Moduli theory . Ref. Stacks and moduli by Jarod Alper ; sites.math.washington.edu/~jarod/moduli.pdf.

Topic 1. Stacks (Generalising schemes). $G \curvearrowright \mathbb{A}^n$, the quotient $\mathbb{A}^n/G$ may not scheme (But for $|G|<\infty$, it's a scheme)

Topic 2. Moduli (GIT quotient: every point is an orbit) Ex. For $G=\mathbb{Z}/2\mathbb{Z} \curvearrowright \mathbb{A}^n$ such $\mathbb{A}^n/G$ is a quotient stack. the action is involution $(x_1 \cdots x_n) \mapsto (-x_1 \cdots -x_n)$

Spirit ① Using some equivalence relation, we move the objects into parametrization space (it's quotient is "= strictly" trivial equivalence)

② Rephase the equivalence by "/G.", and record the stablizers

Riemann's work: classify compact Riemannian surfaces. $\mathcal{M}_g = \{$All compact Riemannian surfaces $|$ genus $g\}/\cong$

$g=0$. sphere; $\mathcal{M}_0=\{pt\}=\{\text{sphere}\}=\{\mathbb{CP}^1\}$ } as set-theortic

$g=1$, torus $\mathbb{C}/\mathbb{Z}^2$; $\mathcal{M}_1=\{\theta\in(0,\pi)\}=(0,\pi)$ } However, as moduli, $\mathcal{M}_0, \mathcal{M}_1$ not exist

they're both complex 0-dim. (Without base-point-free, $\mathcal{M}_1 \cong \mathbb{A}^1_{\mathbb{C}}$) we only define $\mathcal{M}_g(g\geq 2)$.

Riemann shows $\boxed{\dim \mathcal{M}_g = 3g-3}$ complex analytic space

π_1 is surjective by Riemann-Roch

Consider $\mathcal{H}_{d,g}=\{(C, C\xrightarrow{f}\mathbb{P}^1) \mid g(C)=g;\ \deg f=d\}$ the Hurwitz space

$(C,f) \mapsto \{$branch points of $f\}$ (forgetful π_1, π_2)

$\mathcal{M}_g \ni C$; $\boxed{\{\text{Branch points} \in \mathbb{P}^1\}} \subset \mathbb{P}^1$

$\Rightarrow \dim \mathcal{M}_g = \dim \mathcal{H}_{d,g} - \dim \pi_1^{-1}(C)$ $\Longrightarrow$ Hurwitz formula : $\{2d+2g-2$ points $\in \mathbb{P}^1\}$ $\Longrightarrow \boxed{\mathrm{Sym}^{2d+2g-2}(\mathbb{P}^1)}$

$= \dim \mathrm{Sym}^{2d+2g-2}(\mathbb{P}^1) + \dim \pi_2^{-1}(f) - \dim \pi_1^{-1}(C)$ by taking generic fibre

(although we even not define their dimension now)

$= (2d+2g-2) + \boxed{\dim \pi_2^{-1}(f) - \dim \pi_1^{-1}(C)}$ $\Longrightarrow$ It changes into classical problems

Moduli problem. ① Rough moduli problem. Eg 1. Moduli of $\mathbb{P}^n$: $\mathbb{P}^n$ is singleton, What it's be? it's $\{\mathbb{P}^n\}/PGL(n+1)$, $\dim \{\mathbb{P}^n\}/PGL(n+1) = -(n+1)^2+1$

Eg 2. $\mathcal{M}_{0,n}=\{(\mathbb{P}^1, x_1\cdots x_n) \mid x_1\cdots x_n \in \mathbb{P}^1\}$, $\dim \mathcal{M}_{0,n} = \dim \mathcal{M}_0 + n = n-3$ $(n\geq 3)$

② Naive moduli problem in some category (manifold / scheme / stack)

$\mathcal{M} = \{$object in category, satisfying $(*)\}/\cong$

We expect (A) $\mathcal{M}(X)\in Sch$ then, but it's not in general, thus let $\mathcal{M}(X)$ closed to Sch; or at least (A) $\mathcal{M}\in Sch$

relative analogue (Grothendieck):

$\mathcal{M}(X)=\{$relative objects over X, satisfying $(*)$ fibre wise $\}/\cong$
$(Y, Y\to X)$ $\quad \forall x\in X, f^{-1}(x)$ satisfy $(*)$

E.g. (i) $\mathcal{M}_g(X)=\{f: Y\to X$ smooth family of genus g curves$\}$

(ii) $\mathbb{P}^n(X)=\{$All $\mathbb{P}^n$-bundle $\cong \mathbb{P}(\mathcal{E})$ over $X\}$

$\Rightarrow \mathcal{M}=\mathcal{M}(\{pt\})$

$\boxed{(B)}$ $\mathcal{M}=\mathcal{M}(\{pt\})$ determine $\mathcal{M}(X)$, $\forall X\in Sch$, i.e. universal.

(The most important)

Modernly (After Gro & Mumford)

The moduli problem is functorially:

$\mathcal{M}: \mathcal{C} \longrightarrow Sets$ contravariant.

$X \longmapsto \mathcal{M}(X)$

given by $\mathcal{M}(X\to Y) = (\mathcal{M}(Y)\to\mathcal{M}(X))$

$W\to Y \quad Z\to X$

$$\begin{tikzcd} X\times_Y W \arrow[r, dashed] \arrow[d, dashed] & W \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

The fibre preserved

$\Rightarrow$ also satisfy $(*)$ relatively

Let's consider (B): consider $\mathcal{M}(\mathcal{M})$, thus we need to consider $\mathcal{C}$ over $\mathcal{M}$

$\mathcal{C} \xrightarrow{\pi} \mathcal{M}(\{pt\})$ $\quad \pi^{-1}[C] = C \subset \mathcal{C}$

$C \mapsto [C]$

$\mathcal{U}$ the universal covering

then $\forall (f: Y\to X)\in \mathcal{M}(X)$

expect : $\exists X\to \mathcal{M}(\{pt\})$ the fibres determine $\mathcal{M}(X)$ conversely $\Rightarrow$ abstract functorial

(Called moduli map) $x\mapsto f^{-1}(x)$ $\quad \{f^{-1}(x)\}_{x\in X}\subset \mathcal{M}(\{pt\})$

Q. The representable functor?

$(A)+(B) \Longleftrightarrow \mathcal{M}$ representable

and $\mathcal{M}=\mathrm{Mor}(-, \mathcal{M}(\{pt\}))$

For $\mathcal{M}(\mathcal{M}) = \mathrm{Mor}(\mathcal{M},\mathcal{M}) \ni Id$

Id is universal

We call it a fine moduli

(Later the course & good moduli)

$\mathcal{M}(X)=\mathrm{Mor}(X,\mathcal{M})$

$$\begin{tikzcd} X\times \mathcal{U} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ X \arrow[r] & \mathcal{M} \end{tikzcd}$$

$\Rightarrow$ Taking $\mathcal{U}$ the universal covering of $\mathcal{M}$

$\Rightarrow X\times \mathcal{U}\to X$ is recovered in $\mathcal{M}(X)$. $\square$

E.g. Representable: (Grassmannian) Hilbert scheme, Quot scheme

~ moduli of sheaves/bundles

Not representable : $\mathcal{M}_g$

③ Thus the fibres can't recover the whole family

$\to$ It's indeed the only obstruction: the automorphism

Conclusion. Obstruction of $\mathcal{M}$ being representable is the existence of automorphism

① no, they're only parameter space ② Some times

But take Y to be hyperelliptic curve with its automorphism σ

then all the fibres $\cong \mathbb{P}^1$ but the family $\mathcal{Y} \to \mathbb{P}^1$ not the trivial family : σ

E.g. The $\mathcal{M}_g$ and trivial family $Y\times \mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1$ $\quad f^*, \mathbb{P}^1\to \mathcal{M}_g(\{pt\})$

E.g. Without "$/\cong$" (i.e. parameter space instead of moduli space), then it must have no automorphisms

$M_3(\{pt\}) = \{E \subset \mathbb{P}^2 \mid \deg E = 3\}/\cong$; $\mathcal{H}_3(\{pt\}) = \{E \subset \mathbb{P}^2 \mid \deg E = 3\}$ is Hilbert scheme; $M_3(\{pt\})$, however, it's not representable;

$M_3(\{pt\}) = \dfrac{\mathcal{H}_3(\{pt\})}{PGL(\mathbb{P}^2)} = \dfrac{\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))}{PGL(3)} = [redacted] \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ [redacted] is projective

(By a thm due to Noether's father)

When M_g (or more general) isn't fine:

(1) Coarse moduli theory: only consider $\underline{M_g} = M_g(\{pt\}) \in Sch$ and by M_g recover M_g generally partly (The coarse moduli space)

(2) Stack theory: (First, view $X \in Sch$ as representable functor $Mor(-, X)$)

Namely, the part is due to we admit a weaker functorial property: consider moduli function (although it not determine $f: Y \to X$)

Three basic ways:
- Functor (here)
- Geometric object
- Category

(All important!)

Rk. Good moduli is more weaker than coarse moduli

Coarse moduli -- Deligne-Mumford stack

Good moduli --- Artinian stack (Algebraic stack)

M is fine/scheme	$\Leftrightarrow$ M is representable by scheme and sheaf under Zariski topology
M is ~~scheme~~ algebraic space	$\Leftrightarrow$ sheaf under étale topology
M is D-M stack	$\Leftrightarrow$ stack under étale topology
M is Artin stack	$\Leftrightarrow$ stack under smooth topology

Grothendieck topology

Rk. (Zariski $\Rightarrow$ étale)

GAGA: analytic $=$ algebraic but topology not coherent cohomology (or projective) holds. e.g. we can shrinking the open set ~~~~, then the $\boxed{\lim}$

The étale topology ($\hookrightarrow \mathcal{F}$ [redacted], it can't do by algebraic way!)

$=$ analytic topology use the alternative way: ~~be~~ by covering: pullback to a big open set and covering to a smaller one!

