(SN: Seminar notes by myself)

No.

Lagrangian correspondence and the geometry of cotangent bundle.

We admit the basic setting of submfds in §3.1.

Set X^n is $\mathbb{R}$-mfd and $(M=T^*X,\omega)$ is symplectic mfd with canonical symplectic form $\omega = -d\alpha$, α the Liouville 1-form.

Def1. For any symplectic mfd (N^{2n},ω), $Y\subset N$ is isotropic submfd if its submfd and $\forall p\in Y$, $\omega_p|_{T_pY}=0$; is Lagrangian if it's maximal isotropic, i.e. $\dim Y = n$ $\quad$ ($\omega_p: T_pM\times T_pM\to\mathbb{R}$ restrict to T_pY is 0)

Now equip the product $N_1\times N_2$ of two symplectic mfds (N_1^n,ω_1) and (N_2^n,ω_2) a (twisted) symplectic form $\widehat{\omega}=p_1^*\omega_1 - p_2^*\omega_2$

• It does symplectic: $\omega_\lambda = p_1^*\omega_1+\lambda p_2^*\omega_2$, $\forall\lambda\in\mathbb{R}$ is symplectic: closed is obvious by linearity; non-degenerate $\omega_\lambda^{2n} = \binom{2n}{n}(p_1^*\omega_1)^n\wedge\lambda^n(p_2^*\omega_2)^n$

(Here $-^{2n}$ means $2n$ times exterior product) $\quad = \binom{2n}{n}\lambda^n(p_1^*\omega_1)^n\wedge(p_2^*\omega_2)^n \neq 0$ ▨

Thm1. $\varphi: N_1^n \xrightarrow{\text{diffeo}} N_2^n$ is symplectomorphism

$\Leftrightarrow$ its graph $\Gamma_\varphi\subset(N_1\times N_2,\widehat{\omega})$ is Lagrangian

Pf. Lagrangian $\Leftrightarrow \widehat{\omega}|_{\Gamma_\varphi}=0 \Leftrightarrow p_1^*\omega_1|_{\Gamma_\varphi} = p_2^*\omega_2|_{\Gamma_\varphi}$

$$\begin{tikzcd} \Gamma_\varphi \subset N_1\times N_2 \arrow[d, "id"'] \arrow[d, "p_1"] \arrow[dr, "p_2"] & \\ N_1 \arrow[r, "\varphi"] & N_2 \end{tikzcd}$$

By the diagram $\Rightarrow p_1^*\omega_1|_{\Gamma_\varphi} = \omega_1$

$p_2^*\omega_2|_{\Gamma_\varphi} = \varphi^*\omega_2$.

thus $p_1^*\omega_1|_{\Gamma_\varphi} = p_2^*\omega_2|_{\Gamma_\varphi} \Leftrightarrow \omega_1=\varphi^*\omega_2$ ▨

(Restriction is pullback of inclusion)

$\boxed{\text{Weinstein: everything is Lagrangian mfd.}}$

Lagrangian correspondence is a general technique in symplectic geometry developed by Weinstein mainly: correspondence of N_1 to N_2 is a "homology class" $\alpha = \sum \alpha_i[V_i]\in H_{\dim N_2}(N_1\times N_2)$, $V_i\subset N_1\times N_2$ (algebraic cycle)

If V_i are Lagrangian it's called Lagrangian correspondence.

• It's simply extension of morphism $N_1\xrightarrow{f}N_2$ as $[\Gamma_f]$ is a correspondence.

the definition is not rigorous as it's defined in Chow group, here is only an explaination. Similiarly we can define composition $N_1 \xrightarrow{[V]} N_2 \xrightarrow{[W]} N_3$ of $[V]$ and $[W]$ as $[V\circ W]$ and can used as morphism of a suitable category

Weinstein's symplectic category

① $V\times_{N_2} W \longrightarrow W$

$N_1\times N_3 \to N_2\times N_3$

$V \hookrightarrow N_1\times N_2 \to N_2$

$(N_1\times N_2)\times_{N_2}(N_2\times N_3) = N_1\times(N_2\times_{N_2} N_2)\times N_3 = N_1\times N_3$ pt

