Date Reminder on intersection theory

① $\boxed{A^i \text{ can be far from } H^{2i}}$:

A_0(K3 surfaces) is very complicated theory, but H^{2n}(K3 surfaces) $= \mathbb{Z}$;

$A_0(\circledcirc) = \mathbb{Z} \oplus \circledcirc$ but $H^2(\circledcirc) = \mathbb{Z}$;

The cycle class map $A^i(X) \xrightarrow{cl} H^{2i}(X_{\mathbb{C}};\mathbb{Z})$ or $H^{2i}_{et}(X;\mathbb{Z}_l(i))$

thus is not injection,

it's also not surjection in general as Hodge conjecture says the image lies in $H^{1,1}$;

② cl is ring homomorphism, most cases when $A^i(X)$'s product hard to define, H^{2i} suffices to do intersection theory, cup product has the advantage: preserved by deformation.

For example, quantum cohomology has setting in H^* than Chow;

③ The natural group homomorphism $Pic(X) \to A_{n-1}(X)$ is also neither injective nor surjective:

$\boxed{\text{Cartier}}$ $\qquad Cl(X) \leftarrow \boxed{\text{Weil}}$

$\{f_\alpha\} \longmapsto \sum_{W \subset V \text{ codim } 1} ord_W(f_\alpha)[W]$, $f_\alpha \in H^0(U_\alpha, K^\times/\mathcal{O}^\times)$

• Not injection:

$X = \alpha$ a node

$A_0(X) = 0$ as any two points are rational equivalent

(singular degeneration of torus has even simpler Chow!)

$Pic(X) = \mathbb{C}^\times$ can be computed via normalization:

σ: (O_1, O_2) $\longrightarrow$ (O), $\forall p, q \in X$,

$\exists r \in K(X), ord_p(r) = -ord_q(r) = -1$

$\Rightarrow ord_O(r) = ord_{O_1}(\tilde{r}) + ord_{O_2}(\tilde{r}) = 1 + (-1) = 0$

$\Rightarrow r = p/q$, but $\mathcal{O}_{X,O}$ not DVR, can't be determined at O

$\Rightarrow r(O) \in \mathbb{C}^\times$ choices ☑

• Not surjection: X = affine cone $\subset \mathbb{A}^3$ $\qquad Pic(X) = 0$

$A_2(X) = \mathbb{Z}/2\mathbb{Z}$ see [Hartshorne]

We can understand $\mathbb{Z}/2\mathbb{Z}$ by viewing the classes as intersection with hyperplane: , the line is double ☑

Conversely, when $\varphi(X)$ determine operator itself?

When X nonsingular, $End(A_*(X)) \cong A^*(X)$ holds!

④ We can define A^* $\Leftrightarrow$ X is pure-dimensional, i.e. $\boxed{\text{the fundamental class}}$ $[X]$ is well-defined

$\Leftrightarrow \exists$ Poincaré dual and trace map

thus in general setting, A^* not defined and the Chern classes are just operators $c_i \cap -$, acting on lower Chow by a "virtual cap product" (cap product also needs Poincaré duality);

⑤ $\boxed{\text{Definition of Chern}}$:

• $c = s^{-1}$ (abstract nonsense), s is Segre class;

although $s(E)$ depend on rank(E) and dim(X), but $c(E)$ not depend on dim(X)

• Cohomology of Grassmannian (Original Chern), complex coefficient;

• Axioms + c_1

Definition of c_1

• Class satisfy Gauss-Bonnet-Chern / as differential form over $\mathbb{C}$

• Splitting principle + $c_1(\mathcal{L})$, axioms above $\Rightarrow$ splitting principle

Definition of $c_1(\mathcal{L})$

• $c_1(\mathcal{O}(D)) = [D]$

• Using exponential sequences $0 \to \mathbb{Z}(1) \to \mathcal{O} \xrightarrow{exp} \mathcal{O}^\times \to 1$

(or Kummer, with étale cohomology)

$\Rightarrow$ LES $\to H^1(\mathcal{O}^\times) \xrightarrow{c_1} H^2(\mathbb{Z}(1)) \to \cdots$

$c_1(\mathcal{L})$ is topological, $[\mathcal{L}] \mapsto c_1(\mathcal{L})$ is bijection for topological $\mathcal{L}$ when we replace $\mathcal{O}^\times$ by fine sheaf of $\mathbb{C}$, but for $\mathcal{O}$ not bijection;

