb{C}$ = tori/$\mathbb{C}$) as we shown in ② no $E \to \mathbb{P}^1$ algebraic/J-holomorphic, but there $\exists!$ stable map gives the classical part of ⑤ in ②, thus $I_{1,0,1} \neq 0$, what's it is counting? it counts nothing due to no $E \to \mathbb{P}^1$.

• Another example is GW to be negative: in evaluation the result negative is not surprising (negativity $\Leftrightarrow$ rigidity; Fraction $\Leftrightarrow$ automorphism)

Let $X = Bl_0\mathbb{P}^3 \supset E$ exceptional divisor, $\langle [pt], [E]^2 \rangle_{[l]} = -1$, $[l]$ is pullback of line $\subset \mathbb{P}^3$. (*)

(*) is proven by $\langle [pt], E^2, E^2 \rangle_{[l]} \overset{\text{Splitting}}{=} \langle [pt], E, E \rangle_{[l]-[E]} \cdot \langle E^2, E^2, E \rangle_{[E]}$ (Where $[l] = ([l]-[E]) + [E]$ is the only decomposition makes $\langle E^2, E^2, E \rangle_{[E]}$ nonzero) $= 1 \cdot (-1) = -1$ is negative

• Taubes–Gromov–Witten as a generalisation of GW, for S general type surfaces and $g(K_S) = 1 + K_S^2$, $n=0$ marked no pts has T_{GW} is $(-1)^{\chi(\mathcal{O}_S)}$ not enumerative.

More computation tools of GW-theory

We study the reconstruction thm by [KM]; then introduce equivariant tools via Atiyah–Bott localisation by [Givental] & later (about 2005) virtual localisation by [Graber & Vakil] first by Kontsevich

Rk. • One may surprise at the presence that $N_{a,b}$ occurs in the number $E^1_{a,b}$: it means that the 0-dimensional components of $\overline{M}_{1,n}(\mathbb{P}^3, \frac{a+2b}{4}[l])$ have $N^1_{a,b}$ components, but $N^1_{a,b} = E^1_{a,b} + \frac{1-2d}{12} N_{a,b}$

$E^1_{a,b}$ is actually $E^1_{a,b}$-number of 0-dim components

$\frac{1-2d}{12} N_{a,b} (<0)$ is virtual, consists the $\frac{a+2b}{4}$ cover of lines $\subset \mathbb{P}^3$ which is rational (I'm still confused)

• Another example showed later is Quintic threefold in MS, we'll study it in next section.

Q deg $\int [\overline{M}_{0,0}(Q, 2[l])]^{vir} = \underbrace{\frac{4876875}{8}}_{\text{virtual}} = \underbrace{609250}_{\text{true}} + \underbrace{\frac{2875}{8}}_{\text{additional}}$; deg 2

additional part is

$2875 = \int [\overline{M}_{0,0}(Q, [l])]^{vir}$

$= \int [\overline{M}_{0,0}(Q, [l])]$ is degree 1 case comes from 2-cover of $\mathbb{P}^1 \subset Q$.

• Another another is GW-invariant is local for $\mathbb{P}^1$ ⊂ Q: $g=0$ & deg $1/3$; but for $g=0$ & deg $=2$

$GW = -\frac{45}{8} = \frac{3}{8} - 6$

(negative show rigidity)

Thm ([First reconstruction]) X satisfy $\boxed{A^*(X) \text{ generated by } A^1(X)}$ (or $H^*(X)$ generated by $H^2(X)$)

$\{I^X_{0,n,\beta}\}$ is uniquely reconstructed by $\{I^X_{0,3,\beta} \mid (-K_X.\beta) \le 2 + \dim X + 1\}$

Furthermore, is uniquely reconstructed by classes (special values) of $I^X_{0,3,\beta}$ of $A^*(X)$ generated, or more easily taking a basis. this $\beta \in A^1(X)$ concerning

is $\left\{ I^X_{0,n,\beta}(\gamma_1 \otimes \gamma_2 \otimes \gamma_3) \,\middle|\, \begin{array}{l} (-K_X.\beta) \le \operatorname{codim}_X + 1 \\ \Sigma \deg \gamma_i = 2(-K_X.\beta) + 2\dim X,\ \deg \gamma_3 = 2 \end{array} \right\}$

Rk. ① For $X = \mathbb{P}^N$, $H^*(\mathbb{P}^N)$ generated by $H^2(\mathbb{P}^N)$ is obvious, and then all $I^X_{0,n,d}$ can be reconstructed by only $\langle I_{0,2N,1} \rangle ([l]^N, [l]^N) = 1$ (initial value). If $H^*(X)$ generated, then $\beta \in H^2(X)$... the algebraic part $\beta \in A^1(X)$ can be reconstructed ← we call them restricted-GW

② We concern MS thus we look at X Calabi–Yau, i.e. $K_X = 0$, thus we are lucky the relation $(-K_X.\beta) = 0 < 2\dim X + 1$ naturally holds. but unlucky thing is $H^*(X)$ never (why?) generated by $H^2(X)$. Thus we can only do reconstruction for $A^*(X)$-level.

Pf. $\left\{ \begin{array}{l} \boxed{\text{Case I}}\ I^X_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n) \text{ all codimension} \ge 1: \text{By splitting principle} \\ \boxed{\text{Case II}}\ I^X_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n) = \langle \gamma_1, \ldots, \gamma_n \rangle_\beta [X] \text{ codim } 0: \text{reduce to number } \langle \gamma_1 \cdots \gamma_n \rangle_\beta \text{ case} \end{array} \right.$

③ Firstly we note the condition $\deg \gamma_3 = 2$ need to argue ($\Sigma \deg \gamma_i = 2(-K_X, \beta) + 2\dim X = 2\int c_1(X) + 2\dim X$ is degree axiom naturally)

② is hardest, due to it's totally a technical computation, I omit here (See [KM]) (One need a $\boxed{\text{quadratic relation}}$ similar to WDVV)

① For our use, here we partion $\Delta^n = \{1, \ldots, n\}$ into $S_1 \sqcup S_2$, $S = (S_1, S_2) \Rightarrow$ denote $F_S: \overline{M}_{0,S_1 \sqcup \bullet} \times \overline{M}_{0,S_2 \sqcup \bullet} \xrightarrow{\text{glue}} \overline{M}_{0,n}$

we need apply splitting axiom thus need $\beta \neq 0 \Rightarrow \boxed{\bigcap \operatorname{Ker}(F_S^*) = H^{2(n-3)}(\overline{M}_{0,n})}$ (*) (i.e. it only stops at $n=3$, all $n \ge 4$ not meet the kernel, $\exists$ portion S s.t. $\beta \neq 0$)

To prove (*), $\forall a \in H^*(\overline{M}_{0,n}) - H^{2(n-3)}(\overline{M}_{0,n})$ i.e. $\deg(a) < 2(n-3)$

each portion S determines a component of boundary $\partial \overline{M}_{0,n}$ divisor $\Rightarrow$ denoted $D_S \in H_*(\overline{M}_{0,n})$, by Poincaré dual, it has a dual class (volume form of normal bundle $N_{D_S/\overline{M}_{0,n}}$) $d_S \in H^2(\overline{M}_{0,n}) \Rightarrow (F_S)_*(F_S)^*(a) = a \wedge d_S$ is restrict a form to the normal bundle now $\deg(a) < 2(n-3) \Rightarrow \exists$ a nonconstant monomial $d = \prod d_{S_i}$, s.t. $\int_{\overline{M}_{0,n}} a \wedge d \neq 0 \Rightarrow a \notin \bigcap_S \operatorname{Ker}(F_S^*) \supset \bigcap_S \operatorname{Ker}(F_S^*)$

$\Rightarrow a \notin \bigcap_S \operatorname{Ker}(F_S^*)$ ▨

③ Otherwise $\deg \gamma_3 \ge 4$, we repeat ① to γ_3 ▨

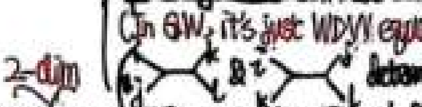
Rk. (Second reconstruction thm) A elegant interpretation of reconstruction thm in physics, also by [KM] relates the CohFT and ACF (abstract system of correlation functions) in tree level: $\boxed{\text{Any tree level ACF consists CF of } \exists! \text{ tree level CohFT}}$ (☆)

Where tree level ACF/K is materials (In GW, it's $\int e_i \cup e_j$)

$(A, \Delta) \left\{ \begin{array}{l} A \text{ is } K\text{-linear superspace (In GW, } A \text{ is the cohomology) equipped a even nondegenerate pairing (respect to basis } (e_i), g_{ij} = \langle e_i, e_j \rangle \text{)} \\ A \text{ is the Casimir element of } \langle -,- \rangle \text{ (i.e. } \Sigma g^{ij} e_i \otimes e_j \text{)} \in A \otimes A \end{array} \right.$ (consists fields in physics) — (M_1)

$(Y_n)_{n \ge 3}$: $Y_n: A^{\otimes n} \to K$ even linear maps — (M_2)

satisfy axioms $\left\{ \begin{array}{l} S_n\text{-covariance of } Y_n, \forall n \ge 3 \\ \text{Coherence axiom } \sum_{S} \sum Y_{|S_1|+1}(\gamma_{i_1} \otimes \cdots \otimes \gamma_{|S_1|} \otimes e_a) g^{ab} Y_{|S_2|+1}(e_b \otimes \gamma_{|S_1|+1} \otimes \cdots \otimes \gamma_n) \\ \text{(In GW, it's just WDVV equation's 3-tuple sum over all } \beta \text{)} \end{array} \right\} \to (A)$



determined $\sum_{\dots} \sum_{a,b} Y_{|S_1|+1}(\gamma_1 \otimes \cdots \otimes \gamma_{|S_1|} \otimes e_a) g^{ab} Y_{|S_2|+1}(e_b \otimes \gamma_{|S_1|+1} \otimes \cdots \otimes \gamma_n)$

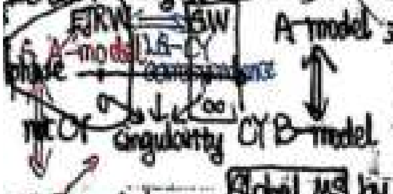
Tree level CohFT (different partion S & S' of Δ^n) (CohFT can be defined in $I_{g,n}$ higher genus, but need adding genus reduction) is same material $(M_1) + (M_2)$ $I_n: A^{\otimes n} \to H^*(\overline{M}_{0,n}, K)$ (in GW, I_n is GW-class & Y_n is GW-number)

(A) is S_n-covariance of I_n, $\forall n \ge 3$ & Splitting instead of Coherence (I omit its precise formula due to it's same as GW)

Correlation functions (CF) of CohFT are $A^{\otimes n} \to H^*(\overline{M}_{0,n}, K) \to K$ satisfy some conditions (See [KM]). Thus (☆) says any pure algebraic datum given by ACF factor through a geometric/topological object $H^*(\overline{M}_{g,n};K)$. [redacted]

Now we can answer: what is the place of GW-invariants in the framework of MS?

[Witten, 2-dim gravity and intersection theory on moduli space] is standard reference/origin of MS of CohFT: it says via the mirror varieties, the A-model CohFT and B-model CohFT interchanges. Where A&B-model are two different CohFTs, A-model is symplectic, i.e. not depend on the deformation of complex structure; B-model is complex & not depend on symplectic form. (B-model is more subtle, the rigidity is more than not depend on ω). GW-model is one of the A-model (although we had shown a perfect algebraic description) and at least it shows the existence (By Hodge theory, B-model's structure is well-known and we don't worry about this). But the most important thing is: all A-model will approximate GW-model.



A-model, where Fan-Jarvis-Ruan-Witten theory in LG side gives another (2-dim) CohFT.

(We can see Kontsevich again, here he is the leading one we LG generalize MS to not CY case)

Global MS by [Chiodo & Ruan, 2013]

It's removable in this case, it's easier

Equivariant techniques (Sketch) Recall equivariant cohomology $H^*_G(X;R)=H^*(X\times_G EG;R)$ [R is G-module / Z[G]-module]

Thus G-action trivial $\Rightarrow H^*_G(X;R)\overset{EG\to BG\text{ is universal family}}{=}H^*(X;R)\otimes R[\lambda]$, λ is the trivial action. $X\times_G EG=(X\times EG)/\sim$, relation $(x,p\cdot g)\sim(gx,p)$ for $p\in EG$, $x\in X$, $g\in G$ action on both X and EG.

G-action free $\Rightarrow H^*_G(X;R)=H^*(X/G;R)$

Equivariant Chow groups $A^G_*(X)=A_*(X\times_G EG)$, also for X projective variety/$\mathbb{C}$.

But it's not true for general scheme, but makes sense in our setting next:

Generally: $A^G_i(X)=A^*(X_{red}\times_G U)[g-l]$ for $\dim G=g$, $\dim V=l$, V is G-rep s.t. $\exists U\subset V$, $G\times U\to U$ [illegible] of $EG\to BG$ finite-dim approximation

[The existence of such G-rep V is a problem and they're built by [Edidin, equivariant intersection theory]]

The issue in algebraic setting to be hard is • scheme structure on G • nonreduced structure on X. Now there to make difficult to original.

Set X cpx mfd or [illegible] with $\mathbb{C}^*\times X\to X$ holomorphic action & $\mathbb{Q}$-coefficient. Next setting we only do for $H^*_{\mathbb{C}^*}$, but they all make sense generally on $A^{\mathbb{C}^*}_*$ also, see [Edidin] has all these construction.

$E\to X$ $\mathbb{C}^*$-principal bundle has equivariant Chern class $C^{\mathbb{C}^*}(E)\in H^*_{\mathbb{C}^*}(X;\mathbb{Q})$

And thus we have famous Atiyah-Bott localisation to connected components $F\subset X$ of the fix locus of $\mathbb{C}^*\curvearrowright X$

$C(E)\in H^*(X;\mathbb{Q})$

Due to $B\mathbb{C}^*=\mathbb{C}P^\infty$, $E\mathbb{C}^*=\mathbb{C}^{\infty}-\{0\}$, $H^*(\mathbb{C}P^\infty)=\mathbb{Q}[\lambda]$ [illegible]

$$C^{\mathbb{C}^*}=\sum_{i:F\subset X}i_*\frac{C^{\mathbb{C}^*}|_F}{C_{top}(N_{F/X})}\ (\text{mod. torsion})$$

or $$[X]=\sum_{i:F\subset X}i_*\frac{[F]}{C_{top}(N_{F/X})}\quad(*)$$

& Here our use is relative localisation

$$\sum_{F^\bullet}(f|_{F^\bullet})_*\left(\frac{C^{\mathbb{C}^*}|_{F^\bullet}}{C_{top}(N_{F^\bullet})}\right)=\frac{(f_*C^{\mathbb{C}^*})|_F}{C_{top}(N_{F/Y})}\quad(**)$$

$X\xrightarrow{f}Y$, $\mathbb{C}^*$-equi means $\mathbb{C}^*$-action on X passes to Y. $f^{-1}(F)\to F$ connected components of fixed locus, $F^\bullet$ connected components of $f^{-1}(F)$

(They can be viewed as "equivariant excess intersection formula")

Stop at here. It's enough to compute via them for g=0 case, as what done by Kontsevich and Givental, separately (they're inspired by MS). I list some (famous) consequences can (! not first may) figure out by (*) without pf due to it's hard computation; I think they're ugly and omitted them.

