1.2. (The Twisted cubic curves) $Y = \{(t, t^2, t^3) \mid t \in k\}$ a cyclic curve.
$(\deg f = 3)$
$= V(\langle f, g \rangle)$

Prove of Dimension.
$\begin{cases} f(x,y,z) = (z - x^3) \\ g(x,y,z) = (x^2 - y) \end{cases} \Rightarrow$ dimension 1

$A(Y) = \dfrac{k[x,y,z]}{I(Y)} = \dfrac{k[x,y,z]}{\langle f, g \rangle}$

$\max \deg(\{f, g\}) = 3 \Rightarrow$ cubic

$\varphi : A(Y) \longleftrightarrow k[x]$, $\Rightarrow A(Y) \cong k[x]$
$(x,y,z) \mapsto x$ $\quad \dim Y = \dim A(Y) = \dim k[x] = 1$ ▨
$(x, x^2, x^3) \mapsto x$

1.3. $Y = V(\{x^2 - yz, x(z-1)\}) = V(\{x^2 - yz, x\}) \cup V(\{x^2 - yz, z-1\})$
$= V(\{y, z\}) \cup V(\{x^2 - yz, z-1\})$
$= \underset{①}{V(\{y\})} \cup \underset{②}{V(\{z\})} \cup \underset{③}{V(\{x^2 - yz, z-1\})}$ ▨

with corresponding prime ideals
$p_1 = I(V(\{y\})) = \sqrt{\langle y \rangle} = \langle y \rangle$, $p_2 = \sqrt{\langle z \rangle} = \langle z \rangle$, $p_3 = \sqrt{\langle x^2 - yz, z-1 \rangle} = \langle x^2 - yz, z-1 \rangle$ by Nullstellensatz ▨

1.4.
Moreover, claim that the Zariski topology in $\mathbb{A}_k^2$ is finer than product topology
$\mathfrak{F}_Z \,/\, \mathfrak{F}_2$

$\forall Z \in \mathfrak{F}_2$, claim $Z \in \mathfrak{F}_Z$: $Z \in \mathfrak{F}_2 \Leftrightarrow Z = \mathbb{A}_k^2 - \bigcup (\mathbb{A}_k^1 - V(\{f_i\}_{i \in n}))^2$
$\Rightarrow \mathfrak{F}_2 \subset \mathfrak{F}_Z \Leftrightarrow \mathfrak{F}_2 \subset \mathfrak{F}_Z$ finer $\quad = \mathbb{A}_k^2 - \bigcup (\mathbb{A}_k^1 - \{x_i\}_{i \in n})^2$
left is showing $\mathfrak{F}_2 \subsetneq \mathfrak{F}_Z$ $\quad = V(\{x - x_i, y - x_i\}_{i \in n}) \in \mathfrak{F}_Z$

$Z_0 \in \mathfrak{F}_Z - \mathfrak{F}_2$ that $Z_0 = V(\{x^2 + y^2 - 1\})$ ▨

1.5. In Remark 1.4.6, we showed that $B \cong A(Y)$ when B is a domain
Proof: $(\Rightarrow)$ $B = \dfrac{k[x_1 \cdots x_n]}{I}$, so B is finitely-generated is trivial
if $B = A(V(I)) \Rightarrow I = I(V(I)) = \sqrt{I}$, left is showing no nilpotent
if $f \in B : f^n = 0$, let $k[x_1 \cdots x_n] \xrightarrow{p} \dfrac{k[x_1 \cdots x_n]}{I}$
$(\Leftarrow)$ left is showing $\sqrt{I} \subset I$ $\quad F \mapsto f$
by $F^n \in I$, if $F \notin I$ $\quad F^n \mapsto f^n \Rightarrow F^n \in \ker p = I = \sqrt{I}$
$\Rightarrow p(F) = f \neq 0$ and $p(F^n) = f^n = 0 \Rightarrow F \in \ker p \Rightarrow f = 0 \Rightarrow$ no nilpotent ▨
$\Rightarrow f$ nilpotent, contradiction

1.7. (a) trivial. The same as the rings/modules' cases
(i) $\Rightarrow$ (ii) Or not, $\forall Z_0 \subset X$, $\exists Z_1 \subsetneq Z_0 \subset X$, $\exists Z_2 \subsetneq Z_1 \subset X$; $\exists Z_n \subsetneq Z_{n-1} \subset X \cdots$
$\Rightarrow$ we find a chain without d.c.c.
(ii) $\Rightarrow$ (i) Any chain $\cdots \subset Z_n \subset Z_{n-1} \subset \cdots \subset Z_0$, let $Z = \{Z_k\}_{k \geq 0}$ has minimum
$Z_{k_0} \Rightarrow$ it stables after k_0
(i) $\Leftrightarrow$ (iii) Any chain $\cdots \subset Z_n \subset Z_{n-1} \subset \cdots \subset Z_0$ d.c.c. $\Leftrightarrow$ Any chain $U_0 \subset \cdots \subset U_n \subset U_{n+1}$ with $\exists Z_i : Z_i = X - U_i$
(ii) $\Leftrightarrow$ (iv) The same ▨

(b) $X = \bigcup_{\alpha \in \Lambda} U_\alpha$ by well-ordered axiom Or apply Zorn's Lemma to $\Sigma = \{\bigcup U_{\alpha_i}\}$ a.c.c.

$U_{\alpha_0} \subset \bigcup_{\alpha_0 \leq \beta \leq \alpha_0 + 1} U_\beta \subset \cdots \subset \bigcup_{\beta \in \Lambda} U_\beta$ has all same expect for finite ones by (iii)
$\Rightarrow X = \bigcup_{\alpha_0 \leq \beta \leq \alpha_0 + n} U_\beta$ ▨
($\alpha_0 + 1$ denote the last one of α_0)

(c) That is, noetherian topological space is hereditarily compact
Any chain $\cdots \subset Z_n \subset Z_{n-1} \subset \cdots \subset Z_0$ in $Y \subset X$
Z_i closed in $Y \Leftrightarrow \exists \tilde{Z}_i$ closed in X: $Z_i = \tilde{Z}_i \cap Y$
$\{\tilde{Z}_i\}_{i \geq 0}$ has minimal one $\tilde{Z}_n$ in Y
$\Rightarrow Z_n = \tilde{Z}_n \cap Y$ is minimal in all $\{Z_i\}_{i \geq 0} \Rightarrow$ d.c.c. $\Rightarrow$ Noetherian for Y ▨

(d) $\{X - V\}_{V \in \mathcal{N}_X}$ has minimal one, denote as $X - V_0$, if $V_0 \neq \{x\}$
$\Rightarrow V_0 = \{x\} \Rightarrow X - \{x\}$ closed $\Rightarrow \{x\}$ open
$\Rightarrow \forall U \subset X$, $U = \bigcup_{x \in U} \{x\}$ open $\Rightarrow$ Discrete ▨

(d) Assume X is irreducible
$\Rightarrow \exists y \neq x$, $y \in V_0$
$\Rightarrow x, y$ can't seperate: X infinite
contradict to T_2 $\quad \forall U \subset X$, open is dense
$\Rightarrow$ contradict Hausdorff
$\Rightarrow$ finite
and finite + T_2

1.8. $Y = V(f_1, \cdots f_{n-r})$, $H = V(F)$ (If $F = \sum a_i f_i \Rightarrow Y \subset H$, contradiction)
$Y \cap H = V(f_1, \cdots f_{n-r}) \cap V(F) = V(\{f_1, \cdots f_{n-r}\} \cup \{F\}) = V(f_1, \cdots f_{n-r}, F)$
$\Rightarrow \dim(Y \cap H) = \dim A(Y \cap H) = \dim \dfrac{k[x_1 \cdots x_n]}{\langle f_1 \cdots f_{n-r}, F \rangle} = n - (n-r+1) = r-1$

As complete proof also in (7.D)'s first step (P48):
Want to prove $\dfrac{k[x_1 \cdots x_n]}{\langle f_1 \cdots f_{n-r}, F \rangle} \cong k[x_1 \cdots x_{r-1}]$ isn't a trivial thing
Let $W \subset Y \cap H$ is an irreducible component
$I(W)$ is the minimal prime principal ideal $p = (f)$ in $A(Y)$
$\left(\begin{array}{l} k[x_1 \cdots x_n] \xrightarrow{p} A(Y) \\ I(W) \mapsto I_Y(W) \text{ by } W \subset Y \end{array} \right) \xRightarrow{\text{Krull's Hauptidealsatz}}$ height$(p) = 1 \Rightarrow \dim A(Y)/p = r-1 = \dim A(W) = \dim W$

ore is computing Claim 2

$\begin{cases} k_{11}f_2+k_{21}g_2 = xz-y^2 \\ k_{12}f_2+k_{22}g_2 = -z^2 \\ k_{13}f_2+k_{23}g_2 = yz \end{cases}$ $\Rightarrow$ f_2 or g_2 has form $a_1xz+a_2y^2+a_3z^2+a_4yz+\cdots$ $(a_1, a_2, a_3, a_4 \neq 0)$

consider degree over t, z^2, yz need reduced by deg 3

$\Rightarrow$ contradiction ▨

Similar Exercise to Ex 1.11 $Q = V(x_0x_2-x_1x_3, x_1^2-x_0x_2, x_1x_3-x_2^2)$
$\neq V(f,g)$

(by contradiction, will find Q has other components, i.e. $V(f,g) = Q \cup Q'$)

9. Let $W \subset Z(\mathfrak{a})$ to be irreducible,

$\dim W = \dim A(W)$, assume $\mathfrak{a} = \langle f_1 \cdots f_r \rangle \rightarrow \langle f_1 \cdots f_k \rangle$ with $k \leq r$

$= \dim \frac{k[x_1 \cdots x_n]}{(f_1 \cdots f_k)} = n-k \geq n-r$ ▨

Left is showing $\dim A(W) = n-k$

$k[x_1 \cdots x_n] \xrightarrow{\varphi} \frac{k[x_1 \cdots x_n]}{\mathfrak{q}}$

$I(W) \mapsto I_{Z(\mathfrak{a})}(W)$ minimal prime ideal with $I_{Z(\mathfrak{a})}(W) = \langle f_1, \cdots, f_k \rangle$

then by induction on Exercise 1.8.

10. (a) Denote $\dim Y = n$, the $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ is maximal in Y with $Z_0 = \{P\}$, $P \in Y$

$\Rightarrow \bar{Z_0} \subset \bar{Z_1} \subset \cdots \subset \bar{Z_n}$ is in $X \Rightarrow n \leq \dim X$ ▨

(b) By (a) $\dim U_i \leq \dim X \Rightarrow \sup \dim U_i \leq \dim X$

Denote $\dim X = n$, the $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ is maximal in Y

we find a U_i let $Z_k \cap U_i \neq \emptyset$ with $\forall k \leq n$

$\Rightarrow U_i \cap Z_0 \subset U_i \cap Z_1 \subset \cdots \subset U_i \cap Z_n$ is maximal in U_i

$\Rightarrow n \leq \dim U_i \Rightarrow \dim X \leq \sup \dim U_i$ ▨

(c) $(X, \mathcal{T}) = (\{0,1\}, \{\emptyset, X, \{0\}\})$ with $\dim X = 1$, $\dim U = \dim \{0\} = 0 \lneq \dim X$ ▨

(d) Denote $\dim X = \dim Y = n$

if $Y \subsetneq X$, for $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ maximal in X, we can let $Z_n \supset Y$

$\Rightarrow Z_0 \cap Y \subset Z_1 \cap Y \subset \cdots \subset Z_n \cap Y$ maximal in Y

$\Rightarrow Z_0 \cap Y \subset Z_1 \cap Y \subset \cdots \subset Z_n \cap Y \subset Y$ also a chain in X

contradict to the maximality $\Rightarrow Y = X$

For maximal chain in Y, add a X in the end to get $\dim X \geq \dim Y$

(e) $A = k[x_1 \cdots x_n \cdots]$ with $p_i \in A$, $p_i = (x_{2^i}, x_{2^i+1}, \cdots, x_{2^{i+1}-1})$

$S = A - \bigcup_{i \in \mathbb{N}} p_i$, $S^{-1}A = R$ is Noetherian of infinite dimension

We always take lower degree to represent better:

11. $I(Y) \neq V(x^3-y^2, y^5-z^4)$ (or other symmetric forms)

Spec(R) also

first, $I(Y) = (x^2-yz, y^2-xz, z^2-x^2y)$

My instinct is that: z^3-y^5 can't represented by x^2-y^3 and y^5-z^4

And x^2-yz, y^2-xz, z^2-x^2y can't represented by lower degree in $I(Y)$, thus can't generated by two element (degree lower, generators less)

Use contradiction to derive a contradiction

If $I(Y) = (f, g)$,

$\boxed{\text{Claim}}$ ① f, g both can't contain deg 1 and constant.

② We can only consider the deg 2 part to derive contradiction

① is trivial, otherwise $f = \cdots + \frac{x^2}{\cdots} + \frac{z}{\cdots}$ not homogenous with t (We take $x^2 - yz$ because it's ...)

2.1. That is, $f \in I(Z(\mathfrak{a})) \Rightarrow f \in \sqrt{\mathfrak{a}}$, that is $I(Z(\mathfrak{a})) \subset \sqrt{\mathfrak{a}}$

It is included in 2.3(d), the projective Nullstellensatz

Proof. $p: (\mathbb{A}_k^{n+1})^* \longrightarrow \mathbb{P}_k^n$, we define the cone (Ex 2.10) $C(Y) = p^{-1}(Y) \cup \{0\}$

and distinct $V(I) \neq \bar{V}(I)$ (I generates $\bar{V}(I) \subset (\mathbb{A}_k^{n+1})^*$) ($I$ generates the $V(I) \subset \mathbb{P}_k^n$)

note that ① $C(V(I)) = \bar{V}(I) \Rightarrow \sqrt{\mathfrak{a}} = I(\bar{V}(\mathfrak{a})) = I(C(V(\mathfrak{a}))) = I(V(\mathfrak{a}))$ ▨

② $I(C(V)) = I(V)$

The two notion is easy to verify ① $C(V(I)) = p^{-1}(V(I)) \cup \{0\}$: $\forall P \in p^{-1}(V(I)) \cup \{0\}$

② Note their coordinates are common ▨

$\Leftrightarrow P = 0$ or $p(P) \in V(I)$

$\Leftrightarrow P \in \bar{V}(I)$ or $P \in \bar{V}(I)$ (because $p(P) = [P_0 : P_1 \cdots P_n]$, $P = (P_0 \cdots P_n)$)

$\Leftrightarrow P \in V(I)$ ▨

2.2. (iii) $\Rightarrow$ (ii)

$\mathfrak{a} \supset S_d \Leftrightarrow \mathfrak{a} \supset (x_1 \cdots x_n)^d$

$\Rightarrow Z(\mathfrak{a}) \subset Z((x_1 \cdots x_n)^d) = \phi$

(i) $\Rightarrow$ (iii) Use the notion same as 2.1

$\sqrt{\mathfrak{a}} = I(\bar{V}(\mathfrak{a})) = I(C(Z(\mathfrak{a}))) = I(Z(\mathfrak{a}) \cup \{0\}) = I(\{0\})$

$\Rightarrow \mathfrak{a} \supset I(\{0\})^d = S_d$, $\exists d > 0$

(ii) $\Rightarrow$ (iii) is contained in the way of (i) $\Rightarrow$ (iii)

(iii) $\Rightarrow$ (ii) is trivial ▨

2.4. (a) {algebraic sets of $\mathbb{P}^n$} $\longleftrightarrow$ {$\mathfrak{a} \trianglelefteq S$ homogenous radical, $\mathfrak{a} \neq S_+$}

$Y \longleftrightarrow I(Y)$

$C(V)$

{cones over $\mathbb{P}^n$} $C(V) \longleftrightarrow \boxed{I(C(V)) = I(V)}$

{algebraic sets of $\mathbb{A}^{n+1}$} $\longleftrightarrow$ {radical ideal $\mathfrak{a} \trianglelefteq k[x_0 \cdots x_{n+1}]$}

By the composition in this graph, we built the bijection of two sets ▨

Notice. $I(V) = I(V(C(V))) = \sqrt{\mathfrak{a}}$ to let $V(\mathfrak{a}) \neq \phi$, we shall let $\mathfrak{a} \neq S$ (trivial) and $\mathfrak{a} \neq S_+$ (Thus S_+ irrelevant)

(b) Same as Coro 1.4 ▨ (Don't depend on the existential properties of $\mathbb{A}_k^n$, 2.5(2) also)

(c) $I(\mathbb{P}^n) = 0 = (0)$ prime ▨

2.5. (1) If not, $\cdots \subset Z_n \subset Z_{n-1} \subset \cdots \subset Z_0 = \mathbb{P}_k^n$ not d.c.c.

$\Rightarrow \exists i: \cdots \subset Z_n \cap U_i \subset Z_{n-1} \cap U_i \subset \cdots \subset Z_0 \cap U_i = U_i$ that $Z_k \cap U_i \neq \phi$

$\Rightarrow$ also, not d.c.c in $U_i \cong \mathbb{A}_k^n$, contradict to when $k \geq N$ $\mathbb{A}_k^n$ Noetherian ▨ (Directly by $k[x_0 \cdots x_n]$ Noetherian)

2.6. By Hint is obvious, namely $\varphi_i: U_i \to \mathbb{A}^n$ homeomorphism

$\Rightarrow \varphi_i|_Y: U_i \cap Y \to Y_i$ homeomorphism

$\Rightarrow \dim(U_i \cap Y) = \dim A(U_i \cap Y) = \dim A(Y_i) = \dim Y_i$

$\{U_i \cap Y\}$ covers Y, Claim $\dim(Y) = \dim(U_i \cap Y)$

$\Rightarrow \dim Y_i = \dim Y$

$\Rightarrow \dim Y = \boxed{\dim A(Y_i) = \dim S(Y)_{x_i} - 1} = \text{height}(x_i) - 1$

$= \dim S(Y) - 1$

$\Rightarrow \dim S(Y) = \dim Y + 1$

By $\dim S(Y)_{x_i} = \dim A(Y_i)[x_i, x_i^{-1}] = \dim A(Y_i) + 1$

Left is proof

① $\dim Y = \dim(U_i \cap Y)$ ($x_i = 1$)

② $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$

Proof. ① follows Ex 1.10(b)

$\dim Y = \sup_i \dim(U_i \cap Y) = \dim(U_i \cap Y)$

② $\varphi: S(Y)_{x_i} \longrightarrow A(Y_i)[x_i, x_i^{-1}]$

$\frac{f}{x_i^m} \longmapsto \frac{\sum_j f_j x_i^{j}}{x_i^m} = \sum_j f_j x_i^{j-m}$ with f_j hasn't variant x_i, thus $\sum_j f_j x_i^{j-m} \in A(Y_i)[x_i, x_i^{-1}]$

Conversely, $\psi: A(Y_i)[x_i, x_i^{-1}] \longrightarrow S(Y)_{x_i}$

$\sum_j f_j x_i^{s_j} \longmapsto \frac{\sum_j f_j x_i^{s_j + s}}{x_i^s}$ ($s_j \in \mathbb{Z}$) with $s = \max\{s_j\} + 1$, $\in S(Y)_{x_i}$ and we need $\forall j, j'$, $\deg(f_j x_i^{s_j+s}) = \deg(f_{j'} x_i^{s_{j'}+s})$

thus we need homogenification on f_j

$\Rightarrow \psi: \sum f_j x_i^{s_j} \longmapsto \frac{\sum_j \beta(f_j) x_i^{s_j+s}}{x_i^s}$

Trivial to verify $\psi \circ \varphi = 1$; $\varphi \circ \psi = 1$ ▨

2.7. (a) $\dim \mathbb{P}^n = \dim S(\mathbb{P}^n) - 1 = \dim k[x_0 \cdots x_n] - 1 = (n+1) - 1 = n$ ▨

(b) $\dim \bar{Y} = \dim \bar{Y}_i = \dim \bar{Y}_i = \dim \overline{U_i \cap Y}$

$\dim \bar{Y} = \dim(\bar{Y})_i = \dim U_i \cap \bar{Y} = \dim \overline{U_i \cap Y} = \dim \overline{U_i \cap Y}$ This directly satisfy

$\Rightarrow \dim Y = \dim \bar{Y}$ ▨

2.8. ($\Rightarrow$) $\dim Y = \dim C(Y) - 1 \Rightarrow \dim C(Y) = n$ in $\mathbb{A}_k^{n+1} \Rightarrow C(Y) = V(f)$

$\Rightarrow \boxed{C(Y) = V(\alpha(f))}$, α is the homogenousification

($\Leftarrow$) $Y = V(f)$, $\boxed{C(Y) = V(\beta(f))}$, β is the dehomogenousification

$\Rightarrow \dim Y = \dim C(Y) - 1 = [(n+1) - 1] - 1 = n - 1$ ▨ I find the "□" two properties not true, we can't do (de)homogenous in $\mathbb{P}_k^n \longleftrightarrow \mathbb{A}_k^n$, I rewrite it below

2.8 ($\Rightarrow$) $\dim \frac{S}{I(Y)} + \text{height}(I(Y)) = n+1$ ($S = k[x_0 \cdots x_n]$)

$\frac{S}{I(Y)} = S(Y)$ has $\dim S(Y) = n$

$\Rightarrow \text{height}(I(Y)) = 1$

$\xRightarrow{\text{Prop 1.12A}} I(Y) = (f)$

($\Leftarrow$) $S(Y) = \frac{S}{(f)}$

$\dim Y = \dim S(Y) - 1 = \dim S - \dim(f) - 1 = (n+1) - 1 - \dim(f) \xlongequal{\text{Thm 1.11A}} n - 1$ ▨

2.9. (a) $\forall f \in I_p(\bar{Y}) = I_p(Z_p(I_A(Y)))$

which we distinct I_A, I_p, Z_A, Z_p to respect operations in $\mathbb{A}_k^n$ or $\mathbb{P}_k^n$

$\Leftrightarrow f$ vanishs in bigger set than

$\forall g$ homogenous vanish in Y, the set of g vanishs

$\Leftrightarrow V(f) \supset \bigcap_{g \in I(Y)} V(g) \Leftrightarrow f = \sum_g k_g g$

$\Leftrightarrow f \in \langle I_p(Y) \rangle = \langle \beta(I_A(Y)) \rangle$ (by $I_p(Y) = \beta(I_A(Y))$)

$\Rightarrow I_A(\bar{Y}) = \langle \beta(I_A(Y)) \rangle$ ▨

(b) $A(Y) = \frac{k[x, y, z]}{\langle y - x^2, z - x^3 \rangle}$, $A(\bar{Y}) = \frac{k[x, y, z, w]}{I(\bar{Y})} = \frac{k[x, y, z, w]}{\langle \beta(I_A(Y)) \rangle} = \frac{k[x, y, z, w]}{\langle \beta(y - x^2, z - x^3) \rangle} = \frac{k[x, y, z, w]}{\langle yw - x^2, zw^2 - x^3, y^2 - zw \rangle}$

2.9.(2) (countinou) Thus $I(Y) \neq \langle \beta(zw^2 - x^3), \beta(yw - x^2)\rangle$ as $\langle y^2 - zw, zw^2 - x^3, yw - x^2 \rangle \neq \langle zw^2 - x^3, yw - x^2\rangle$ ▨

$y^2 - zw$ can be represented by two others

Coro. Although $\langle f_1, f_2, f_3\rangle$ can linearly dependent but $\langle \beta(f_1), \beta(f_2), \beta(f_3)\rangle$ can not

$(y^2 - z^2 = (z+xy)(x^2 - z) + (x^2y + y^2 + xz)(y - x^2))$

2.10.(1) $Y = V(f_i)$, f_i homogenous

$C(Y) = \theta^{-1}(V(f_i)) \cup \{0\}$

$= V(f_i)$

because $f_i(\theta^{-1}(V(f_i))) = 0$ trivial

and $f_i(\{0\})$ follows f_i homogenous (and $\deg f_i \ge 1$) ▨

$I_A(C(Y)) = I(Y)$ is proved in 2.2 ②

We namely prove it again

$\forall f \in I_A(C(Y)) \Leftrightarrow f(C(Y)) = \{0\}$

$\Leftrightarrow f(\theta^{-1}(V(f_i))) = \{0\} \Leftrightarrow \forall P \in V(f_i)$ with $P = [a_0 \cdots a_n]$

and $f(\{0\}) = \{0\} \Leftrightarrow \deg f \ge 1$

$\theta(P) \ni \lambda(a_0 \cdots a_n)$

$f(\lambda(a_0 \cdots a_n)) = f(a_0 \cdots a_n) = 0$

$\Leftrightarrow f$ is homogenous with $f \in I_P Y$ ▨

(2) By (1), $Y = V(f_i)$

$\Leftrightarrow C(Y) = V(f_i)$

Thus $C(Y)$ irreducible $\Leftrightarrow f_i$ all prime

$\Leftrightarrow Y$ irreducible ▨

(3) $\dim C(Y) = \dim A(C(Y)) = \dim \frac{R}{I_A(C(Y))} = \dim \frac{S}{I_P(Y)} = \dim S(Y) = \dim Y + 1$ ▨

2.11.

(1) $I(Y) = \langle l_1 \cdots l_k \rangle \Leftrightarrow V(I(Y)) = V(\langle l_1 \cdots l_k\rangle) = \bigcap V(l_i)$

and Y is variety $\Rightarrow V(I(Y)) = Y = \bigcap V(l_i)$ ▨

(2) $\dim Y = \dim S(Y) = \dim \frac{S}{I(Y)} = n - \dim I(Y) = r \Rightarrow \dim(I(Y)) = n - r$

$\Rightarrow$ at least has $n-r$ generators

Y linear $\Rightarrow$ these generators linear ▨

(3) If $Y \cap Z = \emptyset$

$V(f_k)_{k \le r} \cap V(g_k)_{k \le s} = \emptyset \Leftrightarrow V(\{f_k\} \cup \{g_k\}) = \emptyset$ has dimension 0

$\overset{(2)}{\Rightarrow} n - (r+s) \le 0 \Rightarrow r+s-n \le 0$ ▨

Conversely, for $Y \cap Z \neq \emptyset \Rightarrow V(\{f_k\} \cup \{g_k\}) \neq \emptyset$ linearity is trivial,

$\Rightarrow \dim(Y \cap Z) \ge n - (n-r) - (n-s)$ The equivality comes when $\{f_k\}, \{g_k\}$ linearly indepent Irreducible

$= r+s-n$ ▨

2.12. (1) $\forall f \in a = \ker\theta$ Alternative: S domain $\Rightarrow$ a an ideal $\Rightarrow a \in \operatorname{Spec}(S)$

$\theta(f) = \theta(\sum_r k_r y_0^{r_0} \cdots y_N^{r_N}) = \sum k_r M_0^{r_0} \cdots M_N^{r_N} = 0$ θ is graded $\Rightarrow a$ homogenous ▨

because $r_0 + \cdots + r_N = R$ constant and $\deg(M_0) = \cdots = \deg(M_N) = d$

$\Rightarrow f$ is homogenlous by $\theta(f)$ is $\Rightarrow a$ homogenous ▨

(2) $\operatorname{Im}\rho_d \subset Z(a)$ is trivial by $\forall P \in \mathbb{P}^n$, $\forall f \in a = \ker\theta$, $\theta(f) = 0$

$f(\rho_d(P)) = f(M_0(a) \cdots M_N(a)) = \theta(f)(a) = 0 \Rightarrow \rho_d(P) \in Z(a) \Rightarrow \operatorname{Im}\rho_d \subset Z(a)$

Conversely, $\forall P \in Z(a)$, $\forall f \in a = \ker\theta$, $\theta(f) = 0$, $f(P) = 0$

claim that $P = \rho_d(p)$ for $\exists p$, that is, $P = (b_0 \cdots b_N)$ with $b_i = M_i(a)$, $\exists a = p$

2.12(2) (countinous) $f(P) = \sum_r k_r b_0^{r_0} \cdots b_N^{r_N} = 0$ for $\forall f \in \ker\theta$

that is, for $\forall \{f \in k[y_0, \ldots, y_N]\}$ homogenous

$\Rightarrow f(P) = 0 \Rightarrow b_i = M_i(a)$ for $i \le N$

$\Rightarrow P = \rho_d(a)$ ▨ This a is found by easily renumber

PART I

For M_i, M_j ($i<j$) Namely dictionally order

$M_i = x_0^{n_{i0}} x_1^{n_{i1}} \cdots x_n^{n_{in}}$

that $\exists l$, $n_{is} = n_{js}$; $s \le l$

$n_{it} > n_{jt}$; $t > l$

PART II and projection

Choose the elements $a = (b_0 \cdots b_n)$ after renumber

Verification $P = \rho_d(p)$ with $p = (b_0 \cdots b_N)$

$p = (b_0 \cdots b_n)$

$M_i(b_0 \cdots b_n) = b_0^{n_0} b_1^{n_1} \cdots b_n^{n_n} \overset{?}{=} b_i$ No!

with $\sum_{j \le n} n_{ij} = d$ but $\deg(b_i) = 1$

$M_i(b_0 \cdots b_n) = b_0^{n_0} b_1^{n_1} \cdots b_n^{n_n} = b_i b_0^{d-1}$

Thus, we shall assume $b_0 = 1$ first

$\Rightarrow M_i(b_0 \cdots b_n) - b_i = M_i(b_0 \cdots b_n) - b_i b_0^{d-1} \in a$

(3) Injective is trivial ($\ker\rho_d = 0$)

$Z(a) = \operatorname{Im}\rho_d$, thus surjective

$\Rightarrow$ Bijection,

By ρ_d induced by (2), its just a projection

Thus $\rho_d(V(f_i))$

$= V(f_i)$ f_i delete easily these $x_0 \cdots x_N$

$\Rightarrow \rho_d$ countinous

$\rho_d(V(f_i)) = V(\{f_i\} \cup \{x_0 - x_0^{n_0} \cdots x_n^{n_n}\})$

$\Rightarrow \rho_d$ countinous ▨

$\rho_d: \mathbb{P}^n \to Z(a)$

$V \mapsto Z(I(V))$

$Z(I(W)) \leftarrow W$

(4) $\rho_3: \mathbb{P}^1 \to \mathbb{P}^3$

$(x_0 : x_1) \mapsto (x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3)$

In Ex 2.9: $Y = V(y^2 - zw, zw^2 - x^3, yw - x^2)$

$\operatorname{Im}\rho_3 = V(a) = V(\ldots, y^2 - xw, z^2 - \ldots)$ (By this example, we find (2) trivial)

Thus we find the coordinates' change given by $\left\{ x = z, \ y = y, \ z = x, \ w = w \right.$ ▨

2.13. $\rho_2: \mathbb{P}^2 \to \mathbb{P}^5$

$(x_0 : x_1 : x_2) \mapsto (x_0^2 : x_0 x_1 : x_0 x_2 : x_1^2 : x_1 x_2 : x_2^2)$

$=: (y_0 : \cdots : y_5)$

$Y = \operatorname{Im}\rho_2 = V(a) = V(y_1^2 - y_0 y_3, y_2^2 - y_0 y_5, y_4^2 - y_3 y_5, y_1 y_2 - y_0 y_4)$

$Z \subset Y$ $I(Z) \supset I(Y)$

$\Rightarrow Z = V(I(Z)) = V(I(Y) \cup \{f\})$ claim the only one f because $\dim Y - \dim Z = 1$

$\Rightarrow V(f) \cap Y = Z$ ▨

2.14. $\theta: k[z_{ij}] \to k[x_0 \ldots x_r, y_0 \ldots y_s]$

$z_{ij} \mapsto x_i y_j$

$\sum k_{r_{ij}} \ldots \mapsto 0$

$\ker\theta = \langle (x_i y_j)(x_k y_l) - (x_k y_j)(x_i y_l) \rangle = \langle z_{ij} z_{kl} - z_{il} z_{kj} \rangle$.

Thus claim $\operatorname{Im}\psi = Z(\ker\theta)$

$\operatorname{Im}\psi \subset Z(a)$ is trivial: $Z(z_{ij} z_{kl} - z_{ik} z_{lj})$ trivially 0

Conversely, $\forall P \in Z(a)$, P can be renumbered into a matrix with $\begin{vmatrix} z_{ij} & z_{ik} \\ z_{lj} & z_{lk} \end{vmatrix} = 0$

$\Rightarrow P = a^T b^T$ with two vectors $\Rightarrow z_{ij} = a_i b_j \Rightarrow P \in \operatorname{Im}\psi$ ▨

Ex2.15. (1) $\psi: \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$

$$(x_0, x_1) \times (y_0, y_1) \mapsto \begin{pmatrix} x_0 y_0 & x_0 y_1 \\ x_1 y_0 & x_1 y_1 \end{pmatrix} = (x_0 y_0, x_1 y_1, x_0 y_1, x_1 y_0) = V(xy - zw) \quad \boxed{/\!/}$$

(2) $\psi_M: \mathbb{P}^1 \longrightarrow \mathbb{P}^3$

$(x_0, x_1) \mapsto (x_0, x_1, x_0, x_1)$

$\psi_L: \mathbb{P}^1 \longrightarrow \mathbb{P}^3$

$(y_0, y_1) \mapsto (y_0, y_1, y_1, y_0)$

$\psi_{M_t}: \mathbb{P}^1 \longrightarrow \mathbb{P}^3$

$(x_0, x_1) \mapsto ((x_0, x_1), t)$

Next is verification

$M_t = \mathrm{Im}(\psi_M|_t)$

$\bigcup_t M_t = \mathrm{Im}\, \psi_M$

$L_t = \mathrm{Im}(\psi_L|_t)$

$\bigcup_t L_t = \mathrm{Im}\, \psi_L$

$\psi_{L_t}: \mathbb{P}^1 \longrightarrow \mathbb{P}^3$

$(y_0, y_1) \mapsto (t_0(y_0, y_1))$

$\begin{cases} ① \ M_t \cap M_u = \phi \\ ② \ L_t \cap L_u = \phi \\ ③ \ L_t \cap M_u = \{X\} \text{ singleton} \end{cases}$

③ Trivial by $L_t \cap M_u = V(z_2 - z_0, z_3 - z_1, z_3 - z_0, z_2 - z_1) = (1, 1, 1, 1)$

For ① (② is similiar) $M_t \cap M_u = \{(t_0, t_1, t_0, t_1) = (u_0, u_1, u_0, u_1)\} \overset{\text{if}}{\neq} \phi$

$\Rightarrow [t_0] = [u_0], [t_1] = [u_1] \Rightarrow t = u$, contradiction $\Rightarrow M_t \cap M_u = \phi \ \boxed{/\!/}$

(3) $V(xy - z, zw - x) = \{(x, y, xy, \frac{1}{y})\}$ is a curve by $\dim[V(xy - z, zw - x)]$

but $V(xy - z, zw - x)$ trivially not a line

$= \dim \frac{S}{(xy - z, zw - x)} - 1 = 3 + 1 - 2 - 1 = 1$

(In fact we can test it by let $\mathbb{P}^1 \cap L_t = \phi, \forall t \in \mathbb{P}^1$; $\mathbb{P}^1 \cap M_t = \phi, \forall t \in \mathbb{P}^1$)

$\Rightarrow \mathcal{T}_{\text{Zariski in } Q = V(xy - zw)} \overset{\neq}{\supsetneq} \mathcal{T}_{\text{Im}\,\psi} \approx \mathbb{P}^1_k \times \mathbb{P}^1_k \ \boxed{/\!/}$

(Inheritted by subspace of $\mathbb{P}^3_k$)

*(4) (Additionally exercise) We use the diagonal variety $\Delta \subseteq \mathbb{P}^m_k \times \mathbb{P}^n_k$ for $m = n$

$\Rightarrow \mathbb{P}^n_k \times \mathbb{P}^n_k \xrightarrow{\psi} \mathbb{P}^{(n+1)^2 - 1}_k$

$$\begin{tikzcd} \mathbb{P}^n_k \times \mathbb{P}^n_k \arrow[r, "\psi"] & \mathbb{P}^{(n+1)^2-1}_k \\ \Delta \arrow[u, "i_0"] \arrow[ur] \arrow[ur, phantom, "\psi"] & \end{tikzcd}$$

notice that $\psi|_\Delta: \Delta \xrightarrow{\psi|_\Delta} \mathbb{P}^{(n+1)^2 - 1}_k$, $\Delta \cong \mathbb{P}^n_k$, $\mathbb{P}^n_k \xrightarrow{\varphi} \mathbb{P}^{(n+1)^2-1}_k$ φ (2-Veronese embedding)

thus think $\psi|_\Delta$ as φ

$\Rightarrow \psi(\Delta) \cong \varphi(\Delta) = V(z_{ij} - z_{ji})$ closed

$\Rightarrow \Delta$ is closed in $\mathcal{T}_{\psi}$, but can't $\mathcal{T}_Z$ as it contradicts to Zariski topology usually not T_2

$\Rightarrow \mathcal{T}_Z \subsetneq \mathcal{T}_{\psi} \ \boxed{/\!/}$

Ex2.16. (1) $Q_1 \cap Q_2 = V(x^2 - yw, xy - zw) = V(\underline{x^2 - yw}, xy - zw, \underline{y^2 - xz}, \underline{x^3 - zw^2})$

The underline part is the twisted cubic curve l_1

$\Rightarrow Q_1 \cap Q_2 = l_1 \cup l_2$ with $l_2 = V(x, y - w)$ $\boxed{/\!/}$

(2) $C = V(x^2 - yz)$; $L = V(y)$, $C \cap L = V(x^2 - yz, y) = V(x, y, z) = \{0\} =: P$

but $I(C) = \langle x^2 - yz \rangle$, $I(L) = \langle y \rangle$, $I(P)$ is maximal with $I(P) = S$

$\Rightarrow I(C) + I(L) \subsetneq I(P) \ \boxed{/\!/}$

Ex2.17. (1) $\dim Y = \dim S(Y) - 1 = n + 1 - 1 - \text{height } a = n - \text{height } a$

$\text{height } a \leq q$ is trivial $\boxed{/\!/}$

(2) Strict CI $\Leftrightarrow I(Y) = (f_1 \cdots f_{n-r}) \overset{(\Longrightarrow)}{=} \bigcap_{i \leq n-r} I(V(f_i)) \Leftrightarrow$ Set-theoretic CI

(3) The same as Ex1.11 (In $\mathbb{A}^3_k$) ($\Leftarrow$) $Y = Z(xw - yz, y^2 - xz, z^2 - wy) = Z(z^2 - wy) \cap Z(y^3 - x^2 w)$

(This is In $\mathbb{P}^3_k$) (as its projective closure) $\boxed{/\!/}$

**(4) Unsolved

Ex3.3(2) (Countinous) ($\Longrightarrow$) Notice $S^{-1}\square$ is functorial, thus $\varphi^{-1} =: \psi$ has $\psi^*_{\varphi(p)}$
($\Longleftarrow$) We construct an inverse for φ^* with $\varphi^*_p \psi^*_{\varphi(p)} = (\psi\varphi)^*_{\varphi(p)} = (Id)_{\varphi(p)} = Id$

$$\mathcal{O}(Y) \xrightarrow{\varphi^*} \mathcal{O}(X)$$
$$f \longmapsto f\circ\varphi$$
$$g\circ\psi \longleftarrow g$$

$\psi^*_{\varphi(p)}\varphi^*_p = (\varphi\psi)^*_p = Id_p = Id$ for $\forall p \in X$

only need construct an inverse for φ, by homeomorphism, this's done.

$\psi: Y \to X$ is a morphism (Using Lemma 3.6).
$\varphi(x) \mapsto x$

Proof. $\forall V \subset Y$ open, $V = \varphi(U)$ by homeomorphism
and thus $\langle V, x_i\circ\psi\rangle \sim \langle U, x_i\rangle$ by $\mathcal{O}_{\varphi(p),Y} \cong \mathcal{O}_{p,X}$
$\Rightarrow x_i \circ \psi$ regular function ▨

(3) It's proved in P21 upper that: dominate $\Leftrightarrow \varphi^*$ injective
left is prove φ^* injective $\Rightarrow \forall p \in X, \varphi_p$ injective
Proof. Trivial ▨
Consider the converse, not true, because $\langle U, f\rangle \xrightarrow{\varphi^*_p} \langle V, g\rangle$ injective might for $f_1, f_2 \in \langle U, f\rangle$ with $\varphi^*(f_1) = \varphi^*(f_2)$

Ex3.4. In Ex2.12(2), we done it's a homeomorphism, left is showing ρ_d and $(\rho_d|_{Im\,\rho_d}) =: \overline{\rho_d}$ are morphism. First ρ_d is trivial as it's polynomial map;
$\overline{\rho_d}: Im\,\rho_d = Z(\mathfrak{a}) = Z(\ker\theta) \longrightarrow \mathbb{P}^n$
$\downarrow x_i$
k
polynomial map $\Leftarrow x_i \circ \overline{\rho_d}$
$\Downarrow$
regular map $\xrightarrow{\text{Lemma 3.6}}$ ▨

$\alpha: \mathrm{Hom}(\mathbb{A}^2_k, Y) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{O}(Y), \mathcal{O}(\mathbb{A}^2_k) = k[x,y])$
$\Rightarrow \alpha^{-1}(\varphi)$ is an isomorphism as φ is $\mathcal{O}(X) \to \mathcal{O}(\mathbb{A}^2_k)$
$\alpha^{-1}(\varphi): \mathbb{A}^2_k \to Y \cong \mathbb{A}^2_k$
$(x,y) \mapsto (\varphi(\omega_1)(x,y), \cdots, \varphi(\omega_n)(x,y))$, but $\mathbb{A}^2_k \cong Y$ is above impossible ▨

Ex3.5. By Lemma 4.2 ▨ Ex3.6. $X = (\mathbb{A}^2_k)^*$ obviously open in $\mathbb{A}^2_k \Rightarrow$ Quasi-affine
but not affine (We can't directly by X not closed because it can isomorphic to one)
Proof. $\mathcal{O}(X) = \mathcal{O}(\mathbb{A}^2_k) = k[x,y]$
$\forall f \in k[x,y]$
f regular in X, locally is trivial;
(Here this we shall assume X affine first?)
$\forall g \in \boxed{\mathcal{O}(X) = A(X)}$, $g \in k[x,y] = \mathcal{O}(\mathbb{A}^2_k)$, globally is trivial;
$\Rightarrow X \cong \mathbb{A}^2_k$ if X is affine, by Prop 3.5
$\Rightarrow (\mathbb{A}^2_k)^* \approx \mathbb{A}^2_k$, by topology, it's impossible, contradiction ▨

Ex3.7. (1) Any $V(F)\cap V(G) = V(F, G) \neq \phi$ (Ex2.8)
Claim any two homogenous polynomials have common zeros.
Otherwise, $\deg F = s$, $\deg G = t$
$k[X]_{\deg f \le s+t-1}$
$= \langle F, x_i F, \cdots, x_i^t F, G, x_i G, \cdots, x_i^s G\rangle$
$\Rightarrow$ [illegible] $= a x_i^t F + b x_i^s G$ due to the homogenous, $\exists a, b \in k$
$\Rightarrow G = \frac{a x_i^t}{b x_i^s} F$ [illegible] $\Leftrightarrow$ [illegible] $+ aF = 0 \Leftrightarrow G = \frac{-a}{b} F x_i^{t-s}$ have common zeros
$\Rightarrow$ contradiction ▨
$\mathcal{O}(X) = A(Y) = \mathcal{O}(X) = k \Rightarrow Y = \{*\}, \dim Y = 0$, contradiction ▨
(2) If $Y \cap H = \phi$, $Y \subset \mathbb{P}^n \setminus H$ is both affine and projective.

Ex3.1(1) By Ex1.1(3) any conic either $A(W) \cong k[t] \cong A(\mathbb{A}^1_k) \Leftrightarrow W \cong \mathbb{A}^1_k$
or $A(W) \cong \frac{k[x,y]}{(xy-1)}$
$\mathbb{A}^1_k - \{0\}$ is a variety by Lemma 4.2 with [illegible] $= V(x)$
$\Rightarrow \mathbb{A}^1_k - \{0\} \cong V(1-xy) \subset \mathbb{A}^2_k$
$\Rightarrow A(\mathbb{A}^1_k - \{0\}) \cong \frac{k[x,y]}{(xy-1)} \cong A(W)$ ▨
(2) By (1), $\mathbb{A}^1_k \cong Y\ (Y = V(y - x^2)) \ncong W \cong \mathbb{A}^1_k - \{0\}$
In Zariski topology, $U \subset \mathbb{A}^1_k$ open $\Leftrightarrow U = \mathbb{A}^1_k - V(f_i)$ not connected, or $\phi \Rightarrow U \ncong \mathbb{A}^1_k$ ▨
(Because even not homeomorphic)
(3) Geometric picture: $\mathbb{P}^2_k$ add infinite points

Proof. Let conics to be the homogenousification of conics in $\mathbb{A}^2_k$
We repeat Ex1.1 in $\mathbb{P}^2_k$ (1) $\frac{k[x,y,z]}{(x^2-yz)} \longleftrightarrow k[x,y]$ (2) $\frac{k[x,y,z]}{(xy-z^2)}$ is same as (1); that is, different kind of conics in $\mathbb{A}^2_k$ has same form in $\mathbb{P}^2_k$ ▨
$S(X) \cong S(Y)$
$(x,y,z) \longleftrightarrow (x, y, \frac{x^2}{y}) = (xy^2, x^2) \cong k[x,y]$
$\not\Leftrightarrow X \cong Y$ in $\mathbb{P}^n_k$ $\Rightarrow \frac{k[x,y,z]}{(x^2-yz)} \cong k[x,y]$ ▨
Thus (3) needn't to do (See Ex3.9)
$\Rightarrow S(V) \cong S(\mathbb{P}^1) = k[x,y] \Leftrightarrow V \cong \mathbb{P}^1$ ▨
(4) $\mathbb{A}^2_k \approx U_i \subsetneq \mathbb{P}^2_k$, thus not homeomorphic ▨
(5) Use Thm3.4(a) $X \cong Y \Leftrightarrow \mathcal{O}_X \cong \mathcal{O}_Y = k$
$\cap$ $\cap$
$\mathbb{A}^n_k$ $\mathbb{P}^m_k$
but $\mathcal{O}_X = A(X) = \{a|_X \mid a: \mathbb{A}^n_k \to k \text{ affine map}\} =: A_X = k$
$\Rightarrow X = \{*\} = Y$ ▨

Ex3.2. (1) $\varphi: \mathbb{A}^1_k \to Z = V(y^2 - x^3)$
$t \mapsto (t^2, t^3)$ Note φ is polynomial map $\Rightarrow$ morphism
$\frac{y}{x} \xleftarrow{\psi} (x,y)$ ψ is countinous with add $(0,0) \xmapsto{\psi} 0$ (easy to verify by calculus)
But ψ not a morphism as Lemma3.16 says $Z \xrightarrow{\psi} \mathbb{A}^1_k$
(b) $\varphi: \mathbb{A}^1_k \to \mathbb{A}^1_k$ Trivially ψ isn't a morphism;
$t \mapsto t^p$ φ is polynomial map $\Rightarrow$ morphism
(By $\bar{k} = k$) $\sqrt[p]{t} \xleftarrow{\psi} t$ ψ countinous also trivial ▨
$\frac{y}{x}$ It's not a regular function ▨
What is the difference with $\boxed{\mathrm{char}\, k = 0}$? For $\boxed{\mathrm{char}\, k = p}$, we have Frobenius' formula
$(x+y)^p = x^p + y^p$
$\Rightarrow (x-y)^p = x^p - y^p$ (p prime)
$\Rightarrow$ countinous and homomorphism. ($\Rightarrow$ add)

Ex3.3 (1) $A \xrightarrow{f} B$ arise naturally, thus $X \xrightarrow{\varphi} Y$
$A_m \xrightarrow{} B_m$
$f \circ \varphi \ \mathcal{O}_X \leftarrow \mathcal{O}_Y \ f$
$f\circ\varphi \ \mathcal{O}_{X,p} \leftarrow \mathcal{O}_{Y,\varphi(p)}$ ▨
(2) φ isomorphism $\xLeftrightarrow{\text{Prop 3.5}}$ φ^* isomorphism
$\Downarrow$
φ homeomorphism $\forall p$, φ^*_p isomorphism
left to prove
$g \circ \varphi$ $\frac{f}{g}$

Ex3.13. ③&④ are trivial; ① $\mathcal{O}_{Y,X} = (\mathcal{O}_X)_{P_Y}$ local ring, trivial;

with maximal ideal (only) $m_Y = \{f \mid f|_Y = 0\}$

② $K(Y) = \dfrac{\mathcal{O}_{Y,X}}{m_Y}$

dim $\mathcal{O}_{Y,X} = \max_n \{ p_0 \triangleleft p_1 \triangleleft \cdots \triangleleft p_n \mid p_i \triangleleft \mathcal{O}_{Y,X} \}$

$= \max_n \{ \tilde{p}_0 \triangleleft \tilde{p}_1 \triangleleft \cdots \triangleleft \tilde{p}_n \mid \tilde{p}_i \triangleleft \mathcal{O}_X, \tilde{p}_i \triangleleft m_Y \} = \dim X - \dim Y$ ▨

⑤ ~~if $K(Y) = K(Y)$~~ Consider in $\dfrac{\mathcal{O}_{Y,X}}{m_Y}$'s element $\overline{\langle U, f\rangle} = \bar{0} \Leftrightarrow f(U \cap Y) = 0 \Rightarrow \alpha f + \beta = 0$

but with $\alpha, \beta(U \cap Y) = 0$, it's possible to have $\beta(U \cap Y) \neq 0$

$\Leftarrow (\alpha f + \beta)(U \cap Y) = 0$ ▨

Ex3.14. (1) $\varphi: \mathbb{P}^{n+1} - \{P\} \to \mathbb{P}^n$, assume $\mathbb{P}^n = \{(a_0 : a_1 : \cdots : a_n)\}$

$Q \mapsto L$ $\quad P = (a_0 : a_1 : \cdots : a_n) = (1 : a_1 : \cdots : a_n)$

$= (\beta_0 : \beta_1 : \cdots : \beta_n) \mapsto (\beta_1 - \frac{a_1}{a_0}\beta_0 : \cdots : \beta_{n+1} - \frac{a_{n+1}}{a_0}\beta_0) = (a_0\beta_1 - a_1\beta_0 : \cdots : a_0\beta_{n+1} - a_{n+1}\beta_0)$

$\neq (a_0 : a_1 : \cdots : a_n)$

(Remark. We find L by P, Q, L in one line and $L \in \mathbb{P}^n$, that is, l_0 vanishes)

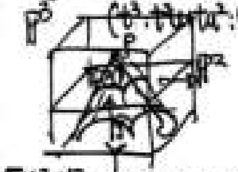
φ is affine map $\leq$ polynomial map $\leq$ morphism ▨

(2) $\rho_3: \mathbb{P}^1 \to \mathbb{P}^3$

$(u:t) \mapsto (t^3 : t^2u : tu^2 : u^3)$ $\quad \mathrm{Im}\,\rho_3 = V(xw - yz, y^3 - x^2w) =: Y$

$\varphi: \mathbb{P}^3 - \{(0,0,1,0)\} \to \mathbb{P}^2$ $\quad \Rightarrow \varphi(Y) = \mathrm{Im}(\varphi\rho_3) = V(y^3 - x^2z) \subset \mathbb{P}^2$

$(t^3 : t^2u : tu^2 : u^3) \mapsto (t^3 : tu^2 : u^3)$



$\| $

$V(y^3 - x^2)$ (projective closure) $\subset \mathbb{A}^2_k$

Thus $\varphi(Y)$ has cuspidal point $(0:0:1)$ ▨

Ex3.15. (1) Otherwise, $X \times Y = Z_1 \cup Z_2$

$X = \{x \mid x \times Y \subset Z_1\} \cup \{x \mid x \times Y \subset Z_2\} =: X_1 \cup X_2$

X_1 and X_2 are both closed, only prove for X_1: let $Z_1 = V(I_1)$

By $\mathcal{O}(X) = A(X)$ regular map has form ... $\Rightarrow X_1 = V(\{f(x, y_0) \mid y_0 \text{ fixed}, f \in I_1\})$ closed

$\Rightarrow X = X_1$ or $X = X_2$, assume $X = X_1 \Rightarrow Z = Z_1$ ▨

Groebner (2) $A(X \times Y) \Longleftrightarrow A(X) \otimes_k A(Y)$ $\quad$ (5) By (2), $\otimes_k$ is product in k-Alg.

φ give by

$\dfrac{k[x_1 \cdots x_{n+m}]}{I(X \times Y)} \longleftrightarrow \dfrac{k[x_1 \cdots x_n]}{I(X)} \otimes_k \dfrac{k[x_1 \cdots x_m]}{I(Y)} \Rightarrow X$ is product in Var_k ▨ (k-Modules) (contravariant)

$x_1^{i_1} x_2^{i_2} \cdots x_{n+m}^{i_{n+m}} \Leftrightarrow x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \otimes x_{n+1}^{i_{n+1}} \cdots x_{n+m}^{i_{n+m}}$ ▨

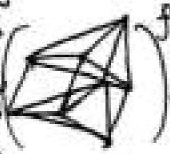
Concretely (upper proof is categorially), see underly

$\dfrac{k[x_1 \cdots x_n] \otimes_k k[x_1 \cdots x_m]}{I(X \times Y)} \underset{\xi}{\cong} \dfrac{k[x_1 \cdots x_n] \otimes k[x_1 \cdots x_m]}{I(X) \otimes I(Y)}$ ▨ $\quad$ See as k-module (k-algebra)

f vanishing in $X \times Y \Leftrightarrow \left.\begin{array}{l} f(x, y_0) \text{ vanish in } X \\ f(x_0, y) \text{ vanish in } Y \end{array}\right\} \Leftrightarrow f(x, y_0) \otimes_k f(x_0, y) \in I(X) \otimes_k I(Y)$

(3)

$$\begin{tikzcd} Z \arrow[r] \arrow[rr, bend left] & X \times Y \arrow[r] & X \\ A(Z) & A(X) \otimes_k A(Y) \arrow[l] & A(X) \arrow[l] \end{tikzcd}$$



It's useless, as $A(\square)$ not functorial.

① $X \times Y \xrightarrow{p_1} X$

$(x, y) \mapsto (x)$ is trivially morphism as it's polynomial map

Y the same;

② $\psi: Z \to X \times Y$

$z \mapsto (\varphi_1(z), \varphi_2(z))$

It's trivially a morphism as $\varphi_1: Z \to X$, $\varphi_2: Z \to Y$ both are

$\Rightarrow$ Exist;

If not unique, it induce $Z \rightrightarrows X \times Y$, $(\varphi_1(z), \varphi_2(z)) \neq (\psi_1(z), \psi_2(z))$

$\varphi_1(z) \neq \psi_1(z)$ or $\varphi_2(z) \neq \psi_2(z)$ contradict to commuting ▨

(4) dim $X \times Y$

$= \dim A(X \times Y) = \dim A(X) \otimes_k A(Y)$

$\stackrel{?}{=} \dim A(X) + \dim A(Y) = \dim X + \dim Y$

left is showing $\dim A(X) \otimes_k A(Y) = \dim A(X) + \dim A(Y)$

$\dim X + \dim Y = \mathrm{trdeg}_k K(X) + \mathrm{trdeg}_k K(Y) = \mathrm{trdeg}_k(\mathrm{Frac}(A(X))) + \mathrm{trdeg}_k(\mathrm{Frac}(A(Y)))$

$= \mathrm{trdeg}_k [\mathrm{Frac}(A(X) \otimes_k A(Y))] = \mathrm{trdeg}_k(\mathrm{Frac}(A(X \times Y)))$

where $A(X \times Y) = \dfrac{k[x_1 \cdots x_{n+m}]}{\langle f_1 \cdots f_{l_1}, g_1 \cdots g_{l_2}\rangle}$, $\langle f_1 \cdots f_{l_1} \cdots g_{l_2}\rangle$ has transcendental degree over $k = \mathrm{trdeg}_k(\mathrm{Frac}A(\cdots$

$(n+m) - (h_1 + h_2) \quad = \quad (n - h_1) + (m - h_2)$

with $h_1 \leq l_1$, $h_2 \leq l_2$ ▨

For (4), we need Noether's Noetherian Lemma to ensure the existence of the algebraic degree h_1 and h_2

3.16. (1) $X \times Y = (X \times \mathbb{P}^m) \cap (\mathbb{P}^n \times Y) \subset \mathbb{P}^n \times \mathbb{P}^m \xrightarrow[\text{embedding}]{\text{Segre}} \mathbb{P}^{(m+1)(n+1)-1} =: \mathbb{P}^N$

also need show $X \times Y$ irreducible (Similar to Ex3.15(1))

$\Rightarrow X \times Y \subset \mathbb{P}^N$, left is showing $X \times Y$ is open subset:

$X \times Y = \underbrace{(X \times \bar{Y})}_{\text{open in } \bar{X} \times \bar{Y}} \cap \underbrace{(\bar{X} \times Y)}_{\text{also open}}$ open ▨

For (2), we can construct $X \times Y = Z(I)$ (if we consider $X \times Y$ as embedded)

$X = Z(f_1(x_0 \cdots x_n), \cdots f_{l_1}(x_0 \cdots x_n))$

(2) By (1), $X \times Y$ is quasi-projective, left is showing $X \times Y$ closed

$\overline{X \times Y} = \overline{(X \times \bar{Y}) \cap (\bar{X} \times Y)} = \overline{X \times \bar{Y}} \cap \overline{\bar{X} \times Y} = \bar{X} \times \bar{Y} = X \times Y$ ▨

$Y = Z(g_1, \cdots g_{l_2})$

$\Rightarrow X \times Y (\psi(X \times Y))$ ...

(3) The difference comes that $X \times Y$ not good to directly use coordinate represent

$= Z\{f_i(z_{0j}, z_{1j} \cdots z_{nj}), g_i(z_{j0} \cdots z_{jm})\}_{n,m}$ closed ▨

thus we composite Segre embedding

$$\begin{tikzcd} Z \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] \arrow[dr] & X \\ Y & X \times Y \arrow[l, "p_2"] \arrow[u, "p_1"'] \arrow[d, "\psi"] \\ & \mathbb{P}^N \end{tikzcd}$$

with $\psi: Z \to X \times Y$

$z \mapsto \psi(\varphi_1(z), \varphi_2(z))$

$p_1: \psi(X \times Y) \to X$

$(z_{00} : z_{01} : \cdots : z_{nm}) \mapsto (z_{0j} : \cdots z_{nj})$; j fixed

Then verify commutative is routine ▨

(4) We prove $\mathcal{J} = \{V(f_i)\}$, with f_i bihomogenous ($f(\lambda x, \mu y) = \lambda^d \mu^e f$, $\deg f = (d, e)$)

Proof. $\mathcal{J} \subset \{V(f_i)\}$ follows $\varphi: \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(m+1)(n+1)-1}$

Thus $V(f)$ closed in $\mathbb{P}$ $\quad V(\tilde{f}) \mapsto V(f)$

f has $\deg(d, d)$ $\in \{V(f_i)\}$: $\deg \tilde{f} = (d, d) \Leftarrow \deg f = d$

Conversely, f has $\deg f = (d, e)$, then $V(f) \xrightarrow{\varphi} V(\tilde{f})$ with $\tilde{f}$ ... $\Rightarrow \{V(f_i)\} \subset \mathcal{J}$ ▨

Left is verification of dominate map's lifting, we do at last after posting globally. (After here we have complete(e)), but we don't stop here)

$\boxed{\text{PART II}}$ locally $\xRightarrow{\text{Sheaf}}$ Globally

Construct $\tilde{Y}_p = \mathrm{Max}[(\mathcal{O}_p)_{K(\tilde{Y})}]$, then Claim2. $\tilde{Y} = \coprod_{p \in Y} \tilde{Y}_p$ is what we need

To show Claim2. We do ① Construct $p: \tilde{Y} \to Y$ ② Proof the universal property

Proof ① $p: \tilde{Y} \to Y$ is morphism

$y \mapsto p$ with $y \in \tilde{Y}_p$

We first construct $\tilde{Y}$'s topology

$\forall U \in \mathcal{T}_p$, U is locally, thus quasi-affine

apply $\boxed{\text{PART I}}$, $\exists\ \varphi_U: V \to U$

$\{\varphi_U^{-1}(p) \mid p \in U\} \overset{\cong}{\Longleftrightarrow} \{Y_p \mid p \in U\}$ (This bijection is given by going-up)

$\varinjlim$ $\qquad$ $\varinjlim$

$V \overset{\cong}{\longleftrightarrow} \coprod_{p \in U} Y_p$ we hope it's even a homeomorphism

Thus $\mathcal{T}_{\tilde{Y}} = \{V \mid V \cong \coprod_{p \in U} Y_p,\ U \in \mathcal{T}_Y\}$

$\Rightarrow$ Under $\mathcal{T}_{\tilde{Y}}$, p is trivially a countinous map; its regular is more trivial ▨

② Only do locally = affine:

$$\begin{array}{ccc} & & \tilde{Y} \\ & \overset{\theta}{\nearrow} & \downarrow p \\ Z & \xrightarrow{\varphi} & Y \end{array} \qquad \begin{array}{ccc} & & A(\tilde{Y}) \\ & \overset{\theta^*}{\swarrow} & \uparrow p^* \\ A(Z) & \xleftarrow{\varphi^*} & A(Y) \hookleftarrow 0 \end{array}$$

(φ dominate $\Leftrightarrow$ φ^* injective)

We construct $\exists!\ \theta^*$, then θ follows Prop3.5.

Indeedly, $\theta^*: A(\tilde{Y}) \to A(Z)$

$f \mapsto \theta^* f =: g$

with $g: z \to k$

$z \mapsto f(\tilde{Y}_{\varphi(z)})$

Claim $g(z)$ is singleton, it's trivial ▨

Ex3.18.

(1) Projectively normal $\Rightarrow S(Y)$ normal

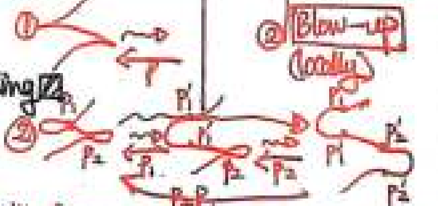
$\Rightarrow (S(Y))_{(m_p)}$ normal $\Leftrightarrow$ Y normal at $\forall p \in Y \Leftrightarrow Y$ normal ▨

$\cong \mathcal{O}_p$

(2) $\mathcal{O}_p \cong (S(Y))_{(m_p)} \cong (k[t,u])_{(m_p)}$ normal follows similiar way as Ex3.17 (4) (Globally)

$S(Y) = \dfrac{k[x,y,z,w]}{(xw-yz,\ y^3-z^2w)}$ not intergallyclosed, how to prove?

Geometric picture of $\boxed{\text{Normalization}}$



(3) $Y \cong \mathbb{P}^1$ is trivial, $\mathbb{P}^1$ normal more trivial;

$(t,u) \mapsto (t^4, t^3u, tu^3, u^4)$ in $\mathbb{P}^3$, (t^2, t^2u, tu^2, u^3) $\Rightarrow$ depend on embedding ▨

Ex3.20. (1) Claim the normal points in a variety are dense set.

$$\begin{array}{ccc} Y - P & \xrightarrow{f} & k \\ i \downarrow & \overset{\tilde{f}}{\nearrow} & \\ Y & & \end{array}$$

Because P is local, we may assume Y is affine

$g|_{V-\varphi(P)}$; $V \hookrightarrow \mathbb{A}^n_k$, g is polynomial map

$U - P \hookrightarrow U \hookrightarrow Y$, $\varphi|_{U} \cong$

$\Rightarrow$ can extend $g|_{V-\varphi(P)}$ into $g|_V$

$\Rightarrow$ We find $h_U: U \to k$

$\Rightarrow h: Y \to k$ is extension of f ▨

(We of course can blowup a series of points same time, but it's out of this book)

My question: Whether $\exists\ \{P_i\} \subset Y$, $Bl_{\{P_i\}} Y \cong \tilde{Y}$?

(2) The conterexample is trivial $g: \mathbb{A}^1 - \{0\} \to k$

$x \mapsto \frac{1}{x}$ ▨

Additional Exs (About Domate map)

Case I. φ countinous

① If $F \neq F_1 \cup F_2$

$\Rightarrow \varphi(E) = (\varphi(E) \cap F_1) \cup (\varphi(E) \cap F_2)$

E irreducible $\Leftrightarrow I(E)$ is prime $\Leftrightarrow \dfrac{k[x_1, \dots, x_n]}{I(E)}$

and by Prop(3.5), (injective) is domain

$\dfrac{k[x_1 \cdots x_m]}{I(F)} \leq \dfrac{k[x_1 \cdots x_n]}{I(E)} \Rightarrow \dfrac{k[x_1 \cdots x_m]}{I(F)}$ is domain

$\Rightarrow I(F)$ is prime $\Leftrightarrow F$ irreducible ▨

② By $\dim E = \dim S(E) - 1 \geq \dim S(F) - 1 = \dim F$ ▨ (due to ①)

Case II, φ rational.

Similarly ▨

Ex3.17.

(1) $\forall V \subset \mathbb{P}^2$ is a conic, $V \cong \mathbb{P}^1$, $\mathcal{O}_p$ (in $\mathbb{P}^1$) $= k$ trivially intergally closed;

$\Rightarrow$ We only to prove $\mathcal{O}_p$'s intergally properties don't change under isomorphism, trivial ▨

(trivial: $V \cong W \Rightarrow \mathcal{O}(V) \cong \mathcal{O}(W) \Rightarrow (\mathcal{O}(V))_{m_p} \cong (\mathcal{O}(W))_{m_{\varphi(p)}}$

$\Leftrightarrow \mathcal{O}_p \cong \mathcal{O}_{\varphi(p)}$ ▨)

(2) $Q_1 = V(xy - zw) \cong \mathbb{P}^1 \times \mathbb{P}^1$

$\mathcal{O}_p = (S(Q_1))_{(m_p)}$ by Thm3.4 $\boxed{\text{is a field}}$, thus it's intergally closed $\Leftrightarrow$ algebraically closed

$(S(Q_1))_{(m_p)} = \left(\dfrac{k[x_0 \cdots x_3]}{\langle x_0x_1 - x_2x_3 \rangle}\right)_{(m_p)} \cong \mathcal{O}_p$ (in $\mathbb{P}^1 \times \mathbb{P}^1$) thus integrally closed?

$Q_2 = V(xy - z^2)$. $\mathcal{O}_p \cong (S(Q_2))_{(m_p)} \cong (k[1, t^2, t]]_{(m_p)} = (k[t])_{(m_p)} = \left\{\dfrac{\sum a_i t^i}{\sum b_i t^i} \;\middle|\; \sum b_i t^i \notin m_p\right\}$

$\sum_{i \in \mathbb{N}} \frac{f_i}{g_i} h^i = 0$, h^n is obvious a form $\frac{f}{g}$ with $g(p) \neq 0$ $\qquad$ $\frac{f}{g}$, $\sum b_i t^i \neq 0$

Left is showing $\deg h = 0$, this easily follows $\deg h^n = \deg \frac{g_1}{f_1} = 0$ ▨

(3) Claim $\mathcal{O}_{(0,0)}$ not integrally closed. Proof is much easier than projective one.

