Selected topics in AG

we using birational way to classify varieties into birational equivalence, and in every class we solving the moduli problem more fine.

Page1

Assume char $\Bbbk = 0$, variety over $\Bbbk$

Curve: smooth, projective $=$ compact Riemannian surface C

Riemann-Roch: $h^0(\mathscr{L}(D)) - h^1(\mathscr{L}(D)) = \deg D + h^0(\mathcal{O}_C)$

$\chi(\mathscr{L}(D)) - h^1(\mathscr{L}(D)) = \deg D + \chi(\mathcal{O}_C)$

The moduli M_g and compactification $\overline{M_g}$, ($M_g \subset \overline{M_g}$)

$\overline{M_g}$ has singularity

(Mumford) $g \geq 23$, $\underline{\dim M_g = 3g-3}$ ($g \geq 1$) and $\overline{M_g}$ is general type variety

Surface: Classification S

Kodaira dimension $\kappa(S) = \begin{cases} -\infty; & \mathbb{P}^2, \text{ ruled surface} \\ 0; & \text{Abelian variety, } K_3 \text{ surface, } E_1\times E_2, E_1 \times E_2 \\ 1; & \text{Elliptic surface, i.e. } S_0 \xrightarrow{f} C \text{ has a generic fibre to be elliptic curve} \\ 2; & \text{the general type} \end{cases}$

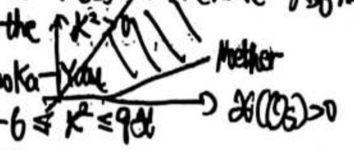
$g=0 \iff C \cong \mathbb{P}^1$ $\quad g=1$, $M_1 \cong \Bbbk$, $[C] \longleftrightarrow j$ (j-invariant)

(MMP) $S \rightsquigarrow S_0$ is the minimal model of S (no (-1)curve)

Describle the singularity

Resolution a singularity, but dominate it to compute its birational invariant

For $\kappa(S)=2$ we have K_{S_0} nef, big



Surface Geography

The moduli $\underline{M_{K^2,\chi}}$ may not irreducible (M_g is irreducible)

Threefold: X

The minimal models of X is X_0 must has singularity

$\kappa(X) = \begin{cases} -\infty \\ 0 \\ 1 \\ 2 \\ 3 \end{cases}$

higher X, $\dim X \geq 4$ more bad. [BCHM]

Setting. S surface, smooth, projective, over $\Bbbk$. $\omega_S = \Omega_S^2$ the canonical sheaf

$\{h^i(\omega_S)\}$ are birational invariant ([Ha] II.8.19, II Ex 8.8)

p_g and q the geometric genus, irregularity

$p_g(S) = h^0(\omega_S) = h^2(\mathcal{O}_S)$, $q(S) = h^1(\omega_S) = h^1(\mathcal{O}_S)$

$\chi(\mathcal{O}_S) = 1 - q + p_g$

$D \geq 0$ a effective divisor. we call D a $\underline{\text{generalised curve on } S}$ (it's a subscheme)

we denote $\mathscr{L}(D)$ as $\mathcal{O}_S(D)$ here)

$0 \to \mathcal{O}_S(-D) = \mathcal{I}_D \to \mathcal{O}_S \to \mathcal{O}_D \to 0$

i.e. $\mathcal{I}_D = \mathcal{O}_S(D)^{\vee}$ (denoted $\mathcal{O}_S(D)^*$ here)

① Normalisation of D irreducible $\underline{\text{reduced}}$, $D>0$, then $\#\mathrm{Sing}(D) < \infty$

($\sum n_i C_i$, $I=1$, $n_i = 1$) for $P \in \mathrm{Sing}(D)$, $\exists\, U = \operatorname{Spec} A \in \mathcal{N}_P$, s.t. $U \cap \mathrm{Sing}(D) = \{P\}$ ($U \subset D$)

then we do normalisation

at $U \Rightarrow \tilde{U} = \operatorname{Spec} \tilde{A}$ smooth (due to when $\underline{\text{dimension } 1}$, normal $\iff$ regular local), then paste, it $\underline{\text{keeps projective}}$ $\tilde{D}$

and $\nu: \tilde{D} \to D$ ($\tilde{D}$ may not contained in S) and it's finite due to $\underline{\text{finite} \iff \text{quasi-finite} + \text{affine}}$ (Trivial) (affine is obvious)

birational ν is obvious

$\Rightarrow 0 \to \mathcal{O}_D \to \nu_*\mathcal{O}_{\tilde{D}} \to \delta \to 0$ due to $\mathcal{O}_D = \nu_*\mathcal{O}_{\tilde{D}}$ except $P \Rightarrow \delta$ is the skyscrapter sheaf at P

② The dualising sheaf ω_D° on D

Lemma. $C = A+B \Rightarrow$

$$\begin{tikzcd}
& & & & 0 \arrow[d] & \\
0 \arrow[d] & 0 \arrow[d] & & & \boxed{\mathcal{K}} \arrow[d] & \\
\mathcal{O}_S(-A) \otimes \mathcal{O}_S(-B) \arrow[r] \arrow[d] & \mathcal{O}_S(-C) \arrow[d] & 0 \arrow[r] & \mathcal{O}_S(-C) \arrow[r] \arrow[d] & \mathcal{O}_S \arrow[r] \arrow[d, equal] & \mathcal{O}_C \arrow[r] \arrow[d] & 0 \\
\mathcal{O}_S \otimes \mathcal{O}_S(-B) \arrow[r] \arrow[d] & \mathcal{O}_S(-B) \arrow[d] & 0 \arrow[r] & \mathcal{O}_S(-B) \arrow[r] \arrow[d] & \mathcal{O}_S \arrow[r] & \mathcal{O}_B \arrow[r] \arrow[d] & 0 \\
\mathcal{O}_A \otimes \mathcal{O}_S(-B) \arrow[r] \arrow[d] & \mathcal{O}_A(-B) \arrow[d] & & \boxed{\mathcal{O}_A(-B)} \arrow[d] & & 0 & \\
0 & 0 & & 0 & & &
\end{tikzcd}$$

Snake Lemma

$\Rightarrow \mathcal{K} \cong \mathcal{O}_A(-B)$ ▢

Coro. Let $A = B$, $C = 2A$ $\Rightarrow \mathcal{O}_A(-A) = \dfrac{\mathcal{I}_A}{\mathcal{I}_A^2}$

Prop. $0 \to \dfrac{\mathcal{I}_C}{\mathcal{I}_C^2} \to \Omega_S^1 \otimes \mathcal{O}_C \to \Omega_C^1 \to 0$ for $C \subset S$ smooth irreducible curve $\Rightarrow \omega_C = \omega_C^\circ$

rank1 rank2 rank1

$\Rightarrow \wedge^2(\Omega_S^1 \otimes \mathcal{O}_C) = \omega_S \otimes \mathcal{O}_C$

$\cong \dfrac{\mathcal{I}_C}{\mathcal{I}_C^2} \otimes \omega_C \overset{\text{Coro}}{=} \mathcal{O}_C(-C) \otimes \omega_C$

$\Rightarrow \omega_C \cong$

$\Rightarrow \mathcal{O}_C(K_C) = \mathcal{O}_C(K_S) \otimes \mathcal{O}_C(C)$

$\Rightarrow \mathcal{O}_C(K_C) = \mathcal{O}_C(K_S + C)$

$K_C \sim (K_S + C)|_C$ (K_C is the canonical divisor, $\mathscr{L}(K_C) = \omega_C$) ▢

then can't defined

Thm. Now we generalise the C generalised curve (now $\omega_C^\circ \neq \omega_C$)

Define. $\omega_C^\circ = \omega_S \otimes \mathcal{O}_C(C)$ is dualizing (verified in [Ha] III. Thm 7.11.)

(RK. This only holds for S surface, C curve) $\Rightarrow H^i(C, \mathscr{F}) \cong H^{1-i}(C, \omega_S \otimes \mathcal{O}_C(C) \otimes \mathscr{F}^\vee)^\vee$

⑤ The degree on generalised curve of $\mathcal{E}$ locally free

When $C \subset S$ irreducible, smooth

then $0 \to \mathbb{Z} \to \hat{\mathcal{O}}_C \xrightarrow{\exp} \hat{\mathcal{O}}_C^* \to 0$ ($\hat{\mathcal{O}}_C$ is the analytic case)

$$\Rightarrow H^1(C,\mathbb{Z}) \to H^1(\hat{\mathcal{O}}_C) \xrightarrow{i} H^1(\hat{\mathcal{O}}_C^*) \xrightarrow{\delta} H^2(C,\mathbb{Z}) \to H^2(\hat{\mathcal{O}}_C)$$

$\Vert$ $\Vert$GAGA $\Vert$GAGA $\Vert$ $\Vert$GAGA

$$H^1(C,\mathbb{Z}) \to H^1(\mathcal{O}_C) \to H^1(\mathcal{O}_C^*) \xrightarrow{\delta} H^2(C,\mathbb{Z}) \to H^2(\mathcal{O}_C)=0$$

$\Vert$ $\Vert$ $\Vert$Grothendieck

$$\boxed{\mathrm{Pic}(C) \xrightarrow{\deg} \mathbb{Z} \to 0}$$

In general, $\begin{cases} \text{singular: normalisation } \nu: \tilde{C} \to C, \text{ let } \deg(\mathcal{E}) = \deg(\nu^*\mathcal{E}) = \deg \det(\nu^*\mathcal{E}) \in \mathrm{Pic}(\tilde{C}) \\ \text{not irreducible: Decompose } C = A+B \Rightarrow \deg \mathcal{E} = \deg(\mathcal{E}|_A) + \deg(\mathcal{E}|_B) \end{cases}$

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Thm (Riemann-Roch)

S smooth projective surface, $C \subset S$ generalised curve

$\mathcal{F}$ is locally free over C, rank r

$\Rightarrow \chi(C,\mathcal{F}) = \sum(-1)^i h^i(C,\mathcal{F}) = \deg \mathcal{F} + r(\chi(\mathcal{O}_C))$

(χ is the Euler characteristic)

Pf. Step 1 When C smooth, irreducible curve

$r=1$, it's classical;

$r>1$, induction

A usual tech: constructing a filtration (in curve)

$\exists H$ a very ample divisor on C, s.t. $H^0(C, H\otimes\mathcal{F}^\vee) \neq 0$

$0 \to \mathcal{O}_C \xrightarrow{\sigma} \mathcal{H} \to \mathcal{Q} \to 0$, $\exists \sigma \in H^0(C, H\otimes\mathcal{F}^\vee)$

$\Rightarrow 0 \to \mathcal{H}^{-1} \to \mathcal{F} \to \mathcal{Q}\otimes\mathcal{H}^{-1} \to 0$ ($\mathcal{H} = H\otimes\mathcal{F}^\vee$)

act by $\mathcal{H}om(\cdot, \mathcal{O}_C)$ then we modify:

$\Rightarrow 0 \to \mathcal{L}_{r-1} \to \mathcal{F} \to \mathcal{M}$ ① We find out the image of $\mathcal{F} \to \mathcal{M}$ to give surjectivity

$\Rightarrow 0 \to \mathcal{L}_{r-1} \to \mathcal{F} \to \mathcal{L}_1 \to 0$ ② Expanding the $\mathcal{L}_{r-1}$ to let the quotient $\mathcal{M}$ locally free.

thus we reduce $\mathcal{F}$ to $\mathcal{L}_{r-1}$ and $\mathcal{L}_1$

$\mathrm{Ker}(\deg) = J(C)$ the Jacobi variety is $\mathbb{C}^g/\Lambda$ torus

by ([Ha] P.222) $\mathrm{Ker}\,\delta = \mathrm{Pic}^0(C)$ the ~~group~~ Picard group variety

the $\mathrm{Pic}^0(C)$ and $J(C)$ is the components containing identity in $H^1(\mathcal{O}_C^*)$ and $\mathrm{Pic}(C)$ seperately) $\Rightarrow \mathrm{Pic}^0(C) \cong J(C)$

then $\det \mathcal{F} = \Lambda^r \mathcal{F} = \Lambda^{r-1}\mathcal{L}_{r-1} \otimes \Lambda^1 \mathcal{L}_1$

$\Rightarrow \deg \mathcal{F} = \deg \mathcal{L}_1 + \deg \mathcal{L}_{r-1}$, then applying to $\deg \mathcal{L}_{r-1}$ //

Step 2. If C not smooth

$\nu: \tilde{C} \to C$ is finite $\Rightarrow$ affine $\Rightarrow R^i\nu_*\mathcal{O}_{\tilde{C}} = 0$

$\Rightarrow$ the cohomology of $\tilde{C}$ and C, $\nu_*(\mathcal{G})$ and $\mathcal{G}$ are same

$\Rightarrow \chi(\nu^*\mathcal{F}) = \chi(\nu_*\nu^*\mathcal{F}) = \chi(\mathcal{F}\otimes\nu_*\mathcal{O}_{\tilde{C}})$

and $\chi(C,\mathcal{F}) = \boxed{\chi(\mathcal{F}\otimes\nu_*\mathcal{O}_{\tilde{C}})} - r\chi(C,S)$. by the

$0 \to \mathcal{O}_C \to \nu_*\mathcal{O}_{\tilde{C}} \to S \to 0 \Rightarrow 0 \to \mathcal{F} \to \mathcal{F}\otimes\nu_*\mathcal{O}_{\tilde{C}} \to S^r \to 0$

$\Rightarrow \chi(C,\mathcal{F}) = \deg\nu^*\mathcal{F} + r\chi(\mathcal{O}_{\tilde{C}}) - r\chi(C,S) = \deg\mathcal{F} + r\chi(\mathcal{O}_C)$ //

Step 3 Generally, let $C = A+B$, $A, B > 0$, assume R-R holds for A & B

then by $0 \to \mathcal{O}_A(-B) \to \mathcal{O}_C \to \mathcal{O}_B \to 0 \Rightarrow \chi(\mathcal{F}) = \chi(\mathcal{F}\otimes\mathcal{O}_A(-B)) + \chi(\mathcal{F}\otimes\mathcal{O}_B)$

$0 \to \mathcal{F}\otimes\mathcal{O}_A(-B) \to \mathcal{F} \to \mathcal{F}\otimes\mathcal{O}_B \to 0$ using R-R to both A, B: $\deg(\mathcal{F}\otimes\mathcal{O}_A(-B)) + r\chi(\mathcal{O}_A) + \deg(\mathcal{F}\otimes\mathcal{O}_B) + r\chi(\mathcal{O}_B)$

due to $\Lambda^r(\mathcal{F}|_A\otimes\mathcal{O}_A(-B))$

$= \Lambda^r(\mathcal{F}|_A)\otimes\mathcal{O}_A(-B)^{\otimes r}$

$\Rightarrow \chi(\mathcal{F}) = \deg(\mathcal{F}|_A) + r\deg(\mathcal{O}_A(-B)) + r\chi(\mathcal{O}_A)$

$+ \deg(\mathcal{F}|_B) + r\chi(\mathcal{O}_B) = \deg\mathcal{F} + r\chi(\mathcal{O}_C)$

② $C_1 \sim C_2 \Rightarrow (C_1 \cdot D) = (C_2 \cdot D)$ ⑤ $(0 \cdot D) = 0$

③ $(D\cdot(D_1+D_2)) = (D\cdot D_1) + (D\cdot D_2)$ ④ $(D_1\cdot D_2) = (D_2\cdot D_1)$

(This axiom may differ from complex Geometry: due to here we'll use the R-R to build:

Thm. If exist $\Rightarrow$ unique

Pf. $\forall D_1, D_2 \in \mathrm{Div}(S)$, $H \in \mathrm{Div}(S)$ is very ample

s.t. n sufficent large, D_1+nH, D_2+nH, nH very ample

take a general member (Bertini's thm)

$C_1 \in |D_1+nH|$, $C_2 \in |D_2+nH|$, $C_1', C_2' \in |nH|$

s.t. C_1, C_2, C_1', C_2' normal crossing

$\Rightarrow D_1 \sim C_1 - C_1'$, $D_2 \sim C_2 - C_2'$

$\Rightarrow (D_1\cdot D_2) = \#(C_1\cap C_2) - \#(C_1\cap C_2') - \#(C_1'\cap C_2) + \#(C_1'\cap C_2')$

is unique ▨

Thm. Exist !

Lemma. $(C\cdot D) = \deg\mathcal{O}_C(D)$, C generalised curve.

Pf. $0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$ $D \in \mathrm{Div}(S)$

$0 \to \mathcal{O}_S(-C-D) \to \mathcal{O}_S(-D) \to \mathcal{O}_C(-D) \to 0$

We define $\mathcal{L} = \mathcal{O}_S(C)$, $\mathcal{M} = \mathcal{O}_S(D)$

and $(C\cdot D) = \chi(\mathcal{O}_S) - \chi(\mathcal{L}^{-1}) - \chi(\mathcal{M}^{-1}) + \chi(\mathcal{L}^{-1}\otimes\mathcal{M}^{-1})$

$\Rightarrow (C\cdot D) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) - [\chi(\mathcal{O}_S(-D)) - \chi(\mathcal{O}_S(-C-D))]$

$= \boxed{\chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C))} - \chi(\mathcal{O}_C(-D))$

$= \chi(\mathcal{O}_C) - \chi(\mathcal{O}_C(-D)) = -\deg\mathcal{O}_C(-D) = \deg\mathcal{O}_C(D)$ ▨

Exercise 1.1. Prove that: $P_m(S) = h^0(mK_S)$ is birational invariant for $m \geq 0$ (For S a variety)

1.2. On a curve C, Recall: special divisor, Clifford thm ([Ha] IV)

Prove that: (1) if $g(C) \geq 2$ and $h^0(C,D) \geq 2 \Rightarrow \deg D \geq 2$

and give a ness & suff condition of the equality holds

(2) $\deg D \geq 2$, C is elliptic curve $\Rightarrow \varphi_{|K_C+D|}$ is birational.

1.3. $P \in C$, a point on a curve $\Rightarrow h^0(P) = 2 \Leftrightarrow C \cong \mathbb{P}^1$

The intersection theory on S: $\mathrm{Div}(S)\times\mathrm{Div}(S) \xrightarrow{(\cdot)} \mathbb{Z}$

s.t. (axiom way) ① C_1, C_2 integral ($\Leftrightarrow$ irreducible + reduced), $C_1 \neq C_2$

$\#(C_1\cap C_2) < \infty$ (with multiplicity counted) $\Rightarrow (C_1\cdot C_2) = \#(C_1\cap C_2)$

$f_1 f_2$ is the local equation of C_1 and C_2 $= \sum_{P\in C_1\cap C_2} \dim_k \mathcal{O}_{S,P}/(f_1,f_2)$

In particular, when simple normal crossing (= transitivity) $\Rightarrow (C_1\cdot C_2) = \#(C_1\cap C_2)$

without multiplicity

Pf of Thm (Existence) ①③④ obvious; ② $D_1, D_2, D_3 \in \mathrm{Div}(S)$ $(D_1.(D_2+D_3))-(D_1.D_2)-(D_1.D_3) = \chi(\mathcal{O}_S)+\chi(\mathcal{O}_S(-D_1))+\chi(\mathcal{O}_S(-D_2))$ Page 3
$+\chi(\mathcal{O}_S(-D_3)) - \chi(\mathcal{O}_S(-D_1-D_2)) - \chi(\mathcal{O}_S(-D_1-D_3))$
$-\chi(\mathcal{O}_S(-D_2-D_3)) + \chi(\mathcal{O}_S(-D_1-D_2-D_3))$

by symmetry $\Rightarrow$ if any of D_1, D_2, D_3 is a curve (effective), by the Lemma, we complete ②;

In general, let $D_1 \sim C_2 - C_1$, $C_2, C_1 \geq 0$

$\Rightarrow C_2 \sim D_2 + C_1 \Rightarrow (D_2 . C_2) = (D_2.(D_1+C_1)) = (D_2.D_1) + (D_2 . C_1) \Rightarrow (D_2 . D_1) = (D_2 . C_2) - (D_2 . C_1)$

$\Rightarrow (D_1.(D_2+D_3)) = (C_2.(D_2+D_3)) - (C_1.(D_2+D_3)) = (C_2.D_2)+(C_2.D_3) - (C_1.D_2) - (C_1.D_3) = (D_1.D_2)+(D_1.D_3)$ ☑

④ $0 \to \mathcal{O}_{C_1}(-C_2) \to \mathcal{O}_{C_1} \to \mathcal{O}_{C_1 \cap C_2} \to 0$

$(C_1.C_2) = \deg \mathcal{O}_{C_1}(C_2) = -\chi(\mathcal{O}_{C_1}(-C_2)) + \chi(\mathcal{O}_{C_1}) = \chi(\mathcal{O}_{C_1 \cap C_2}) = h^0(\mathcal{O}_{C_1 \cap C_2}) = \sum_{p \in C_1 \cap C_2} \dim \frac{\mathcal{O}_{S,p}}{(f_1, f_2)} = \#(C_1 \cap C_2)$ ☑

Rk. $(f^*C, f^*D) = (C, D)$ for $f: S \to T$ surjective

Then we induce $(\cdot\,.\,\cdot): \mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathbb{Z}$

Def. $D \in \mathrm{Div}(S)$ is nef (numerically effective) if $\forall E \in \mathrm{Div}(S)$ and $E \geq 0 \Rightarrow (D.E) \geq 0$

E.g. ① Ample divisor is nef.

② $\mathbb{P}^2$: $K_{\mathbb{P}^2} \sim -3L$, $L^2 = 1$, $L \cdot D = 1 \in \mathrm{Pic}\,\mathbb{P}^2 = \mathbb{Z}$

they're all nef. (We denote $(L.L) = L^2$).

③ $\mathbb{P}^1 \times \mathbb{P}^1$; L, M the generator

$\Rightarrow L^2 = M^2 = 0$, $(L, M) = 1$

$D_1 = aL + bM$, $D_2 = cL + dM \Rightarrow (D_1, D_2) = ad + bc$

(in a family of divisors)

① Néron–Severi thm: algebraic equivalence (I Hail, II V)

$NS(S) = \underline{\mathrm{Pic}(S)}/\sim_{alg} \Rightarrow \mathrm{rank}\, NS(S) < \infty$ the free part

and it's f.g. $\rho(S) = \mathrm{rank}\, NS(S)$ the Picard number

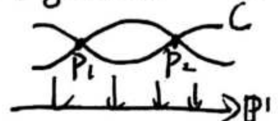
It's not a birational invariant

② Numerical equivalence $D_1 \equiv D_2 \Leftrightarrow \forall D \in \mathrm{Div}(S)$ $(D, D_1) = (D, D_2)$

$\mathrm{Num}(S)_{\mathbb{R}} = \underline{\mathrm{Pic}(S)}/\equiv \otimes \mathbb{R}$

tensoring $\mathbb{R}$ to annihilate the torsion part $\Rightarrow$ E.g. Genus 2 curve

"$\sim_{lin}$" $\Rightarrow$ "$\sim_{alg}$" $\Rightarrow$ "$\equiv$"



deg f times P_1, P_2 the Weierstrass point

$P_1 \nsim P_2$ but $2P_1 \sim 2P_2$

Chow group generalizes the Picard group (all 0, 1 ⋯ cycles).

lower higer dimension

arithmetic hard to study.

Prop. $f: S \to T$ surj (then f is generally finite) then $\begin{cases} ① (f^*D_1, f^*D_2) = \deg f (D_1, D_2) \\ ② (f^*D . E) = 0 \text{ if } E \in \mathrm{Div}(S) \text{ and } f(E) = 0 \text{ (using ①)} \end{cases}$

without proof

Thm. (Riemann–Roch of S) $\chi(D) = \frac{1}{2}(D.(D-K_S)) + \chi(\mathcal{O}_S)$

Pf. $(D.(D-K_S)) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-D)) - \chi(\mathcal{O}_S(K_S-D)) + \chi(\mathcal{O}_S(K_S)) = 2(\chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(D)))$ Serre duality

$\Rightarrow \chi(\mathcal{O}_S(D)) = \frac{1}{2}(D.(D-K_S)) + \chi(\mathcal{O}_S)$ ☑

Cor. When $D = mK_S$: $\chi(\mathcal{O}_S(mK_S)) = \frac{1}{2}m(m-1)K_S^2 + \chi(\mathcal{O}_S)$

$h^0(mK_S) - h^1(mK_S) + h^2(mK_S)$; birational $m \geq 0$; Kodaira vanishing 0; Serre duality $h^0((1-m)K_S) = 0$ when $K_S \geq 0$; when K_S ample, $m \geq 2$; birati[onal]

Cor. When S is minimal, K_S ampl[e] $\Rightarrow K_S^2$ is birational invariant.

