Personal Statement

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August 22, 2025

1 Introduction

This is my personal statement concluding my interest in geometry and topology up to junior and anticipating an increasing deeper understanding in later studies.

Modern geometry and topology have grown to such a large scale with various faces now, these "faces" concern various settings differing with each other greatly, calling for quite different techniques. Hence this PS is for explaining my taste to which faces I prefer, but let's discuss a general philosophy for studying geometry next in a not formal style.

Hence, this introduction is based only on my memory and impression on my mind, it's more close to a story than an article.
Hence, I apologize for some vague or false places in this personal statement.

Different settings come from different additional structures and require methods at different levels of fineness to extract enough data to study them. More additional structures will admit more invariants, which produce more interesting geometry: these structures provide more rigidity in order to detour the difficulty of classification under a too coarse identification, and hence give richer invariants by algebraic or analysis frameworks. These additional structures are motivated by two reasons: perusing for classification in a relatively easier sense, and handling a naturally limited class of geometric objects from different areas. But these so-called "two" reasons are nothing more than your opinion on what is artificial and what is natural.

If we travel through the history of mathematics, then invariants keep track of its development for detecting different isomorphic classes, but later the so-called "invariants" are not numbers or polynomials anymore but lifted into categorical level or factor through some moduli spaces arose from original geometric object. At the same time, the demand of how fine we need the invariant to be, for example, detecting the unknot, is lifted up to the demand of how abundant the geometry of the moduli and the algebraic structure of category are. But now it's still mysterious for mathematicians to determine whether such a given exquisite structure, which may come from physics or by "magic", suffice to reflect back

to original geometry and topology or not.
One more thing worth mentioning is that these two intermediate structures are related with each other closely: one has the moduli space of objects of a given category with additional geometric structure and the groupoid of a geometric object.

Following the trace of the development of geometry, I'll list the "faces" I'm interested, learned or planned to learn later, which will be discussed in depth next, each costs one section. Here we just give some discussion far from rigorous, but state more precise mathematical settings and concrete problems I interested in the next sections.

Low-dimensional topology:
Algebraic topology studies the most general geometry objects–topological spaces, but there is no doubt that we need to add restrictions to it to detour many pathological examples, hence we equip the space with a "good model" to do homological/homotopical algebra, always simplicial. (The Model category may be a parallel trial, but to be honest, I know few about homotopical theory.) These primary invariants motivate the Poincare conjecture, which is our main line in this story, asking how these invariants can determine the geometric object itself if we focus on some kinds of manifold, i.e. how fine these invariants can be. In the view of homotopical theory, manifolds have good cover by geodesic convex neighborhoods, then we have its nerves including all simplicial data. Furthermore, if we want an analogue for CW decomposition, we have handle body decomposition for manifolds.

The Poincare conjecture states that: if a closed n manifold is homotopical to the n sphere, then it's isomorphic to the n sphere. Here we refer to the word "manifold" as topological or smooth, and then the "isomorphic" is referred to as homeomorphic or diffeomorphic, respectively.
It took more than a century to solve the topological case, but the most surprising thing was the first step around the 1960s: the h-corbordism (or Whitney trick) in dimension greater than 5 works directly prove the Poincare conjecture in these cases, showing an unexpected flexibility in higher dimensional geometric topology, one way to cancel such triviality is adding more structures.
The four-dimensional topological h-corbordism was proven later in 1982 by Freedman, but, as I said above, adding the smooth structure will break this, as smooth structures on a given manifold can be too "wild". For example, the four-dimensional Euclidean space admits infinite smooth structures. Hence, the smooth four-dimensional Poincare conjecture is still a mystery.
As for the three-dimensional case, due to a classical theorem saying that any three-dimensional manifold admits unique smooth structure, the story of topological and smooth category are equivalent and both stopped at 2003, when Perelman proved Thurston geometrization theorem via geometric analysis, more precisely, the Ricci flow, which is not my interest yet.

