Deformation theory and applications in moduli space

- Counting 27 lines $\subset$ cubic surface X in $\mathbb{P}^3$:

① Moduli problem: $G(2,4) \cong \mathrm{Hilb}_{P(n)=n+1}(\mathbb{P}^3)$ (line $\mathbb{P}^1 \Leftrightarrow$ Hilbert polynomial $P(n)=n+1$)

Counting lines $\subset X$ determined by given equation F of $\mathbb{C}^4$ (for $G(2,4)$) or $\mathbb{P}^3$ (for $\mathrm{Hilb}_{n+1}(\mathbb{P}^3)$):

$G(2,4)(S) = \left\{ \begin{array}{c} \text{vector bundle of rank 2} \subset \mathbb{C}^4 \times S \\ \downarrow \\ S \end{array} \right\} = \left\{ \begin{array}{l} \mathcal{E} \to \mathcal{O}_S^4 \text{ locally free of rank 2} \\ + \text{inclusion of vector bundles, i.e. } \mathcal{E} \hookrightarrow \mathcal{O}_S^4 \twoheadrightarrow \mathcal{F} \to 0 \\ \mathcal{F} \text{ is locally free but } \mathcal{E} \to \mathcal{O}_S^4 \text{ needn't injective} \end{array} \right\}$

$= \{ \mathcal{E} \to \mathcal{O}_S^4 \twoheadrightarrow \mathcal{F} \to 0 \}$

$\boxed{\mathcal{F} \cong p_* \iota^* \mathcal{O}}$ $\cong \left\{ \begin{array}{c} X \xrightarrow{\iota} \mathbb{P}^3 \times S \\ p \downarrow \text{flat} \\ S \end{array} \middle| \begin{array}{l} X_s \subset \mathbb{P}^3 \text{ has Hilbert} \\ \text{polynomial } n+1 \end{array} \right\} = \mathrm{Hilb}_{n+1}(\mathbb{P}^3)(S)$ due to the S-point

$\Leftrightarrow$ parametrize close points by S

$\Leftrightarrow$ subfamily parametrize objects of moduli by S

And the zero ~~section of~~ of the section $s = p_* \iota^* F$ of $\mathcal{F} \Leftrightarrow$ what we need to count.

② Intersection theory: Compute $\# s^{-1}(0)$ by Euler class $c_{top}(\mathcal{F}) = 27 \in \infty$ (pt)

③ Deformation theory: c_{top} is an expected number, we need show that $s^{-1}(0)$ is exactly 27 pts (transversality)

(i.e. got by topological perturbation to transversal)

$\Leftrightarrow$ not fat pts $\Leftrightarrow$ tangent spaces trivial

But $N = \mathcal{O}(-1)$ negative $\Leftrightarrow$ no deformation ☒

These three theories are motivated by this leading example

- Another counting way instead of counting sheaves are counting maps:

$\{\mathbb{P}^1 \to X \mid \deg 1\}$ or $\{\mathbb{P}^1 \to \mathbb{P}^3 \mid \deg 1, F\}$

$\hookrightarrow$ stable maps $\subset$ quasi-maps

Deformation theory

- Deformation of schemes ~~Def(X) = H¹(X, T_X)~~ $\boxed{X \text{ smooth}}$ $Y \subset X$, $\mathrm{Def}(Y) = H^0(Y, N_Y)$ Pf. Do locally ☒

- Deformation of sheaves i) $\mathrm{Def}(\mathcal{E}) = H^1(X, \mathcal{E}nd(\mathcal{E}))$

ii) $\mathrm{Def}(\mathcal{F}) = \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$

Pf. Definition of Ext^1 ☒

- ii) $\Rightarrow$ i) easily when $\mathcal{F}$ locally free

Pf. Do locally, the local/affine deformation are all trivial, we can get nothing. The deformation here should do for gluing data: ~~...~~ on $X \times_{\mathbb{k}} \mathbb{k}[\varepsilon]$ and $\mathcal{E}_0$ is trivial, the gluing data of $\mathcal{E}'$ Čech 1-cocycle modulo gluing data of $\mathcal{E}_0$ Čech 1-coboundary $\Rightarrow \check{H}^1(X; \mathcal{E}nd(\mathcal{E}))$ ☒

The direct proof to i) is delighting: compare with the scheme case it realized a change from "1" to "2": when we deform objects locally and then we deform the gluing/higher descent datum.

H^2/Ext^2 are obstruction, H^n/Ext^n $(n \geq 3)$ are higher obstructions.

- Deformation of schemes $\mathrm{Def}(X) = H^1(X, T_X)$

$\Rightarrow$ infinitesimal automorphism $\boxed{X \text{ smooth}}$

$H^0(X, T_X)$ is generated by vector fields

(compare the mfd smooth)

Pf. • Local is trivial

- Given deformation X' and trivial deformation $X[\varepsilon]$, choose cover $\mathcal{U}$

$X'|_{U_i} \xrightarrow{\varphi_i} X[\varepsilon]|_{U_i}$, deformation is $\{\varphi_i - \varphi_j\} / \{$changes of φ_i to $\varphi_i \circ f\}$

Čech 1-cocycle Čech 1-coboundary of $\check{H}^1(\mathcal{U}; T_X)$

$\cong \check{H}^1(\mathcal{U}; T_X)$ ☒

- Kodaira–Spencer map $\mathcal{X} \to S$, $X = \mathcal{X}_s$

$T_{s,S} \to H^1(X, T_X)$, viewing as the deformation of cpx structure

- Gluing data not uses smoothness, local uses smoothness heavily as the smooth ~~...~~ infinitesimal extension applies to

$$\begin{array}{c} X[\varepsilon] \xleftarrow{\cong} X' \\ \downarrow \text{smooth} \quad \nearrow \text{affine} \\ \mathrm{Spec}\,\mathbb{k}[\varepsilon] \end{array}$$

As the difference is a derivation $\mathrm{Hom}_B(\Omega'_B, I) = \mathrm{Hom}_B(\Omega_B, B)$

$:= \Gamma(\mathrm{Spec}\,B, T_B)$

$0 \to I \to B' \to B \to 0$

$\| \quad \mathrm{id} + \delta \quad \|$ δ is derivation

$0 \to I \to B' \to B \to 0$

$\mathrm{Spec}\,B = X$, $\mathrm{Spec}\,B' = X'$

- Obstruction to higher deformation

E.g. $\mathrm{Spec}\,\mathbb{k} \xrightarrow{(0,0)} X = \{y^2 = x^3\} \subset \mathbb{A}^2$

$\downarrow$

$\mathrm{Spec}\,\frac{\mathbb{k}[\varepsilon]}{(\varepsilon^2)}$ $(\alpha t, \beta t)$