When C's picture has "✕" $\Rightarrow h^0(\mathcal{O}_C) \geq n$.

$\chi(\omega_C^0) = \deg \omega_C^0 + \chi(\mathcal{O}_C)$ ✕ ✕ $\underbrace{\qquad}_{n \text{ times "✕"}}$

$-\chi(\mathcal{O}_C)$

$\Rightarrow \deg \omega_C^0 = -2\chi(\mathcal{O}_C)$ ($\omega_C^0 \neq \omega_C$ due to here not smooth)

$= 2p_a(C) - 2$ (Compare: when smooth: $\deg \omega_C = 2g(C) - 2$)

then we consider $h^0(\mathcal{O}_C)$ and $h^0(\omega_C^0) = h^1(\mathcal{O}_C)$ together, studing $p_a(C)$

Lemma. C integral (reduced, irreducible) curve $\Rightarrow p_a(C) \geq p_g(C) = p_g(\tilde{C})$ (This Lemma is for Hodge index thm later)

equality holds $\Leftrightarrow$ C smooth

Pf. $0 \to \mathcal{O}_C \to \nu_*(\mathcal{O}_{\tilde{C}}) \to S \to 0 \Rightarrow \chi(\mathcal{O}_C) = \chi(\nu_*(\mathcal{O}_{\tilde{C}})) = \chi(\mathcal{O}_{\tilde{C}}) + \chi(S)$

$\Rightarrow p_a(C) = p_g(C) + h^0(S) \geq p_g(C)$ ☑ $\quad 1 - p_g(C)$ $\quad = 1 - p_a(C) + h^0(S)$

$\Rightarrow$ by definition $p_g(C) = p_g(\tilde{C}) = p_a(\tilde{C})$

Exmp. $\omega_C^0 = \mathcal{O}_C(K_S + C)$, $C > 0 \Rightarrow \deg \omega_C^0 = 2p_a(C) - 2$

Exercise.

$(C.(K_S+C)) \Rightarrow \boxed{p_a(C) = 1 + \frac{1}{2}(C.(K_S+C))}$

2.1. $C = C_1 + C_2 + \cdots + C_r$

reduced (but not irreducible), if $p_a(C) = 0 \Rightarrow p_a(C_i) = 0$

and (1) $i \neq j$, $(C_i, C_j) \leq 1$ (2) C's intersection dualizing diagram (Dynkin diagram)

(3) Not exist ≥ 3 C_i, C_j, C_k have no round

intersect at 1 point.

$\Rightarrow$ if (1 2 3) satisfied, we call it a rational tree

Rk (Dynkin diagram) E.g. C_1 C_2 $\Rightarrow$ C_1 — C_2

Lemma. S smooth projective surface, H ample divisor, $D \in \mathrm{Div}(S)$, s.t. $D^2>0$, $(D.H)>0 \Rightarrow n>>0$, $nD \sim$ a nontrivial effective divisor and $h^0(nD) \approx O(n^2)$

Pf. $\chi(nD) = h^0(nD) - h^1(nD) + h^2(nD) \overset{\text{Riemann-Roch}}{=} \frac{1}{2}(nD)(nD-K_S) + \chi(\mathcal{O}_S) = \frac{1}{2}D^2 \cdot n^2 - \frac{1}{2}n(D.K_S) + \chi(\mathcal{O}_S)$

and $h^2(nD) \overset{\text{Serre}}{=} h^0(K_S - nD) \xrightarrow{n>>0} 0$ $\quad = O(n^2)$ (By taking section $\frac{1}{2}n^2D^2) = \frac{n^2}{2}\underbrace{h^0(D^2)}_{>0} \to n^2$)

due to $H.(K_S - nD) = \underline{H.K_S} - n\underline{(H.D)} \to 0$

$\Rightarrow h^0(nD) = h^1(nD) + O(n^2) \overset{\geq 0}{=} O(n^2)$ ▨

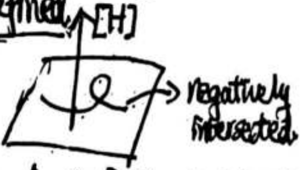
Thm (Hodge index) S, H, D setted, $(H.D)=0 \Rightarrow D^2 \leq 0$, in particular $D^2 = 0 \Leftrightarrow D \equiv 0$

Pf. Assume $D \not\equiv 0$ and $\underline{D^2 \geq 0}$ $\begin{cases} D^2>0 \text{ Let } H' = D + nH \ (n>>0 \text{ s.t. ample}), (H'.D) = D^2 > 0 \Rightarrow \exists m \text{ large}: mD>0 \\ \text{but } (H.mD) = 0 \Rightarrow mD \leq 0, \text{ contradiction ▨} \\ D^2 = 0 \text{ Let } E' = (H^2.E) - (H.E)H, \text{ then we use } D' \text{ reduce to } D'^2>0 \text{ case} \end{cases}$

$D' = E' + nD$ $\quad \begin{cases} ① D'^2>0: D'^2 = 2n(D.E') + E'^2 > 0 \\ ② (D'.H) = 0: D'.H = E.H = 0 \text{ ▨} \end{cases}$

We base change

$\mathrm{Num}(S) = NS(S) \otimes \mathbb{R}$ then turn it to matrix, called the intersection matrix is negatively defined



$\Rightarrow$ **Coro.** Replace H ample by $H^2>0$ also holds ▨

Ex. D nef, $E \geq 0 \Rightarrow (D.E)^2 \geq D^2 . E^2$

Def. C is generalized curve, $C = A+B$ any decomposition $A>0, B>0$. s.t. $(A.B) \geq n \Rightarrow C$ is $\underline{n\text{-connected}}$

E.g. ① Irreducible reduced curve is $\forall n$-connected ② Connected reduced is 1-connected ③ Nef. big is 1-connected

Rk. 1-connected is useful condition, and nef. big is its generalisation in higher dimension) (Nef and $C^2>0$)

Prop 2. C is 1-connected, $L \in \mathrm{Div}(C)$, if $\forall i$, $(L.C_i) = 0$ ($C = \sum m_i C_i$) $\quad$ Ex. Check ③, using Hodge index

$\Rightarrow h^0(S, \mathcal{L}_0) \leq 1$ and $\quad$ ($\mathcal{L}$ is associated invertible sheaf $\mathcal{L}_0 = \mathcal{O}_S(L)$) $\quad$ **Coro.** $D>0$ and 1-connected

$h^0(C, \mathcal{L}_0) = 1 \Leftrightarrow \mathcal{L} \cong \mathcal{O}_C$ $\quad \Rightarrow h^0(\mathcal{O}_D) = 1$.

Pf. Assume $h^0(C, \mathcal{L}_0) > 0$, take $0 \neq h \in h^0(C, \mathcal{L}_0)$, let $D_1 \leq C$ the biggest vanishing locus of h, i.e. $h|_{D_1} \equiv 0$

$\Rightarrow 0 \leq D_1 < C \Rightarrow C = D_1 + D_2$, then:

$$\begin{array}{ccccccccc} 0 \to & \mathcal{O}_{D_2}(-D_1) & \to & \mathcal{O}_C & \to & \mathcal{O}_{D_1} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathcal{L} \otimes \mathcal{O}_{D_2}(-D_1) & \to & \mathcal{L}_0 & \to & \mathcal{L} \otimes \mathcal{O}_{D_1} & \to 0 \\ & h & \mapsto & h & \mapsto & 0 & \end{array}$$

$\Rightarrow h \in H^0(\mathcal{L}_0 \otimes \mathcal{O}_{D_2}(-D_1))$

and $0 \to \mathcal{O}_{D_2} \xrightarrow{(\cdot h)} \mathcal{L}_0 \otimes \mathcal{O}_{D_2}(-D_1) \to Q \to 0$

$\Rightarrow \chi(\mathcal{L}_0 \otimes \mathcal{O}_{D_2}(-D_1)) = -(D_1.D_2) + \chi(\mathcal{O}_{D_2})$ $\quad \overset{\shortparallel}{\chi(\mathcal{O}_{D_2}) + h^0(Q)}$

$\Rightarrow (D_1.D_2) = -h^0(Q) \leq 0 \Rightarrow \begin{cases} D_1 = 0 \\ D_2 = C \\ Q = 0 \end{cases}$

$\Rightarrow \mathcal{O}_C \cong \mathcal{L}_0$

Conversely, if $\mathcal{L}_0 \cong \mathcal{O}_C$

For each i, $h^0(\mathcal{O}_C) \to h^0(\mathcal{O}_{C_i}) \cong k$

We know that a nonzero global section $s \neq 0$, $s|_{C_i} \neq 0$, thus $h^0(\mathcal{L}_0) = h^0(\mathcal{O}_C) = k$ ▨

Coro. If C 1-connected, $p_a(C) \geq 0$

Pf. $p_a(C) = 1 - \chi(\mathcal{O}_C) = 1 - h^0(\mathcal{O}_C) + h^1(\mathcal{O}_C) = h^1(\mathcal{O}_C)$

$= h^0(\omega_C^{\circ}) \geq 0$ ▨ $\quad$ (We need a $\underline{\text{numerical criterion}}$ for $\underline{\text{Fujita}}$ ~~[illegible]~~ $\underline{\text{conjecture}}$, but in $\underline{\text{higher dim no}}$)

Thm (Nakai-Moishezon criterion on ample) $D^2>0$ and $\forall C>0$ $(D.C)>0 \Leftrightarrow D$ ample.

Pf. Let $C = H$ ample $\Rightarrow (D.H)>0 \Rightarrow nD>0$ for $n>>0$ (by the Lemma upper), replace D by nD, we assume $\underline{D \geq 0}$:

Set $\mathcal{L}_0 = \mathcal{O}_S(D)$ $\boxed{\text{Step 1}}$ $\mathcal{L}_0 \otimes \mathcal{O}_D$ is ample over D, by $\underline{\text{Hartshorne III Ex5.7}}$ only prove $\mathcal{L}_0 \otimes \mathcal{O}_{D_{red}}$ ample in $D_{red} = \{D_i\}$ the irreducible component $\Rightarrow$ reduced to $\underline{\text{irreducible reduced case}}$, by normalization $\tilde{D} \xrightarrow{\nu} D$ $\deg(\nu^*\mathcal{L}_0) = (L.D) > 0 \to$ reduced to $\underline{\text{smooth case}}$

$\nu^*\mathcal{L} \leftarrow \mathcal{L}_0 \Rightarrow$ ~~[illegible]~~ $\mathcal{L}_0 \otimes \mathcal{O}_D$ ample

$\Rightarrow \mathcal{L}_0^n \otimes \mathcal{O}_D$ gbgs ($n>>0$)

$\boxed{\text{Step 2}}$ $n>>0$, $|nD|$ is base point free

$$\begin{array}{l} \forall n: 0 \to \mathcal{L}_0^{-1} \to \mathcal{O}_S \to \mathcal{O}_D \to 0 \\ \qquad\qquad \downarrow \qquad \downarrow \qquad \downarrow \\ \quad 0 \to \mathcal{L}_0^{n-1} \to \mathcal{L}_0^n \to \mathcal{L}_0^n \otimes \mathcal{O}_D \to 0 \end{array}$$

$\Rightarrow \boxed{H^0(\mathcal{L}_0^n \otimes \mathcal{O}_D)} \to H^1(\mathcal{L}_0^{n-1}) \to H^1(\mathcal{L}_0^n) \to \boxed{H^1(\mathcal{L}_0^n \otimes \mathcal{O}_D) = 0}$ $n>>0$

$\Rightarrow h^1(\mathcal{L}_0^n) \leq h^1(\mathcal{L}_0^{n-1}) \Rightarrow n>>0$, $h^1(\mathcal{L}_0^n) = h^1(\mathcal{L}_0^{n+1}) = \cdots$

$\Rightarrow H^0(\mathcal{L}_0^n \otimes \mathcal{O}_D) = 0 \Rightarrow H^0(\mathcal{L}_0^n) \to H^0(\mathcal{L}_0^n \otimes \mathcal{O}_D)$ surjective $\Rightarrow \mathrm{Bs}|nD| = \phi$ base point free

$\boxed{\text{Step 3}}$. Complete the proof: D ample.

$\Phi_{|nD|}: S \to \mathbb{P}^N$ isn't a pencil due to $(nD)^2>0 \xRightarrow{(II.8.7)} \Phi_{|nD|}$ is generically finite $\xrightarrow[\text{factorization}]{\text{Stein}}$

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & \Sigma \subset \mathbb{P}^N \\ & f \searrow \quad \nearrow g & \\ & T & \end{array}$$

g finite. T normal, $f_*\mathcal{O}_S = \mathcal{O}_T$

$\Rightarrow nD = \varphi^*(H) = f^*(g^*H)$. I $\underline{\text{claim}}$, $\underline{f \text{ is isomorphism}}$ then $nD = g^*H$ is ample by g finite $\Rightarrow D$ ample

(H ample in $\mathbb{P}^N$)

First the fibre is points

and $\mathcal{O}_{T,t} \cong \mathcal{O}_{S,s} \Rightarrow f$ isomorphism ▨

Hodge thm. Setting, $b_p = \dim H^p_{dR}(X)$ betti-number [Page 5]

De Rham cohomology over $\mathbb{C}$: smooth projective $\Leftarrow$ compact Kähler
$\mathcal{A}^{p,q}$ the (p,q)-differential form sheaf, $H^{p,q}(X) = \dfrac{\mathrm{Ker}(\mathcal{A}^{p,q} \xrightarrow{\bar\partial} \mathcal{A}^{p,q+1})}{\mathrm{Im}(\mathcal{A}^{p,q-1} \xrightarrow{\bar\partial} \mathcal{A}^{p,q})}$ (over $\mathbb{R}$)
Ω^p the p-holomorphic differential form sheaf $\Rightarrow$ define $H^q(X, \Omega^p)$ (sheaf cohomology)
Thm. $H^{p,q}_{\bar\partial}(X) \cong H^q(X, \Omega^p)$ ($d: \Omega^p \to \Omega^{p+1}$ is not related) $\Rightarrow$ but we used it to prove the Hodge
Thm. (Hodge) $H^i_{dR}(X) = \bigoplus_{p+q=i} H^{p,q}(X)$, $\dim H^{p,q}(X) = h^{p,q}$ the Hodge number, $h^{p,q} = h^{q,p}$, $h^{p,q} = h^{n-p, n-q}$
$b_1 = h^{1,0} + h^{0,1} = 2q = 2h^1(\mathcal{O}_S)$; $b_2 = h^{2,0} + h^{0,2} + h^{1,1} = 2p_g(S) + h^{1,1}$
the second Chern class ~ $H^1(\mathcal{O}_S^*) = \mathrm{Pic}(S) \subset H^{1,1} \Rightarrow \rho(S) \le h^{1,1}$
$c_2(S) = \underbrace{\chi_{top}(S)}_{\text{topological Euler-Poincaré char.}} = b_0 - b_1 + b_2 - b_3 + b_4$
$= 2 - 4q + 2p_g(S) + h^{1,1}$
(not birational invariant)

later, we'll introduce the Noether's formula $(K_S)^2 + c_2 = 12\chi(\mathcal{O}_S)$

Rk. (Higher ample criterion)
$\mathrm{Num}(X) \times N_1(X) \Rightarrow$ {numerical equivalence classes space} (set of curves)
(over $\mathbb{R}$) $\overline{NE}(X)$ is a cone

Kleiman criterion.
① D ample $\Leftrightarrow$ $D > 0$ on $\overline{NE}(X) - \{0\}$
② D nef $\Leftrightarrow$ $D \ge 0$ on $\overline{NE}(X)$

Pencil. For D big, take basis $s_0, \dots, s_n \in H^0(\mathcal{O}_X(nD))$
$\Rightarrow X \xrightarrow{g} \mathbb{P}^n$ the image is Σ
$P \mapsto [s_0(P) : \dots : s_n(P)] \neq 0$
$\Rightarrow$ use Stein decomposition to $g: W \to$

$$\begin{array}{ccc} X & \dashrightarrow & \Sigma \\ \uparrow & \nearrow & \\ W & & \end{array}$$

Hironaka $\Rightarrow$ W smooth
$\Rightarrow W \xrightarrow{g} \Sigma$, $f \downarrow$ $\boxed{Z}$ $\nearrow$ finite; Z normal projective
$\dim Z = 1 \Rightarrow |nD|$ composed of a pencil: $Z \cong \mathbb{P}^1 \Rightarrow$ rational; $g(Z) \ge 1 \Rightarrow$ not rational
$f_* \mathcal{O}_W = \mathcal{O}_Z \Rightarrow f$ has fibre connected.
(By consider when not, taking $\lim$ of higher image gives a contradiction [Ha])

Vanishing thm

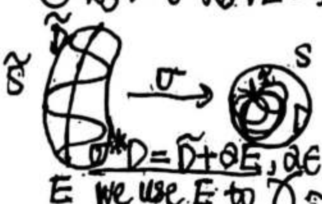
Kodaira's: X variety smooth projective, $\mathcal{L}$ ample $\Rightarrow H^i(X, \omega_X \otimes \mathcal{L}) = H^i(X, \mathcal{L}^{-1})^\vee = 0$
Ramanujam's: S surface, $D \ge 0$ s.t. ① D 1-connected, ② Big ($\exists n > 0$, $h^0(nD) \ge 2$), ③ $|nD|$ isn't non-rational curve family (pencil)
$\Rightarrow H^1(S, K_S + D) = 0$ Pf. [BHPV] ▨ (When big & nef $\Rightarrow$ ①②③)
Ramanujam's': S, $D > 0$, $D \in \mathrm{Div}(S)$ 1-connected, $D^2 > 0 \Rightarrow H^1(S, K_S + D) = 0$
Mumford's: $D \in \mathrm{Div}(S)$ nef & big $\Rightarrow H^1(K_S + D) = 0$ Pf. Covering space ▨
Kawamata–Viehweg's: (Higher generalization of Mumford) $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$ ("$\otimes \mathbb{Q}$" or "$\otimes \mathbb{R}$" is must for higher dim)
nef & big ($D^n > 0$, $n = \dim X$), let $\mathrm{supp}\{\langle D \rangle\}$ the support of fractional part, it's simple normal crossing
$\Rightarrow H^i(X, K_X + \ulcorner D \urcorner) = 0$ (When $\otimes \mathbb{Z}$, it holds auto) (even $\mathbb{Z}$)
Here the notation $a \in \mathbb{Q}$, ① $\llcorner a \lrcorner = [a]$ the round down
② $\ulcorner a \urcorner = -[-a]$ round up
And for $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$
$D = \sum a_i D_i$ ③ $\langle a \rangle = a - [a]$ the fractional part
$\llcorner D \lrcorner = \sum \llcorner a_i \lrcorner D_i$; $\ulcorner D \urcorner = \sum \ulcorner a_i \urcorner D_i$; $\langle D \rangle = \sum \langle a_i \rangle D_i$

This is powerful and meaningful even in complex geometry, using Divisor corresponds to 1-form, using mertic we have a similiar analogue.

The only non-trivial restriction is SNC, when "✕" happens, even blow up at P can't solve due to $H^i(X, K_X + \ulcorner D \urcorner)$ isn't preserved.
The right way is the multiplier ideal sheaf $\mathcal{J}_D$ see left.
(Rk. For $\dim X = 2$ surface, SNC can be simply removed (Sakai))

Nadel's (multiplier ideal sheaf). $H^i(X, \mathcal{O}_X(K_X + \ulcorner D \urcorner) \otimes \mathcal{J}_D) = 0$ Pf. Use Kawamata–Viehweg ▨

MMP of surface. Blow up. $\sigma: \widetilde{S} \to S$ center at x. birationally
① $\sigma^{-1}(x) = E \cong \mathbb{P}^1$; $\sigma|_{\widetilde{S} - E}$ is isomorphism
$\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$;
② $K_{\widetilde{S}} \sim \sigma^* K_S + E \Rightarrow \mathrm{Pic}(\widetilde{S}) = \mathrm{Pic}(S) \oplus \mathbb{Z} \Leftarrow 0 \to \mathbb{Z} \to \mathrm{Pic}(\widetilde{S}) \xrightarrow{\sigma_*} \mathrm{Pic}\, S \to 0$ (σ^*)
$NS(\widetilde{S}) = NS(S) \oplus \mathbb{Z} \Rightarrow \rho(\widetilde{S}) = \rho(S) + 1$
(Recall, Picard number is rank)



$\widetilde{D} = \overline{\sigma^{-1}(D - \{x\})}$ the strict transform
$\sigma^* D = \widetilde{D} + aE$, $a \in \mathbb{Z}$
E we use E to ... several times
$\Rightarrow a$ is called the multiplicity at x $\Rightarrow$ By [Ha] Chap I, §5, Ex, write the local equation at x, $f(u, v) = 0$
and $a = (\widetilde{D}, E) \ge 0$
a is the minium degree of f.

Minimal. Def. if $\pi: S \dashrightarrow S_1$ birational, S_1 smooth
$\Rightarrow \pi$ is isomorphism $\Rightarrow S$ is minimal algebraic surface (E.g. Singular S isn't minimal by blow up)
When S not minimal. $\exists \pi: S \to S_1$, $\pi_* \mathcal{O}_S = \mathcal{O}_{S_1}$, ① if $\forall p \in S_1$, $|\pi^{-1}(p)| < \infty \Rightarrow \pi^{-1}(p) = Q$ single $\Rightarrow \pi$ isomorphism at Q
[By $\mathcal{O}_{S_1,P} \subset \mathcal{O}_{S,Q} \subset K(S)$, and $\mathcal{O}_{S_1,P}$, $\mathcal{O}_{S,Q}$ normal
$\Rightarrow \mathcal{O}_{S_1,P}$, $\mathcal{O}_{S,Q}$ algebraically closed in $K(S) \Rightarrow \mathcal{O}_{S_1,P} = \mathcal{O}_{S,Q}$ ▨]
② Otherwise $\exists P_0 \in S$, s.t. $\pi^{-1}(P_0)$ is curve connected $\supset L$
$L \to P_0$, take H ample on S_1, $\pi^* H$ nef & big $\Rightarrow (\pi^* H, L) = 0$
$\Rightarrow$ By Hodge index, $L^2 < 0$.
Conclude that: if not minimal $\Rightarrow \exists$ a divisor $L^2 < 0$ E.g. $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ is minimal
(It's not a good criterion).
Thm. (Castelnuovo) S minimal $\Leftrightarrow$ No (-1)-curve over S (Recall, (-1) curve C is $(C, C) = -1$)

For proving this hard thm, we prepare.

