Notation: $\hbar = \frac{h}{2\pi}$, $\lambda = \frac{h}{p}$, $k = \frac{2\pi}{\lambda} = \frac{p}{\hbar}$, $\omega = 2\pi\nu = \frac{E}{\hbar}$, $E = h\nu = \hbar\omega$, $\sinh(x) = \frac{e^{x}-e^{-x}}{2}$, $\cosh(x) = \frac{e^{x}+e^{-x}}{2}$ $\Rightarrow \cosh^2(x) - \sinh^2(x) = 1$

[illegible]

BCH: $e^{A}e^{B} = e^{A+B+\frac{1}{2}[A,B]+\cdots}$

We call Ψ probability amplitude and $|\Psi|^2$ the probability density

Rewrite Schrödinger equation as $i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi$ [illegible]

The expected value of observable $\langle A \rangle = \langle \Psi | A | \Psi \rangle$ [illegible] the uncertainty [illegible]

$\sigma_A = \langle A - \langle A \rangle \rangle$ [illegible] $\Rightarrow \sigma_A \sigma_B \geq \frac{1}{2i}\langle [A,B] \rangle$, minimize, we call coherent states. Application [illegible]

Pf. Set $f = (A - \langle A \rangle)\Psi$ and $g = (B - \langle B \rangle)\Psi$ [illegible]

$\Rightarrow \sigma_A^2 \sigma_B^2 = \langle f|f \rangle \langle g|g \rangle \geq |\langle f|g \rangle|^2 \geq \left(\frac{1}{2i}\langle [A,B] \rangle\right)^2$ ▨ Schrödinger

Heisenberg equation: $\frac{d}{dt}\langle A \rangle = \langle \dot{\Psi}|A\Psi \rangle + \langle \Psi|A\dot{\Psi} \rangle + \langle \Psi|\frac{\partial A}{\partial t}\Psi \rangle$ [illegible]

$= \frac{i}{\hbar}\langle HA - AH \rangle + \langle \frac{\partial A}{\partial t} \rangle = \frac{i}{\hbar}\langle [H,A] \rangle + \langle \frac{\partial A}{\partial t} \rangle$ ▨ In Heisenberg picture, A also depends on time, we set time-dependent operator $A_H = U(t,0)^{\dagger} A U(t,0) = e^{iHt/\hbar}\hat{A}e^{-iHt/\hbar}$, $\frac{\partial}{\partial t}A_H(t) = \frac{i}{\hbar}[H, A_H]$

Probability flow: $|\Psi|^2 = \rho$ probability density. Due to the probability is conserved

$\Rightarrow$ we have flow out and into region $[a,b]$ when time changes

$\frac{d}{dt}P_{ab} = \frac{i\hbar}{2m}\int_a^b \frac{\partial}{\partial x}(\Psi^*\Psi_x - \bar{\Psi}_x\Psi)dx = \frac{\hbar}{m}\int_a^b \frac{\partial}{\partial x}\mathrm{Im}(\Psi^*\Psi_x)$

we denote $J(x,t) = \frac{\hbar}{m}\mathrm{Im}(\Psi^*\Psi_x)(x,t)$ the probability flow density

We can use such $\Rightarrow \frac{d}{dt}P_{ab} = \int_a^b [illegible]\,dx$, let $b \to a$, collapse to single pt $\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}J = 0$

probability language to recover physics

Exercises: ① Replace probability by energy, what's the energy flow of string wave equation

② $\int J\,dx = \langle P \rangle$

Pf. ① Conservation equation is $\frac{\partial \varepsilon}{\partial t} + \frac{\partial J}{\partial x} = 0$ similiarly, ε is energy density, then we compute J from it.

ε = potential energy density + momentum energy density [illegible]

② $\int J\,dx = \int \frac{\partial J}{\partial x}x\,dx = [illegible] \Rightarrow \int J\,dx = [illegible]$

Classical limit is: let $\Psi = \sqrt{\rho}e^{iS/\hbar}$ [illegible]

[illegible] real part when $\hbar \to 0$, imaginary part totally degenerates $\Rightarrow$ only $\frac{\partial S}{\partial t} + \frac{1}{2m}|\nabla S|^2 + V = 0$

[illegible] imaginary is Hamilton-Jacobi equation, S is action.

Probability explanation: Here our $J = \rho\frac{\nabla S}{m}$, multiply imaginary part by 2ρ $\Rightarrow \frac{\partial \rho}{\partial t} + \nabla J = 0$ coincide above!