$\downarrow$

$\mathrm{Spec}\,\frac{\mathbb{k}[\varepsilon]}{(\varepsilon^3)}$ $(\alpha t + a_3 t^2, b_2 t + b_3 t^2)$

$(a_2 t + a_3 t^2)^3 \equiv (b_2 t + b_3 t^2)^2 \pmod{t^3} \Rightarrow b_2^2 t^2 \equiv 0 \pmod{t^3} \Rightarrow b_2 = 0$, i.e. the second order must vertical:

On the other hand, if we embed $\operatorname{Spec} k$ to smooth one, the $\operatorname{Spec} k \xrightarrow{(1,1)} X$ has $(a_1 t, b_1 t)$ is the tangent vector

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "(1,1)"] \arrow[d] & X \\ \operatorname{Spec} \frac{k[\varepsilon]}{(\varepsilon^2)} \arrow[ur, "(a_1 t, b_1 t)"'] & \end{tikzcd}$$

(But singular pt $T_{X,0} \cong \mathbb{A}^2$ is flexible)

What it tells? Singularity gives (local) obstruction of the second order deformation. Indeed, the 1st order deformation is always unobstructed.

Now we consider general (Smooth case of above example tells us the 1st deformation is trivial)

Artinian ring: $0 \to J \to A' \to A \to 0$, A' as thicken of A, $m_{A'}$ kills J, called it <u>small</u> if $J = k$

We only consider one setting here, others are similiar:

- Obstruction of vector bundles. ① locally trivial; ② Gluing datum: what we do for 1st deformation is asking whether $\mathcal{E}' \cong \mathcal{E}_0$ by each charts $\mathcal{E}'|_{U_i} \xrightarrow[\cong]{\varphi_i} \mathcal{E}_0|_{U_i}$, but here we are asking the existence of deformation by translation $\mathcal{E}|_{U_{ijk}} \xrightarrow[\cong]{\varphi_{ij}} \mathcal{E}|_{U_{ijk}}$ $\varphi_{ki}^{-1}\varphi_{jk}\varphi_{ij} \in \operatorname{End}(\mathcal{E}|_{U_{ijk}})$

$$\begin{tikzcd} \mathcal{E}|_{U_{ijk}} \arrow[r, "\varphi_{ij}", "\cong"'] & \mathcal{E}|_{U_{ijk}} \arrow[d, "\varphi_{jk}", "\cong"'] \\ & \mathcal{E}|_{U_{ijk}} \arrow[ul, "\varphi_{ki}", "\cong"'] \end{tikzcd}$$

Hence that's: $\{\rho_{ijk} \in \operatorname{End}(\mathcal{E}_0|_{U_{ijk}})\} / \{\psi_{ij} + \varphi_{ij} \mid \text{each } \varphi_{ij}\} = \check{H}^2(X, \operatorname{End}(\mathcal{E}) \otimes J)$

thus our obstruction $\in H^2(X, \operatorname{End}(\mathcal{E}) \otimes J)$

Now, in conclusion, we're asking in such manner: $\boxed{\text{Given an object } Y, \mathcal{E}, \mathcal{F}, X \text{ Sheaves } \cancel{\text{Schemes}} \text{ to be deformed}}$ over $0 \to J \to A' \to A \to 0$

- ① what is its obstruction $\in$ Obstruction space?
- If its obstruction class trivial, then deformation space $= ?$
- Given a deformation family, the automorphism of it $= ?$

And we have

~~Subschemes Vector bundles~~
~~Obs $H^1(X, \ldots)$~~
~~Def $H^1(X, \operatorname{End}(\mathcal{E}))$~~
~~Aut 0~~

	Subschemes	Vector bundles	Sheaves	Schemes
Obs	$H^1(Y, \mathcal{N}_{Y/X} \otimes J)$	$H^2(X, \operatorname{End}(\mathcal{E}) \otimes J)$	$\operatorname{Ext}^2_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes J)$	$H^2(X, T_X \otimes J)$
Def	$H^0(Y, \mathcal{N}_{Y/X} \otimes J)$	$H^1(X, \operatorname{End}(\mathcal{E}) \otimes J)$	$\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes J)$	$H^1(X, T_X \otimes J)$
Aut	0	$H^0(X, \operatorname{End}(\mathcal{E}) \otimes J)$	$\operatorname{Hom}(\mathcal{E}, \mathcal{E} \otimes J)$	$H^0(X, T_X \otimes J)$

Rk1. Aut $= 0$ for subscheme is obvious: two subscheme isomorphic $\Leftrightarrow$ same, as ambient space is strict equality.

Rk2. Subschemes & sheaves & schemes have nontrivial condition on h^i, as otherwise the perturbation will destroy flatness heavily!

Rk3. Obs $\in H^2$ instead of fill all H^2: Def is a G-principal homogeneous space (due to not base pt given) or empty instead of G itself.

E.g. Let's explain $\{\text{Obs}\} \subsetneq H^2$ by conterexamples: line bundle is unobstructed in good cases $\Leftrightarrow$ Picard scheme smooth, as it can be extended by writing as difference of two effective divisors, but $H^2(X, \operatorname{End}(L) \otimes J) = H^2(X, \mathcal{O}_X \otimes J)$ can be nontrivial.

Hence we have $H^2(X, \operatorname{End}(\mathcal{E}) \otimes J) \xrightarrow{tr} H^2(X, \operatorname{Det}(\operatorname{End}(\mathcal{E}) \otimes J))$ trace map (infinitesimal of determinant)

$$ob \longmapsto tr(ob) = 0$$

$\Rightarrow \{obs\} \subset \operatorname{Ker}(tr) =: H^2(X, \operatorname{End}(\mathcal{E}) \otimes J)_{tr} \subsetneq H^2(X, \operatorname{End}(\mathcal{E}) \otimes J)$ the trace-less classes.

Rk4. Now we can answer ①: just as above, ask different question: unique (H^1), existence (H^2) need different gluing data. Sometimes both local (singular) & global (gluing automorphisms) occurs and problem is very complicated. Higher obstruction = Higher gluing datum, but it's harder to explain it! $\boxed{\text{Now, what's "true obstruction"?}}$

Def. An obstruction theory of $0 \to J \to A' \to A \to 0$ for B is V/k, k the residue field of A' and A and (J is k-vector space as $m_{A'} \cdot J = 0$)

$$\varphi: \left\{ \begin{matrix} 0 \to J \to A' \to A \to 0 \\ \uparrow u \\ B \end{matrix} \right\} \longrightarrow V \otimes_k J$$

$$(u) \longmapsto obs(A', u)$$

s.t. ① $obs(A', u) = 0 \Leftrightarrow \exists$ lifting $\tilde{u}: B \to A'$; ② natural.

