~~Theorem~~ In $(\mathbb{R}, d_E)$ [redacted], if m_1 and m_2 coincide in all balls, then $m_1 = m_2$ " This is a note for the seminar of functional analysis (H)

but in (X, d) aribitary? (We can assume all (X, d) is equipped with propability measure next)

The $\boxed{\text{paper}}$ (R Davies) Measures not approximatiable by means of balls proved its wrong!

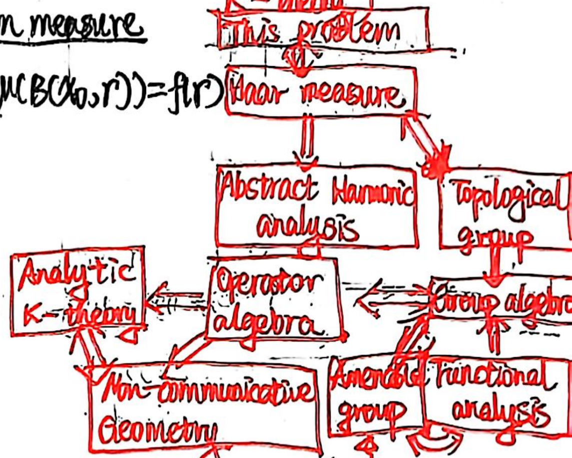
The core is that $\boxed{\text{Proprety (G)}}$ $\forall \mu, d, \varepsilon, \exists \{B_i\}_{i \le p}, d(B_i) \le d$: $\mu(X - \bigcup_{i \le p} B_i) < \varepsilon$
$B_i \cap B_j = \emptyset$

The $\boxed{\text{paper}}$ (Christensen) On some measure analogue to Haar measure gives alternative condition that
① (X, ρ) locally compact
② μ_1, μ_2 both Randon measure
③ $\exists \varepsilon_0 > 0, \forall r < \varepsilon_0, \mu(B(x_0, r)) = f(r)$

This seems not natural, but as the title of this paper, they properties are abstracted from Haar measure. (① locally compact: from definition ② Haar $\le$ Randon ③ The homogenous of topological groups)

Next we namely give the proof.

These problems (even projects) are leading for amenable group (moreover, Analytic K-theory)



Banach-Tarski "paradox"
(In fact not a parad[ox]
$S^2 = \coprod N_i$, every N_i not measurabl[e]

$\boxed{\text{PART I}}$ For Property (G).

Lemma 1. $\mathcal{O} \subset X$ open, $\forall \varepsilon > 0, \exists d > 0$: $\mu(\{x \mid \rho(x, X - \mathcal{O}) \ge d\}) \ge \mu(\mathcal{O}) - \varepsilon$

Proof. Trivial

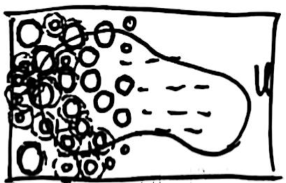
By Carathéodory $\Longrightarrow$ equivalent to prove $\mu(\mathcal{O} - \mathcal{O}_d) \le \varepsilon$

Let $d_1 > d_2 > \cdots > d_n > \cdots \to 0$

by the countinity of μ to complete the proof. $\lim_{n \to \infty} \mu(\mathcal{O} - \mathcal{O}_d) = \mu(\lim_{n \to \infty} (\mathcal{O} - \mathcal{O}_{d_n}))$ [redacted] 0 ▨

Thm 2. For $\mu_1(B) = \mu_2(B)$, $\forall B$ and [redacted] (X, ρ) satisfies (G), then $\mu_1 = \mu_2$.

Proof. It suffices to prove for Borel sets, thus open sets



All balls have upper bound of d

$\forall U \subset X$ open, it surffices to prove $\mu_1(U) = \mu_2(U)$

We choose all balls with $d(B) < d$, use (G), the left places are suffencently small

$\mu_i(U) - \mu_i(\bigcup_{k_i \le p} B_{k_i} \cap U) < \varepsilon$
$\Downarrow$
with $B_{k_i} \cap U \ne \emptyset$

But it still can't restrict to every balls because the boundary cases are quite confusing:

Thus, we use $U_d \subsetneq U$ as $U_d = \{x \mid \rho(x, X - U) \ge d\}$, apply Lemma 1

then:

because $d(B) < d$

$\Rightarrow \forall B_{k_i} \cap U \ne \emptyset$, either $B_{k_i} \subset U_d \overset{\text{denote}}{\Rightarrow} B_{k_j} \subset U_d$
or $B_{k_i} \cap U_d = \emptyset$

then: $\mu_i(\bigcup_{k_i \le p} (B_{k_i} \cap U)) - \mu_i(\bigcup_{k_i \le p} (B_{k_i} \cap U_d)) < \varepsilon$

$= \mu_i(\bigcup_{k_i \le p} (B_{k_i} \cap U)) - \mu_i(\bigcup_{k_j \le p} (B_{k_j} \cap U_d)) < \varepsilon$

$= \mu_i(\bigcup_{k_i \le p} (B_{k_i} \cap U)) - \mu_i(\bigcup_{k_j \le p} B_{k_j}) < \varepsilon$

$\sum_{k_j \le p} \mu_2(B_{k_j})$
$\quad \Vert$

$\Rightarrow \mu_i(U) - \mu_i(\bigcup_{k_j \le p} B_{k_j}) < 2\varepsilon \Rightarrow |\mu_1(U) - \mu_2(U)| \le |\mu_1(U) - \sum_{k_j \le p} \mu_1(B_{k_j})| + |\sum_{k_j \le p} \mu_1(B_{k_j}) - \mu_2(U)|$

$= 2\varepsilon + 2\varepsilon = 4\varepsilon$

$\Rightarrow \mu_1(U) = \mu_2(U)$ ▨

PART II. When not satisfy (Gii), we can construct ~~an~~ a conterexample.

Let $\Omega = \{(x_n)_{n\geq1} \mid \forall n,\ x_n \in E_n\} = \prod_n E_n$

Which $\forall n$, $E_n = \{A_{n,(i,j)} \mid 1 \leq i \leq N_n,\ 0 \leq j \leq N_n\}$ with $|E_n| = N_n(N_n+1)\}$

$A_{n,(i,0)}$ and $A_{n,(i',0)}$ are neiberhood; $A_{n,(i,0)}$ and $A_{n,(i,j)}$ are neiberhood, not translative

thus we can define a metric of Ω as $\rho((x_n)_{n\geq1}, (x_n')_{n\geq1}) = \begin{cases} \frac{1}{2^N}; & x_N \text{ and } x_N' \text{ are neiberhood} \\ \frac{1}{2^{N-1}}; & \text{not} \end{cases}$

(N is the minium index of $x_n \neq x_n'$)

Prop3. (Ω, ρ) does a metric space, and compact by $\max \rho((x_n)_{n\geq1}, (x_n')_{n\geq1}) = 1$

Prop4. $|\{A_{n,(i,0)} \in E_n \mid x_n = A_{n,(i,0)}\}| = N_n$, $|\{A_{n,(i,j)} \in E_n \mid x_n = A_{n,(i,j)}\}| = N_n^2$

Both trivial to prove.

Prop5. Denote the two sets in Prop4 to be $|A| = N_n$, $|B| = N_n^2$

Then restrict $(x_n)_{n\geq1} \in B_{\frac{1}{2^n}}$, $|A_{B_{\frac{1}{2^n}}}| = |B_{B_{\frac{1}{2^n}}}|$

Proof. $A_{B_{\frac{1}{2^n}}} \xleftrightarrow{\text{Bijection}} B_{B_{\frac{1}{2^n}}}$ given by

$$A_{n,(i,0)} \longleftrightarrow A_{n,(i,j(i))}$$

claim that $j(i)$ is uniquely fixed: or not, if $x_n^1 = A_{n,(i,j_1)}$, $x_n^2 = A_{n,(i,j_2)}$ with $j_1 \neq j_2 \Rightarrow x_n^1 \neq x_n^2 \in B_{\frac{1}{2^n}}$ $\Rightarrow \rho((x_n^1), (x_n^2)) \boxed{\geq} \frac{1}{2^n}$ $\to$ contradiction ☒

(The equality only takes when n is minium index $x_n^1 \neq x_n^2$, x_n^1 and x_n^2 are neiberhood)

To define the measure on Ω (Notice that now we don't restrict to problaity measure)

We construct inductively $\alpha_0 = \frac{2}{3} = 2\beta_0$

$\alpha_{n-1} > \beta_{n-1}$, $N_{n-1} > \frac{\alpha_{n-1}}{\beta_{n-1}}$

We define (α_n, β_n) as the solution to $\begin{cases} \boxed{N_n^2 a + N_n b = \alpha_{n-1}} \\ N_n a + N_n^2 b = \beta_{n-1} \end{cases} \Longleftrightarrow \underline{N_n^2 \alpha_n + N_n \beta_n = \alpha_{n-1}}$ (☆)

$\Rightarrow \alpha_n > \beta_n > 0$
$\downarrow \quad \downarrow \quad \downarrow$
$0 = 0 = 0$

Then let $\Omega(x_1 \cdots x_n) = \{(y_m)_{m\geq1} \in \Omega \mid (y_m)_{m\leq n} = (x_m)_{m\leq n}\} =: \Omega_n$ (with $\Omega(\phi) = \Omega$)

$(\Omega, \rho, \mu_1, \mu_2)$ then follows $\mu_1(\Omega_n) = \begin{cases} \alpha_n; & 1 \leq m \leq n,\ x_m = A_{m,(i,0)}, \text{ there are even such } m \\ \beta_n; & \text{odd } m \end{cases}$

$\mu_2(\Omega_n) = \begin{cases} \beta_n; & \text{even } m \\ \alpha_n; & \text{odd } m \end{cases}$

Prop6. Using (☆), we conclude that $\mu_i(\Omega_{n-1}) = \sum_{x_n \in E_n} \mu_i(\Omega_n)$ (either $i=1$ or 2)

