- Milnor conjecture & Grothendieck's pf via ~~[scribble]~~ nearby & vanishing cycle formalisms
- Deligne's monodromy ~~per periodic~~ purity thm — Weil I : ~~reduce to hypersurface~~ proof of Weil
  — Weil II : ~~prove hypersurface~~ more weights (both applied)
  $\boxed{\text{integrality of weight}}$

$(\mathbb{C}^{n+1},0) \xrightarrow{f} (\mathbb{C}^1, 0)$, if $df(0) \neq 0$, locally at 0 it's projection

<u>Milnor's thm/ fibration</u>

$df(0)=0$, $\exists\, \varepsilon, \delta > 0$, s.t. $f^{-1}(\mathring{B}(0,\delta)) \cap B(0,\varepsilon) \to \mathring{B}(0,\delta)$ is (locally) trivial fibration

with fibre F_t, $\forall |t| \in (0,\delta)$ (F_t homeomorphic, not ~~bi~~ biholo.)

$\to \pi_1(S^1) \to GL(H^i(F_t))$

$S^1 \cong \Delta^*$ punctured disk

$S^n \vee S^n \vee \cdots$

F_t called Milnor fibre $= f^{-1}(t) \cap B(0,\varepsilon)$

E.g. ① $\mathbb{C}^1 \xrightarrow[\pi]{n:1} \mathbb{C}^1 \Rightarrow H^0(\pi^{-1}(t)) = \mathbb{Z}\langle \sqrt[n]{t}, \omega\sqrt[n]{t}, \cdots, \omega^{n-1}\sqrt[n]{t} \rangle$

$T^n = id$

$T = \begin{bmatrix} & & 1 \\ 1 & & \\ & \ddots & \end{bmatrix}$ cyclic matrix, eigenvalues are root of unity

② $\mathbb{C}^2 \xrightarrow{\pi} \mathbb{C}^1$, $(x,y) \mapsto xy$

$\pi^{-1}(\lambda) = \{xy=\lambda\} \xleftarrow{\cong} \pi^{-1}(1) \cong \mathbb{C}^*$ canonical iso $\Rightarrow$ (globally) trivial family

$(x, \frac{y}{\lambda}) \xleftarrow{\cong} (x,y)$, $(x,\frac{1}{x}) \leftarrow 1$ $\Rightarrow T = id$.

③ $\mathbb{C}^2 \to \mathbb{C}^1$, $\mapsto \lambda$, $V(y^2 = x^3+\lambda)$

$y^2 = x^3 + \lambda$ is elliptic curve, i.e. if we choose root of unity $\omega^6 = 1$

$\Rightarrow \left(\frac{y}{\omega^3}\right)^2 = \left(\frac{x}{\omega^2}\right)^3 + 1$

$\Rightarrow$ canonically iso $\pi^{-1}(\lambda) \cong \pi^{-1}(1)$ if ω chosen.

$(x,y) \mapsto y^2 - x^3$

$$\Rightarrow \begin{tikzcd} \mathbb{C}^2 - V(y^2=x^3) \arrow[d] \arrow[dr, phantom, "\square"] & \cdot \arrow[l] \arrow[d] \\ \mathbb{C}^* & \mathbb{C}^* \arrow[l, "6:1"] \end{tikzcd}$$

trivial $T = id$

$T^6 = id$ choose ω

Milnor conj: ① eigenvalue of T is roots of unity ✓ by Grothendieck (monodromy rep from geometry is narrowed)

② T is finite order ✗ by A'Campo (even isolated not true)

thm (Grothendieck) $X \xrightarrow{f} S$ global version

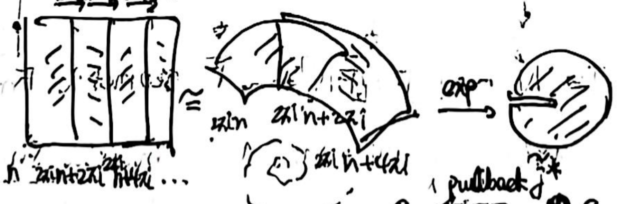
$\exists\, a \in \mathbb{N}$, $(\rho(T)^a - 1)^{i+1} = 0$ ($\rho: \pi_1(S^1) \to GL(H^i(X_t))$)

$\rho(T)$ unipotent, i.e. $\rho(T)$ quasi-unipotent ($1 \mapsto \rho(T)$)

($\Rightarrow$ eigenvalue of $\rho(T)$ is root of unity)

Why?