For B is local, we have canonical smallest obstruction theory $V_B = (I/m_P I)^\vee$, $0 \to I \to P \to B \to 0$, $I \subset m_P^2$

- Check it's obstruction φ: different choices of $\tilde{u}$ have difference a derivation $\Rightarrow f \Leftrightarrow I/m_P I \to J \Leftrightarrow (I/m_P I)^\vee \otimes J$

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & A' \arrow[r] & A \arrow[r] & 0 \\ 0 \arrow[r] & I \arrow[r] \arrow[u, "f"] & P \arrow[r] \arrow[u, "\exists \alpha"] & B \arrow[r] \arrow[u, "u"] & 0 \end{tikzcd}$$

- Smallest: It has geometric meaning $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} P$ with I defining equations, $\operatorname{Spec} P$ regular and always can be extended (infinitesimal lift holds for regular local ring too) for any $0 \to J \to A' \to A \to 0$, the smallest is the defining equation of $\operatorname{Spec} B$ into a smooth space with smallest number of equations. $V_x = \operatorname{Coker}(T_W \otimes k(x) \to E \otimes k(x))$ for $\operatorname{Spec} B = \operatorname{Spec} \mathcal{O}_{X,x}$, $X \hookrightarrow W$ smooth, defined by $s: W \to E$, $X = V(s)$

It's smallest in meaning that $\forall (V, \varphi)$, $V_B \hookrightarrow V$ & commute diagram.

V_B not depend on P: for $0 \to I \to P \to B \to 0$, $0 \to I' \to P' \to B \to 0$, two $V_P \rightleftarrows V_{P'}$ then V_B unique (the proof only uses ①② of (V, φ), not its obstruction to extend)

obstruction to restrict

Deformation as formal local moduli

Due to formal completion reason, we consider pro-representable functors $F: \mathcal{C} \to$ Set $F \cong h_{\hat{C}}$, $\hat{C}$ is pro-object of $\mathcal{C}$, i.e. $\hat{C} = \varprojlim C_i$, $C_i \in \mathcal{C}$, the category completed in $F: \hat{\mathcal{C}} \to$ Set is representable.

Then one can asks, given formal local functor $F: \hat{\mathcal{C}} \to$ Set

- Is F pro-representable $\Leftrightarrow$ $\hat{F}$ representable?
- When F algebraic (on $\mathcal{C}$)?
- Extend to global one? $\to$ harder!

realize as a colimit of $\varprojlim h_{C_i} \cong h_{\varprojlim C_i} \cong \hat{F}$

Take $\mathcal{C}$ the category of Artin local rings and $\hat{\mathcal{C}}$ Artin complete rings, the first question is Schlessinger's criteria.

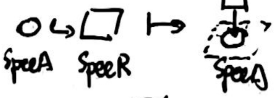
E.g. $\mathbb{P}^1$ has no moduli M, s.t. $\begin{smallmatrix} P \leftarrow \mathbb{P}^1 \\ \downarrow \\ S \end{smallmatrix}$ $\mathbb{P}^1$-bundle $\leftrightarrow \mathrm{Hom}(S, M)$, but its deformation is single pt $\in$ Sch.

Def. A formal family $/R \in \hat{\mathcal{C}}$ is natural transformation $h_R \to F$ ($h_R: \mathcal{C} \to$ Set instead $\hat{\mathcal{C}}$)

$\overset{\text{Yoneda Lemma}}{\Longleftrightarrow} \xi \in \hat{F}(R) \Longleftrightarrow \xi = (\xi_k)$, $F(R/m^k) \leftarrow F(R/m^{k+1})$, $\xi_k = \xi_{k+1}|_{\mathrm{Spec}(R/m^k)} \leftarrow \xi_{k+1}$

The pf of equivalence:

$h_R(A) \to F(A)$ family $/A$ infinitesimal

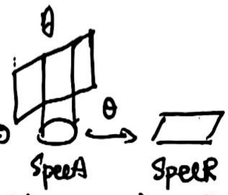


Let $A = R/m^k$ all k, done ▨

Due to nonexistence of global moduli, we should assume weaker correspondence than 1-1

Def. $h_R \to F$ is

- Universal: $h_R \cong F$
- Versal: $h_R \twoheadrightarrow F$ & $h_R \to F$ strongly surjective, i.e. $\forall B \twoheadrightarrow A$, $h_R(B) \twoheadrightarrow h_R(A) \times_{F(A)} F(B)$ $\Leftrightarrow$



We can extend θ to $\mathrm{Spec} B \hookrightarrow \mathrm{Spec} R$ s.t. η family $/\mathrm{Spec} B$

Rk. Versal's pullback is versal $\Rightarrow$ versal can be very big

Versal $\Leftrightarrow$ Formally smooth as $h_R(B) \twoheadrightarrow h_R(A) \times_{F(A)} F(B)$

$$\Leftrightarrow \begin{tikzcd} h_A \arrow[r, "\theta"] \arrow[d] & h_R \arrow[d] \\ h_B \arrow[r, "\eta"] \arrow[ur, dashed, "\exists"] & F \end{tikzcd}$$ lifting property (existence), needn't unique

- Thus universal $\Leftrightarrow$ formally étale $\Leftrightarrow$ isomorphism

- Miniversal: Versal + $h_R(D) \cong F(D)$ for $D = k[t]/(t^2)$

Prop. F versal then $\#F(k) = 1$ and $\forall A' \twoheadrightarrow A$ and $A'' \twoheadrightarrow A$, $F(A' \times_A A'') \twoheadrightarrow F(A') \times_{F(A)} F(A'')$ (Gluing).

Miniversal then $\forall A$, $F(A \times_k D) \xrightarrow{\cong} F(A) \times_{F(k)} F(D)$, let $A = D$, we have ① $F(D)$ is naturally f.d vector space (imagine as tangent space with addition $F(D) \times F(D) \cong F(D \times D) \to F(D)$)

and for $0 \to J \to A' \to A \to 0$ small, $\forall \eta \in F(A)$

② $\exists$ transitive action $F(D) \curvearrowright \{\eta' \in F(A') \mid \eta' \mapsto \eta\}$ (imagine as $H^1 \curvearrowright$ Def) or empty

Universal then $F(A' \times_A A'') \cong F(A') \times_{F(A)} F(A'')$ and the action is principal $F(D)$-space or empty (homogenous)

Rk1. No automorphism $\leftrightarrow$ Universal exists

Automorphism not jump $\leftrightarrow$ Stabilizer of ② exists but not jump $\leftrightarrow$ Miniversal exists

Automorphism jump $\Leftrightarrow$ ①'s vector space destroyed $\leftrightarrow$ Versal only