• Vafa-Intriligator formula $\int_{Gr(m,n)}p(C_1(\mathcal{O}(1)),\dots,C_m(\mathcal{O}(1)))=\frac{(-1)^{\binom{m}{2}}}{m!\,n^m}\sum_{\xi_i^n=1}p(\sigma_1(\xi),\dots,\sigma_m(\xi))\,\sigma_m(\xi)\prod_{i\neq j}(\xi_i-\xi_j)$, σ_j is jth elementary symmetric polynomial

(Write left integral over $\mathbb{C}^*$, then by Bott's residue thm, then left computation. $\int p(C)=\int p(C^{\mathbb{C}^*})=\sum\int\frac{p(C^{\mathbb{C}^*})|_F}{C_{top}(N_F)}$ (See Bertram's note))

Rewrite (*) to virtual: can we have the form [redacted]

$$[X]^{vir}=\sum_{i:F\subset X}i_*\left(\frac{[F]^{vir}}{C_{top}(N^{vir}_F)}\right)\ ?\quad(**)$$

The answer is optimistic. Consider X is moduli space, obvious we can't define it for general Artin stacks, we need conditions ① Algebraic ② $X\xrightarrow{\text{equi}}X$ orbifold/DM ③ $[F]^{vir}$ arises from $\mathbb{C}^*$-fixed part of $E^\bullet|_F$ obstruction theory (Expectation: ①② removable)

Let $X=\overline{M}_{g,n}(\mathbb{P}^1,d)$, (**) is called relative virtual localisation desired. [redacted]

(It's "relative" not come from (**), but $\overline{M}_g(\mathbb{P}^1)$, $\partial^\bullet$ denote it's moduli of relative stable map constructed next)

• Faber conjecture / Faber's type. The tautological ring $R^*(\overline{M}_g)$ is similar to $H^{2*}(X;\mathbb{Q})$, $\exists X$ (g-2)-dim cpx proj mfd.

(Recall structure of $\overline{M}_{g,n}$: kappa class / Mumford-Morita-Miller κ-class $\kappa_i=\pi_*(\psi^{i+1})$, ψ the psi class $=C_1(\mathcal{L})$, $\pi:\overline{C}_{g,n}\to\overline{M}_{g,n}$ & restricts to each curve $\mathcal{L}|_{[C]}=\omega_C$. Then $R^*(\overline{M}_g)=A^*(\overline{M}_g)[\kappa_i]\subset A(\overline{M}_g)$)

① Vanishing: $R^i(\overline{M}_g)=0$, $i>g-2$, $R^{g-2}(\overline{M}_g)\cong\mathbb{Q}$

② Perfect pairing/Poincaré dual: $R^i(\overline{M}_g)\times R^{g-2-i}(\overline{M}_g)\to R^{g-2}(\overline{M}_g)\cong\mathbb{Q}$

③ Intersection: $\frac{(2g-3+n)!(2g-1)!!}{(2g-1)!\prod(2d_j+1)!!}\kappa_{g-2}=\sum_{\sigma\in S_n}\kappa_\sigma:=\sum\left(\prod\kappa_{d_{\alpha_1}+d_{\alpha_2}+\dots+d_{\alpha_j}}\right)$, $\sigma=(\alpha_1\dots\alpha_{i_1})\dots(\alpha_{k_1}\dots\alpha_{k_i})$ cycle decomposition

(For $\overline{M}_{g,n}$, similar)

It's still a conjecture now [illegible] (2024.11), partially [Looijenga; Faber]; On the tautological ring of $\overline{M}_{g,n}$ proved ① by degeneration & use (*); omitted.

Next we describe relative stable map to $\mathbb{P}^1$.

this technique is developed by Jun Li, he has a note on relative GW theory deal these in detail.

The relative stable map is interesting & naturally arose in degeneration: $W\to B$ is family, W_b projective & smooth, $b\neq b_0$, $W_{b_0}=Y_1\cup Y_2$, $Y_1\cap Y_2$ transversally on divisor D, $D\subset Y_1$ & $D\subset Y_2$.

Thus $\overline{M}_{g,n}(W_b/S_b,d)=\coprod\overline{M}_{g,n}(W_b,d)\to S$ proper, restrict to $W^{S^*}=W\times_B(B-b_0)$, $S^*=S-b_0$

family $\Rightarrow$ What's the central fibre of this family?

i.e. Can we extend $\overline{\mathcal{M}}_{g,n}(W^*/S^*;d) \to S^*$ to $\overline{\mathcal{M}}_{g,n}(W/S;d) \to S$?

But it fails: although $W_0 = Y_1 \cup Y_2$ transversally, the central fibre can't have normal crossing singularity, even in virtual sense. The reason is the map send a irreducible component send to $Y_1 \cap Y_2$'s singular loci, thus degenerate. We call they're degenerate stable maps if they $\subset$ singular loci of W_0 and the central fibre is the moduli of relative stable maps.

Construction is by a similar way of X[n] before: by a sequence of blow-up to resolve the singular loci $Y_1 \cup Y_2$, see [11] for details. Here we consider to $\mathbb{P}^1$ case & relative [illegible]. Then the moduli of relative stable maps to $\infty \in \mathbb{P}^1$ & relative GW-invariant construction is similar.

(generally it's relative to Divisor $D \xrightarrow{f} (X, D)$ in the degeneration case above D is then taken to $Y_1 \cap Y_2 \subset Y_1 \cup Y_2$. One put condition of $f^{-1}(D)$ to avoid the degeneration stable maps

can't have $f^{-1}(D) = \text{all } C$)

$(C, p_1 \cdots p_n, q_1 \cdots q_m) \xrightarrow{f} (\mathbb{P}^1, \infty)$, $f^*\infty_T = \sum_i a_i q_i$ $(\sum a_i = d, \vec{a} = (a_1 \cdots a_m))$

$f_1^{-1}(n_j)$ also nodes $\in C$

$$\begin{tikzcd}
(C,p_1\cdots p_n,q_1\cdots q_m) \arrow[r, "f"] \arrow[dr, "f_1"] & (\mathbb{P}^1,\infty) \\
& (T,\infty_T) \arrow[u, "f_2"]
\end{tikzcd}$$

$T_1 \quad T_2 \quad T_3 \quad T_4$ not meet T_{i+1}

$\mathbb{P}^1 = \mathbb{P}^1 - \mathbb{P}^1 = \mathbb{P}^1 = \mathbb{P}^1$

Stability $\Leftrightarrow$ Geometric genus 0 : 3 special pts
Geometric genus 1 : 1 special pt.
send to constant by f_1

(Why geometric genus 1? Absolute case it's arithmetic genus)

$n_j = \infty_{T_j} = 0_{T_{j+1}}$ (A new condition) | $f_1(C_{T_j}) \to T_j$ not all the form $[x:y] \mapsto [x^{g}:y^{g}]$ for some $g \in \mathbb{Z}_{>0}$

Modulo isomorphism given by $(C, p_i, q_i) \cong (C', p_i', q_i')$

$$\begin{tikzcd}
(T,\infty_T) \arrow[r, "\cong"] \arrow[d] & (T',\infty_{T'}) \arrow[d] \\
(\mathbb{P}^1,\infty) \arrow[r, equal] & (\mathbb{P}^1,\infty)
\end{tikzcd}$$

We have $\overline{\mathcal{M}}_{g,n,\vec{a}}(\mathbb{P}^1, D)$ compact then relative GW follows.

Study $\overline{\mathcal{M}}_{g,n}$ for Faber's type conjecture we reduce to $\overline{\mathcal{M}}_{g,n,\vec{a}}(\mathbb{P}^1, D)$ to solve it, mainly

This gives $\mathbb{C}^\times$-action on each $\mathbb{P}^1$ glue to a global $\mathbb{C}^\times$-action, then extend to $\mathbb{C}^\times$-action on f globally

Why we need this?

Now we have two stories: Virtual & Derived. $[X]^{vir} \neq [X]$ most cases; what about the derived thicken $\tilde{X}$? We know $\tilde{X}$ contains all information datum desired, that expect $[X]^{vir}$ access to do!

Further developments & counting curves. X Calabi-Yau 3-fold here & next section

Q. Transversality / Virtual? In symplectic / topological, perturbation is easy (by Hamiltonian) to get the virtual ones; it's not allowed in algebraic case, but an algebraic analogue of symplectic perturbation. Thus virtual number computed, we not concern transversality.

Our problem of counting (virtual) number of curves $\Leftrightarrow$ compactification of $M(X, \beta)$. Stable map is only one of these choices.

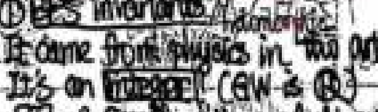
To describe compactification, first we need to describe $C \subset X$ parametrized $C \to X \Leftrightarrow$ worldsheet of string moving in X

We'll introduce some realisations of second way of compactification unparametrized: zero locus of several equations (stable map is the first way)

D-brane embedding in X real curve

Stable pairs

Other methods: unramified stable maps another first way; stable quotient in both ways; tropical method by Gross-Siebert program can be found in [$\frac{B}{2}$]



(They'll be the leading to MS. First, the curious first exploding of how these invariants in MS gives tools in enumerative geometry)

① BPS invariants It came from physics in two articles [Gopakumar & Vafa: M-theory and topological string, I & II]

It's on integer! (GW is $\mathbb{Q}$)

$\mathcal{M}$ the moduli space of D-branes supported on C, $[C] = \beta$ fixed $=$ the moduli space of stable sheaves on C

$[C] = \beta$ fixed, $\chi(E) = 1$ (it means holomorphic)

Such decomposition depends on expected/virtual dim of $\overline{\mathcal{M}}_g(X, \beta)$ is 0, thus it should be $B \subset \overline{\mathcal{M}}_g(X, \beta)$

$B \ni \text{supp} E = C_E \subset X$ curve, $[C_E] = \beta$, $B = \overline{\mathcal{M}}(X, \beta)$ parameter all these curves (Chow varieties)

BPS is not integral in 3-fold! (the definition to β isn't easy to depict)

$\overline{\mathcal{M}}_g(X, \beta)$ decompose to $B = \coprod \overline{\mathcal{M}}_g(X, \beta)$, each $\mathcal{M}_g = \mathcal{M}_g(X, \beta) \times \mathcal{M}$ & $\mathcal{M} = \coprod \mathcal{M}_g$

Thus the Poincaré polynomial $\sum_{g \geq 0} P(\mathcal{M}_g)(y) = y^{-g}(1+y)^{2g}$ for $\mathcal{M}_g(X, \beta)$ smooth case

BPS invariant counts genus g curves; class β is $n_{g,\beta}$, satisfy $\sum n_{g,\beta} y^{-g}(1+y)^{2g} = P(\mathcal{M})$ is some formula given by [Hosono-Saito-Takahashi]

(fibres are Jacobian of C, genus g is real-dim)

BPS conjecture I. We define $n_{g,\beta}$ by (*) $\sum_{g \geq 0} N_{g,\beta}^{GW} u^{2g-2} v^{\beta} = \sum_{g \geq 0, \beta > 0} n_{g,\beta} u^{2g-2} \sum_{d > 0} \frac{1}{d} \left(\frac{\sin(\frac{du}{2})}{\frac{u}{2}} \right)^{2g-2} v^{d\beta}$

u, v are variables formally, $N_{g,\beta}^{GW}$ is counted by GW

then • $n_{g,\beta} \in \mathbb{Z}$ • g large $n_{g,\beta}$ vanishes

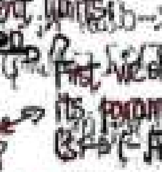
② DT invariants Instead of using g to classify curves, we use χ, i.e. Hilbert scheme to classify them (of course it's far from complete classification, thus it's much coarser than genus case) (This is the most difficulty here

Hilbert scheme is also hard to study (it's reduced pt, but it has stacky issue (orbifold)

Stable map $\Leftrightarrow$ parametrized
Hilbert scheme $\Leftrightarrow$ embedded

Consider their different limits
$C_t = \{t x^2 + t y = 0\} \xrightarrow{t \to 0}$

First, we should write down its parametrized form $(\mathbb{C}^*, t) \to (t s, s^2)$ by a $\mathbb{Z}/2\mathbb{Z}$-stabilizer



$DT(X; n; \beta) = \int 1$ $[I_n(X;\beta)]^{vir}$

Stable map

$I_n(X; n; \beta)$ the Hilbert (sub) scheme parametrised all curves, $\chi(\mathcal{O}_C) = n$ and $[C] = \beta$

Hilbert scheme: $\{x y = 0\}$ of course, its limit

(has a embedded pt on X)

of flat family (Sec II.4.1)

not confused on the "$y < 0$" part, as here it's $x^2 = t y$

(It's a baby case, view as Quot scheme / parameter sheaves $\rightsquigarrow$ parameter derived object. We'll consider as stable locus)

Our problem is deformation / obstruction theory: $\dim X \leq 3$, things are classical & works well (before 2010), the problem is for higher dimension, the obstruction not perfect! After 2010, several papers by Joyce generalises to CY 4-fold: the classical obstruction is perfect amplitude $[-2, 0]$, how to cut? Joyce uses shifted symplectic structure to realize this, limited. Now we only [illegible] 3-fold case (3-fold & 4-fold is contained in Donaldson-Thomas's [illegible] is more [illegible] in MS

Due to I'm not familiar with theory of moduli of sheaves (even Quot scheme), I omit the details.

MNOP conjecture: DT-GW correspondence (A problem open is symplectic side of DT theory is still not clear / Prediction is DT does symplectic invariant?)