Formalism: ① Given $H/\mathbb{C}$ ② Observable $\Leftrightarrow$ (unbounded) self-adjoint operators ③ Measure A contract into an eigenstate, the consequence is its eigenvalue and each eigenstate has probability $P(\lambda_n) = |\langle \Psi_n|\Psi \rangle|^2$, $P(a_n) = \sum |\langle \Psi_n|\Psi \rangle|^2$

Ψ is state before measurement ④ Schrödinger equation ⑤ Uncertainty Principle/Commutator, we can measure basis them once

⑥ $\rho = |\Psi|^2$ is probability density

Path integral formalism: A particle propagates from (q,t) to (q',t') has probability amplitude $K(q',t'|q,t)$ called propagator

In Schrödinger picture, H not depend on t, we have time evolution operator $U(t,t_0)$, $\Psi(t) = U(t,t_0)\Psi(t_0)$, $U(t',t) = e^{-iH(t'-t)/\hbar}$

$K(q',t'|q,t) = \langle q'|U(t',t)|q \rangle = \sum \Psi_n(q')\bar{\Psi}_n(q)e^{-iE_n(t'-t)/\hbar}$ for energy eigenstates Ψ_n. Now partition time (t,t') into small pieces $(t_n, t_n) \Rightarrow$

$K(q',t'|q,t) = \lim \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{N/2}\int dq_1 \cdots \int dq_{N-1} \prod \exp\left[\frac{i}{\hbar}S(q_n,t_n|q_{n-1},t_{n-1})\right] = \int D[q(t)]\exp\left(\frac{i}{\hbar}S[q(t)]\right)$, $q(t)$ are all paths connect (q,t) with (q',t')

A measure on proper path space

Pf. $\Psi(q',t') = \int K(q',t'|q,t)\Psi(q,t)\,dq$, $K(q',t'|q,t) = \int dq_1 \cdots \int dq_{N-1} K(q',t'|q_{N-1},t_{N-1}) \cdots K(q_1,t_1|q,t)$, each $K(q_n,t_n|q_{n-1},t_{n-1})$ compute

$K(q_n,t_n|q_{n-1},t_{n-1}) = \langle q_n|e^{-i\varepsilon(T+V)/\hbar}|q_{n-1} \rangle = [illegible]$

[illegible]

By stationary phase, higher frequency $\Rightarrow$ left classical limit, is $\delta S = 0$, this is minimal action principle

Path integral is topological, in this case without realistic effect, two paths are equivalent if homotopic, thus we can push them to boundary.

Unbounded operator: Hilbert space, for A, $\mathrm{Dom}(A^*) = \{x \in H \mid x \mapsto \langle x, Ay \rangle, y \in \mathrm{Dom}(A)$ is a bounded functional$\}$, $A^*: \mathrm{Dom}(A^*) \to H$ satisfy $\langle A^*x, y \rangle = \langle x, Ay \rangle$ for $\forall y \in \mathrm{Dom}(A)$. A is symmetric if $\forall x, y \in \mathrm{Dom}(A)$, $\langle x, Ay \rangle = \langle Ax, y \rangle \Leftrightarrow A \subset A^*$, self-adjoint if $A = A^*$

Closable operator/Closure $\Leftrightarrow$ The graph has closure = graph of its closure; A essentially self-adjoint if A symmetric + $\bar{A}$ self-adjoint.

A^* closed [illegible] $\Rightarrow A = A^*$ if A symmetric. If A symmetric and $\mathrm{Dom}(A) = H \Rightarrow A$ bounded

Closable $\Rightarrow (\bar{A})^* = A^*$; if A essentially self-adjoint $\Rightarrow \bar{A}$ is the unique self-adjoint extension of A; symmetric $\Rightarrow$ closable

$\mathrm{Dom}(A+B) = \mathrm{Dom}(A) \cap \mathrm{Dom}(B)$; if B bounded, $\mathrm{Dom}(A+B)^* = \mathrm{Dom}(A^*)$ and $(A+B)^* = A^* + B^*$

Spectrum: $A - \lambda I$ bounded below on $\mathrm{Dom}(A) \Rightarrow \mathrm{Ran}(A - \lambda I)$ closed; $\sigma(A) = \mathrm{Res}(A)^c$, $\mathrm{Res}(A) = \{\lambda \in \mathbb{C} \mid \forall y \in H, \exists ! x \in \mathrm{Dom}(A)$ and $(A - \lambda I)x = y$, $B(A - \lambda I)y = y\}$

