# Abelian variety

Def 1. Abelian variety X is complete group variety ($\forall Y$, $p_2: X\times Y \to Y$ is closed) (It's related with properness)
In our setting, fix base $\Bbbk$, note that when char$(\Bbbk)=p$, it's usual occuring non-reduced group scheme.
Eg.1. $GL_n(\Bbbk)\subset M_n(\Bbbk)$ open ⇒ affine ⇒ not Abelian
Eg.2. {Complex tori} = {Abelian var/$\mathbb{C}$}
Multiplication of $x\in X(T)$ ($X(T)$ has group structure naturally), T a $\Bbbk$-variety, X group variety
$r_x: X_T \cong X_T\times_T T \xrightarrow{id\times x} X_T\times_T X_T \xrightarrow{m} X_T$ & $l_x: X_T \cong T\times_T X_T \xrightarrow{x\times id} X_T\times_T X_T \xrightarrow{m} X_T$ are T-isomorphisms.
(In particular taking $T=\Bbbk$)  When communicative, we denote both $t_x$

Prop1. X smooth and canonical trivial
Pf. Smooth loci is dense & translation is $\Bbbk$-iso ⇒ All smooth;
We prove tangent trivial: $\mathcal{T}_X \cong \mathcal{T}_{X,e}\otimes \mathcal{O}_X$, where $\mathcal{T}_{X,e}$ is skycraper sheaf of $\mathcal{T}_X$ at $e\in X$ ($\mathcal{T}_{X,e}=Hom(\mathfrak{m}_e/\mathfrak{m}_e^2,\Bbbk)$)
Let $S=Spec(\Bbbk[\varepsilon]/(\varepsilon^2))$ $\{\tau\in \mathcal{T}_{X,e}\otimes\mathcal{O}_X$ is tangent vector at $e\}$ ⟷ $\left\{\begin{array}{ccc} \tau: S & \to & X \\ \uparrow & & \uparrow \\ Spec\,\Bbbk & \to & e\end{array}\right\}$ by definition ⇒ $\tau\in X(S)$ gives translation $r_\tau\in\mathcal{T}_X$
⇒ $\mathcal{T}_{X,e}\otimes\mathcal{O}_X \to \mathcal{T}_X$ Claim. It's isomorphism
$\tau \mapsto r_\tau$ ⇒ $\mathcal{K}_X\cong\mathcal{O}_X$ ▨
(The claim is by view $\tau$ vector field ⇒ it gives flow/automorphism $X_S\to X_S$, then flow e back to e)

Coro1. $\mathbb{P}^1\to X$ constant.
Pf. Taking image $C\subset X$ (Then $g(C)\le g(\mathbb{P}^1)=0 \Rightarrow g(C)=0$ rational ⇒ $\mathcal{T}_X \geq \mathcal{T}_{\mathbb{P}^1}$ nontrivial ⇒ contradiction ▨)
∃ nontrivial 1-form $\omega\in\Gamma(X,\Omega_X^1)$ & $f^*\omega$ nontrivial, but $\Gamma(\mathbb{P}^1,\Omega_{\mathbb{P}^1})=\Gamma(\mathbb{P}^1,\mathcal{O}(-2))=0$ ▨

Prop2. X, Y Abelian, $f: X\to Y$ is $\Bbbk$-mor preserve identity ⇒ f is morphism between group varieties
Pf. Consider $m_Y\circ[(f\circ m_X)\times(\iota_Y\circ m_Y\circ(f\times f))]: X\times X\to Y$
$(x,y)\mapsto f(m_X(x,y))\,m_Y(f(x),f(y))^{-1}$ contract fibre at e (by preserve identity) ⇒ contract all ⟺ morphism ▨
(Recall $\boxed{\text{rigidity}}$ of complete var)

Coro2. Abelian ⇒ communicative ($f=\iota_X$) Coro3. Fix $x\in X(\Bbbk)$, ∃! Abelian structures s.t. X Abelian with x identity.

Thm1. X Abelian, V smooth, then $\forall\, V \overset{f}{\dashrightarrow} X$ can extend to morphism.
Pf. Assume $\Bbbk=\bar\Bbbk$ ($V_{\bar\Bbbk}\dashrightarrow X_{\bar\Bbbk}$ over $\Bbbk$ defined ⇒ $V_\Bbbk\to X_\Bbbk$ defined), $U=Dom(f)$, V smooth ⇒ $\boxed{codim(U,V)\geq 2}$
Set $F: V\times V\dashrightarrow X$, $(x,y)\mapsto f(x)-f(y)$   $Dom(f)\times Dom(f)\subset Dom(F)$, $\cap\Delta$ Claim. It's equal: $\forall (v,v)\in V(\Bbbk)\times V(\Bbbk)$ defined
$(V\times\{v\})\cap Dom(F)\subset V\times\{v\}$ dense open
⇒ Choose $u\in U$, s.t. $(u,v)\in(V\times v)\cap Dom(F)$
⇒ $(f(u)\times V)\cap Dom(F)\subset f(u)\times V$ open dense
⇒ $f(v)=f(u)-F(u,v)$ defined.
⇒ $Dom(f)\times Dom(f) = Dom(F)\cap\Delta$.
We show $\Delta\subset Dom(F)$ ⇒ $Dom(f)\times Dom(f) \overset{\supseteq\Delta}{=} Dom(F)$ done ▨
$F|_{\Delta\cap Dom(F)}=0$, thus
F regular at $(v,v)\in\Delta$ ⟺ $F^\#: \mathcal{O}_{X,0}\to\mathcal{O}_{V\times V,v\times v}$
Otherwise ∃$\varphi$, $F^\#\varphi\notin\mathcal{O}_{V\times V,v\times v}$, the pole divisor $D=(F^\#\varphi)_\infty=\sum_{P\text{ is pole of }F^\#\varphi} ord_P(F^\#\varphi)P$ intersect $\Delta$ ⇒ $\Delta\cap D\subset\Delta$ divisor ⇒ codim 1, contradiction

Eg.3. (Elliptic curves)
Elliptic curve is complete nonsingular curve E genus 1 over $\Bbbk$ with $P\in E(\Bbbk)$ marked; it induces $\alpha: E\to Pic^0(E)$, $Q\mapsto Q-P$ (deg 0). $\alpha$ is bijection, thus we can make E to a group with same structure as $Pic^0(E)$ via $\alpha$ ⇒ E is group variety.
$\alpha$ is morphism ⇒ E is Abelian
($Pic^0(E)$ is the dual of E, but not ≅ E usual)

Lemma1. (thm of square) $\boxed{X\text{ complete}}$, L, M line bundle on $X\times Y$, $\forall y\in Y$, $L_y:=L|_{X\times\{y\}}$, $M_y:=M|_{X\times\{y\}}$, $L_y\cong M_y$ ⇒ ∃ line bundle N on Y, s.t. $L\cong M\otimes p_Y^*N$
Pf. Let $N=(p_Y)_*(L\otimes M^{-1})$ ▨
Coro4. (See-Saw) $L_y\cong M_y$, $\forall y\in Y$ & $L_{x_0}\cong M_{x_0}$, $\exists x_0\in X$ ⟺ $L\cong M$
Lemma2. $\{y\mid L_y$ trivial on $X\}\subset Y$ closed
Pf. $\{y\mid L_y$ trivial$\}=\{y\mid h^0(L_y)\geq 1\}\cap\{y\mid h^0(L_y^{-1})\geq 1\}$ is intersection of two closed subset ▨
Rk. Here we prove for Y is variety, but it holds for Y scheme; and y is replaced by closed subscheme:
$\exists Y_0\subset Y$ is maximal s.t. $L_{Y_0}$ is trivial (maximal in sense that: $\forall Z\xrightarrow{\varphi}Y$, $(Id_X\times\varphi)^*L$ trivial ⇒ $\varphi$ factor through $Y_0$)
the pf is hard and postpone.
Lemma3. (thm of cube) X, Y complete, if $\exists x_0, y_0, z_0$, s.t. $L_{x_0}, L_{y_0}, L_{z_0}$ trivial ⇒ L trivial.
Rk. Two variety X, Y not true! Consider $\Delta - E\times\{0\}-\{0\}\times E$ on $E\times E$ is nontrivial.
Coro5. X Abelian, L line bundle, $p_i: X\times X\times X\to X$, $p_{ij}: X\times X\times X\to X$, $(x_1,x_2,x_3)\mapsto x_i+x_j$, $p_{123}: X\times X\times X\to X$, $(x_1,x_2,x_3)\mapsto x_1+x_2+x_3$ ⇒ $\Theta(L)=p_{123}^*L\otimes p_{12}^*L^{-1}\otimes p_{13}^*L^{-1}\otimes p_{23}^*L^{-1}\otimes p_1^*L\otimes p_2^*L\otimes p_3^*L$
Coro6. Y $\Bbbk$-scheme, $\forall f,g,h\in X(Y)$ ⇒ $(f+g+h)^*L\otimes(f+g)^*L^{-1}\otimes(g+h)^*L^{-1}\otimes(h+f)^*L^{-1}$ trivial
Pf. Factor f+g+h, ... through X×X×X ▨

Coro7. X Abelian, L line bundle $\Rightarrow (t_{x+y}^*L)\otimes L \cong (t_x^*L)\otimes(t_y^*L)$ Pf. Set $f=Id, g=t_x, h=t_y$ ▨

Thus $\varphi_L : X \to Pic(X)$, $x \mapsto (t_x^*L)\otimes L^{-1}$ is group homomorphism $\Rightarrow$ let $Pic^0(X)=\{L\in Pic(X) \mid \varphi_L=0\}=\{L \mid t_x^*L\cong L \text{ invariant}\}$

$\Rightarrow t_y^*((t_x^*L)\otimes L^{-1})\otimes((t_x^*L)\otimes L^{-1})^{-1}=0 \Rightarrow Im(\varphi_L)\subset Pic^0(X)$

$\Rightarrow$ $t_{x+y}^*L\otimes t_x^*L^{-1}\otimes t_y^*L^{-1}\otimes L$

$0\to Pic^0(X)\to Pic(X)\to Hom(X,Pic(X))$ (by Thm of square), $Pic^0(X) = Ker(\varphi_\square)$, $L\mapsto \varphi_L$

Coro8. $n_X: X\to X \Rightarrow n_X^*L\cong L^{\otimes\frac{n(n+1)}{2}}\otimes(-1)^*L^{\otimes\frac{n(n-1)}{2}}$

Pf. Induction on n, take $f=n_X, g=Id, h=-1$ ▨

When L is $\begin{cases}\text{symmetric}((-1)^*L\cong L) \Rightarrow n_X^*L = L^{\otimes n^2}\\ \text{anti-symmetric} \Rightarrow n_X^*L\cong L^{\otimes n}\end{cases}$

Pf of Lemma3. $\exists$ maximal $Z_0\subset Z$, $L_{Z_0}$ trivial. $Z_0$ is $\begin{cases}\text{closed: done}\\ \text{open}\end{cases}\Rightarrow Z_0=Z$, done

Openness of $Z_0\subset Z$. $\mathcal{I}_{Z_0}\subset\mathcal{O}_Z$, $\forall \zeta\in Z_0$ (scheme-theoretic)

$\mathcal{I}_{Z_0,\zeta}\subset\mathcal{O}_{Z,\zeta}$, Claim $\mathcal{I}_{Z_0,\zeta}=0$ ($\Leftrightarrow$ open). Consider maximal ideal $m\trianglelefteq\mathcal{O}_{Z,\zeta}$

If $\mathcal{I}_{Z_0,\zeta}\neq 0$, due $\bigcap m^n=(0) \Rightarrow \exists n$, $\mathcal{I}_{Z_0,\zeta}\subset m^n$ but $\mathcal{I}_{Z_0,\zeta}\not\subset m^{n+1}$, let $a_1=(\mathcal{I}_{Z_0,\zeta}, m^{n+1})$, $\exists a_2 \begin{cases} m^{n+1}\subset a_2\subset a_1\subset m^n\\ \dim a_1/a_2=1\end{cases}$

Let $Z_1\subset Z_0$ closed defined by $a_1 \Rightarrow L_{Z_1}$ trivial, but $\mathcal{I}_{Z_0,\zeta}\not\subset a_2 \Rightarrow L_{Z_2}$ not trivial

But $L_{Z_2}$ indeed trivial: $Z_1\subset Z_2$, we can lift nonvanishing global section of $L_{Z_1}$ to $L_{Z_2}$ by following:

$X\times Y\times Z_1\subset X\times Y\times Z_2 \Rightarrow 0\to\mathcal{O}_{X\times Y\times\{\zeta\}}\to L_{Z_2}\to L_{Z_1}\to 0 \Rightarrow |Z_1|=|Z_2|$ reduced structure $\Rightarrow$ lift also nonvanishing

Existence of lifting: obstruction $H^1(X\times Y\times\{\zeta\},\mathcal{O}) \underset{\text{Künneth}}{=} H^1(X,\mathcal{O}_X)\oplus H^1(Y,\mathcal{O}_Y)=0$ (as we fixed $\zeta$, here we use assumption)

Thm2. Abelian variety is projective ($\exists$ ample line bundle)

Thm3. $n_X$ is finite flat morphism, $\deg(n_X)=n^{2d}$ ($d=\dim X$), and when $k=\bar{k}$: ① n coprime to char(k) (not Frob) $\Rightarrow n_X$ is étale and $\ker(n_X)=(\mathbb{Z}/n\mathbb{Z})^{2d}$ ② when Frob, $\ker(p_X)=(\mathbb{Z}/p\mathbb{Z})^g$, $g\leq d$

Pf. Take L symmetric & ample (let $(-1)^*L$, it's always possible) $\Rightarrow n_X^*L\cong L^{\otimes n^2} \Rightarrow \deg(n_X)=\frac{(n_X^*L)^d}{(L^d)}=n^{2d}$ ▨

When $k=\bar{k}$ $\begin{cases}① \#\ker(n_X)=\deg n_X=n^{2d}, \text{by Abelian grps' structure thm} \Rightarrow \ker(n_X)\cong(\mathbb{Z}/n\mathbb{Z})^{2d}, \text{and it's unramified} \Rightarrow \text{étale}\\ ② \deg(p_X)=\deg(\text{Frob})=p^d \text{ is } K(X)/K(X)^p, \text{ and we have } K(X)/K(X)^p \text{ inseperable deg} \geq p^d\end{cases}$

$\Rightarrow$ let g be complement

$\Rightarrow [\deg(p_X)]_{sep}=p^g$ ▨

Rk. Both ①② $n_X: X(k)\twoheadrightarrow X(k)$ surj $\Rightarrow$ X is divisible by $\forall n$

Def2. (Mumford bundle) $\Lambda(L)=m^*L\otimes p_1^*L^{-1}\otimes p_2^*L^{-1}$ on $X\times X$

denote $K(L)\subset X$ the maximal subscheme of X, $\Lambda(L)$ restrict $K(L)$ trivial ($\Lambda|_{X\times K(L)}$ is pullback of line bundle on $K(L)$)

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Lemma4. $T\xrightarrow{x} X$, X Abelian, T is k-scheme, then ① x factor through $K(L) \Leftrightarrow t_x^*L_T\otimes L_T^{-1}$ is pullback of a line bundle on T ② $\Lambda(L)|_{X\times K(L)}\cong\mathcal{O}_{X\times K(L)}$ Pf. It's relative analogue of def ▨

Lemma5. $K(L)$ is finite grp scheme, if L ample

Pf. We show $\forall T$, $K(L)(T)\subset X(T)$ as subgrp: apply Lemma4①, $t_x^*L_T\otimes L_T^{-1}$ trivial $\Rightarrow$ grp rules.

And it has dimension=0: $f^*\Lambda(L)$ trivial $\Rightarrow f^*\Lambda(L)=L^{-1}|_Y\otimes((-1)^*L^{-1})|_Y$ is product of ample, can't trivial except dim 0 $\Rightarrow$ finite ▨

Set $K(L)^0=\{$component of $K(L)$ contain unit$\}_{\text{red}}$ is Abelian var

Prop3. X Abelian, $k=\bar{k}$, $f: X\to Y$ with fibres $F_x$ viewed reduced $\Rightarrow F_0$ is Abelian

Pf. $X\times F_x\xrightarrow{m} X\xrightarrow{f} Y$, $\varphi_x$; $\varphi_x(0\times F_x)=f(x)$ contract $\underset{\text{rigidity}}{\Longrightarrow} \varphi_x(z\times F_x)=f(z+F_x)=\{pt\} \Rightarrow z+F_0\subset F_z$ & $-z+F_z\subset F_0$

$\Rightarrow F_z=z+F_0 \Rightarrow F_0$ sub Abelian ▨

Lemma6. $L^{\otimes 2}$ is base pt free (i.e. $\Gamma(X,L^{\otimes 2})\otimes\mathcal{O}_X\twoheadrightarrow L^{\otimes 2}$)

Pf. $\forall y$, we can always let $x$, s.t. $y\notin D\pm x$, D is L's divisor effective ▨

We show $\varphi: X\to\mathbb{P}(H^0(X,L^{\otimes 2}))$ is finite $\Leftrightarrow$ L ample.

Lemma7: $F_0=K(L)^0$, $H(D)^0$, $=\{x\in X(k) \mid t_x^*D=D$ (strict equal)$\}$ Pf. Set $L'=L|K(L)^0$, $L'\otimes(-1)^*L'=0$, two global section tensors to trivial $\Rightarrow L'$ and $(-1)^*L'$ trivial ($\in\mathcal{O}$) $\Rightarrow$ $\varphi_L$ contract $K(L)^0$ to pt $\Rightarrow K(L)^0\subset H(D)^0$

And by Prop3, $\varphi\circ t_x=\varphi$, $\forall x\in F_0 \Rightarrow F_0\subset H(D)^0$, converse inclusion trivial ▨

Coro9. $K(L)$ finite grp scheme $\Leftrightarrow$ L ample (converse of Lemma5 holds) ($\Rightarrow$ proj)

We need use $(X,X)_{\mathbb{P}^{2g}}$ $=c_g(N)$

Pf of Thm2. D divisor, it's complement U quasi-affine, $H(D)\subset U$, $H(D)$ closed & proper ① Understand: if globally complete in

But all lemmas hold true when $k=\bar{k}$

$H(D)$ finite $\Leftrightarrow K(D)$ finite $\Leftrightarrow$ L ample ▨

Generally, $\exists D$ ample on $X_{\bar{k}}$

$\Rightarrow$ D defined over finite extension $k'/k$.