[Maulik, Nekrasov, Okounkov, Pandharipande: GW theory and DT theory, I & II]

The reduced generating function of DT = The generating function of disconnected GW (up to a reparametrization $q = -e^{iu}$)

$$Z_\beta^{red}(q) = \frac{Z_\beta^{DT}(q)}{Z_0^{DT}(q)} = \frac{\sum DT(n,\beta) q^n}{\sum DT(n,0) q^n} \qquad \boxed{q = -e^{iu}} \qquad Z_\beta^{GW}(u) = \sum_g N_{g,\beta}^{GW} u^{2g-2}$$

$Z_\beta^{DT}(q)$ called the DT partition function. $N_{g,\beta}^{GW}$ is the number virtually counted by disconnected GW: without curve connected restrict, but stable map can't contract hole connected component (Also expected dim 0 on X)

the reduced part $Z_\beta^{red}(q)$ makes [MNOP] hard to write geometrically.

$C \longmapsto \mathcal{I}_C$ ideal sheaf, $\det(\mathcal{I}_C) = \mathcal{O}$, $ch(\mathcal{I}_C) = 1 + 0 + (-\beta) + (-n)$ $\in H^0(X) \oplus H^2(X) \oplus H^4(X) \oplus H^6(X)$; $\mathcal{F} = \mathcal{I}_C$; $\exists$ embedding $\mathcal{F} \hookrightarrow \mathcal{O}_X$; is fixed $\Rightarrow$ $\boxed{\text{Stable}}$; $\mathcal{F}$, $ch(\mathcal{F})$ fixed

Deformation is $Ext^1(\mathcal{I}_C, \mathcal{I}_C)_0$

Obstruction is $Ext^2(\mathcal{I}_C, \mathcal{I}_C)_0$

0 means fixed determinant part

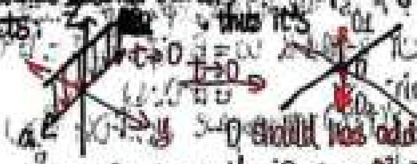
③ Stable pairs. This is the best enumeration for 3-fold. (But same limit as DT case)

DT is flat limit of $\mathcal{I}_{C_t} \xrightarrow{t \to 0} \mathcal{I}_{C_0}$, identify $0 \to \mathcal{I}_{C_t} \to \boxed{\mathcal{O}_X \to \mathcal{F}_t \to 0}$ for some kernel $\mathcal{F}_t$, then take limit of $s: \mathcal{O}_X \to \mathcal{F}_t$ (may not surjective anymore!), then $s: \mathcal{O}_X \to \mathcal{F}_0$ will determine a new limit C_0', differs from C_0 by Hilbert scheme directly

See example comparing with three limits: $C_t = \{x = z = 0\} \cup \{y = z - t = 0\}$ union of two lines $\subset \mathbb{A}^3_{\mathbb{C}}$, what's their limit in three cases?

[redacted]

① Stable map case: Parametrize second factor / just separately can only have nothing: the problem occurs when intersects: ..., this it's



normalization of $\mathcal{O}_{C_0}$ ... $(\mathcal{O}_{xy}, z)$

② Hilbert scheme case.

$\mathcal{I}_{C_t} = (x, z)(y, z - t)$

$= (xy, x(z-t), yz, z(z-t))$

$\xrightarrow{t \to 0} (xy, xz, yz, z^2) = \mathcal{I}_{C_0}$ desired. (nonreduced!)

0 should has additional structure

the $C_0 \hookrightarrow \mathbb{A}^3$ is embedding out 0

③ Stable pair case. $C_t = C_t^1 \cup C_t^2 \Rightarrow 0 \to \mathcal{I}_{C_t} \to \boxed{\mathcal{O}_X \to \mathcal{O}_{C_t^1} \oplus \mathcal{O}_{C_t^2}}$, $0 \longmapsto 0$ is double $\Rightarrow$ two origin O_1 & O_2 (due to viewed as $\mathbb{A}^1 \sqcup \mathbb{A}^1 \to \mathbb{A}^3$ always double)

Thus $\mathcal{O}_{C_0^1} \oplus \mathcal{O}_{C_0^2}$ determines $\{x = z = 0\} \cup \{y = z = 0\}$ is the simplest: Due to we take $\mathcal{F} = \mathcal{O}_{C_t^1} \oplus \mathcal{O}_{C_t^2}$ separate two lines lost intersection datum. (lost datum is the cokernel: $0 \to \mathcal{I}_{C_0} \to \mathcal{O}_X \to \mathcal{O}_{C_0^1} \oplus \mathcal{O}_{C_0^2} \to \boxed{\mathcal{O}_0} \to 0$ is origin)

Precise definition of stable pairs can then be given: $(\mathcal{F}, s)$ is stable pair on X smooth proj 3-fold

iff
- $\mathcal{F}$ is pure, i.e. supp $\mathcal{F}$ is pure dimensional / equidimensional
- $\mathcal{F}$ has dim 1 support C • $s \in H^0(X; \mathcal{F})$, s has zero dim cokernel

These datum gives $0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{F} \to \text{coker}\, s \to 0$

curve C with a zero-dim subscheme ... (not 1-dim) & their ...

Fix $\chi(\mathcal{F})$ and $[C]$ same as DT $\Rightarrow$ moduli of stable pairs $P_n(X; \beta)$ as the example shows, it should be much smaller

For details of construction, see original paper [Pandharipande & Thomas: Curve counting via stable pairs in the derived category]

With similar deformation theory $Ext^1(I^\bullet, I^\bullet)_0$ and obstruction theory $Ext^2(I^\bullet, I^\bullet)_0$, $I^\bullet = [\mathcal{O}_X \xrightarrow{s} \mathcal{F}] \in D^b(X)$, we can define virtual class of $P_n(X; \beta) \Rightarrow P_{n,\beta}^{PT} := \int 1$ the stable pair invariants & $Z_\beta^{PT}(q) = \sum P_{n,\beta} q^n$ stable pair partition function

DT-PT correspondence. $Z_\beta^{PT}(q) = Z_\beta^{red}(q)$ ... $[P_n(X; \beta)]^{vir}$

$\Leftrightarrow \sum_m P_{n-m,\beta} \cdot DT(m; 0) = DT(n, \beta)$ by comparing coefficients: this is equivalent to a wall-crossing formula under changing stability condition. (For example: change phase of Bridgeland stability / θ-stability)

BPS conjecture II. Rewrite $Z^{PT}(q, v) = Z^{PT}_{classical}(q, v) + Z^{PT}_{quantum}(q, v)$

$:= 1 + \sum_{\beta \neq 0} Z_\beta^{PT}(q) v^\beta = \exp\left(\sum_{r, \beta \neq 0} \sum_{d \geq 1} n_{r,\beta} \frac{(-1)^{r-1}}{d} \cdot (-q)^{d(1-r)} (1 - (-q)^d)^{2r-2} v^{d\beta}\right)$

from PT to DT (different moduli, thus different GI) gives $n_{r,\beta}$

On the otherhand, physically BPS state counting gives same but

Conjecture says that $n_{r,\beta} = 0$ for $r < 0$ $\Leftrightarrow$ similarly, rewrite it as $\sum$ ... $\boxed{DT(n, \beta) - P_{n,\beta}}$ is description

wall-crossing ... called a wall in space of stability condition ... codim 1 ... possible

I'd like tell more story on wall-crossing due to ... is an excellent example (quite primary!)

§2.5. Geometry of CY 3-fold (Quintic 3-fold) and Geometry of MS

What is MS? (Not the global sense each CY, here only classical CY case)

A-model	B-model
Symplectic Geometry	Complex algebraic Geometry
Enumerative Geometry	Hodge theory
$F(X)$	$F(X^\vee)$
SYZ	
T-duality	
$\int_{\Sigma} ...$ PT	... PT

Only rigidity of symplectic: this the enumerative is more proper

Kontsevich's HMS – two geometric statements (Due to his statement still vague)

physical

all these in table is 2-dim dual

SYZ concerns special fibres of Lagrangian fibration (but even existence is a problem) singular fibres $\longleftrightarrow$ change of invariants $\longleftrightarrow$ wall crossing, ... codim also have their ... project to ... base and these real-codim-1 walls project to real mfds called tropical disk. thus SYZ modernly is ⊏ Gross–Siebert program

• What we need to do? ... set configuration space ... parameters ... moduli $\mathcal{M} \subset \mathcal{C} \times \Lambda$ sub ... (We can identify $\mathcal{M}$ as ...) $\mathcal{M}_\lambda$ associate some invariant ... depends on λ

"Strongly": $\exists W_i \subset \Lambda$ real codim 1 subvariety walls ... $\Lambda \to \mathbb{R}$ continuous on $\Lambda - \bigcup W_i$ ... locally constant, i.e. constant on connected component on $\Lambda - \bigcup W_i$, called chambers

• Take $\mathcal{C} = \mathbb{R}$, $\Lambda = \mathbb{R}^2$

$\mathcal{M} = \{(t, b, c) \in \mathbb{R} \times \mathbb{R}^2 \mid t^2 + bt + c = 0\}$

$e(b, c) = \#\mathcal{M}_{(b,c)}$ = number of roots

In this section, we'll prove various propertys of CY $X^3 \subset \mathbb{P}^4$ to describe the "local MS", and sketch further developments of MS

Evolution of physics to MS.

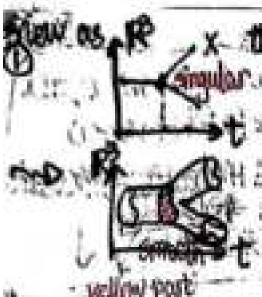
Classical $\xrightarrow[①]{}$ String $\xrightarrow[②]{}$ Superstring: consider particle splitting problem

$\Rightarrow \exists!$ wall $W = \{(b, c) \in \mathbb{R}^2 \mid b^2 - 4c = 0\}$

two chambers $C^{\pm} = \{(b, c) \in \mathbb{R}^2 \mid \pm(b^2 - 4c) > 0\}$

$\Rightarrow e(C^+) - e(C^-) = 2$ is wall-crossing

View as $\mathbb{R}^2$ ①



then $X \in \mathbb{R}^2 \times \mathbb{R}_t$ is singular! But we view particle as strings; we know string is real dim 1 ② String has infinity hard to deal with, they method (classically) is renormalisation ~ a string. Supersymmetry solves this difficulty: bosons ⟷ fermions (this step is the key not recognized by some physicists) Superstring is a SCFT (superconformal FT), the (super)conformal structure on Σ. Sometimes turns out (computation!) motion of fermions (then bosons) has right & left-moving, thus it's a (2,2)-SCFT.

yellow part is worldsheet Σ of real dim 2 / algebraic curve

Construction of superstring as SCFT now known four or five theories

- type I ⟨open & closed string⟩ 1-SCFT ⟲ T-dual
- type IIA ⟨closed string⟩ 2-SCFT
- type IIB ⟨T-dual⟩
- heterotic ⟨SO(32)-gauged ⟩ (2,2) SCFT; $E_8 \times E_8$-gauged ⟩ T-dual

M-theory ?

All of them have their mathematical correspondence in MS.

open string / closed string ⟲ T-dual

boundary condition of open string on D-branes

Rk. M-theory adds S-dual to unify five theories as a correspondence of one-dim higher M-theory. But the statement is still not clear! (Once adding a dual, the energy of theory ↓ a lot; if third symmetry is found, we are possible to ... of type I & $E_8 \times E_8$ ... comes from higher dim)

Heterotic string & their T-dual as MS is the most interesting case concerned by mathematicians: two levels: the mirror pair / model: mirror map

(2,2)-SCFT gives operator $(Q, \bar{Q}): \mathcal{H} \to \mathcal{H}$; $\mathcal{H}$ is the Hilbert space of states

The nonlinear sigma model determined by X CY 3-fold & $\omega = B + iJ$, $B, J \in H^2(X; \mathbb{R})$ (J not complex structure!) ω is Kähler / symplectic class, i.e. (X, ω) pair $\in H^2(X; \mathbb{C})$

In this model, $(Q, \bar{Q}) \leftrightarrow$ their eigenspaces (p,q)-eigenspace $\cong H^p(X, \Lambda^q T_X)$) interchange them? i.e. $\exists (X^\vee, \omega^\vee)$ s.t. their eigenspace have such change?

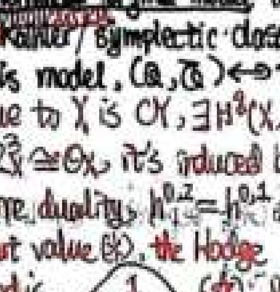
A. Due to X is CY, $\exists H^q(X, \Lambda^p T_X) \cong H^q(X, \Omega_X^{3-p})$ $(3-p, q)$-eigenspace $\cong H^q(X; \Omega_X^{3-p})$

(By $\Omega_X^3 \cong \mathcal{O}_X$, it's induced by cup product) $\Rightarrow h^{p,q} = h^{3-p,q}$ Hodge diamond it's

By Serre duality, $h^{0,2} = h^{0,1}$

$$\boxed{\begin{aligned} h^{0,0} &= \dim H^0(X; \Omega_X^0) = \dim H^0(X, \mathcal{O}_X) = 1 \\ h^{1,0} &= \dim H^0(X; \Omega_X^1) = \text{the first betti number} \overset{CY}{=} 0 \end{aligned}}$$

By start value (※), the Hodge diamond is



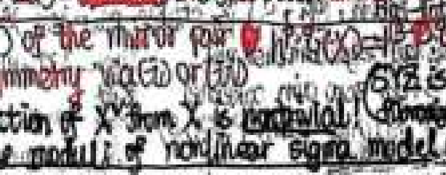
(★) What about (★) of the mirror pair? $h^{1,1}(X) = h^{2,1}(X^\vee)$; symmetry by Serre dual

this's a symmetry via ① or ②

Concrete construction of $X^\vee$ from X is nontrivial! (SYZ is one idea, construct $X^\vee$ as moduli of ... special described before.)

Now look at the moduli of nonlinear sigma model (In mathematic it's precise constructed)

local at (X, ω):



the two foliation by varying X & ω interchanges, due to $T_{(X,\omega)} \mathcal{M} = H^1(X, T_X)$, $T_{(X,\omega)} \mathcal{M} \cong H^1(X, \Omega_X^2)$ (the first varys X as complex moduli, second varys ω as Kähler moduli; the apply to the mirror pair.)