Deck transform



$\tilde{\mathbb{G}}_m \to \mathbb{G}_m$, $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times$

Deck transform $\Rightarrow \pi_1(\mathbb{C}^\times) = \mathbb{Z}(1)$ need choose i

pullback, push forward $\mathcal{E} \oplus \cdots \oplus \mathcal{E}$ and monodromy is $\mathcal{E} \oplus \mathcal{E} \oplus \cdots$

Nearby & vanishing cycle

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] \arrow[d] & X \arrow[d, "f"] & X^* \arrow[l, hook', "j"] \arrow[d] & \tilde{X}^* \arrow[l] \arrow[d] \arrow[ll, bend right, "\tilde{j}"'] \\ \{0\} \arrow[r, hook] & \mathbb{A}^1 & \mathbb{G}_m \arrow[l, hook'] \arrow[ur, phantom, "\square"] & \tilde{\mathbb{G}}_m \arrow[l] \end{tikzcd}$$

$\tilde{\mathbb{G}}_m = \mathbb{C}$ universal cover

$R\Psi_f(\mathcal{F}) = i^* R\tilde{j}_* \tilde{j}^* \mathcal{F}$ nearby (not depend on 0)

$R\Phi_f(\mathcal{F}) = \mathrm{Cone}(i^*\mathcal{F} \to R\Psi_f(\mathcal{F}))$ vanishing (depend on 0 as $i^*\mathcal{F}$ is)

The deck transformation of $\widetilde{\mathbb{G}_m} \to \mathbb{G}_m$ induce monodromy rep of $R\Psi_f(\mathcal{F})$, restrict to stalk is the old monodromy rep

$R\Psi_f(\mathbb{Q})_x = R\Gamma(F_x, \mathbb{Q})$ is the sheafification of Milnor's thm (taking limit of $\varepsilon, \delta \to 0$)

<u>Pf of ①</u> • Resolution of $X \to S$ ⇒ assume it's normal crossing degeneration at $0 \in S \subset \mathbb{A}^1$

i.e. $f(z) = z_1^{a_1} \cdots z_r^{a_r} \cdot (z_{r+1}^0 \cdots z_n^0)$

• Combining examples ① & ② ⇒ Milnor fibre $\{z_1^{a_1} - z_r^{a_r} = \varepsilon\}$

• Then on nearby cycle, $= [\Delta^{n-r} \times (\Delta^*)^{r-1}] \amalg \underbrace{[\Delta^{n-r} \times (\Delta^*)^{r-1}]}\amalg \cdots \amalg [\cdots]$

⇒ $T^a = 1$

$d = \gcd(a_1, \dots, a_r)$, $T^d = 1$ on this fibre

$E_2^{pq} = H^p(X_0, R^q\Psi_f(\mathbb{Q})) \circlearrowleft T$ ⇐ restrict

$\overset{p+q=i}{\Rightarrow} R\Gamma^i(X_0, R\Psi_f(\mathbb{Q})) \overset{\text{Proper}}{=} R\Gamma^i(X_t, R\Psi_f(\mathbb{Q}_\ell)) \circlearrowleft T$

on each E_2^{pq}, the action of T has order d ⇒ eigenvalue is root of unity

⇒ it's preserved under filtration Gr. ...

But Jordan block not preserved ⇒ we let all $(i+1)$ Jordan blocks of T to be 0

$T \circlearrowleft E_2^{0i}$

$T \circlearrowleft E_2^{i0}$

$\Rightarrow (\rho(T)^{a} - 1)^{i+1} = 0$ ☐

(a is maximal d when $\varepsilon \in S$ takes all around 0)

Deligne's

<u>Local version</u>(/ algebraic language)

$\exists K'/K$ finite, $\forall \gamma \in I_{K'}$, $(\rho(\gamma) - 1)^{i+1} = 0$

$(\widetilde{\mathbb{G}_m} \to \mathbb{G}_m)$ $\quad (I_K' \subset I_K \to G_K \xrightarrow{\rho} GL(H^i(X_{\bar\eta})))$

• Thus Hironaka's resolution can't be used (char $k > 0$)

Pf. If we want to repeat pf above, use alteration of resolution ☐

[Berthelot, Bourbaki report on alteration]

Another pf is by $1 \to P_K \to I_K \to I_K^{\text{tame}} \to 1$

wild inertia

pro-p group

tame inertia

$\cong \prod_{\ell \neq p} \mathbb{Z}_\ell(1) = \left(\prod_{\ell \neq p, \ell} \mathbb{Z}_\ell(1)\right) \times \mathbb{Z}_\ell(1)$

use base change (finite extension of residue field) ⇝ • cancel P_K first

↳ not change geometric $\bar\eta$

• cancel $\prod_{\ell \neq p, \ell} \mathbb{Z}_\ell(1)$ part (Geometric part)

• $\mathbb{Z}_\ell(1)$ generated by $T = \rho(\gamma)$

$\sigma T \sigma^{-1} = T^q$ ⇒ T quasi-unipotent ☐

T quasi-unipotent, i.e.