$\mathcal{O}_{(0,0)} = \left(\dfrac{k[x,y]}{(y^2 - x^3)}\right)_{m_{(0,0)}} \cong k[t^2, t^3] \subsetneq k[t]$ with $\overline{k[t^2, t^3]}_{\mathrm{Frac}(k[t^2, t^3])} = k(t) = k[t]$

$\Rightarrow k[t^2, t^3] \neq \overline{k[t^2, t^3]} \Rightarrow \mathcal{O}_{(0,0)}$ not integrally closed ▨

(4) ($\Leftarrow$) $A(Y)$ integrally closed $\Leftrightarrow \mathcal{O}(Y)$ integrally closed $\Rightarrow \forall p \in Y$, $\mathcal{O}_p$ intergally closed (by $\mathcal{O}_p = \varinjlim_{p \in U \subset Y} \mathcal{O}(U)$)

$\Rightarrow Y$ is normal

($\Rightarrow$) Y is normal $\Rightarrow \mathcal{O}_p$ intergally closed $\forall p \in Y \Rightarrow A(Y) = \bigcap_{p \in Y} \mathcal{O}_p$ also intergally closed (by Thm3.2) ▨

(5) The universal problem for $\boxed{\text{Normalization}}$ Example $V(y^2 - x^3) \to \mathbb{A}^1$

$$\begin{array}{ccc} & & \tilde{Y} \\ & \overset{\exists!\theta}{\nearrow} & \downarrow p \\ Z & \xrightarrow{\varphi} & Y \\ & & \downarrow \\ & & 0 \end{array}$$

We construct $\tilde{Y}$ from Y like a bundle: $\tilde{Y} = \coprod_{p \in Y} \tilde{Y}_p = \varinjlim \tilde{Y}_p$

Thus we locally consider $\qquad$ with $p \in Y \leftrightarrow \tilde{Y}_p \subset \tilde{Y}$

$\boxed{\text{PART I}}$ Affine

Claim1. $(K(Y))_{K = A(x_1 \cdots x_n)} = K(\tilde{Y})$ when Y is affine $\qquad$ $\boxed{\text{PART II}}$

This follows (3.9A) as $\overline{K(Y)}_{K=A} = A[f_1 \cdots f_m]$ with $f_i \in K \Rightarrow \overline{(K(Y))}_K = K(\tilde{Y})$

(Or Noether normalization)

Ex 3.21. (4) $\mathrm{Hom}(X, G_a) \longleftrightarrow O(X)$

$f \longmapsto f$ it's by [redacted] $G_a = (\cancel{A}k, \mu) = (k, \mu)$

(5) $\mathrm{Hom}(X, G_m) \longleftrightarrow Z(O(X))$

$f \longmapsto f$ both of them easily by identity $\boxtimes$

Normalization

$$\begin{tikzcd} & \tilde{Y} \arrow[d, "p"] \\ Z \arrow[ur, "\exists!\theta"] \arrow[r, "\varphi"] & Y \arrow[d] \\ & O \end{tikzcd}$$

Completion (of variety) ⟨ distinct rings / We can also do them in schemes

$$\begin{tikzcd} & O \arrow[d] \\ Z \arrow[r, "\varphi"] \arrow[dr, "\exists!\theta"'] & X \arrow[d, "i"] \\ & \bar{X} \end{tikzcd}$$

i is an open immersion

(Nagata's ~~em~~ embedding)

(※Remark. Quite similiar to homotopy lifting/extension property, indeed, to classify such p, i, … (five kinds), we have Model category, both in AT & AG)

Ex4.7 (Continous) $\mathcal{O}_{p,X} \underset{\psi}{\overset{\varphi}{\longleftrightarrow}} \mathcal{O}_{q,Y}$

$\bar{x}_i \overset{\varphi}{\longmapsto} \varphi(\bar{x}_i)$ $(i \le n)$, is regular in $\exists V_1 \subset N_q$

$\psi(\bar{y}_i) \overset{\psi}{\longleftarrow} \bar{y}_i$ $(i \le m)$, $\psi(\bar{y}_i)$ is regular in $\exists U_1 \in N_p$

$\Rightarrow$ Take $U_2 = U_1 \cap \varphi^{-1}(V_1) \Rightarrow U_2 \cong V_2$ by restricting φ and ψ ▨

$V_2 = V_1 \cap \psi^{-1}(U_1)$ (This proof is similiar to Coro4.5(2))

Ex4.8. $\# \mathbb{A}^n = \# \mathbb{P}^n = \# k$ is trivial

For $X \overset{bi}{\sim} H \subset \mathbb{P}^{n+1}$, $\# X = \# U = \# V = \# H$ by density, left is to show $\# H = \# \mathbb{P}^n$

$U \cong V$ — this follows the projection in the Hint

Namely, $\forall (x_0 : \cdots : x_n : 1) \in \mathbb{P}^n$, by solving the equation of f ($H = V(f)$), we find $x_{n+1} \Rightarrow (x_0 : \cdots : x_n : x_{n+1}) \in H$, thus surjective;

We take the point $Q \notin H$, denote $P = (x_0 : \cdots : x_n : 1)$ with $L = L_{PQ}$, $H \cap L$ is finite or whole L by $L \cong \mathbb{A}^1_k \Rightarrow$ is finite to one ▨

Ex4.9. We admit X' does quasi-projective;

This is a proof similiar to (4.9): assume $x_0 \neq 0$, $\{\frac{x_1}{x_0}, \cdots, \frac{x_r}{x_0}\}$ is a separable transcendent basis for $K(Y)/k$. thus it's a single extension, i.e. $K(Y) = k(\frac{x_1}{x_0}, \cdots, \frac{x_r}{x_0})[\frac{x_{r+1}}{x_0}, \cdots, \frac{x_n}{x_0}] = k(\frac{x_1}{x_0}, \cdots, \frac{x_r}{x_0})[\frac{x_{r+1}}{x_0}]$

Take $H_{r+2} \cong \mathbb{P}^{n-1}$, $P = (a_0 : \cdots : a_{r+1} : 1 : \cdots : a_n) \notin X$

$\varphi : X \to H_{r+2}$

$(x_0 : \cdots : x_n) \mapsto (x_0 : \cdots : \widehat{x_{r+2}} : \cdots : x_n)$, $\widehat{x_{r+2}} = 0$

And $\bar{x}_{r+2}$ is algebraic over $K(X) = k(\frac{x_1}{x_0}, \cdots, \frac{x_r}{x_0})[\frac{x_{r+1}}{x_0}]$

$\Rightarrow f(x_0 \cdots x_{r+1}) - x_{r+2} g(x_0 \cdots x_{r+1}) \in I(Y)$, $\exists f, g$

$\Rightarrow \frac{f}{g} = \bar{x}_{r+2}$ regular over $U \subset \operatorname{Im} \varphi$

$\Rightarrow U \cong X \Rightarrow Y \overset{bi}{\sim} \operatorname{Im} \varphi$ ▨

$(x_0 \cdots \widehat{x_{r+2}} \cdots x_n) \mapsto (x_0 \cdots \frac{f}{g}(x_0 \cdots x_{r+1}) \cdots x_n)$

Ex4.10

$$\begin{array}{ccc} Bl_O(Y) & \xrightarrow{\varphi} & Y \\ \downarrow & & \downarrow \\ Bl_O(\mathbb{A}^2) & \xrightarrow{p} & \mathbb{A}^2 \\ \downarrow & \nearrow_{p} & \\ \mathbb{A}^2 \times \mathbb{P}^1 & & \end{array}$$

① $Bl_O(Y) = \overline{\varphi^{-1}(Y - O)}$ $\qquad E = O \times \mathbb{P}^1$

$= \overline{p^{-1}(V(y^2 - x^3))}$ $\qquad = \{(0, 0, z, w) \mid (z, w) \in \mathbb{P}^1\}$

$= V(y^2 - x^3, xw - yz) \cong \mathbb{P}^1$

② w-axis: $w \neq 0 \Leftrightarrow w = 1$, $Bl_O(Y) = V(y^2(1 - yz^3), x - yz)$

$\Rightarrow \tilde{Y} \cap E = \{(0, 0, 1, 0)\}$ $\qquad = V(y, x) \cup V(1 - yz^3, x - yz)$

z-axis: $z \neq 0 \Leftrightarrow z = 1$, $Bl_O(Y) = V(x^2(w^2 - x), y - wx)$

$\Rightarrow \tilde{Y} \cap E = \emptyset$ $\qquad = V(x, y) \cup V(w^2 - x, y - w^3)$

$\Rightarrow \tilde{Y} \cap E = \{(1 : 0)\}$ only one point ▨

③ $\tilde{Y} = Bl_O(Y) = V(y^2 - x^3, xw - yz) \longleftrightarrow \mathbb{A}^1 \Rightarrow \tilde{Y} \cong \mathbb{A}^1$, $\dim Y = 1$

$(x, y, z, w) \longmapsto x$ — by it's polynomial map ▨ $\dim \tilde{Y} = \dim \mathbb{A}^1 = 1$

w-axis: $(x, xz, x, 1) \longleftarrow x$ — $\Rightarrow$ bijective continuous

z-axis: $(x, xz, 1, x^3) \longleftarrow x$ — but $\mathcal{O}(Y) \not\cong \mathcal{O}(\tilde{Y})$ by $\mathcal{O}(Y) = k \subsetneq \mathcal{O}(\tilde{Y})$

Ex4.1. Let $\Sigma = \{U \mid f : U \to k$ is a regular function$\}$

and chain in Σ has maximal element $\bigcup_{U \in \Sigma} U$, thus by Zorn's Lemma, Σ has a maximal ▨

Thus we shall prove we can $\boxed{\text{paste}}$ the define set of f

Proof. Denote the paste of f and g as h; h is countinous by pasting Lemma

$\forall P \in U \cup V$; $U - V$, $V - U$, $U \cap V$ all open

$\Rightarrow \exists W \in N_P$, $W \subset U - V$ or $V - U$ or $U \cap V$ (or U or V is even possible)

$\Rightarrow h = f$ or g thus regular ▨

Ex4.3. (1) $f : \mathbb{P}^2 \longrightarrow k$ $\qquad \varphi : U_0 \longrightarrow k$

$(x_0 : x_1 : x_2) \mapsto \frac{x_1}{x_0}$ defined at U_0; $f = [\varphi]$ $\qquad (1 : x_1 : x_2) \mapsto x_1$ ▨

(2) $\mathbb{P}^2 \xrightarrow{f} \mathbb{A}^1_k \overset{i}{\hookrightarrow} \mathbb{P}^1$

$(x_0 : x_1 : x_2) \mapsto \frac{x_1}{x_0} \longmapsto (1 : \frac{x_1}{x_0}) = (x_0 : x_1)$

$\Rightarrow \varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$

$(x_0 : x_1 : x_2) \longmapsto (x_0 : x_1)$ defined at whole $\mathbb{P}^2$

with $\varphi = [\psi] = [\varphi]$ ▨

Ex4.4. (1) By Ex3.1.(2), conic isomorphic to $\mathbb{P}^1$, thus birational ▨

(2) $\varphi : V \dashrightarrow \mathbb{P}^1$ not defined at $(0, 1)$

$(t^2, t^3) \mapsto (1 : t) \Longrightarrow V - \{(0, 0)\} \cong U_0$ both open $\overset{\text{Coro4.5}}{\underset{(3)}{\Longrightarrow}} V \overset{bi}{\sim} \mathbb{P}^1$ ▨

(3) $\psi : Y \dashrightarrow \mathbb{P}^1$

$(x : y : z) \longmapsto (x : y)$ with $z = 1$ undefined

$\mathbb{P}^1 \cong Y - V(z - 1) \overset{\text{Coro4.5}}{\underset{(3)}{\Longrightarrow}} \mathbb{P}^1 \overset{bi}{\sim} Y - V(z - 1)$ ▨

Ex4.5. $f : Q \dashrightarrow \mathbb{P}^2$

$(x : y : z : w) \longmapsto (x : y : z)$ undefined at $z = 0$

$\Rightarrow (Q - V(z)) \cong \mathbb{P}^2 \overset{\text{Coro4.5}}{\underset{(3)}{\Longrightarrow}} \mathbb{P}^2 \overset{bi}{\sim} Q$ ▨ (by $(x : y : z : w) = (x : y : z : \frac{xy}{z}) = (xz : yz : z^2 : xy) \mapsto (x : y : z)$) (Trivial not isomorphic)

Ex4.6. (1) $\mathbb{P}^2 - \{O_1, O_2, O_3\} \cong \mathbb{P}^2 - \{O_1, O_2, O_3\} \underset{\text{Coro4.5(3)}}{\Longrightarrow} \mathbb{P}^2 \overset{bi}{\sim} \mathbb{P}^2$

($O_1 = (1 : 0 : 0)$; $O_2 = (0 : 1 : 0)$; $O_3 = (0 : 0 : 1)$)

$f : \mathbb{P}^2 - \{O_1, O_2, O_3\} \longrightarrow \mathbb{P}^2 - \{O_1, O_2, O_3\}$

$(a_0 : a_1 : a_2) \overset{f}{\longmapsto} (a_1 a_2 : a_0 a_2 : a_0 a_1)$

$(a_0^2 a_1 a_2 : a_0 a_1^2 a_2 : a_0 a_1 a_2^2)$

$= (a_0 : a_1 : a_2) \Rightarrow f^2 = \mathrm{Id} \Rightarrow \varphi = [f]$ with $\varphi^2 = \mathrm{Id}$ ▨

(2) We done it in (1) ▨

(3) We done it in (1) ▨

Ex4.7: By it's local property, thus we assume X, Y both affine (also by Prop4.3)

Let $X \subset \mathbb{A}^n_k$, $Y \subset \mathbb{A}^m_k$

Then we consider $f(ky,y)$: $f(x,y)=(f_m+f_{m+1}+\cdots+f_d)(x,y)$

① if $f_m(ky,y)=0$, the $l\left(\frac{O_p}{f(ky,y)}\right)$ grows up to $m+1$ (even higher but these k are finite; $f_m=(x-ky)\tilde{f}_m$, f_m at most decomposite into m linear equations)

② if $f_m(ky,y)\neq 0$, the $\frac{O_p}{f(ky,y)}=\frac{O_p}{g(y)}$, thus is of length of m

$\Rightarrow (L\cdot Y)_p=\mu_p(Y)$ ☒

(For $L\not\mid f_{(x,y)}$)

(c) $Y=V(f_d)=V(\sum x^i y^j z^{i_2}\ldots)$ (trivialized)

$L=V(g_d)=V(\alpha x+\beta y+\gamma z)=V(z-(\alpha x+\beta y))$.

consider $Y\cap L=V(f_d(x,y;\alpha x+\beta y))=V(g_d(x,y))$

We claim, $g_d(x,y)$ can do prime decomposition, i.e. $g_d=g_1^{n_1}\cdots g_s^{n_s}$, $n_1+\cdots+n_s=d$

Thus $Y\cap L=\{P_1\ldots P_s\}$, with $P_i=V(g_i,g)$ $\quad (g_i=\alpha_i x+\beta_i y)$

$\Rightarrow \sum_P (L\cdot Y)_P=\sum_{i\in S}\boxed{(L\cdot Y)_{P_i}}=\sum_{i\in S}\boxed{n_i}=d=(L\cdot Y)$ ☒ (a point)

Left is proving ① Claim ② $(L\cdot Y)_{P_i}=n_i$

①: Induction on d $\quad$ ② is trivial by argument in (b)

$d=1$ is trivial

now consider $(d+1)$ case: $g_{d+1}(x,y)=g_d(x,y)(\alpha x+\beta y)$

by algebraically closed ☒ $\quad$ (You may consider $\boxed{p\mid d}$ or not)

Ex5.5. Case I, Char $k=0$ $\quad W_1=V(x^d+y^d+z^d)$

Case II Char $k=p\neq 3$ $\quad W_2=V(x^d+y^d+z^d)$

Case III $\boxed{\text{Char } k=3}$ $\quad W_3=V(x^d+x^{d}y+y^{d-1}z+z^{d-1}x)$ (because if we still take as W_1, W_2, it comes to reducible)

Ex5.6. (a) $Y_1=V(x^4+y^4-x^2)$ Tacnode ~~Cusp~~, $Y_2=V(x^6+y^6-xy)$ Node

$Y_1, Y_2\subset \mathbb{A}^2_k$, now we blow up both.

Picture

From the picture, we see blowup does resolution the singularity (Indeed, it's almost the simplest one way)

Recall. $X\longrightarrow \mathbb{A}^2_k$

$\cap$

$\mathbb{A}^2_k\times\mathbb{P}^1_k$ $\quad$ and $Bl_O(Y_i)=\overline{\varphi^{-1}(Y_i-\{O\})}$

For Y_1 (Cusp): assume $xt=sy$

$s\neq 0$ $\quad s=0$ $\quad \mid x^2=x^4+y^4$

$\Rightarrow s=1$ $\quad\Rightarrow t\neq 0\Rightarrow x=0\Rightarrow y=0$

$Bl_O(Y_1)=V(x[(t^4+1)x^2-1])=V(x,y)\cup V((t^4+1)x^2-1)$

$=E\cup V(y-xt,\ x^4+y^4-x^2)$

For Y_2 (Node), we compute $Bl_O(Y_2)=E\cup V(y-xt,\ (x^4+t^6x^4-t),\ x^6+y^6-xy)$

Ex5.1. By the figure, claim all of them nonsingular expect $\underline{(0,0)}$

We compute their O_{0,X_i} $(i=a,b,c,d)$, first, $\dim O_{O,X_i}=\dim X_i=1$ is trivial

(1) $O_{0,X_1}=\frac{k[x,y]}{(x^4+y^4-x^2)}$ with maximal ideal $m_1=m_{O,X_1}O_{O,X_1}$ (Indeed, by computing Jacobian is trivial find (0,0) singular, but can't distinguish four figures)

$\frac{m_1}{m_1^2}$ has basis is $\{x\}\Rightarrow \dim\frac{m_1}{m_1^2}=1$

$(\forall f\in\frac{m_1}{m_1^2}, f(x,y)=ax+by$, and $x^2=x^4\sim y^4$)

(high degree vanishing m_1^2)

$\Rightarrow$ is Figure "Tacnode" ☒

(2) Similarly $\frac{m_2}{m_2^2}$ has basis $\{x,y\}$, $\dim\frac{m_2}{m_2^2}=2$

$\Rightarrow$ Figure "Node" ☒

(3) $\frac{m_3}{m_3^2}=\langle x\rangle$, $\dim\frac{m_3}{m_3^2}=1\Rightarrow$ Figure "Cusp" (4) $\frac{m_4}{m_4^2}=\langle x,y,xy\rangle$, $\dim\frac{m_4}{m_4^2}$ $\Rightarrow$ Figure "Triple point" ☒

Ex5.2. (a) "Pinch point", singularities $Z(y,z)$ ☒

(b) "Conical double point", singularities (0,0,0) (c) "Double line", singularities $Z(x,y)$

Ex5.3. (a) $\mu_P(Y)=1\Leftrightarrow f_1=ax+by\neq 0\Leftrightarrow \frac{\partial f}{\partial x}=a,\ \frac{\partial f}{\partial y}=b$ both $\neq 0$ ☒

(b) Node: 2 Triple point: 3 $\quad\Leftrightarrow$ rank 2 is $\dim$ $\Leftrightarrow$ non-singular ☒

Cusp: 2 Tacnode: 2 easy analytic Geometry ☒

Ex5.4 (a) First we claim $\mu_P(Y)=l\left(\frac{O_P}{(f)}\right)$, then left is proof $l\left(\frac{O_P}{(f,g)}\right)\geq l\left(\frac{O_P}{(f)}\right)\ldots$

(The Bezout then says that: in $\mathbb{P}^2$ it takes equality)

$\frac{O_P}{(f,g)}$ has the only prime ideal, thus maximal, thus every prime in $\frac{O_P}{(f,g)}$ is maximal (by assume $P=0$)

$\Rightarrow\frac{O_P}{(f,g)}$ is Artinian $\Rightarrow l\left(\frac{O_P}{(f,g)}\right)$ is finite (by Artinian $\Rightarrow$ Noetherian) ☒ $\quad\left(O_P\xrightarrow{p}\frac{O_P}{(f,g)},\ m=\langle x,y\rangle\ p(m)=0\right)$

And we now find a sequence of $\frac{O_P}{(f,g)}$ has length $\mu_P(Y)\mu_P(Z)$ to complete the proof $\quad$ ~~φ of the~~ $\Rightarrow p(m)=0$

$\Rightarrow Y=V(f)=V(\sum_{i\geq m}f_i)=V(\sum_{d\geq i\geq m}\sum_{p\leq i}c_p x^p y^{i-p})$ Denoted $f_{(m)}$

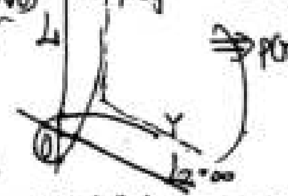
$Z=V(g)=V(\sum_{j\geq n}g_j)=V(\sum_{d_2\geq j\geq n}\sum_{q\leq j}d_q x^q y^{j-q})$

I don't know how to contract such a ~~pair~~ sequence.

(b) $\mu_P(Z)=1$, thus we comes to prove (a) takes equality here

assume $P=0$, $L_1=V(x-ky)$

$\Rightarrow\frac{O_P}{(f,x-ky)}\cong\frac{O_P}{(f(ky,y))}$



(The $\mathbb{P}^2$ holds by the ∞ is included to compute, but $\mathbb{A}^2$ comes not holds by its approximation of Y, Z in ∞, thus now the exceptional lines are all approximation line of Y)

Ex5.9. Otherwise $Z(f) = Z(f_1) \cup Z(f_2)$, then by Ex3.7, $Z(f_1) \cap Z(f_2) \neq \emptyset$

Consider $P \in Z(f_1) \cap Z(f_2)$, if $\mu_P(Y) = 1$, $\underline{f_1 f_2} = \underline{f + \cdots + f^d}$

least degree 2. least degree 1 $\Rightarrow$ Thus absurd ▨

by $f_1(P) = f_2(P) = 0$

Ex5.10. (a) $\dim T_P X = \dim X$

by duality! $\dim \frac{m}{m^2}$ //

Thus for affine case, it's trivial

and both $T_P X$, $\dim X$ are locally defined, thus general case follows ▨

(b)
$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow & \xleftarrow{\varphi^{\#}} & \downarrow \\ T_P X & \xrightarrow{T_P(\varphi)} & T_Q Y \end{array}$$

$\mathcal{O}_P \xleftarrow{\varphi^{\#}} \mathcal{O}_Q$ by duality

$T_P(\varphi) = \varphi_*$

$a \longmapsto \varphi_*(a) = a \circ \varphi^{\#}$ (It's well-defined by m's pull back is still maximal.)

$Z = V(x - y^2)$, $Y = V(y)$, (x, y)

$\varphi: Z \longrightarrow Y$

$(y^2, y) \mapsto y^2$

$\mathcal{O}_{P,Z} \xleftarrow{\varphi^{\#}} \mathcal{O}_{Q,Z}$

$\varphi(y) \longleftarrow \varphi(x)$

but in O is zero ▨

$\forall f \in \mathcal{O} \in m_{\varphi(P)} = m_Q$

$\Rightarrow \boxed{\varphi^{\#} f \in m_P^2}$, $\varphi^{\#}: \frac{m_{\varphi(P)}}{m_{\varphi(P)}^2} \to \frac{m_P}{m_P^2}$

$\bar{f} \mapsto 0$

so f's image is in the kernel ▨

Ex5.11. $Y = V(xz - xz - yw) \cap V(yz - xw - zw)$

$\varphi_P: Y \dashrightarrow \mathbb{P}^2 = V(w)$ $\varphi_P = [\varphi]: Y - P \to \mathbb{P}^2$

$(x:y:z:w) \mapsto (\frac{x}{1-w} : \frac{y}{1-w} : \frac{z}{1-w}) = (x:y:z)$

$\varphi(Y - P) = V(xz - xz) \cap V(yz) = \{(0,0,z)\} \cup \{(x,y,0)\} \cup \{(z,0,z)\}$

$= \{(0,0,1), (0,1,0), (1,0,1)\}$

And, $\bar{Y} = V(y^2 z - x^3 + xz^2) = \{(0,0,1), (0,1,0), (1,0,1), (1,0,-1)\}$

$\Rightarrow Y = \bar{Y} - \{(1,0,-1)\}$ ▨

(This is by: $y^2 z = x(z-x)(z+x)$

① $y = 0$: $\underline{x = 0}$ or $\underline{x = z}$ or $\underline{x = -z}$

② $y = 1$: $z = x(z-x)(z+x) \Rightarrow \underline{x = z = 0}$)

Ex5.12.

(a) Easy linear algebra theory of quadric forms ▨

(b) $\underline{\sum x_i^2}$ is irreducible when char $k \neq 2$, but $\sum x_i^2 = (\sum x_i)^2$ reducible when char $= 2$

$\Rightarrow f$ of Q is irreducible, so does f itself ▨

(if $\sum x_i^2 = (x_0 + a_1 x_1 + \cdots + a_n x_n)(x_0 + c_1 x_1 + \cdots + c_n x_n)$

$\Rightarrow a_i + c_i = 0 \ldots = \frac{c_i + a_i}{1 + c_i} = 0 \Rightarrow$ char 2)

(c) Trivial by Thm5.3: $\mathrm{Sing}\, Q = V(\frac{\partial f}{\partial x_i})$ now f is quadric, its any partial derivative is linear ▨ (The dimension follows) (Indeed $\mathrm{Sing}\, Q = Z = V(x_0, \ldots, x_r)$)

(d) Now assume $f = x_0^2 + \cdots + x_r^2$

Similar with computation in Ex5.5, $f = x_0^2 + \cdots + x_r^2$, $V(f) \subset \mathbb{P}^r$ is non-singular embed $\mathbb{P}^r$ into $\mathbb{P}^n$, then follows, thus $Q \cong C(Q'; Z)$ ▨

Ex5.13. $N = \{P \in Y \mid \mathcal{O}_P$ not intergrally closed$\}$, Then $\bar{\mathcal{O}}_P = \mathcal{O}_P[f_1 \cdots f_r]$ by (3.9A)

$= \{P \in Y \mid \exists g, h \in \mathcal{O}_P, \frac{g}{h} \notin \mathcal{O}_P : \frac{g}{h} = \sum_{\text{for } g} \varphi_i f_i\}$ ($\mathcal{O}_P$-modules) $= \{P \in Y \mid \sum \ldots = 0\}$

$= \{P \in Y \mid \exists f : \sum \ldots, \frac{h}{f} \in m_P, f \notin \mathcal{O}_P\}$

We apply partition of unity

$= \{P \in Y \mid \exists g, h, \varphi_i \in \mathcal{O}_P, h \in m_P : \sum \varphi_i (\frac{g}{h})^i = 0\}$ $\mathcal{O}_P = A(X)$ For some X (locally affine) $\Rightarrow N = X$ ▨

Otherwise, f will be reducible (if it divides its derivative) ▨

Continue Ex5.6(a)

(b) Trivial ▨

Then we show it's regular at O: by partial derivative ▨ (we done it in (a))

(c) Due to my wrong solution in (a), we done the one time blow up of tacnode

$\Rightarrow \varphi^{-1}(O) = V(x, y) \cap Bl_O(Y) = V(x, y) \cap (V((t^4+1)x^2 - 1) \cup V((s^4+1)y^2 - s^2))$

That means after one time blow up, $= V(s^2)$ $\xrightarrow{\text{double}}$ $V(s)$

the two $\varphi^{-1}(O)$ still the same direction, not smooth ▨

$\Downarrow$ node

(d) $\begin{cases} y^3 = x^5 \\ xt = sy \end{cases} \Rightarrow Bl_O(Y) = V(x^2 - t^3) \cup V(s^5 y^2 - 1)$

$\varphi^{-1}(O) = V(x^2 - y^3)$ a double point link the cusp ▨

Ex5.7. (a) $C(Y) = V(f) \cup \{O\}$, thus it at most have $\{O\}$ a singularity

(by $V^{\mathrm{aff}}(f) = V^{\mathrm{aff}}(f)$), now we show $\{O\}$ does a singularity

$X = V(f) \cup V(x, y, z) = V(xf) \cap V(yf) \cap V(zf)$ each vanish at $(\partial f)_a$ ($a = x$ or y or z) in $(x, y, z) = 0$

$\Rightarrow$ by Ex5.3, it's singular ▨

(b) $X - \{O\} \cong \tilde{X} - \varphi^{-1}(O)$, thus by (a), $\varphi^{-1}(O) \cong Y$ also non-singular $\Rightarrow$ all non-singular ▨

Left is proving (c)

(c) $\begin{cases} f(x, y, z) = 0 \\ xt = sy \\ xw = sz \end{cases}$ $s \neq 0$, $\varphi^{-1}(P) \cap D(s) = V(f(x, xt, xw)) \cap V(x, y, z)$

$= V(x^d f(1, t, w)) \cap V(x, y, z)$

$= V(f(1, t, w))$

Similiarly holds for $D(t)$, $D(w) \Rightarrow \varphi^{-1}(P) = V(f(s, t, w))$

$\cong V(f(x, y, z))$ ▨ $= Y$

(Given by $\varphi^{-1}(P) \longleftrightarrow Y$

$((0,0,0), (s, t, w)) \quad (x, y, z)$)

Ex5.8.

We follow the Exercise

(a) Say that we may instinctly consider $\mathbb{P}^n = \mathbb{P}(E)$ ($\dim E = n+1$), the coordinates up to the basis of E.

$M_{n \times r} \subset \ldots (a_0 \cdots a_n)$ has a $(n-r)$-order minor, $\det(M_{n \times r}(a_0 : \cdots : a_n)) \neq 0$

is trivial: their determinant is up to a k^{n-r} $\Leftrightarrow$ $\det(M_{n \times r}(a_0' : \cdots : a_n')) \neq 0$

($k \in k$) time ▨ (If other linear transformation is composited, its only the basis change of E, thus invertible $\Rightarrow$ det not vanish)

(b) By (a), we can this for U_0: then we take $a_0 = 1$ ▨

(c) $\mathrm{rank}\begin{pmatrix} \frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_t}{\partial x_0} & \frac{\partial f_t}{\partial x_1} & \cdots & \frac{\partial f_t}{\partial x_n} \end{pmatrix} = \mathrm{rank}\begin{pmatrix} \boxed{\sum x_i \frac{\partial f_1}{\partial x_i}} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \boxed{\sum x_i \frac{\partial f_t}{\partial x_i}} & \cdots & \frac{\partial f_t}{\partial x_n} \end{pmatrix} = \mathrm{rank}\begin{pmatrix} d f_1 & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ f_t & \cdots & \frac{\partial f_t}{\partial x_n} \end{pmatrix} = n - r$

Ex 5.14 (a) $Y=V(f)$, $Z=V(g)$

$\mu_P(Y)\geq 1$, $\mu_Q(Z)\geq 1$ and $\widehat{\mathcal{O}_P}\cong\widehat{\mathcal{O}_Q}$

if $\mu_P(Y)=p\neq q=\mu_Q(Z)$

$Y=V(f_p+\cdots+f_d)$, $Z=V(g_q+\cdots+g_e)$

$$\widehat{\mathcal{O}_P}=\frac{k[[x,y]]}{(f_p+\cdots+f_d)}\cong\frac{k[[x,y]]}{(g_q+\cdots+g_e)}$$ but $p\neq q$ is absurd. ▨

(b) Induction, we compute g_{s+1} & h_{t+1} by f

$f_{r+1}=g_{s+1}h_t+g_s h_{t+1}$

and notice that g_s, h_t generate m_Q $\Rightarrow \exists g_{s+1}, h_{t+1} \Rightarrow \cdots$ ▨

(c) $f=(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)+\boxed{\text{higher terms}}$, by (b), $f=gh$ with $g=(a_1x+b_1y+c_1)+\boxed{\text{higher terms}}$, $h=(a_2x+b_2y+c_2)+\boxed{\text{higher terms}}$

Thus $\widehat{\mathcal{O}_P}=\frac{k[[x,y]]}{(gh)}$ $\widehat{\mathcal{O}_Q}=\frac{k[[x,y]]}{(gh)}$

$$\cong\frac{k[[x,y]]}{(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)}\cong\frac{k[[x,y]]}{(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)}$$

$\cong$ by a corridate change ▨

The dimension of higher seems same, where the difficulty comes?

(d) In (c), we done the case $r=2$: $(ax+by+c)(a'x+b'y+c')\longleftrightarrow(x+y)(x-y)$

For higher dimension $x^r-y^2\longleftrightarrow(a_1x+b_1y+c_1)\cdots(a_rx+b_ry+c_r)$, some b_r may vanish ▨

Ex 5.15 (a) $\mathbb{P}^N\longleftrightarrow\{V(f)\mid \deg f=d \text{ homogenous}\}$

$(a_0:\cdots:a_N)\longmapsto V(\sum_{i\leq N} a_i x^{i_1}y^{i_2}z^{i_3})$

(coefficients) $\longleftarrow V(f)$ ▨

This need (f) is a radical ideal of $I(V(f))$ thus $f=f_1\cdots f_k$ can't have multiple factor

(b) Irreducible nonsingular curve

$\Updownarrow$ $\boxed{\text{Ex 5.9}}$

Nonsingular curve Y

$\Updownarrow$ $\boxed{\text{Ex 5.8}}$

$\mathrm{Rank}\left\|\frac{\partial f}{\partial x_j}(P)\right\|=1$, $\forall P\in Y$

$\Updownarrow$ Converse

We apply elimination theory (5.7A) to $(f,\frac{\partial f}{\partial x_0},\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2})$.

$\exists g_1\cdots g_t(a_0\cdots a_N)$ $\mathrm{Rank}\left\|\frac{\partial f}{\partial x_j}(P)\right\|=0\Leftrightarrow(a_0\cdots a_N)$ is the common locus of $g_1\cdots g_t$ $\Leftrightarrow P\in V(g_1\cdots g_t)$

$\Updownarrow$ Converse

$\mathrm{Rank}\left\|\frac{\partial f}{\partial x_j}(P)\right\|=1\Leftrightarrow (a_0\cdots a_N)\in D(g_1\cdots g_t)$ open

$\Updownarrow$ $\boxed{\text{Ex I.5}}$

$D(g_1\cdots g_t)\neq\phi$

Finally, Irreducible nonsingular curve $\Leftrightarrow(a_0\cdots a_N)\in D(g_1\cdots g_t)$ open $\subset\mathbb{P}^N$ ▨

continue Ex6.5. Thus left is proving $\overline{X}$ still non-singular

otherwise, $\exists P \in \overline{X} - X : \mathcal{O}_P \neq \mathcal{O}_P$, by Coro 6.6, $\exists U_i : U_i \cap X$ affine

$\exists Q \in U_i \cap X \subset X : \mathcal{O}_Q = \overline{\mathcal{O}_P}$

but it's closed ▨

Ex6.6 (a)

$$\begin{array}{ccc} \mathbb{A}^1_{\infty} & \longrightarrow & \mathbb{A}^1_{\infty} \\ & & \| \\ \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \\ x & \longmapsto & \frac{ax+b}{cx+d} \\ \frac{b-dy}{cy-a} & \longleftarrow & y \end{array}$$

both are morphism as they are rotational map ▨ $\left(PGL_1(k) = \{x \mapsto \frac{ax+b}{cx+d} \mid a,b,c,d \in k, ad-bc \neq 0\}\right)$

(b) By Coro 6.12 (iii) $\cong$ (i)

Notice that now $\mathbb{P}^1 \longleftrightarrow k(x) = K(\mathbb{P}^1)$ ▨

(c) $\mathrm{Aut}(k(x)) = \{\varphi : k(x) \to k(x),\ f(x) \mapsto g(x)\ \text{iso}\} \cong \{\varphi(x) = \frac{g(x)}{f(x)},\ g(x), f(x) \in k[x],\ \underline{\varphi\ \text{iso}}\}$ $= \mathrm{Aut}(\mathbb{P}^1)$

$\Downarrow$ $\deg f, \deg g \leq 1$

$= \{\varphi(x) = \frac{ax+b}{cx+d},\ \varphi\ \text{iso}\}$

$\Downarrow$ $ad - bc \neq 0$

$\cong \{\frac{ax+b}{cx+d} \mid a,b,c,d \in k,\ ad-bc \neq 0\}$

$= PGL_1(k)$

$\Rightarrow \mathrm{Aut}(\mathbb{P}^1) = PGL_1(k)$ ▨

Ex6.7 Assume $r \leq s$, induction on r,

$r = 0$ is trivial by Ex3.1 with induction on s_0; fix s_0. Consider with $r+1$ case, otherwise if $\mathbb{A}^1 - \{P_1 \cdots P_{r+1}\} \cong \mathbb{A}^1 - \{Q_1 \cdots Q_{s_0}\}$ we apply Prop 6.8 to extend to contradict to r case ▨

The converse not true: when $r \leq s$ but $r \geq 4 > 3$, by Ex6.6 we can only moving at most 3 points into where we want ▨

Ex6.1 (a) $Y \simeq \mathbb{P}^1$, $Y \cong U \subsetneq C_{K(t)} \cong \mathbb{P}^1$

By proper change, assume $U \subset \mathbb{A}^1$ ▨

(b) By (a) $Y \cong U = \mathbb{A}^1 - \{a_1 \cdots a_m\} = D(f)$

with $f(x) = (x-a_1)\cdots(x-a_m)$

then $U \cong \{(x, \frac{1}{f(x)}) \mid x \in U\} \subset \mathbb{A}^2$

$= V(\frac{1}{f(x)} - y)$ closed

$\Rightarrow Y \cong U$ an affine variety ▨

(Why $U \cong \{(x, \frac{1}{x-a_1}, \cdots, \frac{1}{x-a_m}) \mid x \in U\} \subset \mathbb{A}^{m+1}$ $= V(\frac{1}{x-a_1} - y_1, \cdots)$ closed ▨)

(c) Recall. Thm 1.12A A Noetherian integral

A is UFD $\Longleftrightarrow$ every $\mathfrak{p}$ with height $\mathfrak{p} = 1$ is $\mathfrak{p} = (f)$ principal

Now consider $A(Y) \cong A(U)$

① Noetherian integral : trivial

② $\mathfrak{p} = (f)$: trivial ▨

Ex6.2 (a) Y nonsingular : $f = y^2 - x^3 + x$, $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})(P) = (-3x^2+1, 2y) \neq 0$ ▨

(If char $k = 2$, $-3x^2+1 = 0 \Rightarrow x = \frac{1}{\sqrt{3}}$ $\Rightarrow$ singular)

$A = \frac{k[x,y]}{(y^2 - x^3 + x)} \cong \mathcal{O}(Y)$ by Coro 6.6 and Thm 6.2A $\Rightarrow$ A is DVR

$\Rightarrow$ A is integrally closed ▨

(b) $x \notin k$ and $\overline{k} = k$ $\Rightarrow x$ is transcendental over $k \Rightarrow k[x]$ is a polynomial ring

$A = \overline{(k[x])}_K$: $\forall \bar{f} \in A$, $\exists k_1 \cdots k_r \in k$: $\underline{\bar{f}^{r+1} + k_r \bar{f}^r + \cdots + k_1} \in \frac{k[x,y]}{(y^2 - x^3 + x)}$ (only has even dimension on each y, thus none of y involves) $= k[x]$

$\Rightarrow \exists g_1 \cdots g_r \in k[x]$: $\bar{f}^{r+1} + g_r \bar{f}^r + \cdots + g_1 = 0$

$\Rightarrow A \subset \overline{(k[x])}_K$

(y integral over $k[x]$ and A is integrally closed $\Rightarrow A = \overline{A} = \overline{(k[x])}_K$ ▨)

Converse is trivial ▨

(c) $N(a)(x,y) = a(x,y) \cdot \sigma(a)(x,y)$

$= a(x,y) \cdot a(x,-y) \in \frac{k[x,y]}{(y^2 - x^3 + x)} = k[x]$ ▨

And : $N(1)(x,y) = 1(x,y) \cdot \sigma(1)(x,y)$

$= (x,y) \cdot (x,-y)$

$= (x,y) = (x,y)$

$N(ab)(x,y) = ab(x,y) \cdot ab(x,-y)$

$= a(x,y) \cdot a(x,-y) \cdot b(x,y) \cdot b(x,-y)$

$= N(a) \cdot N(b)(x,y)$ ▨

(d) $a \in A$ is unit $\Longleftrightarrow N(a)N(a^{-1}) = N(aa^{-1}) = N(1) = 1$

$\Longleftrightarrow N(a)$ is unit $\Longleftrightarrow N(a) \neq 0$ & $N(a) \in k$

$\Longleftrightarrow a \neq 0$ & $a \in k$ ▨

$k[x]$ is UFD $\Rightarrow x, y$ irreducible, but $y^2 = x(x-1)(x+1)$, contradiction $\Rightarrow$ A not UFD ▨

(e) By Ex6.1, if $Y \stackrel{k}{\cong} \mathbb{P}^1$, then either $A(Y) = A$ is UFD or $Y \cong \mathbb{P}^1$

but both choice are absurd: by (d), A not UFD, and Y isn't a projective variety ▨

Ex6.3 (a) $\mathbb{P}^2 - \{\ \} \longrightarrow \mathbb{P}^1$, $(x:y:z) \longmapsto (\ )$ ▨

(You may claim why they can't be extended)

(b) $\mathbb{P}^1 - \{0\} \xrightarrow{\varphi} \mathbb{A}^1$, $(x:y) \longmapsto$ ▨ non-singular

Ex6.4 It's holds when ① Y is projective variety of dimension n, $Y \xrightarrow{\exists \varphi} \mathbb{P}^n$ surj & finite

② Y is quasi-projective variety of dimension, $Y \xrightarrow{\exists \varphi} \mathbb{P}^n$ dominate & generally finite

both contain we want to prove

See Shafarevich for ①, Debarre for ② ▨

Ex6.5 Assume $Y = \mathbb{P}^n$

(as $X \subset Y \subset \mathbb{P}^n$),

Notice : We can't claim $\overline{X}$ also non-singular! Indeed it's why matters : otherwise $\overline{X}$ nonsingular $\Rightarrow \overline{X} \cong C_{K(X)} = C_{K(X)} \supseteq X$ ($X \subset \overline{X}$ open by locally closed)

(You may prove by projection from a point inductively and counting dimension)

Alternative proof using abstract variety.

Y nonsingular-projective $\Rightarrow Y \cong C_K$ $(K = K(Y))$

$Y \xrightarrow{f} \mathbb{A}^1_k$, $\mathbb{P}^1_k$

We define $\varphi : C_K \to \mathbb{P}^1_k$ by determining $\varphi^{-1}(P)$

$\varphi^{-1}(P) \stackrel{\text{def}}{=} \{\eta \in k[\bar{x}] \mid f(P) \neq \infty\}$ finite by $\forall P \in \mathbb{P}^1_k$ Lemma 6.5

1-1 correspond $\{R \in C_K \mid f \in \mathfrak{m}_R\}$

(And $\{R \in C_K \mid f \in \mathfrak{m}_R\} \subset C_K$ dense, only finite points undefined) $\Rightarrow$ We extend φ by Prop 6.8

Ex 1.2 (c) $S(Y\times Z) = S(\delta(Y\times Z)) = k[x_0 y_0 \cdots x_r y_s, \ldots] / I(Y\times Z)$

$P_a(Y\times Z) = (-1)^{r+s}(P_{Y\times Z}(0) - 1)$

by $\dim Y\times Z = \dim Y + \dim Z$

Our main task is defining $I(Y\times Z)$

We give $S(\delta(Y\times Z))$ graded by

$S(Y\times Z) = \bigoplus_{d\geq 0} [S(Y)_d \otimes_k S(Z)_d]$

$\Rightarrow P_{Y\times Z}(l) = P_Y(l)\times P_Z(l)$

$\dim_k S(Y)_d \otimes_k S(Z)_d = (\dim_k S(Y)_d)\times(\dim_k S(Z)_d)$

$\Rightarrow P_a(Y\times Z) = (-1)^{r+s}(P_Y(0)\times P_Z(0) - 1)$

$= [(-1)^r(P_Y(0)-1)(-1)^s(P_Z(0)-1) - 1]$

$+ (-1)^s[(-1)^r(P_Y(0)-1)] + (-1)^r[(-1)^s(P_Z(0)-1)]$

$= P_a(Y)P_a(Z) + (-1)^s P_a(Y) + (-1)^r P_a(Z)$ ▨

Ex 7.3.

① P nonsingular $\Leftrightarrow \exists!\ T_P(Y)$ with $\#(T_P(Y), Y; P) = 1$

② $\operatorname{Reg} Y \longrightarrow (\mathbb{P}^2)^* = \mathbb{P}(\mathbb{P}^2)$ does a morphism

$P \longmapsto T_P(Y)$.

Pf. ① ($\Leftarrow$) Trivial: otherwise $\dim \frac{m_P}{m_P^2} \geq \dim A + 1 = 2$

take $\frac{m_P}{m_P^2}$'s generators $v_1, v_2 \Rightarrow$ non unique.

($\Rightarrow$) $T_P(Y)$: $\frac{\partial f}{\partial x_0}x_0 + \frac{\partial f}{\partial x_1}x_1 + \frac{\partial f}{\partial x_2}x_2 = 0 \Rightarrow$ existence is trivial

Verification: $(T_P(Y), Y; P) = (T_P(Y), Y)_P = l_{\mathcal{O}_P}\left(\frac{\mathcal{O}_P}{(\frac{\partial f}{\partial x_0}x_0 + \frac{\partial f}{\partial x_1}x_1 + \frac{\partial f}{\partial x_2}x_2, f)}\right) = 2$

(or $T_P(Y), Y; P) = \mu_P\left(\frac{S}{(\frac{\partial f}{\partial x_0}x_0 + \frac{\partial f}{\partial x_1}x_1 + \frac{\partial f}{\partial x_2}x_2, f)}\right) = l_{\mathcal{O}_P}\left(\frac{S}{(\frac{\partial f}{\partial x_0}x_0 + \frac{\partial f}{\partial x_1}x_1 + \frac{\partial f}{\partial x_2}x_2, f)}\right)$ (Rk 7.8.1)

$0 \to S(-1) \xrightarrow{T_P(Y)} S \to \frac{S}{(T_P(Y))} \to 0$

$= 2$

$0 \to \frac{S}{T_P(Y)}(-\deg f) \xrightarrow{f} \frac{S}{T_P(Y)} \to \frac{S}{(\frac{\partial f}{\partial x_0}x_0 + \frac{\partial f}{\partial x_1}x_1 + \frac{\partial f}{\partial x_2}x_2, f)} \to 0$

$\parallel$ singleton

Then the uniqueness: otherwise

$\dim \frac{m_P}{m_P^2} \geq 2$, absurd. ▨

② $\varphi: \operatorname{Reg} Y \to (\mathbb{P}^2)^*$

$P \mapsto T_P(Y)$

$(x_0, x_1, x_2) \mapsto \left(\frac{\partial f}{\partial x_0}(P), \frac{\partial f}{\partial x_1}(P), \frac{\partial f}{\partial x_2}(P)\right)$

The $\varphi_0: P \mapsto \frac{\partial f}{\partial x_0}(P)$ is regular function, by Lemma 3.6 ▨

(countinous is trivial, and it's a polynomial by f is)

Ex 7.4. L meets Y exactly $d = \deg Y\cdot \deg L$ points $\overset{\text{Bezout}}{\Longleftrightarrow}$ L meets Y at $P_1 \cdots P_d$ with $i(Y, L; P_i) = 1,\ \forall i$

$\Leftrightarrow L\cap Y = \{P_1 \cdots P_d\}$

P_i not tangent, and smooth.

(closed in $\mathbb{P}^{2*}$ (and proper)) (countinuous, closed set in $\mathbb{P}^{2*}$)

By Hint, we now show $\{L\in(\mathbb{P}^2)^* \mid L\cap \operatorname{Sing} Y \neq \emptyset\}$, $\{L = T_P(Y) \mid P\in \operatorname{Reg}(Y)\}$ are subsets of some closed set in $\mathbb{P}^{2*}$ ▨

but notice the $\operatorname{Sing}(Y)$ it's some points isolated each other

Ex 7.1 (a) $\mathbb{P}^n \xrightarrow{v_d} \mathbb{P}^N$

$v_d(\mathbb{P}^n) = V(\mathfrak{a})$, $\mathfrak{a} = \ker\{\theta: k[y_0 \cdots y_N] \mapsto k[x_0 \cdots x_n]$, $y_i \mapsto M_i = x_0^{i_0}\cdots x_n^{i_n}$, $d = \sum i_k\}$

$P_{v_d(\mathbb{P}^n)} = P_{S/\mathfrak{a}}$

$\cong P_{k[x_0^{i_0}\cdots x_n^{i_n}, \ldots]}$ by Ex 2.12. represented as this, we think such product has deg 1

$\varphi_{k[\cdots x_0^{i_0}\cdots x_n^{i_n}\cdots]}(l) = \dim_k(k[\cdots, x_0^{i_0}\cdots x_n^{i_n}, \cdots])_l = \binom{dl+n}{n} = P_{k[\cdots, x_0^{i_0}\cdots x_n^{i_n}, \cdots]}$

$\deg v_d(\mathbb{P}^n) = \deg P_{k[\cdots, x_0^{i_0}\cdots x_n^{i_n}, \cdots]} = \frac{d^n}{n!}\, n! = d^n$ ▨

(With the relation given by $\mathfrak{a}$, the generators are just degree d!)

(By $\binom{dl+n}{n} = \frac{(dl+n)!}{n!\,(dl)!} = \frac{1}{n!}(dl+n)(dl+n-1)\cdots(dl+1)$ $= \frac{d^n}{n!}(l^n + \cdots)$)

(b) $\delta: \mathbb{P}^r\times\mathbb{P}^s \to \mathbb{P}^N$

$S(\mathbb{P}^r\times\mathbb{P}^s) = Y$, $S(Y)$

$\cong k[x_0y_0 \cdots x_ry_s]$

again, we admit it as degree 1 ($r+s$ generators $x_0, \ldots, x_r$, $y_0, \ldots, y_s$)

$\varphi_{k[x_0y_0\cdots x_ry_s]}(l) = \dim_k(k[x_0y_0\cdots x_ry_s])_l = \binom{2l+r+s}{r+s}$ (degree 2l) $= P_{k[x_0y_0\cdots x_ry_s]}$ X

$= \binom{l+r}{r}\binom{l+s}{s}$

$= \frac{1}{r!}\frac{1}{s!}l^{r+s}$

$\Rightarrow \deg \delta(\mathbb{P}^r\times\mathbb{P}^s) = \frac{(r+s)!}{r!\,s!} = \binom{r+s}{r}$ ▨

(Still not enough small: here includes $x_0x_1 \cdots x_l$ but this's absurd: it should balance the x_i and y_j's number)

Ex 7.2. (a) $P_a(\mathbb{P}^n) = (-1)^n\left(\binom{0+n}{n} - 1\right)$

$= 0$ ▨ (By Prop 7.6(a))

(b) $P_a(Y) = (-1)^1(P_Y(0) - 1) = 1 - P_Y(0)$

$P_Y(l) = \binom{l+2}{2} - \binom{l-d+2}{2} = dl - \frac{d^2-3d}{2}$ (by Prop 7.6(d))

$\Rightarrow P_a(Y) = 1 + \frac{d^2-3d}{2} = \frac{d^2-3d+2}{2} = \frac{(d-1)(d-2)}{2}$ ▨

(c) $P_H(l) = \binom{l+n}{n} - \binom{l-d+n}{n} = \cdots + \frac{n! - (n-d)\cdots(1-d)}{n!}$

$P_a(H) = (-1)^{n-1}\left[\frac{n! - (n-d)\cdots(1-d)}{n!} - 1\right]$

$= (-1)^n \frac{(n-d)(\cdots)(1-d)}{n!} = \binom{d-1}{n}$ ▨

$\frac{(d-1)\cdots(d-n)}{n!}$

(d) $P_Y(l) = \varphi_Y(l)$

$= \dim_k\left(\frac{S}{I_1+I_2}\right)_l$ by complete intersection $I(Y) = I_1 + I_2 = (f, g)$

$= \dim_k\left(\frac{S}{(f,g)}\right)_l$ $\deg f = a$, $\deg g = b$ by Prop 7.6(d)

isn't easy to compute directly.

$0 \to \frac{S}{(f)}(-b) \xrightarrow{g} \boxed{\frac{S}{(f)}} \to \boxed{\frac{S}{(f,g)}} \to 0$; $0 \to S(-a) \xrightarrow{f} S \to \frac{S}{(f)} \to 0$

$V(f)$ $\quad Y$

$\Rightarrow P_Y(l)$ $\qquad \Rightarrow P_{S/(f)}(l) = P_S(l) - P_S(l-a)$

$= P_{S/(f)}(l) - P_{S/(f)}(l-b) = P_S(l) - P_S(l-a) - [P_S(l-b) - P_S(l-a+b)]$

$\Rightarrow P_a(Y) = (-1)^1(P_Y(0) - 1)$, $= \binom{l+3}{3} - \binom{l-a+3}{3} - \binom{l-b+3}{3} + \binom{l-a-b+3}{3}$

$= 1 - \binom{3-a}{3} - \binom{3-b}{3} + \binom{3-a-b}{3} = \frac{1}{6}(3a^2b + 3ab^2 - 12ab - 6) = \frac{1}{2}ab(a+b+4) - 1$ ▨

Ex7.1. (a) Assume $P=(1,0\cdots 0)$

$\forall Q\in Y,\ Q=(q_0\cdots q_n)$

$L_{PQ}=\{(kq_0-k+1 : kq_1 : \cdots : kq_n) \mid \forall k\in A_k\}$

$\varphi_P: Y-\{P\} \longrightarrow H_0=V(x_0)$ is the projection of Y from P

$(x_0\cdots x_n)\mapsto(0,x_1\cdots x_n)$

$\varphi_P(Y-\{P\})=\tilde{Y},\ I(\tilde{Y})=(x_0,f_1\cdots f_r)$

Claim. $X=V(f_1\cdots f_r)$, the verification is trivial, $I(Y)=(F_1\cdots F_s)$, $f_1\cdots f_r$ come from $F_i=\sum_{r\le d} x_0^r F_i^{d-r}$, $f_j=F_i^{d-r}$ corresponds ▨

$\Rightarrow \boxed{\begin{matrix}\dim X\\ \vee\\ \dim Y\end{matrix}} = \dim\tilde{Y}+1\le\dim Y+1 \Rightarrow \dim X=\dim Y+1$ ▨

($\deg Y>1$ ensures this, $\deg Y=1$ then $\dim X=\dim Y$)

(b) Induction on $\dim Y=m$

$m=n-1,\ X=\mathbb{P}^n,\ \deg X=0<d$

$m=n-r$ had done: $\deg X_{n-r}<d$ for $X_{n-r}=J(\{P\},Y_{n-r})$

for $m=n-r-1$, $Y_{n-r-1}=Y_{n-r}\cap H=Y_{n-r}\cap V(F)$

$0\to S(Y_{n-r})(-\deg F)\xrightarrow{F} S(Y_{n-r})\to S(Y_{n-r-1})\to 0,\ \deg Y_{n-r}=d$

$\deg Y_{n-r-1}=(\deg F)(\deg Y_{n-r})=(\deg F)\,d>(\deg F)(\deg X_{n-r})\ge(\deg X_{n-r-1})$

$\Rightarrow \deg X_{n-r-1}<\deg Y_{n-r-1}$ ▨

Continue Ex7.4. Thus we claim. $\{V(a_0y_0+a_1y_1+a_2y_2)\}_{y=(y_0,y_1,y_2)\in \mathrm{Sing}\,Y\ \text{fixed}}$ is a family of locally finite closed subsets

Otherwise $\exists x\in\mathbb{P}^2$ is defined by infinitely lines intersection, but it's absurd $\Rightarrow \bigcup_y V(a_0y_0+a_1y_1+a_2y_2)$ closed $=\{L\in(\mathbb{P}^2)^* \mid L\cap\mathrm{Sing}(Y)\neq\emptyset\}$

And $\{L=T_p(Y)\mid \exists p\in Y\}$

$=\bigcup_{y\in\mathrm{Reg}\,Y}\{(\frac{\partial f}{\partial y_0}(y),\frac{\partial f}{\partial y_1}(y),\frac{\partial f}{\partial y_2}(y))\}$

$\{L=T_p(Y)\mid\exists p\in Y\}=\varphi(\mathrm{Reg}\,Y)\subseteq\mathrm{Reg}\,Y$ closed, by $\mathrm{Reg}\,Y\subset Y$

action $(a_0,a_1,a_2)(x_0,x_1,x_2)\mapsto(a_0,a_1,a_2)$

Thus both of these contained in closed set, now we show it's not the hole $(\mathbb{P}^2)^*$

i.e. find L intersect Y at exact d points

but this trivial: otherwise Y'll have infinite singular points / $\mathrm{Reg}\,Y=\mathbb{P}^2$

both absurd ▨

Ex7.5. (a) By Remark 7.8.1, $\deg Y=d\Leftrightarrow Y=V(f_d)$

assume $P\in Y\cap U_i$, $Y\cap U_i\subset\mathbb{A}^2$ and $Y\cap U_i=V(\tilde{f}_1+\cdots+\tilde{f}_d)$

$\mu_P(Y)<d$ ▨

(b) $\exists P\in Y$, assume $P\in Y\cap U_0$ $\boxed{\subset\mathbb{A}^2}$ Can we prove for higher dimension? Only use

then if $Y=V(f_d(x_0,x_1,x_2))$ $\quad \deg Y=d-1 \Rightarrow g_i(x_1,x_2)\equiv 0$ Prop4.9 for all $i\le d-2$

$=V(\sum_{r\le d} x_0^r g_{d-r}(x_1,x_2))$

$\Rightarrow Y=V(g_d+x_0g_{d-1})$

$Y\cap U_0=V(g_d(x_1,x_2)+g_{d-1}(x_1,x_2))$

$\Rightarrow K(Y\cap U_0)=K(Y)=\mathrm{Frac}\dfrac{k[x_1,x_2]}{(g_d+g_{d-1})}$

$\cong k[x_1]=K(\mathbb{P}^1)$

$\Leftrightarrow Y\cap U_0\simeq\mathbb{P}^1$ ▨

Ex7.6. Induction on r. (By pure we assume irreducible)

$r=n-1$, $\operatorname{codim}Y_1=1$, $I(Y_1)=(L_1)$

then $\deg Y_1=1\Leftrightarrow\deg L_1=1\Leftrightarrow L_1$ is linear

let $r<n-1$ now, we cut Y_{n-r} by H repeatedly $(Y_{n-r}\not\subseteq H)$

$H_{n-r+1}=V(L_{n-r+1})$

$S(Y_{n-r})=\dfrac{S}{I(Y_{n-r})}$, $\quad S(Y_{n-r+1})=\dfrac{S}{I(Y_{n-r+1})}$

$=\dfrac{S}{(Y_1\cdots Y_{n-r})}$ $\quad =\dfrac{S}{(Y_1\cdots Y_{n-r+1})}$

By induction assumption

$\deg Y_{n-r}=1\Leftrightarrow Y_{n-r}$ is linear

$\Updownarrow$ by (P) $\qquad \Updownarrow$

$\deg Y_{n-r+1}=1 \boxed{\Leftrightarrow} Y_{n-r+1}$ is linear ▨

then $0\to S(Y_{n-r})(-1)\xrightarrow{Y_{n-r+1}} S(Y_{n-r})\to S(Y_{n-r+1})\to 0$

$\Rightarrow P_{Y_{n-r+1}}(l)=P_{Y_{n-r}}(l)-P_{Y_{n-r}}(l-1)\cdots(P)$

(c) $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\theta} \mathcal{G}^+$

$\varphi(U): \mathcal{F}(U) \to \mathcal{G}(U) \Rightarrow \{f \mid f \in \mathcal{G}(U)\}$
$\{f \in \mathcal{A}(U) \mid U \subset G\} \to$

$\varphi(U)$ is surjective: $\mathcal{G}(U) = \varphi(U)(\mathcal{F}(U))$, but $\mathcal{G}$ not a sheaf,
after embedding into $\mathcal{G}^+$, not surjective,
but $\mathcal{G}_p = \mathcal{G}^+_p$, thus $\mathcal{F}_p \to \mathcal{G}^+_p$ still surjective, a contradiction ▨

(My insight is notice $\operatorname{Im}\varphi \subset \mathcal{G}$ may not a sheaf, thus we contradict by this way)

Ex1.4. (a) $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$

It's natural to ask $\varphi^+_p \overset{?}{=} \varphi_p$.
If it holds, then $\ker\varphi^+_p = \ker\varphi_p = 0 \Rightarrow \varphi^+$ injective
(in Ex1.2(b)① we showed that $\varphi(U)$ injective $\Leftrightarrow$ φ_p injective)
But $\varphi_p = \varphi^+_p$ is trivial: $\mathcal{F} \mapsto \mathcal{F}_p$, the cube left is commu

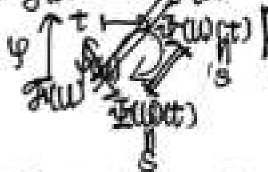
$$\begin{array}{ccc} \mathcal{F}_p & \xrightarrow{\varphi_p} & \mathcal{G}_p \\ \| id & \circlearrowleft & \| id \\ \mathcal{F}^+_p & \xrightarrow{\varphi^+_p} & \mathcal{G}^+_p \end{array} \qquad \begin{array}{ccc} \mathcal{F} & \mapsto & \mathcal{F}_p \\ \downarrow & \circlearrowleft & \downarrow \\ \mathcal{F}^+ & \mapsto & \mathcal{F}^+_p \end{array}$$ ▨

(b) $\operatorname{Im}\varphi \hookrightarrow \mathcal{G}$ as presheaves is naturally injective, (a) $\Rightarrow$ $\operatorname{Im}\varphi \hookrightarrow \mathcal{G}$ injective as sheaves follows ▨

Ex1.5. φ is isomorphism $\Leftrightarrow$ φ has two-side inverses
$\Updownarrow$ trivial $\Updownarrow$ $\varphi_p\psi_p = (\varphi\psi)_p$
$\forall p$, φ_p is isomorphism in $\mathfrak{Ab}$ $\Leftrightarrow$ $\forall p$, φ_p has two-side inverses

The only thing need to do now is only $\forall p$, φ_p has inverse $\Rightarrow$ φ has inverse
Only prove left case: $\mathcal{F}_p \xrightarrow{\varphi_p} \mathcal{G}_p \xrightarrow{\psi^p} \mathcal{F}_p$
We now construct $\Psi: \mathcal{G} \to \mathcal{F}$ be the left inverse of φ

$\Psi(U)$: $\mathcal{G}(U) \to \mathcal{F}(U)$, we let $\Psi(U)(t) = s$, where s is determined by:



$\forall p \in U$, $\langle U, t\rangle \in \mathcal{G}_p$, φ_p iso $\Rightarrow \exists \langle U, s\rangle$: $\langle U, \varphi(U)(s)\rangle = \langle U, t\rangle$
$\psi\langle U, \varphi(U)(s)\rangle = \langle U, s\rangle$
φ_p is isomorphism
$\psi\langle U, t\rangle$

$\Rightarrow \Psi(U): \mathcal{G}(U) \to \mathcal{F}(U)$
$t \mapsto s$ is defined, $\forall U \in \mathfrak{U}$ ▨

Ex1.6. By Ex1.2(c), only show $0 \to \mathcal{F}'_p \to \mathcal{F}_p \to \left(\frac{\mathcal{F}}{\mathcal{F}'}\right)_p \to 0$ exact
This is trivial: $\mathcal{F}'(U) \leq \mathcal{F}(U)$
$\Rightarrow \mathcal{F}'_p \leq \mathcal{F}_p \Rightarrow$ exact ▨ $\qquad \| \ \frac{\mathcal{F}_p}{\mathcal{F}'_p}$

(b) $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact
$\Leftrightarrow 0 \to \mathcal{F}'_p \to \mathcal{F}_p \to \mathcal{F}''_p \to 0$ exact, $\forall p$
$\Leftrightarrow \mathcal{F}''_p \cong \frac{\mathcal{F}_p}{\mathcal{F}'_p} \cong \left(\frac{\mathcal{F}}{\mathcal{F}'}\right)_p$, $\forall p \Leftrightarrow \mathcal{F}'' \cong \frac{\mathcal{F}}{\mathcal{F}'}$ ▨

Ex1.1. Denote the constant presheaf as $\mathcal{C}$
$\mathcal{C}^+ = \mathcal{A}$ is what we want to prove, i.e. $\mathcal{C}^+(U) = \mathcal{A}(U)$, $\forall U \subset X$

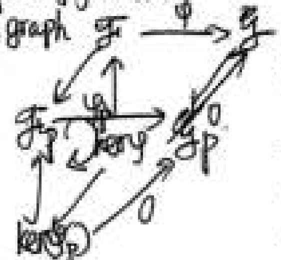
$\mathcal{C}^+(U) = \left\{ s: U \to \coprod_{p \in U} \mathcal{C}_p \;\middle|\; \begin{array}{l} s(p) \in \mathcal{C}_p \\ \exists V \in \mathcal{N}_p, V \subset U, \exists t \in \mathcal{C}(V): \\ \forall a \in V: t_a = s(a) \end{array} \right\}$ And $\mathcal{C}_p = \varinjlim_{U \ni p} \mathcal{C}(U) = \varinjlim_{U \ni p} A = A$

$\mathcal{A}(U) = C(U, A)$ ▨

$\Rightarrow \mathcal{C}^+(U) = \{s: U \to A \mid s(p) \in A$
$= \{s: U \to A \mid \forall a \in V,\ \exists V \in \mathcal{N}_p, V \subset U,\ \exists t \in A$
$s(a) \in A \quad \forall a \in V, t = s(a)$
$= C(U, A)$ ▨

($t_a = t$ not depending)

Ex1.2. (a) Only prove $(\ker\varphi)_p = \ker(\varphi_p)$
then $\operatorname{im}(\varphi_p) = \frac{\mathcal{F}_p}{\ker(\varphi_p)} = \frac{\mathcal{F}_p}{(\ker\varphi)_p} = \left(\frac{\mathcal{F}}{\ker\varphi}\right)_p = (\operatorname{im}\varphi)_p$ ▨
(This has advantage of avoiding sheafification)
the following graph



$\forall \langle U, s\rangle \in (\ker\varphi)_p$
$\varphi_p(\langle U, s\rangle) = \langle U, \varphi(s)\rangle = \langle U, 0\rangle = 0$
$\Rightarrow (\ker\varphi)_p \subset \ker(\varphi_p)$
$\forall \langle U, s\rangle \in \ker\varphi_p$
$\langle U, \varphi(s)\rangle = \langle V, 0\rangle$, by sheaf (iii)
$\Rightarrow \varphi(s) = 0 \Rightarrow s \in \ker\varphi$
$\Rightarrow \langle U, s\rangle \in (\ker\varphi)_p \Rightarrow (\ker\varphi_p) \subset (\ker\varphi)_p$
$\Rightarrow (\ker\varphi)_p = \ker(\varphi_p)$ ▨

(b) (1) (injective)
($\Longrightarrow$) If $\varphi_p(\langle U, s\rangle) = \varphi_p(\langle V, t\rangle)$
$\langle U, \varphi(s)\rangle = \langle V, \varphi(t)\rangle$
$\exists W \subset U \cap V$: $\varphi(s)|_W = \varphi(t)|_W$
$\Rightarrow$ by sheaf (iii) $\Rightarrow \varphi(s) = \varphi(t) \Rightarrow s = t$ by φ injective
$\Rightarrow \langle U, s\rangle = \langle V, t\rangle \Rightarrow \varphi_p$ injective
($\Longleftarrow$) If $\varphi(s) = \varphi(t) \Rightarrow \forall U$, $\langle U, \varphi(s)\rangle = \langle V, \varphi(t)\rangle \Rightarrow \langle U, s\rangle = \langle V, t\rangle$ by φ_p injective
$\varphi_p(\langle U, s\rangle) = \varphi_p(\langle V, t\rangle)$ $\Rightarrow$ $s|_W = t|_W$
$\Rightarrow$ By sheaf (iii), $s = t$
$\Rightarrow \varphi$ injective ▨

(I'm sorry, the upper proof only shows $\boxed{\varphi(U) \text{ inj} \Leftrightarrow \varphi_p \text{ inj}}$)

(2) (Surjective) (although similar, prove again for finding why not holds when $\varphi(U)$) (See Prop 1.2 in my lecture)

φ surjective $\Leftrightarrow$ $\boxed{\operatorname{Im}\varphi = \mathcal{G} \Leftrightarrow (\operatorname{Im}\varphi)_p = \mathcal{G}_p, \forall p}$ $\Leftrightarrow \operatorname{Im}(\varphi_p) = \mathcal{G}_p, \forall p \Leftrightarrow \varphi_p$ surjective $\forall p$ ▨

(3) $\ker\varphi^i = \operatorname{Im}\varphi^{i-1} \Leftrightarrow (\ker\varphi^i)_p = (\operatorname{Im}\varphi^{i-1})_p$, $\forall p$
$\Leftrightarrow \ker(\varphi^i_p) = \operatorname{Im}(\varphi^{i-1}_p)$, $\forall p \Leftrightarrow$ exact ▨

Ex1.3. (a) φ surjective $\Leftrightarrow \operatorname{Im}(\varphi_p) = \mathcal{G}_p$, $\forall p$
($\Rightarrow$) $\forall U \subset X$, $\forall s \in \mathcal{G}(U)$, i.e. $\langle U, s\rangle \in \mathcal{G}_p$, $\forall p \in U$
$\exists \langle V, t\rangle$: $\varphi_p\langle V, t\rangle = \operatorname{Im}(\varphi_p)$
$= \langle V, \varphi_p(V)(t)\rangle = \langle U, s\rangle$, $V \in \mathcal{N}_p$
$\exists W \subset U \cap V$, $W \in \mathcal{N}_p$: $\varphi_p(W)(t)|_W = s|_W \Rightarrow$ We find $\{W_p\}_{p \in U}$ is the covering ▨

($\Leftarrow$) $\forall p$, $\exists U \in \mathcal{N}_p$, $U = \bigcup_{i \in I} U_i$, $p \in U_j$
$\forall \langle U, s\rangle \in \mathcal{G}_p$
$\langle U, s\rangle = \langle U_j, s\rangle = \langle U_j, \varphi(t_j)\rangle$
$= \varphi_p(\langle U_j, t_j\rangle) \in \operatorname{Im}\varphi_p$

Ex1.7. (a) It suffices to prove $0 \to \ker\varphi \to \mathcal{F} \to \operatorname{Im}\varphi \to 0$ exact by Ex1.6(b)

but $0 \to \ker(\varphi)_p \to \mathcal{F}_p \to \operatorname{Im}\varphi_p \to 0$

$\quad\ \ \|\qquad\qquad \|\qquad\qquad \|$

$0 \to \ker(\varphi_p) \to \mathcal{F}_p \to \operatorname{im}(\varphi_p) \to 0$ is obviously exact ▨

(b) Same as (a) ▨

Ex1.8. $0 \to \mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U) \xrightarrow{g(U)} \mathcal{H}(U)$

by Ex1.2(c)(1), (f injective, but g may not surjective, hence comes counterexample)

[illegible] $\operatorname{Im} f(U) = \ker g(U)$: $g(U)\circ f(U)(s) = g_p \circ f_p(\langle s, U\rangle) = 0$ by $0 \to \mathcal{F}_p \xrightarrow{f_p} \mathcal{G}_p \xrightarrow{g_p} \mathcal{H}_p$ exact

Left is showing

$=0$ by sheaf axiom③ ; $\langle s, U\rangle = 0 \Rightarrow s = 0$ $(\forall p \in U)$

Left is showing $\ker g(U) \subset \operatorname{Im} f(U)$

$\forall t \in \ker g(U)$, $\exists \langle s, V_p\rangle \in \mathcal{F}_p : f_p(\langle s, V_p\rangle) = \langle t, U\rangle$, $\forall p$, $V \in \mathcal{N}_p$

by sheaf axiom④, $V_p \in \mathcal{N}_p$ a covering of U, $\exists s \in \mathcal{F}(U)$, $s|_{V_p} = s_p$

with $f_p(\langle s_p, V\rangle) = \langle t, U\rangle = \langle f(U)(s_p), V\rangle$; $f(U)(s_p) = t$ in $W_p \subset V_p$

Sheaf axiom③ $\Rightarrow t = f(U)(s)$ ▨

Ex1.9. ① We verify the sheaf axiom③④ for $\mathcal{F} \oplus \mathcal{G}$. $U = \bigcup V_i$

③ $s \in (\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$, $s = (s_1, s_2)$; $s_1 \in \mathcal{F}(U)$, $s_2 \in \mathcal{G}(U)$

$s|_{V_i} = 0 \Leftrightarrow s_j|_{V_i} = 0$, $\forall i$, $j = 1, 2 \xRightarrow{\text{Sheaf axiom ③}} s_j = 0$ in U, $j = 1, 2 \Leftrightarrow s = 0$ in U

④ $s_i \in (\mathcal{F} \oplus \mathcal{G})(V_i) = \mathcal{F}(V_i) \oplus \mathcal{G}(V_i)$ of $\mathcal{F}$, $\mathcal{G}$

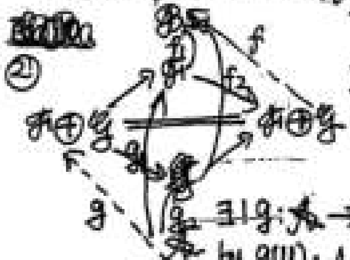
$s_i = (s_1^i, s_2^i)$, $s_1^i \in \mathcal{F}(V_i)$, $s_2^i \in \mathcal{G}(V_i)$

and $s_i|_{V_{ij}} = s_j|_{V_{ij}} \Leftrightarrow s_\delta^i|_{V_{ij}} = s_\delta^j|_{V_{ij}}$, $\delta = 1, 2 \xRightarrow[\text{of } \mathcal{F}, \mathcal{G}]{\text{Sheaf ④}} \exists s_\delta \in \mathcal{F}(U)$: $s_\delta|_{V_i} = s_\delta^i$, $\delta = 1, 2$

$\Rightarrow \exists s = (s_1, s_2) : s|_{V_i} = (s_1|_{V_i}, s_2|_{V_i})$

$= (s_1^i, s_2^i) = s_i$ ▨

②



$\exists!\, f : \mathcal{F} \oplus \mathcal{G} \to \mathcal{H}$

defined by $f(U) : (\mathcal{F} \oplus \mathcal{G})(U) \to \mathcal{H}(U)$

$(s_1, s_2) \mapsto f_1(U)(s_1) + f_2(U)(s_2)$

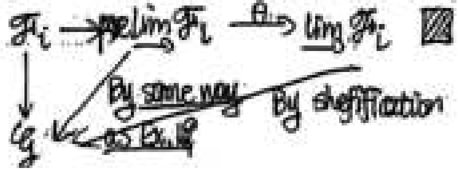
$\exists!\, g : \mathcal{H} \to \mathcal{F} \oplus \mathcal{G}$

by $g(U) : \mathcal{H}(U) \to (\mathcal{F} \oplus \mathcal{G})(U)$

Ex1.10. It's similiar with Ex1.9, $t \mapsto (g_1(U)(t), g_2(U)(t))$ ▨

up to a sheafification, but sheafification has same universal property.