Rk. $\mathcal{M}$ not even locally product of these two foliations! Thus the mirror map's construction is also nontrivial! (Except we fix complex structure)

Where is GW? Recall Yukawa coupling refers to $\langle a_1, a_2, a_3 \rangle = \int a_1 \cdot a_2 \cdot a_3$

$= \int a_1 \cdot a_2 \cdot a_3 + \sum_{\beta \neq 0} N_\beta \left(\int_\beta a_1 \cdot \int_\beta a_2 \cdot \int_\beta a_3 \right)$ (※) ≠ no quantum correction ⟺ it's the renormalisation depicted above

original SFT   holomorphic instantons $+ \sum \langle a_1, a_2, a_3 \rangle_\beta q^\beta$

N_β is the number of rational curve C, $[C] = \beta$, defined as $\int_{[\ldots]} 1$ ... $\omega \in H^2(X, \mathbb{Z})$ also vary, thus we have the A-model correlation function, a twisted SCFT

By Dubrovin formalism (omitted), we have $\langle \theta_1, \theta_2, \theta_3 \rangle = \int \Omega \wedge (\nabla_{\theta_1} \nabla_{\theta_2} \nabla_{\theta_3} \Omega)$, Ω is a holomorphic 3-form on X [illegible] $\theta_i \in H^1(X, T_X)$ is thus called B-model correlation function (it can be [illegible])

In this section, we first state mirror conj & mirror thm for quintic threefold, then discuss concrete construction of mirror ... is unique up to ... $(X^\vee)$ ... a precise statement

Quintic threefold

Def'n (CY) CY variety of dimension d X is normal, compact / $\mathbb{C}$ ... s.t. ① At most Gorenstein canonical singularity; ② ω_X trivial ③ $H^1(X, \mathcal{O}_X) \cong \cdots \cong H^{d-1}(X, \mathcal{O}_X) = 0$ (here ω_X is dualizing sheaf, distinct Ω_X in some cases) I feel uncomfortable about ... Gorenstein canonical singularity is X is Gorenstein (ω_X is line bundle ⟺ K_X is Cartier ⟺ local ring all Gorenstein) & the singularity is canonical. Further if $\mathbb{Q}$-factorial terminal, we call it minimal by MMP; sometimes we call it CY and ... definition

Rk. We have several inequivalent definitions of CY in different texts & articles; such as only ... trivial / restrict in smooth case .. we explain the two cases: we allow canonical / terminal singularity due to ... 's construction of $X^\vee$ is singular Gorenstein due to ω_X behaves well has a good dualizing theory; only ② without ③ due to Beauville–Bogomolov decomposition thm For X satisfy ② only, $\exists$ finite étale covering $(\prod X_i) \times (\prod Y_j) \times (\prod Z_k) = X' \to X$ with X_i is (smooth) CY, Y_j is tori, Z_k is hyperkähler ... we only live in physics now.

E.g. ① What about $d = 1$ CY variety? Should ③ restrict $H^0(X, \mathcal{O}) = 0$?

② As a start pt of CY, K3-surface is CY; ③ Abelian variety satisfy ② but not ③ (consider product of two elliptic curves $E_1 \times E_2$)

④ We had mentioned before, complete intersection $(d_1, \ldots, d_k) \subset \mathbb{P}^{n}$ & $\sum d_i = n+1$ is CY: $\omega_X = 0$ by exercise of [HA], $H^i = 0$ by the Lefschetz hyperplane thm. Same consequence hold for $\prod \mathbb{P}^{n_i}$ and weighted $\mathbb{P}(w_0, \ldots, w_n)$ cases.

⑤ A nontrivial example is elliptic CY threefold, has interesting Geometry and Physics, $X \to S$ has general fibre elliptic curves (why it's Calabi–Yau?) (or more general, general fibre Abelian variety)

Next we'll use the correlation functions of both A-model & B model, then we can have the classical computation of 2875, 609250, 317206375 ... and generalize to the Mirror conj/thm interchange under mirror

Thm6. (MS of quintic threefold) Denoted N_d the number of degree d rational curves on $X \subset \mathbb{P}^4$ quintic threefold = instanton numbers, then by A-model correlation function $\langle H,H,H\rangle = \langle \theta,\theta,\theta\rangle$ B-model correlation function (of $X^\vee$) we have $5+\sum_d N_d d^3 \frac{q^d}{1-q^d} = \frac{5}{(1+5^5 x)\,y_0(x)^2}\left(\frac{q}{x}\frac{dx}{dq}\right)^3$, $y_0(x) = \sum \frac{(5n)!}{(n!)^5}(-1)^n x^n$, $q = \exp(2\pi i \int_\ell \omega)$, ℓ is line class. Mirror map locally $\sim x \exp\left(\frac{5}{y_0(x)} \sum \frac{(5n)!}{(n!)^5}\left(\sum_{k=n+1}^{5n} \frac{1}{k}\right)(-1)^n x^n\right)$ (are defined as period)

Then $5+\sum N_d d^3 \frac{q^d}{1-q^d} = 5 + 2875 \frac{q}{1-q} + 609250 \cdot 2^3 \frac{q^2}{1-q^2} + \cdots$ to have our enumerative formulas. In next computation: the q^d term is important, here [illegible]

Pf. Totally computations:

① A-model correlation function: H the hyperplane class pullback to $X \hookrightarrow \mathbb{P}^4$, $\langle H,H,H\rangle = \int_X H.H.H + \sum N_d (\int H)^3 \boxed{\frac{q^d}{1-q^d}}$

[redacted]

$= 5 \int H.H.H + \sum N_d d^3 \frac{q^d}{1-q^d}$ ~~[illegible]~~ write $\frac{q^d}{1-q^d}$ as q^d simply but

An excellent work [A family of CY varieties and potential automorphic forms, Candelas, ...] $\sum N_d d^3 \frac{q^d}{1-q^d}$ ($\deg X = 5$ in $\mathbb{P}^4$) here we need $\boxed{\frac{q^d}{1-q^d}}$ to comp[illegible]

② Construction of quintic mirror $X^\vee$: $H^{2,1}(X) = \dim H^1(X, T_X) = \frac{[illegible]}{[illegible]} = 101$ ... thus the [illegible] arithmetic. view our mirror as central fibre of an one-parameterized family. The method of [Greene & Plesser, Duality in CY moduli space] is by resolve singularity of $V(x_0^5+\cdots+x_4^5 + t x_0 \cdots x_4)$ to $X_t^\vee$

deformation $= T_X$ ~ $\langle H,H,H\rangle = \boxed{\sum_d \langle H,H,H\rangle_d q^d}$

$\langle H,H,H\rangle = \sum N_d (\int H)^3 q^d$

$\mathbb{P}^4/G$ — Dwork family is $\sum x_i^5 + \psi t \prod x_i$, $\int_t$ form a pencil, called Dwork pencil. or Dwork dilation [illegible]

(a curve in the complexified Kähler moduli) is 1-dim ⟺ base 1-dim

G is finite group $= \{(a_0 \cdots a_4) \in \mathbb{Z}_5^5 \mid \sum a_j \equiv 0 \bmod 5\}/\mathbb{Z}_5$ ($\mathbb{Z}_5$ diagonal $\{(a_0 \cdots a_4) \in \mathbb{Z}_5^5 \mid \sum a_j \equiv 0 \bmod 5\}$)

action on $\mathbb{P}^4$ by $(a_0 \cdots a_4) \times (x_0 \cdots x_4) \mapsto (e^{\frac{2\pi i a_0}{5}} x_0, \cdots, e^{\frac{2\pi i a_4}{5}} x_4)$ fixes $V(x_0^5+\cdots+x_4^5 + t x_0 \cdots x_4)$

It's not contradiction! since [illegible] is the GW-invariant coefficient and other not. $\langle\rangle$ is due to $\langle H,H,H\rangle_d = \sum N_d [illegible]$

Q. Why after resolution it's still CY? Details of the construction in [GP] is physical & not rigorous. later, Batyrev's construction is equivalent and we'll complete this argument.

Q. By dim=3 MMP theory, we know the resolution isn't unique / no canonical choice, but the B-model correlation function not depend on these choice. They effect the Kähler moduli of $X_t^\vee$ via different "flops" physically.

③ Geometry of $X_t^\vee$ (here it's only set up some notations) $(x_0, \cdots, x_4) \mapsto (e^{\frac{2\pi i}{5}} x_0, x_1, \cdots, x_4)$ induce isomorphism $X_t^\vee \cong X^\vee_{e^{2\pi i/5} t}$ through the one parameterized family, thus $t \mapsto t^5$ variable change is nondegenerate, set $x = t^{-5}$ parameterize $(X_x^\vee) = (X_t^\vee) \subset$ complex moduli (i.e. fix the Kähler form / symplectic form invariant) $\rightarrow x$ as local coordinate as the complex moduli, we can describe A-model by MS: the Kähler moduli $\{(X, \omega_t)\} = \{(X, tH)\}$, H the pullback of hyperplane class pullback to X, generates $H^2(X;\mathbb{Z}) = \mathbb{Z}$, $q = e^{2\pi i \int \omega} = e^{2\pi i \int tH} = e^{2\pi i t}$ (Here is due to $\int H = \int H$ pullback to X = in $\mathbb{P}^4$ thus it's 1 not 5!) local parameter of the Kähler moduli with boundary is only $q = 0$. Thus we have next picture

variable change, periodic

Puncture ($0 \ne e^{\frac{2\pi i k}{5}}$), $0 \le k \le 4$ has boundary of the moduli $\Rightarrow 0$, $(\frac{[illegible]}{[illegible]}) = x$, is boundary after reparameterize.

$\boxed{\text{A-model}}$ $\boxed{\text{Kähler moduli}}$ $\cong$ B-model $\boxed{\text{complex moduli}}$

local isomorphism

$\boxed{\text{mirror map}}$

$q \longleftrightarrow x$

$$\boxed{q = -x\exp\left(\frac{5}{y_0(x)} \sum \frac{(5n)!}{(n!)^5}\left(\sum_{k=n+1}^{5n}\frac{1}{k}\right)(-1)^n x^n\right)}$$

Thus our next step is determine (※) the local mirror map's coordinate representation

(※) has form $q = -x \cdot \square$ not $q = -x$ easily, due to $\square$ gives the "quantum corrections" physically.

Q. From this physical point of view, we can write map mirror map to such form: ① First determine boundary pts ② corresponded boundary pts in both model, ③ we can write $q = C(x)$, $\boxed{\text{quantum correlations}}$.

This naturally problem comes to ②: has how why not we can write $q = (x^{\frac{1}{5}} + o(x)) \cdot \square$? How to correspond $q = 0$ & $x = 0$? It's purely Hodge theoretic by look at maximally unipotent monodromy at B-model: it occurs at only $x = 0$ ⟺ it's corresponds to the unique boundary of $q = 0$

④ Compute mirror map (※)

For simplicity of computation, take $\Omega = \mathrm{Res}\left(\frac{t}{x_0^5+\cdots+x_4^5 + t x_0 \cdots x_4}\right) \sum_{0 \le i \le 4} (-1)^i x_i \, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_4$

the holomorphic 3-form, $t = x^{-\frac{1}{5}}$, on $X_t^\vee$, i.e. Ω_x on $X_x^\vee$

Fact1. $q = \exp(2\pi i (\int_{\gamma_1} \Omega / \int_{\gamma_0} \Omega))$ for cycle γ_0 and γ_1 (see ②)

Fact2. All period $\int \Omega =: y$ satisfy Picard–Fuchs equation

Fact3. The monodromy of $\boxed{\gamma_0 \mapsto \gamma_0 \\ \gamma_1 \mapsto \gamma_1 + m\gamma_0}$ (Where y_0, y_1 defined later)

[redacted]

⟹ The period of solution to PF equation $\boxed{y_0(e^{2\pi i} x) = y_0(x) \\ y_1(e^{2\pi i} x) = y_1(x) + 2\pi i [illegible]}$

we process these three facts by Hodge theory

Fact1. (A physical argument see [Bershadsky & Cecotti & Ooguri & Vafa, Kodaira–Spencer theory of gravity and exact results of quantum string amplitudes])

[redacted]

$\mathbb{C}$ [illegible] $\longleftrightarrow$ $\mathbb{C}$ x

$\boxed{\text{maximal unipotent}}$ $x \to 0$ ⟺ [illegible] ⟺ Dwork divisor degenerate to

See [Chap 5, Cox & Katz] $P_1 P_2 \ldots$ is toric degeneration

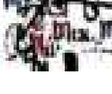
$\boxed{\text{maximally unipotent}}$

We need more about the local canonical coordinates are

$q_i = \exp(2\pi i \int \Omega / \int \Omega)$ (here our Kähler moduli is one-dim)

γ_0, γ_1 ... $m_{ij}\gamma_0$ as, where $\{\gamma_i\} \subset H_3(X;\mathbb{Z})$ chosen to be a basis of it.

$t(H) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ multiplication on the matrix [illegible] symplectic structure [illegible]



$\gamma_i \mapsto \gamma_i + m_{ij}\gamma_0$; if [redacted] has a large complex structure limit (LCSL) ... is essential for this not only dual but generalcase the monodromy action is similar. (I do not prove it due to I have not strong background on ...) [illegible] LCSL ⟺ LCSL [illegible]

Fact2. First we should explain what's period interestingly: Given cycle class γ and differential form Ω, by Poincaré duality Page 65
With a proper condition on their degree ⇒ we have pair $H_*(X) \otimes H^*_{dR}(X) \to \mathbb{C}$ as period $\int_\gamma \Omega \in \mathbb{C}$. (classical Hodge X not K-type)
Our interesting thing is why it's so named. Same thing asked for the period domain in Hodge theory. Algebraic structure on X

period integral in classical is $\int g(z_1 \dots z_n) dz$ for rational function f & g $\{z \in \mathbb{C}^n \mid f(z_1 \dots z_n) \geq 1\}$ with $\mathbb{Q}$-coefficients. consider the classical period integral $\int_0^r \frac{dx}{\sqrt{\dots}}$, $a, b, r \in \mathbb{Q}$, we can rewrite it as a period $\int_\gamma \Omega$ by:

The classical $\boxed{\text{Abelian integral}}$ - genus, Ω is canonical of Riemannian surface

A detail computation on Riemannian surface is [The Origins of Complex Geometry in the 19th century]

Then $U \xrightarrow[\text{map}]{\text{period}} \mathrm{Flag}(d_1 \dots d_k, H^k(X_0; \mathbb{C}))$, $b \longmapsto (F^p H^k(X_b; \mathbb{C}))$, $d_i = F^i H^k(X_0; \mathbb{C})$ (By X_0 Kähler, it has HS weight k) these numbers stays invariant about $b \in U$ for technic.

Central fibre Kähler X_0 ⇒ ∃ $U \subset B$ open (complex topology) s.t. $\forall b \in U$, X_b all Kähler. Here our Teich = Period domain, the image of period map is subset satisfy $F^p H^k(X_0; \mathbb{C}) \oplus \overline{F^{k-p+1} H^k(X_0; \mathbb{C})}$

variation of HS $\xleftarrow[\text{≅ biholomorphic}]{\text{Period map}}$ Period domain $= H^k(X_0; \mathbb{C})$
Griffith transversality $\longleftrightarrow$ Horizontal
Usually period domain is easy (can be described classically) and reflect to HS back.