② $V\times_{N_2} W \subset N_1\times N_3$ can be depicted by drawing a picture

$(W\times N_1)\cap(V\times N_2) \to V\times_{N_2} W$

$N_1\times N_2\times N_3 \to N_1\times N_3$

③ $W\times V \subset N_1\times N_2\times N_2\times N_3$

$N_1\times\Delta_{N_2}\times N_3 \to N_1\times N_3$

$(W\times V)\cap(N_1\times\Delta_{N_2}\times N_3) \to W\times_{N_2} V$

Q. The composite of Lagrangian correspondence is also Lagrangian?

Another example of Lagrangian correspondence is the symplectic reduction ... will be introduced later.

Now consider the cotangent bundle $(M=T^*X, \omega=-d\lambda)$, what is its Lagrangian?

Observation: zero section and fibres are Lagrangian

Consider which section $X_\mu = \mathrm{Im}(s_\mu: X\to T^*X$ section$)$ Lagrangian, here

Thm2 ① $s_\mu^*\lambda=\mu$ the section $s_\mu \leftrightarrow$ 1-form $\mu\in\Omega^1(X)$

② X_μ Lagrangian $\iff \mu$ is closed 1-form.

Pf. ① (Explains why λ is called tautological)

It's almost definition: $X \underset{\pi}{\overset{s_\mu}{\rightleftarrows}} T^*X = M$

$(s_\mu^*\lambda)_x = (ds_\mu)^*_x\lambda_{s_\mu(x)} \overset{\text{definition of }\lambda}{=} (ds_\mu)^*_x(d\pi_{s_\mu(x)})^*\mu_x$ (Denote $s_\mu: x\mapsto(x,\mu_x)$)

Here $\lambda\in\Omega^1(M)$ $= T^*T^*X$ (Here identify $\Omega^1(X)=T^*X\ni y$)

$= (d(\pi\circ s_\mu))^*_x\mu_x$

$= (d\,\mathrm{id}_X)^*_x\mu_x = \mu_x$ ∎

$T^*T^*X \to T^*X$

$T^*X \leftarrow T^*(T^*X)$

$\Omega^1(X) \xleftarrow{s_\mu^*} \Omega^1(T^*X)$

$s_\mu^* = (ds_\mu)^*$ if we identify $X \xrightarrow{s_\mu} T^*X$

② X_μ Lagrangian $\iff \omega|_{X_\mu}=0 \iff d\lambda|_{X_\mu}=0 \iff s_\mu^*(d\lambda)=0$

$\iff d(s_\mu^*\lambda)=0 \iff d\mu=0$ ∎ (Thus zero section is Lagrangian)

Another example is the conormal bundle ($\Rightarrow$ fibre is Lagrangian)

Thm3. $N^*S\subset T^*X$ is Lagrangian

Pf. $\omega|_{N^*S}=0 \iff d\lambda|_{N^*S}=0 \Leftarrow \lambda|_{N^*S}=0 \cdots (*)$ $\quad \lambda=\sum \xi_i dx_i$

We prove (*): We can take local chart of T^*X $(x_1,\dots,x_n,\xi_1,\dots,\xi_n)$, s.t. $S\subset X$ is cutted by $x_{k+1}=\dots=x_n=0$ and thus $\xi_1=\dots=\xi_k=0$ at fibres $N^*_sS\subset T^*_sX$

The composition only works when W and V intersects cleanly/transversally

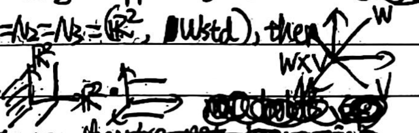
more precisely, $(W\times N_1)$ and $(V\times N_2)$ — the intersection is embedded and

Problem in general is not embedded but immersed

$T_p((W\times N_1)\cap(V\times N_2)) \cdots (*)$
$= T_p(W\times N_1)\cap T_p(V\times N_2)$

Eg. let $M=N_2=N_3=(\mathbb{R}^2,\omega_{std})$, then

the case



not happens as in $(\mathbb{R}^2\times\mathbb{R}^2, \omega^1_{std}-\omega^2_{std})$ immersed case.