$\begin{cases}k'/k \text{ seperable} \Rightarrow \text{assume } k'/k \text{ Galois} \Rightarrow \sum_{\sigma\in Gal(k'/k)}\sigma(D) \text{ desired on X over k}\\ k'/k \text{ purely inseperable, char p} \Rightarrow \exists m, (k')^{p^m}\subset k \Rightarrow p^m D \text{ desired ▨}\end{cases}$

$\Rightarrow N=L_1\oplus\cdots\oplus L_g$, where X is

Coro10. $g\geq 3$, X can't embedded to ...

cutted by zero sections of $(L_i)$

Pf. [Fulton 6.3 Excess intersection]

$\Rightarrow$ ... are local ... ▨

Thm4. X Abelian, dim g $\Rightarrow$ X can't embedded to $\mathbb{P}^{2g-1}$ ($\iota: X\to\mathbb{P}^{2g-1}$) Rk. [Mumford-Horrocks], $\exists X\hookrightarrow\mathbb{P}^4$

Pf. Computing chern class $c(T_{\mathbb{P}^m})=(1+c_1(\mathcal{O}_{\mathbb{P}^m}(1)))^{m+1}$ by Euler sequence $\Rightarrow c(\iota^*T_{\mathbb{P}^m})=(1+\iota^*c_1(\mathcal{O}_{\mathbb{P}^m}(1)))^{m+1}$

$0\to T_X\to\iota^*T_{\mathbb{P}^m}\to N\to 0$. $c(T_X)=1$

$c(N)=c(\iota^*T_{\mathbb{P}^m}) \Rightarrow c_g(N)\neq 0 \Rightarrow rk N\geq g \Rightarrow m\geq 2g$ ▨

Rk. (Embedding problem) Italian school had used following way to down the dim of embedded $\mathbb{P}^N$:

Claim. $X$ smooth & proj $\Rightarrow X \subset \mathbb{P}^{2n+1}_k$ due to $X \subset \mathbb{P}^N$, $\pi_p$ projection from $p \in \mathbb{P}^N - X$, $\pi_p(X) \subset \mathbb{P}^{N-1}$ (Thus Thm4 gives a best estimate)

Let secant variety $\mathrm{Sec}(X)$ be the joint variety of $X$ and itself, i.e. $\mathrm{Sec}(X) = \bigcup_{x_1, x_2 \in X \times X - \Delta} \mathrm{Line}$ $\Rightarrow \dim \mathrm{Sec}(X) \le 2n+1$

When $p \notin \mathrm{Sec}(X)$, it's embedding $\Rightarrow$ down 1-dim $\Rightarrow$ when $N > 2n+1$, such $p$ always exist ☒

Pf of Claim. Otherwise $X \subset \mathbb{P}^{2g}$, denote $h = c_1(\mathcal{O}_{\mathbb{P}^{2g}}(1)) \Rightarrow c(i^* T_{\mathbb{P}^{2g}}) = (1+h)^{2g+1}$, $c(T_X) = 1$, $c(N) = (1+h)^{2g+1} \Rightarrow c_g(N) = \binom{2g+1}{g} h^g$

$\Rightarrow$ by intersection $(X, X)_{\mathbb{P}^{2g}} = \binom{2g+1}{g} h^g$, $[X] \in H^*(\mathbb{P}^{2g}, \mathbb{Z}) = \mathbb{Z}[H]/\langle H^{2g+1} \rangle \Rightarrow [X] = dH^g \Rightarrow (X, X) = d^2$

$\Rightarrow d = \binom{2g+1}{g}$. Then by Grothendieck–Riemann–Roch done $g = 2$ ☒ $\quad d = \deg_{\mathbb{P}^{2g}}(X)$

Dual Abelian variety, char $k = 0$ case (char $k = p$ needs more grp scheme theories)

Thm5. ① $G \le \mathrm{Aut}(X)$ finite grp, $G x \subset$ An affine open subset of $X \Rightarrow Y \simeq X/G$, $\mathcal{O}_Y = (\pi_* \mathcal{O}_X)^G$

② $G \curvearrowright X$ freely $\Rightarrow \{\mathcal{O}_Y\text{-module}\} \longleftrightarrow \{G\text{-inv } \mathcal{O}_X\text{-module}\}$ is category equivalence Pf. Baby case of GIT ☒

Thm6. (Isogeny) We have $\left( \begin{array}{ccc} \mathcal{F} & \longmapsto & \pi^* \mathcal{F} \\ (\pi_* \mathcal{G})^G & \longleftarrow & \mathcal{G} \end{array} \right)$ 1-1 correspondence $\{K \le X \mid K \text{ finite}\} \Leftrightarrow \{\text{isogeny } X \to Y\}$ where isogeny is finite & $\dim X = \dim Y$

Pf. $K \curvearrowright X$ freely by translation, set $Y = X/K$, then it suffices showing that $X/K$ is Abelian:

we first descend to $(X \times X)/(K \times K) \longrightarrow X/K$, then using $(X \times X)/(K \times K) \cong (X/K) \times (X/K)$

Conversely $f: X \to Y$, by categorical quotient, let $K = \ker(f) \Rightarrow \exists X/K \to Y$, when char $k = 0$, we can use Zariski main thm $\Rightarrow X/K \cong Y$ ☒

Set $\mathrm{Pic}^0(X) = \{L \in \mathrm{Pic}(X) \mid t_x^* L = L, \forall x \in X\}$ (/$\mathbb{C}$, it's equivalent to $c_1(L) = 0$), recall we have homomorphism $\varphi_L: X \to \mathrm{Pic}^0(X)$, $x \mapsto t_x^* L \otimes L^{-1}$

Prop4. ① $L \in \mathrm{Pic}^0(X) \Leftrightarrow \Lambda(L)$ trivial on $X \times X$ ② $L \in \mathrm{Pic}^0(X) \Rightarrow (f+g)^* L = f^* L \otimes g^* L$ ③ $L \in \mathrm{Pic}^0(X) \Rightarrow n_X^* L = L^{\otimes n}$

Pf. Exercise ☒ (Hint: For ②, one use ① and $\Lambda(L) = \underbrace{m^* L}_{\text{LHS}} \otimes \underbrace{p_1^* L^{-1} \otimes p_2^* L^{-1}}_{\text{RHS}}$; For ④, use $X$ divisible and $(-1)^* L = L^{-1}$

④ If $L \in \mathrm{Pic}(X)$ finite order ($L^{\otimes m} = \mathcal{O}_X$) $\Rightarrow L \in \mathrm{Pic}^0(X)$ (contains all torsion)

Rk. ① $n_X^* L = L^{\otimes n^2} \otimes Q$, $Q \in \mathrm{Pic}^0(X)$ always: inductively we reduce to $L \otimes (-1)^* L^{-1} \in \mathrm{Pic}^0(X)$, it's obvious by thm of square ■

② ④ says that $\mathrm{Pic}^\tau = \mathrm{Pic}^0$ ($\mathrm{Pic}^\tau$ is $c_1$ is torsion class), it's nontrivial.

⑤ Deformation of line bundle $M_1$ to $M_2 \Rightarrow M_1 \otimes M_2^{-1} \in \mathrm{Pic}^0(X)$ (Trivial if we consider $c_1(M_1) = c_1(M_2)$)

③ Pf of ⑤ is showed here: Let $M$ on $X \times S$, $M_{s_1} = M_1$ and $M_{s_2} = M_2$

it's local on $S \Rightarrow$ shrink $S$, s.t. $M|_{x \times S}$ is trivial, and we can assume $M_2 = \mathcal{O}$

Consider $X \times X \times S \xrightarrow[p_{23}]{p_{13}} X \times S$ and $\mathcal{M} = (m \times \mathrm{id})^* M \otimes p_{13}^* M^{-1} \otimes p_{23}^* M^{-1}$ on $X \times X \times S$, we use thm of cube to show $\mathcal{M}$ trivial $\Rightarrow \mathcal{M}|_{X \times X \times s_1} = \Lambda(M_1)$ trivial, then by ①, done ☒

Prop5. $L \in \mathrm{Pic}^0(X)$, $L \ne \mathcal{O}_X \Rightarrow H^i(X, L) = 0$ Pf. Using $(-1)^* L = L^{-1}$ to give contradiction: when $H^0$ case, $L = \mathcal{O}(D)$

Both $D$ and $(-1)^* D$ are effective divisor $\Rightarrow$ contradiction $(-1)^* L = L^{-1} = \mathcal{O}((-1)^* D)$ except $D = 0$

when $k = \min\{i \mid H^i \ne 0\}$

Let $X \longrightarrow X \times X \xrightarrow{m} X$, $x \mapsto (x, 0) \mapsto x$, pullback $H^k$ to $X \times X$ $\Rightarrow H^k(X \times X, p_1^* L \otimes p_2^* L) = \bigoplus_{p+q=k} H^p \otimes H^q = 0$, contradiction ☒

Thm7. $\varphi_L$ is surjective

Pf. Otherwise $\forall x \in X$, $t_x^* L \otimes L^{-1} \ncong Q$, $\exists Q$

$\Lambda(L)|_{\{x\} \times X} = t_x^* L \otimes L^{-1} \ncong Q$

Set $\tilde{L} = \Lambda(L) \otimes p_2^* Q$ on $X \times X$, we have

Leray–Serre SS ① $H^i(X, R^j p_{1*} \tilde{L}) \Rightarrow H^{i+j}(X \times X, \tilde{L})$, with $p_1^* \tilde{L} \in \mathrm{Pic}^0(X)$

By Prop5, let all 0 and converge to $0 \Rightarrow H^n(X \times X, \tilde{L}) = 0$

② $H^i(X, R^j p_{2*} \tilde{L}) \Rightarrow H^{i+j}(X \times X, \tilde{L})$, with $R^j p_{2*} \tilde{L}$ either trivial or supported on 0-dimensional $y \in K(L)$ $\Rightarrow H^i(X, R^j p_{2*} \tilde{L}) = H^0(X, R^j p_{2*} \tilde{L})$ degenerated at this $E_2$-page: $H^n(X \times X, \tilde{L}) = H^0(X, R^n p_{2*} L)$

By ① & ② $\Rightarrow R^n p_{2*} \tilde{L} = 0$, but when $y \in K(L) \ne \emptyset$, $H^g(\tilde{L}|_{X \times \{y\}}) \cong H^0(\mathcal{O}_X)^\vee = k$ by Serre duality, but $R^g p_{2*} \tilde{L} \otimes \mathcal{O}_y = 0$ ($g = \dim X$) $\cong H^g(\tilde{L}|_{X \times \{y\}})$

contradiction ☒

Thus we have $0 \to K(L) \hookrightarrow X \xrightarrow{\varphi_L} \mathrm{Pic}^0(X) \to 0$, $L$ ample, as grp sequence, this advises that we should give $\mathrm{Pic}^0(X)$ a structure via $X/K(L)$ ($\# K(L) < \infty$)

Thm8. $\exists$ Another abelian variety $\hat{X}$, $\hat{X} \cong \mathrm{Pic}^0(X)$ as grp, with the normalised Poincaré line bundle $P_X$ on $X \times \hat{X}$, s.t. ① $P_X|_{0 \times \hat{X}} \cong \mathcal{O}_{\hat{X}}$; ② $P_X|_{X \times \{a\}} \cong a$; ③ (Universal moduli) $\forall$ normal variety $S$ and $\forall$ line bundle $M$ on $X \times S$, s.t. $M|_{X \times \{s\}} \in \mathrm{Pic}^0$ for $\forall s \in S \Rightarrow \exists f: S \to \hat{X}$, s.t. $M = (\mathrm{id}_X \times f)^* P_X$

Pf. ① When $k = \bar{k}$ and char $k = 0$, we have enough preparation: $K(L) \subset X$ is variety & smooth, finite group var (Cartier) Over $k$ with char $k = 0$, grp scheme must reduced

② Set $\hat{X} = X/K(L)$, then it's obviously Abelian;

③ By Thm5 ②, we descend $\Lambda(L)$ to $P_X$ by $X \times X \xrightarrow{\mathrm{Id}_X \times \varphi_L} X \times \hat{X}$ with action $0 \times K(L) \Rightarrow$ only check $\Lambda(L)$ is $(0 \times K(L))$-equi: $t_{(0,a)}^* \Lambda(L) = t_{(0,a)}^* m^* L \otimes t_{(0,a)}^* p_1^* L^{-1} \otimes t_{(0,a)}^* p_2^* L^{-1} = m^* t_a^* L \otimes p_1^* L^{-1} \otimes p_2^* t_a^* L^{-1} \overset{a \in K(L)}{\cong} \Lambda(L)$ naturally. Naturality is due to we have a canonical choose of isomorphism via condition ① desired

④ We check condition ② and ③, ② is easier. For ③, consider $X \times S \times \hat{X}$ and $p_{12}^* M \otimes p_{13}^* P_X^{-1} = E$, Let $I \subset S \times \hat{X}$, $E|_{X \times e}$ trivial

Thus $\Gamma \subseteq S\times \hat{X}$, view $E$ as family of bundles on $X$, parameterized by $S\times \hat{X}$

$p_1$ $\Rightarrow$ $\Gamma$ is graph of $s \mapsto [M_s]$ $\Rightarrow$ $\Gamma \cong S$ (As in char $=0$ we have Zariski main thm make birational to isomorphism)

⑤ Char $p$ $S$ issues { Quotient by nonreduced $k(2)$: Hold, see [GIT], take geometric quotient with finite degree $=$ length$(\mathcal{O}_{G})$

Check of universality: hard omitted ☒

Prop6. For $f: X\to Y$ we have $\hat{f}: \hat{Y}\to \hat{X}$ as morphism between Abelian varieties

Pf. $(f\times id)^*\mathcal{P}_Y|_{X\times\{y\}} \cong f^*y \in \mathrm{Pic}^0(X)$, by universal of $\mathcal{P}_X \Rightarrow \exists \hat{f}$ ☒ (By $\hat{f}$ sends to pullback, it's unique) ☒

Thm7. $\hat{\hat{X}}\cong X$

Lemma8. $H^i(X\times\hat{X}, \mathcal{P}_X) = \begin{cases} k, & i=\dim X = g\\ 0, & \text{otherwise}\end{cases}$

Pf. For $i\neq \dim X$ it's easy as $H^i(\hat{X}, \mathbb{R}p_{2*}\mathcal{P}_X) \Rightarrow H^{i+j}(X\times\hat{X}, \mathcal{P}_X)$ with $p_{2*}\mathcal{P}_X = \hat{x}\in \mathrm{Pic}^0(X)$ all no cohomology

$\Rightarrow H^n(X\times\hat{X}, \mathcal{P}_X) \cong H^0(\hat{X}, \mathbb{R}^n p_{2*}\mathcal{P}_X)$, by Serre duality $n>\dim X$ all $0$ $\Rightarrow n<\dim X$ all $0$ (By $\mathcal{P}_X^{-1}$ can be viewed as a pullback of $\mathcal{P}_X$: $(1\times(-1))^*\mathcal{P}_X = \mathcal{P}_X^{-1}$ by See-Saw)

Claim. $\mathbb{R}^g p_{2*}\mathcal{P}_X \cong k(0_{\hat{X}})$ (we had known it's supported at $0\in\hat{X}$ and $0$-dim)

Two ways { Compute directly (See [Mumford])

Gro's duality: $(\mathbb{R}f_*)\mathbb{R}\mathrm{Hom}(\mathcal{F}^\cdot, (\mathbb{L}f^*)\mathcal{G}^\cdot\otimes\omega_X[\dim X]) \cong \mathbb{R}\mathrm{Hom}((\mathbb{R}f_*)\mathcal{F}^\cdot, \mathcal{G}^\cdot\otimes\omega_Y[\dim Y])$

(When $f: X\to \mathrm{Spec}\,k$, $\mathcal{F}^\cdot=\mathcal{F}$, it's just Serre duality)

Apply it to $X\times\hat{X}\xrightarrow{p_2}\hat{X}$, $\mathcal{F}^\cdot=\mathcal{P}_X$, $\mathcal{G}^\cdot = (\mathbb{R}p_{2*})\mathcal{P}_X = (A/a) \Rightarrow (\mathbb{R}p_{2*})\mathbb{R}\mathrm{Hom}(\mathcal{P}_X, \mathcal{O}_{X_a}[g]) \cong \mathbb{R}\mathrm{Hom}((\mathbb{R}p_{2*})\mathcal{P}_X, (A/a))$

$\mathbb{L}f^*(A/a) = f^*(A/a) = \mathcal{O}_{X_a}$, $X_a\subset X\times\hat{X}$ defined by $p_2^*a$ $= \mathbb{R}\mathrm{Hom}((\mathbb{R}p_{2*})\mathcal{P}_X[-g], (A/a))$

$\Rightarrow (\mathbb{R}p_{2*})\mathbb{R}\mathrm{Hom}(\mathcal{P}_X, \mathcal{O}_{X_a}) \cong \mathbb{R}\mathrm{End}(A/a)$, so is $(\mathbb{R}p_{2*})\mathbb{R}\mathrm{Hom}(\mathcal{P}_X, \mathcal{O}_{X\times\{0\}}) \cong \mathbb{R}\mathrm{End}(A/m) = k$ $\Rightarrow \mathcal{P}|_{X_a} = \mathcal{O}_{X_a}$ as $\mathcal{P}|_{X\times\{0\}} = \mathcal{O}_X$

$\Rightarrow \mathbb{R}^g p_{2*}\mathcal{P}_X = A/a = A/m = k = k(0_{\hat{X}})$ ☒ then by universal property $\Rightarrow a=m$

Proof of Thm7

We construct $\hat{f}: \hat{B}\to\hat{A}$ as pullback of $\mathrm{Pic}^0$ (pullback of $\mathrm{Pic}^0$ is also $\mathrm{Pic}^0$ is trivial). It's well-defined as $f\times Id_{\hat{B}}: A\times\hat{B}\to B\times\hat{B} \Rightarrow \mathcal{P} = (f\times Id_{\hat{B}})^*\mathcal{P}_B \Rightarrow \exists \hat{f}$, s.t. $\mathcal{P} = (Id_A\times\hat{f})^*\mathcal{P}_A$ ☒ (Rk. Here Nm is the inverse of $f^*$ $\Rightarrow \mathcal{O}_B$ is direct summand of $(f\times Id_{\hat{B}})^*\mathcal{P}_B$)

Prop7. $f: A\to B$ isogeny $\deg f = d \Rightarrow \hat{f}$ also and $\deg\hat{f} = d$

Pf. $\mathrm{Nm}: \mathrm{Pic}(A)\to\mathrm{Pic}(B)$, $f^*L\mapsto L^{\deg(f)} \Rightarrow$ also isogeny. $\chi(\mathcal{P}) = (-1)^g\deg f = (-1)^g\deg\hat{f}$ by two ways $\{\mathcal{P} = (Id\times\hat{f})^*\mathcal{P}_A$ ☒

Pf of Thm7. $X\times\hat{X}\to\hat{X}$, $\mathcal{P}_X = (Id_X\times\varphi)^*\mathcal{P}_{\hat{X}}$ ($\varphi_L: X\to\hat{X}$) $\Rightarrow \exists\varphi$, $\mathcal{P}_X = (\varphi\times Id_{\hat{X}})^*\mathcal{P}_{\hat{X}}$, $\varphi: X\to\hat{\hat{X}}$

{ $\varphi$ is isogeny

$\chi(\mathcal{P}_X) = (\deg\varphi)\chi(\mathcal{P}_{\hat{X}}) \Rightarrow \deg(\varphi)=1$ ☒

Cohomology

Prop8. (Application of HRR) $\chi(X,L) = \frac{1}{g!}(L^g)$ (By $\chi = \int ch(L)$ as todd class is $1$ as $T_X$ trivial)

Prop9. $\deg\varphi_L = (\chi(L))^2$ (when $\varphi_L$ not isogeny, $\deg\varphi_L = \infty$, it says that $\chi(L)=0$, we call it degenerate)

Pf. $H^n(X\times X; \Lambda(L)) \cong H^0(X; \mathbb{R}^n p_{2*}\Lambda(L)) \cong H^0(X; \mathbb{R}^n p_{2*}(m^*L\otimes p_1^*L^{-1}\otimes p_2^*L^{-1})) \cong H^0(X; \mathbb{R}^n p_{2*}(m^*L\otimes p_1^*L^{-1})) \cong H^n(X\times X, m^*L\otimes p_1^*L^{-1})$

$(m^*L\otimes p_1^*L^{-1}) \cong (m\times p_2)^*(p_1^*L\otimes p_2^*L^{-1})$, $(m\times p_2)$ is isomorphism

$\Rightarrow H^n(X\times X, \Lambda(L)) = H^n(X\times X, p_1^*L\otimes p_2^*L^{-1}) = \bigoplus_{i+j=n}(H^i(X,L)\otimes H^j(X,L^{-1})) = \bigoplus_{i+j=n}(H^i(X,L)\otimes H^{g-j}(X,L))$

If we have more than two indices $i_1<i_2$ s.t. $H^i(X,L)\neq 0$, then $\exists j\neq g$, $H^j(X\times X,\Lambda(L))\neq 0$. Contradiction $\Rightarrow$ only $H^i(X,L) = \begin{cases}0, & i\neq i_0\\ \neq 0, & i=i_0\end{cases}$

$\Rightarrow H^g(\Lambda(L)) = (H^{i_0}(L))^2$

$\chi(\Lambda(L)) = (\chi(L))^2$, $\chi(\Lambda(L)) = \deg(\varphi_L)$ ☒

When degenerate case, $X\times X\xrightarrow{Id\times\varphi_L} X\times\hat{X}$, $X\times X/H$

$\forall H\leq K(L)$, $|H|<\infty$,

$(\deg\chi(L))^2 = \chi(m^*L\otimes p_1^*L^{-1}) = \chi((\Phi^*\mathcal{I}^*(\mathcal{P}_X\otimes p^*L^{-1})) = |H|\cdot m$ for some $m$ $\Rightarrow |H|\mid\chi(L)^2 \Rightarrow \chi(L)=0$ ☒

$i_0(L)$ is called the index of $L$

When $L$ ample $\Rightarrow i_0=0$

$L$ anti-ample $\Rightarrow i_0=g$

Prop10. ① $i(L^{-1}) = g-i(L)$; ② $i(t_x^*L) = i(L)$; ③ $i(L_t)$ constant for $t\in S$, $\mathcal{L}|_{X_t} = L_t$, $\mathcal{L}$ on $X\times S$

(Ample case is the Kodaira embedding of Abelian var) (For general $/k$, char $>0$ not holds)

④ $f: Y\to X$ isogeny, $i(f^*L) = i(L)$; ⑤ $i(L_1\otimes L_2) = i(L_1)+i(L_2)$ (All ①-⑤'s line bundles are assumed to be nondegenerate)

Pf. Exo ☒

Thm10. (Mumford index thm) $L$ non-degenerate, $M$ ample, the Hilbert-Poincaré polynomial $\chi(X, L+tM) = \frac{1}{g!}(L+tM)^g$, it has all roots $\in\mathbb{R}$, and $i(L) = \#\{$positive roots$\}$

Eg. $X = E_1\times\cdots\times E_g$, $E_i$ all elliptic curves, $L = L_1\boxtimes\cdots\boxtimes L_g$, $M = \mathcal{O}_{E_1}\boxtimes\cdots\boxtimes\mathcal{O}_{E_g} \Rightarrow \chi(X, L+tM) = \prod(\deg(E_i)+t)$ and $i(L) = \#\{i \mid \deg(E_i)<0\}$

Set $L^{\underline{a}} = L_1^{\otimes a_1} \otimes \cdots \otimes L_r^{\otimes a_r}$, $\underline{a} = (a_1 \cdots a_r)$

<u>Lemma 9</u>. $Y$ proj scheme$/k$, $\dim Y = d \Rightarrow \exists\, C(Y, L_1 \cdots L_r)$, s.t. $h^i(Y, L^{\underline{a}}) \leq C(1+|\underline{a}|^d)$

Pf. Assume $(L_i)$ are very ample (otherwise tensor very ample makes it into very ample), assume $a_r > 0$ ($a_r < 0$ it's same)

pick $D \in |L_r|$ general $\Rightarrow 0 \to L_1^{a_1} \otimes \cdots \otimes L_r^{a_r - 1} \xrightarrow{D} L_1^{a_1} \otimes \cdots \otimes L_r^{a_r} \longrightarrow (L_1^{a_1} \otimes \cdots \otimes L_r^{a_r})|_D \to 0$, then we can run induction

<u>Lemma 10</u>. $L \otimes H$ nondegenerate, $i(L) \neq i(L \otimes H) \Rightarrow \chi(L + tH)$ has root $\in (0,1)$

Pf. Otherwise $\exists\, C$, $|\chi(L+tH)| \geq C$, $t \in [0,1]$ ($t$ can't have zero on $\{0\}$ or $\{1\}$ cos $i(L) = i(H) = 0 \Rightarrow i(L \otimes H) = 0$)

Take $M \in \mathbb{Z}_{>0}$, s.t. $\left.\begin{array}{l} i(L^M) = i(L) \\ i(H \otimes L) = i(H \otimes L)^M) \end{array}\right\}$ (they hold naturally, but here we can prove, only know exist such $M$)

$M$ coprime to the denominator of all rational roots $\Rightarrow L^{\otimes M} \otimes H^{\otimes n}$ are all nondegenerate

$\exists\, n$, s.t. $i(L^M \otimes H^n)$ jumps, i.e. $i(L^M \otimes H^{n-1}) > i(L^M \otimes H^n)$ (denoted $i_1 > i_2$) $\Rightarrow 0 \to H^{i_2}(L^{\otimes M} \otimes H^{\otimes n}) \to H^{i_2}(L^{\otimes M} \otimes H^n)|_Z$

$\Rightarrow h^{i_2}(Z, (L^M \otimes H^n)|_Z) \geq M^g |\chi(L + \frac{n}{M} H)| > CM^g$, contradict to <u>Lemma 9</u> ▨

$\longrightarrow \cdots$

<u>Coro 10</u>. $i(L^m) = i(L)$, $\forall m$.

