Linear representation of compact groups

First, Recall our study in §1–§3 de Serre, ① where depending on the finiteness of G? ② Can these place use a proper alternative to generalise it?

We'll not distinguish $C^*(G)$ and $\mathbb{k}[G]$ next.

I list them next:

① P6, Thm1, "The averaging trick": $\varphi: V \to W$, then $\varphi^0 = \frac{1}{|G|}\sum_{t\in G}(\rho_t^2)^{-1}\varphi\rho_t$ average of φ ⟹ infinite sum i.e. integral on G ⟹ The Mean value functional

② P4, §1.2(b), the regular representation: $\mathbb{k}[G] = \mathrm{span}\langle e_g\rangle_{g\in G}$ ⟹ infinite dimensional algebra $L^2(G)$ (continuous) a Hilbert space

③ P10, §2.1, the character: $\chi_\rho(s) = tr(\rho_s: V\to V)$ ⟹ Now $\dim V = \infty$ Hilbert, the trace even not well-defined.

④ P15, §2.3, the inner product: $\langle \varphi,\psi\rangle = \frac{1}{|G|}\sum_{t\in G}\varphi(t)\overline{\psi(t)}$ i.e. $\rho_\varphi \in S_1(V)$ the non-commutative L^1-space or ② reduce to finite-dimensional ⟹ decomposition

① ⟹ L^2-inner product if we have a proper integral (measure) on G

⑤ P13, Prop4 (Schur's Lemma), for V_1, V_2 irreducible, ① $\mathrm{Hom}_G(V_1,V_2) = \{0\}$; $V_1 \neq V_2$; $\{iso\}$; $V_1 = V_2$ ($\mathrm{End}(V) = \mathrm{Aut}(V)$)

② if $\mathbb{k} = \overline{\mathbb{k}}$, then isomorphism is $\lambda \mathrm{Id}_V$ ⟹ ① not change but ② need the λ is the eigenvalue ⟹ spectrum

⑥ P26, §3.2, the product $G_1\times G_2$'s representation ⟹ "product measure" ① (Hanier) If V has countable basis, it still hold ②

⑦ P28, §3.3, the induced representation ⊕ $W_\sigma = V$ (product measure) using more general description of adjoint functor (we use $L^2(G/H)$ more) ⟸ $|G/H|$ ⟹ Now $|\frac{G}{H}| = \infty$, it comes to be not interesting (Serre said)

For generalising ~~them~~ properly, we prepare something in functional analysis (for $\dim V = \infty$) and abstract harmonic analysis (for Haar)

Setting. G is a compact topological group (in usual, LCH group) and over $\mathbb{k}$, usually to be $\mathbb{C}$.

To define the Mean value functional $M: C(G,\mathbb{k}) \to \mathbb{k}$, we have two approaches.

② Using Hahn-Banach thm, to show the existence $f \mapsto Mf$ and uniqueness. Can be generalise to amenable groups but not all LCH. ① Only Peter-Weyl not holds ② But many pf need to be change. There are many way to give Haar measure, we use this way, but only holds for compact, not LCH.

① choose $G' \subset G$, G' a finite subset, i.e. $G' = \{g_1 \dots g_n\}$ and let $\mu(f, G') = \frac{1}{|G'|}\sum f(g\cdot)$ then $G \to \mathbb{k}$, $g \mapsto \frac{1}{n}\sum f(gg_i)$ the left mean value; (right similar) $H_L(f)$, $H_R(f)$, $M_f(N_f)$ denote all of them (left/right: right is the left mean value of G^{op}, thus properties is parall)

Thm (without pf) ∃! constant function $\mu(f) \in M_f$, then $\mu: C(G,\mathbb{k}) \to \mathbb{k}$ is the mean value. (And then $\mu(f) = \int f\, d\mu$) Here we not focus on this way, but its R-N. The idea of p comes from approach ①

Thm1. ∃ $M: C(G,\mathbb{k}) \to \mathbb{k}$ such that (i) $M(1) = 1$ (ii) if $f\in C(G,\mathbb{k})$ is non-negative, then $M(f) \geq 0$ (iii) M is multiple invariant, i.e. $M(T_a f) = M(f)$ ($(T_a f)(g) = f(ga)$)

(For $\mathbb{k} = \mathbb{C}$ or $\mathbb{R}$, generally the ordered field)

Pf. We have subspace $K = \{f_x: G\to\mathbb{k}, \forall g\in G,\ g\mapsto x \mid x\in\mathbb{k}\} = \mathbb{k}$

And the functional (linear) $M_0: K \to \mathbb{k}$, $f_x \mapsto x$. We have a semi-linear functional $p: C(G,\mathbb{k}) \to \mathbb{k}$, $f \mapsto \inf_{\substack{n\in\mathbb{N}\\ a_1\dots a_n\in G}}\{N(f;a_1\dots a_n) = \sup_{g\in G}\frac{1}{n}\sum_i f(ga_i)\}$

And $M_0 \leq p$ in K, then by Hahn-Banach thm, we extend M_0 to $M \leq p$

The we verify { p does semi-linear; the translation invariant of M }

Due to $p(f+g) \leq N(f+g; a_i, \beta_j) \leq N(f, a_i) + N(g, \beta_j) \leq p(f)+p(g)+2\varepsilon$, then let $\varepsilon \to 0$, thus it suffices to prove $N(f+g; a_i\beta_j) \leq N(f,a_i) + N(g,\beta_j)$ (So chosen a_i's and β_j's) (I don't know how to compute)

And $|M(T_a f) - M(f)| = |M(T_a f - f)| \leq p(T_a f - f) \leq N(T_a f - f; a, a^2, \dots a^n)$
$= \sup_{s\in G}\frac{1}{n}\sum_i [f(sa^{i+1}) - f(sa^i)] = \sup_{s\in G}\frac{1}{n}[f(sa^{n+1}) - f(sa)]$
$\leq \frac{2}{n}\|f\|_\infty \xrightarrow{n\to\infty} 0 \Longrightarrow M(T_a f) = M(f)$ ∎ We can do this due to $\lim \geq \inf$

Rk. The Mean (invariant mean) can't be introduced generally in all LCH group, if a LCH group admit a invariant mean, we said it's amenable (this's used in the Banach-Tarski thm) representatively amenable ⟺ supp of regular representation is $(\Lambda: L^\infty(G) \to \mathbb{R}, \|\Lambda\| = 1, \Lambda \geq 0)$ all irreducible representation

The way of introduce a Haar measure on G is due to Von-Neumann, but it only works on G compact, not all LCH, indeed, even a non-amenable LCH group admit Haar measure (all LCH admit Haar measure), i.e. Haar Measure not depend on mean.

Thm2. ∃ a measure on set G, denoted as μ such that (i) $\mu(G) = 1$ (propability measure) (ii) Translation invariance (i.e. σ-additive $\mu: 2^G \to \mathbb{R}_{\geq 0}$) i.e. $\mu(G') = \mu(L_g G') = \mu(R_g G')$

Pf. Due to Riesz's representation thm, $Mf = \int_G f\,d\mu$, ∃μ and $\mu(G') = \int \chi_{G'}\,d\mu = M(\chi_{G'}) \geq 0$ by Thm1 (ii) additivity due to M is. For (i), $\mu(G) = M(1) = 1$. For (ii), $\mu(L_g G') = \int L_g\chi_{G'}\,d\mu = M(L_g\chi_{G'}) = M(\chi_{G'}) = \mu(G')$ ∎

To be precise, we have a translation invariant (Radon) problabity measure G

Rk. For LCH group, without amenability, the left invariant ⇏ the Right invariant in general, we call left ⟺ right the G is unimodular, thus compact ≤ unimodular; and amenable ≤ unimodular is a conjecture due to A. L. T. Paterson

E.g. $\{\begin{pmatrix} x & y\\ 0 & 1\end{pmatrix} \mid x\in\mathbb{R}^\times, y\in\mathbb{R}\}$ is LCH but not compact, $\frac{dxdy}{|x|^2}$ is left Haar, $\frac{dxdy}{|x|}$ is right Haar, but due to uniqueness up to scalar ⟹ left ⇎ right. thus modular

Thus the proof of compact is easier; you'll be free to change variable directly, but for LCH, you'll multiply $\Delta(g)$ $\Delta(h^{-1})$ always

Prop3 (Uniqueness) The μ in thm2 is unique, if remove (i), then unique up to scalar