E.g. the defining equation $y^n = z^m$, $n, m \in \mathbb{R}$, when $n, m \in \mathbb{Q}$, it's an algebraic space, but when $n, m \notin \mathbb{Q}$, it's more bad, but occurs in study of moduli some times;

Q. Give M the stack structure? $\mathcal{U} \to M$ the $T_x M$ is what?

/the "Geometry"? Fibre $\mathcal{U}_x \mapsto x$ $T_x M = Df_x$: the deformation space of x (locally study)

E.g. (Representable functors) ① $\Gamma: Sch_k \to Set^{op}$

$X \mapsto \Gamma(X, \mathcal{O}_X) = Hom(X, \mathbb{A}_k)$

② Picard functor usually representable

① Need restriction

② May represented by formal scheme

$Pic: Sch_k \to Set^{op}$

$X \mapsto Pic X = H^1(X, \mathcal{O}_X^*)$

and $H^2(X, \mathcal{O}_X^*)$ the Brauer group

③ Grassmannian functor

$Gr_B(r, n): Sch_B \to Set^{op}$, Given V vector bundle on B

$X \xrightarrow{f} \mapsto \{$quotients of f^*V, rank $r\}$

$= \{f^*V \twoheadrightarrow W \to 0 \mid rank(W) = r\}$ coker

ker $\cong \{0 \to W \hookrightarrow f^*V \mid rank(W) = r\}$

(It's the older definition, but functor is worse)

$Gr(r, n) = Gr_{\{pt\}}(r, n)(\{pt\})$; and we usual write $Gr_B(r, V)$ better

Q. Why we not outline using V not n? A. It's representable by a scheme, not depend on [redacted] V!

Notice. Projective and strongly projective morphism

[Ha] Projective: $X \xrightarrow{\pi} B$ is $X \xrightarrow{\pi} B$ it's the strong projective here $\mathbb{P}^n$

$\Leftrightarrow X \xrightarrow{\pi} B$; $E = \pi_*(\mathcal{L}^{\otimes n})$ on B, $\mathbb{P}(E)$

$\Leftrightarrow \exists$ relatively ample line bundle $\mathcal{L}$ on X ($\mathcal{L}|_{X_b}$ ample on X_b) Trivial

The projective (weak projective) $X \xrightarrow{\pi} B$ is fibrewise,

$\Leftrightarrow X_b$ projective space $\Leftrightarrow X_b$ admits ample line bundle

④ $P_{\mathcal{L}}(z) = \chi(\mathcal{L}^{\otimes z}|_{X_b})$ the Hilbert polynomial respect to $\mathcal{L}$

$Hilb_B^{P(z)}: Sch_B \to Set^{op}$, fixed X/B projective flat family

$Y \mapsto \{$Subscheme of $X \times_B Y = X_Y$, flat over Y and relative $P(z)$ $\}$

$= \{Z \subset X_Y$, $P_Z(z) = \chi(Z_b, \mathcal{L}_{Z_b}^{\otimes z})$, $\pi_Y: Z \to Y$ flat, $\rightsquigarrow \mathcal{L}$, $\equiv P(z)\}$

All these ③—⑤ is without quotient $/\cong$, usually it's all admitted representable (no automorphism)

Rk. $P(z)$ isn't essential, essential is the first two. We many replace it by ① invariant in flat family component ② Bounding the family by topological property

$Hilb_k^{P_d(z)}(Spec\, k) = \{$hypersurface of degree $d\} = |\mathcal{O}_{\mathbb{P}^n}(d)| = \mathbb{P}^N$

$Hilb_k^{P_d(z)+N}(Spec\, k) = \{$hypersurface of degree $d \cup \{pt_1\} \cup \cdots \cup \{pt_N\}\}$

$Quot_{X/B}^{P(z)}: Sch_B \to Set^{op}$

$Y \mapsto \{\mathcal{F}_Y \twoheadrightarrow Q \to 0$, Q coherent on X_Y, flat over Y, $P_{Q_y}(z) \equiv P(z)$, $\forall y \in Y\}$

⑤ (Quot scheme) Rk. Hilb scheme is used to build (may intersected) up moduli of ~~~~ varieties (or Chow scheme); but Quot scheme is for schemes/polarized sheaf/vector bundles

Setting. $\pi: X \to B$ (strongly projective $\mathcal{O}(1)$ very ample, $\mathcal{F}$ coherent) it's quotient of $\pi^* V$, V is vector bundle on B

Rk. Quot generalizing Gr and $Hilb$

① Gr: let $\mathcal{F}$ be vector bundle V

② $Hilb$: let $\mathcal{F}$ be (take kernel)

Construction of $\mathcal{M}_g$: $\mathcal{M}_g : Sch \to Set^{op}$

Our then'll prove the representable of Gr, but omit Hilb, Quot and Castelnuovo–Mumford regularity's proof, in order to quickly come into stack.

$C \mapsto \{\mathcal{X} \to C \mid \mathcal{X}$ is family of same genus curves$\}$

$K_{\mathcal{X}/C}|_{\mathcal{X}_t} = K_{\mathcal{X}_t}$, $\forall t \in C$ relatively ample

Furthermore, when $m \geq 3$, it comes to very ample: $\mathcal{X} \xrightarrow{f} C$, $E = f_*(K_{\mathcal{X}/C}^{\otimes m})$, $\mathbb{P}(E)$

(Our first approach is to "forget" the quotient) $\mathcal{X}_C \hookrightarrow |K_{\mathcal{X}_t}^{\otimes m}|$ is very ample linear system, let $m=3$

$\Rightarrow \mathrm{Hilb}^{P_g(z)} = \{$curves, $g(C)=g$, $P(z)=P_g(z)$ and $C \subset \mathbb{P}^N = |K_{\mathcal{X}_t}^{\otimes 3}|\}$ (not dependent)

then the automorphism of C is the automorphism of $\mathbb{P}^N \Rightarrow \mathrm{Hilb}^{P_g(z)} / \mathrm{Aut}(\mathbb{P}^N) \overset{?}{=} \mathcal{M}_g$, $\mathrm{Aut}(\mathbb{P}^N) = PGL_N(\mathbb{C})$ (This's why GIT!)

① the quotient stack $[\mathrm{Hilb}^{P_g(z)} / PGL_N(\mathbb{C})]$

② the GIT quotient scheme $\mathrm{Hilb}^{P_g(z)} /\!/ PGL_N$

open open

Pf. Assume $B = \mathrm{Spec}\, k$ (Geometric points $k = \bar{k}$), we show a stronger consequence: represented by variety (not scheme!)

(Idea. Using representable, open subfunctor to cover Gr, gluing algebraically)

Def. $Gr_I : Sch_k \to Set^{op}$ (Here over k, every bundle trivial $\Rightarrow$ free sheaf)

$T \mapsto \{\mathcal{O}_T^I \hookrightarrow \mathcal{O}_T^{\oplus n} \to Q \to 0\}$

$\mathrm{rank}\, Q = r = |I|$, Gr_I make Gr_r relates to the basis choice

Lemma. Gr_I representable by $\mathbb{A}_{\mathbb{Z}}^{k(n-k)}$

Pf. We identify Gr_I with $\widetilde{Gr}_I^!$: $T \mapsto \{\mathcal{O}_T^{\oplus n} \to \mathcal{O}_T^I \to 0\}$

$= (T \mapsto \prod \Gamma(T, \mathcal{O}_T^{\oplus n}))$ is representable due to $\Gamma(-, \mathcal{O}_-)$ is! ▨

Let's see a example to understand why cover?

E.g. $\mathbb{P}^{n-1} = Gr(1,n)$ covered by $\mathbb{A}^n = \{x_i \neq 0\}$

$= \{1\text{-dim } L \subset V = k^n\} = \{1\text{-dim } L \subset V = k^n \mid L \cap V_i = 0\}$ (V_i is $x_i \neq 0$ in V)

For the rest, $Gr_I \cap Gr_J \Rightarrow (Gr_I)_{|I|=r}$ glued into Gr_r ▨

$\Rightarrow Gr(r,n)$ covered by $\mathbb{A}_I^n = \{1\text{-dim } L \subset V = k^n \mid L \cap V_I = 0\}$, $|I| = r$

Idea of Quot. View $\mathrm{Quot}_B^{P(z)}(\mathcal{F})$ as a locally closed subfunctor of $Gr_B(r, V)$ for some V, by following:

$\forall d \gg 0$, $\mathrm{Quot}_B^{P(z)}(\mathcal{F}) \hookrightarrow Gr(P(d), \pi_*\mathcal{F}(d))$.

$(\mathcal{F}_C \twoheadrightarrow Q) \mapsto ((\pi_C)_*\mathcal{F}_C(d) \to (\pi_C)_*(Q_C(d)))$

Castelnuovo–Mumford regularity

Our core step is locally free $\Leftarrow$ [the $d \gg 0$ vanishing not depend on base B (descent down)]

E.g. $\mathcal{M}_{1,1}$ the moduli of elliptic curves.

$\mathcal{M}_{1,1} : Sch \to Set^{op}$

$C \mapsto \{f : E \to C, 0\}$

Def. An elliptic curve over T is triple $(E, f, 0)$: $\{f : E \to T$, $0 : T \to E\}$ s.t. ① f proper, smooth of relative dim 1. ② $\forall t \in T$, fibre E_t is connected of genus 1 (by ① dim 1 still) ③ $f \circ 0 = \mathrm{Id}$ (0 is section)

(When $T = \mathrm{Spec}\, k$ degenerates to $(E, 0)$)

④ $\mathcal{M}_{1,1}$ not representable due to $\exists (E, 0)$, $\mathrm{Aut}\, E \neq \{\mathrm{Id}\}$ (involution over $\mathbb{C}$) "stacky issue"

Facts ① Over $\mathbb{C}$, E genus 1 $\Leftrightarrow E = \mathbb{C}/\Lambda$

② $(E, 0)$ the elliptic curve $\Rightarrow E \hookrightarrow |30| = \mathbb{P}^2$ (30 very ample)

Such $|30|$ is the all elliptic curves is cubic curve, with $\mathbb{P}^2$ fixed. then elliptic curve $\leftrightarrow$ cubic curve (taking 3-deg point to be 0)

⑧ $(E, f, 0) \cong (E', f', 0')$ means

$$\begin{tikzcd} E \arrow[r, "\cong"] \arrow[d] & E' \arrow[d] \\ T \arrow[u, bend left, "0"] \arrow[r, equal] \arrow[ur, phantom, "\circlearrowleft"] & T \arrow[u, bend right, "0'"'] \end{tikzcd}$$

commutes or strongly, Cartesian diagram

(data of E) (data of E').

⑤ The coarse moduli $M_{1,1} \cong \mathbb{A}^1$

$(E, 0) \mapsto$ the j-invariant of E

⑥ The moduli stack adds the Sch category new object $\mathcal{M}_{1,1}$ $\Rightarrow \mathcal{M}_{1,1}$ is represented by itself

also we need $f : \mathcal{E} \to \mathcal{M}_{1,1}$ and $f : E \to B$ Sch

$\mathcal{F} : \mathcal{E} \to \mathcal{M}_{1,1}$ Enlarged (stack) category $\Rightarrow$ Later, we'll use alternative way to define the stacks.

⑦ $\mathcal{M}_{1,1} \triangleq$ quotient stack $[\mathcal{U}/PGL_3(k)]$

$\mathcal{U} \subset |\mathcal{O}_{\mathbb{P}^2}(3)|$ open subsets

$\mathcal{U} = \{$containing all smooth cubic curves$\}$

$= |\mathcal{O}_{\mathbb{P}^2}(3)| - \Delta$ (discriminant divisor)

$= \{$singular cubic curves$\}$

And satisfy axioms (not too bad).

$$\begin{tikzcd} E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ B' \arrow[r] & B \arrow[r] & \mathcal{M}_{1,1} \end{tikzcd}$$

(One can view as a 2-category) Stable under base change.