Pf. When $L$ degenerate, done; when $L$ nondegenerate, $L = H_1 \otimes H_2^{-1}$, $H_1$ & $H_2$ are ample. $\exists\, M \gg 0$, $\chi(L + tH_1)$ & $\chi(L + tH_2)$ have no roots $\in [0, \frac{1}{M}] \Rightarrow \chi(mL + tH_1)$ & $\chi(mL + tH_2)$, $m > M$ have no roots $\in [0,1] \Rightarrow i(L^{m+1}) = i(L^{m+1} \otimes H_2)$

$= i(H_1^{m+1} \otimes H_2^{-m})$

$= i(L^m \otimes H_1) = i(L^m)$

and we know $\exists\, k$ large, $i(L) = i(L^k)$, take $k > M$ done ▨

$\boxed{\text{Lemma 11}}$. If $\chi(L + tH)$ has unique root $\in [0,1]$ multiplicity $\mu$, then $i(L) \leq i(L \otimes H) + \mu$

Pf. By <u>Coro 10</u> we can assume $H$ very ample, $\exists!\, n$ s.t. $\tau \in [\frac{n-1}{m}, \frac{n}{m}]$, $i(L^m \otimes H^{n-1}) = i_1 = i(L)$

$\hspace{6em} \wedge \hspace{6em} \wedge$

$i(L^m \otimes H^n) = i_2 = i(L \otimes H)$

Take hyperplane section $X = Z_0 \supseteq \cdots \supseteq Z_k \supseteq$

$\Rightarrow 0 \to L^m \otimes H^{g-1} \to L^m \otimes H^g \to (L^m \otimes H^g)|_{Z_1} \to 0$

$\Rightarrow 0 \to H^{i_2}(L^m \otimes H^n) \to H^{i_2}((L^m \otimes H^n)|_{Z_1}) \Rightarrow 0 \to H^{i_2}(L^m \otimes H^n) \to H^{i_2}((L^m \otimes H^n)|_{Z_{(i_1 - i_2)}})$

$\Rightarrow m^g |\chi(L + \frac{n}{m} H)| \leq h^{i_2}(Z_{(i_1-i_2)}, (L^m \otimes H^n)|_{Z_{(i_1-i_2)}}) < Cm^{g-(i_1-i_2)} \Rightarrow \frac{C}{m^{i_1-i_2}} \geq |\frac{n}{m} - \tau|^\mu$

$\Vert$ as <u>Lemma 10</u>.

$m^g (\frac{n}{m} - \tau)^\mu \,|\, C'$

① When $\tau = \frac{a}{b} \in \mathbb{Q}$, $|\frac{n}{m} - \tau| \geq \frac{1}{ma}$

$\Rightarrow \frac{C}{m^{i_1-i_2}} \geq \frac{1}{m^\mu}$, let $m \to \infty \Rightarrow i_1 - i_2 \leq \mu$

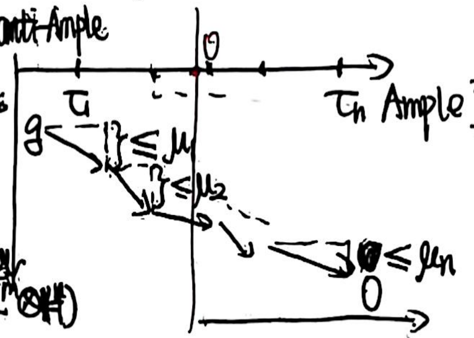
② When $\tau \notin \mathbb{Q}$, using the ~~approximation~~ Kronecker thm,

$\exists\, (\frac{n_i}{m_i})$, s.t. $|\frac{n_i}{m_i} - \tau| \leq \frac{C}{m_i}$ ($m_i \to \infty$), then same as ① ▨

done.

<u>Pf of Thm 10</u>. Real roots $\tau_1 \cdots \tau_n$, $\sum \mu_i \leq g$

$\mu_1 \cdots \mu_n$ multiplicities




$\Rightarrow g \leq \sum \mu_i$

$\Rightarrow g = \sum \mu_i \Rightarrow$ all roots are real ▨

Pf. $\frac{n}{m} \in \mathbb{Q}$ root

$\Leftrightarrow \chi(L^n \otimes H^m) = 0 \Leftrightarrow L^n \otimes H^m$ degenerate

$\Leftrightarrow \varphi_{L^n \otimes H^m}: X \to \hat{X}$ not isogeny

Thus if all roots are $\in \mathbb{Q} \Rightarrow$ all $\varphi$ not isogeny

$\Rightarrow$ its kernel is proper Abelian variety

$\Rightarrow X$ is not simple

But simple abelian var$/\mathbb{C}$ exists when $NS(X) \geq 2$, taking two $L, H$ the generator of $NS(X) \Rightarrow \exists$ root $\tau \in \mathbb{R} - \mathbb{Q}$ ▨

$i(L) = i(L^m)$ thus $= \#\{$positive roots$\}$

$= \sum \mu_i$ ▨

$\tau_i > 0$

<u>Thm 11</u>. (Mukai) $\Phi_{P_X}$ and $\Psi_{P_X}$ are equivalence, $\begin{cases} \Phi \circ \Psi = (-1_X)^* [g] \\ \Psi \circ \Phi = (-1_X)^* [g] \end{cases}$

Where $\Phi_{P_X}: D^b(X) \longrightarrow D^b(\hat{X})$ and $\Psi_{P_X}$ is conversely

$E^\bullet \mapsto Rp_{2*}(p_1^* E^\bullet \otimes P_X)$

<u>Lemma 12</u> (Orlov) $X, Y$ smooth projective, if $\Phi: D^b(X) \xrightarrow{\simeq} D^b(Y)$, then $\exists\, \mathcal{E}^\bullet \in D^b(X \times Y)$, s.t. $\Phi = \Phi_{\mathcal{E}^\bullet}$

Eg 4. ① Shift $[m]$ has kernel $\mathcal{O}[m]$; ② Identity Id has kernel $\mathcal{O}_\Delta$; ③ $f_*$ and $f^*$ has kernel $\mathcal{O}_{\Gamma_f}$ (then former is $\Phi$, latter is $\Psi$)

<u>Lemma 13</u>. $K \in D^b(X \times Y)$, $L \in D^b(Y \times Z) \Rightarrow \Phi_L \circ \Phi_K = \Phi_{K * L}$, the convolution $K * L = Rp_{XZ*}(p_{YZ}^* L \otimes p_{XY}^* K)$

Pf. $Y \times Z \to Y$ smooth, thus $X \times Y \times Z$ cartesian, then are computations ▨

<u>Lemma 14</u>. $\mu: X \times \hat{X} \times X \to X \times \hat{X}$, $X \times Y \quad \square \quad Y \times Z$ $\quad p_{12}^* P_X \otimes p_{23}^* P_X|_{X \times \{\alpha\} \times X} = p_1^* P_\alpha \otimes p_2^* P_\alpha$; $P_\alpha \in \mathrm{Pic}^0(X)$

$(x, L, y) \mapsto (x+y, L)$ $\qquad Y$

then $p_{12}^* P_X \otimes p_{23}^* P_X = \mu^* P_X$ $\qquad \mu^* P_X|_{X \times \{\alpha\} \times X} = m^* P_\alpha$

By See-Saw $\Rightarrow (\mu^* P_X) \otimes (p_{23}^* P_X)^{-1} \otimes p_{12}^* P_X = p_2^* \boxed{?}$, and by restrict to $\{0\} \times \hat{X} \times \{0\}$

$\boxed{?}$ is also trivial $\mathcal{O}_X$, done ▨

<u>Pf of Thm 11</u>. $\Psi_{P_X} \circ \Phi_{P_X} = \Phi_{P_X * P_X}$ as $X \times \hat{X} \times X$ ① is computed before,

$\Phi_{P_X} \quad \Psi_{P_X}$

$X \times \hat{X} \times X \xrightarrow{\mu} X \times \hat{X} \Rightarrow$

$p_{13} \downarrow \qquad \downarrow p_1$

$X \times X \xrightarrow{m} X$

$P_X * P_X = Rp_{13*}(p_{12}^* P_X \otimes p_{23}^* P_X) = Rp_{13*}(\mu^* P_X) = m^* Rp_{1*}(P_X) \overset{①}{=} m^* k_0[-g] \overset{②}{=} (-1)^*[-g]$ ② by $m^* \mathcal{O}_X$ = diagonal

Lemma 11. ① $\Phi_{P_X}(\mathcal{F}\otimes P_a)=t_a^*\Phi_{P_X}(\mathcal{F})\in D^b(\hat{X})$ ② $\Phi_{P_X}(k_x)=k_{\ominus x}[-g]$ ③ The pontryagin product $M * N$, $M,N\in D^b(X)$ is the convolution on $X\times\hat{X}\times X'$, $\varphi_{13}=m$, $p_{12}$...

$\Phi_{P_X}(t_a^*\mathcal{F})=\Phi_{P_X}(\mathcal{F})\otimes P_a\in D^b(\hat{X}) \Rightarrow \Phi_{P_X}(P_a)=t_a^* k_{\ominus\hat{0}}[-g]=k_{-a}[-g]$ $\quad\Phi_{P_X}(M*N)=\Phi_{P_X}(M)\otimes\Phi_{P_X}(N)$

Pf of ①. $X\times\hat{X}$, $X\times\hat{X}$, $\Phi_{P_X}(\mathcal{F}\otimes P_a)=Rp_{2*}(p_1^*(\mathcal{F}\otimes P_a)\otimes P_X)\overset{\text{seesaw}}{=}Rp_{2*}(p_1^*\mathcal{F}\otimes(\mathrm{Id}_X\times t_a)^*P_X)=t_a^*Rp_{2*}p_1^*\mathcal{F}\ldots$

$=t_a^*\Phi_{P_X}(\mathcal{F})$

③ $M*N=Rm_*(p_1^*M\otimes p_2^*N)$, $X\times X\times\hat{X}\xrightarrow{m\times\mathrm{Id}}X\times\hat{X}$, $\Phi_{P_X}(M*N)=Rp_{2*}(p_1^*(M*N)\otimes P_X)$

$=Rp_{2*}(m\times\mathrm{Id})_*(p_{12}^*(p_1^*M\otimes p_2^*N)\otimes(m\times\mathrm{Id})^*P_X)$

$=Rq_{3*}(q_1^*M\otimes q_2^*N\otimes p_{13}^*P_X\otimes p_{23}^*P_X)$ $\quad (m\times\mathrm{Id})^*P_X = p_{13}^*P_X\otimes p_{23}^*P_X$

$\Phi_{P_X}(M)\otimes\Phi_{P_X}(N)$ ☐

④ $f:X\to Y$, $\hat{f}:\hat{Y}\to\hat{X}$ dual

$\Rightarrow \Phi_{P_Y}\circ Rf_*=L\hat{f}^*\circ\Phi_{P_X}$ (Exercise)

Homogeneous bundle $\mathcal{F}$ if $t_x^*E=E$ & semi-homogenous if $\forall x$, $t_x^*E=E\otimes L$

Thm 13. (Mukai) $\Phi_P$ induces equivalence of cat $\quad \exists L\in \mathrm{Pic}^0(X)$

between {homogeneous vectorbundles} $\xrightarrow{\cong}$ $\{\mathcal{F}\in\mathrm{Coh}\mid |\mathrm{supp}\,\mathcal{F}|<\infty\}$

Lemma 15. $\mathcal{F}^\cdot\in D^b(X)$ $\quad \mathrm{rank}\,\Gamma(E)\mapsto$ length$(\mathcal{F})$

$\mathcal{F}^\cdot\otimes P_a=\mathcal{F}$, $\forall a$, then $\mathcal{F}^\cdot$ is finite support (i.e. its cohomology is finite support)

Pf. Assume $\mathcal{F}(\in\mathrm{Coh}(X))=\mathcal{F}^\cdot$, then if $\dim\mathrm{supp}\,\mathcal{F}\neq 0$ otherwise, $\exists C\hookrightarrow A:=\mathrm{supp}\,\mathcal{F}$

$\sigma^*\mathcal{F}$ also not finite supported, kill its torsion we have $\quad \uparrow\sigma$ normalisation

$\sigma^*\mathcal{F}$ vector bundle, $\sigma^*\mathcal{F}\otimes\sigma^*P_a=\sigma^*\mathcal{F}\Rightarrow\sigma^*P_a$ is torsion $\quad \tilde{C}\quad \mathrm{Pic}^0(\tilde{C})$ its Jacobian connected

valued in $\mathrm{Pic}^0(\tilde{C})\Rightarrow$ trivial $\sigma^*P_a=\mathcal{O}_{\tilde{C}}\Rightarrow \tilde{C}\hookrightarrow A$ is constant, contradiction ☐

By $(f\times\mathrm{id})^*P_X|_{C\times a}$ trivial

Pf of Thm 12. Set $Q$ on $\hat{X}$ coh, finite

supported $\Rightarrow \Phi_P(Q)=R^0\Phi_P(Q)$ is homogeneous $\Rightarrow \exists M$ on $\hat{X}$, $(f\times\mathrm{Id})^*P_X=p_2^*M$

vector bundle of rank = length$(Q)$;

Given $E$ homogeneous vector bundle $\Phi_P(E\otimes P_\alpha)=\Phi_P(t_\alpha^*E)=\Phi_P(E)$, by Lemma 15 $\Rightarrow$ finite supported ☐

$\Phi_P(L)=R^{i(L)}\Phi_P(L)[-i(L)]$ for $L\in\mathrm{Coh}$ is line bundle with index $i(L)=i_0$ (thus nondegenerate)

it's semi-homogenous: $\forall a\in\hat{X}$, $t_a^*\Phi_P(L)=\Phi_P(L\otimes P_a)=\Phi_P(t_a^*L)=\Phi_P(L)\otimes P_x$

$\Rightarrow$ so is $R^{i_0}\Phi_P(L)$. in fact we have $\varphi_L^*R^{i_0}\Phi_P(L)=H^{i_0}(L)\otimes L^{-1}$ $(\varphi_L:X\to\hat{X})$:

$$\begin{tikzcd} X\times X \arrow[r, "p_2"] \arrow[d] & X \arrow[d, "\varphi_L"] \\ X\times\hat{X} \arrow[r, "p_2"] & \hat{X} \end{tikzcd}$$

$\Rightarrow \varphi_L^*\Phi_P(L)=Rp_{2*}(p_1^*L\otimes\underbrace{(\mathrm{Id}\times\varphi_L)^*P_X}_{\overset{\text{seesaw}}{=}\Lambda(L)})=Rp_{2*}(m^*L\otimes p_1^*L^{-1}\ldots)=Rp_{2*}(p_1^*L\otimes p_2^*L^{-1})=(H^i(L)\otimes L^{-1})[-i]$ ☐

<u>Atiyah's classification on vector bundles on elliptic curve</u>

Recall (Slope) $\mu(E)=\frac{\deg(E)}{\mathrm{rank}(E)}$, (semi-)stable iff $\mu(E')\le\mu(E)$ for $E'\subset E\Leftrightarrow\mu(E'')\ge\mu(E)$ for $E\twoheadrightarrow E''$

$0\to E'\to E\to E''\to 0$ $\quad$ (Grothendieck) $\mathbb{P}^1$'s vector bundle is $\mathcal{O}_{\mathbb{P}^1}(a_1)\oplus\cdots\oplus\mathcal{O}_{\mathbb{P}^1}(a_n)=E$

$\min\{\mu(E'),\mu(E'')\}\le\mu(E)\le\max\{\mu(E'),\mu(E'')\}$ $\Rightarrow$ semi-stable iff $E=\mathcal{O}_{\mathbb{P}^1}(a)^{\oplus n}$

stable iff $E=\mathcal{O}_{\mathbb{P}^1}(a)$

$\mu(E)>\mu(F)\Rightarrow\mathrm{Hom}(E,F)=\{0\}$ (as otherwise $E\xrightarrow{\varphi}F$

$E,F$ semi-stable $\quad 0\to\ker\varphi\to E\to\mathrm{Im}\,\varphi\to 0$ $\quad\mu(\ker\varphi)\le\mu(E)\le\mu(\mathrm{Im}\,\varphi)\le\mu(F)$ contradiction ☐)

$0\to E'\to E\to E''\to 0$, $\mu(E')=\mu(E'')=\mu(E)$ (Pf. ($\Rightarrow$) Trivial

$E$ semi-stable $\Leftrightarrow E_1$ & $E_2$ are $\quad$ ($\Leftarrow$) $0\to F'\to F\to F''\to 0\Rightarrow\mu(F)\le\max\{\mu(F'),\mu(F'')\}$

On (proj smooth) curve $C$, simple bundles $E$ $\quad\le\max\{\mu(E'),\mu(E'')\}$

(End$(E)=k$) stable bundle are $\quad 0\to E'\to E\to E''\to 0\quad\le\mu(E)$ ☐)

$C/k$ must $\bar{k}=k$ to find a characteristic value of $E_0\to E_0$

Thm 13. (Harder–Narasimhan filtration) $\forall E$ on $C$, $\exists!$ filtration $0=E_0\subsetneq E_1\subsetneq\cdots\subsetneq E_m=E$, s.t. $E_{i+1}/E_i$ semi-stable

and $\mu(E_1/E_0)>\mu(E_2/E_1)>\cdots>\mu(E_m/E_{m-1})$. (Usually for higher dim $X$, not unique as we need choose a polarization

Pf. By induction, we find $E_1$, then repeat it for $E/E_1\cdots$ $\quad$ $C$ has only canonical one

For $E_1$: we take the maximal slope subbundle $\subset E$, only prove bounded above: take $L$ ample

unique maximal one between $\quad\exists n$, $\forall E_1\subset E$, $E_1\otimes L^n$ gbgs $\Rightarrow E_1\hookrightarrow(L^*)^M$, $\exists M$

For all $\mu(E^{(i)})$ maximal $\quad L^n$ semi-stable $\Rightarrow\mu(E)<Mn<\infty$ ☐

$E^{(1)}\oplus\cdots\oplus E^{(k)}$ containing all of them, and $\mu(E^{(1)}\oplus\cdots\oplus E^{(k)})=\mu(E^{(1)})=\cdots=\mu(E^{(k)})$

3. Replace semi-stable by stable, it's Jordan–Hölder filtration but not unique.

Lemma 16. On elliptic curve, HN filtration splittes

Pf. $0 \to E_1 \to E \to E_2 \to 0$, $E_1$ & $E_2$ semi-stable and $\mu(E_1) > \mu(E_2)$