The graph complete the proof:



Ex1.11. Only verify sheaf axiom③④, $U = \bigcup V_i$, $V_0 \subset \bigcup_{i \le 1} V_i \subset \bigcup_{i \le 2} V_i \subset \cdots$

Noetherian $\Rightarrow$ $\bigcup_{i \le n} V_i = \bigcup_{i \le m} V_i = \cdots$

$\Rightarrow U = \bigcup_{i \le n} V_i$ only holds for countable

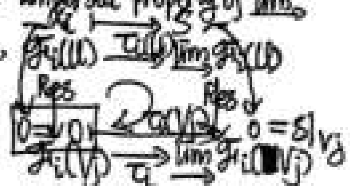
But Noetherian $\Rightarrow U$ is quasi-compact

Now we don't use it until we need it in proving

③ $s \in \varinjlim \mathcal{F}_i(U)$, $s = \tau_i(x_i)$, $x_i \in \mathcal{F}_i(U)$

by the "universal property of $\varinjlim$"

Indeed,



If $s|_{V_j} = 0$, $\forall j$, $s|_{V_j} = \tau_i(V_j)(x_i|_{V_j}) = 0$

But we can't claim $x_i|_{V_j} = 0$ now, thus it seems hard to use "$\mathcal{F}_i$ is a sheaf"

Now fix V_j, consider the sequence $\{\mathcal{F}_i(U)\}_i$, $s|_V = \tau_i(V)(x_i|_V) = \tau_j(V)(x_j|_V)$

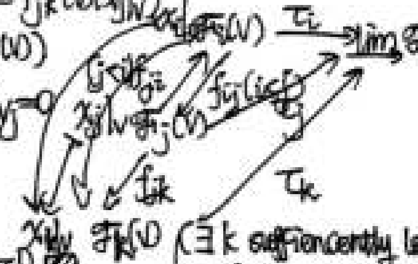
If j not finite, $\exists k$: $f_{ik}(V)(x_i|_V) = f_{jk}(V)(x_j|_V)$

k depend on j may not exist $(f_{ij} : \mathcal{F}_i(V) \to \mathcal{F}_j(V))$

$\Rightarrow \forall j, \exists k : x_k|_{V_j} = 0$

This is where we use Noetherian

$\xRightarrow{\text{The sheaf axiom of } \mathcal{F}_k} x_k = 0$



$\Rightarrow s = \tau_k(U)(x_k) = 0$ ▨

($\exists k$ sufficiently large, comes to a direct system, $K > k$ stable,

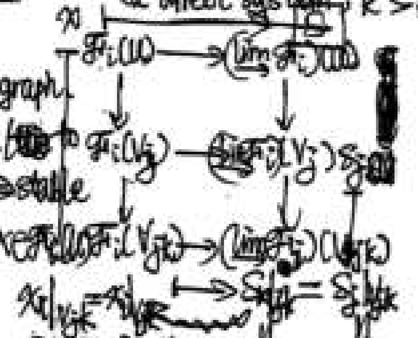
④ $s_i \in (\varinjlim \mathcal{F}_i)(V_i)$, $\forall i$

$s_i|_{V_{ij}} = s_j|_{V_{ij}}$, again consider the graph

the "(A)" use the same tech, $\exists$ [illegible] to make $\{\mathcal{F}_i(V_{jk})\}_i$ a direct system $\Rightarrow$ stable

$\Rightarrow \exists l : x_i|_{V_{jk}} = x_l|_{V_{jk}} \xRightarrow[\text{④ of } \mathcal{F}_l]{\text{Sheaf axiom}} \exists x \in$ [illegible]

$\Rightarrow$ let $s = \tau_l(U)(x)$, we complete the construction ▨



(A) $\tau_i(V_{jk})(x_i|_{V_{jk}}) = \tau_i(V_{jk})(x_i|_{V_{jk}})$

$\tau_i(V_j)(x_i)|_{V_{jk}} = \tau_i(V_j)(x_i)|_{V_{jk}}$

Ex1.12. We outline the difference between $\varinjlim$ & $\varprojlim$

The inverse system doesn't have the stable property of direct sets.

But: Sheafification is left adjoint of natural inclusion functor $({}^{+} \dashv i)$

$\Rightarrow$ It preserves all colimits & finite limits ▨

(commutes) (i.e. $(\varinjlim \mathcal{F}_i)^+ = \varinjlim(\mathcal{F}_i^+) = (\varinjlim \mathcal{F}_i) \Rightarrow$ Sheaf)

Ex1.14. $s_p \in \mathcal{F}_p$, $s_p = \langle U, s\rangle$

$\operatorname{Supp} s = \{p \in U \mid s_p = \langle U, s\rangle \neq 0\} = \{p \in U \mid \forall V \in \mathcal{N}_p, s|_V \neq 0\}$ $\Rightarrow \operatorname{Supp} s^c$ open $\Rightarrow$ Supp closed ▨

$\operatorname{Supp} s^c = \{p \in U \mid \exists V \in \mathcal{N}_p, s|_V = 0\}$, $\forall p \in \operatorname{Supp} s^c$, $\exists V \in \mathcal{N}_p$, $s|_V = 0 \Rightarrow \forall q \in V$, $q \in \operatorname{Supp} s^c$

(d) $f_*\mathcal{F}(U) \xrightarrow{f_{UV}} f_*\mathcal{F}(V)$

$$\begin{tikzcd} f_*\mathcal{F}(U) \arrow[r, "f_{UV}"] \arrow[d, equal] & f_*\mathcal{F}(V) \arrow[d, equal] \\ \mathcal{F}(f^{-1}(U)) \arrow[r] & \mathcal{F}(f^{-1}(V)) \end{tikzcd}$$

notice $V\subset U \Rightarrow f^{-1}(V)\subset f^{-1}(U)$, thus trivial ▨

(e) ① Verification: $\mathcal{G}$ does a sheaf
② $\mathcal{G}$ is flasque ③ $i:\mathcal{F}\to\mathcal{G}$ exist naturally

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r] & \mathcal{G}(V) \end{tikzcd}$$

Pf. ① Verify ③④, $U=\bigcup V_i$

③ $\forall s\in\mathcal{G}(U)$; $s|_{V_i}=0\ \forall i$

$\Rightarrow s|_{V_i}: V_i\to\prod_{p\in V_i}\mathcal{F}_p$, $x_i\mapsto(0,0,\cdots,0)$, thus $\forall p\in U$, $s(p)\in\mathcal{F}_p\subset\prod_{p\in V_i}\mathcal{F}_p\Rightarrow s(p)=0$
$\Rightarrow s=0$ ▨

④ $\forall s_i\in\mathcal{G}(V_i)$; $s_i|_{V_{ij}}=s_j|_{V_{ij}}$, $\forall i,j$
$\Rightarrow$ directly define $s: U\to\prod_{p\in U}\mathcal{F}_p$ by $s(p)\in\mathcal{F}_p\subset\prod_{p\in V_i}\mathcal{F}_p$, let $s(p)=s_i(p)$
by $s_i(p)=s_j(p)$ when $p\in V_{ij}$ $\Rightarrow$ well-defined ▨
(We even not use $\mathcal{F}$ is a sheaf)

② $\mathcal{G}(U)\xrightarrow{\rho_{UV}}\mathcal{G}(V)$, $s\longmapsto s|_V$ $\quad\forall t\in\mathcal{G}(V)$, $t: V\to\prod_{p\in V}\mathcal{F}_p$
can be extended into $\tilde{t}: U\to\prod_{p\in U}\mathcal{F}_p$, $p\mapsto t(p)$, $p\in V$; $p\mapsto 0$, $p\notin V$. naturally, $\tilde{t}|_V=t$ ▨

③ $i(U):\mathcal{F}(U)\to\mathcal{G}(U)$, $s\longmapsto\tilde{s}$, with
$\tilde{s}: U\to\prod_{p\in U}\mathcal{F}_p$, $x\mapsto s_x=\langle s,U\rangle\in\mathcal{F}_x$ naturally,

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r] & \mathcal{G}(V) \end{tikzcd}$$

$s\mapsto(\tilde{s}: x\mapsto s_x)$, $s|_V\mapsto(\tilde{s}: x\mapsto s_x)$ ▨

<u>Ex 1.17</u> ① $i_p(A):\mathfrak{U}\to Ab$
$U\mapsto A$; $U\in N_p$
$U\mapsto 0$; $U\notin N_p$
$[i_p(A)]_q=\{\langle U,s\rangle \mid U\in N_q,\ s\in i_p(A)(U)\}$
$q\in\overline{\{p\}}$, $U\in N_q\Rightarrow U\in N_p\Rightarrow s\in A$
$\Rightarrow[i_p(A)]_q=A$
$q\notin\overline{\{p\}}$, $\forall\langle U,s\rangle\in[i_p(A)]_q$, $\exists\ V\subset U$, $V\notin N_p$ $\Rightarrow\langle U,s\rangle=\langle V,s\rangle=0$ ▨

② $i:\overline{\{p\}}\to X$
$i_*(A):\mathfrak{U}\to Ab$
$U\mapsto A(i^{-1}(U))$
$U\in N_p$, $i^{-1}(U)=\overline{\{p\}}$, $A(i^{-1}(U))=A(\overline{\{p\}})=A$
$U\notin N_p$, $i^{-1}(U)=\emptyset$, $A(i^{-1}(U))=0$ ▨

<u>Ex 1.18</u>. $f^{-1}f_*\mathcal{F}(U)\to\mathcal{F}(U)$, $f^{-1}f_*\mathcal{F}(U)=\varinjlim_{V\supset f(U)}\mathcal{F}(f^{-1}(V))$

By universal property, define
$\varphi_V:\mathcal{F}(f^{-1}(V))\to\mathcal{F}(U)$
$s\longmapsto s|_U$ ($f^{-1}(V)\supset U$ by $V\supset f(U)$)
$\Rightarrow\exists!\varphi$ ▨

<u>Ex 1.15</u>. ① $s_1, s_2\in Hom(\mathcal{F}|_U,\mathcal{G}|_U)$
define $s_1-s_2:\mathcal{F}|_U\to\mathcal{G}|_U$ defined by $(s_1-s_2)(V):\mathcal{F}|_U(V)\to\mathcal{G}|_U(V)$, $x\mapsto s_1(V)(x)-s_2(V)(x)$
$\Rightarrow s_1-s_2\in Hom(\mathcal{F}|_U,\mathcal{G}|_U)$ ▨

$s:\mathcal{F}|_U(V)\to\mathcal{G}|_U(V)$ Sheaf②
Notice. $s|_{V_i}(V)=s(V):\mathcal{F}|_U(V)\to\mathcal{G}|_{V_i}(V)$

② Only verify ③④, $U=\bigcup V_i$

③ $\forall s\in Hom(\mathcal{F},\mathcal{G})(U)=Hom(\mathcal{F}|_U,\mathcal{G}|_U)$; $s|_{V_i}=0$, $\forall i\Rightarrow \boxed{s|_{V_i}(V)}:\mathcal{F}|_{V_i}(V)\to\mathcal{G}|_{V_i}(V)$, $x\mapsto 0$, $\forall V\subset V_i$
$\Rightarrow\forall V_i$, $\forall V\subset V_i$, $s|_{V_i}(V)(x)=0$, $\forall x\in V$ i.e. $\forall V\subset V_i$, $s|_{V_i}(V)(x)=0$
$\Rightarrow\forall V_i$, $\forall x$, $s|_{V_i}(V_i)(x)=0\Rightarrow\forall V_i$, $\forall x\in\mathcal{F}|_U(U)$, $s|_{V_i}(x|_{V_i})=0$ (Sheaf axiom ③ of $\mathcal{G}|_{V_i}$)
$\Rightarrow s(U)(x)=0$, $\forall x\in\mathcal{F}|_U(U)\Leftrightarrow s=0$ ▨ (Sheaf axiom ③ of $\mathcal{G}|_U$)

④ $\forall s_i\in Hom(\mathcal{F},\mathcal{G})(V_i)$; $s_i|_{V_{ij}}=s_j|_{V_{ij}}$, $\forall i,j$, consider the graph:

$$\begin{tikzcd} \mathcal{F}|_U(U) \arrow[r] \arrow[d] & \mathcal{G}|_U(U) \arrow[d] \\ \mathcal{F}|_U(V_i) \arrow[r] \arrow[d] & \mathcal{G}|_U(V_i) \arrow[d] \\ \mathcal{F}|_U(V_{ij}) \arrow[r, "s_i|_{V_{ij}}=s_j|_{V_{ij}}"] & \mathcal{G}|_U(V_{ij}) \end{tikzcd}$$

Let $g_i=s_i|_{V_i}(f)$, then apply Sheaf axiom ④ to $\{g_i\in\mathcal{G}|_U(V_i)\}\Rightarrow\exists g\in\mathcal{G}|_U(U)$
$\forall!$ $g_i|_{V_{ij}}=g_j|_{V_{ij}}$, $g|_{V_i}=g_i$
$\Rightarrow$ thus we find $s: f\mapsto g$, $s|_{V_i}=s_i$ ▨

<u>Ex 1.16</u>. (a) $\mathcal{A}(U)\cong A$ for U connected
and X irreducible $\Rightarrow X$ connected $\Rightarrow\mathcal{A}(U)\to\mathcal{A}(V)$, $A\to A$
$\Rightarrow$ surjective ▨

(b) $0\to\mathcal{F}'\xrightarrow{\varphi}\mathcal{F}\xrightarrow{\psi}\mathcal{F}''\to 0$

$0\to\mathcal{F}'(U)\xrightarrow{\varphi(U)}\mathcal{F}(U)\xrightarrow{\psi(U)}\mathcal{F}''(U)\boxed{\to 0}$

Only thing prove is $\psi(U):\mathcal{F}(U)\to\left(\frac{\mathcal{F}}{\mathcal{F}'}\right)^+(U)$ is surjective
It seems strange how to apply $\mathcal{F}'$ flasque, thus we write as quotient first.
$\forall s\in\left(\frac{\mathcal{F}}{\mathcal{F}'}\right)^+(U)$, $s: U\to\prod_{p\in U}\frac{\mathcal{F}_p}{\mathcal{F}'_p}$
$p\mapsto(0,0,\cdots,s(p),0,\cdots0)$ it's difficult to deal with.

By $0\to\mathcal{F}'_p\to\mathcal{F}_p\xrightarrow{\psi_p}\mathcal{F}''_p\to 0$ exact,
$\forall s\in\mathcal{F}''(U)$, $\langle s,U\rangle=\psi_p(\langle t,V_p\rangle)$, $\forall p\in U$, $\exists V_p\in N_p$
$\bigcup V_p=U$ a covering, now we have $t_p\in\mathcal{F}(V_p)$, we verify their compatibility and apply sheaf axiom ④ of $\mathcal{F}$: $\underbrace{t_p-t_q|_{V_{pq}}}_{\text{(obstruction)}}\xrightarrow{\psi(V_{pq})}0\Rightarrow\exists r\in\mathcal{F}'(V_{pq})$: $\varphi(V_{pq})(r)=(t_p-t_q)|_{V_{pq}}$

<u>Claim. $\varphi(V_{pq})(r)=0$</u>, thus $t_p|_{V_{pq}}=t_q|_{V_{pq}}\Rightarrow\exists t\in\mathcal{F}(U)\Rightarrow\psi(U)(t)=s$ surjective ▨
Now we finally transform the problem into $\mathcal{F}'$, thus we apply flasque
$\forall r\in\mathcal{F}'(V_{pq})$, $r=\tilde{r}|_{V_{pq}}$, $\tilde{r}\in\mathcal{F}'(U)\Rightarrow\varphi(U)(\tilde{r})|_{V_{pq}}=t_p-t_q=0$
by otherwise, $\tilde{r}$ can't be global in U ▨

(c) if $\mathcal{F}'$ flasque, then $\mathcal{F}''$ flasque $\Leftrightarrow\mathcal{F}$ flasque.
Pf. By (b), exact, then

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}'(U) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}''(U) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{F}'(V) \arrow[r] & \mathcal{F}(V) \arrow[r] & \mathcal{F}''(V) \arrow[r] & 0 \end{tikzcd}$$

By <u>weak four Lemma</u> ▨

② When $\mathscr{F}$ flasque, $\forall s \in j_*(\mathscr{F}|_U)(V) = \mathscr{F}|_U(U \cap V)$, we find $\tilde{s} \in \mathscr{F}(V)$
By Ex1.16(b), it's enough to show it (exact at section) $\tilde{s}|_{U\cap V} = s$
But it's trivial: by Ex1.19(b), we extend by zero ▨

Ex1.21. (a) $\mathscr{I}_Y(U) = \Gamma_{Y \cap U}(\mathcal{O}_X|_Y)$, by Ex1.20(a), it's a sheaf ▨
(b) $\mathscr{I}_Y(U) = \mathscr{H}^0_{X-U}(\mathcal{O}_X)$, $j: U \to X$, when $U = Y$, $\mathscr{I}_Y = \mathscr{H}^0_{X-Y}(\mathcal{O}_X)$, $i: Y \to X$
thus by Ex1.20.(b), we have: $0 \to \mathscr{I}_Y \to \mathcal{O}_X \to j_*(\mathcal{O}_X|_U) \dashrightarrow 0$
Claim. ① $\mathcal{O}_X|_Y = \mathcal{O}_Y$ ② $\mathcal{O}_X$ is flasque, then by Ex1.20(b) $\Rightarrow \dfrac{\mathcal{O}_X}{\mathscr{I}_Y} \cong i_*(\mathcal{O}_Y)$ ▨

Pf. ① $\forall U \subset Y$, $(\mathcal{O}_X|_Y)(U)$
$= (i^{-1}\mathcal{O}_X)(U) = \varinjlim_{V \supset i(U)} \mathcal{O}_X(V)$
$= \varinjlim_{V \supset U} \mathcal{O}_X(V) = \mathcal{O}_X(U)$
The regular function on U of X and Y are same
$\Rightarrow \mathcal{O}_X(U) = \mathcal{O}_Y(U)$, $U \subset Y \subset X$
$\Rightarrow \forall U \subset Y$, $(\mathcal{O}_X|_Y)(U) = \mathcal{O}_Y(U)$ ▨

② It's sad to say that it's wrong when X isn't normal, smooth or some other restrictions
EASY COUNTEREXAMPLE.

$$\begin{tikzcd} \mathbb{A}^1 = D(x_0) \arrow[r, hook] \arrow[d, "id"'] & \mathbb{P}^1 \arrow[dl, "can't extended"] \\ k = \mathbb{A}^1_k & \end{tikzcd}$$

can't extended: $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) \to \mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1)$ not surjective ▨

Rk. Precisely, $\mathscr{I}_Y$ is the kernel of $i^\#: \mathcal{O}_X \to i_*\mathcal{O}_Y$, thus (a)(b) all done ▨

We prove without ② is also enough ▨
(c) $0 \to \mathscr{I}_Y \xrightarrow{f} \mathcal{O}_X \xrightarrow{g_1} i_*\mathcal{O}_{P\cup Q}$

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "g_1"] \arrow[dr, "g_2"'] & i_*\mathcal{O}_{P\cup Q} \\ & i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q \arrow[u, "D"] \end{tikzcd}$$

D is injective $\Rightarrow$ not change the kernel.
$\Rightarrow \operatorname{Im} f = \ker g_1 \Rightarrow \operatorname{Im} f = \ker g_2$

$\Rightarrow$ Only prove g_2 surjective
$\mathcal{O}_X(U) \xrightarrow{g_2(U)} (i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q)(U) = \mathcal{O}_P(i^{-1}(U)) \oplus \mathcal{O}_Q(i^{-1}(U))$
$$s \longmapsto (s_1, s_2) = \begin{cases} (0, 0) & ; P, Q \notin U \\ (0, \mathcal{O}_{X,Q}) & ; P \notin U, Q \in U \\ (\mathcal{O}_{X,P}, 0) & ; P \in U, Q \notin U \\ (\mathcal{O}_{X,P}, \mathcal{O}_{X,Q}) & ; P, Q \in U \end{cases}$$

(but not $\operatorname{Im}[g_2(X)]$ $(\mathcal{O}_P(i^{-1}X) \oplus \mathcal{O}_Q(i^{-1}X))$)
obviously not surjective when $X = U$
thus it's a counterexample
$\Gamma(X, \mathcal{O}_X) \cong k$
$\Gamma(X, \mathscr{F}) \cong \cancel{k \oplus} k$

$\mathcal{O}_{X,x} \xrightarrow{(g_2)_x} (i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q)_x = (i_*\mathcal{O}_P)_x \oplus (i_*\mathcal{O}_Q)_x$, $x \in X$
$\Rightarrow (g_2)_x$ is obviously surjective $P, Q \in \mathbb{P}^1 \Rightarrow \overline{\{P\}} = \{P\}$, $\overline{\{Q\}} = \{Q\}$ $\Rightarrow \begin{cases} x = P & ; \mathcal{O}_P \\ x = Q & ; \mathcal{O}_Q \\ x \neq P, Q & ; 0 \end{cases}$
$\forall x$
$\Rightarrow g_2$ follows ▨

(d) ① $\mathcal{O}(U) \xrightarrow{i(U)} \mathscr{K}(U)$
$(s: U \to k) \longmapsto (\tilde{s}: U \to \mathscr{K}; \tilde{s}(U): U \to k \in \mathscr{K})$
injective by $\tilde{s} = 0: U \to \mathscr{K}$, $x \mapsto 0 = s$ $\Rightarrow \tilde{s} = i(U)(s)$, $s = 0$ ▨

② $0 \to \mathcal{O} \to \mathscr{K} \to \mathscr{K}/\mathcal{O} \to 0$
Only to prove exactness of below one
$0 \to \mathcal{O}_Q \to \mathscr{K}_Q \to (\bigoplus i_P(I_P))_Q \to 0$
$(\mathscr{K}_Q = \mathscr{K}$ by const; $(\bigoplus i_P(I_P))_Q = \bigoplus \ldots$ Only not 0 when $P \in \ldots)$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, equal] & \mathscr{K} \arrow[r] \arrow[d, equal] & \mathscr{K}/\mathcal{O} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{O} \arrow[r] & \mathscr{K} \arrow[r] & \bigoplus_{P} i_P(I_P) \arrow[r] & 0 \end{tikzcd}$$

$0 \to \mathcal{O}_Q \to \mathscr{K}_Q \to I_P \to 0$ obvious exact ▨

Ex1.18. continue. $\mathscr{G}(V) \to f_* f^{-1}\mathscr{G}(V)$; $\varinjlim_{U \supset f(f^{-1}V)} \mathscr{G}(U)$; $\varinjlim_{U \supset V} \mathscr{G}(U)$; $\mathscr{G}(V)$
The $\operatorname{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) \cong \operatorname{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$ follows by graph ▨

$$\begin{tikzcd} f^{-1}\mathscr{G} \arrow[r, "\varphi"] \arrow[d] & \mathscr{F} \arrow[d] \\ f^{-1}f_*f^{-1}\mathscr{G} \arrow[r, "f_*\varphi"] & f_*\mathscr{F} \\ \mathscr{G} \arrow[u, "id"] \arrow[ur, "(f^\#)\circ id"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} \mathscr{G} \arrow[r, "\psi"] \arrow[d] & f_*\mathscr{F} \arrow[d] \\ f^{-1}\mathscr{G} \arrow[r, "f^{-1}\psi"] \arrow[dr, "\psi\circ f^{-1}\psi"'] & f^{-1}f_*\mathscr{F} \arrow[d, "\varphi"] \\ & \mathscr{F} \end{tikzcd}$$

Thus take identity ▨

Ex1.20. (a) Verify ③④ $U = \bigcup V_i$
③ $\forall s \in \Gamma_{Z \cap U}(U, \mathscr{F}|_U)$
$s|_{V_i} = 0$, $\forall i \Rightarrow \operatorname{supp}(s|_{V_i}) = 0$
$\operatorname{supp} s = \bigcup \operatorname{supp}(s|_{V_i}) \Rightarrow \operatorname{supp} s = 0$
$\Rightarrow s = 0$
④ $\forall s_i \in \Gamma_{Z \cap V_i}(V_i, \mathscr{F}|_{V_i})$
$s_i|_{V_{ij}} = s_j|_{V_{ij}}$, $\forall i, j$, by Sheaf ④ of $\mathscr{F}|_U$
$\exists s \in \Gamma(U, \mathscr{F}|_U): s|_{V_i} = s_i$
Claim. $s \in \Gamma_{Z \cap U}(U, \mathscr{F}|_U)$
by $\operatorname{supp} s = \bigcup \operatorname{supp} s|_{V_i} \subset \bigcup(V_i \cap Z)$ ▨
$\subset U \cap Z$

Ex1.19. (a) (The same way as we dealing with skyscraper sheaves)
$i_*\mathscr{F}(U) = \mathscr{F}(i^{-1}(U))$
$(i_*\mathscr{F})_p = \{\langle s, U\rangle \mid U \in N_p, s \in \mathscr{F}(i^{-1}(U))\}$
if $P \in Z$, $\exists V \subset U$, $V \subset Z$: $\langle s, U\rangle = \langle s, V\rangle$, $s \in \mathscr{F}(i^{-1}(V)) = \mathscr{F}(V)$
$\Rightarrow (i_*\mathscr{F})_p = \{\langle s, V\rangle \mid V \in N_p, s \in \mathscr{F}(V)\} = \mathscr{F}_p$
if $p \notin Z$, $\exists V \subset U$, $V \cap Z = \emptyset$: $\langle s, U\rangle = \langle s, V\rangle$, $s \in \mathscr{F}(i^{-1}(V)) = \mathscr{F}(\emptyset) = 0$
$\Rightarrow (i_*\mathscr{F})_p = 0$ ▨

(b) ① $(j_!(\mathscr{F}))_p = (\text{pre.} j_!(\mathscr{F}))_p = \{\langle s, W\rangle \mid W \in N_p, s \in \text{pre.} j_!(\mathscr{F})(W)\}$
$P \notin U$: $\exists V \subset W$: $V \cap U = \emptyset$, $\langle s, W\rangle = \langle s, V\rangle$, $s \in \text{pre.} j_!(\mathscr{F})(V) = 0$
$\Rightarrow (j_!(\mathscr{F}))_p = 0$
$P \in U$: $\exists V \subset W$, $V \subset U$, $\langle s, W\rangle = \langle s, V\rangle$, $s \in \text{pre.} j_!(\mathscr{F})(V) = \mathscr{F}(V)$
$\Rightarrow (j_!(\mathscr{F}))_p = \mathscr{F}_p$ ▨

② If $\exists \mathscr{G}$: $\mathscr{G}_p = \begin{cases} \mathscr{F}_p & ; P \in U \\ 0 & ; P \notin U \end{cases}$, we prove $\mathscr{G}: V \mapsto \mathscr{F}(V)$; $V \subset U$, $\mathscr{G} = \mathscr{G}'^+$; $V \mapsto 0$; $V \not\subset U$
$\forall V \subset U$, $\forall s \in \mathscr{G}(V)$, $s: V \to \bigcup_{P \in V} \mathscr{G}_p = \bigcup_{P \in V} \mathscr{F}_p$
$\forall V \not\subset U$, $\forall s \in \mathscr{G}(V)$, $s: V \to \bigcup_{P \in V} \mathscr{G}_p = \bigcup_{P \in V \cap U} \mathscr{F}_p \Rightarrow \mathscr{G} = j_!(\mathscr{F})$ ▨

(c) The stalk sequence
$0 \to (j_!(\mathscr{F}|_U))_p \to \mathscr{F}_p \to (i_*\mathscr{F}|_Z)_p \to 0$, $\forall p$
When $p \in Z$, by (a): $0 \to 0 \to \mathscr{F}_p \to \mathscr{F}_p \to 0$
When $p \in U$, by (b): $0 \to \mathscr{F}_p \to \mathscr{F}_p \to 0 \to 0$
$\Rightarrow$ exact ▨

Ex1.20. (b) ① $\mathscr{H}^0_Z(\mathscr{F}) \to \mathscr{F}$ is trivially injective
only show $\mathscr{H}^0_Z(\mathscr{F}) \xrightarrow{i} \mathscr{F} \xrightarrow{f} j_*(\mathscr{F}|_U)$ has $\operatorname{Im} i = \ker f$ — exact ▨
We define it by $\mathscr{H}^0_Z(\mathscr{F})(V) \xrightarrow{i(V)} \mathscr{F}(V) \xrightarrow{f(V)} j_*(\mathscr{F}|_U)(V) = \mathscr{F}|_U(j^{-1}(V)) = \mathscr{F}|_U(U \cap V)$
$s \mapsto s \longmapsto s|_{U\cap V}$, but $s|_{U\cap V} = 0 \Leftrightarrow \operatorname{supp} s \subset Z$

(c). $(X, \mathcal{O}_X) \to ((\mathcal{O}_X)_{red}) \to (Y, \mathcal{O}_Y)$ is induced by $X \to Y$ and $(\mathcal{O}_X)_{red} \xleftarrow{\varphi} \mathcal{O}_Y$

$\exists!\ \varphi$

given by $\mathcal{O}_X(U)_{red} \xleftarrow{\varphi(U)} \mathcal{O}_Y(U)$ nothing surprising but only the universal $\downarrow p(U)$ properties of quotient ring $\mathcal{O}_Y(U) \xrightarrow{f(U)} \mathcal{O}_X(U)$, $(\mathcal{O}_Y(U))_{red}$ ▨

Ex2.4. a: $\mathrm{Hom}_{Sch}(X, \mathrm{Spec}A) \to \mathrm{Hom}_{Rings}(A, \Gamma(X, \mathcal{O}_X))$

$(f, f^\#) \mapsto f^\#(X)$ ① injective is trivial: $f^\#(X) = 0$

$\boxed{?} \leftarrow\!\!| \ \varphi$ $\Rightarrow f^\#(U) = f^\#(X)|_U = 0 \Rightarrow f^\# = 0$ ▨

② $\varphi: \left(s: \mathrm{Spec}A \to \coprod_{p\in\mathrm{Spec}A} A_p,\ p \mapsto f_p \in A_p\right) \longmapsto \left(\varphi(s): X \to \coprod_{x\in X} \mathcal{O}_{X,x},\ x \mapsto f_x \in \mathcal{O}_{X,x}\right)$

By Prop2.3(b), it has been proven when X is an affine scheme, cover X with affine open sets, then their sheaves are defined. By Ex1.22, we complete the proof. ▨

It induces $f^\#: \mathcal{O}_{\mathrm{Spec}A} \to f_*\mathcal{O}_X$ by $f^\#(U): \mathcal{O}_{\mathrm{Spec}A}(U) \to \mathcal{O}_X(f^{-1}(U))$, $s|_U \mapsto \varphi(s)|_{f^{-1}(U)}$ ▨

Ex2.5. Description $\mathrm{Sp}(\mathrm{Spec}(\mathbb{Z}))$

(2) (3) (5) (p) ∞=(0)

(closed points) (The generic point)

$\mathcal{O}_{\mathrm{Spec}\mathbb{Z}}$ defined by

$\mathcal{O}_{\mathrm{Spec}\mathbb{Z}}(U) = \{s: U \to \coprod_{p\in U} \mathbb{Z}_p = \coprod_{p\in U} \mathbb{F}_p\}$, $p \mapsto 0 \in \mathbb{Z}_p$

And $\mathbb{Z}$ is initial in Ring $\Rightarrow \mathrm{Hom}_{Ring}(\mathbb{Z}, \Gamma(X, \mathcal{O}_X)) = \{*\}$

$\|$ Ex2.4

$\mathrm{Hom}_{Sch}(X, \mathrm{Spec}\mathbb{Z})$ i.e.: exist and unique ⇒ final object ▨

Ex2.6. $\mathrm{Spec}(0) = (0)$

$\forall X \in \mathrm{Obj}(\mathfrak{Sch})$

We define $(X, \mathcal{O}_X) \xrightarrow{(f, f^\#)} (\{(0)\}, \mathcal{O}_0)$ by ① $f: X \to \{(0)\}$, $x \mapsto 0$ obvious $\exists!$ ② $f^\#: \mathcal{O}_0 \to f_*\mathcal{O}_X$

$f^\#(U) = f^\#((0)): \mathcal{O}_0(U) \to \mathcal{O}_X(f^{-1}(U))$, $0 \mapsto 0$ obvious $\exists!$, again ▨

Ex2.7. $\mathrm{Hom}_{Sch}(\mathrm{Spec}K, X) \cong \mathrm{Hom}_{Ring}(k(x), K)$

$(f, f^\#) \xmapsto{\varphi} \forall x \in X$ we have $f^\#_x: \mathcal{O}_{X,x} \to (\mathcal{O}_{\mathrm{Spec}K})_{f^{-1}(x)} = K$

$\downarrow$ $\exists!\ \bar f^\#_x$

$\frac{\mathcal{O}_{X,x}}{m_x} = k(x)$, $(\mathcal{O}_{\mathrm{Spec}K})_y = K$ By Prop2.2(a)

$\varphi(f, f^\#) = \bar f^\#_x$

Conversely, $i: k(x) \to K$, $f: \mathrm{Spec}K \to X$ is naturally by $\mathrm{Spec}K = \{(0)\}$, we assign it to x, i.e. $f((0)) = x \in X$

$f^\#: \mathcal{O}_X \to f_*\mathcal{O}_{\mathrm{Spec}K}$ also natural, ▨

We need define it locally first $\forall x \in X, \exists U \in \mathfrak{N}_x$, $f^\#(U): \mathcal{O}_X(U) \to \mathcal{O}_{\mathrm{Spec}K}(f^{-1}(U))$. For other $U \notin \mathfrak{N}_x$, value 0. This well-defined $f^\#$. And $f^\#_p: \mathcal{O}_{X,p} = \mathcal{O}_{X,x}$ ▨

Ex2.8. Preparations. (Division in $k[\varepsilon]/\varepsilon^2$)

$\frac{a+b\varepsilon}{c+d\varepsilon} = \frac{a}{c} + \frac{bc-ad}{c^2}\varepsilon \Rightarrow (a+b\varepsilon)^{-1} = \frac{a-b\varepsilon}{a^2}$

Ex2.1. ① $\mathrm{Spec}A_f \cong D(f)$, $p \in D(f)$ by $f(p) = \frac{\bar f}{1} \in \frac{A_p}{pA_p}$, it suffices to show $\frac{f}{1} \notin pA_p$

$\mathrm{Spec}A_f \longleftrightarrow D(f)$

$pA_f \leftarrow\!| \ p$ Thus $f(p) \neq 0$ (This is by $p \cap (f) = \emptyset \Rightarrow \frac{f}{1} \notin A_p \subset pA_p$ ▨)

$pA_f = \tilde p \longmapsto p$ It's obviously prime by $\tilde p$ prime

$\{\frac{g}{h} \mid g \in A, h \in A_f\}$ $\{g \mid \frac{g}{1} \in \tilde p, g \in A\}$ Thus it's a bijection;

(Rk. View $\tilde p \in \mathrm{Spec}A_f$ as $\tilde p = pA_f$, $p \cap (f) = \emptyset$ is easier, i.e. $\mathrm{Spec}A_f = \{pA_f \mid p \cap (f) = \emptyset\}$)

Then we show it's both-side continue: $p \mapsto pA_f$, $pA_f \mapsto p$ ▨

② $\mathcal{O}_{\mathrm{Spec}A_f} \cong \mathcal{O}_X|_{D(f)}$: By Prop2.2(b) $\mathcal{O}_X(D(f)) \cong A_f$

$\Rightarrow \mathcal{O}_{\mathrm{Spec}A_f} = \mathcal{O}_{\mathrm{Spec}\mathcal{O}_X(D(f))} = \Gamma(\mathrm{Spec}\Gamma(D(f), \mathcal{O}_X), U) \cong \Gamma(D(f), \mathcal{O}_X) \cong \mathcal{O}_X|_{D(f)}$ By Prop2.2(c) ▨

Ex2.2. First $\mathcal{O}_X|_U|_V = \mathcal{O}_X|_V$ by $i_1^* i_2^* = (i_1 i_2)^*$

$\Rightarrow \forall p \in U \subset X, \exists V \in \mathfrak{N}_p, (V, \mathcal{O}_X|_V)$ affine $\Rightarrow (V \cap U, \mathcal{O}_X|_V|_{V\cap U} = \mathcal{O}|_{V\cap U})$ affine

$\Rightarrow$ i.e. $\forall p \in U, \exists V \in \mathfrak{N}_p, V \subset U: (V, \mathcal{O}_X|_V)$ affine $\Rightarrow (U, \mathcal{O}_X|_U)$ is a scheme ▨

Ex2.3.(a) (⇒) Trivial by $\mathcal{O}_{X,p} = \varinjlim_{U \ni p} \mathcal{O}_X(U)$; If $\exists U, f \in \mathcal{O}_X(U): f^n = 0$ $\Rightarrow (\sigma_U(f))^n = \sigma_U(f^n) = 0$, contradiction (by $\sigma_U: \mathcal{O}_X(U) \to \mathcal{O}_{X,p}$) ▨

(⇐) Also trivial: $[f]^n = 0 \Rightarrow \exists i, f_i^n = 0$, contradiction ▨

(b) ① $\forall p \in X, \exists U \in \mathfrak{N}_p: (U, \mathcal{O}_X|_U) \cong (\mathrm{Spec}A, \mathcal{O}_{\mathrm{Spec}A})$

$\Rightarrow (U, (\mathcal{O}_X|_U)_{red}) \cong (\mathrm{Spec}A, (\mathcal{O}_{\mathrm{Spec}A})_{red})$

The left is proving ⓐ $(\mathcal{O}_X|_U)_{red} \cong (\mathcal{O}_X)_{red}|_U$ ⓑ Assume $(X, \mathcal{O}_X)$ affine

II. ⓐ Trivial by $(\mathcal{O}_X|_U)_{red}(V) = (\mathcal{O}_X|_U(V))_{red} = (\mathcal{O}_X(V))_{red}$, $(\mathcal{O}_X)_{red}|_U(V) = (\mathcal{O}_X)_{red}(V) = (\mathcal{O}_X(V))_{red}$ ▨

ⓑ $(\mathrm{Spec}A, (\mathcal{O}_{\mathrm{Spec}A})_{red})$ is a scheme, it suffices to prove $(\mathcal{O}_{\mathrm{Spec}A})_{red}$ does a sheaf over SpecA, $U = \bigcup V_i$

③ $\forall \bar s \in (\mathcal{O}_{\mathrm{Spec}A})_{red}(U) = (\mathcal{O}_{\mathrm{Spec}A}(U))_{red}$, $\bar s_i \in \mathcal{O}_{\mathrm{Spec}A}(V_i)_{red}$, $\bar s_i = 0, \forall i$

$\Rightarrow s_i^{n_i} = 0$, let $\prod n_i = n \Rightarrow s^n|_{V_i} = 0$ by Sheaf③ of $\mathcal{O}_{\mathrm{Spec}A}$ $\Rightarrow s^n = 0 \Rightarrow \bar s = 0$

④ $\forall \bar s_i \in \mathcal{O}_{\mathrm{Spec}A}(V_i)_{red}$, $\bar s_i|_{V_{ij}} = \bar s_j|_{V_{ij}}$

$\Rightarrow (s_i - s_j)^{n_{ij}} = 0$, let $n = \prod_{i,j} n_{ij}$

$\Rightarrow (s_i - s_j)^n = 0$, take N sufficiently large

$\Rightarrow (s_i - s_j)^N = s_i^N - s_j^N$ in V_{ij}

by Sheaf④ of $\mathcal{O}_{\mathrm{Spec}A}$

$\Rightarrow \exists s \in (\mathcal{O}_{\mathrm{Spec}}(U))$, $s|_{V_i} = s_i^N$

$\overline{s_i^N} = \bar s_i \Rightarrow \bar s|_{V_i} = \bar s_i$ ▨

② $(X, (\mathcal{O}_X)_{red}) \to (X, (\mathcal{O}_X))$ only need given by $(\mathcal{O}_X) \to (\mathcal{O}_X)_{red}$, i.e.

$(\mathcal{O}_X(U))_{red} \leftarrow \mathcal{O}_X(U)$, $(\mathcal{O}_X(V))_{red} \leftarrow \mathcal{O}_X(V)$ It suffices to prove $\bar s_V = \bar s|_V$, but it's trivial ▨

Ex2.11. $\mathrm{Spec}(\mathbb{F}_p[X]) = \{0\} \cup \{(f) \mid f \in \mathbb{F}_p[X]$ irreducible $\}$ (and monic)

For $x=(0)$ the generic point, $\mathbb{F}_p(x) = \frac{\mathcal{O}_{(0)}}{\mathfrak{m}_{(0)}} = (0)$

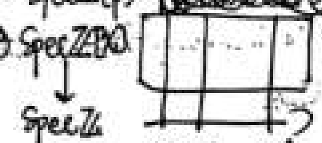
For $x=(f)$, $\mathbb{F}_p(x) = \frac{\mathcal{O}_f}{\mathfrak{m}_f} = \{\frac{g}{h} \mid g,h \notin (f)\}$ ... $\mathbb{F}_{p^d}$ (deg f)

given K, $|\{x \mid k(x)=K\}|$ ⇒ Möbius inverse thm

③ $\mathrm{Spec}\,\mathbb{Z}_{(p)} \longleftrightarrow \mathrm{Spec}\,\mathbb{Z}$, 对 q i.e. $q \neq p$

$\Rightarrow \mathrm{Spec}\,\mathbb{Z}_{(p)} = \mathrm{Spec}\,\mathbb{Z} - \{(q)\}$ ▨

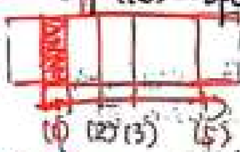
④ $\mathrm{Spec}\,\mathbb{Z}[X]$ Fibre $p^{-1}((p))$ $= \mathbb{F}_p$



$\mathrm{Spec}\,\mathbb{Z}[X] = \{(p)\} \cup \{(p,f) \mid f \in \mathrm{Spec}\,\mathbb{F}_p[X],\ f\neq 0\} = \mathrm{Spec}\,\mathbb{Z} \cup \mathrm{Spec}\,\mathbb{Z} \times (\mathrm{Spec}\,\mathbb{F}_p[X] - \{0\})$.

⇒ Generic point in the component $\mathbb{F}_p$

Fibre $p^{-1}((0)) = \mathrm{Spec}\,\mathbb{Q}[X]$, $f \in \mathrm{Spec}\,\mathbb{Q}[X]$, f reflects orbits of algebraic numbers under the absolute Galois group of $\mathbb{Q}$



(The moduli space)

Finally, $\mathrm{Spec}\,\mathbb{Z}[X]$

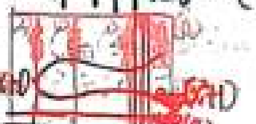
$= \mathrm{Spec}\,\mathbb{Z} \cup \mathrm{Spec}\,\mathbb{Q}[X] \cup (\mathrm{Spec}\,\mathbb{Z} \times \mathrm{Spec}\,\mathbb{F}_p[X])$ ▨

⑤ $\mathrm{Spec}\,\frac{\mathbb{Z}[X]}{(X^2)} \longleftrightarrow \mathrm{Spec}\,\mathbb{Z}[X]$, $p \supseteq (x^2)$, notice (0) not included

$\Rightarrow \mathrm{Spec}\,\frac{\mathbb{Z}[X]}{(x^2)} = \{(x)\}$ ▨ $p \supseteq (x)$ It's Geometry: Sufficiently small ε

⑥ $\mathrm{Spec}\left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right) = \{(p) \mid p \mid n\}$ ▨ (Other (m), $m+n$ unit).

Graph by Mumford (famous)



Ex2.12. (Cocycle condition) ⇒ Gluing

$X = \coprod_{i\in I} X_i / \sim$ $(\forall x \in U_{ij},\ x \sim \varphi_{ij}(x))$ ⇒ Well-defined by cocycle condition $\varphi_{ij}(s_i) = s_j$

Now we only verify ①②③④ and $\mathcal{O}_X(V) = \{(s_i)_{i\in I} \mid s_i \in \mathcal{O}_{X_i}(\psi_i^{-1}(V))\}$

Then $\psi_i : X_i \longrightarrow X$ is just the natural embedding.

By open mapping, $\psi_i(X_i) \approx X_i$ is trivial.

Now we verify its sheaf

$\mathcal{O}_X \Rightarrow \psi_{i*}\mathcal{O}_{X_i}$ $\Rightarrow \mathcal{O}_X(U) \to \mathcal{O}_{X_i}(U \cap X_i)$

$\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X_i}(\psi_i^{-1}(U))$, $U \subset X$, $\psi_i^{-1}(U) = U \cap X_i$ $(s_i)_i \longmapsto s_i$, $(\psi_i(s_i)) \longleftarrow s_i$ ▨

Ex Compute spectrum (and finally)

① $k[X_1, \dots, X_n]$, $k=\bar{k}$

② $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{-5}]$

③ $\mathbb{Z}_p$ (the localization)

④ $\mathbb{Z}[X]$, use this compute $\mathbb{F}_p$, ⑤ $\frac{\mathbb{Z}[X]}{(x^2)}$ ⑥ $\frac{\mathbb{Z}}{n\mathbb{Z}}$

① $\mathrm{Spec}\,k[X_1, \dots, X_n]$ ▨ $= \{0\} \cup \{(x_1 - a_1, \dots, x_n - a_n)\}$

⑦ $\mathrm{Spec}\,\mathbb{Z}[i]$

Spec Z (Spec Z[√-5])

Ex2.8 (continue) $\mathrm{Spec}\left(\frac{k[\varepsilon]}{\varepsilon^2}\right)$'s computation. Eg $(a+b\varepsilon)^{-1} = \frac{a-b\varepsilon}{a^2}$, $\{a=0\} \trianglelefteq \frac{k[\varepsilon]}{\varepsilon^2}$

Recall. $\mathfrak{m} \trianglelefteq R$, if $R - \mathfrak{m}$ all units $\Rightarrow (R, \mathfrak{m})$ local $\Rightarrow \left(\frac{k[\varepsilon]}{\varepsilon^2}, \{a+b\varepsilon\}\right)$ is local ring.

$\mathrm{Spec}\frac{k[\varepsilon]}{\varepsilon^2} \longleftrightarrow \mathrm{Spec}(k[\varepsilon])$ containing $(\varepsilon^2) = (0)$

$\Rightarrow \mathrm{Spec}\frac{k[\varepsilon]}{\varepsilon^2} = \{(\varepsilon)\}$ (Rk. $(\varepsilon - a) \notin \mathrm{Spec}\left(\frac{k[\varepsilon]}{\varepsilon^2}\right)$ as $(\varepsilon-a) = (b\varepsilon^2 - \varepsilon - a)$ not prime)

Now we start our 证. $\mathrm{Hom}_{k\text{-}Sch}\left(\mathrm{Spec}\left(\frac{k[\varepsilon]}{\varepsilon^2}\right) = \{(\varepsilon)\}, X\right) \cong \{(x,v) \in X \times T_x \mid k(x) = k\}$

$(f, f^\#)$ given by Ex2.7. we given $k \longleftrightarrow K$ $(f, f^\#) \longleftrightarrow (x, v)$

$(f, f^\#) \longmapsto (x, i : k(x) \longrightarrow \frac{k[\varepsilon]}{\varepsilon^2})$

Left is showing $i : k(x) \longrightarrow \frac{k[\varepsilon]}{\varepsilon^2}$ induces $v \in \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$, then $v \in T_{(x)}$

$\mathcal{O}_{X,x}$, $\mathfrak{m}_x$ $\frac{\mathcal{O}_{X,x}}{(s,u)} \longmapsto s(x)$ or $s(x)\varepsilon \in \mathfrak{m}_x \Rightarrow$ We determine $s(x)\varepsilon \in \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$ ▨

(but Notice $s(x)$ isn't a good choice: $(s(x)\varepsilon)^2 = 0 \Rightarrow s(x)\varepsilon \in \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$)

Ex2.9. It suffice to prove for X affine and irreducible (Otherwise $U \subset Z \subset X$, $\bar{\xi} \cap U \Rightarrow \bar{\xi} = Z$, we reduce to locally affine scheme U)

The left is proving existence and uniqueness generic point

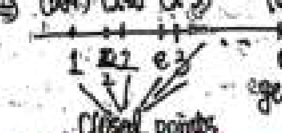
$X = \mathrm{Spec}\,A$, existence is trivial: $(0) \in \mathrm{Spec}\,A$ is generic, $\overline{(0)} = \bigcap V(a) = \bigcap \mathrm{Spec}\,A = \mathrm{Spec}\,A$ ($V(a) \supseteq (0)$, $a \subseteq (0)$, $a \subseteq (0)$, $V(a) = \mathrm{Spec}(A)$)

Uniqueness: $\overline{\{p\}} = \bigcap_{a \subseteq p} V(a) = \mathrm{Spec}(A) = \overline{\{(0)\}}$

$\Rightarrow \mathrm{Spec}(A) \subset V(a)$, $\forall a \subseteq p$

$\Rightarrow \forall a \subseteq p$, $a = (0) \Rightarrow p = (0)$ ▨

Ex2.10. $\mathbb{R}[X]$ is PID, maximal $(x-a)$, and $(0) \in \mathrm{Spec}(\mathbb{R}[X])$ $(a \in \mathbb{R})$



It's a finite-complement topology much coarser than $\mathbb{R}_E$ ▨

For deg $p = 2$ $(n \geq 2)$ Write p as $f(x) = (x-\beta)(x-\bar\beta)$ ... $(x-\beta_n)$ ... $= p_1 \cdots p_n$ reducible

Rk. It's wrong, my communicative algebra is poor: only for k algebraically closed, $\mathrm{Spec}(k[X]) = \{0\} \cup \{(x-a) \mid a \in k\}$; but $\mathbb{R}$ not, indeed, $\mathrm{Spec}(\mathbb{R}[X]) = \{0\} \cup \{(x-a)\} \cup \{(x-\beta)(x-\bar\beta) \mid \beta \in \mathbb{C}\}$

(in particular, $\mathrm{Spec}(\mathbb{C}[X]) = \{0\} \cup \{(x-a) \mid a \in \mathbb{C}\}$)

Ex Compute $\mathbb{Q}[X]$'s spectrum

$\mathbb{Q}[X] = \{0\} \cup \{(f) \mid f \in \mathbb{Q}[X]$ irreducible $\}$ ▨

$f \in \mathbb{Q}[X]$ irreducible can't be depicted same clear as $\mathrm{Spec}(\mathbb{R}[X])$

For deg $p = 2n+1$, by β root $\Leftrightarrow \bar\beta$ root

⇒ has at least 1 real zero;

Notice. Any polynomial of degree 3 or higher must reducible

It fills the fault of $x^2 + bx + c = 0$ is maybe irreducible.

$(x-\beta)(x-\bar\beta)$ is all degree 2 polynomial irreducible in $\mathbb{R}[X]$

(c) $f: \operatorname{Proj} T \to \operatorname{Proj} S$, $\quad \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$

$\sqrt{\langle \varphi(\mathfrak{q}) \rangle} \overset{g}{\leftarrow\!\shortmid} \mathfrak{q}$

[redacted]

$U = \{\mathfrak{p} \in \operatorname{Proj} T \mid \mathfrak{p} \not\supset \varphi(S_+)\} = \{\mathfrak{p} \mid \mathfrak{p} \not\ni x, x \in S_d\}$

$= \{\mathfrak{p}$ [redacted] $\mid \mathfrak{p} \not\supset \varphi(S_+), \mathfrak{p} \not\supset T_+\} \cong \{\mathfrak{p} \mid \mathfrak{p} \not\supset T_d\} = \{\mathfrak{p} \mid \mathfrak{p} \not\supset T_+\}$

$= \operatorname{Proj} T$ ▨

$\forall x \in \boxed{T_+}$, $\deg x \geq$

$\Rightarrow \exists$ [redacted] $n: n\deg x \geq d_0$

$\Rightarrow \boxed{x^n} \in T_{n\deg x} = \varphi(S_{n\deg x}) \subset \varphi(S_+)$

$\boxed{\mathfrak{p} \text{ prime}} \Rightarrow x \in \mathfrak{p} \Rightarrow T_+ \subset \mathfrak{p} \subset \varphi(S_+) \Rightarrow U = \operatorname{Proj} T$

[redacted] $\varphi(\mathfrak{q})$ not even an ideal, we let $\langle \varphi(\mathfrak{q}) \rangle$ to be the ideal it generates and $\sqrt{\langle \varphi(\mathfrak{q}) \rangle}$ for cut to small $d \leq d_0 - 1$'s effection.

We verify ① f is injective. ② g is well-defined, i.e. $\underset{\text{(Trivial)}}{\sqrt{\langle \varphi(\mathfrak{q}) \rangle} \in \operatorname{Proj} T}$; $\varphi^{-1}\left(\sqrt{\langle \varphi(\mathfrak{q}) \rangle}\right) = \mathfrak{q}$

Pf. ① $\varphi^{-1}(\mathfrak{p}) = (0)$

$\forall x \in \mathfrak{p}$, $x^{d_0} \in \mathfrak{p} \Rightarrow$ [redacted] $\varphi^{-1}_{d_0 \deg x}(x^{d_0}) = y$

$y \in (0) \Rightarrow$ [redacted] $\varphi(y) \in$ [redacted] (0)

i.e. $x^{d_0} \in (0) \Rightarrow x = 0$ ▨

② $\varphi^{-1}\left(\sqrt{\langle \varphi(\mathfrak{q}) \rangle}\right) \supseteq \varphi^{-1}(\langle \varphi(\mathfrak{q}) \rangle) \supset \varphi^{-1}\varphi(\mathfrak{q}) = \mathfrak{q}$

Otherhand, $\forall a \in \varphi^{-1}\sqrt{\langle \varphi(\mathfrak{q}) \rangle}$

$\varphi(a) \in \sqrt{\langle \varphi(\mathfrak{q}) \rangle}$, i.e.

$(\varphi(a))^n \in \langle \varphi(\mathfrak{q}) \rangle$, i.e. $\varphi(a^n) = \underset{\text{finite}}{\sum} a_i \varphi(q_i)$

(We assume $n \geq d_0$, otherwise we simply multiply d_0)

$\Rightarrow \varphi(a^n) \in \varphi(\mathfrak{p})$ as $\varphi(q_i) \cong q_i, n \geq d_0$

$\Rightarrow a \in \mathfrak{q} - \varphi^{-1}(\mathfrak{q})$ ▨

Consider their isomorphism sheafwise

$f^\#: \mathcal{O}_{\operatorname{Proj} S} \to f_* \mathcal{O}_{\operatorname{Proj} T}$

We consider locally $D_+(f)$ (assume $\deg f \geq d_0$ by $D_+(f) = D_+(f^n)$)

Thus $D_+(f) \cong f^{-1}(D_+(\varphi_{\deg f}(f)))$

$\Rightarrow \mathcal{O}_{\operatorname{Proj} S, g}$ [redacted] $\mathcal{O}_{\operatorname{Proj} T, f^{-1}(g) = \varphi_{\deg g}(g)}$ is isomorphism by φ itself, $\forall g$

$\Rightarrow f^\#$ also ▨

(d) $t(V) \longleftrightarrow \operatorname{Proj} S$ This is trivial, only consider their sheaves

$\left\{ Z \,\middle|\, \begin{array}{l} Z \subset V \\ \text{closed, irreducible} \end{array} \right\}$

[redacted] $\|$

$\{Z \mid Z = V(\mathfrak{p}), \mathfrak{p} \in \operatorname{Proj} S\} \xrightarrow{\psi} \{\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Proj} S\}$

[redacted]

$V(\mathfrak{p}) \longleftarrow\!\shortmid \mathfrak{p}$

$\psi^\#: \mathcal{O}_{\operatorname{Proj} S} \to \psi_* \mathcal{O}_{t(V)}$

locally, $\psi^\# \Big|_{\mathcal{O}_{\operatorname{Proj} S}(D_+(f))}: \mathcal{O}_{\operatorname{Proj} S}(D_+(f)) \to \psi(D_+(f))^* \mathcal{O}_{t(V)}$

$\mathcal{O}_{\operatorname{Spec} S_{(f)}}$ [redacted] $\to \mathcal{O}_{t(V)}(\psi(D_+(f)))$

$S_{(f)} \; = \!= \; S_{(f)}$ Globally, $\forall f$

▨

Ex2.15. (a) It suffice to prove for affine V, $A = A(V)$

$\forall P \in t(V)$, $k(P) = \dfrac{\mathcal{O}_{V,P}}{\mathfrak{m}_P} = \dfrac{A_P}{\mathfrak{m}_P} \cong k$

By (I.3.2.b), $P \in t(V)$ closed bijection to $P \in V$

Conversely, if $k(P) = k \Rightarrow \dfrac{\mathcal{O}_{t(V),P}}{\mathfrak{m}_P} = \dfrac{A_{\mathfrak{m}_P}}{\mathfrak{m}_P}$

$\Rightarrow P = \mathfrak{m}_P \in \operatorname{Spec} A$ [redacted], $\mathfrak{m}_P$ correspond to closed point ▨

(b) Come, reduce to affine.

By definition $f_P^\#: \mathcal{O}_{Y, f(P)} \to \mathcal{O}_{X,P}$ locally homomorphism, i.e. $f_P^{\#-1}(\mathfrak{m}_P) = \mathfrak{m}_{f(P)}$

thus $\dfrac{\mathcal{O}_{Y,f(P)}}{\mathfrak{m}_{f(P)}} \cong \dfrac{\mathcal{O}_{X,P}}{(f_P^{\#-1}(\mathfrak{m}_P))} \cong \dfrac{\mathcal{O}_{X,P}}{\mathfrak{m}_P}$ ▨

(c) $k \hookrightarrow k(f(P)) \hookrightarrow k(P) = k$

Ex2.12, (Continue) ② $\bigcup_i \psi_i(X_i) = \bigcup_i \left(X_i - \bigcup_j U_{ij}\right) \cup \left(\bigcup_{ij} U_{ij}\right) = \left(\bigcup_i \left(X_i - \bigcup_j U_{ij}\right)\right) \cup \bigcup_{ij} U_{ij}$

$= \left(\bigcup_i \psi(X_i - \bigcup_j U_{ij})\right) \cup \left(\bigcup_{ij} U_{ij}\right) = X$ ▨

③ $\psi_i(U_{ij}) =$ [redacted] $p(i(U_{ij})) = p(U_{ij})$

$\psi_i(U_i) \cap \psi_j(U_j) = p(i(U_i)) \cap p(j(U_j)) = p(X_i) \cap p(X_j) = p(U_{ij})$ ▨

④ $\forall x \in U_{ij}$, [redacted] $x \mapsto \varphi_{ij}(x)$

(We proved it in $\boxed{\text{Ex I.1.}}$ $\psi_i(U_{ij}) = \psi_j(U_{ji})$), $\psi_j \circ \varphi_{ij}(x)$ ▨

Ex2.13. (a) ($\Rightarrow$) $\{U_\alpha\}$ open cover, take $\{U_n\}_{n \geq 0} \subset \{U_\alpha\}$, then $U_0 \subset \bigcup_{i \in I} U_i \subset \cdots \subset \bigcup_{i \in n} U_i \subset$

$\Rightarrow \exists N: U_N = U_{N+1} = \cdots = U_n = \cdots \Rightarrow$ Claim. [redacted] $\{U_n\}_{n \in N}$ is a covering ▨

($\Leftarrow$) Any chain $U_0 \subset \cdots \subset U_n \subset \cdots$, then $\{U_n\}_{n \geq 0}$ [redacted] is a open covering

$\Rightarrow \{U_n\}_{n \in N}$ [redacted] is a finite covering $\Rightarrow U_{N+k} = U_N$ ▨

(b) $\operatorname{Spec} A$ is quasi-compact, but not Noetherian.

$\operatorname{Spec} A = \bigcup_a V(a)^c$, i.e. $\bigcap_a V(a) = \phi$

$\| \qquad \|$

$V(\sum_a a) = V((1)) \iff \sqrt{\sum_a a} = \sqrt{1} = (1) = A$

$\Rightarrow 1 \in \sqrt{\sum_a a} \Rightarrow 1 \in \sum_a a \Rightarrow 1 \in \sum_a a f_a$, a only finite, denote them a_i

$\Rightarrow A = \sqrt{\sum_{i \in n} a_i} = \bigcap_{i \in n} V(a_i) = \phi \Rightarrow \operatorname{Spec} A = \bigcup_{i \in n} (V(a_i))^c$ ▨

Not Noetherian: question comes that in the proof, 1 determine the hole A, but for general $U \subset \operatorname{Spec} A$, we can let $1 \notin I(\operatorname{Spec} A - U)$

Counterexample, $k[x_1, \cdots, x_n, \cdots]$, $V(x_1) \supset V(x_1, x_2) \supset \cdots \supset V(x_1, \cdots, x_n) \supset \cdots$ ▨

(c) Trivial by $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \mathfrak{p}_n = \mathfrak{p}_{n+1} = \cdots$

$\updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow$

$V(\mathfrak{p}_0) \supset V(\mathfrak{p}_1) \supset \cdots V(\mathfrak{p}_n) = V(\mathfrak{p}_{n+1}) = \cdots$ ▨

(d) Due to we can't connect $V(a) \overset{I}{\longleftrightarrow} a$ in Scheme, the converse not true.

The counterexample in $\boxed{\text{Ex I.1.10(e)}}$ is what we need ▨

Ex2.14. (a) Recall (Weak Nullstensatz) [redacted] $V(a) = \phi \iff a$ contains a power of S^+

This is pure algebraic assertion, thus can also apply here

$\operatorname{Proj} S = \phi \iff (0)$ contains a power of S^+

$(V((0))) \iff \forall x \in S^+, \exists d, x^d \in (0)$

$\iff \forall x \in S^+, x \in \sqrt{(0)}$ nilpotent ▨

(b) $U \subset \operatorname{Proj} T$ is trivial

$\operatorname{Proj} T - U = \{\mathfrak{p} \in \operatorname{Proj} T \mid \mathfrak{p} \not\supset T_+, \mathfrak{p} \supset \varphi(S_+)\} = V(\varphi(S_+)_d)$ closed ▨

$f: U \to \operatorname{Proj} S$ $\qquad$ On sheaves, $f^\#: \mathcal{O}_{\operatorname{Proj} S} \to f_* \mathcal{O}_U$

$\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$ (by $\mathfrak{p} \not\supset \varphi(S_+) \Rightarrow \varphi^{-1}(\mathfrak{p}) \not\supset S_+$) [redacted] $f_\mathfrak{p}^\#: S_{(\mathfrak{p})} \to T_{(\varphi(\mathfrak{p}))}$ ▨

Thus it suffice to reduce to each $U \subset U_i$ by sheaf axioms

$\Rightarrow \mathcal{O}_Y|_{U}(U) \cong f^*\mathcal{O}_X|(V)$ it's obviously isomorphic : $f^{-1}(U) \cong U$ as schemes ▨

(b) ($\Rightarrow$) Trivial by $A = A(X)$, by partition of unity ▨ (Take $f = 1$ is easier)

($\Leftarrow$) By (a) and Ex2.16(d)

To show $X \cong \operatorname{Spec} \Gamma(X, \mathcal{O}_X) = \operatorname{Spec} A$, we cover $\operatorname{Spec} A = \bigcup \operatorname{Spec} A_{f_i}$ (By f_i generate 1)

and proving $X_{f_i} \cong \operatorname{Spec} A_{f_i}$ is enough.

and by Ex2.4, It suffice to prove $A_{f_i} \cong \Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}})$ i.e. $(\Gamma(X, \mathcal{O}_X))_{f_i} \cong \Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}})$

Which done in Ex2.16(d) ▨

Ex2.18(c)&(d) φ surjective $\Leftrightarrow$ f homeomorphism and $f^\#$ surjective

($\Rightarrow$) $\operatorname{Spec} A \longleftrightarrow \operatorname{Spec} B$

$\varphi^{-1}(p) \longleftarrow p$ injective is trivial : $\varphi^{-1}(p) = 0 \Rightarrow p = (0)$

$q \longmapsto q + \ker \varphi$

Consider $\ker\varphi \to A \xrightarrow{\varphi} B \to 0 \Rightarrow B \cong \frac{A}{\ker\varphi}$

$\operatorname{Spec} B \cong \operatorname{Spec} A$ containing $(\ker \varphi)$ $= V(\ker\varphi)$

left is show f is an open map

but it's $\operatorname{Spec} B \cong V(\ker\varphi) \subset \operatorname{Spec} A$ embedding $\Rightarrow$ open ▨

$f^\# : \mathcal{O}_{\operatorname{Spec} B} \longrightarrow f^*\mathcal{O}_{\operatorname{Spec} A}$, consider stalks $f^\#_p : \mathcal{O}_{\operatorname{Spec} B, p} \to \mathcal{O}_{\operatorname{Spec} B, p}$

Thus surjective ▨

$$\begin{tikzcd} 0 \arrow[r] & A_{\varphi^{-1}(p)} \arrow[r] & B_p \\ 0 \arrow[r] \arrow[u] & A \arrow[u] \arrow[r] \arrow[ur, phantom, "2"] & B \arrow[u] \end{tikzcd}$$

($\Leftarrow$) $f^\#_p : \mathcal{O}_{\operatorname{Spec} A, \varphi^{-1}(p)} \to \mathcal{O}_{\operatorname{Spec} B, p}$

$A_{\varphi^{-1}(p)} \longrightarrow B_p$ surjective, $\forall p$

$\Rightarrow$ By commutative algebra $\Rightarrow A \xrightarrow{\varphi} B$ subjective ▨

(c) $\operatorname{Hom}_{Var}(V, W) \longleftrightarrow \operatorname{Hom}_{Sch/k}(t(V), t(W))$, assume all these are affine

Thus $\operatorname{Hom}_{Var}(V, W) \underset{\psi}{\overset{\varphi}{\longleftrightarrow}} \operatorname{Hom}_{Sch/k}(\operatorname{Spec} A(V), \operatorname{Spec} A(W))$

This need some explain: injective obviously reducing to affine by take V, W's covering; surjective we reduce to affine schemes as we can gluing sheaves over them.

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d] & W \arrow[d] \\ A(V) & A(W) \arrow[l, "\varphi^*"'] \\ \operatorname{Spec} A_V \arrow[r] & \operatorname{Spec} A_W \end{tikzcd}$$

$(f, f^\#)$

Thus φ is well-defined.

and by β we have $V \underset{\beta_V}{\cong} \operatorname{Max} A(V)$

$W \underset{\beta_W}{\cong} \operatorname{Max} A(W)$

Thus if $\varphi(f, f^\#) = 0 \Rightarrow f|_{\operatorname{Max} A(V)} = 0$

$\Rightarrow \varphi = 0$ is clearly injective;

The hardest is construct ψ: $\operatorname{Hom}_{Sch/k}(\operatorname{Spec} A_V, \operatorname{Spec} A_W) \longrightarrow \operatorname{Hom}_{Var}(V, W)$

$(f, f^\#) \longmapsto (f|_{\operatorname{Max} A_V})_*$

Verification: $\operatorname{Im}(\underbrace{f|_{\operatorname{Max} A_V}}_{g})_* \subset W$

$g : V \longrightarrow W$

$p \longmapsto f(m_p)$ corresponding the point $g(p)$

i.e. proving

$$\begin{tikzcd} \operatorname{Spec} A_V \arrow[r, "f"] & \operatorname{Spec} A_W \\ \operatorname{Max} A_V \arrow[u, phantom, "\cup"] \arrow[r, "f'"] & \operatorname{Max} A_W \arrow[u, phantom, "\cup"] \end{tikzcd}$$

But it's easily by Ex2.17(b) ▨

Ex2.16. (a) $X_f = \{x \in \operatorname{sp} X \mid f_x \notin m_x\}$

$= \{x \in \operatorname{sp} X \mid f(x) \neq 0\} = \operatorname{Supp} f$ (It correspond to maximal $f \notin p \trianglelefteq B_p$)

$U \cap X_f = \operatorname{Spec} B \cap X_f = \{p \in \operatorname{Spec} B \mid f(p) \neq 0\} = D(f)$ ▨

(b) $a|_{X_f} = 0 \Leftrightarrow V(\operatorname{Ann} a) \subset V(f) \Leftrightarrow f \in \sqrt{\operatorname{Ann} a}$ ▨ (After assume $X = \operatorname{Spec} B$)

(c) $X = \bigcup \operatorname{Spec} A_i$

$\Rightarrow \operatorname{Spec} B \cap X_f = \operatorname{Spec}(B_f)$

a zero in $\operatorname{Spec}(B_f) \Leftrightarrow f^{n}a = 1$

(If $\operatorname{Spec} B_i$ the covering $f^{n_i}a = 0$, we take $N = \sum n_i$)

$b_i \in \Gamma(X_f \cap \operatorname{Spec} A_i, \mathcal{O}_{X_f}|_{\operatorname{Spec} A_i})$

$b_i = b|_{\operatorname{Spec} A_i}$, $\operatorname{Spec} (A_i)_f$

$\Rightarrow b|_{\operatorname{Spec} A_i} = \frac{b_i}{f^{n_i}} = \frac{b_i f}{f^n}$ (by take $n = \sum n_i$)

Again, we do this in $U_{ij} = U_i \cap U_j$, consider the destruction $(b_i - b_j)$ (finite; quasi-compact)

$\Rightarrow (b_i - b_j) f^{m_{ij}} = 0$

$(b_i - b_j) f^m = 0$ (by take $m = \sum m_{ij}$)

$\Rightarrow f^n b = b_i$, $f^{m+n} b = b_i$ every where in U_i $\Rightarrow f^{m+n} b$ is what we want globally ▨

(Verification is easy: $f^{m+n} b|_{X_f} = 0$; Thus by zero extension $f^{k}(s - f^{m+n}b) = 0$, $s|_{X_f} = f^{m+n} b$)

(d) $\Gamma(X_f, \mathcal{O}_{X_f}) \longleftrightarrow A_f$

$b \longmapsto s_b$ satisfy $s_b|_{X_f} = f^{m+n} b$ by (c)

$a \longleftarrow \frac{a}{f^n}$

(The injective is by (b): $a|_{X_f} = 0 \Rightarrow f^n \frac{a}{f^n} = a = 0$ ▨)

Ex2.17(b) $\bigcup U_i \approx \bigcup f^{-1}(U_i) = f^{-1}(\bigcup U_i) = f^{-1}(Y) = X$, for their sheaves $\mathcal{O}_Y \xrightarrow{f^*} f^*\mathcal{O}_X$

$\mathcal{O}_Y(\bigcup U \cap U_i) \xrightarrow{f^*|_{U_i}} \mathcal{O}_X(\bigcup f^{-1}(U \cap U_i))$

$\Rightarrow Y \approx X$ ▨

Ex3.4. ($\Leftarrow$) Trivial, again and over again

($\Rightarrow$) $Y=\bigcup V_i=\bigcup \operatorname{Spec} B_i$ and $f^{-1}(\operatorname{Spec} B_i)=\operatorname{Spec} A_i$.

