Research Statement

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November 14, 2025

1 Foreword

This research statement is only for explaining one thing–what kind of mathematical taste do I prefer, not restricted to any fixed setting or problem.

It concludes my interest in geometry and topology up to junior, anticipating an increasing deeper understanding in later studies. For the background and progress of my studies down to earth, whether they exist or not, see my CV or other documents.

Modern geometry and topology have grown to such a large scale with various themes now, these themes concern various settings differing with each other greatly, calling for quite complicated techniques and deep, abstract structures. Hence this research statement is for explaining my taste to which theme I'm interested, but limited to the length of this article, I only discuss general philosophy in a not formal style.
Thus it's just an introduction based only on my memory or impression on my mind than concrete documents, it's more close to a story than an article, I apologize for some vague or false places in this research statement, which may be due to my unfamiliarity. Hence, I plan to fill the fundamental knowledge solidly applied in various brilliant programs during my postgraduate studies.

The main line of my own taste is to find a proper method to extract data from geometric object, such method can be quite abstract, but these "abstract nonsense" can be realized to down-to-earth computation by concrete examples. Hence, both the abstract study of structures, brilliant correspondence, or huge programs and concrete geometries are attractive to me.
You could understand all geometries easily and roughly as two questions: how much data do we need to record the structure; and how easy is it to do operations on the data we extracted. It means that a good theory can not be too much or too little, but in an intermediate level.
Those data are nothing more than the invariants: as a number, a polynomial, a group, and even a category or a moduli space. Invariants can't give us all information, thus we made it to be more and more coherent, but we expect that

we can get as much as we can, for example, before the gauge theory, differential topologists pursued proving something by Kirby calculus, which turned out to be wrong after using gauge-theoretic methods. **Invariants always tell us what we can't instead of what we can in classification problems.**

In terms of concrete themes in geometry, different themes come from different additional structures and require methods at different levels of fineness to extract enough data to study them. More additional structures will produce more interesting geometry, these structures provide more rigidity in order to detour the difficulty of classification under a too coarse identification, and hence new invariants are needed for these additional structures, by algebraic or analysis frameworks.

Another thing worth mentioning here, which occurs everywhere in different themes of geometry, is the deep impression of physics on mathematics. Although we do be able to explain "motivation" cases by cases, we expect a general and uniform reason for such a phenomenon: behaving like the character in a drama, physics under a unified framework comes into the pretty different stages of mathematics. Such entanglements do make a deep impression on me: if two closely related physics phenomena induce different mathematical results, then there should be a bridge indicates a equivalence of theories, as a "homotopy" rather than a single "isomorphism". For example, the correspondence of quantum field theory and string theory (or σ-model) indicates the correspondence of bundles or embedded objects with the maps, which is discussed precisely next in the 6th chapter, discussing enumerative problems.
I suppose that we can't stop at the level of taking physics as black-box, but understanding how it works to motivate related mathematics. Hence, I also plan to penetrate into physics deeper in the later study, meandering through the path of my study of mathematics.

Following the trace of the development of geometry, or by adding increasingly more structure to the most general ones, I'll list the themes I'm interested, which I have learned or planned to learn later in my postgraduate study. Don't forget our main line: let's see the techniques of extracting proper data and see how these techniques changes when the rigidity of different geometries change.

2 Algebraic topology

Algebraic topology studies the most general geometry objects–CW complexes (not only them but most major ones), which has a "good model" to do homological/homotopical algebra by simplicial methods. The Model category may be a parallel trial by axioms in the categorical level of homotopy theory, but here we do not focus on it.
Abstractness is the price of generality. Because of the generality of the algebraic topology, in an early age the study of algebraic topology had delved

into the world of structures deeply to an unbelievably abstract extent.

If we simply divide the algebraic topology into homology and homotopy, then the former can only be viewed as an application of homological algebra, but the latter was really leading the development of techniques and structures of homotopical algebra. The former can be extended to various geometric extents due to the generality of homological algebra, but the generalization of the latter one is more nontrivial, which may be caused by the deep understanding of the structure of simplicial structures, from topological to categorical, extending **everything**.
Simplicial structures turn out to be of pretty general occurrence in various geometric settings, especially algebraic geometry after the Grothendieck school. However, I think the most important technique is the homotopical methods in homological algebra. Theoretically, the simplicial method can be used to compute everything clearly, but besides the computation methods, the structure of chain-level and homology-level is also important.

Roughly speaking, the ∞-category is the usual category enhanced by simplicial sets, as an analog of the category of complexes, with the morphisms, as 0-simplexes in the simplicial complex enhanced, being connected by all higher homotopies. But it's not enough to do homological algebra.
The stable ∞-category is the right condition in chain-level, as it gives rise to a triangulated category in homology-level by taking the homotopy category, modulo the homotopy in the simplicial complex enhanced. Such a procedure is the generalization of the relatively classical homological algebra involving Abelian category and (classical) derived category.
In particular, the E_∞ and A_∞-conditions are the higher generalization of the communicative and associative condition with all the homotopical coherence needed, coming from the monoidal structure addressed in the chain-level categories. The recognition principle realizes them into loop spaces, as one of the models of the realm of ∞-category.

The bar construction is the canonical method to get a simplicial object in category with additional algebraic structures by resolute the relations of the algebraic structure to free one, the simplicial structure is given by the relations, relations between relations... The simplicial free resolution is similar to the telescope construction, by pushing the relations to infinity. For example, the construction of $\mathbb{B}G$ as a simplicial complex.

The K-theory and the cobordism theory are basic generalized cohomologies, concerning the vector bundles and the cobordism classes and expecting them reflecting the data of the original space. Their relationship is given by the classical result of Conner-Floyd. The classical application of K-theory lies in the differential topology and index theorems, I'll mention it in the fourth section; recently, the work of Efimov on the continuous K-theory and dualizable category is also interesting; the cobordism has a deep impression on algebraic topology

itself, with formal group law, as I'll mention below the notation of spectra.
Another interesting thing is that the notation of cobordism gives a categorical formalism to QFT, using the cobordism to represent the involution of space-time and using a functor to address the "fields" on the involution process. Different notations of "fields" define different functors, the advantage of this formalism is that we can define the functor associating the space with de Rham complex, then there is a natural method to make the involution Hamiltonian homotopical to 0, with some tricks in supersymmetry. Hence, we have the topological field theory, which is used in later contexts.

