Setting. Over $\mathbb{C}$, i.e. complex semi-simple Lie algebra $\xrightarrow{\text{Cartan}}$ its representation
$\Longleftrightarrow$ compact Lie group over $\mathbb{C}$ $\xrightarrow{\text{Weyl}}$ its representation cuspidal modular

Origin the

char $= 0$ / algebraically closed ($\Rightarrow$ perfect)

② Langlands

Please distinct with semi-simple Lie group! It's another ample theory

L/K Galois, $G = \mathrm{Gal}(L/K)$ with its $\rho: G \to GL_n(\mathbb{C})$

$L(s,\rho)$ is the Artin's L-function

$\Longrightarrow \exists$ a ① cuspidal ② automorphic representation π_ρ of $GL_n(\mathbb{A}_K)$

$(\mathbb{A}_K = \prod K_v$ the adele$)$ is s.t. $L(s,\pi_\rho) = L(s,\rho)$

(finite left, almost all in $\mathcal{O}_v$ the algebraic integral ring)

$G: \mathbb{K}\text{-Alg} \to \text{Set}$ reductive algebraic group (over $\mathbb{C}$ is perfect)

① cuspidal representation

② Automorphic rep of $G(\mathbb{A}_K)$: irreducible unitary subrep of $L^2(G(\mathbb{A}_K))$ when $G(\mathbb{A}_K)$ is semi-simple

We decompose the G-Rep (as algebraic group) has kernel $L^2_0(G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q}))$ finite; decompose $= \bigoplus V_\pi m_\pi$ irreducible as Hilb. (π, V_π) is cuspidal

② $z \in Z(\mathbb{A}_\mathbb{Q})$ (center)

$L^2_0(G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q}), \omega)$

$f(gz) = f(g)\omega(z)$ $= \{ f \in L^2(G(\mathbb{A}_\mathbb{Q})) \}$ four properties, cuspidal function

Rk. from this, you see, not over $\mathbb{C}$, it's not even defined

③ $sl_2(\mathbb{F})$ turns to $sl_2(\mathbb{C})$

Then (Hard Lefschetz) ① the representation of $sl_2(\mathbb{C})$ is just $H^*(X;\mathbb{C})$ (Serre) the cohomology ring (with $\mathbb{C}$-vect structure) of the compact Kähler manifold X

One generalise to singularity by intersection cohomology

② The highest weight vectors of $sl_2(\mathbb{C}) \to H^*(X;\mathbb{C})$ is turned out to be $P^*(X;\mathbb{C})$ (the primitive cohomology)

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(algebraically, one may see $X \subset \mathbb{CP}^n$ smooth proj variety)

(Hard Lefschetz has many different types)

Lefschetz operator L — Dual Λ — the counting operator H

For Linear $X = V$ case, quickly one can see in Huybrechts Chap 0

$[-,-]$ Lie bracket, $H|_{H^k} = (k - \dim_\mathbb{C} X)$

It's a trivial fact once one know the Lefschetz decomposition $H^k(X;\mathbb{C}) \cong \bigoplus_i L^i P^{k-2i}(X;\mathbb{C})$ (It's important in Hodge theory)

Voisin, section 6 ▨

In General, see Voisin

① $L: \bullet \mapsto \bullet \wedge \omega$ (ω is symplectic form)

② $\Lambda = *^{-1} \circ L \circ *$

③ Ex. Try to determine $H = [L, \Lambda]$ is a multiplication

First let's explain what is semi-simple Lie algebra. Def 1 L semi-simple $\Leftrightarrow$ ① $\sqrt{L} = 0$ ($\sqrt{L} \trianglelefteq L$ is the maximal solvable ideal)

$\Leftrightarrow$ ② $L = \bigoplus L_i$, L_i simple Lie algebra ($[L_1, L_2] = 0$)

①$\Leftrightarrow$② is easy: ($\Leftarrow$) Simple $\Rightarrow$ Semi-simple is by definition and $\mathrm{Rad}(L_1 \oplus L_2) = \mathrm{Rad}(L_1) \oplus \mathrm{Rad}(L_2)$ (Left as Ex2.)

Can't replaced by semi-simple (Ex3)

$\Leftrightarrow$ ③ Killing form non degenerate; $[L, L_i] \neq 0 \Leftrightarrow L_i$ has no nontrivial ideal (later explained)

(Rk. $\oplus$ is same as vect and $L_1 \oplus L_2$ is with $[L_1, L_2] = 0$)

Ex2 (By replacing L_1 by $\frac{L_1}{\sqrt{L_1}}$, L_2 by $\frac{L_2}{\sqrt{L_2}}$) it suffices showing $L_1 L_2$ simple case

Let $J \trianglelefteq L_1 \oplus L_2$ solvable, then $J \cap L_1 = 0$ and $J \cap L_2 = 0$, otherwise L_1/L_2 not simple $\Rightarrow P_1(J) = P_2(J) = 0 \Rightarrow J = 0$ ▨

Rk ① $\Leftrightarrow$ no nontrivial abelian ideal $\Leftrightarrow$ no nontrivial solvable ideal

Ex3. $sl_2(\mathbb{C}) \oplus sl_2(\mathbb{C})$ is semi-simple? (No! $sl_2(\mathbb{C})$ ...)

($\Rightarrow$) L has a collection of ideals $I_1 \cdots I_n \trianglelefteq L \Rightarrow L = I_1 \oplus \cdots \oplus I_n$ ▨

We'll use the Killing form and Cartan's criterion to depict semi-simple Lie algebra. For this, we quickly review. Engel + Lie, and Jordan-Chevalley

Rk. For $L(G)$ the Lie algebra associated with G Lie group ($L(G) = T_e G = T_p G$ ...) $L(G) = L_1 \oplus \cdots \oplus L_n$, if $L_i = L(G_i)$ $\Rightarrow G = G_1 \times \cdots \times G_n$ / ~ a discrete action

Thm 2.A. (Engel) ① $\forall x \in L$ ad-nilpotent $\Rightarrow L$ nilpotent.

② $L \subset gl(V)$, $\dim V < \infty$, $V \neq 0$ if L consist of nilpotent endomorphisms $\Rightarrow \exists\, v \in V$, s.t. $L.v = 0$

Thm 3. (Lie) $L \subset gl(V)$, $\dim V < \infty$, $V \neq 0$ if L solvable $\Rightarrow \exists\, v \in V$, s.t. $L.v = v$

Both form and proof are simply the same, we omit their proof ▨

Ex4. Show the matrix representation of Engel / Lie: Every $x \in L$ is strict upper triangle matrix; Every $x \in L$ is upper triangle matrix

Thm 4 A. (Jordan-Chevalley) $x \in \mathrm{End}\,V \Rightarrow x = x_s + x_n$ { x_n nilpotent, x_s semi-simple and $x_n x_s = x_s x_n$ } Pf. Omitted ▨

A polynomial analogue omitted, but later used

Recall: $x \in \mathrm{End}\,V$ semi-simple $\xLeftrightarrow{\text{def}}$ the roots of its minimal polynomial $\mathrm{End}\,V/\mathbb{C}$ is distinct $\Leftrightarrow$ diagonalizable

Coro 5. ① L solvable, $\exists$ a chain of ideals of L, $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$

$\dim 0 < \dim 1 < \cdots < \dim n$

② L solvable. $x \in [LL] \Rightarrow \mathrm{ad}\,x$ nilpotent

Engel $[LL]$ nilpotent

③ $sl_n(\mathbb{C})$ semi-simple ($sl(V)$ semi-simple)

Recall. $\mathrm{ad}: L \to \mathrm{Der}(L)$, $x \mapsto \mathrm{ad}\,x: y \mapsto [xy]$

④ $\mathrm{ad}\,x = \mathrm{ad}\,x_s + \mathrm{ad}\,x_n = (\mathrm{ad}\,x)_s + (\mathrm{ad}\,x)_n$

Ex. $(x+y)_s = x_s + y_s$, $(x+y)_n = x_n + y_n$ when $[xy] = [y_n x]$ (Ex 4.5)

⑤ $\mathrm{Der}(\mathfrak{U}) \subset gl(\mathfrak{U})$ is stable under the J-C decomposition

Ex5. Over $\mathbb{F}$, $\dim V < \infty$, char $\mathbb{F}$ also holds the Lie thm (Ex4.2)

Pf. ① $\mathrm{ad}(L)$ ..., by Lie, $\mathrm{ad}(L)$ stabilizes $\exists$ flag in V, we pullback this flag into L ($\mathrm{ad}: L \to gl(V)$) $\Rightarrow$ chain of ideal ▨

④ We had known x_n nilpotent $\Rightarrow \mathrm{ad}(x_n)$ nilpotent. First x_s semi-simple $\Rightarrow \mathrm{ad}\,x_s$ semi-simple due to: Taking basis $\{v_1 \cdots v_n\}$ of V, corresponding basis e_{ij} of $gl(V) \Rightarrow e_{ij}(v_k) = \delta_{jk}(v_i)$ s.t. $x_s: V \to V$ is the diagonal $a_1 \cdots a_n$

$\Rightarrow (\mathrm{ad}\,x_s)(e_{ij}) = (a_i - a_j)e_{ij}$ semi-simple, diagonalizable

thus $(\mathrm{ad}\,x_s)$ is diagonalizable $\Rightarrow$ semi-simple // $[\mathrm{ad}\,x_s, \mathrm{ad}\,x_n] = \mathrm{ad}[x_s, x_n] = 0$

② Using ①, let $(x_1 \cdots x_i)$ ($i \le n$) is basis of L_i, using Ex4 (matrix) $\mathrm{ad}\,L$ is upper triangle $\Rightarrow [\mathrm{ad}\,L, \mathrm{ad}\,L] = \mathrm{ad}[LL]$ is strictly upper triangle $\Rightarrow \mathrm{ad}_L x$ is nilpotent for $x \in [LL]$ ▨

Check. S solvable $\Rightarrow S^t$ solvable

$\sqrt{L} \subset B \Rightarrow$ upper triangle

Then by Lie (Ex4) $\Rightarrow$ diagonal ▨

$\sqrt{L} \subset B^t \Rightarrow$ under triangle

③ $Z(L) \subset \sqrt{L}$ is trivial; I claim $\sqrt{L} \subset \{$diagonal matrix$\} \subset Z(L)$ to complete the proof

(Both $\sqrt{L}$, B solvable)

(by Hint of Ex4.1) Taking B the maximal solvable subalgebra $B \subset L$, then $\sqrt{L} \subset B$ (why? $\sqrt{L} + B$ also subalgebra, and solvable)

$\Rightarrow$ I claim $B^t = B$, otherwise, ... $\Rightarrow B \subsetneq S^t$ contradiction $\Rightarrow \sqrt{L} \subset B \cap B^t$

⑤ $\delta \in \mathrm{Der}\,\mathfrak{b}$, $\delta = \sigma + \nu$ ($\sigma = \delta_s$, $\nu = \delta_n$), only show $\sigma \in \mathrm{Der}\,\mathfrak{b}$ then $x, y \in \mathfrak{b}$, let $x \in \mathfrak{b}_a$, $y \in \mathfrak{b}_b$ (then $\nu = \delta - \sigma \in \mathrm{Der}\,\mathfrak{b}$)

$\Rightarrow \sigma(xy) = (a+b)(xy)$ due to $xy \in \mathfrak{b}_{a+b}$

and $\sigma(x)y + x\sigma(y) = a(x)y + x(by) = (a+b)xy$ ▨

We decompose $\mathfrak{b} = \bigoplus_{\lambda \in \sigma(\delta)} \mathfrak{b}_a$, $\mathfrak{b}_a = \{x \in \mathfrak{b} \mid (\delta - \lambda)^k x = 0,\ \exists k\}$, $\sigma(\sigma) = \sigma(\delta)$

$\sigma|_{\mathfrak{b}_a} = a\,\mathrm{Id}$ (a the eigenvalue)

It's a grading: $\mathfrak{b}_{a_1}\mathfrak{b}_{a_2} \subset \mathfrak{b}_{a_1+a_2}$ (Check it by caculation!)