$\Rightarrow \mathrm{Ext}^1(E_2, E_1) \underset{\text{Serre}}{=} \mathrm{Hom}(E_1, E_2 \otimes K_C) \xrightarrow[K_C = \mathcal{O}_E]{\text{Abelian}} \mathrm{Hom}(E_1, E_2) = 0 \Rightarrow E = E_1 \oplus E_2 \Rightarrow \forall E,\ E = \oplus (E_i / E_{i-1})$ splittes ☒

Rk. For genus $g$ curve, their difference should $< 2g-2$ also hold this Lemma

Thm 14 (Atiyah) Fix slope $\mu$, the category of slope $\mu$ vector bundle $\mathrm{Vect}_\mu(C) \underset{T}{\simeq}$ f. coherent sheaf finite supported on $\hat{C}$,

length$(T(E)) = \gcd\{\deg(E), \mathrm{rank}(E)\}$ (When $\mu = 0$, it's just $\Phi_P$, otherwise reduce)

Coro 11. $E$ stable $\Leftrightarrow \gcd\{\deg(E), \mathrm{rank}(E)\} = 1$. The only simple object $\mu = 0$

Pf. $E$ stable $\Leftrightarrow T(E)$ skycraper $\Leftrightarrow$ length$(T(E)) = 1$ ☒ in $\mathrm{Coh}_{\mathrm{Tor}}(C)$ is skycraper sheaf

We set deg/rank $\mu$ for length as they're additive. simple $\Leftrightarrow$ Hom$(E, E) = k \Rightarrow$ Hom$(T(E), T(E)) = k$

(on $\hat{C}$, as $C \cong \hat{C}$)

$C \xrightarrow{\varphi_L} \hat{C}$

$x \mapsto t_x^* L \otimes L^{-1}$

$\Rightarrow \deg(\varphi_L) = \chi(L)^2 = 1$ Isogeny

$\Rightarrow$ Isomorphism

take $h^0(L) = 1$, $L$ ample exists on $C$

☒ We call this condition $(X, L)$ PP Abelian (principal polarized)

Pf of Thm 14. Fourier–Mukai transform $\Phi_P : D^b(E) \to D^b(\hat{E})$ switch rank & deg

$\mathrm{rank}\,\Phi_P(\mathcal{F}) = \deg(\mathcal{F})$

$\deg \Phi_P(\mathcal{F}) = -\mathrm{rank}(\mathcal{F})$

only prove this for vector bundle by additive

$\mathcal{F}$ vector bundle ample $\Rightarrow R^1\Phi_P(\mathcal{F})_\alpha = H^1(E, \mathcal{F} \otimes P_\alpha) \underset{\text{dual}}{\overset{\text{Serre}}{=}} H^0(E, \mathcal{F}^\vee \otimes P_\alpha^\vee)^\vee = 0$

$\Rightarrow \Phi_P(\mathcal{F})$ also vector bundle $= R^0\Phi_P(\mathcal{F})$

$\Rightarrow \mathrm{rank}(\Phi_P(\mathcal{F})) = \chi(E, \mathcal{F}) = \deg \mathcal{F}$

For the other one, using HN filtration on $\hat{\mathcal{F}} = \Phi_P(\mathcal{F})$, it splittes $\hat{\mathcal{F}} = E_1 \oplus \cdots \oplus E_n$

Note that $R^0\Phi_{\hat{P}}(\hat{\mathcal{F}}) = 0$, $R^1\Phi_{\hat{P}}(\hat{\mathcal{F}}) = (-1)^*\mathcal{F}$, by Mukai's thm

$\Downarrow$

$\mu(\hat{\mathcal{F}}) = -\dfrac{1}{\mu(\mathcal{F})}$

| Claim 1. All slopes $\mu(E_i) < 0$ Otherwise $\mu(E_{i_0}) \geq 0 \to R^0\Phi_{\hat{P}}(E_{i_0}) \neq 0$ contradiction ☒ | Claim 2. $\mathcal{F} = F_1 \oplus \cdots \oplus F_n$ HN filtration, all slope $\mu(F_i) > 0$ $\Leftarrow$ It holds for all $\mathcal{F}$ over all elliptic curves. Exercise ☒ |
| --- | --- |

$\Rightarrow \deg \hat{\mathcal{F}} = -\mathrm{rank}(\mathcal{F})$ ☒

$\Downarrow$

$\mathrm{Vect}_\mu(C) \simeq \mathrm{Vect}_0(C)$

By Euclidean algorithm

$\mu = \frac{r}{d} \to d = kr + s$

$\Downarrow$ Euclidean

$\frac{s}{r} \Rightarrow r = ks + s'$

$\Downarrow$ Euclidean

$\downarrow$

$0$

Thus

$\mathrm{Vect}_\mu(C) \cong \mathrm{Vect}_{-\frac{1}{\mu}}(C)$

$= \mathrm{Vect}_{-\frac{d}{r}}(C)$

$\cong \mathrm{Vect}_{-\frac{s}{r}}(C) \cong \cdots \cong \mathrm{Vect}_0(C)$ ☒

(This is where the gcd comes from)

$\mathcal{F}$ semi-stable $\Leftrightarrow \hat{\mathcal{F}}$ semi-stable.

as otherwise $\hat{\mathcal{F}}$ not, $\hat{\mathcal{F}} = E_0 \oplus E_1 \Rightarrow (-1)^*\mathcal{F} = R^1\Phi_{\hat{P}}(E_0) \oplus R^1\Phi_{\hat{P}}(E_1)$

two vector bundles $\Rightarrow$ contradiction ☒

Thus we reduce to the case $\mathrm{Vect}_0(C) \ni \mathcal{F}$

we need to prove $\Phi_P(\mathcal{F})[1] = R^1\Phi_P(\mathcal{F})$ is finite supported/torsion ($R^0\Phi_P(\mathcal{F}) = 0$)

Pf. ① For $\mathcal{F}$ line bundle, $\deg \mathcal{F} = 0 \Rightarrow \mathcal{F} \simeq P_\alpha \Rightarrow$ $\{R^1\Phi_P(\mathcal{F}) = k_{-\alpha}$; $\Phi_P(\mathcal{F}) = k_{-\alpha}[-1]$ done

② Induction on rank$(\mathcal{F})$.

Claim. $\exists Q \in \mathrm{Pic}^0(C)$, $Q \subset \mathcal{F}$

Otherwise $h^0(\mathcal{F} \otimes Q^{-1}) = 0$ for all $Q \to \mathcal{F}$

by RR, $0 = \chi(\mathcal{F}) \overset{RR}{=} h^0 - h^1 \Rightarrow h^0 = h^1 = 0 \Rightarrow R^0\Phi_P(\mathcal{F}) = R^1\Phi_P(\mathcal{F}) = 0$

contradict to $\Phi_P$ is equivalence ☒

$\Rightarrow 0 \to Q \to \mathcal{F} \to Q' \to 0$, $Q'$ is vector bundle as otherwise $\mathcal{F}$ not semi-stable

Thus we reduce to $Q, Q'$, by induction hypothesis, done ☒

Thus we complete our pf, as converse is by decompose into skycraper sheaf, which is then trivial ☒

Coro 12. $\det M_{d,r} \cong \mathrm{Pic}(E)_d$ Pf. It's equivalent to Coro 11 ☒

$[\mathcal{F}] \mapsto \det \mathcal{F}$

Exercise. $\exists!$ vector bundle $\mathcal{F}$ on $E$, s.t. rank $\mathcal{F} = 2$ and $\deg \mathcal{F}$ stable

$\Rightarrow \mathcal{F} \otimes \mathcal{F}$ semi-stable, rank$(\mathcal{F} \otimes \mathcal{F}) = 4$, $\deg(\mathcal{F} \otimes \mathcal{F}) = 4$

$\gcd(4,4) \neq 1$, thus $(4,4) = (r,d)$ not unique.

[illegible] that: what it looks like $\mathcal{F} \otimes \mathcal{F}$ under $T$ concretely

Figure (length 4 torsion sheaf)

Def 2. $\mathcal{F} \in \mathrm{Coh}(X)$ is IT-sheaf of index $i \Leftrightarrow H^j(\mathcal{F} \otimes P) = 0$, $\forall j \neq i$, $\forall P \in \mathrm{Pic}^0(X)$

WIT-sheaf of index $i \Leftrightarrow R^j\Phi_P(\mathcal{F}) = 0$, $\forall j \neq i$

(weak)

Facts: ① $\mathrm{WIT}(i) \supset \mathrm{IT}(i)$;

② $\mathcal{F} \in \mathrm{IT}(i) \Rightarrow h^i(\mathcal{F} \otimes P) = (-1)^i \chi(\mathcal{F} \otimes P) \underset{P \text{ alg. } \equiv 0}{\overset{HRR}{=}} (-1)^i \chi(\mathcal{F})$ not depend on $P \Rightarrow R^i\Phi_P(\mathcal{F})$ is vector bundle on $\hat{X}$

③ $\mathcal{F} \in \mathrm{WIT}(i) \Leftrightarrow \Phi_P(\mathcal{F}) = R^i\Phi_P(\mathcal{F})[-i] \Leftrightarrow R^i\Phi_P(\mathcal{F}) \in \mathrm{WIT}(g-i)$, $g = \dim X$

consider $\mathcal{F} = \mathcal{O}_x$ is counterexample.

Def 3. $V^i(\mathcal{F}) = \{\alpha \in \hat{X} \mid H^i(X, \mathcal{F} \otimes P_\alpha) \neq 0\} \subset \hat{X}$ closed, called $i$th cohomology support loci.

$\mathcal{F}$ is GV-sheaf [Hacon, 2004] $\mathrm{codim}_{\hat{X}} V^i(\mathcal{F}) \geq i$, $\forall i \geq 1$ $\Leftrightarrow \mathrm{codim}_{\hat{X}}(\mathrm{supp}(R^i\Phi_P(\mathcal{F}))) \geq i$

M-regular sheaf [Pareschi–Popa, 2002] $\mathrm{codim}_{\hat{X}} V^i(\mathcal{F}) > i$, $\forall i \geq 1 \Leftrightarrow \mathrm{codim}_{\hat{X}}(\mathrm{supp}\, R^i\Phi_P(\mathcal{F})) > i$

Pf. ($\Rightarrow$) trivial ($\Leftarrow$) Otherwise $\mathcal{F}$ not GV. let $i_0 = \max\{i \mid \mathrm{codim}\, V^i(\mathcal{F}) < i\}$, $Z \subset V^{i_0}(\mathcal{F})$ $\dim > g - i_0$, take $Z \in \mathcal{Z}$ general and $\dim V^{i+1}(\mathcal{F}) \le g - i_0 - 1 \Rightarrow \exists \, V^{i_0+1}(\mathcal{F}) \Rightarrow \exists\, U \ni 2$, s.t. $\forall P \in U$, $H^{i+1}(\mathcal{F}\otimes P_\xi) = 0 \Rightarrow$ surj $\Rightarrow Z \subset \mathrm{Supp}\, R^i\Phi_{\mathcal{P}}(\mathcal{F})$, contradiction

**Thm 15.** $X$ Abelian surface, fix polarization i.e. ample line bundle $L$ on $X$. $\mathcal{F}$ is $\mu_L$-stable vector bundle and $c_1(\mathcal{F}) = 0$, rank $\mathcal{F} \ge 2$.

$\Rightarrow \mathcal{F}$ is $IT(1)$-sheaf & $\hat{\mathcal{F}} = R^1\Phi_{\mathcal{P}}(\mathcal{F})$ also $\mu_{\hat{L}}$-stable and $c_1(\hat{\mathcal{F}}) = 0$.

**Def 3.** $(X, L)$ polarized projective variety, $\mathcal{F}$ torsion-free on $X$, $P_{\mathcal{F}}(m) = \chi(\mathcal{F}\otimes L^{\otimes m})$ is a polynomial of deg $= \dim X$.

$\mathcal{F}$ is Gieseker (semi-)stable iff $\dfrac{P_{\mathcal{F}'}(m)}{\mathrm{rank}(\mathcal{F}')} \underset{(=)}{<} \dfrac{P_{\mathcal{F}}(m)}{\mathrm{rank}(\mathcal{F})}$ for $m \to \infty$ and $\mathcal{F}' \subsetneq \mathcal{F}$

$\mu_L$-(semi-)stable iff $\dfrac{(c_1(\mathcal{F}').L^{d-1})}{\mathrm{rank}(\mathcal{F}')} \underset{(=)}{<} \dfrac{(c_1(\mathcal{F}).L^{d-1})}{\mathrm{rank}(\mathcal{F})}$ for $\mathcal{F}' \subsetneq \mathcal{F}$ (restrict to curves)

$\{\mu_L$-semi-stable$\} \supset \{$Gieseker-semistable$\} \supset \{$Gieseker-stable$\} \supset \{\mu_L$-stable$\}$

For Gieseker semi-stability we have HN filtration too: with $\mu$ replaced by $\dfrac{P_{\mathcal{F}_i/\mathcal{F}_{i-1}}(m)}{\mathrm{rank}(\mathcal{F}_i/\mathcal{F}_{i-1})}$ when $\mathcal{F}$ is free sheaf (all subsheaves supported on the whole space)

**Lemma 17.** $X$ projective, we set $\chi(E,F) := \sum (-1)^i \dim(\mathrm{Ext}^i(E,F)) = \int \mathrm{ch}(E)^\vee \mathrm{ch}(F)\,\mathrm{Todd}(\mathcal{T}_X) \overset{X \text{ Abelian}}{=} [\mathrm{ch}(E)^\vee \mathrm{ch}(F)]_{top}$

Pf. Only prove Abelian case. As $\chi$ is bi-additive assume $E$ & $F$ locally free / vector bundle, then by GRR ▨

**Lemma 18.** $E$ is $WIT(i)$, $F$ is $WIT(j)$, $X$ Abelian var $\Rightarrow \chi(E,F) = (-1)^{i+j}\chi(\hat{E},\hat{F})$

Pf. $\mathrm{Ext}^n(E,F) = \mathrm{Hom}(E, F[n]) = \mathrm{Hom}(\Phi_{\mathcal{P}}(E), \Phi_{\mathcal{P}}(F)[n]) = \mathrm{Hom}(\hat{E}[-i], \hat{F}[-j][n]) = \mathrm{Ext}^{n+i-j}(\hat{E},\hat{F})$ ▨

**Prop 13.** $E, F$ are $WIT(i)$, $WIT(j)$, resp. $X$ Abelian surface $\Rightarrow c_1(E).c_1(F) = (-1)^{i+j}(c_1(\hat{E}).c_1(\hat{F}))$

Pf. $\chi(E,F) = [\mathrm{ch}(E)^\vee \mathrm{ch}(F)]_{top} = \mathrm{rank}(E)\chi(F) + \mathrm{rank}(F)\chi(E) - c_1(E).c_1(F)$

$(-1)^{i+j}\chi(\hat{E},\hat{F}) = (-1)^{i+j}(\mathrm{rank}(\hat{E})\chi(\hat{F}) + \mathrm{rank}(\hat{F})\chi(\hat{E}) - c_1(\hat{E}).c_1(\hat{F}))$

Note the red equality is due to the switch $\begin{cases}\mathrm{rank}(\hat{E}) = (-1)^i\chi(E) : \chi(E) = \sum(-1)^i h^i(E) = \sum(-1)^i h^i(E\otimes P_0) \overset{WIT}{=} (-1)^i \mathrm{rank}(\hat{E}\otimes k) = (-1)^i\mathrm{rank}(\hat{E}) \\ (-1)^i\mathrm{rank}(E) = \chi(\hat{E}) : \text{similar} \end{cases}$ ▨

Pf of Thm 15.

① $\mathcal{F}$ is $IT(1)$-sheaf: Otherwise $\begin{cases} \text{Case I } H^0(\mathcal{F}\otimes P) \neq 0, \exists\, \mathcal{O}_X \hookrightarrow \mathcal{F}\otimes P \Leftrightarrow \exists\, P^{-1} \hookrightarrow \mathcal{F}, \ c_1(P^{-1}) = 0 \\ \Rightarrow c_1(\mathcal{F}) \neq 0, \text{contradiction} ▨ \\ \text{Case II } H^2(\mathcal{F}\otimes P) \neq 0, \text{by Serre duality and } \omega_X = 0, \text{reduce to } H^0(\mathcal{F}^\vee \otimes P^\vee) \\ \mathcal{F}\ \mu_L\text{-stable} \Leftrightarrow \mathcal{F}^\vee\ \mu_L\text{-stable, Done } ▨ \end{cases}$

Rk. ① $WIT(1)$ $\mathcal{F}$ $\Leftrightarrow$ $WIT(1)$ $\mathcal{F}^\vee$

② $IT(1)\,\mathcal{F} \Leftrightarrow IT(1)\,\mathcal{F}^\vee$ when $\mathcal{F}$ vector bundle

② $c_1(\hat{\mathcal{F}}) = 0$:

By Coro 13. $\forall H$ ample line bundle

$(c_1(\hat{\mathcal{F}}).\hat{H}) = (c_1(\mathcal{F}).c_1(H)) \Rightarrow (c_1(\hat{\mathcal{F}}).\hat{H}) = 0, \forall H$

$\|$ $c_1(\mathcal{F})=0$ ... $0$

Rk. $IT(0) \Leftrightarrow$ positivity as ample

$IT(top) \Leftrightarrow$ negativity as anti-ample

for intermediate cases are hard

Exercise $\langle c_1(\hat{H})\rangle \otimes \mathbb{Q} \cong NS(\hat{X})\otimes\mathbb{Q}$

$\Rightarrow (c_1(\hat{\mathcal{F}}), NS(\hat{X})) = 0 \Rightarrow c_1(\hat{\mathcal{F}}) = 0$ ▨

as not $\mu_{L'}$-stable $\Rightarrow$ the HN is nontrivial

It exists as in 2-dim, free $\Leftrightarrow$ torsion-free, and here $\mathcal{F}$ is vector bundle

③ $\hat{\mathcal{F}}$ is $\mu_{\hat{L}}$-stable. Here our $L' := -c_1(\hat{L})$ (as $\hat{L}$ is negative, we add $-$ to make it ample)

Otherwise, it admits a HN filtration ... (it always exists)

$0 \to S \to \hat{\mathcal{F}} \to Q \to 0 \Rightarrow 0 \to R^0\Phi_{\hat{\mathcal{P}}}(Q) \to R^0\Phi_{\hat{\mathcal{P}}}(\hat{\mathcal{F}}) \to R^0\Phi_{\hat{\mathcal{P}}}(Q) \to R^1\Phi_{\hat{\mathcal{P}}}(S) \to R^1\Phi_{\hat{\mathcal{P}}}(\hat{\mathcal{F}}) \to R^1\Phi_{\hat{\mathcal{P}}}(Q) \to R^2\Phi_{\hat{\mathcal{P}}}(S) \to 0 \ (*)$

$\| \quad 0 \qquad (-1)^*\mathcal{F} \qquad \hat{Q}$ obvious contradict

By $(*)$, $Q$ is $WIT(1)$

and $\boxed{R^2\Phi_{\hat{\mathcal{P}}}(S) \text{ is finitely supported}}$ (P)

$\Rightarrow (c_1(R^1\Phi_{\hat{\mathcal{P}}}(S)).L) = (c_1(\mathcal{F}) - c_1(\hat{Q}).L) \overset{\text{Coro}}{=} -(c_1(\hat{\mathcal{F}}).L') + (c_1(Q).L')$ ($\mathcal{F}$ is $IT(1)$, $L$ is $IT(0)$)

Case I $R^1\Phi_{\hat{\mathcal{P}}}(S) = 0$ Case II $r(R^1\Phi_{\hat{\mathcal{P}}}(S)) \neq r(\mathcal{F})$

Only to prove (P):

$\overset{②}{=} (c_1(Q).L') > 0$ as $L'$ ample

contradict to $\mathcal{F}$ is $\mu_L$-stable ▨

$\# V^2(S) \le \mathrm{rank}(S)$ as $\forall x \in V^2(S)$, $H^2(S\otimes P_x) \neq 0 \overset{\text{Serre dual}}{\Leftrightarrow} H^0(S^\vee \otimes P_x^\vee) \neq 0$

then induction, it stops before than $\mathrm{rank}(S)$ $\Leftrightarrow \exists\, S \to P_{x}$, take its kernel

Case I, then by (P) $\Rightarrow 0 \to (-1)^*\mathcal{F} \to \hat{Q} \to$ finite-supp, an extend of vector bundle must be by divisor $\Rightarrow$ contradict

$\Rightarrow V^2(S) \subset \{x_0\} \cup \{x_1\} \cup \cdots \cup \{x_{\mathrm{rank}}\}$ ▨

$\Rightarrow$ ... $V^i(\mathcal{G}) \subset V^i(S) \cup$ ...