The unification of (co)homology theories and generalized (co)homology theories in algebraic topology is realized by the notation of spectra (Different with the realm of unifying the Weil cohomology in algebraic geometry by motive.), stated by the Brown representability theorem in stable homotopy theory, by associating different spectra to different (co)homology theories. Although the fundamental groups for objects in algebraic geometry can't be directly defined as topological ones, the homotopical method still works when we concerning cohomology.
The chromatic homotopy theory is named by Ravenel, as the different periodicities of some resolutions are similar to different colors of the rainbow. The main result is that the functor induced by the cobordism from the homotopy category of spectra to the category of quasi-coherent sheaves on the moduli space of formal group law is an equivalence. Different formal group law gives different spectra or different (co)homology theories such as the Morava K-theory and the Lubin-Tate spectra, which have pretty general applications in various topological and arithmetic fields. But I don't know enough details of these stories.

3 Low-dimensional topology

The primary invariants in algebraic topology, i.e. the fundamental and homology groups motivated the Poincare conjecture when we restrict to the world of (differential) manifolds, which is our main line in this story, asking how these invariants can determine the geometric object itself if we focus on some kinds of manifolds.
Other impact of homotopical theory is finding model similar to algebraic topology: manifolds have good cover by geodesic convex neighborhoods, then we have its nerves including all simplicial data. Furthermore, if we want an analog of the CW decomposition, we have handle body decomposition for manifolds.

The Poincare conjecture states that: if a closed n manifold is homotopical to the $\mathbb{S}^n$, then it's isomorphic to the $\mathbb{S}^n$. Here we refer to the word "manifold" as topological or smooth, and then the "isomorphic" is referred to as homeomorphic or diffeomorphic, respectively.
It took more than a century to solve the topological case, but the most sur-

prising thing was the first step around the 1960s: the h-cobordism (or Whitney trick) in dimension greater than 5 works directly prove the Poincare conjecture in these cases, showing an unexpected flexibility in higher dimensional geometric topology, one way to cancel such triviality is adding more structures.
The four-dimensional topological h-cobordism was proven later in 1982 by Freedman, but, as I said above, adding the smooth structure will break this, as smooth structures on a given manifold can be too "wild". For example, the four-dimensional Euclidean space admits infinite smooth structures. Hence, the smooth four-dimensional Poincare conjecture is still a mystery due to the breaking of the four-dimensional smooth h-cobordism.
As for the three-dimensional case, due to a classical theorem saying that any three-dimensional manifold admits unique smooth structure, the story of topological and smooth category are equivalent and both stopped at 2003, when Perelman proved Thurston geometrization theorem via geometric analysis, more precisely, the Ricci flow, which is not my major interest yet.

What attracts me most are the concrete surgeries in this story, which occur everywhere and serve as a main technique. They're full of geometric insight, may tighten by analysis languages or not, including moves of knots, handle decompositions or Kirby calculus, and so on. These fundamental blocks make up the topological toolbox, which would become a"cameo" in the next story.
The method to extract data is direct by looking at the geometric object directly, but when we apply it to some kind of geometric analysis, it becomes assistant to the indirect techniques, which is more powerful.

An interesting observation emerges in our insight when we compare the two kinds of Poincare conjecture in the topological and smooth categories. That is, when the rigidity is given more to the geometric category, admissible surgeries are less due to more structures need to be preserved, for example, in the symplectic category we can't do cutting and gluing so rashly, but we can use Luttinger surgery; in algebraic or complex analytic category surgery is even more impossible, but only blowing up can be used. Hence, less so-called "geometric insight" via flexible operations can be found under more rigidity.
Essentially, the so-called "rigidity" and "softness" are nothing more than the small and large scales of automorphism groups. For example, the isometry group is usually finite-dimensional, or even zero-dimensional, i.e. discrete; but the diffeomorphism group or symplectomorphism is usually infinite-dimensional. In the moduli space's viewpoint, automorphism groups determine to what extent we can deform such a manifold with additional structure. For example, the Riemannian surface with genus g and marking point n has the automorphism group whose dimension can be negative, when the genus is larger, the dimension decreases until a finite group, thus the rigidity increases; the same thing happens when n larger, thus adding markings is called rigidification in order to collapse the scale of automorphism.
Thus, when the automorphism is smaller, the local shape of the moduli space can determine more global properties, which is just what we mean by "rigidity"

for a **single** geometric object. In conclusion, "rigidity" and "softness" behave the same in the single and moduli levels uniformly.
When structures become increasingly abundant, the concepts adapted in different settings behave like "upgrading" in games. For example, the monodromy in topological settings is upgraded into the holonomy on curved Riemannian manifolds, consisting of two counterparts of obstructions, which corresponds to the Berry phase in physics.

My favorite example is the real two-dimensional, or complex one-dimensional objects, as it predicts higher-dimensional general cases. Even this case itself can reflect abundant and interesting geometry: compact surfaces are decomposed into $\mathbb{S}^2$, $\mathbb{T}^2$ and $\mathbb{RP}^2$ via connected sum; as for the oriented case we needed to make it compatible with complex or algebraic settings (Which is called the "**cornerstone**" in Griffith and Harris!), only $\mathbb{S}^2$ and $\mathbb{T}^2$ used. Then we can classify them into "no holes", "one hole" and "more than one hole", equivalent to classification under many different settings: Fano, Calabi-Yau and general type; or positive, zero and negative; or spherical, Euclidean and hyperbolic; or abelian, non-abelian and anabelian... Although we can't expect these classifications to hold for higher dimension anymore, these classifications are separately useful. Additionally, the theory of real two-dimensional case also contains abundant theory, including global analysis, topology, hyperbolic geometry, algebraic geometry, geometric class field theory, the Riemann hypothesis over finite fields, and so on. What is highly nontrivial and interesting is the analog and difference compared with higher-dimensional cases, leading to a variety of general theories.