Then we introduce the Killing form, it's the core technique in Killing–Cartan's classification of semi-simple Lie algebra, then we start study the representation of Lie algebra.

Rk. But it fails for char $\neq 0$ case: then such bilinear form can be degenerate

Rk. Such a consequence of classification is quite discrete. It means in a moduli space is discrete points or simply in parameter space, the components (isomorphism classes; algebra) (Due to Weyl's complete reducibility)

Def 6. (Killing form)

$\kappa: L \times L \to \mathbb{C}$

$(x, y) \mapsto \mathrm{tr}(\mathrm{ad}\,x\,\mathrm{ad}\,y)$

and $L\times L \times L \xrightarrow{(\mathrm{ad},1)} L \times L$

(same) i.e.

$$\begin{array}{ccc} L\times L & \xrightarrow{\mathrm{ad}} & \mathrm{End}(L)\times\mathrm{End}(L) \\ \downarrow \kappa & ② & \downarrow m \ \cdots ① \\ \mathbb{C} & \xleftarrow{\mathrm{tr}} & \mathrm{End}(L) \end{array}$$

$$\begin{array}{ccc} L\times L\times L & \xrightarrow{(1,\mathrm{ad})} & L \times L \\ \downarrow (1,\mathrm{m}) & ② & \downarrow \kappa \ \cdots ③ \\ L \times L & \xrightarrow{\kappa} & \mathbb{C} \end{array}$$

(associative) $\xLeftarrow{\text{Proved by}}$

$$\begin{array}{ccc} L\times L\times L & \xrightarrow{(\mathrm{ad})} & L \\ \mathrm{mean}\ ② & & \downarrow \mathrm{tr} \cdots ② \\ L & \xrightarrow{\mathrm{tr}} & \mathbb{C} \end{array}$$

Ex6. Using diagram ① to prove κ is symmetric first

Eg. Consider V parameters all Lie structure on $\mathbb{C}^n$, it forms a variety, $GL_n(\mathbb{C}) \curvearrowright V$, the $S(g) = \mathrm{Aut}(g)$, $O(g) =$ the isomorphism class $\ni g$ (S, O represent the stabilizer, orbit), $\forall g \in V$

By Ratner's thm, $O \subset \bar{O}$ is open ($O = O(g)$)

$\Rightarrow \bar{O} - O$ is orbits of lower dimension $\Leftrightarrow$ stablizer of higher dim

I claim, $\mathrm{Aut}(g)$ depend only on $\dim g$ (also discrete)! to complete the proof ▨

It needs more. I omit it (Procesi, P301, Lie Groups: An Approach through Invariants and Reps, Thm B.)

It's nondegenerate:

Def 7. $\beta: L\times L \to \mathbb{C}$ a symmetric bilinear form is nondegenerate if $S = \bigcap_{y\in Y} \mathrm{Ker}(\beta(-, y)) = 0$ (the radical of β)

κ is nondegenerate $\Leftrightarrow$ $|(\kappa(x_i, x_j))_{ij}| \neq 0$ first writing as matrix

Using the Cartan's criterion, we can prove the equivalent condition noted at first:

(Cartan's second criterion: κ nondegenerate $\Leftrightarrow$ L semi-simple)

Thm 8. (Cartan's criterion) $L \subset \mathfrak{gl}(V)$ subalgebra, $\dim V < \infty$

$\mathrm{tr}(xy) = 0$ for $\forall x \in [LL]$, $y \in L \Leftrightarrow L$ solvable

For proving this, we need

Cor 9. If $\kappa|_{[LL]\times L} = 0 \Rightarrow L$ solvable

Pf. $\mathrm{Ker}(\mathrm{ad}) = Z(L)$, and $Z(L)$ solvable $\Rightarrow L$ solvable

For $\mathrm{ad}: L \to \mathfrak{gl}(V)$, $\mathrm{ad}(L) \subset \mathfrak{gl}(V) \Rightarrow \mathrm{tr}(\mathrm{ad}x\,\mathrm{ad}y) = 0$

$\Rightarrow \mathrm{Ker}(\mathrm{ad})$ solvable due to $\mathrm{ad}(L)$ solvable by Cartan ▨

$\kappa(x, y)$ for $\mathrm{ad}x \in [\mathrm{ad}L\,\mathrm{ad}L]$ only for $\mathrm{ad}y \in \mathrm{ad}L$ $\Rightarrow x \in [LL], y \in L$

Lemma 10. $A \subset B \subset \mathfrak{gl}(V)$ subspace, $\dim V < \infty$

$M = \{x \in \mathfrak{gl}(V) \mid [xB] \subset A\}$, $x \in M$ s.t. $\mathrm{tr}(xy) = 0$, $\forall y \in M$

$\Rightarrow x$ nilpotent

Pf. By J–C decomposition $x = s + n$, choice a basis $\{v_1 \cdots v_n\}$ of V, s.t. $s: V \to V$ is the diagonal matrix $\mathrm{diag}(a_1 \cdots a_n)$

Let E be a $\mathbb{Q}$-vector space, $E = \mathrm{span}_{\mathbb{Q}}\{a_1 \cdots a_n\}$, I claim, $E = 0 \Rightarrow s = 0 \Rightarrow x = n$ nilpotent

We prove $\forall f: E \to \mathbb{Q} \in E^*$ is zero: let $y \in \mathfrak{gl}(V)$ is the diagonal matrix $\mathrm{diag}(f(a_1) \cdots f(a_n))$, $\{e_{ij}\}$ the corresponding basis of $\mathfrak{gl}(V)$

$\Rightarrow \mathrm{ad}\,s(e_{ij}) = (a_i - a_j)e_{ij}$; $\Rightarrow \exists\, r(X) \in \mathbb{C}[X]$, s.t. $r(a_i - a_j) = f(a_i) - f(a_j)$ ($\Leftarrow$ We hope show $y \in M$ applying $\mathrm{tr}(xy) = 0$)

$\mathrm{ad}\,y(e_{ij}) = (f(a_i) - f(a_j))e_{ij}$ is by Lagrange polynomial $\Rightarrow r(X)$ without constant term

We use the another Jordan–Chevalley: $s = p(x)$ for $p(X) \in \mathbb{C}[X]$]

$\Rightarrow r(X)$ is of $\mathrm{ad}\,x \Rightarrow \exists\, r'(X)$ is of $\mathrm{ad}\,s$; $\mathrm{ad}\,x: B \to A$

so is for $\mathrm{ad}\,y \Longrightarrow \mathrm{ad}\,y: B \to A \Rightarrow y \in M$

$\Rightarrow \mathrm{tr}(xy) = 0 \Leftrightarrow \sum a_i f(a_i) = 0$

$\Rightarrow \sum f(a_i)^2 = 0$, $f(a_i) \in \mathbb{Q}_{\geq 0}$ $\Rightarrow f(a_i) = 0 \Rightarrow f = 0$ ▨

$\mathrm{tr}([xy]z)$ $\|$ $\mathrm{tr}(x[yz]) = \mathrm{tr}([xy]z) = 0$

$[xy]$ generate all of $[LL]$ ▨

Pf of thm 8 ($\Rightarrow$) Take $A = [LL] \subset B = L \subset \mathfrak{gl}(V)$, $M = \{x \in \mathfrak{gl}(V) \mid [xL] \subset [LL]\} \supset L$, if we generalise $y \in L$ to $y \in M$, we apply the Lemma. (Check it!)

by Lemma $\Rightarrow \forall x \in [LL]$ nilpotent $\Rightarrow$ $\boxed{[LL] \text{ nilpotent} \Rightarrow L \text{ solvable}}$ ▨

($\Leftarrow$) (The converse of Thm 5 ②: check definition)

(Rk. Some texts replace $[LL]$ by L itself then "$\Leftarrow$" not holds: Eg1. Verify that L is 1-dim solvable Lie algebra $\subset \mathfrak{gl}(\mathbb{C})$ the $\{\lambda\,\mathrm{Id}\}$ but $\mathrm{tr}(xy) \neq 0$ $x, y \in L$)

$\mathrm{ad}\,L \subset \mathfrak{gl}(L)$ is a solvable subalgebra $\Rightarrow \forall x \in [LL]$ is a strictly upper martix by Lie $\Rightarrow \mathrm{tr}(xy) = 0$ ▨

(L is upper triangle by Lie $\Rightarrow [LL]$ strictly)

Lemma 11. $I \trianglelefteq L$ ideal, κ_I is the Killing form in I

$\Rightarrow \kappa_I = \kappa|_{I\times I}$ (i.e. $\kappa_I(x, y) = \kappa(x, y)$, $\forall x, y \in I$)

Pf. Take a basis of I, extends to L

$x \in I \Rightarrow \mathrm{ad}\,x: L \to I \subset L$ is the form $\begin{pmatrix} A_x & B_x \\ 0 & 0 \end{pmatrix}$, $y \in I \Rightarrow \mathrm{ad}\,x\,\mathrm{ad}\,y$ is $\begin{pmatrix} A_xA_y & B_xB_y \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \kappa(x, y) = \mathrm{tr}(A_xA_y) = \kappa_I(x, y)$ ▨

Ex8. Can we always do this, s.t. the Killing form of L is the form $\begin{pmatrix} \kappa_I & 0 \\ 0 & * \end{pmatrix} = \kappa$?

Eg. ① $\mathfrak{sl}_2(\mathbb{C})$ $\kappa = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$

② $\mathfrak{sl}_n(\mathbb{C})$ $\kappa(x, y) = 2n\,\mathrm{tr}(xy)$

③ $\mathfrak{gl}_n(\mathbb{C})$ $\kappa(x, y) = 2n\,\mathrm{tr}(xy) - 2(\mathrm{tr}\,x)(\mathrm{tr}\,y)$

Ex9. Check them.

Finally. We prove the Cartan's second criterion (semi-simple $\Leftrightarrow$ κ non-degenerated)

Pf. ($\Rightarrow$) S denote the radical of $\kappa \Rightarrow \kappa(x, y) = \mathrm{tr}(\mathrm{ad}x\,\mathrm{ad}y) = 0$, $x \in S$, $y \in [SS]$

$\Rightarrow \mathrm{ad}\,S$ solvable $\Rightarrow S$ solvable $\Rightarrow S \subset \mathrm{Rad}\,L = 0 \Rightarrow S = 0 \Rightarrow \kappa$ non-degenerated ($\Leftarrow$) $S = 0$ now, I claim S is maximal

$\forall I \trianglelefteq L$, I abelian, $x \in I$, $y \in L \Rightarrow L \xrightarrow{\mathrm{ad}\,y} L \xrightarrow{\mathrm{ad}\,x} I \Rightarrow (\mathrm{ad}x\,\mathrm{ad}y)^2: L \to [II] = 0$ (by Abelian) $\Rightarrow \mathrm{ad}x\,\mathrm{ad}y$ nilpotent $\Rightarrow 0 = \mathrm{tr}(\mathrm{ad}x\,\mathrm{ad}y)$

Then we study the representation theory of Lie algebra (Recall $\rho: L \to gl(V)$ is a rep of L)

Def 12 (L-module) $V \in \mathrm{GVect}_{\mathbb{C}}$ is L-module if linearity and $[xy].v = x.y.v - y.x.v$ (for action $L \times V \to V$)

It's not a surprising thing, it's just modify the concrete formula when chaning $xy \rightsquigarrow [xy]$. Ex 9. Is here category equivalence $L\text{-Mod} \cong L\text{-Rep}$?