A self-adjoint $\Rightarrow \sigma(A) \subset \mathbb{R}$, closed, $\sigma(A) = \{\lambda \in \mathbb{R} \mid \exists x_n \in \mathrm{Dom}(A) - \{0\}, \lim \frac{\|(A - \lambda I)x_n\|}{\|x_n\|} = 0\}$

When A symmetric, A essentially self-adjoint $\Leftrightarrow \mathrm{Ran}(A - iI)$, $\mathrm{Ran}(A + iI)$ dense in $H \Leftrightarrow A^* + iI$, $A^* - iI$ injective on $\mathrm{Dom}(A^*)$

Self-adjoint $\Leftrightarrow \mathrm{Ran}(A \pm iI) = H$; $f: X \to \mathbb{C}$ measurable, μ projection-operator-valued measure on X, $B_f = \{x \in H \mid \int |f|^2 d\mu_{x,x} < \infty\}$

$\Rightarrow \exists ! A = \int f\,d\mu$, s.t. $\langle x, Ay \rangle = \int f\,d\mu_{x,y}$ and $\mathrm{Dom}(A) = B_f$; Conversely, any self-adjoint operator $A = \int \lambda\,d\mu^A$, $\exists !\,\mu^A$

Ex. [illegible] if $\mathrm{Dom}(P)$ chosen to be C^{∞} [illegible] $\Rightarrow P$ is symmetric but not self-adjoint, not essential [illegible]

④ However, self-adjoint behave bad for addition: let $V(x)=-x^4$, then $\hat{H}=\frac{\hat{p}^2}{2m}+\hat{V}$ is not essentially self-adjoint and $\hat{H}^*$ not symmetric even!
pf. ① Symmetry is trivial, but our problem is the boundary condition $\psi(0)=\psi(1)$ restrict $\hat{p}\psi=\lambda\psi$ has only trivial $\psi \Rightarrow$ no eigenvalue, hence the space too small to make self-adjoint ▨; ② Here we can let $|x|\to\infty$, hence taking closure suffices ▨; ③ Symmetric obvious, one use the definition of $Dom(\hat{A}^*)$ to prove then omitted ▨; ④ $\hat{H}$ symmetric obvious, but if $\hat{H}$ essentially self-adjoint $\Rightarrow \hat{H}^*=\hat{H}$ is self-adjoint $\Rightarrow \hat{H}^*$ is symmetric $\Rightarrow \forall a, \forall\psi\in Dom(\hat{H}^*)$ $(\hat{H}^*-ia\hat{I})$ below-bounded by a, but we can find ψ_a not hold this! ▨