$\Rightarrow f^{-1}(V\cap V_i)=\bigcup_{\varphi\in B}\operatorname{Spec}(A_i)_{f^\#\varphi} \Rightarrow f^{-1}(V)=\bigcup\bigcup_{\varphi\in B}\operatorname{Spec}(A_i)_{f^\#\varphi}=\bigcup_{i,\varphi\in B}\operatorname{Spec}(A_i)_{f^\#\varphi}$

By A_i is finite B_i-module, we take generators $\{\varphi_{ij}\}_j$ finite for each i

$\Rightarrow f^{-1}(V)=\bigcup_{j\le N_i}^{i}\operatorname{Spec}(A_i)_{f^\#\varphi_j}$. Claim. It's affine open set. Thus we need use Ex2.17. (Affine criterion)

Pf of Claim. The question is not finite, can't generated to unit finite, but we have $\operatorname{Spec}(A_i)_{f^\#\varphi_j}\xrightarrow{f}\operatorname{Spec}(B_i)_{\varphi_j}$, $\uparrow$ $(A_i)_{f^\#\varphi_j}\xleftarrow{f^\#}(B_i)_{\varphi_j}$ $\uparrow$

$\forall i$, $\{\varphi_j\}_{j\le N_i}$ generates the unit $\Rightarrow$ pass $f^\#$, $\{f^\#\varphi_j\}_{j\le N_i}$ generates the unit, which is finite $\Rightarrow$ affine ▨

Thus $f^{-1}(V)=\operatorname{Spec} A=\bigcup_{i,j\le N_i}\operatorname{Spec}(A_i)_{f^\#\varphi_j}$, thus left is verification A is ~~finite~~ B-module.

Verification $\operatorname{Spec} A\xleftarrow{i}\operatorname{Spec}(A_i)_{f^\#\varphi_j}$ (A_i is obvious finite A-module generated by $\{f^\#\varphi_j\}_{j\le N_i}$ (useless))

$A\xrightarrow{i^\#}(A_i)_{f^\#\varphi_j}\xleftarrow{f^\#}(B_i)_{\varphi_j}$, localization map, by universal property, $A_i\xleftarrow{f^\#}B_i\leftarrow B$. We need ~~[illegible]~~ $f^\#: B\longrightarrow A$ factored through $f^\#$:

$B\longrightarrow A$, $B\to B_i\longrightarrow A_i$, r

But I think such r may not exist?

Recall (Algebra) Take $f_1,\dots,f_n\in B$ generate $1\in B$, then A_{f_i} is finite generated B_{f_i}-module $\Rightarrow A$ is finite generated B-module ▨

Ex3.5. (a) $\forall y\in Y$, $\exists V\ni N_y$ open affine, $f^{-1}(V)=\operatorname{Spec} A$, $V=\operatorname{Spec} B$, $f^{-1}(y)\in\operatorname{Spec} A$ (Rk. If a morphism is finite, we easily reduce into affine)

Thus it's a pure algebraic question: A is a finite B-module, $\varphi: B\to A$ given $q\trianglelefteq B$ prime, there ~~are at most finite~~ $p\trianglelefteq A$ prime, $\varphi^{-1}(p)=q$ ▨

(b) Again, take $X=\operatorname{Spec} A$, $Y=\operatorname{Spec} B$

$\forall V(a)\subset X$, claim: $f(V(a))=V(f(a))\cap\operatorname{Im} f$

Trivial to prove ▨

Thus suffice to prove $\operatorname{Im} f$ is closed, but it's not trivial.

Claim. $\operatorname{Im} f$ closed, i.e. f surjective

Pf. Trivial, as any prime in B $p=\varphi^{-1}(q)$, by A is a B-module ▨

The proof directly

$$\begin{array}{ccc}\operatorname{Spec} A\otimes_B k(q) & \longrightarrow & \operatorname{Spec} A\\ \tilde f\downarrow & \square & \downarrow f\\ \operatorname{Spec} k(q) & \longrightarrow & \operatorname{Spec} B\end{array}$$

By base change, $\tilde f$ also finite, but $\operatorname{Spec} k(q)=\{*\}$ $\Rightarrow\operatorname{Spec} A\otimes_B k(q)$ a finite set

The proof for algebraic. Recall (Fibre)

$\Leftarrow$ $A\otimes_B k(q)$ has finite prime ideal, by $A\otimes_B k(q)$ is a finite $k(q)$-module with the stable under base change

$$\begin{array}{ccc}A\otimes_B k(q) & \leftarrow & A\\ \uparrow & \square & \uparrow\varphi\\ k(q) & \leftarrow & B\end{array}$$

finite $\Leftarrow$ finite

i.e. finite-dim $k(q)$-vectorspace $\Rightarrow$ Artinian $\Rightarrow$ finite primes ▨

Ex3.1. ($\Leftarrow$) By definition, it's trivial

($\Rightarrow$) $Y=\bigcup V_i$, $V_i=\operatorname{Spec}(B_i)$

Consider $V=\bigcup(V\cap V_i)$. For each $V\cap V_i=\bigcup_\varphi D(\varphi)=\bigcup_\varphi\operatorname{Spec}(B_i)_\varphi$

$f^{-1}(V)=\bigcup_{i,\varphi}f^{-1}(\operatorname{Spec}(B_i)_\varphi)$, thus now consider singleton i, denote $V\cap V_i=V$ (we don't restrict any finite covering condition here)

$\Rightarrow f^{-1}(V)=\bigcup_\varphi f^{-1}(\operatorname{Spec}(B_i)_\varphi)\subset\bigcup\operatorname{Spec} A_{ij}$, A_{ij} is f-g-B-algebra ▨

(Rk. Indeed, at last we pass that: $\underbrace{A_{ij}\ \text{f-g}\ B_j}_{\Uparrow\ \text{By condition}}$, $\underbrace{B_j\ \text{f-g}\ B}_{\Uparrow}\Rightarrow A$ f-g B ▨)

$V\cap V_i=\operatorname{Spec} B\cap\operatorname{Spec} B_i$

$\bigcup_{\substack{D(f)\subset V\\ f\in B_i}}\operatorname{Spec}(B_i)_f\overset{?}{=}\bigcup_{\substack{D(f)\subset V\\ f\in B_i}}D(f)$

$\bigcup\operatorname{Spec}(B_i)_f\longrightarrow\operatorname{Spec} B$ $\Leftrightarrow B\longrightarrow(B_i)_f$ finite type by algebra

Ex3.2. (Similar with Ex3.1.)

($\Leftarrow$) Trivial

($\Rightarrow$) $Y=\bigcup V_i=\bigcup\operatorname{Spec}(B_i)$

$f^{-1}(V_i)=f^{-1}(\operatorname{Spec} B_i)$ quasi-compact

i.e. $f^{-1}(\operatorname{Spec} B_i)=\bigcup_{j\le N_i}\operatorname{Spec} A_{ij}$

then $\forall V\subset Y$, $V=\bigcup(V\cap V_i)=\operatorname{Spec} B$

and $V=\bigcup_{\varphi\in B}D(\varphi)=\bigcup_{\varphi\in B}\operatorname{Spec} B_\varphi\Rightarrow V\cap V_i=\bigcup_{\varphi\in B}D(\varphi)\cap\operatorname{Spec} B_i=\bigcup_{\varphi\in B}\operatorname{Spec}(B_i)_\varphi$

Recall. Affine $\overset{\text{Ex2.13}}{\Rightarrow}$ Quasi-compact. By Example 3.2.6.

$\Rightarrow V=\bigcup_{k\le N}\operatorname{Spec} B_{\varphi_k}\to$ it suffices to prove each $V\cap V_i=\bigcup_{k\le N_i}\operatorname{Spec}(B_i)_{\varphi_k}$

$\to f^{-1}(V\cap V_i)=\bigcup_{k\le N}f^{-1}(\operatorname{Spec}(B_i)_{\varphi_k})=\bigcup_{k\le N}\operatorname{Spec}(A_i)_{f^\#\varphi_k}$, and $\operatorname{Spec}(A_i)_{f^\#\varphi_k}$ is affine

$\Rightarrow$ Quasi-compact again, then $f^{-1}(V\cap V_i)$ is finite union of quasi-compact, thus quasi-compact ▨ (We used $f^{-1}(\operatorname{Spec} B_i)$ quasi-compact for restrict f into each $\operatorname{Spec} A_{ij}$, i.e. $f^\#\varphi_k=f_j^\#\varphi_k$)

Ex3.3. (a) Finite type $\overset{①}{\Leftrightarrow}$ locally finite type + $f^{-1}(V_i)$ covered by finite affine open sets and finite algebra

$\overset{②}{\Leftrightarrow}$ Locally finite type + $f^{-1}(V_i)$ quasi-compact

$\overset{③}{\Leftrightarrow}$ Locally finite type + Quasi-compact ▨

①③ are easy but ② need notice that ($\Leftarrow$) not obvious: we need covering the finite algebra, we do that following way:

($\overset{②}{\Leftarrow}$) $f^{-1}(V_i)=\bigcup_{j\le N}\operatorname{Spec} A_{ij}$, and $f^{-1}(V_i)=\bigcup\operatorname{Spec} A_{i\alpha}$, $A_{i\alpha}$ is finite B_i-algebra

We only take the subcover of $\{\operatorname{Spec} A_{i\alpha}\}_\alpha$ ▨ (It's trivial, I'm wrong at first)

(b) By Ex3.1, Ex3.2, Ex3.3(a) ▨

(c) We have graph

$$\begin{array}{ccccc}\operatorname{Spec} A & \longrightarrow & \operatorname{Spec} A_i & \longrightarrow & \operatorname{Spec} B\\ \uparrow & & \uparrow & & \uparrow\\ A & \longleftarrow & A_i & \xleftarrow[\text{generated}]{\text{finite}} & B\end{array}$$

It suffices to prove A is a finite generated A_i-algebra

It suffices to prove for

But A_i is finite generated B-algebra, take generators $\{p_{ij}\}$

$U\cap U_i=\operatorname{Spec} A\cap\operatorname{Spec} A_i=\bigcup_{\varphi\in A_i}\operatorname{Spec} A\cap D(\varphi)=\bigcup_{\varphi\in A_i}\operatorname{Spec} A_\varphi\overset{?}{=}\bigcup_{j\le N}\operatorname{Spec} A_{p_{ij}}$ ▨

We done the first step, as an analogue with we done in variety.
Then we use the equivalent description to prove:
Recall (Atiyah): A is finite B-algebra + Integral $\iff$ A is finite B-module
take $U = \operatorname{Spec} B$, $f^{-1}(U) = \operatorname{Spec} A$ ▨

Ex3.8. $\boxed{\text{PART I}}$ Affine case. Only to prove the universal property

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & \tilde{A} \arrow[dl, "\tilde{f}"] \\ B & \end{tikzcd}$$

We define $\exists ! \tilde{f}: \tilde{A} \to B$ by $x \in \tilde{A}$, $\sum a_i x^i = 0$
$\Rightarrow \tilde{f}: x \mapsto$ the radical of $\sum_{i \in d} f(a_i) x^i = 0$ "
due to $\tilde{Z}$ integral $\Rightarrow$ it exists and unique

Then $\operatorname{Spec} A \leftarrow \operatorname{Spec} \tilde{A}$ ▨

$\boxed{\text{PART II}}$ Paste (By Ex2.12, Thm3.3 Step3)

$$\begin{tikzcd} X & & \tilde{X} \arrow[ll, "\varphi"'] \\ \operatorname{Spec} A \arrow[u] & \operatorname{Spec} \tilde{A} \arrow[l, "\varphi_i"'] \arrow[ur] & \\ & \operatorname{Spec} B \arrow[ul] \arrow[u] & \\ & Z \arrow[uuul] \arrow[uuur] & \end{tikzcd}$$

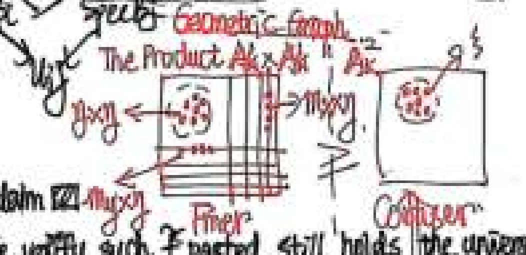
We paste $\operatorname{Spec} B$ at first, then $\operatorname{Spec} \tilde{A}$ (with f, $\tilde{f}$) ~~...~~ $\operatorname{Spec} \tilde{A}$ follows

For $\{U_i\}$ covering of Z, $U_i = \operatorname{Spec} B_i$
in $U_{ij} = \operatorname{Spec} B_i \cap \operatorname{Spec} B_j$
Claim. $\tilde{f}_i = \tilde{f}_j \Rightarrow \tilde{f}_i = \tilde{f}_j$
Pf. $\operatorname{Spec} A \Rightarrow \operatorname{Spec} A = \operatorname{Spec} B_i \times \operatorname{Spec} B_j$

$$\begin{tikzcd} & \operatorname{Spec} A \arrow[dl] \arrow[dr] & \\ \operatorname{Spec} B_i \arrow[dr] \arrow[rr, phantom, "\Diamond"] & & \operatorname{Spec} B_j \arrow[dl] \\ & U_{ij} & \end{tikzcd}$$

▨

This we can paste into Z;
$\{U_j\}$ covering of X,
$U_j = \operatorname{Spec} A_j$, then $\tilde{U}_j = \operatorname{Spec} \tilde{A}_j$
the covering of $\tilde{X}$. consider
$\varphi_j : \operatorname{Spec} \tilde{A}_j \to \operatorname{Spec} A_j$
$\varphi_j|_{U_{ij}} = \varphi_i|_{U_{ij}}$: similar with the claim ▨



$\boxed{\text{PART III}}$ Verification: $X \xleftarrow{\varphi} \tilde{X}$, f, Z, $\tilde{f}$. We verify such $\tilde{f}$ pasted still holds the universal properties. But it's trivial ▨

Ex3.9. (a) ①

$$\begin{tikzcd} & \mathbb{A}^2_k \arrow[dl] \arrow[dr] & \\ \mathbb{A}^1_k \arrow[dr] \arrow[rr, phantom, "\Diamond"] & & \mathbb{A}^1_k \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}$$

i.e.

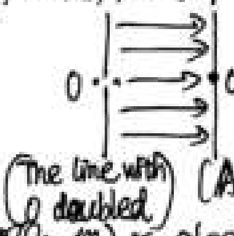
$$\begin{tikzcd} & \operatorname{Spec} k[x,y] \arrow[dl] \arrow[dr] & \\ \operatorname{Spec} k[x] \arrow[dr] \arrow[rr, phantom, "\Diamond"] & & \operatorname{Spec} k[y] \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}$$

We verify its universal properties by $k[x] \otimes_k k[y] = k[x,y]$ ▨

Namely $p = (xy-1)$ not in the product space of points ▨

② Claim. $\operatorname{sp}(\mathbb{A}^2_k) \neq \operatorname{sp}(\mathbb{A}^1_k) \times \operatorname{sp}(\mathbb{A}^1_k)$ as the $(0) \neq (0) \times (0) \Rightarrow \xi$ not a product

Ex3.5(c). Consider $X \xrightarrow{f} Y$. It's not finite but quasi-finite



$f^{-1}(Y)$ not affine $|f^{-1}(y)| \leq 2$, generically 1
Surjective is trivial
For finite type: We cover the preimage by itself if not contain 0, otherwise cover by two both affine, and ring is $k[x]$ itself ▨

Ex3.6. ① $(\mathcal{O}_\xi, m)$ is a local ring finite algebra ▨
It suffices to show $m = m_\xi \mathcal{O}_\xi = 0$
$\forall (f, U) \in m_\xi \mathcal{O}_\xi$, $f(\xi) = 0$, claim. $f \equiv 0$, then we complete the proof.
Pf. Consider locally, $\forall U \subset X$ affine, $\mathcal{O}_X(U)$ integral, i.e. $U = \operatorname{Spec} A$, A integral
$\Rightarrow \boxed{(0) \text{ is prime}, \xi \text{ correspond it, exist}}$ (That's why we need integral)
(By Ex2.9) $\Rightarrow \mathcal{O}_X(U)_{(0)} = \mathcal{O}_\xi$, but $\mathcal{O}_X(U)_{(0)} = \operatorname{Frac} \mathcal{O}_X(U) \Rightarrow$ Field ▨
(RK. The Generic point contains more than all closed points: $f(p) = 0$ for all closed p $\not\Rightarrow f \equiv 0$, but $f(\xi) = 0 \Rightarrow f \equiv 0$)
② Done in ① ▨

Ex3.7. By Hint, we first prove:
Lemma. Generic finite $\iff$ Finite field extension: $[K(X) : K(Y)] < \infty$
Pf. We reduce to affine case.
$X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "f"] & \operatorname{Spec} B \\ A \arrow[u] & B \arrow[l, "\varphi"'] \arrow[u] \end{tikzcd}$$

(By Ex3.6, we know $\mathcal{O}_\eta = K(Y)$, $\mathcal{O}_\xi = K(X)$ only depend on any of their affine open sets)

A is B-algebra, and
① A, B are integral domain (By integral schemes).
② $f^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$ (By dominate).
③ A is a finite B-algebra (By finite type and ②)
Then by these basic observations, we can deal scheme as variety
By induction, assume $A = B[x]$ for some $x \in A$
then ① When x is algebraic over $K(Y)$
We choose the minimal polynomial of x, $G(t) \in K(Y)[t]$, $G(t) = \sum_{i \in d} a_i t^i$, $a_d = 1$, $a_i \in K(Y)$
after multiply a denominator, we have $G(t) = \sum_{i \in d} b_i t^i$, $b_i \in B$
thus $\forall y \in Y$, $f^{-1}(y)$ has at most d elements, i.e. $|f^{-1}(y)| \leq d$ (By $K(Y) = \operatorname{Frac}(B)$)
② When x is transcendental of $K(Y)$
$X = Y \times \operatorname{Spec} A$
Now Generic finite $\iff$ all x_i generators of A over B is algebraic and by ① $|f^{-1}(y)| \leq [K(X) : K(Y)]$, in particular $\Leftarrow$ finite extension ▨

3.2.11 (a) $Y \times_X X' \longrightarrow Y$ For affine case, recall, f is closed immersion $\iff \varphi$ is surjective

$$\begin{tikzcd} Y\times_X X' \arrow[r] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "e"] & X \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec}(B\otimes_A A') \arrow[r] \arrow[d, "f'"] & \operatorname{Spec} B \arrow[d, "f"] & B \arrow[l, mapsto] \\ \operatorname{Spec} A' \arrow[r, "e"] & \operatorname{Spec} A & A \arrow[l, mapsto] \arrow[u, "\varphi"] \end{tikzcd}$$

Now for general case! It's the standard technique of pasting for a pullback graph:

thus $\varphi : A \to B$ surjective $\Longrightarrow \varphi' : A' \to A'\otimes_A B$ surjective } implies what we need

X' or Y first, X then, at last $Y\times_X X'$ ▨

Recall. Tensor is right exact: (Tor)

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\ A'=A'\otimes_A A \arrow[r] & A'\otimes_A B \arrow[r] & 0 \end{tikzcd}$$

(b) We had shown $\operatorname{Spec}\frac{A}{\mathfrak a}$ is closed subscheme of $\operatorname{Spec} A$. (Example 3.2.6)
now we show: every closed subscheme has this form

(Rk (Sheaves of ideals, Ex 1.21(a)) $0 \to \mathcal I_Y \to \mathcal O_X \to \mathcal O_Y \to 0$
Thus we should built up a correspondence between $\mathcal O_Y$-module and $\frac{A}{\mathfrak a}$-module
then $\mathcal I_Y = \mathfrak a A$, $\mathcal O_Y = \frac{A}{\mathfrak a}$, and this is Lemma 5.5)

We proceed this proof by hint

④ is simplest: we use Affine critison (Ex 2.17(b)) $\Rightarrow Y = \operatorname{Spec} B$

and $Y \overset{f}{\hookrightarrow} X$ $\Rightarrow B = \frac{A}{\ker f}$ by Ex 2.18(d) $\Longrightarrow$ let $\mathfrak a = \ker f$ and we complete the proof. thus it suffice to prove ①②③ to derive the restriction of Affine critition ▨

$$\begin{tikzcd} Y \arrow[r, hook, "f"] & X \\ B \arrow[u] & A \arrow[l, "f"] \arrow[u] & \ker f \arrow[l] \end{tikzcd}$$

① $Y = \bigcup_{f\in A} (Y \cap D(f))$

(Rk. closed $\Rightarrow$ compact need the hole space compact (In AG, called quasi-compact)
By $X = \operatorname{Spec} A$ affine $\Rightarrow$ quasi-compact $\Rightarrow Y$ is compact as $\operatorname{sp}(Y)$ closed)

$\Rightarrow Y = \bigcup_{j\in N} (Y \cap D(f_i))$ ▨

② $\operatorname{sp}(Y)$ closed in $\operatorname{sp}(\operatorname{Spec} A) \to \operatorname{sp}(Y)^c$ open i.e. $(\operatorname{sp}(Y))^c = D(\mathfrak a)$, and by $X = \operatorname{Spec} A$ affine, again $\Rightarrow \mathfrak a$ can be finitely generated $\operatorname{sp}(Y)^c = \bigcup_{j\in M} D(g_j)$

$\Rightarrow \{D(f_i)\}_{i\in N} \cup \{D(g_j)\}_{j\in M}$ is what we want covering of X ▨

③ Trivial. ▨

(c) In Example 3.2.6, we had done for affine case $\boxed{\operatorname{Spec}\frac{A}{\sqrt{\mathfrak a}}} \to \boxed{\operatorname{Spec} A}$ given by $\frac{A}{\sqrt{\mathfrak a}} \leftarrow A$, $\operatorname{Spec}\frac{A}{\mathfrak a}$, $\frac{A}{\mathfrak a}$

For general case, proceeding pasting as usual:
First $\operatorname{sp}(Y) = \operatorname{sp}(Y')$, then $X = \operatorname{Spec} A$ doesn't need glue ▨

(d) (Rk $\boxed{\text{Scheme}}$-theoretic ⋯ means outline such a structure of such a object's property; set-theoretic means forget all structure)

Ex 2.7(b). $\operatorname{Spec} k(s) \times_{\operatorname{Spec} k} \operatorname{Spec} k(t) = \operatorname{Spec}(\underline{k(s)\otimes_k k(t)})$

$$\begin{tikzcd} & \operatorname{Spec} k(s)\times_{\operatorname{Spec} k}\operatorname{Spec} k(t) \arrow[dl] \arrow[dr] & \\ \operatorname{Spec} k(s) \arrow[dr] \arrow[rr, phantom, "\Diamond"] & & \operatorname{Spec} k(t) \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}$$

$= k[s,t]_{(s,t)}$ localisation, thus its prime ideals are not trivial to be $\operatorname{Spec} k[s,t]$ generated by not inverted elements is a subspace of $\mathbb A^2_k$ ▨

(Rk. The product is the pullback, may have complex structure; it's due to the complex of tensor $X\times_Y \operatorname{Spec} k(y)$)

Ex 3.10. (a)

$$\begin{tikzcd} X\times_Y \operatorname{Spec} k(y) \arrow[r] \arrow[d] & \operatorname{Spec} k(y) \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Let $Y = \bigcup \operatorname{Spec} B_j$
$y \in \operatorname{Spec} B_j$
$\Rightarrow k(y) = \frac{B_y}{\mathfrak m B_y} = \operatorname{Frac}(B) = (\frac{B}{\mathfrak m})_y$
$\Rightarrow X\times_Y \operatorname{Spec} k(y) \cong X \otimes_{\operatorname{Spec} B} \operatorname{Spec}\operatorname{Frac}(B)$

and by local, it equals to

$f^{-1}(\operatorname{Spec} B) \times_{\operatorname{Spec} B} \operatorname{Spec}(\operatorname{Frac}(B))$

$= \bigcup_i \operatorname{Spec} A_i \times_{\operatorname{Spec} B} \operatorname{Spec}\operatorname{Frac}(B)$

$= \bigcup_i \operatorname{Spec}(A_i \otimes_B \operatorname{Frac} B) = \bigcup_i (f|_{\operatorname{Spec} A_i})^{-1}(y) = f^{-1}(y)$

i.e. the topology of $f^{-1}(y)$ induced from X
$\iff$ the topology of $f^{-1}(y)$ induced from each $\operatorname{Spec} A_i$, $X = \bigcup \operatorname{Spec} A_i$
$\iff$ the topology of the affine scheme (fibre) $\operatorname{Spec} A_i \times_{\operatorname{Spec} B} \operatorname{Spec} k(y)$

Left is proving $\operatorname{Spec}(A_i \otimes_B k(y)) = (f|_{\operatorname{Spec} A_i})^{-1}(y)$

Thus we reduce to affine case.
Now proving the affine case

Pf.

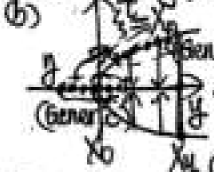
$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "f"] & \operatorname{Spec} B \\ \operatorname{Spec}(A\otimes_B k(y)) \arrow[r] \arrow[u] & \operatorname{Spec} k(y) \arrow[u] \end{tikzcd}$$

We assume $y = \mathfrak p$, $f^{-1}(\mathfrak p)$ not $\varnothing$
i.e. $\mathfrak p = \varphi^{-1}(\mathfrak q)$ for some $\mathfrak q$
$\Rightarrow f^{-1}(\mathfrak p) = \{\mathfrak q \mid \mathfrak p = \varphi^{-1}(\mathfrak q)\}$

$\operatorname{Spec}(A\otimes_B k(y)) \cong \operatorname{Spec}(A\otimes_B (\frac{B}{\mathfrak p})_{\mathfrak p}) \cong \operatorname{Spec}(A_{\mathfrak p}\otimes_B \frac{B}{\mathfrak p}) \cong \operatorname{Spec}(\frac{A_{\mathfrak p}}{\mathfrak p A_{\mathfrak p}}) = f^{-1}(\mathfrak p)$ ▨

[Recall (Algebraic) $A\otimes B_{\mathfrak p} = (A\otimes B)_{\mathfrak p} = A_{\mathfrak p}\otimes B$]

(b) Similar with Example 3.3



① $X_y = \operatorname{Spec}\frac{k[s,t]}{(s-a, a-t^2)}$, $k(y) = k$ by construction $= \{(a, t_1), (a, t_2)\}$ t_1, t_2 is the two radical of $t^2 = a$. ▨

② $X_\eta = \operatorname{Spec}(\frac{k[s,t]}{(s-t^2)}) \cong \operatorname{Spec}\frac{k[t]}{t^2} \Rightarrow$ non-reduced ▨

③ $X_\eta = \operatorname{Spec}(\frac{k[s,t]}{(s-t^2)} \otimes_{k[s]} k(\eta)) = \operatorname{Spec}(\frac{k[s,t]}{(s-t^2)} \otimes_{k[s]} K(\operatorname{Spec} B)) = \operatorname{Spec}(\frac{k[s,t]}{(s-t^2)} \ldots)$
$= \operatorname{Spec}(k[t^2,t] \otimes_{k[s]} k(s)) = \operatorname{Spec}(k[t]\otimes_{k[t^2]} k(t^2)) = \operatorname{Spec}(k(t)) \quad \operatorname{Frac}(k[t])$
$k(t^2)$ a field $\Rightarrow$ singleton point, then $k(X_\eta) = K(\operatorname{Spec} k(t^2)) = k(t^2)$, & obvious a deg 2 extension ▨

Ex3.13 (f). $\mathrm{Spec}B \xrightarrow[g|_{\mathrm{Spec}B}]{} Z$ then $g(\mathrm{Spec}B) = \bigcup_{\text{finite}} \mathrm{Spec}C_i$ may not finite

and $f^{-1}(\mathrm{Spec}B) = \bigcup_{j\in M} \mathrm{Spec}A_j$ by quasi-compact

by $g\circ f$ finite type with $\mathrm{Spec}C_i \xleftarrow{g\circ f} \mathrm{Spec}A_j \Rightarrow A_j$ is finite generated C_i-algebra

then we have factor through B

$$\begin{array}{ccc} \mathrm{Spec}C_i & \xleftarrow{g\circ f} & \mathrm{Spec}A_j \\ \uparrow & \circlearrowleft & \uparrow \\ C_i & \longrightarrow & A_j \end{array}$$

$$\begin{array}{ccc} A_j & \longleftarrow & C_i \\ & \nwarrow \quad \swarrow & \\ & B & \end{array} \Longrightarrow A_j \text{ is finite generated } B\text{-algebra} \ \blacksquare$$

(g) Take $U_0 = U_1 \subset \cdots \subset U_n \subset \cdots$ in X, $U_i = \mathrm{Spec}A_i$

Then $\{U_i\}_{i\geq 0}$ is a covering of any subset of X

Consider $\forall V\subset Y$ affine open, $V = \mathrm{Spec}B$

$f^{-1}(V) = \bigcup_{i\geq 0} (U_i \cap f^{-1}(V))$, by $f^{-1}(V)$ is open, $U_i\cap f^{-1}(V)$ open $\Rightarrow$ we can apply quasi-compact

$\Rightarrow \exists N: U_N \cap f^{-1}(V) = U_{N+1}\cap f^{-1}(V) = \cdots = \cdots$

By Y Noetherian, $V_0 \subset V_1 \subset \cdots \subset V_n = V_{n+1} = \cdots = Y$, $V_i = \mathrm{Spec}B_i$

$\forall V \subset Y$, $V = \bigcup_{i\leq n} (V\cap V_i)$ can be covered by affine finitely

$\Rightarrow U_N \cap f^{-1}(Y) = U_N \cap (\bigcup V_i) = \bigcup_{i\leq n} (U_N \cap V_i) = \bigcup_{i\leq n} (U_{N+1}\cap V_i) = U_{N+1}\cap f^{-1}(Y)$

$f^{-1}(Y) = X \Rightarrow U_N = U_{N+1} = \cdots \Rightarrow X$ Noetherian $\blacksquare$

Ex3.14. Let $X = \mathrm{Spec}A$

$$\left(\begin{array}{l} \text{If } X = \bigcup \mathrm{Spec}A_i \\ \overline{\{x \mid x \text{ closed}, x\in \mathrm{Spec}A_i\}} = \mathrm{Spec}A_i \\ \qquad\qquad \cap \\ \overline{\{x \mid x \text{ closed}, x\in X\}} = X \end{array}\right)$$

$$\begin{array}{ccc} \mathrm{Spec}A & \longrightarrow & \mathrm{Spec}\,k \\ \updownarrow & & \updownarrow \\ A & \longleftarrow & k \end{array}$$

$A = \frac{k[x_1\cdots x_n]}{\mathfrak{a}} \to k[x_1\cdots x_n]$

Then closed point in $\mathrm{Spec}\frac{k[x_1\cdots x_n]}{\mathfrak{a}} \subset \mathrm{Spec}\,k[x_1\cdots x_n] = \mathbb{A}^n_k$

it suffice to prove for $A = k[x_1\cdots x_n]$, then trivial $\blacksquare$

Counterexample $X = \mathrm{Spec}(R) = \{\xi, m\}$, the closed point is singleton $\blacksquare$

Ex3.15. (a) (iii) $\Rightarrow$ (ii) & (i) is trivial

Consider the graph

$$\begin{array}{ccccc} X\times_k \bar{k} & \longrightarrow & X\times_k k_s & \longrightarrow & X \\ \downarrow & \square & \downarrow & \square & \downarrow \\ \bar{k} & \longrightarrow & k_s & \longrightarrow & k \end{array}$$

If $X\times_k \bar{k}$ irreducible $\Leftrightarrow X\times_k k_s$ geometric irreducible and X geometric irreducible

Left is showing (i)/(ii) $\Rightarrow$ (iii), but also by base change $\blacksquare$

base change

finite type

Thus (i) $\Leftrightarrow$ (iii) ...

(b) Still $\mathbb{F} \to k_0 \to k$, thus it's same $\blacksquare$

Ex3.11 (d) By the in particular part, we claim, $Y = \overline{f(Z_{red})}$

Then the in particular part follows, only verify Y's universal property

Pf of Claim.

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \exists!\, f_0 \downarrow & \exists! f' \searrow \ \nearrow & \uparrow i' \\ Y & \underset{\exists!\, \tilde{f}_0}{\dashrightarrow} & Y' \end{array}$$

i.e. Y has initial property

Then obvious: $Z_{red} \xrightarrow{\exists!} Z'$ ($\mathrm{sp}(Z_{red}) = \mathrm{sp}(Z')$)

$f(Z_{red}) \xrightarrow{\exists!} f(Z')$ by Ex2.3(c)

to let Y be closed, we take closure $\blacksquare$

Ex3.12 (a) The difference with Ex2.14(c) is only have φ surjective not isomorphism

(but this not effects proof of $U = \mathrm{Proj}\,T$ $\blacksquare$)

and the conclusion is $f: \mathrm{Proj}\,T \to \mathrm{Proj}\,S$ closed immersion

notice the proof of injective is trivial, thus we even can only use the conditio[n]

"$\exists d_0: d > d_0$, φ_d is surjective" $\blacksquare$

(b) Similiar with affine case, projective case also have $\{$all closed subscheme$\} \overset{1:1}{\longleftrightarrow} \{$all homogenous ideal$\}$

① Now we only show "$\Leftarrow$" part, "$\Rightarrow$" part left to ②

Pf. $S \to \frac{S}{I}$ surjective, follow (a) $\blacksquare$ (You can see from Ex3.11(b) that affine had been so complex)

② The small $d < d_0$ doesn't matter the structure of scheme (i.e. twist-stable)

We prove $\mathrm{Proj}\frac{S}{I} \cong \mathrm{Proj}\frac{S}{I(d_0)}$

Pf. By Ex2.14(c) $\blacksquare$

Ex3.13. (a) $\forall V = \mathrm{Spec}B \subset X$, $i: X' \to X$ closed immersion

$\Rightarrow i^{-1}(V) = V\cap X'$. (Recall. $X'\cap V$ closed in V, V affine $\Rightarrow$ quasi-comp[act]

$\Rightarrow X'\cap V$ quasi-compact in V

$\Rightarrow i^{-1}(V) = \bigcup_{i\in N} \mathrm{Spec}A_i$ finite by Recall

left is showing A_i is finite generated B-algebra

But by Ex3.11(b)* $\Rightarrow A_i = \frac{B}{\mathfrak{a}_i}$ $\blacksquare$

(b) $\forall V = \mathrm{Spec}B \subset X$, $i: U \to X$ open immersion (we can assume $U\subset X$)

$\Rightarrow i^{-1}(V) = V\cap U$, now we can't prove as (a) it has finite covering, but it's give[n]

$\Rightarrow i^{-1}(V) = V\cap U = \bigcup_{i\in N} \mathrm{Spec}A_i$, left is showing A_i is finite generated B-algebra

On the other hand, $V\cap U = \mathrm{Spec}B \cap \bigcup_{f_i\in B} D(f_i) = \bigcup_{i\in N} \mathrm{Spec}B_{f_i}$

$B_{f_i} = A_i$ obvious finite generated B-algebra $\blacksquare$

(c) $X \xrightarrow{f} Y \xrightarrow{g} Z$, $\forall V = \mathrm{Spec}\,C \subset Z$, $g^{-1}(V) = \bigcup_{i\in N} \mathrm{Spec}B_i$

and $f^{-1}(\mathrm{Spec}B_i) = \bigcup_{j\in M} \mathrm{Spec}A_{ij} \Rightarrow (g\circ f)^{-1}(V) = \bigcup_{i\in N, j\in M} \mathrm{Spec}A_{ij}$

with A_{ij} is finite generated B_i-algebra, B_i is finite generate C-algebra $\Big\} \Rightarrow A_{ij}$ is finite generated C-algebra $\blacksquare$

$=$ [redacted] or $(X-U_2)\cap U_1$ or $(X-U_1)\cap U_2$ or $(X-U_1)\coprod(X-U_2)$ ☒

$U_1\cap U_2$ still open, locally closed

(b) By Ex3.17(a), it suffices to prove the second assertion

$C=\coprod_{i\in N}(U_i\cap F_i)\supset\coprod_{i\in N}(U_i\cap \mathring{F}_i)$ open and nonempty ☒ ($\mathring{F}_i\neq\emptyset$ by Ex3.17(b))

(c) The two assertions are same, we prove the closed one:

(⟹) Trivial (⟸) $C=\coprod_{i\in N}(U_i\cap F_i)$, $\forall x_0\in C$, $x_1\in\overline{\{x_0\}}$ the specialisation of x_0

Now $x_1\in\overline{\{x_0\}}\subset C$, i.e. $\forall x_0\in$ [redacted] $\overline{\{x_0\}}\subset C\Rightarrow\overline{U_i\cap F_i}\subset C$

$\Rightarrow C=\coprod_{i\in N}U_i\cap F_i\supset\coprod_{i\in N}\overline{U_i\cap F_i}=\overline{\coprod_{i\in N}U_i\cap F_i}=\overline{C}\Rightarrow C=\overline{C}$ ☒

(d) $f^{-1}(C)=\coprod_{i\in N}f^{-1}(U_i)\cap f^{-1}(F_i)$ Trivial ☒

Ex3.19. (a) ① We can do restriction: for $C\subset X$ constructible

(Chevalley)

$X\xrightarrow{f}Y$, $C\xrightarrow{f|_C}f|_C(C)$ Thus reducing to proving $f(X)$ constructible in $\overline{f(X)}$ and f dominate

② Then we can reduce to affine case: $X\xrightarrow{f}\overline{f(X)}$, $\operatorname{Spec}A\to f(\operatorname{Spec}A)$

[redacted]

$\exists U\subset f(X)$, then $f(X)=U\coprod U^c$

By induction of $\dim X$ $=U\coprod f(f^{-1}(U^c))$

and $f^{-1}(U^c)$ closed in $X\Rightarrow$ [redacted] $\dim f^{-1}(U^c)\le\dim X-1$

$\Rightarrow f^{-1}(U^c)$ is constructible, by induction hypothesis, $ff^{-1}(U^c)$ also

③ ⟹ left is showing $U\subset f(X)$ constructible ⟹ we assume $Y=\overline{f(X)}$ affine

Then $X=f^{-1}\overline{f(X)}=\bigcup_{i\in N}\operatorname{Spec}A_i$ by finite type ⟹ we assume X affine

Left is showing it suffice to deal with integral, ⟺ $Y=\operatorname{Spec}A$ with A, B integral

We only replace A & B by their quotient of nilpotent radical, $X=\operatorname{Spec}B$

$A'=\frac{A}{\sqrt{(0)}}$, $B'=\frac{B}{\sqrt{(0)}}$, due to the assertion is about topological, this affects nothing ☒

(Now we almost reduce to variety) (Separable not)

(b) We first admit this algebraic consequence (Other restrictions are natural by reduction)

Then $A\xrightarrow{\varphi}B$ by f dominate ⟺ φ injective, and by finite type ⟹ B finite generated

thus we apply the algebraic consequence when $b=1\in B$, A-algebra

$\operatorname{Spec}A\xleftarrow{f}\operatorname{Spec}B$ $\exists a\in A$: $A\xrightarrow{\psi}K$, $\psi(a)\neq0$ Thus [redacted]

$\Rightarrow f(\operatorname{Spec}B)\subset\operatorname{Spec}A$ contained ☒ $\exists!$: $\psi(b)\neq0$, $A_a\longrightarrow B_b=B$ the localisation

Ex3.21. Zariski space $\overset{def}{\iff}$ Noetherian & irreducible subset $\exists!\ \xi$ generic

(Pure point-set topology) (Recall any scheme $\exists!\ \xi$ generic in component, this restricts)

(a) $sp(X)$ is Noetherian by definition; We apply Noetherian induction (Ex3.16) is stronger

Thus it suffices to prove $\forall Y$ closed in $sp(X)$ any proper closed subsets $\exists!$ generic point (with all irreducible)

Thus every irreducible component Y of $sp(X)$

If $Y'\subset Y$, Y' irreducible closed with $\exists!$ generic point ⟹ Y $\exists!$ generic point as Y is irreducible ⟹ ☒

Noetherian induction

(b) Minimal ⟹ irreducible ⟹ $\exists!\ \eta\in M$ generic and $\overline{\{\eta\}}=M=\overline{M}$

$\forall x\in M$, $\overline{\{x\}}=M\Rightarrow\eta=x$ is the only element in M ☒

(c) Done ☒ (d) $\forall U\subset X$ open, $U\subset\overline{\{\xi\}}=X\Rightarrow\overline{U}\subset\overline{\{\xi\}}=X$

If it's strict ⟹ $V=X-\overline{U}\supset X-\overline{\{\xi\}}$ open but without generic point, impossible

⟹ $\overline{U}=X\Rightarrow\{\xi\}\subset U$ ☒

(e) ① If x_0 not closed ⟺ it's not the minimal nonempty closed set, i.e. [redacted] $\exists x'$: $\overline{\{x'\}}\subset\overline{\{x_0\}}$, i.e. $x'\in\overline{\{x_0\}}$, i.e. $x'\rightsquigarrow x_0$, contradict to its minimal partial order ☒

② If x_0 not generic $\overline{\{x_0\}}\neq X=\overline{\{\xi\}}\Rightarrow x_0\rightsquigarrow\xi$, contradict to maximality ☒

③ $\forall x_0\in E$ closed ⟹ $\forall x_1\in\overline{\{x_0\}}\subset E$ ☒ (By $\overline{\{x_0\}}\subset E\Rightarrow\overline{\{x_0\}}\subset E$)

④ $\forall x_0\in U$ open ⟹ $\forall x_1\in\overline{\{x_0\}}\subset U$ ☒ (By $\overline{\{x_0\}}\subset\overline{\{\xi\}}\subset U$ (by (d)))

(f) It suffices to prove the second assertion

(If we admit it, then X noetherian ⟹ [redacted] $t(X)\cong X$ Noetherian and by (a) ⟹ $t(X)$ Zariski space ☒)

(⟸) Done (⟹) [redacted] $\mathcal{F}(t(X))\longleftrightarrow\mathcal{F}(X)$ Thus it suffices to show injectivity

$t(Y)\longleftrightarrow Y$, closed, closed. $t(Y_1)=t(Y_2)$, we prove $Y_1=Y_2$. Assume $Y_1\subsetneq Y_2$, claim: $\exists Z$: $Y_1\subset Z\subset Y_2$

Then $\xi_1\in Y_1$, $\xi_2\in Y_2$, $\overline{\{\xi_1\}}=Y_1\subsetneq\overline{\{\xi_2\}}$ ⟹ by T_0 ⟹ $\exists Z=\overline{\{\xi\}}$ ☒

⟹ $t(Y_1)\neq t(Y_2)$ ☒

Ex3.18. (a) (⟸) $=\coprod_{i\in N}(U_i\cap F_i)$

$=X-\bigcap_{i\in N}(X-(U_i\cap F_i))$

$=X-\bigcap_{i\in N}((X-U_i)\cup(X-F_i))$

$=X-\bigcap_{i\in N}((X-U_i)\cup(V_i))=X-\bigcap(X-(U_i\cap(X-V_i)))\in\mathcal{F}$ ☒ intersection $\in\mathcal{F}$

(⟹) $C\in\mathcal{F}$, due to operation (a)(c)(d) is finite, thus we can induction, it suffices to deal C generated by two U_1,U_2

$C=U_1\cap U_2$ or $U_1\cap U_2^c$ or $U_1^c\cap U_2$ or $U_1^c\cap U_2^c$ or $(U_1\cap U_2)^c$ [redacted]

(d) Then we prove the algebraic convergence

By induction, let $B = A[t]$

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\psi"'] & K \\ A[t] \arrow[ur, "\tilde\varphi"'] \end{tikzcd}$$

When ① t transcendental over A: $\mathrm{Spec}\, A[t] \cong \mathrm{Spec}\, A \times \mathbb{P}^1_k$ Nothing to do ▨

② t algebraic over A, trivial: $\sum a_i t^i = 0$

we define $\tilde\varphi(\sum b_j t^j) = \sum \varphi(b_j)\, \tilde t^j$

where $\tilde t$ is the radical of $\sum \varphi(a_i) t^i = 0$ ▨

(c) By Noetherian induction, it suffices to prove if $\forall$ [redacted] $\subset Y = f(\mathrm{Spec}\, B)$ satisfy constructible, then Y, but F is closed $\Rightarrow$ constructible ▨

Ex3.20 & Ex3.21 (we'll verify (a)(d)(e) for $\mathrm{Spec}\, R[x]$, R a DVR ~~fails~~ false)

(a) $\dim X = n$, i.e. $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X$

For $X = \mathrm{Spec}\, A$: $V(a_0) \subsetneq V(a_1) \subsetneq \cdots \subsetneq V(a_n) = X$

(Otherwise, due to dim is local, take $U \in N_p$)

$a_0 \supsetneq a_1 \supsetneq \cdots \supsetneq a_n = \emptyset$ i.e. $\dim A \Rightarrow \dim X = n = \dim A$

If $X = \mathrm{Spec}\, A$ integral $\Leftrightarrow$ A integral $\Rightarrow \mathcal{O}_p = A_p$ thus $\dim \mathcal{O}_p = \dim A_p = \dim A = n$ ▨

Rk. If not integral, we can't localisation; and $\dim A = \dim$ ... $\boxed{\text{height } p}$ $\neq \dim A_p$, $\neq \dim p^{-1}(m)$

(e) Done in (a)

Now consider why (3.21) not true in (a) & (e)

First, we depict $\mathrm{Spec}\, R[x] =$ ... $\downarrow$ $\mathrm{Spec}\, R$, $= m$, $\xi = (0)$

$\dim \mathrm{Spec}\, R[x] \neq \dim p^{-1}(\xi)$, thus (e) wrong, and $\dim \mathcal{O}_p < \dim \mathrm{Spec}\, R[x]$, thus (a) wrong ▨

(b) By (e), again, and Ex3.6

$\Rightarrow K(X) = K(\mathrm{Spec}\, A) = \mathrm{Frac}(A)$

$\Rightarrow \mathrm{trdeg}\, K(X)/k = \mathrm{trdeg}\, \mathrm{Frac}(A)/k = \mathrm{trdeg}\, \mathrm{Frac}(A_m)/k = \dim A_m = \dim \mathcal{O}_m = \dim X$ ▨

(d) Again and again, $X = \mathrm{Spec}\, A$ affine integral finite type

$\Rightarrow \mathrm{codim}(Y, X) = \dim X - \dim Y$ ▨

Now consider why (3.21) not true in (d) [redacted]: $\dim \mathrm{Spec}\, R[x] - \dim \mathrm{Spec}\, R$

(c) By Ex3.11 (Of course, first use (e) again) $= \dim \mathrm{Spec}\, R[x] = 2$

$\mathrm{Spec}\frac{A}{a} \subset \mathrm{Spec}\, A \Rightarrow \mathrm{codim}(\mathrm{Spec}\frac{A}{a}, \mathrm{Spec}\, A)$ [redacted] $= n$

But $Z \subset \mathrm{Spec}\, R[x]$ closed $\Rightarrow Z \subset p^{-1}(m)$, $\dim p^{-1}(m) = 1 < \dim \mathrm{Spec}\, R[x] = 2$ contradiction ▨

$Y \subset X$

$\mathrm{Spec}\frac{A}{a} = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n = \mathrm{Spec}\, A$

$V(a_0) \subsetneq V(a_1) \subsetneq \cdots \subsetneq V(a_n)$

$a_0 \supsetneq a_1 \supsetneq \cdots \supsetneq a_n$ contained in a $\Leftrightarrow n = \mathrm{height}\, a = \inf_{p \supset a} \mathrm{height}\, p = \inf(\dim \mathcal{O}_{p,X})$ ▨

*Ex3.22 (The last one, on fibres' dimension)

(a) $X \xrightarrow{f} Y$. f dominate $\Leftrightarrow f^\#$ injective

$\Leftrightarrow f^\#_y$ injective, $\forall y \in Y$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \mathcal{O}_X \arrow[d] & \mathcal{O}_Y \arrow[l, "f^\#"'] \arrow[d] \\ \mathcal{O}_{X,p} & \mathcal{O}_{Y,f(p)} \arrow[l, "f^\#_p"'] \end{tikzcd}$$

$\Rightarrow \mathcal{O}_{Y,f(p)} \subset \mathcal{O}_{X,p} \Rightarrow \dim \mathcal{O}_{Y,f(p)} \le \dim \mathcal{O}_{X,p}$

Then by Ex3.20(d) ▨

(b) We hope find Y' is a component only contain y, that's why we take $Y' = \overline{\{y\}}$ (by Hint)

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\{y\}} \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

i.e. $f^{-1}(\overline{\{y\}}) = X_y$ (as component)

$\Rightarrow \mathrm{codim}(X_y, X) \le \mathrm{codim}(\overline{\{y\}}, Y)$

$\Leftrightarrow \dim X - \dim X_y \le \dim Y - \dim \overline{\{y\}} = \dim Y$

$\Leftrightarrow e \le \dim X_y$ ▨

(c) Notice (b) & (a) told us, when the fibre not stable, it must in almost all other points regular; Furthermore, when smooth, the fibre when jump up, it can jump up to finite points but at last dimension also up (Zariski's main thm)

Pf. It suffice to do with affine (Take $U \subset X$ is local. Then $f(U)$ is always contained in $V \subset Y$; $f(U) \subset V \subset Y$: we only collapse U)

$$\begin{tikzcd} \mathrm{Spec}\, A \arrow[r] & \mathrm{Spec}\, B \\ A \arrow[u] & B \arrow[l] \arrow[u] \end{tikzcd}$$

By finite type $\Rightarrow A = B[t_1 \cdots t_e]$ (Differ to the Hint, we needn't t_i to be transc...)

By induction, it suffices to consider $A = B[t_1]$

by Ex3.7. It suffices to prove generically finite $\Rightarrow$ [redacted] $\exists U$: $f^{-1}(U) \to U$ is finite

Consider { ① When t algebraic over B. Nothing to prove, only by definition

② When t transcendental over B $\Rightarrow \mathrm{Spec}\, B = \mathrm{Spec}\, A \times k$ also generically fin... ▨

Now we have $\exists U$: $f^{-1}(U) \to U$ finite

Then we claim such $f^{-1}(U)$ is our desired construction

$\forall y \in U$, $\dim f(U)_y = \dim f^{-1}(U) = \dim B - \dim A = \dim X - \dim Y = e$ ▨

(c)' (Now we use the Hint's way) $A = \frac{B[t_1 \cdots t_e]}{a}$

then we have graph: $\mathrm{Spec}\, A \to \mathrm{Spec}\, B[t_1 \cdots t_e] \to \mathrm{Spec}\, B$, and $\mathrm{Spec}\, B[t_1 \cdots t_e] = (\mathrm{Spec}\, B)^e$

$A \leftarrow B[t_1 \cdots t_e] \leftarrow B$

$X \to X_1$ generically finite, $X_1 \to B$ trivially finite

$\Rightarrow X \to Y$ generically finite, then the same way

Rk. Indeed just same: we store the "algebraic part" into a, and separate them by factor

Ex4.1. We reduce to affine: $\begin{array}{ccc}\operatorname{Spec}K & \longrightarrow & \operatorname{Spec}A \\ \downarrow & \circlearrowleft & \downarrow \\ \operatorname{Spec}R & \longrightarrow & \operatorname{Spec}B\end{array} \Longleftrightarrow \begin{array}{ccc}K & \xleftarrow{v} & A \\ \uparrow & \overset{u}{\swarrow} & \uparrow \\ R & \longleftarrow & B\end{array}$ Finite B-module

Pf of Claim. (Pure algebraic)
A is finite $\Rightarrow$ integral over B
R is valuation ring $\Rightarrow$ R integrally closed
$u(B) \subset R \Rightarrow v(A) \subset R$ ▨

v the valuation of $A \to K$
Claim $\operatorname{Im} v \subset R$, then complete the proof

Ex4.2. ~~Non~~Reduced $\xrightarrow{f}$ ~~non~~ Separated, then f determined by open dense set
(a) Take $X = Y = \operatorname{Spec}\frac{k[x,y]}{(x^2,xy)}$, and $X \xrightarrow{f} Y$; $X \xrightarrow{g} Y$
$f = Id_X = Id_Y$ g send closed point identically, but ~~f~~ the zero to reduced
$\Rightarrow f|_{X-0} = g|_{X-0}$ but $f \neq g$ ▨
(b) Take $X = \mathbb{A}^1_k$
Y = doubled origin, then
X = X $\xrightarrow{f, g}$ Y of course not equal
but $f|_{X-0} = g|_{X-0}$ ▨

Pf. $h: X \to Y \times_S Y$ Claim. $\operatorname{Im}(h) \subset \Delta$
$x \mapsto (f(x), g(x))$ Notice that $\Delta \subset Y \times_S Y$ closed due to Y separated
and $\operatorname{Im}(h|_U) \subset \Delta \Rightarrow h^{-1}(\Delta) \supset U \Rightarrow X \subset h^{-1}(\Delta)$
$\Rightarrow \overline{h^{-1}(\Delta)} \supset \overline{U} = X$ $\Leftrightarrow \operatorname{Im}(h) \subset \Delta$ ▨

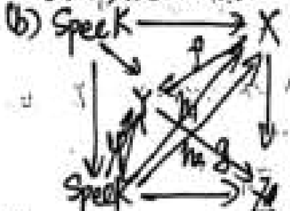
Ex4.3. Just to be the doubled origin: $U = \mathbb{A}^1_k$, $V = \mathbb{A}^1_k$
Then $U \cap V = \mathbb{A}^1_k - \{0\}$ can't be affine ▨

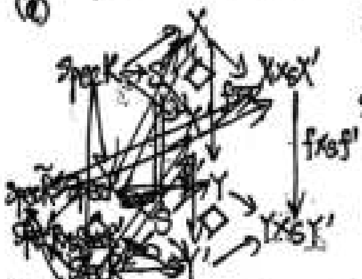
by commutativity, f' surjective
and $W \subset T \times_S f(Z)$, $g(W) = g(f'(f'^{-1}(W)))$
$= (g \circ f')(f'^{-1}(W))$ closed as $g \circ f$ is universally closed ▨

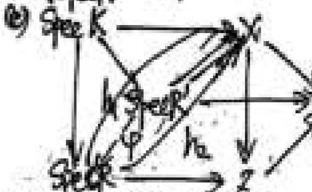
Ex4.4. $X \xrightarrow{f} Y$, $Z \xrightarrow{i} X$ is seperated as it's a closed immersion
U U $\Rightarrow$ Z is seperated, and so does f(Z), thus it suffices to show
Z $f|_Z$ f(Z) that ① $f(Z) \subset Y$ closed ② $f(Z)$ is universally closed
(Notice: we can see $f(Z) \hookrightarrow Y \Longleftrightarrow \operatorname{Spec}\mathbb{Z}$ seperated by composition but Y not proper, that can do directly.)
① $f = p_2 \circ \Gamma_f$
and $\Gamma_f = (1_X, f)_S = f^* \Delta_{Y/S}$ is the closed immersion: $\Delta_{Y/S}$ is closed immersion as Y is separated (Indeed, any $(p, g)_S$ is closed immersion if Δ is) ▨
② $f(Z) \xrightarrow{i'} Y$ To show i' closed immersion, we do for Hom(■T, I) functor:
$f(Z)(T) \to Y(T)$ in Set category, then we have the pullback as
$X(T) \to Y(T)$ Thus closed ▨
By the reduced structure $\exists! \tilde{i}$
$(f(Z), y', \tilde{i}) \mapsto y'$
$\Rightarrow i' = \tilde{i} \circ p \Rightarrow$ closed immersion

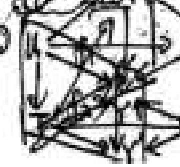
Ex4.5.
(a) $\begin{array}{ccc}\operatorname{Spec}K & \to & X \\ \downarrow & \exists! & \downarrow \\ \operatorname{Spec}R & \to & \operatorname{Spec}k\end{array}$ If $\exists x: (\mathcal{O}_{x,X}, m_x) \le (R_0, m_0)$
then we can define the $f: \operatorname{Spec}R \to X$
thus by valuation criterion $\Rightarrow$ unique f and generic to generic ▨
$\Rightarrow$ only $\exists! x \in X$ ▨
Thus we build a map f_x: $\operatorname{Spec}R \to X$ it map the only closed point to $x \in X$

The proof of Coro4.6-(c)
(a) Open/Closed Immersion $\Rightarrow$ seperated (In this case, Δ is even isomorphism)
Pf. Trivial, as open and closed are both stable under special isolation ▨
(f) ($\Longrightarrow$) Notice that $f^{-1}(V) = f^{-1}(V) \cap X = V \times_Y X$ by (c), it follows
($\Longleftarrow$) It's equivalent that $\forall V \subset Y$, $f^{-1}(V) \to V$ seperated (by Ex3.1~Ex3.4)
Then similar as Lemma 4.2, (For V affine open)
the diagonal usually local, thus pasting is rountine ▨
(b)(d)(e) are similar

(b) 
Composition: $fh_1 = fh_2$ denote φ
Then $\operatorname{Spec}K \to X \Rightarrow \tilde{h} = h_1 = h_2$ ▨

(d) 
By universal property of pullback of following subgraph
Such h_1, h_2 can be pull back to $\operatorname{Spec}R \rightrightarrows X$
and $\operatorname{Spec}R' \rightrightarrows X$
▨

(e) 
$h_1\varphi = h_2\varphi$, and by universal property of:
$R \xleftarrow{} K \Rightarrow h_1 = h_2$ ▨
(R' can be pullback to R)

The proof of Coro4.8-(e)
(a) It suffices show universally closed, i.e. $f^{-1}(Z)$ closed, i.e. $f(Z)$ stable under specialization
thus trivial by f is closed immersion ▨
(f) It suffices show universally closed, i.e. if $f^{-1}(Z \cap \operatorname{Spec}A_i)$ closed for $\forall i \Leftrightarrow$ stable under specialization
then $f^{-1}(Z)$ closed $\Longleftrightarrow$ (locally, by restricting some)
▨
(b)  Thus we find f_2 ▨
(c) $U \to X' \to X$ same as Coro4.6 ▨
$T \to Y' \to Y$
(d)  $X \times_S X'$ $f_1 \times f_2$ is what we want ▨

Ex4.5 (b) is same as (a) ▨ As we build the corresponding $\{x \text{ center} / (\mathcal{O}_{X,x}, m_x) \le (R_v, m_v)\}$

(c)* $\operatorname{Spec} K \to X$ if $(\mathcal{O}_{X,x}, m_x) \le (R_v, m_v)$ $\exists!$ $\iff \{f_x : \operatorname{Spec} R \to X\}$

$\operatorname{Spec} R \to \operatorname{Spec} k$ $\Rightarrow f_x : \operatorname{Spec} R \to X$ can be defined ▨

(d) $a \in \Gamma(X, \mathcal{O}_X) - k$, then $v : K \to G$, as $a \in \Gamma(X) \Rightarrow v(a) \le 0$ taken proper v
and $v(a) = 0$ when $a \in k$ $\Rightarrow v(a) < 0$

$\Rightarrow v(\frac{1}{a}) > 0 \Rightarrow a^{-1} \in m_R$

then such valuation has no center:

If $\exists x \in X : (\mathcal{O}_{X,x}, m_x) \le (R_v, m_R)$ $a^{-1} \in m_R$, $a \in \Gamma(X, \mathcal{O}_X)$

$\Rightarrow a^{-1} \in m_x$, i.e. $a^{-1}(x) = 0$, a contradiction $\Rightarrow a \in \Gamma(X, \mathcal{O}_X) \Rightarrow a^{-1} \in \mathcal{O}_{X,x}$

Ex4.6.
$$\begin{array}{ccc} \operatorname{Spec} K & \to & \operatorname{Spec} A \\ \downarrow & \exists!\varphi & \downarrow f \\ \operatorname{Spec} R & \to & \operatorname{Spec} B \end{array} \iff \begin{array}{ccc} K & \leftarrow & A \\ \uparrow i & \varphi^\# & \uparrow f^\# \\ R & \xleftarrow{g} & B \end{array}$$ ▨ proper

We need to show A is a finite generated B-module = finite type + integral

We take K containing A, and by (II.A) $\Rightarrow \bar{A} = \bigcap_{R \supset A} R$

and for $\forall v$, R_v $\exists! \varphi^\# : A \to R$ $\quad \varphi^\#(A) \supseteq g(B)$

if $R \supset A \Rightarrow \varphi^\# = i$

thus it suffices to show $A \subseteq \overline{g(B)} = \bigcap_{R \supset g(B)} R$

then $R \supset g(B)$, by $i : A \to R \Rightarrow A \subset R$, thus obvious ▨

Ex4.7. (a) $X \xrightarrow{\sigma} X \xrightarrow{\sigma} X$ $\forall x_1 \in X$, denote $x_2 = \sigma(x_1)$

then $\exists\, U \in N_{x_1} \cap N_{x_2}$ affine: $U = \operatorname{Spec} A$

Thus $x_1 = p_1$, $x_2 = p_2 \subset A$,

and $\alpha^2 = \mathrm{Id}$ by commutativity $\Rightarrow \alpha$ is the conjugate of $\mathbb{C}$

$$\begin{array}{ccc} \operatorname{Spec} A & \xrightarrow{\alpha} & \operatorname{Spec} A \\ \downarrow \varphi & & \downarrow \varphi \\ \operatorname{Spec} \mathbb{C}[x] & \xrightarrow{-} & \operatorname{Spec} \mathbb{C}[x] \end{array} \iff \begin{array}{ccc} A & \xleftarrow{\alpha} & A \\ \uparrow \varphi^\# & & \uparrow \varphi^\# \\ \mathbb{C}[x] & \xleftarrow{-} & \mathbb{C}[x] \end{array}$$

take $A_0 = \{\frac{\varphi^\#((x-\xi_1)\cdots(x-\xi_n)\lambda) + \varphi^\#((x-\bar\xi_1)\cdots(x-\bar\xi_n)\bar\lambda)}{2}$
$\forall f(x) = (x-\xi_1)\cdots(x-\xi_n)\lambda \in \mathbb{C}[x]\}$

$\Rightarrow A = A_0 \otimes_{\mathbb{R}[x]} \mathbb{C}[x] \Rightarrow \operatorname{Spec} A = \operatorname{Spec} A_0 \times_{\mathbb{R}} \mathbb{C}$ ▨

(b) ($\Leftarrow$) By the construction of A_0, then obvious ($\Rightarrow$) Trivial ▨

(c) ($\Rightarrow$) $f : X_0 \times_{\mathbb{R}} \mathbb{C} \to Y_0 \times_{\mathbb{R}} \mathbb{C}$ then $f \circ \sigma_X(x_0, z) = f(x_0, \bar z) = (f_0(x_0), \bar z)$

($\Leftarrow$) $f_0 : X_0 \to Y_0$ $\quad (x_0, z) \mapsto (f_0(x_0), z)$ $\quad \sigma_Y f(x_0, z) = \sigma_Y(f_0(x_0), z) = (f_0(x_0), \bar z)$

$\forall x_0 \in X_0$, $\exists\, U \in N_{x_0}$ affine: $f_0 : \operatorname{Spec} A_0 \to \operatorname{Spec} B_0$

$p_0 \subset A_0 \subset A$ $\qquad p_0 = x_0 \mapsto f_0(p_0)$

let $p = (p_0 + i p_0)^e$ and $f_0(p_0) = \frac{f(p) + \sigma_Y f(p)}{2}$ (as ideal) $\trianglelefteq_{\text{prime}} B_0$ as $f(p) \trianglelefteq_{\text{prime}} B$

$\Rightarrow f_0$ is defined for $\forall p_0 = x_0 \in X_0$ ▨

(d) $X = \operatorname{Spec} \mathbb{C}[x] = \operatorname{Spec}(\mathbb{R}[x] \otimes_{\mathbb{R}} \mathbb{C}) = X_0 \times_{\mathbb{R}} \mathbb{C}$ and $X_0 = \operatorname{Spec} \mathbb{R}[x] = \mathbb{R}$ ▨

(e) If $X = X_0 \times_{\mathbb{R}} \mathbb{C}$ and $X_0 \not\cong \mathbb{P}^1_{\mathbb{R}}$, then we prove $X_0 \cong \operatorname{Proj} \frac{\mathbb{R}[x_0, x_1, x_2]}{(x_0^2 + x_1^2 + x_2^2)}$

$X = \mathbb{P}^1_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[x_0, x_1] \cong X_0 \times_{\mathbb{R}} \mathbb{C}$ $\quad$ Let $u = (x_0 + i x_1)$

$P_0 \leftrightarrow (x_0, z)_{\mathbb{R}}$ $\quad v = (x_0 - i x_1)$

Next we show that if $X_0 \not\cong \mathbb{P}^1_{\mathbb{R}} = \operatorname{Proj} \mathbb{R}[x, y]$ $\quad w = x_2$

then $X_0 \cong \operatorname{Proj} \mathbb{C}[t, \frac{1}{t}]$ $\quad \operatorname{Proj} \frac{\mathbb{R}[x_0, x_1, x_2]}{(x_0^2 + x_1^2 + x_2^2)} \cong \operatorname{Proj} \frac{\mathbb{R}[u, v, w]}{(uv + w^2)}$

This means that $\sigma : X \to X$ has no fixed points, then its complexification and then $\cong \operatorname{Proj} \mathbb{R}[t, \frac{1}{t}]$, $\operatorname{Proj} \mathbb{C}[t, \frac{1}{t}]$

consider $X - \{x_0, \sigma(x_0)\} = U$

and $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$, $f \in \operatorname{Aut}(\mathbb{C}_\infty) \cong SL_2(\mathbb{C})$

$\Rightarrow$ it sends $(x_0, \sigma(x_0)) \mapsto (0, \infty)$

$\Rightarrow U \cong \operatorname{Spec} \mathbb{C}[t, \frac{1}{t}] \Rightarrow X \cong \operatorname{Proj} \mathbb{C}[t, \frac{1}{t}]$ ▨

Ex4.8.

(d)
$$\begin{array}{ccc} & X_1 \times Y_2 & \\ {}^{\mathrm{Id} \times g} \nearrow & & \searrow {}^{f \times \mathrm{Id}} \\ X_1 \times X_2 & \xrightarrow{f \times g} & Y_1 \times Y_2 \end{array}$$

$\Rightarrow (f \times g) = (f \times \mathrm{Id})(\mathrm{Id} \times g)$

and $f, g \in \mathcal{P}$

(c) $\Rightarrow f \times \mathrm{Id}, \mathrm{Id} \times g \in \mathcal{P} \xRightarrow{(b)} f \times g \in \mathcal{P}$ ▨

(e) $X \xrightarrow{g \circ f} Z$ $\quad f = p_2 \Gamma_f = ((g \circ f) \times_Z \mathrm{Id}) \Gamma_f \xRightarrow{(d)} p_2 \in \mathcal{P}$

$\Gamma_f \downarrow \uparrow$ $\quad \Gamma_f = f(\Delta)$, $\Delta \in \mathcal{P}$ by (a) $\xRightarrow{(c)} \Gamma_f \in \mathcal{P}$

$X \times_Z Y \xrightarrow{p_2} Y$ $\quad \Rightarrow f \in \mathcal{P}$ ▨

(f) $X \xrightarrow{f} Y$ by universal property of Y_{red} $\exists!\, \tilde f : X \to Y_{red}$

$i_1 \uparrow \quad \uparrow i_2$

$X_{red} \xrightarrow{f_{red}} Y_{red}$ $\quad \Rightarrow \tilde f \in \mathcal{P} \Rightarrow f_{red} = \tilde f \circ i_1 \in \mathcal{P}$ ▨

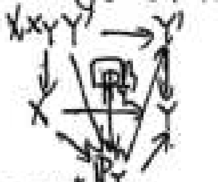
$i_1 \in \mathcal{P}$ (a)

Ex4.9.

where $\tilde p$ given naturally and $\tilde i$ given by $Y \to Z$ a closed immersion and the Segre map also

$\Rightarrow i = \tilde i \circ i$, $p = p_2 \circ \tilde p$ ▨

Then by Ex4.8, it suffices to prove (c), but it's trivial ▨



Ex4.10* (Chow's Lemma)

(a) $X = X_1 \cup X_2 \cup \cdots \cup X_n$, if $\forall X_i$, $\exists X'_i \xrightarrow{g_i} X_i$ and $U_i \subset X_i$: $g_i^{-1}(U_i) \cong U_i$ $\quad$ ∃ closed immersion then Spec ...

then $X' = X'_1 \cup X'_2 \cup \cdots \cup X'_n$ and $g : X' \to X$ pasted, $U = U_1 \cup U_2 \cup \cdots \cup U_n$

Thus it suffices to consider $X_i \cap X_j$, for this, only induction on $\dim X$ ▨

(b) $X = \bigcup_{i \in I} U_i$ is trivial ① Affine $\le$ quasi-projective: $\operatorname{Spec} A$, $A \leftarrow B$ finite type $\Rightarrow A = \frac{B[x]}{I}$

② proper $\Rightarrow$ finite type $\Rightarrow$ finite affine cover ▨ $\Rightarrow \operatorname{Spec} A \to$ ...

Ex4.12. (a) I don't know (Algebraic)

(b) (1) $\mathcal{O}_{x_1,X} \cong A_{p_1}$, where $\operatorname{Spec} A \in N_{x_1}$ and $x_1 \leftrightarrow p_1 \trianglelefteq A$
and x_1 is a curve $\Rightarrow$ height $p=1 \Rightarrow \dim \mathcal{O}_{x_1,X} = \dim A_{p_1} = \text{height } p_1 = 1$
And X nonsingular $\Rightarrow A$ integrally closed $\Rightarrow A_{p_1}$ integrally closed $\Leftrightarrow A_{p_1}$ is DVR ▨

(2) f birational $\Rightarrow$ We can assume X' also nonsingular $\xRightarrow[(1)]{} R$ is DVR
then $(\mathcal{O}_{x_0,X}, m_{x_0}) \leq R$ is trivial ▨

(3) $R \geq R_0 \geq \mathcal{O}_{x_0,X_0} = \mathcal{O}_{x_0,X}$ trivial ▨

(R_0 is a DVR $\Leftrightarrow R_0$ is UFD and unique irreducible element
$\Leftrightarrow$ at least $x_0, x_1 \cdots$ can't be singular?)