⑥ $\boxed{\text{Chern as obstruction}}$: $c_1(E)$ and $c_{top}(E)$ as obstruction are well-known, intermediate terms $c_k(E)$? $\quad s_1 \wedge \cdots \wedge s_r$

$s_1 \wedge \cdots \wedge s_k$ degeneracy locus D

rank $\Big(\overbrace{\qquad\qquad}^{r}\Big)\}k$ $< k \Leftrightarrow$ all $\binom{r}{k}$ $(k \times k)$-minors has det $= 0$

$\Rightarrow$ "dim" $= r \times k - \binom{r}{k}$ $\cdots$ ①

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But the form is $\begin{pmatrix} \boxed{\text{flexible}} \\ \boxed{S_k} \end{pmatrix}$ $\}k-1$, $\}1$ $\dim = r(k-1)+(k-1) = (r+1)(k-1)$

S_k linear combination of $(k-1)$ coefficients

$\binom{r}{k} > (r-k+1)$ is over-counting defining equation, i.e. these det $=0$ has linear dependent relations (concrete nonsense)

the difference can be formulated in excess intersection:

$\mathcal{O}^{\oplus k} \xrightarrow{(s_1 \dots s_k)^T} E$, $\operatorname{rank}\begin{pmatrix}s_1\\ \vdots \\ s_k\end{pmatrix}_x \le 1$ for $\forall x\in X$

$\Leftrightarrow$ $\forall x \in X$, $\exists V_x \subset E_x$, $\dim V = 1$

s.t. $\mathcal{O}^{\oplus k}|_x \xrightarrow{(s_i)} E|_x \to E|_x/V$ (0)

$\Leftrightarrow D = \pi(\{(x,l)\in Gr(k-1,E) \mid \varphi|_{(x,l)} = 0\})$ (Here Gr is projective notation, thus $k-1$)-dim than k)

where we lift to $Gr(k-1,E) \xrightarrow{\pi} X$

we have sheaves on $Gr(k-1,E)$:

$\mathcal{O}(-1) \to \pi^*E \to Q$

$\uparrow \pi^*(s_i)$, $\pi^*\mathcal{O}^{\oplus k}$ denote composition as φ

vector bundle E	projective $\mathbb{P}(E)$
cutted by sections $s \in E$ or π^*E	cutted by section φ of Q

(Importance of Q)

thus what is it means?

$\operatorname{codim}\{(x,l) \in Gr(k-1,E) \mid \varphi|_{(x,l)}=0\} = \binom{r}{k}$

but $\operatorname{codim} D = r-k+1$ is good

as the excess intersection incoded in the fibre of $Gr(k-1,E) \to X$, cancelled after projection.

this is MacPherson's Graph Construction, choose of order or rows to do Gauss cancellation row by row

the general result is Thom-Porteous's formula

$\cap\, \alpha = \pi_*(c_{top}((\pi^*(\mathcal{O}^{\oplus k}))^\vee \otimes Q) \cdot \pi^*\alpha)$

$c_{r-k+1}(E)$ (Abstract nonsense)

For $k=2$, i.e. $Gr(k-1,E)=\mathbb{P}E$, it explains the graph construction of by rescaling:

⑦ Cones can be concluded by $\boxed{\text{representibility of sheaves}}$

vector bundle $\mathcal{E}: \mathfrak{U} \to \mathrm{Set}$ is represented by its geometric vector bundle E

$U \mapsto \mathcal{E}(U) = \{\text{section of } E|_U\}$, but skyscraper sheaf not representable!

We have $\operatorname{Spec}(\operatorname{Sym} S^1) \subset$ Cones $\operatorname{Spec}(\oplus S^\bullet)$

vector bundles $\subset$ Abelian cones $\subset$ Coherent sheaves

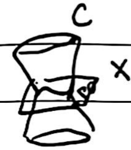
$\operatorname{Spec}(\operatorname{Sym} \mathcal{E}^\vee)$ $\operatorname{Spec}(\operatorname{Sym} \mathcal{F})$ $\mathcal{F}$

note that $\operatorname{Spec}(\operatorname{Sym}\mathcal{F})$ represents $\mathcal{F}^\vee$

$\mathcal{F}^\vee$ has addition and it's Abelian, thus has better representibility.