**Thm 16.** (Generic vanishing) $X$ Abelian, $\mathcal{F} \in \mathrm{Coh}(X)$, $(\mathcal{F}^\vee) = R\mathcal{H}om(\mathcal{F}, \omega_X)$, then

② $\mathcal{F}$ is GV-sheaf $\Leftrightarrow \Phi_{\mathcal{P}}(\mathcal{F}^\vee) = R^g\Phi_{\mathcal{P}}(\mathcal{F}^\vee)[-g]$

Pf. By Grothendieck duality

$\Phi_{\mathcal{P}}(\mathcal{F}^\vee) \overset{\text{def}}{=} Rp_{2*}(p_1^*(R\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)) \otimes \mathcal{P}_X)$

$= Rp_{2*}(R\mathcal{H}om(p_1^*\mathcal{F}, \mathcal{O}_{X\times\hat{X}}) \otimes \mathcal{P}_X)$

$X \times \hat{X}$, $p_1 \swarrow$ $\searrow p_2$, $X$ $\quad \hat{X}$

① $\mathcal{F}$ M-regular sheaf $\Leftrightarrow \hat{\mathcal{F}}$ torsion-free

Pareschi-Popa

③ GV $\Leftrightarrow H^i(\mathcal{F}\otimes L) = 0$, $i > 0$, $N \gg 0$, $L$ an isogeny of $N$ time, associated to $L$

$= Rp_{2*}(R\mathcal{H}om(p_1^*\mathcal{F}\otimes\mathcal{P}_X^\vee[g], \mathcal{O}_{X\times\hat{X}}[g])) \overset{\text{Gro's dual}}{=} R\mathcal{H}om_{\hat{X}}(Rp_{2*}(p_1^*\mathcal{F}\otimes\mathcal{P}_X^\vee[g]), \mathcal{O}_{\hat{X}}) = R\mathcal{H}om((-1)^*\Phi_{\mathcal{P}}(\mathcal{F})[g], \mathcal{O}_{\hat{X}})$ [Huybrechts]

$\Rightarrow \boxed{\Phi_{\mathcal{P}}(\mathcal{F}^\vee)[g] = R\mathcal{H}om((-1)^*\Phi_{\mathcal{P}}(\mathcal{F}), \mathcal{O}_{\hat{X}})}$ $(*)$ All are sheaf-$\mathcal{H}$om

②) This tells $(*)$'s left is a sheaf $=: Q$, $R\mathcal{H}om(Q, \mathcal{O}_{\hat{X}}) = (-1)^*\Phi_{\mathcal{P}}(\mathcal{F})$ (bi dual) $\Rightarrow R^i\Phi_{\mathcal{P}}(\mathcal{F}) = \mathcal{E}xt^i(Q, \mathcal{O}_{\hat{X}})$

$\Rightarrow$ has support by codim $\ge i$ ▨

①) $Q$ torsion free, $\mathrm{codim}_{\hat{X}}(\mathrm{supp}\, \mathcal{E}xt^i(Q, \mathcal{O}_{\hat{X}})) \ge i+1 > i$ ▨

②) codim supp $R^i\Phi_{\mathcal{P}}(\mathcal{F}) \ge i \Rightarrow \mathcal{E}xt^i((-1)^*\Phi_{\mathcal{P}}(\mathcal{F}), \mathcal{O}_{\hat{X}}) = 0$ ... via induction on $i$

All these homological algebra can be founded in [Geometry of moduli of sheafs, HL] the SS $\mathcal{E}xt^i((-1)^* R\Phi_{\mathcal{P}}(\mathcal{F}), \mathcal{O}_X) \Rightarrow R^{g+i-j}\Phi_{\mathcal{P}}(\mathcal{F}^\vee) = Q[i-j]$, and $Q$ supported on $[-g, g]$, as $<g$ all $0$

$\Rightarrow Q$ concentrated on $\deg = g$ ▨

$(\Leftarrow)$ is similar ▨

② $H^i(\hat{X}, \hat{\mathcal{F}} \otimes L) = \mathrm{Ext}^i(\hat{\mathcal{F}}^\vee, L) = \mathrm{Hom}(\hat{\mathcal{F}}^\vee, L[i]) = \mathrm{Hom}(\Phi_{\mathcal{P}}(\hat{\mathcal{F}}^\vee), \Phi_{\mathcal{P}}(L)[i]) = \mathrm{Hom}(Q, \hat{L}[i])$

($\mathcal{F}$ vector bundle)

$= \mathrm{Hom}(Q, \hat{L}[i])$ --- (**)

Using ①, we have

$(\Rightarrow)$ $\mathrm{Hom}(Q, \hat{L}[i]) = 0$ Now we use $M \gg 0$, $\hat{L}$ is sufficient negative

$(\Leftarrow)$ By (**), obvious $\forall i > 0$ all $0$ ▨ as $Q$, $L$ sheaves not complex.

$\mathrm{Hom}(\hat{L}, Q[-i]) = 0 \overset{\text{SD dual}}{=} H^{-i}(Q \otimes \hat{L}^{-1}) \overset{\text{Serre vanishing}}{=\!=} H^{-i}(Q \otimes \underbrace{\hat{L}^{-1}}_{\text{sufficient ample}}) = 0$ ▨

<u>Generic vanishing</u>

- <u>Cpt Kähler mfd $X$</u>, if $g(X) = h^1(X, \mathcal{O}_X) = h^1(X^{an}, \mathcal{O}_{X^{an}}) = \frac{1}{2} h^1(X^{an}; \mathbb{C}) > 0$

$\Rightarrow$ Albanese morphism

$a_{x_0}: X \to A_X = H^0(X, \Omega_X^1)^\vee / H_1(X; \mathbb{Z})$

$x \mapsto \int_{x_0}^x \omega$

Vanishing loci $V^i(K_X) = \{P \in \mathrm{Pic}^0(X) \mid H^i(X, K_X \otimes P) \neq 0\}$

<u>Thm 11.</u> (Green–Lazarsfeld) ① $\mathrm{codim}_{\mathrm{Pic}^0(X)} V^i(K_X) \geq i - (\dim X - \dim a_{x_0}(X))$; ② $V^i(K_X)$ is union of torsion translates of <u>subtorus</u> (or sub Abelian variety) of $\mathrm{Pic}^0(X)$

Pf. They used deformation$/\mathbb{C}$, ②'s $p$-char is open now ▨

<u>Algebraic proof</u>: ① We need use a

> <u>blackbox by Kollár</u>: (a) (Kollár splitting) $Rf_* \omega_X = \bigoplus R^i f_* \omega_X[-i]$ [Chen]
> $f: X \to Y$ surj, $X$ smooth proj, (b) (Kollár vanishing) $L$ nef & big, $H^i(R^j f_* \omega_X \otimes L) = 0$
> $Y$ normal proj

$H^i(X, K_X \otimes P) \overset{(a)}{=} \bigoplus_{j \leq i} H^j(X, R^{i-j}(a_{x_0})_* (K_X \otimes P))$

$\Leftrightarrow V^i(K_X) = \bigcup V^j(R^{i-j}(a_{x_0})_* (K_X))$

Claim. $R^j(a_{x_0})_* K_X$ all GV-sheaf, then by Def, done ▨

<u>Pf of claim</u> By [Hacon], we prove $H^i(A_{x_0}, \hat{\mathcal{F}} \otimes L) = 0$, $M \gg 0$, this is <u>Kollár vanishing (b)</u> ▨

Coro 13. $a_{x_0}: X \to A_{x_0}$ generically finite $\Rightarrow \chi(X, K_X) \geq 0$, $R^k(a_{x_0})_* K_X$

Pf. $\chi(K_X) = \chi(K_X \otimes P) = h^0(K_X \otimes P) \geq 0$ ▨ (It's a special case of <u>Beauville conjecture</u>)

Coro 14. $X$ Abelian, $\mathcal{F}$ GV, we have ① $V^0(\mathcal{F}) \supset V^1(\mathcal{F}) \supset \cdots \supset V^g(\mathcal{F})$; ② $\chi(\mathcal{F}) \geq 0$; ③ $Z \subset V^0(\mathcal{F})$ component with maximal dim,

Pf. ① $P_\alpha \notin V^i(\mathcal{F}) \Rightarrow H^i(\mathcal{F} \otimes P_\alpha) = 0 \Rightarrow \boxed{R^{g-i-1}\Phi_{\mathcal{P}}(\mathcal{F}^\vee) \otimes k_\alpha \to H^{g-i-1}(\mathcal{F}^\vee \otimes P_\alpha)}$ denote $\mathrm{codim}\, Z = d \Rightarrow Z \subset V^d(\mathcal{F})$ also component

$H^{g-i}((\mathcal{F}^\vee \otimes P_\alpha^\vee)^\vee) = 0$ [EGA III 7.7]

$\Rightarrow P_\alpha \notin V^{i+1}(\mathcal{F})$ ▨ Left $= 0$ as $\Phi_{\mathcal{P}}(\mathcal{F}^\vee) = R^g\Phi_{\mathcal{P}}(\mathcal{F}^\vee)[g] \Rightarrow H^{g-i-1}(\mathcal{F}^\vee \otimes P_{-\alpha}) = 0 = H^{i+1}(\mathcal{F} \otimes P_\alpha)$

② Same as Cor 13. ▨

③ If $Z$ is component of $V^0(\mathcal{F})$ minimum codim $\Rightarrow Z \subset \mathrm{Supp}(\hat{\mathcal{F}})$ a component, then by homological support ▨

<u>Lemma 19.</u> $\mathcal{F}$ is M-regular $\Rightarrow V^0(\mathcal{F}) = \hat{X}$ and $\chi(\mathcal{F}) > 0$

Pf. $\mathcal{F}$ M regular $\Leftrightarrow \Phi_{\mathcal{P}}(\mathcal{F}^\vee)[g] = R^g\Phi_{\mathcal{P}}(\mathcal{F}^\vee)$ torsion-free $\Rightarrow \forall \alpha \in \hat{X}$ general, $R^g\Phi_{\mathcal{P}}(\mathcal{F}^\vee)_\alpha = H^g(\mathcal{F}^\vee \otimes P_\alpha) \overset{\text{Serre dual}}{\Longrightarrow} H^0(\mathcal{F} \otimes P_\alpha) \neq 0$

$\Rightarrow V^0(\mathcal{F}) = \hat{X}$ ▨

($\chi(\mathcal{F})$ can used to measure positivity of $\mathcal{F}$ on Abelian variety, but not general $X$)

We can $\mathcal{F}$ is continuously globally generated (CGG) if $\bigoplus (H^0(X, \mathcal{F} \otimes P_\alpha) \otimes P_\alpha^{-1}) \twoheadrightarrow \mathcal{F}$, $\forall U \subset \mathrm{Pic}^0(X)$ Zariski open.

CGG $\Rightarrow V^0(\mathcal{F}) = \hat{X}$ obviously: otherwise we take complement to be $U$, then not surj ▨

Polarized Abelian variety $(X, L)$, <u>$L$ ample</u> $\Rightarrow L$ is CGG due to $\boxed{\bigcap_{\alpha \in U} \mathrm{Bs}|L \otimes P_\alpha| = \emptyset}$ [Ex], Bs is base pts of linear system.

<u>Thm 18.</u> (Pareschi–Popa) M regular $\Rightarrow$ CGG

Pf. (Schnell) We have a <u>contravariant FM transform</u> $FM_X: D^b(X) \to D^b(\hat{X})$, then

$\mathcal{F} \mapsto \Phi_{\mathcal{P}}(\mathcal{F}^\vee[g])$

$\begin{cases} \mathcal{F}\ \mathrm{GV} \Leftrightarrow FM(\mathcal{F}) \text{ coherent} \\ \mathcal{F} \text{ M-regular} \Leftrightarrow FM(\mathcal{F}) \text{ torsion-free} \\ FM_{\hat{X}} \circ FM_X = \mathrm{Id}_X \\ FM_Y \circ Rf_* = Lf^* \circ FM_X \\ FM_X \circ Lf^* = R\hat{f}_* \circ FM_Y \\ \Rightarrow \begin{cases} FM_X \circ (t_x)_* = (P_x \otimes -) \circ FM_X \\ FM_X \circ (P_\alpha \otimes -) = (t_\alpha)_* \circ FM_X \end{cases} \end{cases}$

e: $\mathcal{O}_X \hookrightarrow \hat{X}$, then $FM(\mathcal{F})|_\alpha = Le^* \circ (t_{-\alpha})_* (FM(\mathcal{F}))$

$= Le^* \circ FM(\mathcal{F} \otimes P_{-\alpha})$

$\Updownarrow$ dual

$= FM_{\mathrm{Spec}\,k}\, R\varphi_*(\mathcal{F} \otimes P_{-\alpha})$

$\varphi: X \to \mathrm{Spec}\, k$

$= \mathrm{Hom}_k(R\varphi_*(\mathcal{F} \otimes P_{-\alpha}), k) = \mathrm{Hom}_k(H^0(\mathcal{F} \otimes P_{-\alpha}), k)$

We have dual

evaluation map: $H^0(\mathcal{F} \otimes P_\alpha) \overset{\otimes \mathcal{O}}{\to} (\mathcal{F} \otimes P_\alpha) \Rightarrow (*)\ H^0(\mathcal{F} \otimes P_\alpha) \to (\mathcal{F} \otimes P_\alpha)_x$, $\forall x \in X$

Thus

(A) $\bigoplus H^0(\mathcal{F} \otimes P_\alpha) \to (\mathcal{F} \otimes P_\alpha)_x$ dual to (B) $H^0(\hat{X}, FM(\mathcal{F}) \otimes P_{-x}) \to (FM(\mathcal{F}) \otimes P_{-x})|_\alpha$

$\Updownarrow$ dual

$\bigoplus H^0(\hat{X}, FM(\mathcal{F}) \otimes P_{-x}) \to \prod FM(\mathcal{F}) \otimes P_{-x}$

(A) surj $\Leftrightarrow$ (B) inj obvious as $FM(\mathcal{F})$ torsion free ▨

when for $L$ ample, $\mathcal{I}_p \otimes L$ is GV

$\quad$ | $\mathcal{I}_p \otimes L$ is M-regular $\Leftrightarrow |L|$ no base divisor

As $0 \to \mathcal{I}_p \otimes L \to L \otimes k_p \to 0$

$\alpha \in V^1(\mathcal{I}_p \otimes L) \Leftrightarrow H^0(L \otimes P) \to k_p$ not surj

$\Leftrightarrow p \in Bs|L \otimes P|$ proper subset

$\Rightarrow \operatorname{codim} V^1(\mathcal{I}_p \otimes L) \geq 1$ ☑

We can use these to prove a classical result

Thm 17. $L$ ample $\Rightarrow$ ① $L^{\otimes 2}$ base pt free (proven)

② $L^{\otimes 3}$ very ample

③ $|L|$ no base divisor $\Rightarrow L^{\otimes 2}$ very ample

Pf. ② $\forall x$, $\mathcal{I}_x \otimes L^{\otimes 3}$ globally generated $\Rightarrow L^{\otimes 3}$ globally generated $\Rightarrow L^{\otimes 3}$ very ample.

$0 \to \mathcal{I}_x \otimes L^{\otimes 2} \to L^{\otimes 2} \to k_x \to 0$

it's surjective & surjective after $\otimes P$ (as $\exists\, \alpha^2 = P$) $\Rightarrow \mathcal{I}_x \otimes L^{\otimes 2}$ is IT(0) $\Rightarrow$ M-regular $\xRightarrow{PP}$ CGG

From $L^2$ to $L^3$, we consider

$$\begin{array}{ccc} \bigoplus H^0(\mathcal{I}_x \otimes L^2 \otimes P) \otimes H^0(L \otimes P^{-1}) \otimes \mathcal{O}_X & \longrightarrow & \bigoplus H^0(\mathcal{I}_x \otimes L^3 \otimes P) \otimes (L \otimes P^{-1}) \\ \downarrow & & \downarrow \\ H^0(\mathcal{I}_x \otimes L^3) \otimes \mathcal{O}_X & \longrightarrow & \mathcal{I}_x \otimes L^3 \end{array}$$

we need to show this surj

By upper triangle ☑

③ Same

$\mathcal{I}_p \otimes L$ is M regular too, replace all $L^{\otimes 2}$ to $L$ ☑

Rk. (GV-index) $\operatorname{gv}(\mathcal{F}) := \min\{\operatorname{codim}_X V^i(\mathcal{F}) - i\}$

$\operatorname{gv} \geq 0 \Leftrightarrow$ GV-sheaf, $i > 0$

$\operatorname{gv} \geq 1 \Leftrightarrow$ M-regular sheaf, $i > 1$

$\operatorname{gv} = \infty \Leftrightarrow$ IT(0)

$\operatorname{gv}(\mathcal{F}) \geq m \Rightarrow \chi(\mathcal{F}) \geq m$ (Cor of Thm 20)

Def 4. $L$ is $k$-very ample if $H^0(X, L) \to L \otimes \mathcal{O}_\xi$ surj, $\xi \subset X$ finite of length $k+1$

0-very ample $\Leftrightarrow \mathcal{O}_\xi = k_p \Leftrightarrow$ globally generated; 1-very ample $\Leftrightarrow$ very ample

take $\xi$ = two closed pts

take $\xi$ = pt & tangent vector at it

then surj separates them

$L$ is $k$-jet ample if $H^0(X, L) \to L \otimes \mathcal{O}_X / (\mathfrak{m}_{x_1}^{a_1} \cdots \mathfrak{m}_{x_r}^{a_r})$, $\sum a_j = k+1$

Then 0-jet ample / 1-jet ample $\Leftrightarrow$ 0/1-very ample

$\boxed{k\text{-jet ample} \Rightarrow k\text{-very ample} \Rightarrow \varphi(k)\text{-very ample}, \ \varphi(k) \text{ is maximal s.t. } \binom{\varphi(k)+d}{d} \leq k+1}$ Exercise

Thm 18. $L^{\otimes(k+2)}$ is $k$-jet ample

$\quad$ | $|L|$ no base divisor $\Rightarrow L^{\otimes(k+1)}$ is $k$-jet ample.

Thm 20* (PP) $\operatorname{gv}(\mathcal{F}) \geq m \Leftrightarrow FM_X(\mathcal{F})$ $m$-syzygy, i.e. $0 \to FM_X(\mathcal{F}) \to \mathcal{E}_m \to \cdots \to \mathcal{E}_1 \to K \to 0 \Rightarrow \chi(\mathcal{F}) \geq m$

$(m < \infty)$

(1-syzygy = torsion-free, $\infty$-syzygy = locally free

2-syzygy = reflexive)

($m = \infty$ not true: a line bundle is IT(0), but $\chi(L) \in \mathbb{Z}$ can be any $\mathbb{Z}$)

In origin paper of [PP], it's called Higher Castelnuovo–DeFranchis Lemma, why? (we omit pf of Thm 20)

Classical CD Lemma is $p_g(S) < 2q(S) - 3 \Rightarrow \exists$ fibration $f: S \to C$, s.t. $g(C) \geq 2$

Consider $\mathbb{P}(\Lambda^2 H^0(\Omega^1_S)) \dashrightarrow \mathbb{P}(H^0(K_S))$

$\quad \uparrow$ Plücker

$\operatorname{Gr}(2, H^0(\Omega^1_S))$

$\dim \operatorname{Gr}(2, H^0(\Omega^1_S)) = 2(q(S) - 2) > p_g(S) - 1 = \dim \mathbb{P}(H^0(K_S))$, by dimension reason, it's not morphism only rational map (otherwise the pullback of divisor must ample as $\operatorname{Pic}(\operatorname{Gr}) = \mathbb{Z}[H]$, $H$ ample but by dim reason pullback of divisor not divisor, contradiction)

$\Rightarrow \exists\, \omega_1 \wedge \omega_2 = 0$

$\omega_i \in H^0(\Omega^1_S) \Rightarrow$ these two $\omega_i$ comes from same curve $C$, $\omega_i \in f^* H^0(K_C)$ ☑

Hodge theory

Then consider Albanese map $a_X: X \to A_X$ and assume $A_X$ simple Abelian variety

$\operatorname{gv}(a_{X*}\omega_X) = \min_{k>0}\{\operatorname{codim}_{A_X} V^k(\omega_X) - k\} = q(X) - n$ by Griffith–Lazarsfeld $V^k(\omega_X)$ is translated abelian variety.

$\Rightarrow \chi(\omega_X) \geq \operatorname{gv}(a_{X*}\omega_X) = q(X) - n$, let $n = 2$ surface $\Leftrightarrow p_g \geq 2q - 3$ degenerate to classical case

Pf of Thm 19

Lemma 20. TFAE ① $L$ $k$-jet ample; ② $\forall x_1, \cdots x_{k+1} \in X$ $(\mathcal{I}_{x_1} \otimes \cdots \otimes \mathcal{I}_{x_{k+1}}) \otimes L$ IT(0); ③ $(\mathcal{I}_{x_1} \otimes \cdots \otimes \mathcal{I}_{x_k}) \otimes L$ globally generated.