Rk2. A subtly is $A' \times_A A''$ not tensor not corresponds to geometric pullback/ fibre product, but geometric gluing. But this just what we need now: for example $\frac{k[t_1]}{(t_1^2)} \otimes_k \frac{k[t_2]}{(t_2^2)} = \frac{k[t_1, t_2]}{(t_1^2, t_2^2)}$ isn't what we want as $t_1 t_2$ meaningless, but $\frac{k[t_1]}{(t_1^2)} \times_k \frac{k[t_2]}{(t_2^2)} = \frac{k[t_1, t_2]}{(t_1^2, t_2^2, t_1 t_2)}$ makes sense with all $at_1 + bt_2$ is addition of tangent vectors

Thm (Schlessinger's criteria) Miniversal $\Leftrightarrow \#F(k) = 1$ & Gluing map surj for $0 \to J \to A' \to A \to 0$ small & $F(A \times_k D) \cong F(A) \times_{F(k)} F(D)$

Universal $\Leftrightarrow$ Miniversal & principal homogenous space (② & ①)

E.g. X_0 affine / proj with isolated singularities (otherwise two many sings $\Rightarrow F(D)$ infinite dim)

$\Rightarrow$ Miniversal exists as $F(D) = H^1(T_{X_0})$ + singular part and we can glue schemes

But universal not exists, even when X_0 smooth! One asks we have principal homogenous, why it's not true? This is due to difference of deformation functors: we respect X_0 when we consider its geometric def, but respect $X'|_{A'}/A$ for X'/A' in the above criteria, this is strictly different (later auto strictly modulo less than former)

Universal $\Leftrightarrow \forall X'/A'$, $\mathrm{Aut}_{A'}(X'/X_0) \twoheadrightarrow \mathrm{Aut}_A(X'|_A/X_0) \Leftarrow H^0(X_0, T_{X_0}) = 0$, X_0 smooth

E.g. Def of $g \geq 1$ curves has universal. Pf. $g \geq 2$ by $H^0(X_0, T_{X_0}) = 0$, $g = 1$ by $\mathrm{Aut}_{A'}(X'/X_0) \twoheadrightarrow \mathrm{Aut}_A(X'|_A/X_0) = A' \twoheadrightarrow A$ as elliptic curve (with base) has auto trivial (or more general Abelian Vars)

$\Rightarrow$ Aut = section of $\mathrm{Spec} A$ or A' ▨

Rk 3. Continuing Rk 2. above, given a deformation problems we need choose an equivalence to modulo. The two deformation functors above are modulo X_0 and $X_i'|_A$, fix larger result the equivalence more relaxed and deformation space larger. Different equivalence have their uses respectively, their arose are due to

$$X_1' \cong X_2' \text{ over } X_0 \text{ and } \operatorname{Spec}A' \quad \subset \quad X_1' \cong X_2' \text{ over } X_i'|_A \text{ and } \operatorname{Spec}A'$$

$$\mathrm{Def} \twoheadleftarrow \mathrm{Def}'$$

as if you fix X_0, one can use $\operatorname{Aut}(X_0')$ to justify the second diagram to make $X_i'|_A$ fixed

It's vague to say about "deformation of X" "formal neighbourhood of $x \stackrel{\Delta}{=} [X] \in \mathcal{M}$" when $\mathcal{M}$ is stacky and delicated fix automorphism are used, $\Leftrightarrow x = [X, \operatorname{Aut}(X)]$ to ensure the moduli stack do record enough datum of automorphisms

E.g. Modulo equivalence $X_1' \cong X_2'$ over $\operatorname{Spec}A'$, we get DM stack's formal theory, it means that we fix nothing but only record all automorphism

$\Leftrightarrow$ We have $G \curvearrowright X$ itself instead of X/G (or $X/\!/G \cdots$)

E.g. $\mathbb{Z}/2\mathbb{Z} \curvearrowright \mathbb{A}^2$ with bad pt 0, the result quotient looks like a cone with vertex 0, stack doesn't see the singular 0, $T_0 \cong \mathbb{A}$ but stacky $(0, \mathbb{Z}/2\mathbb{Z})$ the only pt with nontrivial automorphism $\mathbb{Z}/2\mathbb{Z}$

this verifies the <u>hidden smooth principle</u>!

E.g. The quotient stack $[\mathbb{C}^2/\mathbb{C}^*] : S \mapsto \operatorname{Hom}(S, [\mathbb{C}^2/\mathbb{C}^*]) = \{(\mathcal{L}, x, y) \mid \mathcal{L} \in \operatorname{Pic}(S),\ x, y \in H^0(S, \mathcal{L})\}$

The quotient by $\mathbb{C}^*$ here means different trivialization of $\mathcal{L}$ up to a rescaling

Our functor is $F : \mathcal{C} \to \mathrm{Set}$, $F(A) = \{(\mathcal{L}, x, y, \theta) \mid \mathcal{L} \in \operatorname{Pic}(\operatorname{Spec}A),\ x, y \in H^0(\operatorname{Spec}A, \mathcal{L}),\ xy \equiv 0 \bmod m\}/\sim$ modulo equi respecting all structure

$\Rightarrow F(D) = \{(at, bt) \mid a, b \in \mathbb{C}\}/\sim \ \cong \{(x, y) \mid xy \in m \subset A\}/\sim$ $\quad (x_1, y_1) \sim (x_2, y_2) \Leftrightarrow \exists\, h \in 1 + m,\ x_1 = hx_2,\ y_1 = hy_2$

$= \{(at, bt) \mid a, b \in \mathbb{C}\} = \mathbb{C}^2$

(all second order cancelled, $\sim$ is strict equal)

is miniversal but not universal (other tangent space are $\mathbb{C}$ / computation shows in $A = \frac{k[t]}{(t^3)}$, second order term fails)

Now we answer the last two questions, i.e. <u>algebraization</u>: family $\{\mathfrak{X}_n\}$ over A_n $\Leftrightarrow$ $\mathfrak{X} \to \operatorname{Spf}A$ $\Leftarrow$ $X' \to \operatorname{Spec}A$ $\rightsquigarrow$ $X \to S$ finite type $/\Bbbk$

$F_n \Leftrightarrow \hat{F}$ $\quad \boxed{\text{algebraization}}$ not always possible

Thm. (Grothendieck)

- If $\exists\, \mathcal{L}$ on $\mathfrak{X}$, $\mathcal{L}_0 = \mathcal{L}|_{X_0}$ ample $\Rightarrow \mathfrak{X} \to \operatorname{Spf}R$ effective (i.e. $\exists\, X \to \operatorname{Spec}R$ finite type and $\hat{X} \to \operatorname{Spf}R$ s.t. $\hat{L} \cong \mathcal{L}$ and $\hat{X} \cong \mathfrak{X}$)
- (Formal GAGA) $\operatorname{Coh}(X) \simeq \operatorname{Coh}(\hat{X})$, X proper!