The moduli problem that arose from these extreme easy but nontrivial examples can be pretty interesting, for example, the Teichmuller spaces, as a synthetic place of hyperbolic geometry, algebraic geometry, dynamic systems, representation theory, even anabelian geometry. Hyperbolic geometry, as a fundamental and pretty important part of geometric topology, serves as "general" objects in geometric topology, which has interesting phenomena in lower dimensional cases, distinct from 2 and ≥ 3 dimensional cases, with rigidity theorem holds or not.

The basic fact that the hyperbolic isometry is given by the conformal transform of its boundary is developed into several highly nontrivial programs.
One is the relations with the number theory. First it's used in the theory of modular forms, which indicates that the symmetry of hyperbolic spaces is related to the symmetry given by Galois representations by classical Langlands programs. Second, we have the correspondence between prime geodesics in hyperbolic 2-spaces and the prime numbers, which is from the analogy between the Galois groups and the Deck transformation groups, i.e. the fundamental groups. Such two theories are both some sides of geometric Langlands program, for example, some works by Beilison and Drinfeld on some algebraic variants of conformal field theory indicate that there is a correspondence between the modular form and the vertex algebra.

Another is the holography principle in physics. The baby fact above leads to the AdS-CFT duality in physics, which indicates a correspondence between gravity in space-time and conformal field theory in its boundary. To be honest, I know nothing about its details.

4 Gauge theory

The gauge theory is motivated from physics, more precisely, instanton theory or Yang-Mills theory, but used to deal with differential topology.
When we focus on "wild" smooth four-dimensional world, the direct method is replaced by indirect technique of constructing moduli spaces via global analysis, which was first discovered by Donaldson in his Ph.D thesis. It generalizes global analysis on manifolds to vector bundle valued, enriching the target of usual functions to vector bundle valued in order to grab more data.

To understand why the indirect method of global analysis on vector bundles is so powerful, we go back to classical geometric analysis. The relationship of global analysis and geometry and topology is a story much earlier, starting from the Gauss-Bonnet formula in differential geometry, which used analysis to compute topological invariants.
But for differential topology, we need finer invariants, thus it's asking for a more coherent technique. The story of classical geometric analysis is only the study of its tangent bundle, but the more general vector bundle can encode more differential structure data by the connection structure on it, thus the moduli of special connections will reflect more delicate differential structure of the base manifold, just analogous to that parameterizing vector bundles themselves can reflect topological invariants of the base manifold. Hence, these moduli spaces are believed to be useful when we need to construct some differential invariants.

A side story: when the value is lifted to vector bundles, these formulas are developed to index theorems for computing dimension, or other invariants of moduli spaces now. In philosophy, it's an "infinitesimal" viewpoint to study the global geometry.
It can be formulated as the pair between K-homology and K-cohomology (K-theory) compatible with the pair between periodic cyclic homology and periodic cyclic cohomology, where the former is given by the analytic index and the latter is the topological index, forming a communicative diagram connected by the Chern character from the K side to the cyclic side. Such a formalism of using commutative diagrams to represent formulas is also used in the brother of the index formula–the Grothendieck-Riemann-Roch formula, but settings are replaced by K-theory and Chow.
The story above can be lifted to the categorical level by considering the K-theory of a triangulated category, and then the result formula will compute the dimension of the moduli of objects of this category.

From the viewpoint of path integral formalism, it's not surprising that we can use physics to detect geometric structure: the "holes" of base manifold make two paths with same start and end points have different propagators; hence, this topological datum can be seen in the same manner as some kinds of homology theory, for example, the BRST (Becchi-Rouet-Stora-Tyutin) homology, or anomaly in physical language. And with additional structure put, it's detected by changing the Lagrangian in path integral.
Various quantum field theories (QFT) induced by varying the additional structure in base manifolds are believed and verified to be powerful in different branches of mathematics. As mathematicians, we can simply view physics as geometries of well-chosen moduli spaces with physical meaning. For example, the spin structure, originally came from particle physics or QFT, which essentially can be interpreted in representation theory, is a kind of "2-oriented" condition, i.e. making the moduli become orientable; the σ-model is the moduli of maps, and so on.

When we study the moduli spaces, there are many properties needed to be checked, two of the most important properties are the transversality and the compactness. The compactification is given by stability conditions: the boundary need to be added is collection of the degeneration, or equivalently, the limited ones of its interior. As a modification to Donaldson theory later, Seiberg-Witten theory, or monopole theory, came into our stage with an occasionally compact property of the moduli because of the good analytic form of the Seiberg-Witten equations. Due to this automatic closeness, who came from behind is now more popular.
In this story, one shall note that the stability of category corresponds to the compactness of moduli of stable objects. But the notion of stability here is not compatible with the stability in "stable" homotopy theory: the former comes from the stable critical points in analysis, while the latter comes from taking limit to infinity.

Another main line of gauge theory is its application to GIT (Geometric invariant theory), by the Hitchin-Kobayashi correspondence, saying that the gauge-theoretic moduli space is isomorphic to the moduli space of vector bundles with poly-stable conditions, which is a bridge of symplectic reduction and GIT quotient in infinite dimensional cases, called Kempf-Ness theorem in finite dimensional cases.
Strengthening the stability to stable we can get submoduli spaces called Mumford-Narashimhan-Seshadri, which are more familiar with algebraic geometers, via another bridge of Riemann-Hilbert related to some kinds of representation of fundamental group. Such a style of results concerning the representation of fundamental groups is this main line of gauge theory, leading to more studies on Higgs bundles and non-abelian Hodge correspondences.
This algebraic geometric interpretation made it more applicable to the geometric Langlands program as the duality of abelian gauge theory ought to Witten, for example, the electric-magnetic duality, which is just an extension of the basic

idea of gauge theory as representation theory.