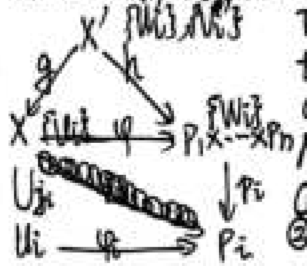
(a) (Algebraic) By (I.6.2A), it suffices to show that m_R is principal
as R is Noetherian local dimension 1
Valuation ring $K|k$ dimension 1

We proceeding this by: take $s_0 \in m_R$, $(s_0) = m_R$ nothing to do
or not, take $s_1 \in m_R - (s_0)$, Claim: $(s_0) \subset (s_1)$
then repeat $\Rightarrow (s_0) \subset (s_1) \subset \cdots \subset (s_N) \subset \cdots \subset m_R$
by Noetherian $\Rightarrow m_R = (s_N)$, complete the proof

Pf of Claim: i.e. $s_0 \in (s_1)$
first s_0 transcendental over k: otherwise $\sum a_i s_0^i = 0 \Rightarrow a_0 = s_0 \sum_{i \in n} a_i s_0^{i-1} \neq 0$
but a_0 is unit $\Rightarrow (s_0) = (1)$, contradiction $\Rightarrow a_0 \in (s_0)$
Thus $K | k(s_0)$ is finite algebraic extension
$\Rightarrow s_1$ is algebraic over $k \Rightarrow \sum_{i \in n} b_i s_1^i = 0$, $b_0 = \sum_{i \in n} b_i s_1^{i+1} \neq 0$
$b_0 = \frac{f(s_0)}{g(s_0)} = s_1 \sum_{i \in m} a_i s_1^{i-1} \Rightarrow f(s_0) \in (s_1)$
and I claim: f has no constant: otherwise $c \in (s_1)$ and $c \neq 0$ unit $\Rightarrow (s_1) = (1) = m_R$
$\Rightarrow s_0 \in (s_1)$ ▨

Ex4.10 (b) (continue) thus U_i quasi-proj over $S \Rightarrow U_i \longrightarrow S$ ▨
$\varphi_i \downarrow$ open, $P_i \xhookrightarrow{\text{closed}} \mathbb{P}^N_S$

(c) $h: X' = f(U) \longrightarrow P_1 \times \cdots \times P_n$
We show h is a closed immersion (hardest)
We reduce to a covering of X', i.e. the closed immersion is locally in target (Ex3.11 Ex3.4)
This is trivial: take the covering to be affine, then by Ex2.18.



Thus $p_i^{-1}(\varphi_i(U_i))$ is an open covering of $P_1 \times \cdots \times P_n$, denoted as $\{W_i\}$
then $h^{-1}(p_i^{-1}(\varphi_i(U_i)))$ covering X', denoted as $\{U_i'\}$
and $\{g^{-1}(U_i)\}$ denoted as $\{V_i\}$
Now locally, we prove for $U_i' \longrightarrow W_i$ is a closed immersion
Claim ①: We have map $U \xrightarrow{u_i} X \times_S W_i$ and $u_i(U) = U_i'$ (scheme-theoretic)
② $j_i: W_i \to X \times_S W_i$, $\supset X'$
then u_i factor through $U_i'^{\text{red}}$
$\Rightarrow \operatorname{Im} u_i = U_i' = \operatorname{Im} j_i \Rightarrow U_i = \operatorname{Im} j_i$ closed in $X \times_S W_i$
then we complete the proof (as it's a immersion of product)

Next we show the two claims (which are both trivial)
① & ② $U \longrightarrow U_i' \longrightarrow W_i$. Thus it suffices to verify the commutativity
(the image on W_i at p_i component).
$X \times P_1 \times \cdots \times P_n \longrightarrow X \times_S W_i$ ▨

Ex4.11. (a) (Algebraic)

① Assume if $L = K[x_1 \cdots x_n]$ is done, now $L' = K(x_1 \cdots x_n) = \operatorname{Frac} L$
Then we extend the $L \xrightarrow{\nu} \mathbb{Z}$ given by $\nu(\frac{f}{g}) = \nu(f) - \nu(g)$
the valuation ring has
$(R', m_R') \geq (R, m_R) \geq (\mathcal{O}, m)$ ▨

② If $\forall$ basis $(x_1 \cdots x_n) = m$, $(x_i) = (1) \subset \mathcal{O}[\frac{x_2}{x_1} \cdots \frac{x_n}{x_1}] = \mathcal{O}'$
i.e. $1 \in (x_1)$, $1 = x_1 p(\frac{x_2}{x_1} \cdots \frac{x_n}{x_1})$ for some $p \in \mathcal{O}'$
then $p(\frac{x_2}{x_1} \cdots \frac{x_n}{x_1}) = \frac{1}{x_1}$, $= \frac{g(x_2 \cdots x_n)}{x_1^M}$
$\Rightarrow (x_2 \cdots x_n) = (1)$ $\Rightarrow m$ also maximal in $\mathcal{O}'$
and $m \trianglelefteq \mathcal{O}$ maximal $\Rightarrow (\mathcal{O}', m)$ local as $a \subset m$
$\Rightarrow m(\frac{1}{x_1}) \trianglelefteq \mathcal{O}'$ maximal $\Rightarrow a \subset m^2$, but it contradict to
$m(\frac{1}{x_1}) = m(\frac{g(x_2 \cdots x_n)}{x_1^M}) = m(1)$ the fact $\mathcal{O}'$ is a polynomial ring
$\Rightarrow \exists$ basis $(x_1 \cdots x_n) = m$, $(x_1) \subsetneq (1)$ ▨

③ $\mathcal{O}$ Noetherian $\xRightarrow[\text{basis}]{\text{Hilbert}} \mathcal{O}'$ Noetherian $\Longrightarrow \mathcal{O}_p'$ Noetherian local $(\mathcal{O}_p', m_p) \geq (\mathcal{O}, m)$ is obvious
It suffices to prove $\dim \mathcal{O}_p' = 1$, trivial as it localises in a p ▨ ($\dim \mathcal{O}_p' = \text{height } p = 1$)

④ Directly by K-A thm, we can apply, $\mathcal{O}_p' \xrightarrow{i} \widetilde{\mathcal{O}_p'}$ and $\operatorname{Frac} \widetilde{\mathcal{O}_p'} | \operatorname{Frac} \mathcal{O}_p' \subseteq L | K$ ▨

(b) ($\Rightarrow$) Trivial
($\Leftarrow$) seperatedness: (DVR (+Noetherian) $\Rightarrow$ VR) (Properness is the same)
Where we use VR? Only to dominate $\mathcal{O}_{x_0}$ in Ex4.5, but by (a), all DVR does ▨

Ex5.5(b) Finite, closed immersion are local, thus let $f: \operatorname{Spec}A \to \operatorname{Spec}B$, A is finite-generated B-module as $A = \frac{B}{b}$ ▨

(c) First by Prop 5.8(c), as X noetherian $\Rightarrow f_*\mathcal{F}$ quasi-coherent
it's local in Y, let $Y = \operatorname{Spec}B \Rightarrow f_*\mathcal{F} = \widetilde{M}$, M a B-module
It suffices to show M is finite generated B-module
and $\mathcal{F} = f^*f_*\mathcal{F}$ coherent, i.e. $X = \bigcup \operatorname{Spec}A_i$, then $M \otimes_B A_i$ is finite-generated A_i-module and $\mathcal{F}|_{\operatorname{Spec}A_i} = \widetilde{M \otimes A_i} \Rightarrow M$ is finite generated B-module as A_i is finite generated B-module (finite morphism) ▨

Ex5.6, (a) $\operatorname{Supp} m = \{p \in \operatorname{Spec}A \mid m_p \neq 0\} = \{p \in \operatorname{Spec}A \mid \exists a \notin p : m \cdot a \neq 0\}$
$= \{p \in \operatorname{Spec}A \mid p \supset \operatorname{Ann} m\} = V(\operatorname{Ann} m)$ ▨

(b) $\operatorname{Supp}\mathcal{F} = \{p \in \operatorname{Spec}A \mid \mathcal{F}_p \neq 0\}$
$= \{p \in \operatorname{Spec}A \mid M_p \neq 0\} = \{p \in \operatorname{Spec}A \mid p \supset \operatorname{Ann} M\} = V(\operatorname{Ann} M)$ ▨

(c) It had done locally in (b), by Noetherian $\to$ cover finite $\Rightarrow$ finite closed set union is closed ▨

(d) $\mathcal{F} = \widetilde{M}$ quasi-coherent $\Rightarrow \mathcal{F}|_{U = X - Z}$ quasi-coherent
$\Rightarrow f_*(\mathcal{F}|_U)$ quasi-coherent as $U \subset \operatorname{Spec}A$ is Noetherian by A noetherian. (Prop 5.8)
$\Rightarrow \mathcal{H}^0_Z(\mathcal{F}) = \operatorname{Ker}(\mathcal{F} \to f_*(\mathcal{F}|_U))$ quasi-coherent;
Then $\Gamma(X, \mathcal{H}^0_Z(\mathcal{F})) = \Gamma_{X \cap Z}(X, \mathcal{F}) = \Gamma_Z(X, \mathcal{F}) = \Gamma_Z(X, \widetilde{M})$
$= \{m \in M \mid \operatorname{Supp} m \subset Z\}$
$\overset{(a)}{=} \{m \in M \mid V(\operatorname{Ann} m) \subset V(a)\}$
$= \{m \in M \mid \operatorname{Ann} m \supset \sqrt{a}\}$
and $\Gamma_a(M) = \{m \in M \mid \exists n > 0 : a^n m = 0\}$
$= \{m \in M \mid \operatorname{Ann} m \supset \sqrt{a}\}$ ▨

(e) Done in (d) for quasi-coherent; for coherent by locally, $\Gamma^{V(a) = V(a)}$
M f-g A-module $\Rightarrow \Gamma_a(M)$ also f-g A-module ▨

Ex5.7.
(a) Assume $X = \operatorname{Spec}A$, $\mathcal{F} = \widetilde{M}$
Now $M_{p_x} \cong A_x^n$ and M is f-g A-module
I don't know how to do it pure algebraically ▨

Algebraic proof. M generators $m_1 \cdots m_n$ and $M_p \cong A_p(x_1 + \cdots + x_n)$ for some section x_i. Then $\mathcal{F}_x \cong \mathcal{O}_{X,x}^n$, $x_i \in \Gamma(D(f), \widetilde{M})$ $\Rightarrow \mathcal{F}|_U \cong \mathcal{O}_X^n|_U$ complete the proof, as coherent $\overset{\text{Ex5.4}}{\Longrightarrow}$ f-p
and $M \to M_p = A_p(x_1 + \cdots)$
$m_i \mapsto \frac{m_i}{1} = \sum_{j \leq n} \frac{a_{ij}}{b_{ij}} x_j$
thus m_i is spanned by $\{x_i\}$, $= \frac{1}{b}\sum a'_{ij} x_j$ in $D(f_i b)$, let $h = f_b$, $f = \prod f_i$ then $D(h)$ is the desired U ▨

Lemma. $\mathcal{F}_x \cong \mathcal{G}_x \Longrightarrow \exists U \in N_x : \mathcal{F}|_U \cong \mathcal{G}|_U$
$\mathcal{F}, \mathcal{G}$ finitely presented $\mathcal{O}_X$-module (This is the way of EGA I)

Pf of Lemma: let $f: \mathcal{F}_x \to \mathcal{G}_x$
$g: \mathcal{G}_x \to \mathcal{F}_x$
and by $\mathcal{H}om(\mathcal{F}, \mathcal{G})_x \cong \operatorname{Hom}(\mathcal{F}_x, \mathcal{G}_x)$
$\tilde{f} \longleftarrow f$
i.e. $\exists U \in N_x$: $u \in \Gamma(U, \mathcal{H}om(\mathcal{F}, \mathcal{G}))$; $u_x = \tilde{f} \leftrightarrow f$
Namely we have V, v, take $U \cap V \neq \phi$, and u, v inverse to each other
$\Rightarrow \mathcal{F}|_{U \cap V} \cong \mathcal{G}|_{U \cap V}$ ▨

Ex5.1. (a) $\mathcal{E}^{\vee\vee} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X), \mathcal{O}_X)$
Let $X = \bigcup U_\alpha$ and $\mathcal{E}|_{U_\alpha} = \mathcal{O}_{U_\alpha}^n$, locally, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{O}_X) \cong \mathcal{O}_X^n$, i.e. $\check{\check{\mathcal{E}}} \cong \mathcal{E}$
Globally, we paste the isomorphism in U_α ▨ (by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_X) \cong \bigoplus_n \mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{O}_X^{\oplus n}$) ($\mathcal{H}om(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}$)

(b) $\mathcal{H}om(\mathcal{E}, \mathcal{F}) \cong \mathcal{H}om(\mathcal{E}, \mathcal{O}_X) \otimes \mathcal{F} = \check{\mathcal{E}} \otimes \mathcal{F}$
the first is given by $\operatorname{Hom}(\mathcal{E}|_U, \mathcal{O}_X|_U) \otimes \mathcal{F}(U) \cong \operatorname{Hom}(\mathcal{E}|_U, \mathcal{F}|_U)$
then paste the isomorphism ▨

$\sum f_i \otimes s_i \longmapsto (g : e \mapsto \sum f_i(e) \otimes s_i)$
$f \otimes s(e) \longleftarrow g : e \mapsto g(e) \in \mathcal{F}|_U$
$g(e) = f(e) \otimes s \in \mathcal{F} \otimes \mathcal{O}_X|_U$

(c) $\operatorname{Hom}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G})$
$\cong \operatorname{Hom}(\mathcal{F}, \check{\mathcal{E}} \otimes \mathcal{G}) \cong \operatorname{Hom}(\mathcal{F}, \mathcal{H}om(\mathcal{E}, \mathcal{G}))$
The first is given by locally $\operatorname{Hom}(\mathcal{O}_X^n \otimes \mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}(\mathcal{F}, \mathcal{O}_X^n \otimes \mathcal{G})$
$\cong \operatorname{Hom}(\mathcal{F}, \check{\mathcal{O}}_X^n \otimes \mathcal{G})$ ▨ (By $\bigoplus_n \operatorname{Hom}(\mathcal{F}, \mathcal{G})$ isomorphic to both of them)

(d) locally again, LHS $= f_*(\mathcal{F} \otimes f^*\mathcal{O}_Y^{\oplus n}) = f_*(\mathcal{F} \otimes \mathcal{O}_X^{\oplus n}) \cong f_*(\mathcal{F}^{\oplus n})$
$\cong (f_*\mathcal{F})^{\oplus n} \cong f_*\mathcal{F} \otimes \mathcal{O}_Y^n \cong f_*\mathcal{F} \otimes \mathcal{E} =$ RHS; the Global follows ▨

Ex5.2. (a) $X = \operatorname{Spec}R$ only has two points: one closed and one generic
the only open set $U \subset X$ is $U = \{\eta\}$, the any $\mathcal{O}_X$-module $\mathcal{F}$, $\mathcal{F}|_U = \mathcal{F}_\eta$ is $\mathcal{O}_{\eta,X}$-module $= K$-module
$\Rightarrow \mathcal{F}$ is quasi-coherent, thus by Prop 5.4 $\Rightarrow$ it's equivalent ▨
(b) ρ isomorphism $\Leftrightarrow \mathcal{O}_X(X) \cong \mathcal{O}_X(\{\eta\})$ to give $f : \mathcal{O}_X(X) \to \mathcal{O}_X(\{\eta\}) \otimes R$ the restriction map and tensor $\otimes \mathcal{F}|_\eta$
$\Leftrightarrow \mathcal{F} = \widetilde{M}$ ▨

Ex5.3. $\operatorname{Hom}_A(M, \Gamma(\operatorname{Spec}A, \mathcal{F})) \cong \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F})$
$\Gamma(\operatorname{Spec}A, f) \overset{\Xi}{\longleftarrow} f$ the injectivity of Ξ is trivial
$\varphi \overset{\Phi}{\longmapsto} (\varphi_p : M_p \to \Gamma(\operatorname{Spec}A)_p = \mathcal{F}_p)_{p \in \operatorname{Spec}A}$ determine the sheaf map $\widetilde{M} \to \mathcal{F}$ ▨ (Prop 1.1)

Ex5.4. The question is local, assume $X = \operatorname{Spec}A$
($\Rightarrow$) $\mathcal{F}$ quasi-coherent $\Leftrightarrow \mathcal{F} = \widetilde{M} \Rightarrow$ there is $A^I \to A^J \to M \to 0$
as $\sim \dashv \Gamma \Rightarrow \sim$ is exact / Γ also
($\Leftarrow$) Given $\mathcal{O}_X^I \to \mathcal{O}_X^J \to \mathcal{F} \to 0$

$$\begin{tikzcd}
\mathcal{O}_X^I \arrow[r] \arrow[d] & \mathcal{O}_X^J \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
A^I \arrow[r] & A^J \arrow[r] & \Gamma(X, \mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
A^I \arrow[r] \arrow[d] & A^J \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
\widetilde{A^I} \arrow[r] \arrow[d, equal] & \widetilde{A^J} \arrow[r] \arrow[d, equal] & \widetilde{M} \arrow[r] \arrow[d, equal] & 0 \\
\widetilde{A^I} \arrow[r] \arrow[d, equal] & \widetilde{A^J} \arrow[r] \arrow[d, equal] & \widetilde{M} \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{O}_X^I \arrow[r] & \mathcal{O}_X^J \arrow[r] & \widetilde{M} \arrow[r] & 0
\end{tikzcd}$$

Also by $\mathcal{F}' \to \mathcal{F}'' \to \mathcal{F}$ by Prop 5.7 (Five) $\mathcal{F}'$, $\mathcal{F}''$ coherent $\Leftarrow$ free $\Rightarrow \mathcal{F}$ coherent ▨

$\Rightarrow \mathcal{F} = \widetilde{\Gamma(X, \mathcal{F})}$ ▨
For coherent, take $I, J < \infty$ ▨

Ex5.5. (a) $\mathbb{A}^1 \to \mathbb{P}^1$ (variety) then $\mathcal{O}_{\mathbb{A}^1} = k[x]$ f-g k-module
$x \mapsto (1, x)$ $\Rightarrow$ but $f_*(\mathcal{O}_{\mathbb{A}^1}) \cong k[x]$ not ▨

As the first two steps are same, it suffices to show $S_d \cong S'_d$

by (5.19) again, $\forall s' \in S'$, $\exists d \gg 0$: $x_0^d s' \in S$ thus $\exists d \geq d$ (take max deg of generators of S' over A): $S_d = S'_d$ ▨

(c) $\mathrm{Coh}(\mathrm{Proj}\, S) \cong S\text{-mod}/\approx$ We need to show ① (well-defined) $M \approx M' \Rightarrow \widetilde{M} \cong \widetilde{M'}$

② (composite) $\Gamma_*(X, \widetilde{M})$ are $\Gamma(X, \mathcal{F}) = \mathcal{F}$

② is direct: $\Gamma(X, \widetilde{M}) \cong \Gamma(X, \widetilde{M'})$ where M' f-g S-module by ①

$\Rightarrow \Gamma(X, \widetilde{M'}) \cong M' \approx M$, the latter is proven in text ▨

① It suffices to prove for M f-g S-module

The we repeat the pf of Ex2.14(c), with then replace T, s by M, M' ▨

Ex5.10 (a) $I \cap S_d = \{s \in S \mid \deg s = d, x_i^{n_i} s \in I, \forall i\}$

$= \{s \in S \mid \deg s = d, x_i^{n_i} s \in I_{d+n_i}, \forall i\}$

We take $N = \max n_i$ then $I \cap S_d \subset I_{d+N}$ (up to isomorphism)

let $\bar{I}_d = I \cap S_d$, then $\bar{I} = \bigoplus \bar{I}_d$ as $\forall s \in \bar{I}$, $s = \sum_{i \in d} s_i \in \bigoplus_{i \in d} \bar{I}_i$

taking $N = \max_{i \in d} n_i$ ▨

(b) It suffices to show for $I_2 = \bar{I}$, $I_1 = I$, then we prove $\mathrm{Proj}\,\frac{S}{\bar{I}} \cong \mathrm{Proj}\,\frac{S}{I}$

This is due to $(S/\bar{I})_d \cong (S/I)_d$ for $d \gg 0$, the reason is just we proved in Ex5.9(b)

($\Leftarrow$) If $\mathrm{Proj}\,\frac{S}{I_1} \cong \mathrm{Proj}\,\frac{S}{I_2}$ and $I_1 \subset I_2$ assumed

and indeed, $V(\bar{I}) = V(I)$ $\Rightarrow V(I_1) \supset V(I_2)$

$\Rightarrow \mathrm{Proj}\,\frac{S}{I_1} \cong \mathrm{Proj}\,\frac{S}{\bar{I}_1} \cong \mathrm{Proj}\,\frac{S}{\bar{I}_2} = \mathrm{Proj}\,\frac{S}{I_2}$

$\Rightarrow \bar{I}_1 = \bar{I}_2$ ▨

pf. $V(\bar{I}) \subset V(I)$

$\forall P \in V(I)$

i.e. $f(P) \equiv 0$

$\exists x_0^n f \in I \Rightarrow x_0^n f(P) = 0$

$\Rightarrow P \in V(\bar{I})$ ▨

(c) $\forall s \in \bar{I}_*(\mathcal{I})$, i.e. $x_i^{n_i} s \in \Gamma_d(\mathcal{I})$, $\forall i$, $\exists n_i$

assume s is homogeneous, then $x_i^{n_i} s \in \Gamma(X_i, \mathcal{I}(N+d))$, $\forall i$

$\deg s = d$ (by $s = \sum s_d$) (We may cover $\Gamma_*(\mathcal{I})$ first and do for each n_i)

We need to prove $s \in \Gamma(X, \mathcal{I}(d))$, by tensoring $x_i^{-N} \in \mathcal{O}_X(-N)$ ▨

(d) Given by $I \mapsto \mathrm{Proj}\,\frac{S}{I}$

$\bar{I} \longleftrightarrow Y = \mathrm{Proj}\,\frac{S}{\bar{I}}$ ▨

Ex5.11 ① $\mathrm{Proj}(S \times_A T) \longleftrightarrow \mathrm{Proj}\, S \times_A \mathrm{Proj}\, T$

We show the universal property, i.e.

$$\begin{tikzcd} \mathrm{Proj}(S \times_A T) \arrow[r] \arrow[d] & \mathrm{Proj}\, T \arrow[d] \\ \mathrm{Proj}\, S \arrow[r] & \mathrm{Spec}\, A \end{tikzcd}$$

given $X \xrightarrow{f} \mathrm{Proj}\, T$, $X \xrightarrow{g} \mathrm{Proj}\, S$

We need to induce $X \xrightarrow{(f,g)} \mathrm{Proj}(S \times_A T)$

given $f(x) \trianglelefteq T$, $g(x) \trianglelefteq S$, then let $(f,g)(x) = \bigoplus_d (f(x)_d \otimes g(x)_d)$ is the desired map ▨

② $\mathcal{O}(1) = \widetilde{S \times_A T(1)}$

$= \bigoplus_{d \geq 1} S_d \otimes T_d$

$\cong \bigoplus_{d \geq 1} S_d \otimes \bigoplus_{d} T_d$

$= p_1^*(\mathcal{O}_X(1)) \otimes p_2^*(\mathcal{O}_Y(1))$ ▨

Ex5.12 (c) We'll reduce to affine local case and use Nakayama to solve it

First is a basic observation: ($\Rightarrow$) Then $\mathcal{F} \otimes \mathcal{F}^{\vee} \cong \mathcal{O}_X$, locally, $\mathcal{O}_X \otimes \mathcal{O}_X \cong \mathcal{O}_X$

($\Leftarrow$) It even holds for X is ringed space and $\mathcal{F}$ finite type

Now $\exists \mathcal{G}$: $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$

It suffices to show: $\mathcal{F}$ locally free: as then locally $\mathcal{O}_X^n \otimes \mathcal{G} \cong \mathcal{O}_X$ $\Rightarrow n = 1$ and $\mathcal{G} \cong \mathcal{O}_X$

by (b). It suffices to show, $\mathcal{F}_x$ is free to complete proof

$\mathcal{O}_X$-module / $\mathcal{F}_x \cong \mathcal{O}_{X,x}$ locally

To show this, let $X = \mathrm{Spec}\, A$ and (A, m) local, given M a A-module and $\exists M'$ also A-module: $M \otimes M' \cong A$, we prove $M \cong A$ (pure algebraic)

pf. $\frac{A}{m} \otimes (M \otimes M') \cong \frac{A}{m} \otimes A \cong \frac{A}{m}$

$\cong$

$\frac{M}{mM} \otimes_{\frac{A}{m}} \frac{M'}{mM'}$ the tensor of vector space $\Rightarrow \dim \frac{M}{mM} = \dim \frac{M'}{mM'} = 1$

$\Rightarrow \forall m \in M - mM$, $M = Am + mM$ i.e. $M = Am$ by Nakayama

and m is "good" i.e. $\mathrm{Ann}\, m = \mathrm{Ann}(M \otimes M') = \mathrm{Ann}\, A = 0 \Rightarrow M \cong A$ ▨

Ex5.8 (a) $\{x \in X \mid \varphi(x) \geq n\} = \{x \in X \mid \dim_{k(x)} \mathcal{F}_x \otimes k(x) \geq n\}$

closed is local property, let $X = \mathrm{Spec}\, A$ and $\mathcal{F} = \widetilde{M}$

$= \{p \in \mathrm{Spec}\, A \mid \dim_{\frac{A_p}{m_p A_p}} M_p \otimes_{A_p} \frac{A_p}{m_p A_p} \geq n\}$ to show $= V(a)$ for some a is hard

thus consider $\{p \in \mathrm{Spec}\, A \mid \dim_{\frac{A_p}{m_p A_p}} M_p \otimes_{A_p} \frac{A_p}{m_p A_p} < n\}$, with $M_p \otimes \frac{A_p}{m_p A_p} = \frac{M_p}{m_p M_p}$ has basis $m_1 \cdots m_r$

and M is finite generated A-module by $m'_1 \cdots m'_s$, Nakayama tells us: $r = s < n$

then by proper choice of basis $\Rightarrow m_i = m'_i$, then we repeat the algebraic proof of Ex5.7(a) $\Rightarrow$ $\forall p \in$ the complementary set, $\exists h$: $p \in D(h) \subset$ the set

$\Rightarrow$ open ▨

(b) locally free + connected + coherent $\Rightarrow \mathcal{F}_x \cong \mathcal{O}_x^n$ everywhere, $n < \infty$

$\Rightarrow \varphi = n$ ▨

(c) local problem, assume $X = \mathrm{Spec}\, A$, A is reduced and (without nilpotent) $\varphi(x) = \dim_{\frac{A_x}{m_x A_x}} \mathcal{F}_x \otimes_{\mathcal{O}_X} \frac{A_x}{m_x A_x} = n$

We need to show

$\mathcal{F}_x \cong A_x^n$, then apply Ex5.7(b) to complete the proof

But it's trivial by $\varphi = n$ (and no nilponent) ▨

Ex5.9 (a) $\Gamma_*(\widetilde{M}) = \bigoplus \Gamma(X, \widetilde{M(n)}) = \bigoplus \Gamma(X, \widetilde{M(n)}) = \bigoplus M(n)$

then $M = \bigoplus M_n \to \bigoplus M(n)$ the canonical injection $M_n \hookrightarrow M(n)$ ▨

(b) We imitate (5.19) Step 1. Nothing to do Step 2. Reduce to $M = S$ case, $S' = \Gamma_*(\widetilde{S})$

Step 3. Show $S_d \cong S'_d$

and $S' = S \Longrightarrow S = \bigoplus \Gamma(X, \mathcal{O}_X(n)) \Rightarrow S_n = \Gamma(X, \mathcal{O}_X(n))$

the map is $\Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \longrightarrow \Gamma(X, \mathcal{O}_X(n))$
$$\begin{matrix} \parallel & & \parallel \\ k[x_0 \cdots x_r]_n & \longrightarrow & S_n \end{matrix} \text{ surjective}$$

$\Longleftarrow$ By (a)

it has $\Gamma(X, \mathcal{O}_X(n)) = S'_n$ and $k[x_0 \cdots x_r]_n \xrightarrow{\varphi} \left(\frac{k[x_0 \cdots x_r]}{I}\right)_n$ is surjective

but the induced map φ has $\operatorname{Im}\varphi = \left(\frac{k[x_0 \cdots x_r]}{I}\right)_n$ as it's a quotient

$\Rightarrow S_n = S'_n \Rightarrow S = S'$ ▨

Ex 5.15. (a) We have the algebraic fact: For M an A-module, then $M = \bigcup_{\substack{M_\alpha \subset M \\ \text{finite-generate} \\ A\text{-module}}} M_\alpha$

take $\sim$ functorially ▨

(b) $i: U \to X$, and X noetherian, U open $\Rightarrow U$ noetherian

5.8(c) $\Rightarrow i_* \mathcal{F}$ quasi-coherent $\overset{(a)}{\Longrightarrow} i_* \mathcal{F} = \bigcup \mathcal{F}_\alpha$, $\mathcal{F}_\alpha$ coherent over X

I claim $\bigcup \mathcal{F}_\alpha|_U = \bigcup_{\substack{\alpha \in \Lambda \\ \Lambda \text{ finite}}} \mathcal{F}_\alpha|_U + \bigcup_{\alpha \notin \Lambda} \mathcal{F}_\alpha|_U = \bigcup_{\alpha \in \Lambda \text{ finite}} \mathcal{F}_\alpha|_U$ coherent
$$\begin{matrix} \parallel & & \parallel \\ i_*\mathcal{F}|_U & & 0 \end{matrix}$$
$\cong \mathcal{F}|_U$ thus it suffices to show: only finite α has $\mathcal{F}_\alpha|_U \neq 0$

but it's obvious; otherwise $\mathcal{F}$ not coherent over U ▨ (Thus the affine case is done)

(c) $\mathcal{F} \subseteq \mathcal{G}|_U \Rightarrow i_* \mathcal{F} \subset i_*(\mathcal{G}|_U)$, let $\rho^{-1}(i_* \mathcal{F}) \subset \mathcal{G}$ over X

$\Rightarrow \rho^{-1}(i_* \mathcal{F}) = \bigcup \mathcal{F}_\alpha$, $\mathcal{F}_\alpha$ coherent over X, let $\mathcal{F}' = \bigcup_{\text{finite}} \mathcal{F}_\alpha$ coherent

and $\rho^{-1}(i_* \mathcal{F})|_U \cong i_* \mathcal{F}|_U \cong \mathcal{F}|_U$, and $\bigcup_{\text{finite}} \mathcal{F}_\alpha \subset \rho^{-1}(i_* \mathcal{F}) \subset \mathcal{G}$ ▨

(d) $X = \bigcup_{\text{finite}} U_i$ by Noetherian and $U_i = \operatorname{Spec} A_i$ affine

We extend over one of them at a time by Hint, namely, take U_i each i

then $\mathcal{F}|_{U \cap U_i}$ can be extend to $\mathcal{F}'_i \subset \mathcal{G}|_{U_i}$ in U_i and we can paste $\mathcal{F}'_i$ with $\mathcal{F}$ as they are same in $U \cap U_i$, as they all same with $\mathcal{F}$

$\Rightarrow$ it extended to global ▨

(e) $\forall U \subset X$, $\forall s \in \Gamma(U, \mathcal{F})$, then s generate a sheaf $\mathcal{G}$ over U as
$$\mathcal{G}_s: V \longrightarrow \mathcal{G}(V) \qquad \Rightarrow \mathcal{G}_s \subset \mathcal{F}|_U$$
$$t \longmapsto \langle s|_V(t) \rangle \text{ generated as a subgroup of } \mathcal{F}(V)$$
then we extend to $\mathcal{G}'_s$ over X, $\mathcal{G}'_s \subset \mathcal{F} \Rightarrow \mathcal{F} = \bigcup_{s \in \Gamma(U, \mathcal{F})} \mathcal{G}'_s$ ▨

and $\mathcal{G}'_s$ is coherent obviously

Ex 5.16. (a) $T^r(\mathcal{F})|_U = T^r(\mathcal{F}|_U) = T^r(\mathcal{O}_X|_U^n) = (\mathcal{O}_X|_U^n)^{\otimes r} = \mathcal{O}_X|_U^{n^r}$, $U \subset X$

$S^r(\mathcal{F})|_U = S^r(\mathcal{F}|_U) = S^r(\mathcal{O}_X|_U^n)$, $\mathcal{O}_X|_U^n$ is free $\mathcal{O}_X$-module of rank n

$\Rightarrow S(\mathcal{O}_X|_U^n) \cong \mathcal{O}_X|_U[e_1 \cdots e_n] \Rightarrow S^r(\mathcal{O}_X|_U^n)$ the r-homogeneous polynomial

$\Rightarrow$ rank is $\binom{n+r-1}{n-1}$

$\Lambda^r(\mathcal{F})|_U = \Lambda^r(\mathcal{F}|_U) = \Lambda^r(\mathcal{O}_X|_U^n)$, and $\Lambda^r(\mathcal{O}_X^n(V)) \ni s_{i_1} \otimes \cdots \otimes s_{i_r}$

with $1 \le i_1 < \cdots < i_r \le n \Rightarrow$ rank is $\binom{n}{r}$ ▨

Ex 5.11. ③④ are trivial ▨

Ex 5.12. (a)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & \mathbb{P}^r_Y & \mathbb{P}^r_Y \times \mathbb{P}^s_Y \arrow[l, "p_1"'] \arrow[dl, "p_2"] \\ & \mathbb{P}^s_Y & \end{tikzcd}$$
and $f^*(\mathcal{O}(1)) \cong \mathcal{L}$, $g^*(\mathcal{O}(1)) \cong \mathcal{M}$

the $\mathbb{P}^r_Y \times \mathbb{P}^s_Y \xrightarrow{S} \mathbb{P}^{rs+r+s}_Y$ is the Segre embedding

$\Rightarrow S^*(\mathcal{O}(1)) \cong f^*(\mathcal{O}(1)) \otimes g^*(\mathcal{O}(1)) \cong \mathcal{L} \otimes \mathcal{M}$ thus very ample

thus it suffices to show the choice of S can make the isomorphism

by Ex 5.11, we reduce to $S^*(\mathcal{O}(1)) \cong S^*(p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{O}(1))) \cong (p_1 \circ S)^*(\mathcal{O}(1)) \otimes (p_2 \circ S)^*(\mathcal{O}(1))$

$= f^*(\mathcal{O}(1)) \otimes g^*(\mathcal{O}(1))$ ▨

(b)
$$\begin{tikzcd} \mathbb{P}^r_Y \arrow[r, "g"] & \mathbb{P}^n_Z \\ X \arrow[u, "\varphi"] \arrow[ur, "\Phi"] \arrow[r, "f"] \arrow[dr, "\Psi"'] & Y \arrow[r, "g"] \arrow[d, "\psi"] & Z \\ & \mathbb{P}^m_Z & \end{tikzcd}$$
Thus $\mathcal{L} \cong \varphi^*(\mathcal{O}(1)) = \varphi^* g^*(\mathcal{O}(1)) = \Phi^*(\mathcal{O}(1))$

and $f^*\mathcal{M} \cong f^* \psi^*(\mathcal{O}(1)) = \Psi^*(\mathcal{O}(1))$

then by (a) ▨

Ex 5.13. ① Define $\mathfrak{p}^{(d)} = \bigoplus \mathfrak{p}^d_{S_d} = \bigoplus \mathfrak{p}_{nd}$, for $\mathfrak{p} \le S$, then $\mathfrak{p}^{(d)} \le S^{(d)}$

$\operatorname{Proj} S \longrightarrow \operatorname{Proj} S^{(d)}$ is an isomorphism of spectra (not scheme)

$\mathfrak{p} \longmapsto \mathfrak{p}^{(d)}$ injective

$\mathfrak{q} \longleftarrow \mathfrak{q}$ ▨

Let us cover X by $D_+(f)$, $f \in S_1$

and cover $\operatorname{Proj} S^{(d)}$ by $D_+(f)$, $f \in S_d = S_1^{(d)}$

~~thus both~~ is covered by $D_+(f)$, $f \in S_d$

~~then~~ in each $D_+(f)$, the identity is the isomorphism;

$\widetilde{S^{(d)}(1)}_{(f)} \cong S(d)_{(f)} \Rightarrow \widetilde{S^{(d)}(1)} \cong \widetilde{S(d)}$ ▨

② Then $\widetilde{S^{(d)}(1)} \cong \widetilde{S(d)}$ is obvious ▨

③④ are trivial ▨

Ex 5.14. (a) $S = \frac{k[x_0 \cdots x_r]}{I}$ is domain is trivial,

i.e. normal $\Longrightarrow$ irreducible $\Rightarrow$ prime $\Rightarrow$ integral

otherwise $\exists \mathfrak{p} \in X$: $\mathfrak{p} \in X_1 \cap X_2$, give $X = X_1 \cup X_2 \cup \cdots \cup X_n$

$\Rightarrow \mathcal{O}_{\mathfrak{p},X}$ isn't integral: the zero divisor through X_1 is all elements in X_2 ▨

And, to show $S' = \bar{S}$, we imitate (steps of pf of (5.19))

we show that ① $S'' = S'$ ② $S' \subset \bar{S}$: directly by (5.19)

① By Hint $S' = \bigoplus \Gamma(X, \mathcal{O}_X(n)) = \Gamma(X, \bigoplus \mathcal{O}_X(n)) = \Gamma(X, \mathcal{S})$

It suffices to show $\mathcal{S}$ is integrally closed sheaf, checking on stalks:

$\mathcal{S}_\mathfrak{p} = \bigoplus \mathcal{O}_X(n)_\mathfrak{p} = (\bigoplus \widetilde{S(n)})_\mathfrak{p} = \bigoplus S_{(\mathfrak{p})}(n)$, $\forall s$ integral over $\mathcal{S}_\mathfrak{p} \Rightarrow$ integral over $S_{(\mathfrak{p})}$

, and $S_{(\mathfrak{p})} = \mathcal{O}_{\mathfrak{p},X}$ integrally closed by normality $\Rightarrow s \in \mathcal{S}_\mathfrak{p} \Rightarrow S' = \bar{S}$ ▨

(b) $S_d \underset{\text{Ex 5.9(b)}}{\cong} \Gamma(X, \widetilde{S(d)}) = \Gamma(X, \mathcal{O}_X(n)) = S'_d$ ▨

(c) For $d > 0$, by (b) $S_d \cong S'_d \Rightarrow S_1^{(d)} \cong S_1^{(d)'}$ the generator

$\Rightarrow S^{(d)} \cong S^{(d)'} = \overline{S^{(d)}} \Rightarrow S^{(d)}$ integrally closed ▨

Then directly by Ex 5.14, d-tuple embedding is projective normal ▨

(d) $(\Longrightarrow)$ Normal is clear: the localisation of coordinate ring is the stalk, and localisation of integral is still integral;

Ex5.17 (c) $X = f^{-1}(Y) = f^{-1}(\bigcup V_\alpha) = \bigcup f^{-1}(V_\alpha) \cong \bigcup \mathrm{Spec}(\mathcal{A}(V_\alpha))$ $\quad V_{\alpha\beta} = \bigcup \mathrm{Spec} A_{f_i}$

Consider $V_{\alpha\beta} = V_\alpha \cap V_\beta = \mathrm{Spec} A_\alpha \cap \mathrm{Spec} A_\beta$, cover it with $D(f_i)$, $f_i \in A_\alpha$

$f^{-1}(V_\alpha) \cong \mathrm{Spec}(\mathcal{A}(V_\alpha))$, $f^{-1}(V_\beta) \cong \mathrm{Spec}(\mathcal{A}(V_\beta))$, we can assume $\mathcal{A}|_{V_\alpha} = \widetilde{M_\alpha}$, M_α a A_α-module

$\mathcal{A}|_{V_\beta} = \widetilde{M_\beta}$, M_β a A_β-module

$\Rightarrow f^{-1}(V_\alpha) \cong \mathrm{Spec} M_\alpha$, $f^{-1}(V_\beta) \cong \mathrm{Spec} M_\beta$

furthermore, a $A_\alpha(A_\beta)$-algebra

thus $f^{-1}(V_{\alpha\beta}) = \mathrm{Spec} M_\alpha \cap \mathrm{Spec} M_\beta$ corresponding to $\bigcup \mathrm{Spec}(M_\alpha)_{f_i}$, $(M_\alpha)_{f_i}$ a $(A_\alpha)_{f_i}$-algebra, for A_β and g_i, it's the same

$\Rightarrow$ it's compatible, we can paste, and by this way the $\mathcal{O}_{\mathrm{Spec}\mathcal{A}}$ is also clear ▨

(d) $\mathrm{Coh}(X) \cong \mathrm{Coh}(\mathcal{A}, f_*\mathcal{O}_X)$ Thus it suffices to ① construct $\widetilde{M}$

$\mathcal{F} \longmapsto f_*\mathcal{F}$ ② $f_*\widetilde{M} = M$ and $\widetilde{f_*\mathcal{F}} = \mathcal{F}$

$\widetilde{M} \longleftarrow M$

Pf. ① Assume $Y = \mathrm{Spec} A$, then $f: \mathrm{Spec}\mathcal{A}(\mathrm{Spec} A) \to \mathrm{Spec} A$ is induced by $A \to \mathcal{A}(\mathrm{Spec} A)$ as an A-algebra

$\Rightarrow \mathcal{O}_X = \mathcal{A}(\mathrm{Spec} A)$, it's obvious that given a $\mathcal{A}$-module $M \Longleftrightarrow$ given a $\mathcal{A}(Y)$-module $M(Y)$ (Global section)

thus $\widetilde{M} = M$ locally given ▨

② By the local construction, both are identity ▨

Ex5.18 (a) We verify ① The translation function linear ② Doesn't depend on $\mathcal{E}|_U$'s basis

Pf. ① $U = \mathrm{Spec} A_1$, $V = \mathrm{Spec} A_2$, $U \cap V \neq \emptyset$

then $\psi_1: f^{-1}(U) \to \mathbb{A}^n_U$ $\quad \psi_2: f^{-1}(V) \to \mathbb{A}^n_V$

$\mathrm{Spec}(S(\mathcal{E}(\mathrm{Spec} A_1))) = \mathrm{Spec} A_1[x_1 \cdots x_n]$ $\quad \mathrm{Spec}(S(\mathcal{E}(\mathrm{Spec} A_2))) = \mathrm{Spec} A_2[x_1 \cdots x_n]$

and $W \subset U \cap V$, i.e. $A_1 \to B$, $A_2 \to B$, $W = \mathrm{Spec} B$, and $\psi_2 \circ \psi_1^{-1}$ of $\mathrm{Spec} B[x_1 \cdots x_n]$ is given by $\psi_2 \circ \psi_1^{-1}(b) = b$ ▨

② ψ send the base elements of $\mathcal{O}_U^n = \mathcal{E}_U$ to $x_1 \cdots x_n$, but the $x_1 \cdots x_n$ has no meaning concretely ▨

(b) $S(X|Y)(U) = \{s: U \to X \mid f\circ s = \mathrm{Id}_U\} = \{s: U \to f^{-1}(U) \mid f \circ s = \mathrm{Id}_U\}$

$= \{s: U \to \mathbb{A}^n_U \mid f\circ s = \mathrm{Id}_U\} \cong \mathcal{O}_Y(U)[e_1 \cdots e_n]$ by taking base of $\mathbb{A}^n_U$

$\Rightarrow S(X|Y)$ is $\mathcal{O}_Y$-module, local free of rank n ▨

(c) We prove $S(U) \cong \mathcal{E}^\vee(U)$

$S(X|Y)(U) \cong \mathrm{Hom}(\mathcal{E}|_U, \mathcal{O}_U)$

By Hint, it's $\mathcal{O}_V = \mathcal{E}(\mathcal{O}_V)$

The inverse is obvious by construction, we only verify that $f\circ s' = \mathrm{Id}_V$

the $\theta: A[x_1 \cdots x_n] \to A$

$s = \theta^*: \mathrm{Spec} A \to \mathbb{A}^n_{\mathrm{Spec} A}$

$(s': V \to f^{-1}(V)) \longleftarrow (s: \mathcal{E}|_V \to \mathcal{O}_V)$

$(s: V \to f^{-1}(V)) \to (s^*: S(\mathcal{E}|_V) \to \mathcal{O}_V)$ $\quad (\tilde{s}, S(\mathcal{E}|_V) \to \mathcal{O}_V)$

i.e. $f\circ s'(p) = f(s'^{-1}(p)) = f(\theta^*(s'(p))) = p$ ▨

Ex5.16 (b) local problem, thus consider $\Lambda^r \mathcal{O}_X^n \longleftrightarrow \mathrm{Hom}(\Lambda^{n-r}\mathcal{O}_X^n, \Lambda^n\mathcal{O}_X^n)$

and $\Lambda^n\mathcal{O}_X^n$ is single generated, thus let the generator to be $e = e_1 \wedge e_2 \wedge \cdots \wedge e_n$

$\Rightarrow f \in \mathrm{Hom}(\Lambda^{n-r}\mathcal{O}_X^n, \Lambda^n\mathcal{O}_X^n)$, $f(\sum a_i e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{n-r}}) = \sum a_i f(e_{i_1} \wedge \cdots \wedge e_{i_{n-r}})$

thus given f, it's equivalent to give a complement $= \beta e$

i.e. $\omega \in \Lambda^r\mathcal{O}_X^n$ ▨ (Given in section in aribitary section on $U(X)$)

(c) Let $\mathcal{F}^i = \bigoplus (S^i(\mathcal{F}') \otimes S^{r-i}(\mathcal{F}''))$, we can do this as the local free sheaves exact sequences are all locally split $\Rightarrow \mathcal{F}^0 = \bigoplus_{i \leq r} S^i(\mathcal{F}') \otimes \bigoplus S^{r-i}(\mathcal{F}'') = S^r(\mathcal{F}' \oplus \mathcal{F}'') = S^r(\mathcal{F})$ ▨

(d) The filtration is samely done;

then $\mathcal{F}^p/\mathcal{F}^{p+1} \cong \Lambda^p\mathcal{F}' \otimes \Lambda^{n-p}\mathcal{F}''$, and take $p = n'$, $n' + n'' = n$ by rank is an additive function $\Rightarrow \mathcal{F}^{n'}/\mathcal{F}^{n'+1} \cong \Lambda^{n'}\mathcal{F}' \otimes \Lambda^{n''}\mathcal{F}''$

$\Lambda^n\mathcal{F}$ by setting $n = r$ ▨

(e) ~~[illegible]~~

To use these filtrations, we use induction for n, only prove for $f^*(T^n\mathcal{F})$ ~~[illegible]~~ is the easiest

~~[illegible]~~, we have $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$

$\Rightarrow 0 \to f^*\mathcal{F}' \to f^*\mathcal{F} \to f^*\mathcal{F}'' \to 0$

$= f^{-1}(T^n\mathcal{F}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \cong (f^{-1}(\mathcal{F}) \otimes f^{-1}(T^{n-1}\mathcal{F})) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ and it done, it suffices

(as $T^n\mathcal{F} = \mathcal{F} \otimes \mathcal{F} \otimes \cdots \otimes \mathcal{F}$, f^{-1} commute with $\otimes$) to show $f^*(S^r(\mathcal{F}') \otimes S^{n-r}(\mathcal{F}''))$

$\cong f^*(\mathcal{F}) \otimes f^*(T^{n-1}\mathcal{F})$ induction $f^*(\mathcal{F}) \otimes T^{n-1}f^*(\mathcal{F})$ $= f^*S^r(\mathcal{F}') \otimes f^*S^{n-r}(\mathcal{F}'')$

$\cong f^*(\mathcal{F}) \otimes T^{n-1}(f^*\mathcal{F}) \cong T^n(f^*\mathcal{F})$ ▨ need to prove $S^r(f^*\mathcal{F}') \otimes S^{n-r}(f^*\mathcal{F}'')$

For this, we take the short exact sequence to be $\mathcal{O}_X$-free, by Hilbert syz it's finite, i.e. we have $0 \to \mathcal{O}_X^{a_1} \to \mathcal{O}_X^{a_2} \to \cdots \to \mathcal{O}_X^{a_N} \to \mathcal{F} \to 0$ ▨

Ex5.17 (a) ($\Leftarrow$) Trivial ($\Rightarrow$) Assume $Y = \bigcup V_i$, V_i affine, each affine is done

then $\forall V \subset Y$, $V = \bigcup (V \cap V_i)$, let $V_i \cap V \subset V = \mathrm{Spec} A$ to be $\mathrm{Spec} A_{f_i} \Rightarrow f^{-1}(V) = \bigcup f^{-1}(\mathrm{Spec} A_{f_i})$

(b) ~~Quasi-compact~~ is trivial

$\forall V \subset Y$, $f^{-1}(V)$ is affine by Ex2.17, f_i's generates $1 \Rightarrow f^{\#}f_i$ generate 1

$\Rightarrow f^{-1}(V)$ quasi-compact ▨ $\quad \Rightarrow$ also affine.

$\Delta^c \subset X \times_Y X$, we prove it's open then Δ closed to complete the proof

Thus we reduce to $Y = \mathrm{Spec} A$ affine and cover $\{D(f_i)\}$ done then $\forall V \subset Y$ and $V = \mathrm{Spec} B$, $f^{-1}(V) = \bigcup f^{-1}(\mathrm{Spec} B_{f_i})$ thus affine by Ex2.17, again ▨

$\forall (x_1, x_2) \in \Delta^c$, $x_1 \neq x_2$ and $f(x_1) = f(x_2)$, take $V \in \mathcal{N}_{f(x_1)}$ then $f^{-1}(V)$ is affine, $f^{-1}(V) \times_Y f^{-1}(V) \subseteq \Delta^c$ and is contained in $\Delta^c \Rightarrow \Delta$ open ▨

Finite $\Rightarrow$ affine

~~Finite~~ Affine is local in target, let $Y = \mathrm{Spec} A$, then $f^{-1}(V)$ covered by finite $\mathrm{Spec} B_i$, with $V = D(\varphi)$, then B_i is finite ~~A-module~~ ▨

$\Rightarrow B_{f^*(\varphi)}$ also $\Rightarrow$ generated by finite A_φ

is a chain $\Rightarrow l\left(\frac{S}{I_V+I_X}(f_i)\right) \geq n$, and by the localization we are the same thing for I Ex 5.4 (compare with Prop 6.6.)

(d) ① We identify Cartier divisors $\Rightarrow l\left(\frac{S}{I_V+I_X}(f_i)\right) = n$ ▨
and Weil divisor, again

Then: $\operatorname{Div}\mathbb{P}^n \xrightarrow{\Phi} \operatorname{Div} X$

$D \longmapsto D.X$

$\Gamma(\mathbb{P}^n, \mathcal{K}^*/\mathcal{O}^*) \xrightarrow{\Phi} \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$

principal $\qquad$ principal

$\{(\mathbb{P}^n, f)\} \longmapsto \{(X, f|_X)\}$

Thus we need show $\forall D$ principal on $X \Rightarrow D = V.X$, $\exists V$

i.e. Φ surjective at subgroup of principal

$D \in \operatorname{Div} X$ principal $\Rightarrow D = (f)$, $f: X \dashrightarrow k$ rational

Claim. We extend f to $\tilde{f}: \mathbb{P}^n \dashrightarrow k$ rational and $D = (\tilde{f}).X$.

Pf of Claim. I don't know

② $\deg D = \deg(\tilde{f}).X = \deg(\tilde{f}) \cdot \deg X = 0 \cdot \deg X = 0$ ▨ (in $\mathbb{P}^n$, poles = zeros)

Ex 6.5* (a)

① $\bar{X} - P \longrightarrow V$

$(x_1, \dots, x_{n+1}) \longmapsto \left(\frac{x_1}{x_{n+1}} : \dots : \frac{x_n}{x_{n+1}}\right)$

Thus let U_i be $V_i \cap V$, V_i the standard

$\Rightarrow \pi^{-1}(U_i) = U_i \times \mathbb{A}^1_k$ by copy x_i at last coordinate $n+1$ ▨ (Thus $\bar{X}-P$ has a $\mathbb{A}^1$-bundle structure over V)

② We have

$$\begin{array}{ccccccccc} 0 \to & \mathbb{Z} & \to & Cl(V) & \to & Cl(U_i) & \to & 0 \\ & \| & & \downarrow \pi^* & & \downarrow \pi^* & & \| \\ 0 \to & \mathbb{Z} & \to & Cl(\bar{X}-P) & \to & Cl(\pi^{-1}(U_i)) & \to & 0 \\ & \| & & \| & & \| & & \| \\ 0 \to & \mathbb{Z} & \to & Cl(\bar{X}-P) & \to & Cl(U_i \times \mathbb{A}^1_k) & \to & 0 \\ & \| & & \| & & \| & & \| \\ 0 \to & \mathbb{Z} & \to & Cl(\bar{X}-P) & \to & Cl(U_i) & \to & 0 \end{array}$$

By Prop 6.5, Prop 6.6 and Ex 6.7(1) (injective)

Then by Five lemma

$\Rightarrow Cl(V) \cong Cl(\bar{X}-P)$ ▨

(b) $0 \to Cl(\mathbb{P}^n) \xrightarrow{f_V} Cl(V) \xrightarrow{\pi^*} Cl(X) \longrightarrow 0$ $\quad$ We prove ① π^* surjective ② $\ker \pi^* = \operatorname{Im} f_V$

$nH_\infty \mapsto n(V.H_\infty) \mapsto n(\pi^*(V.H_\infty))$

$Y \longmapsto \pi^{-1}(Y)$

$V \xleftarrow{\pi} X = C(V)$

Pf. ① Compare with the proof of Prop 6.6. Type 2 is impossible in a variety $\Rightarrow$ nothing to prove ▨

② $\pi^*(n(V.H_\infty)) = n\pi^*(\sum n_i V_i) = \sum n n_i \pi^{-1}(V_i)$

V_i is the component of $V \cap H_\infty$, $\pi^{-1}(V_i) \subset \pi^{-1}(V \cap H_\infty) \subseteq \pi^{-1}(H_\infty) = 0$

$\Rightarrow \operatorname{Im} f_V \subset \ker \pi^*$, Converse same: $\pi^*(D) = 0 \Rightarrow \sum n_i \pi^{-1}(D_i) = 0$

$\Rightarrow \pi^{-1}(D_i) = 0 \Rightarrow D_i \subset H_\infty \Rightarrow D_i = V \cap H_\infty$ ▨

(By the discussion below, we know $\mathbb{P}^n$ can't be replaced easily).

① Only verify $X \times \mathbb{P}^n_k$ satisfy smooth at codimension 1.

This follows Prop 6.6. $X \times U_i$ smooth $\Longrightarrow X \times \mathbb{P}^n_k$ also ▨

② $Cl(X) \times \mathbb{Z} \xrightarrow{\Psi} Cl(X \times \mathbb{P}^n)$ $\quad$ We verify ① $[D \times nH_0] \in Cl(X \times \mathbb{P}^n)$

$([D], n) \mapsto [D \times nH_0]$ $\quad$ ② $D_1 \sim D_2 \Rightarrow D_1 \times nH_0 \sim D_2 \times nH_0$

③ Injective ④ Surjective

Pf. ① By right, it suffices to prove ① ②: well-defined

① By the topology of $X \times \mathbb{P}^n$ can be defined by bihomogeneous polynomial

$\Rightarrow D \times nH_0$ is codimension 1 subscheme

② $D_1 - D_2 = (f)$

$\Rightarrow (D_1 \times nH_0) - (D_2 \times nH_0) = (f) \times nH_0$

$[H_0] = (g) \Rightarrow$ principal ▨

By Prop 6.5.

$$\begin{array}{ccccccc} \mathbb{Z} & \to & Cl(X \times \mathbb{P}^n) & \to & Cl(X \times U_i) & \to & 0 \\ \| & & \| & & \| & & \\ \mathbb{Z} & \to & Cl(X \times \mathbb{P}^n) & \to & Cl(X \times \mathbb{A}^n) & \to & 0 \\ \| & & \| & & \| & & \\ \mathbb{Z} & \to & Cl(X \times \mathbb{P}^n) & \to & Cl(X) & \to & 0 \end{array}$$

$\mathbb{Z} = \mathbb{P}_2(Cl(X))$ $\quad$ $X \times \mathbb{P}^n \supset \dots \cap (X \times \mathbb{A}^n)$

▨ $0 \to Cl(\mathbb{P}^n) \to Cl(X \times \mathbb{P}^n) \underset{\Psi}{\rightleftarrows} Cl(X) \to 0$

split $\Rightarrow Cl(X \times \mathbb{P}^n) \cong Cl(X) \oplus \mathbb{Z}$ ▨

We still verify ① $X \times \mathbb{P}^n \xrightarrow{p_2} \mathbb{P}^n \longrightarrow 0$

$Cl(X \times \mathbb{P}^n) \xleftarrow{p_2^*} Cl(\mathbb{P}^n) \leftarrow 0$

$X \times nH_0 \longleftarrow [nH_0]$

② $\varphi p_1^* = \operatorname{Id}_{Cl(X)}$ $\quad$ If $[X \times nH_0] = 0$

$\varphi p_1^*([D])$ $\quad \Rightarrow [nH_0] = 0 \Rightarrow p_2^*$ injective ▨

$= \varphi([D] \times \mathbb{P}^n)$

$= [D \times \mathbb{A}^n] \cong [D]$ ▨

Ex 6.2*. We pullback the divisor $V \subset \mathbb{P}^n$ into $X \subset \mathbb{P}^n$,

X is smooth $\Rightarrow Cl(X) = \operatorname{Pic}(X)$

Thus we do (a) & (b) by considering pullback the invertible sheaf.

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}^n \\ \downarrow & \square & \downarrow \\ \operatorname{Pic}(X) & \xleftarrow{p} & \operatorname{Pic}(\mathbb{P}^n) \end{array}$$

$\{(U_i \cap X, g_i|_{U_i \cap X})\} \longleftarrow \{(U_i, g_i)\}$

It's always well-defined, but we should verify it's same as defined in (a)

$\operatorname{Pic}(\mathbb{P}^n) \xrightarrow{p} \operatorname{Pic}(X)$

$\{(U_i, g_i)\} \longmapsto \{(U_i \cap X, g_i|_{U_i \cap X})\} = \{(V_i, \dots)\}$

$Cl(\mathbb{P}^n) \xrightarrow{A} Cl(X)$

$\{(U_i, g_i)\}$ Cartier $\qquad \{(V_i, g_i|_{V_i})\}$ Cartier

$D = \sum_{Y \cap U_i \neq \emptyset} v_Y(g_i) Y$ Weil $\qquad D = \sum_{Y \cap V_i \neq \emptyset} v_Y(g_i|_{V_i}) Y$

By (a)'s definition $\quad \sum_{Z_i} v_{Z_i}(g_i|_{Z_i}) Z_i = \sum_{\substack{Y \cap U_i \neq \emptyset \\ Y \subset X}} v_Y(g_i|_{V_i}) Y$

$Z_i \subset Y \cap X$ component

$\Rightarrow A(D) = \sum_{Y \cap U_i \neq \emptyset} v_Y(g_i) (Y.X)$

$= \sum_{Y \cap U_i \neq \emptyset} v_Y(g_i) \sum_{\substack{Z_i \subset Y \cap X \\ \text{component}}} v_{Z_i}(g_i|_{Z_i}) Z_i$

$= \sum_{Z_i \subset X} v_{Z_i}(g_i|_{Z_i}) Z_i$ $\quad$ Replace Z_i by Y, complete

(c) $\deg(D.X) = \deg\left(\sum_i n_i V_i . X\right)$

$= \sum_i n_i \deg(V_i . X)$

$(\deg D) \cdot (\deg X) = \sum_i n_i (\deg V_i)(\deg X)$

$\Rightarrow$ Assume D is prime $D = V$

By (a) $\Rightarrow \deg(D.X) = \sum_i v_{Y_i}(f_i) \deg Y_i$

$= \sum_i i(X, V; Y_i) \deg Y_i$

Bezout $\deg X \deg V$ ▨

Left is showing $v_{Y_i}(f_i) = i(X, V; Y_i)$

$i(X, V; Y_i) = l_{\mathcal{O}_{Y_i}}\left(\frac{S}{I_X + I_V}\right)$

$= l\left(\frac{S}{I_X + I_V + (f_i)}\right)$ First let $n = v_{Y_i}(f_i)$

$\Rightarrow \forall f \in \frac{S}{I_X+I_V}(f_i)$, $f = g f_i^n \Rightarrow (f) \dots$

Ex6.3*(c) It's equivalent to say $Cl(\mathrm{Spec}\,S(V))=0 \Leftrightarrow Cl\,V\cong\mathbb{Z}$ (By Prop 6.2)

$\Leftrightarrow \mathrm{Spec}\,S(V)\cong V\times\mathbb{P}^m$ for proper m

We are dealing variety, thus $\mathrm{Spec}\,S(V)=\mathrm{Max}\,S(V)=X \subset \mathbb{P}^{n+1}$

$\Rightarrow X\cong V\times\mathbb{P}^1$ ▨

(d)
$$\begin{array}{ccc} \mathrm{Spec}\,\mathcal{O}_p & \longrightarrow & X \\ \downarrow & & \downarrow \\ Cl(\mathrm{Spec}\,\mathcal{O}_p) & \longleftarrow & Cl(X) \\ D\cap \mathrm{Spec}\,\mathcal{O}_p & \longmapsto & D \end{array}$$

① injective: $D_1\sim D_2 \Leftrightarrow D_1.\mathrm{Spec}\,\mathcal{O}_p\sim D_2.\mathrm{Spec}\,\mathcal{O}_p$

② surjective: $\forall D\in Cl(\mathrm{Spec}\,\mathcal{O}_p)\Rightarrow D=D'\cap\mathrm{Spec}\,\mathcal{O}_p$

We reduce to affine: $\exists\, U=\mathrm{Spec}\,A\in N_p$

Thus
$$\begin{array}{ccc} A & \longrightarrow & A_p \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,A & \longleftarrow & \mathrm{Spec}\,A_p \\ \downarrow & & \downarrow \\ Cl(\mathrm{Spec}\,A) & \longrightarrow & Cl(\mathrm{Spec}\,A_p) \end{array}$$

X is variety $\Rightarrow \mathrm{Spec}\,A$, $\mathrm{Spec}\,A_p$ all satisfy (*)

then it's algebraic ▨

Ex6.4. (An algebraic preparation of Ex6.5) By Hint, $A=\overline{k[x_1\cdots x_n]}^{\,k(x_1\cdots x_n)[z]/(z^2-f)}$

$\forall \alpha\in K$, $\alpha=g+hz$, $g,h\in k(x_1\cdots x_n)$

The minimal polynomial of α is $x^2-2gx+(g^2-h^2f)$

Thus α integral over A $(g+hz)^2-2g(g+hz)+g^2-h^2z^2$

$\Leftrightarrow 2g\in k[x_1\cdots x_n]\Rightarrow g\in k[x_1\cdots x_n]\Rightarrow h\in k[x_1\cdots x_n]$

Converse is trivial $\Rightarrow A=k[x_1\cdots x_n]$ ▨

Thus we apply Ex6.3

$X=\mathbb{P}_k\times\mathbb{P}_k$

$\Rightarrow Cl(X)\oplus\mathbb{Z}=Cl(\ldots)$

$\Rightarrow Cl(X)=\mathbb{Z}$ ▨

Ex6.5*. (a) Take $f=-\sum x_i^2$, apply Ex6.4 ▨

(b) let $\tilde{x}_0=x_0+\sqrt{-1}x_1$; $\tilde{x}_1=-x_0+\sqrt{-1}x_1$ by k algebraically closed

(1) $r=2$, nothing different with (6.5.2)

(2) $r=3$, By I Ex3.4 $\boxed{x_0x_1=x_2^2+x_3^2}=Cl(\mathbb{P}_k^1)$ $\left(\cong x_0x_1=x_2x_3\cong\mathbb{P}_k\times\mathbb{P}_k\right.$ in $\mathbb{P}_k^3$, but not $\mathbb{A}_k^2\left.\right)$

By Ex6.3. $\Rightarrow Cl(C(\mathbb{P}_k^1))=Cl(\mathbb{P}_k\times\mathbb{A}_k)\cong Cl(\mathbb{P}_k)=\mathbb{Z}$ ▨

(3) $r\geq 4$ $\forall D\in \mathrm{Div}X$, we show it principal. By Prop 6.2, done ▨

(By Nagata's thm $\Rightarrow \mathcal{O}_{X,0}$ UFD $\Rightarrow X$ locally factored $\Rightarrow Cl(X)=0$ when $r\geq 4$

(c) (2) We done in (6.6.1) ▨

(1) We have $0\to\mathbb{Z}\to Cl(Q)\to\mathbb{Z}/2\mathbb{Z}\to 0$ by Ex6.2* and Ex6.5*(b)(1).

$nH_0\mapsto 2n(Q.H_0)\mapsto 0$

$n(Q.H_0)\mapsto 0$ or 1

$\Rightarrow Cl(Q)=\mathbb{Z}$ ▨

(3) $0\to\boxed{\mathbb{Z}\to Cl(Q)}\to 0\to 0$ $\Rightarrow \mathbb{Z}\cong Cl(Q)$ ▨

(d) The Hint is done by Nagata's thm. $S(Q)$ is UFD

Then $i(V,Q,Y)=\nu_Y(f)$, f is the local equation of V

We need show $\exists f$, then V is constructed conversely: $\nu_Y(f)=1$

This is trivial: let f be the locally parameter ($r\geq 4$ ensures $\mathcal{O}_{X,p}$ is UFD) ▨ $\Rightarrow$ DVR

b) This solves the classification of the group structure on elliptic curves.
Alternative way is by direct computation on Weierstrass forms)

② $\operatorname{Pic}(X) = \operatorname{Cl}(X-Z)$, Z is the cuspidal point.
Now we apply normalization to X get $\tilde{X}$, $\operatorname{Pic}(\tilde{X})$ is computed by
$0 \to \mathbb{Z} \to \operatorname{Pic}(\tilde{X}) \to \operatorname{Pic}(\tilde{X}-\tilde{Z}) = \operatorname{Pic}(X-Z) = \operatorname{Pic}(\mathbb{A}^1_k) = 0 \to 0$
$\Rightarrow \operatorname{Pic}(\tilde{X}) = \mathbb{Z}$ (we can't directly apply to X due to X not non-singular)
$\Rightarrow 0 \to \bigoplus_{P \in X} \mathcal{O}^*_{\tilde{P}}/\mathcal{O}^*_P \to \operatorname{Pic}(X) \to \mathbb{Z} \to 0$

The direct sum gives it natural addition $\Rightarrow$ left is showing $\mathcal{O}^*_{\tilde{P}}/\mathcal{O}^*_P$ corresponding point in $\mathbb{G}_a$ naturally.
But it's same as $P \mapsto P - P_0$ ☒

③ Same way, but the nodal point can't taken, corresponding to $\mathbb{G}_m$ ☒

Ex6.10 (a) $K_0(X)$ is generated by constant sheaf, i.e. $\forall \mathcal{F} \in \operatorname{Coh}(X)$
$0 \to \mathcal{O}_X \to \mathcal{O}_X^n \to \mathcal{F} \to 0$
for $X = \mathbb{A}^1_k$ in particular
$\Rightarrow \mathcal{F} = \tilde{M}$ and $M = k[x_1 \dots x_n]/I$
$\Rightarrow K_0(X) \cong \mathbb{Z}$ ☒
$([\mathcal{O}_X^m] \leftrightarrow m)$

(b) First $\operatorname{rank}(\mathcal{F} + \mathcal{G})$
$= \dim_K (\mathcal{F}+\mathcal{G})_\xi = \dim_K \mathcal{F}_\xi + \dim_K \mathcal{G}_\xi = \operatorname{rank}\mathcal{F} + \operatorname{rank}\mathcal{G}$
and $\forall n \in \mathbb{Z}_{\geq 0}$, let $\mathcal{F} = \mathcal{O}_X^n$, for $n<0$, let $\mathcal{F} = \mathcal{O}_X^{\oplus(-n)}$ ☒

(c) By Ex5.15, $\forall [\mathcal{F}] \in K(X-Y)$, $\mathcal{F}$ can be extend to $\mathcal{F}' \in \operatorname{Coh}(X)$ as $X-Y \subset X$ is open subset;
$\Rightarrow [\mathcal{F}'] \mapsto [\mathcal{F}]$ is surjective //
For the middle, it's chain complex is obvious $\mathcal{F}|_{X-Y} = 0$
it suffices to prove: given $\mathcal{F} \in \operatorname{Coh}(X)$ and $\mathcal{F}|_{X-Y} \cong 0$
then $\mathcal{F} \sim j_! \mathcal{F}'$ for some $\mathcal{F}' \in \operatorname{Coh}(Y)$.
Pf: $\mathcal{F}|_{X-Y} \sim 0 \Rightarrow \mathcal{F} \sim j_*(\mathcal{F}|_Y) + i_!(\mathcal{F}|_{X-Y}) \sim j_*(\mathcal{F}|_Y) + 0$ ☒ (By Ex2.19, we have $0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0$)

Ex6.11 (a) ① We show $\mathcal{O}_D \sim \sum n_i k(P_i)$
$D = \sum P_i n_i \Rightarrow \mathcal{O}_D = \sum \mathcal{O}_{P_i, D} = \sum \mathcal{O}_{P_i, D}(P_i) n_i$ it's a subsheaf of $\sum n_i k(P_i)$
as for $\forall P_i$, $\mathcal{O}_{P_i, D} \subset k$, then define $\sum n_i \frac{k}{\mathcal{O}_{P_i,D}}(P_i) = \mathcal{F}$
$\Rightarrow 0 \to \mathcal{O}_D \to \sum n_i k(P_i) \to \sum \frac{k}{\mathcal{O}_{P_i,D}}(P_i) \to 0$
then I claim $\sum \frac{k}{\mathcal{O}_{P_i,D}}(P_i) \sim 0$, this is direct: X nonsingular $\Rightarrow \mathcal{O}_{P_i,X} \cong k[x]$ and $\mathcal{O}_{P_i,D} = k$ ☒
② By (6.18) $\mathcal{O}_D \cong \mathcal{L}(-D)$
For $D \sim D'$, $\mathcal{L}(-D) = \mathcal{L}(-D') \Rightarrow \mathcal{O}_D \cong \mathcal{O}_{D'} \Rightarrow \psi(D) = \psi(D')$ ☒
(b) ① is local, let $X = \operatorname{Spec} A$, X nonsingular $\Rightarrow A$ is regular local ring (I, Thm5.1) $\Rightarrow$ By a thm de Serre, $\operatorname{pd} M \leq \dim A$, and $\mathcal{F} = \tilde{M}$
thus $\operatorname{pd} M \leq 1$ and $\mathcal{F}$'s projective resolution $\Leftrightarrow$ M's projective resolution $\Rightarrow \operatorname{pd}\mathcal{F} \leq 1$ ☒

Ex6.6 (a) It's smooth, thus we identify the group with the one in Example 6.11.3
then P, Q, R is colinear by computation (It's wrong: we now can't do this until Ex6.7*)
(a) ($\Rightarrow$) l passes $P, Q, R \Rightarrow l$ only passes P, Q, R by Bezout $\Rightarrow P+Q+R \sim 3P_0$
$\Leftrightarrow P+Q+R = 0$
($\Leftarrow$) let l pass P, Q, l intersects at $T \in X$
$\Rightarrow P+Q+T \sim 3H_0 \sim P+Q+R$ by its group structure we can eliminate $\Rightarrow T = R$ ☒
(b) ($\Rightarrow$) $P+P = 0 \Leftrightarrow P+P+P_0 \sim 3P_0$
$\Rightarrow$ The line pass P and P intersect P_0
The tangent line at P
($\Leftarrow$) Then
$P+P+P_0 \sim 3P_0 \Rightarrow P+P = 2P_0 = 0$ ☒
(c) $3P = 0 \Leftrightarrow P+P+P \sim 3P_0 \Leftrightarrow P \sim P_0$
($\Rightarrow$) If the tangent line pass $Q \Rightarrow P+P+Q \sim 3P_0 \sim P+P+P$
$\Rightarrow P = Q \Rightarrow$ The tangent line pass P intersect $X \geq 3$
($\Leftarrow$) By Thm de Bezout $\Rightarrow$ equal 3 $\Rightarrow P+P+P \sim 3P_0$ ☒
(d) Claim: The only $\mathbb{Q}$-points are $(0,1,0), (0,0,1), (1,0,1), (1,0,-1)$
Thus just $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$ ☒
(The proof of Claim is hardest: This particular example is so proper, if you can solve it, then you can use some arithmetic way to prove Mordell-Weil thm!)