Hodge □ above, we recover upper period by composite the □ above, we take cpx structure ⇔ Period domain, i.e. Teich $\xrightarrow[\text{map}]{\text{period}}$ Period domain ∋ elements of period domain gives HS, thus the pair $H_*(X) \otimes H^*_{dR}(X) \to \mathbb{C}$
(In this we know: Hodge goes 'from' than de Rham)

Classically, these period studies the multivalue functions

Then consider the PF equation satisfied by $y = \int \Omega$, relative to varible x of cpx moduli, $\theta = x\frac{d}{dx}$, then PF equation is $\theta^4 y_0 = 5^5 x(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4) y$
as $\theta^4 y_0 = \theta^4 \sum \frac{(5n)!}{(n!)^5} x^n = \sum \boxed{(n+1)^4 \frac{(5n)!}{(n!)^5}} x^{n+1} = \sum \boxed{5(5n+1)(\dots)(5n+4)} x^{n+1}$
$5x(5\theta+1)\cdots(5\theta+4) y = \sum 5x(5n+1)(\cdots)(5n+4) x^n$ ▨
(expand this to $(\theta^4 y) + \frac{2 \cdot 5^5 x}{1+5^5 x}(\theta^3 y) + \frac{7 \cdot 5^5 x}{1+5^5 x}(\theta^2 y)$ + by Fact1 other period proportional to $\frac{\dots 5^5 x}{1+5^5 x}(\theta y) + \frac{\dots}{1+5^5 x}(y) = 0$ (**)

Fact3. By analysis of ODE (*), we know it has two solutions, one is y_0, another denoted $y_1(x)$, then it's tautological computation to have (*) $q = y_1(x) = \boxed{\dots}$ done. The statement of monodromy is contained in Fact1. ▨

③ B-model correlation function $\langle \theta, \theta, \theta \rangle = \int_{X^\vee} \Omega \wedge \nabla_\theta \nabla_\theta \nabla_\theta \Omega = \int_{X^\vee} \Omega \wedge \Omega''' = Y$ (Set notation by θ - operator)

Observation. Y satisfy the ODE $\boxed{\theta Y = \frac{-5^5 x}{1+5^5 x}}$ (***)

To prove (***)
$$Y' = \left(\int_{X^\vee} \Omega \wedge \Omega'''\right)' = \int_{X^\vee} \Omega' \wedge \Omega''' + \int_{X^\vee} \Omega \wedge \Omega'''' \underset{\boxed{G}}{=} \int_{X^\vee} \Omega' \wedge \Omega''' + \int_{X^\vee} \Omega \wedge \Omega'''' = \frac{1}{2} \int_{X^\vee} \Omega \wedge \Omega'''' \underset{\mathrm{PF}+\boxed{G}}{=} \frac{\dots 5^5 x}{1+5^5 x} Y \text{ done}$$

where $\boxed{G}$ is Griffith transversality tells us that Ω, Ω', Ω'' lower degree have no (0,3) component ⇒ $\int \Omega \wedge \Omega = \int \Omega \wedge \Omega' = \dots$
thus $\left(\int_{X^\vee} \Omega \wedge \Omega''\right)' = \int \Omega'' \wedge \Omega' + 2\int_{X^\vee} \Omega' \wedge \Omega'' + \int_{X^\vee} \Omega \wedge \Omega''' = 0 + 2\int \Omega' \wedge \Omega'' + \int_{X^\vee} \Omega \wedge \Omega''' = 0 \Leftrightarrow \int_{X^\vee} \Omega \wedge \Omega'' = -\frac{1}{2}\int_{X^\vee} \Omega \wedge \Omega''' \quad \boxed{\int_{X^\vee} \Omega \wedge \Omega'' = 0}$

$\mathrm{PF} \Leftrightarrow 0 = \Omega'''' + \frac{2 \cdot 5^5 x}{1+5^5 x}\Omega''' + \boxed{\dots} \Leftrightarrow 0 = \int \Omega'''' \wedge \Omega + \int_{X^\vee} \frac{2 \cdot 5^5 x}{1+5^5 x} \Omega''' \wedge \Omega + \boxed{\dots}$ gives desired property ⇒ (***) ▨

$Y' = \frac{-5^5 x}{1+5^5 x} Y$ has obvious solution $Y = \langle \theta, \theta, \theta \rangle = \frac{C}{1+5^5 x}$ for constant C (by $\boxed{G}$)

Taking a normalisation of Ω (due to Ω is only unique up to scaloring), make $\int_{\gamma_0} \Omega = 1$ i.e. $y_0 = 1$ ⇒ $\Omega = \Omega / y_0$
⇒ $\langle \theta, \theta, \theta \rangle = \frac{C}{(1+5^5 x) y_0^2}$, C will later determined by compare coefficient of power series in A&B-model.

④ Computation.

$\langle H, H, H \rangle = \langle \theta, \theta, \theta \rangle$ to translate (**) into what we expect, first we need have bothside uses same varible q
$\boxed{5 + \sum_{d \geq 1} N_d d^3 \frac{q^d}{1-q^d} = \frac{C}{(1+5^5 x) y_0^2}}$ (**) this's ensured by (*) the mirror map, then expand rightside w.r.t. varible $\frac{q^d}{1-q^d}$ to complete our pf ▨ (It's a complicated calculus, nothing to tell)

Rk. Although we have such a computable method (very complicated!), it's still unknown $N_d \in \mathbb{Z}$? but stay at $N_d \in \mathbb{Q}$-level, this a problem for numericalists, not geometrists.

We only state the MS conjecture & MS thm in general, motivated by Thm6 in quintic threefold case:

Thm6' (Mirror symmetry thm) Main results in classical MS contains next several thms, all of them comes from interchange SCFTs:

① (Mirror conjecture / A-variation of HS) $\mathcal{X}^\vee \to B$ with fibre X_t naturally as a VHS in B-model.
The mathematical mirror pair $(X, X^\vee)$ is defined as interchange A-model VHS of $(X, \omega_t) \subset$ Kähler moduli and (B-model, i.e. usual) VHS of $(X_t, \omega) \subset$ complex moduli via the mirror map
and the A-model VHS is VHS of $H^*(X; \mathbb{C})$, whose connection is given by A-model connection $\nabla = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & Y(q) & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $Y = \langle \theta, \theta, \theta \rangle$ is the B-model Yukawa coupling, normalised w.r.t. $\int_{\gamma_0} \Omega = 1$ above, and rewriten as $Y(q) = Y(x)$, has q as varible.
Mirror conjecture states that
Mathematical mirror pair = (physical) mirror pair (By rotate Hodge diamond / interchange SCFT) It's a physical approach I

② (Givental) J = I (J is A-model given by gravitational correlators/generalised GW; I is B-model), two cohomology-valued func omit it, but it's equivalent to ③ [LLY]'s approach

③ [B. Lian, K. Liu & Yau, Mirror principle I] Q = P (P&Q is B&A-model Euler datum)
Euler datum is a "linear approximate/linearization" of the moduli of stable maps $M_{0,0}(\mathbb{P}^n, d)$
code datum of B&A-model correlation functions resp. They're also cohomology-valued function (Both ② & ③ are numerical)
Precisely, the $\boxed{\text{linearization}}$ is nonlinear sigma model ⟶ linear sigma model
linear space

More details are $\overline{M}_{0,n}(\mathbb{P}^4,d)$ isn't enough due to the stacky issue of reparameterization $Aut(\mathbb{P}^1)$, here use $\overline{M}_{0,n}(\mathbb{P}^1\times\mathbb{P}^4,(1,d))$ [illegible]
then reparameterization not change image $\subset \mathbb{P}^1\times\mathbb{P}^4$. Its linear sigma model is $\mathbb{P}(H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(d)^5))$ (a complete linear system)
The Euler data comes from the equivariant (with natural torus action as we mentioned in equivariant method in compute GW) [illegible]
theory on $\overline{M}_{0,n}(\mathbb{P}^1\times\mathbb{P}^4,(1,d))$ and $\mathbb{P}(H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(d)^5))$, and P is totally determined by GW invariants. See [LLY] for details.

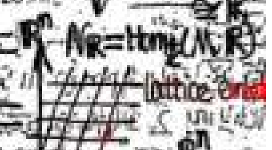
<u>Construction of mirror varieties</u>
In my view, MS consists three parts related with each other closely.
① Construction of mirror pair : Batyrev, Voisin-Borcea, SYZ ⇔ Gross-Siebert
② Enumerative geometry of mirror pair : Mirror [redacted] then
③ Explicit geometry of mirror pair (more than rotation of Hodge diamond!) : Mirror conjecture
HMS ⇔ SYZ
Most work of MS classical done near (Intrinsic [illegible] of ②) 1990 due to the evolution of Yau's school, and nowadays (2024) nothing left to tell about ① and ② except big breakthrough, thus our main work focus on ③ i.e. the HMS and SYZ / Gross-Siebert projects.

Due to the need of tropical geometry, we'll first study the combinatorial method, mainly <u>the toric geometry</u>, then Batyrev's construction (VS construction is near (but not equivalent) to Batyrev's in some cases, see [VS]'s original paper). Then focus on Gross-Siebert project via Gross's survey [Mirror Symmetry and the SYZ conjecture]

<u>Toric geometry</u> (here it's a quick tour, see [Fulton, Introduction to toric varieties] for details)
$M\cong\mathbb{Z}^n$ free Abelian grp of rank n, $N=M^\vee=Hom(M,\mathbb{Z})$, $N_\mathbb{R}=N\otimes\mathbb{R}\supset\sigma$ rational polyhedral cone $=\{\sum\lambda_i v_i \mid \lambda_i\in\mathbb{R}_{\geq0}, v_i\in N\}$
σ is <u>strongly convex</u> if $\sigma\cap(-\sigma)=\{0\}$ (subset of $Hom(M,\mathbb{Z})\otimes\mathbb{R}$ has natural grp structure)

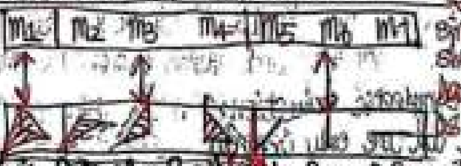
<u>Dual cone</u> $\sigma^\vee=\{m\in M\otimes_\mathbb{Z}\mathbb{R}=M_\mathbb{R} \mid \langle m,v\rangle\geq0, \forall v\in\sigma\}$
$=\{\sum\lambda_i m_i \mid \lambda_i\in\mathbb{R}_{\geq0}, m_i\in M\}$ also rational polyhedral cone $\subset M_\mathbb{R}$ by linear algebra.
Thus all definition & property of σ translate to $\sigma^\vee$
A <u>face</u> of σ is $\tau(m)=\{v\in\sigma \mid \langle m,v\rangle=0\}$ for $m\in M\cap\sigma^\vee$ (integral pt in $\sigma^\vee$)


σ standard
pair determined by $(a,b),(c,d)$ [illegible]
Q1. Why not [illegible]
A1. It's a problem of drawing
σ is infinite.
$\sigma\in\Sigma\subset\mathbb{R}^n$ $N_\mathbb{R}=Hom_\mathbb{Z}(M,\mathbb{R})$




$\sigma^\vee$ $M\cap\sigma^\vee=\{m_1,m_2,m_3\}\overset{\sim}{=}\{(0,1),(1,0),(1,1)\}=\mathbb{Z}^2\cap\sigma^\vee$
And consists [illegible] $\sigma^\vee$ but linear dependence
(Only $\boxed{\{0\}}$ is face by [illegible]
Q2. Why not single pt. ? A2. Even m [illegible] not its face, thus it doesn't contradict next fact.
Not embed to $\mathbb{R}^n$, it's lattice not discrete pts.

<u>Fact.</u> Face of face is face
<u>Fan</u> Σ in $N_\mathbb{R}$ is a family of σ above, s.t.



simple examples shows the σ has similar geometry to simplicial

① If $\sigma\in\Sigma\Rightarrow$ [illegible] $\subset\Sigma$; ② $\sigma,\sigma'\in\Sigma\Rightarrow\sigma\cap\sigma'$ is face of σ & σ', but the face of faces also face of σ
(③ analogous to the topology / the complex) thus $\sigma\cap\sigma'\in\Sigma$)
The support of Σ is $|\Sigma|=\bigcup\sigma$, $\Sigma(d)=\{\sigma \mid \dim\sigma=d$ [redacted] $, \sigma\in\Sigma\}\subset\Sigma$ subset (it's not a fan due to face of it $\notin\Sigma$)
$\rho\in\Sigma(1)$ are called 1-skeleton of the fan, they "generate" Σ by $\forall\sigma\in\Sigma$, $\sigma=\{\sum_{\rho\in\sigma}\lambda_\rho v_\rho \mid \lambda_\rho\in\mathbb{R}_{\geq0}\}$, $v_\rho\in\rho\cap N$ the generator
We identify ρ and v_ρ, then $v_1,\dots,v_r$ ($r=\#\Sigma(1)$) generates $\forall\sigma\in\Sigma$
$\boxed{\text{As } \sigma \text{ the affine chart}}$ locally, gluing by axiom of fan Σ ① & ②, we have
$X_\sigma=Spec(\mathbb{C}[M\cap\sigma^\vee])$ glued together to X_Σ (over $\mathbb{C}$, variety)
$\boxed{\text{affine toric variety} \xrightarrow{\text{glue}} \text{Toric variety}}$
ρ intersects N to [illegible] the $v_\rho=$ [illegible]
Why it's "toric"? Let $\sigma^\vee=M$, s.t. i.e. dual of $\{0\}\Rightarrow X_{\{0\}}=Spec(\mathbb{C}[M])$, M is lattice
$\Rightarrow X_{\{0\}}=T(N)\cong N\otimes\mathbb{C}^\times$ the complex torus of lattice $N=M^\vee$ (Then $X_{\{0\}}\overset{\sim}{=}T(N)\curvearrowright X_\Sigma$ natural torus action by multiplication)
<u>Prop A.</u> ① $|\Sigma|=N_\mathbb{R}\Leftrightarrow X_\Sigma$ complete/compact
② Σ is smooth $\Leftrightarrow X_\Sigma$ smooth, A fan is smooth $\Leftrightarrow\forall\sigma\in\Sigma$, then its generators $(v_\rho)_{\rho\subset\sigma}\subset\mathbb{Z}$-basis of N
③ Σ is simplicial $\Leftrightarrow$ orbifold: A fan is simplicial $\Leftrightarrow\forall\sigma\in\Sigma$, then its generators $(v_\rho)_{\rho\subset\sigma}$ linearly independent
[Orbifold is a variety, singularity $p\in X$ locally [redacted] to [redacted] $0\in U/G$, $G\subset GL_n(\mathbb{C})$
(locally quotient singularity) biholomorphic $U\subset\mathbb{C}^n$ is G-stable]
<u>Prop B.</u> (Intersection theory of X_Σ) We always assume X_Σ is $\boxed{\text{cpt \& orbifold}}$
① $Pic(X_\Sigma)\subset A^1(X_\Sigma)$ is a finite index subgroup (when smooth obvious equal), $Pic(X_\Sigma)$ is torsion-free;
② $0\to M\to\mathbb{Z}^{\Sigma(1)}\to A^1(X_\Sigma)\to0$ & $1\to Hom_\mathbb{Z}(A^1(X_\Sigma),\mathbb{C}^*)\to(\mathbb{C}^*)^{\Sigma(1)}\to T(N)\to1$ SES, [illegible] of this is $Aut(X_\Sigma)$ can be computed
The second totally describe global geometry of X_Σ i.e. can define $\boxed{X_\Sigma=((\mathbb{C}^{\Sigma(1)}-Z(\Sigma))/G}$ [illegible] determined by ideal of coordinate ring
by the second SES, $G=Hom_\mathbb{Z}(A^1(X_\Sigma),\mathbb{C}^*)$ and the quotient is $\boxed{\text{categorical quotient}}$ in GIT. This give X_Σ [illegible]
③ $A^1(X_\Sigma)$ spanned by $(D_\rho)_{\rho\in\Sigma(1)}$, given by the bijective correspondence $\rho\in\Sigma(1)\longleftrightarrow T(N)$-invariant divisor $D_\rho\subset X_\Sigma$
comes from the torus $T(N)$-action ; orbit closures $\leftrightarrow\sigma\in\Sigma$ and the exactness of ② first is
orbit's pt. closure $\leftrightarrow\rho\in\Sigma(1)$ $\sum\langle m,v_\rho\rangle D_\rho=div(\chi^m)$ principal ~0
<u>Pf of Prop A & B. Omitted</u>
Next we'll introduce another approach $T(N)$-invariant divisors $\chi^m:T(N)\to\mathbb{C}^*$ character of $T(N)$-rep is rational function
to X_Σ : <u>polytopes</u> and <u>reflexive polytopes</u> ; its main use of [illegible] Batyrev's method [illegible]
But before it, I'd like to introduce <u>the symplectic reduction method</u>, motivated by (※) & GKZ decomposition as a special [illegible]
Batyrev's construction later. We summarize these via a [illegible]