Angular momentum and Spin $\hat{L}=\hat{r}\times\hat{p}=-i\hbar(\hat{r}\times\nabla) \Rightarrow \hat{L}_x=y\hat{p}_z-z\hat{p}_y$, $\hat{L}_y=z\hat{p}_x-x\hat{p}_z$, $\hat{L}_z=x\hat{p}_y-y\hat{p}_x=-i\hbar\frac{\partial}{\partial\varphi}$, where components
$\Rightarrow [\hat{L}_x,\hat{L}_y]=i\hbar\hat{L}_z$, $[\hat{L}_y,\hat{L}_z]=i\hbar\hat{L}_x$, $[\hat{L}_z,\hat{L}_x]=i\hbar\hat{L}_y$ $\quad = i\hbar(\sin\varphi\frac{\partial}{\partial\theta}+\cot\theta\cos\varphi\frac{\partial}{\partial\varphi})$, $i\hbar(-\cos\varphi\frac{\partial}{\partial\theta}+\cot\theta\sin\varphi\frac{\partial}{\partial\varphi})$ $\quad \hat{p}_x=-i\hbar\partial_x$
quantum bits $|0\rangle$ and $|1\rangle$ given by spin-$\frac{1}{2}$ $|\uparrow\rangle$ and $|\downarrow\rangle$
$\hat{L}^2=\hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2$ (distinguish $\hat{L}$ and L), L is called total angular momentum
$\Rightarrow [\hat{L}^2,\hat{L}]=0$ (or write in three components) and $\sigma_{L_x}^2\sigma_{L_y}^2 \geq (\frac{1}{2i}\langle i\hbar L_z\rangle)^2=\frac{\hbar^2}{4}\langle L_z\rangle^2 \Rightarrow \sigma_{L_x}\sigma_{L_y}\geq\frac{\hbar}{2}|\langle L_z\rangle| \Rightarrow$
$|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$ ... Bloch sphere
Ladder operators $\hat{L}_\pm=\hat{L}_x\pm i\hat{L}_y$, $[\hat{L}_+,\hat{L}_-]=2\hbar\hat{L}_z$, $[\hat{L}^2,\hat{L}_\pm]=0$, $[\hat{L}_z,\hat{L}_\pm]=\pm\hbar\hat{L}_\pm \Rightarrow \hat{L}_z(\hat{L}_\pm Y)=(\mu\pm\hbar)(\hat{L}_\pm Y)$
EPR paradox: entangled state ... entanglement states ... can't write as direct ⊗
We can compute the eigenstates only from abstract commutation relation then: adding/minus $\hbar$ eigenvalue
Bell states have maximal entanglement
$\langle L_z^2\rangle\leq\langle L^2\rangle \Rightarrow \mu\pm\hbar$ has below bounds and upper bound, after computation, we
due to it's not linear
Y is eigenstate of $\hat{L}^2$ and $\hat{L}_z$ both
have $\mu_{max}-\mu_{min}=n\hbar \Rightarrow \mu_{min}=-l\hbar$, $\mu_{max}=l\hbar$, $l=\frac{n}{2}\in\frac{1}{2}\mathbb{Z}$, eigenstate $|l,m\rangle$
$\Rightarrow \hat{L}_\pm Y$ also eigenstate of $\hat{L}_z$, eigenvalue shifted
$\frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B\pm|1\rangle_A|0\rangle_B)$
$l=0,\frac{1}{2},1,\frac{3}{2},2\ldots$ weights and $m=-l,-l+1,\ldots,l$ basis of eigenspace
$\hat{L}_\pm Y$ also eigenstate of $\hat{L}^2$, same eigenvalue
and operator action $\boxed{\text{principal quantum numbers}}$ are $\hat{L}_\pm|l,m\rangle=\hbar\sqrt{l(l+1)-m(m\pm1)}\,|l,m\pm1\rangle$, $|l,m\rangle$ is both eigenstate of $\hat{L}^2$ and $\hat{L}_z$
• For angular momentum, we can use concrete analysis compute $|l,m\rangle=$ spherical harmonic functions $Y_l^m(\theta,\varphi)$ $\quad$ eigenvalue $l(l+1)\hbar^2$ and $m\hbar$
and $l\in\mathbb{Z}$, but for spin, we allow $l\in\frac{1}{2}\mathbb{Z}$, this corresponds to $Spin(3)\xrightarrow{2:1}SO(3)$. $= \epsilon\sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}}P_l^m(\cos\theta)e^{im\varphi}$, $\epsilon=(-1)^m$, $m\geq0$; 1 otherwise
Total angular momentum (denote S here) thus $\hat{J}=\hat{L}\otimes\hat{1}_S+\hat{1}_L\otimes\hat{S}$ have same commutation relations. Legendre polynomial, and the eigenstates
$|j,m\rangle$, $j=|l-s|,\ldots,l+s$.
... $|1\rangle\to|\bar{b}\rangle$, $U_{NOT}=\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\quad$ Phase-shift-gate: $R_\varphi=\begin{pmatrix}1&0\\0&e^{i\varphi}\end{pmatrix}$, R_π is Pauli Z-gate
is rotation of Bloch sphere
$\hat{S}_z|\downarrow\rangle=-\frac{\hbar}{2}|\downarrow\rangle$
Spin-$\frac{1}{2}$ case: particles in our life are Fermions, $s=\frac{1}{2}$ case has $|\uparrow\rangle=|\frac{1}{2},\frac{1}{2}\rangle$ and $|\downarrow\rangle=|\frac{1}{2},-\frac{1}{2}\rangle \Rightarrow \hat{S}^2|\uparrow$ or $\downarrow\rangle=\frac{3}{4}\hbar^2|\uparrow$ or $\downarrow\rangle$ and $\hat{S}_z|\uparrow\rangle=\frac{\hbar}{2}|\uparrow\rangle$
called Pauli X-gate
denote $|\uparrow\rangle=\begin{pmatrix}1\\0\end{pmatrix}$ and $|\downarrow\rangle=\begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \hat{S}^2=\frac{3}{4}\hbar^2\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\hat{S}_+=\hbar\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $\hat{S}_-=\hbar\begin{pmatrix}0&0\\1&0\end{pmatrix} \Rightarrow \hat{S}_x=\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\hat{S}_y=\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$, we denote them Pauli matrices $\hat{S}=\frac{\hbar}{2}\sigma$
Black body ... Boltzmann distribution ... partition function ...
under $\hat{S}_x$-eigenstates (or eigenspinors called): $|\hat{S}_x-\lambda I|=0 \Rightarrow \lambda=\pm\frac{\hbar}{2} \Rightarrow$ basis $|\uparrow_x\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $\boxed{\hat{S}_z=\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}}$ and $|\downarrow_x\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$, eigenvalue $\frac{\hbar}{2}$ & $-\frac{\hbar}{2}$
... energy density ...
Write as state $e^{i\gamma}\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\varphi}\end{pmatrix}=e^{i\gamma}(\cos\frac{\theta}{2}|\uparrow\rangle+\sin\frac{\theta}{2}e^{i\varphi}|\downarrow\rangle) \Rightarrow \langle\hat{S}\rangle_\chi=\langle\chi|(\hat{S}_x,\hat{S}_y,\hat{S}_z)|\chi\rangle=\frac{\hbar}{2}(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)\in$ Bloch sphere
Tensor two spin-$\frac{1}{2}$ particles: $|\uparrow\uparrow\rangle,|\uparrow\downarrow\rangle,|\downarrow\uparrow\rangle,|\downarrow\downarrow\rangle = |s,m\rangle$, $m=1,0,0,-1$ is obvious. We determine s by $|m|\leq s \Rightarrow s=0$ or 1
$\Rightarrow |s,m\rangle=|0,0\rangle$ or $|s,m\rangle=|1,-1\rangle,|1,0\rangle,|1,1\rangle$, how to correspond these four states, i.e. $|s,m\rangle=\sum_{m_1+m_2=m} C^{s_1s_2s}_{m_1m_2m}|s_1,m_1\rangle\otimes|s_2,m_2\rangle$, $C^{s_1s_2s}_{m_1m_2m}$ called
Clebsch-Gordan coefficients. They're determined by ladder operators: $|1,1\rangle=|\uparrow\uparrow\rangle$ and $|1,-1\rangle=|\downarrow\downarrow\rangle$ as their m is highest/lowest, and
$\hat{S}_-|1,1\rangle=(\hat{S}_{1-}+\hat{S}_{2-})|\uparrow\uparrow\rangle \propto |\downarrow\uparrow\rangle+|\uparrow\downarrow\rangle$, then normalize $\Rightarrow |1,0\rangle=\hat{S}_-|1,1\rangle=\frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle+|\uparrow\downarrow\rangle)$ and $|0,0\rangle=\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle)$ due to orthogonal ▨