Pf. Due to $\mu(G') = M(\chi_{G'})$, it suffices to prove the uniqueness of invariant mean

Now given two M, M' satisfy (i), (ii), (iii) in Thm1, then we prove $Mf = M'f$: Let $H_L(f) = \mathrm{co}\{gf \mid g\in G\}$ and $K_f = \overline{H_L(f)}$

I claim, K_f exist and unique element to be constant function, then due to $Mf\, 1_G, M'f\, 1_G \in K_f \Rightarrow Mf = M'f$ (indeed, it's an equivalent definition of Mf: $M: C(G,\mathbb{k}) \to \mathbb{k}$, $f \mapsto$ the unique number of K_f)

Pf of the claim (Hard) Set $a1_G \in H_L(f)$, $b1_G \in H_R(f)$, then $a = b$

$\forall \varepsilon > 0$, $\exists \{a_j\}, \{s_j\}, \{b_k\}, \{t_k\}$, $a_j, b_k > 0$, $\sum a_j = 1 = \sum b_k$, $\{s_j\}\{t_k\} \subset G$ such that $|\sum a_j f(s_j g) - a1_G(g)|, |\sum b_k f(g t_k) - b1_G(g)| < \varepsilon$, then we have:

[redacted] $|a-b| \le \|a 1_G(g) - b 1_G(g)\| \le |a 1_G(g) - \sum_k \sum_j a_j b_k f(s_j g t_k)| + |\sum_j \sum_k a_j b_k f(s_j g t_k) - b 1_G(g)| \le \sum_k b_k |\sum_j a_j f(s_j g t_k) - a 1_G(g t_k)| + \sum_j a_j |\sum_k b_k f(s_j g t_k) - b 1_G(s_j g)| \le \varepsilon(\sum_k b_k) + \varepsilon(\sum_j a_j) = 2\varepsilon$

and $\varepsilon \to 0 \Rightarrow a = b$ ☒

However, the red underline why?, i.e. why we can say that: the mean value can be uniformly approximated by $M_L(f)$ and $M_R(f)$, this needs the Mazur thm and Krein-Milman thm, thus we omit its proof and accept it (Besides, we omit that Haar measure is Radon) Coro4. We can define $L^p(G)$.

(We use these to show ① ⟺ ② and then trivial)

(Indeed, if we choose approach ① at first, the uniqueness is imitated: due to it just define the mean this way, however, then its existence isn't trivial.

Now due to G not finite, we need discuss [redacted] $\dim V = \infty$, or topological vector space, for simplicity, let V to be ~~H~~ a Hilbert space, then $GL(H) = \mathrm{End}(H) = \mathcal{B}(H)$, a representation $\rho: G \to \mathcal{B}(H)$, thus it's natural to consider

① When $\rho: G \to \mathcal{K}(H)$?

② When $\rho: G \to \mathcal{B}(L^2(G))$?

③ The translation L_g, R_g is representation?

(given by myself)

① is not interesting: $\mathcal{K}(H) = \overline{\mathcal{F}_r(H)}$, thus it's nothing more than $\dim H < \infty$ ② answer the shape of a regular representation.

②③ will lead us to Peter-Weyl thm

Now we build the basic thms in §1-§3 first.

① The averaging turns out to be $\frac{1}{|G|}\sum_{g\in G} (\rho_g^{\prime})^{-1} \varphi \rho_g^{-1} \Rightarrow \int_G (\rho_g^{\prime})^{-1} \varphi \rho_g^{-1} dg$; $\varphi \mapsto \int_G (\rho_g^{\prime})^{-1} \varphi \rho_g^{-1}(v) dg$ Here the integral of $G \to \mathcal{B}(V,W)$, $g \mapsto (\rho_g^{\prime})^{-1} \varphi \rho_g$ value on a Banach space, and $\int: L^1(G, [redacted] B) \to B$ for any Banach space, it's well-defined due to

$\|\int_G (\rho_g^{\prime})^{-1} \varphi \rho_g^{-1} [redacted] dg\| \le \mu(G) \|(\rho_g^{\prime})^{-1} \varphi \rho_g\| = \|(\rho_g^{\prime})^{-1} \varphi \rho_g\| \le \|\rho_g^{\prime}\|^{-1} \|\varphi\| \|\rho_g\|$ bounded ☒

Later (Thm11), we'll see its generality

~~pre~~ hermitian form (unitary)

② Recall our $k[G] = \mathrm{span}_{g\in G}\{e_g\}$ and $\rho: G \to \mathrm{End}(k[G])$ here we define $\rho: G \to \mathcal{B}(L^2(G))$ to be the regular representation of compact group G.

$g \mapsto L_g$ (Notice $L_g: f \mapsto f(g^{-1}\square)$!)

And Biregular $\mathrm{Reg}: G\times G \to \mathcal{U}(L^2(G))$ ($\mathcal{U}$ means preserve) $\mathrm{Reg}(g_1, g_2) = L_{g_1} R_{g_2} = R_{g_2} L_{g_1}$

(Indeed it's the left regular representation $g \mapsto L_g$, but for G compact we not concern it) Thus we can [redacted] easily write as L (or R) as regular representation

Later we'll prove that $R(G) = \{f \in C(G, k) \mid f$ is G-finite: $\langle R_g f \rangle_{g\in G}$ is finite dimensional$\}$

Thm5. (Peter-Weyl) ① $R(G)$ is dense in $L^2(G)$ and $C(G)$ ② The two-side biregular representation [redacted] of $G\times G$, $\mathrm{Reg}(G\times G) = L(G\times G) \cap R(G\times G)$ can be decomposed: $\boxed{\mathrm{Reg} [redacted] \cong \widehat{\bigoplus} (\pi_\sigma \otimes \pi_\sigma^*)}$, each component is space of matrix coefficient (explained later) which is finite dimensional: this shows that ① Regular representation density, i.e. important ② Regular representation is easy to study

(We use $\widehat{\bigoplus}$ to show it's Hilbert-space-theoretic) $\sigma \in \widehat{G}$

③ Consider $S^1(H)$ the Banach space of the trace class operator (This answer might acquire our functional analysis about spectrum)

My thought: if a representation (ρ, H) such that $\mathrm{Im}\rho \subset S^1(H)$, then $\rho_s \in S^1(H) \Leftrightarrow \exists T_1, T_2 \in S^2(H)$: (Hilbert-Schmidt operator): $\rho_s = T_1 T_2$, and $S^2(H) \subset \mathcal{K}(H) \Rightarrow \boxed{S^1(H) \subset \mathcal{K}(H)} \Rightarrow \mathrm{Im}\rho \subset \mathcal{K}(H)$, thus it's essentially not interesting

Thus we study more on decomposition into finite not directly study ρ

Def (Trace class group) We extend $\pi: G \to \mathcal{B}(H)$ to $\pi: C(G) \to \mathcal{B}(H)$, then if $\forall f \in C(G)$, $\pi(f) \in S^1(H)$

$f \mapsto \int_G f(g)\pi(g) dg \Rightarrow$ this unitary representation π is of trace class

(By identify g and δ_g).

If every unitary irreducible representation of G is trace class, then G is a trace class group (Later we'll show compact [redacted] ⟹ only irreducible is enough)

For LCH groups, we extend to $C_c(G)$, for Lie groups, we extend to $C_c^\infty(G)$

Recall. It means that H has a Hermitian form [redacted] (When Hilbert, it equipped one) then ρ_g preserve it

Then the character $\chi_\pi: G \to k$ is continuous, indeed $\chi_\pi: C(G) \to k$

$g \mapsto \mathrm{tr}(\pi_g)$ $\quad f \mapsto \mathrm{tr}(\pi_f)$

also Prop6 Hilbert space $\langle \cdot, \cdot \rangle$ isn't the definition of trace class, but we have the category ~~and~~ Hilb with $\mathrm{Hom}_k \xrightarrow{\sim} \mathrm{Hilb}_k$, a canonical conclusion

E: $\mathrm{tr}(\pi_f) = \lim_{\substack{\to \\ \text{finite}}} \sum_i \langle \pi_f e_i, e_i \rangle$ and let the $E \subset I$ to be the finite [redacted] subset. $\Rightarrow \mathrm{tr}(f) = \lim_{\to E \subset I} \sum \langle \pi_f e_i, e_i \rangle$ give the completion of metric direct limit

Thus well-defined $\sup$ this generalised character

$\sup_E |\pi_E(f)| \le \sup_E |\sum_E \langle \pi_f e_i, e_i \rangle| < \infty$

However, this road is seldomly used, as ~~due to the Peter-Weyl ②~~, the irreducible representation is all finite-dimensional, and we used the decomposition to derive the ~~[redacted]~~ Weyl Character Formula (by techs as root systems, maximal torus theory ...)