What attracts me most are the concrete surgeries in this story, which occur everywhere and serve as a main technique. They're full of geometric insight but may tightening by analysis languages, including moves of knots, handle decompositions or Kirby calculus, and so on. These fundamental blocks make up the topological toolbox, which would realize a change from "protagonist" to "cameo" in the next story.
An interesting thing is that, the rigidity given more to the geometric category, admissible surgeries are less due to more structures need to be preserved, for example, in symplectic category we can't do cutting and gluing so rashly but we can use Luttinger surgery; in algebraic or complex analytic category surgery is even more impossible but only blowing up can be used. Hence, less so-called "geometric insight" via flexible operations can be found under more rigidity.

The so-called "rigidity" and "softness" are nothing more than the small and large scales of automorphism groups. For example, the isometry group is usually finite-dimensional, or even zero-dimensional, i.e. discrete; but the diffeomorphism group or symplectomorphism is usually infinite-dimensional. In the moduli space's viewpoint, automorphism groups determine to what extent we can deform such a manifold with additional structure. For example, the Riemannian surface with genus g and marking point n has the automorphism group whose dimension can be negative, when the genus is larger, the dimension decreases until a finite group, thus the rigidity increases; same thing happens when n larger, thus adding markings is called rigidification.
Thus, when the automorphism is smaller, the local shape of the moduli space can determine more global properties, which is just what we say "rigidity" for **single** geometric object. In conclusion, "rigidity" and "softness" behave the same in the single and moduli levels harmonically.

Another reason is that lower-dimensional cases provide basic examples, to predict or to embed into higher-dimensional cases. Even in the two-dimensional case, i.e. surfaces case, can reflect abundant and interesting geometry: compact surfaces are decomposed into $\mathbb{S}^2$, $\mathbb{T}^2$ and $\mathbb{RP}^2$ via connected sum; as for the oriented case we needed, only $\mathbb{S}^2$ and $\mathbb{T}^2$. Then we can classify them into "no holes", "one hole" and "more than one hole", equivalent to classification under many different settings: Fano, Calabi-Yau and general type; or positive, zero and negative; or spherical, Euclidean and hyperbolic; or abelian, non-abelian and anabelian... Although we can't expect these classifications to hold for higher dimension anymore, these classifications are separately useful.

Gauge theory:
When we focus on "wild" smooth four-dimensional world, the direct method, served as "protagonist" above, is replaced by indirect way of constructing moduli spaces via global analysis, which was first magically discovered by Donaldson in his Ph.D thesis, motivated by physics, more precisely, instanton theory or Yang-Mills theory. Such a path does show our philosophy of collecting data in a more coherent way: it generalizes global analysis on manifolds to vector

bundles, enriching the target of usual functions to vector bundle valued in order to grab more data. Later, the Seiberg-Witten theory, or monopole theory, came into our stage with an occasionally and naturally closed property discovered in the moduli. Due to this automatic closeness, who came from behind is now more popular.

The relationship of global analysis and geometry and topology is a story much earlier, starting from the Gauss-Bonnet formula in differential geometry, and developed to index theorems for computing dimension, or other invariants, of moduli spaces now. Generally, it can be viewed as the pair between K-homology and K-cohomology (K-theory) compatible with the pair between periodic cyclic homology and periodic cyclic cohomology, where the former is given by analytic index and the latter is the topological index, forming a communicative diagram connected by Chern character from the K side to the cyclic side. Such a formalism of using commutative diagrams to represent formulas is also used in the Grothendieck-Riemann-Roch formula, but settings are replaced by K-theory and Chow.
The story above can be lifted to the categorical level by considering the K-theory of a triangulated category, and then the result formula will compute the dimension of the moduli of objects of this category.