Rk. However, the topology of $\mathcal{M}_{1,1} = \mathcal{U}/PGL_3(\mathbb{C})$ can never be Zariski! (the not discrete action is quite analytic (orbifold))

(It can't be really "covering" (surjective) or be criterioned)

Sites and stacks

Three ways { Moduli functor — Yoneda Lemma, Representable, enlarge to add stacks; Geometric object; Category itself — topology of Geometry object induce a $\mathcal{U}$ open set category }

(We enlarge $\mathcal{U}$ to carry more, to recover Geometry)

Def. (Grothendieck topology) Fix $\mathcal{C}$ category, give data: $\forall X \in \mathcal{C}$ (object), $\exists\, \mathrm{Cov}(X)$ a "set": $\mathrm{Cov}(X) = \{f_i : U_i \to X\}_{i \in I}$

s.t. ① (Identity) $(f : Y \cong X) \in \mathrm{Cov}(X)$ ② (Restriction) $\{f_i : U_i \to X\} \in \mathrm{Cov}(X)$, $Y \to X \Rightarrow \{U_i \times_X Y \to Y\} \in \mathrm{Cov}(Y)$

③ (Composition) $\{f_i : U_i \to X\} \in \mathrm{Cov}(X)$, $\{g_{ij} : U_{ij} \to U_i\} \in \mathrm{Cov}(U_i)$ $\Rightarrow \{f_i \circ g_{ij} : U_{ij} \to X\} \in \mathrm{Cov}(X)$

$\Rightarrow (\mathcal{C}, \{\mathrm{Cov}(X)\}_X)$ is a site.

E.g. Zariski site (small) of scheme X, the open sets category:

$\mathrm{Top}(X_{Zar}) = \{$open immersions into X | morphism natural$\}$

E.g. Étale sites (small) $\mathrm{Top}(X_{\acute{e}t}) = \{$étale morphism into $X \mid$ morphism natural$\}$

small smooth sites ... or $\boxed{fpqc, fppf}$ importantly (French)

faithfully flat (fp)	faithfully flat (fp)
quasi-compact (qc)	locally finite type (p)
locally finite presentation	finite presentation (f)
(More strong)	

E.g. Big Zariski site / Big Zariski site

Set $\mathcal{C} = Sch$, $X \in Sch$, $Cov(X) = \{$small sites on $X\}$

If { Let small Zariski $\Rightarrow Sch_{Zar}$; Let small Étale $\Rightarrow Sch_{\acute{e}t}$ } Both big sites.

Others: Big smooth sites Sch_{sm}, Sch_{fpqc}, Sch_{fppf} (it automorphically big)

Def. (Sheaf on site) presheaf is just $F: \mathcal{C} \to Set^{op}$.

Sheaf is $F: (\mathcal{C}, Cov(X))_{X \in \mathcal{C}} \to Set^{op}$ (Left is injective)

s.t. $F(X) \to \prod F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j)$ is equalizer

for $\forall \{f_i : U_i \to X\} \in Cov(X)$ and $\forall X \in \mathcal{C}$

Consider the category of sheaves on a site fixed.

Morphism of sheaves: ($\hookrightarrow$ Grothendieck topos)

Def. Presheaf is just natural transformation; sheaf also. $sh \dashv F$ (forgetful)

We need determine $M \in Sh(Sch_{\acute{e}t})$ whether representable. $sh: Psh(\mathcal{C}) \to Sh(\mathcal{C})$

Thm. $\mathcal{P}$ is a property of morphism. $X \to Y$ surjective & smooth

$F: Sch_{\acute{e}t} \to Set^{op}$ sheaf. $F_X \xrightarrow{\varphi} X = Mor(-,X)$ if F_X scheme, φ has $\mathcal{P}$

(Usually, we can take affine to test)

$$\begin{array}{ccc} F_X & \xrightarrow{\varphi} & X = Mor(-,X) \\ \downarrow & \square & \downarrow \\ F & \xrightarrow{\varphi} & Y = Mor(-,Y) \end{array} \Rightarrow F \text{ scheme, } \varphi \text{ has } \mathcal{P}$$

Pf. Descent □

Rk. When fine moduli $M \cong Mor(-, M): Sch_{Zar} \to Set^{op}$ is a sheaf $\Rightarrow$ thus sheaf is generalization of representable.

over Sch_{Zar}, $\boxed{Sch_{\acute{e}t}, Sch_{fppf}}$ $\Rightarrow$ need descent theory (Better)

For $M_g, M_{1,1}$ moduli of sheaves not representable; For $M_g, M_{1,1}$: Using étale base change to kill the automorphisms.

[fpqc $\geq$ smooth $\geq$ étale $\geq$ Zariski; thus if M is sheaf for stronger, then weaker also] Conversely, if weaker representable, stronger

Thm. (Sheafification) $F \mapsto F^{sh}$

Pf. Set $F^{sh}(X) = \{((U_i \to X), s_i) \mid (U_i \to X) \in Cov(X), s_i \in F(U_i)\}$

s.t. $s_i|_{U_i \times_X U_j} = s_j|_{U_i \times_X U_j}$

The equivalence "$\sim$" is $((U_i \to X), (s_i)) \sim ((U'_j \to X), (s'_j))$ when $s_i|_{U_i \times_X U'_j} = s'_j|_{U_i \times_X U'_j}$ □

$M_g(X) \to \prod M_g(U_i)$

but isotrivial $\not\Rightarrow$ globally trivial (locally trivial)

$\Rightarrow M_g$ not sheaf on Sch_{Zar}

(Gluing also not holds) $\Rightarrow$ it's not desired.

If use sheafification, due to $M: Sch \to Set^{op}$ we lose $M_g(X) = \{\mathcal{C} \to X \mid \mathcal{C}$ genus $g\}/\cong$ lost too much; $M_g(X) = \{\mathcal{C} \to X \mid \mathcal{C}$ genus $g\}/\cong$ locally

We need memorize more by groupoid (with automorphism)

$M: Sch \to Groupoid$ (Force to be isomorphism, we keep track on all isomorphisms, thus we need all diagram "=" strictly)

$X \mapsto (\{f: \mathcal{C} \to X\}, Aut(f))\}/\cong$

Sch is pasted by affine representable functor in $\boxed{Sch_{Zar}}$. Modification of $M: \boxed{Sch} \to Set^{op}$ $\Rightarrow$

$\rightsquigarrow$ in $Sch_{\acute{e}t} \Rightarrow$ Algebraic space category. Furthermore $\rightsquigarrow$ Generalisation

Equivalently, $X \cong Y/\mathcal{R}$ is algebraic space, Y scheme, $\mathcal{R}$ relation $\mathcal{R} \subset Y \times Y$ is étale on $\mathcal{R}$

$$\begin{array}{ccc} & \mathcal{R} & \\ \swarrow & & \searrow \\ Y & & Y \end{array}$$

Rk. ① dim ≤ 2, proper algebraic space $Y \Rightarrow Y$ scheme ② (Hironaka) $\exists$ 3-fold algebraic space not a scheme

Def. (Category fibred in Groupoids) (CFG / prestack) (Generalisation of presheaf)

$(\mathcal{C}, F: \mathcal{C} \to \mathcal{B})$ is a CFG over $\mathcal{B}$

if (i) Existence of pullback $\boxed{a \to b} \in \mathcal{C}$, $F \downarrow \ \square\ \downarrow F$, $\boxed{S \to T} \in \mathcal{B}$

(ii) Universal property of pullback

$b \to c \in \mathcal{C}$ ... $\in \mathcal{B}$, $f \circ g$ ... compatible (one may use higher category to rewrite).

Def. Now $\mathcal{C} \to \mathcal{B}$ CFG, $T \in \mathcal{B}$

$\mathcal{C}(T) = \{$over T's objects; over 1_T's morphisms$\}$

is category, called fibred category (Check: it's a groupoid)

E.g. Scheme is CFG over Sch category

$X: Sch \to Set^{op} \Rightarrow \mathcal{C}_X \to Sch \ni T$

$\mathcal{C}_X(T) = \{T \to X\} = Mor(T, X)$

[We not distinguish $\mathcal{C}_X$ and X later]

Def. (Morphism of CFG) A functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ over $Id_{\mathcal{B}}$ strictly

Def. (Diagram/2-morphism of CFG) (We use this for pullback) A natural transformation $\mathcal{C} \Downarrow \mathcal{C}'$ over $Id_{\mathcal{B}}$ strictly.

Def. (Stack) (Note these "restriction" is pullback omitted)

A stack is prestack $\mathcal{C} \to \mathcal{B}$ over $\mathcal{B}$ a site

s.t. for $\forall X \in \mathcal{B}$, $\forall \{U_i \to X\} \in Cov(X)$

① (Glue morphism) $\forall a, b \in \mathcal{C}$ over $X \in \mathcal{B}$, and $\varphi_i: a|_{U_i} \to b \Rightarrow \exists!\ \varphi: a \to b$, $\varphi|_{U_i} = \varphi_i$

s.t. $\varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}}$

② (Glue object) a_i over U_i, $a_i|_{U_{ij}} \cong a_j|_{U_{ij}}$, $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = 0$ (cocycle) on U_{ijk}

Rk. Why we define stack ①② ① is $=$, ② is $\cong$? due to automorphisms happens in objects

E.g. $M_g: Sch \to Set^{op}$ consider a fibred category $M_g \to Sch$ CFG, $M_g(B) = \{f: \mathcal{C} \to B \mid$ family of smooth curves genus $g\} \subset Sch(B)$ [it's a groupoid]

It forgets the automorphism if you compare $M_g: B \mapsto M_g(B)$ (distinguish the notions)

E.g. Presheaf is CFG: $F: \mathcal{B} \to Set^{op}$

then $\mathcal{C}_F \to \mathcal{B}$ is CFG

by defining $\mathcal{C}_F(T) = \{b \in F(T)\}$, morphism over 1_T

Precisely, $\mathcal{C}_F \to \mathcal{B}$

$\{(b, T) \mid T \in \mathcal{B}, b \in F(T)\}$, morphism $(b', T) \to (b, T')$

(Remember this "correspondence", it's the ideal of we construct both sides: Functor (sheaf) $\Longleftrightarrow$ CFG)

(Notice that F is contravariant! thus $b \in F(T)$ is over T: $b \to b'$ over $T \to T'$)

Thm. Sheaf is stack. (Scheme ⇝ sheaf ⇝ stack; Algebraic space, presheaf ⇝ prestack) ☐

By think stack as 2-sheaf

Descent. Given $R \to A$, a descent datum of it is (N, φ), N is A-module, $\varphi : N \otimes_R A \xrightarrow{\sim} A \otimes_R N$ as $A \otimes_R A$ module, φ satisfy cocycle condition $(N \otimes_R A) \otimes A \to A \otimes_R (N \otimes_R A)$ ... $(A \otimes_R N) \otimes_R A$

Prop. Descent datum is effective when $R \to A$ is faithfully flat

Pf. Omitted ☐ (Faithfully flat ⟺ flat + injective)

Taking spec, $R \otimes_R A \to A \otimes_R A \Rightarrow$ injective. the affine got. $\downarrow m \Rightarrow R \to A$ injective. and the general scheme are glued ⇒ also effective

Stackification. $\mathcal{M} \to \mathcal{C}$ prestack.

$\mathcal{C}$ site ⟹ ∃ a canonical stack $\mathcal{M}^{st} \to \mathcal{C}$ and $\mathcal{M} \to \mathcal{M}^{st}$ is stackification functor s.t. $Mor(\mathcal{M}, -) \cong Mor(\mathcal{M}^{st}, -)$ (adjoint to the forgetful functor: a stack (sections) ⇝ section / 2-section)

Pf. Step 1 Add isomorphisms to satisfy ①'

Sheafification the $Isom(x, y)$, done //

Step 2 Add objects/sections

Add all descent datum to $\mathcal{M}(U)$ i.e. $\mathcal{M}^{st}(U) = \{$All descent datum over $U_i\}$ ☐

Rk. Step 2 not add arrow due to $Isom(x,y)$ had been sheaf

What we need in AG is ≤ Artinian < Stack in general ... < D-M stack < Artin (étale ↓ < smooth)

E.g. $\mathcal{M}_g$

E.g. $\mathcal{C} = Sch_{ét} / Sch_{fppf}$ $\{V \to U\} \in Cov(Sch_{fpqc})$

single due to faithfully flat ⟺ surjective + flat.