Proof. Let $i=1$, for $n-1$ case, there are even m

$\Rightarrow$ LHS $= \alpha_{n-1}$ (with symmetry, other cases are similiar)

$\Rightarrow$ RHS $= \sum_{x_n = A_{n,(i,0)}} \mu_1(\Omega_n) + \sum_{\substack{x_n = A_{n,(i,j)} \\ j\neq 0}} \mu_1(\Omega_n) = \sum_{N_n} \beta_n + \sum_{N_n^2} \alpha_n = N_n^2 \alpha_n + N_n \beta_n$ $\xrightarrow{(☆)}$ proved ☒

Thm7. Inductively by Prop6, we conclude that μ_i are well-defined measure (although not probability)

To let it comes to probability measure, we let $\tilde{\Omega} = \Omega \times \{0,1\}$ with $\tilde{\mu}_i$ defined as $\tilde{\mu}_1(\Omega_n \times \{0\}) = \tilde{\mu}_2(\Omega_n \times \{1\}) = \mu_1(\Omega_n)$, $\tilde{\mu}_1(\Omega_n \times \{1\}) = \tilde{\mu}_2(\Omega_n \times \{0\}) = \mu_2(\Omega_n)$

$\tilde{\rho}$ defined as $\tilde{\rho}((x_n), 0), (y_n), 1)) = 1$

Thus we conclude that in $(\tilde{\Omega}, \tilde{\rho})$, $\tilde{\mu}_i$ coincides in balls but not coincides in all subsets of $\tilde{\Omega}$, a conterexample ☒

PART III Assume (X, ρ, μ) satisfy condition ③ always, μ_1, μ_2 coincides in Balls

$\exists\, \mathcal{O} \subset X$ open bounded with $\underline{\mu_1(\overline{\mathcal{O}}-\mathcal{O}) = \mu_2(\overline{\mathcal{O}}-\mathcal{O}) = 0}$ (#)

Review. In Real analyse, we use transfinite induction to find X's Borel sets' family can be formed by $(\mathcal{F}, \square^c, \coprod_{\alpha<\omega_1})$ countable

Proof. Let $\mathcal{S} = (\mathcal{F}, \square^c, \coprod_{\alpha<\omega_1})$, $\mathcal{B} = (\mathcal{F}, -, \bigcup_{\alpha<\omega_1})$

$\mathcal{S} \subset \mathcal{B}$ is trivial.

Now let $\mathcal{S}_\beta = \{\coprod E_n, E_1 - E_2 \mid E_n \in \mathcal{S}_\alpha, \alpha < \beta < \omega_1\}$

$\mathcal{S} = \bigcup_{\alpha<\omega_1} \mathcal{S}_\alpha \supset \mathcal{B}$ ▨

Thus one way think wether μ_1, μ_2 coincides in open sets, the answer is true for $\mathcal{O}$ satisfies property (#)

Lemma 8. $g_n : X \times \mathcal{O} \to \mathbb{R}_{\geq 0}$, then $\int_{X\times\mathcal{O}} g_n \, d(\mu_1 \otimes \mu) = \int_{X\times\mathcal{O}} g_n \, d(\mu_2 \otimes \mu)$

$(x, y) \mapsto \frac{1}{f(\frac{1}{n})}$; $\rho(x,y) < \frac{1}{n}$

$\quad\searrow 0$; otherwise

Proof. Note that $\operatorname{Im} g_n \geq 0$, thus using Tonelli's thm ▨

Prop 9. $\mu_1(\mathcal{O}) = \mu_2(\mathcal{O})$

Proof. In proof of Lemma 8, we use Fubini-Tonelli as $\int_{\mathcal{O}} \mu(dy) \int_X g_n(x,y)\, \mu_1(dx) = \int_{\mathcal{O}} \frac{\mu_1(B(y, \frac{1}{n}))}{f(\frac{1}{n})} \mu(dy) = \int_{\mathcal{O}} \frac{\mu_2(B(y,\frac{1}{n}))}{f(\frac{1}{n})} \mu(dy)$

$= RHS$

Now we use other way as

$$\int_X \mu_1(dx) \int_{\mathcal{O}} g_n(x,y)\, \mu_0(dy) = \Big(\int_{X-\overline{\mathcal{O}}} + \underbrace{\boxed{\int_{\partial\mathcal{O}}}}_{=0} + \int_{\mathcal{O}}\Big) \mu(dx) \int_{\mathcal{O}} g_n(x,y)\, \mu_0(dy) = \Big(\int_{\mathcal{O}} + \int_{X-\overline{\mathcal{O}}}\Big) \boxed{\frac{\mu(B(x,\frac{1}{n}) \cap \mathcal{O})}{f(\frac{1}{n})}} \mu(dx)$$

$n \to \infty \downarrow$ a.e. pointwise

$1_{\mathcal{O}} \Rightarrow$ "$\int_{X-\overline{\mathcal{O}}}$" vanishes

Thus, $n \to \infty$, we find $\mu_1(\mathcal{O}) = \mu_2(\mathcal{O})$ ▨

~~But we still can't find $\mu_1 = \mu_2$~~, in fact, if we admit continuum hypothesis, we have conterexample: $([0,1], \mathcal{F}_{\text{Discrete}})$

Thus the balls in it only has singleton or $[0,1]$ itself. $\mu_1([0,1]) = 1 = \mu_2([0,1])$ is natural

Thus we must have $\mu_1(\{x\}) = \mu_2(\{x\}) = 0$, or not, $\mu_i([0,1] \cap \mathbb{Q}) \to \infty > 1$, contradiction

then $|[0,1]| \gg |\mathbb{R}| = \aleph_1$, thus the existence is an open problem until we admit continuium hypothesis

(We don't prove it now by it's more set-theoretic) (In fact $|[0,1]| > 2^c$, $|[0,1]| > 2^{2^c} \cdots$)

If we throw away the existence, we easily construct different μ_1 & μ_2 by μ_1 normally defined

μ_2 normally defined in $[0,2]$

then zip up to $[0,1]$ ▨

Thm 10. We need ①②③ to ensure $\mu_1 = \mu_2$

Proof. Too hard ▨

$\underbrace{\text{Classical harmonic analysis}}$: we consider $\boxed{\begin{matrix} L^1(\mathbb{T}) \\ L^2(\mathbb{T}) \end{matrix}}$

Thesis two It's about the basic theory of convolution, fourier series, we ignore the basic matirals, which had learnt in real analysis or mathmatical...

the motivating problem $\begin{cases} ① \ f*g \text{ is differentible?} \\ ② \ \varphi : S^1 \to S^1, \text{ can we use } S_N\varphi \text{ to respresent } \deg(f) \end{cases}$

$\deg(f)$ $\begin{cases} \text{"Wing number"; called in complexo analysis} \\ \text{"mapping degree"; called in topology} \end{cases}$

PART I

① is easy to have $f \in C^k(\mathbb{T}), g \in L^1(\mathbb{T}) \Rightarrow f*g \in C^k(\mathbb{T})$ & $(f*g)^{(i)} = f^{(i)} * g$

The left thing is only construct a conterexample.

This conterexample is delighting! We want to find $f*g$ not differentible at 0 (surffiently large) but not effect other points.

We apply the technique of partion of unity, thus cutting $\mathbb{T} = [-1,1]$ into $\bigcup_{i\geq 0} I_i$ with $I_i = [\frac{1}{2^i}, \frac{1}{2^{i-1}}]$

$\Rightarrow$ We do a sum, thus $\begin{cases} x \neq 0 : \text{only finite sum of bump funotions} \\ x = 0 : \text{infinite sum of bump funotions} \Rightarrow \text{Left is give the concrete number to let } x=0 \nearrow \infty \end{cases}$

Lemma 1. $\forall I \subset \mathbb{T}$ with $\mu(I) = 1$, $\forall n \geq 0$, $\exists \varphi, \psi \in C^\infty(\mathbb{R})$ bounded in $[0,1]$ ① $\operatorname{supp}\varphi, \operatorname{supp}\psi \subset I$

② $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$

③ $\mu(\{x \in I \mid \varphi(x) = \psi(x - \frac{1}{2n}) = 1\}) = \frac{1}{4}$

Proof. Trivial, by the use of bump function, we can cut I into n parts

every part has:  $\frac{1}{n}$ ▨

Thm2. Take $f(x)=x^2\sum_{i\ge 0}\varphi_i(x)$

$g(x)=x^2\sum_{i\ge 0}\psi_i(-x)$ which φ_i,ψ_i is defined as in the Lemma1 with $I=I_i$, $n=n_i=\boxed{2^{114514i}-1}$

(Over the proof, you'll see we can take numbers we like.)

Then by our discussion, $f*g$ differentible in $x\neq 0$

Claim. $f*g$ not differentible in $x=0$

Proof. $(f*g)'(0)=\lim_{t\to 0}\frac{1}{t}[(f*g)(t)-(f*g)(0)]$

Take $t=t_i=\frac{l_i}{2n_i}=\frac{1}{2^{114514i}}\xrightarrow{i\to\infty}0 \Rightarrow \lim_{i\to\infty}\frac{1}{t_i}[(f*g)(t_i)-\underset{\substack{\text{n}\\ 0}}{\boxed{(f*g)(0)}}]=\lim_{i\to\infty}\frac{1}{t_i}\int x^2(x-t_i)^2[\underbrace{\sum\varphi_i(x)\sum\psi_i(x-t)}_{\Downarrow}]dt \ge \lim_{i\to\infty}\frac{1}{t_i}\left(\frac{1}{2i}\right)^4\frac{1}{2^{i}}\ldots$ $(x=t_i)$

This is why we multiply x^2 first

This is the function we construct in Lemma1 in I

$=\lim_{i\to\infty}2^{114514i}\left(\frac{1}{2i}-\frac{1}{2^{i}}\right)$

$\approx\lim_{i\to\infty}2^{114513i}-1\to\infty$ ▨

Thus we only need take $n_i=2^k$

with $\begin{cases}k\le -(i+1)\\ -k+1>i\end{cases}\Rightarrow k\ge(i+1)$ is OK

PART II ② is trivial to have $S(f(\mathbb{D}))=\pi\sum_{n\ge 0}n|c_n|^2$ when $f=\sum_{n\in\mathbb{Z}}c_nz^n$ injective holomorphic

Review our winding number formula $\deg f=\int\frac{f'}{f}dz$ and S is the area (or complex measure)

We claim it still holds when $\boxed{f \text{ absolutely countinous}; f'\in L^2(S^1,\mu)}$ (Condition(☆))

Then notice that $f(z)=\sum_{n\in\mathbb{Z}}c_nz^n=\sum_{n\in\mathbb{Z}}c_ne^{in\theta}$ is surprisingly a fourier series, thus we can respresent $\boxed{\deg f=\sum_{n\in\mathbb{Z}}n|c_n|^2}$ (Formula(✳))

The only thing prove (☆)$\rightsquigarrow$(✳) is proving $\deg f=\int\frac{f'}{f}dz$ holds when (☆). this is you learnt in Complex analysis.