$\Rightarrow \mathrm{tr}\,\pi(f) = \lim_{\to E \subset I} \pi_E(f)$ continuous ☒

④ $\langle \varphi, \psi \rangle = \int \varphi(t)\overline{\psi(t)} dt$ is trivially generalised.

⑤ I state and prove the Dixmier's thm (infinite-dimensional Schur Lemma), assume over $\mathbb{C}$ (algebraically closed) due to our Thm7. H is irreducible, at most countable basis (indeed this tells us $H \cong l^2(\mathbb{C})$ or $\mathbb{C}^n$, but we not use it) study of spectrum is quite

$\Rightarrow$ [redacted] $\mathrm{Hom}_G(H, H) = \{\lambda \mathrm{Id}_H \mid \lambda \in \mathbb{C}\}$ (The first part of Schur Lemma not change)

E.g. Construction of ODE without solution

Pf. Apply the first version, we show that, $\exists \lambda \in \mathbb{C}$, i.e. $\mathrm{Hom}_G(H, H) = \mathrm{Aut}_G(H)$ and $\mathrm{Hom}_G(H_1, H_2) = 0$)

[redacted] $-\lambda I$) not invertible for $T \in \mathrm{Hom}_G(H, H)$, then $T - \lambda I = 0 \Rightarrow T = \lambda I$ differ before we can take $\lambda \in \sigma(T) - \{0\}$, but it may not exist

Otherwise, $(T - \lambda I)$ invertible for $\forall \lambda \in \mathbb{C} \Rightarrow p(T)$ invertible for $\forall p \in \mathbb{C}[X]$ due to $p(T) = z(T - p_1 I)(\cdots)$ $\boxed{T = p(T)}$ by $R \mapsto \boxed{p(T)}$

[redacted] $= \mathrm{Frac}\,\mathbb{C}[X]$ is uncountable [redacted] dimension over $\mathbb{C}$ (Zariski's Lemma) $R(x) = p(x)/q(x)$, and [redacted] $\mathbb{C}(x) \hookrightarrow \mathrm{Hom}_G(H, H)$

And $P(T)$ is also invertible. I claim ev_T is injective: $\frac{p(T)}{q(T)} = 0 \Rightarrow p(T) = 0 \Rightarrow p(x) \equiv 0$, i.e. $R(x) = 0$ ($q(T)$ invertible)

thus $\dim \mathrm{Hom}_G(H, H) \geq \dim_{\mathbb{C}} \mathbb{C}(x)$, uncountable

$\Rightarrow$ H's dimension also uncountable $\Rightarrow$ contradiction ▨

[Rmk 3] In general (uncountable) we have a functional result. Thm 1' $\mathcal{F} \subseteq \mathcal{B}(H)$ s.t. $T \in \mathcal{F} \Rightarrow T^*$ [illegible] and $\mathcal{F}$ is topologically irreducible (not nontrivial invariant subspace under $\mathcal{F}$), then: $\Rightarrow$ All $S \in \mathcal{B}(H)$ commutes with all $T \in \mathcal{F}$ is scalar; more usually: Coro: (π, V) unitary, $\pi(G)$ topologically irreducible $\Leftrightarrow \mathrm{Hom}_G(\pi, \pi)$ is scalar

⑥ The construction of product measure on $G \times G$ is same as in $\mathbb{R}$, thus like the Fubini thm just follows like Euclid case.

For generality, I state it precisely on LCH groups $G_1 \times G_2$ (similarly for finite product) $= \prod G_i$ (G_i equipped μ_i Haar)

We have $\Phi: C_c(G) \to \mathbb{K}$ functional. Using approximation, $\int_{G_1}\left(\int_{G_2} f\, d\mu_2\right) d\mu_1 = \int_{G_2}\left(\int_{G_1} f\, d\mu_1\right) d\mu_2$. By definition, Φ is translation invariant $\Rightarrow$ so is the μ

$f \mapsto \int_{G_1}\left(\int_{G_2} f\, d\mu_2\right) d\mu_1$ then due to Riesz $\Rightarrow \exists \mu: \Phi(f) = \int f\, d\mu$

⑦ The induced representation

(then $(G_1 \times G_2)$-Rep $\cong \mathbb{K}[G_1 \times G_2]$-mod $\cong \mathbb{K}[G_1] \otimes \mathbb{K}[G_2]$-mod $\cong G_1$-Rep $\otimes$ G_2-Rep) uniqueness $\Rightarrow \mu$ is the Haar measure of $G_1 \times G_2$ denoted by $\mu_1 \otimes \mu_2$

$H \leq G$ closed subgroup. At first, we do prepare work: the Haar measure on $\frac{G}{H}$ equipped with the quotient topology, it's non-trivial (due to we need $L^2(\frac{G}{H})$ due to the regularity is the biggest)

H inherit a Haar measure from G naturally, denoted as μ_H, we define that $\forall f \in C(G, \mathbb{K})$, $f^H(g) := \int_H f(gh)\, d\mu_H(h) \xRightarrow{\text{translation invariance of } \mu_H} f^H(gh) = f^H(g)$, for $\forall h \in H$ and $\forall g \in G \Rightarrow G \xrightarrow{f^H} \mathbb{K}$ indice $f^\# \in C(\frac{G}{H}, \mathbb{K})$: $f^\#(gH) = f^H(g)$

thus we have map $\#: C(G, \mathbb{K}) \to C(\frac{G}{H}, \mathbb{K})$, $f \mapsto f^\#$ $\quad G \xrightarrow{} \frac{G}{H} \xrightarrow{f^\#}$

(This taken makes this proof more closed to LCH case. We can always for compact case we can take this for C compact)

Props. $\#$ is surjective and bounded.

Pf. $\boxed{\text{Surjectivity}}$: $\forall F \in C(\frac{G}{H}, \mathbb{K})$, we need $\boxed{\text{the partition of unity}}$ tech. (it holds for G (or $\frac{G}{H}$) due to the Urysohn's Lemma)

i.e. $\exists \varphi \in C(G, \mathbb{K})$: $\varphi(g) = 1$ for $C \subset G$ compact, and $0 \leq \varphi(g) \leq 1$ for all $g \in G \Rightarrow \varphi^\# \in C(\frac{G}{H}, \mathbb{K})$ and $\varphi^\# \geq 0$ on $p(C) = \mathrm{supp} F$

let $\psi \in C(\frac{G}{H}, \mathbb{K})$ to be $\psi(gH) = \begin{cases} \frac{F(gH)}{\varphi^\#(gH)} & ; gH \in p(C) \\ 0 & ; gH \notin p(C) \end{cases}$ then $f(g) = \psi(gH)\varphi(g)$ is the desired construction. $C = p^{-1}(\mathrm{supp} F)$ is preimage of compact, it's also compact due to proper

due to $f^\#(gH) = f^H(g) = \int_H f(gh)\, d\mu_H = \int_H \psi(gH)\varphi(gh)\, d\mu_H = \psi(gH)\varphi^\#(gH) = F(gH)$ ▨

Bounded: i.e. $\|f^\#\| \leq C$

$\|\#\| = \sup_{f \in C(G, \mathbb{K})} \frac{\|f^\#\|_\infty}{\|f\|_\infty}$, i.e. $\forall f \in C(G, \mathbb{K})$, $\|f^\#\|_\infty \leq C\|f\|_\infty$. let $g \in G$, $h \in H$ (for LCH group, replace $C(G, \mathbb{K})$ with $C_c(G, \mathbb{K})$ and using the fact that a quotient map is proper)

$\Rightarrow |f^\#(gH)| \leq \int_H |f(gh)|\, d\mu_H(h) \leq \mu_H(H)\|f\|_\infty \Rightarrow \|f^\#\|_\infty \leq \mu_H(H)\|f\|_\infty$ ▨

For generalising "a sum of $\bigoplus_{\gamma \in \frac{G}{H}}$", our way depict $\gamma \in \frac{G}{H}$ is the modular quasi-character Δ in general case (LCH group)

