No.
CSN

Differential and de Rham theory in Algebraic Geometry

Tangent bundle $\Leftarrow$ Normal bundle

$\updownarrow$ dual $^\vee$ $\qquad\qquad$ $\updownarrow$ dual $^\vee$ (☆) Careful for dual. $\mathcal{H}om(-, \mathcal{O}_X)$ in AG!

cotangent bundle $\Leftarrow$ conormal bundle. We need to provide finite presentation / Noetherian $\Rightarrow$ result is $\in \mathrm{QCoh}$. Case relative $X \to S$ is easy

$\|$ differential (not closed is $X \xrightarrow{\mathcal{I}} U \subseteq Y$) discussion on absolute to generalise.

Set $X \to Y$ closed immersion with ideal sheaf $\mathcal{I}$, all of our

Normal bundle $N_{X/Y}$ ($\to X$) "is"  Y from topology

our idea is get it from

the infinitesimal neighborhood $\mathcal{I}/\mathcal{I}^2$, thus set $N_{X/Y} = \mathbb{V}(\mathcal{I}/\mathcal{I}^2)$

(Recall our functor of total space $\mathbb{V}$: {Locally free} $\xrightarrow{\sim}$ {Vector bundles}
$\mathcal{F} \longmapsto \underline{\mathrm{Spec}}(\mathrm{Sym}(\mathcal{F}^\vee))$)

• We have no reason to claim $\mathcal{I}/\mathcal{I}^2$ is locally free in general!

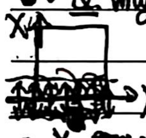
Thus we define the conormal sheaf $\mathcal{C}_{X/Y} = \mathcal{I}/\mathcal{I}^2$

(Why not normal sheaf $\mathcal{N}_{X/Y} := \mathcal{I}/\mathcal{I}^2$ as [EGA] and $\mathcal{C}_{X/Y} = (\mathcal{I}/\mathcal{I}^2)^\vee$?
Let's recall the T^*X on mfd: we believe object of "ω_i" is (equivalent) classes of functions 0-forms / function germ)

Hence then we deduce tangent bundle from normal bundle:

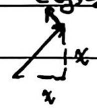
$N_\Delta \cong: T_X$
($\Delta: X \hookrightarrow X \times X$ diagonal embedding)

• Q. Why we not use a zero embedding $0: X \hookrightarrow X \times X$?



A. ① Obviously one reason is not canonical/natural: we can identify conormal vectors and tangent vectors of Δ in $X \times X$ canonically via $X \times X \longrightarrow X \times X$ of Δ $(x, x) \mapsto (x, -x)$;

② Another explaination is for $(x, x) \in \Delta$, its normal vector $(x - \varepsilon y, x + \varepsilon y)$ $= (x', x' + 2\varepsilon y)$ do depict the tangent vector $\lim_{\varepsilon \to 0} \frac{x' + 2\varepsilon y - x'}{\varepsilon}$, not that y is normal vector of Δ; for (✦) we do not have such description.



First is $x' \in X$
second is tangent vector $x' + 2\varepsilon y \in T_{x'}X$

(✱) Rk (functorial description of $\mathcal{T}_X$) Recall $T_{X,x} = \mathrm{Hom}(\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}, k)$, we rewrite it globally for sheaf-level $\mathcal{T}_X$: $\mathrm{Hom}(Y[\varepsilon], X) \cong \mathrm{Hom}(Y, \underline{\mathrm{Spec}}_X \mathcal{O}_X[\varepsilon]/(\varepsilon^2)) \to \mathrm{Hom}(Y, X)$, $Y[\varepsilon] = \underline{\mathrm{Spec}}(\mathcal{O}_Y[\varepsilon]/(\varepsilon^2))$ is infinitesimal neighborhood

$\Leftrightarrow$ T_X represent the functor
$T_X: \mathrm{Sch}^{op} \to \mathrm{Set};\ Y \mapsto \mathrm{Hom}(Y[\varepsilon], X)$

Thus is reasonable to have definitions as in textbooks:

Def 1. For X, we have tangent sheaf $\mathcal{T}_X = (\mathcal{I}_\Delta/\mathcal{I}_\Delta^2)^\vee$
and differential sheaf $\Omega_X^1 = \mathcal{I}_\Delta/\mathcal{I}_\Delta^2$

Thus we'll build properties: ① Fundamental sequences; ② Der sheaf and other properties; ③ de Rham theory.