thus recent developments by Fukaya is directly to

We define the conormal bundle of $S\hookrightarrow X$ submfd, $N^*S\subset T^*X$ as the dual of $NS\subset TX$ ($\dim NS = \operatorname{rank} NS + \dim S = (\dim X-\dim S)+\dim S$ $= \dim X$ not depend on $\dim S$)

$\Rightarrow \forall p\in N^*S$, $(\lambda|_{N^*S})_p = \lambda_p|_{T_p(N^*S)} = (\sum \xi_i dx_i)_p|_{T_p(N^*S)} = (\sum_{i=k+1}^{n}\xi_i dx_i)|_{T_p(N^*S)} = 0$

(3) Compatible Triples in linear case

Fix $(2n)$-dimensional vector space $V/\mathbb{R}$, we have three structures can be put:

- ω symplectic structure
- J (almost) complex structure ($J \in \mathrm{End}(V)$, $J^2 = -\mathrm{Id}$)
- g Riemann/Inner product/positive-definite quadric form

Given any two we can get the left one: with compatibility

(ω, J)	Compatible $\begin{cases}\omega(Ju,Jv)=\omega(u,v)\\ \omega(u,Ju)>0, u\neq 0\end{cases}$	$g(u,v)=\omega(u,Jv)$ symmetric, positive definite
(J, g)	Orthogonal $g(Ju,Jv)=g(u,v)$	$\omega(u,v)=g(Ju,v)$
(ω, g)		J by polar decomposition

Rk. For global mfd case, the problem of result third one occurs nontrivially, this is left to next speaker.

Prop1. Given (ω, g) arbitrarily, $\exists$ canonical (not depend on choice of basis) J almost cpx.

Pf. By polar decomposition; it's based on the idea "J is a rotation"

Now g induces $V \xrightarrow[\cong]{g} V^*$ by their nondegeneracy: $v \mapsto g(v,-)$

$\exists$ canonical (iso) ; Given any $u \in V$, $\omega(u,-) \in V^*$

$\Rightarrow \exists A: V \to V$, s.t. $\omega(u,v) = g(Au,v)$, i.e. $V \xrightarrow{\tilde{\omega}} V^* \xrightarrow{(g)^{-1}} V$, $A = g^{-1}\circ\tilde{\omega}$

(hence A is also unique! Same is J next!)

Then, A is skew-symmetric, AA^* symmetric & positive w.r.t. g

(I omit their verification)

$\Rightarrow AA^*$ has polar decomposition $A = \sqrt{AA^*}\cdot\sqrt{AA^*}^{-1}A$

$= \sqrt{AA^*}\cdot J$

Then check J is orthogonal, skew-adjoint ($J^* = -J$), compatibility (omitted) ($JJ^* = \mathrm{Id}$, i.e. $g(Ju,Jv)=g(u,v)$) easily ▨

To understand the compatibility, we have

Prop2. (ω, J) is compatible $\Leftrightarrow$ symplectic basis can in form $v_1 \cdots v_n, Jv_1 \cdots Jv_n$

$\Leftrightarrow J$ is ω-tamed ($\omega(u,Ju)>0$, $u\neq 0$) and L Lagrangian $\Rightarrow JL$ Lagrangian

Pf. ($\overset{1}{\Rightarrow}$) We need to check $\omega(v_i,v_j)=\omega(Jv_i,Jv_j)=0$ & $\omega(v_i,Jv_j)=\delta_{ij}$ w.r.t. $\omega(-,J-) = g(-,-)$ Orthogonal

First there exist Lagrangian subspace $\Lambda \subset (V,\omega)$, take its basis $\{v_1 \cdots v_n\}$ by Lagrangian, $\omega|_\Lambda = 0 \Rightarrow \omega(v_i,v_j)=0 \Rightarrow \omega(Jv_i,Jv_j)=\omega(v_i,v_j)=0$

& $\omega(v_i,Jv_j)$ is inner product $g(v_i,v_j)$, by orthogonal done.