Consider the internal automorphism: $\mathrm{Int}_{g_0}: G \to G$ i.e. $\mathrm{Int}_{g_0} = (L_{g_0} R_{g_0})^{-1}$

We have a measure on G: $\mu(S) = \mu_G(\mathrm{Int}_{g_0}(S))$, $g \mapsto g_0 g g_0^{-1}$, due to Int_{g_0} is automorphism & Homeomorphism $\Rightarrow$ such μ also translation invariant, thus $\exists c_{g_0} \in \mathbb{K}$: $\mu(S) \cdot c_{g_0} = \mu(\mathrm{Int}_{g_0}(S))$, assume μ is left-invariant $\Rightarrow \mu(S) c_{g_0} = \mu(S g_0)$. Due to in general LCH left $\not\Leftrightarrow$ right, i.e. modular case, thus $\Delta: G \to \mathbb{R}^\times_{\geq 0}$ is the modular function, it's a quasi-character due to Δ is continuous homomorphism $G \to \mathbb{C}^\times$; it's importance is $g_0 \mapsto \Delta(g_0) = c_{g_0}$ [illegible] def

that: $\Delta(g)\, d\mu(g)$ is right Haar measure if μ is left one, this is the usual technique to reduce the proof of modular case to unimodular case (But for compact case, $\forall g \in G$, $\Delta(g) = 1$, thus we can change variable freely)

Lemma 9. $\varphi \in L^1(G, \mathbb{K})$ and invariant under H, i.e. $\varphi(gH) = \varphi(g)$, $f \in C(G, \mathbb{K})$ s.t. $f^\# = 0$, then $\int f\varphi\, d\mu = 0$

Pf. Again we use partition of unity, $\exists \Phi \in C(\frac{G}{H}, \mathbb{K})$: $\Phi(gH) = 1$ for $gH \in p(\mathrm{supp} f)$, and $\exists \Psi \in C(G, \mathbb{K})$: $\Phi = \Psi^\#$ due to the surjectivity $\Rightarrow \Psi^H(g) = 1$ for $g \in \mathrm{supp} f$ and by $f^\# \equiv 0 \Leftrightarrow f^H(g) \equiv 0$, $\forall g \in G \Rightarrow \int \varphi \Psi f^H\, d\mu = 0$

expand the $f^H \Rightarrow \int_G \int_H \varphi(g)\Psi(g) f(gh)\, d\mu(g)\, d\mu_H(h) = 0$ $\quad \int_G \int_H \varphi(gh)\Psi(gh) f(g)\, d\mu(g)\, d\mu_H(h) = \int_G \varphi(g) f(g) \Psi^\#(gH)\, d\mu(g)$

‖ Fubini (due to G, H compact). $\quad$ ‖②

$\int_H \int_G \varphi(g)\Psi(g) f(gh)\, d\mu(g)\, d\mu_H(h) \underset{①}{=} \int_H \int_G \varphi(gh^{-1})\Psi(gh^{-1}) f(g)\, d\mu(g)\, d\mu_H(h)$ $\quad = \int_G \varphi f\, d\mu$ (by definition of $\Psi^\#$)

thus $\int_G f\varphi\, d\mu = 0$, the step ①② the change of variable $g \mapsto gh^{-1}$; $h^{-1} \to h$ is the multiply of modular function, which not occur for compact case ▨

(Hint: You compute all things out, keep mind on multiply the modular function then find property to make equality takes)

Rk. For LCH, what will (P) comes to be? (Exercise), and prove it.

Thm 10. $\exists$ a positive G-invariant measure π on $\frac{G}{H}$, such that: $\int_{\frac{G}{H}} \int_H f(gh)\, d\mu_H(h)\, d\pi(g) = \int_G f(g)\, dg$, $\forall f \in C(G, \mathbb{K})$ (It's up to a constant if the equation holds, then unique)

Pf. Again we use the Riesz Representation thm: $\forall F \in C(\frac{G}{H}, \mathbb{K})$ [illegible]

$\exists f \in C(G, \mathbb{K})$: $f^\# = F$, for $\forall \varphi \in L^1(G, \mathbb{K})$ invariant under H, we define a linear functional $\lambda_\varphi: C(\frac{G}{H}, \mathbb{K}) \to \mathbb{K}$. $\quad F \mapsto \int_G f\varphi\, d\mu$

Let $\varphi \geq 0 \Rightarrow \lambda_\varphi \geq 0 \xRightarrow{\text{Riesz}} \exists \mu_\varphi$ on $\frac{G}{H}$: $\lambda_\varphi(F) = \int_{\frac{G}{H}} F\, d\mu_\varphi(g)$ and $\mu_\varphi \geq 0$

Let $d\mu(g) = d\mu_\varphi(g)$, then we verify the G-invariant: $\forall x \in G \Rightarrow \int_{\frac{G}{H}} f^\#(x^{-1}g)\, d\mu(g) = \int f(x^{-1}g)\, d\mu(g) = \int f(g)\, d\mu(g) = \int f^\# \ldots$

(Rk. For LCH group, it exists $\Leftrightarrow \Delta(h) = \Delta_H(h)$ for $\forall h \in H$) It means that $|\varphi(g)|$ is constant in each coset of $\frac{G}{H} \Rightarrow$ their L^2 ▨

For this part, ref. [KT] E. Kaniuth and K. Taylor, Induced representation of LC groups, Cambridge, 2013

Def. (Induced representation) $\tau: H \to V$ a representation of H, the $\mathrm{Ind}_H^G(\tau) = \{f: G \to V \mid f \in L^2(\frac{G}{H}), f(gh^{-1}) = \tau(h) f(g)\}$

(That's why we need to depict μ on $\frac{G}{H}$: depict $L^2(\frac{G}{H})$), then we build theories as usual (finite case) (for compact, it can be omitted, integral is finite. [illegible] compact support)

Now we start to prove some interesting (big) thms about compact groups G, they are following:

Thm 11. Compact group G, every irreducible representation (π, V) of G is unitary, furthermore, every representation is unitary. Precisely, G compact, (π, H) is continuous representation, then H has inner product itself, $\exists \langle \cdot,\cdot \rangle_1$ s.t. $\langle \cdot,\cdot\rangle_1 \sim \langle \cdot,\cdot \rangle$ (i.e. their norm is equivalent)

Thm 5. (Peter-Weyl) and we can study a set of further consequences (Frobenius reciprocity, ...)

Thm 12. Weyl character formula for Lie group and his maximal tori theory (omitted). After these we introduce some theories related to Arithmetic.

But at first, we consider some examples

E.g 1'. Recall our E.g 1, $G = GL_2(\mathbb{R})$, let $H \leq G$, the group of upper triangle matrices in G.

G unimodular, H modular (due to we only concern compact case, we omit it) $\Rightarrow \frac{G}{H}$ has no G-invariant measure!

And furthermore, $\frac{G}{H} \cong P_{\mathbb{R}}^1$, the Lebesgue measure of $\mathbb{R}$ induces a measure (quasi-invariant), but not G-invariant.

E.g 2. The group D_∞ is the group of rotations and reflections of plane, preserving the origin, i.e. $D_\infty = \{r_\alpha \mid \alpha \in [0, 2\pi)\} \cup \{s\}$, $s^2 = 1$, $s r_\alpha s = r_\alpha^{-1}$

(D_n is similiar: $D_n \cong D_\infty \times I$, left to the reader) I claim. $D_\infty \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, then I give a pure algebraic view:

Just write $D_\infty = \langle r, s \mid srs = r^{-1}, s^2 = 1\rangle \cong \langle rs, s \mid (rs)^2 = 1, s^2 = 1\rangle \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ $\boxtimes$

I find nothing about the Haar measure on a free product, but I think view Rep functorially, $Rep(D_\infty) = \coprod Rep_{i,j}$, due to every representation of D_∞ corresponding to two representations on $\frac{\mathbb{Z}}{2\mathbb{Z}}$, by (rs) and s seprately; the $Rep_{i,j}$ denoting the subset of $Rep\, D_\infty$, restricted to $i \in Rep(\frac{\mathbb{Z}}{2\mathbb{Z}})$ on (rs) and j on s ($i, j \in Rep \frac{\mathbb{Z}}{2\mathbb{Z}}$)

Thus we may reduce our study on $Rep(D_\infty)$ to $Rep(\frac{\mathbb{Z}}{2\mathbb{Z}})$, which is a finite group