(thus now we have Four approaches to X_Σ: gluing, categorical quotient, symplectic reduction, polytopes)

Ex. [Toric Variety via symplectic reduction & GKZ decomposition] (What the "GKZ" means? I think they're not names of persons)

By the categorical quotient, we're adviced to have symplectic quotient $\mu^{-1}(a)/G$ via:

$1 \to G \to (\mathbb{C}^*)^{\Sigma(1)} \to T(N) \to 1$. Denote $r = \#\Sigma(1)$ $\Rightarrow 0 \to M \to (\mathbb{Z}^r)^\vee \to A^1(X) \to 0$ $\Rightarrow$ moment map is the composition $\mu_\Sigma : \mathbb{C}^r \xrightarrow{\mu_{\mathbb{C}^r}} (\mathbb{R}^r)^\vee \xrightarrow{p} \mathfrak{g}^\vee_{\mathbb{R}}$

$G \supset G_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{Z}}(A^1(X_\Sigma), U(1))$ maximal cpt subgrp: $0 \to M_{\mathbb{R}} \to (\mathbb{R}^r)^\vee \to A^1(X) \otimes \mathbb{R} \to 0$

$\mathfrak{g}_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{Z}}(A^1(X_\Sigma), \mathbb{R})$ associated Lie algebra $\mathrm{Lie}(G_{\mathbb{R}})$ $0 \to M_{\mathbb{R}} \to \boxed{(\mathbb{R}^r)^\vee \to \mathfrak{g}^\vee_{\mathbb{R}}} \to 0$ $\Rightarrow$ the Hamiltonian action $G_{\mathbb{R}} \curvearrowright \mathbb{C}^r$ descend to $G_{\mathbb{R}} \curvearrowright \Sigma$; the momentum μ_Σ projection $\mathbb{R}^r \to \mathbb{R}^{r-n}$ constant on every orbits

(Nontrivial!) $\mu_\Sigma^{-1}(a)/G_{\mathbb{R}} \cong X_\Sigma$ orbifold diffeomorphism

$\forall a \in \mathfrak{g}^\vee_{\mathbb{R}} = A^1(X_\Sigma) \otimes \mathbb{R} = H^{1,1}(X_\Sigma, \mathbb{R})$ is Kähler class & X_Σ projective. Such approach leads to (GLSM) Gauged linear sigma model in physics

We not prove but (must) explain it: $\mu_\Sigma^{-1}(a) \subseteq \mathbb{C}^r - Z(\Sigma)$ induce $\mu_\Sigma^{-1}(a)/G_{\mathbb{R}} \to (\mathbb{C}^r - Z(\Sigma))/G \cong X_\Sigma$ for X projective & smooth & toric $\Rightarrow A^1(X) = H^2(X;\mathbb{Z}) = H^{1,1}(X;\mathbb{Z})$

GKZ decomposition (Gelfand–Kapranov–Zelevinsky) (we'll later see this by an example) for X projective & simplicial & toric $\Rightarrow A^1(X;\mathbb{R}) = H^2(X;\mathbb{R}) = H^{1,1}(X;\mathbb{R})$

Consider the Kähler cone of all Kähler classes $K(X_\Sigma) \subset H^{1,1}(X_\Sigma;\mathbb{R}) = A^1(X_\Sigma;\mathbb{R})$:

$K(X_\Sigma) = \mathrm{int}(\mathrm{cpl}(\Sigma)) = \mathrm{int}\left(\left\{ a \in A^1(X) \otimes \mathbb{R} \;\middle|\; \begin{array}{l} a \text{ generated by } [D_\rho]: a = \sum a_\rho [D_\rho],\ \rho \in \Sigma(1) \text{ indexed by } 1 \sim r \\ a \text{ convex}: \forall \sigma \in \Sigma,\ \forall \rho,\ \langle m_\sigma, v_\rho \rangle \geq -a_\rho,\ m_\sigma \text{ is chosen as } \exists m_\sigma \in M_{\mathbb{R}},\ \langle m_\sigma, v_\rho \rangle = -a_\rho \text{ for } v_\rho \in \sigma \end{array} \right\}\right)$

$= \left\{ a \in A^1(X) \otimes \mathbb{R} \;\middle|\; \begin{array}{l} a = \sum a_\rho [D_\rho] \\ a \text{ strictly convex}: \langle m_\sigma, v_\rho \rangle > -a_\rho \end{array} \right\}$ can be computed combinatorially.

GKZ decomposition says that the analogous computation for "complexified" Kähler moduli $K_{\mathbb{C}}(X_\Sigma)$ we can describe by enlarging $K(X_\Sigma) = \mathrm{int}(\mathrm{cpl}(\Sigma))$ via gluing an adjacent Kähler cone in such way:

$K_{\mathbb{C}}(X_\Sigma)$ (in next example we'll see it satisfies our expectation)

Construction: Fix finite strongly convex rational 1-dim cones v_i, $\mathcal{S}_r = \{v_i \mid i = 1, \dots, r\}$. Σ is fan s.t. $\Sigma(1) = \mathcal{S}_r \Rightarrow X_\Sigma$ only depend on $\mathcal{S}_r$

$A^1(\mathcal{S}_r) := A^1(X) \otimes \mathbb{R} \supseteq$ effective classes generated by $[D_\rho]$

$A^+(\mathcal{S}_r)$ only depends on $\mathcal{S}_r$ (but $A(\mathcal{S}_r)$ not!) $\Rightarrow \mathrm{cpl}(\Sigma) \subset A^+(\mathcal{S}_r)$ (depend on Σ)

$A(\mathcal{S}_r) := A(X_\Sigma) \otimes \mathbb{R} \cong \mathbb{R}^{r-n}$

Varying Σ all fans satisfy $\Sigma(1) = \mathcal{S}_r$ & X_Σ proj orbifold.

$\Rightarrow \bigcup \mathrm{cpl}(\Sigma) \subset A^+(\mathcal{S}_r)$ Q. Equal? A. No! (later example shows) But we should consider $\Sigma(1) \subset \mathcal{S}_r$ (also proj & simp)

$\Sigma(1) = \mathcal{S}_r$ (GKZ decomposition) $\mathrm{cpl}(\Sigma)$s & their faces form a fan supported exactly $A^+(\mathcal{S}_r)$

Such decomposition of $A^+(\mathcal{S}_r)$, the effective Kähler cone coincides with the conjectured Mori chamber decomposition in [Hu, Keel (2000)], thus toric variety (and quotient by it) is Mori dream space.

Next we relate it with symplectic reduction: Another thing is GKZ decomposition of $A^+(\mathcal{S}_r)$ extends to decomposition of whole $A(\mathcal{S}_r)$, we'll also show by example next.

Due to the SES above all are only depend on $\Sigma(1) = \mathcal{S}_r$, so is the momentum $\mu_{\mathcal{S}_r} := \mu_{\mathcal{S}_r}: \mathbb{C}^{\mathcal{S}_r} \to A(\mathcal{S}_r)$ image restrict to $A^+(\mathcal{S}_r) \Rightarrow G(\mathcal{S}_r) \curvearrowright \mathbb{C}^{\mathcal{S}_r}$ and as before

for $\mathrm{cpl}(\Sigma) \subset A^+(\mathcal{S}_r) \Rightarrow \mu_{\mathcal{S}_r}^{-1}(a)/G(\mathcal{S}_r)_{\mathbb{R}} \overset{\text{(nontrivial)}}{\cong} X_\Sigma$

But surprisingly, even $\Sigma(1) \subsetneq \mathcal{S}_r$ case, $\mu_{\mathcal{S}_r}^{-1}(a)/G(\mathcal{S}_r)_{\mathbb{R}} \cong X_\Sigma$ still! Thus GKZ decomposition can be constructed such way.

Now consider an important example: $\mathbb{P}(1,2,2,2,2)$ useful in MS. Its GKZ decomposition is clear to explain all details we promised: (i) $X_\Sigma = (\mathbb{C}^{\Sigma(1)} - Z)/G$; (ii) $\mu_\Sigma^{-1}(a) \subset \mathbb{C}^{\Sigma(1)} - Z$; (iii) its GKZ decomposition: $\mathrm{cpl}(\Sigma) \subsetneq A^+(\mathcal{S}_r)$ for $\Sigma(1) = \mathcal{S}_r$ only;

(iv) extend GKZ decomposition to $A(\mathcal{S}_r) = \mathbb{R}^2$; Same construction for other $\mathbb{P}(a_0 \dots a_n)$

$N = \mathbb{Z}^4$ the dual lattice, $v_0 = (-1,-2,-2,-2) \in N$, $v_i = e_i$, $i = 1,2,3,4$ $\Rightarrow 1 v_0 + 1 v_1 + 2 v_2 + 2 v_3 + 2 v_4 = 0$, $\mathbb{P}(1,1,2,2,2)$ is the toric variety X_Σ for the fan $\Sigma = \{\langle v_0 \rangle, \langle v_1 \rangle, \dots, \langle v_0, v_1 \rangle, \dots, \langle v_0 \dots v_4 \rangle\}$ generated by all subsets of $\{v_0 \dots v_4\}$

$\Rightarrow \Sigma(1) = \{v_0 \dots v_4\} \Rightarrow$ the SES is

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & \mathbb{Z}^{\Sigma(1)} \arrow[r] \arrow[d, equal] & A^1(X_\Sigma) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}^4 \arrow[r] & \mathbb{Z}^5 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

$(a_0 \dots a_4) \mapsto a_0 + a_1 + 2a_2 + 2a_3 + 2a_4$

$(a_1 \dots a_4) \mapsto (-a_1 - 2a_2 - 2a_3 - 2a_4, a_1, a_2, a_3, a_4)$

Σ is simplicial but not smooth, here we give a toric resolution of X_Σ by operation on the fan Σ: Its singular issue of Σ is v_0 can be spaned by $v_1 \dots v_4$ not $\mathbb{Z}$-basis, thus take the singular one $\sigma = \langle v_1, \dots \rangle \in \Sigma$, adding the new generator $v_5 = \frac{v_0 + v_1}{2}$ $\Rightarrow \sigma' = \langle v_0, \dots, v_4 \rangle$ now can be rewrite $= (0, -1, -1, -1) = \langle v_1 \cdots v_5 \rangle$ linearly independent ($\mathbb{Z}$-basis) $\Rightarrow \Sigma$ smooth ☑

Replaced by its toric resolution, X_Σ is smooth now (abuse notation)

$G = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, \mathbb{C}^*) = (\mathbb{C}^*)^2 \curvearrowright \mathbb{C}^6 = \mathbb{C}^{\Sigma(1)}$ (Here $0 \to \mathbb{Z}^4 \to \mathbb{Z}^6 \to \mathbb{Z}^2 \to 0$)

$(\lambda, \mu) \cdot (x_0 \dots x_5) = (\lambda x_0, \lambda x_1, \mu x_2, \mu x_3, \mu x_4, \dots x_5)$ by construction of toric resolution.

We can determine Z in (✱) (as we said, it's the most computable way): $Z = Z(\Sigma)$ is the locus of "primitive collections" $\{v_0, v_1\}$, $\{v_2, v_3, v_4, v_5\}$ (they have same "action manner", multiply λ & μ, resp.)

$\Rightarrow Z(\Sigma) = \{x_0 = x_1 = 0\} \cup \{x_2 = \dots = x_5 = 0\}$ & $X_\Sigma = (\mathbb{C}^6 - Z)/(\mathbb{C}^*)^2$ …(i)

and $\mu_\Sigma : \mathbb{C}^6 \to \mathbb{C}^{6-4} = \mathbb{C}^2$

$(z_0 \dots z_5) \mapsto \left(\frac{1}{2}(|z_0|^2 + \dots + |z_5|^2), \frac{|z_2|^2 + |z_3|^2}{2} - |z_5|^2\right)$

thus $\mu^{-1}(a) = \dots \subset \mathbb{C}^6 - Z$ is obvious

$\forall a$ Kähler class is similar not intersect with $Z(\Sigma)$ …(ii)

Extend $\mathcal{S}_r = \Sigma(1) = \{v_0 \dots v_5\}$ to $\mathbb{R}^{\mathcal{S}_r} = \dots \times \{1\}$, they have relations $(1, \dots) + (1, \dots) = 2(v_5, 1)$, $(v_5, 1) + \dots = 4(\dots)$

$A^1(\Sigma(1)) = \mathbb{Z}^2 \otimes \mathbb{R} = \mathbb{R}^2$

$A^+(\Sigma(1)) \subsetneq \mathbb{R}^2$ by GKZ contains two parts (and their faces) Σ of X_Σ, Σ' of $\mathbb{P}(1,1,2,2,2)$ without toric resolution

$\mathrm{cpl}(\Sigma') \subsetneq A^+(\Sigma(1))$ …(iii)

$\Rightarrow$ the relation matrix $\left(\begin{array}{cccccc|c} 0 & 0 & 1 & 1 & 1 & 1 & -4 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 \end{array}\right)$ ... (iv) ☒

Look at columns!