But the story of classical geometric analysis is only the study of its tangent bundle and the exponential map, replace the tangent bundle by some specific vector bundles, then we expect that we get a finer theory.
Hence further, for vector bundles with more structures depending on the smooth structure of the base manifold as connections here, parameterizing these structures is expected to reflect smooth structures of the given base manifold, just analogous to that parameterizing vector bundles themselves can reflect topological invariants of the base manifold.
Although such a leading principle, or even a slogan, may not be rigorous and convincing, at least it's convincing that such method is effective in constructing refined invariants from these moduli spaces, for example, Donaldson and Seiberg-Witten invariants for four-dimensional manifolds.

Various quantum field theories are believed and verified to be powerful in different branches of mathematics, indeed viewing spaces physicists preferred as base manifold, then we can view these theories as the studies of well-chosen moduli spaces with physical meaning. It shows that the symmetry of particles in our real world is also natural in the mathematical world, for example, the condition of spin is a kind of "2-oriented" condition, i.e. making the moduli become orientable. These physics occurs everywhere, even out of geometry, which may be unbelievable at first discovery, making a deep impression on mathematics without a general reason. Although the eagerness of finding such "general reason" may be meaningless, these entanglements do make a deep impression on me.

Now the moduli of connections modulo gauge equivalence need to be compact,

the compactification is given by stability conditions: the boundary need to be added is collection of the degenerate, or equivalently, the limited ones of its interior. Hence, one can understand the moduli of monopoles are automatically compact is a pretty rare and surprising thing.
In this story, one shall note that the stability of category corresponds to the compactness of moduli of stable objects.

However, just as the same meaning as Atiyah said to Bott after he discovered index formula, we only know how to prove and compute it, but don't know why it's true, there should be a deeper discovery to explain this. The reason why the four-dimensional differential topology is so strange and challenging is still mysterious.

Symplectic geometry:
The starting point is the equivalence between handle decomposition and Morse theory, and different Morse functions related topologically by handle slide, thus viewing them built by handle bodies or level sets is essentially expected as the same thing. This also reflects in homology level, Morse homology is usually same as topological ones. Now we take the Morse viewpoint into our use in some moduli space coming from a symplectic manifold, with some functional as "Morse function", which varies in different problems, and consider its Morse homology. Recall that Morse-type homology has three ingredients: critical points of Morse function, an index for grading and an equation for flow lines, they're chosen differently and cleverly in abundant scenes.
Then these Morse-type homologies used in the lower-dimensional topology are renamed differently, as "XX" Floer homology to memorize Floer's first discovery of infinite-dimensional Morse theory. Just as the Morse homology can be viewed as a coherent algebraic form of Morse theory, these Floer homology can be viewed as a coherent algebraic form of gauge theory.
Counting the J-holomorphic curves can be viewed as a simplified case of counting flow lines in Floer theory to defining the differential, but with homogeneous Cauchy-Riemann equations.

It's Weinstein's philosophy that "All are Lagrangians". By choosing a certain type of Lagrangian in symplectic moduli spaces, all topological Floer theories in gauge theory are expected to coincide with Lagrangian Floer theory. However, such equivalence can be hopeless to be proven now, for example, the Atiyah-Floer conjecture, but it sometimes works for some simpler moduli spaces, too.
Unfortunately, with too many technical restrictions, such as the compactness and transversality of the moduli needed when defining differential, of Lagrangian Floer theory put in both the symplectic manifold and its Lagrangians, defining a category with objects consisting of Lagrangians of given type and their morphisms consisting of their correspondence or intersection, more precisely, their Lagrangian Floer theory, is still hopeless. For example, for the noncompact Lagrangians, the wrapped ones are needed, motivated by imitating the cotangent bundle. These categorifications of the Floer theory are called Fukaya categories.