E.g. Algebraic analysis usual involves D-module. Def. X smooth variety, over $\mathbb{C}$

Ex 10 Consider $X = \mathrm{Spec}\,\mathbb{C}[X] = \mathbb{A}^1_{\mathbb{C}}$ the line. D_X is the sheaf globally generated by $\Gamma(X, \mathcal{O}_X)$ (vector fields)

D_X-module $\mathbb{C}[X]\log(x) + \mathbb{C}[X, \frac{1}{x}] = M$, by Nagata. D_X-module M is $\mathcal{O}_X$-module s.t. (They're classified; in particular, linear reductive = reductive)

prove: $D_X[\log(x)] = M$ (Then we call a semi-simple Lie algebra is linearly reductive)

$D_X \times M \to M$

$(v, m) \mapsto \nabla_v m$

(D1) $\nabla_{fv+gw} m = f\nabla_v m + g\nabla_w m$

(D2) $\nabla_v (fm+gn) = \nabla_v(fm) + \nabla_v(gn)$

(D3) $\nabla_{[v,w]}(m) = \nabla_v\nabla_w - \nabla_w\nabla_v (m)$

The relative notions are inherited from representation theory. Schur Lemma holds still. Our next task is proving Weyl's $\boxed{\text{completely reducible}}$ thm (decompose to irreducible) $\Leftrightarrow$ existence of complement $W \oplus W' = V$

Except (D2), other two holds it as L-module

Thus M is D_X-module $\Leftrightarrow$ M induce a flat connection (curvature $= 0$) $\nabla: M \to \Omega^1_{X/k} \otimes_A M$ ($\Omega^1_{X/k} = \Omega^1_{D_{X/k}}$)

Thm 13. The rep of semi-simple Lie algebra is complete reducible

Rk. Why we not outline finite-dimensional here? Semi-simple $\Leftrightarrow$ compact $\xrightarrow{\text{Peter-Weyl}}$ Factor into finite dim case.)

Then Thm 13 allows us to study simple Lie algebra / connect compact Lie group enough. For proving, we use the Casimir (a people's idea)

Lemma 14. $\rho: L \to gl(V)$ representation $\Rightarrow \rho(L) \subset sl(V)$

Pf. When L simple, $[LL] = L$ by ideal $\neq 0$; L semi-simple, by decompose $L = L_1 \oplus \cdots \oplus L_n \Rightarrow [LL] = L$ also

$L \subset gl(V) \Rightarrow [LL] = [gl(V)\, gl(V)] = sl(V) \Rightarrow L \subset sl(V)$ ▨

Our study on non-degenerate bilinear form $\beta: L \times L \to \mathbb{C}$ taking basis $(x_1 \cdots x_n)$, $\exists (y_1 \cdots y_n)$ another, s.t. $\beta(x_i, y_j) = \delta_{ij}$

then for $\forall \varphi: L \to gl(V)$ faithful rep, let $c_\varphi(\beta) = \sum_i \varphi(x_i)\varphi(y_i) \in \mathrm{End}(V)$; let $\beta(x, y) = tr(\varphi(x)\varphi(y))$ Ex 11 Check it's non-degenerate

we denote c_φ then the Casimir element of φ (injective) Ex 12 $L = sl_2(\mathbb{C})$, $V = \mathbb{C}^2$ and $Id: L \to gl(V) \Rightarrow c_{Id} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} = \frac{3}{2}I$ it motivates that

Prop 15. ① c_φ (or $c_\varphi(\beta)$) commuting with $\varphi(L)$ ② $tr(c_\varphi) = \dim L$ ③ When φ irreducible, $c_\varphi: x \mapsto \frac{\dim L}{\dim V} x$, i.e. $c_\varphi = \frac{\dim L}{\dim V}$

Pf. ① $\forall x \in L$, $[\varphi(x), c_\varphi(\beta)] = \sum [\varphi(x)\varphi(x_i)]\varphi(y_i) + \sum \varphi(x_i)[\varphi(x)\varphi(y_i)] = \sum a_{ij}\varphi(x_j)\varphi(y_i) + \sum b_{ij}\varphi(x_i)\varphi(y_j)$

(a_{ij}, b_{ij} is the structure constants) and $a_{ij} = \sum a_{ik}\beta(x_k, y_j) = \beta([x x_i], y_j) = -\beta([x_i x], y_j) = \beta(x_i, -[x y_j]) = -\sum b_{jk}\beta(x_i, \ldots) = -b_{ji}$

$\Rightarrow [\varphi(x)\, c_\varphi(\beta)] = 0$ ▨ ② $\sum tr(\varphi(x_i)\varphi(y_i)) = \sum \beta(x_i, y_i) = \dim L$ ▨ ③ By Schur ▨

For not faithful case, we modify $\varphi \to \mathrm{Ker}\varphi \hookrightarrow L \to gl(V)$, $\mathrm{Ker}\varphi \subset L \Rightarrow \mathrm{Ker}\varphi = L_1 \oplus \cdots \oplus L_r$

$\Rightarrow \varphi: L' \to gl(V)$ faithful, thus the Prop 15 ① provide us to using L' replace L. $L = L_1 \oplus \cdots \oplus L_r \oplus \underline{L_{r+1} \oplus \cdots \oplus L_s} = \mathrm{Ker}\varphi \oplus L'$

Ex 13 ① Classify all the 2-dim rep of 2-dim L. ② Discuss: the V, V^* appear in which class?

Pf of Thm of Weyl (I think it's long and $\boxed{\text{a little technical}}$) $\Rightarrow$ later we'll give a cohomological way to prove this (indeed, this is just showing $H^1(L, V) = 0$)

$\forall W \subset V$, we need find its complement, consider $0 \to W \to V \to \frac{V}{W} \to 0$ $\boxed{\text{Step 1}}$ Reduce to codim 1

and $\mathcal{V} \subset \mathrm{Hom}(V, W)$ s.t. $\mathrm{Hom}(W, W) \leftarrow \mathrm{Hom}(V, W) \leftarrow \mathrm{Hom}(\frac{V}{W}, W) \leftarrow 0$

$\mathcal{W} \subset \mathcal{V}$

$\mathcal{W} = \mathrm{Ker}(i^*) = \mathrm{Hom}(\frac{V}{W}, W)$

Thus we reduce to this case: $\mathcal{V} = \mathrm{Ker}(i^* - \mathbb{C}Id) \supset \mathcal{W} \Rightarrow \mathrm{codim}(\mathcal{V}, \mathcal{W}) = 1$ i.e. $0 \to \mathcal{W} \to \mathcal{V} \to \mathbb{C} \to 0$

$W \subset V$ irreducible of codimension 1 ($= \bigcup_{\lambda \in \mathbb{C}} \mathrm{Ker}(i^* - \lambda Id)$) Thus we can assume $W \subset V$ is of codimension 1

and faithful representation (to use c_φ) (Why? Once give $\mathcal{V} = \mathcal{W} \oplus \mathbb{C}(f)$ this case, $V = W \oplus \ker f$, check it)

(Faithful by the modification upper, irreducibility is by why?)

$\boxed{\text{Step 2}}$ Reduce to irreducible (For Schur)

$W' \subsetneq W$ is nonzero sub-L-module $\Rightarrow 0 \to \frac{W}{W'} \to \frac{V}{W'} \to \mathbb{C} \to 0$, induction on dimension of $\frac{W}{W'}$ (or $\dim \frac{V}{W'} = \dim \frac{W}{W'} + 1$)

$\Rightarrow$ it spilts (due to the lower dimension ($\frac{W}{W'}$) spilts $\Rightarrow$ higher dimension (W) spilts as following arugment) $\frac{W}{W'} \oplus \mathbb{C} \cong \frac{V}{W'}$

$\Rightarrow 0 \to W' \to \tilde{W} \to \mathbb{C} \to 0$ spilt again $\Rightarrow W'$ has complement in W, thus $\Rightarrow W \oplus \mathbb{C} \cong V$

$\boxed{\text{Step 3}}$ Prove the reduced case (by dimension) $\Rightarrow$ We fine complement of W in V if we find complement of W' in V

$$\begin{array}{ccc} W & \xrightarrow{c} & W \\ \downarrow & & \downarrow \\ V & \xrightarrow{c} & V \\ \downarrow & & \downarrow \\ \frac{V}{W} & \xrightarrow{c} & \frac{V}{W} \end{array}$$

Scalar (By Schur)

$0 \to \mathrm{Ker}\,C \to V$, $Tr(C) = \dim L$

dim 1 part. $Tr(\bar{C}) = 0$

$C: \beta \mapsto \sum \varphi(x_i)\varphi(y_i)$ trivial on $\frac{V}{W}$ and $\varphi(L) \curvearrowright \frac{V}{W}$ trivial

$\dim \mathrm{Ker}\,C = 1$ $\Rightarrow \mathrm{Ker}\,C \cap W = 0$

$\Rightarrow \mathrm{Ker}\,C \oplus W = V$ ▨

Rk. This pure algebraic pf is given by Brauer, the original proof is using the Lie group to prove (their correspondence)

We'll introduce the work of Weyl: weight, maximal vector, maximal torus, root system, the following (maximal toral)

Rk. All these definition has their analogue in theory of Lie group, later we'll follow the both sides (denote L Lie algebra, G Lie group)

Def16. (Reductive) L is reductive $\Leftrightarrow Z(L) = \sqrt{L}$
$\Leftrightarrow L = L_1 \oplus L_2$, s.t. L_1 semi-simple, L_2 abelian

Ex14. Prove this equalence and show, reductive although weaker than semi-simple, it holds the Weyl's completely reducible

Def17. (Toral subalgebra) $T \leq L$ is toral $\Leftrightarrow$ T contains only semi-simple elements

Thm18. For L semi-simple (even reductive), we have Toral subalgebra $\Leftrightarrow$ Cartan subalgebra

Rk. (Existence) Both char $= 0$, algebraically closed removed will broke the Existence!

$C \leq L$ is self-normalizing
if $[x, y] \in C$, $\forall x \in C$ $\Rightarrow y \in C$

(Rk. There are more fine definitions, such as Cartan 1-algebra/2-algebra)

Pf. Cartan $\Leftrightarrow$ diagonal action but coincides for semi-simple $\Leftrightarrow$ only semi-simple (the last ($\Leftarrow$) is due to once not diagonal by some operation will turns to nilpotent) (We used Engel). ▨

[Ex17. Study to pf of thm18 in detail and pf case of reductive]

Def19. (Maximal toral subalgebra) Tatological! $H \leq L$

Def20. (Root, Root space, Root vector)

$L = \bigoplus_{\alpha \in H^*} L_\alpha = \{x \in L \mid [h x] = \langle \alpha, h \rangle (x), h \in H\}$ $\boxed{C(H) \supset H}$

$H^* \supset \Phi = \{\alpha \in H^* \mid L_\alpha \neq 0\}$ the root $L_0 \supset H$

thus the decomposition of partion into two parts: $H^* — \Phi$, 0 $\Rightarrow$ the Cartan/root space decomposition $L = C(H) \oplus \coprod_{\alpha \in \Phi} L_\alpha$, L_α is the root space corresponding to α.