Ex6.7* Recall $\mathbb{G}_m = (\mathbb{A}^1_k)^*, X)$
We define $\mathbb{G}_m \to X$
$(\operatorname{Spec} k[t])^* \ni t \mapsto (1, \sqrt{t \dots}, t)$ ☒
(It's clear why we can't let $t=0$)

Ex6.8 (a) $f^*: \operatorname{Pic} Y \to \operatorname{Pic} X$ ① Well-defined: $\mathcal{L}_1 \cong \mathcal{L}_2 \Rightarrow f^*\mathcal{L}_1 \cong f^*\mathcal{L}_2$
$\mathcal{L} \mapsto f^*\mathcal{L}$ ② $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1} \mapsto f^*\mathcal{L}_1 \otimes f^*\mathcal{L}_2^{-1}$
This's by Prop5.2(e)
Another thing is $f^*\mathcal{L} \in \operatorname{Pic} X$
Pf: ① Trivial by definition of sheaves ☒ ② $f^*(\mathcal{L}_1 \otimes \mathcal{L}_2^{-1})(U)$
$= (\mathcal{L}_1 \otimes \mathcal{L}_2^{-1})(f(U))$
$= \mathcal{L}_1(f(U)) \otimes (\mathcal{L}_2(f(U)))^{-1}$
$= f^*\mathcal{L}_1 \otimes f^*\mathcal{L}_2^{-1}(U)$ ☒
(b) Done in Ex6.2* ☒ (c) also)

Ex6.9* (a) $0 \to \frac{\pi_*\mathcal{O}^*_{\tilde{X}}}{\mathcal{O}^*_X} \to \frac{\mathcal{K}^*}{\mathcal{O}^*_X} \to \frac{\mathcal{K}^*}{\pi_*\mathcal{O}^*_{\tilde{X}}} \to 0$
exact is trivial
and $\Gamma\left(X, \frac{\pi_*\mathcal{O}^*_{\tilde{X}}}{\mathcal{O}^*_X}\right) = \bigoplus_{P \in X} \left(\frac{\pi_*\mathcal{O}^*_{\tilde{X}}}{\mathcal{O}^*_X}\right)_P = \bigoplus_{P \in X} \frac{(\pi_*\mathcal{O}^*_{\tilde{X}})_P}{(\mathcal{O}^*_X)_P} = \bigoplus_{P \in X} \frac{\tilde{\mathcal{O}}^*_P}{\mathcal{O}^*_P}$ by the local ring of $\tilde{X}$ just $\tilde{\mathcal{O}}^*_P$
$\Gamma\left(X, \frac{\mathcal{K}^*}{\mathcal{O}^*_X}\right) = \operatorname{CaCl}(X) = \operatorname{Pic}(X)$
$\Gamma\left(X, \frac{\mathcal{K}^*}{\pi_*\mathcal{O}^*_{\tilde{X}}}\right) \cong \Gamma\left(\tilde{X}, \frac{\mathcal{K}^*}{\mathcal{O}^*_{\tilde{X}}}\right) = \operatorname{CaCl}(\tilde{X}) = \operatorname{Pic}(\tilde{X})$
left is showing π^* surjective $\tilde{X} \xrightarrow{\pi} X \to 0$ $\forall \{(U_i, f_i)\} \in \operatorname{CaCl}(\tilde{X})$
We can recover f_i over X from
$0 \leftarrow \operatorname{Pic}(\tilde{X}) \xleftarrow{\pi^*} \operatorname{Pic}(X)$ $f_i \in \mathcal{K}(U_i) = \mathcal{K}(\pi(U_i))$ $\Rightarrow \exists f_i \in \mathcal{K}(U_i) \Rightarrow \{(U_i, f_i)\}$ ☒

Ex6.12. By Ex6.11, we have $K(X) \xrightarrow{\det} \mathrm{Pic}(X) \longrightarrow \mathbb{Z}$

then $\mathrm{Coh}(X) \xrightarrow{\mathcal{F}} K(X) \xrightarrow{\det} \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z}$ is the desired construction (To distinct, denoted as $\underline{\mathrm{Deg}}$)

(1) $\mathrm{Deg}\,\mathcal{L}(D) = \deg(\det(\varphi(\mathcal{L}(D))))$

$\overset{(a)}{=} \deg(\det(\varphi(D)))$ (a) is obvious: all of this is additive ▨

$\overset{(b)}{=} \deg(D)$ ▨

(b) det $(\varphi(D))(U_i)$ $= (\Lambda^{n_i}\varphi(D))^{\mathrm{sgn}(n_i)}(U_i)$

If $n_i > 0$, then $\Lambda^{n_i}\varphi(D)(U_i)$ generated by $P_i \in U_i$ determined $\frac{1}{f_i}$ in Cartier divisor; $n_i < 0$ is similar

$\Rightarrow \det(\varphi(D))(U_i) = \mathcal{L}(D)(U_i)$ ▨

(2) $\mathcal{F}$ torsion $\Rightarrow \mathcal{F} \simeq 0 \Rightarrow \mathcal{F} \cong \mathcal{L}(D)$ with some D (as it only skyscraper in finite generic point $X-U$) $\Rightarrow \mathcal{F}_{P_i} = k^{n_i}$, with $D = \sum_i n_i P_i$

$\Rightarrow \mathrm{Deg}\,\mathcal{F} = \sum_i n_i = \sum \mathrm{length}\,\mathcal{F}_{P_i}$ ▨

(4) Run induction, by (3) the inductive step is done

thus we reduce to invertible sheaf, then $\mathrm{Deg}\,\mathcal{L} = \deg D$ is unique, as $\mathcal{L}$ is associated with a line bundle ▨ (by (1))

② Trivial, we check it locally: $\det \mathcal{F}|_U = \Lambda^r(\mathcal{E}|_U) \otimes \Lambda^r(\mathcal{G}|_U)$

$= \Lambda^r(\mathcal{O}_X|_U^m) \otimes \Lambda^r(\mathcal{O}_X|_U^n)$ ▨

(3) ① Let's tensor $\mathcal{F}$ with an ample invertible sheaf $\mathcal{L}$

$\Rightarrow \mathcal{F} \otimes \mathcal{L}^n$ generated by global sections: $s_1 \cdots s_m$

$\Rightarrow \mathcal{O}_X^m \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \longrightarrow 0$ (Now we can see why we do this: it doesn't change the stalks/local information)

$\downarrow \qquad\qquad \downarrow$

$\mathcal{O}_{X,\eta}^m \longrightarrow \mathcal{F}_\eta \otimes \mathcal{L}_\eta^n \longrightarrow 0$ And they are vector-spaces

$\| \qquad\qquad \|$ $\Rightarrow \exists\, r < m : \mathcal{O}_{X,\eta}^r \cong \mathcal{F}_\eta$

$\mathcal{O}_{X,\eta}^m \longrightarrow \mathcal{F}_\eta$ (as $\mathcal{L}_\eta = \mathcal{O}_\eta$) $\to 0$

$\Rightarrow \mathcal{O}_X^r \overset{\text{generic}}{\cong} \mathcal{F} \otimes \mathcal{L}^n \Rightarrow \exists\, U : \mathcal{O}_U^r \cong \mathcal{F}|_U \otimes \mathcal{L}^n|_U$

and $X-U$ closed, by X is curve $\Rightarrow X-U$ is finite

I claim: $\varphi: \mathcal{O}_X^r \longrightarrow \mathcal{F} \otimes \mathcal{L}^n$ is injective

Pf. $\ker\varphi$ is skyscraper sheaf at $X-U$ finite

$\Rightarrow \forall p \in X-U$, we take the stalk: $0 \to (\ker\varphi)_p \to (\mathcal{O}_X^r)_p \to (\mathcal{F} \otimes \mathcal{L}^n)_p \to \mathrm{coker}\,\varphi$

$\Rightarrow (\ker\varphi)_p \trianglelefteq \mathcal{O}_{X,p}^r$, which is free, as over $\mathcal{O}_{X,p}$ is a PID $\Rightarrow (\ker\varphi)_p$ also free, however $(\ker\varphi)_p$ is skyscraper $\Rightarrow (\ker\varphi)_p = 0$

$\Rightarrow \ker\varphi = 0$ ▨

$\Rightarrow$ We have $0 \to \mathcal{O}_X^r \to \mathcal{F} \otimes \mathcal{L}^n \to \mathrm{coker}\,\varphi \to 0$

tensor with $\mathcal{L}^{-n} \Rightarrow 0 \to \mathcal{O}_X^r \otimes \mathcal{L}^{-n} \to \mathcal{F} \to \mathrm{coker}\,\varphi \otimes \mathcal{L}^{-n} \to 0$

I claim: It's just the desired sequence: ① $\mathcal{O}_X^r \otimes \mathcal{L}^{-n} \cong \mathcal{L}(D)^r$ for some D; ② $\mathrm{coker}\,\varphi \otimes \mathcal{L}^{-n}$ is torsion

Pf. ① $\mathcal{L}^n$ is line bundle $\Rightarrow$ so is $\mathcal{L}^{-n}$

$\Rightarrow \mathcal{L}^{-n} \cong \mathcal{L}(D)$ for some $D \Rightarrow \mathcal{O}_X^r \otimes \mathcal{L}(D) = \mathcal{L}(D)^r$ ▨

② Denote it as $\mathcal{T}$, then $0 \to \mathcal{L}(D)_\eta^r \to \mathcal{F}_\eta \to \mathcal{T}_\eta \to 0$

and $\dim_{K(X)} \mathcal{F}_\eta = \dim_{K(X)} \mathcal{L}(D)_\eta^r = r \Rightarrow$ the isomorphic as $K(X)$-vector space

$\Rightarrow \mathcal{T}_\eta = 0$ ▨

② is easier: $\mathcal{F} - \mathcal{O}_X^r \sim \mathrm{coker}\,\varphi = \mathcal{T} \otimes \mathcal{L}^n$ correspond to a divisor D, as $\mathcal{L}^n$ is and $\mathcal{T}$ is torsion ▨

3.2.4(b) Let's describe $\mathcal{L}$ over X by $(\mathcal{L}|_{U_1}, \mathcal{L}|_{U_2})$
denoted $(\mathcal{L}_1, \mathcal{L}_2)$, $\mathcal{L}_i \in \mathrm{Pic}(\mathbb{A}^1_k)$, $\mathcal{L}_1|_{\mathbb{A}^1_k \setminus 0} = \mathcal{L}_2|_{\mathbb{A}^1_k \setminus 0}$
$\mathrm{Pic}(\mathbb{A}^1_k)=0 \Rightarrow (\mathcal{L}_1, \mathcal{L}_2) \cong (\mathcal{O}_{\mathbb{A}^1_k}, \mathcal{O}_{\mathbb{A}^1_k}) = (\mathcal{O}_{U_1}, \mathcal{O}_{U_2})$
The isomorphism however can be not canonical on X
thus $(\mathcal{L}_1, \mathcal{L}_2) \overset{\text{can}}{\cong} (\mathcal{O}_{\mathbb{A}^1}, \mathcal{L}_2 \otimes \mathcal{L}_1^{-1})$, we consider $\mathcal{L}_2 \otimes \mathcal{L}_1^{-1}$
$\mathcal{L}_2 \otimes \mathcal{L}_1^{-1}|_{\mathbb{A}^1_k \setminus 0} = \mathcal{O}_{\mathbb{A}^1_k \setminus 0}$, but $\underline{(\mathcal{L}_2 \otimes \mathcal{L}_1^{-1})_0 \in \mathbb{Z}} \Rightarrow (\mathcal{L}_2 \otimes \mathcal{L}_1^{-1})_0$ is an isomorphism class
$\Rightarrow \mathrm{Pic}(X) = \mathbb{Z}$ ▨ (Due to it corresponds to $n[0]$ a divisor Weil.)
$\Gamma(X, \mathcal{L}) = \Gamma(X, \mathcal{L}(n[0])) \ni (s,t)$, with $s|_{U_1} = t|_{U_2}$
For $\mathcal{L}_x$, $x \neq 0$, it's obvious that and
generated by $s_x = t_x \Leftrightarrow n \geq 0$, thus $n<0$ is impossible
(This is because in $\mathbb{A}^1 = \mathrm{Spec}\, k[t]$, for $n<0$, $\Gamma(\mathbb{A}^1, \mathcal{L}(n[0])) = 0$).
For $n=0$, it's trivial that globally section generated
For $n>0$, consider any of $0 \in U_i$, let $i=1$
$\Rightarrow (k[t])_0 = \mathcal{O}_{0,X} \cong \frac{1}{t^n} k[t]$, it can't be simply generate, thus also impossible
$\Rightarrow$ Only $n=0$ ▨
Thus nonample ▨ (Due to that: if $\mathcal{L}(n[0])$ ample
$\Rightarrow \exists n: \mathcal{L}(n[0]) \otimes \mathcal{L}(n_0[0])^m = \mathcal{L}(n+mn_0[0])$
is generated by global sections (gbgs), contradiction ▨)

Ex7.5 (a) $\mathcal{O}_X^I \to \mathcal{F} \otimes \mathcal{L}^n \to 0$
and $\mathcal{O}_X^J \to \mathcal{M} \to 0$
$\Rightarrow \mathcal{O}_X^{I+nJ} \to \mathcal{F} \otimes (\mathcal{L}^n \otimes \mathcal{M}^n) \to 0$ thus $\mathcal{L} \otimes \mathcal{M}$ is ample ▨
(b) $\mathcal{O}_X^I \to \mathcal{F} \otimes \mathcal{L}^{n_1} \to 0$
$\mathcal{O}_X^J \to \mathcal{M} \otimes \mathcal{L}^{n_2} \to 0$
$\Rightarrow$ we can let $n_1 = dn_2$ by $n_1 \gg 0$ (as $n_1 + d$, the left is a power of $\mathcal{L}$, gbgs)
then $\mathcal{O}_X^{I \times J} \to \mathcal{F} \otimes (\mathcal{M} \otimes \mathcal{L}^{n_2+n_1})^d \to 0$
$\Rightarrow \mathcal{M} \otimes \mathcal{L}^{n_2+n_1}$ ample ▨
(c) $\mathcal{O}_X^I \to \mathcal{F} \otimes \mathcal{L} \to 0$
$\mathcal{O}_X^J \to \mathcal{F} \otimes \mathcal{M} \to 0$
$\Rightarrow \mathcal{O}_X^{I \times J} \to (\mathcal{F} \otimes \mathcal{L}^{d_1} \otimes \mathcal{M}^{d_2}) \to 0$
$\mathcal{O}_X^I \to \mathcal{F} \otimes (\mathcal{L} \otimes \mathcal{M})^d \to 0 \Rightarrow \mathcal{L} \otimes \mathcal{M}$ ample ▨
$\mathcal{O}_X^I \to \mathcal{F} \otimes \underline{(\mathcal{L} \otimes \mathcal{M})^d \otimes \mathcal{L}^{d_1-d} \otimes \mathcal{M}^{d_2-d}} \to 0$
(All has gbgs)
(d) $\mathcal{L} = \varphi_1^*(\mathcal{O}(1))$
$\mathcal{M} = \varphi_2^*(\mathcal{O}(1))$
i.e. $X \xrightarrow{\varphi_1} \mathbb{P}^n_{\mathbb{Z}}$ the immersion
$\xrightarrow{\varphi_2} \mathbb{P}^m_{\mathbb{Z}}$ a morphism
and a Segre map $\mathbb{P}^n_{\mathbb{Z}} \times \mathbb{P}^m_{\mathbb{Z}} \hookrightarrow \mathbb{P}^N_{\mathbb{Z}}$ is closed immersion
$\Rightarrow X \xrightarrow{(\varphi_1, \varphi_2)} \mathbb{P}^N_{\mathbb{Z}}$ and $\varphi^*(\mathcal{O}(1)) = \varphi^*(p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{O}(1))) = \varphi_1^*(\mathcal{O}(1)) \otimes \varphi_2^*(\mathcal{O}(1))$
$= \mathcal{L} \otimes \mathcal{M}$ ▨ (Same as Ex5.12)

Ex7.1. It's equivalent to prove $f_p: \mathcal{L}_p \to \mathcal{M}_p$ surjective $\Rightarrow f_p$ is isomorphism
and $\mathcal{L}_p, \mathcal{M}_p$ are $\mathcal{O}_{p,X}$-module of rank 1 (stalk is locally)
$\Rightarrow \mathcal{L}_p \cong \mathcal{O}_{p,X} \cong \mathcal{M}_p$, and locally, $\mathcal{O}_{p,X} = A_p$

We claim.
$$\begin{tikzcd} A_p \arrow[r] & A_p \arrow[r] & 0 \\ A \arrow[u] \arrow[r, Rightarrow] & A \arrow[u] \end{tikzcd}$$

Pf of injective:
$$\begin{tikzcd} & A_p \arrow[r] & A_p \\ 0 \arrow[r] & A \arrow[u] \arrow[r, "f"] & A \arrow[u] \arrow[r] & 0 \end{tikzcd}$$
$\frac{a}{1} \mapsto \frac{f(a)}{1}$
$\Rightarrow uf(a) = 0$, $\exists u \notin p$
$\Rightarrow f(v)f(a) = 0$, $\exists v \notin p$
$\Rightarrow va = 0$, $\exists v \notin p$
$\Rightarrow \frac{a}{1} = 0$ in $A_p \Rightarrow$ injective ▨

Ex7.2.
$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathbb{P}^n_k \arrow[d] \\ \mathbb{P}^m_k & \mathbb{P}^m_k - L \arrow[l] \end{tikzcd}$$
$x \mapsto (s_0(x) : \cdots : s_n(x))$
$(t_0(x) : \cdots : t_m(x))$ $(S_0(x), \cdots, S_n(x))$

By the theory of matrix $\begin{pmatrix} t_0(x) \\ \vdots \\ t_m(x) \end{pmatrix}$ $A_{m \times n}$ (RK. It has form $m \times n$ due to $m \geq n$)
Then the matrix can be decomposited into $A = B_{m \times n} \times C_{n \times n}$
$\Rightarrow$ it just the composition, what we want ▨
$C_{n \times n}$: Automorphism of $\mathbb{P}^n$; By Example 7.1.1.
$B_{m \times n} = \begin{array}{c} \begin{pmatrix} B_{n \times n} \\ 0 \end{pmatrix} \end{array} \begin{array}{l} n \\ m-n \end{array}$ L generated by the bottom all zero and the form is just projection

Ex7.3.
(a) By Thm7.1. it corresponds to $\mathcal{O}(d)$ invertible over $\mathbb{P}^n_k$, and a set of $s_0 \cdots s_m \in \Gamma(\mathbb{P}^n_k, \mathcal{O}(d))$
$\varphi(\mathbb{P}^n_k) = \{(s_0(x), \cdots s_m(x)) \mid x \in \mathbb{P}^n_k\}$
$\mathcal{O}(d) = \varphi^*(\mathcal{O}(1))$ $\Rightarrow$ these degree d polynomial $s_0 \cdots s_m$ have no common zero in $\mathbb{P}^n_k \Rightarrow n > m$ is impossible: in this case they must have,
$\Rightarrow$ if not $s_i = c_i$ constant, then $m \geq n$ must
and no common zero $\Rightarrow \dim \varphi(\mathbb{P}^n_k) = n$ ▨
(b) Observation. $\dim \Gamma(\mathbb{P}^n_k, \mathcal{O}(d)) = \binom{n+d}{n}$ corresponds to the d-uple map (Veronese)
Then
$$\begin{pmatrix} y_0 \\ \vdots \\ y_m \end{pmatrix} \overset{(s_0(x) \cdots s_m(x))}{A_{n \times m}} = \begin{pmatrix} y_0 \\ \vdots \\ y_N \end{pmatrix} A_{n \times N} \begin{pmatrix} B_{N \times m} \\ 0 \end{pmatrix} C_{m \times m}$$
The Veronese map "free" the $\Gamma(\mathbb{P}^n_k, \mathcal{O}(d))$, then we restrict to span$\{s_0 \cdots s_m\}$
by $B_{N \times m}$ a projection ▨
Ex7.4 (a) X has ample sheaf $\mathcal{L} \overset{(7.6)}{\Longleftrightarrow} \mathcal{L}^m$ very ample, i.e. $X \xrightarrow{\varphi} \mathbb{P}^n_A$ φ immersion and $\mathcal{L}^m = \varphi^*(\mathcal{O}(1))$
then $X \underset{\text{open immersion}}{\hookrightarrow} \bar{X} \underset{\text{closed immersion}}{\hookrightarrow} \mathbb{P}^n_A \xrightarrow{f} \mathrm{Spec} A$ is composite of separated map ▨

③ $\bigcup_{\frac{b}{a}\in k} V(x_0=(\frac{b}{a})^2x_1, x_2=-\frac{b}{a}x_3)=\mathbb{P}^1_k$, thus $f(L) \to X$ ▨

Ex 7.8. $\{\sigma: X\to \mathbb{P}(\mathcal{E}) \mid \pi\sigma = \mathrm{Id}\} \longleftrightarrow \{\mathcal{L} \text{ invertible} \mid \mathcal{L}=\mathcal{E}/\mathcal{E}'\}$

Let $\operatorname{rank}(\mathcal{E})=n+1$

$\Longrightarrow \{\sigma_U: U\to \mathbb{P}^n_U \mid \pi|_U\sigma_U=\mathrm{Id}_U\}_{U\subset X} \longleftrightarrow \{\mathcal{L} \text{ invertible} \mid \mathcal{L}=\mathcal{E}/\mathcal{E}'\}$

$(\sigma_U)_{U\subset X} \longmapsto \mathcal{L}$ pasted by $\mathcal{L}|_U=\mathcal{L}_U$

(7.1) $\mathcal{L}$ over U invertible & $s_0 \dots s_n \in \Gamma(U,\mathcal{L}_U)$

$(\sigma_U: U \to \mathcal{L}(U)\subset \mathbb{P}^n_U) \longmapsto \mathcal{L}$.

It suffices verify that ① $(\Longrightarrow)$ $\mathcal{L}=\mathcal{E}/\mathcal{E}'$, $\exists \mathcal{E}'\subset\mathcal{E}$

②③ are trivial by definition — ② Paste is possible; ③ $(\Longleftarrow)$, $\pi|_U\sigma_U=\mathrm{Id}_U$

For ① is local, $\mathcal{L}_U=\mathcal{O}^n_U/\mathcal{E}'_U$ for $\mathcal{E}'_U\subset\mathcal{O}^n_U$ is obvious; as it's coherent ▨

or using Prop 7.12, with $Y=X \xrightarrow{Id} X$

Ex 7.9. (a) We have $\mathbb{P}(\mathcal{E}) \longrightarrow X \xrightarrow{Id} 0$

As locally $\mathbb{P}^n_U \longrightarrow U \longrightarrow 0$ ($n\geq 1$ as $\operatorname{rank}\mathcal{E}\geq 2$)

$\operatorname{Pic}\mathbb{P}^n_U=\operatorname{Pic}U\times\mathbb{Z}$ is obvious and the isomorphism of sheaves is local $\Longrightarrow \operatorname{Pic}\mathbb{P}(\mathcal{E})=\operatorname{Pic}X\times\mathbb{Z}$ ▨

The gluing using regularity to given the local ring reduced and regular in codimension 1 allows us to apply

(Lemma) $\operatorname{Pic}\mathbb{P}^n_A=\operatorname{Pic}(\operatorname{Spec}A)\times\mathbb{Z}=\mathbb{Z}$, thus $\operatorname{Pic}\mathbb{P}^n_U=\operatorname{Pic}U\times\mathbb{Z}$)

(b) $(\Longleftarrow)$ $\mathcal{E}'\cong\mathcal{E}\otimes\mathcal{L} \Longrightarrow \mathbb{P}(\mathcal{E}')(U)=\operatorname{Proj}(S(\mathcal{E}')(U))=\operatorname{Proj}(S(\mathcal{E}\otimes\mathcal{L})(U))$

$=\operatorname{Proj}(S(\mathcal{E}(U)\otimes\mathcal{L}(U)))=\operatorname{Proj}(S(\mathcal{E}(U)))$

$=\mathbb{P}(\mathcal{E}(U))$ trivial

$(\Longrightarrow)$ By (a), we write the invertible sheaves over $\mathbb{P}(\mathcal{E})$ (or $\mathbb{P}(\mathcal{E}')$)

as $(\mathcal{L},n)=\pi^*\mathcal{L}\otimes\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)$ (This's the direct construction in (a))

thus now $\pi_*(\mathcal{O}(1))=\mathcal{E}$, given $\mathbb{P}(\mathcal{E}) \xrightarrow[f]{\cong} \mathbb{P}(\mathcal{E}')$

$\Rightarrow \mathcal{E}=\pi_*(f^*(\mathcal{O}(1)')\otimes\pi^*\mathcal{L})$, up to a $\pi^*\mathcal{L}$ due to $f^*(\mathcal{O}(1)')=\mathcal{O}(1)\otimes\pi^*\mathcal{L}$, $\exists\mathcal{L}$:

$\pi^*\mathcal{E}'=(\pi'f)^*\mathcal{E}'=f^*\pi'^*\mathcal{E}'=\dots$

$\Rightarrow \mathcal{E}'\cong\mathcal{E}\otimes\mathcal{L}$ ▨

Ex 7.10. (a) Nothing to do //

(b) $\mathbb{P}(\mathcal{E}) \underset{\text{locally}}{\cong} \mathbb{P}^n\times U$ is done, it suffices to check the translation function. however, it's due to, in $U\cap V$, $\psi_U\circ\psi_V^{-1}$ is the change of sections, by (7.1.1), then up to an $\operatorname{Aut}(\mathbb{P}^n)$, thus linear ▨

Ex 7.5 (e) $\mathcal{L}$ ample $\Rightarrow \mathcal{L}^n$ very ample $\Rightarrow \forall m>0$, $\mathcal{L}^{n+m}=\underbrace{\mathcal{L}^n}_{\text{very ample}}\otimes\underbrace{\mathcal{L}^m}_{\text{gbgs}}$

thus very ample by (d) ▨

Ex 7.6 (a) We show $l(nD)=P_X(n)$ for D very ample divisor

$P_X(n)=P_{S(X)}(n)=\dim_k S(X)_{(n)}=\dim_k(k[x_0\dots x_n]/I_X)_{(n)}$ (Here we used $\Gamma(X,\mathcal{O}_X(n))\cong\Gamma(X,\mathcal{O}(1)^n)$ for $n\gg 0$ [See pf of Thm 5.19])

$\Gamma(X,\mathcal{L}^n)=\Gamma(X,\varphi^*(\mathcal{O}(1))^n)$ is gbgs by $\{\varphi^*x_i\}$, $\langle\varphi^*x_i\rangle\to\langle\varphi^*x_i\rangle$

$\Longrightarrow \dim\Gamma(X,\mathcal{L}^n)=\dim_k\langle\varphi^*x_i\rangle=\dim_k(k[x_0\dots x_n]/I_X)_{(n)}$ ▨

(b) $rD\cong 0 \Longleftrightarrow rD=(f)$ for $f\in\Gamma(X,\mathcal{L})$

if $r\mid n \Rightarrow |nD|=|krD|=|0|$, then $\dim|0|=0$ is obvious as $l(0)=1$

otherwise $|nD|=\emptyset \Rightarrow \dim|nD|=-1$ ▨

Let's show $l(nD)=0$
$\Rightarrow l(nD)=0 \Longleftrightarrow nD$ isn't effective
Otherwise nD effitive and nonzero
$\Rightarrow nD\sim D'>0$, D' effective
$\Rightarrow rnD\sim rD'>0$, but $0\sim rnD>0$
contradiction ▨

Ex 7.7. (a) By (7.1.1) or (7.8.1)

It send the generating sections to the coordinate of $\mathbb{P}^5$, i.e., $x^2\mapsto x_0$; $y^2\mapsto x_1$; $z^2\mapsto x_2$; $xy\mapsto x_3$; $xz\mapsto x_4$; $yz\mapsto x_5$

thus it's just the Veronese map ▨

(b) We prove by (7.8.2) ① $\mathfrak{d}$ separate points: $\forall P_1=(x_1:y_1:z_1)\neq P_2=(x_2:y_2:z_2)\in\mathbb{P}^2$

We need to find $D\in\mathfrak{d}$, $P_1\in\operatorname{Supp}D$, $P_2\notin\operatorname{Supp}D$

as $D=\sum n_i(s_i)_0 \Rightarrow \operatorname{Supp}D=\bigcup X_{s_i}$ (By Prop in Pic), where $s_i=x^2,y^2,z^2,xy,xz$ & yz

And here $X_{s_i}=\mathbb{P}^2-V(s_i)$

Hence, it suffices to prove $\cap X_{s_i}=\emptyset$, but it's trivial

② $\mathfrak{d}$ separate vectors: $\forall(x:y:z)\in\mathbb{P}^2$ — $T_P(D)=T_P(\operatorname{Supp}D)$

and $T_P(\mathbb{P}^2)\cong\mathbb{P}^2$ — if $s_i=x^2,y^2,z^2$, $T_P(\operatorname{Supp}D)\cong\operatorname{Supp}D$ directly by ①

We show that $D=(y(x-z))_0$ and $P\in V(x-z)$ case, $t\in T_P(\mathbb{P}^2)$, $t=(t_1:t_2:t_3)$

① others are similiar then.

$T_P(D)\cong D$ due to the factorization is linear, still same ▨

(c) ① Let $P=(1:0:0)$

$\Rightarrow\mathfrak{d}$ is defined by (y^2,z^2,xy,xz,yz)

We directly complite $\tilde{\mathbb{P}}^2_k$ by ways in (I, §4) (However $\tilde{X}\subset\mathbb{P}^5_k$ as $\tilde{X}\subset\mathbb{P}^2_k\times\mathbb{P}^1_k\subset\mathbb{P}^5_k$)

$\tilde{X}=V(x(yw-zv), y(xw-zu), z(xv-yu))$

$\deg\tilde{X}=3$ is obvious even project to $\mathbb{P}^4_k$ ▨

② Lines pass P can't be seen in $\tilde{X}\subset\mathbb{P}^5_k$, due to the projection isn't clear

Consider $\tilde{X}=f(U)$, and $f: (y^2,z^2,xy,xz,yz)\mapsto(x_0:x_1:\dots:x_4)$

$\{\text{Lines pass } P\} \longleftrightarrow \{V(ay+bz)\}_{(a,b)\in\mathbb{P}^1_k}$ then $f(V(ay+bz))$.

Not meet is obvious: $\pi^{-1}(U)\cong U$ — P not meet also $=V(x_0=(\frac{b}{a})^2x_1, x_2=-\frac{b}{a}x_3)\cap\hat{X}$

$\Rightarrow$ can meet in not P, and after blowing-up line by counting dimensions ▨

Pf of the claim. It suffices to prove $\operatorname{codim}(Z-f^{-1}(U))=1$
this is due to it generic fibre is of dimension 1 ▨

Ex7.12. let $\mathscr{L}=\oplus(\mathscr{I}_Y+\mathscr{I}_Z)^d$
$\mathscr{L}_1=\oplus(\mathscr{I}_Y+\mathscr{I}_Z)\cdot\mathcal{O}_Y)^d$ $\mathscr{L}_2=\oplus(\mathscr{I}_Y+\mathscr{I}_Z)\cdot\mathcal{O}_Z)^d \Rightarrow \mathscr{L}_1=\oplus(\mathscr{I}_Y+\frac{\mathscr{I}_Z}{\mathscr{I}_Y})^d$ consider $\mathscr{I}_Y+\mathscr{I}_Z\subset\mathcal{O}_X$ an ideal sheaf, $\mathscr{L}_2=\oplus(\mathscr{I}_Y+\mathscr{I}_Z)^d$ namely
We show that: $\operatorname{Proj}\mathscr{L}_1\cap\operatorname{Proj}\mathscr{L}_2=\emptyset$
As $(Y\cap Z)^c$ induce an isomorphism, only take $U\subset Y\cap Z$
let $P\in\operatorname{Proj}\mathscr{L}_1\cap\operatorname{Proj}\mathscr{L}_2$, then $\pi(P)$ is contained in such $U=\operatorname{Spec}A$
$\Rightarrow \pi^{-1}(U)=\operatorname{Proj}(\mathscr{L}_1(U))\cap\operatorname{Proj}(\mathscr{L}_2(U))=\operatorname{Proj}(\mathscr{L}_1(U)+\mathscr{L}_2(U))$
thus let $\mathscr{I}_Y(U)=I_Y\trianglelefteq A$ $=\operatorname{Proj}\oplus(\mathscr{I}_Y+\mathscr{I}_Z)^d(U)$
$\mathscr{I}_Z(U)=I_Z\trianglelefteq A$
by the closed scheme $\Rightarrow Y\cap U=\operatorname{Spec}\frac{A}{I_Y}$ and $Z\cap U=\operatorname{Spec}\frac{A}{I_Z}$
$\Rightarrow \pi^{-1}(Y\cap U)=\operatorname{Proj}\oplus((\mathscr{I}_Y+\mathscr{I}_Z)(U)\frac{A}{I_Y})^d$, $\pi^{-1}(Z\cap U)=\operatorname{Proj}\oplus((I_Y+I_Z)\frac{A}{I_Z})^d$
Now it changes into pure algebra:
We prove that: $\exists\, p\trianglelefteq\oplus(I_Y+I_Z)^d$; $p\supset\oplus((I_Y+I_Z)\frac{A}{I_Y})^d$ containing both
Pf. (Algebraic) Just by these two generating $\oplus((I_Y+I_Z)\frac{A}{I_Z})^d$ to complete proof the $\oplus(I_Y+I_Z)^d$ ▨

Ex7.13 (a) Proper is local on bases, let $U=\mathbb{P}^1-\{0\}$
$\Rightarrow \pi^{-1}(U)\longrightarrow U$ it's obvious proper ▨
$C\times(\mathbb{P}^1-\{0\})\to\mathbb{P}^1-\{0\}$ (Due to it's restriction of projective morphism $\mathbb{P}^2_{\mathbb{A}^1_k}\times\mathbb{A}^1_k\to\mathbb{A}^1_k$)
(However, our final motivation is show it's not projective!)
(b) First consider the blowing-up of $C\times\mathbb{A}^1$ in $(P,\mathbb{A}^1)$ a line ($(C\times\mathbb{A}^1)^*$ is similar)
$\Rightarrow\widetilde{C\times\mathbb{A}^1}$, I claim. It's just its normalization: normal is obvious and the universal property just by one of blowing-up
And $\bigoplus_{P\in C\times\mathbb{A}^1}\frac{\mathscr{O}^*_P}{\mathscr{O}^*}=\bigoplus_{P\neq(P,0)}\frac{\mathscr{O}^*}{\mathscr{O}^*}=\ldots\mathbb{G}_m$ same as Ex6.9
(The only difference of $C\times\mathbb{A}^{1*}$ is here: here we have $A[u,u^{-1}]^*=A[u]^*\times\mathbb{Z}$ thus the at last "up to $\times\mathbb{Z}$")
Then $0\to\mathbb{G}_m\to\operatorname{Pic}(C\times\mathbb{A}^1)\to\operatorname{Pic}(\widetilde{C\times\mathbb{A}^1})\to 0$, we hope to show that:
① Split ② $\operatorname{Pic}(\widetilde{C\times\mathbb{A}^1})=\mathbb{Z}$ (Done)
For split, Pf. I don't know //
(c) ① $\mathbb{G}_m\times\mathbb{Z}\xrightarrow{\deg}\mathbb{Z}$ We prove $\ker\deg=\mathbb{G}_m$
$r(\mathbb{G}_m\times\mathbb{Z})\xrightarrow{\deg}$ as r injective $\Rightarrow\ker\deg=\ker\deg=\mathbb{G}_m$ ▨
$\mathbb{G}_m\times\mathbb{Z}\xrightarrow{\deg}$
$(r:\mathbb{G}_m\times\mathbb{Z}\to\mathbb{G}_m\times\mathbb{Z}\times\mathbb{Z})$
② $\varphi\in\operatorname{Aut}(C\times\mathbb{A}^{1*})$ given by $\langle P,u\rangle\mapsto\langle h(P),u\rangle$, inducing $\operatorname{Pic}(C\times\mathbb{A}^{1*})\to\operatorname{Pic}(C\times\mathbb{A}^{1*})$

Step 1 Extend $\mathscr{L}_U$ over $U\times\mathbb{P}^n$ to $\mathscr{L}$ over $\mathbb{P}(\mathscr{E})$
For $\varphi_{ij}:\mathscr{L}_{U_i}|_{U_{ij}}\to\mathscr{L}_{U_j}|_{U_{ij}}$ given by $\varphi_{ij}(V):\mathscr{L}_{U_i}(V)\to\mathscr{L}_{U_j}(V)$
as over $V\subset U_{ij}=U_i\cap U_j$ $\mathcal{O}_{U_i}(1)(V)\to\mathcal{O}_{U_j}(1)(V)$
$\mathcal{O}_{U_i}$ is over $\pi^{-1}(U_i)\cong U_i\times\mathbb{P}^n$
and $U_{ij}\times\mathbb{P}^n\to U_{ij}\times\mathbb{P}^n$ is linear, inducing $\mathcal{O}_{U_i}(1)(V)\to\mathcal{O}_{U_j}(1)(V)$
as the matrix multiplication satisfy the cocycle condition $\Rightarrow$ glue is possible ▨
Step 2 $\pi_*\mathscr{L}\cong\mathscr{E}\cong\pi_*(\mathcal{O}(1))$, then by Ex7.7(a) to complete the proof
This isomorphism is local, thus let P $\mathbb{P}(\mathscr{E})$ $\mathcal{O}(1)$
$\pi_*\mathscr{L}\cong A^n$, $X=\operatorname{Spec}A$ X $\mathscr{E}\cong\pi_*(\mathcal{O}(1))$
$\pi_*\mathscr{L}(\operatorname{Spec}A)=\mathscr{L}(\pi^{-1}(\operatorname{Spec}A))$
$=\mathscr{L}(\operatorname{Spec}A\times\mathbb{P}^n)$
$\cong\mathcal{O}(P)\cong A[x_0,\ldots,x_n]_1\cong A^n$ ▨
The regularity is used for the reduced structure over $U\subset X$
thus can be weaken as no nilpotent/reduced ▨
(d) $\mathbb{P}^n\text{-}B(X)\longleftrightarrow$ locally free $\mathcal{O}(X)$ rank n
$P\longmapsto\mathscr{E}$ by $P\cong\mathbb{P}(\mathscr{E})$ as (c)
$\mathbb{P}(\mathscr{E})\longleftarrow\mathscr{E}$
And ① $P\cong\mathbb{P}(\mathscr{E})\cong\mathbb{P}(\mathscr{E}')$, then $\mathscr{E}\sim\mathscr{E}'$ Thus it's a category isomorphism
② $\mathbb{P}(0)=0$ and $\mathbb{P}(\mathscr{E})=0$ for $\mathscr{E}\sim 0$ pass to the equivalence class
Pf. ① By Ex7.9(b) ② Trivial ▨
Ex7.11 (a) $\mathscr{L}=\oplus\mathscr{I}^d$, then $\widetilde{X}_{\mathscr{L}}=\operatorname{Proj}(\mathscr{L})$, then this is the relative analoge of Ex5.13 ▨
$\widetilde{X}_{\mathscr{L}(\ldots)}=\operatorname{Proj}\mathscr{L}(\ldots)$
(b) First, we need to define what the $\mathscr{I}\cdot\mathscr{J}$ is
Due to $\mathscr{J}$ invertible $\Rightarrow$ we think $\mathscr{I}$ as $\mathscr{J}$-algebra,
the $\mathscr{I}\cdot\mathscr{J}=\operatorname{Id}(\mathscr{I}\cdot\mathscr{J})$ to be the image of $\mathscr{I}$ over the identity map $\mathscr{J}\to\mathscr{J}$
the $\mathscr{I}\cdot\mathscr{J}=\mathscr{I}\otimes\mathscr{J}\Rightarrow\operatorname{Proj}\mathscr{I}^d\cong\operatorname{Proj}(\mathscr{I}\otimes\mathscr{J})^d$ by Lemma7.9.
as $\oplus(\mathscr{I}\otimes\mathscr{J})^d=\oplus((\oplus\mathscr{I}^d)_d\otimes\mathscr{J}^d)=(\oplus\mathscr{I}^d)*\mathscr{J}$ ▨
(c) (7.17) only give a $\mathscr{I}$ and corresponding $Y\subset X-U$ (otherwise $f^{-1}U\ncong U$) as there i
What (c) do now is by (a)(b) to make $\mathscr{I}$ to $\mathscr{I}'$, making point of $x\in U\cap Y$ is blown-up
$Y=X-U$ be maximal, then: such a isomorphic to blow-up preserves all inform of this birational map, that's non-trivial! (Again regular is for reduced
Pf. To make Y maximal, it's equivalent to make $\mathscr{I}_Y$ "smallest"
Claim. The Weil divisor $D=Z-f^{-1}(U)$ is well-defined in Z (i.e. in birational geometry, calling it $\operatorname{Exc}(f)$)
then $f_*\mathscr{L}(D)=\mathscr{I}_Y$ we want to construct, largest is obvious, then complete the proof.

the deg $=$ ~~nd~~ doesn't change

<u>Claim</u>. For each $u \in k^*$, ~~...~~ $u=1$ induce all Picard-level morphism, then t fixed

(The claim is easy: by $u=1$ generating the group of units)

At last

$$\begin{array}{ccccccccc} 0 \to & G_m \times \mathbb{Z} & \to & G_m \times \mathbb{Z} \times \mathbb{Z} & \to & \mathbb{Z} & \to 0 \\ & (t, ~~\cdot~~) \downarrow & & \downarrow & & n \ \| & \\ 0 \to & G_m \times \mathbb{Z} & \to & G_m \times \mathbb{Z} \times \mathbb{Z} & \to & \mathbb{Z} & \to 0 \\ & & & & & n & \end{array}$$

$(t, d+n) \swarrow \mathbb{Z}$

Given by the map $A[u,u^{-1}]^* = A^* \times \mathbb{Z}$

<u>Why $d \mapsto d+n$</u>? I don't know //

(c) $0 \to \mathrm{Cacl}^0 ~~C~~ \to \mathrm{Pic}(C) \to \mathbb{Z} \to 0$
$\nwarrow \quad \uparrow r$
$\mathrm{Pic} X$

Given by

$$\begin{array}{ccc} \langle t, n\rangle \quad \mathrm{Pic} X & \xrightarrow{I} & \mathrm{Pic}(C \times \mathbb{A}^1) \ \langle t, n \rangle \\ r \downarrow & \swarrow r & \downarrow r \quad\quad I \downarrow \\ \langle t, n \rangle \ \mathrm{Pic}(C \times \{0\}) & \leftarrow & \mathrm{Pic}(C \times \mathbb{A}^{1*}) \ \langle t, 0, n \rangle \end{array}$$

~~...~~ but $\mathrm{Im}\, r = \mathrm{Ker}\, \deg$ by exactness

$\Rightarrow \mathrm{Im}\, r \cong G_m = \mathrm{Pic}^0 C_0 = \mathrm{Pic}(C \times \{0\})$ $\boxed{/\!/}$

<u>Ex 7.14</u> (a) $X = \mathbb{P}^1$, $\mathcal{E} = \mathcal{O}(-1)$

then $\mathcal{O}(1)$ over ~~...~~ $\mathbb{P}(\mathcal{E})$ isn't ~~a~~ very ample:

Otherwise $\mathbb{P}(\mathcal{E}) \longrightarrow$ ~~...~~ $\mathbb{P}^n$ but $\mathbb{P}(\mathcal{E}) = \mathbb{P}^1$ as $\mathcal{E}$ invertible

$\varphi \searrow \mathbb{P}^n_{\mathbb{P}^1} \nearrow$ and local isomorphism inducing $\mathbb{P}^1 \xrightarrow{\pi = \mathrm{Id}} \mathbb{P}^1$

$\mathbb{P}^n_{\mathbb{P}^1} = \mathbb{P}^1 \times \mathbb{P}^n$ and $\mathcal{O}_1(1) = \varphi^* \mathcal{O}(1)$ ~~...~~ $= \varphi^*(p_1^* \mathcal{O}(1) \times p_2^* \mathcal{O}_2(1))$

$\|$

$\mathcal{O}_1(-1)$

$\Rightarrow$ the pullback of global section gives $\Gamma(\mathcal{O}_1(-1), \mathbb{P}^1) \neq 0$, absurd $\boxed{/\!/}$

(b) (<u>Rk</u>. For the example in (a), $n \geq 2$: $\mathcal{O}_p(1) = \mathcal{O}(-1)$
$\Rightarrow \mathcal{O}_p(1) \otimes \mathcal{L}^2 = \mathcal{O}(1)$ very ample)

By <u>Prop 7.10</u>, $\exists n$: $\mathcal{O}_p(1) \otimes \pi^* \mathcal{L}^n$ very ample

and $\forall n' > n$, $\mathcal{O}_p(1) \otimes \pi^* \mathcal{L}^{n'} = (\mathcal{O}_p(1) \otimes \pi^* \mathcal{L}^{n}) \otimes \pi^* \mathcal{L}^{(n'-n)}$

~~π~~ $\mathcal{L}$ ample $\Rightarrow \mathcal{L}^N$ very ample for $n \geq N \Rightarrow$ let $n' > n + N$

$\Rightarrow \pi^* \mathcal{L}^{(n'-n)}$ very ample $\xRightarrow[\text{(a)}]{\text{Ex 5.12}}$ $\mathcal{O}_p(1) \otimes \pi^* \mathcal{L}^{n'}$ very ample $\boxed{/\!/}$

8.4. (a) (⇒) By (I, Ex 2.7), $Y = H_1 \cap \cdots \cap H_r$
and $I = \Gamma_*(\mathcal{I}_Y) \Rightarrow \mathcal{I}_Y = \tilde{I}$, Now $I = \langle s_1, \cdots, s_r \rangle$
$\Rightarrow \mathcal{I}_Y = \tilde{I} = (\tilde{s_1}) + \cdots + (\tilde{s_r})$ corresponding to $H_1 \cdots H_r$
(⇐) $\Gamma_*(\mathcal{I}_Y) = \Gamma_*(\mathcal{I}_{H_1}) + \cdots + \Gamma_*(\mathcal{I}_{H_r}) = (f_1) + \cdots + (f_r) = \langle f_1 \cdots f_r \rangle = I$ ▨

This is where the unmixedness thm applies to: $\Gamma_*(\mathcal{I}_{H_1}) \cdots \Gamma_*(\mathcal{I}_{H_r})$ has no primary components of co-dimension ≥ 2
$\Rightarrow (\Gamma_*(\mathcal{I}_{H_1}), \cdots \Gamma_*(\mathcal{I}_{H_r})) = \Gamma_*(\mathcal{I}_Y) \cap J$
and $J = 0$ as codim $J > 2$ by Bezout ▨

(b) $Y \subset \mathbb{P}^n$
$C(Y) \subset \mathbb{A}^{n+1}$ and $\mathcal{I}_Y = \mathcal{I}_{C(Y)}$
$\Rightarrow S(C(Y)) = S(Y)$
by (8.23b), it suffices to show
$C(Y)$ is regular in codimension 1
but Y normal $\Rightarrow$ regular in codimension 1
$\Rightarrow C(Y)$ also (due to codim Sing $Y \geq 2$ and codim Sing $C(Y)$ = codim π^{-1}(Sing Y) = codim Sing $Y \geq 2$) ▨

(c) $\Gamma(H_1 \cap \cdots \cap H_r, \frac{\mathcal{O}_X}{\mathcal{I}_{H_1} + \cdots + \mathcal{I}_{H_r}}(l))$, induction on r ($S(l) \to \frac{S}{I}(l)$ is obvious surjective)
$r = 1$: $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l)) \longrightarrow \Gamma(H_1, \frac{\mathcal{O}_X}{\mathcal{I}_{H_1}}(l))$
$lH \longmapsto l(H \cap H_1)$, and every $l\tilde{H} \in \Gamma(H_1, \frac{\mathcal{O}_X}{\mathcal{I}_{H_1}}(l))$
due to $H_1 \cong \mathbb{P}^{n-1}$, $\tilde{H} = H \cap H_1$ for some H
thus surjection
Inductively: $\Gamma(H_1 \cap \cdots \cap H_{r-1}, \frac{\mathcal{O}_X}{\mathcal{I}_{H_1} + \cdots + \mathcal{I}_{H_{r-1}}}(l))$
$\longrightarrow \Gamma(H_1 \cap \cdots \cap H_r, \frac{\mathcal{O}_X}{\mathcal{I}_{H_1} + \cdots + \mathcal{I}_{H_r}}(l))$ is surjective
same as $l\tilde{H} = l(H \cap H_r)$ ▨
For $l = 0$, [illegible] $\longrightarrow$ [illegible]
$\Gamma(Y, \mathcal{O}_Y)$ by Liouville
$\Rightarrow \dim \Gamma(Y, \mathcal{O}_Y) \leq 1$
$\Rightarrow \Gamma(Y, \mathcal{O}_Y) = k$, but Liouville again $\Rightarrow$ only one component ▨

(d) By the old definition of (I, Ex 5), $H_i = V(f_i) \Rightarrow Y = V(f_1, \cdots, f_r)$
with $(\frac{\partial f_i}{\partial x_j})$ non-degenerate ▨

(e) Let $Y_i = H_1 \cap \cdots \cap H_i$, then $Y = Y_r \subset Y_{r-1} \subset \cdots \subset Y_1 = H_1$
$\Rightarrow \omega_Y = \omega_{Y_r}$, $\omega_{Y_1} = \omega_{H_1} = \mathcal{O}_Y(d_1 - n - 1)$
inductively, by $Y_{i+1} = Y_i \cap H_{i+1} \overset{(8.20)}{\Rightarrow} \omega_{Y_{i+1}} = \omega_{Y_i} \otimes \mathcal{L}(Y_{i+1}) \otimes \mathcal{O}_Y$
$= \omega_{Y_i} \otimes \mathcal{O}_Y(d_{i+1})$
$\Rightarrow \omega_Y = \mathcal{O}_{Y_1} \otimes \mathcal{O}_Y(d_2) \otimes \cdots \otimes \mathcal{O}_Y(d_r)$
$= \mathcal{O}_Y(\sum d_i - n - 1)$ ▨

(f) $p_g(Y) = \dim \Gamma(Y, \omega_Y) \overset{(e)}{=} \dim \Gamma(Y, \mathcal{O}_Y(d - n - 1)) = \dim \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-n-1)) - \dim \Gamma(Y, \mathcal{I}_Y(d-n-1))$
Due to $\deg Y = d > d - n - 1$
$= \binom{d-1}{n} - 0 = \binom{d-1}{n}$
by $0 \to \mathcal{I}_Y \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Y \to 0$
$0 \to \Gamma(Y, \mathcal{I}_Y(d-n-1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-n-1)) \to \Gamma(Y, \mathcal{O}_Y(d-n-1)) \to \cdots$
It suffices to show $\Gamma(Y, \mathcal{I}_Y(d-n-1)) = 0$
by $\Gamma(Y, \mathcal{I}_Y) = 0$ ▨

(g) Again as (f) $\Rightarrow p_g(Y) = \dim(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d+e-4)) - \dim \Gamma(Y, \mathcal{I}_Y(d+e-4)) = \binom{d+e-1}{3} -$ [illegible]

The next is compute $\dim \Gamma(Y, \mathcal{I}_Y(d+e-4)) \neq 0$
as $Y = H_1 \cap H_2$, the global section induced by H_1 and H_2 separately: $h_1 f + h_2 g \in \Gamma(Y, \mathcal{I}_Y(d+e-4))$
$\deg h_1 = e - 4$
$\deg h_2 = d - 4$
$\Rightarrow p_g(Y) = \binom{d+e-1}{3} - \binom{e-4+3}{3} - \binom{d-4+3}{3}$
$= \frac{1}{2} de(d+e-4) + 1$ ▨

Ex 8.1 (a) The only need $k = k(B)$ in (8.7) is factor $B = k(B) \oplus m = k \oplus m$
here, we only replace $k(B)$ with K the representatives for $B \Rightarrow B = K \oplus m$
if B is complete local ring, but it has no reason, thus by Hint, we pass to $\frac{B}{m^2}$
Assume $\frac{B}{m^2}$ is done $\Rightarrow \mathrm{Hom}(\Omega_{\frac{B}{m^2}/k} \otimes k(\frac{B}{m^2}), k) \longrightarrow \mathrm{Hom}(\frac{m}{m^2}, k) \longrightarrow 0$
and $\mathrm{Hom}(\Omega_{B/k} \otimes k, k) \longrightarrow \mathrm{Hom}(\Omega_{\frac{B}{m^2}/k} \otimes k(\frac{B}{m^2}), k)$ is also surjective, thus complete this proof
$\mathrm{Der}(B, k) \to$ [illegible] $\mathrm{Der}(\frac{B}{m^2}, k) \oplus \mathrm{Hom}(k(\frac{B}{m^2}), k)$, with $\mathrm{Hom}(k(\frac{B}{m^2}), k) = 0$
and $\mathrm{Der}(B, k) \longrightarrow \mathrm{Der}(\frac{B}{m^2}, k) \longrightarrow 0$ is trivial ▨
(Due to $\mathrm{Hom}_k(k(\frac{B}{m^2}), k) = \mathrm{Hom}_k(K, k)$, $k \subset K$)
(b) (⇐) $\mathrm{rank}\, \Omega_{B/k} = \dim B + \mathrm{trdeg}\, k(B)/k$
$= \dim B + \mathrm{rank}\, \Omega_{k(B)/k}$ (8.6A)
and by (a), $\mathrm{rank}\, \Omega_{B/k} = \dim \frac{m}{m^2} + \mathrm{rank}\, \Omega_{k(B)/k}$
$\Rightarrow \dim B = \dim \frac{m}{m^2}$
(⇒) $\dim \frac{m}{m^2} = \dim B = \mathrm{rank}\, \Omega_{B/k} \otimes k(B) - \mathrm{trdeg}\, k(B)/k$
$\mathrm{rank}\, \Omega_{B/k} \otimes k(B) = \dim B + \mathrm{trdeg}\, k(B)/k$, by Lemma 8.9, it suffices do again
for $L = \mathrm{Frac}\, B$ $\mathrm{rank}\, \Omega_{B/k} \otimes L = \dim \Omega_{L/k} = \mathrm{trdeg}\, L/k \underset{\text{(tower formula)}}{=} \mathrm{trdeg}\, L/k(B) + \mathrm{trdeg}\, k(B)/k$
$= \dim \Omega_{L/k(B)} + \mathrm{trdeg}\, k(B)/k = \dim B + \mathrm{trdeg}\, k(B)/k$ ▨
(c) $\mathcal{O}_{x,X}$ regular local $\overset{(b)}{\Longleftrightarrow} \underbrace{\Omega_{\mathcal{O}_{x,X}/k}}_{(\Omega_{X/k})_x}$ is free of rank $= \dim \mathcal{O}_{x,X} + \mathrm{trdeg}\, k(x)/k$
$= n + \mathrm{trdeg}\, k/k = n$
nothing to prove! ▨
(d) $U = \{x \in X \mid (\Omega_{X/k})_x$ free rank $n\}$
$\overset{(\text{Ex } 5.7)}{=} \bigcup_{x \in X} U_x = \{x \in X \mid U \in \mathcal{N}_x, \Omega_{X/k}|_U$ free rank $n\}$ open dense ▨
Ex 8.2 ① $I = \{(x, s) \in X \times V \mid s_x \in m_x \mathcal{E}_x\} \subset X \times V$
$X \xleftarrow{f} I \xrightarrow{g} V$ (or $\dim I = m + \dim g^{-1}(s) = m + \dim \{x \in X \mid s_x \in m_x \mathcal{E}_x\} < n + m$ by Zariski closed proper)
$\dim I = n + \dim f^{-1}(x) = n + \dim \{s \in V \mid s_x \in m_x \mathcal{E}_x\} < n + m = \dim X \times V$
$\Rightarrow \exists s : (X, s) \cap I = \emptyset \Rightarrow$ such s is the desired section ▨
② Now by ①, we fix s_0
For every stalks: $0 \to \mathcal{O}_{x,X} \to \mathcal{E}_x$, $f_x \mapsto f_x s_{0x}$, then $\frac{\mathcal{E}_x}{\mathcal{O}_{x,X}}$ is free $\mathcal{O}_{x,X}$-module
by Ex 5.7(b), we complete the proof ▨
Ex 8.3 (a) $\Omega_{X \times Y/S} \overset{(8.11)}{=} p_1^* \Omega_{X/S} \oplus \Omega_{X \times Y/X} \overset{(8.10)}{=} p_1^* \Omega_{X/S} \oplus p_2^* \Omega_{Y/S}$
or $\Omega_{X \times Y/S} \overset{(8.11)}{=} p_2^* \Omega_{Y/S} \oplus \Omega_{X \times Y/Y} \overset{(8.10)}{=} p_2^* \Omega_{Y/S} \oplus p_1^* \Omega_{X/S}$ ▨
(b) By (a) and Ex 5.16(d) ▨
(c) $p_a(X) = p_a(Y \times Y) \overset{(\text{Ex } 7.2(d))}{=} p_a(Y)^2 - 2 p_a(Y) \overset{(\text{Ex } 7.2(b))}{=} \left(\frac{(3-1)(3-2)}{2}\right)^2 - 2 \frac{(3-1)(3-2)}{2} = -1$
$p_g(X) = p_g(Y \times Y) = \dim \Gamma(Y \times Y, \omega_{Y \times Y}) = \dim \Gamma(Y \times Y, p_1^* \omega_Y \otimes p_2^* \omega_Y)$
$= \dim \Gamma(Y, \omega_Y)^2 = p_g(Y)^2 = p_a(Y)^2 = 1$ ▨

Ex8.6 (b) ③ Trivial: $P \longrightarrow B'$, b_i by surjectivity

$x_i \in P$, $\bar{x}_i \in A$, $A \longrightarrow B$, $B \to B'$; then $b_i = h(x_i)$ is desired map ☒

(c) ① It suffices to show surjectivity $Hom(\Omega_{P/k}, I) \to Hom_A(\mathcal{I}/\mathcal{I}^2, I) \to 0$ by $Hom(-, I)$ is left exact

$Der_k(P, I)$, θ, h

This due to $\Omega_{A/k}$ free $\Longleftarrow$ A is nonsingular ☒

② $h'(f) = (g-\theta)(f) = h(f) - h(f) = 0$ by (b)'s commuting ☒

Ex8.7 $X = Spec A$ nonsingular

Step1 Change it into algebraic problem.

We hope to prove $Ext(X, \mathcal{F}) \cong \mathcal{O}_X \oplus \mathcal{F}$, $\forall \mathcal{F}$

$(X', \mathcal{I})$ given by $\mathcal{I}^2 = 0$ and $(X, \mathcal{O}_{X'}/\mathcal{I}) \cong (X, \mathcal{O}_X)$, $\mathcal{I} \cong \mathcal{F}$

$\Longleftrightarrow \forall$ M f-g A-module

$Ext(A, M) \cong A \oplus M$

(A', I) given by $I^2 = 0$ and $(A', \frac{A'}{I}) \cong (A, A)$, $I \cong M$

i.e. $\forall M$ f-g A-module, if $\frac{A'}{I} \cong A$ and $I^2 = 0 \Longrightarrow A' \cong A \oplus M$, $I \cong M$

Step2 Apply Ex8.6.

$A \xrightarrow[f]{\cong} A'/I$, $g: A \to A'$, $p: A' \to A'/I$, and composite $f^{-1} p = g^{-1}$ $\Longrightarrow$ A is a direct summand of A' and $A \cong \frac{A'}{I} \Longrightarrow A \oplus I \cong A'$

$A \oplus M$ ☒

Ex8.8 (a)(4) where use ω_X?

① Pullback by $\Omega_{X/k}$, then induce pullback of ω_X

This holds for $\wedge^q \Omega_{X/k} = \Omega^q_{X/k}$ and $\omega_X^{\otimes n}$ also

② Use ω_X invertible, indeed, locally free $\omega_X|_U = \mathcal{O}_U$

This holds for $\omega^{\otimes n}$ as $\omega^{\otimes n}|_U = \mathcal{O}_U$ also makes sense

For $\wedge^q \Omega_{X/k} = \Omega^q_{X/k}$, by Ex5.16(a), it's also locally free of rank $\binom{\dim X}{q}$ ☒

Ex8.5 (a) We have $0 \to \mathbb{Z} \to Pic\tilde{X} \to Pic(\tilde{X} - Y') \to 0$ split due to $Pic(\tilde{X} - Y') \cong Pic(X - Y)$ and codim $Y \geq 2$ $\Longrightarrow Pic(X - Y) \cong Pic X$ (Weil)

$Pic\tilde{X} \underset{\pi^*}{\leftrightarrows} Pic(X)$

and the exactness only need to prove the:

$0 \to \mathbb{Z} \to Pic\tilde{X}$, $n \mapsto n[Y']$. If $\exists n: nY' \sim 0$ i.e. $nY' = (f)$, f along Y' of vanishing order $= n$, then f induce f along $Y \subset X$ through π

$\Longrightarrow nY$ comes to be a divisor ~ 0, but codim $Y \geq 2$

$\Longrightarrow \forall n$, nY not a divisor, contradiction $\Longrightarrow \nexists n: nY' \sim 0 \Longrightarrow$ injection ☒

(b) By (a), $\omega_{\tilde{X}} \in Pic\tilde{X} = Pic X \oplus \mathbb{Z} \Longrightarrow \omega_{\tilde{X}} = \pi^* M \otimes \mathcal{L}(nY')$

$\Longrightarrow \omega_{\tilde{X}}|_{\tilde{X}-Y'} = \pi^*(M_{X-Y}) \otimes \mathcal{L}(nY')|_{\tilde{X}-Y'}$

$\omega_{\tilde{X}}|_{\tilde{X}-Y'} = \pi^*(M_{X-Y})$ due to $(\tilde{X}-Y')$ is isomorphism

$\Longrightarrow \omega_{\tilde{X}}|_{\tilde{X}-Y'} \cong M_{X-Y} \Longrightarrow \omega_X \cong M \Longrightarrow \omega_{\tilde{X}} = \pi^*\omega_X \otimes \mathcal{L}(nY')$

$\omega_{\tilde{X}}|_{\tilde{X}-Y'} \cong \omega_X|_{X-Y}$

Next we compute $n = r - 1$: $\omega_{Y'} \overset{(8.20)}{=} \omega_{\tilde{X}} \otimes \mathcal{L}(Y') \otimes \mathcal{O}_{Y'}$

$= \pi^*\omega_X \otimes \mathcal{L}((n+1)Y')$

$= \pi^*\omega_X \otimes \mathcal{O}_{Y'}(-n-1)$

let $y \in Y$

$Z = \pi^{-1}(y) = \tilde{X} \times_X Spec k(y) = Y' \times_Y Spec k(y)$ as $\pi^{-1}(y) \subset Y'$, $y \in Y$

$\omega_Z \overset{Ex8.3(b)}{=} p_1^* \omega_{Y'} \otimes p_2^* \omega_{Spec k(y)}$

$= p_1^*(\pi^*\omega_X \otimes \mathcal{O}_{Y'}(-n-1)) \otimes p_2^*(k)$

$= (\pi \circ p_1)^* \omega_X \otimes \mathcal{O}_Z(-n-1)$

$= \mathcal{O}_Z(-n-1)$ but $\omega_Z = \omega_{\mathbb{P}^{r-1}} = \mathcal{O}_Z(-r) \Longrightarrow -n-1 = -r$, $n = r - 1$ ☒

Ex8.6 (a) ① First $\theta(a) \in I$ (Algebraic) as $(g - g')(a) = g(a) - g'(a)$

$pg(a) - pg'(a) = f(a) - f(a) = 0$

$\Longrightarrow \overline{g(a)} = \overline{g'(a)}$ in $\frac{B}{I} \cong B'$

$\Longrightarrow (g - g')(a) = \theta(a) \in I$

And Leibniz rule: $\theta(a_1 a_2) = g(a_1 a_2) - g'(a_1 a_2)$

$= g(a_1)g(a_2) - g(a_1)g'(a_2) + g(a_1)g'(a_2) - g'(a_1)g'(a_2)$

$= g(a_1)\theta(a_2) + \theta(a_1)g'(a_2)$ ☒

② $pg'(a) = p(g(a) + \theta(a)) = pg(a) + \underbrace{p\theta(a)}_{\in I} = f(a) + 0$ well-defined ☒

(b) ② is easy: by ①, then we extend to $I \to I$ then $\mathcal{I}/\mathcal{I}^2 \to I/I^2 = I$ ☒

Ex9.4. $0 \to \mathfrak{F}' \to \mathfrak{F} \to \mathfrak{F}'' \to 0$

$0 \to \Gamma(\mathfrak{X},\mathfrak{F}') \to \Gamma(\mathfrak{X},\mathfrak{F}) \to \Gamma(\mathfrak{X},\mathfrak{F}'') \to 0$ by Ex9.3

then $\Gamma(\mathfrak{X},\mathfrak{F}')$ and $\Gamma(\mathfrak{X},\mathfrak{F}'')$ f-g module $\Rightarrow \Gamma(\mathfrak{X},\mathfrak{F})$ also

$\Rightarrow \mathfrak{F} = M^\Delta$ coherent ▨

Ex9.5. $\mathcal{O}_{\mathfrak{X}}^n \to \mathfrak{F} \to 0$, then proceed same as Ex9.4 to Mod-category ▨

Ex9.6. (a) $\Gamma(Y_n,\mathcal{O}_{Y_n})^* \subset \Gamma(Y_n,\mathcal{O}_{Y_n})$

due to $(\Gamma(Y_n,\mathcal{O}_{Y_n}))$ is (ML) $\Leftrightarrow$ any unit has preimage

and its preimage can't be not an unit: as it belongs to kernel mapping to 0

$\Rightarrow$ any unit has an unit preimage $\Leftrightarrow (\Gamma(Y_n,\mathcal{O}_{Y_n})^*)$ is (ML) ▨

(b) Due to the isomorphism (φ_n) isn't natural, $\varprojlim \varphi_n$ may not an isomorphism

① $\frac{\mathfrak{F}}{\mathfrak{J}^{n'}\mathfrak{F}} \xrightarrow[\cong]{\varphi_{n'}} \mathcal{O}_{Y_{n'}}$ by (ML), $\varphi_{n'n}(\frac{\mathfrak{F}}{\mathfrak{J}^{n'}\mathfrak{F}})$ is stable image

$\downarrow \varphi_{n'n}$ $\quad \downarrow \psi_{n'n}$ replace $\frac{\mathfrak{F}}{\mathfrak{J}^n\mathfrak{F}}$ by this, then the identity is compatible with (φ_n)

$\frac{\mathfrak{F}}{\mathfrak{J}^n\mathfrak{F}} \xrightarrow[\cong]{\varphi_n} \mathcal{O}_{Y_n}$ the same is for $(\mathcal{O}_{Y_n})$, then $\varprojlim \varphi_n = \varphi$ makes sense ▨

② $\mathrm{Pic}\,\mathfrak{X} \to \varprojlim \mathrm{Pic}\, Y_n$

$\mathfrak{F} \mapsto \varprojlim \frac{\mathfrak{F}}{\mathfrak{J}^n\mathfrak{F}} \overset{\text{locally}}{\cong} \mathcal{O}_{Y_n}$ due to $\mathfrak{F} \overset{\text{locally}}{\cong} \mathcal{O}_{\mathfrak{X}}$

then if $\varprojlim \frac{\mathfrak{F}}{\mathfrak{J}^n\mathfrak{F}} \overset{\text{locally}}{\cong} \mathcal{O}_{Y_n} = 0 \Rightarrow \varprojlim \mathcal{O}_{Y_n} = 0$ each local open set

$\Rightarrow \mathcal{O}_{\mathfrak{X}} = 0$ locally $\Leftrightarrow \mathfrak{F} = 0 \Rightarrow$ injective ▨

(c) ① We have natural map $\mathcal{O}_{Y_{n+1}} \to \mathcal{O}_{Y_n}$

thus $\underset{\mathcal{L}_{n+1}}{\underset{\|}{\mathcal{L}_{n+1} \otimes \mathcal{O}_{Y_{n+1}}}} \to \mathcal{L}_{n+1} \otimes \mathcal{O}_{Y_n} = \mathcal{L}_n$, then $\mathcal{L}_{n'} \to \mathcal{L}_n$ inductively ▨

② Local question, assume $\mathcal{L}_n = \mathcal{O}_{Y_n}$, then $\varprojlim \mathcal{O}_{Y_n} = \mathcal{O}_{\mathfrak{X}}$ obviously

and the paste need careful: we can't directly claim the paste of U_n, $\varprojlim$ compatible with $\varprojlim$, but again take the stable image, we can do this as (b) ▨

③ $\mathrm{Pic}\,\mathfrak{X} \to \varprojlim \mathrm{Pic}\,Y_n$

$\forall (\mathcal{L}_n) \in \mathrm{Pic}\,Y_n$, $\varprojlim \mathcal{L}_n = \mathcal{L} \in \mathrm{Pic}\,\mathfrak{X}$ by ② ▨

(d) ① $\mathfrak{X} = \mathrm{Spf}\, A$, then $Y_n = (\mathfrak{X}, \frac{\mathcal{O}_{\mathfrak{X}}}{\mathfrak{J}^n}) = (\mathrm{Spf}\,A, \varprojlim (\frac{A}{I})^n)$, by discussion in P196

$\Rightarrow \mathfrak{J} = (I^n)^\Delta \Rightarrow Y_n = (\mathrm{Spf}\,A, \frac{\mathcal{O}_{\mathfrak{X}}}{(I^n)^\Delta}) = (\mathrm{Spf}\,A, (\frac{A}{I^n})^\Delta)$

and $\Gamma(Y_n,\mathcal{O}_{Y_n}) = \frac{A}{I^n}$, thus (ML) by A is noetherian ▨

③ By Liouville, $(\Gamma(Y_n,\mathcal{O}_{Y_n})) = (k)$ ▨

[illegible] by (8.17) $0 \to \mathcal{J}/\mathcal{J}^2 \to \Omega_{X/k} \otimes \mathcal{O}_Y$

and by (8.13) $0 \to \Omega_{X/k} \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0 \Rightarrow \Omega_{X/k} \subseteq \mathcal{O}_X(-1)^{n+1}$

$\Rightarrow \Omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_Y = \mathcal{O}_X(-1)^{n+1} \otimes_{\mathcal{O}_X} \mathcal{O}_Y = \mathcal{O}_Y(-1)^{n+1}$

$\Rightarrow \mathcal{J}/\mathcal{J}^2 \hookrightarrow \mathcal{O}_Y(-1)^{n+1}$ desired map ▨

(b) We have $\frac{\mathcal{J}^r}{\mathcal{J}^{r+1}} \subset \cdots \subset \frac{\mathcal{J}}{\mathcal{J}^2} \subset \mathcal{O}_Y(-1)^{n+1}$

and section de $\mathcal{O}_Y(-1)$ is 0 $\Rightarrow \Gamma(Y, \frac{\mathcal{J}^r}{\mathcal{J}^{r+1}}) = 0$ ▨

(c) Induction on r: Now $\Gamma(Y,\mathcal{O}_X/\mathcal{J}^r) = k$

$\Rightarrow 0 \to 0 \to \Gamma(Y,\mathcal{O}_X/\mathcal{J}^{r+1}) \to k \to 0 \Rightarrow \Gamma(Y,\mathcal{O}_X/\mathcal{J}^{r+1}) = k$

For r=1, we need to show $\Gamma(Y,\mathcal{O}_X/\mathcal{J}) = k$

by (8.21A, e) $\Rightarrow \mathcal{O}_X/\mathcal{J}[t_1\cdots t_n] \cong \bigoplus_{m\ge 0} \frac{\mathcal{J}^m}{\mathcal{J}^{m+1}}$, for $\mathcal{J}$ is generated by $s_1\cdots s_n$

$\Rightarrow \Gamma(Y, \frac{\mathcal{O}_X}{\mathcal{J}}[t_1\cdots t_n]) = k$

$= \Gamma(Y, \bigoplus \frac{\mathcal{J}^m}{\mathcal{J}^{m+1}}) = \Gamma(Y, \frac{\mathcal{O}_X}{\mathcal{J}}) \oplus 0 = k \oplus 0 = k$ ▨

(d) Taking limit, by communitivity (Prop 9.2) ▨

Ex9.2. Assume f dominate (replace Z by $\overline{f(X)}$ (scheme-theortic))

$\Rightarrow f^\#: \mathcal{O}_Z \to f_*\mathcal{O}_X$ is injective

it induce $\mathcal{O}_{\hat{Z}} \to \hat{f}_*\mathcal{O}_{\hat{X}}$ injective by exactness

then $\Gamma(\hat{Z},\mathcal{O}_{\hat{Z}}) \to \Gamma(\hat{Z}, \hat{f}_*\mathcal{O}_{\hat{X}})$ also injective

$\| \quad \Gamma(\hat{X},\mathcal{O}_{\hat{X}}) = k$

$\Rightarrow \Gamma(\hat{Z},\mathcal{O}_{\hat{Z}}) = k$, then I claim $Z = \{P\}$

Pf. Otherwise $\dim Z \ge 1$, but by Ex7.3, $\mathbb{P}^n \to Z \to \mathbb{P}^{\dim Z}$, $\dim Z = \dim \overline{f(X)} \le n$

$\Rightarrow$ it collapses to a point ▨

Ex9.3. (a) By Prop 9.5, (b) and (a), let us assume $\mathfrak{J}$ is maximal, i.e. $\mathfrak{J} = \hat{I}$, I the reduce structure of $Y = \mathrm{Spec}\frac{A}{I}$

then $0 \to \mathfrak{F}'/\mathfrak{J}^n\mathfrak{F}' \to \mathfrak{F}/\mathfrak{J}^n\mathfrak{F} \to \mathfrak{F}'' \to 0$

$\| \qquad \| \qquad \|$

$0 \to (\frac{M'}{I^nM'})^\sim \to (\frac{M}{I^nM})^\sim \to M''^\sim \to 0$ thus is a $\mathcal{O}_Y$-module exact sequence and $(\frac{M'}{I^nM'})^\sim$ is quasi-coherent

(5.6) $0 \to \Gamma(\mathfrak{U}, \frac{\mathfrak{F}'}{\mathfrak{J}^n\mathfrak{F}'}) \to \Gamma(\mathfrak{U}, \frac{\mathfrak{F}}{\mathfrak{J}^n\mathfrak{F}}) \to \Gamma(\mathfrak{U},\mathfrak{F}'') \to 0$

exact ▨

(b) By (9.1), $\varprojlim$ also exact; by (9.2), Γ commutes with $\varprojlim$; by (9.6), $\mathfrak{F}' \cong \varprojlim \frac{\mathfrak{F}'}{\mathfrak{J}^n\mathfrak{F}'}$

$\Rightarrow 0 \to \Gamma(\mathfrak{X},\mathfrak{F}') \to \Gamma(\mathfrak{X}, \varprojlim \frac{\mathfrak{F}}{\mathfrak{J}^n\mathfrak{F}'}) \to \Gamma(\mathfrak{X},\mathfrak{F}'') \to 0$

it suffices to prove $\varprojlim \frac{\mathfrak{F}}{\mathfrak{J}^n\mathfrak{F}'} \cong \mathfrak{F} \cong \varprojlim \frac{\mathfrak{F}}{\mathfrak{J}^n\mathfrak{F}}$, i.e. $\varprojlim \mathfrak{J}^n\mathfrak{F}' = \varprojlim \mathfrak{J}^n\mathfrak{F}$

Now we change to an pure algebraic problem

$M \subseteq M'$ are $\frac{A}{I}$-module and M' is f-g generated, then $\varprojlim I^nM' = \varprojlim I^nM$

This obvious as I is their annihilator ▨ (The last step depend our assume $\mathfrak{J} = \hat{I}$ reduce

(e) Just by (d) $0 \to \Gamma_Y(X,\mathcal{F}) \to \Gamma(X,\mathcal{F}) \to \Gamma(U,\mathcal{F})$ in general $\Rightarrow$ applying the LES ▨

(f) Now we have two effaceable functors $\Gamma(U,\mathcal{F}|_U)$ and $\Gamma_Y(U,-|_U) = \Gamma_Y(X,-)$ obviously $\Gamma_Y(U,-|_U)$ and $\Gamma_Y(X,-)$, we consider their right satellites to complete the proof, thus we show they do effaceable: just for $\forall \mathcal{F}$, ~~embedding~~ $\mathcal{F}$ to $\mathcal{F}|_U$.