5 Symplectic geometry

The starting point is the equivalence between handle decomposition and Morse theory, more precisely, different Morse functions are related topologically by handle slide, thus viewing them built by handle bodies or level sets is essentially expected as the same thing. In homology level, Morse homology is usually as same as topological ones. Now we take the Morse viewpoint into our use in some moduli space coming from a symplectic manifold, with some functional as "Morse function", which varies in different problems, and consider its Morse homology as an indirect invariant for the original symplectic manifold. Recall that Morse-type homology has three ingredients: critical points of Morse function, an index for grading and an equation for flow lines, they're chosen differently and cleverly in abundant scenes.
The Morse-type homologies used in the lower-dimensional topology are renamed differently, as "XX" Floer homology to memorize Floer's first discovery of infinite-dimensional Morse theory. Just as the Morse homology can be viewed as a coherent algebraic form of Morse theory, these Floer homology can be viewed as a coherent algebraic form of gauge theory, where the symplectic manifolds are just those moduli spaces in gauge theory.
Counting the J-holomorphic curves can be viewed as a simplified case of counting flow lines in Floer theory to defining the differential, but with homogeneous Cauchy-Riemann equations. This special case is equivalent to the virtual counting in algebraic geometry, indicated by mirror symmetry.

It's Weinstein's philosophy that "All are Lagrangians". By choosing a certain type of Lagrangian in symplectic moduli spaces, all topological Floer theories in gauge theory are expected to coincide with Lagrangian Floer theory. However, such equivalence can be hopeless to be completely proven now, for example, the Atiyah-Floer conjecture, but it sometimes works for some simpler moduli spaces, too.
The difficulty lies on the too many technical restrictions of Floer theory, as we need the compactness and transversality of the moduli of the solution to Floer equations to define the differential d of the Floer chain complex with $d^2 = 0$, i.e. without bubbles. For example, for the noncompact Lagrangians, the wrapped ones are needed, motivated by imitating the cotangent bundle. Or for excluding the bubbles, we often acquire conditions like monotone symplectic manifolds or exact Lagrangians.

These categorifications of the Floer theory are called Fukaya categories. On these categories, we can apply algebraic techniques in algebraic topology.
For example, an interesting study of the Floer theory is the homotopy type, similar to the classical works of the de Rham homotopy type. From stable homotopy theory we pursue the correspondence between the homotopy and co-

homology, especially here we have a cohomology theory of the Morse type, which is analytic instead topological in technical level.
Another, and the most important application of the algebraic topology to the algebraic structure of Fukaya category is the A_∞-structure, which arises naturally in Morse theory as chain-level, with these higher associativity being given by counting "broken flow lines", which is the limit of flow lines. With respect to the complexity of the A_∞-structure, there is no doubt that the chain-level contains more data than the homology-level, thus one enhanced the classical Donaldson-Fukaya category whose morphisms are Floer homology into the Fukaya category whose morphisms are chains. As we expected, it has better properties, for example, finding generators.
In such categorical level, we have many representation-theoretic methods applied to these category, as they usually occur as some module category. For example, how can we pass chain-level categories to homology-level categories by localizations, with additional structure of chain-level ones preserved? And how can we find the split generators of the category if exists? These categorical operations are motivated by module category studied in representation theory, which is also useful in the study of algebraic derived categories, and these generators are expected to behave well under mirror symmetry with precise geometric meaning. Such algebraic work in the Fukaya category corresponding to geometric operations is viewed as a noncommunicative geometry method.

The study of the sheaf category is another approach toward Fukaya category, started at Kashiwara and Schapira's microlocal study to sheaves, more precisely, the microsupport of some kinds of sheaves on cotangent bundle is coisotropic. It has ample algebraic theory, related to D-module, cluster algebra, and so on, thus which is expected to provide an alternative to tighten the construction of Fukaya category. It had be generalize to a kind of Weinstein manifolds which are close to cotangent bundles, to be equivalent to the wrapped ones.
From this viewpoint, homological mirror symmetry can be explained as the correspondence of the constructible sheaves on a real manifold and the coherent sheaves on the mirror of its cotangent bundle. This can be viewed as a correspondence between topology (constructible) and geometry (coherent).
It turns out that sheaf-theoretic method is powerful when we handle problems about Lagrangian filling of Legendrian knots, as the Legendrians are served as the intersection of Lagrangians with the sphere cotangent bundle in a cotangent bundle. Its correspondence to cluster algebra is attractive to me.

The generality of symplectic geometry lies four reasons: the first is that the moduli space usually carries a symplectic form given by the cup product of cohomology, which is just the tangent space of moduli by deformation theory; the second is the Lagrangian correspondence, or specially, the symplectic reduction, which works in the construction of moduli space, for example, the Kempf-Ness theorem in GIT mentioned above; the third is indicated by the mirror symmetry, as the other side is algebraic, which doesn't need to attach any importance to it; the last is the symplectic geometry on the cotangent bundle corresponds to the

topology on the manifold, for example, $\mathcal{L}\wr\rfloor_G(M) \cong Bun_G(T^*M)$ by Hitchin.
Due to these motivations, my interest on symplectic geometry doesn't lie in explicit geometry and topology of certain types of symplectic manifolds majorly but general abstract formalism, especially the algebraic methods, which are expected to be applied into some place where symplectic form is discovered naturally, and the problem we concern can be rewritten as Lagrangian correspondence. For example, the well-known Heegaard-Floer theory focuses on the symplectic manifold coming from the symmetric product of the knot surface, and its two special Lagrangians are products of generators of the first homology of two handlebodies divided by the knot surface; this Lagrangian Floer theory is relatively simple and close to combination.

As for my interest in mirror symmetry, I postpone it to the next story. One has claimed that all the above story, including the mirror symmetry, can only indicate the "rigid" part of symplectic geometry, I can't agree more, as there is concrete example showing that the Fukaya category can't recover the symplectic manifold itself. However, now my interest in this area is still in the "rigid" part and mirror symmetry.