Without (☆), we have conterexample $S^1 \overset{f}{\underset{g}{\rightrightarrows}} S^1$ which $\sum n|f_n|^2=\sum n|g_n|^2$ but $\deg f\neq\deg g$

These works are done by Kahane, Nirenberg, Bourgain, Brezis (VMO & BMO spaces in harmonic analysis)

掃描全能王 創建

$B(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ bounded}\}$ ... we first and generalize to $L^\infty(\mathbb{R})$

1. $(B(\mathbb{R}), \|\cdot\|_\infty)$ is banach. 「Trivial ☒」

2. $p : B(\mathbb{R}) \to \mathbb{R}$

$$f \mapsto \inf_{n, a_1 \cdots a_n \in \mathbb{R}} \{ N(f; a_1 \cdots a_n) \overset{def}{=} \sup_{s \in \mathbb{R}} \left(\frac{1}{n} \sum_{k \le n} f(s + a_k) \right) \}$$

(The discrete analogue of convolution) (Notice this then 4 trivial)

(Such translation had used the group structure)

p is a semi-linear functional

「Only prove $p(f+g) \le p(f) + p(g) \Leftrightarrow \forall a_1 \cdots a_m \in \mathbb{R},\ \beta_1 \cdots \beta_n \in \mathbb{R} : p(f+g) \le N(f; a_1 \cdots a_m) + N(g; \beta_1 \cdots \beta_n)$

$\Leftarrow \exists \gamma_1 \cdots \gamma_N \in \mathbb{R} : N(f+g; \gamma_1 \cdots \gamma_N) \le N(f; a_1 \cdots a_m) + N(g; \beta_1 \cdots \beta_n)$

Only take $a_1+\beta_1, a_1+\beta_2, \cdots, a_m+\beta_n$ ☒

(Notice: without measure condition)

(Then is simple verification, but notice that we use the communicative of a_i and β_j in $\mathbb{R}$, in other groups can we do?)

(Abelian is trivially holds, others?)

3. Use Hahn-Banach, $\exists M : B(\mathbb{R}) \to \mathbb{R}$ ⓪ linear such that ① it's translation invariant ② $M(G_1) = 1$

$(P) \Rightarrow (P')$ ③ $\forall f \ge 0,\ M(f) \ge 0$

Pf of ③ $\forall f \le 0,\ M(f) \le p(f) \le 0 \Rightarrow M(f) \ge 0$ ☒

「2 gives enough hints: we take the $C = \{f \equiv G_x\} \le B(\mathbb{R})$

$M_0 : C \to \mathbb{R}$. Iso($\mathbb{R}$)-invariant

$G_x \mapsto x$ has $M_0 = p \xRightarrow{H-B} \exists M : B(\mathbb{R}) \to \mathbb{R}$, ② follows.

Coro. Exist a ② probablity ③ measure and ② translation invariant ④ (positive) ① finite additive, Definited on $\boxed{\forall X \subset \mathbb{R}}$

Pf. $m(A) = M(\chi_A)$ ☒

We verify ①: $M(T_a f - f) = 0$ ($T_a : B(\mathbb{R}) \to B(\mathbb{R})$, $f \mapsto f(a + \square)$)

Generalise to all Ab, in partigular, take $\boxed{S^1}$, we have: Coro. Exist a translation and symmetry invariant finite additive measure, defined on $\boxed{\forall X \subset \mathbb{R}}$, and it's a extension of Lebesgue's

$$M(T_a f - f) \le p(T_a f - f) \le \left| N(T_a f - f, a, 2a, \cdots n a) \right| = \left| \sup_{s \in \mathbb{R}} \left\{ \frac{1}{n} \sum_{k \le n} \underbrace{[f(s + (k+1)a) - f(s + ka)]}_{\text{Can be reduced}} \right\} \right|$$

$$\left| \sup \left\{ \frac{1}{n} f(s + (n+1)a) - f(s+a) \right\} \right| \le \frac{2\|f\|_\infty}{n} \to 0 \text{ ☒}$$

Rk. In history, the extension of Lebesgue's in $\mathbb{R}^n$

1914 Hausdorff: $n \ge 3$ ✗ & Hausdorff paradox

1923 Banach: $n = 1, 2$ ✓ with giving insight of H-B thm

1924 Banach-Tariski paradox

1929 von-Neumann summarize them, find amenability

4. $\boxed{g_n = \frac{1}{2n} \chi_{[-n,n]}}$, $\forall f \in L^\infty(\mathbb{R})$

$G_n : f \mapsto (g_n * f)(0)$ smoothing, then $\lim_{n\to\infty} G_n(T_a f) - G_n(f) = 0$ 「Trivial ☒」

Notice we only need to take $\{g_n\}$ satisfy $\|g_n - \tau_b g_n\|_1 \xrightarrow{n\to\infty} 0$ and $\|g_n\|_1 = 1,\ \forall n$

Iso($\mathbb{R}^n$) amenability $n \le 2$, not $n \ge 3$

$F : \frac{\mathbb{R}}{n} \to \frac{\mathbb{R}}{n+\mathbb{Z}} \to \mathbb{R}^1$, §1.

5. Use Banach limit, $\exists M : L^\infty(\mathbb{R}) \to \mathbb{R}$ linear satisfy (P)

「$L_0 = \{f \in L^\infty(\mathbb{R}) \mid \lim_{n\to\infty} G_n(f) \text{ exists}\}$

$M_0 : L_0 \to \mathbb{R}$, $f \mapsto \lim_{n\to\infty} G_n(f)$.

$M(f) \le \|f\| \xRightarrow{H-B} \exists M : L^\infty(\mathbb{R}) \to \mathbb{R}$, ② follows; ① follows 4 ☒

define $m(A) = \sum m(A \cap [n, n+1))$ but it's not invariant about symmetry $\Rightarrow$ let $m(A) = \frac{m(A) + m(-A)}{2}$ ☒

(Dual space, indeed G_n is the pairing of dual space) (convolution)

Now we consider non-abelian cases: $\mathbb{R} \rtimes \mathbb{R}_{>0}$ for example. $(\mathbb{R} \rtimes \mathbb{R}_{>0} = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}_{>0}\})$

6. $T_{(b,a)} : B(\mathbb{R} \rtimes \mathbb{R}_{>0}) \to B(\mathbb{R} \rtimes \mathbb{R}_{>0})$. $\exists M : B(\mathbb{R} \rtimes \mathbb{R}_{>0}) \to \mathbb{R}$ satisfy (P')

$f \mapsto f(b + a\square, a\square)$. (replace ② by $T_{(b,a)}$ invariance)

「Only need define $p : B(\mathbb{R} \rtimes \mathbb{R}_{>0}) \to \mathbb{R}$,

$$f \mapsto \inf_{n, a_1, \beta_1, a_2, \beta_2, \cdots a_n, \beta_n \in \mathbb{R}} \{ N(f, a_1, \beta_1 \cdots a_n, \beta_n) \overset{def}{=} \sup_{(s,t) \in \mathbb{R} \rtimes \mathbb{R}_{>0}} \left(\frac{1}{n} \sum_{k \le n} f(b_k + a_k s, a_k t) \right) \}$$

then we verify $p(f+g) \le p(f) + p(g)$.

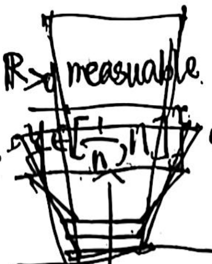
Only: $\forall \alpha_1 \beta_1 \cdots \alpha_n \beta_n, \tilde\alpha_1, \tilde\beta_1 \cdots \tilde\alpha_m, \tilde\beta_m$: $N(f+g, \alpha_i \tilde\alpha_j, \beta_i + \tilde\beta_j) \le N(f, \alpha_i, \beta_i) + N(g, \tilde\alpha_i, \tilde\beta_i)$

Translation invariance: Same as 3.

(Consider $(\tilde s, \tilde t) = \left(\frac{1}{\tilde\alpha_j}\left(s - \frac{\tilde\beta_j}{\tilde\beta_j}\right), \frac{1}{\tilde\alpha_j} t\right)$, then trivial. ☒

7. ~~[illegible]~~ $A \subset \mathbb{R} \rtimes \mathbb{R}_{>0}$ measurable, $\lambda(A) \overset{def}{=} \int_A \frac{dx\,dy}{y^2} = \int_{aA+(b,0)} \frac{dx\,dy}{y^2}$ Trivial ☒

First: Why we define measurable on $\mathbb{R} \rtimes \mathbb{R}_{>0}$



8. $A_n = \{(x,y) \mid |x| \le ny, y \in [\frac{1}{n}, n]\} \subset \mathbb{R} \rtimes \mathbb{R}_{>0}$

$g_n = \frac{1}{\lambda(A_n)} \chi_{A_n}$

$l_n : L^\infty(\mathbb{R} \rtimes \mathbb{R}_{>0}) \longrightarrow \mathbb{R}$

$f \longmapsto \int_{\mathbb{R} \rtimes \mathbb{R}_{>0}} g_n(x,y) f(x,y) \frac{dx\,dy}{y^2}$

The Haar measure

Semi-direct product

Def. (Semi-product) $H \curvearrowright G$

$\alpha : H \to \mathrm{Aut}(G)$

$G \rtimes H$ has multiply

$(g_1, h_1)(g_2, h_2) = (g_1 \alpha_{h_1}(g_2), h_1 h_2)$.