Pf. ① $\Leftrightarrow$ ②: $0 \to \mathcal{I}_{x_1} \otimes \cdots \otimes \mathcal{I}_{x_{k+1}} \otimes L \to L \to (\mathcal{O}_X / \mathcal{I}_{x_1} \otimes \cdots \otimes \mathcal{I}_{x_{k+1}}) \to 0$

twist $\forall\, k \geq 0$, all surj $\Leftrightarrow H^{k+1}(\mathcal{I}_{x_1} \otimes \cdots \otimes \mathcal{I}_{x_{k+1}}) \otimes L) = 0$

$\Leftrightarrow$ IT(0) ☑

② $\Leftrightarrow$ ③: it's same, consider $0 \to (\mathcal{I}_{x_1} \otimes \cdots \mathcal{I}_{x_k} \otimes \mathcal{I}_{x_{k+1}}) \otimes L \to (\mathcal{I}_{x_1} \otimes \cdots \otimes \mathcal{I}_{x_k}) \otimes L \to (\mathcal{I}_{x_1} \otimes \cdots \otimes \mathcal{I}_{x_k}) \otimes L|_{x_{k+1}} \to 0$

again we have last is IT(0) $\Leftrightarrow (\mathcal{I}_{x_1} \otimes \cdots \mathcal{I}_{x_k}) \otimes L$ globally generated ☑

$\forall\, x_k$

Set the M-regular index of ample line bundle $L$ on $X$: $m(L) := \max\{l \mid L \otimes (\mathcal{I}_{x_1} \otimes \cdots \otimes \mathcal{I}_{x_l})$ is M-regular, $\forall x_1 \cdots x_l \in X\}$

$m(L) \geq 1 \Leftrightarrow L \otimes \mathcal{I}_x$ M-regular $\Leftrightarrow Bs|L|$ has codim $\geq 2$ (no base divisor)

(This is due to $\varphi_L : X \to \hat{X}$ defines $\varphi_L^{-1}(V(L \otimes \mathcal{I}_x)) = \{\alpha \in X \mid x \in \alpha + Bs|L|\}$ is translation, contradiction ☒)

We prove that $\forall L_1 \cdots L_{k+1-m(L)}$, $L \otimes L_1 \otimes \cdots \otimes L_{k+1-m(L)}$ k-jet ample (then take them as $L$ itself's proper power, done)

Pf. By lemma we need to show surjective of $H^0(L \otimes L_1 \otimes (\mathcal{I}_{x_1} \otimes \cdots \otimes \mathcal{I}_{x_m})) \xrightarrow{\text{ample}} (L \otimes L_1) \otimes (\mathcal{I}_{x_1} \otimes \cdots \otimes \mathcal{I}_{x_m})$ to show $L \otimes L_1$ is m-jet ample, then induction:

$(L \otimes L_1) \otimes (\mathcal{I}_{x_1} \otimes \cdots \otimes \mathcal{I}_{x_{m+1}})$ IT(0) $\Rightarrow$ M-regular $\Rightarrow$ this inductively $(L \otimes L_1) \otimes L_2$ can use it ☒.

Surj: $\bigoplus_{p \in \hat{X}} H^0(L \otimes (\mathcal{I}_{x_1} \otimes \cdots \otimes \mathcal{I}_{x_m}) \otimes P) \otimes H^0(L_1 \otimes P^{-1}) \otimes \mathcal{O} \twoheadrightarrow \bigoplus_{p \in \hat{X}} H^0(L \otimes (\mathcal{I}_{x_1} \otimes \cdots \otimes \mathcal{I}_{x_m}) \otimes P) \otimes L_1 \otimes P^{-1}$

$$\begin{array}{ccc} \downarrow & \circlearrowleft & \downarrow \\ H^0(L \otimes L_1 \otimes (\mathcal{I}_{x_1} \otimes \cdots \otimes \mathcal{I}_{x_m})) & \longrightarrow & (L \otimes L_1) \otimes (\mathcal{I}_{x_1} \otimes \cdots \otimes \mathcal{I}_{x_m}) \end{array}$$ ☒

Def. $L$ is projectively normal if $\forall m \geq 1$, $S^m H^0(X, L) \to H^0(X, L^{\otimes m})$

$L$ ample, then $\Rightarrow$ very ample as $L^{\otimes k}$ very ample $\Rightarrow X \hookrightarrow \mathbb{P}H^0(X, L^{\otimes k})$

Due to upper is embedding, so is left ☒

(Mumford) Further it calls normally presented if $\varphi_L : X \hookrightarrow \mathbb{P}H^0(X, L)$ has $\varphi_L(X)$ cutted out by quadric scheme-theoretically.

$\mathbb{P}H^0(X, L) \xrightarrow{\text{Segre embedding}} \mathbb{P}S^k H^0(X, L)$

Rk. (Green's View) Consider $S_L = \bigoplus S^m H^0(X, L)$ polynomial, $R_L = \bigoplus H^0(X, L^{\otimes m})$ is $S_L$-module. f.g.

By Hilbert's syzygy thm, $\exists$ l(?) resolution of $R_L$: $\to E_1 \to E_0 \to R_L$, $E_i = \bigoplus S_L(-a_{ij})$ then

$L$ projective normal $\Leftrightarrow E_0 = S_L$

$L$ normally presented $\Leftrightarrow E_1 = S_L(-2)$

Thm 21. ① (Koizumi), ② (Ohbuchi) If $L = A^{\otimes n}$ $(n \geq 3) \Rightarrow L$ projectively normal (char $k \neq 2$)

③ (Mumford) If $L = A^{\otimes n}$ $(n \geq 4) \Rightarrow L$ normally presented. (char $k = 0$??)

$A$ is ample on $X$ Abelian.

Pf. $0 \to M_L$ (kernel bundle) $\to H^0(X, L) \otimes \mathcal{O}_X \to L \to 0$

$\Leftarrow S^m H^0(X, L) \twoheadrightarrow H^0(X, L^m) \Leftarrow H^0(X, L) \otimes H^0(X, L^m) \to H^0(X, L^{m+1})$, $\forall m \geq 1$

$\Leftarrow H^1(M_L \otimes L^m) = 0$, $\forall m \geq 1$, it suffices to prove $M_L \otimes L$ is IT(0)

Observation. $M_L \otimes L^{-1} = \varphi_L^* \Phi_{\mathcal{P}}(L \otimes \mathcal{I}_{e_X})$ ($\Rightarrow M_L \otimes L^m$ are also IT(0) done)

($e_X \in X$ the unit)

Due to apply FM transform $\varphi_L^* \Phi_{\mathcal{P}}(-)$ to $0 \to L \otimes \mathcal{I}_{e_X} \to L \otimes k_{e_X} \to 0$

$\Rightarrow 0 \to M_L \otimes L^{-1} \to H^0(L) \otimes L^{-1} \to \mathcal{O}_X \to 0$, done.

Lastly, $H^i(X, M_L \otimes L \otimes P) = H^i(X, \varphi_L^* \Phi_{\mathcal{P}}(L \otimes \mathcal{I}_{e_X}) \otimes L^{\otimes 2} \otimes P) = H^i(X, \Phi_{\mathcal{P}}(L \otimes \mathcal{I}_{e_X}) \otimes (\varphi_L)_*(L^{\otimes 2} \otimes P))$

$= H^i(X \times \hat{X}, p_1^* \Phi_{\mathcal{P}}(L \otimes \mathcal{I}_{e_X}) \otimes p_2^* (\varphi_L)_*(L^{\otimes 2} \otimes P) \otimes \mathcal{P}_X) = H^i(X, (L \otimes \mathcal{I}_{e_X}) \otimes \Phi_{\mathcal{P}}((\varphi_L)_*(L^{\otimes 2} \otimes P))) = H^i(X, (L \otimes \mathcal{I}_{e_X}) \otimes \varphi_L^* \Phi_{\mathcal{P}}(L^{\otimes 2} \otimes P)$

$= H^i(X, m_2^*(L \otimes \mathcal{I}_{e_X}) \otimes m_2^* \varphi_L^* \Phi_{\mathcal{P}}(L^2 \otimes P)) = H^i(X, m_2^*(L \otimes \mathcal{I}_{e_X}) \otimes H^0(L^2 \otimes P) \otimes L^{-2} \otimes P^{-1}) = H^i(X, m_2^*(A \otimes \mathcal{I}_{e_X}) \otimes m_2^*(A^3) \otimes L^{-2} \otimes P^{-1})$

or $A$ base pt free

$= 0$ ☒ GV $\quad \underbrace{A^8 + A^{-6}}$ still ample

### Ampleness of M-regular

We call $\mathcal{F}$ vector bundle ample if $\mathcal{O}_{\mathbb{P}\mathcal{F}}(1)$ ample. Recall Kleiman thm is $L$ ample $\Leftrightarrow (L^d \cdot V) > 0$, $\forall V \subset X$ sub variety of dim $d$.

Prop 12. ① $\mathcal{F}$ ample $\Leftrightarrow \forall \mathcal{G}$ coherent, $\exists n$, $\mathcal{G} \otimes S^n \mathcal{F}$ globally generated;

② $\mathcal{F}$ ample $\Rightarrow \forall$ quotient ample; ③ $\mathcal{F}$ ample $\Leftrightarrow S^n \mathcal{F}$ ample; ④ $\mathcal{F}$ globally generated, IT(0)

$\mathcal{F}$ ample $\Leftrightarrow \forall C$ irreducible curve, $\mathcal{F}|_C$ admits no trivial quotient ($\mathcal{F}|_C \not\twoheadrightarrow \mathcal{O}_C$)

Pf. ①②③ are easy. ④ ($\Rightarrow$) $\mathcal{F}$ ample $\Rightarrow \mathcal{F}|_C$ ample, obvious. ($\Leftarrow$) Globally generated $\Leftrightarrow H^0(\mathcal{F}) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{F}$

Thm 22. (Debarre) M-regular sheaf is ample.

Lemma 20. Continuously globally generated $\Leftrightarrow \exists$ isogeny $X' \xrightarrow{\pi} X$ $\pi^*(\mathcal{F} \otimes P)$ is globally generated, $\forall P \in \mathrm{Pic}^0 X$

$$\begin{array}{ccc} \mathbb{P}(\mathcal{F}) & \hookrightarrow & \mathbb{P}(H^0(\mathcal{F}) \otimes \mathcal{O}_X) \\ \downarrow & & \| \\ \mathbb{P}(H^0(\mathcal{F})) & \longleftarrow & X \times \mathbb{P}(H^0(\mathcal{F})) \end{array}$$

$\varphi$ can't contract any fibre $\Rightarrow \varphi$ finite $\Leftrightarrow \mathcal{F}$ ample ☒

Pf. ($\Rightarrow$) CGG $\Leftrightarrow \exists U \subset \hat{X}$ open dense, s.t.

$\bigoplus_{i \leq N} H^0(\mathcal{F} \otimes P_i) \otimes P_i^{-1} \twoheadrightarrow \mathcal{F}$, $\mathcal{F}$ coherent (finite generated) $\Rightarrow \bigoplus H^0(\mathcal{F} \otimes P_i) \otimes P_i^{-1} \twoheadrightarrow \mathcal{F}$, $P_i \in U$

$(P_1, \cdots, P_N) \in U^N \subset \hat{X}^N$, denote $V \subset U^N$ open dense.

Consider torsion pts $\bigcup \hat{X}[p^n]$ is dense ($\bigcup \hat{X}[p^n] = \hat{X}$ is nontrivial for char $k = p$)

Using torsion line bundles $\in \hat{X}$

$(Q_1, \cdots, Q_N) \in V$ torsion line bundles (torsion pts $\in \hat{X}$), they induces $\pi_i : Y_i \to X$, then they're all isogeny. Set $X' = \bigcup X_i$ ☒

($\Leftarrow$) trivial ☒

Lemma 21. CGG $\Rightarrow$ ample

Pf. By Lemma 20, CGG $\Rightarrow \exists$ isogeny $\pi \Rightarrow \mathcal{F}$ ample $\Leftrightarrow \pi^* \mathcal{F}$ ample (finite), to show latter, we use Prop 12 ④, Otherwise $\pi^* \mathcal{F}|_C \twoheadrightarrow \mathcal{O}_C$

$\forall Q \in \mathrm{Pic}^0(X)$, $\pi^* \mathcal{F}|_C \otimes Q \twoheadrightarrow Q|_C$, former globally generated $\Rightarrow$ latter $Q|_C$ also $\Rightarrow Q|_C = \mathcal{O}_C$, then contradiction ☒

Thus M-regular $\Rightarrow$ CGG $\Rightarrow$ ample ☒

Similarly, GV-sheaf is nef: $\mathcal{F}$ GV $\Leftrightarrow \pi_{M}^{*}\mathcal{F}\otimes H$ is IT(0) $\Rightarrow$ M-regular $\Rightarrow \pi^{*}\mathcal{F}\otimes H$ ample $\Rightarrow \mathbb{P}(\varphi_{M}^{*}\mathcal{F})$ ... $\mathbb{P}(\mathcal{F})$
$\Rightarrow f_{M}^{*}\mathcal{O}(1)\otimes p^{*}H$ ample $= f_{M}^{*}(\mathcal{O}(1)\otimes p^{*}M^{2}H) \Rightarrow \mathcal{O}(1)\otimes p^{*}M^{-2}H$ ample
$\Rightarrow \mathcal{O}(1) = \lim_{t\to 0} \mathcal{O}(1) + t\,p^{*}M^{-2}H$ is limit of ample in NS-group $\Leftrightarrow$ nef ▨

$$\begin{array}{ccc} \mathbb{P}_M & \xrightarrow{f_M} & \mathbb{P} \\ \downarrow & & \downarrow \varphi \\ X & \xrightarrow{\pi_M} & X \end{array}$$

An application:

Prop 13. $\pi: Y \to X$ finite cover, $X$ abelian, $\pi_{*}\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{E}$, we can prove a Lefschetz-type thm: $H^{k}(Y,\mathbb{C}) \cong H^{k}(X,\mathbb{C})$ $k<n$

Pf. By Gro's dual $\pi_{*}\omega_Y \cong \mathcal{O}_X \oplus \mathcal{E}^{\vee}$, it suffices to prove $\mathcal{E}^{\vee}$ ample. For this we prove it's M-regular.
We use generic vanishing, $V^{i}(\mathcal{E}^{\vee}) \subseteq V^{i}(\pi_{*}\omega_Y) - V^{i}(\omega_X)$ is finite pts $\Leftrightarrow$ codim $V^{i}(\mathcal{E}^{\vee}) > i$
Only codim $= g = i$ need be considered: Claim. $V^{g}(\mathcal{E}^{\vee}) = \emptyset$ $\quad 0 \neq H^{g}(\pi_{*}\omega_Y \otimes P) = H^{g}(\omega_Y \otimes \pi^{*}P) \cong H^{0}(\pi^{*}P^{-1})$
$\Rightarrow \pi^{*}P^{-1}$ global section $\Rightarrow \pi^{*}P = \mathcal{O}_Y = \pi^{*}P$, contradiction ▨

We extend CGG to continuous generated by $\mathcal{Z} = \{Z_i \mid Z_i \subset \hat{X}$ subvariety$\}_{i=n}$
if our take $U \subset \hat{X}$ satisfy $U \cap Z_i \neq \emptyset$
[CGG $=$ CG$\{\hat{X}\}$, $\mathcal{O}_X$ not CGG but CG$\{\mathcal{O}_X\}$]

As $\otimes$ is right exact, $\mathcal{F}$ CG $\mathcal{Z}$, $\mathcal{G}$ CG $\mathcal{Z}' \Rightarrow \mathcal{F}\otimes\mathcal{G}$ CG$\{Z_i + Z_j'\}_{i,j}$ ▨ denoted as

Thm 23. (Pareschi) $\mathcal{F}$ CG $\mathcal{Z}$, $\mathcal{Z}$ span/generate $\hat{X}$ (i.e. $\forall Z_i$, $\exists n_i < \infty$, $n_i Z_i = \hat{X}$, $\hat{X} = \langle Z_i \rangle$)
$\Rightarrow \mathcal{F}$ ample

Pf. Let $M > n_i$, $\forall i \Rightarrow \mathcal{F}^{\otimes M}$ is CG$\{Z_{i_1} + \dots + Z_{i_M}\}$ $=$ CG$\{\hat{X}\}$ $\Rightarrow \mathcal{F}^{\otimes M}$ CGG $\Rightarrow \mathcal{F}^{\otimes M}$ ample $\Rightarrow \mathcal{F}$ ample ▨

E.g. 2. $C \hookrightarrow J(C)$, $g(C) = g$
then $J(C) = \langle C \rangle$ and $gC = J(C)$ (Jacobian is Abelian variety)

> Exercise $\exists Y \subset X$ Abelian subvariety s.t. $nZ \subset Y + x$ and $n \gg 0$ $\Rightarrow nZ = Y + x$

Theta group of line bundle

$\mathcal{G}(L) = \{(x,\varphi) \mid x \in X, \varphi: L \xrightarrow{\cong} t_x^{*}L\}$ with group law $(x_1,\varphi_1)(x_2,\varphi_2) = (x_1+x_2, t_{x_2}^{*}\varphi_1 \circ \varphi_2)$

identity $(0, \mathrm{Id})$
$\Rightarrow \mathcal{G}(L)$ is group scheme, $0 \to \mathbb{G}_m \to \mathcal{G}(L) \to K(L) \to 0$

$$\begin{array}{ccc} \mathcal{G}(L) & & \\ \downarrow p & \searrow p_1 & \\ X & \supset & K(L) \end{array}$$

Commutator pairing $\mathcal{G}(L)\times\mathcal{G}(L) \to \mathcal{G}(L)$, $(g_1,g_2) \mapsto g_1g_2g_1^{-1}g_2^{-1}$, $K(L)\times K(L) \xrightarrow{e^L} \mathbb{G}_m$

Prop 14. ① $e^{L}(x,x) = 1$, $e^L$ is alternating; ② $e^{f^{*}L}(-,-) = e^{L}(f(-),f(-))$; ③ $e^{L\otimes M} = e^{L}e^{M}$; ④ $e^{L}|_{\mathrm{Pic}^{0}(X)} = \mathrm{Id}$ ⑤ $e^{L^{\otimes n}}(x,y) = e^{L}(x,ny)$

Thm 24. $f: X \to Y$ both Abelian ($f$ isogeny)
$\{(M,\varphi) \mid M \in \mathrm{Pic}(Y), f^{*}M \xrightarrow{\varphi} L\} \xleftrightarrow{\cong} \{\mathrm{Ker} f \to \mathcal{G}(L)\} \Leftrightarrow \left\{\begin{array}{l}\mathrm{Ker} f \subset K(L) \\ e^{L}|_{\mathrm{Ker} f} = \mathrm{Id}\end{array}\right\}$

Pf. By descent ($L$ descends to $Y$)
$\{M, f^{*}M \xrightarrow{\cong} L\} \leftrightarrow \{\mathrm{Ker} f \curvearrowright L\} \leftrightarrow \{\mathrm{Ker} f \hookrightarrow \mathcal{G}(L)\}$ ▨ (latter one acquires more facts)

Thm 25. $L \cong M^{\otimes n} \Leftrightarrow K(L) \supset X[n]$ ($n$-torsion pts)
Pf. ($\Rightarrow$) $\varphi_L = \varphi_{M^{\otimes n}} = n_X \circ \varphi_M = \varphi_M \circ n_X \Rightarrow K(L) = \mathrm{Ker}(\varphi_L) = \mathrm{Ker}(\varphi_M \circ n_X) \supset \mathrm{Ker}(n_X) = X[n]$
($\Leftarrow$) We prove $L^{\otimes n} = n_X^{*}N$ (then $L^{\otimes n} = N^{\otimes n^2} \otimes P \Rightarrow (L\otimes N^{\otimes -n})^{\otimes n} = P \Rightarrow L = M^{\otimes n}$ and $P = M \otimes N^{-1}$) $= \mathrm{Hom}(N,N)$

Thm 26. We say $\mathcal{G}(L)$ nondegenerate if $\sigma: K(L) \to K(L)^{\nabla} = \mathrm{Hom}(K(L),\mathbb{G}_m)$ isomorphism ($0 \to \mathbb{G}_m \to \mathcal{G}(L) \to K(L) \to 0$)
[$L$ nondegenerate $\Leftrightarrow \mathcal{G}(L)$ nondegenerate] $K(L)$ is finite group scheme

We set the complement of $H \subset K(L)$ by $H^{\perp} = \{h' \in K(L) \mid e^{L}(h',h) = 1, \forall h \in H\}$
$0 \to H^{\perp} \to K(L) \to H^{\nabla}$, $H^{\perp}$ is kernel of the composite map $K(L) \to K(L)^{\nabla} \to H^{\nabla}$

For $\pi: X \to Y$, $H = \mathrm{Ker}\,\pi \Rightarrow H \subset H^{\perp}$ as $e^{L}|_{H\times H} = \mathrm{Id}$
$\pi^{*}M \cong L$, $M$ $\Rightarrow$ we have its quotient $H^{\perp}/H$