$\exists!\, H_\alpha \in H$, the root vector for α, s.t. $\langle \alpha, h \rangle = \kappa(H_\alpha, h)$, $\forall h \in H$.

Φ is a root system, i.e. $\Phi \subset E$ s.t. (Here $E = H^*$)

(R1) $|\Phi| < \infty$, Φ span E, $0 \notin \Phi$; (R2) $\alpha \in \Phi$ and $\lambda\alpha \in \Phi \Rightarrow \lambda = \pm 1$; (R3) $\alpha \in \Phi$, the reflection $\sigma_\alpha : \beta \mapsto \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$ is invariant on Φ

(Rk. The reflection is over $\mathbb{R}$, for this, forget the almost complex structure of L)

(R4) $\langle \alpha, \beta \rangle \in \mathbb{Z}$, $\alpha, \beta \in \Phi$

This is our next task, for proving this

Prop22. ① $C(H) = H$; ② $\exists!$ root vector for $\forall \alpha \in \Phi$ ③ Φ holds the (R1) $\sim$ (R4) Pf. Left to the next speaker ▨

Rk. ① By Prop22①, we can recover L by Φ and H; ② The root vectors has their weight and maximal vectors are used in their representation; ③ $\alpha \in \Phi$ is simple root $\Leftrightarrow \alpha \in \Phi^+$ and $\alpha \neq \beta + \gamma$ for $\forall (\beta, \gamma) \in \Phi^+ \times \Phi^+$

Def23. Irreducible root system if it can't be partitioned into two orthogonal parts

Ex18. Let L be the Lie algebra associates to $G = SU_3(\mathbb{C})$, show that ① Compute Φ of L $\Phi^+ = \{\alpha \in \Phi \mid \langle \alpha, x_0 \rangle > 0\}$ ($\exists x_0 \in H$, s.t. $\langle \alpha, x_0 \rangle \neq 0$, $\forall \alpha \in \Phi$)
② $x_0 = \mathrm{diag}(1, 0, -1) \in L$ s.t. $\langle \alpha, x_0 \rangle \neq 0$, $\forall \alpha \in \Phi$
③ Compute Φ^+ and Δ (the all simple roots)

A. Ex18, $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$, $\Delta = \{\alpha_1, \alpha_2\}$
Ex19, ① indeed $\kappa(H_\alpha, H_\beta) < 0$
② Right!

Ex19. By Ex18 consider these general facts:
① $\alpha \neq \beta \in \Delta \Rightarrow \kappa(H_\alpha, H_\beta) \geq 0$
② Δ is a basis of T^*
If right, prove it; false, modify it

Ex20. Φ, Φ' are two root systems irreducible, respect to E, E' over $\mathbb{R}$, then prove: $\Phi \cong \Phi' \Leftrightarrow \exists \lambda \in \mathbb{R}$ and $\varphi : E \xrightarrow{\sim} E'$, s.t. $\varphi(\Phi) = \Phi'$ and $\langle \varphi(\alpha), \varphi(\beta) \rangle = \lambda \langle \alpha, \beta \rangle$

Def24. (Weyl chambers and Weyl group (Lie algebra))
The reflecting hyperplanes $P_\alpha = \{\beta \in E \mid \langle \beta, \alpha \rangle = 0\}$ is the reflection in (R3), they cut E into finite reigons: $E - \bigcup_{\alpha \in \Phi} P_\alpha = \bigcup_{i \in N} E_i$, E_i called the Weyl chambers of E

$\mathcal{W} \subset GL(E)$ generated by $(\sigma_\alpha)_{\alpha \in \Phi}$ as group, permute the set Φ, called Weyl group (it's a Lie group & algebraic group)

Ex21. ① Weyl group send one Weyl chamber to another ② Weyl group transform different basis of Φ ③ $\langle \sigma_\alpha \rangle_{\alpha \in \Phi} = \mathcal{W} = \langle \sigma_\alpha \rangle_{\alpha \in \Delta}$ ④ $\alpha, \beta \in \Phi$, $\alpha \neq \pm\beta$, $\langle \sigma_\alpha, \sigma_\beta \rangle \subset \mathcal{W}$ is a dihedral group

Def16' (Reductive) G is reductive $\Leftrightarrow T_0 G = \mathfrak{g}$ is reductive

Rk. A reductive Lie group isn't same as reductive algebraic group
For over $\mathbb{C}$, $\mathcal{G}$ a connected linear algebra group
let $G = \mathcal{G}(\mathbb{R})$ the $\mathbb{R}$-rational points of $\mathcal{G} \Rightarrow G$ is Lie group
If $\mathcal{G}$ is unipontent ($\forall g \in \mathcal{G}$, $(g - I)$ nilpontent), G can be reductive but $\mathcal{G}$ can't be reductive. (this is non-trivial, but an easy part is:
Ex15. Show unipontent algebraic group $\Rightarrow$ not reductive

Def17' (Torus)
$T \subset G$ is torus $\Leftrightarrow T$ is compact connected abelian subgroup
$\Leftrightarrow T \cong \mathbb{T}^n = \mathbb{R}^n / \Lambda$

(When G compact connected $\Leftrightarrow L$ simple, they coincide)

Def19' (Maximal torus)
A maximal torus T is the maximal tori of G.

Ex16. $T \subset G$ maximal torus, $N_G(T) = \{g \in G \mid g t g^{-1} \in T, t \in T\}$ and $W = N_G(T)/T$ is the Weyl group of T (the analogue in Lie algebra will I use the root system) (the normalizer/the centerlizer, I see it last occurs in AT, about transform the covering)
Prove: ① Any two maximal tori conjudgate ② $\forall g \in G$, it must $\in$ $\exists T \subset G$ maximal ③ $|W| < \infty$ ($gTg^{-1} = T$)

Def20'. The defition is paralleled by considering the $\mathfrak{g} = T_0 G$ associates to G.

Coro21. By T is torus $\Rightarrow T$ is communicative $\Rightarrow T_0 T = H$ is abelian Lie algebra

Rk. Over $\mathbb{R}$ case (complexification) let $L^{\mathbb{C}} = L \oplus iL$, $H^{\mathbb{C}} = H \oplus iH$
$L_{\mathbb{R}} = iL$ $H_{\mathbb{R}} = iH$
$\Rightarrow L^{\mathbb{C}} = C(H^{\mathbb{C}}) \oplus \coprod_{\alpha \in \Phi} L_\alpha^{\mathbb{C}}$

Pf of decomposition. Give L the inner product $\langle\, ,\, \rangle : L \times L \to \mathbb{C}$; $(\alpha, \beta) \mapsto \kappa(\bar{\alpha}, \beta)$

(Rk. The $\bar{\alpha}$ in L is given by involution in $\mathbb{C}$)
$\Rightarrow \{\mathrm{ad}\, x \mid x \in H\}$ is a space of commuting self-adjoint operators on $L \Rightarrow$ decompose into simultaneous eigenspaces ▨

Rk. A interseting corollary: Pf that: Orthogonal matrix is diagonal
Pf. $\forall g \in U_n(\mathbb{C})$ $\exists T \ni g$, and T_0 the diagonal matrix also maximal $\Rightarrow hTh^{-1} = T_0$ $\Rightarrow hgh^{-1} \in T_0$

Summary. I had introduced that
① Description of semi-simple; ② The rep of semi-simple Lie algebra: Weyl's completely irreducible; ③ A quick overview of Weyl's study on rep of Lie group/Lie algebra

One may study later: ① Weyl's integral/charistic formulas about Lie group ② More about the weights, maximal vectors, root system (Chap III) $\Rightarrow$ using to classify (Killing-Weyl) and Geomertic meaning)

I omitted topics ① Pf of Lie's thm; ② Abstract Jordan decomposition and it coincides usual one when semi-simple; ③ $sl_2(\mathbb{C})$'s representation.

Problems. Prob1. (2004, Hämmerli) $H^1(W;T)$ is the cohomology with coeffiecent in T maximal torus
$\Longrightarrow \mathrm{Out}(N_G(T)) \simeq H^1(W;T) \rtimes \mathrm{Out}(G)$ (outautomorphism is auto − innerauto)

Prob2. (2008, Morita) Σ_g is closed oriented surface, $H = H_1(\Sigma_g;\mathbb{Z})$, $H_{\mathbb{Q}} = H \otimes \mathbb{Q} (= H_1(\Sigma_g;\mathbb{Q}))$

① $x \in H$ the 1-homological class $\longleftrightarrow$ 1-dim submanifold of Σ_g is the core idea of Doldason school
(dually, the [L.Hal III Ex7.4] tells us that 1-cohomological class $\leftrightarrow$ 1-codim subvariety

Then define the pairing $H \times H \xrightarrow{\omega} \mathbb{Z}$, $(u,v) \mapsto (u,v)$ the intersection number

Pf. ω is a symplectic perfect pairing, and by perfect, it induce $H \cong H^*$, $H_{\mathbb{Q}} \cong H_{\mathbb{Q}}^*$
coincides the Poincare duality.

② L_g is the graded Lie algebra generated by H, $L_g^{\mathbb{Q}} = L_g \otimes \mathbb{Q}$, compute $L_g(n)$; in particular, show that $L_g(2) = \Lambda^2 H_{\mathbb{Q}} \ni \omega_0$ (ω_0 the symplectic class)

③ Define two basic Lie algebra $\mathfrak{a}_g = \bigoplus_{n\geq 0} \mathfrak{a}_g(n) = \bigoplus_{n\geq 0} \{D \in \mathrm{Hom}(H_{\mathbb{Q}}, H_{\mathbb{Q}}^{\otimes(n+1)}) \mid D(\omega_0) = 0\}$; $\mathfrak{a}_g^+ = \bigoplus_{n\geq 1} \mathfrak{a}_g(n)$

$\mathfrak{h}_g = \bigoplus_{n\geq 0} \mathfrak{h}_g(n) = \bigoplus_{n\geq 0} \{D \in \mathrm{Hom}(H_{\mathbb{Q}}, L_g^{\mathbb{Q}}(n+1)) \mid D(\omega_0) = 0\}$; $\mathfrak{h}_g^+ = \bigoplus_{n\geq 1} \mathfrak{h}_g(n)$

Prove that $H_1(\mathfrak{a}_g^+) \cong \Lambda^3 H_{\mathbb{Q}} / \omega(H)$

Prob3 (Moment map, GIT/McDuff)
Set. G a Lie group, L its associated Lie algebra, X ~~...~~ $\subset \mathbb{P}^n_{\mathbb{C}}$ nonsingular [redacted] projective variety ($\Rightarrow$ Kähler $\Rightarrow$ symplectic)

Def. (Moment map $\mu: X \to L^*$) $G \curvearrowright X$ preserve symplectic form ω on X, it induces the infinitesimal action $L \curvearrowright X$ by the Lie algebraic homomorphism $L \to \mathfrak{X}(X) = \Gamma(X, \Theta_X)$ (vector fields), then $\mu: X \to L^*$ is a G-equivariant
$l \mapsto (x \mapsto l_x)$
and $d\mu(x)(\xi)(l) = \omega_x(\xi, l_x)$ generalized to vector bundle

Now G act on X by a Lie group representation $\rho: G \to GL_{n+1}(\mathbb{C})$, then prove: Hitchin-Kobayashi correspondence
($X \subset \mathbb{P}^n$, and $\mathrm{Aut}(\mathbb{P}^n) = PGL_{n+1}(\mathbb{C})$) ① $\exists \mu: X \to L^*$ this case ② the symplectic quotient $\mu^{-1}(0)/X \cong X/\!/G$ the categorical quotient. Then $=$ GIT quotient
(Kempf-Ness, 1980s)

Talk2 Next we'll study more Lie algebra, mainly associated an algebraic group, for this, we introduce some premiliary:

Mainly. ① The cohomology of Lie algebra and applications; ② Basic algebraic group; ③ Lie algebra associated to an affine algebraic group
④ Correspondence of algebraic group − Lie algebra; ⑤ the Geometry of action and quotient.