Examples important: ① Hydrogen atom: $V(r)=-\frac{e^2}{4\pi\varepsilon_0 r}$, hence we can decompose into radical and angular parts: $(-\frac{\hbar^2}{2m}\nabla^2-\frac{e^2}{4\pi\varepsilon_0 r})\psi(r,\theta,\phi)=E\psi(r,\theta,\phi)$
E are eigenenergy. Set Bohr radius $a_0=\frac{4\pi\varepsilon_0\hbar^2}{e^2 m}$, $\rho_0=\frac{2}{\kappa a_0}$, $\kappa=\sqrt{-2mE/\hbar^2} \Rightarrow (\tilde{\nabla}^2+\frac{\rho_0}{\rho})\varphi=\varphi$, $\tilde{\nabla}^2=\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\frac{\partial}{\partial\rho})-\frac{L^2}{\hbar^2\rho^2}$, assume
Radical: $(\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\frac{\partial}{\partial\rho})+\frac{\rho_0}{\rho})R(\rho)=R(\rho)$, normalize $R(\rho)$ to $u(\rho)=\rho R(\rho) \xRightarrow{\rho=\kappa r} \frac{\partial^2}{\partial\rho^2}u(\rho)=(1-\frac{\rho_0}{\rho})u(\rho)$, note that $\quad \varphi(r,\theta,\phi)=\varphi(\rho,\theta,\phi)=R(\rho)\cdot Y(\theta,\phi)$
when $\rho\to\infty$, $u(\rho)\to e^{-\rho}$, thus all physical solutions are given by $u(\rho)=e^{-\rho}\sum c_k\rho^k \Rightarrow c_{k+1}=\frac{2k-\rho_0}{k(k+1)}c_k$ converges $\Leftrightarrow 2k=\rho_0$ for some k, denoted
the n-th eigenstate $\Rightarrow \kappa_n=\frac{1}{na_0} \Rightarrow E_n=-\frac{\hbar^2\kappa_n^2}{2m}=-\frac{m}{2\hbar^2}(\frac{e^2}{4\pi\varepsilon_0})^2\frac{1}{n^2}$, coincide with experiment and used to check Schrödinger equation in history ▨
Angular: $\hat{L}^2Y=\lambda\hbar^2 Y$, let $Y(\theta,\phi)=\Theta(\theta)\Phi(\phi)$ taken into $[\sin\theta\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})+\lambda\sin^2\theta]+\frac{\partial^2}{\partial\phi^2}Y=0$
$\Rightarrow \frac{1}{\Theta}(\sin\theta\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Theta}{\partial\theta})+\lambda\sin^2\theta\Theta)=\mu=-\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} \Rightarrow \mu=m^2$, $\Phi(\phi)=e^{im\phi}$; $\Theta(\theta)=AP_l^m(\cos\theta)$; $l=0,1,2\ldots$ & $m=-l\ldots l$, combine into spherical harmonic function ▨