E.g. $\mathcal{M}_g \to Sch$ is prestack. $\mathcal{M}_g(S) = \{f : \mathcal{X} \to S$ fibre smooth, projective; dim 1; genus $g\}$

The pullback
$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$
base change not effect fibre ⇒ exist

Rk. If restriction about the totally space, base change may not exists

It's also a stack in $Sch_{ét} / Sch_{fpqc}$

①⟸①' Covering $V \to U$ ⇒ $\mathcal{X}_{V \times_U V} \rightrightarrows \mathcal{X}_V \to \mathcal{Y}$, $V \times_U V \rightrightarrows V \to U$, by descent holds ☐

Summary. $\mathcal{C} \to \mathcal{B}$ fibred ⟺ $\mathcal{C}_i : \mathcal{B} \to$ Groupoid, $B \mapsto \mathcal{C}(B)$ the fibre

let $\mathcal{C} = \mathcal{M}$, $\mathcal{B} = Sch$ is stack and topology

Use this, one rewrite the stack axiom as: $F : \mathcal{M}(U) \to \mathcal{M}(\{U_i \to U\})$ of fibred category

$$\begin{tikzcd} x \arrow[d] & x|_{U_i} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & x \arrow[d] \\ U \arrow[r, mapsto] & U_i \arrow[r] & U \end{tikzcd}$$

then ①② hold ⟺ F is category equalience

⟺ ①' $\forall U \in \mathcal{C}$, $x, y \in \mathcal{M}(U)$ (Groupoid) $Isom(x,y) : \mathcal{C}_U \to Set$ is a sheaf

$\{V \to U \mid V \in \mathcal{C}\}$, $V \to U \mapsto Isom(x|_V, y|_V) = Hom(x|_V, y|_V)$

②' Every descent datum is effective (generalize scheme)

A descent datum of $\{U_i \to U\}$ covering in $\mathcal{C}$ is $\{(x_i, \varphi_{ij})\}$, s.t. $x_i \in \mathcal{M}(U_i)$, $\varphi_{ij} : x_i|_{U_{ij}} \cong x_j|_{U_{ij}}$ satisfy cocycle (= descent) condition

effective is $\exists x \in \mathcal{M}(U)$ s.t. $x_i = x|_{U_i}$

Algebraic stack. ($\mathcal{M}_g$ is stack now, we'll show it's algebraic)

Def. $\mathcal{M} \to Sch$ is a stack, it's algebraic if: ① the diagonal map $\Delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is representable by an algebraic space (Usually $Sch_{fpqc} / fppf$ / ⟸ smooth) ⟸ $Isom(X, X)$ ⟸ The automorphism is good $= Autom(X)$

② ∃ A scheme X, and a morphism $X \to \mathcal{M}$ (call representation) surjective smooth (think as functor of stacks) and étale $X \to \mathcal{M}$

(E.g. $\mathbb{P}^n \to \mathbb{P}^n / Aut(\mathbb{P}^n)$ not depend on ét/fpqc-!)

E.g. D-M stack is ... over $Sch_{ét}$ ...

Rk. ① Both Algebraic stack/space, ①⟸②, thus we may omit ①.

② Algebraic space ⟺ $U/G \cong X$ and $G \curvearrowright U$ stablizer trivial;

D-M stack ⟺ $U/G \cong X$ and $G \curvearrowright U$ stablizer finite.

Then we see how $G \curvearrowright Y$, Y a scheme, $|G| < \infty$

E.g. (We know dim ≤ 2, proper ⇒ quotient also scheme. thus we take a not proper one) $Y = $ —·— ; $G = \mathbb{Z}/2\mathbb{Z}$, then $G \curvearrowright Y$ by $(-1)x = -x$ and $(-1)O_1 = O_2$

then $\mathbb{A}^1 \subsetneq Y/G \subsetneq \mathbb{A}^1 + \{O_2\}$

there is not a such scheme

E.g. (Hironaka) Proper not projective threefold

Details can be found in [Hal]

$G = \mathbb{Z}/2\mathbb{Z}$; $\exists y \in Y$, s.t. the orbit $O(y) = G.y$ not contained in any affine G-inv

⇒ $Y/G = \{O(y)\}$, if it's a scheme, we have affine covering ⇒ containing $O(y)$, contradiction ☐

Thm. Y projective, $|G| < \infty \Rightarrow Y/G$ projective

Pf. $Y = Proj(\bigoplus H^0(Y, \mathcal{O}(n)))$ ⇒ $Y/G = Proj(\bigoplus H^0(Y, \mathcal{O}(n))^G)$

Details can be found in [GIT] ☐

②⟹②' Gluing family is globally, we need algebraic geometry

$$\begin{tikzcd} \mathcal{X}_i \arrow[r, hook] \arrow[d, "f_i"] & \mathbb{P}(E_i) \arrow[d] \\ U_i \arrow[r, equal] & U_i \end{tikzcd}$$
$E_i = (f_i)_*(\omega^{\otimes 3}_{\mathcal{X}_i/U_i})$ very ample

thus we embed each $(\mathcal{X}_i \to U_i)$ to $\mathbb{P}(E_i)$ projective bundle

Gluing bundle (more generally, sheaves) can always done by descent fpqc ☐

Recall. Def. (Artin) algebraic space X is a sheaf over $Sch_{ét}$ s.t. ① $\Delta : X \to X \times X$ is representable by an scheme;

② ∃ A scheme U, and a morphism $U \xrightarrow{P} X$, surjective étale (P call the representation of X)

Def. Now set $\mathcal{C} = Sch$ or Sch_S

X, Y (pre) stack over $\mathcal{C}$, $F : X \to Y$ representable by a scheme if, $\forall T$ scheme, s.t. $X \times_Y T \to T$ and $X \times_Y T$ is a scheme

And we say F has $\mathcal{P}$ (smooth, ...) $\begin{tikzcd} X \times_Y T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$ when F representable by a scheme, and the base change $X \times_Y T \to T$ has $\mathcal{P}$ (locally check) (Usually $\mathcal{P}$ is stable under base change)

Site. Categorical equivalence

the Axiom of Choice in class. Categorical Isomorphism

Usually, directly say representable means representable by a algebraic space.

Thm (Recall) Algebraic space X ⟺ ∃ U scheme, $R \subset U \times U$ is subscheme we call R étale equalience if $R \subset U \times U$ are étale map $\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ U & & U \end{tikzcd}$ s.t. $X = U/R$

Eg. (Quotient stack) $S \in Sch$ and G is a group object of Sch, action $G \times X \to X$ finite type map

Def. A (principal) G-bundle over S is $P \to S$, s.t. $\exists S'$ étale S, s.t. $P \times_S S'$ has fibre $G \times_S S'$, ($G \curvearrowright P$ fibrewise naturally.)

Def. $[X/G]$ the quotient stack over Sch, the fibred stucture is $[X/G](S) = \left\{ \begin{array}{c} P \xrightarrow{G\text{-equivariant}} X \\ \downarrow \text{principal } G\text{-bundle} \\ S \end{array} \right\}$

Eg. let $S = \operatorname{Spec} k$ naively $\Rightarrow P = \{G(x)\}_{x \in X}$ and act of $G \times X \to X$, fix $x \in X$ in P gives $G \to X$ G-equivariant $\Rightarrow \left\{ \begin{array}{c} G \to X \\ \downarrow \\ \operatorname{Spec} k \end{array} \right\}$

Thm. $[X/G]$ is algebraic stack.

Pf. ① $[X/G]$ is stack (check)

② Representation map: $X \xrightarrow{F} [X/G]$ smooth by

$F(S): X(S) \to [X/G](S)$

$\{S \xrightarrow{f} X\} \mapsto \left\{ \begin{array}{c} P \to X \\ \downarrow \\ S \end{array} \right\}$

$\mapsto \left\{ \begin{array}{c} S \times G \xrightarrow{f \times id} X \times G \xrightarrow{act} X \\ \downarrow \text{trivial proj} \\ S \end{array} \right\}$

Rk. $G \curvearrowright X$

X/G	traditional	
X/G	categorical quotient	may not exist
$X/\!/G$	GIT quotient	but exist $\Rightarrow$ equal all
$[X/G]$	quotient stack	

Representable thm

Thm. X algebraic space, ② $\Rightarrow$ ①

Pf. (Descent) Check $X \times_{X \times X} T$ is scheme $=: Y$

Y is a sheaf over Sch is clear. $\boxed{\text{Sheaf} \overset{?}{\Rightarrow} \text{Scheme}}$

Consider ① $U \xrightarrow{\Delta_U} U \times U$, étale, étale, $X \xrightarrow{\Delta_X} X \times X$

② $Y \to T$, $X \xrightarrow{\Delta} X \times X$

③ $T \times_{X \times X} (U \times U) \to U \times U$, étale $\square$ étale, $T \to X \times X$

Recall a Thm we omitted the proof by descent:

$S \to T$ étale, surj, schemes

if $F \times_T S$ is scheme $\Rightarrow F$ is scheme

T scheme $\Rightarrow T \times_{X \times X} (U \times U)$ scheme

④ $Y \times_T T \times_{X \times X} (U \times U) \to T \times_{X \times X} (U \times U)$, $\square$ étale, $Y \to T$

by the Recall, it suffices show that:

$Y' = Y \times_T T \times_{X \times X} (U \times U)$ is a scheme

$= (X \times_{X \times X} T) \times_T (T \times_{X \times X} (U \times U))$ $\underset{\text{magic square}}{=}$ $X \times_{X \times X} (T) \times_{X \times X} (U \times U) = Y \times_{X \times X} (U \times U)$

Pf. (Conclusion) (Right)

$Y' \to T \times (U \times U)$, $U \to U \times U$, $Y \to T$, étale

Step 1 $T \times_{X \times X} (U \times U)$, U, $U \times U$ all Sch $\Rightarrow$ Y' also

Step 2 Using "Recall" $\Rightarrow Y$ scheme

① Δ Algebraic stack, represent Δ by algebraic space

② X D-M stack, representable by scheme

Pf: By local scheme, ② is done, ① we use the smooth

Then ① $U \times U \leftarrow Y' \Rightarrow Y'$ is a scheme, étale $\square$, $X \times X \leftarrow Y$

by a base extension but it will broke some extension

This proof is wrong

We can use D-M stack to cover (Artin) stack

Eg. ① DM stack is locally $[Y/G]$, $|G| < \infty$

② Artin stack is locally $[Y/G]$

①.5 Good algebraic stack is locally $[Y/G]$, G reductive. Take $H \leq G$ and H-slice

Slicing thm. $Y \to [Y/G]$, take slices to make finite $\Rightarrow$ the existence of H-slice.