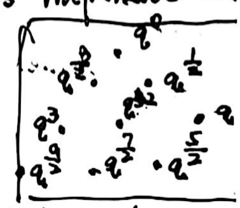
<u>Coro</u>. $(T^a - 1)^b = 0$, we can define $\log T := \frac{1}{a} \log[1 + (T^a - 1)]$ is nilpotent =: N

Set $1 \to I \to \mathrm{Gal}(\bar\eta/\eta) \to \hat{\mathbb{Z}} \to 1$ $\quad \overset{\text{reduce}}{\rightsquigarrow} \quad 1 \to \mathbb{Z}_\ell(1) \to \mathrm{Gal}(\bar\eta/\eta) \to \hat{\mathbb{Z}} \to 1$

σ : Frob

$T \rightsquigarrow \boxed{\sigma T \sigma^{-1} = T^q} \leftarrow \sigma$ of $\mathbb{F}_q$

$\log T = N \rightsquigarrow \boxed{NF = qFN} \leftarrow F$ geometric Frob $= \sigma^{-1}$



<u>Thm</u> (Jacobson–Morozov) Given N nilpotent

$\exists\, sl_2$-triple (X, Y, H) s.t. $X = N$

<u>Integrality</u>: We're proving integrity of Frob eigenvalue $Fv = \lambda v$, v is interia fixed $\in V^I$

⇒ $N(Fv' - q\lambda v') = 0 \overset{N \text{ is isomorphism}}{\Rightarrow}$ Frob eigenvalue of v' is $q\lambda$ ... $\overset{N}{\to} \overset{N}{\to} \overset{N}{\to}$ $q^3\lambda \quad q^2\lambda \quad q\lambda \quad \lambda$ ⇒ $v \in \operatorname{Ker} N$ ⇒ $Nv' = v$

$|q^i\lambda| = |\lambda| \cdot q^{\frac{ai}{2}}$, suffices do for λ & $\operatorname{Ker} N$. take $V \otimes V$ ⇒ $|\lambda| \le q^{\frac{i}{2}}$, $V \otimes V^\vee$ ⇒ $|\lambda| \ge q^{\frac{i}{2}}$ ⇒ $|\lambda| = q^{\frac{i}{2}}$ integral ☐

A linear algebra result

$\left(\cancel{\ } \; SL_2(\mathbb{C}) = sl_2(\mathbb{C}) = SU_2 \; \begin{matrix} \text{algebraic} & \text{holomorphic} & \text{Lie algebra} & \text{Continuous} \\ \text{-rep} & \text{-rep} & \text{-rep} & \text{-rep} \end{matrix} \right)$

Thm. G semi-simple Lie group $\Rightarrow \forall\, \rho(g) \curvearrowright V$ is semi-simple

$sl_2(\mathbb{C})$ has basis

$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

- General stategy: ① choose $\underset{\sim}{1}$ basis $e \overset{\rho}{\curvearrowright} V$. and decompose V by irreducible eigensubspaces

② Consider how other basis act on these subspaces

$\circlearrowleft \oplus \circlearrowleft \oplus \circlearrowleft \oplus \circlearrowleft = V$

$\boxed{\text{If } \rho(e) \in GL(V) \text{ is semi-simple}}$

③ How to choose such a special basis e make $\rho(e)$ semi-simple?