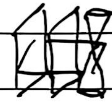
- compare normal cone $\operatorname{Spec}(\oplus I^k/I^{k+1})$
  blow up $\operatorname{Proj}(\oplus I^k)$
  exceptional divisor $\operatorname{Proj}(\oplus I^k/I^{k+1})$
  projectivization

⑧ In Fulton's textbook, $C \oplus 1$ is a $\boxed{\text{notation issue}}$, one set $\mathcal{O} \to S^0$, i.e. supported on a closed subscheme.



but if one define $\mathcal{O}=S^0$, then $C\oplus 1 = C$

such as [Behrend-Fantechi]



all fibres nontriv

⑨ The $\boxed{G_m\text{-action}}$ on cone is by rescaling, algebraically:

$G_m = \operatorname{Spec} k[t,t^{-1}]$, $\mathbb{A}^1 = \operatorname{Spec} k[t]$ (abstract nonsense)

$\operatorname{Spec} R^\bullet \times G_m \to \operatorname{Spec} R^\bullet$

$R^\bullet[t,t^{-1}] \leftarrow R^\bullet$

$\sum f_i t^i \mapsfrom f = \sum f_i$

If $R^\bullet$ is $\mathbb{Z}_{\ge 0}$-graded $\Rightarrow$ extends to $\mathbb{A}^1$-action

⑩ $\boxed{\text{Graph of deformation to normal cone}}$:

deformation to normal cone $\frac{s}{t}$, $t \to 0$

Y (not "" due to algebraically closed)

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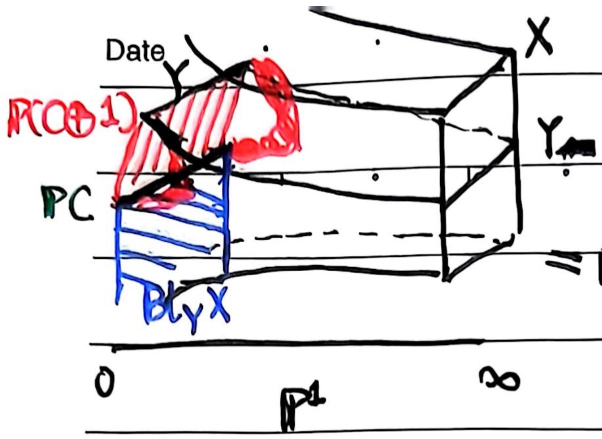


the $\mathbb{P}C$ is infinity of $\mathbb{P}(C\oplus 1)$, $\mathbb{P}(C\oplus 1) - \mathbb{P}C \cong C$

also exceptional divisor of $Bl_Y X$

$\cong Bl_{X\times\{0\}}(Y\times\mathbb{P}^1)$

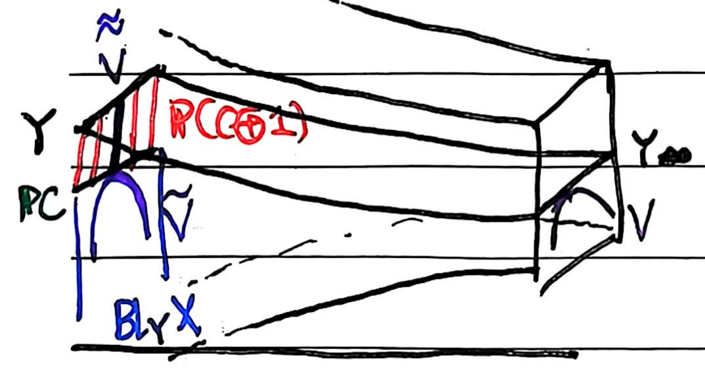
the total exceptional divisor is hole $\mathbb{P}(C\oplus 1) \to$

The $\mathbb{G}_m$-action ~~is~~ are:

the $\mathbb{G}_m$-fixed loci ~~is~~ are three parts:

- $Y\times\{\infty\}$
- $\mathbb{P}C$ the infinity part
- $Bl_Y X$

The intersection of Y with some $V\subset X$:



- V proper transform to $\tilde{V}\subset Bl_Y X$, $\tilde{V}\cong$
- $\tilde{\tilde{V}} = C_{\tilde{V}\cap Y} Y = C_{V\cap Y} Y$ is double line in this picture $\subset$ normal bundle $\mathbb{P}(C\oplus 1) - \mathbb{P}C$ is given by Gysin pullback $0^!$
- Let Y intersect with $\tilde{\tilde{V}}$, we have ~~two~~ double poin

$\Rightarrow$

$0^!_{N_Y X|_{V\cap Y}}(C_{V\cap Y} Y) = (V.Y) \in A^*(V\cap Y)$ $\boxed{//}$

(abstract nonsense)