The Graph of birational map. $\varphi: X \dashrightarrow Y$

E.g. When φ is blow up a point $\Rightarrow \Gamma = \Gamma_0 \cong \tilde{X}$

$\Gamma = \overline{\Gamma_0} = \overline{\{(x, \varphi(x)) \mid x \in U_0 \text{ the definition}\}}$ the graph

The domain of definition has rigidity: extend to a maximal domain. let U_0 upper is maximal uniquely. Precisely, U_0, V_0 are defined, if $\exists W_0 \subset U_0 \cap V_0$ s.t. $(\varphi|_{U_0})|_{W_0} = (\varphi|_{V_0})|_{W_0} \Rightarrow \varphi|_{W_0}$ extend to $U_0 \cup V_0$ (Pf is easy)

Consider φ^{-1}. $\Gamma_0 \subsetneq \Gamma$

$D \subset X$ closed subset, $\varphi(D) := p_2 p_1^{-1}(D)$ the total transform of D, in particular, we take $D = \{P\}$, $P \in X$ if P not defined $\Rightarrow$ call P the fundamental point

Lemma. $X \overset{\varphi}{\dashrightarrow} Y$ birational, X, Y proj, X normal, U_0 the domain of definition (maximal) $\Rightarrow \operatorname{codim}(X - U_0) \geq 2$

Pf. We can define φ at codim 1 point P (scheme-theoretic)

Let $K = K(X) = K(Y)$, $\mathcal{O}_{X,P}$ is 1 dimension local ring + normal $\Rightarrow$ DVR, $k \subset \mathcal{O}_{X,P} \subset K$, we use the valuation criterion of properness:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & Y \arrow[d, "\text{proper over } k"] \\ \operatorname{Spec}\mathcal{O}_{X,P} \arrow[r] \arrow[ur, dashed, "\exists ! T"] & \operatorname{Spec} k \end{tikzcd}$$

We extend T to a neiborhood of P by algebraic ways: $\operatorname{Spec}\mathcal{O}_{X,P} \to \operatorname{Spec} B \subset Y$ $\Rightarrow B \to A_p \subset K$, B finite generated $\exists f \in A$, $f \notin p$, s.t. $p \in \boxed{A_f}$ open neiborhood $\subseteq U_0$

$\Rightarrow \varphi$ defined at a neiberhood at P, for $\forall P$ codim 1 ▨

Thm. (Zariski main thm)

$\varphi: X \dashrightarrow Y$, X normal, X, Y proj, $p \in X$ fundamental point $\Rightarrow \varphi(p)$ connected and $\dim \varphi(p) \geq 1$

birational

Pf. By Stein decomposition of $p_1: \Gamma \to X$ $\Rightarrow p_1^{-1}(P)$ is connected

$\dim p_1^{-1}(P) = 0$: single point $Q \Rightarrow \mathcal{O}_{X,P} \subset \mathcal{O}_{\Gamma, Q} \subset K$ $\Rightarrow \varphi(P)$ defined, contradiction! DVR $\Rightarrow \mathcal{O}_{X,P} = \mathcal{O}_{\Gamma, Q} \Rightarrow$ local iso

$\dim p_1^{-1}(p) \geq 1 \Rightarrow \dim \varphi(p) = \dim p_1^{-1}(p) \geq 1$ ▨

Then we turn back to surfaces: decompose a birational map.

Thm. (X, X' smooth surface) $f: X' \to X$ birational, $f^{-1}: X \dashrightarrow X'$, p is fundamental point, then f factor through the blow up center at p

Rk. Due to Γ need a closure, Γ may worse than X: even not normal)

(And then $X' \simeq \tilde{X}$ birational, we can apply it again to $X' \to \tilde{X}$ and stop at finite times

Rk. One can think for dimension higher, local computation is impossible

Pf. Let $T = \pi^{-1} \circ f: X' \dashrightarrow \tilde{X}$, if T isn't morphism $\Rightarrow \exists p' \in X'$ is fundamental point of $T \Rightarrow$ by the blowing up $\Rightarrow f(p') = p$ (only $p \in X$ is fundamental)

Using to graph of $T: X' \to \tilde{X}$: E $\Rightarrow T(p') = E$ the only exceptional curve of $\tilde{X}$ $\Rightarrow \exists Q \in E$, s.t. $\boxed{T(p') \ni Q}$ $(T(p')$ is the strict transform of $p)$

$\Rightarrow$ Then we study $T^{-1}: \tilde{X} \dashrightarrow X'$ it has finite fundamental points $\Rightarrow \exists Q \in E$, s.t. T^{-1} defined at Q and a neiberhood of Q, U_0 $\boxed{T^{-1}(Q) = P'}$ [The rigidity of birational map $\tilde{X} \dashrightarrow X' \dashrightarrow \tilde{X}$

Then it's isomorphism of U_0

The local computation: (there is surface)

locally $\boxed{y=0}$ (t, y) $\boxed{x = ty}$

$\Rightarrow E$ has local equation $\begin{cases} y=0 \\ ty - x = 0 \end{cases}$

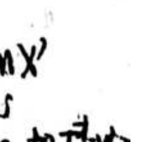
$Q \in E$, $Q = ((0,0), [0,1])$ (x, y, t, u)

And $\mathcal{O}_{X,P} \subset \mathcal{O}_{\tilde{X}, Q}$ is a local dominated $m_{X,P} = m_{\tilde{X}, Q} \cap \mathcal{O}_{X,P}$

let $x, y \in \mathcal{O}_{X', p'}$, then $x, y \in (z) \Rightarrow x = az$, $y = bz$ $a, b \in \mathcal{O}_{X', p'}$, and t, y is local coordinate of $Q \Rightarrow y \notin m_Q - m_Q^2$ $\Rightarrow y \notin m_{p'}^2 \Rightarrow b \in \mathcal{O}_{X', p'}^{*}$ $(b \neq 0)$

$U \quad U$ but $Q \in E$ exceptional, contradiction $U_0 \to V \to U_0$

$\Rightarrow t = \frac{x}{y} = \frac{a}{b} \in \mathcal{O}_{X', p'}$ by $b \neq 0$ $\Rightarrow t \in m_{X',p'}$ and $T(p') = E \Rightarrow \forall \omega \in m_{X', p'}$, vanishing on the whole E, i.e. $\omega \in (t)$ the definition of $E \Rightarrow$ let $\omega = t \in m_{X', p'}$ $\Rightarrow \boxed{t = cy}$, $c \in \mathcal{O}_E$, but it's a local coordinate! Contradiction ▨



By $f^{-1}(p)$ has $\dim \geq 1$, let $C \subset f^{-1}(p)$ irreducible containing p' With local equation $z = 0$, $f^{-1}(p)$ has local equation $\begin{cases} x=0 \\ y=0 \end{cases}$

Cor. $f: X' \to X$, denote $n(f) = \#\{C \subset X'$ irreducible curve $\mid f(C) = \exists p \in X$ a point$\}$ $\Rightarrow f$ can be factored as $n(f)$ composition of blowing-up.

Rk. $n(f) < \infty$ always Due to: C collapse to $p \Rightarrow C^2 < 0$, and every such C linearly independent and in the normal space of $[H]$ $\Rightarrow$ finite dimensional (See Hodge-index) ▨

Thm. $T: X \dashrightarrow X'$ birational, then (Decomposition) T can factored as a finite collection of blow-up and blow-down (inverse of blow-up)

Pf. Induction. When $n(f) = 0$, by Stein decomposition $\Rightarrow$ isomorphism (smooth + bijection = isomorphism)

Now let $C' \subset X'$ collapse C' to $p \in X$ $\Rightarrow p$ is f^{-1}'s fundamental point (Normal is enough) Otherwise. $\to \bullet$ Zariski $/ \to \bullet \Rightarrow$ collapse a curve $\Rightarrow n(f) \geq 1$ contradiction

$\Rightarrow$ f factor through the blowing-up

$$\begin{tikzcd} & \tilde{X} \arrow[d] \\ X' \arrow[ur, "f_1"] \arrow[r, "f"] & X \end{tikzcd}$$

And $n(f_1) = n(f) - 1$ then by induction hypothesis ▨

Pf. H' very ample divisor on X' T^{-1} has fundamental point; (otherwise, apply thm before to T^{-1}) U_0' is the domain of definition of T^{-1} $|X' - U'| < \infty$, take a generic element $C' \in |2H'| \Rightarrow C' \subset U'$, $C = \varphi(C') \subset X$ $\Rightarrow C'$ is normalization of C

Case I $m(T) > 0$, C has singularity, we blow-up the P

$$\begin{tikzcd} & \tilde{X} \arrow[d] \arrow[dr, "\tilde{T}"] & \\ X \arrow[rr, dashed, "T"] & & X' \end{tikzcd}$$

and $p_a(\tilde{C}) < p_a(C)$ $\Rightarrow m(\tilde{T}) < m(T)$ $\Rightarrow \tilde{C} \to \tilde{C} \to C$ Zariski

$C = T(C')$ the strict transformation

let $m(T) = p_a(C) - p_a(C') = p_a(C) - \ldots$ only depend on H' and T ≥ 0

Thus we down $m(T)$ to $m(T')$ until $m(T) = 0$ — case II

Case II $m(T) = 0$. The graph X''

$$\begin{tikzcd} & X'' \arrow[dl] \arrow[dr, "T'"] & \\ X \arrow[rr, dashed, "T"] & & X' \end{tikzcd}$$

We prove T' is morphism: otherwise, let $P' \in X''$ the fundamental point $\Rightarrow E \subset T'(P')$ connected curve, s.t. $(E, H') > 0$ (strict transformation) $\Rightarrow (E, C) \geq 2$

Then

$X' \quad X'' \quad X''$

$\Rightarrow$ We factor $X \xrightarrow{2} X'$, repeat. ▨

By rigidity $\Rightarrow \varphi''(E') = P'$
similiarly, $(T')^{-1}(E) = P'$ by using the graph of $(T')^{-1}$.
$\Rightarrow m(T') = P_a(C') - P_g(C) > 0$ due to p has multiplicity 2 in C'
$\Rightarrow$ contradiction $\Rightarrow T'$ is morphism!

Step 1 $|M|$ is base point free. $\Rightarrow \Phi_{|M|}: X \to \mathbb{P}^N$ birational, image $\Sigma \subset \mathbb{P}^N$

For $i<k$: $0 \to \mathcal{O}_X(H+(i-1)E) \to \mathcal{O}_X(H+iE) \to \mathcal{O}_E(H+iE) \to 0$

LES $\to H^1(E, \mathcal{O}_E(k-i)) \to \boxed{H^1(H+(i-1)E) \to H^1(H+iE)} \to 0$

$\Rightarrow H^1(X,H) = H^1(X, H+E) = H^1(X, H+2E) = \cdots = H^1(H+kE)$

$\Rightarrow H^0(\mathcal{O}_X(M)) \to H^0(\mathcal{O}_E) \to 0 \Rightarrow \mathrm{Base}\,|M| = \phi$ //

Step 2 $\Phi_{|M|}$ gives a blow-down: $\Phi_{|M|}|_{X-E}: X-E \to \Sigma - \Phi(E)$ isomorphism

$(M,E)=0$, take Stein decomposition of Φ: $X \xrightarrow{\Phi} \Sigma \subset \mathbb{P}^N$, $E \subset X \xrightarrow{f} X_0 \xrightarrow{S} \Sigma$, S 1-1, finite, normal, $f_*\mathcal{O}_X = \mathcal{O}_{X_0}$. We only need to prove X_0 smooth.

Locally, we prove X_0 smooth:

We use the completion technique: $\mathcal{O}_{X_0,p}$ regular $\Leftrightarrow \hat{\mathcal{O}}_{X_0,p}$ regular. By formal function (Ha. III, 11)

$\hat{\mathcal{O}}_{X_0,p} = \varprojlim H^0(nE, \mathcal{O}_{nE})$, we compute all $H^0(nE, \mathcal{O}_{nE})$

$\Rightarrow f \cong$ the blow up τ centered at $p \in X_0$ by decomposition thm

$X \xrightarrow{f} X_0$ isomorphism, $X \xrightarrow{\tau} X_0$ $\Rightarrow$ local ring equal

$0 \to \mathcal{I}_E^n / \mathcal{I}_E^{n+1} \to \mathcal{O}_{(n+1)E} \to \mathcal{O}_{nE} \to 0$ $(H^0(E, \mathcal{O}_E) = k$ clearly$)$

$\parallel$ $(\mathcal{O}_{\mathbb{P}^1}(n))$

$\Rightarrow$ inductively $H^0(\mathcal{O}_{2E}) = k \oplus (kx \oplus ky) = \dfrac{k[x,y]}{(x,y)^2}$

$\Rightarrow H^0(\mathcal{O}_{nE}) = \dfrac{k[x,y]}{(x,y)^n} \Rightarrow \hat{\mathcal{O}}_{X_0,p} = k[[x,y]]$ regular ▨

Kodaira dimension $\kappa(S) = \begin{cases} -\infty & ; \forall m \in \mathbb{Z}_{>0}, h^0(mK_S) = 0 \\ 0 & ; \forall m \in \mathbb{Z}_{>0}, h^0(mK_S) \le 1 \text{ and can take equality} \\ \max_m \{\dim \Phi_{|mK_S|}(S)\} & \text{, otherwise } (1 \text{ or } 2) \end{cases}$

It's obviously birational by compare moduli, we find

$\kappa(S) = 2$ is of general type

Lemma. $\kappa(S) \ge 0$, S minimal $\Leftrightarrow K_S$ nef.

$\left(\begin{array}{l}\text{Recall.} \quad \Leftrightarrow E^2 = (E, K_S) = -1 \\ E \text{ is } (-1)\text{-curve} \Leftrightarrow E \cong \mathbb{P}^1, E^2 = -1\end{array}\right)$

Pf.

$(\Leftarrow)$ K_S nef, $\exists E$, (-1)-curve $\Rightarrow E^2 = (K_S, E) = -1$, contradiction

$(\Rightarrow)$ Otherwise, K_S not nef $\Rightarrow \exists E$ irreducible, s.t. $(K_S, E) < 0$

$\kappa(S) \ge 0 \Rightarrow \exists m > 0$, s.t. $mK_S \sim \sum m_i E_i$, $m_i > 0$, E_i irreducible

$\Rightarrow (\sum m_i E_i, E) < 0 \Rightarrow \exists i_0$, s.t. $E^2 = (E, E_{i_0}) < 0 \Rightarrow \left|\begin{array}{l}(K_S,E) < 0 \\ E^2 < 0\end{array}\right.$

Claim. $\left|\begin{array}{l}(K_S,E) < 0 \\ E^2 < 0\end{array}\right. \Leftrightarrow E^2 = -1$ to complete the proof

Pf. $(K_S, E) + E^2 = 2P_a(E) - 2 < 0 \Rightarrow P_a(E) = 0 \Rightarrow E \cong \mathbb{P}^1$ ▨

Thus $f_1(E)$ not a point $\Rightarrow E$ isn't exceptional of f_1

But $(K_S, E) = -1 < 0$, and $(K_S, E) = (f_1^* K_{S_1} + \underbrace{\sum n_i F_i}_{\text{exceptional locus of } f_1}, E) \ge (f_1^* K_{S_1}, E) \ge 0$

$\Rightarrow 0 < 0$, contradiction ▨

Study surface. Fibration, covering, we'll introduce them.
(curves) (singularity)

Def. (Fibration/algebraic fibre space) $f: X \to S$ proper, S normal, $\dim X > \dim S$, $f_*\mathcal{O}_X = \mathcal{O}_S$ ($\Leftrightarrow$ fibre connected), $\exists U \subset S$, $f|_{f^{-1}(U)}: f^{-1}(U) \xrightarrow{\text{smooth}} U$ ($\Leftrightarrow X_s$ smooth, a generic fibre smooth)

Eg. (Blow up the Lefestz pencil) X smooth proj, $\exists H$ very ample, $\Lambda \subset |H|$, $\dim \Lambda = 1$ $\Rightarrow \Lambda$ is a pencil (Lefschetz)

Λ Base $\Lambda \neq \emptyset$, see the Fig 1.

Fig 1.

Blow up the Base $\Lambda \Rightarrow$ See Fig 2

Fig 2.

$\Rightarrow f$ is fibration

We turn back to $f: S^2 \to C^1$

$F = S_t$ is a generic fibre

the genus of F, $g(F)$ not depend on F, denoted $g(f)$

Thm (Castelnuovo)

X has (-1)-curve $E \Rightarrow \exists X_1$ and $\sigma: X \to X_1$ birational s.t. σ is blow-up center at $p \in X_1$, exceptional curve E. (The converse is trivially holds)

in particular, $\boxed{X \text{ has no } (-1)\text{-curve} \Leftrightarrow \text{minimal.}}$

Pf. $\exists$ very ample divisor H, s.t. $H^1(X,H) = 0$

(By Kodaira $H^1(X,H) = H^1(X, K_X + (H - K_X)) = 0$)

$\Rightarrow (H, E) > 0$, denoted k, let $M = H + kE$ isn't ample (Due to $(M, E) = k + kE^2 = 0$)

Coro. $\forall S$, it has minimal model (existence)

Pf. It procedure is finite times, due to the Picard number strictly down ▨

What about uniqueness?

E.g. $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ has no (-1)-curve $\Leftrightarrow$ minimal

$\mathbb{P}^2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ but $\mathbb{P}^2 \not\cong \mathbb{P}^1 \times \mathbb{P}^1$

$\rho(\mathbb{P}^2) = 1 < \rho(\mathbb{P}^1 \times \mathbb{P}^1) = 2$

E_p, E_Q are clearly (-1)-curve

$\tilde{L}$ also (-1)

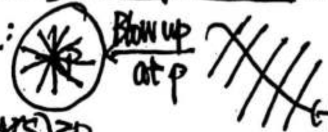
$\mathbb{P}^1 \times \mathbb{P}^1$ only 3 (-1)-curve

$\tilde{L}$ is (-1)-curve due to $L^2 = 1$, blow up $P, Q \Rightarrow$ down $2 \Rightarrow \tilde{L}^2 = -1$

Ex Check it does $\mathbb{P}^1 \times \mathbb{P}^1$

For uniqueness. ① $\kappa(S) = -\infty$, the minimal model of S is $f: S \to B$ smooth surface; $f_*\mathcal{O}_S = \mathcal{O}_B$, $\forall t \in B$, $f^{-1}(t) \cong \mathbb{P}^1$ (See [Ha.]) and exceptional case: $\mathbb{P}^2$ and Hirzebruch surface

F_1:



$= F_1$ (Not unique! we call relative MM)

② $\kappa(S) \ge 0$

Thm. $\kappa(S) \ge 0 \Rightarrow$ the minimal model of S unique

Pf. let S_0, S_1 both minimal:

$$\begin{tikzcd} & S \arrow[dl, "f_0"'] \arrow[dr, "f_1"] & \\ S_0 \arrow[rr, dashed] & & S_1 \end{tikzcd}$$

by minimality, both sides of $S_0 \to S_1$ and $S_1 \to S_0$ can't be isomorphism $\Rightarrow f_0, f_1$ both not isomorphism.

We factor f_0: $\tilde{S} \xleftarrow{\sigma} S \supset E$, $f_1(E) \ni Q$, Assume $f_1(E) = Q$

the last blowup σ, center at $P \in S_0$, E is (-1)-curve of S

$$\begin{tikzcd} \tilde{S} & S \arrow[l, "\sigma"'] \arrow[dl, "f_0"'] \arrow[d, "f_1"] \\ S_0 \arrow[r, dashed] & S_1 \ni Q \end{tikzcd}$$

Case I f_1 collapse ≥ 2 curves

$\exists C'' \ne E$, $f_1(C'') = $ a point.

$f_1 \circ \sigma^{-1}$, $\tilde{S} \dashrightarrow S_1$ and $S_0 - \{P\} \to S_1 - f(E)$ is morphism

And by rigidity $\Rightarrow$ except a finite points isomorphism $f_1 \circ \sigma^{-1}$, but C'' is exceptional $\Rightarrow$ contradiction

Case II f_1 only collapse E. It's also not true! due to then $\tilde{S} \cong S_1$ $\Rightarrow$ replace S by $\tilde{S}$, and continue. Then after a finite step, we back to Case I or $f_1(E) \ne Q$. //

Lemma. $f: S \to C$ fibration, then ① $\mathcal{O}_F(F) (\cong \mathcal{L}(F)) \cong \mathcal{O}_F$, and give F_0 a singular fibre $\Rightarrow (\Gamma_i, F) = 0$, $\forall i$

② $D = \sum a_i \Gamma_i$ ($a_i \in \mathbb{Z}$) $\Rightarrow D^2 \le 0$, and $D^2 = 0 \Leftrightarrow D = rF_0$, $r \in \mathbb{Q}$, $F_0 = \sum m_i \Gamma_i$ ($m_i \ge 0$)

Pf. ① $F = f^*P$, and $P \sim H_1 - H_2$ very ample in C, $H_1 \cap H_2 = \emptyset$, $P \notin H_1 \Rightarrow \mathcal{O}_F(F) \cong \mathcal{O}_F \Rightarrow F^2 = 0 \Rightarrow F$ is nef $\Rightarrow (F, F_0) = 0$

② Otherwise if $D^2 > 0 \Rightarrow (D, F) = 0$, $F \sim f^*H_1 - f^*H_2$

by Hodge index thm $\Rightarrow F^2 \lneq 0 \Rightarrow F^2 < 0$,

contradiction ▨

$D^2 = 0 \Rightarrow \exists$ minimal r, s.t. $D + rF_0 \ge 0$ (by comparing coefficient) $(F, \Gamma_i) = 0$, F_0's ...

if $D + rF_0 \ne 0$ (i.e. rF_0 kills some components, but not kill all) $\Rightarrow \exists i$, $\Gamma_i \notin$ supp$\{D + rF_0\}$, s.t. $(\Gamma_i, D + rF_0) > 0$ (Γ_i killed by $rF_0 \Rightarrow \Gamma_i \notin$ Supp, but $(\Gamma_i, D) > 0$)

$\Rightarrow (\Gamma_i, D) > 0$

$N \gg 0$, $(ND + \Gamma_i)^2 = \Gamma_i^2 > 0$, but supp$(ND + \Gamma_i) \subset$ supp D, it must have negative self-intersection, contradiction ▨ (Converse trivial)

Cor. If F_0 not irreducible $\Rightarrow \forall \Gamma$ the irreducible component $\Gamma^2 < 0$

Lemma. $F_1 \cdots F_k$ is all not irreducible singular fibre

denote $l(F_i)$ the number of component $\Rightarrow \rho(S) \ge \sum (l(F_i) - 1) + 2$

Pf. $\exists H$ very ample on S, generic fibre, $F_i = F_{i,1} \cup F_{i,2} \cup \cdots \cup F_{i,l(F_i)}$

Claim. $\{H, F, \{\Gamma_{ij}\}\}$ is numerical independent (linear independent in N_S) $_{(j \le l(F_i) - 1)}$

then it's clear $\rho(S) =$ generators of $N_S \ge \sum l(F_i) - 1 + 2$

Let $\forall a, b, c_{ij} \in \mathbb{R}$, $D_{abc} = aH + bF + \sum_i \sum_j c_{ij} \Gamma_{ij} \equiv 0$

$(D_{abc}, F) = 0 \Rightarrow a = 0$, $D_{abc} = 0 \Rightarrow \sum_i (\sum_j c_{ij} \Gamma_{ij})^2 = 0$

$\Rightarrow \forall i$, $(\sum_j c_{ij} \Gamma_{ij})^2 = 0$, by ② $\begin{cases} c_{ij} = 0 \\ \sum_j c_{ij} \Gamma_{ij} = r \sum_j c_{ij} \Gamma_{ij} \end{cases}$ impossible!

$\Rightarrow c_{ij} = 0$ ▨ $\quad \boxed{j \le l(F_i) - 1} \quad \boxed{j \le l(F_i)}$

Lemma. $f: S \to C$ relatively minimal

$\Gamma \subset F_0$ is one of component of a not irreducible fibre

$\Rightarrow (K_S, \Gamma) \ge 0$, and $(K_S, \Gamma) = 0 \Leftrightarrow \Gamma$ is (-2)-curve, i.e. $\Gamma \cong \mathbb{P}^1$

Pf. If $(K_S, \Gamma) < 0 \Rightarrow \Gamma^2 < 0 \Rightarrow 0 > (K_S, \Gamma) + \Gamma^2 \ge 2p(\Gamma) - 2$ ... $(K_S, \Gamma) = 0$

$\Rightarrow (K_S, \Gamma) + \Gamma^2 = -2 \Rightarrow (K_S, \Gamma) = \Gamma^2 = -1 \Rightarrow \Gamma$ is (-1)-curve, contradiction ▨

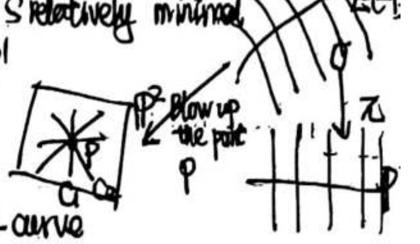
and $(K_S, \Gamma) = 0 \Rightarrow \Gamma^2 = -2 \Rightarrow p(\Gamma) = 0 \Rightarrow \Gamma \cong \mathbb{P}^1$ ▨

Given two fibration $S \dashrightarrow^{\pi} S'$ $\Rightarrow$ birational equivalence, $f: S \to C$, $f': S' \to C$

Def. If f has not fibre containing (-1)-curve, we call f is the relative minimal fibration

S minimal $\underset{\not\Leftarrow}{\Rightarrow}$ S relatively minimal

Eg. $X' \to \mathbb{P}^1$, $X' \to \mathbb{P}^2$, $\mathbb{P}^2$ blow up the pole P



all fibres are 0-curve

but X' has "horizontal" (-1)-curve

Observation. ① $g(E) = 0$ for E is (-1)-curve

② By Hurwitz, genus 0 can't dominate genus > 0

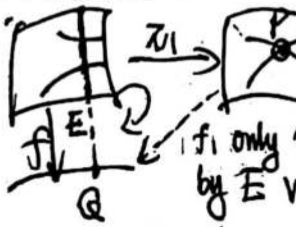
③ When $g(C) > 0$, minimal $\Leftrightarrow$ relative minimal ▨

Thm. Any fibration f has a relatively minimal model

i.e. $S \xrightarrow{\pi} S_0$ and if $g(C) \ge 1 \Rightarrow$ unique.