Setting over $\mathbb{C}$ still.

[Ex1. More generally, L over R not field, U_L the universal eveloping algebra, show that $H^p(L,M) = \mathrm{Ext}^p_{U_L}(R, M)$]

Def1. (Chevalley-Eilenberg complex) $\mathrm{Hom}_{\mathbb{C}}(\Lambda^{\cdot} L, M)$ the C-E complex (M is L-module)
i.e. $C^p(L,M) = \{\alpha: L\times \cdots \times L \to M$ multilinear skew symmetric$\}$ and (co)boundary map
$d: C^p(L,M) \to C^{p+1}(L,M)$; $(d\alpha)(l_1 \cdots l_{p+1}) = \sum (-1)^{i+1} l_i \cdot \alpha(l_1 \cdots \hat{l_i} \cdots l_{p+1}) + \sum (-1)^{i+j} \alpha([l_i, l_j], l_1 \cdots \hat{l_i} \cdots \hat{l_j} \cdots l_{p+1})$

By this (co)chain, we define $H^p(L,M)$ as usual [Ex2. By concrete computation in lower degree, show that
① $H^0(L,M) = M^L = \{m \in M \mid lm = 0, \forall l \in L\}$ ② $H^1(L,M) = \mathrm{Der}(L,M) / \mathrm{IDer}(L,M)$ (Inner der)
Once you do some caculation (E.g. Ex2), you'll find Lie cohomology is hard to compute! Even ③ $H^2(L,M) = \mathrm{Ext}^1(L,M) = e(L,M)$ (I think this can be found in Hilton)
(extension $0 \to M \to L' \to L \to 0 / \simeq$)

checking: $C^{\cdot}(L,M)$ does a cocomplex is a hard work, thus we
① Define some notion for caculation; ② Checking $d^2 = 0$; ③ Prove some vanishing.

Def2. $\forall x \in L$, define $\theta(x): C^p(L,M) \to C^p(L,M)$ $\quad$ $\iota(x): C^p(L,M) \to C^{p-1}(L,M)$
$\theta(x)(\alpha)(l_1 \cdots l_p) = x\alpha(l_1 \cdots l_p) - \sum \alpha(l_1 \cdots [x, l_i] \cdots l_p)$ $\quad$ $\iota(x)(\alpha)(l_1 \cdots l_{p-1}) = \alpha(x, l_1 \cdots l_{p-1})$
i.e. $\theta: L \to \mathrm{End}_{\mathbb{C}}(C^p(L,M))$, $\iota: L \to \mathrm{Hom}_{\mathbb{C}}(C^p(L,M), C^{p-1}(L,M))$
and set $f^*: \alpha \mapsto f \circ \alpha$, $f: M \to N$ as usual)

Prop3. ① $\theta(x) = \iota(x)d + d\iota(x)$; ② $\iota([x,y]) = [\theta(x)\iota(y)]$; ③ $\theta([x,y]) = [\theta(x)\theta(y)]$; ④ $[d\theta(x)] =$ [redacted] 0; ⑤ if f is not only a $\mathrm{Vect}_{\mathbb{C}}$ map, but also L-Mod map $\Rightarrow df^* = f^*d$.

Pf. I only prove ③ (The hardest), others left to reader: [Ex3. Prove ①②④, generalise ⑤ into f only $\mathrm{Vect}_{\mathbb{C}}$ map]
③ Induction on p, $p=0$ is evident;
$\iota(z)\theta([x,y]) = \theta([x,y])\iota(z) - \iota([[x,y],z])$ by ② ~~by induction~~
(induction due to $\iota(z)$ down 1-dimension)
~~induction~~
$[\theta(x)\theta(y)]\iota(z) - \iota([[x,y],z])$ = $\boxed{\theta(x)\theta(y)\iota(z)} - \boxed{\theta(y)\theta(x)\iota(z)} - \iota([[x,y],z])$ $\xrightarrow[\text{together}]{\text{adding}}$ $\iota([x[yz]]) + \iota([y[zx]]) + \iota([y[zx]])$
$\|$ by ② $\qquad$ $\|$ by ② again similiarly $\quad = \iota(z)[\theta(x)\theta(y)] + \iota(z)[\theta(x)\theta(y)]$
$\Rightarrow \forall z \in L$, $\quad \theta(x)(\iota([yz]) + \iota(z)\theta(y))$ $\quad \iota([y[xz]]) + \iota([xz])\theta(y) + \iota([xy])\theta(y) + \iota(z)\theta(y)\theta(x)$
$\iota(z)\theta([xy])$ $\quad = \theta(y)\iota([xz]) + \theta(x)\iota(z)\theta(y)$
$-\iota(z)[\theta(x)\theta(y)]$ $\quad = \iota([x[yz]]) + \iota([yz])\theta(x) + \iota([xz])\theta(y) + \iota(z)\theta(y)\theta(x)$
$\Rightarrow \theta([xy]) = [\theta(x)\theta(y)]$ ▨

Prop4. $C^{\cdot}(L,M)$ is a cochain complex Pf. $\iota(x)d^2 = \iota(x)dd = (\theta(x) - d\iota(x))d = \theta(x)d - d\iota(x)d = d\theta(x) - d\iota(x)d = d(\theta(x) - \iota(x)d)$
$= d\theta(x) - d(\theta(x) - d\iota(x)) = d\theta(x) - d\theta(x) + d^2\iota(x) = d^2\iota(x) \Rightarrow \iota(x)d^2 = d^2\iota(x)$
then induction on p ▨

Thm5. (Vanishing of Lie algebra cohomology)
① Fix $p \geq 0$, if $\forall M$ irreducible L-module, $H^p(L,M) = 0$ $\Rightarrow H^p(L,W) = 0$, for all W finite-dim L-module.
② L semi-simple, M irreducible and $M \neq k$ $\Rightarrow H^p(L,M) = 0$, $\forall p > 0$
③ L semi-simple, M finite-dim $\Rightarrow H^p(L,M) = 0$, $p = 1, 2$ (Whitehead)

Pf. ① Induction on $\dim W$; $W = \mathbb{C}$, nothing to prove;
$\dim W = n$ and W not irreducible $\Rightarrow \exists\, W_1 \subsetneq W$ nonzero and $0 \to W_1 \to W \to \frac{W}{W_1} \to 0$ induce LES $\boxed{H^p(L, W_1)} \to H^p(L, W) \to \boxed{H^p(L, \frac{W}{W_1})}$
W_1 irreducible; W_1 irreducible; induction: 0
$\Rightarrow H^p(L, W) = 0$ ☒

② $\rho: L \to gl(V)$ a irreducible representation, and Casimir element $c_\rho: V \to V \to 0$, and $[d, c_\rho^*] = 0$ by Prop 3 ⑤
Choose basis $(x_1 \cdots x_n)(y_1 \cdots y_n) \Rightarrow c_\rho = \sum \rho(x_i)\rho(y_i)$; take $\alpha \in Z^p(L, V) \subset \ker(d)$ and $\alpha_1 \in C^{p-1}(L, V)$ by $\alpha_1 = \sum \rho(x_i)^* (y_i)\alpha$
$\Rightarrow d\alpha_1(l_1 \cdots l_p) = \sum\sum (-1)^{j+1} [\rho(l_j)\rho(x_i)] \cdot \alpha(y_i, l_1 \cdots \hat{l_j} \cdots l_p) + \sum_i \rho(x_i)\rho(y_i)\alpha(l_1 \cdots l_p) - \sum_i\sum_j \rho(x_i)\alpha(l_1 \cdots [y_i l_j] \cdots l_p)$
(Check this step! Using Ex3 ⑤ Generalize: $[d, f^*](\alpha)(l_1 \cdots l_p) = (df)\alpha(l_1 \cdots l_p) - f^*(d\alpha)(l_1 \cdots l_p) = \sum_i (-1)^{i+1} l_i f(\alpha)(l_1 \cdots \hat{l_i} \cdots l_p)$
and $\rho(x_i)^* = f^*$ to give this formula!)
$= \sum_j \sum_i [l_j x_i]\,\alpha(l_1 \cdots l_{j-1}, y_i, l_{j+1} \cdots l_p) + (c_\rho)^*(\alpha)(l_1 \cdots l_p) - \sum\sum x_i \alpha(l_1 \cdots [y_i l_j] \cdots l_p) = (c_\rho)^*(\alpha)(l_1 \cdots l_p) + \sum_j\sum_i ([l_j x_i]\alpha(l_1$
$\cdots l_{j-1}, y_i, l_{j+1} \cdots l_p) + x_i \alpha(l_1 \cdots l_{j-1}, [l_j y_i], l_{j+1}, \cdots l_p)) = (c_\rho)^*(\alpha)(l_1 \cdots l_p) \Rightarrow d\alpha_1 = (c_\rho)^*(\alpha)$
$\Rightarrow d((c_\rho^*)^{-1}\alpha_1) = \alpha \Rightarrow \alpha \in B^p(L, V)$

[Ex4. Using we learnt fact about semi-simple Lie algebra in Talk 1, prove " = " = 0] $\Rightarrow B^p(L, V) = Z^p(L, V) \Rightarrow H^p(L, V) = 0$ ☒

③ By ①②, it suffices to prove $M = \mathbb{C}$ case, for $p = 1, 2$, this is a direct computation for lower dimension: [Ex5. Do same computation for $p = 3$ case to find $H^3(L, \mathbb{C}) = 0$ or not.]
We only prove $\underline{p = 2 \text{ case}}$: $\forall \alpha \in Z^2(L, \mathbb{C})$, $(d\alpha)(l_1, l_2, l_3) = -\alpha([l_1 l_2], l_3) + \alpha([l_1 l_3], l_2) - \alpha([l_2 l_3], l_1)$
$\tilde{\alpha} \in C^1(L, L^*)$; $\tilde{\alpha}(x)(y) = \alpha(x, y) \Rightarrow d\tilde{\alpha}(l_1, l_2)(y) = (l_1 \tilde{\alpha}(l_2))(y) - (l_2 \tilde{\alpha}(l_1))(y) - \tilde{\alpha}([l_1 l_2])(y)$
$= -\alpha(l_2, [l_1 y]) + \alpha(l_1, [l_2 y]) - \alpha([l_1 l_2], y) = \alpha([l_1 y], l_2) - \alpha([l_2 y], l_1) - \alpha([l_1 l_2], y) = 0$
$\Rightarrow d\tilde{\alpha} = 0$, by $L^* \neq \mathbb{C} \Rightarrow H^1(L, L^*) = 0 \Rightarrow \tilde{\alpha} \in B^1(L, L^*) \Rightarrow \exists\, \beta \in C^0(L, L^*)$, $d\beta = \tilde{\alpha} \Rightarrow x\beta = d\beta(x) = \tilde{\alpha}(x)\ \forall x \in L$
$\Rightarrow -\beta([xy]) = \tilde{\alpha}(x)(y) = \alpha(x, y)$, then we consider β as $C^1(L, \mathbb{C}) \Rightarrow d\beta = \alpha \Rightarrow \alpha \in B^2(L, \mathbb{C}) \Rightarrow Z^2(L, \mathbb{C}) = B^2(L, \mathbb{C})$
$\Rightarrow H^2(L, \mathbb{C}) = 0$ ☒