$1 \to G \to G \rtimes H \to H \to 1$ exact.

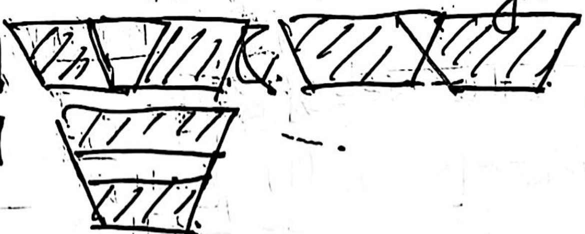
$\mathbb{R}_{>0} \curvearrowright \mathbb{R}$

Then $\lim_n [l_n(T_{(b,a)} f) - l_n(f)] = 0$ Such $T_{(b,a)}$ coincides with the additive structure of $\mathbb{R} \rtimes \mathbb{R}_{>0}$

Case I

$\lambda(A_n) = 4n\log n$, $|l_n(T_{(b,a)} f - f)| \le \frac{2\|f\|_\infty}{\lambda(A_n)} b(n - \frac{1}{n}) = \frac{\|f\|_\infty b(n - \frac{1}{n})}{2n\log n} \longrightarrow 0$

9. $T_{(b,a)}$ is composite by $\left\{ \begin{array}{l} a=0 \text{ Case I} \\ b=0 \text{ Case II} \end{array} \right.$



We do M in $\mathbb{R}$ & $\mathbb{R}_{>0}$

$1 \to \boxed{\mathbb{R}} \to \mathbb{R} \rtimes \mathbb{R}_{>0} \to \boxed{\mathbb{R}_{>0}} \to 1$

$\forall f \in B(\mathbb{R} \rtimes \mathbb{R}_{>0})$

$f_y : \mathbb{R} \to \mathbb{R}$

$x \mapsto f((0,y),(x,1))$

Case II

$|l_n(T_{(b,a)} f - f)| \le \frac{\|f\|_\infty}{\lambda(A_n)} 4n|\log a| = \frac{\|f\|_\infty |\log a|}{\log n} \to 0$

$a_y = M_{\mathbb{R}}(f_y) \in \mathbb{R}$ induces $a : \mathbb{R}_{>0} \to \mathbb{R}$ ☒

9. Similiar as 5, we finally have $\underset{\text{mean}}{\boxed{M}} : L^\infty(\mathbb{R} \rtimes \mathbb{R}_{>0}) \longrightarrow \mathbb{R}$ satisfy (p') ☒ let $M(f) = M_{\mathbb{R}_{>0}}(a)$ ☒ $y \mapsto a_y$

A functional on $B(G)$: $M : B(G) \to \mathbb{R}$ is a $\boxed{\text{invariant mean}} \iff (p')$ or (p) holds

find/construct a M on $\forall G$ is always difficult (especially abelian, we had seen)

thus we use approximation way: $L^\infty(G) = B(G)$ in 4 & 5 & 7 & 8 & 9

$M : L^\infty(G) \to \mathbb{R}$ an $\boxed{\text{almost invariant mean}}$

$\|g_n - T_a(g_n)\|_1 \xrightarrow{n\to\infty} 0$, $g_n \in L^1(G)$

$\|g_n\|_1 = 1$

Thm (From almost invariant mean to invariant mean). $\exists M : L^\infty(G) \to \mathbb{R}$. if $\exists \{g_n\}$ ...

$L^1(G) \xrightarrow{\ J\ } (L^1(G))^{**}$ isometry $\Rightarrow \|J(g_n)\| = 1 \Rightarrow J(g_n) \in B(1)$.

By Banach-Alaoglu $\Longrightarrow (J(g_n))$ has w^*-converge net

limit is M ☒

Remark. For G more general, apply L^1, L^∞ with Haar's measure (but still need locally compact T_2 topological group)

def-Thm (Haar) $\exists$ regular Borel measure μ_G ($\not\equiv 0$) is left invariant and unique up to $\pm c$ called Haar's measure

(Then supp $\mu_G = G$)

(To deal with right invariant, we need modular function)

Example. ① $G = GL_n(\mathbb{R})$

$d\mu_G(X) = (\det X)^{-n} \prod_{i,j} dx_{ij}$

② $G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}$

$d\mu_G = dx\,dy\,dz$ Heisenberg group

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The Schwarz space and Fourier transform

$\mathcal{S} \overset{def}{=} \{f \in C^\infty(\mathbb{R}) \mid \forall m,n, \sup_{x\in\mathbb{R}} |x^m \frac{d^n f}{dx^n}| < \infty\}$

We admit fact in harmonic analyse: $f \in \mathcal{S} \iff \lim_{x\to\infty} |x^m \frac{d^n f}{dx^n}| = 0, \forall m,n \iff |\frac{d^n f}{dx^n}| \le C_{k,n} \frac{1}{(1+x^2)^k}$

Then $\mathcal{S} \subset L^p(\mathbb{R}), \forall 1 \le p < \infty$, and admit $f, g \in \mathcal{S} \Rightarrow f * g \in \mathcal{S}$ and $(f*g)^{(n)} = f * g^{(n)}$

Thm1 $\overline{\mathcal{S}} = L^p(\mathbb{R})$. Pf. $C_c^\infty(\mathbb{R}) \subset \mathcal{S} \subset L^p(\mathbb{R})$

$\Rightarrow \overline{C_c^\infty(\mathbb{R})} \subset \overline{\mathcal{S}} \subset L^p(\mathbb{R})$; $\overline{C_c^\infty(\mathbb{R})} = L^p(\mathbb{R})$ ☒

Then we consider the fourier transform

$(\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)dx$, for $f \in \mathcal{S}$

Thm2 $\mathcal{F}f \in \mathcal{S}$

Pf. By def, we need compute $(\mathcal{F}f)'$, then $(\mathcal{F}f)^{(n)}$

By integral by part $\Rightarrow (\mathcal{F}f)'(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} -ixe^{-ix\xi} f(x)dx$

$\Rightarrow (\mathcal{F}f)^{(n)}(\xi) = \frac{(-i)^n}{\sqrt{2\pi}} \int_{\mathbb{R}} x^n e^{-ix\xi} f(x)dx$

and $(\mathcal{F}f')(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f'(x)dx \xrightarrow[\text{by part}]{\text{integral}} i\xi(\mathcal{F}f)(\xi)$

$\Rightarrow (\mathcal{F}f^{(n)})(\xi) = i^n \xi^n (\mathcal{F}f)(\xi)$

$\Rightarrow \xi^n \frac{d^n \mathcal{F}f}{d\xi^n} = \frac{(-i)^n}{i^m\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \underline{(x^n f(x))^{(m)}} dx \in \mathcal{S}$ ☒

Coro3. $\mathcal{F}$ is a isometry in $\mathcal{S} \subset L^2(\mathbb{R})$

Admit: ($\mathcal{F}^{-1}$ the Fourier inverse transform)

$\mathcal{F}^{-1}\mathcal{F} = \mathcal{F}\mathcal{F}^{-1} = I$ ☒

Pf. Let $h(x) = \overline{g(-x)}$

$\Rightarrow \langle f, g\rangle = (f*h)(0) = (\mathcal{F}^{-1}\mathcal{F}(f*h))(0)$

$= \sqrt{2\pi}\, \mathcal{F}^{-1}(\mathcal{F}f \cdot \mathcal{F}h)(0) = \langle \mathcal{F}f, \mathcal{F}h\rangle = \langle \mathcal{F}f, \mathcal{F}g\rangle$ ☒

Thm4. Then $\mathcal{F}$ can extended into $L^2(\mathbb{R})$, and unitary.

Pf. By $\overline{\mathcal{S}} = L^2(\mathbb{R})$, extension is trivial (by approximation)

Left is show $\mathcal{F}^* = \mathcal{F}^{-1}$ on $\mathcal{S}$: $\langle \mathcal{F}f, g\rangle = \int_{\mathbb{R}} \mathcal{F}f(x)\overline{g(x)}dx = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(y)e^{-ixy}dy\right)\overline{g(x)}dx$

$= \int_{\mathbb{R}} \overline{\int_{\mathbb{R}} g(x)e^{ixy}dx}\, f(y)dy = \langle f, \mathcal{F}^{-1}g\rangle$ ☒

Thm5. $\sigma(\mathcal{F}) = \sigma_p(\mathcal{F}) = S = \{\pm 1, \pm i\}$

Pf. The Herimite functions are the eigen vector of $\mathcal{F}$: $\mathcal{F}\varphi_n(x) = (-i)^n \varphi_n(x)$

(Recall. $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$

$\varphi_n(x) = H_n(x) e^{-\frac{x^2}{2}}$ is what we want)

by dealing with Cauchy's integral formula

$\Rightarrow \sigma_p(\mathcal{F}) \supset S$

For the converse, consider $(\mathcal{F}^4 f)(x) = \mathcal{F}^2(\mathcal{F}\mathcal{F}^{-1}(f))(-x) = (\mathcal{F}^2 f)(-x) = f(x)$

$\Rightarrow \mathcal{F}^4 = I \Rightarrow \forall \lambda \in \mathbb{C} - S$ (by $(\mathcal{F}f)(x) = (\mathcal{F}^{-1}f)(-x)$).