$f: S \to C$, $f_0: S_0 \to C$

Pf. $\boxed{\text{Existence}}$ $E_1 \subset S$ a vertical (-1)-curve, by Castelnuovo, we blow down E_1, $S \xrightarrow{\pi_1} S_1$, $f: S \to C$, $f_1: S_1 \to C$



f_1 only P not defined, $\rho(S_1) < \rho(S)$

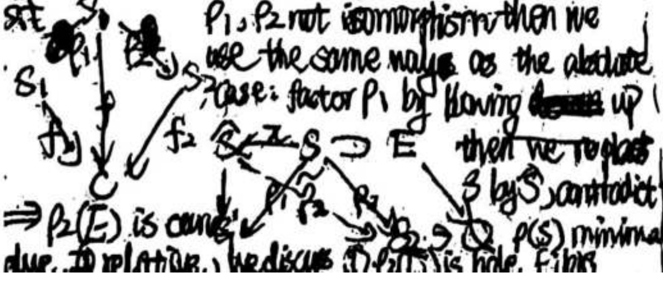
by E vertical, we just

extend $f_1(P) = Q$ defined, then run inductively ▨

(rigidity) $\boxed{\text{Precisely, } f_1 \text{ defined by linear system } \Lambda = f^*D, f_1 = \Phi_\Lambda \text{, thus morphism}}$

$\boxed{\text{Uniqueness}}$

Let S to be the minimal picard number surface s.t. S_1, S_2; p_1, p_2 not isomorphism, then we use the same ways as the absolute case: factor p_1 by blowing up $\tilde{S} \xrightarrow{\pi} S \supset E$, then we replace S by $\tilde{S}$, contradict $\rho(S)$ minimal



$\Rightarrow p_2(E)$ is curve,

due to relative, we discuss ① $p_2(E)$ is hole fibre

Then ② $\Leftrightarrow p_2(E) \subset F_0$, not irreducible

by the Lemma $\Rightarrow 0 \le (K_{S_2}, p_2(E)) = (p_2^* K_{S_2}, E)$

$\le ((p_2^* K_{S_2} + \sum E_j), E)$ (E_j are exceptional curves of π) $= (K_S, E) = -1$, contradiction

① $\Rightarrow p_2(E) \equiv rF_2$ ($r \in \mathbb{Q}$) by $p_2(E)^2 = 0$ due to it's a hole fibre $\Rightarrow (K_{S_2}, F_2) + F_2^2 = 2g(F) - 2$

$= 2g(F) - 2$ (by birational not change generic fibre) ≥ 0, but $(K_{S_2}, p_2(E)) = r(K_{S_2}, F_2) < 0$

(and $F_2^2 = 0$) $\Rightarrow$ contradiction ▨

$\boxed{\text{Ex.}}$ Show that p_2 is morphism. (by same way of proving existence)

Eg. (higher dimensional base is bad) $W^3 \to \mathbb{P}^1$, $W \to S^2$

and blow up P

$\tilde{W} \to W$

The fibre dimension can be bad.

Def. (Exceptional curve) S smooth, $\exists S'$ normal surface, birational $\pi: S \to S'$, s.t. (i) $E = \sum E_i \subset S$, $\pi(E)$ is one point $\in S'$

(ii) $\pi|_{S-E}: S - E \to S' - \pi(E)$ isomorphism.

$\Rightarrow E$ exceptional curve

Lemma. Intersection matrix

If $E = \sum E_i$ is $(E_i \cdot E_j)_{m \times m}$

When E exceptional, it's negatively defined.

In particular, $\forall r_i \in \mathbb{Q}$, $(\sum r_i E_i)^2$

$\Rightarrow E_i^2 < 0$ ...

Pf. $(H, \sum E_j^2) < 0 \Rightarrow$ negatively defined

Let H very ample on S'

$\Rightarrow \pi^* H$ nef, big.

$\Rightarrow \begin{cases} (\pi^* H, \sum r_i E_i) = 0 \\ \pi^* H^2 > 0 \end{cases}$

$\Rightarrow (\sum r_i E_i)^2 \le 0$ ▨

Lemma. $E = \sum E_j$ exceptional curve, $\forall m_i \ge 0$, $h^0(\sum m_i E_i) = 1$

Pf. Induction on $\sum m_i$

① $\sum m_i = 1$, $h^0(E_i) = 1$, $\forall D \in |E_i| \Rightarrow 0 > E_i^2 = (D, E_i) \ge 0$, $\Rightarrow |E_i| = \{E_i\}$

② $D \in |\sum m_i E_i| \Rightarrow (D, \sum m_i E_i) < 0$

$\Rightarrow \exists i$, $(D, E_i) < 0 \Rightarrow E_i \in$ Supp D

$\Rightarrow D - E_i \ge 0 \Rightarrow D - E_i \in |\sum m_i E_i - E_i|$

by induction hypothesis

$\Rightarrow D \cdot E_i = \sum m_j E_j \cdot E_j$

$\Rightarrow D = \sum m_j E_j \Rightarrow h^0(\sum m_j E_j) = 1$ ▨

(It holds for higher dimension)

Def. (Semi-ample) L is semi-ample if $m \gg 0$, $|mL|$ base-point free, $h^0(mL) \geq 2$

Thm. (Zariski's main thm) (It's the basis of Mori's work)

S smooth proj surface (for higher dimension holds), L on S semi-ample, big $\Leftarrow$ For Z to be normal

$\Rightarrow \exists$ normal proj surface Z and birational $\pi: S \to Z$, H ample on Z, $\pi^* H \sim L$

Pf. $\Phi_{|mL|}: S \to \mathbb{P}^N$, using stein decomposition (due to Φ depends on m)

$$\begin{tikzcd} S \arrow[r, "\Phi"] \arrow[dr] & \mathbb{P}^N \\ & Z \arrow[u] \end{tikzcd}$$

$\pi_* \mathcal{O}_S = \mathcal{O}_Z$, Z normal

$\Rightarrow \exists H_m$ ample on Z, $mL \sim \pi^* H_m$

However, we hope (π, Z) not depend on m

$\boxed{\begin{array}{l} \text{Ex. } \pi: S \to Z \text{ is uniquely determined by} \\ ① \ \forall C \subset S,\ \pi(C) \text{ point} \Leftrightarrow (L \cdot C) = 0 \\ ② \ \pi_* \mathcal{O}_S = \mathcal{O}_Z,\ Z \text{ normal} \\ \text{(Kenji Matsuki Prop 1.2.16 P34)} \end{array}}$

Then let $p, q \gg 0$, $(p, q) = 1$

$\Rightarrow rp + sq = 1$

$\Rightarrow L = (rp+sq)L \sim \pi^*(rH_p + sH_q) = \pi^* H$ ▨

Chow group & Chow ring & Chern class

Let $Z^i(X)$ denote the i-cycle (divisor is 1-cycles)

$\deg: Z^n(X) \to \mathbb{Z}$, $\sum m_i P_i \mapsto \sum m_i$ set X smooth or normal

Def. $f: X \to X'$ induces $f_*: Z^i(X) \to Z^i(X')$

$$\begin{array}{ccc} \cup & & \cup \\ Y & \to & \overline{f(Y)} \end{array} \qquad Y \mapsto f_* Y$$

$$f_* Y = \begin{cases} 0; & \dim \overline{f(Y)} \neq \dim Y \\ [K(Y) : K(\overline{f(Y)})]; & \dim \overline{f(Y)} = \dim Y \end{cases}$$

Def. (Rational equivalence) $Y_1, Y_2 \in Z^i(X)$

if $\exists V \subset X$ subvariety and $Y_1, Y_2 \subset V$, $Y_1, Y_2 \in Z^1(V)$

$\sigma: \tilde{V} \to V$ normalization

$\begin{cases} D_1 \to \sigma_* D_1 = Y_1 \text{ and } D_1 \sim_{linear} D_2 \\ D_2 \to \sigma_* D_2 = Y_2 \Rightarrow Y_1 \sim_{rational} Y_2 \end{cases}$

then $i = 1$, linear equivalence = rational equivalence.

Def. (Chow group) $A^i(X) = Z^i(X) / \sim_{rational}$

$A(X) = \bigoplus A^i(X)$ is Chow group

Eg. $A^0(X) = \mathbb{Z}$;

$A^n(X)$:

$P \sim_{rat} Q \Leftrightarrow$ It can be connected by finite rational curves (rational cycle connected)

in particular, rational connected

$\Rightarrow A^n(X) = \mathbb{Z}$ ($\forall P, Q$, $\exists \gamma$ rational connected P & Q)

(Rk. Rational connected is important; usual its $K(X) < 0$)

To give a ring structure, we need use intersection in higher dimension (For $\mathrm{Div}(X)$ is done)

Intersection theory

Set $\mathbb{V}$ a family of varieties, s.t. $\exists$ axioms and all $X \in \mathbb{V}$

(A1) $A(X)$ is a communicative graded ring with 1. ($i > n$, $i < 0$ set $A^i(X) = 0$)

$(\cdot : A^r(X) \times A^s(X) \to A^{r+s}(X)$, $(Y, Y') \mapsto (Y \cdot Y'))$

(A2) $(g \circ f)^* = f^* \circ g^*$

(A3) $(g \circ f)_* = g_* \circ f_*$

(A4) (Projection formula) $f: X \to X'$, $f_*(Y \cdot f^*(Y')) = (f_* Y) \cdot Y'$

Def. $f: X \to X'$; $f^*: A(X') \to A(X)$ is group homomorphism but not a graded group homomorphism

By: $X \xrightarrow{i = \Gamma_f} X \times X' \xrightarrow{p_2} X'$; $f^*: Y \mapsto p_{1*}(p_2^*(Y))$ preserve ~ not not preserve dim

(A5) (Diagonal) $(Y \cdot Y') = \Delta^*(Y \times Y')$

(A6) (Local property) Normal intersection Y, Y' two cycles in $A(X)$, let $Y'' \subset Y \cap Y'$ irreducible component $\Rightarrow (Y \cdot Y') = \sum i(Y, Y'; Y'') Y''$ (Y'' has same codimension $=$ codim Y + codim Y')

(A7) Z effective divisor on X, $Y \subset X$ subvariety $\Rightarrow (Y \cdot Z) = i^* Z = Z|_Y$, $i: Y \hookrightarrow X$

Thm. Intersection theory (A1)–(A7) exist and unique in $\mathbb{V}$ the family of nonsingular quasi-projective. Pf. Omitted ▨

Eg. $C \subset X$, C curve, D is $\mathbb{Q}$-cartier ($\exists m \in \mathbb{Q}$, mD is Cartier), we define $(C \cdot D)$ on X by ① pullback mD to C; $i^*(mD)$ on C ② $\sigma: \tilde{C} \to C$ normarization and $\sigma^* i^*(mD)$ has degree ③ $(C \cdot D) = \frac{1}{m} \deg(\sigma^* i^*(mD))$

Thus the intersection theory can be more than nonsingular case.

Def. (Chern class) rank $\mathcal{E} = r$, X smooth projective, $\pi: \mathbb{P}(\mathcal{E}) \to X$

$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is relatively (fiberwise) ample invertible sheaf $\Rightarrow \exists\, \xi \in Z^1(\mathbb{P}(\mathcal{E}))$, $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\xi)$

then $[\xi] \in A^1(\mathbb{P}(\mathcal{E}))$, by following thm, $\xi^r = \sum_{i \leq r-1} a_i \xi^i$ each a_i not homogeneous, thus take the homogeneous part of $\deg = r$

Thm. (Chevalley 1958, [Har] P429 A.1) $A(\mathbb{P}(\mathcal{E})) \cong A(X) \oplus A(X)\xi \oplus \cdots \oplus A(X)\xi^{r-1}$

$\Rightarrow \xi^r = b_0 + b_1 \xi + \cdots + b_{r-1} \xi^{r-1}$, $\deg b_i = (r-i)$ pure

$\Rightarrow \sum_{i \leq r} (-1)^i b_i \xi^{r-i} = 0$, let $b_i = \pi^*(c_i(\mathcal{E}))$, $c_i \in A^i$

$\Rightarrow c_i(\mathcal{E})$ called the ith Chern class of $\mathcal{E}$.

Eg. ① $\mathcal{L}_0$ line bundle, $c_1(\mathcal{L}_0) = [L] \in \mathrm{Pic}(X)$ $\left(\text{due to } \mathbb{P}(\mathcal{L}_0) \xrightarrow{\cong} X,\ \xi - c_1(\mathcal{L}_0) = 0 \right)$ Pf. Omitted ▨

② $f: X \to X'$ nontrivial ($f \neq 0$) and "good" (such as smooth)

$\mathcal{E} = f^*(\mathcal{E}')$

then $c_i(\mathcal{E}) = f^*(c_i(\mathcal{E}'))$ once f^* preserve the graded structure. (By uniqueness).

$$\begin{tikzcd} \mathbb{P}(\mathcal{E}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{P}(\mathcal{E}') \arrow[d] \\ X \arrow[r] & X' \end{tikzcd}$$

And $c(X) = c_i(\mathcal{T}_X)$ the tangent sheaf. Here the pullback means intersection, confused about it

$c_n(X)$ is a number.

$c(X) = \sum c_i(X) \in A(X)$ totally Chern class

$c_t(X) = \sum c_i(X) t^i \in A(X)[t]$ Chern polynomial

($c(\mathcal{E})$, $c_t(\mathcal{E})$ similiar) (Force to preserve degree)

Prop. ① $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$, then $c_t(\mathcal{E}) = c_t(\mathcal{E}') c_t(\mathcal{E}'')$

② Filtration $\mathcal{E} = \mathcal{E}_0 \supset \mathcal{E}_1 \supset \cdots \supset \mathcal{E}_r = 0$, each $\mathcal{L}_i = \mathcal{E}_{i-1} / \mathcal{E}_i$ invertible $\Rightarrow c_t(\mathcal{E}) = \prod c_t(\mathcal{L}_i) = \prod (c_0(\mathcal{L}_i) + c_1(\mathcal{L}_i)t) = \prod (1 + [L_i]t)$ (splitting formula)

Pf. ① is trivial by Def, ② By ① ▨

Thm. (Splitting principle) $\exists\, f: \tilde{X} \to X$, s.t. $f^*(\mathcal{E})$ is splitting as ②, applying f_* $\Rightarrow c_t(\mathcal{E}) = \prod (1 + a_i t)$ (But f_* not preserve $a_i = [L_i]$)

Pf. Omitted ▨

Def. (Exponentical character) $ch(\mathcal{E}) = \sum e^{a_i}$; Def. (Todd class) $td(\mathcal{E}) = \prod \frac{a_i}{1 - e^{-a_i}} = \prod \left(1 + \frac{1}{2}a_i + \frac{1}{12}a_i^2 - \frac{1}{720}a_i^4 + \cdots\right)$

These are technical, mearly used to computation: $ch(\mathcal{E}) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1 c_2 + 3c_3) + \cdots$

$td(\mathcal{E}) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1 c_2 - \frac{1}{720}(\cdots) + \cdots$

Thm. (Hirzebruch-Riemann-Roch)

$\chi(\mathcal{E}) = \deg_n (ch(\mathcal{E}) \cdot td(\mathcal{T}_X))$ the deg of intersection (a set of points)

Eg. ① X curve case: $ch(\mathcal{E}) = 1+[D]$, $td(\mathcal{T}_X) = 1-\frac{1}{2}[K_X] \Rightarrow \deg(ch(\mathcal{E})\,td(\mathcal{T}_X)) = \deg([D]-\frac{1}{2}[K_X]) = \deg D + \chi(\mathcal{O}_X)$
$\mathcal{E} = \mathcal{O}_X(D)$, rank 1, $\mathcal{T}_X = \mathcal{O}_X(-K_X)$ here) $\quad$ ‖ HRR $\quad$ $\chi(\mathcal{O}_X(D))$ — Usual RR.

② X surface, $r=1$ case: $ch(\mathcal{E}) = 1+[D]+\frac{1}{2}[D]^2$
$\Rightarrow td(\mathcal{T}_X)$ $\quad$ $td(\mathcal{T}_X)$: need use splitting here:
$= 1-\frac{1}{2}[K_X]+\frac{1}{12}([K_X]^2+c_2(X))$ $\quad$ $0 \to \mathcal{L}_1 \to \mathcal{T}_X \to \mathcal{L}_2 \to 0$
$\Rightarrow \chi(\mathcal{O}_X(D))$ $\quad$ $c_1(\mathcal{T}_X) = c_1(\mathcal{L}_1)+c_1(\mathcal{L}_2) = c_1(\mathcal{L}_1\otimes\mathcal{L}_2)$
HRR $\deg_2\left((1+[D]+\frac{1}{2}[D]^2)\right.$ $\quad$ $= c_1(\bigwedge^2\mathcal{T}_X) = c_1(\omega_X^{-1})$
$\left.(1-\frac{1}{2}[K_X]+\frac{1}{12}([K_X]^2+c_2))\right)$ $\quad$ $= -[K_X]$
$c_2(\mathcal{T}_X)$ can't computed easily, denoted c_2.
$= -\frac{1}{2}(D.K_X)+\frac{1}{2}D^2+\frac{1}{12}(K_X^2+c_2)$, in particular, $\boxed{\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2+c_2)}$ the Noether formula.
and $\chi(\mathcal{O}_X(D)) = \frac{1}{2}D(D-K_X)+\chi(\mathcal{O}_X)$

Ex. Write out $n=3$, $r=1$ case (smooth projective threefold) [Hint: $\chi(\mathcal{O}_X) = \frac{1}{24}c_1c_2$, $\chi(\mathcal{O}_X(D)) = \frac{1}{12}D(D-K_X)(2D-K_X)+\frac{1}{12}(D, c_2)+\chi(\mathcal{O}_X)$
$\chi(\mathcal{O}_X(mK_X)) = \frac{1}{12}m(m-1)(2m-1)K_X^3 - (2m-1)\chi(\mathcal{O}_X)$]

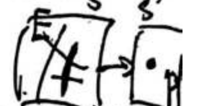
Then we study the singularity.

Def (Rational singularity) $p\in S'$ normal, if $\exists$ resolution of singularity $\pi: S \to S'$ (birational, S smooth, $\pi|_{S\setminus\pi^{-1}(p)}$ isomorphic), s.t. $R^i\pi_*\mathcal{O}_S = 0$, $i\geq 1 \Leftrightarrow p$ is rational singularity.

(For not smooth case, the right need add a singularity part) $\quad$ (not birational: it may hold for minimal but for 3-fold smooth minimal model may not exist.)

Rk. ① Normal is for p is single; not occurs. For $\dim \geq 2$, it's not so easy, we need modify definition of resolution to resolute hole E. $\quad$ (all singular)

Def. $C = \sum C_i$ exceptional curve $\subset S$, $S \xrightarrow{\pi} S'$, s.t. $\forall i$, $C_i \cong \mathbb{P}^1$ and $C^2 = -2$, $\pi(C) = p$, then we call C is A-D-E curve.



② Smooth points are obvious rational singularity; ③ Different with blowing-up, here the $E = \pi^{-1}(p)$ the exceptional curve can be not irreducible / not reduced.
④ The quotient singularity are almost all rational; ⑤ A-D-E singularity are rational (Will be proven!)

Thm. (H. Grauert). $p\in S'$ rational (Eg. $p\in \mathbb{C}^2/G$, $|G|<\infty$) (The A-D-E curve collapse to one point) ($\Longleftrightarrow$ It's analytic neiberhood collapse to C)
and $C\subset S$ exceptional curve, $\exists\, \mathcal{U}$ the strictly pseudo-convex neiberhood of C, $\pi(\mathcal{U})$ is neiberhood of p, s.t. For $\forall \mathcal{F}$ coherent over $\mathcal{U}$,
$\exists k_0>0$, $\forall k\geq k_0$, $H^i(\mathcal{U},\mathcal{F}) \to H^i(kC, \mathcal{F}|_{kC})$ is injective, $i>0$ (View $\mathcal{F}|_{\mathcal{U}}$ analytic coherent sheaf) Pf. Omitted ▨ locally free

Cor. $H^1(\mathcal{O}_{\mathcal{U}}) \cong H^1(\mathcal{O}_{kC})$ when $k\geq k_0$. Pf. $0\to\mathcal{O}_{\mathcal{U}}(-kC)\to\mathcal{O}_{\mathcal{U}}\to\mathcal{O}_{kC}\to 0$ inducing LES: $0 \to \boxed{H^1(\mathcal{O}_{\mathcal{U}}) \to H^1(\mathcal{O}_{kC})} \to H^2(\mathcal{O}_{\mathcal{U}}(-kC)) \to H^2(\mathcal{O}_{\mathcal{U}}) \to$
Similiarly $H^i(\mathcal{F}) \cong H^i(kC, \mathcal{F}\otimes\mathcal{O}_{kC})$ $\quad$ $0\to\mathcal{F}\otimes\mathcal{O}_{\mathcal{U}}(-kC)\to\mathcal{F}\to\mathcal{F}\otimes\mathcal{O}_{kC}\to 0$ Let $\mathcal{F}=\mathcal{O}_{\mathcal{U}}$ (By Grauert) ▨ $H^2(kC, \mathcal{O}_{\mathcal{U}}(-kC)\otimes\mathcal{O}_{kC})$ by $\dim kC = 1$ ‖ $\dim 2$ $= 0$

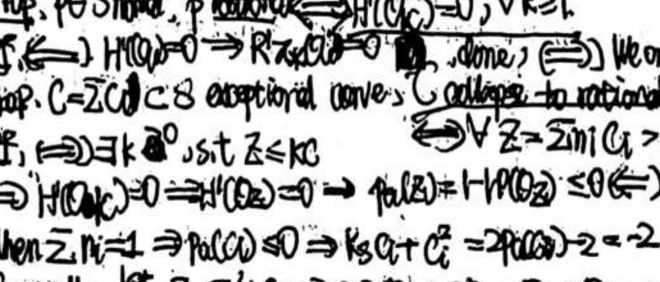
The Leray spectral sequence can be precisely write out due to degenerate:
① $j\geq 2$: $H^j(\pi(\mathcal{U}), \mathrm{pre}(R^i\pi_*(\mathcal{O}_{\mathcal{U}})) = H^j(\mathcal{O}_{\mathcal{U}}) = 0 \Rightarrow \mathrm{pre}\,R^i\pi_*(\mathcal{O}_{\mathcal{U}}) = 0 \Rightarrow R^i\pi_*\mathcal{O}_{\mathcal{U}} = 0$;
② $i>0$, $j>0$, by $\mathrm{supp}(R^i\pi_*\mathcal{O}_{\mathcal{U}}) = p$, $\dim p < i \Rightarrow H^j(R^i\pi_*\mathcal{O}_{\mathcal{U}}) = 0$
$\Rightarrow$ the Leray SS gives just 7 elements
$0\to H^1(S',\mathcal{O}_{S'})\to H^1(S,\mathcal{O}_S)\to H^0(S', R^1\pi_*\mathcal{O}_S)\to H^2(S',\mathcal{O}_{S'})\to H^2(S,\mathcal{O}_S)\to H^1(S', R^1\pi_*\mathcal{O}_S)\to 0$
Cor. $q(S') = q(S)$, $P_g(S') = P_g(S)$

Prop. $p\in S'$ normal, p rational $\Longleftrightarrow H^1(\mathcal{O}_{kC}) = 0$, $\forall k\geq 1$.
Pf. ($\Leftarrow$) $H^1(\mathcal{O}_{\mathcal{U}}) = 0 \Rightarrow R^1\pi_*\mathcal{O}_{\mathcal{U}} = 0$, done; ($\Rightarrow$) We only know $k\geq k_0$ holds, by $0\to\mathcal{O}_C(-kC)\to\mathcal{O}_{(k+1)C}\to\mathcal{O}_{kC}\to 0$
$\Rightarrow H^1(\mathcal{O}_{(k+1)C}) \to \boxed{H^1(\mathcal{O}_{kC})} \to 0$, ‖ 0 $\quad$ inductively 0. ▨

Prop. $C = \sum C_i \subset S$ exceptional curve, C collapse to rational singularity $\Longleftrightarrow \forall Z = \sum n_i C_i > 0$, $p_a(Z) \leq 0$
Pf. ($\Rightarrow$) $\exists k>0$, s.t. $Z\leq kC$ $\quad$ (It tells us why called "rational": all irreducible components are rational)
$\Rightarrow H^1(\mathcal{O}_{kC}) = 0 \Rightarrow H^1(\mathcal{O}_Z) = 0 \Rightarrow p_a(Z) = 1-\chi(\mathcal{O}_Z) \leq 0$ ($\Leftarrow$) induction on $\sum n_i \geq 1$
When $\sum n_i = 1 \Rightarrow p_a(C_i) \leq 0 \Rightarrow K_S C_i + C_i^2 = 2p_a(C_i) - 2 = -2 \Rightarrow C_i \cong \mathbb{P}^1 \Rightarrow H^1(\mathcal{O}_{C_i}) = 0$;
Generally, let $Z = Z' + C_i \Rightarrow 0 \to \mathcal{O}_{C_i}(-Z') \to \mathcal{O}_Z \to \mathcal{O}_{Z'} \to 0 \xRightarrow{LES} \boxed{H^1(\mathcal{O}_{C_i}(-Z'))} \to H^1(\mathcal{O}_Z) \to H^1(\mathcal{O}_{Z'}) \Rightarrow H^1(\mathcal{O}_Z) = 0$ ▨
‖ $C_i \cong \mathbb{P}^1$ $\quad$ ‖ by induction hypothesis 0
$H^1(\mathcal{O}_{\mathbb{P}^1}(-(C_i, Z')))$
Serre duality $\boxed{(C_i, Z') \leq 2}$ $\Longleftarrow p_a(Z) = 1+\frac{1}{2}(K_S.Z+Z^2)$ $\quad$ Otherwise, $(C_i, Z') \geq 2$, $\forall i$, then:
$H^0(\mathcal{O}_{\mathbb{P}^1}((C_i, Z')-2)) = 0$
$= 1+\frac{1}{2}(K_S.\sum n_i C_i + Z.\sum n_i C_i)$
$\geq 1+\frac{1}{2}(\sum n_i (K_S C_i) + \sum n_i C_i^2 + \sum n_i \ldots)$
(By $(C_i, Z) \geq 2 + C_i.C_i = 2+C_i^2$)
$\geq 1+\frac{1}{2}\sum n_i(-2+2) \geq 1$ contradiction!