A interesting application: reproof the Weyl's complete reducibility thm (By Chevalley & Eilenberg)

Pf. $0 \to V \to W \to \frac{W}{V} \to 0$

But $0 \to \mathrm{Hom}_L(\frac{W}{V}, V) \to \mathrm{Hom}_L(\frac{W}{V}, W) \to \mathrm{Hom}_L(\frac{W}{V}, \frac{W}{V}) \to \boxed{H^1(L, \mathrm{Hom}(\frac{W}{V}, V))}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \; 0$

$0 \to \mathrm{Hom}(\frac{W}{V}, V) \to \mathrm{Hom}(\frac{W}{V}, W) \to \mathrm{Hom}(\frac{W}{V}, \frac{W}{V}) \to$ $\Rightarrow 0 \to \mathrm{Hom}_L(\frac{W}{V}, V) \to \mathrm{Hom}_L(\frac{W}{V}, W) \to \mathrm{Hom}_L(\frac{W}{V}, \frac{W}{V}) \to 0$
But not $\mathrm{Hom}_L(\frac{W}{V}, -)$! $\Rightarrow f: \frac{W}{V} \to V$ split the original exact sequence $\exists f \mapsto \mathrm{Id}$ $\Rightarrow W = \frac{W}{V} \oplus V$ ☒

Before coming to algebraic group, Let's see a interesting case: over char p Lie algebra

Def6. (p-Lie algebra / restricted Lie algebra) The definition is vaild in any field (even ring), but what's non-trivial is when over $\mathbb{k}$ with char p, the natural additional structure "p-th power operation", called p-map $[p]: L \to L$, s.t.
① $(\lambda l)^{[p]} = \lambda^p l^{[p]}$; ② $\mathrm{ad}(l^{[p]}) = (\mathrm{ad}(l))^p$; ③ $(l_1 + l_2)^{[p]} = l_1^{[p]} + l_2^{[p]} + \sum s_i(l_1, l_2)$
when s_i is polynomial, s.t. $i s_i(l_1, l_2)$ is λ^{i-1}-coeffecient of $\mathrm{ad}(\lambda l_1 + l_2)^{p-1}(l_1)$, i.e. $\mathrm{ad}(l_1\lambda + l_2)^{p-1} l_1 = \sum i s_i(l_1, l_2)\lambda^{i-1}$

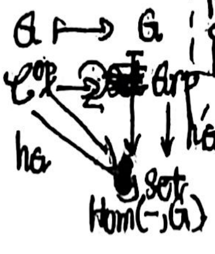
Such addition structure fit $\varphi(l^{[p]}) = (\varphi(l))^{[p]}$ naturally

Ex6. ① compute out $\mathbb{k} = \mathbb{F}_5$ case, what (s_i) are for p-map;
(Hint. $s_1(l_1, l_2) = [[[[l_1 l_2] l_2] l_2] l_2]$, $s_2(l_1, l_2) = \frac{1}{2}([[[[l_1 l_2] l_1] l_2] l_2] + [[[[l_1 l_2] l_2] l_1] l_2] + [[[[l_1 l_2] l_2] l_2] l_1])$, $s_3(l_1, l_2) = \frac{1}{3}([[[[l_1 l_2] l_1] l_1] l_2] + [[[[l_1 l_2] l_1] l_2] l_1] + [[[[l_1 l_2] l_2] l_1] l_1])$, $s_4 = \frac{1}{4}[[[[l_1 l_2] l_1] l_1] l_1]$)
② Guess the form of $\mathbb{k} = \mathbb{F}_p$ case.

E.g. (Important) (Restricted Lie algebra of derivations)
$\mathrm{Der}(L) \subset gl(L)$
$\mathrm{Der}(L) = \{D \in gl(L) \mid D^n([l_1 l_2]) = \sum \binom{n}{i} [D^i(l_1) D^{n-i}(l_2)]\}$
$\Rightarrow \mathrm{Der}(L)$ is the restrice Lie algebra of derivations of $gl(L)$
Replace $\mathrm{Gal}(E/F)$ by proper $\mathrm{Der}(L)$ in Galois theory (※) (Jacobson, 1937)
Let E/F be a finite purely inseperable extension, s.t. $E^p \subset F$
then $\mathcal{D}_{E/F}$ is a restricted Lie algebra over E, s.t. (Differential module)
$\{$the intermediate fields in E containing $F\} \longleftrightarrow \{$the restricted sub-Lie-algebra of $\mathcal{D}_{E/F}\}$

The AG theory of (p-) Lie algebra. (※)
Let $\mathcal{L}$ is Lie-$\mathcal{O}_X$-algebra, then we define it's associated vector bundle $L = \mathrm{Spec}(\mathrm{Sym}(\mathcal{L}^\vee)) \Rightarrow L$ is a Lie-X-algebra
Now let S the base scheme of char p, then L has p-map similarly defined but existence is nontrivial!
$\mathcal{L}ie_n$ is the moduli stack of n-dim Lie algebras; L_n the coase moduli space
$\Rightarrow \mathcal{L}ie_n = [L_n / GL_n]$ the quotient stack
then we study the Algebraic group.

Def7. (Group object, FGA) Fix a category $\mathcal{C}$
$G \in \mathrm{obj}(\mathcal{C})$ is a group object, together with $\mathcal{C}^{op} \xrightarrow{G} \mathrm{Grp}$
$\Leftrightarrow \forall C \in \mathrm{obj}(\mathcal{C})$
$h_G = G: \mathcal{C}^{op} \to \mathrm{Set}$
$C \mapsto \mathrm{Hom}(C, G) = G(C) \in \mathrm{Grp}$
(also morphisms)
$\Leftrightarrow$ Write as serval diagrams
[Ex7. Using 5 graphs to depict group object]



Def8. (Action of group objects, FGA)
A left action of $G: \mathcal{C}^{op} \to \mathrm{Grp}$ on $F: \mathcal{C}^{op} \to \mathrm{Set}$ is a natural transformation $G \times F \to F$, s.t. $\forall\, C \in \mathrm{obj}(\mathcal{C})$, induce $G(C) \times F(C) \to F(C)$ is the usual group action
$\Leftrightarrow$ $G(C_1) \times F(C_1) \to G(C_2) \times F(C_2)$
$F(C_1) \to F(C_2)$
[Ex8. Using 2 diagram on functors to depict left action]

Still setting over $\mathbb{C}$ for later discussion of reductive. Now (affine) algebraic group is simply the Group object in (Aff$_{\mathbb{C}}$) Var$_{\mathbb{C}}$

The reason of relationship between Lie algebra and algebraic group is by functor $G \mapsto \mathrm{Lie}(G)$, G connected algebraic group. It's faithful but not fully, thus we modify to a new functor $\mathrm{Rep}(G) \to \mathrm{Rep}(L)$, it's fully faithful $\Rightarrow$ the study of G — $\mathrm{Rep}(G)$ = Morita equivalence: the study of $\mathrm{Rep}(G)$ — the study of $\mathrm{Rep}(L)$ — Tannaka duality — tensor subcategory $\subset \mathrm{Rep}(L)$

Let's explain the Tannaka duality.

Def 9. (Monoidal category) $(\mathcal{C}, \otimes, 1)$ s.t. ① $(-\otimes-)\otimes- \cong -\otimes(-\otimes-)$ ② $-\otimes 1 \cong - \cong 1\otimes -$

($1 \in \mathrm{Obj}(\mathcal{C})$ unit, $\otimes: \mathcal{C}\times\mathcal{C} \to \mathcal{C}$)

$(A, \mu, e) \in \mathrm{Obj}(\mathcal{C}, \otimes, 1)$ is a monoidal object, s.t. $e: 1 \to A$, $\mu: A\otimes A \to A$ satisfy the unitality and associativity

[Ex 9. Using Graph to descript unitality and associativity] The morphism of (A, μ_A, e_A) and (B, μ_B, e_B) is $f: A \to B$, s.t.

$$\begin{array}{ccc} A\otimes A & \to & B\otimes B \\ \mu_A\downarrow & \circlearrowleft & \downarrow \mu_B \\ A & \xrightarrow{f} & B \end{array} \quad \& \quad \begin{array}{ccc} 1 & \to & A \\ & \searrow & \downarrow f \\ & & B \end{array}$$

The functor of $(\mathcal{B}, \otimes_{\mathcal{B}}, 1_{\mathcal{B}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ is lax monoidal functor $(F, \varepsilon: 1_{\mathcal{D}} \to F(1_{\mathcal{B}}), \ldots)$ s.t. unitality (A generalization of Fontaine in duality) and associativity

Furthermore, when ε is iso $\Rightarrow$ lax monoidal functor turn to monoidal functor

Def 10. (Left module) A left module of a monoidal object A is $(N, \rho: A\otimes N \to N)$ s.t. unitality, associativity; duality, the comodule defined

The $\mathrm{Hom}_{A\text{-Mod}}(M, N)$ can be defined similiarly.

$$\begin{array}{ccccccc} 1\otimes N & \to & A\otimes N & (A\otimes A)\otimes N & \to & A\otimes(A\otimes N) & \to A\otimes N \\ & \searrow \circlearrowleft & \downarrow & \downarrow & \circlearrowleft & & \downarrow \\ & & N & A\otimes N & \longrightarrow & & N \end{array}$$

[Ex 10. Define the monoidal transformation between two monoidal functors]

Def 11. (Symmetry)

$S: \mathcal{C}\times\mathcal{C} \to \mathcal{C}\times\mathcal{C}$, a braiding β on $\otimes$ is a monoidal natural isomorphism (Ex 10) $\mathcal{C}\times\mathcal{C} \xrightarrow{\otimes} \mathcal{C}$, $\Downarrow\beta$, $\mathcal{C}\times\mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ (and two diagrams on the associativity omitted)

$(A, B) \mapsto (B, A)$ if $\beta_{BA}\beta_{AB} = \mathrm{Id}_{A\otimes B} \Rightarrow \mathcal{C}$ is symmetric monoidal category

Ex 11. Define the (semi-)simple in $\mathcal{C}$ abelian category ($\Rightarrow$ monoidal) and prove the Schur's Lemma in $\mathcal{C}$

$\Rightarrow$ It's easy to see, such as $\mathrm{Rep}(G)$, $k[G]$-Mod, $\mathrm{Rep}(L)$ ... are all "good" (closed, symmetry) monoidal category with their tensor.