Eg. When proj, use Bertini thm

A point is $x: \operatorname{Spec} k \to \mathcal{X}$ in $\mathcal{X}$ stack is a fibre category $\mathcal{X} \to Sch$

Def. Automorphism group / stablizer of x: $\operatorname{Aut}_{\mathcal{X}(\operatorname{Spec} k)}(x, x) := \underline{\operatorname{Isom}}_{\mathcal{X}(\operatorname{Spec} k)}(x, x)$ only depend on $\operatorname{Spec} k$ (point) and $\underline{\Delta}$ $= \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \operatorname{Spec} k$.

Taking local neiberhood of x, $[X/G] \Rightarrow \operatorname{Aut}_{\mathcal{X}(k)}(x, x) \cong G_x$ the stablizer.

Rk. $\operatorname{Aut}_{\mathcal{X}(k)}(x, x)$ is scheme / algebraic space if Δ is, i.e. $\mathcal{X}$ is D-M / Artin

Thm. TFAE ① $\mathcal{X}$ is DM; ② $\operatorname{Aut}_{\mathcal{X}(k)}(x, x)$ finite, reduced, $\forall k$

③ Δ unramified (Algebraicity assumed)

Thm. TFAE ① $\mathcal{X}$ is algebraic space; ② Δ is monomorphism (Algebraicity assumed)

Using Δ, we have (without check the representibility)
- closed immersion: scheme
- immersion: prescheme
- monomorphism: algebraic space
- unramified: D-M stack

Inertia stack. $I_{\mathcal{X}} = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$

$= \{(x, \alpha), \alpha \in \operatorname{Aut}(x, x)\}$

the stacky point (x, α), equipped an automorphism

Eg. $\mathcal{X} = [X/G]$

$I_{\mathcal{X}} \to \mathcal{X}$, G_X, $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$, $G \times X \to X \times X$

is the Inertia stack.

We show M_g is D-M

Step 1. M_g is algebraic (hard)

Step 2. M_g is D-M (easy)

(In general holds for polarized varieties)

Step 1

Idea: M_g is quotient stack. $\omega_{D/S}^{\otimes 3}$ very ample $\Rightarrow$ fix $\omega_{D/S}^{\otimes 3}$ vector bundle, we embedding $D \hookrightarrow \mathbb{P}(f_* \omega_{D/S}^{\otimes 3})$ then we use Hilbert scheme to depict it

each fibre is $\mathbb{P}H^0(D_s, \omega_{D_s}^{\otimes 3})$ $\underset{\text{use R-R}}{\overset{\dim}{=}}$ $\deg(\omega_{D_s}^{\otimes 3}) + 1 - g = 6g - 6 + 1 - g = 5g - 5$

$\Rightarrow D_s \hookrightarrow \mathbb{P}^{5g-6}$

(determine genus g)

and $P(n) = \chi(\omega(n)) = 6n(g-1) + 1 - g = (6n-1)(g-1)$, let $\mathcal{H} = \operatorname{Hilb}_{P}(\mathbb{P}^{5g-6})$

$PGL(5g-5) \curvearrowright \mathcal{H}$ by coordinate change. $M_g \subset \mathcal{H}$ due to M_g only smooth and connected

$\Rightarrow$ let $\mathcal{H}' \subset \mathcal{H}$ parametize only smooth connected curves, precisely the universal family $\mathcal{C}_{\mathcal{H}} \to \mathcal{H}$ s.t. fibre wise satisfy

$\Rightarrow \mathcal{H}' \subset \mathcal{H}$ open subscheme by smooth loci (and connected loci) are open (by Stein Factorization)

Claim. $M_g \subset [\mathcal{H}/PGL(5g-5)] \Rightarrow$ **Claim'.** $M_g \cong [\mathcal{H}'/PGL(5g-5)]$ — (almost done now)

To show $[H/PGL(5g-5)] \simeq \mathcal{M}_g$, consider $F: [H/PGL(5g-5)]^{pre} \to \mathcal{M}_g$ forgetful ↳ We forget the embedding into $\mathbb{P}^{5g-6}$. Thus the recover is intristintic, the embedding unique up to a $PGL(5g-5)$, thus canonical. ▨

To be precisely show what forget: $F: \{\mathcal{Y} \to S\} \mapsto \{\mathcal{Y} \to S\}$

$\Rightarrow$ descend to $[H/PGL(5g-5)]^{pre}$.

One may check fully faithful and essentially surjective

the "$/PGL(5g-5)$" is due to embedding $\mathcal{C} \hookrightarrow \mathbb{P}(E)$ depend on coordinate ▨

For polarised variety, $\mathcal{M}_\Sigma = \{(X, L) \mid L \text{ ample}\}$, $\Sigma = \{(X, L)\}$ / deformation equivalence (not change the $P(n)$)

$\Rightarrow \mathcal{M}_\Sigma$ is quotient stack. Sch, $Pic(X)$ — replace the ω_X

The proof same: embedding to H (However, for Calabi-Yau, ample not exist)

E.g. $U \to \mathcal{M}_g$ a open subscheme of Hilb; let $[C] \in V \subset U$, V is a deformation space of $[C]$ and V is locally universal family by factor through the presentation morphism.

Deformation of $[C]$: $Y \supseteq C$, $0 \in B$ is "shrinking the ball" $B \ni 0$

① Analytically, let radius $\to 0$, it's impossible!

② Algebraically, we use infinitesimal deformation (first order) ($\mathcal{X}(\operatorname{Spec} k)$) $B = D = \operatorname{Spec} \frac{k[\varepsilon]}{(\varepsilon^2)}$

Step 2. We need show $Aut(X) = Aut(C)$ is finite, reduced ($[Ha]$ done in Chap II)

$Aut(C) \cong (\Gamma_f)_{f \in Aut C} \subset H$ subschemes of $C \times C$; $P(n)$ fixed due to $\Gamma_f \cong C$

$\Rightarrow Aut(C)$ is scheme. And smooth $\Rightarrow$ reduced

Claim. $\dim T_{[id]} Aut(C) = 0 \Rightarrow$ discrete $\Big\} \Rightarrow$ finite ▨ We expect also $\overline{\mathcal{M}}_g$ is D-M (stable curves of genus g)

(By deformation, later) also compact — preserve polarisation in $\mathbb{P}$. In fact $\overline{\mathcal{M}}_g \cong$ Quotient stack also holds.

Artin's criterion of algebraicity (six axioms) and deformation. All local! (scheme theory back)

Given prestack $\mathcal{X} \to Sch_k$

(A1) It's stack; (A2) $\mathcal{X}$ is linearly preserving i.e. $\operatorname{colim} \mathcal{X}(S_i) = \mathcal{X}(\operatorname{colim} S_i)$; (A3) (Deformation) $\forall x \in \mathcal{X}(k)$ "closed point" it has a versal deformation;

(A4) Every formal object is effective; (A5) Open versality; (A6) Δ representable by algebraic space

(A3–A5) concerns deformation

$\boxed{(A1-A6) \Leftrightarrow \mathcal{X} \text{ is algebraic stack}}$

Formal deformation $\operatorname{Spec}\left(\frac{k[x]}{(x^{n+1})}\right)$ n-th order

E.g. $\mathcal{X}_0$ to be parameterize space of all hypersurface. $\mathcal{X}_n$

$\mathcal{X}_1 = \{f(z) + \varepsilon f_1(z) = 0\}$; $\mathcal{X}_n = \{\sum f_i(z) \varepsilon^i = 0\}$

Although also one equation, it has different coefficients i.e. the underlying field/ring

Def. Let Art_k denote k-Artinian local algebra, completion $\widehat{Art}_k$ Noetherian complete local ring.

$F: Art \to Set^{op}$ is pro-representable if $\exists R \in \widehat{Art}_k$, s.t. $F \cong h_R|_{Art_k}$

Def. (Local deformation) $\mathcal{X} \to Sch_k$ prestack. $\forall x_0 \in \mathcal{X}(k)$

$Def_{\mathcal{X}, x_0}(A) := \left\{(x, \alpha) \middle| \begin{array}{l} x \in \mathcal{X}(A) \\ \alpha: x_0 \cong x|_k \end{array}\right\}$ for $A \in Art_k$ (in particular for $D = \frac{k[\varepsilon]}{(\varepsilon^2)}$ is the formal) ($k = \operatorname{Spec} k$)

$Def_{\mathcal{X}, x_0}: Art \to Set^{op}$, i.e. $x \leftarrow x_0 \cong x|_k$, $\operatorname{Spec} A \leftarrow \operatorname{Spec} k$

Def. $\mathcal{X}, x_0$ is $\boxed{\text{universal}}$ if it's pro-representatable.

formally / locally not: universal $\Rightarrow$ $\boxed{\text{versal}}$

Given a geometric deformation (i.e. family over a base) $\exists$ formal deformation at 0 ⟲ algebraic. For base has higher dimension by ... $\mathcal{X}$ exist? $\operatorname{Spec} k[[x]] \to \operatorname{Spec} k[x]$

Def. (Versal deformation)

A versal deformation of $x_0 \in \mathcal{X}(k)$ is a deformation $\boxed{\text{unique}}$ exist

if $\hat{x} = (x_n) \in \mathcal{X}(R)$, $\exists R \in \widehat{Art}_k$, s.t. $\forall$ surjection $A \to k$, $\forall y \in \mathcal{X}(A)$

with $y|_k \cong x_0$ ($\cong x_n|_k$ for some n); $\exists \frac{R}{m^{n+1}} \cong A$ extension and $\alpha: x_n \xrightarrow{\cong} y$

i.e. the complete R dominate all (but not unique) A) (each $x_n \in \mathcal{X}\left(\frac{R}{m^{n+1}}\right)$)

($\Leftarrow$) tells us that $\exists U \to \mathcal{X}$ presentation $\Rightarrow$ formal deformation exist

Spec R is ball; Spec A are arrows

Artin answers "$\Leftarrow$"! (A3–A5)

Def. (Effective) A formal deformation $(x_n) \in \mathcal{X}(R)$ is effective if:

(x_n) is in the image of $\mathcal{X}(R) \to \operatorname{colim} \mathcal{X}\left(\frac{R}{m^{n+1}}\right)$

Def. (Open versality) $\forall$ affine scheme U locally finite type over k, $x \in \mathcal{X}(U)$, if $\exists u_0 \in U$, $x|_{u_0}$ has versal deformation

$\Rightarrow \exists V \subset U$ open, s.t. $\forall u \in V$, $x|_u$ has versal deformation (natural)

Pf. ($\Rightarrow$) Trivial (except A4/A5); ($\Leftarrow$) We do local and gluing by (A2) $\Rightarrow$ preserve fibre product, Hilb

Idea. $\forall x_0 \in \mathcal{X}(k)$, $\exists U_{x_0} \in Sch$, finite type, smooth $U_{x_0} \to \mathcal{X}$ containing x_0 (Let U pastes all U_{x_0})

By (A3) & (A4) $\exists$ a versal effective deformation

$\hat{x} \in \mathcal{X}(R) \Rightarrow \operatorname{Spec} R \to \mathcal{X}$

Then it suffices to show $\operatorname{Spec} R$ is algebraic, thus we use $\boxed{\text{Artin's algebraization}}$ ▨

Artin's algebraization

Thm. $\exists$ finite type scheme U_{x_0} s.t. $\exists U_{x_0} \xrightarrow{f} \mathcal{X}$, $R \cong \hat{\mathcal{O}}_{U_{x_0}, x_0}$ and $P|_{\frac{R}{m^{n+1}}} \cong \hat{x}|_{\frac{R}{m^{n+1}}}$, $\forall n$.