The Serre's view to it is straightforward, $\frac{d\alpha}{4\pi} = d\mu(\alpha)$ is the Haar measure, and the integral is computed by factor to adding (r)-component and (sr)-component, i.e. $\int_G f(t)\, dt = \frac{1}{4\pi}\int_0^{2\pi} (f(r_\alpha) + f(sr_\alpha))\, d\alpha$, then the following computation is obvious. (Prop 14)

Pf of Thm 11. Construct $\langle v, w\rangle_1 = \int_G \langle \pi(g)v, \pi(g)w\rangle dg$, it does a Hermittian form, and domination: $\|v\|_1^2 = \int_G \|\pi_g v\|^2 dg \leq \sup_{g\in G} \|\pi_g v\|^2 < \infty$ (compact)

and $\langle \pi(h)v, \pi(h)w\rangle_1 = \int_G \langle \pi(g)\pi(h)v, \pi(g)\pi(h)w\rangle dg = \int_G \langle \pi(gh)v, \pi(gh)w\rangle d(gh)$, thus $\langle\cdot,\cdot\rangle_1$ is unitary

Apply the Banach-Steinhaus thm to $\{\pi_g\}_{g\in G}$, $\forall v \in H$, $\sup_{g \in G} \|\pi_g v\| < \infty$ by the compactness of $G \Rightarrow \sup_{g\in G} \|\pi_g\| < \infty$ $\boxed{R}$. Although equivalent, there may exist non-unitary one

then $\|\cdot\|_1 \leq (\sup_{g\in G}\|\pi_g\|)\|\cdot\|$, then $\|\cdot\|_1$ is complete, then by the norm equivalence $\Rightarrow \|\cdot\|_1 \sim \|\cdot\|$ $\boxtimes$

Then we prove the Thm 5 and

① 's idea: $E = R(G)^\perp$, consider $E = 0$. Although the equivalence holds, we always idenitfy every G-Rep to unitary one

Pf of Thm 5 (Peter-Weyl) by quotients it by $R(f)$, contradicts, but we can't directly say it is unitary. Note, Here $R(G)$ isn't mean $Im(R|_G)$ but defined later

① $R(G)$ dense in $L^2(G)$, ② also $C(G)$

Recall that. $R: C(G) \to B(L^2(G))$

$f \mapsto (R(f): \varphi \mapsto (g \mapsto \int_G f(h)\varphi(gh)\, d\mu(h)))$

and $R|_G(g) = R(g) = (R_g: \varphi \mapsto R_g\varphi)$

Then R (and L) are both unitary regular representation

① unitary: $\|R(g)\varphi\|_2^2 = \int |\varphi(hg)|^2 d\mu(h) = \int |\varphi(h)|^2 d\mu(h) = \|\varphi\|_2^2$

② Continuity: $R|_G: G \xrightarrow{\alpha} B(L^2(G))$
$g \mapsto R_g$

fix $\forall g_0 \in G$ and $\varphi \in L^2(G)$, $\exists \psi \in C(G)$: $\|\varphi - \psi\|_2 \leq \frac{\varepsilon}{3}$

then $\|R(g)\varphi - R(g_0)\varphi\|_2 \leq \|R(g)(\varphi-\psi)\|_2 + \|(R(g)-R(g_0))\psi\|_2 + \|R(g_0)(\psi - \varphi)\|_2$
$\leq 2\|\varphi-\psi\|_2 + \|(R(g)-R(g_0))\psi\|_2 < \frac{2}{3}\varepsilon + \frac{\varepsilon}{3} = \varepsilon$

$\Rightarrow \|(R(g) - R(g_0))\psi\| < \varepsilon$ when g, g_0 suffiencently closed, i.e. $\exists U \ni g, g_0$ and $\forall g \in U \Rightarrow \|(R(g) - R(g_0))\psi\|_2 < \frac{\varepsilon}{3}$

def Continuous rep: $G \times V \to V$ continous $\Leftrightarrow$ $G \to V$ continous, $g \mapsto \rho g v$ for $\forall v \in V$ $\Leftarrow$ $G \to V$ for $\forall g \in G$

Now the $R(G)$ show the finiteness of continous function:

$R(G) = \{\varphi \in C(G) \mid \varphi$ is G-finite$\}$. Thus if φ is invariant under $R(G)$ $\Rightarrow$ φ is G-finite

def G-finite $\Leftrightarrow$ left (or Right) finite $\Leftrightarrow$ $\langle R_g \varphi \rangle_{g\in G}$ is finite-dimensional

$\Leftrightarrow \exists a_i, b_i \in C(G)$; $\varphi(gh) = \sum_{\text{finite}} a_i(g) b_i(h)$. (or $\langle L_g \varphi\rangle$)

The equivalence is easy: the last step is symmetric, and proven by taking basis.

Then $R(G) \leq L^2(G)$ a subalgebra: we just write as the form

$(\varphi + \psi)(gh) = \sum a_i(g)b_i(h) + \sum c_j(g)d_j(h)$, $(\varphi\psi)(gh) = \sum_{i,j} (a_i c_j)(g)(b_i d_j)(h)$

And $R(G)$ invariant under $R|_G$ is clear, furthermore, $R(G)$ invariant under R (L also): $\forall f \in C(G)$, $R(G)$ invariant under $R(f)$: this is due to $\forall \varphi \in R(G)$

$R(f)\varphi(g) = \int_G f(h)\varphi(gh)\,d\mu(h) = \int_G f(h)\sum a_i(g)b_i(h)\,d\mu(h) = \sum \int_G f(h)b_i(h)\,d\mu(h) \cdot a_i(g)$

Then we prove the density: let $E = R(G)^\perp \subset L^2(G)$; $R(G)$ is invariant $\Rightarrow E$ also invariant. Before are the prepare work

consider the operator $R(f^*) \circ R(f)$, where $f^*(g) = \overline{f(g^{-1})}$

denoted T. $T|_E: E \to E$ is positive, selfadjoint

due to $T^* = (R(f^*) \circ R(f))^* = R(f)^* \circ R(f^*)^* = R(f^*) \circ R(f)$.

and $R(f)^* = R(f^*)$ is due to

$\langle R(f)\varphi, \psi\rangle = \int_G R(f)\varphi(g)\overline{\psi(g)}\,d\mu(g) = \int_G \int_G f(h)\varphi(gh)\,d\mu(h)\overline{\psi(g)}\,d\mu(g)$
$= \int_G f(h)\int_G \varphi(gh)\overline{\psi(g)}\,d\mu(g)\,d\mu(h) = \int_G f(h)\int_G \varphi(g)\overline{\psi(gh^{-1})}\,d\mu(g)\,d\mu(h)$
$= \int_G \varphi(g)\int_G \overline{f^*(h)\psi(gh)}\,d\mu(h)\,d\mu(g) = \int_G \varphi(g)\overline{\int_G f^*(h)\psi(gh)\,d\mu(h)}\,d\mu(g)$
$= \int_G \varphi(g)\overline{R(f^*)\psi(g)}\,d\mu(g) = \langle \varphi, R(f^*)\psi\rangle$

I claim. $\|T|_E\| = 0$ i.e. $T|_E \equiv 0$

then $\|R(f)\psi\|_2^2 = \langle R(f)\psi, R(f)\psi\rangle = \langle R(f)^* R(f)\psi, \psi\rangle = 0$

for $\forall \psi \in E \Rightarrow R(f)\psi \equiv 0$, in particular, $R(f)\psi(e) = \int_G f(h)\psi(h)\,d\mu(h)$

i.e. $\langle \psi, f\rangle = 0$, i.e. $\psi \perp f$, for $\forall f \in C(G)$

$C(G)$ dense in $L^2(G) \Rightarrow \psi = 0 \Rightarrow E = \{0\}$

Pf of the Claim. Otherwise $\|T|_E\| > 0$, by $T|_E$ is positive, selfadjoint ① compact operator, it's $\sigma(T|_E) = \sigma_p(T|_E)$ and $|\lambda_1| < \cdots < |\lambda_n|$, $\|T|_E\|$ is one of its point spectrum (eigenvalue), choose a eigenvector ψ. Riesz-Schauder

$\Rightarrow \psi$ also eigenvector of $T = R(f^*)R(f)$

$\Rightarrow \psi \in Im(R(f^*)) \subset C(G)$, and ψ is invariant under R (or L) $\Rightarrow \psi \in R(G) \Rightarrow \psi \in E \cap R(G) = 0$, contradiction

① $T|_E$ is compact is due to $\|R(f)\varphi\| \leq \|f\|\|\varphi\|_2$ ...(A)

thus it map closed ball to a bounded set

and $|(R(f)\varphi)(g_1) - (R(f)\varphi)(g_2)| < \varepsilon \Rightarrow$ equicontinuous

By A-A $\Rightarrow$ it sends closed ball to the relatively sequence set $\Rightarrow T|_E$ compact. By (A), $\|g - g'\|$ ...