Def 12. (Tensor category and Tensored category)

$(\mathcal{C}, \otimes, 1)$ is tensor if $\otimes$ linear on Hom, 1 is simple obj

$(\mathcal{C}, \otimes, 1)$ is tensored if it's over $(\mathcal{D}, \otimes', 1')$ and $\forall a \in \mathcal{D}$, $m \in \mathcal{C}$, $a, m \in \mathcal{C}$, s.t. $\forall n \in \mathcal{C}$, $\mathcal{C}(a, m, n) \cong \mathcal{D}(a, \mathcal{C}(m, n))$

Thm 13. (Tannaka duality) It has many version, now I choose only one:

For $F: \mathrm{Rep}(G) \to \mathrm{Vect}_{\mathbb{C}}$ the forgetful functor $\Rightarrow G = \mathrm{Aut}(F^{\otimes})$

$\Rightarrow L = \{T \in \mathrm{End}(F) \mid \forall V, W \in \mathrm{Rep}(G),$

$$\begin{array}{ccc} F(V)\otimes F(W) & \xrightarrow{T(V)\otimes e + e\otimes T(W)} & F(V)\otimes F(W) \\ \| & & \| \\ F(V\otimes W) & \xrightarrow{T(V\otimes W)} & F(V\otimes W) \end{array} \}$$

Pf. The first "$\Rightarrow$" is the truly part of Tannaka duality. Only prove second "$\Rightarrow$": $A = \mathcal{O}(G)$ the regular function ring, the comodule $\Delta_V: F(V) \to F(V)\otimes A$, compatible with $\mu: A\times A \to A$

$$\Rightarrow \begin{array}{ccc} F(V)\otimes F(W) & \xrightarrow{\Delta_V\otimes\Delta_W} & F(V)\otimes F(W)\otimes A\otimes A \\ \downarrow \wr\| & \circlearrowleft & \downarrow \mathrm{Id}\otimes\mu \\ F(V\otimes W) \xrightarrow{\Delta_{V\otimes W}} F(V\otimes W)\otimes A & \cong & F(V)\otimes F(W)\otimes A \end{array}$$

and give map $\varphi, \varepsilon: A \to \mathbb{C}$

① $\varphi: A \to \mathbb{C}$ s.t. $F(V) \xrightarrow{\Delta_V} F(V)\otimes A \xrightarrow{\varphi} F(V)$

② $\varepsilon: A \to \mathbb{C}$ s.t. the counit.

$$\begin{array}{ccc} F(V)\otimes F(W) & \longrightarrow & F(V)\otimes F(W)\otimes A\otimes A \\ \downarrow \Delta_{V\otimes W} & \circlearrowleft & \downarrow \mathrm{Id}\otimes(\varphi\otimes\varepsilon + \varepsilon\otimes\varphi) \\ F(V\otimes W)\otimes A & \xrightarrow{\mathrm{Id}\otimes\varphi} & F(V)\otimes F(W) \end{array}$$

$\Rightarrow \varphi(xy) = \varphi(x)\varepsilon(y) + \varepsilon(x)\varphi(y)$ $\square$

Then we return back to concrete.

Def 14. A (affine) algebraic group is

① Group object in (Aff$_{\mathbb{C}}$) Var$_{\mathbb{C}}$; ② Representable functor $G: \mathrm{Alg}_{\mathbb{C}} \to \mathrm{Set}^{opp}$ by a f.g. $\mathbb{C}$-algebra (only affine)

③ An algebraic group with a multiplication, inverse. (The action is done categorically!) $\Updownarrow$ linear

E.g. (Easy but important!) $\mathbb{G}_a$ and $\mathbb{G}_m$

$\mathbb{G}_a = (\mathbb{A}^1_{\mathbb{C}}; \mu: \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}; i: \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}})$; $(a, b) \mapsto a+b$, $a \mapsto a^{-1}$; $\mathbb{G}_m = ((\mathbb{A}^1_{\mathbb{C}})^*; \mu: (\mathbb{A}^1_{\mathbb{C}})^{*2} \to (\mathbb{A}^1_{\mathbb{C}})^*; i: (\mathbb{A}^1_{\mathbb{C}})^* \to (\mathbb{A}^1_{\mathbb{C}})^*)$; $(a, b) \mapsto ab$, $a \mapsto a^{-1}$

They're basic material for affine GIT; $\mathbb{T}^n = \mathbb{G}_m^n$ is the algebraic torus

E.g. $GL_n(\mathbb{C})$ is an algebraic group. [It's basic due to $\forall$ affine algebraic group G $\exists j: G \to GL_n(\mathbb{C})$ closed immersion, group morphism]

Note. $\det: M_n(\mathbb{C}) \to \mathbb{C}$ is polynomial

$\Rightarrow GL_n(\mathbb{C}) = M_n(\mathbb{C}) - \det^{-1}(0)$ is affine open $\Rightarrow \Gamma(GL_n(\mathbb{C}), \mathcal{O}) = \mathbb{C}[X_{11}, X_{12} \cdots X_{nn}, \frac{1}{\det(X_{ij})}]$

then the product of matrix makes $GL_n(\mathbb{C})$ to an algebraic group

[Ex 12. Show that $SL_n(\mathbb{C})$; $B_n(\mathbb{C})$ (invertible upper triangular); $D_n(\mathbb{C})$ (invertible diagonal); $U_n(\mathbb{C})$ (unipotent upper triangular) are algebraic groups. (Hint. $B_n(\mathbb{C}) \cong \mathbb{A}^{\frac{n(n-1)}{2}} \times (\mathbb{C}^*)^n$, $D_n(\mathbb{C})$ is torus, $U_n(\mathbb{C}) \cong \mathbb{A}^{\frac{n(n-1)}{2}}$)]

A collection of definitions:

Def 15. ① $H \leq G$ closed immersion is a closed subgroup (algebraic subgroup) of G, $G\times G \to G$ induce $H\times H \to H$ by restriction; (An abstract subgroup is ...)

② The homomorphism just as the homomorphism of group objects; ③ G is semi-simple as in tensor category with group structure

④ G is reductive $\Leftrightarrow$ The only subgroup of G is torus; ⑤ G is solvable $\Leftrightarrow$ if it has a filtration of normal subgroups, quotients ...

Ex 13. ① $U, V \subset G$ open $\Rightarrow G = UV$; ② $H \leq G$ abstract subgroup $\Rightarrow \bar{H} \leq G$ algebraic subgroup; ③ $H \leq G$ abstract subgroup, H constructible ...

④ $\varphi \in \mathrm{Aut}(G)$, $H \leq G$ algebraic subgroup, if $\varphi(H) \subset H \Rightarrow \varphi(H) = H$; give a counterexample $\Rightarrow H \leq G$ algebraic subgroup s.t. $\varphi \in \mathrm{End}(G)$ but $\varphi(H) \subsetneq H$;

Then we comes to build relation of Algebraic group and Lie algebra.

At first, due to we need an analogue of tangent structure of Lie group, first recall things about Zariski tangent space and differential sheaves.

This is one way to dealing; but I take also an approach more abstract: Hopf algebra. (It's role is just differential, keeping in mind!)

coassociativity + counitality (They're useful for dual representation)

Def 16. (Coalgebra) $\mathbb{C}$-coalgebra is $(C, \Delta: C \to C\otimes C, \varepsilon: C \to \mathbb{C})$ s.t. $C \xrightarrow{\Delta} C\otimes C$ ... comultiplication, counit

Morphism of coalgebra is $C \xrightarrow{f} D$ ... $C \xrightarrow{f} D$ (It's pure monoidal categorical) Def 17. (Convolution $*$) $f, g \in Hom_k(C, A)$ C coalgebra, A algebra, $f*g \in Hom_k(C,A)$ $(f*g)(c) = \sum f(c_1)g(c_2)$

By coassociativity, we have Δ^2 & ε

This is nothing new but a representation if one act Conv ...

the Sweedler's notation for $C \otimes C \to D \otimes D$

simplicity: $\Delta^n = (\Delta \otimes id \cdots \otimes id)\Delta^{n-1}$

A coalgebra + algebra $(\varepsilon, \Delta$ is C-algebra homomorphism) is a bialgebra.

Def 18. A bialgebra C, with $Id_C \in End_k(C)$ is invertible under $*$ (the category $\mathcal{H}_C$) $\Leftrightarrow$ C is Hopf algebra. Note that Id_C may non-trivial, denoted S_C the antipode of C

this is by present $\Delta: C \to C \otimes C$, $c \mapsto (c_1, c_2)$
$\Rightarrow c = \sum \varepsilon(c_1)c_2 = \sum c_1\varepsilon(c_2)$ and $f(c) = \sum f(c_1) \otimes f(c_2)$

Def 19. The C-comodule is purely monoidal categorical, we write again here

$M \xrightarrow{\varphi} M \otimes C$ and $M \to M \otimes C$ & M ... Id

We also write φ^n simply

$M \otimes C \to M \otimes C \otimes C$, $M \otimes C \to M$

the morphism of comodule similar or equivalently $C \xrightarrow{\Delta} C \otimes C$, $f*g$, $f \otimes g$, $A \leftarrow A \otimes A$

We have the dual (Mumford ... GIT)
Representation of ... $\rightsquigarrow$ Dual representation of G
coalgebra $\Gamma(G, \mathcal{O}_G) \rightsquigarrow$ action $G \to Spec(R)$
dual action
$\Rightarrow$ this motivates our Thm 21

We denote ${}^C Mod$, Mod^C for left/right C-comodule category. Ex14. Check that ${}^C Mod$, Mod^C are abelian, and C is injective-comodule of itself.

Turn back to G affine algebraic group, $k[G]$ defined

$\Delta: k[G] \to k[G \times G]$ and $\varepsilon: k[G] \xrightarrow{eval} k$ ($k = \bar{k}$) $\Rightarrow (k[G], \Delta, \varepsilon)$ is a Hopf algebra

and the inverse of $Id_{k[G]}$ is $S: k[G] \to k[G]$, $f \mapsto (S(f): x \mapsto f(x^{-1}))$

$k[G] \otimes k[G]$

Prop 20. Check upper facts. ② $G \xrightarrow{\varphi} H$ induce $\varphi^*: k[H] \to k[G]$ of bialgebra.

Pf. ① coassociative: $f((xy)z) = f(x(yz))$
$f((xy)z) = \sum f_1(xy) f_2(z) = \sum f_{11}(x) f_{12}(y) f_2(z)$; $f(x(yz)) = \sum \ldots$
$\Rightarrow \sum f_{11}(x) f_{12}(y) f_2(z) = \sum f_1(x) f_{21}(y) f_{22}(z)$ the coassociative

Thus prop 20 gives us a functor, denote $\mathcal{G}$ the category of affine algebraic group

$k[-]: \mathcal{G} \to \mathcal{H}^{op}$

$G \mapsto k[G]$; but a amazing fact is the following

Thm 21. $k[-]$ is an equivalence of category, with inverse is the $Max(-): \mathcal{H}^{op} \to \mathcal{G}$

$k[G] \mapsto \boxed{Max(k[G]) = G}$ and $\boxed{k[Max(A)] = A}$

antipode: $f = \sum \varepsilon(f_2) f_1$ is trivial;
$f(e) = f(y^{-1}y) = \sum f_1(y^{-1}) f_2(y) = \sum S(f_1)(y) f_2(y) = (\sum S(f_1) f_2)(y)$
$\Rightarrow \sum S(f_1) f_2 = f(e) = \varepsilon(f) \Rightarrow (S * id)(f) = \varepsilon(f)$ ▨

② Let $f \in k[H]$; $x, y \in G \Rightarrow \sum \varphi^*(f_1)(x) \varphi^*(f_2)(y) = \sum f_1(\varphi(x)) \ldots$
$f(\varphi(xy)) = f(\varphi(x)\varphi(y)) = f(\varphi(xy)) = \varphi^*(f)(xy) = \sum (\varphi^* f)_1(x) (\varphi^* f)_2(y)$
$\Rightarrow \sum \varphi^*(f_1) \otimes \varphi^*(f_2) = \sum (\varphi^*(f))_1 \otimes (\varphi^*(f))_2$ ▨

Rk. One learnt theory of C^*-algebra will familiar with these two formula! They're nothing new but Gelfand-Naimark thm $\Rightarrow$ In fact, Gelfand's theory is just a representation theory

Ex 15. Review the G-N thm and modify its proof to prove Thm 21.