① We not prove it as it's same as their local form

Thm 1. (First FS) $f^*\Omega_Y^1 \to \Omega_X^1 \to \Omega_{X/Y}^1 \to 0$, $f: X \to Y$ of $\mathcal{O}_X$-modules

(Second FS) $\mathcal{I}/\mathcal{I}^2 \to \Omega_X^1|_Y \to \Omega_Y^1 \to 0$, $\mathcal{I}$ is ideal sheaf of $Y \hookrightarrow X$ closed embedding / immersion of $\mathcal{O}_Y$-modules

$\mathcal{I}/\mathcal{I}^2$ are naturally given

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ \downarrow & & \downarrow \\ Y \times Y & \hookrightarrow & X \times X \\ \Omega_Y^1 & & \Omega_X^1 \end{array}$$

Thm 2. (Euler sequence) $0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}(1)^{n+1} \to \mathcal{T}_{\mathbb{P}^n} \to 0$ on $\mathbb{P}^n$

(Or $0 \to \Omega_{\mathbb{P}^n}^1 \to \mathcal{O}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0$)

Pf. See [Ha] / After the discussion on Grassmannian, it as Corollary [7.4] in [GW] ▨

We explain it's rough idea here; (where is Euler's contribution: Let $\mathbb{P}^n / \mathbb{C}$ for simplicity, then the standard vector field $(\frac{d}{dz_i})_{0 \le i \le n}$ on $(\mathbb{C}^{n+1})^\times$ descend to $\mathbb{P}^n_{\mathbb{C}}$, thus the section level it's
$\mathcal{O}(1)^{n+1} \to \mathcal{T}_{\mathbb{P}^n} \to 0$
$(v_0, \dots, v_n) \mapsto \sum v_i \frac{d}{dz_i}$ with kernel $(z_1, \dots, z_n) \in \mathcal{O}_{\mathbb{P}^n}$
as $\sum z_i \frac{d}{dz_i}$ on $(\mathbb{C}^{n+1})^\times$ descend to 0 (due to $\mathbb{P}^n_{\mathbb{C}}$ all vector has radius same)

(Euler vector field)

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Thus $0\to\mathcal{O}_{\mathbb{P}^n}\to\mathcal{O}_{\mathbb{P}^n}(-1)^{n+1}\to T_{\mathbb{P}^n}\to 0$ (section ~~$\not\Leftarrow$~~ sheaf exact, thus we need an argument to glue) $(d=\deg f)$

Recall our Euler formula in mathematical analysis $(\sum x_i \frac{\partial}{\partial x_i}) f = df$ for f is homogenous polynomial, this's where it named!

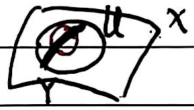
Prop3. (a) $pr_1^*\Omega^1_X \oplus pr_2^*\Omega^1_Y \cong \Omega^1_{X\times Y}$;

(b) $h: X'\to X$, $h^*\Omega^1_X \cong \Omega^1_{X'}$ natural.

Pf. See [GW] ▨

Thm4. (a) $Y\hookrightarrow X$ regular $\Rightarrow N_{Y/X}$ is locally free.

[A closed immersion $Y\xrightarrow{i} U\subset X$ is regular if $\forall y\in Y$, $\exists$ open affine nbhd $V\subset X$, s.t. $V\cap Y\subset V$ ideal is generated by regular sequence



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(b) X smooth, then diagonal embedding is regular.

Thus we can define tangent bundle well in smooth case.

Rk. X regular / k not perfect, Δ may not regular)

Pf. (a) Locally I/I^2 is a free A/I-module (locally set $X=\mathrm{Spec}A$ and $Y=\mathrm{Spec}(A/I)$)
as $(A/I)^{\oplus r}\to I/I^2$ isomorphism
$(a_1\cdots a_r)\mapsto \sum a_i x_i$ if I is regular sequence $(x_1\cdots x_r)$ ▨

(b) $\Delta: X \rightleftarrows X\times X: pr_1$ or pr_2, Δ is section of smooth map $\Rightarrow$ regular.