The domination $\|R(f)\varphi\| \le \|f\|_1 \|\varphi\|_2$ is due to: $\|R(f)\varphi(g)\| \le \int |f(h)| \, |\varphi(gh)| d\mu(h) \le \|f\|_1 \|\varphi\|_2$ ... $\le \|f\|_1 \int |\varphi(h)| d\mu(h) \le \|f\|_1 \|\varphi\|_2$

The proof of ① is direct, it's detail is non-trivial

② $(R(f) L(g)\varphi)(h) = \int f(k) (L_g \varphi)(hk) d\mu(k)$
$= \int_G f(k) \varphi(g^{-1}hk) d\mu(k) = (R(f)\varphi)(g^{-1}h) = L(g) R(f) \varphi(h)$

For $C(G)$ we have a simpler way: Stone–Weierstrass. The only non-trivial fact to apply S-W is showing that $R(G)$ seperate points: due to the group structure, it suffices to seperate e and $\forall h \in G$ (for g and g' ⇒ reduce to e and $g^{-1}g'$)

$\forall h \in G, \exists U \in N_e : U = eU, hU \in N_h, U \cap hU = \emptyset$.

By partion of unity, $\exists \varphi \in C(G) : \varphi|_U = 0 ; \varphi|_{hU} = 1$

(indeed, it's so-called $T_{3\frac{1}{2}}$ property, we can do due to it's group ⇒ T_4.)

⇒ This part is also type of Peter–Weyl

construct a φ, then let $\psi = L(h^{-1})\varphi$

$\Rightarrow \|R_g\varphi - \varphi\| = \int_{U \cup hU} (\varphi(hg) - \varphi(h))^2 d\mu(h) \ge$ ...

$\Rightarrow R_g \neq Id_{R(G)} \Rightarrow R_g|_{R(G)} \neq Id_{R(G)}$ due to density

thus we can choose such $\varphi \in R(G)$ ☒

③ Recall. Our define $G \times G \xrightarrow{Reg} U(L^2(G))$ unitary
$(g_1, g_2) \longmapsto (\varphi \longmapsto (x \mapsto \varphi(g_2^{-1} x g_1)))$

i.e. $Reg(g_1, g_2) = L_{g_2} R_{g_1} = R_{g_1} L_{g_2}$, is unitary due to both R, L are, and continuous due to R, L are. we call it to be the biregular representation

$\hat{G}$ dual has many equivalent def: Set of characters; Pontryagin dual; $\hat{G} \cong Max(L^1(G))$ by Fourier transform

$\hat{G}$ the unitary dual to G, is the set of isomorphism classes of G's (finite dimensional) irreducible unitary representation

(Due to irreducible + unitary ⇒ finite dimensional)

This proof is "pure algebraic", although not hold in General

Ⅰ Due to the $\bigoplus M(\pi_\sigma) \xrightarrow{\Phi} C(G)$ is injective (by orthogonality), it suffices to prove that $\operatorname{Im}\Phi \subset C(G)$ dense, in L^2-topology (sub) $\Rightarrow \bigoplus_{\sigma \in \hat{G}} M(\pi_\sigma) \cong C(G)$, then by the interwining, we get the $G \times G$-rep formula of Peter–Weyl. we only prove that: the case ① of G admit a faithful rep

Our proof is by: Step 1. $\mathbb{C}[G] \subset C(G)$ dense, L^2-topology

Step 2. Factor $\mathbb{C}[G] : \mathbb{C}[G] = \bigcup_{m \ge 0} Fil_m(\mathbb{C}[G])$ by Grading structure

Step 3. Composite $Fil_m(\mathbb{C}[G]) \longrightarrow M(\pi_\sigma) \xrightarrow{\Phi} C(G)$

then $\mathbb{C}[G] \to \bigoplus M(\pi_\sigma) \xrightarrow{\Phi} C^0(G)$ is just to inclusion and dense ⇒ the second Φ is dense.

Indeed, Peter–Weyl ① ensure this by Portion of unity

Step 1 let $a_{ij} = \mu_\rho(e_j \otimes e_i^*) \in C^0(G)$ the matrix coefficients of $\rho : G \to GL_n(\mathbb{C})$ finite dimensional irreducible ② (such ρ exist due to irreducible + unitary ⇒ finite dimensional)

Due to iso to, we replace by $\rho \cong \rho$ unitary

$\mathbb{C}[G] \subset C(G)$, is the image of $\mathbb{C}[x_{ij}, y_{ij}]_{(i,j) \le (n,n)} \xrightarrow{\Psi} C^0(G)$, $x_{ij} \longmapsto a_{ij} ; y_{ij} \mapsto \bar{a}_{ij}$

by Stone–Weierstrass, $\mathbb{C}[G]$ dense, $\|\cdot\|_\infty$ topology, we can apply the S-W due to ρ is faithful ⇔ seperate $e \neq h$

(① ② hold, both may not hold general!, thus we need to restrict at first)

and $\|\varphi\|_2^2 \le \mu(G) \|\varphi\|_\infty^2 = \|\varphi\|_\infty^2 \Rightarrow \|\varphi\|_2 \le \|\varphi\|_\infty$, by norm-equivalence $\Rightarrow L^2$-dense

Step 2 $\forall m \in \mathbb{Z}_{\ge 0}$, define $Fil_m \mathbb{C}[G]$ be the image of $\mathbb{C}[x_{ij}, y_{ij}]_{(i,j) \le (n,n)}$, $\deg \varphi \le m$

$\Rightarrow \mathbb{C}[G] = \bigoplus_{m \in \mathbb{Z}_{\ge 0}} Fil_m \mathbb{C}[G]$ is obvious (or $\bigoplus$ and graded structure)

and $\forall m \in \mathbb{Z}_{\ge 0}$, $Fil_m \mathbb{C}[G] \subset C(G)$ is R, L invariant, and finite dimensional

Step 3. $R : G \to \mathcal{B}(\mathbb{C}[G])$ restrict to $R_m : G \longrightarrow End(Fil_m \mathbb{C}[G])$
$g \mapsto R_g$ (induce) $g \longmapsto (R_g|_{Fil_m \mathbb{C}[G]})$ (R_m is of ...)

And. $M(R_m) \hookrightarrow C(G)$ can be factored into $M(R_m) \hookrightarrow \bigoplus M(\pi_\sigma) \hookrightarrow C(G)$ of σ to verify:

Also we need prepare some knowledge about martix coefficients. Page 5

Def. Now (ρ, H) is continous-representation of G, $H^* = \mathcal{B}(H, \mathbb{C})$ then $\forall f \in H^*, v \in H$, $g \longmapsto f(\rho_g v)$ is a martix coefficient of ρ, by Riesz ⟺ $g \mapsto \langle \rho_g v, w \rangle$ is a martix coefficient of

the martix coefficient space of an irreducible representation π, $M(\pi)$ is $M(\pi) \subset L^2(G)$ spanned by martix coefficent of π (or $L^2(G, \pi)$)

⟺ the direct sum $M(\pi) = \bigoplus V_\rho$, the direct sum is vector-space-theoritic (i.e. purely algebraic)

The equivalence holds when π is trace class

$(\pi, H) \cong (\pi', H')$ irreducible $(\rho, \psi) \subset (\pi, H)$

We not prove it due to we not use this

This for finite dim it can be identified as a matrix (a_{ij})

i.e. $M(\pi) = \operatorname{Im}(\mu_\pi) \subset L^2(G)$, $\mu_\rho : H \otimes H^* \to$, $v \otimes w^* \longmapsto \langle \rho_g v, w \rangle$

Then for π_1, π_2 (irreducible) $\Rightarrow \begin{cases} M(\pi_1) = M(\pi_2) \text{ iff } \pi_1 \cong \pi_2 \sim \text{Case I} \\ M(\pi_1) \perp M(\pi_2) \text{ iff } \pi_1 \not\cong \pi_2 \sim \text{Case II} \end{cases}$