($\overset{2}{\Leftarrow}$) Tame is by $\omega(\Sigma(a_iv_i+b_iJv_i), \Sigma(a_jv_j+b_jJv_j)\cdot J)$

$= \sum_{i,j}\omega(a_iv_i+b_iJv_i, Ja_jv_j - b_jv_j)$

$= \Sigma\, \omega(a_iv_i, a_iJv_i) + \omega(b_iJv_i, -b_iv_i)$

$= \Sigma(a_i^2+b_i^2) > 0$

$\forall u, v \in L$

$\omega(u,v) = \omega(\Sigma(a_iv_i+b_iJv_i), \Sigma(c_jv_j+d_jJv_j)) = \Sigma\, a_ic_i + b_id_i = 0$

$\omega(Ju,Jv) = \omega(\Sigma(-b_iv_i+a_iJv_i) + \Sigma(-d_jv_j+c_jJv_j)) = \Sigma\, b_id_i + a_ic_j$

$\Leftrightarrow JL$ Lagrangian

($\Leftarrow$) We need to prove $\omega(Ju,Jv)=\omega(u,v)$

Otherwise g not symmetric, $\exists u_0, v_0$, $\omega(u_0,Jv_0) \neq \omega(v_0,Ju_0)$

set $w_0 = u_0 - \frac{\omega(v_0,Ju_0)}{\omega(u_0,Ju_0)}v_0 \Rightarrow \omega(v_0,Jw_0) = 0 \neq \omega(w_0,Jv_0)$

$\exists \Lambda \ni v_0, Jw_0 \longrightarrow w_0, Jv_0 \notin J\Lambda$ also Lagrangian

contradiction ▨

Now we denote the space of (ω, J, g) respectively:

$\mathcal{J}(V)\times\Omega(V)$ $\quad \Omega(V)\times\mathrm{Met}(V)$ $\quad \mathrm{Met}(V)\times\mathcal{J}(V)$ $\quad \mathcal{J}(V,\omega)$ means all J compatible with ω, $\mathcal{J}(V,g)$, $\Omega(V,J)$... Similar

$\cup$ $K(V)$ (compatible) $\quad O(V)$ (orthogonal)

$\mathcal{J}(V,\omega)$ $\quad \mathcal{J}(V,g)$ $\quad \Omega(V,J) = \Omega(V)$ $\quad \mathrm{Met}(V) \supseteq \mathrm{Met}(V,\omega)$

These various (but obvious) relations have may homotopy equivalence, here we only focus the only useful picture in symplectic topology:

(☆)
$$\begin{tikzcd} & \mathcal{C}(V) \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] \arrow[dd, hook, "\simeq"] & \\ \Omega(V) & & \mathcal{J}(V) \\ & \mathcal{C}_\tau(V) \arrow[ul, "\simeq"] \arrow[ur, "\simeq"'] & \end{tikzcd}$$

all homotopy equivalences. fix ω or J, $\Omega(V,J) \subsetneq \Omega_\tau(V,J)$

$\mathcal{J}(V,\omega) \subsetneq \mathcal{J}_\tau(V,\omega)$ are contractible

$\mathcal{C}_\tau(V) = \{(\omega,J) \mid \text{tame}\}$ (as fibres of four projections)

What this picture tells us?

① When study a symplectic mfd (M,ω), the different choices of J not effect our invariants, esp, in homological level they invariant under homotopy.

E.g. Floer homology defined by counting some special J-$\mathbb{C}$.$\mathbb{R}$ curves associated to Lagrangians as the differential of Floer cpx, but different J_0 & J_1 gives different J_0-$\mathbb{C}$.$\mathbb{R}$ curves and J_1-$\mathbb{C}$.$\mathbb{R}$ curves, but the (J_t) connecting them defining a quasi-isomorphism between their Floer chain complexes $\Rightarrow$ Floer homology isomorphic $\Rightarrow$ A_∞-category isomorphic between Fukaya cats.

Rk. This motivates us to give an approach to build Fukaya cats without choice of J or J-$\mathbb{C}$.$\mathbb{R}$ curves, this is the belief of microlocal sheaf category.

② Most time we can weaken compatible to tame, compatible is something too strong. For example Gromov optness for moduli of J-$\mathbb{C}$.$\mathbb{R}$ curves only need tame.