Had we abstracted our mind and lifted our eyes several times by grabbing geometric data increasingly coherent, there is no doubt that our target should be algebraic.
Starting at here, I'd like to lift all homological algebra, both chain-level and homology-level, into categorical settings coherently. The A_∞-structure arises naturally in Morse theory as chain-level, with these higher associativity being given by counting "broken flow lines", which is compactification of the counting of flow lines, i.e. defining the differential. With respect to the complexity of the A_∞-structure, there is no doubt that the chain-level contains more data than the homology-level, thus one developed the classical Donaldson-Fukaya category whose morphisms are Floer homology into the Fukaya category whose morphisms are chains. As we expected, it has better properties, for example, finding generators.
In such categorical level, we have many representation-theoretic methods applied to these category, as they usually occur as some module category. For example, how can we pass chain-level categories to homology-level categories by localizations, with additional structure of chain-level ones preserved? And how can we find the split generators of the category if exists? These categorical operations are motivated by module category studied in representation theory, which is also useful in the study of algebraic derived categories, and these generators are expected to behave well under mirror symmetry with precise geometric meaning. Such algebraic work in the Fukaya category corresponding to geometric operations is viewed as a non-communicative geometry method.

The study of the sheaf category is another approach toward Fukaya category, started at Kashiwara and Schapira's microlocal study to sheaves, more precisely, the microsupport of some kinds of sheaves on cotangent bundle is coisotropic. It has ample algebraic theory, related to D-module, cluster algebra, and so on, thus which is expected to provide an alternative to tighten the construction of Fukaya category. It had be generalize to a kind of Weinstein manifolds which are close to cotangent bundles, to be equivalent to the wrapped ones.
From this viewpoint, homological mirror symmetry can be explained as the correspondence of the constructible sheaves on a real manifold and the coherent sheaves on the mirror of its cotangent bundle.

The generality of symplectic geometry lies in two roads: the first is that the moduli space usually carries a symplectic form given by the cup product of cohomology, which is just the tangent space of moduli by deformation theory; the second is the Lagrangian correspondence, or specially, the symplectic reduction, which works in the construction of moduli space, for example, the Kempf-Ness theorem. One can also pursue deeper reasons such as the mirror symmetry, as the other side is algebraic, which doesn't need to attach any importance to it, but this is another story.
Due to this motivation, my interest on symplectic geometry doesn't lie in explicit geometry and topology of certain types of symplectic manifolds but gen-

eral abstract theories, especially the algebraic methods, which are expected to be applied into some place where symplectic form is discovered naturally, and the problem we concern can be rewritten as Lagrangian correspondence. For example, the well-known Heegaard-Floer theory focuses on the symplectic manifold coming from the symmetric product of the knot surface, and its two special Lagrangians are products of generators of the first homology of two handlebodies divided by the knot surface; this Lagrangian Floer theory is relatively simple and close to combination.

As for my interest in mirror symmetry, I postpone it to the next story. One has claimed that the mirror symmetry can only indicate the "rigid" part of symplectic geometry, I can't agree more, as there is concrete example showing that the Fukaya category can recover the symplectic manifold itself. However, now my interest in this area is still in the "rigid" part and mirror symmetry.

Algebraic geometry:
The fundamental problem of algebraic geometry is classification under algebraic isomorphism, but due to its complexity, we study classification under birational equivalence, which is coarser. Geographically, fixing several birational invariants as the coordinates of a map, we have two questions to ask. Is a point on the map empty? If not, what does the moduli modulo isomorphism look like? Hence, the central problem of "geometric" algebraic geometry is studying the moduli space, besides the imitation of topological theories into the algebraic framework with higher rigidity, such as the intersection theory on algebraic setting and so on.

Mirror symmetry predicted such a correspondence via the A-model and B-model in the S-duality occurring in superstring theory, on pairs of Calabi-Yau manifolds: enumerative geometry and Hodge theory, symplectic geometry and complex algebraic geometry, derived Fukaya category and derived category of coherent sheaves, coming into their stage in different levels. Outside the Calabi-Yau cases, there are also non-geometric objects with potential serving as the mirror of non-Calabi-Yau manifolds. Just as what we said about the classification into Fano, Calabi-Yau and general type, generalizing the subtle Calabi-Yau case to more general cases and coming back to this special case is FJRW(Fan-Javis-Ruan-Witten) theory.