6 Algebraic geometry

The fundamental problem of algebraic geometry is classification under algebraic isomorphism, but due to its complexity, we study classification under birational equivalence, which is coarser. Geographically, fixing several birational invariants as the coordinates of a map, we have two questions to ask. Is a point on the map empty? If not, what does the moduli modulo isomorphism look like? Hence, the central problem of "geometric" algebraic geometry is studying the algebraic moduli space.
Another topic, used in studying the moduli, is the imitation of "topological" theories into the algebraic framework with higher rigidity, such as the intersection theory on algebraic setting and so on, intersection theory on moduli spaces gives arise to the enumerative geometry. One thing to note is that the deformation in algebraic geometry is different from topological ones, as it can preserve the topological shape but deform the algebraic or complex structure. The deformation theory is the local study of moduli spaces.
These needs, including the classification and translation of topological tricks into algebraic nature, motivate the more abstract formalism by expanding the category, which makes modern algebraic geometry a language.

Mirror symmetry predicted such a correspondence via the A-model and B-model in the S-duality occurring in superstring theory, on pairs of Calabi-Yau manifolds: enumerative geometry and Hodge theory, symplectic geometry and complex algebraic geometry, derived Fukaya category and derived category of coherent sheaves, coming into their stage in different levels.
Compared with the symplectic side, the B-model derived category can recover

and determine more geometry of the geometric object itself due to its rigidity. We can begin this story from the Abel-Jacobi theory studying the complex, or algebraic structures on topological objects: the Hodge structure on cohomologies corresponds to the algebraic structure on geometric objects; then the Torelli-type theorems are lifted to categorical level by expecting that the derived category can recover almost all data of geometry itself, many works are by Orlov and so on.
Such a type of results are also representation-theoretic or noncommutative methods. For example, the Beilison theorem says that the exceptional sequence of $\mathbb{P}^n$ is $\mathcal{O}$, $\mathcal{O}(1)$, $\ldots$, $\mathcal{O}(n)$, and the tilting object is given by the direct sum of these, and we can also give a quiver to serve as the path algebra of the endomorphism of the tilting module. Hence, it gives a simple description of the derived category by tilting theory.

Outside the Calabi-Yau cases, there are also non-geometric objects with potential serving as the mirror of non-Calabi-Yau manifolds. Just as what we said about the classification into Fano, Calabi-Yau and general type, generalizing the subtle Calabi-Yau case to more general cases and coming back to this special case is FJRW (Fan-Javis-Ruan-Witten) theory, by varying the phase. The FJRW theory, for extending the Calabi-Yau cases, majorly states a correspondence of the LG (Landau-Ginzburg) models by relating the A-model with LG potential with the matrix factorization category by Chern character, which is easier to compute, and giving a correspondence to the B-model category.

The harmony of algebraic and analytic is presented in not only conclusions but also techniques. For example, when we construct and prove good properties of moduli spaces, the fundamental question is taking limits, in algebraic language, the valuation criteria, then it's realized by degeneration of an algebraic family; the boundary of moduli space in analytic cases is formalized by real codimension 1 boundary which is not admitted in algebraic setting, but replaced by boundary divisors. Another important example is the analog between algebraic groups and Lie groups with their actions and quotients, respectively.
However, it's admitted that the algebraic side, thanks to the abstract framework, has much more advantages than the analytic story: the infinitesimal in algebraic moduli problems has precise description modeled by the ring of dual numbers or other Artinian algebras, although we still have to face technical problems of compactness and transversality. Equivalently, the derived structures on algebraic objects are simpler than what we expected to exist in different geometry, hence the try of so-called "derived differential geometry" instead of derived algebraic geometry is still an unending story.

Pursuing a well-behaved moduli space acquires three conditions: automorphisms not too bad, compact or proper, and transversal or good deformation properties. The first is realized by GIT methods, for example, the rigidification of the curves by marking points. A pleasing viewpoint of this rigidification is **viewing markings as punctures**, which diffeomorphic to the real dimension 1 boundary can be placed in Lagrangians in the symplectic setting, which is said to have made a deep impression on Grothendieck's anabelian geometry. The second is discussed briefly above, and the third is too hard and too technical in most cases.

Counting problem is also occurring in both areas, which acquires enough rigidity to make the number finite, in symplectic setting it's Cauchy-Riemann equations, in algebraic setting it's the algebraic nature. From the algebraic viewpoint, it's important to have a viewpoint of switching our mind of counting embedded curves to counting parametrized curves, more precisely, counting maps. The former is more considered in algebraic geometry as we view the embedded sub-object as the ideal sheaf, the latter is more considered in differential geometry due to the lack of sheaf language. Here our counting of stable maps in GW theory is a harmony of algebraic and symplectic worlds too, which both fallen into the A-model side, while counting sheaves, for example, the DT (Donaldson-Thomas) theory, are more believed to fall into the B-model side. I suppose these two viewpoints are linked closely by DT-GW correspondence, or MNOP (Maulik-Nekrasov-Okounkov-Pandharipande) conjecture, which can also be thought of as a version of mirror symmetry.
Such a switching of viewpoint in physics corresponds to the equivalence of QFT and string theory (or nonlinear σ model of QFT), i.e. the fields on spacetime and worldsheets. For example, the worldsheet is real 2 dimensional spanned by one dimension of time and one dimension of real dimension 1 string, thus may be fitted into algebraic curves or J-holomorphic curves.
I prefer strings or worldsheets as they are much more concrete and **drawable (and funny)** than bundles or sheaves.
The main difference between symplectic geometry and algebraic geometry in counting can be viewed as the difference between open-string and closed-string. We can simply take the open case as bounded while the closed case doesn't have boundary, thus in symplectic geometry, the open string is more usual as we need the boundary condition lies in the Lagrangians, serving as D-branes ("D" means Dirichlet boundary condition.); but the GW (Gromov-Witten) theory, both algebraic and symplectic, is closed, their B-model D-branes are more complicated. The closed-to-open map maps from the quantum cohomology to the Floer cohomology, after categorification, it's to the Hochschild cohomology of Fukaya category, which is roughly realized by "cutting" the closed string (loop) to open one (segment) by D-branes.