Ex2.4. It suffices to prove $0 \to \Gamma_{Y_1 \cap Y_2}(X,I) \to \Gamma_{Y_1}(X,I)$ $f(U): \mathcal{F}(U) \to \mathcal{F}(U \cap V)$ for $I^\bullet$ the injective resolution, the by the $\oplus \Gamma_{Y_2}(X,I) \to \Gamma_{Y_1 \cup Y_2}(X,I)$ i.e. $f = \ldots$ ▨ LES to complete the proof (Recall our pf in AT, replace cycles with sections.)

$0 \to \Gamma_{Y_1 \cap Y_2}(X, I_j) \to \Gamma_{Y_1}(X,I_j) \oplus \Gamma_{Y_2}(X,I_j) \to \Gamma_{Y_1 \cup Y_2}(X,I_j) \to 0$ only show the surjective due others are trivial

$s \mapsto (s,-s) \longmapsto 0$

$(s,s') \longmapsto (s+s')$

i.e. $\forall t \in \Gamma_{Y_1 \cup Y_2}(X,I_j), \exists$ $s_i \in \Gamma_{Y_i}(X,I_j): t = s_1 + s_2$

But it's also trivial (recall the simplicial case!), let $s_i = j_i(t|_{Y_i})$, where $j_i: Y_i \hookrightarrow X$ ▨

Ex2.5. We repeat our pf of the excision: it suffices to prove ① $\Gamma_P(X,-)$ and $\Gamma_P(X_P, j^*-)$ are both effaceable ② $\Gamma_P(X,\mathcal{F}) = \Gamma_P(X_P, j^*\mathcal{F})$

① Just let $U = X - \{P\}$ and restrict ▨

② $\Gamma_P(X,\mathcal{F}) \leftrightarrow \Gamma_P(X_P, j^*\mathcal{F})$

$s \longmapsto s|_{X_P}$

$j(s) \longleftarrow s$ ▨

Ex2.6. Our gap is that: $\mathrm{Hom}(-, \varinjlim \mathcal{G}_\alpha)$

$\cong \varinjlim \mathrm{Hom}(-, \mathcal{G}_\alpha)$

$\Leftarrow$ it act of finite presented sheaf.

Thus we use the Hint: $\mathcal{G}$ injective $\Leftrightarrow \forall U \subset X$, $\forall \mathcal{R} \subset \mathbb{Z}_U$, $\forall f: \mathcal{R} \to \mathcal{G}$, $\exists \bar{f}: \mathbb{Z}_U \to \mathcal{G}$, i.e. $0 \to \mathcal{R} \to \mathbb{Z}_U \to \frac{\mathbb{Z}_U}{\mathcal{R}} \to 0$

Pf. $\mathcal{G} = \prod_{P \in X} j_*(\mathcal{G}_P)$ by Prop2.2.

thus $\mathcal{G}$ injective $\Leftrightarrow$ $\mathcal{G}_P$ are all injective $\mathcal{O}_P$-modules $\Leftrightarrow$ (Baer criterion) $0 \to R \to \mathbb{Z} \to \frac{\mathbb{Z}}{R} \to 0$

Thus ($\Leftarrow$) is clear

($\Rightarrow$) is due to $\forall p \in X$, we extend f into some $U \in \mathcal{N}_p$ ▨

$\mathcal{R} \subset \mathbb{Z}$ now $f.g \Rightarrow$ $\Rightarrow \mathrm{Hom}(\mathcal{R}, \varinjlim \mathcal{G}_\alpha) = \mathrm{Hom}(\mathcal{R}, \mathcal{G}_{\alpha_0})$

This is due to $\mathcal{R}(U) \ldots \to \mathcal{G}_{\alpha_0} \to \cdots$ now for $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, let α be the and take a minimum α maximal one of $\alpha_0', \alpha_0, \alpha_0'' \Rightarrow \mathrm{Hom}(-, \varinjlim \mathcal{G}_\alpha)$

Due to the universal property of lim, induce $\mathcal{G}_\alpha \to \varinjlim \mathcal{G}_\alpha$ ▨ (Later we'll use Čech to decompose (Mayer-Vietoris)) $= \mathrm{Hom}(-, \mathcal{G}_\alpha)$ exact ▨

Ex2.7. (a) It's not easy to use M-V sequence for sheaf cohomology: we can't claim that $H^i(X,\mathcal{F}) = H^i(Y,\mathcal{F})$ for $X \simeq Y$ (due to $\mathcal{F}$ contains more information than $\mathbb{Z}$ or $\mathbb{R}$)

We need to find a resolution: $0 \to \mathbb{Z} \to \bigoplus_{P \in S^1} i_P(\mathbb{Z}_P) \to \bigoplus_{P \in S^1} i_P\left(\frac{I_{0,P}}{\mathbb{Z}}\right) \to \bigoplus_{P \in S^1} i_P\left(\frac{I_{1,P}}{I_{0,P}}\right)$

$\Rightarrow H^1(S^1,\mathbb{Z}) = \frac{\mathrm{Ker}\, d^2}{\mathrm{Im}\, d^1}$

$0 \to \mathbb{Z} \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2$

$0 \to \Gamma(S^1,\mathbb{Z}) \to \Gamma(S^1, I_0) \xrightarrow{d^1} \Gamma(S^1, I_1) \xrightarrow{d^2} \Gamma(S^1, I_2)$

$= \{f: S^1 \to \coprod I_{1,P}$ locally make constant$\} / \{f: S^1 \to \coprod I_0(S_P)$ locally make constant$\}$

(I_0 is nature to construct due to prop2.2., later I_i are a standard construction, remember it! (the Godement's way))

$= \mathbb{Z}$, due to their difference is given globally: around 2π, it differs a time $n \in \mathbb{Z}$ ▨ (Recall our proof of de Rham is similar, using $d\theta$ around S^1)

I give alternative proof, which we'll proceed again in (c)

Ex2.7. (b) $0 \to \mathbb{Z}_U \to \mathbb{Z} \to i_*(\mathbb{Z}) \oplus i_*(\mathbb{Z}) \to 0$

$0 \to \Gamma(X,\mathbb{Z}_U) \to \Gamma(X,\mathbb{Z}) \to \Gamma(X, i_*(\mathbb{Z}) \oplus i_*(\mathbb{Z})) \to H^1(X,\mathbb{Z}_U) \to \cdots$

$0 \to \mathbb{Z} \to \boxed{\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}} \to \boxed{H^1(X,\mathbb{Z}_U)} \to \cdots$

It can't be surjective! $\Leftrightarrow H^1(X,\mathbb{Z}_U) \neq 0$ ▨

(Furthermore, from the morphism we have $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, $n \mapsto (0,b)$)

(b) Consider what $\mathbb{Z}_Y$ looks like and I claim, for a generic $\xi \in$; $V(H_i)$ s.t. $Y = H_1 \cup \cdots \cup H_q$ suitably taken i.e. s.t. $H^n(X, \mathbb{Z}_U) = \mathbb{Z}$

To avoid the dealing with generic point, we accept now Y is general position well

$\Rightarrow H^n(X,\mathbb{Z}_Y) = \mathbb{Z}$, but $H^n(X,\mathbb{Z}) = 0$ (due to $\mathbb{Z}$ constant $\Rightarrow$ acyclic)

$\Rightarrow 0 \to \mathbb{Z} \to H^n(X,\mathbb{Z}_U) \to 0 \Rightarrow H^n(X,\mathbb{Z}_U) = \mathbb{Z}$ to complete the proof

Now we suffice to prove $H^n(X,\mathbb{Z}_Y) = \mathbb{Z}$, first let's us see why general

$\underbrace{H^n(X,\mathbb{Z}_Y) = H^n(Y,\mathbb{Z})}_{\text{(sheaf case)}} = H^n(Y,\mathbb{Z})$ (singular case) $= H^n(S^n,\mathbb{Z})$ since $H^n(S^n,\mathbb{Z}) = \mathbb{Z}$ ▨

(E.g. $\mathbb{A}^2_k$ and three lines general the $\triangle$ is S^1-like and cohomology (singular) $\mathbb{Z}$! (2-simplex) not general)

Another way is take $Y = \bigcup H_i \Rightarrow 0 \to \mathbb{Z}_Y \to \bigoplus \mathbb{Z}_{H_i} \to \bigoplus_{i<j} \mathbb{Z}_{H_i \cap H_j} \to \cdots \to \bigoplus_{i_0 < \cdots < i_n} \mathbb{Z}_{H_{i_0} \cap \cdots \cap H_{i_n}} \to 0$ is the Čech resolution of $\mathbb{Z}_Y$, it's finite due to there are $H_1 \cap \cdots \cap H_{n+1} = \emptyset$ due to the generality

Due to the H^i commutes the $\oplus \Rightarrow$ it's a flasque resolution (acyclic)

$\Rightarrow H^n(X,\mathbb{Z}_Y) = H^n(\Gamma(X,-)(\mathbb{Z}_Y^\bullet)) = \mathbb{Z} \oplus \left(\bigoplus H^n(\Gamma(X,\mathbb{Z}_{Y-H_i}))\right) = \mathbb{Z} \oplus 0 = \mathbb{Z}$ ▨

Ex2.2. $\mathcal{K}$ is constant $\Rightarrow$ flasque

and $\frac{\mathcal{K}}{\mathcal{O}_X}$ is flasque due to Ex1.21 II, $\frac{\mathcal{K}}{\mathcal{O}_X} = \bigoplus_{P \in X} i_*\left(\frac{K}{\mathcal{O}_P}\right)$, and $H^i\left(X, \frac{\mathcal{K}}{\mathcal{O}}\right) = \bigoplus_{P \in X} H^i\left(X, i_P\left(\frac{K}{\mathcal{O}_P}\right)\right) = 0$

$\Rightarrow \underbrace{H^{i-1}\left(X, \frac{\mathcal{K}}{\mathcal{O}}\right)}_{=0} \to H^i(X,\mathcal{O}) \to \underbrace{H^i(X,\mathcal{K})}_{=0} \to \cdots \Rightarrow H^i(X,\mathcal{O}) = 0$ (only for $i>1$)

detail: $H^0\left(X, \frac{\mathcal{K}}{\mathcal{O}}\right) = \frac{\Gamma(X,\mathcal{K})}{\Gamma(X,\mathcal{O})} = \frac{k}{k} = 0$ ▨

Ex2.3. (a) $\Gamma_Y(X,-): \mathrm{Ab}(X) \to \mathrm{Ab}$ is obviously a functor

$\Gamma_Y(X,\mathcal{F}) \to \Gamma_Y(X,\mathcal{G})$ ▨

$s \mapsto f(X)(s)$ and if $\mathrm{supp}\, s \subset Y \Rightarrow \mathrm{supp}(f(X)(s)) \subset Y$ due to $f(X)(s)_p = f(X)(s_p) = 0$

For left exactness: $0 \to \mathcal{F}' \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{F}'' \to 0$ if $f(X)(s) = 0 \Rightarrow s = 0$ by the Γ left exact and $\mathrm{Ker}\, g(X)|_{\Gamma_Y} = \{s \in \Gamma_Y(X,\mathcal{F}) \mid s_p = 0$ for $\forall p \notin Y$, $s \in \mathrm{Ker}\, g(X)\}$

$0 \to \Gamma_Y(X,\mathcal{F}') \xrightarrow{f(X)} \Gamma_Y(X,\mathcal{F}) \xrightarrow{g(X)} \Gamma_Y(X,\mathcal{F}'')$

$s \mapsto f(X)(s) \mapsto g(X)(t)$ $\Rightarrow \mathrm{Ker}\, g(X)|_{\Gamma_Y}$

(b) Similarly, due to $\mathrm{Im}\, f(X)|_{\Gamma_Y} = \mathrm{Im}\, f(X) \cap \Gamma_Y(X,\mathcal{F})$

$0 \to \Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{F}'') \ldots$ $= \Gamma_Y(X,\mathcal{F}) \cap \mathrm{Ker}\, g(X)$

$= \Gamma_Y(X,\mathcal{F}) \cap \mathrm{Im}\, f(X) = \mathrm{Im}\, f(X)|_{\Gamma_Y}$

$= \Gamma(X,\mathcal{F}'') \cap \Gamma_Y(X,\mathcal{F}'') = \Gamma_Y(X,\mathcal{F}'')$ ▨

(c) $0 \to \mathcal{F} \to \mathcal{G} \to \frac{\mathcal{G}}{\mathcal{F}} \to 0$ and $\to H^{i-1}(X,\mathcal{G}) \to \boxed{H^{i-1}\left(X,\frac{\mathcal{G}}{\mathcal{F}}\right) \to H^i(X,\mathcal{F})} \to \underbrace{H^i(X,\mathcal{G})}_{=0} \to \cdots$

$\Rightarrow H^i(X,\mathcal{F}) \cong H^{i-1}\left(X,\frac{\mathcal{G}}{\mathcal{F}}\right)$, induction on i due to $\frac{\mathcal{G}}{\mathcal{F}}$ also flasque ▨

(d) $0 \to \mathcal{H}_Y^0(\mathcal{F}) \to \mathcal{F} \to j_*(\mathcal{F}|_{X-Y}) \to 0$ for $\mathcal{F}$ flasque due to Ex1.20(c) II

then by (b) $\Rightarrow 0 \to \Gamma(X,\mathcal{H}_Y^0(\mathcal{F})) \to \Gamma(X,\mathcal{F}) \to \Gamma(X, j_*(\mathcal{F}|_{X-Y})) \to 0$ i.e. the desired one ▨

We embedding $0\to\mathbb{Z}\to\mathcal{F}\to\frac{\mathcal{F}}{\mathbb{Z}}\to 0$, $\mathcal{F}$ flasque

$$0\to\underbrace{\Gamma(S^1,\mathbb{Z})}_{=\mathbb{Z}}\to\boxed{\Gamma(S^1,\mathcal{F})\to\Gamma(S^1,\tfrac{\mathcal{F}}{\mathbb{Z}})}\to\boxed{\underbrace{H^1(S^1,\mathbb{Z})}_{=\mathbb{Z}}}\to\underbrace{H^1(S^1,\mathcal{F})}_{=0}\to\cdots$$

We compute its cokernel

Now we cover S^1 by $U_1=S_{up}$ and $U_2=S_{down}$ due to Ex1.3, II $\Rightarrow \Gamma(S^1,\mathcal{F})\ni s$ is given by $s_i\in\Gamma(U_i,\mathcal{F})$, s.t. $(s_1-s_2)|_{U_1\cap U_2}\in\underbrace{\Gamma(U_1\cap U_2,\mathbb{Z}_{U_1\cap U_2})}_{=\mathbb{Z}}$, we denote it as $n\in\mathbb{Z}$ thus one give

Now we do this again for $\frac{\mathcal{F}}{\mathbb{Z}}$, corresponding to $m\in\mathbb{Z}$

now $\{(s_1,U_1),(s_2,U_2)\}\in\Gamma(S^1,\mathcal{F})\xrightarrow{\Phi}\Gamma(S^1,\frac{\mathcal{F}}{\mathbb{Z}})\ni\{(\bar{s}_1,U_1),(\bar{s}_2,U_2)\}$

$s_1-s_2=n$ $\quad\quad$ $\bar{s}_1-\bar{s}_2=m$

due to we quotient $\mathbb{Z}$, $\bar{s}_1-\bar{s}_2$ can be any $m\in\mathbb{Z}$ not depend on $n\Rightarrow\frac{\Gamma(S^1,\frac{\mathcal{F}}{\mathbb{Z}})}{\mathrm{Im}\,\Phi}=\mathbb{Z}$ due to $\{(\bar{s}_1,U_1),(\bar{s}_2,U_2)\}$ given by $\{(s_1,U_1),(s_2+n,U_2)\}$ ▨

(b) We embedding $0\to\mathcal{C}\to\mathcal{D}\to\frac{\mathcal{D}}{\mathcal{C}}\to 0$

where $\mathcal{D}$ is $\mathcal{C}$ without continuous

and $\mathcal{D}$ is flasque

$\Rightarrow H^1(S^1,\mathcal{C})=\mathrm{coker}(\Phi:\Gamma(S^1,\mathcal{D})\to\Gamma(S^1,\frac{\mathcal{D}}{\mathcal{C}}))$

We show Φ surjective

$\Phi:\Gamma(S^1,\mathcal{D})\to\Gamma(S^1,\frac{\mathcal{D}}{\mathcal{C}})$, again take a covering (U_1,U_2)

$((s_1,U_1),(s_2,U_2))\to((\bar{s}_1,U_1),(\bar{s}_2,U_2))$ now $s_1-s_2=n$

We need to choose $s_1',s_2'\in\bar{s}_1,\bar{s}_2$ respectively $\bar{s}_1-\bar{s}_2=m$ and $m-n=f_1-f_2$,

and $s_1'-s_2'=m$, this is by $s_1'=s_1+f_1$ $\quad$ f_1,f_2 are continuous on S^1.

$s_2'=s_2+f_2$ ▨

and here we replace M with $\frac{M}{(x_1 \cdots x_{n-1})M}$ for simplicity

It suffices to show that $H^i_{\mathfrak{a}}(\frac{M}{x_n M}) = 0$, then by LES and induction

but it's trivial: for $i < n-1$, $H^i_{\mathfrak{a}}(\frac{M}{x_n M}) = 0$ due to $\mathrm{depth}_{\mathfrak{a}}(\frac{M}{x_n M}) \geq n$, and by Grothendieck's vanishing and induction hypothesis;

$(\Leftarrow)$ $H^i_{\mathfrak{a}}(M) = 0$ for $i \leq n-1 \Rightarrow \mathrm{depth}_{\mathfrak{a}} M \geq \mathrm{depth}_{\mathfrak{a}} \frac{M}{x M} + 1 \geq (n-1)+1 = n$ ▨

$\Updownarrow$ $H^i_{\mathfrak{a}}(\frac{M}{x_n M}) = 0$ for $i < n-1$

Ex3.5. $\mathrm{depth}\, \mathcal{O}_p = \mathrm{depth}_{m_p} \mathcal{O}_p \geq 2 \Leftrightarrow H^i_{m_p}(\mathcal{O}_p) = 0$, $i \leq 1$.

Extend a function is local on target $\Rightarrow$ assume X affine and $U = X = \mathrm{Spec} A$

$H^i_{m_p}(\mathcal{O}_p) = H^i_p(X, \widetilde{\mathcal{O}_p})$ by Ex3.3 $\Rightarrow H^0_{m_p}(\mathcal{O}_p) = \Gamma_p(X, \widetilde{\mathcal{O}_p}) = 0$ hold naturally due to $m_p = \{s \in \Gamma(X, \mathcal{O}_p) \mid s = 0\}$

$\Rightarrow \mathrm{depth}\, \mathcal{O}_p \geq 2 \Leftrightarrow H^1_{m_p}(\mathcal{O}_p) = 0$

$\Leftrightarrow$ The $\boxed{\text{obstruction}}$ of extend the section of $\Gamma(X - p, \mathcal{O}_{X-p})$ to $\Gamma(X, \mathcal{O}_X)$ vanishing

Consider $0 \to m_p \to \mathcal{O}_p \to k(p) \to 0$ and $0 \to m_p \mathcal{O}_p \to \mathcal{O}_p \to k(p) \to 0$, $0 \to 0 \to i_p(\mathcal{O}_p) \to i_p(k(p)) \to 0$

due to $0 \to \mathcal{H}^0_p(\mathcal{O}_X) \to \mathcal{O}_X \to \mathcal{O}_{X-p} \to 0$ and $H^1_{m_p}(\mathcal{O}_p) = H^1_p(X, \widetilde{\mathcal{O}_p}) = H^1_p(X, \mathcal{H}^0_p(\mathcal{O}_X)) = 0$

thus the preimage is the extension $\Rightarrow \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_{X-p}) \to 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \quad \Gamma(X - p, \mathcal{O}_{X-p})$

and uniqueness due to $\Gamma_p(X, \widetilde{\mathcal{O}_p}) = 0 \Rightarrow$ injective ▨

Ex3.6. (a) Consider the exactness of the functor

$\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_X}(-, \bigoplus f_{i*}(\widetilde{I_i})) = \bigoplus_{\text{finite}} \mathrm{Hom}_{\mathcal{O}_X}(-, f_{i*}(\widetilde{I_i})) = \bigoplus_{\text{finite}} \mathrm{Hom}_{\mathcal{O}_{U_i}}(-, \widetilde{I_i})$

$\cong \bigoplus_{\text{finite}} \mathrm{Hom}_{A_i}(-, M_i)$, the last step due to the category equivalence

and M_i is injective A_i-module $\Rightarrow$ exact ▨

(b) Cover X by $\{U_i = \mathrm{Spec} A_i\}$ (Precise by $\{U_i \xrightarrow{f_i} U\}$ in view of Grothendieck topology)

$\mathcal{G}$ injective $\Rightarrow \mathcal{G}|_{U_i} = \widetilde{M_i}$ injective $\Rightarrow M_i$ injective $\Rightarrow M_i$ flasque $\Rightarrow \mathcal{G}|_{U_i}$ flasque

$\Rightarrow \mathcal{G}(U) \xrightarrow{\mathrm{Res}} \mathcal{G}(V) \Leftrightarrow \bigoplus j_{i*} \mathcal{G}|_{U_i}(U_i \cap U) \xrightarrow{\mathrm{Res}} \bigoplus j_{i*} M_i(U_i \cap V) \to 0 \Rightarrow \mathcal{G}$ flasque

thus by Hint we prove $\mathcal{G}$ injective $\Rightarrow \forall U \subset X$, $\mathcal{G}|_U$ injective in $\mathrm{Qco}(X)$

For this we assume $U = \mathrm{Spec} A$ affine, thus $\mathcal{G}|_U = \widetilde{M}$ and we prove M injective

It suffices to show $j_*(\mathcal{G}|_U)$ injective, then $0 \to \mathcal{H}_{X-U}(\mathcal{G}) \to \mathcal{G} \to j_*(\mathcal{G}|_U) \to 0$

then take injective resolution of all of the three and by horseshoe Lemma

a subobject of injective

injective is injective

$\Rightarrow H^1(j_*(\mathcal{G}|_U)) = 0$ ▨

Ex3.7 (a) (Deligne) We have $0 \to \Gamma_{\mathfrak{a}}(M) \to \Gamma(X, \widetilde{M}) \to \Gamma(U, \widetilde{M}) \to H^1_{\mathfrak{a}}(M) \to H^1(X, \widetilde{M}) \to \cdots$

and $M_{\mathfrak{a}} \xrightarrow{f} N$ induce by all $(\varphi_i) \in (\mathrm{Hom}_A(\mathfrak{a}^n, M))_i$, $f_i : \varphi_i \mapsto m_i$, $\varphi_i = \varphi_i \otimes m_i$

by $f : \frac{m}{a} \mapsto \sum f_i(a \otimes m)$ such that every $f \circ i : a \otimes m$ where $\mathrm{Hom}(\mathfrak{a}^n, M) = \mathfrak{a}^n M$ $= \mathfrak{a} \otimes M$

$\Rightarrow M_{\mathfrak{a}}$ is a inclusion ▨

(b) Now we show $\Gamma(X, \widetilde{I}) \to \Gamma(U, \widetilde{I})$ is surjective

$\mathrm{Hom}(\mathfrak{a}^n, -)$ is fully functor for injective

$\qquad\qquad \| \qquad\qquad \|$

$\varinjlim \mathrm{Hom}(\mathfrak{a}^n, I)$

it's surjective due to $\varinjlim$ exact and $I \to \mathrm{Hom}(\mathfrak{a}^n, I)$ surjective ▨

Ex3.1. $(\Rightarrow)$ $X = \mathrm{Spec} A \Rightarrow X_{red} = \mathrm{Spec}(A_{red})$, A_{red} is $\frac{A}{\mathfrak{a}}$ for a radical ideal

$(\Leftarrow)$ $X_{red} = \mathrm{Spec}(A_{red})$, A_{red} has no nilpotent, now we need to prove $H^1(X, \mathcal{F}) = 0$

To apply induction on filtration, we need to show: $\exists n : \mathcal{N}^n \mathcal{F} = 0$

take affine covering of X, we assume $U_i = \mathrm{Spec} A_i \Rightarrow \mathcal{N}^n \mathcal{F}(U_i) = \mathcal{N}^n(U_i) \mathcal{F}(U_i) = \mathrm{nil}(A_i)^n \widetilde{M_i}$

$\Rightarrow \exists n : \mathrm{nil}(A_i)^n M_i = 0$ and due to quasi-compact

and A_i; M_i noetherian due to Noetherian property is local on target

$\Rightarrow \exists N : \mathcal{N}^N \mathcal{F} = 0$ ▨ Let N^0 to be the smallest

$\Rightarrow \mathcal{F} \supset \mathcal{N}\mathcal{F} \supset \cdots \supset \mathcal{N}^{N_0} \mathcal{F} = 0 \Rightarrow H^1(X, \mathcal{N}^{N_0} \mathcal{F}) = 0$

$\forall i$, consider $0 \to \mathcal{N}^{i+1} \mathcal{F} \to \mathcal{N}^i \mathcal{F} \to \frac{\mathcal{N}^i \mathcal{F}}{\mathcal{N}^{i+1} \mathcal{F}} \to 0$, by induction, it suffices to show

$H^j(X, \frac{\mathcal{N}^i \mathcal{F}}{\mathcal{N}^{i+1} \mathcal{F}}) = 0$, $\forall i, j$, but note that $\frac{\mathcal{N}^i \mathcal{F}}{\mathcal{N}^{i+1} \mathcal{F}} \subset \mathcal{O}_X$ is ideal sheaf, thus sufficient

show for $i = 1$, i.e. $H^1(X, \frac{\mathcal{N}^i \mathcal{F}}{\mathcal{N}^{i+1} \mathcal{F}}) = 0$ (due to $\frac{\mathcal{N}^i \mathcal{F}}{\mathcal{N}^{i+1} \mathcal{F}} \subset \mathcal{N} \subset \mathcal{O}_X$)

$\frac{\mathcal{N}^i \mathcal{F}}{\mathcal{N}^{i+1} \mathcal{F}} = \mathcal{N}^i \cdot \frac{\mathcal{F}}{\mathcal{N}\mathcal{F}}$, and $\mathcal{O}_{X_{red}} = \frac{\mathcal{O}_X}{\mathcal{N}} \Rightarrow H^1(X, \frac{\mathcal{N}^i \mathcal{F}}{\mathcal{N}^{i+1} \mathcal{F}}) = H^1(X, \mathcal{N}^i \frac{\mathcal{F} \cdot \mathcal{O}_X}{\mathcal{N}^i \cdot \mathcal{N}})$

$= H^1(X_{red}, \mathcal{N}^i \frac{\mathcal{F} \cdot \mathcal{O}_{X_{red}}}{\mathcal{N}^i \cdot \mathcal{N}})$ by view it as quasi-module

$= 0$ ▨

Ex3.2. $(\Rightarrow)$ $X = \mathrm{Spec} A$, A have $\mathfrak{p}_1 \cdots \mathfrak{p}_r$ minimal primes

$\Rightarrow X = X_1 \cup \cdots \cup X_r$ s.t. $X_i = \mathrm{Spec} A_{\mathfrak{p}_i}$

$(\Leftarrow)$ $X = X_1 \cup \cdots \cup X_r$, $X_i = \mathrm{Spec} A_i$, it's not easy to construct the $X = \mathrm{Spec} A$, A out, thus we use Thm3.7 (i) $\Leftrightarrow$ (iii)

$\mathcal{J} \subset \mathcal{O}_X$ an ideal sheaf and filtration $\mathcal{J} \supset \mathcal{J} \cdot \mathcal{J}_1 \supset \cdots \supset \mathcal{J}(\mathcal{J}_1 \cdots \mathcal{J}_r)$, where $\mathcal{J}_i$ is the ideal sheaf of $X_i \Rightarrow H^1(\mathcal{J}(\mathcal{J}_1 \cdots \mathcal{J}_r), X) = H^1(X, 0) = 0$ due to $\mathcal{J}_1 \cdots \mathcal{J}_r = 0$

here we need to reduce: $\mathcal{J}_1 \cdots \mathcal{J}_r$ vanishing in all of X $\Rightarrow \mathcal{J}_1 \cdots \mathcal{J}_r$ is nilpotent $\Rightarrow \mathcal{J}_1 \cdots \mathcal{J}_r = 0$ ▨

and $H^1(X, \frac{\mathcal{J}\mathcal{J}_1 \cdots \mathcal{J}_i}{\mathcal{J}(\mathcal{J}_1 \cdots \mathcal{J}_{i+1})}) = H^1(X_{i+1}, \frac{\mathcal{J} \cdot \mathcal{J}_1 \cdots \mathcal{J}_i}{\mathcal{J} \cdot (\mathcal{J}_1 \cdots \mathcal{J}_{i+1})}) = 0$

then by induction ▨

Ex3.3. (a) we have category equivalence:

$$\begin{array}{ccc} A\text{-Mod} & \xrightarrow{\Gamma_{\mathfrak{a}}} & A\text{-Mod} \\ \sim \downarrow \wr & & \wr \downarrow \sim \\ \mathrm{QCoh}(\mathrm{Spec} A) & \xrightarrow{\mathcal{H}^0_{V(\mathfrak{a})}} & \mathrm{QCoh}(\mathrm{Spec} A) \end{array}$$

and $\Gamma_{\mathfrak{a}}(M) = \Gamma_{V(\mathfrak{a})}(X, \widetilde{M}) \Rightarrow$ exact left, both $\Gamma_{\mathfrak{a}}$ and $\mathcal{H}_{V(\mathfrak{a})}$ due to $\Gamma_{V(\mathfrak{a})}$ is left exact ▨

(b) Now $\Gamma_{\mathfrak{a}}(M) = \Gamma_Y(X, \widetilde{M})$, it suffices to show both of them are effaceable, this is trivial due to Γ_Y is ▨

(c) Due to $\Gamma_Y(X, \widetilde{\Gamma_Y(X, \widetilde{M})}) = \Gamma_Y(X, \widetilde{M})$ and effaceable $\begin{pmatrix} M \mapsto \Gamma_{\mathfrak{a}}(M) \\ M \mapsto \Gamma_Y(X, \widetilde{M}) \end{pmatrix}$ ▨

$\qquad \| \qquad\qquad \|$

$\Gamma_{\mathfrak{a}} \Gamma_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(M)$

Ex3.4. (a) $\Gamma_{\mathfrak{a}}(M) = \{m \in M \mid \exists n, \mathfrak{a}^n m = 0\}$ now $\mathrm{depth}_{\mathfrak{a}} M \geq 1$, i.e. $\exists x_i \in \mathfrak{a}$: x_i isn't zero divisor

$\Rightarrow \forall m \in M, \forall n, \exists x_i \in \mathfrak{a} : x_i^n m \neq 0 \Rightarrow \Gamma_{\mathfrak{a}}(M) = 0$

Conversely, $\Gamma_{\mathfrak{a}}(M) = 0 \Leftrightarrow \mathfrak{a} \not\subset$ [illegible] $\Leftrightarrow \mathrm{depth}_{\mathfrak{a}} M \geq 1$

if $\exists \mathfrak{p} \in \mathrm{Spec} A$: $\mathfrak{a} \subset \mathrm{Ann}(m) \Rightarrow \Gamma_{\mathfrak{a}}(M) = \{m \in M \mid \exists n, \mathfrak{a}^n m = 0\} \ni m \Rightarrow \Gamma_{\mathfrak{a}}(M) \neq 0$, contradiction

and $M = (m_1, \cdots m_r)A$, $\forall m \in M$, $\mathfrak{a} \not\subset \mathrm{Ann}(m) \Leftrightarrow \forall m_i$, $\mathfrak{a} \not\subset \mathrm{Ann}(m_i)$

$\Rightarrow \mathfrak{a} \not\subset \bigcup \mathrm{Ann}(m_i)$ the set of all zero divisor

$\Rightarrow \mathfrak{a}$ contains all non-zero-divisors

and $\mathfrak{a} \neq \emptyset \Rightarrow \exists x \in \mathfrak{a}$: x isn't zero divisor

$\Rightarrow \mathrm{depth}_{\mathfrak{a}} M \geq 1$ ▨

(b) In (a) we proven the case of $n = 1$ $(\Rightarrow)$ for induction, consider the filtration

$M \supset x_1 M \supset \cdots \supset (x_1 \cdots x_n) M \supset (x_1 \cdots x_{n+1}) M$, where $x_1 \cdots x_{n+1}$ is regular, then $0 \to M \xrightarrow{x_n} M \to$

② is similar as the Step 2 in ①, replace ρ by β

Consider sheaf of ideal $\mathcal{F} \subset \mathcal{O}_Y$ and $\mathcal{F}$ is torsion-free due to Y integral

$\mathcal{O}_Y^r \xrightarrow{\beta} f_*\mathcal{M} \to \mathcal{J} \to 0$, the $\mathcal{J} = \mathrm{coker}\,\alpha$ the torsion sheaf (i.e. supp $\mathcal{J} \subsetneq Y$)

Claim, then β is injective. Pf. Due to supp $\mathcal{J} \cap V = \phi$

$\Rightarrow 0 \to f_*\mathcal{G} \to \mathcal{F}^r \to \mathrm{Coker}\,\beta \to 0 \Rightarrow 0 \to$ ... $\boxed{\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{J}, \mathcal{F})} \to f_*\mathcal{G} \xrightarrow{\beta} \mathcal{F}^r$

Due to $\mathcal{J}$ is torsion but $\mathcal{O}_Y$, $\mathcal{F}$ is torsion free ▨

then $f_*\mathcal{G}$, Coker support not whole Y (Same reason as Step 2)

$\Rightarrow H^1(Y, f_*\mathcal{G}) = H^1(\mathrm{Coker}\,\beta) = 0 \Rightarrow H^1(Y, \mathcal{F}^r) = H^1(Y, \mathcal{F}^r) = 0 \Rightarrow H^1(Y, \mathcal{F}) = 0$,

then by Serre's Criterion ▨

Ex 4.3. Take $V_1 = V(x)^c$, $V_2 = V(y)^c$ both affine open, $\mathcal{U} = \{V_1, V_2\}$

$\Rightarrow H^i(\mathcal{U}, \mathcal{O}_U) = H^i(\mathcal{U}, \mathcal{O}_U)$, $\mathcal{O}_U \subset \mathcal{O}_X$ is coherent

Computation. $C^0 = \Gamma(V_1, \mathcal{O}_U) \times \Gamma(V_2, \mathcal{O}_U) = \Gamma(V_1, \mathcal{O}_X) \times \Gamma(V_2, \mathcal{O}_X) = k[x, \frac{1}{x}, y] \times k[x, \frac{1}{y}, y]$

$C^1 = \Gamma(V_1 \cap V_2, \mathcal{O}_U) = \Gamma(V_1 \cap V_2, \mathcal{O}_X) = k[x, \frac{1}{x}, y, \frac{1}{y}]$

and $C^0 \xrightarrow{d} C^1$ given by $k[x, \frac{1}{x}, y] \times k[x, y, \frac{1}{y}] \to k[x, \frac{1}{x}, y, \frac{1}{y}]$, $(f_1, f_2) \mapsto f_1 - f_2$

Ind

$= \{f(x, y, \frac{1}{x}) - g(x, y, \frac{1}{y}) \mid f, g$ are polynomial$\}$

$\Rightarrow H^1(\mathcal{U}, \mathcal{O}_U) = k[x, \frac{1}{x}, y, \frac{1}{y}] / \ldots = k[\frac{1}{x}, \frac{1}{y}]$ ▨

(And $H^0(\mathcal{U}, \mathcal{O}_U) = \mathrm{Ker}\,d = k[x, y]$ is clear, verified the correctness)

Ex 4.4. (a) In the Chain level $\check{C}(\mathcal{U}, \mathcal{F}) \to \check{C}(\mathcal{B}, \mathcal{F})$

and the naturality is clear

Commute with d

$\Rightarrow \varphi = (\varphi^p)$ induce the Cohomology level ▨

$\prod f_*(\mathcal{F}|_{U_{i_0 \ldots i_p}}) \xrightarrow{\varphi^p} \prod f_*(\mathcal{F}|_{V_{j_0 \ldots j_p}})$

$\varphi^p(W): \prod_{i_0 < \cdots < i_p} \Gamma(W \cap U_{i_0 \ldots i_p}, \mathcal{F}) \to \prod_{j_0 < \cdots < j_p} \Gamma(W \cap V_{j_0 \ldots j_p}, \mathcal{F})$

is the natural restrictions' product

(b) Chain-Level:

thus cohomology level: $H^i(\mathcal{B}, \mathcal{F}) \to H^i(X, \mathcal{F})$ ▨, $H^i(\mathcal{U}, \mathcal{F})$

$$\begin{array}{l} 0 \to \mathcal{F} \to \mathcal{C}^\cdot(\mathcal{B}, \mathcal{F}) \\ 0 \to \mathcal{F} \to \mathcal{C}^\cdot(\mathcal{U}, \mathcal{F}) \end{array}$$

(c) We have

$0 \to C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F}) \to \prod H^1(U_{i_0 \ldots i_p}, \mathcal{F}) \to \cdots$

and $\mathcal{U} \to \infty \Rightarrow U_{i_0 \ldots i_p} \to \emptyset$ (It's not strict) for p small, i.e. $p = 0, 1, 2$

$\Rightarrow \varinjlim C^\cdot(\mathcal{U}, \mathcal{F}) = \varinjlim \mathcal{F}^\cdot(\mathcal{U})$, i.e. $0 \to C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F}) \to 0$

then by LES and same as the proof of Thm 4.5 ▨

Ex 4.5. $H^1(X, \mathcal{O}_X^*) = \varinjlim H^1(\mathcal{U}, \mathcal{O}_X^*) = \varinjlim \{\mathcal{L} \mid \mathcal{L}$ trivial on $\mathcal{U}\}$

then we verify that $\mathcal{F} = \varinjlim \mathcal{F}_{\mathcal{U}}$

has kernel of isomorphism class $\Rightarrow H^1(X, \mathcal{O}_X^*) \cong \mathrm{Pic}(X)$

We'll see that ① cocycle condition of ... $\Rightarrow$ cocycle elements in $Z^1(\mathcal{U}, \mathcal{O}^*)$

thus complete the ② $\mathcal{L} \cong \mathcal{L}'$, then their difference of section $\in B^1(\mathcal{U}, \mathcal{O}^*)$

proof: ① φ_{ij} ... $= \varphi_i^{-1} \circ \varphi_j : \mathcal{O}_{U_i \cap U_j} \to \mathcal{O}_{U_i \cap U_j}$ and $g_{ij} = \varphi_{ij}(1) \in \mathcal{O}(U_i \cap U_j)$

$\Rightarrow g_{ij} g_{jk} g_{ki} = 1$ and ... $S_i - S_j = \ldots = 0$ ▨

Due to here is $\mathcal{O}_X^*$, we turn it to multiplicative language

Ex 4.6. By Thm 4.5. $H^1(X, \mathcal{F}) = \check{H}^1(f^{-1}\mathcal{U}, \mathcal{F})$

we prove $\check{H}^1(f^{-1}\mathcal{U}, \mathcal{F}) \cong \check{H}^1(\mathcal{U}, f_*\mathcal{F})$; this is by chain level: $\check{C}^p(f^{-1}\mathcal{U}, \mathcal{F})$

$= \prod_{i_0 < \cdots < i_p} \Gamma(\mathcal{F}|_{f^{-1}(V_{i_0 \ldots i_p})})$

$= \prod_{i_0 < \cdots < i_p} \Gamma(f_*\mathcal{F}|_{V_{i_0 \ldots i_p}})$

Ex 4.2. (a) Let $x \in X$, $y \in Y$ to be generic and $V \subset N_y$ a open neiberhood, then $\mathcal{O}_Y^r|_V \cong f_*\mathcal{M}|_V$, one can let $\mathcal{M} = \mathcal{G}$ as a priori, $\mathcal{L}$ invertible due to X affine.

Consider the fibers to apply the finiteness: (generic fibre)

$f^{-1}(y) = X \otimes_Y \mathrm{Spec}\,k(y) = \mathrm{Spec}\,\mathcal{O}_X$, and $\mathcal{O}_X / \mathcal{O}_Y$ has basis $s_1 \ldots s_r \in A$

$(\mathcal{O}_X = \mathrm{Frac}\,A$, $X = \mathrm{Spec}\,A$, A integral), then $\exists\, U \subseteq N_y$ affine, s.t. $x \in X_g \subset f^{-1}(U)$, $g \in \Gamma(X, \mathcal{L})$

thus by II Lemmas 5.3(b) $\Rightarrow \exists\, n: (s_i|_{X_g})\, g^n$ is global $\in \Gamma(X, \mathcal{L}^n)$

$\Rightarrow \alpha: \mathcal{O}_Y^r \to f_*\mathcal{L}^n$ is defined due to $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y^r, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{F})^{\oplus r}$

due to we gives out the global sections $\cong \bigoplus \mathcal{F}(Y)$

Verification (and collapse U to V)

$\mathcal{L}^n|_{f^{-1}(U)} = \widetilde{M}$, M finite type A-module, and $U = \mathrm{Spec}\,B$: $\varphi: B \to A$

$\Rightarrow f_*\mathcal{L}^n|_U = \widetilde{\varphi_* M} \Rightarrow f_*\mathcal{L}^n$ is coherent due to it's locally coherent

$\Rightarrow \alpha = \widetilde{\theta}$, $\theta: B^r \to M$ by categorial equivalence and $\alpha_y = \theta_y = \ldots$

is isomorphism at generic y ▨

(b) (II.5.1e) gives the equivalence of categories $\mathrm{Qco}(\mathcal{O}_X)$ and $\mathrm{Qco}(f_*\mathcal{O}_X)$

$$\begin{array}{ccc} f_*\mathcal{M} & \xleftarrow{\alpha} & \mathcal{O}_Y^r \\ \downarrow & & \downarrow \\ \mathcal{H}om(f_*\mathcal{M}, \mathcal{F}) & \xrightarrow{\beta} & \mathcal{H}om(\mathcal{O}_Y^r, \mathcal{F}) \\ \| & & \| \\ f_*\mathcal{H}om(\mathcal{M}, \mathcal{F}) & & \mathcal{F}^n \end{array}$$

thus we have $\beta: f_*\mathcal{G} \to \mathcal{F}^r$, with $\mathcal{G} = \mathcal{H}om(\mathcal{M}, \mathcal{F})$.

and stalks commute all of these

$\Rightarrow$ take stalk at y, all isomorphisms preserved ▨

(c) ① Assume $Y = Y_{red}$ is clear, due to Y_{red} affine $\xRightarrow{\text{Ex 3.1}}$ Y affine (Step 1)

also we can assume $X = X_{red}$ due to ..., we can directly assume X is integral, i.e. replace X with X_i, X_i is the underlying space of X's irreducible component.

Because: As a proiri, we assume Y integral (later we prove it irreducible, and irreducible + reduced = intergral), then $f: X \to Y$ restrict to $f_i: X_i \to Y$ and $\exists\, i_0$: f_{i_0} dominant

$\Rightarrow$ dominant + finite $\Rightarrow$ surjective $\Rightarrow$ we replace X by X_{i_0} ▨ (Due to Ex 3.2.)

Now take $Y' \subset Y$ to be a irreducible component and $Y'' = Y - Y'$

then $Y' = Y'_{red}$ due to $Y = Y_{red}$ and $j: Y' \to Y_{red}$, then we have $f^*\mathcal{F} \to j_*j^*\mathcal{F}$ naturally

now we need to prove $H^1(Y', \mathcal{F}) = 0 \Rightarrow H^1(Y, \mathcal{F}) = 0$ then by Serre criterion Y affine.

by Noetherian induction (singleton holds), it suffices to prove any $\mathcal{F}$ s.t. supp $\mathcal{F} \subset Y'$

$\Rightarrow H^1(Y, \mathcal{F}) = 0$

Consider $j_*j^*\mathcal{F}$ coherent due to $j_*\mathcal{O}_{Y'} = \frac{\mathcal{O}_Y}{\mathcal{I}_{Y'}}$ and j^* preserve to coherence

$\Rightarrow 0 \to \mathrm{Ker}\,\rho \to \mathcal{F} \to \mathrm{Im}\,\rho \to 0$, y' the generic of Y', then $\begin{cases} \text{① Supp Ker}\,\rho \not\ni y' \\ \text{② Supp Im}\,\rho \subset Y' \subsetneq Y \end{cases}$

$\Rightarrow H^1(Y, \mathrm{Ker}\,\rho) = H^1(Y, \mathrm{Im}\,\rho) = 0$

$\Rightarrow H^1(Y, \mathcal{F}) = 0 \Rightarrow Y$ affine ▨

Ex4.9. ① Otherwise by Ex3.8e $\Rightarrow cd(Y^c) \le 2$, i.e. $H^3(Y^c, \mathcal{O}_X) = 0$
But by Čech cohomology $H^3(Y^c, \mathcal{O}_X) \neq 0$, a contradiction to complete the proof
Computation. It's hard to find an affine cover for Y^c directly, thus we use $H^3(X-Y, \mathcal{O}_{X-Y}) \cong H^4_Y(X, \mathcal{O}_X) \cong H^4_{\cap Y_i}(X, \mathcal{O}_X) = H^4_P(X, \mathcal{O}_X)$, and cover $X - P = (\mathbb{A}^4_k)^*$
By Ex2.3d $\Rightarrow \to \boxed{H^3(X, \mathcal{O}_X)} \to H^3(X-Y, \mathcal{O}_{X-Y}) \to H^4_Y(X, \mathcal{O}_X) \to \boxed{H^4(X, \mathcal{O}_X)} \to \cdots$ (Serre vanishing) by $D(x_i)$, $i \le 4$ (Serre vanishing) 0
$\Rightarrow H^3(X-Y, \mathcal{O}_{X-Y}) \cong H^4_Y(X, \mathcal{O}_X)$
and by M-V sequence: $\to \boxed{H^4_{Y_1}(X, \mathcal{O}_X) \oplus H^4_{Y_2}(X, \mathcal{O}_X)} \to H^4_Y(X, \mathcal{O}_X) \to H^4_{Y_1 \cap Y_2}(X, \mathcal{O}_X) \to \boxed{H^5_{Y_1}(X, \mathcal{O}_X) \oplus H^5_{Y_2}(X, \mathcal{O}_X)} \to \cdots$ ~~due to here we assume~~
and $H^q_{Y_i}(X, \mathcal{O}_X) = H^{q-1}(X - Y_i, \mathcal{O}_{X-Y_i}) = 0$ due to $cd(X - Y_i) \le 1$ as it's an complete intersection (set-theoretic) of codim 2, for $q = 3, 4$, $i = 1, 2$
Now it suffices to compute $H^3(X-P, \mathcal{O}_{X-P})$ by $\mathfrak{U} = \{D(x_i)\}$
the Čech complex is clear (Repeat Ex4.3)
$\bigoplus_{i \le 4} k[x_1, \ldots, x_4, \frac{1}{x_i}] \xrightarrow{d^0} \bigoplus_{i<j} k[x_1, \ldots, x_4, \frac{1}{x_i}, \frac{1}{x_j}] \xrightarrow{d^1} \cdots \bigoplus_{i \le 4} k[x_1, \ldots, \hat{x_i}, \ldots, x_4, \frac{1}{x_1 \cdots x_4}]$ (1) (2) (3)
$\Rightarrow \check{H}^3(X-P, \mathcal{O}_{X-P}) = k\langle x_1^i x_2^j x_3^k x_4^l \mid i, j, k, l < 0 \rangle \Rightarrow$ at least $\check{H}^3(X-P, \mathcal{O}_{X-P}) \neq 0$ ▨
② If $Y = H_1 \cap H_2$ $\Rightarrow$ project to $\mathbb{A}^4_k \Rightarrow Y$ also contradiction ▨
Ex4.10* To relate infinitesimal extension with $\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$, we use the Ex8.6
we have $\mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{J}) \ldots \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$
where h' is lifting of h $\quad \boxed{\theta} + h$ (the obstruction of lifting) $= h'$
By Ex8.7, the affine's infinitesimal extension is trivial, thus take an affine covering $\mathfrak{U}$ of X (underlying space), i.e. $U_i \in \mathfrak{U}$, $\mathcal{J}_{U_i} = \mathcal{O}_{U_i} \oplus \mathcal{F}|_{U_i}$
Now every infinitesimal extension $\Longleftrightarrow \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{J})$ by $h'-h$
$\Longleftrightarrow \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{J}) \Longleftrightarrow H^1(X, \mathcal{F} \otimes \mathcal{T}_X)$ by same way in Ex4.5 ▨
$= \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) \otimes \mathcal{J}$ $\quad$ (4.5) $H^1(X, \mathcal{F} \otimes \mathcal{T}_X)$
by basis extension $h: \mathcal{O}_X \to \mathcal{O}_{X'}$
$\Rightarrow$ equal to $\mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) \otimes \mathcal{F}$
$= \mathcal{T}_X \otimes \mathcal{F}$
Ex4.11. (Leray) Compare the proof of Thm4.5, embed $\mathcal{F} \subset \mathcal{G}$ flasque, where we using affine?, it's ① $H^i(U_{i_0 \ldots i_p}) = 0 \Rightarrow \Gamma(U_{i_0 \ldots i_p}, -)$ exact } both holds here ▨
② inductively $H^p(U_{i_0 \ldots i_p}) = 0$

② Consider $\mathcal{L} \cong \mathcal{L}'$ and $\mathcal{L} \otimes \mathcal{L}'^{\vee}$ (Recall $\mathcal{F}^{\vee} = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$, for invertible $\mathcal{F}$, $\mathcal{F} \otimes \mathcal{F}^{\vee} = \mathcal{O}_X$)
$\Rightarrow \mathcal{L} \otimes \mathcal{L}'^{\vee}$ has translation $\varphi_{ij} = \frac{\psi_{ij}}{\psi'_{ij}}$
for $\forall \tau \in B^1(X, \mathcal{O}_X^*)$, $\exists \sigma \in C^0(X, \mathcal{O}_X^*)$ s.t. $d\sigma = \tau$, and $d\tau_{ij} = \sigma_j - \sigma_i \Rightarrow \tau_{ij} = \frac{\sigma_j}{\sigma_i}$
$\Rightarrow \mathcal{L} \otimes \mathcal{L}'^{\vee} \in B^1(X, \mathcal{O}_X^*) \Rightarrow$ then have same cohomology class ▨
Ex4.6. By LES and Ex4.5, ② is trivial by ①
① $0 \to \mathcal{J}_p \to \mathcal{O}^*_{X', p} \to \mathcal{O}^*_{X, p} \to 0$, for $\forall p \in X$, injective is trivial: $1 + f = 1 \in \mathcal{O}^*_{X', p}$ $\Rightarrow f = 0$
$f \mapsto 1 + f \mapsto \overline{1+f}$
$g \mapsto \bar{g}$
surjective also trivial: $\mathcal{O}_{X', p} \to \frac{\mathcal{O}_{X', p}}{\mathcal{J}_p}$ surjective and induce $\mathcal{O}^*_{X', p} \to \left(\frac{\mathcal{O}_{X', p}}{\mathcal{J}_p}\right)^* = \frac{\mathcal{O}^*_{X', p}}{\mathcal{J}_p}$ also surjective
For the exactness in middle:
$\overline{1+f} = 0 \cdot \bar{f} = 0$
and conversely $\bar{g} = 0 \Rightarrow \bar{g} = 0 \cdot \bar{g} = \overline{1+g}$ ▨
Ex4.7. $H^0(X, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = k$ by Liouville $\Rightarrow \dim H^0(X, \mathcal{O}_X) = 1$
$\Gamma(U, \mathcal{O}_X) = \frac{k[\frac{x_0}{x_1}, \frac{x_2}{x_1}]}{(f(\frac{x_0}{x_1}, 1, \frac{x_2}{x_1}))}$, $\Gamma(V, \mathcal{O}_X) = \frac{k[\frac{x_0}{x_2}, \frac{x_1}{x_2}]}{\ldots}$, $\Gamma(U \cap V, \mathcal{O}_X) = \frac{k[\frac{x_0}{x_1}, \frac{x_1}{x_2}, \frac{x_2}{x_1}]}{\ldots}$ (note $\frac{x_0}{x_2} = \frac{x_0}{x_1} \cdot \frac{x_1}{x_2}$)
thus $d: \frac{k[\frac{x_0}{x_1}, \frac{x_2}{x_1}]}{(f(\frac{x_0}{x_1}, 1, \frac{x_2}{x_1}))} \oplus \frac{k[\frac{x_0}{x_2}, \frac{x_1}{x_2}]}{(f(\frac{x_0}{x_2}, \frac{x_1}{x_2}, 1))} \to \frac{k[\frac{x_0}{x_1}, \frac{x_1}{x_2}, \frac{x_2}{x_1}]}{(f(\ldots))}$
$(g(\frac{x_0}{x_1}, \frac{x_2}{x_1}), h(\frac{x_0}{x_2}, \frac{x_1}{x_2})) \mapsto g(\frac{x_0}{x_1}, \frac{x_2}{x_1}) - h(\frac{x_0}{x_1} \cdot \frac{x_1}{x_2}, \frac{x_1}{x_2})$
$\ker d = (k, k) \in \Delta_k \cong k = H^0(X, \mathcal{O}_X)$
$\operatorname{coker} d = H^1(X, \mathcal{O}_X)$, I claim $\operatorname{coker} d = \{f \in \Gamma(U \cap V, \mathcal{O}_X) \mid f(\frac{x_0}{x_2}, \frac{x_1}{x_2}) = \sum_{\substack{1 \le i < d \\ -i < j < 0}} a_{ij} x^i y^j\}$
Compute the cokernel: then $\dim H^1(X, \mathcal{O}_X) \le \frac{1}{2}(d-1)(d-2)$
$\Gamma(U \cap V, \mathcal{O}_X) = \ldots$ and takes equality due to $f(\frac{x_0}{x_2}, \frac{x_1}{x_2}, 1)$
and $\operatorname{Im} d = \{g(\frac{x_0}{x_1}, \frac{x_2}{x_1}) - h(\frac{x_0}{x_2}, \frac{x_1}{x_2})\}$, $= x^d + f(\frac{x_0}{x_2}, \frac{x_1}{x_2}, 1)$ then $\exists\, i \ge d$, thus $x^i y^j$ can't exists $\Rightarrow \dim H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2)$ ▨
$\Rightarrow x^i y^j \in \operatorname{coker} d$, then $\forall j \ge 0$, $x^i y^j = 0$ ($x = \frac{x_0}{x_2}$, $y = \frac{x_1}{x_2}$)
and $j \ge i$, consider $\left(\frac{x_0}{x_1}\right)^i \left(\frac{x_2}{x_1}\right)^{j-i} = \ldots \mapsto x^i y^j$
for $\forall j \ge i$, thus only $1 \le i < d$, $-i \le j < 0$ non zero ▨
Ex4.8. (a) $\mathcal{F} = \bigcup \mathcal{F}_\alpha$ by II Ex5.15(e), and $\bigcup \mathcal{F}_\alpha = \varinjlim \mathcal{F}_\alpha$ gives a proper order
(where $\forall \mathcal{F} \in \mathrm{Qco}(X)$, $\mathcal{F}_\alpha \in \mathrm{Coh}(X)$) and (2.9) gives $H^i(X, \mathcal{F}) = \varinjlim H^i(X, \mathcal{F}_\alpha)$
thus now we denote $cd_{\mathrm{Qco}}(X)$ and $cd_{\mathrm{Coh}}(X)$ seperately, $cd_{\mathrm{Qco}}(X) = cd_{\mathrm{Coh}}(X)$ due to $H^i(X, \mathcal{F})$ vanishing for all $\mathcal{F} \in \mathrm{Qco}(X) \Longleftrightarrow \mathcal{F} \in \mathrm{Coh}(X)$ ▨
(b) For X projective, by Coro5.18 II, $0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{E} \to 0$ where $\mathcal{E}$ locally free (of finite rank), thus if $H^i(X, \mathcal{E}) = 0 \Longrightarrow H^i(X, \mathcal{F}) \cong H^{i+1}(X, \mathcal{K}) = 0$ by Grothendieck vanishing ▨
(c) Trivial: $H^{r+1}(\mathfrak{U}, X) = 0$ due to $C^{r+1}(\mathfrak{U}, X) = 0$ ▨ And reduce to projective
(d)* $X \xrightarrow{\text{open}} X' \xrightarrow{\text{closed}} \mathbb{P}^r_k$, cover $\mathbb{P}^r_k$ with standard affine covering and pullback by the two immersion, then also open; and also affine holds for projective

(2) For (1) $\Rightarrow$ (2), it's a easy computation: $\chi(\mathcal{O}_{\mathbb{P}^i}(n)) = \chi(\mathcal{O}_L(n)) = \binom{i+n}{i}$
thus $P: K(X) \to \mathbb{Q}[n]$ is injective due to $\binom{i+n}{i} \equiv 0 \Leftrightarrow \mathcal{O}_{\mathbb{P}^i}(n) \equiv 0$
and in (a), we shown that $\mathbb{Q}[n]$ determine in this case, this completes our proof.
Computation. $\chi(\mathcal{O}_{\mathbb{P}^i}(n)) = \sum_{0 \le p \le i} (-1)^p \dim_k H^p(X, \mathcal{O}_{\mathbb{P}^i}(n)) = \sum_{0 \le p \le i} (-1)^p \dim_k H^p(\mathbb{P}^i, \mathcal{O}_{\mathbb{P}^i}(n))$
and $0 < p < i$, by Thm 5.1(b), $H^p(\mathbb{P}^i, \mathcal{O}_{\mathbb{P}^i}(n)) = 0 \Rightarrow \chi(\mathcal{O}_{\mathbb{P}^i}(n)) = \dim_k \Gamma(\mathbb{P}^i, \mathcal{O}_{\mathbb{P}^i}(n))$
Ex 5.5. (a) By induction on $\operatorname{codim}(Y, X)$, it suffices to show $= \binom{i+n}{i}$ ▨
the case of $Y = H$ a hyperplane and It is complete intersection, due to we can compose them finally; [redacted]. it's must: $H = V(f)$, and $I(H) = (f)$
Thus $\Gamma(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k}(n)) \to \Gamma(H, \mathcal{O}_H(n))$ is singly generated
induced by $0 \to (f) \to S \to \frac{S}{(f)} \to 0$
let $H = V(f)$ and $f = \deg d$, then we have $0 \to \mathcal{O}_{\mathbb{P}^r_k}(n-d) \to \mathcal{O}_{\mathbb{P}^r_k}(n) \to \mathcal{O}_H(n) \to 0$
and $H^1(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k}(n-d)) = 0 \overset{\text{LES}}{\Longrightarrow}$ surjective ▨ $\quad s \mapsto s|_H$
(b) We use the Liouville thm, if $H^0(Y, \mathcal{O}_Y) = k$, then Y connected (due to $\operatorname{rank} H^0(Y, \mathcal{O}_Y) =$ components' numbers), now $H^0(X, \mathcal{O}_X) = k \to H^0(Y, \mathcal{O}_Y)$ surjective
and $H^0(Y, \mathcal{O}_Y)$ is k-vect $\Rightarrow H^0(Y, \mathcal{O}_Y) = k$ or 0, but 0 is impossible ▨
(c) Again by the LES, $H^i(X, \mathcal{O}_X(n)) \cong H^i(H, \mathcal{O}_H(n)) = 0$ ▨ (d) Done ▨
Ex 5.6. (a) Now we have $Y \subset Q \subset X = \mathbb{P}^3_k$, we construct proper exact sequence of twisting sheaf $\mathcal{O}_Y, \mathcal{O}_Q, \mathcal{O}_X$ and consider their cohomology by LES
(1) $b = a$: $H^1(Q, \mathcal{O}_Q(a,a)) = H^1(Q, \mathcal{O}_Q(a)) = 0$ due to $0 \to \mathcal{O}_X(a-2) \to \mathcal{O}_X(a) \to \mathcal{O}_Q(a) \to 0$
then by LES; (As Q is $\deg Q = 2$ and hypersurface)
$b = a \pm 1$: it suffices to deal with $b = a-1$ case due to the symmetry
thus consider $0 \to \boxed{\mathcal{O}_Q(a, a-1)} \to \mathcal{O}_Q(a,a) \to \mathcal{O}_Y(a) \to 0$ ($\mathcal{O}_Q(a-1,a)$ and $\mathcal{O}_Q(a,a-1)$ is same cohomology)
(by $0 \to \mathscr{L}(Y)(a) \to \mathcal{O}_Q(a) \to \mathcal{O}_Y(a) \to 0$; $\mathscr{L}(Y)(a)$ here $(-Y)$ is a divisor)
thus by LES complete the proof ▨
(Here $H^i(\mathcal{O}_Y(a)) = 0$, due to $Y = \coprod \mathbb{P}^1_k$ and H^i factored into pieces)
(2) consider $0 \to \boxed{\mathcal{O}_Q(a, a+n)} \to \mathcal{O}_Q(a,a) \to \mathcal{O}_Y(a) \to 0$, where $a+n = b$, $n \in \mathbb{Z}$
induced by $0 \to \underbrace{\mathcal{O}_Q(-a, -(a+n))}_{\mathscr{I}(-a)} \to \mathcal{O}_Q(-a, -a) \to \mathcal{O}_Y(-a) \to 0$, for all positive
Thus by LES, $H^1(Q, \mathcal{O}_Q(a,b)) = H^1(Q, \mathcal{O}_Q(a)) = 0$ ▨
(3) $0 \to \boxed{\mathcal{O}_Q(-a, 0)} \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$ by $\mathcal{O}_Q(-a, 0) = \mathscr{L}(-Y) = \mathscr{I}(Y)$
and $\Gamma(Q, \mathcal{O}_Y) = k^{\oplus a}$ due to here Y is union of a $\mathbb{P}^1_k$, and by Liouville's thm, thus by LES, replace a by $-a$, and $a \le -2$
$\Rightarrow H^1(Q, \mathcal{O}_Q(a, 0)) = k^{\oplus(-a-1)} \neq 0$ (if $a = -1$, then equal to (1))
due to $0 \to k \to k^{\oplus(-a)} \to H^1(Q, \mathcal{O}_Q(a,0)) \to 0$
$H^0(Q, \mathcal{O}_Q(a,0)) \to H^0(Q, \mathcal{O}_Q) \to H^0(Q, \mathcal{O}_Y) \to H^1(Q, \mathcal{O}_Q(a,0)) \to H^1(Q, \mathcal{O}_Q) \to \cdots$ ▨