We study the A-D-E singularity

Lemma. C is A-D-E $\Rightarrow$ C's Dynkin diagram is A-D-E
i.e. $A_n(n\geq 1)$, $D_n(n\geq 4)$; E_6, E_7, E_8. $\quad$ $C = \sum C_i$, Eg.



Pf. $i\neq j$, $(C_i+C_j)^2 < 0$
$\Rightarrow (C_i, C_j) \leq 1$ and $C_i \cong \mathbb{P}^1$ $\quad$ $\Rightarrow$ not A-D-E!
$p_a(\sum C_i) = 1+\frac{1}{2}(K_S.\sum C_i + (\sum C_i)^2) \leq 0$
$\sum C_i$ connect reduce $\Rightarrow$ h^0 connected $\Rightarrow p_a(\sum C_i) \geq 0$ $\Rightarrow \boxed{p_a(\sum C_i) = 0} \Rightarrow$ Combinational $\sum C_i$ is rational connected tree $\Rightarrow$ Combinational $\sum C_i$'s Dynkin diagram is A-D-E ▨
Cor. (By upper Lemma & Prop) A-D-E collapse to rational singularity. $\quad$ ② $Z \cong \oplus H^1(C_i, \mathbb{Z})$ topologically retraction $S|_{\mathcal{U}} \simeq \ldots$

Fundamental circle, We have $H^1(\mathcal{O}_{\mathcal{U}}) = H^1(\mathcal{O}_C)$ for rational, then consider LES of $0\to\mathbb{Z}\to\mathcal{O}_{\mathcal{U}}\to\mathcal{O}_{\mathcal{U}}^*\to 0 \Rightarrow \boxed{\mathrm{Pic}(\mathcal{U}) = H^1(\mathcal{O}_{\mathcal{U}}^*) \cong H^2(\mathcal{U}, \mathbb{Z})}$ ($\dim_{\mathbb{R}} C = 2$)
More precisely, $D\in \mathrm{Div}(\mathcal{U}) \to [D]$ is C^1 1-form $\in H^2(\mathcal{U}, \mathbb{Z}) \mapsto (D.C_1)\ldots(D.C_n) \in \oplus\mathbb{Z}$

Write analytically $\Rightarrow (D.C_i)-(D.C_n)=(\int_{C_i} j_i^*[D])_i$, $j_i: C_i \hookrightarrow U$. the formula $(D.C)=\int_C j^*[D]$ is idea of quantum cohomology & intersection

Prop. C is A-D-E $\Rightarrow \omega_U \cong \mathcal{O}_U$ (ω_U $\pm(D_U^2)$ defined)

Pf. $(K_S.C_i)=(K_U.C_i) \Rightarrow K_U \sim \mathcal{O}_U \Rightarrow \omega_U=\mathcal{O}_U(K_U)=\mathcal{O}_U$

Def. $D\in \mathrm{Div}(U)$ the holomorphic function divisor if $D\geq 0$ and $\mathcal{O}_U(D)\cong\mathcal{O}_U$

Def. We can have a minimal cycle $Z_0=\sum m_iC_i$, s.t. $Z_0>0$ ☒ and $(Z_0.C_i)\leq 0$, we call Z_0 the fundamental cycle of rational points P

Prop. $C=\sum C_i$ collapse to rational, $Z=\sum n_iC_i>0$, then Z is holomorphic function divisor restrict to $Z \Leftrightarrow \forall i, (Z.C_i)\leq 0$

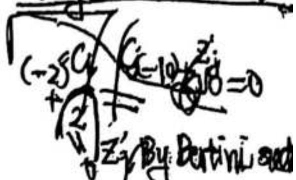
Pf. ($\Rightarrow$) $\exists f\in\mathcal{O}_U$ s.t. $\mathrm{div}(f)=Z+\sum D_j$, $D_j\neq C_i, \forall i,j \Rightarrow (C_i.D_j)\geq 0$

$\Rightarrow (C_i.(Z+\sum D_j))=0$ ~ 0

and $(Z.C_i)=-(C_i.\sum D_j)\leq 0$

($\Leftarrow$) We add some D_j to C: $\forall i, (Z.C_i)\leq 0$, adding Z', s.t. $(Z+Z',C_i)=0$

$\Rightarrow Z+Z'$ is holomorphic function divisor ☒



By Bertini and Z_i curve (hypersurface) exists

Cor. The fundamental cycle of A-D-E curves:

by a concrete computation of each

$\Rightarrow Z_0^2=-2$, $K_SZ_0=0$

the singularity theory is ample but out of date, $\Rightarrow p_a(Z_0)=0$

Thm. Z_0 fundamental cycle of P, $m_P \trianglelefteq \mathcal{O}_{X,P}$ maximal

$\Rightarrow \forall k\geq 1$, $\pi^* m_P^k \cong \mathcal{O}_S(-kZ_0)$

Pf. $\forall g\in m_P \Rightarrow \pi^*g\in\mathcal{O}_U$ gives $\mathrm{div}(\pi^*g)\geq Z_0$ by minimality $\Rightarrow \pi^*g\in H^0(\mathcal{O}_U(-Z_0)) \Rightarrow \pi^*m_P\subseteq\mathcal{O}_S(-Z_0)$

Conversely, $\forall f\in\mathcal{O}_U(-Z_0)\subset\mathcal{O}_U$, $\pi_*f\in\mathcal{O}_{S'}=\pi_*\mathcal{O}_S$

$f=\pi^*\pi_*f$ and $\pi_*f\in m_P \Rightarrow \mathcal{O}_U(-Z_0)\subset\pi^*m_P$ ☒ ($k\geq 1$ is same)

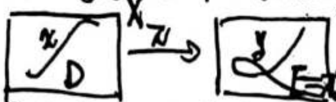
then we using covering to reeducate A-D-E = rational double singularity

Finite covering

Def. For normal varieties X, Y, $\pi: X\to Y$ finite surjective $\Rightarrow \pi$ is finite covering, $\deg\pi=[K(X):K(Y)]$

Given $D\in\mathrm{Div}(X)$ prime divisor, the generic point $x\in D \Rightarrow \mathcal{O}_{X,x}$ is 1-dimensional local normal ring $\Rightarrow$ DVR

Define $R=\sum(e(D)-1)D$, D prime, $D\in\mathrm{Div}(X)$



$x\mapsto y$ also generic in $\pi(D)$

$E=\pi(D)$ $\pi^*: \mathcal{O}_{Y,y}\hookrightarrow\mathcal{O}_{X,x}$ and $\pi^*m_y\subset m_x$

Taking out local equations $D=(f)$, $\pi(D)=(g)$

$\Rightarrow \pi^*g\in m_x=(f) \Rightarrow \pi^*g=cf^{\nu}$, $c\in\mathcal{O}_{X,x}^*$ the ramification divisor

Define $\nu=: e(D)$ the ramification index

Check. R is well-defined

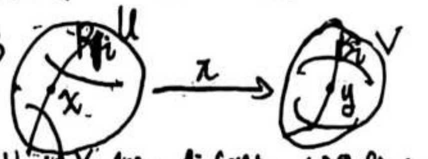
Pf. π flat $\Rightarrow \exists U\subset X$, $V\subset Y$, $U=\pi^{-1}(V)$: the smooth locus

i.e. $\pi: U\to V$ smooth $\Rightarrow \pi: D|_U\to\pi(D)|_V$ is smooth $\Rightarrow$ a topological covering $\Rightarrow e(D)=1 \Rightarrow$ finite sum ☒

Only when $D\subset X-U$, $e(D)-1\neq 0$

Thm. X, Y smooth projective (can be removed), $\pi: X\to Y$ finite $\Rightarrow \omega_X\cong\pi^*(\omega_Y)\otimes\mathcal{O}_X(R)$ (In particular for curve, it's Hurwitz)

Pf. Taking $U\subset X$, $V\subset Y$, $x\in U$, $y\in V$, $\pi(x)=y$, $R=\sum R_i \longrightarrow \sum B_i=B$



let local coordinate in y good (smooth locus in R)

$t_1\cdots t_m$, $B_i=(t_1=0)$ locally $\Rightarrow$ we extend the rational differential form in V to Y: $\omega_Y=\mathrm{div}(a(t_1\cdots t_m)dt_1\wedge\cdots\wedge dt_m)$

$\Rightarrow$ the local coordinate in x is then $s, \pi^*t_2\cdots\pi^*t_d$, $R_i=(s=0)$; By $\omega_Y|_V=\mathrm{div}(a(t_1\cdots t_n))\Rightarrow\omega_X|_U=\mathrm{div}(\pi^*(a(t_1\cdots t_n))dt_1\wedge\cdots$

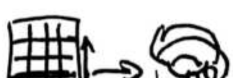
$=\pi^*\mathrm{div}(a(t_1\cdots t_n))+\mathrm{div}(\pi^*(dt_1\wedge\cdots\wedge dt_m))$; and $\pi^*t_1=c(s,\pi^*t_2\cdots\pi^*t_n)\cdot s^{e(D)}$ $\Rightarrow \mathrm{div}(\pi^*(dt_1\wedge\cdots\wedge dt_m))=\mathrm{div}(\pi^*(d(cs^{e(D)})\wedge d(\pi^*t_2)$

$\cdots\wedge d(\pi^*t_m)))=\mathrm{div}(s^{e(D)-1}(ce(D)+s\frac{\partial c}{\partial s})ds\wedge d(\pi^*t_2)\wedge\cdots\wedge d(\pi^*t_m))=\mathrm{div}(c's^{e(D)-1}ds\wedge d(\pi^*t_2)\wedge\cdots\wedge d(\pi^*t_m))$

$\Rightarrow \omega_X|_U=\pi^*(\omega_Y)|_U+(e(D)-1)B_i$ denoted c'

Cor. X, Y smooth, $\pi: X\to Y$ finite, $\deg\pi=n$. if $\exists\mathcal{L}\in\mathrm{Pic}(Y)$, $\pi^*\mathcal{L}=\mathcal{O}_X \Rightarrow \mathcal{L}^{\otimes n}\cong\mathcal{O}_Y$

Eg. Tori



Pf. Projection formula $\Rightarrow \pi_*\pi^*\mathcal{L}=\pi_*\mathcal{O}_X$ and $\mathcal{L}\otimes_{\mathcal{O}_Y}\pi_*\mathcal{O}_X\cong\pi_*\mathcal{O}_X\cdots$ ①

locally algebraic fact $\mathcal{L}\otimes\pi_*\mathcal{O}_X$ $\pi_*\mathcal{O}_X$ is flat over $\mathcal{O}_Y$

(local ring + flat $\Rightarrow$ free)

$\mathbb{C}/\Lambda$ integral point after $\otimes n$ come back $\Rightarrow$ rational equivalent to 0 (torsion)

$\Rightarrow \pi_*\mathcal{O}_X=\mathcal{O}_Y^{\oplus n}$, then taking Λ^n at ① $\Rightarrow \mathcal{L}^{\otimes n}\otimes_{\mathcal{O}_Y}\Lambda^n(\pi_*\mathcal{O}_X)\cong\Lambda^n(\pi_*\mathcal{O}_X)$ invertible ☒

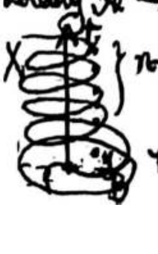
Cyclic covering. Y smooth, $\mathcal{L}_0=\mathcal{O}_Y(S)$, $H^0(Y,\mathcal{L}_0^{\otimes n})\cong H^0(Y,\mathcal{O}_Y(nS))\neq 0$, taking $s\in H^0(Y,\mathcal{L}_0^{\otimes n})$ (motivation)

$B=\mathrm{div}(s)\geq 0$. Consider $\mathcal{A}=\mathcal{O}_Y\oplus\mathcal{L}_0^{-1}\oplus\mathcal{L}_0^{-2}\oplus\cdots\oplus\mathcal{L}_0^{-(n-1)}$, it has multiplication cyclic graded by locally $\mathcal{L}_0^{\otimes n}|_V\cong\mathcal{O}_V$

$\Rightarrow \mathcal{O}_X\cong\mathcal{L}^{-n}$ by the multiplication $\mathcal{L}_0^{\otimes n}|_V\xrightarrow{s}\mathcal{O}_V$ ($\mathcal{L}|_V\cong\mathcal{O}_V$, $\mathcal{L}_0^{\otimes n}|_V\cong\mathcal{O}_V$), $\mathcal{A}$ is f.g. as both algebra & module; (locally think)

Set $X=\mathrm{Spec}\,\mathcal{A}\Rightarrow X\to Y$ naturally a covering (only check X does variety)

locally $\mathcal{A}=A\oplus At\oplus At^2\oplus\cdots\oplus At^{n-1}\cong\frac{A[t]}{(t^n-s)}$ for $\mathcal{L}_0^{-1}|_V=At$ does f.g. & $A\hookrightarrow\frac{A[t]}{(t^n-s)}$ is finite field extension $\Rightarrow$ finite



n-points $B\subset X\cap Y$ the zero of s

and $B\subset X$ just the ramification divisor $\pi^*B=nD$ ($n=\deg\pi$)

$t(t=0)$ We call $(B,\mathcal{L})$ the cyclic covering.

$\mathrm{Sing}X\subset\mathrm{Sing}B$ $X\to Y$

Question. X smooth? We study the Jacobi matrix: $\frac{\partial(t^n-s(x_1\cdots x_n))}{\partial t\,\partial x_1\cdots\partial x_n}=(nt^{n-1}\ \frac{\partial s}{\partial x_1\cdots\partial x_n})\neq 0\Rightarrow$ When $t\neq 0$, smooth; When $t=0$, B smooth $\Rightarrow$ smooth

Case ① B nonreduced $\Rightarrow$ X not normal (normal has singular locus codim ≥ 2, but B is completely not smooth).

② B reduced $\Rightarrow$ By Serre's (S2) condition, $\left\{\begin{array}{l} X \text{ locally complete intersection} \Rightarrow \text{CM local ring} \\ \operatorname{codim}(\operatorname{Sing} X) \geq 2 \ (S2) \end{array}\right\} \Rightarrow$ normal.

$\Rightarrow$ <u>Reduced $\Leftrightarrow$ normal.</u>

③ Y projective $\Rightarrow$ X projective.

Pf of ③ $X \xrightarrow{\pi} Y$ π finite $\Rightarrow$ proper, Y projective $\Rightarrow$ proper $\Rightarrow$ X proper, we pullback H ample Y also ample $\Rightarrow (\pi^* H)^{\otimes n}$ very ample $\Rightarrow$ embedded in $\mathbb{P}^N \Rightarrow$ projective ☒

Eg, ① $Y = \mathbb{P}^1$, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(a)$, $\mathcal{L}^{\otimes 2} = \mathcal{O}_{\mathbb{P}^1}(2a)$, $B \sim \sum_{2a} P_i \Rightarrow \pi: C \to \mathbb{P}^1$ is a degree 2-covering

with $K_C \sim \pi^* K_{\mathbb{P}^1} + \sum P_i \Rightarrow \omega_C \cong \mathcal{O}(2a-4)$ (not rigorious, one write $\omega_C \cong \pi^* \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}(\sum P_i)$)

$\Rightarrow 2g-2 = \deg \omega_C = 2a-4 \Rightarrow \boxed{g = a-1}$; In particular $\left\{\begin{array}{l} a=1, \mathbb{P}^1 \text{ (rational)} \\ a=2, \text{ elliptic curve} \end{array}\right.$

and $\pi^* \mathcal{O}(-2) = \mathcal{O}_C(-4)$ in deg $= 2$

Consider $\Delta = Q_1 - Q_2$ a <u>2-torsion divisor</u>: $\Delta \nsim 0$ when $a \geq 2$; but $2\Delta = 2Q_1 - 2Q_2 \sim \pi^*(P_1 - P_2) \sim 0$

② T = Abelian surface $\Rightarrow K_T \sim 0$, $p_g(T) = 1$, $q(T) = 2 \Rightarrow T = \mathbb{C}^n/\Lambda$ complex torus (as $\chi(T) = 1-2+1 = 0$)

$\forall \eta \in \operatorname{Pic}^0(T)$ a n-torsion divisor. ($h^1(\mathcal{O}_T) = 2$ has enough 1-form)

$\Rightarrow H^0(T, n\eta) = \mathbb{C}$, $B = n\eta = 0$, the n-covering $(n\eta, \eta)$ $\pi: S \to T \Rightarrow \pi^* K_T = K_S$ <u>the Étale covering</u> (no ramification divisor)

Here is more good: even no ramification point. K_T nef $\Rightarrow K_S$ nef $\Rightarrow S$ minimal

$\Rightarrow K_S^2 = nK_T^2 = 0 \Rightarrow K_S \sim 0 \Rightarrow p_g(S) = 1 \Rightarrow \chi(\mathcal{O}_S) = \sum \chi(\mathcal{O}_T(-i\eta)) = n\chi(\mathcal{O}_T) = 0 \Rightarrow q(S) = 2 \Rightarrow S$ also Abelian surface;

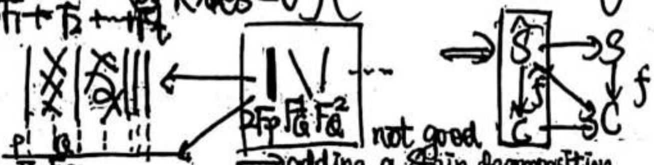
③ Consider a <u>fibration</u> $f: S \to C$

$\mathcal{L} = f^*(D) = f^*(\sum m_i P_i)$, $H^0(S, \mathcal{L}^{\otimes 2})$ $\left(\begin{array}{l}\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{\tilde S}) \\ \text{By } K_S^2 \cdot \mathcal{O}_S = 0\end{array}\right)$ $\left(\text{By R-R, } \chi(\mathcal{O}(i\eta)) = \frac{1}{2}(-i\eta)(-i\eta - K_T) + \chi(\mathcal{O}_T) = 0 \right)$

$= H^0(S, 2f^*(D))$, $2f^*(D) \sim F_1 + F_2 + \dots + F_{2n}$

$\Rightarrow$ 2-covering $\tilde{S} \xrightarrow{\pi} S$

Singular fibre split to 2 singular fibres, smooth fibre multiplicity 2



not good $\Rightarrow$ adding a Stein decomposition

$K_{\tilde S} = \pi^* K_S + \sum_{i=1}^{2n} F_i$

S minimal, K_S nef $\Rightarrow K_{\tilde S}^2 = (\pi^* K_S + \sum F_i)^2 = 2K_S^2 + 2(2K_S \sum F_i) = 2K_S^2 + 8a(2g(F)-2)$

$\Rightarrow \chi(\mathcal{O}_{\tilde S}) = \chi(\mathcal{O}_S) + \chi(\mathcal{O}_S(-f^*(D))) = \chi(\mathcal{O}_S) + \chi(\mathcal{O}_S(-aF)) \overset{R-R}{=} \chi(\mathcal{O}_S) + a(g(F)-1) + \chi(\mathcal{O}_S) = 2\chi(\mathcal{O}_S) + a(g(F)-1)$

④ Singularity case: when B is nodal.

$\tilde{E} = \pi^* E \Rightarrow \tilde{E}^2 = 2E^2 = -2$

thus $\hat{P} = \pi^{-1}(P)$ is a <u>singularity collapsed by a (-2)-curve</u>, it's the <u>simplest</u> singularity: <u>A_1-singularity</u> or <u>rational double singularity</u> (due to (-1) collapse to smooth)

<u>Double covering of surface</u>

It's a <u>2-cyclic covering</u> $(B, \mathcal{L})$ of Y smooth, B reduced.

$Q \in B$ singular in B:

$\mathcal{O}_{P,X} = \frac{\mathbb{C}[x,y,w]}{(w^2 - s)}$

$s(x,y)$ defining B locally at Q

By a proper locally coordinate change and $s(x,y) = s_m(x,y) + \dots + s(x,y)$

$\Rightarrow x \mid s_m(x,y)$ by let Q is m-multiplicity

i.e., $s_m(x,y) = cy^m$ (due to Y smooth $\Rightarrow$ locally $\mathbb{A}^2_{\mathbb{C}}$)

By upper disscussion, we resolate singularity in Y and pullback

$$\begin{array}{ccc} X_1 & \longrightarrow & X \\ \downarrow \pi_1 & \square & \downarrow \pi \\ Y_1 & \xrightarrow{\sigma_1} & Y \\ Q_1 & \longmapsto & Q \end{array}$$

the equations in Q_1: $\sigma_1^* s(x,y) = x^m \tilde{S}(x,t)$ setting $y = xt$, by $x \mid s_m(x,y) \Rightarrow \tilde{S}(x,t)$ containing t^m

$\Rightarrow \mathcal{O}_{X_1, P_1} \cong \frac{\mathbb{C}[x,t,\omega]}{(\omega^2 - x^m \tilde S(x,t))}$

However $\mathcal{O}_{X_1,P_1}$ is nonreduced $\Rightarrow$ normalisation $\tilde{\mathcal{O}}_{X_1,P_1} = \frac{\mathbb{C}[x,t,\omega']}{(\omega'^2 - x^{m-2[\frac{m}{2}]}\tilde S(x,t))}$

here $(\omega')^2 = \left(\frac{\omega}{x^{[\frac{m}{2}]}}\right)^2 = x^{m-2[\frac{m}{2}]} \tilde{S}(x,t)$ $\left\{\begin{array}{l} m \text{ odd, adding a new singularity} \\ m \text{ even, done} \end{array}\right.$

thus in m odd case, repeating this two steps until smooth. It stops in finite due to $B_1 = \tilde B + (m - 2[\frac{m}{2}])E_1$, $B_2 = \tilde B_1 + (m_1 - 2[\frac{m_1}{2}])E_2 \dots \Rightarrow \exists B_N$ smooth, done.

$$\begin{array}{ccccccccccc} X_N & \to & \bar X_N & \to & X_{N-1} & \to & \bar X_{N-1} & \to \cdots \to & \bar X_1 & \to X_1 \to & X \\ & \searrow & \downarrow & \square & \searrow & \downarrow & \square & & \searrow & \square & \downarrow \\ & & Y_N & \longrightarrow & & Y_{N-1} & \longrightarrow \cdots & \longrightarrow & & Y_1 & \to Y \end{array}$$

It calls the <u>even-resolution</u>, finally $\tilde{X} = X_N$ is the minimal model.