The harmony of algebraic and analytic is presented in not only conclusions but also techniques. For example, when we construct and prove good properties of moduli spaces, the fundamental question is taking limits, in algebraic language, the valuation criteria, then it's realized by degeneration of an algebraic family. Another example is the analogue of algebraic groups and Lie groups, and their actions and quotients, respectively.
Counting problem is also occurring in both areas, which acquires enough rigidity to make the number finite, in symplectic setting it's Cauchy-Riemann equations, in algebraic setting it's algebraic condition. From the algebraic viewpoint, it's

important to have a process of switching our mind of counting embedded curves to counting parametrized curves, more precisely, counting maps. The former is more considered in algebraic geometry as we view the embedded sub-object as the ideal sheaf structure, the latter is more considered in differential geometry due to the lack of sheaf language, here our counting of stable maps in GW(Gromov-Witten) theory is a harmony of algebraic and symplectic worlds too, which both fallen into the A-model side, while counting sheaves, for example, the DT(Donaldson-Thomas) theory, are more believed to fall into the B-model side. I suppose these two viewpoints are linked closely by DT-GW correspondence, or MNOP(Maulik-Nekrasov-Okounkov-Pandharipande) conjecture, which can also be thought of as mirror symmetry.

Out of the world of classical geometry of "spaces", it's Grothendieck's philosophy that we can consider geometric objects over more general objects, if we put the family of objects side away but consider over non-algebraic fields. Then with Galois actions equipped, the study of a given object is divided into two parts: the geometric part and the Galois action of its algebraic closure part. This can be more precisely shown in the etale fundamental group written as an "entanglement" of the topological fundamental group and the absolute Galois group. From here we first glimpse into the arithmetic geometry, which I know few things about.

My interest in algebraic geometric side majorly lies in the analogous, or essentially intersected, or dual part of geometric or topological theories, which is more down-to-earth, as otherwise the rigidity of algebraic world will make it harder to imagine through concrete examples, which are either trivial or too complicated. But what makes algebraic geometry more attractive than classical geometry is the use of abstract and coherent languages. It provides a completely new and higher viewpoint to revisit geometry, might via functors, might via sheaves and so on which helps a lot in the dream of unification thanks to Grothendieck. With such a challenging task of switching our mind in different levels of perspective, I prefer the theory of moduli spaces and enumerative problems; I also like concrete models such as toric models, but it's something hard for me to keep the geometric insight in my mind when I study related topics on the minimal model program, thus I can only take the birational geometry as something technical in construction of moduli spaces.

Derived geometry:
Due to under both the algebraic or analytic settings, the singular and stacky issues occur naturally, the derived formalism is expected to deal with them by detouring along higher structures which allow us to detect higher equivalence relations. This is what Kontsevich said in his famous hidden smoothness principle. Such a principle can also be used to explain why the theory is powerful in intersection theory, for example the construction of virtual fundamental classes: smoothness of moduli is the same thing as the perturbation toward transversal, thus adding derived structure is expected to give hidden perturbation data to

it, i.e. hidden intersection class and so on.
Motivated by the asymptotic expansion in analysis or physics, I suppose that the derived formalism should be as powerful as the Taylor expansion in analysis in a pure algebraic setting, which serves as the “approximation” by well-behaved ones–chosen as smooth ones here and via simplicial approximation language.

However, restricted by my poor knowledge to the language of simplicial methods, I can’t give any detail of this theory. I plan to study its application to moduli stack of bundles in the geometric Langlands program and intersection theory. The former one is motivated by the fact that the representation theory had taken increasingly important participation in the modern study of geometry with group action and quotient everywhere.

Starting from here, I’d like to discuss more “mathematics” instead “philosophy”, and present examples I know.

2 Low-dimensional topology

To be continued ...

3 Gauge theory

4 Symplectic geometry

5 Algebraic geometry