② Potential well & wall: scattering in QM (or M) $\leftrightarrow$ given incident wave into scattering center (with potential $V(x)$ given), how it behaves after scattering
Here we consider infinite square potential well first: $V(x)=0$; $0<x<a$; ∞ otherwise, separable of varibles $\psi(x,t)=\phi(x)e^{-iEt/\hbar}$ due to
$V(x)$ not depend on time, $\phi(0)=\phi(a)=0 \Rightarrow k=\frac{\sqrt{2mE}}{\hbar}=\frac{n\pi}{a}$, $n=1,2,3\ldots \Rightarrow$ eigenenergy $E_n=\frac{n^2\pi^2\hbar^2}{2ma^2}$ and eigenstate $\phi_n(x)=\sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})$

stationary state is single $\phi_n e^{-iE_nt/\hbar}$, superposition state is linear combination + normalization. $E_1>0$ by uncertainity principle!
$\Delta E=0 \Rightarrow \Delta E\Delta t\geq0 \Rightarrow \Delta t=\infty$ $\quad$ $\hat{H}$ or E is uncertain $\quad \Rightarrow x\to\infty$, $\psi(x)\sim Ae^{ikx}+Be^{-ikx}$ can't normalized ... but not achieve minimum of ...
When we consider finite height, n larger, E_n larger $\Rightarrow$ some energy go out; scattering states, otherwise bounded states. $\quad F=-\frac{dE}{da}=\frac{n^2\pi^2\hbar^2}{ma^3}$
For the tunneling effect makes sense, our wall can't be infinite height, square case $V(x)=V_0>0$; $0\leq x\leq L$, 0 otherwise, $E<V_0$
$\psi_I(x)=Ae^{ikx}+Be^{-ikx}$, $k=\frac{\sqrt{2mE}}{\hbar}$; $\psi_{II}(x)=Ce^{\kappa x}+De^{-\kappa x}$, $\kappa=\frac{\sqrt{2m(V_0-E)}}{\hbar}$ expotential decay; $\psi_{III}(x)=Fe^{ik(x-L)}$. $\quad R=|\frac{B}{A}|^2$ $\quad$ D is expotential decay,
Boundary of ψ and $\psi'_x \Rightarrow \frac{B}{A}=\frac{(k^2+\kappa^2)\sinh(\kappa L)}{2ik\kappa\cosh(\kappa L)+(k^2-\kappa^2)\sinh(\kappa L)}$ and $\frac{F}{A}=\frac{2ik\kappa}{2ik\kappa\cosh(\kappa L)+(k^2+\kappa^2)\sinh(\kappa L)}$ are reflection and transmisivity $\quad$ omitted in I $\quad T=|\frac{F}{A}|^2$
When $L\to\infty$, $R\to1$, $\frac{B}{A}=e^{i\varphi}$, we can compute the depth of tunneling: $e^{i\varphi}=e^{i\pi-2ik\lambda} \Rightarrow \lambda=\frac{1}{k}\operatorname{arccot}(\frac{\kappa}{k})$ $\quad$ reflect in depth
The key of computation $=\frac{k-i\kappa}{k+i\kappa}$ $\quad$ Similiarly, all scatting problem (φ should be π ($e^{i\pi}=-1$) if reflect at $x=0$, but the phase change means
is boundary condition. $\quad$ can define R and T.
$V(x)=-a\delta(x)$, $a>0$, ① $E<0\leq V(x) \Rightarrow \psi_I(x)=Ae^{\kappa x}$, $\psi_{II}(x)=Be^{-\kappa x}$ ($\psi_I(x)\neq Ae^{\kappa x}+A'e^{-\kappa x}$ due to $A'e^{-\kappa x}$ can't be normalised!)
$\Rightarrow$ bounded state $E=-\frac{\hbar^2\kappa^2}{2m}=-\frac{ma^2}{2\hbar^2}$ and $\psi(x)=\sqrt{\kappa}e^{-\kappa|x|}$, $\kappa=\frac{ma}{\hbar^2}$ $\quad A=B$
② Scattering state $\psi_I(x)=Ae^{ikx}+Be^{-ikx}$, $\psi_{II}(x)=Ce^{ikx}\ldots \Rightarrow \frac{B}{A}=-\frac{ma}{i\hbar^2k+ma}$, $\frac{C}{A}=\frac{ik\hbar^2}{ik\hbar^2-ma}$ ▨