• $pr: X\times X\to X$ is smooth due to
$$\begin{array}{ccc} X\times X & \to & X \\ \downarrow & \lrcorner & \downarrow \\ X & \to & \mathrm{Spec}\,k \end{array}$$
is stable under base change

• Not easy to prove, we admit it ▨ [SP, Lemma 31.22.8]

We define $\mathcal{O}_X$-module $Der(\mathcal{O}_X,\mathcal{F})$ by $\Gamma(U, Der(\mathcal{O}_X,\mathcal{F}))$
for $U\subset X$ Zariski open $=\mathrm{Spec}A$ $= Der(\mathcal{O}_U,\mathcal{F}|_U)$
and $\mathcal{F}$ as $\mathcal{O}_X$-module restrict to M, $= Der(A, M)$ ← satisfy Leibniz rule

Thus by local properties, we quickly have:

Prop5. $Der(\mathcal{O}_X,\mathcal{F})\cong \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X,\mathcal{F})$ isomorphism
$u\circ d_X \leftarrow\!\shortmid u$, $d_X:\mathcal{O}_X\to\Omega^1_X$ is the Kähler differential

Thus $T_X=\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/S},\mathcal{O}_X)$ (defined by locally)
$= Der(\mathcal{O}_X,\mathcal{O}_X)$ is $\mathcal{O}_X$-Lie algebra (k the constant sheaf locally)
as sections $Der(\mathcal{O}_X,\mathcal{O}_X)$ is $\mathcal{O}_X$ Lie algebra.

Thm6. (Representability)
$$\begin{tikzcd} Hom(Y[\varepsilon],X) \arrow[rr, leftrightarrow, "\cong"] \arrow[dr] & & Hom(Y,T_X) \arrow[dl] \\ & Hom(Y,X) \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

Pf. $\forall\, Y\to X$, $Hom(-[\varepsilon],X) \xLeftrightarrow{\cong} Hom(-,T_X)$, $\searrow Hom(-,X) \swarrow$ is commutative of functors

$\xLeftrightarrow{\text{Yoneda Lemma}}$ $Hom(-[\varepsilon],X)\xLeftrightarrow{\cong} Hom(-,T_X)$, $\searrow X \swarrow$, thus it's equivalent to work in category Sch/X $\Leftrightarrow$ $Hom_X(Y[\varepsilon],X)\cong Hom_X(Y,T_X)$

$$\begin{tikzcd} & Y[\varepsilon] \arrow[r, "\tilde g"] & X \\ & Y \arrow[u, "\iota_Y"] \arrow[ur, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r] \arrow[dr, "g"'] & T_X \arrow[d] \\ & X \end{tikzcd}$$
($\tilde g\in Hom(Y,X)$ … g)

thus we need a bijection between
$\{\tilde g\in Hom(Y[\varepsilon],X)\mid \tilde g\circ\iota_Y=g\}\leftrightarrow Hom_X(Y,T_X)$

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$$\left\{ \begin{tikzcd} Y[\varepsilon] \arrow[r, "\tilde g"] & X \\ & Y \arrow[ul, "\iota_Y"] \arrow[u, "g"'] \end{tikzcd} \right\} \cong \mathrm{Der}(\mathcal{O}_X, g_*\mathcal{O}_Y)$$

by local affine case (abuse notation) $\mathrm{Der}(\mathcal{O}_X, g_*\mathcal{O}_Y) = \mathrm{Hom}_{\mathcal{O}_X}$

$$\left\{ \begin{tikzcd} B[\varepsilon]/(\varepsilon^2) \arrow[dr, "\iota_Y"'] & A \arrow[l, "\tilde g"'] \arrow[dl, "g"] \\ & B \end{tikzcd} \right\} \cong \mathrm{Der}(A, B)$$ by checking locally.