Def 22. G affine algebraic group, $L(G) = T_e(G)$ is the associated Lie algebra, with $[-,-]: L(G) \wedge L(G) \to L(G)$ is given by: $\tau, \sigma \in D_e(k[G]) = L(G)$ (The relation of Hopf algebra $k[G]$ lies that: $T_eG = \frac{m_{e,G}}{m_{e,G}^2} = D_e(k[G])$ (the dual ring)

$[\sigma \tau] = \sigma * \tau - \tau * \sigma \in D_e(k[G])$

Check. $[\sigma \tau](fg) = \sum \sigma((fg)_1) \tau((fg)_2) - \tau((fg)_1)\sigma((fg)_2) = \sum \sigma(f_1 g_1) \tau(f_2 g_2) - \tau(f_1 g_1) \sigma(f_2 g_2) = \sum (f_1(e)\sigma(g_1) + g_1(e)\sigma(f_1))(f_2(e)\tau(g_2) + g_2(e)\tau(f_2)) - \sum (f_1(e)\tau(g_1) + g_1(e)\tau(f_1))(f_2(e)\sigma(g_2) + g_2(e)\sigma(f_2)) = f(e)[\sigma, \tau](g) - g(e)[\sigma, \tau](f)$ Lebiniz rule ▨

Rk. $G \xrightarrow{\varphi} H$ induce $L(G) \xrightarrow{d\varphi} L(H)$ obvious. We then prove that $L(GL(V)) \cong gl(V)$, although one might known it in theory of (when $GL(V)$ think as a Lie group), we need some prepare work for this:

Ex 16. k with char $k = p$ and the Frobenius map $F: \mathbb{G}_m \to \mathbb{G}_m$, $x \mapsto x^p$, compute $dF: k \to k$ (the differentials)

Prop 23. $L(G) \oplus L(H) \cong L(G \times H)$ (Check $G \times H$ is well-defined)

Pf. Set $\Phi: L(G) \oplus L(H) \to L(G \times H)$
$(\sigma, \tau) \mapsto \sigma \otimes \varepsilon + \varepsilon \otimes \tau$

Injectivity: $(\sigma \otimes \varepsilon + \varepsilon \otimes \tau)(f \otimes 1) = \sigma(f)$, $(\sigma \otimes \varepsilon + \varepsilon \otimes \tau)(1 \otimes g) = \tau(g)$. Surjectivity, Counting dimension

the homomorphism can be checked easily ▨

Ex 17. What about the $L(G \rtimes H)$ looks like?

Lemma 24. $d_{(1,1)}(m): L(G) \oplus L(G) \to L(G)$
$d_e(\iota): L(G) \to L(G)$, $(\sigma, \tau) \mapsto \sigma + \tau$
$\sigma \mapsto -\sigma$ (m, ι is the multiplication and inverse)

Pf. $d_{(1,1)}(m)(\sigma, \tau)(f) = (\sigma, \tau)(f \circ m) = (\sigma \otimes \varepsilon + \varepsilon \otimes \tau)(\sum f_1 \otimes f_2)$
$= \sum \sigma(f_1) f_2(e) + f_1(e) \tau(f_2) = \sigma(f) + \tau(f)$, ι is similarly done ▨

Def 25. (Rational module and some representation)

Why rational G-module turns out is due to the representation of G isn't finite dimensional. (e.g. regular representation)

Def 26. (Locally finite representation) A G-vector space V, $G \xrightarrow{\rho} GL(V)$ s.t. $\forall v \in V$, $\exists W \subset V$, W is G-invariant and $\dim W < \infty$, $W \ni v$ $\Rightarrow$ called the (ρ, V) is locally finite

Def 27. (Representation function) (ρ, V) a representation of G,
$\forall \lambda \in V^*$, $v \in V$, the function $G \xrightarrow{\rho(\lambda, v)} k$, $x \mapsto \langle \lambda, x, v \rangle$ is the representive function

$\Rightarrow$ A rational G-module is locally finite and $\rho(\lambda, v) \in k[G]$, thus given a rational G-module M, we may consider it as a $k[G]$-comodule by $\Delta_\varphi: M \to M \otimes k[G]$ induced by action $\varphi: G \times M \to M$

Def 28. M is rational G-module, action $G \times M \to M$, $\Delta_\varphi: M \to M \otimes k[G]$

Ex 18. Verify φ° does an action $\Rightarrow$ the differential of φ, denoted $\varphi^\circ: L(G) \otimes M \to M$, this defines $L(G) \to M$, $\tau \otimes m \mapsto \sum m_0 \tau(m_1)$

② This coincide the infinitesimal action we learnt in talk 1.

Thm 29. $L: L(GL(V)) \to gl(V)$, $L(\tau)(v) = \tau v$ is isomorphism. (I can't understand the proof!)

证. By a trivial dimension-counting, it suffices to prove the injectivity: $\tau.v=0 \Rightarrow \forall \alpha\in V^*, v\in V, \tau(\alpha|v)=0$
$\Rightarrow (\alpha|v-\alpha(v)\cdot e)$ generates the maximal ideal $m_e \leq k[G]$, then if $(\alpha|v)(x)-\alpha(v)=0 \Rightarrow x.v-v=0, \forall v\in V$
$\Rightarrow x=e\in GL(V)$

$\alpha(x.v)-\alpha(v)$

a polynomial

$\Rightarrow \tau|_{m_e}=0 \Rightarrow \tau=0 \Rightarrow (\omega(\tau)=0 \Rightarrow \tau=0) \Rightarrow$ injective ▨

Ex19. Show that:

$$\begin{tikzcd} L(GL_n(\mathbb{C})) \arrow[r, "\det"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & L(GL_1(\mathbb{C})) \arrow[d, "\wr"] \\ gl_n(\mathbb{C}) \arrow[r, "tr"] & \mathbb{C} \end{tikzcd}$$

Ex20. The adjoint representation $Ad: GL(V)\times gl(V) \to gl(V)$, $(x, M) \mapsto xMx^{-1}$

Then we focus on the action & representation of algebraic group. We except to prove:

Thm30. ① (Kostant–Rosenlicht) U affine algebraic group and unipotent $U \curvearrowright X$ regularly $\Rightarrow \forall p\in X, \mathcal{O}(p)\subset X$ is closed
② $\dim G=\dim \mathcal{O}(x)+\dim S(x), \forall x\in X$ ③ the orbit map $G \xrightarrow{\pi} \mathcal{O}(p)$ is seperated $\Rightarrow \mathcal{O}(p)\cong G/\!/S(p)$ (Geomertic quotient)

$d_e\pi: L(G) \to T_p(\mathcal{O}(p))$ surjective $\Leftrightarrow \mathrm{Ker}(d_e\pi)=L(G_p)$
$\Leftrightarrow \pi$ is submersive (or open map) and $\mathcal{O}_{\mathcal{O}(p)} \cong \mathcal{O}_G^{S(p)}$ (the definition of Geomertic quotient)

Def31. (Categorical quotient and Geomertic quotient) Set $G \curvearrowright X$

① Y is categorical quotient if $G \curvearrowright Y$ trivial and satisfy the universal properties: $X \xrightarrow{f} Z$ for $\forall Z$ s.t. $G \curvearrowright Z$ trivial

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] \arrow[dr, phantom, "\circlearrowleft"] & Z \\ Y \arrow[ur, dashed, "\exists! \tilde{f}"'] & \end{tikzcd}$$

② Y is a geomertic quotient if $\pi: X\to Y$ surjective, submersive (open), its fibres are all G-orbit; $\mathcal{O}_Y \cong \mathcal{O}_X^G$. (then Y can be considered as the orbit space)

Prop32. A Geomertic quotient may not exist, once it exist, it's unique.

证. By showing Geomertic $\Rightarrow$ Categorical (Due to the time limit, we omit the proof of Thm30)
This is by covering X affine ▨

Ex21. $\mathbb{G}_m \curvearrowright \mathbb{A}^2_{\mathbb{C}}$ by $t(x,y)=(tx, t^{-1}y)$
What about its categorical quotient?
(Hint, Exist and $\pi: \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ is $(x,y)\mapsto xy$)

Ex20 (Categorical quotient also may not exist) $\mathbb{G}_a \curvearrowright \mathbb{A}^4_{\mathbb{C}}=M_2(\mathbb{C})$ by
$\lambda\cdot\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{pmatrix} 1 & \lambda \\ 0 & 1\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{pmatrix} a+\lambda c & b+\lambda d \\ c & d\end{pmatrix}$
Prove, this action can't have a categorical quotient!

(Guess there're more facts later: homogenous space / G-principal bundle / G-torsor, the affine GIT, projective GIT, moduli problem... can be learnt then.

Prob1. (Moduli of p-Lie algebra) (Bouillot, 2022) Set $L\to S$ is the Lie algebra vector bundle
① Define functor $F: Sch_S \to Set^{op}$, $T\mapsto \{p \text{ mapping on } L\times_S T\}$. Show that F is representable by an affine scheme X. X is formally homogenous (i.e. $E\times_S X \to X\times_S X$, $(e,x)\mapsto (e.x, x)$ is isomorphism) under $E=\mathrm{Hom}_{\mathrm{Frob}}(L, Z(L))$ (Frob means the Frobenius map $x\mapsto x^p$)
② Set $L'\subset L$ invertible $\Rightarrow \alpha: L\to \mathrm{End}(L')\cong \mathbb{G}_a$ by $x\mapsto (ad(x)|_{L'} \mapsto \alpha(x))$ is map of vector bundles

It induces $\tilde{\alpha}: L\to L$, $x\mapsto \alpha(x)^{p-1}x$. Prove that: $\tilde{\alpha}$ is p-mapping and if $\mathrm{rank}(L)=2 \Rightarrow \alpha$ unique

Prob2 (Tannaka duality for Geomertic stacks) (Lurie, 2004)
A (artinian) stack X is Geomertic if it's quasi-compact and the diagonal map $X \xrightarrow{\Delta} X\times_{\mathrm{Spec}\,\mathbb{Z}} X$ is representable affine.
The category ~~…~~ $\mathcal{O}_S$-Mod ~~and~~ defined, the category ~~…~~ $QCoh(X)$ (Think X category) (Think X geometric object)
Now set $(S, \mathcal{O}_S)$ a ringed topos, local for étale topology ($\forall E\in S, \forall \{\mathcal{O}_S(E)\to R_i\}_{i\in I}, |I|<\infty \Rightarrow \mathcal{O}_S(E)\to \prod R_i$ faithfully flat $\Rightarrow \exists E_i \to E$ in S and factorization $\mathcal{O}_S(E)\to R_i \to \mathcal{O}_S(E_i)$ and $\coprod E_i \to E$ is epimorphism)
Prove. ① $QCoh(X)$, $\mathcal{O}_S$-Mod are complete abelian (tensor) category; ② Categorical equivalence $\mathrm{Hom}(S, X) \to \mathrm{Hom}_{\otimes}(QCoh(X), \mathcal{O}_S\text{-Mod})$, $f\mapsto f^*$

Prob3 (Abelian varieties)
An abelian variety is a connected, complete (projective) algebraic group, when over $\mathbb{C}$ it's just a complex torus $\mathbb{C}^n/\Lambda$
① One dimension abelian variety $\Leftrightarrow$ elliptic curve; ② The moduli $\mathcal{M}_{1,1}$ of elliptic curves $\cong \mathbb{H}^2_{\mathbb{R}}/SL_2(\mathbb{Z})$ ($\mathbb{H}^2_{\mathbb{R}}$ is the upper ~~…~~-half plane)
③ The moduli $\mathcal{A}_g$ of genus g abelian variety $\cong \mathcal{U}/PSL_{4g}(\mathbb{Z})$, which is quasi-projective, for the $\mathcal{U}$ Seidel upper-half plane (Arapura, 2012)