③ Harmonic oscillator: $V(x)=\frac{1}{2}m\omega^2x^2$ parabolic form $\Rightarrow \hat{H}\psi=(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega^2x^2)\psi(x)=E_n\psi_n(x)$, we do triangular decomposition to $\hat{H}$: $\hat{H}=\frac{1}{2m}(\hat{p}^2+(m\omega\hat{x})^2)$ $\quad$ in upper triangular matrix $\hat{a}_+$ raise, lower $\hat{a}_-$
$\hat{a}_\pm=\frac{1}{\sqrt{2m\hbar\omega}}(\mp i\hat{p}+m\omega\hat{x}) \Rightarrow \hat{a}_-\hat{a}_+=\hat{H}+\frac{1}{2}$, $\hat{a}_+\hat{a}_-=\hat{H}-\frac{1}{2}$ by $[\hat{x},\hat{p}]=i\hbar \Rightarrow [\hat{a}_-,\hat{a}_+]=1$, $\hat{a}_\pm\psi_n$ are still eigenstate of $\hat{H}$, with $E_{n\pm1}=E_n\pm\hbar\omega$
$\hat{a}_-\psi_0=0 \Rightarrow$ ground state $\psi_0(x)=(\frac{m\omega}{\pi\hbar})^{1/4}e^{-m\omega x^2/2\hbar}$, $E_0=\frac{1}{2}\hbar\omega \Rightarrow E_n=(n+\frac{1}{2})\hbar\omega$, $|n\rangle=$ normalisation of $(\hat{a}_+)^n|0\rangle \Rightarrow \hat{a}_+\hat{a}_-|n\rangle=n|n\rangle$, $\hat{a}_-\hat{a}_+|n\rangle=(n+1)|n\rangle$
$\Rightarrow \hat{a}_+|n\rangle=\sqrt{n+1}|n+1\rangle$, $\hat{a}_-|n\rangle=\sqrt{n}|n-1\rangle \Rightarrow |n\rangle=\frac{1}{\sqrt{n!}}\hat{a}_+^n|0\rangle$, the ground state has $\Delta x\Delta p=\frac{\hbar}{2}$. Both harmonic oscillator and angular momentum use $\pm$,
Schiwinger Boson model is two independent harmonic oscillators $\hat{a}_\pm$ and $\hat{b}_\pm$, $[\hat{a}_\pm,\hat{b}_\pm]=0 \Rightarrow \hat{J}_x=\frac{\hbar}{2}(\hat{a}_+\hat{b}_-+\hat{b}_+\hat{a}_-)$, $\hat{J}_y=\frac{\hbar}{2i}(\hat{a}_+\hat{b}_--\hat{b}_+\hat{a}_-)$, $\hat{J}_z=\frac{\hbar}{2}(\hat{a}_+\hat{a}_--\hat{b}_+\hat{b}_-)$
all are "string" oscillators! QFT view: harmonic oscillator of electromangatic field is Boson, of mechanical oscillation is photon, here is phonon and the
eigenstate $|n\rangle$ has n phonon, the phonon number operator $\hat{n}|n\rangle=\hat{a}_+\hat{a}_-|n\rangle=n|n\rangle$, thus $\hat{a}_\pm$ essentially translation of Boson!
Classical states $|n\rangle$ are stable w.r.t. time, coherent state $\psi(x)=e^{i\Delta p x/\hbar}|0\rangle$ given a momentum change $=e^{\lambda(\hat{a}_+-\hat{a}_-)}|0\rangle$, $\lambda=i\sqrt{\frac{m\omega}{2\hbar}}\frac{m b}{\hbar}$ $\quad$ $\hat{a}_-$ are eigenstates of
$\hat{a}_-|\psi\rangle=\lambda|\psi\rangle$ and $|\psi\rangle=e^{-\frac{|\lambda|^2}{2}}\sum_n\frac{\lambda^n}{\sqrt{n!}}|n\rangle$, due to $\hat{x}=\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_++\hat{a}_-)$, $\hat{p}=i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+-\hat{a}_-) \Rightarrow \langle x\rangle=\sqrt{\frac{2\hbar}{m\omega}}\lambda\cos(\omega t)$ & $\langle p\rangle=-\sqrt{2\hbar m\omega}\lambda\sin(\omega t)$ similiar to classical