$\Rightarrow (\lambda I - \mathcal{F})^{-1} = \frac{\mathcal{F}^3 + \lambda\mathcal{F}^2 + \lambda^2\mathcal{F} + \lambda^3 I}{\lambda^4 - 1}$ ☒

Next, we'll prove the Fuglede-Putnam thm: Thm1. On a Hilbert space H, M, N normal and $MT = TN \Rightarrow M^*T = TN^*$ (Recall $MM^* = M^*M$, $NN^* = N^*N$)

Lemma2. $\mathcal{B}(H)$ complete then $e^X = \sum \frac{X^n}{n!}$ exists, $e^{X+Y} = e^X e^Y$ we commute

then $e^M T = Te^N$

Pf. Trivial ☒

Coro3. $U_1 = e^{M-M^*}$, $U_2 = e^{N-N^*} \Rightarrow U_1TU_2 = e^{M^*}\underset{\boxed{\text{The core}}}{Te^{-N^*}}$ $\|U_1\| = \|U_2\| = 1$ are unitary any operator

Pf. Trivial ☒

Let $f(\lambda) = e^{\lambda M^*} Te^{-\lambda N^*}$, Claim. $\|f(\lambda)\|$ bounded. (the functional caculate.)

Then we apply Liouville thm $\Rightarrow \forall g \in (\mathcal{B}(H))^*$, $gf(\lambda) \equiv C$

Thm4. $e^{\lambda M^*} T = Te^{\lambda N^*}$, $\forall \lambda \in \mathbb{C}$ Pf. Trivial ☒

Pf of Thm1. By derivative ☒ (First verify it's well-defined)

$\boxed{\text{Bergman space}}$ $\Omega \subset \mathbb{C}$, $H^2(\Omega) = \{ f \in \mathcal{O}(\Omega) \mid \int |f|^2 d\sigma < \infty \}$ a Hilbert space (proved later)

Lemma 1. $f \in H^2(\Omega)$, $z_0 \in \Omega$, $r < d(z_0, \partial\Omega)^2$, $B(z_0, r) \subset \Omega \Longrightarrow |f(z_0)| \le \frac{1}{\sqrt{\pi} r} \|f\|$

[Pf. $|f^2(z_0)| = \left| \frac{1}{2\pi i} \int_{|\xi - z_0| = \rho} \frac{f^2(\xi)}{\xi - z_0} d\xi \right| \le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|^2 d\theta \le \frac{1}{\pi r^2} \|f\|^2$ ▨]

Thm 2. $H^2(\Omega)$ is complete

[Pf. By showing $\lim_{n} f_n(z)$ is compactly uniformly converge ▨]

Prop 3. $\left\{ \frac{\sqrt{n+1}}{\sqrt{\pi}} z^n \right\}_n$ is the complete normal orthongtal system of $H^2(\mathbb{D})$

[Pf. Complete: power-seires-expansion, $\langle z^n, z^m \rangle = 0 \Rightarrow$ orthongal ▨]

Prop 4. $h: H^2(\mathbb{D}) \longrightarrow l^2$

$\left(\sum a_n z^n \right) \longmapsto \left(\frac{\sqrt{\pi}}{\sqrt{n+1}} a_n \right)$ is isometry

[Pf. Isomorphic is trivial

Preserve metric: by Parseval equality ▨]

Prop 5. $F: H^2(\Omega) \to \mathbb{C}$, by Riesz, $F(f) = \langle f, g \rangle$

and $g = \sum \langle g, f_n \rangle f_n = \sum \overline{F(f_n)} f_n$, for any $\{f_n\}$ is complete normal orthangal system ▨

Thm-Def 6. (Bergman kernel) $\exists!\, z \mapsto K(z, \xi)$, $\forall f \in H^2(\Omega)$: $f(\xi) = \int f(z) K(z, \xi) dx dy$

[Pf. $F_\xi(f) = f(\xi)$, $F_\xi(f) = \langle f, g_\xi \rangle \Longrightarrow K(z, \xi) = g_\xi(z)$ ▨] (It's the reproduce kernel in $H^2(\Omega)$)

Prop 7. ① $|K(z, \xi)|^2 \le |K(z, \bar{z})|^2 |K(\xi, \bar{\xi})|$ ② $\overline{K(z, \xi)} = K(\xi, z)$. ③ $K(z, \xi) = g_\xi(z) = \sum \overline{F_\xi(f_n)} f_n = \sum \overline{f_n(\xi)} f_n(z)$

[Pf. ① follows ③, ② by Cauchy's inequality ▨]

by Thm-Def 6, it doesn't depend on $\{f_n\}$.

Prop 8. $\Omega = \mathbb{D}$, $K(z, \xi) = \frac{1}{\pi (1 - z\bar{\xi})^2}$

[Pf. Only by Prop 3, Prop 7 ③ ▨]

*Prop 9. $\Omega = \mathbb{D}$, $F_\xi' : H^2(\mathbb{D}) \longrightarrow \mathbb{C}$

$f \longmapsto f'(\xi)$, $\forall \xi$ is countinue

and $\|F_\xi'\| = \frac{2(1+2|\xi|^2)}{\pi (1 - |\xi|)^4}$

[Pf. Expansion as power series ▨]

$\Omega = \mathbb{D}$

Coro 10. $\max |f'(\xi)| = \|F\|$, take when $f = \frac{e^{i\theta} g}{\|F\|}$ ▨

Prop 11. $x^* \in H^2(\Omega)^*$, $X = \{ f \in H^2(\Omega) \mid x^*(f) = 1 \}$, $\min_{f \in X} \|f\| = \frac{1}{\|x^*\|}$, take equality when $f = \frac{1}{\|x^*\|^2} \sum \overline{x^*(f_n)} f_n$

[Pf. Same as Coro 10 ▨]

Prop 12. $\xi \in \Omega$, $X_\xi = \{ f \in H^2(\Omega) \mid f(\xi) = 1 \}$ $f_\xi(z) = \frac{K(z, \xi)}{K(\xi, \xi)}$ takes minimal $\frac{1}{\sqrt{K(\xi, \xi)}} = \|f_\xi\|$ ▨

$\boxed{\text{Liouville number}}$ omitted.

$\boxed{\text{An assertion on Hilbert's space}}$ (Real). The complex case is similiar

Lemma. $\{ x \in H \mid \langle c_i, x \rangle \ge 0 \} \subset \{ x \in H \mid \langle b, x \rangle \ge 0 \} \Longleftrightarrow b = \sum \lambda_i c_i$

(The only hard is proving Y is convex and closed)

[Pf. ($\Leftarrow$) Trivial ($\Rightarrow$) Otherwise, denote $Y = \{ \sum a_i c_i \mid \forall \lambda_i \}$. if $b \notin Y \Rightarrow$ (By Geomertic H-B-thm) $\exists\, a \in H: \langle a, c_i \rangle \ge 0 > \langle a, b \rangle$

which is a contradiction ▨]

Now we consider a coercive bounded bilinear form a as Lax-Milgram thm

We proving: $\forall b \in H, \exists!\, x \in H: \forall y \in H: a(x, y) = \langle b, y \rangle$

and more over, $b \mapsto x$ is countinous (A representation thm by Lax-Milgram)

(Dixmier). For a Hilbert space $(H, \{e_i\}_{i \ge 0})$, there are none complete one, i.e. $\overline{\text{span}\langle e_i \rangle_{i \ge 0}} \subsetneq H$

(This involving the naive set theory, we omit it) orthogonal system

Haar measure

We'll prove main thm

Thm 1. (Existence of Haar measure) G compact, Hausdorff topological group, the $\exists!$ positive, regular Borel measure m: ① $m(G)=1$ ② $U\subset G$ open $\Rightarrow m(U)>0$ ③ $A\subset G$ a Borel set, $\forall x\in G$: $m(A)=m(xA)=m(Ax)=m(A^{-1})$

(Rk. ③ + ① $\Rightarrow$ ②)

We prove this by ① construct m by Riesz: $(C(G))^* \cong M(G)$

② Construct such a functional $M: C(G)\to G$ (the $\boxed{\text{mean}}$ function) with properties mentioned before.

$\Uparrow$ convex linear combination

Lemma 2. $\exists \sigma(C(G)) \supset \{L_s \mid s\in G\}$ [Trivial ▨]

Lemma 3. $\{L_s f \mid s\in G\}$ compact.

Pf. It suffices to prove $L: G\to C(G)$ continuous

Or not, $\exists s, \exists \varepsilon>0, \forall U\in N_s$ $s\mapsto L_s f$

$\exists s_U\in U, x_U\in G: |f(x_U s_U)-f(x_U s)|\geq\varepsilon$

$\Rightarrow$ Two net $\{x_U\}, \{s_U\}\to s$ $\} \Rightarrow s_{U_\alpha}x_{U_\alpha}\to sx$, contradiction with f continuous ▨

$\cap$ G compact $\Rightarrow \{x_{U_\alpha}\}\to x$

(Recall. (Mazur's thm) X Banach, $K\subset X$ compact $\Rightarrow \overline{co(K)}$ compact.)

Now let $K_f=\overline{H_L(f)}=\overline{co\{L_s f \mid s\in G\}}$ compact

$\Omega=\{H\subset K_f$ nonempty closed convex $\mid \forall s\in G, L_s(H)\subset H\}$, by Hausdorff maximal principal (admit axiom of choice)

(Rk. Thm 4 is most hard part)

$\Omega_0\subset\Omega$ it's the maximal chain, let $Q=\bigcap_{H\in\Omega_0} H$

Claim. Thm 4 $Q\in\Omega$ and Q is singleton

(Then $\{g\}=Q\Rightarrow L_s g=g\Rightarrow g\equiv$ Constant).

Coro 5. $\exists$ at least 1 constant can be uniformly approximate by $H_L(f)$ ▨

Similiarly, we do it for $H_R(f)$

Then we have $M: f\mapsto c$ $C(G)\to G$

(where $c\in K_f$)

Thus we guest that ① M well defined ② M is left/right invariant and inverse invariant.