E.g. (Noneffective) $X_0 \subset \mathbb{P}^3$ quartic smooth (hyper) surface $\Rightarrow$ K3, Hodge diamond is $\begin{smallmatrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{smallmatrix}$ ($H^2(\mathcal{O}_X)$), 20 abstract def, 19 polarized def / embedded are algebraic $+1$ formal in $\mathbb{P}$

$H^1(\mathcal{O}_{\mathbb{P}^3}(1)|_{X_0}) = 0 \Rightarrow \mathcal{L}_0 = \mathcal{O}_{\mathbb{P}^3}(1)|_{X_0}$ can't deform (i.e. $\operatorname{Pic}(X_0) = H^2(\mathbb{Z}) \cap H^{1,1}(\mathbb{C})$, due to $H^2(\mathcal{O}_X) \neq 0$ obstructed $\Rightarrow$ the integral class $\in \operatorname{Pic}(X_0)$ comes to be not integral)

<u>Claim.</u> $\forall\, \mathcal{L}_0$, $\exists$ direction $X_0 \hookrightarrow X \to \operatorname{Spec}D$, s.t. $\mathcal{L}_0$ can't lift to X $\Rightarrow$ not effective!

<u>Pf.</u> We need deform pair (X, L), controlled by $H^i(T_X)$ and $H^i(\mathcal{O}_X)$, they're entangled as the 1st and constant term of asymptotic expansion, we have bundle of principal part $J_L^1 = p_*(p^*\mathcal{L} \otimes \mathcal{O}_{X \times X}/\mathcal{I}_\Delta^2)$ (or 1st-Jet). It's natural in geometry as no given direction for X, but in $\Delta_X \subset X \times X$, there is a canonical one, but there is some algebraic ambiguity in definition. $\Rightarrow 0 \to \Omega^1 \otimes L \to J_L^1 \to L \to 0$

$\exists$ canonical $\Bbbk$-linear (not $\mathcal{O}_{X \times X}$-linear) section $0 \to \mathcal{I}_\Delta/\mathcal{I}_\Delta^2 \to \mathcal{O}_X/\mathcal{I}_\Delta^2 \to \mathcal{O}_\Delta \to 0$ (imagine it as $f \mapsto f + df = f + f'dx$)

$\Rightarrow \exists$ canonical $\Bbbk$-linear (not $\mathcal{O}_X$-linear) section $0 \to \Omega^1 \otimes L \xrightarrow{\iota} J_L^1 \to L \to 0$ satisfy Leibniz rule

$\Rightarrow$ bijection $\{\nabla : L \to \Omega \otimes L$ algebraic connection$\} \leftrightarrow \{\mathcal{O}_X$-linear section $\varphi : L \to J_L^1\}$

$\nabla \longmapsto \varphi = \sigma - \iota \circ \nabla$

$\sigma - \varphi \longleftarrow \varphi$

$\Leftrightarrow H^1(X, \Omega^1) \cong \operatorname{Ext}^1_{\mathcal{O}_X}(L, \Omega_X \otimes L) \ni a$ the obstruction to algebraic connection, called Atiyah class

Prop. $d\log : \mathcal{O}_X^* \to \Omega_X^1$

$\Rightarrow d\log : H^1(\mathcal{O}_X^*) \to H^1(\Omega_X^1)$

$[L] \mapsto d\log[L] = a$

$H^1(X, \Omega^1) \hookrightarrow H^2(X; \mathbb{C}) \stackrel{\text{Hodge}}{=} H^0(X, \Omega^2) \oplus H^1(X, \Omega) \oplus H^2(X, \mathcal{O}_X)$ send a to $c_1(L)$

$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0 \Rightarrow$ induce LES $\cdots \to H^1(\mathcal{O}_X^*) \xrightarrow{c_1} H^2(\mathbb{Z})$, $\operatorname{Pic}(X) \xrightarrow{c_1} H^2(\mathbb{Z}) \cap H^{1,1}$

$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \cdots$, c_1 in $H^{1,1}$ coincide with a

(analytic topology) $\Rightarrow$ de Rham resolution by comparing the upper and lower rows of the diagram $\square$

Rk. $a \Leftrightarrow$ splitting $\Leftrightarrow$ obstruction of σ be linear $\Leftrightarrow \exists$ nowhere vanishing section of L $\Leftrightarrow c_1(L)$

$\Rightarrow 0 \to \mathcal{O}_X \to \mathcal{H}om(J_L^1, L) \to T_X \to 0 \Rightarrow$ induce LES $\cdots \to H^1(\mathcal{O}) \to H^1(\mathcal{H}om(J_L^1, L)) \to H^1(T_X) \to \cdots$, $H^1(\mathcal{H}om(J_L^1, L)) = \mathrm{Def}(X, L)$

$\to H^0(T_X) \to H^1(\mathcal{O}_X) \to H^1(\cdot\cdot) \to H^1(T_X) \to H^2(\mathcal{O}_X) \to \cdots$ not inj & surj, this can be understood as:
the deformation of L ($H^1(\mathcal{O}_X)$) some part comes from pullback vector field on X, this part is cancelled as it's
& the deformation of X ($H^1(T_X)$) not comes from a pair $\iff$ obstruction of deformation of L along, same as $H^1(T_X)$;

Rk. Due to the non-effectiveness of abstract deformation, we always consider moduli of polarized objects (i.e. given embedding to $\mathbb{P}^n$)

Thm. (Artin) $\hat{X} \to \operatorname{Spec} R$, $\exists\, X \to S$, S finite type $/k$, $s_0 \in S$ has fibre X_0
$\hat{X} = X \times_S \operatorname{Spec} \hat{\mathcal{O}}_{S,s_0}$, and it's étale locally unique.