This is by restrict to maximal torus $\mathbb{G}_m^n \hookrightarrow G$, $\rho(\mathbb{G}_m^n)$ is always semi-simple, here $\mathbb{G}_m^1 = \langle H \rangle$

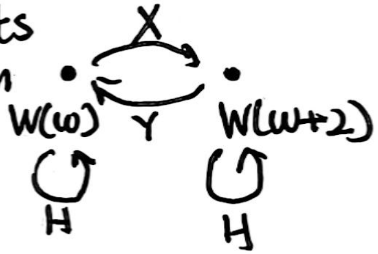
- $sl_2(\mathbb{C})$-triple: we apply our method: $tH = \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \middle| t \in \mathbb{C} \right\} \Longleftarrow \mathbb{G}_m^1$ maximal torus in $sl_2(\mathbb{C})$

$\rho(H) \curvearrowright W$ semi-simple

$\Rightarrow W = \bigoplus\limits_{w \in \mathbb{C}} W(w)$, w are eigenvalues of $\rho(H)$ $= T_e \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \middle| t \in \mathbb{C} \right\} = T_e \mathbb{G}_m^1 \subset T_e SL_2(\mathbb{C})$

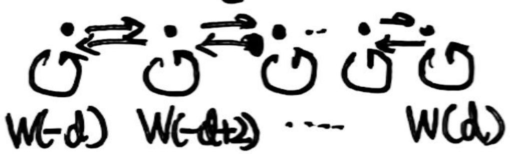
called weights

the weight decomposition, then



Thm. All rep of $SL_2(\mathbb{C})$ are $\mathrm{Sym}^d W$

$\Leftrightarrow$ the quiver rep of quiver



Thm. (Jacobson-Mirvsov)

$\forall$ N nilpotent operator

$\exists$ sl_2-triple (X, H, Y), s.t. $N = X$

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Reminder. Lastly, we need to show degeneration, weight ↘ by Deligne's semi-continuity, know $\chi_{\bar{t}} \Rightarrow$ know $\chi_U(\overset{0}{R}j_*\mathbb{Q}_\ell)$ or generic $\chi_{\bar{\eta}}$

It left to show $R^0 j_* \mathbb{Q}_\ell \Rightarrow \chi_0$ case!

Pf. $\mathbb{D}$ swaps $[\pm n]$ and swaps $!$ and $*$

thus ① $\xleftrightarrow{\mathbb{D}}$ (2.1), but for $*.2$, this is due to self-dual $\mathbb{D}\mathcal{E} = \mathcal{E}^\vee[2\dim U](\dim U)$, $\mathcal{E}/U$ local system

$\Rightarrow$ self dual concentrated at -1

② $\Leftrightarrow \mathcal{H}^i(i^*\mathcal{F}) = 0$, for $i \geq 0$ → at -1 ☐

(2.2) $\Leftrightarrow \mathcal{H}^i(j^*\mathcal{F}) = 0$, for $i \leq -2$

(Beilison's gluing construction)

By excision step either proj or affine, here for Artin vanishing, Verdier dual, use affine $\mathbb{D}$

① perverse connective: (1.1) $i^*\mathcal{F}$ connective; (1.2) $j^*\mathcal{F}[-1]$ connective

② perverse coconnective: (2.1) $i^!\mathcal{F}$ coconnective; (2.2) $j^*\mathcal{F}[-1]$ coconnective

if dim 1, $i: \Sigma \hookrightarrow C \hookleftarrow U: j$

For dim ↗ glue $-1, -2, -3$ ....

Degeneration step (Affine hypersurfaces)

$f=0$ singular, $g=0$ smooth: $\{tg + (1-t)f\}$ = $\mathcal{X}$

WRH? WRH ✓

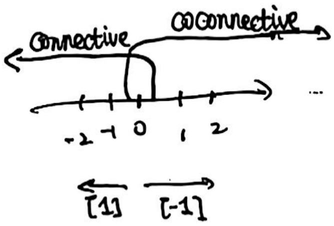
$$\mathcal{X} \xrightarrow{\text{proper, not smooth}} \mathbb{A}^1_t$$

$$\{0\} \xrightarrow{i} \mathbb{A}^1_t \xleftarrow{j} U$$

$\exists$ represent $\mathcal{F}$ in $\mathrm{Comp}^b_c(C)$, supported on $(-\infty, 0]$ or $[0,\infty)$

⇕

$\mathcal{H}^i(\mathcal{F})$ supported on ~



① + ② = perverse.