Let's prove (☆): and $\Omega_\tau(V,J)$

Thm 3. ① $\Omega(V,J)$ is convex $\Rightarrow$ contractible $(\Rightarrow \mathcal{C}(V) \simeq \mathcal{C}_\tau(V))$ (using linear homotopy)

② $\mathcal{J}(V,\omega)$ is contractible (Hence by $\mathcal{C}(V) \simeq \mathcal{C}_\tau(V)$, $\mathcal{J}_\tau(V,\omega)$ also)

Pf. ① Trivial by linearity

② $\mathcal{J}(V,\omega) \rightleftarrows \mathrm{Met}(V)$ With $J \mapsto g_J \mapsto \boxed{J_{g_J,\omega} = J \text{ is identity}}$ (☆☆)

$J \longmapsto g_J : (u,v) \mapsto \omega(u,Jv)$

$\boxed{J_{g,\omega} \xleftarrow[\text{Prop 1}]{} g}$ Polar decomposition

and $\mathrm{Met}(V) \to \mathrm{Met}(V)$ is always homotopy $g \mapsto g_{J_{g,\omega}}$ to identity due to $\mathrm{Met}(V)$ is convex, we can connect by linear homotopy, done!

$\mathcal{J}(V,\omega) \simeq \underbrace{\mathrm{Met}(V) \simeq \{\mathrm{Id}\}}_{} = \{pt\}$

(By $P \mapsto tP + (1-t)\mathrm{Id}$, all positive definite & symmetric)

(☆☆) is due to

$\begin{cases} g_J(u,v) = \omega(u,Jv) \\ \omega(u,v) = g_J(Au,v),\ A = \sqrt{AA^*} \cdot J_{g_J,\omega} \end{cases}$

$\Rightarrow g(Au,Ju) = |u|^2 = \|Ju\|^2,\ \forall u \in V$

$\xRightarrow{\text{nondegenerate}} A = J$ (by linear algebra)

$\Rightarrow -A^2 = \mathrm{Id} \xRightarrow[A=-A^*]{} A = J_{g_J,\omega}$ (polarization identity, omitted)

$\Rightarrow J = J_{g_J,\omega}$ ☒

Rk. ① A direct proof of $\mathcal{J}_\tau(V,\omega)$ is contractible is harder but meaningful, as one of the key step is: from ω only tame, we can get compatible one. But it's still not obvious. $\tilde{\omega} = \frac{\omega + J^*\omega}{2}$ (pullback 2-form) (see [MS, Prop 2.5.13, Proof 1], she/he given many proofs on these conclusions)

② Concrete computation shows that we have

$$\begin{tikzcd} \frac{GL(2n;\mathbb{R})}{GL(n;\mathbb{C})} \arrow[d, "\cong"] & \frac{GL(2n;\mathbb{R})}{U(n)} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d, "\cong"] \arrow[dl, phantom, "\circlearrowright"] \arrow[dr, phantom, "\circlearrowleft"] & \frac{GL(2n;\mathbb{R})}{Sp(2n)} \arrow[d, "\cong"] \\ \mathcal{J}(V) & \mathcal{C}(V) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \Omega(V) \end{tikzcd}$$

Here we avoid the linear groups...

A interesting thing is $\pi_1(Sp(2n)) \cong \mathbb{Z}$ can be given explictly by Maslov index. $M: \mathrm{Hom}(\mathbb{R}/\mathbb{Z}, Sp(2n)) \to \mathbb{Z}$

SN

Date . .

Moment map, symplectic reduction and quantization

First we can see moment map from physical momentum and angular momentum:

Eg 1. Our phase space is $T^*\mathbb{R}^3 = \mathbb{R}^6 = \{(x,\xi) \mid x \in \mathbb{R}^3, \xi \in T_x^*\mathbb{R}^3\}$

$\mathbb{R}^3 \curvearrowright T^*\mathbb{R}^3$ by translation on x	$SO(3) \curvearrowright T^*\mathbb{R}^3$ by rotation both x and ξ centred at 0
$\updownarrow$	$\updownarrow$
$\mu: T^*\mathbb{R}^3 \xrightarrow{\cong} \mathbb{R}^6 \xrightarrow{pr_\xi} \mathbb{R}^3 = (T_e\mathbb{R}^3)^*$	$\mu: T^*\mathbb{R}^3 \xrightarrow{\cong} \mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{\times} \mathbb{R}^3 \cong (\mathfrak{so}(3))^*$
send (x,ξ) to its momentum ξ	send (x,ξ) to its angular momentum $x \times \xi$