Thm 6. $c1_G\in K_f=\overline{H_L(f)}$

$c'1_G\in \overline{H_R(f)}$ $\Rightarrow a=b$

$\Rightarrow K_f$'s constant function is unique.

Pf. ① $Q\in\Omega$

Q is the intersection of closed set $\Rightarrow$ closed in compact $\Rightarrow$ compact

and Q is intersection of convex set $\Rightarrow$ convex

and $L_s(Q)\subset\bigcap A=Q$.

By dual definition of G compact $\Rightarrow \neq\emptyset$ ▨

② Q singleton. If $Q\ni g_1,g_2$, let $h=\frac{g_1+g_2}{2}\in Q$ by convex

$\Rightarrow \{L_s h \mid s\in G\}\subset Q\Rightarrow \overline{co\{L_s h\mid s\in G\}}\subset Q$, by minimality

$\Rightarrow \overline{co\{L_s h\mid s\in G\}}=Q$

Claim. $\text{ext}\,Q\subset\overline{\{L_s h\mid s\in G\}}$ (By Krein-Milman's thm, $\text{ext}\,Q\neq\emptyset$)

Then $z\in\text{ext}\,Q\subset H$, $\exists L_{s_n}h\to z$

$\frac{L_{s_n}g_1+L_{s_n}g_2}{2}\to\frac{x+y}{2}$

But $z\in\text{ext}\,Q$ $\Rightarrow x=y$

$\Rightarrow L_{s_n}(g_1-g_2)\to 0$

but $\|L_{s_n}(g_1-g_2)\|=\|g_1-g_2\|$

$\Rightarrow g_1=g_2$ ▨

Pf of Claim. Otherwise $\exists p\in\text{ext}\,Q-H$

$d=\text{dist}(p,H)$, $B=\{x \mid \|x\|<\frac{d}{3}\}$

$\Rightarrow (p+B)\cap(H+B)=\emptyset\Rightarrow p\notin \overline{H+B}$

and $\exists y_1,\dots,y_n\in H: H\subset\bigcup(y_k+B)$

let $Q_k=\overline{co}(H\cap(y_k+B))$, then $\boxed{Q=\overline{co}(H)=co(Q_1\cup\dots\cup Q_n)}$ (*)

$p\in Q\Rightarrow p=\sum\lambda_j h_j$, $h_j\in Q_{k_j}$

$\exists k: p\in Q_k\Rightarrow \|p-y_k\|<\frac{d}{3}$, contradiction ▨

Pf of Condition (*) $co(H)\subset co(Q_1\cup\dots\cup Q_n)\subset Q$

$\rightarrow$ it suffices to prove $co(Q_1\cup\dots\cup Q_n)$ closed

$Q_k\subset Q$ closed, Q compact $\Rightarrow Q_k$ compact

$x_m\in co(Q_1\cup\dots\cup Q_n)$, $x_m=\sum a_j^m h_j^m\to x$

$h_j^m\in Q$, $\sum a_j^m=1$, we can let $\{h_j^m\}$, $\{a_j^m\}$ both converge (subsequence) $\Rightarrow x\in co(Q_1\cup\dots\cup Q_n)$ ▨

Pf of Thm6. $\forall \varepsilon>0$, $\exists \{a_j\}\{s_j\}\{b_i\}\{t_i\}$ $\left|\sum_j a_j f(s_j x)-a 1_G(x)\right|<\varepsilon$

$\sum a_j=\sum b_i=1$, $\{s_j\},\{t_i\}\subset G$ $\left|\sum_i b_i f(x t_i)-b 1_G(x)\right|<\varepsilon$

$\Rightarrow \left|\sum_i\sum_j a_j b_i f(s_j x t_i)-a 1_G(x)\right|<\varepsilon$, similiarly replacing $a1_G(x)$ by $b1_G(x)$

$\Rightarrow |a-b|<2\varepsilon\to 0$ ▨

Thus M is well-defined.

Lemma7. ① $M(L_s f)=M(R_s f)=M(f)=M(f^\#)$ ($f^\#(x)=f(\frac{1}{x})$)

② M is positive $\begin{cases} M(f+g)=M(f)+M(g) \\ M(af)=aM(f)\end{cases}$ Then $\mu(G)=C(G)^*\Rightarrow$ we inducing m by M

Pf of Thm1. It is verification ①+③, and deriving ②

also uniqueness ▨

Pf of Lemma7

② $M(af)=aM(f)$ is trivial

M positive also trivial

$M(f+g)=M(h)$, $h(x)=\sum_i b_i(f+g)(x t_i)$

and $|M(h)-Mf-Mg|<2\varepsilon\to 0$ ▨

① $H_L(f)=H_L(L_s f)\Rightarrow M(f)=M(L_s f)$,

R_s is same,

For $M(f)=M(f^\#)$, $\left|\sum_j a_j f^\#(y s_j^{-1})-c1_G(y)\right|<\varepsilon$

$\Rightarrow |Mf^\#(y)-Mf(y)|<\varepsilon\to 0$ ▨

Function on compact topological group.

In particular, we consider $G=\{a+bi+cj+dk \mid a^2+b^2+c^2+d^2=1\}\leq \mathbb{H}$

with its finite dimensional unity representation $\pi: G\to U(n)$ and its Haar measure μ

Then we can assumption (Or anti-representation) in general and convuduction on $L^2(G,\mu)$, $C(G)$

The Fredoholmoperator $T_K: f\mapsto \int K(g,h)f(h)\,d\mu$ is still compact, even Fredoholm

---

Peter-Weyl thm: In $L^2(G,\mu)$, the finite dimensional representation has a decomposition

---

Dixmier trace.

Lemma1. $\exists\, Lim: l^\infty\to\mathbb{R}$ satisfy that $\begin{cases}(a_n)\geq 0\Rightarrow \lim(a)\geq 0\\ Lim\, 1=1\\ Lim(a_1,a_1,a_2,a_2\cdots)=Lim(a)\ (*)\\ (a_n)\in c_0,\ \lim(a)=0\end{cases}$

Pf. By Hahn-Banach ▨

(Different the Banach Limit slightly)

(Hint. $p(a)=\overline{\lim_{n\to}}\left(a_n+\sum_{i\leq n}\frac{\sum_{2^{i-1}\leq j\leq 2^i-1}a_j}{2^{i-1}}\right)/n$, this construction comes from the verification of $(*)$)

Prop2. H complex Hilbert, $C(H)=\{T\geq 0\mid T\in K(H)\}$, then $T\in C(H)$, we have a chain $\cdots\leq\lambda_n(T)\leq\cdots\leq\lambda_1(T)\leq\lambda_1$

then we have: $\lambda_i(T)=\max_{E_i}\min_{x\in E_i,\|x\|=1}\langle Tx,x\rangle=\min_{E_{i-1}}\max_{x\in E_{i-1}^{\perp},\|x\|=1}\langle Tx,x\rangle$

($E_i\in Gr(i,H)$)

Rk. $T\geq 0 \overset{\text{Denote}}{\Longleftrightarrow} T$ is a positive operator $\Leftrightarrow \langle Tx,x\rangle\geq 0$

$\Updownarrow$ def

$\sigma(T)\subset\mathbb{R}\Rightarrow \sigma(T)\subset\mathbb{R}_{\geq 0}$

$\Updownarrow$

T is self-adjoint, apply §4.4

Pf. $T=\sum_i\lambda_i\langle\cdot,e_i\rangle e_i$, the pair (e_i,λ_i) the eigenvalue and eigenvector ($\sigma_p(T)=\sigma(T)$)

① "$\leq$" $E_i=\operatorname{span}\{f_1\cdots f_{i-1}\}$

$\exists v\in\operatorname{span}\{e_1\cdots e_i\}$, $v\perp E_{i-1}$ and $\|v\|=1$

We let $(a_{ij})=\langle e_i,f_j\rangle$ a martix A

Then $\max_{x\in E_{i-1}^{\perp},\|x\|=1}\langle Tx,x\rangle\geq\langle Tv,v\rangle\geq\lambda_i$, another is similiar

② "$\geq$" Trivial ▨

Thm3 (Dixmier trace $tr_D(T)=\sum\lambda_i(T)$)

$\lambda_1(T)+\cdots+\lambda_n(T)=O(\log n)$ (P)

Let $\mathcal{D}=\{T \text{ satisfy}(P),\ T\in\mathcal{L}(H)\}$, then U unitary

$\Rightarrow UTU^{-1}\in\mathcal{D}$

Pf. $\lambda_i(UTU^{-1})=\lambda_i(T)$, with eigenvalue is Uv ▨

Prop4. $0 \le S \le T$, $T \in \mathcal{J} \Rightarrow S \in \mathcal{J}$

Pf. ① $S \in \mathcal{C}(H)$ by $Sx_n \xrightarrow{\omega} 0 \Rightarrow \|Sx_n\| \to 0$

② Claim. $\lambda_i(S) \le \lambda_i(T)$

By $\lambda_i(S) \le \lambda_i(T)$
$\| \text{Prop2} \quad \| \text{Prop2}$
$\max \min \langle Sx, x\rangle \le \max \min \langle Tx, x\rangle$ ☒

Next, we show the property of Dixmier trace.