Besides the works on moduli spaces, a pretty interesting thing is the global section-spectrum correspondence. What it attracts me most is not the algebraic setting itself but the unification of the "spectral theory" in different areas of mathematics via the representation theory; for example, consider the spectrum of $\mathbb{C}[T]$, which are primes $(T-\lambda)$; when we extend an operator T on vector space V by functional calculus to a $\mathbb{C}[T]$-representation on V, then V as $\mathbb{C}[T]$-module is equivalent to the coherent sheaf on $\mathbb{A}^1$ with fiber at λ is $V/Ker(T-\lambda)$, which is nothing more than the co-eigensubspace of operator T with eigenvalue λ. What does this baby example tell us? It gives us one way

to identify the spectrum in algebraic geometry and the spectrum of operators, which is meaningful in the Cartier duality, in general saying that, for G good, the representation of G corresponds to the sheaf on $\check{G} = Hom(G, \mathbb{G}_m)$, i.e. the equivalence between categories $\mathcal{R}\rceil_{\surd}(G) \simeq QCoh(\check{G})$, as a baby case of the geometric Langlands program.
Recently, there are some higher works on the generalization of the section-spectrum, from the affine case only to global geometry. Roughly, it's called 1-affineness by Gaitsgory originally, showing the ability of Tannakian reconstruction from algebra (symmetric monoidal category) to geometry by taking *Spec* functor in categorical level. But this notation has very interesting behavior for stacks: for example, $B^i\mathbb{G}_m$ is 1-affine when $0 \leq i \leq 3$, but not 1-affine when i is larger. This is out of our original expectation: due to $QCoh(BG) \simeq \mathcal{R}\rceil_{\surd}(G)$ for G reductive, it's not surprising that BG are all 1-affine in good cases, but higher delooping acquire higher affineness.
Apply the spectrum of symmetric monoidal categories above by Tannakian reconstruction, the two equivalence above induce the isomorphism up to shift 1, called the shift 1 Cartier duality, i.e. the delooping functor shift 1 in some category large enough.

The (geometric) Langlands program says that there is a relationship between two (function spaces on) moduli spaces, as a generalization of Plancherel equality in Fourier theory, which is a version of the generalization of Fourier-Mukai transform, also called "the 4-dimensional analog of mirror symmetry". But what is more attractive for me is the more challenging arithmetic side: we study the more complicated side–the Galois representation by something with abundant symmetries. It shows that symmetry is at the crossroads of the harmony of number theory and geometry.
The two sides of Langlands program are also those so-called A-side and B-side by Ben-Zvi, while the A-side is the automorphic one and the B-side is the arithmetic one. The A-side is the geometric extension of modular forms by a special eigenfunction space on the bi-quotient space by lattice and intrinsic symmetry, which is referred to the local symmetric space. The B-side is precisely the collection of Galois or fundamental group representations; but one should note that there is a difference between the Galois and fundamental group cases– Galois groups are much harder than the A-side while the fundamental group of Riemannian surfaces is well-known. I suppose this is also why we think the geometrization of the Langlands program is important: we can use the geometric B-side to study the arithmetic B-side via the A-side. It's motivated by the Galois representation arising from geometry, serving as the cohomology of some geometric objects, whose fundamental example is the Tate module on elliptic curves.

Another pretty interesting thing about the Langlands program is the answer to why it's an equivalence of 4-dimensional TQFT (topological quantum field theory) than the 2-dimensional TQFT of mirror symmetry via the arithmetic

topology in the B-side. In the mirror symmetry we consider the worldsheet, i.e. the moving of string by time, which forms algebraic curves or J-holomorphic curves, are two dimensional; but the number theoretic there is an analog of number fields with the 3-manifolds, adding the time axis or the cobordism issue, it's 4-dimensional. The analog above, due to Mazur, is an enhanced version of Weil's geometric analog of number fields and the algebraic curves: we found that the residue fields with Frobenius have nontrivial absolute Galois group $\hat{\mathbb{Z}}$, this non-geometric issue causes a $\mathbb{S}^1$-fibration on the geometric analogous curve above, whose analog then is just a 3-manifold. The analog, which is far from rigorous, may be hopefully reformulated rigorously in the (geometric) Langlands program and build the bridge to physics in a safer setting than geometers expected.
The TQFT method is originally used in geometric Langlands in order to deal with the non-abelian case, which is also the reason of the requirement of the complicated form of automorphic form instead of the cleaner D-modules or eigensheaves in the Riemann-Hilbert correspondence.

My interest in algebraic geometric side majorly lies in the analogous, or essentially intersected, or dual part of geometric or topological theories, which is more down-to-earth, as otherwise the rigidity of algebraic world will make it harder to imagine through concrete examples, which are either too trivial or too complicated. But what makes algebraic geometry more attractive than classical geometry and topology is the use of abstract and coherent languages. It provides a completely new and higher viewpoint to revisit geometry, might via functors, might via sheaves, might via categories, and so on, which helps a lot in the dream of unification thanks to Grothendieck. With such a challenging task of switching our mind in different levels of perspective, I prefer the theory of moduli spaces and enumerative problems; I also like concrete models such as toric models, tropical method applied to classical intersection theory and modern mirror symmetry, and so on, but it's something hard for me to keep the geometric insight in my mind when I study related topics on the minimal model program, thus I can only take the birational geometry as something technical in construction of moduli spaces.

7 Arithmetic geometry

It's hard to totally separate the so-called "algebraic geometry" and "arithmetic geometry", thus some related topics have been mentioned in the above section, by the analog between function fields (geometric cases) and number fields (arithmetic cases). Here we focus on the latter, but using geometric method over them.
Starting at Weil's proof to Fermat's last theorem (FLT), number theorists began to focus on the relationship between the elliptic curves and Galois representation induced from them and the modularity, which can be viewed as the origin of the Langlands program, which is linked with geometry naturally. It asks for

a general approach to algebraic geometry over more general fields during that time, more precisely, FLT is over Q, which has less problem; but Weil conjecture is over $\mathbb{F}_q$, which is more nontrivial.
Historically, Weil developed his masterpieces at that time after World War II for his proof of curve case in prison, then replaced by Grothendieck's general work later for the general case of Weil conjecture. There is no doubt that the Weil conjecture motivates the development of modern algebraic geometry and arithmetic geometry.