$(a \mapsto g(a) + D(a)) \leftarrow D$

$\tilde g \mapsto (D : a \mapsto \tilde g(a) - g(a))$

thus $\{\tilde g \in \mathrm{Hom}(Y[\varepsilon], X) \mid \tilde g \circ \iota_Y = g\} \cong \mathrm{Der}(\mathcal{O}_X, g_*\mathcal{O}_Y)$

$\cong \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, g_*\mathcal{O}_Y) \cong \mathrm{Hom}_X(Y, T_X)$ ▨

Def 2 ③ The de Rham complex is $\Omega^\bullet_X = \Lambda_{\mathcal{O}_X}\Omega^1_X = \bigoplus_{p\geq 0} \Lambda^p_{\mathcal{O}_X}\Omega^1_X$

with Kähler differential

$d : \Omega^\bullet_X \to \Omega^\bullet_X[1]$; $f\,da_1 \wedge \cdots \wedge da_p \mapsto df \wedge da_1 \wedge \cdots \wedge da_p$ locally

($f, a_1, \dots, a_p \in \Gamma(U, \mathcal{O}_X)$ locally)

Then the de Rham cohomology $H^*_{dR}(X) = H^*(R\Gamma(X, \Omega^\bullet_X))$

Eg 1. $A = R[T_1 \cdots T_n] \to R$, $\Omega^1_{A/R} = R[T_1 \cdots T_n]$ free A-module with basis (dT_i)

$\Rightarrow \Omega^p_{A/R}$ has basis $(dT_{i_1} \wedge \cdots \wedge dT_{i_p})_{I = (i_1 \cdots i_p)}$ ordered $\Leftrightarrow$ affine case $\mathbb{C}^n$ for $R = \mathbb{C}$ ▨

• $\det(\mathcal{E}) = \Lambda^{top}\mathcal{E}$ determinant, $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ exact induces $\det(\mathcal{E}') \otimes \det(\mathcal{E}'') \cong \det(\mathcal{E})$ obviously

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Complement talk: CY hypersurface.

- We have Euler sequence

$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0 \rightsquigarrow \Lambda^{n+1}\mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \cong \Lambda^1 \mathcal{O}_{\mathbb{P}^n} \otimes \Lambda^n T_{\mathbb{P}^n}$

$\mathcal{O}_{\mathbb{P}^n}(n+1) \cong \mathcal{O}_{\mathbb{P}^n} \otimes \omega_{\mathbb{P}^n}^{\vee}$

$\Rightarrow \omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(n+1)^{\vee} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$

- Then for $Y \subset X$ subvariety we have, $\underbrace{\ }_{\operatorname{codim} r = N-n}$ X smooth (ensures left exactness)

$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \to 0 \rightsquigarrow \Lambda^N(\Omega_X \otimes \mathcal{O}_Y) = \Lambda^n \Omega_Y \otimes \Lambda^r \mathcal{I}/\mathcal{I}^2$

$\omega_X \otimes \mathcal{O}_Y = \omega_Y \otimes \Lambda^r (N_{X/Y})^{\vee}$

$\Rightarrow \boxed{\omega_Y \cong \omega_X \otimes \Lambda^r N_{X/Y}}$ ∴ adjunction formula

- When $Y \subset X$ is hypersurface, then it's a Weil divisor $[Y] \leftrightarrow \mathcal{O}([Y]) = \mathcal{I}^{\vee}$ line bundle

$\Rightarrow \omega_Y \cong \omega_X \otimes (\mathcal{O}([Y]) \otimes \mathcal{O}_Y)$

$\Rightarrow (\mathcal{I}/\mathcal{I}^2)^{\vee} = \mathcal{O}([Y])/\mathcal{O}([Y])^2 \cong \mathcal{O}([Y]) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$

- Thus for $H \subset \mathbb{P}^n$ of degree d, $H = V(f)$, $\deg f = d$

$\omega_H \cong \omega_{\mathbb{P}^n} \otimes (\mathcal{O}(\operatorname{div} f) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_H) = \omega_{\mathbb{P}^n} \otimes \mathcal{O}(d) = \mathcal{O}_{\mathbb{P}^n}(d-n-1)$ ☑

For $d = (n+1)$ hypersurface we call it CY

- E.g. $\{t(x_0^5 + \cdots + x_4^5) + \lambda \ldots$ $\subset \mathbb{P}^4$ called Dwork pencil $\mathcal{X}$ is the first example

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