At first I expect I can use a graph to describle the equivalence above physically, ~~but~~ I find it impossible to describle $L(X)\omega$, thus let's accept it. (Only a simple computation is used after giving a definition)

Def 1. A Hamiltonian G-space is $(M, \omega, G, \mu$ or $\mu^*)$ where ⓪ (M,ω) is symplectic

① $G \curvearrowright (M,\omega)$ symplectically ($G \xrightarrow{\psi} \mathrm{Symp}(M,\omega)$ or $\omega \in \Omega^2(M)^G$)

② and Hamiltonian: its infintesimal action $\mathfrak{g} \to \mathfrak{X}(M)$ determines a Hamiltonian vector field $\xi \mapsto \frac{d}{dt}\big|_{t=0} \psi_{\exp(t\xi)} =: X_\xi$

(Recall $L_X\omega$ exact $\Leftrightarrow X$ Hamiltonian & $L_X\omega \overset{\text{closed}}{\Longleftrightarrow} X$ symplectic)

$L_{X_\xi}\omega =: dH_\xi$, ③ $\mu^*: \mathfrak{g} \to C^\infty(M)$ is G-equivariant

(unique up to constant) $\xi \mapsto H_\xi$ (i.e. $H_\xi \circ \psi_g = H_{g^{-1}\xi g}$ invariant under adjoint action)

Or dually ③ $\mu: M \to \mathfrak{g}^*$ $\qquad \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$, μ is G-equivariant

$p \mapsto \mu(p) \qquad (\mu(p), \xi) \mapsto H_\xi(p)$ (i.e. $\langle \mu \circ \psi_g(p), \xi\rangle = \langle \mu(p), g^{-1}\xi g\rangle$)

We call ⓪~② is weak Hamiltonian space

⓪-③ or ③ is G-Hamiltonian space, μ & μ^* are moment & comoment map

(Rk. Let ④ be: μ^* is Lie algebra homomorphism, i.e. $\mu^*[\xi_1,\xi_2] = \{H_{\xi_1}, H_{\xi_2}\}$, then ③ $\Rightarrow$ ④, and when G connected, ③ $\Leftrightarrow$ ④

Proof see [MS, Lemma 5.2.1]. $\qquad \mathfrak{g}$ is quantization of $C^\infty(M)$)

For the existence and uniqueness of such Hamiltonian space for a symplectic action $G \curvearrowright (M,\omega)$, it's other's presentation. (Via Lie cohomology)

No.

Date

to each $G = \mathbb{R}^2$

$t, t \mapsto e^{2\pi i t}$

Eg2. Consider S^1-action:

$S^1 \curvearrowright (\mathbb{R}^{2n+2}, \omega_{std}) = (\mathbb{C}^{n+1}, \omega_{std}, J_{std})$ by rotation/multiplication

then it's Hamiltonian by observing the Hamiltonian function is

$H = -\frac{1}{2}|z|^2$ (Here $\mathfrak{g} = \mathbb{R}$ and $H = H_1$), up to constant:

Check: This action induces vector field $X = \sum \frac{\partial}{\partial \theta_i}$

$\omega_{std} = \sum dx_i \wedge dy_i = \sum r_i\, dr_i \wedge d\theta_i$: reparametize each $\mathbb{R}^2 = \mathbb{C}$ by (r_i, θ_i)

$\Rightarrow \iota_X \omega = -\sum r_i dr_i = -\frac{1}{2}\sum dr_i^2 = d(-\frac{1}{2}\sum r_i^2) = d(-\frac{1}{2}|z|^2) = dH$ done

Now as a leading example to symplectic reduction, consider $\mu^{-1}(-\frac{1}{2})$ is the $S^{2n+1} \subset \mathbb{R}^{2n+2}$ and S^1-action restricted. The result sp

$\mathbb{CP}^n = S^{2n+1}/S^1 = \mu^{-1}(-\frac{1}{2})/S^1$ should equip a descent symplectic form by