④ Wave packet: A particle in free space has form of planar waves: $-\frac{\hbar^2}{2m}(\frac{d}{dx})^2\varphi=E\varphi \Rightarrow Ae^{ipx/\hbar}$, $p=\sqrt{2mE}=\hbar k$, $k_n=\frac{2n\pi}{L}$ as $\varphi(0)=\varphi(L)$, $n\in\mathbb{Z}$
$\Rightarrow \psi(x,t)=A\varphi_n(x)e^{-iE_nt/\hbar}=\sqrt{\frac{1}{L}}e^{ik_nx}e^{-iE_nt/\hbar}$, $E_n=\hbar^2k_n^2/2m=\frac{2n^2\pi^2\hbar^2}{L^2m}$. When $L\to\infty$ free, k_n are continuous, so is $E_n \Rightarrow \psi(x,t)=e^{i(kx-\frac{\hbar k^2}{2m}t)}$, $k=\pm\frac{\sqrt{2mE}}{\hbar}$
all these eigensolutions are non-physical, can't be normalized $\Rightarrow$ parameterised by $k\in\mathbb{R}$. $\Psi(x,t)=\frac{1}{\sqrt{2\pi\hbar}}\int\varphi_k e^{i(p_kx-E_kt)/\hbar}dk$, normalized $\Leftarrow$
$\int_{\mathbb{R}}|\varphi_p|^2dp=1$ ($p=\hbar k$), $\langle p\rangle=\int_{\mathbb{R}}p|\varphi|^2dp$. Perturbation expansion $E_p=E_0+E'_0(p-p_0)+O(p-p_0) \Rightarrow \Psi(x,t)=e^{i(p_0x-E_0t)/\hbar}\frac{1}{\sqrt{2\pi\hbar}}\int\varphi_p e^{i(p-p_0)(x-E'_0t)/\hbar}dp$
$=: e^{i(p_0x-E_0t)/\hbar}W(x-E'_0t)$, W not change shape, translation with $v_{group}=E'_0$ group vecolity; phase vecolity $v_{phase}=E_p/p$; when E_p term considered, the
dispersion happens; the length of $W(x,t)$ increasing by time: $\Psi(x,t)=e^{i(p_0x-E_0t)/\hbar}\frac{1}{\sqrt{2\pi\hbar}}\int\varphi_p\exp(-\frac{i}{2\hbar}E''_0t(p-p_0)^2)\exp(\frac{i}{\hbar}(p-p_0)(x-E'_0t))dp=:e^{i(p_0x-E_0t)/\hbar}W(x,t)$, the
$W(x,t)=\frac{1}{\sqrt{2\pi\hbar}}\int(\frac{A}{\sqrt{2}})^{\frac{1}{2}}\exp(-\frac{1}{2\sigma^2}(1+i\frac{E''_0t\sigma^2}{\hbar})(p-p_0)^2)\cdot\exp(\frac{i}{\hbar}(p-p_0)(x-E'_0(t)))dp \propto \exp(-\frac{(x-E'_0t)^2}{2\sigma^2+\frac{i}{\hbar}E''_0t})\to$ length $\Delta x(t)=\sigma\sqrt{1+(\frac{E''_0t}{\hbar\sigma^2})^2}$ ↗ ▨ Planar: ... Wave packet: ...
① leads to identical particles; ② leads to WKB; ③ leads to QFT; ④ leads to perturbation.