C smooth, affine, geometrically connected. $P \in \underset{\text{constructible}}{D^b_c(C)}$ perverse coconnective / cosupport condition

$\mathcal{H}^i(P|_U)$ local systems, $\forall i$ ⇐ $\Leftrightarrow \mathcal{H}^i(P|\text{strata})$ local systems, $\forall i$ $\Leftrightarrow P|\text{strata}$ local systems (as we change representative in D^b) (SGA IV + Verdier dual $\pi_* \rightsquigarrow \pi_!$)

We can embed "WRH(0)" to "WRH(U)" by Artin vanishing

$R\pi_!(\mathbb{Q}_\ell[n])$, if $\mathbb{Q}_{\ell,\mathcal{X}}$ on $\mathcal{X}$ is perverse

is perverse coconnective

Take $P = R\pi_!(\mathbb{Q}_\ell[n])$

(☆) and $\boxed{\text{Lemma}}$ $H^1(P_s) \hookrightarrow i^* \underbrace{R^0 j_*(\mathcal{H}^{-1}P|_U)}_{\text{local monodromy invariant part / inert}}$

respect Frob in both side

$\Rightarrow H^{n-1}_c(\mathcal{X}_0, \mathbb{Q}_\ell) \circlearrowleft$

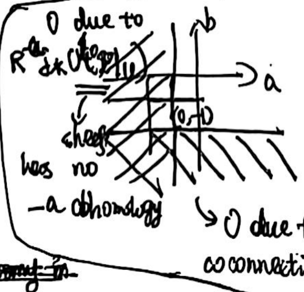
$\hookrightarrow H^{n-1}_c(\mathcal{X}_U, \mathbb{Q}_\ell) \circlearrowleft$, weight smaller, done ☐

$\mathcal{E}/U$ local system, the monodromy inv parts:

$i^! j_! \mathcal{E}[1]$ coconnective

$\Rightarrow \mathbb{D} i^! j_! \mathcal{E}[1]$ connective

$\Rightarrow i^* R j_* \mathcal{E}^\vee[1]$ connective



0 due to coconnective

Pf. $i_* i^! P \to P \to R j_* j^* P$, act $\underline{\underline{i^*}}$, take cohom at -1 ↘ turns to vector space, thus H not $\mathcal{H}$

$\Rightarrow H^{-1}(i^! P) \to H^{-1}(P_s) \to H^{-1}(i^* R j_* j^* P)$

$i^! \mathcal{F}$ coconnective ‖ ‖ ‖

$0 \longrightarrow H^{-1}(P_s) \longrightarrow i^* \mathcal{H}^{-1}(R j_* P|_U)$

$\mathcal{H}^{-1}(R j_* P|_U) = R^0 j_*(\mathcal{H}^{-1} P|_U)$

Grothendieck SS, by $P|_U[-1]$ coconnective $\Rightarrow \mathcal{H}^i(P|_U) = 0$ for $i \leq -2$

$R^a j_* \mathcal{H}^b \Rightarrow R^{a+b} j_* = \mathcal{H}^{a+b}(R j_*)$, but $a+b=-1$ only $(a,b)=(0,-1)$ survive, done ☐

~~$R j_* \mathcal{E}$ is local monodromy in~~

$j_* \mathcal{E}[1]$ perverse iff $j_! \mathcal{E}[1]$ perverse, if holds,

$j_! \mathcal{E}[1] \longrightarrow R j_* \mathcal{E}[1]$

↘ $R^0 j_* \mathcal{E}[1]$ ↗ also perverse, called intersection cpx / intermediate extension, advantage: it's pure!

essentially the Weil conj is about intersection cohomology part

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Thm. (Deligne's semi-continuity of weight)

$j: U \hookrightarrow C$, $\mathcal{E}$ local system $/U$, $\forall\, x \in \underline{U}$, $\mathcal{E}_{\bar{x}}$ weight $\leq w$ (act on fibre than cohomology)

$\Rightarrow (R^0 j_* \mathcal{E})_{\bar{x}}$ weight $\leq w$ for $\forall\, x \in \underline{C \setminus U}$

Pf. (Deligne's L-function method)

$0 \to j_! \mathcal{E} \to R^0 j_* \mathcal{E} \to \mathcal{G} \to 0 \cdots (*)$ $\Rightarrow L(R^0 j_* \mathcal{E}, t) = L(\mathcal{E}, t) \prod_{x \in C \setminus U} \det(1 - t\,\mathrm{Frob}_x |_{(R^0 j_* \mathcal{E})_x})^{-1}$ multiplicity