For the basic settings, it's Grothendieck's philosophy that we can consider "geometric" objects over more general objects and its non-geometric points. Then with Galois actions equipped on X if it's over a non-algebraically closed field k, the study of a given object is divided into two parts: the geometric part and the Galois action of its algebraic closure part. For example, the k-points of X are the fixed points of the Frobenius on $X_{\bar{k}}$, this is a process of "folding" when we base change to algebraically closed fields, where transform the intrinsic symmetry into something like the quotients. Viewing this folding as the "arithmetic covering spaces", this can be more precisely shown in the etale fundamental group written as an "entanglement" of the topological fundamental group and the absolute Galois group.
I'd like to use such "entanglement" as a leading example to depict a taste of mathematics which I pursue: when we appreciate the analog between quite different stages, we must ask if there is possibility making them essentially the same in a higher viewpoint of mathematics for the rigorousness. Oort said in his lecture note: place yourself in the situation that you know these correspondences in Galois theory and in the topological theory of coverings, and try to find a general theory of which these two are special cases. It's exciting that one day we can create a "mountain" high enough to overview, making all analog sub-theories of some general theory.

The finite fields, or the closed points on universal $Spec(\mathbb{Z})$, usually work in the technique of specialization and generalization to the generic point over $\mathbb{Q}$ to simplify our geometric arguments, which makes it easier to make sense for algebraic geometers. When we're working over them, it's easy to compare with algebraic geometry, as we can base change $\mathbb{F}_q$ to some power of it to get some geometric properties.
However, those p-adic ones, or more general non-archimedean local fields and other classical "arithmetic" objects, turning out also to be useful in various geometry settings, is pretty surprising: besides its own importance in number theory served as a general place, as far as I know, the family Floer theory by Abouzaid and the quantum D-modules by Iritani school, which is helpful in a try of Kontsevich toward the irrationality of cubic fourfolds. It's just due to the simple technique reason: the convergence behaves much better in local and non-archimedean case than in the analytic settings.

When we come to the land of Weil conjecture, or the Riemann conjecture for

varieties over finite fields, many new techniques blow up.
One of the most important techniques motivated by Weil conjecture is the etale cohomology, as an imitation of topological one, this makes us to face an important problem directly, which is mentioned above as the rigidity and softness. That is, what is the difference between topology and geometry? I was confused with this meaningless problem about two years during my undergraduate study, but here is a so simple answer by sheaf language–the geometry of X is the module category over the structure sheaf $\mathcal{O}_X$, while the topology of X is the module category over the constant sheaves $\mathbb{Z}_X$ or $\mathbb{Q}_X$.
Hence, the etale cohomology is built on the sheaves in topology, then one might view all techniques to be nothing more than the algebraic topology, with the arithmetic analog of homological and homotopy groups. There are many other general techniques based on this language, which is used in the proof to Weil conjecture, such as the Lefschetz theory (Deligne's proof), monodromy theorem, the nearby and vanishing cycles formalism (Schroll-Katz's proof), yoga of weights, and so on, which are the origin of his brilliant pursuit after Weil II. For example, the study of perverse sheaves in BBD and the study of mixed weight cohomology in Hodge I-III are both attractive to me.
The idea of the proof of Weil conjecture by Schroll and Katz is: first reduce to top dimension by Poincare duality and weak Lefschetz; then reduce to hypersurface by birational equivalence to hypersurface, and apply the excision sequence; then reduce to smooth hypersurface by degeneration and the local monodromy around the singular fiber; then reduce to a special hypersurface, for example, the diagonal hypersurface, which is proven by Dwork, by deformation and Rankin's trick. Differently from the proof by Deligne, which is by reducing to curve case by dimension induction via Lefschetz pencil.

There are too many interesting topics in arithmetic which I know nothing about, such as the deformation of Galois representations and crystals by Witt vectors, various rigid cohomologies and prisms, the p-adic Hodge theory and Fontaine-Fargues curves, even the arithmetic mirror symmetry and so on. They're also full of geometric taste and attractive for me.

8 Noncommutative geometry

We had used the philosophy of "commutative geometry" a lot in these sections above, the noncommutative geometry is for keeping in track on the geometric data when the algebraic structure is not commutative, where the "good duality" is broken.
Historically, noncommutative geometry is motivated by quantum mechanics, in mathematical language, we can deform the function ring of the cotangent bundle $\mathcal{O}(T^*X)$ to the ring of differential operators $\mathcal{D}_X$, i.e. the quantization is a procedure like deformation. For example, let $X = \mathbb{A}^1_{\mathbb{C}}$, the family of rings $\mathbb{C}\langle x,p\rangle/(xp - px = i\hbar)$ parameterized by $\hbar$ are $\mathcal{O}(T^*X)$ when $\hbar = 0$, are $\mathcal{D}_X$ when $\hbar = 1$.

From here, one can notice that one of the generalities of the noncommutative setting is the deformation or perturbation create noncommutative nature, analogous to the differential operators and Taylor expansion, which can also be viewed as a motivation of derived language, indicated by works of Kontsevich.

For various algebraic structures on the algebra of some kinds of functions for different geometric structures: metric, smooth, and so on, we can move our eyes from geometry to algebra safely if our algebraic structure is chosen well. Then we use these algebraic structures to replace our concept of "space".
Such motivation may have different philosophical explanations, but for mathematics, what makes a deep impression on me is the thesis of Connes, who told us that the non-communication creates an additional "god-given" time axis, via the classification of factors of von Neumann algebras. In one word, it says that: ***commutativity indicates geometry, noncommutativity indicates time.***

The most formulation of noncommutative geometry, by Connes and so on, might be close with functional analysis and operator algebra, and those later formulations, such as the study of the derived category in algebraic geometry and the study of A_∞-category in symplectic geometry, are also called noncommutative geometry. Thus we are not talking about a concrete construction of technique, but a general and highly important philosophy.
All so-called noncommutative geometries are using the algebra of functions to study the space itself, due to Grothendieck's sheaf-to-function dictionary, it's also natural to consider the algebraic structure of all sheaves, i.e. the derived category with tensor product. For the commutative cases, we can have the spectrum functor applied to rings, or the Tannakian reconstruction in categorical level, or some classical theorem like the Gelfand-Naimark theorem or the Serre-Swan theorem. But for noncommutative cases, the road form algebra to geometry is nontrivial, although we refer the study of noncommutative algebra as the "noncommutative geometry", it can't always be realized as classical geometric objects, but used some other constructions, for example, the groupoids, this means that the deformation of algebraic structure creates new nontrivial geometric issue. Hence, someone simply study the algebraic structure themselves without referring to the concrete geometric constructions.