> Dual isogeny
> $X \xrightarrow{f} Y \mapsto \hat{f}: \hat{Y} \to \hat{X}$ constructed by $(f\times\mathrm{Id}_{\hat{Y}})^{*}\mathcal{P}_Y = (\mathrm{Id}_X \times \hat{f})^{*}\mathcal{P}_X$
> For $X \xrightarrow{\varphi_L} \hat{X}$ itself, $\hat{\hat{X}} \xrightarrow{\hat{\varphi}_L} \hat{X}$ is just $\hat{\varphi}_L = \varphi_L$ due to $(\varphi_L\times\mathrm{Id})^{*}\mathcal{P}_{\hat{X}} = \Lambda(L)$ Mumford bundle $\Rightarrow$ symmetric

Lemma 22. $K(M) = H^{\perp}/H$ Pf. $\pi(x) \in K(M) \Leftrightarrow t_{\pi(x)}^{*}M \cong M \Leftrightarrow t_x^{*}L \xrightarrow{\varphi} L \Leftrightarrow (x,\varphi) \in \mathcal{G}(L) \Leftrightarrow e^{L}(x,h) = 1 \Leftrightarrow x \in H^{\perp}$ ▨

Thm 26'. $L$ nondegenerate, $H \subset K(L)$ maximal totally isotropic subgroup ($e^{L}|_{H\times H} = 1$) $\Rightarrow H = H^{\perp}$ & $|H|^2 = |K(L)|$

Prop 15. $\varphi: X \to \hat{X}$ isogeny $\Leftrightarrow \hat{\varphi} = \varphi$ ($\varphi = \varphi_L$)

Observation $\Rightarrow |K(M)| = 1$, $\exists M$, $L \cong \pi^{*}M$.
$\varphi_M: x \mapsto \varphi(x) + \varphi(x) = 2\varphi(x)$

Pf. ($\Leftarrow$) Done ($\Rightarrow$) $M := (\mathrm{Id}_X, \varphi)^{*}\mathcal{P}_X$ on $X \Rightarrow$ $X \xrightarrow{\varphi_M} \hat{X}$, $(\mathrm{Id}_X,\varphi)$, $\mathrm{Id}+\varphi$, $X \times X \xrightarrow{\varphi\times\varphi} X\times\hat{X}$

Observation. $\varphi_{\mathcal{P}_X} = \tau$ (switch $\hat{X}\times X$ to $X\times\hat{X}$)
See-Saw $t_{(x,\alpha)}^{*}\mathcal{P}_X \otimes \mathcal{P}_X^{-1}|_{X\times\beta} = P_\beta \otimes P_{\alpha+\beta} = P_\alpha$ and ...
Due to $t_{\dots}^{*}$ ... $= P_\alpha \boxtimes P_\alpha$, this ...

then $\varphi_M = 2\varphi \Leftrightarrow K(M) \supset X[2] \Leftrightarrow \exists L, L^{\otimes 2} \cong M \Leftrightarrow \varphi = \varphi_L$ ☒

Pf of Thm 24. $\mathcal{G}^{\vee}|_{H\times H} = 1 \Rightarrow 0 \to \mathbb{G}_m \to \mathcal{G}(L) \xrightarrow{p} K(L) \to 0$ splits $\Rightarrow \exists \pi: X \to X/H = Y, \exists M$ on $Y$, $\pi^* M \cong L$

Otherwise $|K(M)| = (\chi(M))^2 \geq 2^2$ $\Rightarrow$ it suffices to show $|K(M)| = 1 = |\chi(M)|$

$\Rightarrow \exists M' \subset K(M)$ subgroup, $|M'|$ prim $\Rightarrow 0 \to \mathbb{G}_m \to \mathcal{G}(M) \to K(M) \to 0$

$\Rightarrow e^M|_{M'\times M'} = 1 \Rightarrow e^L = e^M_{\circ(\pi,\pi)} \Rightarrow \pi^*(M') \supsetneq H$ contradict to maximality ☒

$\Rightarrow \chi(L) = (\deg \pi)\chi(M) = |H| \cdot 1 \Rightarrow |K(L)| = \chi(L)^2 = |H|^2$ ☒

Coro 15. Polarized Abelian variety $(X, L) \Rightarrow \exists$ isogeny $\pi: X \to Y$, s.t. $(Y, \Theta)$ is a principal polarization ($\chi(\Theta) = 1$) & $\pi^*\Theta = L$

(PPAV is natural. $J(C)$ / Intermediate Jacobian of Fano 3-fold)

Heisenberg group

Give $H = \bigoplus_{i\leq g} \mathbb{Z}/d_i\mathbb{Z} \to H^{\vee} = \mathrm{Hom}(H, \mathbb{G}_m) = \bigoplus \mu_{d_i}$ (We must have $g = \dim X$, otherwise we take some $d_i = 1$)

Our Heisenberg group $\mathcal{H} = \mathbb{G}_m \times H \times H^{\vee}$ with grp law $(\lambda_1, e_1, \chi_1)(\lambda_2, e_2, \chi_2) = (\lambda_1\lambda_2\chi_2(e_1), e_1+e_2, \chi_1\chi_2)$

$\Rightarrow 0 \to \mathbb{G}_m \to \mathcal{H} \to H \times H^{\vee} \to 0$ identity $(1, 0, 1)$

Schrödinger representation $W = \{f : H \to k\} \Rightarrow \mathcal{H} \to \mathrm{End}(W)$

$(\lambda, e, \chi) \mapsto (f \mapsto (h \mapsto \lambda\chi(h) f(e+h)))$ (Check it's well-defined)

$\mathcal{H}$ is a canonical theta grp $\dim(d_1 \cdots d_g)$

And it's irreducible & unique one s.t. $\mathbb{G}_m \curvearrowright W$ is scalaring (i.e. unique weight 1 rep)

consider $0 \to \mathbb{G}_m \to \mathcal{G}(L) \xrightarrow{p} K(L) \to 0$

Chark. $\chi(L) = 1$. We call $\tilde{H} \leq \mathcal{G}(L)$ level subgrp if $\tilde{H} \cap \mathbb{G}_m = \mathrm{Id}$ (i.e. $p(\tilde{H}) \cong \tilde{H}$)

Coro 16. $L$ nondegenerate $\Rightarrow r: K(L) \to K(L)^{\vee}$ isomorphism

Pf. $A = \ker(r) \Rightarrow A \subset H = H^{\perp}$ and $(H/A)^{\vee} \subset H^{\vee} \Rightarrow 0 \to H^{\perp} \to K(L) \to H^{\vee} \Rightarrow \mathrm{Im}(q \circ r) = (H/A)^{\vee}$

We show that, surprisingly, all theta group $\cong$ Heisenberg group. $r|_{K(L)} \nearrow q \Rightarrow |K(L)| \leq |H^{\perp}| |(H/A)^{\vee}| = \frac{|H|^2}{|A|}$

Lemma 23. $H \leq K(L)$, TAFE ① $H$ maximal isotropic; ② $H$ totally isotropic ($H \subset H^{\perp}$) and $|H|^2 = |K(L)|$ $\Rightarrow |A| \leq 1 \Rightarrow A$ trivial ☒

③ $H = H^{\perp}$

We call them Lagrangian subgroup of $K(L)$

Pf. ① $\Rightarrow$ ② is thm 26. ③ $\Leftrightarrow$ ② $\Rightarrow$ ① by Coro 16 $\Rightarrow K(L)/H^{\perp} \cong H^{\vee}$ ☒

We call $\tilde{H} \leq \mathcal{G}(L)$ level subgrp Lagrangian if $p(\tilde{H}) \subset K(L)$ is Lagrangian

A Lagrangian decomposition is $K(L) \cong H_1 \times H_2$, $H_i$ Lagrangian, s.t. $r$ induces (restrict to $H_1 \cong H_2^{\vee}$)

Lemma 24. If $K(L)$ admittes Lagrangian decomposition $K(L) = H_1 \times H_2 \Rightarrow \mathcal{G}(L) \cong \mathcal{H} = \mathbb{G}_m \times H_1 \times H_2$ (as $H_1 \cong H_2^{\vee}$)

Pf. $\mathcal{H} \to \mathcal{G}(L)$ where $\delta_1: H_1 \to \tilde{H}_1 \subset \mathcal{G}(L)$ lift to $\mathcal{G}(L)$ via splitting

$(\lambda, e, \chi) \mapsto \alpha\delta_1(e)\delta_2(v^{-1}(\chi))$ $\qquad \delta_2: H_2 \to \tilde{H}_2 \subset \mathcal{G}(L)$

Thm 27. $\exists H$, s.t. $\mathcal{G}(L) \cong \mathcal{H}$ $\qquad v: H_2 \xrightarrow{\cong} H_1^{\vee}$ ☒ (Thus we call $(d_1 \cdots d_g)$ ($H = \bigoplus \mathbb{Z}/d_i\mathbb{Z}$) the polarization type of $L$)

and $\mathcal{G}(L) \cong \mathcal{H} \to \mathrm{End}(H^0(L))$ is Schrödinger rep

Pf. We need find a Lagrangian decomposition (linear algebra) $\Rightarrow$ first done ☒

Using uniqueness, $\mathbb{G}_m \circlearrowright H^0(L)$ is scalaring (weight 1) $\Rightarrow$ Schrödinger ☒ (as $\dim H^0(L) = \dim W = d_1 \cdots d_g$)

> Pf of uniqueness
>
> Let $W/k$ is weight 1 rep of $\mathcal{G}(L)$, $H \leq \mathcal{G}(L)$ maximal isotropic subgrp, lift to $\tilde{H} \leq \mathcal{G}(L) \Rightarrow \tilde{H} \curvearrowright W \Rightarrow W = \bigoplus_{\chi \in \mathrm{Hom}(\tilde{H}, \mathbb{G}_m) = \tilde{H}^{\vee}} W_\chi$
>
> and $g \in \mathcal{G}(L)$ permutes $\mathrm{Hom}(\tilde{H}, \mathbb{G}_m)$: $g W_\chi = W_{e(g^{-1}\cdot)\chi}$ ($g^{-1} = p(g^{-1})$)
>
> $\Rightarrow \chi \in \mathrm{Hom}(\tilde{H}, \mathbb{G}_m)$ passes all subrepresentation of 1-dimensional $\Rightarrow W$ is unique ☒ (dimension reason)

Coro. If $L = M \otimes N$ both ample $\Rightarrow H^0(L) = \sum H^0(M \otimes P_\alpha) \otimes H^0(N \otimes P_\alpha^{-1})$

For more, see [Mumford 1966].

> Fact. Smooth proj var $X$, $h^1(\mathcal{O}_X) = \dim X = g$, $h^0(K_X) = h^0(2K_X) = 1$
> $\Rightarrow X$ Abelian for chark $= 0$
> [Rong Kai Chen, Hacon]
> (By Albanese map $\cong$)

Endomorphism ring

Thm 28. $f: Y \to X$ étale covering, $X$ Abelian $\Rightarrow Y$ abelian & $f$ is seperable isogeny

Pf. $\Gamma \subset X \times X \times X$ let $\Gamma'' \subset \Gamma'$ component containing $(y_0, y_0, y_0)$

$\uparrow \square \uparrow f\times f\times f$ take $y_0 \in f^{-1}(0_X)$

$\Gamma' \subset Y \times Y \times Y$ then we show that $\Gamma'' \xrightarrow{p_{12}} Y \times Y$ is isomorphism $\Rightarrow$ $\boxed{Y \times Y \xrightarrow{p_{12}^{-1}} \Gamma'' \xrightarrow{p_3} Y \text{ is group law}}$, done

($\Gamma$ graph of $m: X \times X \to X$) By $\Gamma' \to \Gamma$ $\Rightarrow \Gamma'' \xrightarrow{p_{12}} Y \times Y$ also étale, and we need to show $\deg(p_{12}) = 1$:

étale $f^2 \downarrow p_{12} \cong$ Denote a picture $\Gamma'' \xrightarrow{p_{12}} Y \times Y$ we have section of $\delta_{1,2}$ by lift

$Y \times Y \xrightarrow{\text{étale}, f\times f} X \times X$ $\qquad \delta_{1,2} \nearrow Y \Downarrow p_{12}$ to $y \mapsto (y_0, y, y)$ or $(y, y_0, y)$, $\sigma_1$ and $\sigma_2$

Pf of Claim.

$h: X \to Y$ smooth proper, $X, Y$ irreducible thus we prove $p_{12}^{-1}(Y \times y_0) = \sigma_1(Y)$. Claim: $p_{12}^{-1}(y_0)$ irreducible

If $h$ admittes of section $\Rightarrow h$ irreducible fibre

Pf. Stein factorization $X \to \tilde{Y} \to Y$, section gives $Y \to X \to \tilde{Y}$ $p_{12}^{-1}(Y \times y_0) \to Y$ irreducible fibre ☒ $\Rightarrow p_{12}^{-1}(y_0) \subset \sigma_1(Y)$ irreducible must equal

Then we need to check $p_3\circ p_{12}^{-1}$ is desired group law: $p_3\circ p_{12}^{-1}(y_0 \sigma y) = y = p_3\circ p_{12}^{-1}(y, y_0)$ by definition of $\sigma_1$ and $\sigma_2$

Then it's enough to say $Y$ is Abelian variety, due to $y_0 y = y = y y_0 \Rightarrow$ others properties if $Y$ is complete (one may had seen such argument in Lie groups ... as the behaviour around "origin" determines all if compact)

Our key is rigidity Lemma: (here we write as $(X, e)$)

$\varphi: X\times X \to X\times X$ surjective, $\forall x\in X$, $\exists x'\in X$ s.t. $x'x = e$

$(x,y) \mapsto (xy, y)$

$\Gamma = \{(x,y)\in X\times X \mid xy = e\}$, $\Gamma' \subset \Gamma$ irreducible component dominant $p_2|_{\Gamma'}$

Then associativity

$\Rightarrow \psi: \Gamma\times X \to X$ it contract fibre $\psi(\Gamma\times e) = e$

$(x', x)\times y \mapsto x'(xy) \Rightarrow$ all fibre $(x', x)\, y = y$ contracts

$\Phi: \Gamma\times X\times X \to X$

$(x', x)\times y\times z \mapsto x((x'y)z)$

is inverse law $(x^{-1}x = xx^{-1} = e)$

contract fibre $\Phi(\Gamma\times e\times e) = e$

$\Rightarrow$ all fibres $\Phi(\Gamma\times y\times z) = yz \Rightarrow x(x'(yz)) = yz$

$\Rightarrow (x'y)z = x'(yz)$ $\square$

$\Rightarrow$ for $(x'', x'), (x', x) \in \Gamma$: $x''(x'(xx')) = x''(x'x)x'$

$= x''x' = e = xx'$

$\Rightarrow x'x = xx'$

We define $\mathrm{Hom}(X, Y)$ as group $X, Y$, thus $\mathrm{Hom}(X,Y)$ group $\Rightarrow \mathrm{End}(X)$ ring with composition, we denote $\mathrm{Hom}^0(X,Y) = \mathrm{Hom}(X,Y)\otimes\mathbb{Q}$ & $\mathrm{End}^0(X) = \mathrm{End}(X)\otimes\mathbb{Q}$

If $X \xrightarrow{f} Y$ is isogeny $\Rightarrow \exists$ isogeny $Y\to X$, s.t. $g\circ f = n_X$, $f\circ g = n_Y$, $n_X$ kills $\ker(f)$ / $n_Y$ kills $\ker(g)$ $\Rightarrow$ isogeny is equivalence relation

Thm 29. $Y \subset X$ subAbelian $\Leftrightarrow \exists$ "complement" ! if we fix a polarization)

$\Rightarrow \exists Z \subset X$ subAbelian, s.t. $Y\cap Z$ finite & $Y+Z = X \Rightarrow Y\times Z \to X$ isogeny, $Z$ unique up to polarization.

$(y,z) \mapsto y+z$

Pf. Fix $L$ ample on $X$ / polarization

$$\begin{tikzcd} X \arrow[r, "\varphi_L"] & \hat X \arrow[d, "\hat i"] \\ Y \arrow[u, "i"] \arrow[r, "\varphi_{i^*L}"] & \hat Y \end{tikzcd}$$

isogeny. $\Phi$ surj & $(\ker\Phi)_{\mathrm{red}}^0$ the component containing identity $=: Z$ desired.

$\dim Z = \dim X - \dim Y = \dim X - \dim \hat Y$, $(\ker\Phi)^0 \cap Y = \ker(\varphi_{i^*L})$ finite

$\Rightarrow Y+Z = X$ $\square$

Poincaré's reducibility thm

We thus call Abelian variety simple if none nontrivial subAbelian variety. Thus we have decomposition $X \simeq \prod X_i^{n_i}$ into simple Abelian subvariety by Thm 29 $\square$

Thm 30. $\mathrm{End}^0(X) \cong \bigoplus M_{n_i}(D_i)$, $D_i/\mathbb{Q}$ division ring.

Pf. When $X$ simple, $\forall f\in \mathrm{End}(X)$ is isogeny, thus invertible $\Rightarrow$ tensor $\otimes\mathbb{Q}$ $\mathrm{End}^0(X)/\mathbb{Q}$ division

Otherwise $X \simeq \prod X_i^{n_i} \Rightarrow \forall f\in\mathrm{End}(X) = \mathrm{End}(\prod X_i^{n_i})$ = (Different $X_i$ has no isogeny) $\bigoplus \mathrm{End}(X_i^{n_i}) = \bigoplus M_{n_i}(D_i)$, and $\otimes\mathbb{Q}$, $D_i$ division $\square$

$\bigoplus M_{n_i}(\mathrm{End}(X_i))$

Q. Is $D_i/\mathbb{Q}$ f.d.? When $/\mathbb{C}$ lattice it's easy to compute what $D_i$ are. But for $/k$, char $k = p$, we need Tate module

Prop 16. $\mathrm{End}(X) \xrightarrow{\deg} \mathbb{Z}$ can be extend to $(\mathrm{End}^0(X)[T])_{\deg} \to \mathbb{Z}$ as a degree $2g$ polynomial

$\varphi \mapsto \deg\varphi$ ($\varphi$ isogeny)

$0$ , otherwise

Pf. $\deg(n\varphi) = \deg(n_X)\deg\varphi = n^{2g}\deg\varphi \Rightarrow P(n) = \deg(n\varphi + \psi)$ is polynomial deg $(2g)$ extend $\deg(\varphi)$

$P(n) = \dfrac{\chi((n\varphi+\psi)^*L)}{\chi(L)} = \dfrac{\chi(L_n)}{\chi(L)}$ for $L$ ample & symmetric, we need $4(\ldots)^{-1}$ show such $P(n)$ has degree $2g$.