$\mathcal{G} = \bigoplus_{x \in C \setminus U} (R^0 j_* \mathcal{E})_x$

$$\Rightarrow \prod_{x \in C \setminus U} \det(1 - t\,\mathrm{Frob}_x |_{(R^0 j_* \mathcal{E})_x})^{-1} = \frac{L(R^0 j_* \mathcal{E}, t)}{L(\mathcal{E}, t)} \underset{\text{formulae}}{\overset{\text{trace}}{=}} \frac{\dfrac{\det(1 - t\,\mathrm{Frob}) |_{H^1_c(C, R^0 j_* \mathcal{E})}}{\det(1 - t\,\mathrm{Frob}) |_{H^2_c(C, R^0 j_* \mathcal{E})}}}{\dfrac{\det(1 - t\,\mathrm{Frob}) |_{H^1_c(U, \mathcal{E})}}{\det(1 - t\,\mathrm{Frob}) |_{H^2_c(U, \mathcal{E})}}}$$

$H^2_c(U, \mathcal{E}) = H^2_c(C, R^0 j_* \mathcal{E})$ by LHS of $(*)$

$$= \frac{\det(1 - t\,\mathrm{Frob}) |_{H^1_c(C, R^0 j_* \mathcal{E})}}{\det(1 - t\,\mathrm{Frob}) |_{H^1_c(U, \mathcal{E})}}$$

$\Rightarrow$ ~~pole of~~ we need to prove $\det(1 - t\,\mathrm{Frob}_x |_{(R^0 j_* \mathcal{E})_x})^{-1}$ converge / no pole for $|t| \leq q^{-\frac{w}{2}}$

$\Leftarrow$ pole ~~...~~ of $\det(1 - t\,\mathrm{Frob}) |_{H^1_c(C, R^0 j_* \mathcal{E})}$
& zero of $\det(1 - t\,\mathrm{Frob}) |_{H^1_c(U, \mathcal{E})}$ $\Big\}$ ...

This is by trivial estimate: X^d, $\mathcal{F}$ weight $\leq w \Rightarrow L(\mathcal{F}, t)$ converge on $|t| < q^{-(\frac{w}{2} + d)}$

by write the generating function $L(\mathcal{F}, t) = \exp(\Sigma \Sigma \cdots)$

$\Rightarrow \frac{L'(\mathcal{F}, t)}{L(\mathcal{F}, t)}$ log derivative $= \sum_{e \geq 1} \sum_{x \in X(\mathbb{F}_{q^e})} \mathrm{tr}(\mathrm{Frob} |_{\mathcal{F}_x}) \underset{\text{trivial bound}}{\leq} \# X(\mathbb{F}_{q^e}) \cdot \max_{x \in X} \dim_{\overline{\mathbb{Q}}_\ell} \mathcal{F}_x$

$|\alpha| \leq (q^e)^{\frac{w}{2}}$

$= C q^{ed} \cdot q^{\frac{ew}{2}}$ ~~$\max_{x \in X} \dim \ldots$~~ done $\boxed{}$

$\Rightarrow (d = 1)$ $|t| < q^{-\frac{w}{2} + 1}$ converges

by Rankin method again, done $\boxed{}$

Thus we complete Weil I / proof of Weil conj $\boxed{}$

Application. By Weil I $\Rightarrow \# X(\mathbb{F}_q) = \sum_{i \leq 2n} (-1)^i \mathrm{tr}(\mathrm{Frob} |_{H^i_c(X)})$

(Lang-Weil Bound) $\Rightarrow$ we have ~~...~~ estimate (trivial)

$$|\# X(\mathbb{F}_q) - q^n| = \Big| \sum_{i < 2n} (-1)^i \mathrm{tr}(\mathrm{Frob} |_{H^i_c(X)}) \Big| \text{ as } H^{2n}_c(X) = \mathbb{Q}_\ell(-n)$$

Generally.

$$\leq \sum_{i \leq 2n} |\mathrm{tr}(\mathrm{Frob} |_{H^i_c(X)})| \leq \sum_{i < 2n} \dim H^i_c(X)\, q^{\frac{i}{2}} \leq \Big(\sum_{i < 2n} \dim H^i_c(X) \Big) q^{n - \frac{1}{2}}$$

Counting solution of algebraic equation

$\Leftrightarrow \# X(\mathbb{F}_q) \xleftrightarrow{\text{Weil}}$ Betti number $/ \mathbb{F}_q \underset{\text{comparison}}{\xleftrightarrow{\text{Artin's}}}$ Betti number $/ \mathbb{C}$ purely topological

- Both side is useful in partical problems!
- Due to most case half cohomology H^{n-1}_c is hard, we need use $\chi_{top} = \int e - \Sigma$ all other Betti to get it