$\omega_{std} \in \Omega^2(\mathbb{R}^{2n+2})^{S^1} \xrightarrow{i^*} \Omega^2(S^{2n+1})^{S^1} \xrightarrow{\text{descent}} \Omega^2(\mathbb{CP}^n)$

by standard construction, it's just the FS form

In particular, let $n = 1$, we have $S^1 \curvearrowright \mathbb{R}^4$ and $\mathbb{CP}^1 = S^2 = \mu^{-1}(-\frac{1}{2})/S^1 = S^3/S^1$

is the Hopf fibration. $S^1 \to S^3$

$\downarrow$

S^2

Q. Hopf fibration comes from the magnetic monopole in $\mathbb{R}^3 \supset \mathbb{R}^3 - \{0\} \simeq S^2$

at $\{0\}$, which a unit point magnetic charge removed, can we explain the

moment map & symplectic reduction physically as Eg1?

Rk: When $\mathfrak{g} = \mathbb{R}$ case, μ is a function on M, it's natural to

ask about its critical pts. In S^1-action case, μ is always Morse-Bott

(Generalize Morse function $\Leftrightarrow$ nondegenerate critical pts, but allowing critical pts

can be not isolated but critical submflds.

Then we preview symplectic reduction, again first we study some physics:

We always expect 0 is regular value to take $\mu^{-1}(0)$

& Restrict $G \curvearrowright \mu^{-1}(0)$ is $\boxed{\text{proper}}$ & $\boxed{\text{free}}$

closed orbit. trivial stabilizer

Slice thm $\Rightarrow$ MWM quotient Idea. From orbit of one pt to its neibourhood U

$M = \mu^{-1}(0) \curvearrowleft G$ cpt $\Rightarrow$ proper $\dim U + \dim \mathcal{O}_p = \dim M$

$\mathcal{O}_p$ orbit ($\mu^{-1}(0)$ is G-inv) free transversal

extend $\mathcal{O}_p$ to $\mathcal{O}_p \times U$ need $\boxed{\text{free}}$

otherwise two nearby orbits intersect

How to understand $\mu^{-1}(0)$? (By a constant, always take 0) $\mathcal{O}_p$ singular

It should be understood as solution to an equation (differential/algebraic)

$\Leftrightarrow$ critical pt of a functional: this is based on minimal action principle

$\mu^{-1}(0)$ describe the motion of particle in real world: momentum can

be viewed derivative/perturbation of action. We'll see it later in gauge theory.

(Dimension analysis: action $\sim$ momentum $\times$ length $\sim$ angular momentum $\times$ angle by space/angle/time
$\sim$ energy $\times$ time

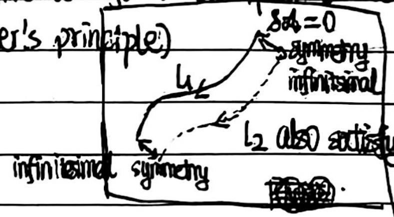
$\mu^{-1}(0)/G$ modulo G-action is the process of reduction, this's based

on Noether's principle, slogan is $\boxed{\text{symmetry} \leftrightarrow \text{conservation} \quad (G \longleftrightarrow \text{Hamiltonian})}$, the rigorous

mathmatical formulation is others' presentation

Here we give an informal description: (indeed it can be modified to a rigorous

pf to Noether's principle)



$\mathfrak{g}$ - infinitesimal symmetry

$\downarrow \exp_0$ generate

G - symmetry (G connected)

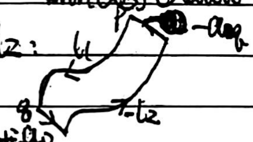
$\delta A = 0$ by symmetry & $A(L_1) = \dots$

Extended phase space $T^*M \times \mathbb{R}$

with (x, ξ, t) allow time (symmetry

Symmetry/G-action can determine two

orbits $\subset T^*M \times \mathbb{R}_t$ connecting L_1 and L_2:



then the action $A(\alpha_1) = A(\alpha_2)$

this means $\forall \tilde{p} \in L_1$, $A(\alpha_{\tilde{p}})$ is conservation

Symmetry is: time $\Leftrightarrow$ Energy conservation

Translation $\Leftrightarrow$ Momentum conservation

Rotation $\Leftrightarrow$ Angular momentum conservation in classical mechanics

Thus after reduction, we call $\mu^{-1}(0)/G$ the reduced phase space in physics

it has conserved quantities constant.