(thus the R comes from $\mathcal{O}_{U_{x_0}, x_0}$, then it's algebraic

For smoothness: $\forall y \in U_{x_0}$ with versal deformation $\Rightarrow U_{x_0} \to \mathcal{X}$ is smooth (By (A5)) The "biggest" $\Rightarrow$ dim "almost" = dim Tangent.)

Versal deformation of schemes (Kodaira–Spencer map), more concrete.

Setting, $\mathcal{X} \supset X_0$, Δ is scheme (E.g. Spec of Artinian local k-algebra).

$$\begin{array}{ccc} \mathcal{X} & \supset & X_0 \\ \downarrow \pi & & \downarrow \\ \Delta & \ni & 0 \end{array}$$

π is flat, locally finite type.

Motivated, then we have:

Thm. ① X_0 smooth affine, then its deformation is locally trivial.

i.e. def: $X \to \operatorname{Spec} \frac{k[\varepsilon]}{\varepsilon^2}$ 1st-order deformation is trivial,

i.e. $X \cong X_0 \times \operatorname{Spec} \frac{k[\varepsilon]}{\varepsilon^2}$

② $\operatorname{Def}^1(X_0) \cong H^1(X_0, T_{X_0})$ when X_0 smooth.

Pf. ① By [Ha II §8, Ex 8.7–Ex 8.8] Infinitesimal Lifting

$$\begin{array}{ccccc} X_0 & \to & X & \dashrightarrow & X_0 \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k & \to & \operatorname{Spec} \frac{k[\varepsilon]}{\varepsilon^2} & \to & \operatorname{Spec} k \end{array}$$

exist $\Leftarrow$ smooth $\Rightarrow \varphi: X \to X_0 \times_k \frac{k[\varepsilon]}{\varepsilon^2}$ morphism over k. extend $\varphi|_{X_0} = \mathrm{Id}$.

By affine, X, $X_0 \times_k \frac{k[\varepsilon]}{\varepsilon^2}$ are $\frac{k[\varepsilon]}{\varepsilon^2}$-module

by quotient maximal, we turn to $X_0 \Rightarrow \mathrm{Id}$ isomorphism

$\Rightarrow \varphi$ isomorphism, by the following Lemma. ▨

Lemma. $M \to N$ over A Artinian, $\frac{M}{\mathfrak{m}M} \xrightarrow{\cong} \frac{N}{\mathfrak{m}N} \Rightarrow M \cong N$

Pf. Take the kernel/cokernel $B \Rightarrow \frac{B}{\mathfrak{m}B} = 0 \Rightarrow B = \mathfrak{m}B = \mathfrak{m}^2 B = \cdots$

by Artinian $\Rightarrow B = 0$ ▨

② By ① we cover X_0 by $\{U_i\}$ locally trivialize $\varphi_i: U_i' \xrightarrow{\cong} U_i \times_k \frac{k[\varepsilon]}{\varepsilon^2}$.

assume $U_{ij} = U_i \cap U_j$ also affine (smooth affine component), and φ_{ij} defined.

Analytically use K-S map the proof is by the computation and solving differential equations and compare the Jacobians (corresponding to the datum (θ_{ij}))

Eg (Kodaira–Spencer map) A family of manifold over $\mathbb{C}$

$\mathcal{X} \xrightarrow{\pi} B$, π smooth naturally and satisfy flat + finite type.

Observation. When B is simply connected $\Rightarrow \mathcal{X} \cong B \times X_0$ as manifold.

Thus $\boxed{\text{deformation of complex manifold} \Leftrightarrow \text{deformation of complex structure.}}$ The moduli of complex structure is the Teichmüller space.

Eg. The Riemannian surfaces have clear topology, but adding complex structure isn't such easy.

$0 \to T_{X_0} \to T_{\mathcal{X}}|_{X_0} \to (\pi^* T_B)|_{X_0} \to 0$ the LES $\Rightarrow$

$\rho: H^0(X_0, (\pi^* T_B)|_{X_0}) \to H^1(X_0, T_{X_0})$ the K-S map.

$T_{B,0}$

Let B enough large: universal / miniversal deformation, we expect that ρ isomorphism (dominate dimension same)

$T_{B,0} = \operatorname{Hom}(X_0, \frac{k[\varepsilon]}{(\varepsilon^2)}) = \operatorname{Def}^1(X_0)$

(And versal may have "double or more cover" to $H^1(X_0, T_{X_0})$, $T_{B,0}$ more large. We use this to depict!)

In each U_{ij}, φ_{ij} 1-1 corresponds to $\phi_{ij}: \frac{R[\varepsilon]}{\varepsilon^2} \to \frac{R[\varepsilon]}{\varepsilon^2}$

and $\phi_{ij}|_{\varepsilon=0} = \mathrm{Id}$ set. $\phi_{ij}(x + y\varepsilon) = x + \varepsilon(\theta(x) + y)$

Claim. θ is derivative (converse is same)

By let $y = 0$, $\phi_{ij}(x_1 x_2) = x_1 x_2 + \varepsilon(\theta(x_1 x_2))$

$\phi_{ij}(x_1)\phi_{ij}(x_2) = (x_1 + \varepsilon\theta(x_1))(x_2 + \varepsilon\theta(x_2))$ ▨

Then $(\theta_{ij}) \in (H^0(U_{ij}, T_{U_{ij}}))_{ij}$ satisfy cocycle condition. Then by Čech cohomology (with affine cover $\cong$ sheaf cohomology)

$\Rightarrow (\theta_{ij}) \in \check{H}^1(X_0, T_{X_0}) = H^1(X_0, T_{X_0})$ thus $\operatorname{Def}^1(X_0) \cong (\operatorname{Def}^1(U_{ij}))_{ij} \cong (\operatorname{Def}^1(U_{ij} \times \frac{k[\varepsilon]}{\varepsilon^2}))_{ij} \cong (H^0(U_{ij}, T_{U_{ij}}))_{ij}$

$\cong \check{H}^1(X_0, T_{X_0}) \cong H^1(X_0, T_{X_0})$ ▨

Rk. Using (A3)–(A5) we can prove ① in versal case, and 1st-order deformation determines all versal deformation in scheme (not manifold: polynomial is more rigid)

Obstruction theory asks about this: ① $\operatorname{Def}^1(X_0) \neq \emptyset$? ; ② $\forall \alpha^n \in \operatorname{Def}^n(X_0)$, $\exists \alpha^n \in \operatorname{Def}^n(X_0)$, s.t. restricted to α^1 ; ③ α^1 lifts to $\alpha^n \in \operatorname{Def}^n$? ; ④ $\lim\limits_{n\to}$, lifts to formal deformation ? ; ⑤ Can we algebraization the formal deformation ?

Always: We set X_0 smooth, we have singularity case also;

Usually ③ suffices do for $n=2$ case due to obstruction lies in $\operatorname{Obs}(X_0) = H^2(X_0, T_{X_0})$ the obstruction space (but not all elements are obstruction!)

E.g. X Abelian, it has no obstruction but $H^2(X_0, T_{X_0}) \neq 0$

Embedding deformation. $\mathcal{X} \subset Y \times B$, i.e. $X_0 \subset Y$, the formal 1st-order ... samely defined.

$$\begin{array}{ccccc} X_0 & \subset & \mathcal{X} & \subset & Y \times B \\ \downarrow & & \downarrow & \swarrow & \\ 0 & \in & B & & \end{array}$$

Eg. $H = \mathrm{Hilb}^{P(z)}(\mathbb{P}^N)$, its embedding of $Y_0 \subset \mathbb{P}^N$

then by deformation is tangent to moduli

$\Rightarrow T_{H,0} \cong \boxed{\operatorname{Def}(X_0 \hookrightarrow \mathbb{P}^N)} \Leftarrow$ How to depict it in general?

Thm. $\operatorname{Def}^1(X_0 \hookrightarrow Y) \cong H^0(X_0, N_{X_0/Y})$

Pf. By LES of $N_{X_0/Y}$ and $\mathcal{T}_{X_0}$, $\mathcal{T}_Y$ bundle locally free ▨

Thm. $\operatorname{Obs}(X_0 \hookrightarrow Y) \cong H^1(X_0, N_{X_0/Y})$ when locally free ▨

Eg. $X_0 \subset Y$, Y is quartic surface in $\mathbb{P}^3$, X_0 is smooth rational curve on Y (by Mori's work, exist) $N_{X_0/Y} = \mathcal{O}(-2)$ by counting the 1st-Chern class and $N_{X_0/Y}$ is line bundle $\Rightarrow N_{X_0/Y}$ is just its 1st-Chern class:

1st. Chern: $\quad 0 \qquad -2$

$0 \to \mathcal{T}_{X_0} \to \mathcal{T}_Y \to N_{X_0/Y} \to 0 \Rightarrow H^0(\mathbb{P}^1, \mathcal{O}(-2)) = 0$ by $X_0 \cong \mathbb{P}^1$ $\Rightarrow X_0$ can't deform in Y

Furthermore, when X_0 not smooth, normalization $\sigma: \mathbb{P}^1 \to X_0$ Collect all maps, then deform the morphism:

Here $\mathbb{P}^1$ deform to itself, thus deform Y, i.e. deform the embedding $\mathbb{P}^1 \xrightarrow{f} Y$ (it's the usual way to deform morphism) $\Rightarrow$ contradiction $\Rightarrow$ can't deform

Thm. $\operatorname{Def}^1(f) \cong H^0(\mathbb{P}^1, f^*\mathcal{T}_Y) \cong$ tangent space of Kontsevich moduli space ▨

Geometry of algebraic stacks $\mathcal{X} \to \mathrm{Sch}$ algebraic

① Topology ② Geometry ③ $\mathcal{X}(k)$ the coarse/good moduli.

Eg. M_g (fine) projective and $\overline{M}_g$ both smooth, irreducible, separated D-M stacks; and admit coarse moduli space but not smooth

smooth...

Can we pass ⑨ from $\mathcal{X}$ to $\mathcal{X}(k)$

(For Fano variety may not, only good moduli)

M_g or $\overline{M}_g$ smooth at $[C] \iff 3g-3 = \dim \overline{M}_g$ (or M_g) $= \dim H^1(C, T_C)$, but $M_g(k)/\overline{M}_g(k)$ not $(3g-3)$

For quotient stack ① $\mathcal{X}$ smooth $\iff$ $\mathcal{X}(k)$ smooth'' $\iff$ action is free.

Geometry of M_g ① ⇒ universal properties of curves, ①.

② ↳ substacks of M_g ⇒ universal properties of special curves (Eg. hyperelliptic, Hurwitz..)

③ ↳ Local properties ⇒ deformation space or limit process

(Smoothness, dimension) (properness/compact).

M_g: { dim $3g-3$, smooth, irreducible, quasi-projective, separated, proper X ⇒ $\overline{M}_g$ Compactification [taking limit and take closure, singularity in Fano case]

It's a highly nontrivial in general: esp. dim ↗, the singularity can't be dominated. Eg. (Birkar) He dominates the finite type

Turn back to general setting: ↳ Fortunately, M_g can be depicted concrete

$$\begin{array}{ccc} \mathcal{X}\times U & \to & U \\ \downarrow & \square & \downarrow \\ \mathcal{X} & \to & \mathcal{Y} \end{array}$$ With $\mathcal{P}$

Eg. (Mordell conjecture)

① Naively, C/K number field, $g(C) \geq 2 \Rightarrow |C(K)| < \infty$ (Faltings) K-rational points

② (Uniform Mordell conjecture) $\forall C \in M_g(K)$, $|C(K)| < N$, $N = N(g, K)$ not depends on ...