(You might treat the P-W ③ by language of martix coeffiecent subspace)

Case I, $H_1 \xrightarrow[\cong]{h} H_2$ induce that: $H_1 \otimes H_2^* \xrightarrow{Id \otimes h^*} H_1 \otimes H_1^*$, $h \otimes Id$, μ_{π_1}

$\Rightarrow M(\pi_1) = \operatorname{Im}(\mu_{\pi_1})$
$= \operatorname{Im}(\mu_{\pi_1} \circ (Id \otimes h^*)) = \operatorname{Im}(\mu_{\pi_2} \circ (h \otimes Id))$, $H_2 \otimes H_2^* \xrightarrow{\mu_{\pi_2}} L^2(G)$ ($\pi_1 \circ f = f \circ \pi_2$)
$= M(\pi_2)$

Case II. $M(\pi_1) \xhookrightarrow{i_1} L^2(G) \xrightarrow{p_2} M(\pi_2)$; we need to prove $p_2 \circ i_1 = 0$

this is due to Schur's Lemma: we only to show i_1 and p_2 are $G \times G$ representation-theortic, i.e. intertwining, but it's trivial due to $M(\pi)$ is translation (L and R)-invariant: $M(\pi_1) \hookrightarrow L^2(G) \longrightarrow M(\pi_2)$, $\pi_1 g$, $Reg(g_1, g_2)$, $\pi_2 g$, $M(\pi_1) \hookrightarrow L^2(G) \longrightarrow M(\pi_2)$ ☒

And μ_π induce a map of representation due to μ_π intertwines with $\pi \otimes \pi^*$ and Reg

$\Rightarrow \mu_\pi : \pi \otimes \pi^* \longrightarrow Reg$, and $\pi \otimes \pi^*$ is irreducible. $G \times G$-Rep

Def. We say (ρ, H) is faithful ⟺ if $g \neq e \Rightarrow \rho_g \neq Id_H$ (The upper one: preparation)

due to R_m is finite-dimensional: we decompose it into $\bigoplus M(\pi_\sigma)$, a collection of irreducible representation (canonical). $\sigma \in \hat{G}' \subset \hat{G}$

Then:

$$\begin{array}{ccccccccc} Fil_m \mathbb{C}[G] & \xrightarrow{\iota_m} & M(R_m) & \hookrightarrow & \bigoplus M(\pi_\sigma) & \hookrightarrow & C^0(G) & \xrightarrow{\varepsilon} & \mathbb{C} \\ \varphi & \longmapsto & \mu_{R_m}(\varphi \otimes \varepsilon) & & & & \varphi & \longmapsto & \varphi(e) \end{array}$$

then we get a map $Fil_m \mathbb{C}[G] \to C^0(G)$ composite

Check: it does inclusion map: $\iota_m(\varphi)(g) = \mu_{R_m}(\varphi \otimes \varepsilon)(g) = \varepsilon(R_m(g)(\varphi))$
$= R_m(g)(\varphi)(e) = \varphi(eg) = \varphi(g)$, i.e. $\iota_m(\varphi) = \varphi$

Thus take a sum on this map to complete the proof ☒

Due to the proof, we see that $L^2(G) = \bigoplus_{\rho \in \hat{G}} L^2(G, \rho)$

beautifully (it's a usual type of Peter–Weyl)

Coro 12. (Peter–Weyl again) (A decomposition analogue) (Next are applications)

(π, V) is unitary ⟹ $V = \bigoplus V_\sigma$ is a direct sum of finite-dimensional irreducible representation (as Hilbs)

Pf. We prove by Zorn Lemma $\Sigma = \{(\rho', V') \text{ irreducible subrep } (\rho', V') \text{ pairwise orthogonal}\}$ dominated by inclusion

We show that ① $\Sigma \neq \emptyset$

② Every chain upper bounded: trivial due to we can union over the chain

For ①, take $\langle \cdot, \cdot \rangle$ G-invariant equipped on V Hilbert

let $T_w : V \to C(G)$, $v \mapsto \langle \pi_x v, w \rangle$ ⟹ $\|T_w\| \le \|w\|$ (indeed "=") bounded

and T_w is G-Reg-theortic: $T_w(\pi_x v)(g) = \langle \pi_g \pi_x v, w \rangle = T_w(v)(gx) = (\rho_x T_w(v))(g)$, where ρ is the regular one

and $T_w(w)(e) = \|w\|^2 \neq 0$, thus T_w non-trivial

$T_w(w) \in C(G) \subset L^2(G)$, $\exists \sigma : T_w(w) \subset L^2(G, \sigma)$ by the factor of $L^2(G)$, we project $P_\sigma : V \to L^2(G, \sigma)$ is the desired construct $\in Hom_{G\text{-Rep}}((\rho, L^2(G)), (\rho, L^2(G, \sigma)))$

Thus $P_\sigma \circ T_{ij} \in \mathrm{Hom}_{G\text{-Rep}}(\pi_i, (\rho, L^2(G,\sigma)))$, and $L^2(G,\sigma)$ finite dimensional, identified with a subrepresentation of V

$\Rightarrow \Sigma \neq \emptyset \Rightarrow \exists$ a maximal element $S = ((\pi_1, V_1) \cdots (\pi_i, V_i) \cdots) = (\sigma_\alpha, V_\alpha)_{\alpha \in \Lambda}$

I claim, the finite linear span of S, $\langle S \rangle$ is dense in V ($\langle S \rangle = \{v \mid v = \sum (v_i, v_i \in V_{\alpha_i})\} \Rightarrow V = \widehat{\bigoplus_{\alpha\in\Lambda}} V_\alpha$.

Consider the complement $\langle S \rangle^\perp = V'$, V' closed (by def) and G-invariant $\Rightarrow$ a representation

if $V' \neq 0$, again by $\Sigma \neq \emptyset \Rightarrow V'$ contains a finite-dimensional irreducible subrepresentation W, append W to S, contradiction ☒

Some applications, and more on the induce–Restrict duality (Frobenius).

By Peter-Weyl, it makes sense to define the projection to a irreducible unitary (thus finite-dimensional) representation

Def. (π_1, V_1), (π_2, V_2) are unitary G-rep and π_2 irreducible, let $V_1^{\pi_2}$ to be the lagest subrep of (π_1, V_1), s.t. every subrep irreducible of $(\pi_1, V_1^{\pi_2})$ is equivalent to (π_2, V_2)

Such $V_1^{\pi_2}$ is the π_2-isotypic subspace of (π_1, V_1). Finite dimension $\Rightarrow$ trace and χ well-defined

(By finiteness, the χ_{π_i} is well-defined)

Lemma B (Convolution) (π_1, V_1), (π_2, V_2) two irreducible unitary G-rep, then $\chi_{\pi_1} * \chi_{\pi_2} = \begin{cases} (\dim V_1)^{-1} \chi_{\pi_1} & \pi_1 \cong \pi_2 \\ 0 & \text{otherwise} \end{cases}$

Pf. $(\chi_{\pi_1} * \chi_{\pi_2})(x) = \int_G \chi_{\pi_1}(g)\, \mathrm{tr}\,\pi_2(g^{-1})\pi_2(x)\, dg = \int_G \chi_{\pi_1}(g) \left(\sum_{i,j} \langle v_j, \pi_2(g^{-1}) v_i \rangle \langle \pi_2(x) v_i, v_j \rangle \right) dg = \sum_{i,j} \left(\int_G \chi_{\pi_1}(g) \langle \pi_2(g) v_i, v_j \rangle dg \right) \langle \pi_2(x) v_i, v_j \rangle$

Thus if $\pi_1 \not\cong \pi_2$, by the orthogonality of matrix coefficients, due to χ_{π_1} is a sum (choosing a basis $\{v_i\}$ of V_2) of matrix coefficients $\Rightarrow$ the first integral $= 0$

if $\pi_1 = \pi_2$, replace $\pi_2(g)$ and $\pi_2(x)$ by $\pi_1(g)$, $\pi_1(x)$

$= \sum_{i,j,k} \left(\int_G \langle \pi_1(g) v_k, v_k \rangle \langle \pi_1(g) v_i, v_j \rangle dg \right) \langle \pi_1(x) v_i, v_j \rangle$