It's obvious that our mathematical formulation is much more general

than classical physical symmetries: $\mathfrak{g}$ and G infinite-dimensional also

possible and important (esp. in Gauge)

Date

Before coming into more interesting gauge theory, we list the general facts of symplectic reduction, which built by next (and...) presentation by others:

① Marsden–Weinstein–Meyer quotient: just as Eg2, $\mathbb{CP}^n$ as reduced space with descent symplectic form as Fubini–Study form. In general we discuss $\boxed{\text{when the quotient is good, then it's symplectic}}$;

(Orbifold case postpone to later)

② Duistermaat–Heckman localization: I know nothing on equivariant.

③ Atiyah–Guillemin–Sternberg convexity: as a generalization of S^1–action, the toric action $T^n = G_m^n$ induce moment map has image a convex polytope, each vertex are symplectic submfds which μ is constant; (Our textbook focus on this and $\boxed{\text{disjoint}}$ contains more toric case)

④ Kempf–Ness theorem: GIT quotient is a quotient modify back to good by delete away orbits that not closed & not finite stabilizers, denoted $X/\!/G \cong \mu^{-1}(0)/K$ homeomorphism (not semi-stable / not prestable) "add" by closure, two orbits intersect, $\boxed{\text{delete}}$

$K \subset G$ maximal cpt subgp

X smooth projective cpx variety (thus Kähler)

(GIT) $\boxed{\text{Stability} = \text{Stability}}$ (of critical pt of functional)

(Algebraic $\longleftrightarrow$ symplectic)

Q. Realize all "GIT" stability as "critical pt" of a "functional" in AG side? (algebraic) $\updownarrow$ minimal point of a $\boxed{\text{function}}$ on a proper moduli space

A. I think it's possible: K-stability has analytic nature, but works by Xu and ... gives both a pure algebraic description and realization it as a minimal pt of a function by birational geometry.

From here we start $\boxed{\text{quantization}}$: What's quantization? There're too many formulation, but our leading principle is simple: from space to space of functions/sheaves (of deformation?), from communicative function algebra to noncommutative operator algebra; from finite-dim to infinite-dim; from classical path to path-integral...

(Weyl's deformation $*_W$, small quantum product $*$... classical part $+\square e^{\int\square}\square$)

Date

⑤ Geometric quantization commute with symplectic reduction. YM connection + 4-dim on the unique

These reduction we can do for moduli of ASD connections / $\mathcal{G}$ (Gauge symmetry) and the construction is a symplectic reduction $\mu^{-1}(0)$ due to Atiyah–Bott (by others' presentation) $\longleftrightarrow$ critical of YM functional

Combining ④ + ⑤, we have a quantized analogue of the algebra–symplectic correspondence:

⑥ (A big picture) X is curve / Riemann surface, G cpt connected simple

Character variety (Representation of fundamental gp)	$\mathrm{Hom}(\pi_1(X), G)/\!/G$	$\underset{\text{Riemann–Hilbert}}{\cong}$ (Monodromy rep of connection)	$\{\text{Flat connection}\}/\mathcal{G}$ — Symplectic reduct part
• Take cotangent bundle of RH we have non-abelian Hodge	$\cup$ $\{\text{Irreducible unitary projective rep}\}/\!/G$		$\{\ \}/\mathcal{G}$
	Mumford– $\|$ Narasimhan–Seshadri		Hitchin $\|$ Kobayashi
righthand is Higgs bundle	$\{\text{Stable vector bundles}\}/\cong$ $\boxed{\text{GIT quotient part}}$	$\subset$	$\{\ \text{stable vector bundles}\}$ Kempf–Ness type

It connects representation theory, Gauge theory and algebraic geometry. (polystable = ⊕ stable)

// Taking care of these equivalences are biholomorphic / algebraic

⑦ Ele cohomology and BRST cohomology