① done by counting heights in C, N depends on C

② done by counting heights in $M_g(K)$, and using Hodge bundle on $M_g/\mathbb{Z}$ ~ (...)

Def. $F: \mathcal{X} \to \mathcal{Y}$ has $\mathcal{P}$ iff F representable by schemes U and $\mathcal{X} \to \mathcal{Y}$

Def. $\mathcal{P}$ is étale/smooth local, $\mathcal{X}$ DM/Artin stack, then $\mathcal{X}$ has $\mathcal{P}$ if a presentation $U \to \mathcal{X}$ has $\mathcal{P}$

The étale/smooth local ensures it not depends on the choice of U: $U \times_{\mathcal{X}} U'$ and covering U, U'

Def. $|\mathcal{X}|$ the underlying topological space $|\mathcal{X}| = \{\mathrm{Spec}\,k \to \mathcal{X}\}/\cong$, $(\mathrm{Spec}\,k_1 \to \mathcal{X}) \cong (\mathrm{Spec}\,k_2 \to \mathcal{X})$ iff $\exists K/k_1, k_2$, s.t.

$$\begin{array}{ccccc} \mathrm{Spec}\,K & \to & \mathrm{Spec}\,k_1 & \to & \mathcal{X} \\ & \searrow & & \nearrow & \\ & & \mathrm{Spec}\,k_2 & & \end{array}$$

with topology $\{|U| \subset |\mathcal{X}| \mid U \subset \mathcal{X} \text{ open substack}\}$

Rk. We call it irreducible, connected, quasi-compact ... iff $|\mathcal{X}|$ has; (Hilb connected)

M_g irreducible by back to the Hurwitz space, all bases are fibres irreducible; $M_g = [W/PGL(5g-5)]$ connected

Other way: $M_g = [T_g/\text{Mapping class group}]$, T_g is Teichmuller space over Σ_g, but connected here only over $\mathbb{C}$

(diffeomorphisms) (all complex structure)

$M_g \to M_{g,\mathbb{C}}$, reduction (mod p) $M_{g,\mathbb{F}_p}$

Observation. $M_{g,\mathbb{Z}}$ has smallest point. Using different bases $\mathbb{C}/\mathbb{F}_p/\mathbb{Z}$ we can't derive topologically properties.

Def. $x \in |\mathcal{X}|$, the local dimension $\dim_x \mathcal{X} = \dim_u U$, $U \to \mathcal{X}$ étale presentation, u = the preimage. $\dim \mathcal{X} = \sup \dim_x \mathcal{X}$

Def. For smooth presentation, − the dimension of fibre.

Eg. $\mathcal{X} = [U/G]$, G affine smooth, U irreducible scheme

$\dim \mathcal{X} = \dim_x \mathcal{X} = \dim U - \dim G$, $\dim \in \mathbb{Z} \cup \{\pm\infty\}$

negative case: $\mathbb{P}^n = [pt/PGL(n+1)]$, $\dim M_0 = -3$

For turn to positive, one can $M_{0,n} = \{(\mathbb{P}^1, x_1, \dots, x_n) \mid x_i \in \mathbb{P}^1\}$

$\Rightarrow \dim M_{0,n} = n-3$ $(n \geq 3)$.

Coro. $\dim M_{0,3} = 0 \Rightarrow \forall x_1, x_2, x_3 \in \mathbb{P}^1$ can be coordinate change to $(0, 1, \infty)$

Def. The Zariski tangent space $x: \mathrm{Spec}\,k \to \mathcal{X} \Rightarrow T_{x,\mathcal{X}} = \{\mathrm{Spec}\,k[\epsilon]/(\epsilon^2) \to \mathcal{X}\}$

the $\cong$ is natural equivalence of graphs.

Observation ① $T_{x,\mathcal{X}} \cong \mathrm{Def}^1(\mathcal{X}, x)$, ② $\dim_x \mathcal{X} = \dim_x \mathrm{Def}_x(\mathcal{X})$

③ By ①② $\Rightarrow \dim T_{x,\mathcal{X}}$ 1st $\xrightarrow{\text{restrict}}$ universal $\geq \dim_x \mathcal{X}$, take equality iff smooth/regular

($\mathrm{Def}^2(\mathcal{X}, x) = \mathrm{Obs}(\mathcal{X}, x) \oplus \mathrm{Def}_x(\mathcal{X})$)

Eg. $\mathcal{H}(S) \supset M_g(S)$ $\Rightarrow \mathrm{Def}^1(\mathcal{X}, Y) = H^1(Y, T_Y)$ & $\mathrm{Obs}(\mathcal{X}, Y) = H^2(Y, T_Y)$. Thus the vanishing of tangent bundle ⇒ smooth

{X over k smooth canonically polarized i.e. K_X ample}

Eg. M_g: $\dim T_{M_g, C} = h^1(C, T_C) = h^0(T_C) - \chi(T_C) = -\chi(T_C)$

R-R: $-(\deg(T_C) + 1 - g) = 3g-3$, and $H^2(C, T_C) = 0 \Rightarrow \dim M_g = 3g-3$

On the other hand $\dim M_g = \dim W - \dim PGL(5g-5)$

$\dim_C W = \dim \mathrm{Def}_C(W) = h^0(C, N_{C/\mathbb{P}^N}) = h^1(C, N_{C/\mathbb{P}^N}) + \chi(T_{\mathbb{P}^N}|_C) - \chi(T_C)$

Using Euler sequence to compute h^1: Notice: it's an embedding deformation. $\mathrm{Obs}(W)$

$\to H^1(T_{\mathbb{P}^N}|_C) \to H^1(N_{C/\mathbb{P}^N}) \to 0$

↳ 0

By $0 \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_{\mathbb{P}^N}(1)^{\oplus N+1} \to T_{\mathbb{P}^N} \to 0$ restrict to C

(Hausdorff) $H^1(C, \omega_C)^{\oplus N+1} \to H^1(C, T_{\mathbb{P}^N}|_C) \to 0$

$= 0$

Then at last computation we have also $3g-3$.

Def. $f: \mathcal{X} \to \mathcal{Y}$ over Sch_k.

① Universally closed: $\forall Y' \to \mathcal{Y}$, s.t. $\mathcal{X} \times_{\mathcal{Y}} Y' \to Y'$ closed

② Separatedness: $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ i.e. $|\mathcal{X} \times_{\mathcal{Y}} Y'| \to |Y'|$ closed is always proper, f separated iff $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ proper

③ When $\mathcal{X} \to \mathcal{Y}$ representable, $\forall T \to \mathcal{Y}$, $\mathcal{X} \to \mathcal{Y}$ proper iff $\mathcal{X} \times_{\mathcal{Y}} T \to T$ proper as Sch_k;

④ Properness = separated + universally closed + finite type.

When representable ③ = ④

Valuation criterion

Thm. $f: \mathcal{X} \to \mathcal{Y}$ quasi-compact & locally Noetherian, and quasi-compact

① f is universally closed $\iff \exists$ lifting

② f is separated $\iff$! lifting (if exists)

③ f is proper $\iff \exists!$ lifting.

Different from scheme, we adding a extension of DVR to resolve the automorphism obstruction,

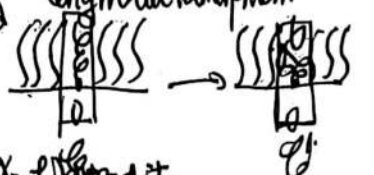
$$\begin{array}{ccccc} \mathrm{Spec}\,K' & \to & \mathrm{Spec}\,K & \to & \mathcal{X} \\ \downarrow & & \downarrow & \nearrow_{\text{lifting}} & \downarrow \\ \mathrm{Spec}\,R' & \to & \mathrm{Spec}\,R & \to & \mathcal{Y} \end{array}$$

$R \to R'$ extension, $K' = (\mathrm{Frac}(R')/K)$ is finite transcendental "$\bullet \to \bullet\!\to$"

DVR is a infinitesimal: "$\bullet\!\to$" a point and an arrow, and $\mathrm{Spec}\,K \to \mathrm{Spec}\,R$; analogously it's the disks $\triangle \to \triangle$, thus the completion is $K \rightsquigarrow R = \mathrm{Spec}$

Eg. Consider seperatness of M_g. $\mathcal{C} \to B \hookrightarrow \overline{B}$, $\mathcal{C}' \to \overline{B}$ $\Rightarrow$ $\mathcal{C} \overset{\text{birational}}{\dashrightarrow} \mathcal{C}'$ over $\overline{B}$. Outside $\overline{B}-B$ origin all isomorphism.

As $\mathcal{C} \simeq \mathcal{C}'$ smooth (projective) surfaces $\Rightarrow \mathcal{C} \cong \mathcal{C}'$ i.e. equal in M_g $\Rightarrow$ seperated



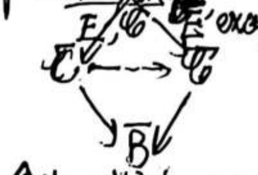
In general similiarly,

Thm. Given two families of regular polarized schemes $X_1 \to \operatorname{Spec} R \leftarrow X_2$, $(X_1, \mathcal{L}_1)$, $(X_2, \mathcal{L}_2)$, $K = \operatorname{Frac}(R)$, DVR with k residue. and $\mathcal{L}_i$ are relatively ample (fibrewise) $(X_1, \mathcal{L}_1)|_{\operatorname{Spec} K} \simeq (X_2, \mathcal{L}_2)|_{\operatorname{Spec} K} \Rightarrow (X_1, \mathcal{L}_1) \cong (X_2, \mathcal{L}_2)$ extend it.

except: $X_i|_{\operatorname{Spec} k}$ ruled! ($X_i \cong \mathbb{P}^1 \times Y$) (Matsusaka–Mumford)

Coro. When canonical polarized, the thm holds, in particular M_g holds (K relspec is the polarization)

The right example shows when not scheme, it not holds.

Skeleton: 

E exceptional. $\mathcal{C}$ the relatively minimal. E over points $\Rightarrow$ rational $\Rightarrow$ genus $0 < g \Rightarrow$ can't dominate $\Rightarrow$ contradiction $\Rightarrow E$ not exists $\Rightarrow$ isomorphic ▨

Eg. Consider (1,4) deg-hypersurface in $\mathbb{P}^1 \times \mathbb{P}^3 \supset S = V(sf + tg)$, $(s,t) \in \mathbb{P}^1$ and f, g have degree 4. $S \to \mathbb{P}^1$

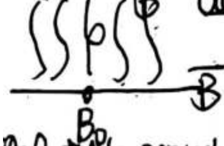
Observation ① Every fibre has at most 1 node singular, and ② S is Lefschetz pencil of quartic surface. (all is 60~69)

Then let $\mathbb{P}^1 \to \mathbb{P}^1$ 2-covering pullback to $\widetilde{S}$. And blowing-up $\widetilde{S}$ pointwise in each fibre to give $\widehat{S} \to \widetilde{S} \to S$.