First six gives GKZ of $\mathbb{P}(\ldots)$

Last one enlarge GKZ to $A(\ldots)$

Then we start Batyrev's theory of polytopes: It's the dual to cones $\subset N$, due to N is abstract maps but M is concrete lattice, we describe it more easily: $\Delta \subset M_{\mathbb{R}}$ is polytope if $\Delta = \mathrm{conv}(p_1 \ldots p_n)$, $p_j \in M_{\mathbb{R}}$; it's integral iff $p_j \in M$; it's facets are the codim=1 face of Δ.

It's realised as a dual notation by: $\forall D = \sum a_\rho D_\rho$ Cartier divisor, we can let $\Delta_D = \{m \in M_{\mathbb{R}} \mid \langle m, v_\rho \rangle \geq -a_\rho, \forall \rho\}$ is polytopes.

D generated by global sections (gbgs) $\Leftrightarrow \langle m_\sigma, v_\rho \rangle \geq -a_\rho, \forall \rho \not\subset \sigma \Leftrightarrow \Delta_D = \mathrm{conv}\{m_\sigma \ldots\}$

D ample $\Leftrightarrow \langle m_\sigma, v_\rho \rangle > -a_\rho, \forall \rho \not\subset \sigma$ n-dim cone $\Leftrightarrow \Delta_D$ is n-dim polytope with vertex $\{m_\sigma \mid \sigma \in \Sigma(n)\}$

Let $n=1$, we have

$\boxed{\text{facets of } \Delta \xleftrightarrow{\text{1:1}} \Sigma(1)}$ (※) It's the reason of why X_Σ is called toric: a usual tori will have same characters' behaviors.

The construction of X_Σ via polytopes is the most hard one, but we can believed on it due to (※) above; via the duality (second method is representation-theoretic via $H^0(X, \mathcal{O}(D)) = \bigoplus \mathbb{C}\chi^m$ decomposed into characters of T_N-action on $H^0(X, \mathcal{O}(D))$, I omit.

Batyrev's mirror construction. Assume all Δ is n-dim integral. We denote the result projective variety $X_\Sigma = \mathbb{P}_\Delta$

We start $X \subset \mathbb{P}_\Delta$, X is CY hypersurface $\subset$ toric variety (we know all weighted projective spaces are toric), then X arises from reflexive polytope Δ, then construction of mirror reduces to combinational operation of the duality)

Then for $X \subset \mathbb{P}_\Delta$ CY complete intersections, we need do operations of different $\Delta_1 - \Delta_d$. We prepare the two need things first:

- $\Delta \subset M_{\mathbb{R}}$ is called reflexive if ① $\Gamma \subset \Delta$ facets are supported by affine hyperplane $\{m \in M_{\mathbb{R}} \mid \langle m, v_\Gamma \rangle = -1\} \subset M_{\mathbb{R}}$; (Due to $\forall \Gamma \subset \Delta \Leftrightarrow \rho \in \Sigma(1)$ by (※), thus we simply write v_ρ as v_Γ);

② $\mathrm{Int}(\Delta) \cap M = \{0\}$

Why it's reflexive called? The dual notation $\Delta^\circ \subset N_{\mathbb{R}}$ polar polytope, $\Delta^\circ = \{v \in N_{\mathbb{R}} \mid \langle m, v \rangle \geq -1, \forall m \in \Delta\}$, then $(\Delta^\circ)^\circ = \Delta$ is obvious for all (not only reflexive) $\Delta \Rightarrow \Delta$ reflexive $\Leftrightarrow \Delta^\circ$ reflexive (identify $\Delta^\circ \subset N_{\mathbb{R}}$ the reflexive is same defined.

Δ reflexive $\Leftrightarrow \mathbb{P}_\Delta = \mathrm{Proj}(S_\Delta) = \mathrm{Proj}(\mathbb{C}[t_0 \chi^m])$ the graded $\mathbb{C}$-algebra generated by $t_0^k \chi^m$ of degree k (t_0 is the formal variable) is Fano (i.e. canonical bundle is anti-ample) ($K_{\mathbb{P}_\Delta} = -\sum D_\Gamma$ for Γ facets of Δ°)

(A interesting corollary of this is let $\mathbb{P}(a_0 \ldots a_n) = \mathbb{P}_\Delta = X_\Sigma \Rightarrow \Sigma$ has one generators $v_0 \ldots v_n$, $\sum a_j v_j = 0$)

Thus $\mathbb{P}(a_0 \ldots a_n)$ Fano $\Leftrightarrow \Delta$ reflexive $\Leftrightarrow a_j \mid a, \forall j$ for $a = \sum_j a_j$ ☒

- We have two equation given $\Delta_1 \ldots \Delta_d$: $\mathrm{Conv}(\Delta_1 \ldots \Delta_d) = \mathrm{Conv}(\Delta_1 \cup \ldots \cup \Delta_d)$ and $\sum_j \Delta_j = \{\sum m_j \mid m_j \in \Delta_j\}$ Minkowski sum $d\Delta = \sum_d \Delta = \{dm \mid m \in \Delta\} \Rightarrow \Delta_{dD} = d\Delta_D$ and $\Delta_{D + \mathrm{div}(\chi^m)} = \Delta_D - m$ translation, i.e. principal divisor not change the shape

The maximal projective subdivision of Δ reflexive is defined next, we modify $X_\Sigma \xrightarrow{f} \mathbb{P}_\Delta$ birationally due to $\mathbb{P}_\Delta$ too singular

① Σ refines normal fan of Δ; ② $\Sigma(1) \subset (\Delta^\circ \cap N) - \{0\}$; ③ X_Σ proj orbifold called projective subvision, maximal if ② equal.

We associate Δ a fan by $\forall F \subset \Delta$ a face (not codim 0), $\Sigma_\Delta = \{\sigma_F \mid F \neq \emptyset\}$

$\sigma_F = \{\lambda(m - m') \mid m \in \Delta, m' \in F, \lambda \geq 0\} \subset M_{\mathbb{R}}$

$\Rightarrow \Sigma_\Delta$ is called normal fan of Δ

(Σ refines Σ_Δ just $\Sigma_\Delta \subset \Sigma$)

maximal one exist but not unique due to GKZ decomposition

each maximal projective subdivison Σ of Δ
$\Updownarrow$
each cone of GKZ decomposition $\mathrm{cpl}(\Sigma)$ of $\Xi = \Sigma(1) = \{\text{facet of } \Delta\}$
$\Updownarrow$
each phase of gauged linear sigma model

Given projective subdivision Σ, we can say many things to the geometry of $X_\Sigma \xrightarrow{f} \mathbb{P}_\Delta$: ① K_{X_Σ} is anti-semi-ample ($-K_{X_\Sigma}$ gbgs & $(-K_{X_\Sigma})^n > 0$, $n = \dim X_\Sigma$)

$K_{X_\Sigma} = -\sum D_\Gamma$ for $\Gamma \subset \Delta$

i.e. $K_{X_\Sigma} \cong K_{\mathbb{P}_\Delta}$ via f, i.e. $f^* K_{\mathbb{P}_\Delta} = K_{X_\Sigma}$

We call each f crepant.

② X_Σ has Gorenstein canonical singularities (orbifold), and Σ maximal $\Leftrightarrow X_\Sigma$ minimal (i.e. ($\mathbb{Q}$-factorial) terminal singularities)

Now why we need use K_{X_Σ} and $K_{\mathbb{P}_\Delta}$? By Fano, $-K_{X_\Sigma}$ ample and the complete linear system $|-K_{X_\Sigma}|$ has general member hypersurface we have the general members are CY varieties,

and a family analogue of ② is ②' next (as Kontsevich said, MS naturally should be a statement between dual families not single variety)

②' The general member $\in |-K_X|$ is CY orbifold $\Leftrightarrow \Sigma$ is projective subdivision

minimal $\Leftrightarrow \Sigma$ maximal

$$\begin{tikzcd} X_\Sigma \arrow[r, "g"] & \mathbb{P}_\Delta \\ V \arrow[u, phantom, "\cup"] \arrow[r, "g"] & \nabla \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Thus $V \in |-K_X|$ are proper transform of $\nabla \in |-K_{\mathbb{P}_\Delta}|$, i.e. $V = g^{-1}(\nabla)$ (and $K_V = g^* K_\nabla$) (Due to the choice of Σ not unique)

Σ maximal $\Leftrightarrow V$ minimal model of ∇, called MPCP-desingularization of ∇ (maximal projective crepant partial)

When $\dim X_\Sigma \leq 3$, X_Σ is smooth, i.e. mirror of CY mfd also CY mfd (smooth)!

Then $\boxed{\text{MS of } V \subset X_\Sigma \Leftrightarrow \text{the reflexive property of } \Delta \,\&\, \Delta}$

A problem is not unique in both side, we can choose different Σ and Σ' is their GKZ decomposition respectively. It's allowed, called multiple mirror.

Thus repeat these steps for reflexive polytope $\Delta^\circ \Rightarrow V'$ desired Batyrev mirror.

Toric complete intersection is due to Batyrev & Borisov, via nef partition:

(another generalised approach can also founded in their article) $V \subset \mathbb{P}_\Delta$ (global) complete intersection is cutted by $\mathcal{O}(E_1) \ldots \mathcal{O}(E_r)$, $\sum E_j = -K_{\mathbb{P}_\Delta}$ a partition $\Delta = \sum \Delta_j$ Minkowski sum $\Leftrightarrow \Sigma(1) = \sqcup I_j$ partition, $E_j = \sum D_\rho$, each $\mathcal{O}(E_j)$ gbgs $\ldots \nabla_j$ dual nef partition $\Delta^\circ = \sum \nabla_j \Leftrightarrow \Delta^\circ = \mathrm{Conv}(\cup \nabla_j)$ Convex hull $\ldots$ vertices of Δ, $\nabla_j = \mathrm{Conv}(\ldots$ reflexive

From SYZ toward GS and tropical geometry

SYZ also comes from string theory via D-brane theory: the modern space-time $\mathbb{R}^{1,3}\times X$ where $\mathbb{R}^{1,3}$ is classical Minkowski space-time and X is CY threefold; D-branes $D\subset\mathbb{R}^{1,3}\times X$ submfds and additional datum (i.e. Dirichlet boundary condition of open string equation) string

D-branes can split into $D'\subset X$ can be classified into two kinds, exactly corresponds to A&B-model of MS: Holomorphic D-branes (B-model): $D_B\subset X$ holomorphic, and additional datum is a holomorphic line bundle; dual $\Rightarrow$ MS; their moduli both matched

Special Lagrangian D-branes (A-model): $D_A\subset X$ special Lagrangian and additional datum is flat $U(1)$-principal bundle

Thm (SYZ conjecture, not rigorous) $(\dim_{\mathbb{R}} B=\dim_{\mathbb{C}} X=\dim_{\mathbb{C}} X^\vee)$

$(X,X^\vee)$ is a mirror pair $\Leftrightarrow \exists$ fibrations $X\xrightarrow{f} B\xleftarrow{f^\vee} X^\vee$ fibres are special Lagrangian, i.e. Lagrangian & $\mathrm{Im}\,\Omega|_{M}=0$ $(\dim X=\dim X^\vee=n)$ General fibres are n-torus, others singular. Ω and $\Omega^\vee$ is nowhere vanishing holomorphic n-form on X and $X^\vee$ resp.

We'll call such fibration SYZ-fibration if $f, f^\vee$ is dual: $X_b=H^1(X_b^\vee;\mathbb{R}/\mathbb{Z})$ and $X_b^\vee=H^1(X_b;\mathbb{R}/\mathbb{Z})$

Generally, it's known to be False! by Joyce, but special cases such as quintic threefold is proven.

Rk ① How to understand the duality? Consider the deformation of Lagrangian submfd $M\subset(X,\omega)$ & $H^1(M;\mathbb{R})$ and flat $U(1)$-principal bundle $/\sim$ gauge $=H^1(M;\mathbb{R})/H^1(M;\mathbb{Z})$ is torus expectation

$\Rightarrow X$ is family of torus of form $H^1(M;\mathbb{R})/H^1(M;\mathbb{Z})=H^1(T;\mathbb{R}/\mathbb{Z})$, then the Mirror fill into ☑

② Where is the next step? $X\to B$ special Lagrangian fibration with all smooth fibres $\Rightarrow B$ will naturally have (two) affine structure; this's where affinoid & tropical geometry comes in (affinoid is Λ-coefficient with Novikov ring is valuation & non-Archimedean, here for only $\mathbb{C}$-coefficient we don't need this; But in HMS, Fukaya cat defined by Λ-coefficient Floer homology we can't avoid it.)

Defn (Affine & Tropical structure)

A mfd of dim n has translation function are — homeomorphism $\supseteq$ piecewise linear (PL) $\supseteq$ diffeomorphism $\supseteq$ affine ($\{PL\}\cap\{Diff\}$) $\supseteq$ tropical $\supseteq$ integral

then it's topological $\supseteq$ PL $\supseteq$ smooth $\supseteq$ affine $\supseteq$ tropical $\supseteq$ integral mfds

Write as groups are $\mathrm{Homeo}(\mathbb{R}^n)\supseteq PL_n(\mathbb{R})\supseteq \boxed{\mathrm{Aff}_n(\mathbb{R})\supseteq\mathbb{R}^n\rtimes GL(\mathbb{Z}^n)}\supseteq\mathrm{Aff}_n(\mathbb{Z})$; $\mathrm{Diff}(\mathbb{R}^n)\supseteq$; $\mathbb{R}^n\rtimes GL(\mathbb{Z}^n)$ has translation functions have integral linear part $x\mapsto kx+b$, $k\in\mathbb{Z}$, $b\in\mathbb{R}$ (and $\mathrm{Aff}_n(\mathbb{Z})$ has $b\in\mathbb{Z}$ also)

Rk It's interesting to study its related algebraic topology via let $\mathrm{Homeo}=\varinjlim\mathrm{Homeo}(\mathbb{R}^n)$ and consider their $B\mathrm{Homeo}$, BPL, $B\mathrm{Diff}$, $\Longrightarrow$ the tangent space ("the geometry") $\in[M,B\mathrm{Homeo}]$, $[M,BPL]$, ... resp.