By thm of cube applied to $(n\varphi+\psi), \varphi, \varphi \Rightarrow L_{n+2}\otimes L_{n+1}^{-1}\otimes L_{n+1}^{-1}\otimes 2\varphi^*L^{-1}\otimes L_n\otimes\varphi^*L\otimes\varphi^*L = \mathcal{O}_X$

$\Rightarrow L_{n+2} \cong L_{n+1}^{2}\otimes L_n\otimes(\varphi^*L)^2$, then induction to give $L_{n+2} \cong H^{\binom{n+2}{2}}\otimes\ldots$, top degree $(n^2)^g = n^{2g}$ $\square$

Thm 31. $\mathrm{Rank}(\mathrm{Hom}(X,Y)) \le 4\dim X\dim Y$, $\mathrm{Hom}(X,Y)$ is free $\mathbb{Z}$-module (thus $\mathrm{End}^0(X)$ is f.d. semi-simple $\mathbb{Q}$-algebra)

Coro 18. $\mathrm{End}^0(X)$ f.d. central simple algebra

Coro 19. $\mathrm{Rank}(NS(X)) \le 4(\dim X)^2$, $NS(X)$ is free $\mathbb{Z}$-module (By $NS = \mathrm{Pic}(X)/\mathrm{Pic}^0(X) \hookrightarrow \mathrm{Hom}(X, \hat X)$)

$L \mapsto \varphi_L$ $\square$

Rk. $/\mathbb{C}$ case Coro 19 is easy: $1\to\mathbb{Z}\to\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$, $X = \mathbb{C}^g/\Lambda$

$$\begin{tikzcd} H^1(X;\mathbb{Z}) \arrow[r] \arrow[d, "\shortparallel"] & H^1(X;\mathcal{O}_X) \arrow[r] \arrow[d, "\shortparallel"] & \mathrm{Pic}(X) \arrow[r, "c_1"] & H^2(X;\mathbb{Z}) \arrow[r] & \cdots \\ \Lambda^\vee \arrow[r] & \mathbb{C}^g & \mathrm{Pic}^0(X) \arrow[u] & & \\ & & 0 \arrow[u] & & \end{tikzcd}$$

SES as $\ker(c_1) = \mathrm{Pic}^0(X)$

$\Rightarrow NS(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X) = \mathrm{Pic}(X)/\mathrm{Im}(\ldots)$

$\Rightarrow \mathrm{rank}(NS(X)) = \mathrm{rank}(\Lambda^2 H^1(X;\mathbb{Z})) = \dfrac{2g(2g-1)}{2} = g(2g-1) \le 4g^2$ $\square$

Our idea of pf of Thm 31 is found a replacement of lattice $H^1(X;\mathbb{Z})$: this is Tate module (Another possible way is replace the LES above by étale directly)

Eg. $X$ Abelian surface$/\mathbb{C}$, By Lefschetz (1,1) $\Rightarrow \mathrm{Im}(c_1) \subset H^{1,1}(X) \subset H^2(X;\ldots)$, further $\mathrm{Im}(c_1) = H^{1,1}(X)\cap H^2(X;\mathbb{Z})$

$\Rightarrow NS(X)$ can have any rank $1,2,3,4$: 1: trivial & generic; 2: $E_1\times E_2$; 3: $E\times E$; 4: $(E_1\times E_1)$ or $(E\times E)$ with complex multiplication. $E_1 = \mathbb{C}/\mathbb{Z}\oplus\ldots$, $E = \mathbb{C}/\mathbb{Z}\oplus\rho\mathbb{Z}$ $(\rho^3=1)$

Where $E_1, E_2$ are two (only) elliptic curves

then $NS(E_1\times E_2) \cong \ldots$

$NS(E_i \times E_i) = \langle O_{E_i} \boxtimes L_i, O_{L_i} \boxtimes O_{E_i}, \Delta, \Gamma_i \rangle$, where $\Gamma_i$ is graph of automorphism of $E_i$ (only E1 & E2 have automorphism among all elliptic curves $\Rightarrow$ complex multiplication by $i$ & $\rho$)

Def 6 (Tate module) $X/k$, $k = \bar{k}$, $(\mathrm{char}\, k, l) = 1$ prime. We take inverse limit of all $l$-torsion pts $X[l^n] \xrightarrow{l} X[l^{n-1}]$ ; $T_l X = \varprojlim X[l^n]$ as a replacement of fundamental group

As only group, $T_l X = \varprojlim (\mathbb{Z}/l^n\mathbb{Z})^{2g} \cong \mathbb{Z}_l^{2g}$ as $\mathbb{Z}_l$-module (a special case of etale one in Abelian variety)

$T_l X$ has additional arithmetic structure such as a Galois representation but here we only stop as group-level

Set $V_l X = T_l X \otimes \mathbb{Q}_l$ is a dimensional $2g$ $\mathbb{Q}_l$-vector space,

$\mathrm{Hom}(\mathbb{Q}_l/\mathbb{Z}_l, X) = \mathrm{Hom}_{\mathbb{Z}_l}((\varinjlim l^{-n}\mathbb{Z}_l)/\mathbb{Z}_l, X) = \varprojlim \mathrm{Hom}(l^{-n}\mathbb{Z}_l/\mathbb{Z}_l, X) = \varprojlim_n X[l^n] = T_l X$;

$\Rightarrow 0 \to \ker f \to X \xrightarrow{f} Y \to 0$ isogeny $\Rightarrow$ $\underline{\mathrm{Hom}(\mathbb{Q}_l/\mathbb{Z}_l, \ker f)} = 0 \to T_l X \to T_l Y \to \mathrm{Ext}^1(\mathbb{Q}_l/\mathbb{Z}_l, \ker f)$

As a finite group, $\ker f = N_1 \times N_2$ (killed by $n_X$ for some $n$, thus $l$ large $= 0$) $\quad \mathrm{Ext}^1(\mathbb{Q}_l/\mathbb{Z}_l, N_1) \times \mathrm{Ext}^1(\mathbb{Q}_l/\mathbb{Z}_l, N_2)$

$N_1$ relative prime to $l$, $|N_2| = l^k$.

To prove Thm 31, we need

Thm 32 $0 \to \mathrm{Hom}(X,Y) \otimes \mathbb{Z}_l \xrightarrow{\Phi} \mathrm{Hom}_{\mathbb{Z}_l}(T_l X, T_l Y) \xrightarrow{\Psi} \underset{\text{torsion free}}{Q} \to 0$ $\quad$ ($\Rightarrow \mathrm{Hom}(X,Y)$ free & rank $\leq$ rank $\mathrm{Hom}(T_l X, T_l Y)$ done Thm 31)

$f \otimes c \mapsto c T_l(f)$ $\quad$ $\underline{\mathbb{Z}_l \text{ module of rank} = 4 (\dim X)(\dim Y)}$

Pf. torsion free is easier part: let $\varphi \in \mathrm{Hom}_{\mathbb{Z}_l}(T_l X, T_l Y)$ $\quad$ $l$-torsion pts are dense $\subset X$

s.t. $l^m \varphi = \sum c_i T_l(f_i)$, i.e. $l^m \varphi \in \mathrm{Im}\,\Phi$ $= \mathrm{coker}\,\Phi$, we'll show that $l^m \varphi = 0$, it suffices show $\varphi|_{X[l^m]} = 0$ due to then it's obvious ▨ (due to $Q$ $\mathbb{Z}_l$-module, if it has torsion it only have $l$-torsion)

To show injectivity, assume $X = Y$; set $Z = X \times Y$. $\mathrm{Hom}(X,Y) \xrightarrow{T_l} \mathrm{Hom}_{\mathbb{Z}_l}(T_l X, T_l Y)$ thus if $X = Y$ done, general done.

Otherwise $\exists\, r$ minimal s.t. $\sum_{j \leq r} c_j T_l(f_j) = 0$

$f_j \in \mathrm{End}(X)$, $c_j \in \mathbb{Z}_l^{*}$ $\quad$ $\mathrm{Hom}(Z,Z) \xrightarrow{T_l} \mathrm{Hom}_{\mathbb{Z}_l}(T_l X, T_l Y)$ passes injectivity from down to up.

Set $B_L : \mathrm{End}(X) \times \mathrm{End}(X) \to \mathbb{Z}$

$(f,g) \longmapsto (f+g)^* L \otimes f^* L^{-1} \otimes g^* L^{-1}) \cdot L^{g-1}$ is $\underline{\text{bilinear positive definite quadric form}}$

By linear algebra, we can assume $B(f_i, f_j) = 0$, $\forall i, j$ (theorem of cube) $((2f)^* L \otimes f^* L^{-1})^2 = f^* L^2$ thus intersect to $L^{g-1}$ (variable change $c_i' = \sum_j B(f_i, f_j) c_j$ and $f_i' = B(f_1, f_1) f_j - B(f_j, f_1) f_1$)

$\forall m$, $\exists\, n_i \in \mathbb{Z}$ $n_i \equiv c_i \bmod l^m \Rightarrow g = \sum n_i f_i \in \mathrm{End}(X)$, then by torsion free part, $T_l(g)$ divisible by $l^m$, i.e. $T_l(g) = l^m T_l(h)$

$\Rightarrow B_L(g, f_1) = n_1 B(f_1, f_1)$ divisible by $m \Rightarrow c_1$ divisible by $m$, $\forall m \Rightarrow c_1 = 0$, contradiction (to minimality) ▨ $\quad g = l^m h$

$\underline{\text{Rosati involution}}$, $\varphi_L : X \to \hat{X}$ $\quad \psi_L \circ \varphi_L = n\,\mathrm{Id}$, then $' : \mathrm{End}(X) \to \mathrm{End}^0(X)$

$\psi_L : \hat{X} \to X$ $\quad f \mapsto f' = \varphi_L^{-1} \circ \hat{f} \circ \varphi_L \, (\tfrac{1}{n}\psi_L)$

Thm 33. $\forall l$, $P_{l,f}(T)$ the characteristic polynomial of $T_l(f)$ over $\mathbb{Z}_l$ $= P_f(T) = \deg(T \cdot \mathrm{Id} - f)$ $\deg = 2g$ not depend on $l$

Pf. It's equivalent to say that $\forall l$, only one determinant (all character is a power of determinant $\mathrm{End}^0(X) \to \mathbb{Q}_l$ due to $\mathrm{End}^0(X)$ can be factored by matrix algebra.

Two norms $N_1 : \mathrm{End}^0(X) \to \mathbb{Q}$ & $N_2 : \mathrm{End}^0(X) \to \mathbb{Q}_l$ coincides by considering their $\mathbb{Q}_l$-valuation:

$(N_i(f_1 f_2) = N_i(f_1) N_i(f_2))$ $\quad f \mapsto \deg f$ $\quad f \mapsto \det(T_l f)$ $\quad 0 \to \ker f \to X \xrightarrow{f} X \to 0$, $\ker f = N_1 \times N_2$

Thm 34. ① $\mathrm{tr}(f f') = 2g \dfrac{(L^{g-1} \cdot f^* L)}{(L^g)} > 0$ $\quad \Rightarrow v_l(f) = v_l(\deg f)$ $\quad (|N_1|, l) = 1$ & $|N_2| = l^k$

② $\mathrm{End}^0(X) \times \mathrm{End}^0(X) \to \mathbb{Q}$ is symmetric & positive definite. $\quad = v_l(|\ker f|) = v_l(N_2) = k$

$(f,g) \longmapsto \mathrm{tr}(f g')$ $\quad$ and $0 \to T_l X \xrightarrow{T_l f} T_l X \to \mathrm{Ext}^1(\mathbb{Q}_l/\mathbb{Z}_l, \ker f) \to 0$

(Here $P_f(T) = T^{2g} - \mathrm{tr}(f) T^{2g-1} + \cdots + \mathrm{Nm}(f)$ for polynomial $f$'s characteristic polynomial. Here $f \in \mathrm{End}^0(X)$) $\quad 0 \to \mathrm{Hom}(\mathbb{Q}_l/\mathbb{Z}_l, \mathbb{Q}_l) \to \mathrm{Hom}(\mathbb{Q}_l/\mathbb{Z}_l, N_2) \to \mathrm{Ext}^1(\mathbb{Q}_l/\mathbb{Z}_l, N_2) \to 0$

thus same valuation $|N_2| = k$ ▨

Pf. ① $\varphi_{f^* L} = \hat{f} \circ \varphi_L \circ f : X \to \hat{X}$, $\deg(\varphi_{n\cdot L - f^* L}) = \deg(n\varphi_L - \varphi_{f^*L}) = \deg(n\varphi_L - \hat{f} \circ \varphi_L \circ f) = \deg(\varphi_L \circ (n - \varphi_L^{-1} \hat{f} \varphi_L f))$

$= \deg(\varphi_L) \deg(n - f' f)$

$\chi(L^n \otimes f^* L^{-1})^2$ $\quad$ $\underline{\text{Compare their coefficient}}$ $\quad = \chi(L)^2 P_{f' f}(n)$

$\dfrac{g!}{g!} P(n)^2$ $\quad$ in $(2g-1)$ degree

$\Rightarrow \mathrm{tr}(f f') = \dfrac{2g (L^{g-1} \cdot f^* L)}{(L^g)} > 0$ ▨ $\quad \Rightarrow \left(\dfrac{L^g}{g!}\right)^2 \mathrm{trace}(f f') = \dfrac{1}{g!} 2g \left(\dfrac{L^g}{g!}\right) (L^{g-1} \cdot f^* L)$

② Omitted due to time ▨

Rk. All these study of endomorphism ring is useful in arithmetic, and it's used in the moduli of Abelian Var with endomorphism

Jacobian. $C/k$, $k = \bar{k}$ irreducible smooth proj curve, $g = h^1(O_C) = h^0(\Omega_C)$

Relative Picard functor $\mathrm{Pic}_{C/k} : \mathrm{Sch}/k \to \mathrm{Ab}$, $T \mapsto \mathrm{Pic}_T(X_T) = \mathrm{Pic}(X_T)/\mathrm{Pic}(T)$. For X smooth proj, Pic is representable by a group scheme $(\mathrm{Pic}(X), P) \subset \mathrm{Pic}(X)$ has infinite complete components, $\mathrm{Pic}^0(X)$ netrual component) Page

Fact: ① $x_0 \in X$ fixed, $\exists!$ Poincare line bundle $P_X$ on $X \times \mathrm{Pic}^0(X)$, s.t. ⓘ $P_X|_{x_0 \times \mathrm{Pic}^0(X)} \cong \mathcal{O}_{\mathrm{Pic}^0(X)}$ (may not Abelian var, as it can be nonreduced)

ⓘⓘ $k$-scheme $T$, $\mathcal{L} \to X \times T$, s.t. $\mathcal{L}|_{X \times t}$ algebraically equivalent to $\mathcal{O}_X \Rightarrow \exists! f_{\mathcal{L}}: T \to \mathrm{Pic}^0(X)$, s.t. $\mathcal{L} \cong (id_X \times f_{\mathcal{L}})^* P_X$

② $T_0 \mathrm{Pic}^0(X) \cong H^1(X, \mathcal{O}_X)$, for char $k = 0$ / $\dim X = 1$ / $X$ Abelian $\Rightarrow \mathrm{Pic}^0(X)$ Abelian

Let $X = C$ smooth projective curve, $\deg L_0 = \deg \mathcal{O}(\mathcal{L}) = \int c(L)$, if $\deg L_1 = \deg L_2 \Rightarrow L_1$ algebraically equivalent to $L_2$

$\Rightarrow \mathrm{Pic}(C) = \bigsqcup \mathrm{Pic}^n(C) \cong \mathbb{Z}$ as group (numerical equivalence / rational equivalence)

Prop 17. $\varphi: C \to \mathrm{Pic}^1(C)$ is embedding.

$c \mapsto [\mathcal{O}_C(c)]$

Pf. Injective: $p \neq q \in C$ but $\mathcal{O}(p) \cong \mathcal{O}_C(q) \Rightarrow \mathcal{O}_C(p-q)$ trivial

$\Rightarrow (p-q) = \mathrm{div}(f) \Rightarrow f: C \to \mathbb{P}^1 \Rightarrow C \to \mathbb{P}^1$ deg 1, contradiction! ($p \mapsto 0$, $q \mapsto \infty$)

Tangent level:

$T_p C \xrightarrow{d\varphi_p} T_{\varphi(p)} \mathrm{Pic}^1(C)$ injective

$\| \quad H^1(C, \mathcal{O}_C)$

$\Updownarrow$ Serre dual

$H^0(C, K_C)$: $H^0(C, K_C) \otimes \mathcal{O}_C \to K_C$ surjective

$\Leftrightarrow H^0(C, K_C)$ base pt free is well-known

A fundamental pf is by RR:

$h^0(L) - h^1(L) = 1 + \deg L - g$ $\quad \exists D_0$ divisor, $\deg D_0 = m \gg 0$

$h^1$ Serre $\quad (m + d \geq 2g - 1)$

$h^0(C, K \otimes L^{-1})$

$\Rightarrow \deg(L \otimes \mathcal{O}(D_0)) = m + d \geq 2g - 1$

$\Rightarrow \exists 0 \neq s_i \in H^0(L(D_0))$, $\mathrm{div}(s_i) = \sum a_i z_i$, $z_i \in C$

$\Rightarrow$ We construct the deformation space to be on $C^{(n)} = \underbrace{C \times \cdots \times C}_{n} / S^n$, $\sum a_i = m + d \geq 2g - 1$

$\mathcal{O}_{C \times C} \supset D_n = \{(z, t \in C) \mid z \in C^{(n)}\}$

gives $\mathcal{L} = \mathcal{O}_{C^{(n)} \times C}(D_n) \Rightarrow \mathcal{L}|_{z \times C} = \mathcal{O}_C(z)$, thus $\mathcal{L}$ is the deformation space ▨

We have Abel-Jacobi map $j^{(n)}: C^{(n)} \to \mathrm{Pic}^n(C)$

$[z] \mapsto [\mathcal{O}_C(z)]$

Thm 25. (Abel's thm)

$\forall L \in \mathrm{Pic}^n(C)$, $(j^{(n)})^{-1}(L)$ ~~scheme-theortic inverse image~~ $= \mathbb{P}(H^0(L))$

Pf. For naive inverse image / set-theortic, the inverse image is obviously due to $H^0(\mathcal{O}_C(z)) = \mathbb{Z}$

Rk. ① for $j^{(g-1)}: C^{(g-1)} \xrightarrow{\simeq} \Theta \hookrightarrow \mathrm{Pic}^{(g-1)}(C)$ ② ~~Torelli thm~~ is birational to a divisor of $\mathrm{Pic}^{(g-1)}(C)$ ② $\Theta$ is prinipal divisor

Coro 20. $j^{(g)}: C^{(g)} \xrightarrow{\simeq} \mathrm{Pic}^g(C)$ birational. (very complicated structure)

$C_1 \cong C_2 \Leftrightarrow (\mathrm{Pic}^{g-1}(C_1), \Theta_{C_1}) \cong (\mathrm{Pic}^{g-1}(C_2), \Theta_{C_2})$

$C^{(n)}$

$h^0(L) = \{z \in |L|$ effective$\}$

Pf. By RR $h^0(L) - h^1(L) = 1 + g - g = 1 \Rightarrow \exists D$ effective, $\mathcal{O}(D) \cong L \Rightarrow j^{(g)}$ surj

Omit the general case ▨

Then we show $h^1(L) = 0$, i.e. $h^1(L) = h^0(C, K \otimes L^{-1}) = 0$

This's due to $\varphi: C \to \mathbb{P}H^0(K_C) = \mathbb{P}^{g-1}$ canonical map $\Rightarrow \exists c_1 \cdots c_g \in C$, $\varphi(c_1) \cdots \varphi(c_g)$ spanns $\mathbb{P}^{g-1} \Rightarrow L \cong \mathcal{O}_C(c_1 + \cdots + c_g)$

(due to $K_C$ is base pt free) $\Rightarrow h^0(K_C \otimes L^{-1}) = h^0(K_C - \sum c_i) = 0$ ▨

(For $g \geq 1$, either $|K_C|$ base pt free or $C$ hyperelliptic & $\varphi$ is 2 to 1)

Albanese: $V$ irreducible / $k$, $k = \bar{k}$, set $f: V \dashrightarrow X$ rational, $X$ Abelian. $(V, f)$ generates $X$ if $\exists n$, $V^n \xrightarrow{f^n} X^n \xrightarrow{\text{addition}} X$ dominant rational map

Rk. We know if $V$ further smooth, $\forall f: V \dashrightarrow X$ can be extended into morphism.

but here we have conterexample if not smooth:

Consider $E_1 \times E_2$ product of two ellptic curves and $(E_1 \times 0) \cup D \xrightarrow{f} E_1 \times E_2$, $D$ sufficiently ample, s.t. $E_1^2 < 0$ in $E_1 \times 0 \cup D$

then $E_1$ contract to pt $x_1 \in V$, but $f(x_1)$ can't be defined! (contracts $E_1$; $V \dashrightarrow f$)

An Albanese variety of $V$ is $f: V \dashrightarrow X$ ① $(V, f)$ generates $X$ ▨

② $\forall g: V \dashrightarrow Y$, $\exists V \xrightarrow{f} X$, $g = h \circ f + c$

$\exists! h$ (up to translation on $Y$), and $f$ is called Albanese map.

Thm 36. $\forall V$, $\exists$ Albanese map / variety

Lemma: $j: C \to \mathrm{Pic}^1(C)$ is Albanese of $C$

Pf. $C^g \longrightarrow \mathrm{Pic}^1(C)^g$

$C^{(g)} \cong \mathrm{Pic}^g(C)$

birational $\Rightarrow$ dominant

and then $(C, j^{(1)})$ generates $\mathrm{Pic}^1(C)$

For universal property $C \to B \Rightarrow C^n \to B^n$ descend $C^n \to B$ to $C^{(n)} \to B$, then to $\mathrm{Pic}^n(C) \to B$ and this is just $j^{(n)}$ commutes with ▨

Lemma: $f: V \dashrightarrow X$ generates $X$, $\exists g$ only depends on $V$, $\dim X \leq g$

Pf. Assume $V$ is projective (By compactification & Chow lemma proper $\rightsquigarrow$ proj)

$H$ very ample, $D_i \in |H| \Rightarrow D_1 \cap \cdots \cap D_{n-1} = C \hookrightarrow V$ irreducible curve, $\tilde{C} \to C$ normalization

$\Rightarrow \dim X \leq h^1(\tilde{C}, \mathcal{O}_{\tilde{C}})$ due to $\tilde{C} \to X$ is Albanese by upper Lemma ▨

Pf of Thm 36. Construct $f: V \dashrightarrow X$ s.t. $\dim X$ maximal (by upper Lemma, it's bounded), claim it's Albanese!

$V^n \dashrightarrow Y \to X$, $K(X) \subset K(Y) \subset K(V^n)$ the maximal function field extension, then it's canonical $V \hookrightarrow \Gamma \to Y \times Z$, $\forall Z$

Stein factorization

canonical by construct $\Gamma = \mathrm{Im}(\alpha, g)$, $(\alpha, g): V \dashrightarrow Y \times Z$ ▨ Both Abelian

Rk. When $V$ smooth proj it's easier: $P = \mathrm{red}(\mathrm{Pic}^0(V))$ representable Abelian var $\Rightarrow V \to \hat{P}$ is Albanese.

or not Kähler $V \to H^0(\Omega^1)^\vee / H_1(X, \mathbb{Z}) = $ cpx torus $T$, $z \mapsto (\omega \mapsto \int_{z_0}^{z} \omega)$