Thm. Given $\pi: X \to Y$ 2-cyclic covering by (B, L), $B \geq 2L$

$B \geq 0$ reduced $\Rightarrow \exists \tilde\pi: \tilde X \to \tilde Y$ smooth 2-cyclic covering by $(\tilde B, \tilde L)$ Pf. done in right ☒

and $K_{\tilde X} = \tilde\pi^* \sigma^* (K_Y + L) + \sum (1 - [\frac{m_i}{2}]) \tilde\pi^* \tilde E_i$

m_i the multiplicities in each step, $\sigma: \tilde Y \to Y$ blowing ups, E_i the preimage of exceptional curves

Then $K_{\tilde X} = \tilde\pi^* \sigma^* (K_Y + L) \Leftrightarrow m_i = 2$ or 3

and $\chi(\mathcal{O}_{\tilde X}) = 2\chi(\mathcal{O}_Y) + \frac{1}{2} L(K_Y + L) - \frac{1}{2} \sum [\frac{m_i}{2}]([\frac{m_i}{2}] - 1)$

Pf. It suffies show σ is single blowing up: $\tilde B = \sigma^* B - mE \Rightarrow \hat B = \tilde B + (m - 2[\frac{m}{2}])E = \sigma^*(B) - 2[\frac{m}{2}]E = 2(\sigma^*(L) - [\frac{m}{2}]E)$

$\Rightarrow K_{\tilde X} = \tilde\pi^*(K_{\tilde Y} + \tilde L) = \tilde\pi^*(\sigma^* K_Y + E + \sigma^*(L) - [\frac{m}{2}]E) = \tilde\pi^* \sigma^*(K_Y + L) + (1 - [\frac{m}{2}]) \tilde\pi^* E$ ☒ (Rk. By $(E_i \cdot E_j) = 0$, one can induction to adding directly)

$\tilde\pi_* \mathcal{O}_{\tilde X} = \mathcal{O}_{\tilde Y} \oplus \mathcal{O}_{\tilde Y}(-\tilde L) \Rightarrow \chi(\mathcal{O}_{\tilde X}) = \chi(\tilde\pi_* \mathcal{O}_{\tilde X}) = \chi(\mathcal{O}_{\tilde Y}) + \chi(\mathcal{O}_{\tilde Y}(-\tilde L)) \overset{R-R}{=} \chi(\mathcal{O}_{\tilde Y}) + \frac{1}{2}(-\tilde L)(-\tilde L - K_{\tilde Y}) + \chi(\mathcal{O}_{\tilde Y})$ ...

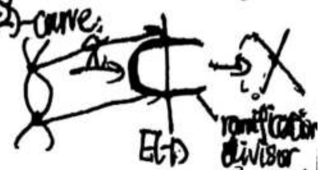
[illegible]

Rk. Too much blowing up makes computation hard, thus we need a criteria for (-1)-curve in $\tilde X$ to stop in time.

Claim. Pullback the (-2)-curve $\subset \tilde B$ gives (-1)-curve in $\tilde X$: $\tilde X \xrightarrow{2:1} \tilde Y$, $E(-2)$, F, $\pi^* E = 2F$, $(\pi^* E)^2 = 4F^2 \Rightarrow F^2 = -1$ ☐, $2E^2 = 2\cdot(-2) = -4$

Surface fibration, if has a pair of rational 4-points $\Rightarrow$ existence of (-2)-curve.

Eg. Pullback (-1)-curve gives (-2)-curve.



We return back to ADE-singularities:

$A_n (n\ge 1)$: $w^2 + x^2 + y^{n+1} = 0$

$D_n (n\ge 4)$: $w^2 + y(x^2 + y^{n-2}) = 0$

E_6: $w^2 + x^3 + y^4 = 0$

E_7: $w^2 + x(x^2 + y^3) = 0$

E_8: $w^2 + x^3 + y^5 = 0$ (analytic isomorphic to such singularity in the hypersurfaces)

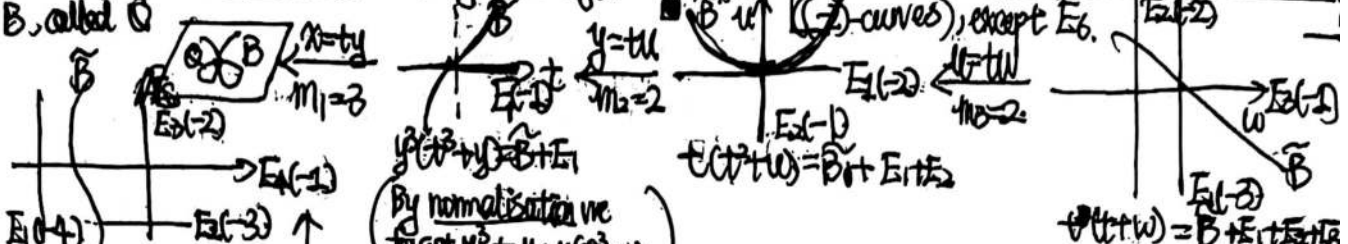
They classify all rational double points.

Rk. Not all singularity can be written out as can't be cutted by hypersurfaces.

Thus view these hypersurface as 2-cyclic covering of curves C. A_n: $w^2 + y^{n+1} = 0$, $y(x^2 + y^{n-2}) = 0$, we can resolution by this way.

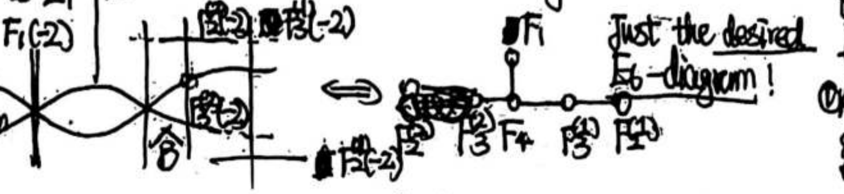
Ex. Refer to [BHPV] P90, checking all A-D-E singularities and gives out all A-D-E curves ((-2)-curves), except E_6.

Eg. The canonical even resolution of E_6: $w^2 = x^3 + y^4 = S(x,y)$, a three-fold singularity at B, called Q.



$\tilde B + E_1 + E_2 + E_3 + E_4$

Here the ramification divisor is E_1, thus covering $\tilde \pi$ $\Rightarrow$ all have 6 (-2)-curves $F_i(-2)$. Consider the intersection of them by Dykin diagram:



Summary of classification of surfaces (then we can to higher dimension)

① $\kappa(S) = -\infty$, $S \cong \mathbb{P}^2$ or $S \xrightarrow{f} C$ ruled surfaces, each fibre $\cong \mathbb{P}^1$. Here $S \cong C \times \mathbb{P}^1$ analytically $\Rightarrow p_g(S) = 0$, $g(S) = 0$, $q(S) = g(C)$, $K_S^2 = 8(1 - g(C)) = 8\chi(\mathcal{O}_S)$ [Hai Chap 5] $\kappa(S) = -\infty \Leftrightarrow P_{12}(S) = 0$ ($P_{12}(S) = h^0(12K_S)$)

② $\kappa(S) = 0$, Abelian surface $K_S \sim 0$, $p_g(S) = 1$, $q(S) = 2 \Leftrightarrow$ torus; K3 surfaces. $K_S \sim 0$, $p_g(S) = 1$, $q(S) = 0$; Enriques surface $K_S \equiv 0$, $K_S \nsim 0$, $p_g(S) = q(S) = 0$, $2K_S \sim 0$, 2-cyclic covering it gives K3; Bi-elliptic surface $S \cong E \times F / G$, $G \curvearrowright E$ translation, $|G| < \infty$, $g(E) = g(F) = 1$ elliptic, $F/G \cong \mathbb{P}^1$, $p_g(S) = 0$, $q(S) = 1$, admittes an analytic fibration to...
Rk. $\kappa(S) = 0 \Rightarrow \chi(\mathcal{O}_S) \ge 0$, $K_S^2 = 0$ (volume) [Beauville]

③ $\kappa(S) = 1$, $P_m(S) = O(m) \Rightarrow |mK_S|$ is base point free by K_S not big & mK_S pencil $\Rightarrow$ (Stein decomposition of $\varphi_{|mK_S|}$) it admittes a fibration $f: S \to C$ with generic fibres are elliptic curves (by $K_S F + F^2 = 0 \Rightarrow g(F) = 1$). Classify the singular fibres by Japanese school [BHPV] using canonical divisor formula, called all elliptic surface. $K_S^2 = 0$. Using 2-cyclic covering and taking big, many examples can be given. $g(C) \le q(S) \le g(C) + 1$, when $g(C) + 1 = q(S) \Rightarrow$ locally trivial.

④ $\kappa(S) = 2$, General type, $K_S^2 > 0$ nef & big, $\chi(\mathcal{O}_S) > 0$, i.e. $p_g(S) \ge q(S)$. $c_2(S) = \chi_{top}(S) > 0$

Noether inequality & Miyaoka-Yau inequality gives volume estimate. $2\chi(\mathcal{O}_S) - 6 \le K_S^2 \le 9\chi(\mathcal{O}_S)$, this slope gives moduli (Noether's father) Thm. (Gieseker) $\forall x, y \in \mathbb{Z}_{>0}$, $K_S^2 = x$ & $\chi(\mathcal{O}_S) = y$, all these surfaces forms a quasi-projective scheme $M_{x,y}$, compactification $\overline{M}_{x,y}$ projective.

(it needs more degeneration, with not occurs in $\overline{M}_g$, thus it's much harder)

Rk. the inequalities are given by the stability of vector bundle. Or by geometric analysis way $\frac{n}{2(n+1)}c_1 c_2 - c_1^3) K^{n-2} \ge 0$, taking $n=2$ one gives the Yau's side.

(Bombieri, 1973; Reider, later) ① $m \ge 5$, $|mK_S|$ base point free, $\varphi_{|mK_S|}$ birational
$m = 4$, $|4K_S|$ base point free, $\varphi_{|4K_S|}$ birational $\Leftrightarrow$ (K_S^2, $p_g(S)$) $\ne$ (1,2)
$m = 3$, $|3K_S|$ base point free, $\varphi_{|3K_S|}$ birational $\Leftrightarrow$ (K_S^2, $p_g(S)$) $\ne$ (1,2) & (2,3)

② $K_S^2 \ge 5$ or $p_g \ge 2$ $\Rightarrow |2K_S|$ base point free (Prof. Jiang Zhi)

Rk. For $\kappa(S) = 1$, the m can be very large (Using stability). Finally we prove Noether inequality

Thm. S minimal & general type $\Rightarrow K_S^2 \ge 2p_g(S) - 4$

Pf. Assume $p_g(S) \ge 3$ (otherwise trivial) $\Rightarrow \varphi_{|K_S|}: S \dashrightarrow \mathbb{P}^{p_g(S)-1}$, set $d = \dim \overline{\varphi_{|K_S|}(S)}$, $1 \le d \le 2$

$\Rightarrow$ (1) $K_S^2 = \pi^* K_S^2 \ge M^2 \ge \deg \varphi \cdot \deg \varphi(\Sigma) \ge \deg \varphi (p_g(S) - 2)$, $\deg \varphi \ge 2$, done. $\deg \varphi = 1 \Rightarrow \varphi_{|K_S|}$ birational, also done by ... $h^0(S)-1$ and $|\pi^* K_S| = |M| + Z$; (Beauville) $\deg D \ge 2h^0(D) - 4 \Rightarrow$ let $C \in |M|$, $D = M|_C \Rightarrow M^2 = \deg D$

⟹ $N^2 = \deg D \geq 3h^0(L) - 4 \geq 3h^0(m) - 7 = 3p_g(S) - 7 \geq 2p_g(S) - 4$ ▨ (2) $d=1$ case is easy, left as Exercise ▨

Noether's type inequality is a sharpened, take equality in infinite classes. It exists in any dim (Chen & Jiang)

Higher birational Geometry (MMP.) { Italian school: dim 2 done. Only 3-dim given out precisely by Chen
Japanese school: generalise surface to higher (Kenji Ueno). fails.

Normalisation of curves in space.

$C \subset \mathbb{P}^N$ non degenerate, $N=2$ done

$N \geq 3$, projection to $\mathbb{P}^2$ birationally by choosing projection point properly

$$\begin{array}{ccccc} \mathbb{P}^N & \supset & C & \xleftarrow{\mu} & \tilde{C} \\ \downarrow & & \downarrow & \square & \downarrow \\ \mathbb{P}^2 & \supset & C' & \xleftarrow{\mu'} & \tilde{C}' \end{array}$$

Define $(D.C) = \deg_{\tilde C} \mu^*(\mathcal{O}_C(D))$ the intersection theory

Q. Depending on choice of projection?

(D) is Cartier on X means ℚ-Cartier; $\dim X = d$ normal projective

Chow group & Chow ring defined as usual. But in dimension higher, ℝ-coefficient is preferred: $N_1(X) = Z_1(X)/\equiv_{num} \otimes \mathbb{R}$; $N^1(X) = Z_{d-1}(X)/\equiv_{num} \otimes \mathbb{R}$. ↳ second one need X ℚ-Cartier

Pairing $N_1(X) \times N^1(X) \to \mathbb{R}$ (in ℚ-factorial)

$NE(X) \subset N_1(X)$ effective cone, generated by all irreducible curves

$\overline{NE}(X)$ closed ⟺ only finite generators, thus always taking $\overline{NE}(X)$

Recall. (Kleiman) X smooth, proj, $D \in \mathrm{Div}(X)$ ample ⟺ $(D.x) > 0$ $\forall x \in \overline{NE}(X) - \{0\}$ (or ℚ-factorial)

Idea of proof

→ all >0 ⟹ Using minimax principle ⟹ all >0 ⟹ ample ▨

If $\mathcal{F} \cong \mathcal{F}^{\vee\vee}$, we call it reflexive, when X normal, $\mathcal{F}$ coherent. (Here it's same as Functional analysis, $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ canonical, must iso)

Lemma ([Ha, Math Ann. 254 (1980), 121–176, Prop 1.6]) TAFE when X normal & $\mathcal{F}$ coherent: ① $\mathcal{F}$ reflexive line bundle; ② $\exists X^0 \subset X$ smooth, $\mathrm{codim}(X, X^0) \geq 2$, $X^0 \overset{i}{\hookrightarrow} X$, $\mathcal{F}|_{X^0}$ invertible and $\mathcal{F} = i_*(\mathcal{F}|_{X_0})$; ③ $\forall X^0 \subset X$ smooth ....

Cor. X normal, the $Z_{d-1}(X)/\sim_{lin} \longleftrightarrow \{\mathcal{F} \mid \mathcal{F}$ reflexive line bundle$\}/\cong$
$D \longmapsto \mathcal{O}_X(D)$

Pf. and by $\mathcal{O}_X(D)(U) = \{f \in K(X) \mid v_{D_j}(f) + n_j \geq 0\}$, $D = \sum n_j D_j$, thus $\mathcal{O}_X(D)$ determined up to isomorphism, by Lemma②, restrict to $(X - U) \cap X^{sm}$; converse taking U and D is coding, determine all (D_j, n_j) by U chosen ▨

Def. $\omega_X := i_* \omega_{X^0}$ is reflexive canonical sheaf of X ($X^0 \subset X^{sm}$ not depending choice by Lemma)

$\exists K_X \in \mathrm{Div}(X)$, $\omega_X \cong \mathcal{O}_X(K_X)$ canonical divisor. (CM is related with singularities. Eg. If X only has rational singularity ⟹ X is CM)

[Ha] When X is CM (Cohen-Macaulay) ⟹ $\omega_X = \omega_X^{\circ}$

Rk. $\forall r > 0$, $\omega_X^{[r]} = \mathcal{O}_X(rK_X)$, usually $\omega_X^{[r]} \neq \omega_X^{\otimes r}$ as $\omega_X^{\otimes r}$ not reflexive, thus we prefer $\omega_X^{[r]}$ here.

Def. X normal CM, ω_X invertible ⟺ X Gorenstein; $\omega_X^{[r]}$ invertible ⟺ X is r-Gorenstein; ($r > 1$)

② X has canonical singularity at P ⟺ X is r-Gorenstein, $\exists r$, ∃ resolution of singularity $\pi: Y \to X$, E_j exceptional divisor of π, s.t. $rK_Y \sim \pi^*(rK_X) + \sum a_j E_j$, $a_j \geq 0$ (i.e. $K_Y \sim_{\mathbb{Q}} \pi^*(K_X) + \sum \frac{a_j}{r} E_j$)

Terminal singularity if $a_j > 0$, $\forall j$

Rk. a_j's positivity not depends on π, thus we can replace "∃" as "∀"

By
$$\begin{array}{ccc} & Y_1 & \\ W & & \xrightarrow{\pi_1} X \\ & Y_2 & \xrightarrow{\pi_2} \end{array}$$
Finding a both resolution covering π_1 and π_2 ▨

Eg. ① S surface with only ADE singularity, C smooth, $X = S \times C$ normal with a curve not smooth (codim $= 2$).

Resolution $\mu: T \to S$

⟹ $T \times C \xrightarrow{\pi} S \times C$, $K_T = \mu^* K_S$

$$\begin{array}{ccc} T \times C & \xrightarrow{\pi} & S \times C \\ \| & & \| \\ Y & \xrightarrow{\pi} & X \end{array}$$

⟹ ADE singularity is canonical & Gorenstein

⟹ All singularities in X is canonical, X Gorenstein.

② S general type, smooth, minimal

⟹ $\exists m \geq 4$, $|mK_S|$ base point free, $\phi_{|mK_S|}: S \to T$ stein decomposition it, assume T is normal; $mK_S = \phi^*(H)$, $\exists H$

Claim. $H = mK_T$; by $K_S \sim \phi^* K_T + \sum m_j E_j$, $\Delta_+ - \Delta_- = \Delta$, Δ_+, Δ_- supported in different (−2)-curves (Here Δ_+, Δ_- by $m_j \geq 0, m_j < 0$)

⟹ $(K_S.(\Delta_+ - \Delta_-)) = 0$

⟹ $(\Delta_+ - \Delta_-)^2 \leq 0$ by Hodge index thm.

⟹ $\Delta_+ - \Delta_- \equiv 0$ ⟹ all $m_j = 0$ ⟹ $K_S \sim \phi^* K_T$

⟹ $mK_S = \phi^*(mK_T)$ ▨

Hence we call T the canonical model of S; $T = \mathrm{Proj} \bigoplus_{m \geq 0} H^0(S, mK_S)$.

③ $X \subset \mathbb{A}^n$ hypersurface by f, $\deg f = d$, $P \in X$

P is general k-point (General singularity means only one blow-up) X Gorenstein by complete intersection

$K_X = (K_{\mathbb{A}^n} + X)|_X$

Blowing-up $\mu: B \to \mathbb{A}^n$, $E_B|_Y = E_\pi$, $K_B = \mu^* K_{\mathbb{A}^n} + (n-1)E_B$

$$\begin{array}{ccc} B & \xrightarrow{\mu} & \mathbb{A}^n \\ \cup & & \cup \\ Y & \xrightarrow{\pi} & X \end{array}$$

and $\mu^* X = Y + kE_\pi$ & $K_Y = (K_B + Y)|_Y$ ⟹ $K_Y = (\mu^* K_{\mathbb{A}^n} + (n-1)E_B + \mu^* X - kE_B)|_Y = \mu^*(K_{\mathbb{A}^n} + X)|_Y + (n-1-k)E_\pi = \pi^*(K_X) + (n-1-k)E_\pi$

thus { P canonical; $n \geq k+1$
P terminal; $n > k+1$
P noncanonical; $n < k+1$ } is quite easy ▨

④ (Cyclic quotient singularity) $\mu_r = \mathbb{Z}/r\mathbb{Z}$, $r > 1$, ε generator, $\varepsilon \curvearrowright \mathbb{A}^3$ by $\varepsilon(x, y, z) = (\varepsilon^a x, \varepsilon^b y, \varepsilon^c z)$; $a, b, c \in \mathbb{Z}/r\mathbb{Z}$, $X = \mathbb{A}^3/\mu_r$ naively. $0 \in X$ singularity (isolated) ⟺ $(a,r) = (b,r) = (c,r) = 1$. Resolute π is a toric resolution, omitted [Reid, YPG] not unique & canonical. (Denote $0 \in X$ this case by $\frac{1}{r}(a,b,c)$)

Terminal Lemma $\frac{1}{r}(a,b,c)$ terminal $\iff \forall 0<j<r, \sum \overline{j a_i} > r$ (Here view $a_i \in \mathbb{Z}/r\mathbb{Z}$ as $a_i \in \mathbb{Z}_{\le r}$)

$\frac{1}{r}(a,b,c)$ canonical $\iff \forall 0<j<r, \sum \overline{j a_i} \ge r$ Holds for $(a_1 \cdots a_n)$ any dimension

$\Rightarrow$ The 3-dimensional terminal cyclic singularity is $\frac{1}{r}(a,-a,1)$ or $\frac{1}{r}(1,-1,a)$, $(a,r)=1$, $a<r$.

[Reid, Canonical 3-fold] ① Canonical singularity of threefold is rational ($\Rightarrow \omega_X = \omega_X^{\circ}$)

② If X only has terminal singularity $\Rightarrow \operatorname{codim}(X_{sing}, X) \ge 3$ ($\iff$ the terminal singularity in 3-fold is isolated)

Thm. X normal, TFAE ① $P \in X$ canonical singularity; ② $\exists r>0$, rK_X is Cartier & $\exists$ Resolution of singularities $\pi: Y \to X$

Pf. ① ($\Rightarrow$) $\omega_Y^{\otimes r} = \pi^*\omega_X^{[r]} \otimes \mathcal{O}_Y(\sum a_i E_i)$; $a_i \ge 0 \Rightarrow \pi_*\omega_Y^{\otimes r} \cong \omega_X^{[r]} \otimes \pi_*(\mathcal{O}_Y(E))$ s.t. $\pi_*(\omega_Y^{\otimes r}) = \omega_X^{[r]}$

($\Leftarrow$) $\omega_Y^{\otimes r} = \pi^*\omega_X^{[r]} \otimes \mathcal{O}_Y(\sum m_i E_i)$, we need show $m_i \ge 0$ $\quad = \mathcal{O}_X$ by $\mathcal{O}_X \subset \pi_*(\mathcal{O}_Y(E))$

By $\pi_*\omega_Y^{\otimes r} = \omega_X^{[r]} \otimes \pi_*\mathcal{O}_Y(\sum m_i E_i) = \omega_X^{[r]} \Rightarrow \pi_*\mathcal{O}_Y(\sum m_i E_i) = \mathcal{O}_X \Rightarrow m_i \ge 0$ ▨ and $\forall s \in \mathcal{O}_X$, $\pi^* s \in \mathcal{O}_Y$

$$\begin{array}{ccccc} 0 \to & \mathcal{O}_Y & \to & \mathcal{O}_Y(E) & \to \cdots \\ & \downarrow \pi_* & & \downarrow \pi_* & \\ 0 \to & \mathcal{O}_X & \to & \pi_*(\mathcal{O}_Y(E)) & \to \cdots \end{array}$$

$\operatorname{div}(\pi^* s) + E \ge 0 \Rightarrow \pi^* s \in \Gamma(U, \mathcal{O}_Y(E))$ done.

We'll classify terminal singularity, introduce KLT singularity & MMP, in particular, $\dim = 3$.