(But what happens $x\mapsto\frac{k}{l}x$ also translation & $\frac{k}{l}\in\mathbb{Q}$ $\frac{k}{l}=1$? I'm confused!)

Now we describe the two affine structure (and one metric) of B from $f:X\to B$ smooth special Lagrangian fibration, by McLean's thm: $\iota(v)\mathrm{Im}\,\Omega=-*\iota(v)\omega$ $(*)$; $v\in\Gamma(M,N_{M/X})$ normal vector field of special Lagrangian $M\subset(X,\omega)$

Do things for both sides of $(*)$, ω and $\mathrm{Im}\,\Omega$: we construct local coordinates of $U\subset B$

$\gamma_1,\dots,\gamma_n\subset f^{-1}(U)$ 1-cycles s.t. $\{\gamma_i\cap X_b\}$ is $\mathbb{Z}$-basis of $H_1(X_b;\mathbb{Z})$, $\forall b\in U\Rightarrow y_i:=\int_{\gamma_i}\iota(v)\omega\Rightarrow\delta_x(\omega|_{\gamma_i})$ closed $=dy_i$

$\Gamma_1,\dots,\Gamma_n\subset f^{-1}(U)$ $(n-1)$-cycles s.t. $\{\Gamma_j\cap X_b\}$ is $\mathbb{Z}$-basis of $H_{n-1}(X_b;\mathbb{Z})$, $\forall b\in U\Rightarrow \check{y}_j:=\int_{\Gamma_j}\iota(v)\mathrm{Im}\,\Omega=\dots$ closed $=d\check{y}_j$

By $\mathbb{Z}$-basis, we know (y_i) and $(\check{y}_j)$ are tropical, and they do local coordinates due to (y_i) (or $(\check{y}_j)$): $U\to\mathbb{R}^n$ is homeomorphism by $T_bB\longrightarrow H^1(X_b,\mathbb{R})$, $v\mapsto\iota(v)\omega$ is isomorphism (also a consequence of McLean) (we choose U is small enough)

By Hitchin, they're same up to a Legendre transform (or $-\iota(v)\mathrm{Im}\,\Omega$)

$T_bB\cong\Gamma(X_b,N_{X_b/X})$ as the picture of deformation: lift v to $\tilde{v}$ is isomorphism

the McLean metric $g: TB\otimes TB\to\mathbb{R}$

$(v_1,v_2)\longmapsto-\int_{X_b}(\iota(v_1)\omega)\wedge(\iota(v_2)\mathrm{Im}\,\Omega)$

induced from the Hodge metric $g: H^1(X_b;\mathbb{R})\otimes H^1(X_b;\mathbb{R})\to\mathbb{R}$, $\alpha\otimes\beta\longmapsto\int\alpha\wedge*\beta$

has local representation $(\partial_{ij}K)_{ij}$ (i.e. $g(\partial/\partial y_i,\partial/\partial y_j)=\partial_{ij}K$) under basis $\partial/\partial y_i$ for $K:U\to\mathbb{R}$

Then in smooth fibre cases, SYZ is realised via construction $X(B,\check{B},K)$, $X^\vee(B,\check{B}^\vee,K^\vee)$, (B) SYZ-fibrations with smooth fibres

$\Rightarrow$ the Legendre transform $\check{K}(\check{y}_1,\dots,\check{y}_n)=\sum\check{y}_iy_i-K(y_1,\dots,y_n)$ s.t. $y_i=\frac{\partial\check{K}}{\partial\check{y}_i}$ & $(\partial_{ij}\check{K})_{ij}$ is local representation under basis $(\partial/\partial\check{y}_i)$ ▨

$\check{B}$ & $\check{B}^\vee$ are $\boxed{B\text{-fields}}\in H^2(X^\vee,\mathbb{R}/\mathbb{Z})$ & $H^2(X,\mathbb{R}/\mathbb{Z})$ (named from nonlinear sigma model) $= H^1(B,\Lambda_{\mathbb{R}}/\Lambda)$ & $H^1(B,\check{\Lambda}_{\mathbb{R}}/\check{\Lambda})$

$\Lambda\subset\mathcal{T}_B$ lattices locally generated by $(\partial/\partial y_i)$

$\check{\Lambda}\subset\mathcal{T}_B$ ... by $(\partial/\partial\check{y}_i)$

constructs $X(B,\check{B},K)$ by locally $\check{B}$ is Čech 1-cocycle (U_{ij},β_{ij}) on B is gluing condition of a mfd. locally B is $\Lambda_{\mathbb{R}}/\Lambda$ glued together $=T_U/\Lambda$

There're many analytic details on the McLean metric we omitted ▨

For singular case, no analytic tools as $\mathcal{T}_B$ can be used, but luckily ☒ tells us use the combinatorial tools is able! Our next task is deal via tropical method to have mirror construction via toric degeneration

Tropical Geometry, Why?

An affine mfd with singularities is topological with $\exists B_0\subset B$ open, B_0 is affine, and $\Gamma=B-B_0=\bigcup B_\alpha$, B_α locally closed and union is locally finite (i.e. ⫩⫩, ..., ✓, ✳, ×).

Same as tropical/integral.

To motivate next definition via adding combinatorial datum to Γ, consider special case with $B=\partial\Delta\subset\Delta\subset M_{\mathbb{R}}$, Δ is reflexive, $\dim M=n\Rightarrow B=\bigcup\sigma$ factored into polytopes as a fan supported on B ... Now we can describe $\Gamma\subset B=\partial\Delta$ as a discriminant locus,

We explain ☒ picture

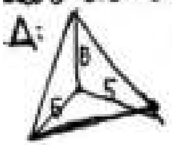
is important.

Our next step is toric/log tropical geometry

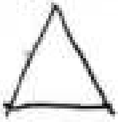
$\Gamma := |\mathrm{Bar}(\mathscr{B})|$ the support of the first barycentric subdivision of $\mathscr{B}$ (why we must need this not only $|\mathscr{B}|$ will showed later) $\boxed{\text{Page 11}}$
and $B_0 = B - \Gamma$ has affine structure by open cover $\{\mathrm{Int}\,\sigma \mid \sigma \in \mathscr{B} \text{ maximal}\} \amalg \{\bigcup_{\substack{\tau \in \mathrm{Bar}(\mathscr{B}) \\ v \in \tau}} \mathrm{Int}\,\tau \mid v \in \mathscr{B} \text{ vertex}\}$ Called discriminant loci,
and affine chart $\begin{cases} \mathrm{Int}\,\sigma \hookrightarrow A_\sigma \subset N_{\mathbb{R}} & \text{inclusion map into } A_\sigma \text{ the affine hyperplane of } N_{\mathbb{R}}\text{, uniquely containing } \sigma \\ \bigcup_{v \in \tau} \mathrm{Int}\,\tau \twoheadrightarrow M_{\mathbb{R}}/\mathbb{R}v & \text{projection map} \end{cases}$

Let's see how this affine chart works: take $\Delta \subset \mathbb{R}^4$. $\Delta = \mathrm{Conv}\{(-1,-1,-1,-1),(4,-1,-1,-1),(-1,4,-1,-1),(-1,-1,4,-1),(-1,-1,-1,4)\}$

Δ:

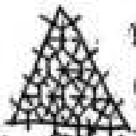


$\partial\Delta$: $\overset{\shortparallel}{B}$

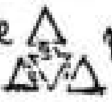


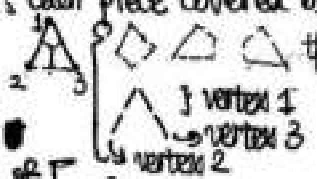
$\cup$ …other 2-dimensional faces

decomposed by red lines into $\mathscr{B}$ (or it's barycentric subdivision. it not changes the "form")
and then the dashed black lines are just Γ



$\Rightarrow B_0 = B - \Gamma$ are △ pieces, each piece covered by affine charts of $\mathscr{B}$ as (take △ for example)





An important feature of affine mfd (with singularities) is the monodromy around the loops of Γ.
In the slight of Hodge theory, as △ we need state SYZ by a degeneration next. For this example one can compute monodromy precisely (omitted) to motivate it:

Thm 7. (Topological SYZ) B a 3-dim tropical affine mfd with singularity, s.t. $\Gamma \subset B$ has monodromy same as upper example or its dual (take the matrix form transpose & inverse). then $X(B_0) \dashrightarrow X(B) \vee \check{X}(B_0) \dashleftarrow \check{X}(B_0)$ as their compactification ('without singularities' case $X(B_0)$ & $\check{X}(B_0)$ is done)

$$\begin{tikzcd} X(B_0) \arrow[r, dashed] \arrow[d] & X(B) \arrow[d] & \check{X}(B) \arrow[dl] & \check{X}(B_0) \arrow[l, dashed] \arrow[dll] \\ B_0 \arrow[r, hook] & B & B_0 \arrow[l, hook'] & \end{tikzcd}$$

$\exists$ topological fibration

Note that under this procedure, our tropical mfd lost and only satisfy at topological level.
It's originally proven by Gross, called topological MS.
However, the problem is it loses $\boxed{\text{all}}$ enumerative datum! It means we need additional combinatorial datum to carry enumerative datum. As summerize, $\boxed{\begin{array}{l}\text{topological} \longrightarrow \text{toric datum} \\ \text{algebraic/enumerative} \longrightarrow \text{tropical datum}\end{array}}$. The second method is even before Gross-Siebert project, due to Mikhalkin… apply tropical method solve many enumerative problems, and related with LG-LG MS by Sarkisov. (LG model of a geometric object is a Laurent polynomial, next we can see it's naturally a tropical structure) Finally by GS focus on SYZ's statement, they translate both A&B-model into combinatorial setting, and state MS by tropical ways:

The primary but we omit it. It's also a interesting conterpart of enumerative geometry.

$$\begin{tikzcd} \boxed{\text{A-model}} \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & \boxed{\text{Tropical Geometry}} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \boxed{\text{B-model}} \arrow[dl, leftrightarrow] \\ & \boxed{\text{Log Geometry}} & \end{tikzcd}$$

(Mirror symmetry)

Let's state the tropical geometry's basic ideas and consider counting tropical curves to solve enumerative problems to end this part (next part is HMS, then all ends)

Log Geometry (occurs naturally in degeneration/GH-limit)

Tropical analysis and geometry

"Tropical" is named by French mathematicians in honor of Imre Simon born in Brazil, a place located in tropical in earth. Not be confused with $\mathbb{R}$, due to complex tori also $\mathbb{R}^2/\mathbb{Z}^2$ start at $\mathbb{R}$, and $\mathbb{R} = B \Leftrightarrow \mathbb{T}^2 = X(B)$
The tropical semiring $(\underline{\mathbb{R}}, \oplus, \odot)$ is $a \oplus b = \min(a,b)$, $a \odot b = a + b$, and polynomial rings over it $(\mathbb{R}[x_1 \cdots x_m], \oplus, \odot)$
Note that it not equal to $\mathbb{T}[x_1 \cdots x_m]$ as we not allow the coefficent to be $-\infty$ (Here $\underline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, we denote this semi-ring as $\mathbb{T}$) (some time denote $\mathbb{R} \cup \{+\infty\}$ and replace $\oplus$ of max by min)
Thus we can have a hypersurface defined by a tropical polynomial $F = \sum C_{p_1 \cdots p_m} x_1^{p_1} \cdots x_m^{p_m}$. $V(F) \subset \mathbb{R}^m$, let $m=2$, we have tropical curve
How is it applied to enumerative study?
① It's easy to study due to the "ring" structure of $\mathbb{T}$, and polynomial ring is closed to classical algebraic case, and almost all arguments hold well after not-hard modifications. such as Bézout, RR… has their tropical analogues. (Later, more proceeding such as a tropical Brill-Noether thm is striking, and gives a proof to original one)
(Compare with classical one. due to its $\boxed{\text{polynomials are all interpreted as linear equality + inequality}}$, we can study it easier (later we'll see the geometry of tropical curves)
② From tropical back: originally idea given by Kontsevich, again, and completed a lot by Mikhalkin.

Tropical curves (Let's show "□" by examples!)
The problem of tori cases are the subspace constructed by toric method are only subtori, however, let $B = \mathbb{R}^2$, and a "curve" given by

$L \subset B$
$\shortparallel$
$l_1 \cup l_2 \cup l_3$
(with "boundary b")

, then we realise $X(L) \subset X(B)$ as gluing three "tori" $\subset (\mathbb{C}^*)^2$ (this allow us achieve higher genus)
and $L \subset B$ defined by $\{x_1 \geq b_1, x_2 = b_2, x_3 = b_3\} \cup \cdots \cup \cdots$
$= V(0 \oplus x_1 \oplus x_2)$ (if $b = 0$)

And

has same topological structure but different algebraic structure thus tropical is more than toric

Summary: $B \longmapsto X(B)$ "Quantization"

- Lattice $\subset \mathbb{R}^n \longmapsto$ Toric MS, topological
- Tropical variety $\subset \mathbb{R}^n \longmapsto$ MS, enumerative/algebraic

Here we see the SYZ done by back a "quanization" back to combinational curves in algebraic side (B-model) and without this process, direct to A-model. HMS also says this:

Basic ideas of HMS

Due to I had learnt a basic couse on HMS, we don't state it ~~and~~, but only expect to explain how HMS related with classical MS (It's my sought on the "data" this statement carries: morphism of Fuk(X) is Floer, ~~&~~ then differential are GW, thus it's are horizonal categoricalification.

HMS is the $\boxed{\text{most}}$ coheret mathmatical formalization of MS: we can recover our classical enumerative results from it

$$\begin{tikzcd}
\boxed{\mathrm{Fuk}(X) \underset{\substack{\text{derived}\\ \text{equivalence}}}{\cong} D^b(X^\vee)} \arrow[d] \arrow[d, shift left=12ex] & \\
GW \quad \underset{\nwarrow}{=} \quad \text{Period integral} &
\end{tikzcd}$$

, X cpt CY smooth variety (All of these three had been some way generalised) But indeed, it's expected that SYZ ($\in$ T-duality) $\Rightarrow$ HMS $\Rightarrow$

GW = Period integral (GW is had but Period integral is easy)

(They're derived invariant)

But both two not proven now (2025.1)

Rk. Originally, Kontsevich got his idea from physics (A & B-models of) superstring theory $=$ Classical one Representation theory of (different) A_∞-cat and derived equivalence gives same physics.

Top3. Derived Geometry

It's designed to introduce basic derived techniques in intersection theory & various geometry by Toen & --- But due to my own default on homological algebra, I think it's not a good time to learn it now (As a Junior)