Cotangent complex $X \xrightarrow{f} Y$ needn't smooth. $\mathbb{L}_{X/Y} \in D^{\le 0}(X)$, we'll control def of $X \to Y$ by $\operatorname{Ext}^{\bullet}(\mathbb{L}_{X/Y}, J)$
[J still data of a small extension $\in D^b(X) \subset D^+(X)$, $\mathbb{L} \in D^{\le 0}(X) \subset D^-(X) \Rightarrow \operatorname{Hom}^\bullet$ defined
& J is injective (ideal) in $D^+(X)$ $\Rightarrow \operatorname{Ext}^\bullet$ defined]

Recall:
① $\operatorname{Ext}^1(\mathbb{L}_{X/Y}, J) = H^1(R\mathcal{H}om(\mathbb{L}_{X/Y}, J)) = H^0(R\mathcal{H}om(\mathbb{L}_{X/Y}, J[1])) = \operatorname{Hom}_{D(X)}(\mathbb{L}_{X/Y}, J[1])$
tensor is similar. $\operatorname{Tor}^1(E,F) = H^1(E \overset{L}{\otimes} F) = H^0(E \overset{L}{\otimes}_{\mathcal{O}_X} F[1])$
② $\boxed{?} \to I/I^2 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to 0$ if f closed embedding
$\boxed{?} \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$ if f general

Fact. ① $\mathbb{L}_{Y/S} \to \mathbb{L}_{X/S} \to \mathbb{L}_{X/Y} \xrightarrow{+1}$ is distinguished triangle in $D^{\le 0}(X)$;
② $H^0(\mathbb{L}_{X/Y}) = \Omega^1_{X/Y}$ (thus we have second SES), by the first one, if f closed embedding, $\mathbb{L}_{X/Y} \in D^{\le -1}(X)$ as $\Omega^1_{X/Y} = 0$)
③ $X \to Y$ étale (e.g. open embedding), $\mathbb{L}_{X/Y} = 0$;
④ $X \to Y$ smooth, $\mathbb{L}_{X/Y} = \Omega^1_{X/Y}$ centred at 0;
⑤ $X \to Y$ is l.c.i $\overset{def}{\iff} \exists W$, $X \xrightarrow[I]{regular} W \xrightarrow{smooth} Y \overset{Quillen}{\iff} \mathbb{L}_{X/Y}$ has flat resolution in finite length
$\iff \mathbb{L}_{X/Y} = [I/I^2 \to \Omega^1_{W/Y}|_X]$ length 2 (independent W, up to q.is)
Thus when singular, any flat resolution is infinite!
Luckily, we here only use the first three terms of $\mathbb{L}_{X/Y}$, i.e. $\tau^{\ge -1}\mathbb{L}_{X/Y} =: \mathbb{L}^{\ge -1}_{X/Y}$ called naïve cotangent cpx
⑥ $X \xrightarrow[closed]{} W \xrightarrow[smooth]{} Y$, $\mathbb{L}^{\ge -1}_{X/Y} = [I/I^2 \to \Omega_{W/Y}|_X]$ $= [L^{-1}/d(L^{-2}) \to L^0]$ good truncation
$\Rightarrow X \xrightarrow[closed]{I} Y$, $\mathbb{L}^{\ge -1}_{X/Y} = (I/I^2)[1]$ concentrated at (-1)
To prove them, simplicial definition to $\mathbb{L}_{X/Y}$ is unavoidable, we omit! Then we use it to deformation:

Deformation of Maps

Problem
$$\begin{tikzcd} X \arrow[r, phantom, "F"] & \\ T \arrow[u, "f"] \arrow[r, hook, "J"] & T' \arrow[ul, dashed, "?\tilde f"'] \end{tikzcd}, \quad J^2 = 0 \iff \begin{tikzcd} X\times \operatorname{Spec} A' & X \times \operatorname{Spec} A \arrow[l] \\ T' \arrow[u, dashed, "?\tilde F"] \arrow[d] & T \arrow[l, hook] \arrow[u, "F"] \arrow[d] \\ \operatorname{Spec} A' & \operatorname{Spec} A \arrow[l, hook] \end{tikzcd}, \quad J^2 = 0$$
equivalent to extension as old
(Here we use $T \hookrightarrow T'$ introduce additional obstruction of gluing datum of T, if T not affine)

- no automorphisms
- deformation / extension is pseudo-torsor of $\operatorname{Hom}_T(f^*\Omega^1_X, J) = \operatorname{Ext}^0_T(f^*\mathbb{L}_X, J)$
- obstruction $\in \operatorname{Ext}^1_T(f^*\mathbb{L}_X, J)$

Pf. ③ Obstruction is given by: $f^*\mathbb{L}_X \to \mathbb{L}_T \to \mathbb{L}_{T/X} \xrightarrow{+1}$ $\Rightarrow (f^*\mathbb{L}_X \to J[1]) \in \operatorname{Hom}_T(f^*\mathbb{L}_X, J[1])$
$= \operatorname{Ext}^1_T(f^*\mathbb{L}_X, J)$
is the obstruction class $\boxed{//}$

Why? $i^*\mathbb{L}_{T'} \to \mathbb{L}_T \to \mathbb{L}_{T/T'} \xrightarrow{+1}$, $\mathbb{L}_{T/T'} \to \mathbb{L}^{\ge -1}_{T/T'} \cong J[1]$

Rk. If $\exists W$, $X \overset{I}{\hookrightarrow} W$ smooth $\Rightarrow$
$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ T' \arrow[r, dashed, "g"] \arrow[ur, dashed, "\tilde f"] & W \end{tikzcd}$$
(g may not exist if T not affine)
$\Rightarrow I/I^2|_T \xrightarrow{\varphi} \Omega^1_W|_T \to \Omega^1_X|_T \to 0$, $\Omega^1_W|_T \xrightarrow{g^*} J$

$X \xrightarrow{\pi} S$
$\Rightarrow \pi^{-1}\mathcal{O}_S[\mathcal{O}_X] \to \mathcal{O}_X$
$\Rightarrow (X, \mathcal{O}_X) \hookrightarrow (X, \pi^{-1}\mathcal{O}_S[\mathcal{O}_X]) =: W$

Rk. When T not affine, we take W the affine space $/T$;
When T not affine, take W (not scheme but ringed space) as the bar resolution of X

In derived category, $[g^*] = 0$
$\Downarrow$
We can adjust g^* to make α to be 0, and we can extend

Recall that: $T' \xrightarrow{\tilde f} X$ exists $\iff \Omega_W|_T \to J$ exists
Now $[I/I^2|_T \to \Omega^1_W|_T] = \mathbb{L}_X|_T = f^*\mathbb{L}_X$, $[J \to 0] = J[1]$, $(\Omega^1_W|_T \to J) \in \operatorname{Ext}^1_T(f^*\mathbb{L}_X, J)$
then by a fact of homological algebra $\Rightarrow \exists \alpha$, $\alpha \circ \varphi + g^* = 0$
thus iff, we can extend, i.e. α is obstruction class, the obstruction
existence of $W \iff T$ not affine
existence of $g \iff X$ singular
~~we can adjust g^* to make~~
$[g^*]$ = obstruction class $= 0$

Local-to-global SS is the Grothendieck SS of composite functor $\Gamma(\mathcal{E}xt) = \operatorname{Ext}$ here, it explains the two part of obstruction:
(From local to global also in geometry)
Ext/Tor ($\mathcal{E}xt/\mathcal{T}or$) Set here our SS to be $\downarrow\!\to$ direction, read the red part with convergence to $\operatorname{Ext}^1_T(f^*\mathbb{L}_X, J)$ and its filtration $\Rightarrow$

