K-theory and index theory, review

I know nothing about higher K-groups, it might be computed by $\pi_i(BU)$ and shift to different index by B and Ω, all K-theory (K-spectra) here're K_0 or K^0, even then encode ample geometric data, especially in index theory.

- Here we should note the contrast between K-theory and a choice of cohomology theory;
- "Higher index theory" doesn't the same "high" in algebraic topology

We have many sorts of K-theories:

$K^0(X)$: vector bundles / X (topological)

$K_0(X)$: cohérent sheaves / X (algebraic)

or Fredholm differential operators / X (analytic)

$K_0(A)$: projective module / A (non-commutative generalization of above)

$K^0(A)$: Fredholm module / A

are objects from some derived category, come from the triangulated structures, so is the induced morphisms

We're discussing how $RR \subset HRR \subset AS \subset$ family AS, $\subset$ GRR

- Setting of GRR: we choose cohomology is Chow

$ch: K^0(X)\otimes\mathbb{Q} \to A.(X)\otimes\mathbb{Q}$, if X pure-dim, $A.(X) \leftarrow End(A.(X))$

if X nonsingular, $\varphi[X] \leftarrow \varphi$

it's isomorphism $K^0\otimes K_0 \cong K^0$

$\Rightarrow ch: K^0(X)\otimes\mathbb{Q} \to A.(X)\otimes\mathbb{Q}$, but here due to nonsingular $K^0\cong K_0$

$$\begin{array}{ccc} K^0(X)\otimes\mathbb{Q} & \xrightarrow{f_*} & K^0(Y)\otimes\mathbb{Q} \\ \downarrow ch\,\boxed{\cdot} & & \downarrow ch\,\boxed{\cdot} \\ A.(X)\otimes\mathbb{Q} & \xrightarrow{f_*} & A.(X)\otimes\mathbb{Q} \end{array}$$

$f_* = \sum(-1)^i Rf_*$ inherited from derived cat, thus let $Y=pt$ $\Rightarrow \chi(E)$ is HRR and f_* is some sort of index map

Setting of AS: we choose cohomology is de Rham

$$\begin{array}{ccc} K^0(X)\otimes K_0(X) & \xrightarrow{Ind_a} & \mathbb{Z} \\ \downarrow ch\,\boxed{\cdot} & & \downarrow \\ H^*_{dR}(X)\otimes H^{n-*}_{dR}(X) & \xrightarrow[\text{Poincaré}]{Ind_{top}} & \mathbb{Z} \end{array}$$

$\Leftrightarrow Ind_a(D) = \int ch\,\boxed{\cdot}$

(the def of Ind_{top} is not direct)

Setting of noncommutative AS: we choose cohomology is periodic cyclic

$$\begin{array}{ccc} K^0(A)\otimes K_0(A) & \xrightarrow{ind} & \mathbb{Z} \\ \downarrow ch & & \downarrow \\ HC^*(A)\otimes HC_*(A) & \xrightarrow[\text{dual}]{\text{Poincaré}} & \mathbb{C} \end{array}$$

, let $A=\overline{C_0(X)}$, $HC^*(\overline{C_0(X)}) = H^*_{dR}(X)$

(Q. What's the closure? it needs some operator algebra)

Setting of family AS: we choose same cohomology as above

Replace ind by functorial induced morphism.

Observation ① For different cohomology theory, there are different correction term for Chern character, it can be Todd genus, $\hat{A}$ genus, Witten genus ... or in the definition of noncommutative Chern character directly redefine it;

② Chern class is topological (not depend on algebraic structure of X and E), thus Chern character & its corrections are also topological;

K^0 is always topological as it consists vector bundles

$\Rightarrow$ we can replace Chow by singular cohomology in GRR

③ X is curve, $Ind(D) = \dim(Ker\,\Delta^+) - \dim(Ker\,\Delta^-)$

$= \dim H^0 - \dim H^0(\otimes\omega)$

$\overset{\text{Serre dual}}{=} \dim H^0 - \dim H^1 \rightsquigarrow RR$

for general HRR, it's similiar.

The computation idea is find what the correction term is, by expressing it as Chern roots.

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