Lemma6. Let $(a_n(T))_n = \left(\frac{1}{\log n}(\lambda_1(T)+\dots+\lambda_n(T))\right)_n$

$f(T) = \lim a_n(T) \Rightarrow f(UTU^{-1}) = f(T)$ for U unitary

Pf. Trivial ☒

Prop5. $\sum_{i\le n}\lambda_i(T_1+T_2) \le \sum_{i\le n}\lambda_i(T_1)+\sum_{i\le n}\lambda_i(T_2)$ for $T_1, T_2 \in \mathcal{C}(H)$

Pf. [redacted] (Hard)

(Cor of $\le$: if $T_1, T_2 \in \mathcal{J}$, then $T_1+T_2 \in \mathcal{J}$)

Claim. For $T \in \mathcal{C}(H)$, we have $\sum_{i\le n}\lambda_i(T) = \sup_P\{tr(PTP) \mid P \text{ is the projections, } \operatorname{rank}(P) \le n\}$

Pf of claim. $\le$ is trivial (indeed, it's $\max_P$)

$\ge$ by $tr(PTP) = \sum_j \lambda_j \, tr(P\langle -, e_j\rangle e_j P)$ → (**)

Pf of (**). Nontrivial ☒

(Rk. $Tr(PTP) = Tr(TPP) = Tr(TP) = Tr(PT)$
You can use any one you like)

Lemma7. $f \not\equiv 0$

Pf. Let $T = \sum_{n\ge1}\frac{1}{n}\langle -, e_n\rangle e_n$ a diagonal operator $\Rightarrow \|T\| = 1$, $\|T_n\| = \|1 - \frac{1}{n+1}\|$

$\|T - T_n\| = \frac{1}{n+1} \to 0$ compact ☒

Prop8. $f(T_1+T_2) \le f(T_1)+f(T_2)$, $T_1, T_2 \in \mathcal{J}$

Pf. By Prop5 ☒

Prop9. $\sum_{i\le n}\lambda_i(T_1)+\sum_{i\le n}\lambda_i(T_2) \le \sum_{i\le 2n}\lambda_i(T_1+T_2)$, $T_1, T_2 \in \mathcal{C}(H)$

Pf of Prop9. (Hard).
Nontrivial ☒

Cor10. $T_1, T_2 \in \mathcal{J}$, $f_\omega(T_1)+f_\omega(T_2) \le f(T_1+T_2) \Rightarrow f_\omega(T_1)+f_\omega(T_2) = f(T_1+T_2)$ (Linearity)

Pf. Done in before ☒

(Notice: we use (*) to have $a_{2n}(T) \simeq a_n(T)$)

Thm11. We extend that $f: \mathcal{C}(H) \to \mathbb{R}_\infty$

$$\begin{cases} A \mapsto f(A); & A \in \mathcal{J} \\ A \mapsto \infty; & A \notin \mathcal{J} \end{cases}$$

Verification. Unitary invariant, homogenous, positive

Ex1. $c_0=\{(c_n)_{n\geq 1} \mid \lim_{n\to\infty} c_n = 0\}$; $a=(a_1,a_2,\cdots)\in l^\infty$

$T_a: l^2 \to l^2$ the multiply operator. Here l^2 means over $\mathbb{C}$, (or not (5) doesn't make sense)

$(x_1,x_2,\cdots)\mapsto(a_1x_1, a_2x_2,\cdots)$

Claim: $H\cong l^2_{\mathbb{C}}$ for H over $\mathbb{C}$, separable, infinite dimensional

Pf. $H\longrightarrow l^2$, $x\longmapsto(\langle x,h_1\rangle,\langle x,h_2\rangle,\cdots)$

$l^2\longrightarrow H$: $x=(x_1,x_2,\cdots)=(\langle h,x_1\rangle,\langle h,x_2\rangle,\cdots)\longmapsto\sum x_i\chi_i$ ▨

Rk. It's easy to generalise to any separable ~~[illegible]~~ infinite dimensional Hilbert space (l^2 is their model) By Riesz, we have

(1) $T_a\in\mathscr{B}(l^2,l^2)$, compute $\|T_a\|$

It suffice to compute $\|T_a\|<\infty$, then $\|T_ax\|\leq\|T_a\|\|x\|_2<\infty \Rightarrow T_ax\in l^2$ well defined

Pf. Claim: $\|T_a\|=\|a\|_\infty$: $\|T_a\|=\sup_{\|x\|_2=1}\|T_ax\|_2=\sup_{\|x\|_2=1}\left(\sum|a_ix_i|^2\right)^{\frac12}\leq\sup_{\|x\|_2=1}\left(\sum(\|a\|_\infty x_i)^2\right)^{\frac12}=\|a\|_\infty\|x\|_2=\|a\|_\infty$

and $\|a\|_\infty=\sup_{i\geq1}|a_i| \Rightarrow \exists\ \{i_j\}_j$: $\lim_{j\to\infty}|a_{i_j}|=\|a\|_\infty$ (Or by $T_ae_i=a_i,\forall i\Rightarrow\|T_a\|\geq\|a\|_\infty$ ▨)

Then let $x_j=(0,\cdots,\underset{\text{The } i_j \text{ th component}}{1},0\cdots)=e_{i_j}\Rightarrow\left(\sum_i(a_i(x_j)_i)^2\right)^{\frac12}\xrightarrow{j\to\infty}\lim_{j\to\infty}|a_{i_j}|=\|a\|_\infty$ $\quad\|a\|_\infty=\sup_i|T_ae_i|\leq\|T_a\|$

$\Rightarrow\|T_a\|=\|a\|_\infty$ ▨ 2) When $T_a\in\mathscr{F}_r(l^2)$?

(2) $T_a\in\mathscr{F}_r(l^2)\Leftrightarrow T_a(x)=\sum_{i\leq n}f_i(x)y_i$, $y_i\in l^2$, we take $y_i=e_i$ because T_al^2 is finite dimension, after a reorder, we can always do this $\Leftrightarrow T_a:(x_1,x_2,\cdots)\longmapsto(a_1f_1(x),\cdots f_n(x),0\cdots0)\Leftrightarrow$ There are only finite a_i not vanishing ▨

(3) $T_1=Id$ isn't compact operator ▨ (any finite rank $T=\sum\langle\cdot,e_i\rangle e_i$, we have $\|Id-\sum_{i\leq n}\langle\cdot,e_i\rangle e_i\|\geq1$ (By $\|e_{n+1}-\sum\langle e_{n+1},e_i\rangle e_i\|=1$))

$\Rightarrow Id\notin\overline{\mathscr{F}_r(l^2)}$, by Ex2(1) $\Rightarrow$ Id not compact

(4) We prove $\boxed{a\in c_0\Leftrightarrow T_a\in\mathcal{K}(l^2)}$, which is stronger than obvious conclusion. (§4.2 Ex5)

Pf. ($\Rightarrow$) $a\in c_0$, let $a_n=p_na$ the projection of the n first components, then $\|T_a-T_{a_n}\|=\|T_{a-a_n}\|=\|a-a_n\|_\infty$

$=\sup_{i\geq n+1}|a_i|\to0\Rightarrow T_{a_n}\to T_a$, by (2), $T_{a_n}\in\mathscr{F}_r(l^2)\Rightarrow T_a\in\mathcal{K}(l^2)$

($\Leftarrow$) Otherwise $a\notin c_0$: $\exists\{n_i\}\subset\{n\}$, $\{n_i\}$ infinite : $a_{n_i}\neq0$ project l^2 to these $\{n_i\}$ components, also infinite

Then a bounded set in l^2 $\underset{\text{for } n=n_i,\exists i}{B(0,\frac{1}{|a_{n_1}|})}\times\cdots\times\underset{\text{for } n\neq n_i\ \forall i}{\{0\}}\times\cdots\times B(0,\frac{1}{|a_{n_i}|})\times\cdots\times\{0\}\times\cdots\overset{\text{denote}}{=}B$, $T_aB=B(0,1)\times\cdots\times B(0,1)\times\{0\}\times\cdots\{0\}\times\cdots$

is a infinite unit ball, not sequencely compact $\Rightarrow T_a$ not map bounded set to sequencely compact

$\Rightarrow T_a$ not compact, contradiction ▨

(5) Consider T_a^*, $\|T_a^*\|$, T_a^* is normal, when T_a^* self-adjoint?

Pf. $\langle T_ax,y\rangle=\langle x,T_a^*y\rangle\Rightarrow T_a^*=T_{\bar a}$ ($\bar a=(\bar a_1,\bar a_2,\cdots)$)

$\sum a_nx_n\bar y_n=\sum x_n\overline{\bar a_ny_n}$ $\quad$ Thus $\|T_a^*\|=\|T_{\bar a}\|=\|\bar a\|_\infty=\sup|\bar a_i|=\sup_i|a_i|=\|a\|_\infty=\|T_a\|$

$T_aT_a^*=T_aT_{\bar a}=T_{\bar a}T_a=T_a^*T_a$ normal

and $T_a^*=T_a\Leftrightarrow T_{\bar a}=T_a\Leftrightarrow a=\bar a\Leftrightarrow a\in l^\infty_{\mathbb{R}}$ ▨

Now $a\in c_0$, then $T_a\in\mathcal{K}(l^2)$ (setting)

Then apply Riesz-Schauder prove

(6) T_a's eigenvalue & eigen-subspace, the non-zero eigenvalue's eigen-subspace is closed, orthogonal

(7) $\dim\ker(I-T_a)<\infty$, $\mathscr{R}(I-T_a)$ closed $\dim\frac{l^2}{\mathscr{R}(I-T_a)}<\infty$

Pf. The second assertion of (6) is Thm 4.3.1 (iii), it suffice compute eigenvalue & eigen-subspace of T_a

(6) $(\lambda I-T_a):(x_1,x_2\cdots)\mapsto((\lambda-a_1)x_1,(\lambda-a_2)x_2\cdots)$

if not injective $\Rightarrow\exists i:\lambda=a_i$

(What about $\lambda=0$? When $0\notin\{a_i\}$ It's injective, thus 0 not eigenvalue : $T_ax=0\Rightarrow a_ix_i=0,\forall i\Rightarrow x_i=0$ as $a_i\neq0$)

$\Rightarrow$ The eigenvalue is $\{a_i\}_{i\geq1}$, the eigenvector of a_i is spaned by $\{e_i\}$ ▨ The second assertion then also obvious directly

(7) If 1 not eigenvalue $\Rightarrow\ker(I-T_a)=0$, nothing to prove : $(I-T_a)(l^2)\cong l^2$ closed