$H^0(\mathcal{E}xt^2)$	$\cdots$	$\cdots$
$H^0(\mathcal{E}xt^1)$	$H^1(\mathcal{E}xt^1)$	$\cdots$
$H^0(\mathcal{E}xt^0)$	$H^1(\mathcal{E}xt^0)$	$H^2(\mathcal{E}xt^0)$

$0 \to H^1(T, \mathcal{E}xt^0_T(f^*\mathbb{L}_X, J)) \to \mathrm{Ext}^1_T(f^*\mathbb{L}_X, J) \to \mathrm{Ker}(H^0(T, \mathcal{E}xt^1_T(f^*\mathbb{L}_X, J)) \to H^2(T, \mathcal{E}xt^0_T(f^*\mathbb{L}_X, J))) \to 0$

global obstruction (gluing pieces of T is $H^1(T, -)$) — local obstruction, $H^0(T, \mathcal{E}xt^1_T(f^*\mathbb{L}_X, J))$ (If $T = \mathrm{Spec}\, k \hookrightarrow \mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2) = T'$, $\mathcal{E}xt^1$ is local obstruction, contribute by X singular)

if local obstruction $= 0$, we can lift to global obstruction by this SES, coincide with geometric insight.

thus: ① If local obstruction $\neq 0$, can't lift, done; (locally, $H^i(\text{affine}) = 0$ for $i > 0$, only $i = 0$ considered)

② If local obstruction $= 0$, lift to global obstruction, if $\neq 0$, obstruction exists: two $U_i, U_j \subset T$ local extension of U_i, U_j has overlap in $U_i \cap U_j \Rightarrow$ can't glued to T

③ If $= 0$ and lift to $0 \Rightarrow$ no obstruction $\Leftrightarrow \mathrm{Ext}^1_T(f^*\mathbb{L}_X, J) = 0$ $\boxed{/}$

Deformation of schemes

Similarly for $\mathrm{Ext}^2_X(\mathbb{L}_X, J)$, we use the local-to-global SS in $H^2(\mathcal{E}xt^0)$ global, $H^1(\mathcal{E}xt^1)$, $H^0(\mathcal{E}xt^2)$ local, converge to $\mathrm{Ext}^2_X(\mathbb{L}_X, J)$

$H^0(\mathcal{E}xt^2) = $ local obstruction $= \mathcal{E}xt^2$

$H^1(\mathcal{E}xt^1) = H^1 + \mathcal{E}xt^1 \triangleq$ Gluing (isomorphic) local deformation / extension

$H^2(\mathcal{E}xt^0) = H^2 + \mathcal{E}xt^0 = $ Cocycle condition + different gluing method of local deformation

Problem $X \overset{I}{\hookrightarrow} X'$, $\pi \downarrow$ flat $\downarrow$, $S \hookrightarrow S'$, $J^2 = 0$; $X' \to S'$ flat $\Leftrightarrow$ the I is pullback of J, thus we always acquire $X' \to S'$ flat (check it locally): $I = \pi^* J$

- Automorphism $= \mathrm{Ext}^0_X(\mathbb{L}_{X/S}, \pi^* J)$
- Deformation $= \mathrm{Ext}^1_X(\mathbb{L}_{X/S}, \pi^* J)$ — pseudo torsor
- Obstruction class $\in \mathrm{Ext}^2_X(\mathbb{L}_{X/S}, \pi^* J)$

Locally, it's $0 \to N \to B' \to B \to 0$, given $I \to N$ map of A-modules and N is B-module (without any flatness), as if $A \to B$ flat, then so is $A' \to B'$)

(Not affine we use bar resolution) $0 \to I \to A' \to A \to 0$, $I^2 = 0$

Lemma. Given two such problems $\Rightarrow 0 \to N_2 \to B_2' \to B_2 \to 0$, it's a problem of extension of maps

$0 \to N_1 \to B_1' \to B_1 \to 0$
$0 \to I_1 \to A_1' \to A_1 \to 0$ $\Rightarrow$ obstruction class $\in \mathrm{Ext}^1_{B_1}(\mathbb{L}_{B_1/A_1}, N_2)$

Thus deformation is proven: by definition of torsor, one compare two problems $\in \mathrm{Ext}^1$, surjective need. Similar to Extension of modules

Our advantage to move out the flatness is if $I = 0$, then the trivial extension $N \oplus B$ always exists, but highly non-flat

$\Rightarrow 0 \to N \to B' \to B \to 0$; The existence of B' ($\Rightarrow$ base is given $\Rightarrow$ not only torsor but iso)

$0 \to I \to A' \to A \to 0$ $\Leftrightarrow$ given $\xi \in \mathrm{Ext}^1_A(\mathbb{L}_{A/A'}, I)$, the existence of map $A' \to B'$ (view the upper two row both problem over lower one!)

$0 \to 0 \to A' \to A' \to 0$ $\Leftrightarrow$ given $\xi \in \mathrm{Ext}^1_A(\mathbb{L}_{A/A'}, I)$, $\exists \xi' \in \mathrm{Ext}^1_B(\mathbb{L}_{B/A}, N)$

$\Leftrightarrow$ given $\xi \in \mathrm{Ext}^1_A(\mathbb{L}_{A/A'}, I)$ ~~...~~ $\exists \xi' = \xi''$ ~~...~~ canonical and $\xi \mapsto \xi''$

$0 \to N \to B' \to B \to 0$ $\}\xi'$ ~~...~~ $\mathrm{Ext}^1_B(\mathbb{L}_{B/A}, N)$ map to $\xi' = B'$ ~~...~~ B' by pushout ~~...~~

$0 \to N \to \boxed{B} \to A \to 0$ (if $\xi' = \xi''$, then it's extension $/ 0 \to I \to A' \to A \to 0$ not only $/ 0 \to 0 \to A' \to A' \to 0$)

$0 \to N \to \boxed{B} \to A \to 0$ $\}\xi'' \longleftarrow \xi$ all are extension $/ 0 \to 0 \to A' \to A' \to 0$

$0 \to I \to A' \to A \to 0$

$\Leftrightarrow \mathrm{Ext}^1_B(\mathbb{L}_{B/A'}, N) \xrightarrow{\rho} \mathrm{Ext}^1_A(\mathbb{L}_{A/A'}, N) \leftarrow \mathrm{Ext}^1_A(\mathbb{L}_{A/A'}, I)$

Thus:

- Deformation is $\rho^{-1}(\xi'')$, $\forall \xi'' \in \mathrm{Ext}^1_A(\mathbb{L}_{A/A'}, N)$, i.e. pseudo torsor of $\rho^{-1}(0)$ ($\xi' \mapsto \xi'' \longleftarrow \xi$)
- Obstruction class $\in \mathrm{coker}(\rho)$

Then by $\mathbb{L}_{A/A'} \otimes B \to \mathbb{L}_{B/A'} \to \mathbb{L}_{B/A} \xrightarrow{+1} \Rightarrow$ LES: $\cdots \to \mathrm{Ext}^1_B(\mathbb{L}_{B/A'}, N) \xrightarrow{\rho} \mathrm{Ext}^1_B(\mathbb{L}_{A/A'} \otimes B, N) \to \mathrm{Ext}^2_B(\mathbb{L}_{B/A}, N) \to \cdots$

$\mathrm{Ext}^1_B(\mathbb{L}_{A/A'} \otimes B, N) \overset{\text{HS}}{=} \mathrm{Ext}^1_A(\mathbb{L}_{A/A'}, N)$

$\Rightarrow \mathrm{coker}(\rho) \subset \mathrm{Ext}^2$, then ~~locally~~ done $\boxed{/}$

~~(Globally, ...) gluing pieces of ... ~~
~~then use local-to-global SS.~~

Extension of sheaves $\mathcal{F}$ ($\mathcal{G}$ dominates kernel)

Problem. $X \overset{I}{\hookrightarrow} X'$, $\mathcal{F}$, $\mathcal{G}$, given $I \otimes \mathcal{F} \xrightarrow{c} \mathcal{G}$, then $\exists ?\ \mathcal{F}' / X'$, s.t. $0 \to \mathcal{G} \to \mathcal{F}' \to \mathcal{F} \to 0$

- Automorphism $= \mathrm{Ext}^0_X(\mathcal{F}, \mathcal{G})$
- Deformation $= \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{G})$ — pseudo torsor
- Obstruction class $\in \mathrm{Ext}^2_X(\mathcal{F}, \mathcal{G})$ Pf. Same (omit) $\boxed{/}$

Stacks

Recall the $Isom(S_1, S_2) = \{(s_1, s_2, \varphi) \in S_1 \times S_2 \times Isom(M_{s_1}, M_{s_2})\} \xrightarrow{ét} S_2$ ← scheme

$$\begin{array}{ccc} & & S_2 \\ \downarrow ét & & \downarrow \\ S_1 & \to & M \end{array}$$

scheme

scheme

Scheme ⟷ Functor + Gluing / Algebraic condition
Yoneda
!!

but all these (gluing) maps are étale than Zariski
⟹ M not scheme in general!

Samely for sheave / big Zariski site
algebraic / over / space, weak latter; for stack, change target & weak latter

• Algebraic space: $F: Sch^{op} \to Set$ s.t. sheaf over big étale site & $\exists$ U scheme, $h_U \to F$ étale & $\Delta_F: F \to F \times F$ representable

E.g. (Algebraic space but not scheme)

Construction: $X = \mathbb{P}^1 \times C$, C is torus, gluing $\mathbb{P}^1$ into cubic nodal curve = ⊘
the gluing point set the fibre $C_0 \xrightarrow{\sim} C_\infty \rightsquigarrow \tilde{X}$
$x \mapsto x+a$, a has infinite order in C as group (irrational, ergodic)

densed / ergodic of orbit • $C \curvearrowright C$ by "+a"

⟹ due to irrationality, the gluing not algebraic
⟹ $\tilde{X}$ not scheme;

But $\tilde{X}$ is algebraic space by gluing two étale covers by
$$\begin{array}{ccc} ? & \to & X \\ \downarrow & \square & \downarrow ét \\ X & \xrightarrow{ét} & \tilde{X} \end{array}$$
gluing two base $\mathbb{P}^1$ by "banana" (does algebraic curve)

the result glued space has $\mathbb{Z}/2\mathbb{Z}$-action / involution without fixed points
⟹ it's double cover to $\tilde{X}$ / $\tilde{X}$ is its quotient ☒

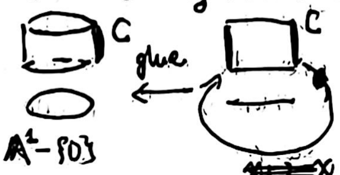
with x glued to x (0 to 0)
x+a glued to x+a (∞ to ∞)

The quotient is by functor $[X/G]$
$S \mapsto \{G \curvearrowright P \to X, P \downarrow S\}$, as $Hom(S, [X/G]) = \{S \to [X/G]\}$
$$\cong \left\{ \begin{array}{ccc} S & \to & [X/G] \\ \uparrow & \square & \uparrow \\ P & \to & X \end{array} \right\}$$
P is G-torsor / S, i.e.
$$\begin{array}{ccc} G \times P & \xrightarrow{a} & P \\ pr \downarrow & \square & \downarrow \\ P & \to & S \end{array}$$

E.g. (Stack but not algebraic space)
$\widehat{DM}$

M_g isn't algebraic space as there $\exists$ auto, i.e. $\exists$ family nontrivial

(P → S base change to P ⟺ find $(s, g) \in P_S$ for each fibre a preferred element
⟺ globally every fibre are G trivial after base change
⟹ Pullback is $G \times P$

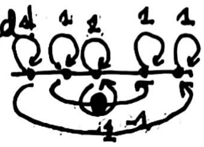


$\mathbb{A}^1 - \{0\} = \mathbb{C}^\times$

$\varphi \in Aut(C) - \{e\}$, $g(C) \le 1$

Rk: DM stack doesn't have infinitesimal auto
⟺ finite and reduced
but finite ⇏ reduced in char $= p \neq 0$

Abstract nonsense of stacks: fibred groupoids ... → omitted

E.g. (What's stack?) ① $[\mathbb{A}^1 / (\mathbb{Z}/2\mathbb{Z})] =$  $\mathbb{A}^1$, Cartesian square

$$\begin{array}{ccc} \mathbb{A}^1 \times \mathbb{Z}/2\mathbb{Z} & \to & \mathbb{A}^1 \\ \downarrow & & \downarrow 2:1 \\ \mathbb{A}^1 & \to & [\mathbb{A}^1 / G] \end{array}$$

étale 2-to-1 even in 0
coarse
$\mathbb{A}^1 / G$.
(0 is "1-to-1/2")

② $[*/G]$: $S \mapsto \{G \curvearrowright P \to *, P \downarrow S\}$
$= \{G \curvearrowright P \downarrow S\}$ ⟹ $[*/G] = BG$

Coherent with algebraic topology: $BG = EG / G$, EG contractible $\simeq *$
⟹ $BG \simeq * / G$