$= \sum_{i,j,k} \left(\frac{\langle v_k, v_i \rangle \overline{\langle v_k, v_j \rangle}}{\dim V_1} \right) \langle \pi_1(x) v_i, v_j \rangle$

(and $i \neq j$ vanish, also $i \neq k$, $j \neq k$)

$= \sum_i \frac{\langle v_i, v_i \rangle \overline{\langle v_i, v_i \rangle}}{\dim V_1} \langle \pi_1(x) v_i, v_i \rangle$

$= \sum_i \frac{1}{\dim V_1} \langle \pi_1(x) v_i, v_i \rangle = (\dim V_1)^{-1} \mathrm{tr}(\pi_1(x)) = (\dim V_1)^{-1} \chi_{\pi_1}$ (orthogonal) ☒

Prop 14. (π, V) is unitary G-rep, (σ, W) is an irreducible unitary G-Rep, $V^\sigma \neq 0$, let $\boxed{\bar{e}_\sigma = \dim W\, \overline{\chi_\sigma}}$ $\Rightarrow$ we have a projection $V \to V^\sigma$ given by $\pi(e_\sigma)$

(e_σ called a dirac functional with $(\dim W)^{-1}\chi_\sigma$ directly)

$v \mapsto (\dim W) \int_G \overline{\chi_\sigma(g)}\, \pi(g) v\, dg$

② in particular, for $V = L^2(G)$ regular, $V \to V^\sigma$ is given by convolution.

Pf. ① $e_\sigma(g) = e_\sigma^*(g)$ by definition (recall $f^*(g) = \overline{f(g^{-1})}$) and $e_\sigma * e_\sigma = e_\sigma$ by the upper Lemma B $\Rightarrow \pi(e_\sigma)$ does a projection due to $\pi(e_\sigma^*) = \pi(e_\sigma)^* = \pi(e_\sigma)$

Thus it suffices to show $\mathrm{Im}(\pi(e_\sigma)) = V^\sigma$.

Let (π, V') be any irreducible G-subrep, equivalent to (σ, W), if $(\pi, V') \subset \mathrm{Im}(\pi(e_\sigma))$ and $\pi(e_\sigma * e_\sigma) = \pi(e_\sigma)^2 = \pi(e_\sigma)$

this is due to choosing basis of V', $\{v_i\}$ $\Rightarrow V^\sigma \subseteq \mathrm{Im}(\pi(e_\sigma))$

$\pi(e_\sigma) v_i = \int_G (\dim W) \overline{\chi_\sigma(g)}\, \pi(g) v_i\, dg = (\dim W) \sum_j \left(\int_G \overline{\chi_\sigma(g)} \langle \pi(g) v_i, v_j \rangle dg \right) v_j = (\dim W) \sum_{j,k} \left(\int_G \overline{\langle \pi(g) v_k, v_k \rangle} \langle \pi(g) v_i, v_j \rangle dg \right) v_j$

$= (\dim W) \sum_{j,k} \frac{\langle v_i, v_k \rangle \overline{\langle v_j, v_k \rangle}}{\dim W} v_j = v_i$ (due to $i = j = k$) $\Rightarrow \pi(e_\sigma) v_i = v_i$, i.e. $v_i \in \mathrm{Im}(\pi(e_\sigma))$

Then by the maximality of V^σ, we show that every irreducible subrep of $(\pi, \pi(e_\sigma) V)$ equivalent to σ, i.e. any irreducible G-subrep not equivalent to σ is must in the $\mathrm{Ker}(\pi(e_\sigma))$, for this, again choose basis $\{v_i\}$ of V'

$\Rightarrow \pi(e_\sigma) v_i = \sum_j \int_G (\dim W) \overline{\chi_\sigma(g)} \langle \pi(g) v_i, v_j \rangle dg\, v_j = 0$ ☒

② $f * \bar{e}_\sigma = \rho(e_\sigma) f$ Due to $\sigma \not\cong \pi$, by orthogonality of matrix coefficients $\Rightarrow$ equal to 0

(ρ the representation), and $\bar{e}_\sigma = (\dim W)^{-1} \chi_\sigma$, complete the proof. ☒

$f * \bar{e}_\sigma(x) = (\dim W)^{-1} \int_G f(g) \chi_\sigma(g^{-1}x) dg = (\dim W)^{-1} \int_G f(xg) \chi_\sigma(g^{-1}) dg = (\dim W)^{-1} \int_G \overline{\chi_\sigma(g)} (\rho(g) f)(x) dg = \int_G e_\sigma(g) (\rho(g) f)(x) dg = (\rho(e_\sigma) f)(x)$

Def. $\boxed{\mathrm{Res}_H^G V}$ is the restriction of (ρ, V) to H, i.e. $(\rho_{\mathrm{Res}_H^G V}, \mathrm{Res}_H^G V) = \mathrm{Res}_H^G V$, $H \leq G$ closed, $\rho_{\mathrm{Res}_H^G V} = \rho|_H$

Def. $\boxed{\mathrm{Ind}_H^G W}$ is the induced of (ρ, W) to G, i.e. $(i_{\mathrm{Ind}_H^G W}, W) = \mathrm{Ind}_H^G V$, $H \leq G$ closed, $\mathrm{Ind}_H^G V = \{ f: G \to W$ continuous, $f(gh) = \rho(h)^{-1} f(g) \}$

(It coincides to old one definition, just replace to $C(\frac{G}{H})$ to $L^2(\frac{G}{H})$)

Thm 15. (Frobenius Reciprocity) $H \leq G$ closed, (ρ, V) is G-Rep, (π, W) is H-Rep, then ① $\mathrm{Hom}_{G\text{-Rep}}(V, \mathrm{Ind}_H^G W) \cong \mathrm{Hom}_{H\text{-Rep}}(\mathrm{Res}_H^G V, W)$

i.e. an adjunction pair $(\mathrm{Ind}_H^G \dashv \mathrm{Res}_H^G)$ (continuously)

② The projection formula $\mathrm{Ind}_H^G(\mathrm{Res}_H^G W \otimes V) \cong V \otimes \mathrm{Ind}_H^G W$ (G-Rep-theortic)

(Rk: $\mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G$ is not true, we'd define coInd_H^G to be functor s.t. $\mathrm{Res}_H^G \dashv \mathrm{coInd}_H^G$)

Pf. ① $\mathrm{Hom}_{H\text{-Rep}}(\mathrm{Res}_H^G V, W) = \mathrm{Hom}_{H\text{-Rep}}(V, W) \cong \mathrm{Hom}_{K[H]\text{-Mod}}(V, W) \cong \mathrm{Hom}_{K[G]\text{-Mod}}(V, K[G] \otimes_{K[H]} W) \cong \mathrm{Hom}_{G\text{-Rep}}(V, K[G] \otimes_{K[H]} W)$

I claim, $K[G] \otimes_{K[H]} W \cong \mathrm{Ind}_H^G V$, and here $K[G](K[H]) = L^2(G)(L^2(H))$ (then we also prove ②)

$\mathrm{Ind}_H^G V = L^2(G, W) = L^2(G) \otimes_H W_0 = L^2(G) \otimes_{L^2(H)} W$, the $f(gh) = \rho(h)^{-1} f(g)$ ensure the intertwining ☒

and $\mathrm{coInd}_H^G(W) = \mathrm{Hom}_{H\text{-Rep}}(L^2(G), W)$, in pure categorical nature, It's a Grothendieck's six functor

($f_! \dashv f^! \dashv f_*$, i.e. $\mathrm{Ind} \dashv \mathrm{Res} \dashv \mathrm{coInd}$, induced by $f = i: H \hookrightarrow G$. In Kontsevich, we show it for the sheaf mod categories)

Def. (Hecke algebra)

$\mathrm{Hecke}(G, H) = \mathrm{Hom}_G(\mathrm{Ind}_H^G(1_H), \mathrm{Ind}_H^G(1_H)) = (\mathrm{Ind}_H^G(1_H))^H$, it's a $\mathbb{C}$-algebra

(Ex. Writing out the projection formula for $\mathrm{coInd}_H^G \dashv \mathrm{Res}_H^G$)

(You might be familiar with it's G_1-mod analogue: $\mathrm{Ind}(\mathcal{F}_1 \otimes f^* \mathcal{G}) \cong \mathcal{F}_1 \otimes \mathcal{G}$ ... Ind is the $f_!$ functor and $\mathcal{G}$ locally free finite rank.)