If 1 is an eigenvalue, by Thm 4.3.5 (ii), $\dim\frac{l^2}{\mathscr{R}(I-T_a)}=\dim\ker(I-T_a)$

it suffices to deal with the first two assertions

$\ker(I-T_a)=\{x\in l^2\mid x=T_ax\}=\operatorname{span}\{e_i\}_{(a_i=1)}$

Then $\dim\ker(I-T_a)<\infty$ as $a\in c_0\Rightarrow\exists N:\forall n>N,|a_n|<1$

And $\ker(I-T_a)^*=\ker(T_{1-a})^*=\ker(T_{\overline{1-a}})=\ker(I-T_{\bar a})=\operatorname{span}\{e_i\}_{(\bar a_i=1)}=\operatorname{span}\{e_i\}_{(a_i=1)}$

$\Rightarrow{}^\perp\ker(I-T_a)^*=\operatorname{span}\{e_i\}^\perp_{(a_i=1)}$

and $\mathscr{R}(I-T_a)=\{(x_1,x_2\cdots)\mid x_i=0\text{ for }a_i=1\}=\operatorname{span}\{e_i\}^\perp_{(a_i=1)}\xRightarrow[\text{Range thm}]{\text{Closed}}\mathscr{R}(T)$ closed ▨

(8) $\sigma(T_a)=\{1,\frac12,\frac13\cdots\}\cup\{0\}$ for $a=(1,\frac12,\frac13,\cdots)$
and by Riesz–Schauder Thm 4.3.1 (iii) and (6) $\Rightarrow$ eigenvalue just $\{1,\frac12,\frac13\cdots\}$ ▨

(9) $\sigma(T_a)=\{0,1,\frac12,\frac13\cdots\}\cup\{0\}=\{0,1,\frac12,\frac13\cdots\}=\sigma_p(T_a)$ ▨ (Rk. For $L^2(\Omega)\xrightarrow{T_g}L^2(\Omega)$, $f\mapsto gf$ is something more technical)

Ex2. l^2, $X_n=\operatorname{span}\{e_j\}_{1\le j\le n}^{\perp}$, $T\in\mathcal{B}(l^2,l^2)$

$\varkappa_n(T)\overset{\text{def}}{=}\sup_{x\in X_n,\|x_n\|=1}\|Tx\|$

What we'll prove in (1) is that $\overline{\mathcal{F}_r(l^2)}=\mathcal{K}(l^2)$, which not holds for general Banach space

Counterexample. Pisier, Gilles Counterexamples to a conjecture of Grothendieck. Acta Math 151 (1983), no. 3-4, 181–208.

(1) $T\in\mathcal{K}(l^2)\Leftrightarrow\varkappa_n(T)\to0$.

(1) ($\Leftarrow$) $\varkappa_n(T)\to0\Rightarrow$ let $T_n=P_nT$ the $P_n:X\to X_n^{\perp}=\operatorname{span}\{e_1\cdots e_n\}$
$\|T-T_n\|=\varkappa_n(T)\to0\Rightarrow T$ approximated by $T_n\in\mathcal{F}_r(l^2)\Rightarrow T\in\mathcal{K}(l^2)$

($\Rightarrow$) First $\varkappa_n(T)\downarrow$ and ≥0, thus $\lim_{n\to\infty}\varkappa_n(T)$ exists $=\alpha$
Prove by contradiction: if $\alpha>0$
$\forall n,\exists x_n:\|x_n\|=1$ and $x_n\in X_n$ and $\|Tx_n\|\ge\frac{\alpha}{2}$, then $\{x_n\}$ is a bounded sequence, $\exists\{x_{n_j}\}\xrightarrow{\text{strongly}}x_0$
i.e. $\|Tx_{n_j}-y\|\xrightarrow{j\to\infty}0\Rightarrow\|Tx_{n_j}-y\|^2=\|Tx_{n_j}\|^2+\|y\|^2-2\langle Tx_{n_j},y\rangle=\|Tx_{n_j}\|^2+\|y\|^2-2\langle x_{n_j},T^*y\rangle$

I claim: $\langle x_{n_j},T^*y\rangle\xrightarrow{j\to\infty}0$, $\|y\|=0$

Then $\lim_{j\to\infty}\|Tx_{n_j}\|^2\to0$, a contradiction. The claim is proven by: $|\langle x_{n_j},T^*y\rangle|\le\|x_{n_j}\|\,\|P_{n_j}T^*y\|\xrightarrow{j\to\infty}0$
$\|Tx_{n_j}\|^2\ge(\frac{\alpha}{2})^2$ ($P_{n_j}:X\to X_n$)

I don't know where the (1) used in following?

(By $|\langle x_{n_j},T^*y\rangle|=|\langle P_{n_j}x_{n_j},T^*y\rangle|=|\langle x_{n_j},P_{n_j}^*T^*y\rangle|=|\langle x_{n_j},P_{n_j}T^*y\rangle|$)

And $|\langle Tx_{n_j},y\rangle|\ge|\langle\frac{\alpha}{2}x_{n_j},y\rangle|$, $\|x_{n_j}\|=1\Rightarrow y=0$ ▨

(2) $T^*T\ge S^*S$, $T\in\mathcal{K}(l^2)\Rightarrow S\in\mathcal{K}(l^2)$

Pf. $T^*T\ge S^*S\Leftrightarrow T^*T-S^*S\ge0$
$\Leftrightarrow\langle(T^*T-S^*S)x,x\rangle\ge0$
$\Leftrightarrow\|Tx\|^2-\|Sx\|^2\ge0\Leftrightarrow\|Tx\|\ge\|Sx\|$
$\Rightarrow\|S(x_n-x)\|\le\|T(x_n-x)\|$

Let $x_n\xrightarrow{w}x$, T compact $\Leftrightarrow\|T(x_n-x)\|\to0$
$\Rightarrow\|S(x_n-x)\|\to0\Leftrightarrow S$ compact ▨ (Prop 4.2.11)

(Rk. $T\ge0\Leftrightarrow T=T^*$ and $\sigma(T)\subset\mathbb{R}_{\ge0}$
$\Updownarrow$
$\langle Tx,x\rangle\ge0$ Pf. (Rudin P330 Thm 12.32)

(3) $T\ge0$, $T\in\mathcal{K}(l^2)\Rightarrow\sqrt{T}\in\mathcal{K}(l^2)$
(It's obvious that the inverse is also true)

Pf. $T=\sum_{n\ge1}\lambda_n\langle-,x_n\rangle x_n$, x_n is the eigenvector of T
(by Thm 4.4.5), with $\lambda_1>\lambda_2>\cdots>\lambda_n>\cdots>0$, $\lambda_n\to0$
$\Rightarrow\sqrt{T}=\sum_{n\ge1}\sqrt{\lambda_n}\langle-,x_n\rangle x_n=\lim_{m\to\infty}\sum_{n\le m}\sqrt{\lambda_n}\langle-,x_n\rangle x_n$ (Riesz–Schauder Thm 4.3.1 (v))
$\|\sum_{n\le m}\sqrt{\lambda_n}\langle-,x_n\rangle x_n-\sqrt{T}\|$ $\in\mathcal{K}(l^2)$ ▨
$=\|\sum_{n\ge m+1}\sqrt{\lambda_n}\langle-,x_n\rangle x_n\|\le\sqrt{\lambda_m}\to0$.

Rk. $\sqrt{T}=\sum_{n\ge1}\sqrt{\lambda_n}\langle-,x_n\rangle x_n$ is by $\sqrt{T}\sqrt{T}x=\sum\sqrt{\lambda_n}\langle\cdots,x_n\rangle x_n$
$=\sum_{n,m\ge1}\sqrt{\lambda_n}\sqrt{\lambda_m}\langle x,x_m\rangle\langle x_m,x_n\rangle x_n=\sum_{n\ge1}\lambda_n\langle x,x_n\rangle x_n$ ▨
(Only $n=m$ not zero)

(4) $T\ge S\ge0$ self-adjoint, $T\in\mathcal{K}(l^2)\Rightarrow S\in\mathcal{K}(l^2)$

Pf. $T\ge S\Leftrightarrow\sqrt{T}\sqrt{T}\ge\sqrt{S}\sqrt{S}$ $\sqrt{T}\in\mathcal{K}(l^2)$
$(\sqrt{T})^*(\sqrt{T})\ge(\sqrt{S})^*(\sqrt{S})\overset{(2)}{\Rightarrow}\sqrt{S}\in\mathcal{K}(l^2)$ (by (3))
$\Rightarrow S=\sqrt{S}\sqrt{S}\in\mathcal{K}(l^2)$ ▨

Pf. ($\Leftarrow$) T self adjoint, apply spectral decomp
$T=\int_{\sigma(T)}\lambda\,dE\Rightarrow\langle Tx,x\rangle=\int_{\sigma(T)}\lambda\,dE_{x,x}$
$E_{x,x}\ge0$, $\lambda\ge0\Rightarrow\langle Tx,x\rangle\ge0$
($\Rightarrow$) $\langle Tx,x\rangle=\langle x,Tx\rangle=\langle x,Tx\rangle\ge0$
$\Rightarrow T=T^*\Rightarrow\forall\lambda<0$,
$\lambda I-T=-(T+(-\lambda)I)$, i.e. $\lambda>0$, $T+\lambda I$
I claim. $(T+\lambda I)$ invertible for $\lambda>0$
by $\|(T+\lambda I)x\|\ge\frac{1}{\|x\|}\langle(T+\lambda I)x,x\rangle\ge\frac{1}{\|x\|}\langle\lambda x,x\rangle$

$(T+\lambda I)$ invertible $\xleftarrow{\text{Inverse Operator Thm}}$ $(T+\lambda I)^*$ surjective $\xleftarrow{\text{closed Graph}}$ Thus $(T+\lambda I)$ below bounded, and
▨ $T+\lambda I$ injective $\Leftarrow\ker(T+\lambda I)=\operatorname{Im}(T+\lambda I)^{*\perp}$