This provides us another way to understand the concept of "stacks": its also an entanglement between geometric part and algebraic part, as a classical geometric object equipped with a group action, just as what we have done in defining the arithmetic fundamental groups, it can be viewed as a decomposition into a single point with automorphism and an automorphism-free classical geometric object. Such an entanglement is not the quotient, which creates singularities, but follows the hidden smoothness principle, which is a special case of the general hidden smooth principle in derived geometry, as it's indicated in the next section. The standard example is the classifying stack $\mathbb{B}G = [\bullet/G]$ as a quotient stack, it doesn't have any "geometry" under the classical meaning.

Either in the study of operator algebra or the study of derived categories, or even Lie algebras, we use the techniques from representation theory, i.e. study the category of modules. These techniques are for the core problem is this program–Can we reconstruct the geometric completely form the algebra structure arising form the functions on it? If not, how much data can we get?
From the sections above, we have seen that the general answer is not, but luckily, constructing invariants is possible.

9 Derived geometry

Due to under both the algebraic or analytic settings, the singular and stacky issues occur naturally, the derived formalism is expected to deal with them by detouring along higher structures which allow us to detect higher equivalence relations. This is what Kontsevich said in his famous hidden smoothness principle, meaning that the higher structure is the price of smoothness. Such a principle can also be used to explain why the theory is powerful in intersection theory, for example the construction of virtual fundamental classes: smoothness of moduli is the same thing as the perturbation toward transversal, thus adding derived structure is expected to give hidden perturbation data to it, i.e. hidden intersection class and so on.
This language is well developed in algebraic settings, but there are also some trials in smooth or symplectic settings. But except these lack (might) of useful systematic work, the topics about the shift symplectic structure, derived *dg*-geometry and Lie ∞-groupoid balabala are focused in derived geometry. Different settings give additional conditions on the simplicial resolution.

Motivated by the asymptotic expansion, or the homological perturbation theory, or quantization, developed in *dg*-geometry settings, i.e. BRST-BV (Batalin-Vilkovisky) formalism, in analysis or physics, I suppose that the derived formalism should be as powerful as the Taylor expansion in analysis in the algebraic setting, which serves as the “approximation” by well-behaved ones–chosen as smooth ones here and via simplicial approximation language. The BRST complex is an example of derived critical locus in the derived language, which is much later than the physical concepts, thus it's hopeful to rewrite the physics by derived geometry. The *dg*-manifolds can be viewed as a generalization to the notation of supermanifolds in physics.

The role of stable homotopy theory in derived geometry, or the role of the so-called homotopical algebra in homological algebra mentioned above, is to provide the basic objects by spectrum (in stable homotopy theory), which naturally extend the classical algebraic geometric objects, for example, the “spectrum of spectrum” $Spec(\mathbb{S})$ which extends the universal $Spec(\mathbb{Z})$. We can consider its Bousfield localization to each prime and many other analogous operations to algebraic geometry, while the simplicial enrichment makes us able to detect all higher structures of geometric objects over it.

Sadly, even to describe the object we study acquires a huge amount of details of simplicial languages in categories, which I'm still not familiar with.

I plan to study its application the geometric Langlands program modernly, with derived structure, which developed the stacky work of Beilison and Drinfeld into the derived setting, as a derived equivalence of the category of quasi-coherent sheaves on the geometry of the B-side and the category of D-modules on the geometry of the A-side. My eagerness is motivated by the fact that the representation theory has taken an increasingly important participation in the modern study of geometry with group action and quotient everywhere. Another reason is the "relation" with homological mirror symmetry: using the theory of microlocal sheaves, we can identify the A-side to some Fukaya category on its cotangent bundle, then it might become a version of homological mirror symmetry. However, this rough idea is quite unbelievable, as these moduli are highly stacky and singular; and a reduction from 4-dimensional TQFT to 2-dimensional TQFT is highly nontrivial, although we know the reduction from 2-dimensional chiral QFT to 1-dimensional QFT.
Another one for intersection theory is more familiar with geometers in constructing virtual fundamental class mentioned above, which means recovering the intersection data in general case even when the real case is exceptional, as the deformation acquired is controlled by complexes, or cohomology, which is derived in nature.

10 Summarization

In conclusion, there are so many analogs between mathematics and physics; and between different specific fields in mathematics. These analogs, which are called the "***metaphor***" by Manin, can be seen as a combination of the similarity of philosophy and the essential difference of natural settings or languages. I suppose what attracts mathematicians the most is the spirit, existing and staying invariant when our journey passed different worlds in mathematics, like the pillar in the middle of a spiral staircase. Hence, who pursuing mathematics itself shouldn't be manacled masochistically by any particular field, but try our best to dig and appreciate the hole mathematics we prefer: although they're not essentially same, we can revisit them together in their spirit is the same.

As for my taste in geometry in general, I prefer geometry satisfying two conditions: one is the beauty–it should present a brilliant correspondence between things seem to be unrelated at first, for example, the geometric Langlands program and homological mirror symmetry; another is the concreteness–I'm always scared about the fact that sometimes we can only lose our geometric insight and sell our soul to the machine of algebra or analysis to prove something obvious in geometric viewpoint. Such a preference doesn't depend on the setting of mathematics, but on its style and technique.

This statement is far away satisfactory for myself, those materials, which are harmonic in my mind, are placed not coherent enough for me here; what's worse is that I may get some topic lost, which is really annoying. I'll update it consistently in my later studies.