The Birational Geometry for threefold

Def. X normal, $D \in Z_{d-1}(X) \otimes \mathbb{Q}$, $d = \dim X$, (X,D) is $\begin{cases} \text{Pair}, & D \ge 0 \\ \text{Subpair}, & D \text{ general} \end{cases}$ (Boundary divisor)

If: ① $K_X + D$ is $\mathbb{Q}$-Cartier; ② $\exists$ log resolution $\mu: X' \to X$ s.t. $K_{X'} \sim_{\mathbb{Q}} \mu^*(K_X+D) + \sum a_j E_j$, X' smooth & $\mu^* D$ is SNC (or $\operatorname{Supp}(\mu^* D)$)

E_j is: (i) $\operatorname{Supp}(\mu^{-1}(D))$ (strict transformation) (ii) Exception divisors.

satisfy that $\forall j$, $a_j > -1$ (or ≥ -1)

$\Rightarrow (X,D)$ is log terminal (or log canonical)

Rs. Moderly, log terminal $\begin{cases} \text{KLT}; & D \ge 0 \\ \text{sub KLT}; & D \text{ general} \end{cases}$

taking $D = 0$, the log terminal (canonical) is worse than terminal (canonical)

$\Downarrow$

But it's still good: KLT is rational $\Rightarrow$ CM $\Rightarrow \omega_X = \omega_X^{\circ}$

Ref. [KMM] (by Kawamata) and [KM] (Kollár–Mori).

Multiplier ideal (From complex Geometry, subtle in singular case)

Gives pair $(X,D) \Rightarrow \exists$ log resolution $\pi: Y \to X$ (existence in char $=0$ by Hironaka) $\Rightarrow K_Y = \pi^* K_X + E$, $E > 0$

$\lfloor \pi^* D \rfloor > 0 \Rightarrow 0 \to \mathcal{O}_Y(-\lfloor \pi^* D \rfloor) \to \mathcal{O}_Y \to \mathcal{O}_{\lfloor \pi^* D \rfloor} \to 0 \Rightarrow \pi_*\mathcal{O}_Y(E - \lfloor \pi^* D \rfloor) \subset \mathcal{O}_X$ is a ideal sheaf, denoted $\mathcal{J}(D)$

$$\begin{array}{ccccccc} 0 \to & \mathcal{O}_Y(E - \lfloor \pi^* D \rfloor) & \to & \mathcal{O}_Y(E) & \to & \mathcal{O}_{\lfloor \pi^* D \rfloor}(E) & \to 0 \\ & \downarrow & & \downarrow & & & \\ 0 \to & \pi_*\mathcal{O}_Y(E - \lfloor \pi^* D \rfloor) & \to & \pi_*\mathcal{O}_Y(E) = \mathcal{O}_X & & & \end{array}$$

$\mathcal{J}(D)$ not depend on choice of π: $\pi_*\mathcal{O}_Y(E - \lfloor \pi^* D \rfloor) = \pi_*\mathcal{O}_Y(K_Y - \lfloor \pi^* D \rfloor)$ and by choicing a same bigger resolution dominating two, done ▨

Nadel vanishing

Thm. (X,D) pair and X smooth

L is big & $L - D$ is nef & big $\Rightarrow \forall i > 0$, $H^i(X, \mathcal{O}_X(K_X+L) \otimes \mathcal{J}(D)) = 0$

Pf. (Using Kawamata vanishing, and Nadel not need SNC, it's used more)

Taking $\pi: Y \to X$

By Kawamata–Viehweg vanishing $\Rightarrow \forall j > 0$, $R^j\pi_*\mathcal{O}_Y(K_Y + \lceil \pi^*(L-D) \rceil) = 0$ by Leray–Serre SS

$\Rightarrow \forall i > 0$, $0 = H^i(Y, K_Y + \lceil \pi^*(L-D) \rceil) = H^i(X, \pi_*\mathcal{O}_Y(K_Y + \lceil \pi^*(L-D) \rceil)) = H^i(X, \pi_*\mathcal{O}_Y(\pi^*(K_X+L) + \lceil K_{Y/X} - \pi^*(D) \rceil))$

$= H^i(X, \mathcal{O}_X(K_X + L + \pi_* \lceil K_{Y/X} - \pi^*(D) \rceil))$

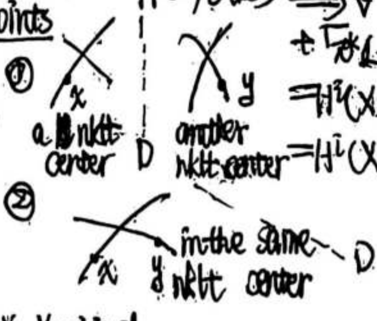
$= H^i(X, \mathcal{O}_X(K_X+L) \otimes \mathcal{J}(D))$ ▨

$\mathcal{J}(D)$ is the ideal sheaf of nonklt center: $\operatorname{Nklt}(X,D) = \operatorname{supp}(\mathcal{O}_X/\mathcal{J}(D))$

Using the D & $\mathcal{J}(D)$, we can distinguish two points

Due to K_X+L has positivity.

If is the ②, we can always choice another D to return back to ①



MMP in threefold (1990)

Thm. $\dim V = 3$, V smooth proj, by finite step of (i) flip; (ii) contraction birationally $\Rightarrow V \dashrightarrow X$, X minimal.

and has finite $\mathbb{Q}$-factorial terminal singularities isolated.

And $\begin{cases} \text{When } K(V) = -\infty, X \text{ is Mori fibre space: } \text{① } -K_X \text{ ample, ② Picard number } \rho(X) = 1; \\ \text{When } K(V) \ge 0, K_X \text{ nef.} \end{cases}$

③ Del pezzo fibration $X \xrightarrow{f} Z$ fibres are Delpezzo surfaces, $\rho(F) = 1$;

④ Conic bundle $X \xrightarrow{g} Z'$ fibres are conic curves.

Classification

① Smooth Fano threefold (Mori–Mukai)

One singularity: $\mathbb{Q}$-Fano threefold (open)

② General type ~~smooth~~ threefold

③ Calabi–Yau (can't do now, 2024)

Eg. Blowing-up (≤ 8 points)

$S \to \mathbb{P}^2$ is a Del-pezzo surface & $-K_S$ ample.

$h^0(-K_S) \ne h^0(-K_{\mathbb{P}^2})$ ($h^0(K_S)$ is birational, but $h^0(-K_S)$ not)

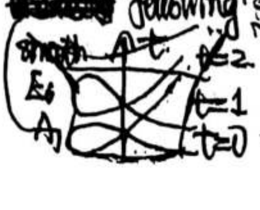
$\begin{matrix} \| \\ 0 \text{ here} \end{matrix}$

The singularity of X can by sufficiently large. Construction: taking X with $g(X) > 0$, and cyclic covering with order large ▨

Def. n-dim Hyper-quotient singularity is $Q \in Y = V(F) \subset \mathbb{A}^{n+1}$, $\mu_r = \langle \xi \rangle$ cyclic group $\curvearrowright \mathbb{A}^{n+1}$ by $(\xi^{a_0} x_0, \cdots, \xi^{a_n} x_n)$, Y is μ_r-invariant (i.e. F is characteristic polynomial), after action, $Q \in Y \subset \mathbb{A}^{n+1}/\mu_r$ is hyper-quotient

Def. (CDV singularity) Compound Du Val singularity is analytically isomorphic to following singularities of hypersurface $V(F) \subset \mathbb{A}^4$, $F = f(x,y,z) + t g(x,y,z,t)$, where $V(f) \subset \mathbb{A}^3$ is ADE singularity.

(i.e. a 1-dim deformation of ADE) Almost all CDV singularities are terminal.



CDV in surface, $= $ ADE

Thm. (Mori) 3-dim terminal singularities is hyper-quotient singularity, and it's classified into 6 classes:

① Cyclic quotient singularity, type $\frac{1}{r}(1,-1,b,0,0)$, equation (analytic local) $xy+g(z^r, t)=0$, $\deg g \geq 2$, $\gcd(r,b)=1$

② Noncyclic quotient singularity, type $\frac{1}{4}(1,1,3,2,2)$, equation $xy+z^2+g(t)$ or $x^2+z^2+g(y,t)=0$, $\deg g \geq 3$;

③ Type $\frac{1}{3}(0,2,1,1,0)$, equation $x^2+y^3+z^3+t^3$ or $x^2+y^3+z^3+y\,g(z,t)+h(z,t)=0$, $\deg g \geq 4$, $\deg h=6$;

④ Type $\frac{1}{2}(0,1,1,1,0)$, equation $x^2+ty^2+g(z,t)$, $\deg g=4$;

⑤ $\frac{1}{2}$-singularity, type $\frac{1}{2}(1,0,1,1,0)$, equation $x^2+y^3+yzt+g(z,t)$ or $x^2+yzt+y^n+g(z,t)$ or $x^2+yz^2+y^n+g(z,t)$,

⑥ $\frac{1}{2}$-singularity, type $\frac{1}{2}(1,0,1,1,0)$, equation $x^2+y^3+xg(z,t)+h(z,t)$, $\deg g \geq 3$, $\deg h=4$, $h\neq 0$. $n\geq 3$, $\deg g=4$

Notice. And all g, h in ①–⑥ is general! (One can take g or h bad to give ①–⑥ not terminal)

Such local information of singularity returns back to global through Riemann-Roch (Chern class only introduced in ≤ 3 dim when singular)

Thm. (Reid-Riemann-Roch) [YPG, Reid] X at most canonical singularity varieties

$\Rightarrow \exists$ Reid Basket of singularities $B_X=\{\frac{1}{r_i}(1,-1,b_i) \mid i=1,\dots,s,\ r_i\geq 1,\ \forall i,\ 0<b_i,\ \gcd(b_i,r_i)=1\}$ ($\subset$ "class ①")

s.t. $\forall m>0$, $\chi(\mathcal{O}_X(mK_X))=\frac{1}{12}m(m-1)(2m-1)K_X^3-(2m-1)\chi(\mathcal{O}_X)+\underline{l(m)}$ (about singularity); $l(m)=\sum_i\sum_j \frac{\overline{jb_i}(r_i-\overline{jb_i})}{2r_i}$

For Weil divisor D general, $\chi(\mathcal{O}_X(D))=\cdots$ complexer. $\quad l(m)\leq 0$ is done, we prefer $l(m)\geq 0$

Cor. ① X Gorenstein $\Rightarrow B_X=\emptyset$ ② X general type $\Rightarrow B_X$ is birational invariant; ③ X $\mathbb{Q}$-Fano $\Rightarrow B_X$ must not birational

④ X $\mathbb{Q}$-Fano or weak $\mathbb{Q}$-Fano [By $\chi(\mathcal{O}_X(mK_X))$ birational $\Rightarrow l(m)$ birational $\Rightarrow B_X$ birational)

⑤ X general type, $m\geq 2$
$P_m=\frac{1}{12}m(m-1)(2m-1)K_X^3-(2m-1)\chi(\mathcal{O}_X)+l(m)$ (Prof. Chen)

Def. X has terminal singularity
$-K_X$ ample $\Leftrightarrow$ $\mathbb{Q}$-Fano
$-K_X$ nef & big $\Leftrightarrow$ weak $\mathbb{Q}$-Fano

then $P_{-(m-1)K_X}=-\frac{1}{12}m(m-1)(2m-1)K_X^3+(2m-1)\chi(\mathcal{O}_X)-l(m)$
(Prof. Chen & Prof. Chen Jiang anti-canonical Geometry)

$\hookrightarrow \chi(\mathcal{O}_X(mK_X)) \overset{\text{Serre duality}}{=} -\chi(\mathcal{O}_X((1-m)K_X))$
$=-\chi(\mathcal{O}_X(-(m-1)K_X)) \overset{\text{Kawamata-Viehweg}}{=} -h^0(-(m-1)K_X)$
$=: -P_{-(m-1)}(X)$
called the anti-canonical genus ☒

Kollár's way: using P_m to give global Geometry.

Thm. X smooth proj. general type 3-fold, $P_m=h^0(mK_X)\geq 2$
$\Rightarrow \phi_{|(m+5)K_X|}$ is birational (canonical volume $\mathrm{vol}(K_X)\geq 2$)
$\Rightarrow \mathrm{vol}(K_X)\geq 2/k^3$ below bounded

Idea $V \xrightarrow{\phi_{|mK|}}$
$\pi\downarrow$
$X \xrightarrow{\phi_{|mK_X|}} \mathbb{P}^{P_m-1}$

(Volume estimate is hard when $\chi(\mathcal{O}_X)\leq 0$ and K_X^3 is small, then we must estimate $l(m)$)

To study Kollár's way (one final task), we study the Geometry of linear system

Applying Stein decomposition. Assume moving part of $|mK_X|$ is base point free

Eg. (dim=1) $C \to |K_C|$ two cases
{ embedding: others.
{ 2-covering: hyperelliptic curve

taking pencil $\Lambda \subset |mK_X|$, $H_2 \in |mK_X|$; blowing up $\Lambda \Rightarrow W \xrightarrow{f} \mathbb{P}^1$, $|mK_X|$

$\mathcal{O}_{\mathbb{P}^1}(1) \hookrightarrow f_*\mathcal{O}(\omega^m)$
$\Rightarrow \mathcal{O}_{\mathbb{P}^1}(1)\otimes f_*(\omega_{W/\mathbb{P}^1}^{\otimes 5}) \hookrightarrow f_*\mathcal{O}(\omega_W^{m+5})$

Setting. V smooth variety (proj. always)
$h^0(V,D)\geq 2$ and basis $f_0 \cdots f_N$, $N=h^0(V,D)-1$
$\Rightarrow \phi_{|D|}: V \dashrightarrow \mathbb{P}^N \Rightarrow \dim \phi_D(V)>0$

(Fujita) f fibration $\Rightarrow \exists N$ (here $N=5$)
$f_*\omega_{W/\mathbb{P}^1}^{\otimes 5}$ is semi-positive (i.e. $=\mathcal{O}(a_1)\oplus\cdots\oplus\mathcal{O}(a_n)$)
Now ① Each fibre of f is desired general type surface } $\Rightarrow$ section gives ☒
② $\mathcal{O}_{\mathbb{P}^1}(1)\otimes f_*\omega_{W/\mathbb{P}^1}^{5}$ gbgs $\Rightarrow$ local section lift to $f_*\omega_W^{m+5}$ the desired birational map

Q. ① $\phi_{|D|}$ is regular
② $\phi_{|D|}$'s geometric property (fibres...)
③ $\phi_{|D|}$ is birational equivalence/embedding

(It's strictly proven by Prof. Chen) Kollár's way $(|m+5|)$ can't sharpen or loose
even $(|m+6|)$ can't be proven in this way.
Chen develop a differ way to deal with
By fix degree $\xrightarrow{\cong}$ then the ① gives out the Chow variety

$|D|=|M|+Z$ the moving part/movable part + fixed part
$\Rightarrow \phi_{|D|}=\phi_{|M|}$; ② $\Leftrightarrow$ $|M|$ has no base point, solved
All of this three questions highly depend in the positivity of D

③ $\phi_{|D|}$ birational $\Leftrightarrow \exists U \subset V$, $\phi_{|D|}|_U: U \to \mathbb{P}^N$
$\forall x,y\in U$, $\phi_{|D|}(x)\neq\phi_{|D|}(y)$
$\phi_{|D|}$ embedding
$\Leftrightarrow$ replace x,y by their infinitesimal neiberhoods.
(They're refered as criteria later)

We introduce that (both open)

P. ① (Fujita conjecture) L ample line bundle $\Rightarrow K_V+mL$ very ample for $m\geq \dim V+3$ (dim2 done, dim 3,4,5 done in base point free)

② (Matsusaka's last thm) L ample, $\exists N(L)$, $\forall m\geq N$, mL very ample (The best estimate of $N(L)$ given by Siu)

(It's not concerned in MMP, as ample line bundle seldom makes sense in minimal (always behaves singularity, not ...)
instead we consider nef & big $\mathbb{Q}$-Cartier divisor, then very ample problems are meaning less

Tankeev principle (We separate not only two points, but two subvarieties)

Def. $\varphi: V \dashrightarrow \mathbb{P}^N$ rational (nontrivial), with base set $\Sigma\subset V$, $W_1, W_2\subset V$ subvarieties. If (i) $W_i \not\subseteq \Sigma$; (ii) $\overline{\varphi(W_1)} \not\subseteq \overline{\varphi(W_2)}$, $\overline{\varphi(W_1)} \not\supseteq \overline{\varphi(W_2)}$
$\Rightarrow \varphi$ can separate W_1 and W_2

Given another linear system $|L|$ on V, base point free $\Rightarrow \phi_{|L|}: V \to W \subset \mathbb{P}^r$, denote $d_L = \dim \phi_{|L|}(V) \to \Gamma = W$, $\Sigma \dim V - n$ [Case I]

[Case I] $d_L \geq 2$, i.e. not a pencil, then by Bertini $\Rightarrow$ a general $N \in |L|$ is irreducible & smooth $\Rightarrow$ Stein decomposition [normal fibration]

[Case II] When $|L|$ is pencil, by Γ $\Rightarrow$ a general $N \in |L|$ ($\hookrightarrow$ Generic irreducible element $\in |L|$)

Then we can give out $V \to W$ consists $\sqcup$ (Fibres) i.e. $N = \sum F_i$ (Another Bertini thm), then the generic fibre [illegible]

Thm. (Tankeev principle) Setting as upper,

If ① $\phi_{|D|}$ seperate different generic irreducible elements of $|L|$; ② $\phi_{D}|_F$ is birational, $\forall F \in$ Generic irreducible elements of L

$\Rightarrow \phi_{|D|}$ is birational. (Pf, Ex. Convenient [K. Matsuki] [Chen])

Prop① (i) If $|L|$ not pencil or $|L|$ is pencil and $\Gamma \cong \mathbb{P}^1$

then $D \geq L \Rightarrow$ ① holds;

(ii) When $|L|$ is pencil and $\Gamma \not\cong \mathbb{P}^1$, then if $H^0(V,D) \to H^0(F_1, D|_{F_1}) \oplus H^0(F_2, D|_{F_2}) \to H^1(V, D-F_1-F_2)$

then ① also holds.

Any $x \neq y$ separated by F_1, F_2 except $p \in F_1 \cap F_2$ but by generic, we can modify to distinct

Prop② If $H^1(V, D-F) = 0 \Rightarrow$ ② holds.

$\parallel (\forall F_1 \neq F_2)$ 0

(Rk. Prop ① & Prop ② are a induction method in order to down dimension)

The "good" of this classical way seperate points is we don't need any positivity of D. If D has positivity, a modern approach can be done (Using vanishing thm)

Estimate of canonical volume (below bounded)

Set D further nef & big, we study $|K_X + mD|$ and $\phi_{|K_X+mD|}$ birational.

Rk. Three cases: Fano, general type, Calabi-Yau of MMP's terminal cases all satisfy this setting

To reduce to Tankeev principle's setting, blow up a base points of $|D| = Mov|D| + Fix|D|$, $\pi: V' \to V$, s.t.

① V' is smooth, projective; ② $Mov(\pi^* Mov(D))$ have no base point.

③ Supp $E_j \cup \pi^*(D)$ is SNC (E_j is exception divisor of π).

thus blowing up moving part also have fixed part

$$\begin{tikzcd} V' \arrow[r, "f"] \arrow[d, "\pi"] & \Gamma \arrow[d, "\cong"] \\ V \arrow[r, dashed, "\phi_{|D|}"] & \mathbb{P}^N \end{tikzcd}$$

Consider $|K_{V'} + \pi^*(mD)| \supset |\pi^*(K_V + mD)|$ and dimension are same $\Rightarrow$ they have same moving part, i.e. they are birationally equivalent.

$K_{V'} = \pi^* K_V + \sum E_j$, $\pi_*(\mathcal{O}_{V'}(K_{V'} + \pi^*(mD))) = \mathcal{O}_V(K_V + mD)$

$\Rightarrow Mov|\pi^*(D)|$ is base point free and $\pi^*(D)$ nef & big $\Rightarrow$ Tankeev principle holds. All can be paralleled carried out in $\mathbb{Q}$-factorial terminal case in right, but we need avoid the term of singular bracket, thus set $\exists m_i > 0$, $\dim \phi_{m_i D}(X) \geq i$, or adding that

Singular case ($\mathbb{Q}$-factorial terminal)

We modify ② into

② $Mov|\lfloor \pi^*(D) \rfloor|$ is base point free; also exists

Consider $|K_{X'} + \lceil m\pi^*(D) \rceil| \supset |\pi^*(K_X + mD)|$

④ $Mov|\lfloor \pi^*(m_iD) \rfloor|$ is base point free $\Rightarrow \pi^*(m_iD) \sim_{\mathbb{Q}} M_{m_i} + Z_{m_i}$ with M_{m_i} base point free and Z_{m_i} is effective, and ⑤ All involving divisors are SNC (for K-V) $\Rightarrow M_{m_i}$ nef & big

and dimension is still same by $\pi_* \mathcal{O}_{X'}(K_{X'} + \lceil m\pi^*(D) \rceil) = \mathcal{O}_X(K_X + mD)$ thus also birationally equivalent, study $|K_{X'} + \lceil m\pi^*(D) \rceil|$ finally.

thus ④ & ⑤ provides a way to down the dimension inductively.

taking $S \in |M_{m_0}|$ generic irreducible element ($m_0 = 1$ is always possible).

$M_{m_0} \equiv a_0 S$ ($a_0 \geq h^0(D) - 1$ by a_0 is the multiplicity of the fibres), taking $m_0 = 1 \Rightarrow \pi^*(D) \equiv a_0 S + Z_{m_0}$, by $m\pi^*(D) - S - \frac{1}{a_0} Z_{m_0} \equiv (m - \frac{1}{a_0}) \pi^*(D)$ nef & big

by Kawamata-Viehweg vanishing $\Rightarrow H^1(K_{X'} + \lceil m\pi^*(D) - S - \frac{1}{a_0} Z_{m_0} \rceil) = 0 \Rightarrow$ we have a surjection

$H^0(X', K_{X'} + \lceil m\pi^*(D) - \frac{1}{a_0} Z_{m_0} \rceil) \twoheadrightarrow H^0(S, K_S + \lceil m\pi^*(D) - S - \frac{1}{a_0} Z_{m_0} \rceil|_S)$, $\dim S = \dim X - 1$

By SNC $\Rightarrow |K_{X'} + \lceil m\pi^*(D) \rceil|\,|_S \geq |K_{X'} + \lceil m\pi^*(D) - \frac{1}{a_0} Z_{m_0} \rceil|\,|_S = |K_S + \lceil m\pi^*(D) - S - \frac{1}{a_0} Z_{m_0} \rceil|_S| \underset{SNC}{=} |K_S + \lceil (m\pi^*(D) - S - \frac{1}{a_0} Z_{m_0})|_S \rceil|$

$D_{S,m} \equiv (m - \frac{1}{a_0}) \pi^*(D)|_S$ is nef & big, then consider m_1, we have $|D| = |(M_{m_1})|_S|$ and $\pi^*(m_1 D)|_S \sim_{\mathbb{Q}} (M_{m_1})|_S + (Z_{m_1})|_S$ denoted $D_{S,m}$

By $H^0(M_{m_1} - S) \to H^0(M_{m_1}) \to H^0(S, (M_{m_1})|_S)$, then $(M_{m_1})|_S$ is codim 2

Only by check $+(Z_{m_1})|_S$

effective, moving + moving $\Rightarrow$ moving

Lets denote $S^{(1)} = S$, $S^{(2)} = \cdots$ all effective $(M_{m_1})|_S$ is moving

where $(M_{m_1})|_S \equiv a_1 S^{(2)} + (Z_{m_1})|_S$, $a_1 \geq h^0((M_{m_1})|_S) - 1 \geq 1$

$\boxed{K_{S^{(1)}} + \lceil D_{S^{(1)}, m} \rceil}$

$\Rightarrow D_{S^{(1)}, m} - S^{(2)} - \frac{1}{a_1} (Z_{m_1})|_{S^{(1)}} \equiv (m - \frac{m_1}{a_1} - \frac{m_2}{a_2}) \pi^*(D)|_{S^{(1)}}$ nef & big, thus by Kawamata-Viehweg $\Rightarrow$ we have again a surjection

$H^0(S^{(1)}, K_{S^{(1)}} + \lceil D_{S^{(1)}, m} - \frac{1}{a_1}(Z_{m_1})|_{S^{(1)}} \rceil) \to H^0(S^{(2)}, K_{S^{(2)}} + \lceil D_{S^{(2)}, m} \rceil)$

Such inductive step stops at $(M_{m_{n-2}})|_{S^{(n-2)}}$

$\Rightarrow$ We have $|K_{X'} + \lceil m\pi^*(D) \rceil|\,|_{S^{(1)}} \geq |K_{S^{(1)}} + \lceil D_{S^{(1)}, m} \rceil|\,|_{S^{(2)}} \geq \cdots$

$\geq |K_{S^{(n-1)}} + \lceil D_{S^{(n-1)}, m} \rceil| = |K_{S^{(n-1)}} + \lceil (m - \sum \frac{m_i}{a_i}) \pi^*(D)|_{S^{(n-1)}} \rceil|$ finally a curve, in curve $S^{(n-1)}$, $\deg D_{S^{(n-1)}, m} \geq (m - \sum \frac{m_i}{a_i}) \deg_{S^{(n-1)}} (\pi^*(D))$

If $\deg_{S^{(n-1)}}(\pi^*(D))$ is "good", then we can use Tankeev each dimension to left.