5.2(b). Trivial, $\dim_k$ is additive function and by LES, with Grothendieck's vanishing $\Rightarrow$
$0 \to H^0(X, \mathscr{F}') \to \cdots \to H^n(X, \mathscr{F}'') \to 0 \Rightarrow \sum (-1)^i \dim_k (H^i(X, \mathscr{F}) - H^i(X, \mathscr{F}') - H^i(X, \mathscr{F}'')) = 0$
$\Rightarrow \chi(\mathscr{F}) - \chi(\mathscr{F}') - \chi(\mathscr{F}'') = 0$ ▨ (Why we need projective? I guess only Noetherian topological space)
Ex 5.2. (a) $i: X \hookrightarrow \mathbb{P}^r_k$, we replace X by $\mathbb{P}^r_k$ due to the pushforward not changing the dimension of cohomology
Induction on $N = \dim \operatorname{supp} \mathscr{F}$, $N = 0 \Rightarrow P(n) = \chi(\mathscr{F}(n)) = 0$ thus numerical
[redacted] for $\mathscr{F}(-1) \xrightarrow{x_i} \mathscr{F}$ multiply some $x_i \in \Gamma(X, \mathcal{O}_X(1))$, we write as Hints
$0 \to \mathscr{R} \to \mathscr{F}(-1) \xrightarrow{x_i} \mathscr{F} \to \mathscr{Q} \to 0$, $\mathscr{R} = \operatorname{Ker}(x_i)$, $\mathscr{Q} = \operatorname{coker}(x_i)$ and we need to show
(1) $\dim \mathscr{R} = 0$ (2) Then $\chi(\mathscr{F}) - \chi(\mathscr{F}(-1)) = \chi(\mathscr{Q})$, and $\chi(\mathscr{F}(n)) - \chi(\mathscr{F}(n-1)) = \chi(\mathscr{Q}(n))$
It suffices to show that $\chi(\mathscr{Q}(n))$ is numerical, i.e. $\operatorname{supp} \mathscr{Q} \subsetneq \operatorname{supp} \mathscr{F}$, $P(n) - P(n-1)$, thus we use the induction hypothesis to it
by (I, 7.3(b))
Pf. (1) Recall the pf of Thm 5.1(b), this is where we use $X = \mathbb{P}^r_k$ ▨
(2) thus $\mathscr{Q} \cong \frac{\mathscr{F}}{(x_i)\mathscr{F}}$ and $\operatorname{supp} \mathscr{Q} \subsetneq \operatorname{supp} \mathscr{F}$ ▨
(b) I.E. $\chi(\mathscr{F}(n)) = \dim(M_n) = \dim_k H^0(X, \mathscr{F}(n))$, thus we prove $\sum_{i \ge 1} (-1)^i \dim_k H^i(X, \mathscr{F}(n)) = 0$
$\sum (-1)^i \dim_k H^i(X, \mathscr{F}(n))$
but by Serre vanishing, $n \gg 0$ $\Rightarrow H^i(X, \mathscr{F}(n)) = 0$ for $i \ge 1$ ▨
Ex 5.3. (a) X integral over k, and projective $\Rightarrow X$ is a variety
thus we can discuss to old definition (I, Ex 7.2) (Recall Integral, seperated, finite-type) in this nature (and $X \cong X' \subset \mathbb{P}^n$, $p_a(X) = p_a(X')$)
(a) $H^0(X, \mathcal{O}_X) = k$ is due to Liouville thm, as here X is projective variety
$\Rightarrow p_a(X) = (-1)^r \left(\sum (-1)^i \dim_k H^i(X, \mathcal{O}_X)\right) = (-1)^r \left(\sum_{1 \le i \le r} (-1)^i \dim_k H^i(X, \mathcal{O}_X)\right)$
$= \sum_{1 \le i \le r} (-1)^{i+r} \dim_k H^i(X, \mathcal{O}_X) = \sum_{0 \le i \le r-1} (-1)^i \dim_k H^{r-i}(X, \mathcal{O}_X)$ ▨
(b) It's due to Ex 5.2(b): $P_X(0) = \chi(\mathcal{O}_X(0)) = \chi(\mathcal{O}_X) \Rightarrow p_a(X) = (-1)^r (\chi(\mathcal{O}_X) - 1)$
(c) It's due to for nonsingular projective curve, $p_a(X) = p_a(X) = (-1)^r (P_X(0) - 1)$ ▨
Alternatively, in (I, Ex 7.2(b)), we proved that $= \frac{1}{2}(d-1)(d-2)$ and p_a is birational ▨
birational equivalence $\Leftrightarrow$ isomorphic for nonsingular projective curve ▨
Ex 5.4. (a) $P: K(X) \to \mathbb{Q}[z]$ and we extend $P_{\mathscr{F}}$ to arbitrary z over $\mathbb{Q}$
$\mathscr{F} \mapsto P_{\mathscr{F}}: n \mapsto \chi(\mathscr{F}(n))$ (2) Show that extension is unique.
(1) By Ex 5.2(a) (3) By Ex 5.1. (3) Show that for $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$,
(2) If $P(z) = Q(z) \Rightarrow (P - Q)(z) = 0$ for $z \in \mathbb{Z}$ and by (I, 7.3(b)) $\quad P$ is additive homorphism, thus $P(\gamma(\mathscr{F})) = P(\gamma(\mathscr{F}'))$ for $\mathscr{F} \sim \mathscr{F}'$
$\Rightarrow \exists f(z) = (P - Q)(z) = 0$, infinite zero locus $\Rightarrow f = 0 \Rightarrow P = Q$ ▨
(b) (1) Induction on r, for $r = 0$, $i = 0$, nothing to prove, then for $\forall r$, we have
$K(\mathbb{P}^{r-1}_k) \to K(\mathbb{P}^r_k) \to \boxed{K(\mathbb{P}^r_k - \mathbb{P}^{r-1}_k)} \to 0$ thus the generator of $K(\mathbb{P}^r_k)$ adding $\gamma(\mathcal{O}_{L^{r-1}})$
$\langle \gamma(\mathcal{O}_{L^{r-1}}) \rangle \quad = K(\mathbb{A}^r_k) = \gamma(\mathcal{O}_{\mathbb{A}^r_k}) \cdot \mathbb{Z}$ by (Ex 6.10(a), II) due to extension by $\gamma(\mathcal{O}_{\mathbb{A}^r_k}) \cdot \mathbb{Z}$ and free ▨
$\mathcal{O}_{L^{r-1}} \mapsto j_! \mathcal{O}_{L^{r-1}} \mapsto j_! \mathcal{O}_{L^{r-1}}|_{\mathbb{P}^r_k - \mathbb{P}^{r-1}_k}$ (indeed also left exact)

Thus by LES, with tensoring $\mathcal{L}^n \Rightarrow H^i(X, \mathcal{F}\otimes\mathcal{L}^n) = H^i(X_i, [illegible](\mathcal{F}_i)\otimes\mathcal{L}^n)$

(d) $H^i(X, \mathcal{F}\otimes\mathcal{L}^n) = 0 \Leftrightarrow H^i(Y, g_*(f^*\mathcal{L})^n) = 0 = H^i(X_i, i_*(\mathcal{F}_i)\otimes\mathcal{L}^n) = 0$

this is due to ($\Rightarrow$) $H^i(Y, \mathcal{G}\otimes(f^*\mathcal{L})^n) = H^i(X, f_*(\mathcal{G}\otimes\mathcal{L}^n))$ projection formula

($\Leftarrow$) $H^i(X, \mathcal{F}\otimes\mathcal{L}^n)$ ... $H^i(X, f_*\mathcal{G}\otimes\mathcal{L}^n)$

$\Rightarrow H^i(Y, f^*\mathcal{F}\otimes(f^*\mathcal{L})^n) = 0$ due to this case both f_* and f^* preserve the coherence

But here ① and ② we need f to be affine map or $R^if_* = 0$ (Ex 8.1) ▨

Ex 5.8. (a) Nothing to prove //

(b) X is $\boxed{\text{proper}}$: ~~separatedness~~ and universal closed, f factor through

Using valuation criterion for proper, $X^{\prime}_{\mathrm{red}}$ = closed ▨

And nonsingular is local. let $X = \operatorname{Spec} A$ affine, then integrality + integrally closed $\Rightarrow$ regular local ring by algebraic

$X \xrightarrow{\Delta} X\times_k X$, $\tilde{X} \xrightarrow{\Delta} \tilde{X}\times_k\tilde{X}$ factor through $p \Rightarrow f' = (p\times_k p)\circ\Delta'$

X closed $X\times_k X \Rightarrow \exists!\, f$ due to ... closed f

$(p\times_k p)(\Delta') = \Delta$

$\Rightarrow \Delta = \Delta'\circ p$ closed

due to $(p\times_k p)\tilde{\Delta}' = \Delta$ closed

Because I remember a wrong universal property: $Z \to X$

Why we can't using this way? is right

①②③ $X_i \xrightarrow{i} \mathbb{P}^n_k \Rightarrow \mathcal{L} = \tilde{f}^*\mathcal{O}(1) = f^* i^*\mathcal{O}(1) = f^*\mathcal{L}_0$

$\mathcal{L}_0 = i^*\mathcal{O}(1)$ is very ample invertible

$H^0(\mathcal{L}(D)) = \mathcal{L}$ is an intersection $X \subset \mathbb{P}^n_k$ with hyperplane H and by Bertini's thm $\Rightarrow \exists H$: $X\cap H$ is smooth $\Rightarrow D$ is nonsingular points ▨

④ Our $\boxed{\text{core}}$: proper + very ample $\Leftrightarrow$ projective

now X has $\mathcal{L}_0$ invertible and $\mathcal{L} = f^*\mathcal{L}_0$ ample, by Ex 5.7(d) $\Rightarrow \mathcal{L}_0$ ample

$\Rightarrow \exists n: \mathcal{L}^n$ very ample ▨

(c) $\operatorname{Pic} X \to \oplus \operatorname{Pic} X_i \to 0$, then due to (b), choose $\mathcal{L}_i$ over X_i to be ample $\Rightarrow \exists \mathcal{L}_i = \mathcal{L}$ is ample on X due to Ex 5.7(c) $\Rightarrow \mathcal{L}^n$ very ample $\Rightarrow X$ projective

Thus it suffices to show the surjectivity: induction on r, it suffices to deal $X = X_1\cup X_2$, consider $1 \to \mathcal{O}_X^* \to \mathcal{O}_{X_1}^*\oplus\mathcal{O}_{X_2}^* \to \mathcal{O}_{X_1\cap X_2}^* \to 1$ by Ex 4.6,

LES $\to H^1(X, \mathcal{O}_X^*) \to H^1(X_1, \mathcal{O}_{X_1}^*)\oplus H^1(X_2, \mathcal{O}_{X_2}^*) \to \boxed{H^1(X_1\cap X_2, \mathcal{O}_{X_1\cap X_2}^*)} \to \cdots$

$\to \operatorname{Pic}(X) \to \operatorname{Pic}(X_1)\oplus\operatorname{Pic}(X_2) \longrightarrow 0$

It's where we using the dimension 1: $X_1\cap X_2$ is points, thus by Grothendieck vanishing $\Rightarrow H^1(X_1\cap X_2, \mathcal{O}_{X_1\cap X_2}^*) = 0$ ▨

(d) Same argument as (c), it suffices to show surjectivity due to Ex 5.7(b)

let $\mathcal{N} \subset \mathcal{O}_X$ the nilradical $\Rightarrow \mathcal{N}^r = 0$,

and by a thm of infinitesimal extension, $\mathcal{N}^2 = 0$

thus by Ex 4.6, $1 \to \mathcal{N} \to \mathcal{O}_X^* \to \mathcal{O}_{X_{\mathrm{red}}}^* \to 1$

LES $\to H^1(X, \mathcal{O}_X^*) \to H^1(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}}^*) \to H^2(X, \mathcal{N}) \to \cdots$ ▨

Ex 5.9. ① Trivial: $\delta_{ij}\delta_{jk}\delta_{ki} = \frac{x_j}{x_i}\frac{x_k}{x_j}\frac{x_i}{x_k}$, $d\left(\frac{x_j}{x_i}\right) d\left(\frac{x_k}{x_j}\right) \cdots = 1$ ▨

④ Then $\delta: \mathbb{Z} \to k$ can be only injective. It's calculated locally by Čech cohomology (see texts) ▨

② Locally, by (II Ex 5.16(b)), $\eta \in \Omega^n_{\mathbb{P}^n/k}$: $\mathcal{N}\Lambda^-: \Omega \to \omega$

i.e. $\eta \in \Omega^n_{\mathbb{P}^n_k} \cong \mathcal{H}om_{\mathcal{O}_X}(\Omega_{\mathbb{P}^n}, \omega) \cong \omega\otimes\mathcal{O}$ ▨

③ Now $H^n(X, \omega) = H^n(\mathbb{P}^n_k, \mathcal{O}_X(-n-1))$ ... counting dimension $1 \Rightarrow H^n(X, \omega) = k$ ▨

Ex 5.6 (b) (1) $0 \to \mathcal{O}_Q(-a, -b) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$, $a, b < 0$

and $H^1(Q, \mathcal{O}_Q(-a, -b)) = 0$ by (a)(2) ..., and $\Gamma(Q, \mathcal{O}_Q(-a, -b)) = 0$ is clear

$\Rightarrow H^0(Y, \mathcal{O}_Y) = H^0(Q, \mathcal{O}_Y) \cong H^0(Q, \mathcal{O}_Q) = k$ by Liouville's thm ▨

(2) II Example 7.6.2 gives us the construction: $Q \cong \mathbb{P}^1\times\mathbb{P}^1 \to \mathbb{P}^a\times\mathbb{P}^b$ by (a, b)-tuple embedding, then pullback $\mathcal{O}_{\mathbb{P}^n}(1)$ gives (a, b)-type. Segre $\mathbb{P}^a\times\mathbb{P}^b \to \mathbb{P}^N$

By Bertini (II 8.18) $\Rightarrow \exists H$: $\operatorname{Im} s_{ab}\cap H \subset \mathbb{P}^N$ pullback gives a curve, and by Bertini, it's nonsingular, irreducible ▨

(3) By Ex 8.4(b) II $\Rightarrow$ Due to Q is complete intersection (hypersurface) $\Rightarrow$ normal $\Rightarrow$ projective normal $\Rightarrow H^1(Q, \mathcal{O}_Q(n)) = 0$

and normal is local, locally $Q|_U \cong \mathbb{A}^1\times\mathbb{A}^1 = \mathbb{A}^2$ thus normal

then $0 \to \mathcal{O}_Q(-a, -b) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$ and LES give:

Thus Y projective normal $\Leftrightarrow H^1(Y, \mathcal{O}_Y(n)) = 0 \xleftrightarrow{\text{LES}} H^1(Q, \mathcal{O}_Q(n-a, n-b)) = 0$

$\xLeftrightarrow{(a)} |a-b| \le 1$ ▨

(4) We have the way to factor $\mathcal{O}_Q(a, b)$ now: $\mathcal{O}_Q(a, b) = \mathcal{O}_Q(a, 0)\otimes\mathcal{O}_Q(0, b)$, due to type (a, b) Y is of form $Y_1 \amalg Y_2$, Y_1 of type $(a, 0)$, Y_2 of type $(0, b)$ due to local principal, thus we can calculate them seperatedly, by LES: $0 \to \mathcal{O}_Q(0, -b) \to \mathcal{O}_Q \to \mathcal{O}_{Y_2} \to 0$ and $0 \to \mathcal{O}_Q(-a, 0)\otimes\mathcal{O}_Q(0, -b) \to \mathcal{O}_Q\otimes\mathcal{O}_Q(0, -b) \to \cdots$

and ... simultaneously

$\to \mathcal{O}_{Y_1}\otimes\mathcal{O}_Q(0, -b) \to 0 \to \mathcal{O}_Q(-a, -b) \to \mathcal{O}_Q(0, -b) \to \mathcal{O}_{Y_1}\otimes\mathcal{O}_Q(0, -b) \to 0$

(1)(2) tells us $H^1(Q, \mathcal{O}_Q(a, -b)) = k^{(a-1)(b-1)}$ and H^2 vanishing

thus for ③'s LES, we have $0 \to 0 \to 0 \to k \to H^1(Q, \mathcal{O}_Q(-a, -b)) \to \oplus \cdots$

$\to H^2(Q, \mathcal{O}_Q(-a, -b)) \to 0 \to 0 \to 0$

H^0 $\qquad H^1$

and by (a)(2) H^2: $\Rightarrow H^1(Q, \mathcal{O}_Q(-a, -b)) = 0 \Rightarrow H^2(Q, \mathcal{O}_Q(-a, -b)) = k^{(a-1)(b-1)}$

$= k^{(a-1)(b-1)} \Rightarrow p_a(Y) = \dim_k H^1(Y, \mathcal{O}_Y) = (a-1)(b-1) = ab - a - b + 1$ ▨

Ex 5.7. (a) $H^i(Y, \mathcal{F}\otimes\mathcal{L}^n) = H^i(X, i_*\mathcal{F}\otimes\mathcal{L}^n) = 0$ due to $\mathcal{F}$ coherent on Y

$\Rightarrow i_*\mathcal{F}$ coherent on X, then by Prop 5.3 ▨

(b) Due to $f: X_{\mathrm{red}} \to X$ and $f^*\mathcal{L} = \mathcal{L}\otimes\mathcal{O}_{X_{\mathrm{red}}}$, thus same as (a) ▨

($\Leftarrow$) $\mathcal{N} \subset \mathcal{O}_X$ is the nilradical ideal, thus we can recover $\mathcal{L}$ by $\mathcal{L}_{\mathrm{red}}$ and ...

$\forall \mathcal{F} \in \mathrm{Coh}(X)$, the the filtration $\mathcal{F} \supset \mathcal{N}\mathcal{F} \supset \cdots \supset \mathcal{N}^r\mathcal{F} = 0$

and $0 \to \mathcal{N}^{i+1}\mathcal{F} \to \mathcal{N}^i\mathcal{F} \to \frac{\mathcal{N}^i\mathcal{F}}{\mathcal{N}^{i+1}\mathcal{F}} \to 0$, due to $\frac{\mathcal{N}^i\mathcal{F}}{\mathcal{N}^{i+1}\mathcal{F}}$ has reduced group structure

view it as $\mathcal{O}_{\mathrm{red}}$-module, thus by $\mathcal{L}_{\mathrm{red}}$ ample $\Rightarrow H^i(X_{\mathrm{red}}, \frac{\mathcal{N}^i\mathcal{F}}{\mathcal{N}^{i+1}\mathcal{F}}\otimes\mathcal{L}^n_{\mathrm{red}}) = 0$

$\Rightarrow H^i(X, \frac{\mathcal{N}^i\mathcal{F}}{\mathcal{N}^{i+1}\mathcal{F}}\otimes\mathcal{L}^n) = 0$, then by LES $\Rightarrow H^i(X, \mathcal{N}^i\mathcal{F}\otimes\mathcal{L}^n) = 0$ by induction on i, conversely $i = r$ holds $\Rightarrow i = 0$, i.e. $H^i(X, \mathcal{F}\otimes\mathcal{L}^n) = 0$ ▨

(c) ($\Rightarrow$) $j_i: X_i \hookrightarrow X$, then $\mathcal{L}\otimes\mathcal{O}_{X_i} = j_i^*\mathcal{L}$, then by (a)

($\Leftarrow$) $0 \to \mathcal{J}_{X_1} \to \mathcal{O}_X \to \mathcal{O}_{X_1} \to 0$, now $H^i(X_i, \mathcal{F}|_{X_i}\otimes(\mathcal{L}\otimes\mathcal{O}_{X_i})^n) = 0$

and $0 \to j_{i!}\mathcal{F}|_{U} \to \cdots \to 0$

And $\mathcal{E}xt^i(-,\mathcal{G})$ is effaceable due to it vanishing for $\mathcal{L}$ locally free
i.e. $\mathcal{E}xt^i(\mathcal{L},\mathcal{G}) = \mathcal{E}xt^i(\mathcal{O}_X,\mathcal{G})^{\oplus n} = 0$ ▨

Ex6.5. (a) (⇒) Trivial (⇐) $\mathcal{E}xt^1(\mathcal{F},\mathcal{G})=0 \Rightarrow \mathcal{E}xt^1(\mathcal{F},\mathcal{G})_x = \mathrm{Ext}^1_{\mathcal{O}_x}(\mathcal{F}_x,\mathcal{G}_x)=0$
$\Leftrightarrow \mathcal{F}_x$ is projective over $\mathcal{O}_x$ + $\mathcal{F}_x$ is f-g $\Rightarrow \mathcal{F}_x$ is free over $\mathcal{O}_x$
i.e. $\mathcal{F}_x \cong \mathcal{O}_x^n \Rightarrow \exists U \in N_x: \mathcal{F}|_U \cong \mathcal{O}_X|_U^n$ ▨
(b) (⇒) due to $\mathcal{E}xt^i(-,\mathcal{G}) = h^i(\mathcal{H}om(-,\mathcal{G}))$ in $\mathrm{Coh}(X)$
(⇐) Prove by induction on n: $0\to\mathcal{K}\to\mathcal{E}\to\mathcal{F}\to 0$ appling the LES
$\Rightarrow \cdots \to \mathcal{E}xt^i(\mathcal{E},\mathcal{G}) \to \mathcal{E}xt^i(\mathcal{K},\mathcal{G}) \to \mathcal{E}xt^{i+1}(\mathcal{F},\mathcal{G}) \to \mathcal{E}xt^{i+1}(\mathcal{E},\mathcal{G}) \to \cdots$
$\Rightarrow \mathcal{E}xt^{i+1}(\mathcal{F},\mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{K},\mathcal{G})$, $\mathcal{K}\in \mathrm{Coh}(X)$
thus if $\mathcal{E}xt^i(\mathcal{F},\mathcal{G})=0$ for $i>n \Rightarrow \mathcal{E}xt^i(\mathcal{K},\mathcal{G})=0$ for $i>n-1$
$\Rightarrow hd(\mathcal{K}) \le n-1 \Rightarrow hd(\mathcal{F}) \le n$ due to the resolution of $\mathcal{K}$ is pasted with
$\to \mathcal{L}_\bullet \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0 \Rightarrow hd(\mathcal{F}) \le n-1+1 = n$ ▨
(c) $\forall x\in X$: $pd_{\mathcal{O}_x}\mathcal{F}_x \le hd(\mathcal{F})$ due to any resolution $\mathcal{L}_\bullet$ of $\mathcal{F}$, taking stalk
at $x\in X \Rightarrow pd_{\mathcal{O}_x}\mathcal{F}_x \le \mathrm{length}(\mathcal{L}_\bullet) \le hd(\mathcal{F}) \Rightarrow \sup_x pd_{\mathcal{O}_x}\mathcal{F}_x \le hd(\mathcal{F})$
Conversely, if $hd(\mathcal{F}) \ge pd_{\mathcal{O}_x}\mathcal{F}_x + 1$, $\forall x\in X$, then $\mathrm{Ext}^i(\mathcal{F}_x, N)=0$ for $\forall i \ge hd\mathcal{F}$ $\forall x\in X$
$\Rightarrow \forall \mathcal{G}$, $\mathcal{E}xt^i(\mathcal{F},\mathcal{G}) = 0$, $i \ge hd(\mathcal{F})$, i.e. $i > hd(\mathcal{F})-1$ $\forall N$ over $\mathcal{O}_x$
(b) $\Rightarrow hd(\mathcal{F}) \le hd(\mathcal{F})-1$, contradiction ▨

Ex6.6. (a) (⇒) Trivial by our homological algebra (or by Hilt.)
(⇐) $\mathrm{Ext}^i(M,A)=0 \Rightarrow \mathrm{Ext}^i(M,N)=0$, let $A=N$,
then for $0\to K\to A^n\to N\to 0$ applying LES $\Rightarrow \cdots \to \mathrm{Ext}^i(M,A^n) \to \mathrm{Ext}^i(M,N)$
$\to \mathrm{Ext}^{i+1}(M,K) \to \mathrm{Ext}^{i+1}(M,A^n) \to \cdots \Rightarrow \mathrm{Ext}^{i+1}(M,K) \cong \mathrm{Ext}^i(M,N)$
Descending induction $\Rightarrow \mathrm{Ext}^1(M,N)=0$ (same technique in Ex6.5(b))
Let $\cdots \Rightarrow \cdots \mathrm{Ext}^0(M,A^n) \to \mathrm{Ext}^0(M,M) \to 0$
thus we factor $M \to M$ through A^n, thus $\mathrm{Hom}(M,A^n) \to \mathrm{Hom}(M,M) \to 0$
$M\oplus P = A^n$ ▨ $\Rightarrow M$ projective
(b) $pd(M) \le n$
$\Leftrightarrow \mathrm{Ext}^i(M,N)=0, \forall i>n \Leftrightarrow \mathrm{Ext}^i(M,A)=0\ \forall i>n$ ▨

Ex6.7.
We prove these by (By (a) same technique)
resolute on M, let $(A^{n_i})_i$ be a free resolution of M, then $(\widetilde{A^{n_i}})_i$ is a locally free resolution of $\tilde M$, the we can compute
① $\mathcal{E}xt^i_X(\tilde M,\tilde N) = h^i(\mathcal{H}om_X(\widetilde{A^{n_\bullet}},\tilde N))$ by categorical equivalence
$\widetilde{\mathrm{Ext}^i_A(M,N)} = h^i(\mathrm{Hom}_A(A^{n_\bullet},N)^\sim)$ it suffices to show effaceable ▨
it's trivial due to vanishing on free object
② It's the same way, due to $\mathcal{H}om_X(\widetilde{A^{n_i}},\tilde N) \cong \mathrm{Hom}_A(A^{n_i},N)^\sim$ ▨

Ex6.1. (The Ext and extension of short exact sequence)
$e(\mathcal{F}'',\mathcal{F}') \longleftrightarrow \mathrm{Ext}^1(\mathcal{F}'',\mathcal{F}')$ is a pure homological proof, we use way of Rotman
$[\xi] \longmapsto \bar\partial_1$ $\xi: 0\to\mathcal{F}'\to\mathcal{F}_1\to\mathcal{F}''\to 0$
the class of extension   the homological class of ξ: $0\to\mathcal{F}'\to\mathcal{X}_2\to\mathcal{F}''\to 0$
map $\partial_1: \mathcal{P}_1\to\mathcal{F}'$
$d_2 \to \mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{F}'' \to 0$ a projective resolution of $\mathcal{F}''$
thus due to the commute of right graph $\Rightarrow$ such map not depend on $\xi \cong \xi'$
(It corresponds to $\mathrm{Ext}^1(\mathcal{F}'',\mathcal{F}')$ accidentally!)
Conversely, $\mathrm{Ext}^1(\mathcal{F}'',\mathcal{F}')\to e(\mathcal{F}'',\mathcal{F}')$ is due to the filling of graph, i.e. given graph: $0\to\mathcal{L}_1\to\mathcal{L}_0\to\mathcal{F}''\to 0$, $\mathcal{L}_1\to\mathcal{F}'$, $\mathcal{F}''=\mathcal{F}''$
$\exists \mathcal{F}: 0\to\mathcal{L}_1\to\mathcal{L}_0\to\mathcal{F}''\to 0$ commutes, then $[h\xi]\in e(\mathcal{F}'',\mathcal{F}')$
$h\xi: 0\to\mathcal{F}'\to\mathcal{F}_1\to\mathcal{F}''\to 0$
Consider $0\to \mathcal{P}_1/\mathrm{Im}\,d_2 \to \mathcal{P}_0\to\mathcal{F}''\to 0$ to be β, then it gives $\mathcal{P}_1 \xrightarrow{\partial_1} \mathcal{F}'$, $\mathcal{P}_1/\mathrm{Im}\,d_2 \xrightarrow{\bar\partial_1} \mathcal{F}'$
$\partial_1 \to \bar\partial_1 \longmapsto h\beta \longmapsto [h\beta]$ is the converse
①② needs to prove longly, omit here ▨

Ex6.2. (a) We have $\mathcal{P}\to\mathcal{O}_X\to 0$, $\mathcal{O}_X\to i_*(k(x))\to 0$
thus composite gives $\mathcal{P}\to i_*(k(x))\to 0 \Rightarrow \mathcal{P}(U)\to k(x)\to 0$
On the other hand $\mathcal{P}(U)\to\mathcal{O}_X(U)\xrightarrow{\text{restriction}}\mathcal{O}_X(U\cap V)=\mathcal{O}_X(U)\to k(x)\to 0$
The two compositions are same; but now we can take $U\subset \mathbb{P}^1_k$: $\mathcal{O}_X(U)=0$ but take $x\in U$: $k(x)\neq 0$
$\Rightarrow \mathcal{P}(U)\to\mathcal{O}(U)=0$, $\mathcal{P}(U)\to k(x)\to 0$ contradiction ▨
(b) It suffices to prove: if $\mathcal{P}\in \mathrm{Coh}(X)$ is projective, then it's projective in $\mathrm{Mod}(X)$
but it's may not true? and needs more categorical discussions
Similarly, we have $(\mathcal{L}\otimes\mathcal{P})(U)\to k(x)\otimes\mathcal{L}\to 0$ and $(\mathcal{L}\otimes\mathcal{P})(U)\to\mathcal{L}|_V(U)\to k(x)\otimes\mathcal{L}\to 0$
coincides, and for $\mathcal{L}$ invertible, $k(x)\otimes\mathcal{L}$ is indeed $\in \mathrm{Coh}(X)$
now let $\mathcal{L}=\mathcal{O}_X(n) \Rightarrow k(x)\otimes\mathcal{L}=k(x)$ and
$\Rightarrow \mathcal{O}_V(U)(n)=0$, $\mathcal{P}(U)(n)\to k(x)\to 0$ contradiction ▨

Ex6.3. Coherent/Quasi-Coherent are local thus let $X=\mathrm{Spec}A$, $\mathcal{F}=\tilde M$, $\mathcal{G}=\tilde N$
(a) Now M, N are both finitely-generated A-module, then $\mathcal{E}xt^i(\mathcal{F},\mathcal{G}) = \mathcal{E}xt^i(\tilde M,\tilde N) = \mathrm{Ext}^i(M,N)^\sim$
by Ex6.7, requires M finitely-generated $\Rightarrow \mathrm{Ext}^i(M,N)$ is also f-g
by our homological algebra ▨
(b) Again by Ex6.7. ▨

Ex6.4. $\mathcal{E}xt^i(-,\mathcal{G})$ is δ-functor is obvious and locally free resolution $0\to\mathcal{L}_\bullet\to\mathcal{F}$
Taking LES For $0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0$, $\mathcal{E}xt^i(-,\mathcal{G})$ is a δ-functor

Ex6.1 (b) $f_* \mathcal{H}om_X(\mathcal{F}, f^! \mathcal{G}) \cong \mathcal{H}om_Y(f_*\mathcal{F}, f_* f^! \mathcal{G}) \cong \mathcal{H}om_Y(f_*\mathcal{F}, f_* \mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{G}))$ Ex5.1(b)
$\cong \mathcal{H}om_Y(f_*\mathcal{F}, \mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{G})) \cong \mathcal{H}om_Y(f_*\mathcal{F}, \mathcal{G})$ ▨

These two isn't isomorphism but their composition is
① We have $f_* \mathcal{H}om_X(\mathcal{F}, f^!\mathcal{G}) \to \mathcal{H}om_Y(f_*\mathcal{F}, f_* f^!\mathcal{G})$
for U, $\mathcal{H}om_X(\mathcal{F}, f^!\mathcal{G})(f^{-1}U) \to \mathcal{H}om(f_*\mathcal{F}, f_*f^!\mathcal{G})(U)$
$(\varphi_V): \mathcal{F}(f^{-1}V) \to f^!\mathcal{G}(f^{-1}V)$ (U)
② $\cong$ induced by $\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{G}) \to \mathcal{H}om_Y(\mathcal{O}_Y, \mathcal{G}) = \mathcal{G}$
and locally check that finally
$f_* \mathcal{H}om_X(\mathcal{F}, f^!\mathcal{G}) \cong \mathcal{H}om_Y(f_*\mathcal{F}, \mathcal{G})$ by algebraic way: obvious by tensor and Hom ▨

(c) $f_* f^! \mathcal{G} \cong \mathcal{G}$ naturally is just by (II, Ex5.1(e)), thus it suffices to
ind $f_* \mathcal{E}xt^i_X(\mathcal{F}, f^!\mathcal{G}) \to \mathcal{E}xt^i_Y(f_*\mathcal{F}, f_* f^!\mathcal{G})$
i.e. $f_* h^i \mathcal{H}om_X(\mathcal{F}, f^\bullet) \to h^i \mathcal{H}om_Y(f_*\mathcal{F}, f_* f^\bullet)$
For this, we show that
① Both are δ-functors
② $f_* \mathcal{H}om_X(-, f^\bullet)$ is effaceable $\Rightarrow$ universal
then $\exists!$ map
② is trivial due to vanishing for $\mathcal{E}$ locally free
①: Due to f_*, $\mathcal{H}om(-, f^\bullet)$ are both δ ▨
(d) Now it suffices to show $\mathcal{H}om_Y(f_*-, f_*f^\bullet)$ is also universal, however f_* inside $\mathcal{H}om$ isn't easy to deal with, thus we consider the Hint: using resolution of $\mathcal{F}$ instead of $\mathcal{G}$: ① $i=0$ done in (b) ② $\mathcal{F} = \mathcal{O}_X$: $\mathcal{E}xt^i_X(\mathcal{O}_X, f^!\mathcal{G}) = 0 = \mathcal{E}xt^i_Y(f_*\mathcal{O}_X, \mathcal{G})$
③ Same as ② for $\mathcal{F} = \mathcal{E}$, check locally $f_*\mathcal{E}$ is also locally free
④ then generally, $0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$, due to f_* ⊣ $f^!$, thus (Due to it's locally free) both of them are exact $\Rightarrow 0 \to f_*\mathcal{K} \to f_*\mathcal{E} \to f_*\mathcal{F} \to 0$
$\mathcal{H}om(-, f^!\mathcal{G})$ $\to \mathcal{E}xt^i_X(\mathcal{K}, f^!\mathcal{G}) \to \mathcal{E}xt^i_X(\mathcal{E}, f^!\mathcal{G}) \to \mathcal{E}xt^i_X(\mathcal{F}, f^!\mathcal{G}) \to \cdots$
by ③
By Hom coincides $\Rightarrow \mathcal{E}xt^i_Y(f_*\mathcal{K}, \mathcal{G}) \to \mathcal{E}xt^i_Y(f_*\mathcal{E}, \mathcal{G}) \to \mathcal{E}xt^i_Y(f_*\mathcal{F}, \mathcal{G}) \to \cdots$
$\Rightarrow \mathcal{E}xt^1$ coincides $\xRightarrow{\text{induction}}$ all coincides ▨
Five Lemma

Ex6.8. (a) $\forall x \in X$, $\forall U \in N_x$, we prove: $\exists \mathcal{L}$, $s \in \Gamma(X, \mathcal{L})$: $X_s \subset U$, let $Z = X - U$
If it holds for $\forall U \in N_x$, s.t. Z irreducible, then for $Z = Z_1 \cup \cdots \cup Z_r$, consider the open set $U_i = X - Z_i$, $i \le r$, $\exists \mathcal{L}_i, s_i$: $X_{s_i} \subset U_i$, then $\cap X_{s_i} \subset U$ also satisfy the condition of basis, and $\cap X_{s_i} \ne \varnothing$, thus we reduce to Z irreducible, let $\xi \in Z$ be generic and $f \in K(X)$: $f \in \mathcal{O}_{\xi} - \mathcal{O}_x$, such f exists due to let f regular at x but pole outside U, then by Hints: $D = (f)$ and $\mathcal{L} = \mathcal{L}(D)$, $s \in \Gamma(X, \mathcal{L}(D))$ is given by (Prop 6.13 II), a global section $s \in \Gamma(X, \mathcal{L}(D))$ corresponding to $D = (f)$, then $X_s = \{P \in X \mid s_x \notin m_x \mathcal{L}_x\}$, $\mathcal{L}_x = \mathcal{O}(D)_x = \frac{1}{f_x}$, thus $X_s \subset U$ due to $\forall x \in X_s$, f_x not vanishing, i.e. f_x will not have a pole, i.e. $X_s \cap \overline{\{\xi\}} = Z$, i.e. $x \in U$ ▨
(b) $\forall s \in \Gamma(X, \mathcal{F})$, we hope to give $\bigoplus \Gamma(X, \mathcal{L}_i)^{\oplus n_i} \to \Gamma(X, \mathcal{F}) \to 0$ by (a)
however, it's only local by algebra theory, that's why we using $\mathcal{F} \in \mathrm{Coh}(X)$
② Although holds, recovering to $\mathcal{L}_i^{\oplus n_i} \to \mathcal{F} \to 0$ need to coherence
Taking two covering of X: $\{U_i = \operatorname{Spec} A_i\} \succ \{X_{s_{ij}}\}$ by (a)
then $\mathcal{F}|_{U_i} = \tilde{M_i}$, M_i f-g over A_i (Here $\{X_{s_{ij}}\}_j$ covering U_i)
$\Rightarrow A_i^{n_i} \to M_i \to 0$, then we posting by (II, 5.14)
Now $\forall s \in \Gamma(X, \mathcal{F})$, it's generated by $m_i \in M_i$, finer: $m_{ij} \in M_i$, the number of j depending on i, then by (II, 5.14) $\Rightarrow \exists n_i$: $s_{ij}^{n_i} m_{ij}$ extend to globally that gives the preimage in $\Gamma(X, \mathcal{L}_i^{\oplus n_i})$, thus surjective on sections $\Gamma(X, \mathcal{L}_i^{n_i})$
$\Rightarrow$ surjection on sheaves due to locally equivalence ▨
Ex6.9. (a) By Ex6.8 $\Rightarrow \mathcal{E}_\bullet \to \mathcal{F} \to 0$ locally free resolution
due to $\mathcal{F}_x$ is f-g $\mathcal{O}_x$-module $\Rightarrow \mathrm{pd}_{\mathcal{O}_x} \mathcal{F}_x \le n_x < \infty$ due to (Ex 6.11(A))
$\Rightarrow \mathrm{hd}(\mathcal{F}) = \sup_{x \in X}(\mathrm{pd}_{\mathcal{O}_x} \mathcal{F}_x)$, and by Noetherian, $\sup_{x \in X} n_x < \infty$
$\Rightarrow \mathrm{hd}(\mathcal{F}) \le n \Rightarrow$ Finite locally free resolution ▨
(b) ① (Ex6.5(b))
$0 \to \mathcal{K} \to \mathcal{E}_0 \to \mathcal{F} \to 0 \Rightarrow \delta(\mathcal{F}) = \gamma(\mathcal{E}_0) - \delta(\mathcal{K})$
$0 \to \mathcal{K}' \to \mathcal{E}_0' \to \mathcal{F} \to 0 \Rightarrow \delta(\mathcal{F}) = \gamma(\mathcal{E}_0') - \delta(\mathcal{K}')$
Consider induction on the chain length, we have $\Rightarrow \delta(\mathcal{K})$ not depend on $\mathcal{E}_\bullet$, $\mathcal{K}$, i.e. $\delta(\mathcal{K}) = \delta(\mathcal{K}')$ and $\gamma(\mathcal{E}_n) = \gamma(\mathcal{E}_0)$ by definition
② $K(X) \underset{\varepsilon}{\overset{\delta}{\rightleftarrows}} K_1(X)$ due to ①, δ is well-defined under quotient, homomorphism is trivial ▨
$\gamma(\mathcal{E}) \xleftarrow{\varepsilon} \gamma(\mathcal{E})$
$\gamma(\mathcal{F}) \xmapsto{\delta} \delta(\mathcal{F})$
③ $\varepsilon\delta(\gamma(\mathcal{F})) = \sum(-1)^i \gamma(\mathcal{E}_i) = \gamma(\mathcal{F})$, and $\delta\varepsilon(\gamma(\mathcal{E})) = \gamma(\mathcal{E})$ ▨
Ex6.10 (a) By (Ex5.17(e)) we have category equivalence $f_*\mathcal{O}_X$-Mod and $\mathcal{O}_X$-Mod for f affine map, but due to finite $\Rightarrow$ affine $\Rightarrow$ it holds, now it suffice to show $\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{G})$ is quasi-coherent: it's local, thus let $Y = \operatorname{Spec} B$ and $\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{G})$
$= \mathrm{Hom}_A(M, N)$ due to $f_*\mathcal{O}_X$, $\mathcal{G}$ are both $\in \mathrm{Coh}(Y)$ ▨

Ex 7.4* (b) consider the map $H^{n-p}(Y, \Omega_Y^{n-p})^\vee \longrightarrow H^p(X, \Omega_X^{n-p})^\vee = k$.

we prove it send 1 to $(\deg Y) \cdot 1$

i.e. $H^{n-p}(X, \Omega_X^{n-p}) \xrightarrow{(\deg Y)} H^{n-p}(Y, \Omega_Y^{n-p})$, thus consider $H^{n-p}(X, \Omega_X^{n-p}) \longrightarrow H^{n-p}(Y, \Omega_Y^{n-p})$

$k \xrightarrow{(\deg Y)} k$, $\swarrow$ Id

We have $Y \cap H_1 = Y_1$ nonsingular. $\dim Y_1 = p-1$

and $Y_1 \cap H_2 = Y_2$ nonsingular; $\dim Y_2 = p-2$ ...

then for Y_p we can prove, by induction on p conversely we prove the Y_i:

$\Rightarrow Y_p \subset Y_{p-1} \subset \cdots \subset Y_1 \subset Y \subset X$ by Bertini

① Y_p, done as (a)

and Y_p is finite points $= \deg Y$ (Y_{p-r}, $r \geq 2$ is connected)

② now Y_i done, consider Y_{i-1}

by Bezout's thm $(\deg Y) \cdot 1 = \sum_{|Y_p|} 1 \cdot 1$

$(Y_i = Y_{i-1} \cap H_i)$ i.e. we prove that $\Rightarrow |Y_p| = \deg Y$

$H^{i}(Y, \Omega_Y^{i}) \cong H^{i-1}(Y \cap H_1, \Omega_{Y \cap H}^{i-1})$, then $H^{n-p}(X, \Omega_X^{n-p}) \to H^{n-p}(Y, \Omega_Y^{n-p}) \cong H^0(Y_p, \Omega_{Y_p}^0)$ is the $\deg Y \cdot$ Id.

this is due to: Ex 7.3 and induction

$\Rightarrow$ all them are k ☒

(c) Nothing to prove ☒

(d) I don't know //

Apply Prop 2.2 Matsumura [1] $\gamma(Y) = \mathrm{tr}_Y(\lambda)$ is by the λ the image under the map $H^{n}(X) \to H^{n-p, n-p}(Y)$ restriction

$\Rightarrow \delta(\mathrm{tr}_Y(\lambda)) = c(D) \cdot \lambda$, $\forall D$, $\forall \lambda$.

let $D = \delta(Y) \Rightarrow$ then $\delta : \mathrm{tr}_Y(\lambda) \mapsto \mathrm{tr}(\delta) \cdot \lambda$

$\Rightarrow \mathrm{tr}_Y(\lambda) = c(D) = c(\delta(Y))$

$\gamma(Y)$ ☒

Ex 7.2. We'll show the Deligne's proof of the existence of a dualizing sheaf in this Ex using Ex 6.10, generalizing to the Grothendieck duality

(a) Now embed $X \hookrightarrow \mathbb{P}^N_k$ and $\omega^\circ_{\mathbb{P}^N}$ exists by Thm 7.1: $\omega^\circ_{\mathbb{P}^N} = \omega_{\mathbb{P}^N} = \mathcal{O}_{\mathbb{P}^N}(-N-1)$

and assume f is flat, finite map and using Ex 6.10, to show that $\mathrm{Ext}^i_X(\mathcal{F}, f^!\omega_{\mathbb{P}^N}) \cong H^{n-i}(X, \mathcal{F})^\vee$

(b) f is finite is easy, (affine)

it left to show that: $f: X \to Y$ is finite; X, Y noetherian; Y nonsingular

then f flat $\Leftrightarrow$ X is C-M scheme

(c) We have two duality now ① $\mathrm{Ext}^i(\mathcal{F}, \omega_X) \cong H^{n-i}(X, \mathcal{F})^\vee$ for X proj over k

We generalize both of them ② $\mathrm{Ext}^i_X(\mathcal{F}, f^!\mathcal{G}) \cong \mathrm{Ext}^i_Y(f_*\mathcal{F}, \mathcal{G})$ for X affine relative to Y

to Grothendieck duality: X proj over k and $f: X \to \mathrm{Spec}\, k$ has $f_*\mathcal{F} \cong \Gamma(X, \mathcal{F})$

then $\mathrm{Ext}^i_X(\mathcal{F}, f^!\mathcal{G}) \cong \mathrm{Ext}^i_Y(f_*\mathcal{F}, \mathcal{G}) \Rightarrow \mathrm{Ext}^i_X(\mathcal{F}, \omega_X = f^!(k)) \cong \mathrm{Ext}^i_k(f_*\mathcal{F}, k) = H^i(X, \mathcal{F})^\vee$ (nothing to prove) ☒

Pf of (a) & (b):

(a) $f^!\omega_{\mathbb{P}^N} = \mathcal{H}om_{\mathbb{P}^N}(f_*\mathcal{O}_X, \omega_{\mathbb{P}^N}) \Rightarrow \mathrm{Hom}_X(\mathcal{F}, \omega_X) \cong \mathrm{Hom}_{\mathbb{P}^N}(f_*\mathcal{F}, \omega_{\mathbb{P}^N})$

then it's the case $p = 0$, $\cong H^N(\mathbb{P}^N, f_*\mathcal{F})^\vee$ by Thm 7.1

for $p \geq 0$ case we only show coeffaceable, $= H^N(X, \mathcal{F})^\vee$ due to the category equivalence

or by Ex 6.10 directly ☒

(b) It's local on target: let $y \in Y$ and $A = \mathcal{O}_{Y,y}$, and due to finite $\Rightarrow$ quasi-finite

$\Rightarrow f^{-1}(y)$ is finite, let $B = \varinjlim \mathcal{O}_{f^{-1}(U)}$ is semi-local ring, then B is free over A

due to the flat $\Leftarrow (f_*\mathcal{O}_X)_y$ locally free (finite maximal) $\Leftrightarrow \mathrm{hd}_A B = 0$

Generic freeness $(f_*\mathcal{O}_X)_y$ free A-module (due to A local $\Rightarrow$ its projective module are all free)

$\Leftrightarrow \mathrm{hd}_A B = 0 \Leftrightarrow \mathrm{depth}\, B = \dim A = \dim B \Leftrightarrow B$ C-M ring

$\Leftrightarrow X$ C-M scheme ☒

Ex 7.1.

$\Gamma(X, \mathcal{L}^\vee) = \Gamma(X, \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)) = \mathrm{Hom}(\mathcal{L}, \mathcal{O}_X)$

but globally $f: \mathcal{L} \to \mathcal{O}_X$ has $f(X) = 0$ due to $\mathcal{O}_X(X) = k$ by Liouville theorem (integral $\Rightarrow$ irreducible) but $\mathcal{L}(X) \neq k$ due ampleness $\Rightarrow \mathrm{Hom}(\mathcal{L}, \mathcal{O}_X) = 0$ ☒

Ex 7.2. (a) Proved in Ex 7.0 (a) ☒

(b) Due to the universal property of $\Omega_{Y/k}$

$\Rightarrow f_*\Omega_{X/k} \to \Omega_{Y/k}$, then due to $\dim X = \dim Y = n$

$\Rightarrow f_*(\wedge^n \Omega_{X/k}) \to f_*(\Omega_{Y/k})$ is desired map ☒

Ex 7.3. $H^q(X, \Omega^p_X) \cong \mathrm{Ext}^{n-q}(\wedge^p \Omega_{X/k}, \omega)^\vee$

$\cong \mathrm{Ext}^{n-q}(\omega \otimes (\wedge^{n-p} \Omega_{X/k})^\vee, \omega)^\vee$

$\cong \mathrm{Ext}^{n-q}(\mathcal{O}, \Omega^{n-p}_X)^\vee \cong H^{n-q}(X, \Omega^{n-p}_X)^\vee$

thus ① $q > p$, $H^q(X, \Omega^p_X) = 0$

② $n-q > n-p$, by Ex 6.5 $\Rightarrow \mathrm{Ext}^{n-q}(\omega, \omega \otimes \Omega^{n-p})^\vee = 0$

③ $p = q$, $H^{p}(X, \Omega^p_X) = k$ ☒

Ex 7.4*. (a) $t_X: H^n(X, \omega_{\mathbb{P}^n}) = k \longrightarrow k$ due to $\Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-1)) = 0$

and $\Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-1)) \longrightarrow \Gamma(X, \Omega_{X/k}) \cong H^n(X, \omega_{\mathbb{P}^n}) \Rightarrow$ map to 1 ☒

($\mathcal{O}_X \otimes \mathcal{L}^n$ no globs

$\Rightarrow \exists n_1: (\mathcal{O}_X \otimes \mathcal{L}^{n})^{\otimes n_1}$

$= \mathcal{L}^{n n_1}$ has enough many globs

$\Rightarrow \dim \Gamma(X, \mathcal{L}^m) \geq \dim X + 1$

then if $\Gamma(X, \mathcal{L}^{-1}) \neq 0$

$\Rightarrow \Gamma(X, \mathcal{L}^{-m}) \neq 0$

$\Rightarrow \Gamma(X, \mathcal{L}^m \otimes \mathcal{L}^{-m})$

$= \Gamma(X, \mathcal{O}_X)$ have many globs, contradict to $\Gamma(X, \mathcal{O}_X) = k$ ☒

Ex 8.1. $\Gamma(X,\mathcal{F}) = \Gamma(Y, f_*\mathcal{F})$ and effaceable by $R^i f_*(\mathcal{F}) = 0$ ▨

Rk. In view of S-S we have

$0 \to \mathcal{F} \to \mathcal{J}^\cdot$ --- ①

$0 \to f_*\mathcal{F} \to f_*\mathcal{J}^\cdot$ --- ②

$0 \to R^1 f_*\mathcal{F} \to R^1 f_* \mathcal{J}^\cdot$ $=0$

$=0$

thus ② is exact

Ex 8.2. Continuity is trivial

it suffices to show that $R^i f_*(\mathcal{F}) = 0, \forall i > 0$

It's local, let Y affine $\Rightarrow R^i f_*(\mathcal{F}) = \widetilde{H^i(X,\mathcal{F})}$

but due to $X = f^{-1}(Y)$ affine and Serre criterion

$\Rightarrow H^i(X,\mathcal{F}) = 0$ ▨

Ex 8.3. By (II, Ex 5.1), $f_*(\mathcal{F} \otimes f^*\mathcal{E}) \cong f_*\mathcal{F} \otimes \mathcal{E}$

For $R^i f_*$, it suffices proving to effaceable

by Prop 8.1, we change to $H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)} \otimes (f^*\mathcal{E})|_{f^{-1}(V)})$

and $H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)} \otimes (f^*\mathcal{E})|_{f^{-1}(V)}) = H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)} \otimes \mathcal{E}(V))$

$\cong H^i(\Gamma(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}) \otimes \Gamma(V, \mathcal{E}|_V)) \cong H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}) \otimes \Gamma(V, \mathcal{E}|_V)$

$= H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}) \otimes \mathcal{E}(V)$ ▨

Ex 8.4. (a) By (II, 7.11(a)), we have

$\pi_*(\mathcal{O}(l)) = \begin{cases} 0 & ; l < 0 \\ \mathcal{O}_Y & ; l = 0 \\ S^l(\mathcal{E}) & ; l > 0 \end{cases}$

Then to prove ① $R^i \pi_*(\mathcal{O}(l)) = 0$ for $0 < i < n$

it suffices to prove Y affine, i.e. $R^i \pi_*(\mathcal{O}(l)) = H^i(X, \mathcal{O}(l))$

and for Y affine, X is locally projective space, then follows the Thm 5.1(b) ▨

② Again let $Y = \operatorname{Spec} A$

$X = \mathbb{P}^n_A \Rightarrow$ by Thm 5.1(d)

$\Gamma(X, \mathcal{O}(-l-n-1)) = 0 = \operatorname{Hom}(H^n(X, \mathcal{O}(l)), A)$

$\Rightarrow H^n(X, \mathcal{O}(l)) = 0$ ▨

(b) By the Euler sequence (II, 8.13) $\Rightarrow 0 \to \Omega_{X/Y} \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0$ locally for $Y = \operatorname{Spec} A$ and $X = \mathbb{P}^n_A$, thus we prove $\mathcal{O}_X(-1)^{n+1} \cong \pi^*\mathcal{E}(-1)$ naturally (to paste)

this is due to $\pi^*(\mathcal{E}(-1)) = \pi^*(\mathcal{O}_Y(-1)^{n+1}) = (\pi^*(\mathcal{O}_Y(-1)))^{n+1} = \mathcal{O}_X(-1)^{n+1}$ ▨

②

Using (II Ex 5.16(d)) $\wedge^{n+1}(\pi^*\mathcal{E}(-1)) = \mathcal{O}_X \otimes \wedge^n \Omega_{X/Y} = \omega_{X/Y}$

$\pi^*(\wedge^{n+1}\mathcal{E}(-1)) = \pi^*(\wedge^{n+1}\mathcal{E})(-(n+1))$ ▨

③ $R^n \pi_*(\omega_{X/Y})$

$\cong R^n \pi_*(\pi^* \wedge^{n+1}\mathcal{E} \otimes \mathcal{O}_X(-1)^{n+1})$

$= R^n \pi_*(\mathcal{O}_X(-1)^{n+1}) \otimes \wedge^{n+1}\mathcal{E}$ (by (a), $-1 > -n-1$)

$= \wedge^{n+1}\mathcal{E} = \mathcal{O}_Y$ ▨

(c) $\begin{cases} l > -n-1 & ; \text{both } 0 \text{ by (a)} \\ l = -n-1 & ; R^n \pi_*(\mathcal{O}(-n-1)) \cong (\pi_*\mathcal{O}_X)^\vee \otimes (\wedge^{n+1}\mathcal{E})^\vee \cong \mathcal{O}_Y \\ l < -n-1 & ; S^{-l-n-1}\mathcal{E} \otimes \wedge^{n+1}\mathcal{E} = \text{RHS} \end{cases}$

and $R^n \pi_*(S^{-l-n-1}\mathcal{E} \otimes \mathcal{O}_X(l)) \overset{\text{projection formula}}{=} R^n \pi_*(\mathcal{O}_X(l)) \otimes S^{-l-n-1}(\mathcal{E})$

then we associate $\mathcal{O}_X(l)$ with $\wedge^{n+1}\mathcal{E}$ by perfect pairing given in Thm 5.1(d), that is:

$S^{-l-n-1}(\mathcal{E}) \otimes R^n \pi_*(\mathcal{O}_X(l)) \to R^n \pi_* \mathcal{O}_X(-n-1) = (\wedge^{n+1}\mathcal{E})^\vee$ ▨

(d) $p_g(X) = \dim H^0(X, \omega_X)$, locally $H^0(X, \omega_X) = H^0(\mathbb{P}^n_A, \omega_{\mathbb{P}^n_A}) = 0$ (II, 8.20.1)

$(-1)^n p_a(Y)$ ▨ $\sum (-1)^i \dim H^{n-i}(Y, \mathcal{O}_Y)$ due to vanishing

$\Rightarrow p_g(X) = 0$ ▨

By Ex 8.1 $\Rightarrow H^i(X, \mathcal{O}_X) = H^i(Y, \mathcal{O}_Y) \Rightarrow p_a(X) = \sum (-1)^{i} \dim H^{n-i}(X, \mathcal{O}_X) = \sum (-1)^i \dim H^{n-i}(Y, \mathcal{O}_Y)$

then proof. Verify ① and ②:

① [illegible] $-\otimes M_Q = (-\otimes M_P)\otimes B_Q$ is exact due to M_P, B_Q flat
$\Rightarrow M_Q$ flat $\Rightarrow Q \in U$

② Let $A = \frac{B}{p}$, $Q \in V(P) \Rightarrow pB_Q \subset \sqrt{B_Q}$ by $\sqrt{pB_Q} \subset \sqrt{B_Q}$,
thus by a flat criterion:

M_Q flat over $A \Leftrightarrow \frac{M_Q}{pM_Q}$ flat over $\frac{A}{p}$ and $\mathrm{Tor}^A_1(M_Q, \frac{A}{p}) = 0$ $\quad pB_Q\,(p \subset Q)$

Ex 9.5.
(a) We need imitate Example 9.8.4 and note that $I(C(X)) = I(X)$ and, $C(X) \subset \boxed{\mathbb{A}^{n+1} \subset \mathbb{P}^{n+1}}$ embedded

$\exists\, a \in A_P$ s.t. $\frac{M_a}{pM_a}$ free over $(\frac{A}{p})_a$, thus flat ▨

$\mathrm{Tor}^A_1(M_Q, \frac{A}{p}) = \boxed{\mathrm{Tor}^{A_1}_1(M, \frac{A}{p})} \otimes B_P$ holds for $U \in N_P$, s.t. $\mathrm{Tor}_1(M_a, \frac{A}{p})$, $a \in U$ due to $\mathrm{Tor}_1(M, \frac{A}{p})$ is finite, let length to be longer

thus we only need add a coordinate to $I(X)$ to give $I(C(X))$ (homogenenization)
This is the question comes: it suffices to give a family $\Rightarrow$ the bounding of $\mathbb{A}^{n+1}$
Let $I(X) = \langle xz - ty, yz - txy, z^2 - t^2 y \rangle$ here $t=0$ is impossible by $C(\mathbb{A}^{n+1})$
$\Rightarrow I(X) = \langle xz - ty, yz - txy, z^2 - t^2 y, xy - xy^2 \rangle \Rightarrow$ flat
But embed $X \subset \mathbb{P}^{n+1}$
$\Rightarrow I(X) = \langle xz, yz, \boxed{z^2}, xy - xy^2 \rangle \Rightarrow$ not flat ▨ $\boxed{t=0}$

Such $X = (X_t)_{t \in \mathbb{A}^1}$
$X_t = \{[1:1:t]\}$ ▨

(b) Nothing to prove! (c) By Thm 9.9 (X_t) flat over $T \Leftrightarrow P(m) = \dim_{k(t)} H^0(X_t, \mathcal{O}_{X_t}(m))$ not depend on the $t \in T$

$\dim_{k(t)} H^0(X_t, \mathcal{O}_{X_t}(m)) = \dim_{k(t)} I_{X_t}(\mathcal{O}_{X_t}(m))$
$= \dim_{k(t)} (\frac{S_t}{I_t})_m$ not depend on t ▨

And for $C(X_t)$, $\dim_{k(t)} H^0(C(X_t), \mathcal{O}_{C(X_t)}(m)) = \dim_{k(t)} (\frac{C(S)}{I_{C(X)}})_m = \dim_{k(t)} (\frac{S_t}{I_t})_m$ due to the cone has same S and I ▨

(d) We strengthen restriction and conclusion in thm 9.11. By thm 9.11 $\Rightarrow X_{t_0} = X_t$ [illegible]
(normal $\Rightarrow$ projectively normal $\Rightarrow$ flat $\Rightarrow$ very flat) then we use some consequence to give the Hilbert polynomial not depend on t ▨

Ex 9.6. $(\Rightarrow)$ $Y = H_1 \cap \cdots \cap H_r$ scheme-theoretic
$\Rightarrow Y' = H_1 \cap \cdots \cap H_r \cap \mathbb{P}^{n-1}$ set-theoretic and it suffices show $\mathcal{J}_{Y'} = \mathcal{J}_{H_1 \cap \mathbb{P}^{n-1}} + \cdots + \mathcal{J}_{H_r \cap \mathbb{P}^{n-1}}$
$= (H_1 \cap \mathbb{P}^{n-1}) \cap \cdots \cap (H_r \cap \mathbb{P}^{n-1})$
due to each codim $H_i = 1$, this's trivial;

By $0 \to \mathcal{J}_{H_i}(-1) \to \mathcal{J}_{H_i} \to \mathcal{J}_{H_i \cap \mathbb{P}^{n-1}} \to$ (tensoring $\mathcal{J}_{H_i}$ to $0 \to \mathcal{O}_P \to \mathcal{O} \to \mathcal{O}_{\mathbb{P}^{n-1}} \to$)

$(\Leftarrow)$ Now we recover the $\mathcal{J}_{H_i}$ by $\mathcal{J}_{H_i \cap \mathbb{P}^{n-1}}$
Section (hypersurface) $S_1 \cdots S_r \in \mathcal{J}_{H_i \cap \mathbb{P}^{n-1}}$ defining Y'
a general collection $(S_1 \cdots S_r) \mapsto (S_1' \cdots S_r')$ has just r elements generated $\mathcal{J}_{H_i}$ (Bertini)
(Why we need (9.12)?) ▨

Ex 9.7. We have $0 \to \mathcal{J}/\mathcal{J}^2 \to \Omega_{X} \otimes \mathcal{O}_Y \to \Omega_{Y/k} \to 0$ $\Rightarrow 0 \to N_{Y/X} \to$ [illegible] $\mathcal{T}_X \to \mathcal{T}_Y$
infinitesimal extension for $f: X \& Y \Rightarrow$ [illegible] $N_{Y/X} \to \mathcal{H}om_{\mathcal{O}_Y}(\ldots)$ [illegible]
thus by the absolute case (9.13.2) $H^1(X, \mathcal{F}(\ldots))$ [illegible] classify the [illegible]

Ex 9.1. By Chevalley's thm (Finite type, Noetherian) $\Rightarrow f(U)$ constructible
and by II, Ex 3.18(c) $\Rightarrow$ it suffices show that stable under generalization
(Recall, it means that $\forall\, y_0 \in \overline{\{y_1\}} \cap f(U) \Rightarrow y_1 \in f(U)$)
$y_0 \in f(U) \Rightarrow \exists\, x_0 \in U$: $y_0 = f(x_0)$, let take $V \in N_{x_0}$, $V \subset U$ affine, $V = \mathrm{Spec} A$
due to finite type, let $f(\mathrm{Spec}(A)) = \mathrm{Spec} B$, A is flat B-module by $\varphi: B \to A$
(By fix B and let $V = \mathrm{Spec} A \subset f^{-1}(\mathrm{Spec} B)$) $\boxed{\text{Flat} + \boxed{\text{Finite type}} \Rightarrow \text{integral}}$
then we use the going up: we need to find preimage of y_1
and $\overline{\{y_1\}} \ni y_0$ means a chain $p_1 \subset p_0$ in B, with p_0 has preimage in A It's enough ✓
thus we use the going up to give p_1's preimage ▨

Ex 9.2. $X_1 = \{(t^3, t^2u, tu^2, u^3)\} \subset \mathbb{P}^3$, $X_0 = \{(t^3, t^2u, \ldots)\} \subset \mathbb{P}^3$
$X_a = \{(t^3, t^2u, atu^2, u^3)\} \subset \mathbb{P}^3$ and $I(X_a) = (a^3y^3w - z^3x, xz - ay^2, yz - axw)$
$\Rightarrow I(X_0) = (z^3x, xz, yz) \Rightarrow$ embedded point 0 ▨

Ex 9.3. (a) $\mathcal{O}_{y,Y}$ locally, $\mathcal{O}_{x,X}$ is f.g. $\mathcal{O}_{y,Y}$-algebra $\Rightarrow$ it suffices to show $\mathcal{O}_{x,X}$ is free
(Finite $\Rightarrow$ quasi-finite i.e. $f^{-1}(y) < \infty \Rightarrow$ dimension of fibres are $0 \Rightarrow$ flat ▨)
it's by $\otimes \frac{\mathcal{O}_{y,Y}}{m_{y,Y}} \Rightarrow$ the vector space injection $(\frac{\mathcal{O}_{y,Y}}{m_{y,Y}})^N \to (\frac{\mathcal{O}_{x,X}}{m_{x,X}}) \to 0$ ▨ [Ex 6.7]

(b) Counting dimension ▨

(c) $X_{red} = \mathrm{Spec}\left(\frac{k[x,y,z,w]}{(z, \ldots)^2}\right)$

$(z, 0) \mapsto z$ the 0 isn't flat
$(z, -z) \mapsto z$ ▨
$\Rightarrow \mathrm{Spec}\, k[x,y] = Y$.

An embedded point at least to be singular, then $J(X) = \begin{bmatrix} 0 & 0 & 2z & 0 \\ 0 & 0 & w & z \\ 0 & 0 & 0 & 2w \\ z & -w & x & -y \end{bmatrix}$
thus any points closed or dim 1 are singular (dim ≥ 2 is nonsingular)
then we consider whether localisation has nilpotent?
Not flat is clear: generic fibre is dim 2 but 0's fibre is dim 1 ▨

Ex 9.4. Covering X, we prove for $X = \mathrm{Spec} A$ and then taking union of flat locus
I claim (Algebraic) $\{p \in \mathrm{Spec} A \mid B_p \ (M_p)$ is flat over $A_p\}$ is open in $\mathrm{Spec} A$
This is the open loci problem, in general rephrased as $\{p \in \mathrm{Spec} A \mid$ Property P holds for $A_p\} = $ open, for $P = \{$regular local$\}$, we done in §1.5 T)
Others are due to Grothendieck.

Lemma (Nagata's criterion) If $P(\frac{A}{p}) \supset U$ non-empty open $\Rightarrow P(A) \subset \mathrm{Spec} A$ open
I prove the flat case (For more, see the Matsumura, Chap 8.)

Lemma (Topological Nagata's criterion)
($U \subset \mathrm{Spec} A$ open $\Leftrightarrow$ ① $Q \in \mathrm{Spec} A$, $p \in U$, $p \supset Q \Rightarrow Q \in U$ (generalisation!)
② [illegible]
Pf. Done by rephrasing $p \in U \Rightarrow U \supset V$, V non-empty open subset $\subset V(p)$
into generalisation of points.

Ex9.7. I forget, we should take locally affine open set and consider Spec A, A a quotient of $k[x_1,\dots,x_n]$

Ex9.8*. I had proven in Ex4.10* by Ext^1 classification, here it's a dual proof, replace Ext^1 by Tor^1 ☒

Ex9.9.

$$\frac{(x,y)\cap(z,w)}{((x,y)\cap(z,w))^2} \to \Omega_{k[x,y,z,w]/k}\otimes\frac{k[x,y,z,w]}{(x,y)\cap(z,w)} \to \Omega_{\frac{k[x,y,z,w]}{(x,y)\cap(z,w)}/k} \to 0$$

$$\frac{(xz,xw,yz,yw)}{(xz,xw,yz,yw)^2} \to \frac{k[x,y,z,w]}{(x,y)\cap(z,w)}[dx,dy,dz,dw] \to \frac{k[x,y,z,w]}{(x,y)\cap(z,w)}[dx,dy,dz,dw] \to 0$$

$$\mathrm{Tor}^1(A) \leftarrow \frac{((xz)^*,(xw)^*,(yz)^*,(yw)^*)}{((xz)^*,\dots)^2} \xrightarrow{\Phi} \frac{k[x,y,z,w]}{(xz,xw,yz,yw)}[dx^*,dy^*,dz^*,dw^*]$$

⇒ (We write $(xz)^*\dots$ for the value of xz of map determined)

It send four generators to the corresponding four

i.e. $\Phi(dx^*)$; $xz \mapsto (xz)^*$ ⇒ surjective ☒

More precisely, we need find the preimage of $((xz)^*,(xw)^*,(yz)^*,(yw)^*)$: (b_1,b_2,b_3,b_4)

⇒ $b_1 = \frac{x}{y}b_3 + b_1'$, $b_1'\in(z,w)$

$= \frac{z}{w}b_2 + b_1''$, $b_1''\in(x,y)$

By take both plane. $k[x,y]$ section

⇒ $b_1 = \frac{z}{w}b_2 + \frac{x}{y}b_3$

$b_2 = \frac{x}{y}b_4 + \frac{w}{z}b_1$

$b_3 = \frac{y}{x}b_1 + \frac{z}{w}b_4$

$b_4 = \frac{y}{x}b_2 + \frac{w}{z}b_3$ ☒

Ex9.10 (a) $H^1(\mathbb{P}^1_k, \mathcal{T}_{\mathbb{P}^1_k}) \underset{(II.8.20.1)}{=} \boxed{H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(2)) = 0} \xRightarrow{(9.13.2)}$ No infinitesimal extension ☒

By LES: Euler sequence: $0\to\Omega_{\mathbb{P}^1_k/k}\otimes\mathcal{O}_{\mathbb{P}^1_k}(2)\to\mathcal{O}_{\mathbb{P}^1_k}\to\mathcal{O}_{\mathbb{P}^1_k}(2)\to 0$

$\to H^1(\Omega_{\mathbb{P}^1_k}\otimes\mathcal{O}(2)) \to H^1(\mathbb{P}^1_k,\mathcal{O}_{\mathbb{P}^1_k}(2)) \to H^1(\mathbb{P}^1_k,\mathcal{O}_{\mathbb{P}^1_k}) \to \cdots$

$\Gamma(\mathbb{P}^1_k,\Omega_{\mathbb{P}^1_k}(2)) = 0$; $= 0$

⇒ $H^1(\mathbb{P}^1_k,\mathcal{O}_{\mathbb{P}^1_k}(2)) = 0$ ☒

Rk, A more convenient way is $H^1(\mathbb{P}^1_k,\mathcal{O}_{\mathbb{P}^1_k}(n)) = [x_0^{i}x_1^{j}]$, $i+j=n$

⇒ $H^1(\mathbb{P}^1_k,\mathcal{O}_{\mathbb{P}^1_k}(2)) = [x_0^{-1}, x_0x_1, x_1^{-1}] = 0$

(b) I don't know. (But it shows the Zariski open set is too big)

(b) Now I know: $\operatorname{Spec}\mathbb{C}[x,x^{-1},y] \xrightarrow{f} \operatorname{Spec}\mathbb{C}[x]$ is a double covering ⇒ not trivializing due to $\mathbb{Z}/2\mathbb{Z}$ ($f: -\to 2$) has a affine line can't be punctured by Zariski but in étale/complex, we only think it as a rank 2-vector bundle ☒

(c)* (This shows the étale open $\{T'\to T\}$ is a good replace)

$$\begin{array}{ccccc}\mathbb{P}^1\times_T T' & \cong & X\times_T T' & \longrightarrow & X\\ \downarrow & & \downarrow & \square & \downarrow\\ T' & = & T' & \xrightarrow{g} & T\end{array}$$

by flat base change not change fibre-wise

⇒ take T' be "neiberhood" of t_0 s.t. $\forall t'\in T'$, $g(t')$ has fibre $\mathbb{P}^1_k$ ☒

Ex9.11. We can always embed curve into $\mathbb{P}^3_k$

⇒ $Y\subset\mathbb{P}^3_k \to \mathbb{P}^2_k$, then we have a family of $(Y_t)_{t\in\mathbb{P}^1}$ and $Y_0 = Y$, $Y_1\subset\mathbb{P}^2_k$, for Y_1, $p_a(Y) = \frac{(d-1)(d-2)}{2}$ is clear and ~~degree~~ this family flat, reduced, then by (9.13)

We need check flatness (reduced is clear), done in (9.8.2) ☒