However, the deg is something subtle, we need consider more concrete cases.

(Such general argument is Tsuji method later applied to dim = 3 case.)

Taking $D = \pm K_X$, we can give volume estimate. ($D = K_X$: general type multi-canonical, $D = -K_X$: $\mathbb{Q}$-Fano anti-canonical)

Multi-canonical linear system: set X minimal & general type or ② X is $\mathbb{Q}$-Fano (weak), with $\mathbb{Q}$-factorial terminal singularity ($\mathbb{Q}$FT)

Recall. When ① & $P_{m_0}(X) \geq 2$ ($h^0(X, m_0 K_X) \geq 2$) $\Rightarrow \phi_{m_0}: X \dashrightarrow \mathbb{P}^{h^0 - 1}$ ($\hookrightarrow$ ($-K_X$) nef & big (or ample)) is nontrivial and $d_{m_0} = \dim \phi_{m_0}(X)$ called the m_0-canonical dimension $\Rightarrow 1 \leq d_{m_0} \leq 3$ (Here all three-fold), define $r_s(X) = \min\{p \mid \forall m \geq p,\ \phi_{m,X}$ birational$\}$ called the canonical stability index of X, by Kollár's consequence, $r_s(X) \leq 11 m_0 + 5$.

Blowing up (First blow up singularities, then blow up to make base point free)

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr] & \Gamma \text{ (normal)} \\ X & \mathbb{P}^{h^0 - 1} \end{tikzcd}$$

Setting ① ② [illegible] $Mov|\lfloor m_0 \pi^*(K_X) \rfloor|$ is base point free

Exercise Show that $|M_{m_1}| = \mathrm{Mov}|m_1 K_{X'}|$

Hint. Showing that $H^0(m_1 K_{X'}) = H^0(\mathrm{Mov}|m_1 K_{X'}|)$
$= H^0(\mathrm{Mov}|\pi^*(m_1 K_X)|) = H^0(\mathrm{Mov}|m_1 \pi^*(K_X)|)$
$= H^0(m_1 K_X)$ first

At last, we need ① At S & C, using $Q_{m,S}/Q_{m,C}$ distinguish different subvarieties (subtle but not hard)

② Estimate $\deg(Q_{m,C})$

$|m K_{X'}||_S|_C \supset |K_S + \lceil Q_{m,S}\rceil|_C \supset |K_C + Q_{m,C}|$

We have $\deg(|m\rangle|_C) \geq \deg(|K_C + Q_{m,C}|)$ (if $\deg Q_{m,C} \geq 2$)

$\Rightarrow m\xi \geq 2g(C) - 2 + \deg Q_{m,C}$
$\geq 2g(C) - 2 + \lceil (m-1-\frac{m_0}{a_{m_0}} - \frac{m_1}{a_{m_1}})\xi \rceil$

Let $m >> 0$, $\deg Q_{m,C} \geq 2$ holds obviously

$\Rightarrow \boxed{\xi \geq \frac{2g(C)-2}{1+\frac{m_0}{a_{m_0}}+\frac{m_1}{a_{m_1}}}}$

Set $m_0 \pi^*(K_X) = M_{m_0} + E_{m_0}'$
$m_0 K_{X'} = M_{m_0} + Z_{m_0}$
$K_{X'} = \pi^*(K_X) + E_\pi$ (both eff. $\mathbb{Q}$-divisor)

and the generic irreducible element

$S = \begin{cases} \in |M_{m_0}| & ; \dim \Gamma > 1 \\ \text{the generic fibre} & ; \dim \Gamma = 1 \end{cases}$

Recall. then we have $M_{m_0} \equiv a_{m_0} S$

$a_{m_0} = \begin{cases} 1 & ; \dim \Gamma > 1 \\ \geq P_{m_0} - 1 & ; \dim \Gamma = 1 \end{cases}$

and $m_0 \pi^*(K_X) \equiv a_{m_0} S + E_{m_0}'$

to apply the general induction argument before to here we restrict $|m K_{X'}|$ & $|m K_{X'}|$ to surface S

We restrict $|m_0 K_{X'}|$ & $|m_0 K_{X'}|$ to curve C (again generic irreducible elements)

and $Q_{m,S} = (Q_{m,S} - C - \frac{1}{a_{m_1}} E_{m_1}')|_S$

$\equiv (m-1-\frac{m_0}{a_{m_0}} - \frac{m_1}{a_{m_1}})\pi^*(K_X)|_S$ nef & big

$H^0(S, K_S + \lceil Q_{m,S} - \frac{1}{a_{m_1}} E_{m_1}|_S \rceil) \to H^0(C, K_C + \lceil Q_{m,C} \rceil)$

$Q_{m,C} = Q_{m,S}|_C$ also nef & big $\Rightarrow (C, K_C + \lceil Q_{m,C} \rceil)$

desired pair $\Rightarrow \deg Q_{m,C} \geq (m-1-\frac{m_0}{a_{m_0}} - \frac{m_1}{a_{m_1}}) \underbrace{\pi^*(K_X)|_S \cdot C}_{\xi}$

to apply the Takeov principle, we need a numerical formula

$\Uparrow$ (With more assumption (but it can be done) by modify π) $\Rightarrow |M_{m_1}| = \mathrm{Mov}|m_1 \pi^*(K_X)|$

Then run it, we have (K-V vanishing)

$H^0(X', K_{X'} + \lceil (m-1)\pi^*(K_X) - \frac{1}{a_{m_0}} E_{m_0}' \rceil) \to H^0(S, \boxed{K_S + \lceil Q_{m,S} \rceil}) \to 0$

where $Q_{m,S} := ((m-1)\pi^*(K_X) - S - \frac{1}{a_{m_0}} E_{m_0}')|_S$
$\equiv (m-1-\frac{m_0}{a_{m_0}})\pi^*(K_X)|_S$ nef & big

$(S, |K_S + \lceil Q_{m,S} \rceil|)$ reduced, and it's studied well by Japanese school before. Prof. Chen studied it by restrict to curve (again the general argument as before)

$\geq \frac{a_{m_0}}{m_0} \frac{1}{m_1} (\pi^*(K_X)|_S \cdot m_1 \pi^*(K_X)|_S)$ (it's minimal model) $= \frac{a_{m_0} a_{m_1}}{m_0 m_1} (\pi^*(K_X)|_S \cdot C) \geq \frac{a_{m_0} a_{m_1}}{m_0 m_1} \left(\frac{2g(C)-2}{1+\frac{m_0}{a_{m_0}}+\frac{m_1}{a_{m_1}}} \right)$ ▨

Coro. We have volume estimate $\mathrm{Vol}(X) \geq \frac{2 a_{m_0} a_{m_1} (g(C)-1)}{m_0 m_1 (1+\frac{m_0}{a_{m_0}}+\frac{m_1}{a_{m_1}})}$ (it's best some cases)

Pf. $\mathrm{Vol}(X) = \mathrm{Vol}(X') = K_X^3 = \frac{1}{m_0} \pi^*(K_X)^2 \cdot m_0 \pi^*(K_X) \geq \frac{a_{m_0}}{m_0} (\pi^*(K_X)^2 \cdot S)$

Using m_0 determines m_1 (best some cases) (Kollár told us $m_1 = 11 m_0 + 5$, it's too big) When $|m_0 K_X|$ isn't pencil, taking $m_1 = m_0$ trivial

Assume $|m_0 K_X|$ pencil, then we acquire $|K_S + \lceil Q_{m,S} \rceil|$ moving, i.e. $h^0(K_S + \lceil Q_{m,S} \rceil) \geq 2$, comparison $\pi^*(K_X)|_S$ and $K_S \Rightarrow \pi^*(K_S)|_S \leq K_S$

$\Rightarrow \boxed{2\sigma^*(K_{S_0}) \leq \pi^*(K_X)|_S} \leq K_S$, where $\sigma: S \to S_0$. Then using Kawamata's extension thm: $H^0(m(K_{X'} + S)) \to H^0(m K_S) \to 0$

$(\exists a \in \mathbb{Q}) \Uparrow$ $\Rightarrow m(\pi^*(K_X) + m_0 \pi^*(K_X))|_S \geq \mathrm{Mov}|m(K_{X'} + S)||_S \geq \mathrm{Mov}|m K_S| = m\sigma^*(S_0)$

Kodaira lemma. L nef & big $\Rightarrow \exists m_0$ s.t. $mL - H \geq 0$ for H fixed.

$m(m_0+1)\pi^*(K_X)|_S$ when $m >> 0$

Pf. $H^0(mL - H) \to H^0(mL) \to H^0(H) \to \cdots$
$\quad\quad \Downarrow$
$\quad\quad 0$
m^n order $\quad m^n$ order
$\Rightarrow$ when m large
$H^0(mL - H) \neq 0 \Rightarrow mL - H \geq 0$ ▨

Thus we have the following comparison inequality $\pi^*(K_S)|_S \geq \frac{1}{m_0+1} \sigma^*(K_{S_0})$

apply the thm of Xiao Gang with $\pi^*(K_S)|_S \geq \frac{1}{(m_0+1)} 2\sigma^*(K_{S_0})$

Conclude. $\boxed{m_1 \leq 2m_0 + 2}$

Thm. (Xiao Gang) For S_0 general type & minimal, $|2K_S|$ pencil $\Leftrightarrow P_2(S) > 2 \Leftrightarrow K^2 = 1$, $P_g = 0$ is impossible.

Eg. X is 3-fold general type, $P_g \geq 2$

then ① $\mathrm{Vol}(X) \geq \frac{1}{3}$ ② $r_S(X) \leq 8$; ③ $r_S(X) = 8 \Leftrightarrow P_g(X) = 2$ & $\mathrm{vol}(X) = \frac{1}{3}$

When $X = X_{16} \subset \mathbb{P}(1,1,2,3,8)$ (weighted projective space), all equalities hold (X_{16} is deg $= 16$ and general type hypersurface)

And it's the unique example taking equality. And we can study it precisely the $M_{\frac{1}{3},2}$

Let's compute ① the volume. $P_g \geq 2 \Rightarrow m_0 = 1$.

Observation. Easy cases $\begin{cases} d_1 = 3 \Rightarrow K_X^3 \geq 2 \\ d_1 = 2 \Rightarrow K_X^3 \geq 1 \\ d_1 = 1, g(C) > 0 \Rightarrow K_X^3 \geq 1 \end{cases} > \frac{1}{3}$

Only case $d_1 = 1$, $\Sigma \cong \mathbb{P}^1$ is hard

When S is $(1,2)$-type; When S not $(1,2)$-type
$(K_{S_0}^2 = 1, P_g = 2) \Rightarrow \mathrm{vol}(X) \geq \frac{1}{2} > \frac{1}{3}$

$\Rightarrow g = 0$ by Leray-Serre. consider the choice of m_1:

$H^0(K_{X'} + \lceil \pi^*(K_X) \rceil + S) \to H^0(K_S + \lceil \pi^*(K_X)|_S \rceil)$

$\Rightarrow |3K_X|$ is base point free, $M_3|_S \geq \mathrm{Mov}|K_S|$

thus $m_1 \leq 3$, we can take $m_1 = 2$ due to the comparison inequality: $\sigma^*(K_{S_0})$ had been moving here $\Rightarrow m_0 + 1 \geq m_1$

$\Rightarrow m_0 = 1$ & $m_1 = 2$ $(a_0 = a_1 = 1)$

then compute $\xi = (\pi^*(K_X)|_S \cdot C) \geq \frac{2g-2}{1+1+2} = \frac{1}{2}$

taking $m = 7$, $(7-1-1-2)\frac{1}{2} > (2g(C)-2) = \frac{3}{2} > 1$

$\Rightarrow 7\xi \geq 2 + 2 \Rightarrow \xi \geq \frac{4}{7}$ is sharpen.

Repeat the sharpen and taking the limit $\Rightarrow \xi \geq \frac{2}{3}$

$\Rightarrow K_X^3 \geq \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$ ▨

Coro. $(8-4)\frac{2}{3} = \frac{8}{3} > 2 \Rightarrow \varphi_{8,X}$ birational

Rk. (Difficulty in $\mathbb{Q}$-Fano) $m_0 \leadsto m_1$? It's harder and a complete different method. Prof. Chen and the PHD Thesis of Prof. Jiang tell $|m_1 K_X|$ not pencil $\Rightarrow m \geq 97$, birational (Later $m \geq 39$ .. hard example shows 33 possible)

Prof. Chen can use the old method show $(-K_X)^3 \geq \frac{1}{330}$ (best)

Next question: Finding m_0 (Reid-Riemann-Roch)

$P_{m_0} = h^0(m_0 K_X) = \chi(m_0 K_X) = \frac{1}{12} m_0(m_0-1)(2m_0-1) K_X^3 - (2m_0-1)\chi(\mathcal{O}_X) + l(m_0)$

Prof. Chen notice that it's impossible to compute directly

And using weighted singular basket, then taking partial order to compute $l(m_0) \geq 2$ (hard part)

Weighted singular basket theory on threefolds

Recall the singularity basket $B = \{(b_i, r_i)\}$ $\begin{cases} i = 1, \ldots, s \\ 0 < b_i \leq r_i, \gcd(b_i, r_i) = 1 \end{cases}$ $(b_i, r_i) := \frac{1}{r_i}(1, -1, b_i)$

set $\sigma(B) = \sum b_i$, $\sigma'(B) = \sum (b_i^2 / r_i)$, $\forall n > 0$; $\delta = \lfloor \frac{b_i n}{r_i} \rfloor \Rightarrow \frac{\delta}{n} < \frac{b_i}{r_i} \leq \frac{\delta+1}{n}$

$\Delta^n(b_i, r_i) = \delta b_i n - \frac{\delta(\delta+1)}{2} r_i \in \mathbb{Z}_{\geq 0}$

$\Delta^n(B) = \sum \Delta^n(b_i, r_i)$

Partial order on baskets by squeezing singularities: $\{(1,2), (2,5)\} \geq \{(3,8)\}$

The canonical sequence of singularity basket: $S^{(0)} = S^{(n_0)} \cup \{\frac{1}{n} | 1 < i \leq \frac{n}{2}, \gcd(i,n) = 1\}$ and $S^{(0)} = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

Eg. $S^{(4)}=S^{(3)}=S^{(2)}=S^{(1)}=S^{(0)}$, $S^{(5)}=S^{(0)}\cup\{\frac{2}{5}\}$, $S^{(6)}=S^{(5)}$, $S^{(7)}=S^{(6)}\cup\{\frac{2}{7},\frac{3}{7}\}$; then it partion $(0,\frac{1}{2}]$ into $\bigcup[\omega_{j+1}^{(n)},\omega_j^{(n)}]$ $=\bigcup[\frac{q_{j+1}^{(n)}}{p_{j+1}^{(n)}},\frac{q_j^{(n)}}{p_j^{(n)}}]=\bigcup[\frac{q_{j+1}^{(n)}}{p_{j+1}^{(n)}},\frac{q_j^{(n)}}{p_j^{(n)}})$

We define the canonical sequence $\{B^{(n)}\}$ of B by induction, for $n=0$ case, $\exists!\, n_0$ $\frac{1}{n_0+1}\le\frac{b_i}{r_i}<\frac{1}{n_0}$

Set $B_i^{(0)}:=\{(n_0b_i+b_i-r_i)\times(1,n_0),(r_i-n_0b_i)\times(1,n_0+1)\}$ here $\times$ meaning squeeze $\Rightarrow B_i^{(0)}\ge B_i$, $B^{(0)}:=\bigcup B_i^{(0)}$

$\Rightarrow B^{(0)}\ge B$, then repeating. We finally have $\frac{b_i}{r_i}\in[\frac{q_{j+1}^{(n)}}{p_{j+1}^{(n)}},\frac{q_j^{(n)}}{p_j^{(n)}})=[\frac{q_1}{p_1},\frac{q_2}{p_2})$ $\begin{cases}\boxed{\text{Case I}}\ \frac{b_i}{r_i}\in S^{(n)}\Rightarrow B_i^{(n)}=\{\frac{b_i}{r_i}\}\\ \boxed{\text{Case II}}\ \text{Otherwise}\Rightarrow B_i^{(n)}=\{(r_iq_2-b_ip_2)\times(q_1,p_1),(r_iq_1-b_ip_1)\times(q_2,p_2)\}\end{cases}$

$\Rightarrow B^{(n)}=\bigcup B_i^{(n)}\ge B$ and $B^{(n)}\xrightarrow{n\to\infty}B$. $B^{(0)}\ge B^{(5)}\ge\cdots\ge B^{(n)}\ge\cdots\ge B$

Eg. $B=\{\frac{3}{8}\}$, $\frac{1}{3}<\frac{3}{8}\le\frac{2}{5}\Rightarrow B^{(0)}=\{(1,3),2\times(1,3)\}\ge B^{(5)}=\{(1,3),(2,5)\}=B^{(8)}=B$.

Prop. ① $j=3,4$, $\Delta^j(B^{(0)})=\Delta^j(B)$; ② $j<n$, $\Delta^j(B^{(n-1)})=\Delta^j(B^{(n)})$; ③ $\Delta^n(B^{(n-1)})=\Delta^n(B^{(n)})+\varepsilon_n$; ④ $\Delta^n(B^{(n)})=\Delta^n(B)$. $(K^3(B))$

After combinational setting, we can define weighted singularity basket:

Def. $\mathbb{B}=\{B,\chi_2,\chi\}$, B is basket and $\chi,\chi_2\in\mathbb{Z}$ called WSB. Set $\chi_3(\mathbb{B})=-\sigma(\mathbb{B})+10\chi+5\chi_2\in\mathbb{Z}$ and $\mathrm{Vol}(\mathbb{B})=\sigma'(\mathbb{B})-4\chi-3\chi_2+\chi_3$ $=-\sigma+\sigma'+6\chi+2\chi_2\in\mathbb{Q}$, inductively we define $\chi_{m+1}(\mathbb{B})=\chi_m(\mathbb{B})+\frac{m^2}{2}(\mathrm{Vol}(\mathbb{B})-\sigma'(\mathbb{B}))+\frac{m}{2}\sigma(\mathbb{B})-2\chi+\Delta^m(\mathbb{B})$ $(m\ge3)$ $\in\mathbb{Z}$

Geometric meaning: when X ~~is~~ canonical singular threefold with B to Reid's basket and $\chi_2=\chi(\mathcal{O}_X(2K_X))$ and $\chi=\chi(\mathcal{O}_X)$, then $\mathrm{Vol}(\mathbb{B})=\mathrm{Vol}(X)$ and $\chi_m=\chi(\mathcal{O}_X(mK_X))$. Thus we turn the numerical computation of RRR into a general combinational setting:

Prop. Vol and χ_m preserve under the partial ordering;

Combinational	**Geometric**
$\mathbb{B}=(B,\chi_2,\chi)$ pure combinational	$\mathbb{B}=(B,\chi_2,\chi)$ realize induced by Reid datum.
vol, χ_m; inverse problem	vol χ_m
Using $\mathbb{B}$ return back B	

Idea: using $B^{(0)}\ge B^{(5)}\ge\cdots\ge B$ and Prop to approximate.

Computation of inverse problem

If $B^{(0)}=\{n_{1,2}^0\times(1,2),n_{1,3}^0\times(1,3)\cdots\}$

We can determine these n by

$n_{1,2}^0=\Delta^3(B^{(0)})$, $2n_{1,2}^0+n_{1,3}^0=\Delta^4(B^{(0)})$, ...

and $\Delta^3,\Delta^4\cdots$ can be computed by $\chi,\chi_2\cdots$ by datum of $\mathbb{B}$.

(Eg. $\Delta^{12}=230\chi+180\chi_2-16\chi_3-\chi_{12}+\chi_{13}$)

$\Rightarrow\begin{cases}n_{1,2}^0=5\chi+6\chi_2-4\chi_3+\chi_4\\ n_{1,3}^0=4\chi+2\chi_2+2\chi_3-3\chi_4+\chi_5\\ \cdots\end{cases}$ and repeat this for $B^{(5)}$...

And we can find the key inequality:

$(*)$ $2\chi_5+3\chi_6+\chi_8+\chi_{10}+\chi_{12}\ge\chi+10\chi_2+4\chi_3+\chi_9+\chi_{11}$ $+\chi_{13}+R\ge\chi+10\chi_2+4\chi_3+\chi_7+\chi_{11}+\chi_{13}$.

Applications ① For minimal general type 3-fold $\Rightarrow\chi_m=P_m(X)$ $\begin{cases}\chi<0\Leftrightarrow\chi(\mathcal{O}_X)<0, \text{ by RRR}, P_2(X)\ge4, \text{ we can use Kollár's way, done}\\ \chi>0\ \begin{cases}\exists\, 0<m_0\le12, \text{ s.t. } P_{m_0}\ge2, \text{ then Kollár's way also done}\\ \forall m_0, P_{m_0}\le1, \text{ by }(*)\Rightarrow P_{13}=\chi_{13}\text{ has upper bound}\\ \qquad\Rightarrow B^{(12)}\text{ has finite possible cases}\end{cases}\end{cases}$

then consider the $B_{min}\le B^{(12)}$ also finite cases.

Then $B^{(12)}\ge B_X\ge B_{min}\Rightarrow K^3(X)=K^3(\mathbb{B}(X))\ge K^3(B_{min})$ can be computed a below bound.

Main thm ① $P_{12}>0$, $P_{24}\ge2$; ② $r_S(X)\le57$; ③ $K_X^3\ge\frac{1}{1680}$

② Weighted complete intersection three fold (Iano-Fletcher), we can completely classify them.

Consider generic $X_{d_1\cdots d_c}\subset\mathbb{P}^{(a_1\cdots a_{c+3})}$, c is the codimension of X in the weighted projective space. Generic $\Rightarrow X$ has terminal singu... Fletcher used computer to give $K^3=\frac{1}{420}$'s example, which it's minimal now: $X_{46}\subset\mathbb{P}^{(4,5,6,7,23)}$ Chen Chen Chen's work classifying all ... dealing the Fletcher types conjectures. (Chen Zhengjie, Chen Rongkai, Chen Meng)

③ Anticanonical Geometry of $\mathbb{Q}$-Fano 3-fold.

Main thm ① $-K_X^3\ge\frac{1}{330}$ is best below bound, example given by Fletcher ② (With Chen Jiang) $m\ge97$, φ_{-m} birational equivalence (now ≥59)

Rk. Such method can't applyed to Kodaira dimension 1 & 2 — Chen Meng.

Open problems ① The best bound is $v_3=\frac{1}{420}$ and $r_3=27$. (Jiang Zhi says the former may have counterexample when $\chi(\mathcal{O}_X)=2$ or 3, $h^2(\mathcal{O}_X)=1$ or 2)

② Weak $\mathbb{Q}$-Fano best bound $-K_X^3$? (Jiang Cheng ≤1300)

③ $n\ge4$ cases finding $m>0$ s.t. $P_m\ge2$, estimate $h^0(mK_X)$ below bound; ④* Calabi-Yau, 3 dimension, finite deformation classes? (Now due to it's open, we even can't define moduli)