However, I claim $\widehat{S} \to \mathbb{P}^1$ not projective! This is by $\mathcal{O}_{\widehat{S}}(1)$ not relatively ample: $\widehat{S}_t \to \widetilde{S}_t$ but $E_t \cdot \mathcal{O}_{\widehat{S}}(1) = 0$ by E_t exceptional $\Longrightarrow \mathcal{O}_{\widehat{S}_t}(1)$ not ample (due to ample $\Leftrightarrow$ all intersection >0) ▨

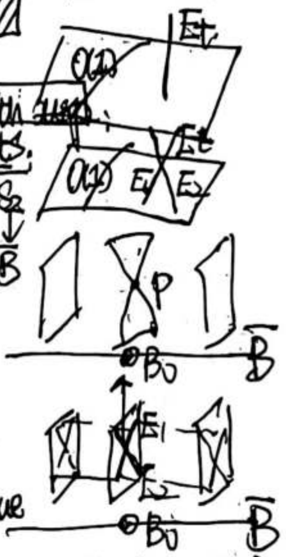
Eg. M_g not proper. $\exists$ family $\mathcal{C} \subset \operatorname{Hilb}(\mathbb{P}^N) \times \mathbb{P}^N$, s.t. $\mathcal{C}|_B$ is relatively smooth but $\overline{\mathcal{C}}|_{\overline{B}-B}$ not. $\downarrow \overline{B}$

Claim. $\mathcal{C}|_{\overline{B}-B}$ consists of stable curves: nodal, $|\operatorname{Aut}(\mathcal{C})| < \infty$



This advices us the nodal is good (one blowup)

Aside. Using this way, replace node with $\mathbb{P}^1$ $\Rightarrow E_t$ has two irreducible components. $\Rightarrow \exists$ two algebraic space $\overline{S}_1 \supset S \subset \overline{S}_2$ over $\overline{B} \supset B \subset \overline{B}$, s.t. $\overline{S}_1|_{\overline{B}-B} \simeq \overline{S}_2|_{\overline{B}-B}$ but $\overline{S}_1 \not\cong \overline{S}_2$



And stable curves' family has "uniqueness": $\nexists$ $\mathcal{C}'$ s.t. $\mathcal{C}'|_B \simeq \mathcal{C}$, $\overline{\mathcal{C}} \supset \mathcal{C} \subset \overline{\mathcal{C}}' \Rightarrow \overline{\mathcal{C}} \cong \overline{\mathcal{C}}'$ over $\overline{B} \supset B \subset \overline{B}$, $\overline{B}$ smooth. i.e. $\overline{M}_g$ is seperated (easy)

Then we formally contact stable curves (semi-stable reduction)

Stability in algebraic geometry

① stable singularity/degeneration

$\mathcal{X}_0$ singular but $\mathcal{X}_t (t \neq 0)$ smooth and $\mathcal{X}$ is stable family (i.e. $\mathcal{X} \cong \mathcal{X}'$ over $\Delta^* = \Delta^*$ then over Δ also same)

② GIT stability ① Orbit closed ② stabilizer finite.

① = ② for stable curves.

$\hookrightarrow$ in curves, it's just stable reduction

Precisely, statement is: For $X \to \operatorname{Spec} K$ (or Δ^*), $K = \operatorname{Frac}(R)$. We hope extend K to R, but by "semi", we don't of DVR do it directly, but let $R \to R'$ finite $\Leftarrow K'/K$ finite s.t. $X' = X \times_{\operatorname{Spec} K} \operatorname{Spec} K'$ admits an extension $\overline{X}' \leftarrow \overline{X}$ (not unique) and $\overline{X}'|_{\operatorname{Spec} k}$ is SNC (simple normal crossing) over $\operatorname{Spec} R' \to \operatorname{Spec} R$

(k is special fibre 0, R is hole $\overline{B}$, K is general fibre $\overline{B}-B$)

Here we need $\boxed{\operatorname{char} k = 0}$ (p open)

For curves, ① It's stable: unique and directly for (K, R) ② For any char holds. (In GIT, semi-stable isn't orbit closed but stable is automorphism not finite.)

Pf. Pure birational ▨

Def. (Stable curve) C at-most nodal proj curve over k, C is stable if

① Each geometric connected component of C has genus > 1 (arithmetic)

② Each geometric irreducible component of C, $g > 1$; $g = 1$ and at least 1 singularity; $g = 0$ and at least 3 singularity

(It's deformation invariant)

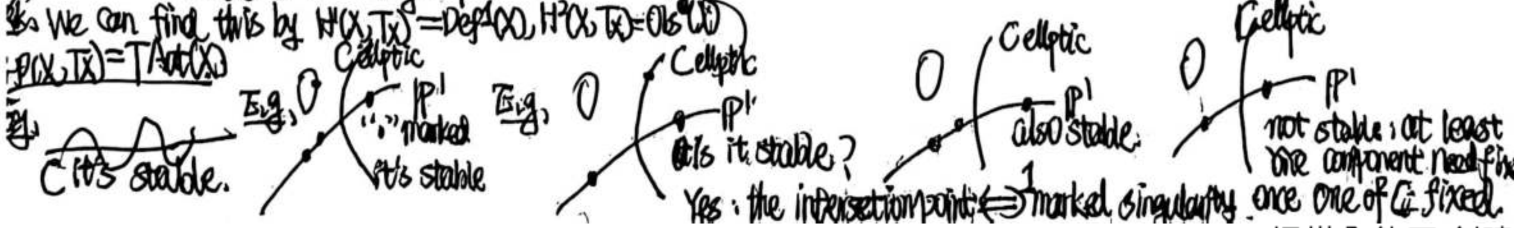
Prop. $g \geq 2$, C stable $\Leftrightarrow |\operatorname{Aut} C| < \infty \Leftrightarrow \omega_C^0 = \omega_C$ ample. (relatively it's $\omega_{C/S}$)

We can use stable marked curves $(C, p_1 \cdots p_n)$ singularity $\Rightarrow |\operatorname{Aut}(C)| < \infty$

This's by $\dim \operatorname{Aut}(\text{elliptic}) = 1$, adding 1 pt; $\dim \operatorname{Aut}(\mathbb{P}^1) = 3$, adding 3 pts ▨

$\overline{M}_g$ is DM-stack; Thm. $\overline{M}_g \cong [U / PGL(5g-5)]$, $U \subset$ Hilb parameter stable curves (It's even projective)

So we can find this by $H^1(X, T_X) = \operatorname{Def}^1(X)$, $H^2(X, T_X) = \operatorname{Obs}(X)$, $H^0(X, T_X) = T\operatorname{Aut}(X)$



Pf. $\forall (\mathcal{C} \to S) \in \overline{\mathcal{M}}_g(S)$, $f^*\omega^{\otimes 3}_{\mathcal{C}/S}$ is locally free on S $\Rightarrow$ we have a PGL-bundle $\mathcal{C} \to \mathbb{P}^N \times S \to S$ and $PGL \curvearrowright U$ has finite stablizer ☐

$\Rightarrow$ D-M stack. $\subset \mathrm{Aut}(C)$ finite.

Thm. $\overline{\mathcal{M}}_g$ is proper (compact) Pf. Separated is done the same as $\mathcal{M}_g$ by an argument on minimal surface.

Universally closed $\Leftrightarrow \forall \mathcal{C}^* \to \Delta^*$, $\exists \Delta' \to \Delta$ finite extension, s.t. $\mathcal{C}^* \times_{\Delta^*} (\Delta')^* \to \mathcal{C}'$ extending ($\Delta^* = \mathrm{Spec} K$ & $\Delta = \mathrm{Spec} R$) ($\Delta' = \mathrm{Spec} R'$ ...) (The stable reduction)

Five steps to construct family $\mathcal{C}' \to \Delta'$

Step 0 Reduce to smooth: by desingularisation $\tilde{\mathcal{C}}^* \to \mathcal{C}^*$

then $\tilde{\mathcal{C}} \supset \tilde{\mathcal{C}}^* \to \mathcal{C}^* \subset \mathcal{C}$

$\Delta \supset \Delta^* = \Delta^* \subset \Delta$

Step 1 Extend $\mathcal{C}^*$ to $\mathcal{C}$: by the properness of Hilbert scheme

By valuation critira of Hilb $\Rightarrow$ lifts to $\mathcal{C} \to \Delta$

$\mathcal{C}^* \downarrow$ $\Delta^* \to \mathrm{Hilb} \in \mathrm{Sch}$ ☐

Step 2 Blow up cuspidal ... to nodal in $\mathcal{C} \to \Delta$ set-theortic ($\mathcal{C}_0^{red}$) ☐ thus here it's not nodal in fact:

Step 3 Reduced to $\mathcal{C}_0^{red} = \mathcal{C}_0$

$\mathcal{C}_0^{red} \subset \mathcal{C}$ is a Cartier divisor cutted by one equation, $\forall p \in \mathcal{C}_0^{red}$ $(x,y) = 0$

locally $\mathcal{C} \to \Delta$ near p

$(x,y) \mapsto x^n$ or $x^n y^m$

smooth or singular, or

$\Rightarrow \mathcal{C}$ is locally normal crossing (not single!)

To make it single. a base change / coordinate change $t \mapsto t^n$ ☐

Δ $\{x \mapsto x^n, \; y \mapsto y^m$ makes sense in analytic disk allow it to define $-\frac{1}{n}$ radical. (normalisation)

but in algebraic, only t one parameter take $t \mapsto t^{\mathrm{lcm}(m,n)}$

Step 4 $\mathcal{C}_0$ is stable: The by Bezout $\Rightarrow am - bn = 1$, $\exists (a,b)$ $C \subset \mathcal{C}_0$ irreducible component; base change $\mathcal{C}' = \mathcal{C} \times_\Delta \Delta'$ again

$g(C) > 1$ done; $g(C) = 1$ done by intersection; $g(C) = 0$, i.e. $C = \mathbb{P}^1$ {1 special point; 2 special points}

s.t. $\mathcal{C}' \underset{\text{locally}}{\cong} \mathrm{Spec}\left(\frac{R[x,y]}{(x^m y^n - t^{mn})}\right)$ then normalisation of $R[x,y]$ is $R[u,v]/(uv - t)$ $\{u = x^m \ldots, \; v = y^n$ ($x^m y^n - t^n$) ☐

But $(uv - t^n)$ has A_{n-1} singularity $\Rightarrow$ blow up last time

to be more precise, $\mathrm{Spec} R' \to \mathrm{Spec} R$ induced by $R \to R'$, $t \mapsto t^{\ldots}$, local equation $x^n y^m - t^{\ldots}$ by $-\frac{1}{d}$, $d = \gcd(m,n)$, assume n, m relatively prime

Rk. Step 0 & 1 & 2 valid for $\mathbb{P}^1$ char $= 0$ or p; but for higher dimension (even $= 2$) is unknown now (2024) (Although $n \le 3$ ... is done in char p).

Eg. Stable reduction of cusp: $\mathcal{C} = \{(x,y,t) \mid y^2 = x^3 + t\}$

We blow up first in affine nature $\Rightarrow \{x_1 = x/y, \; y_1 = y$ $\Rightarrow x_1^2 y_1^2 = x_1^3 + t$ at $(0,0,0)$

thus $\mathcal{C}_0 = x_1^2(y_1^2 - x_1)$ is

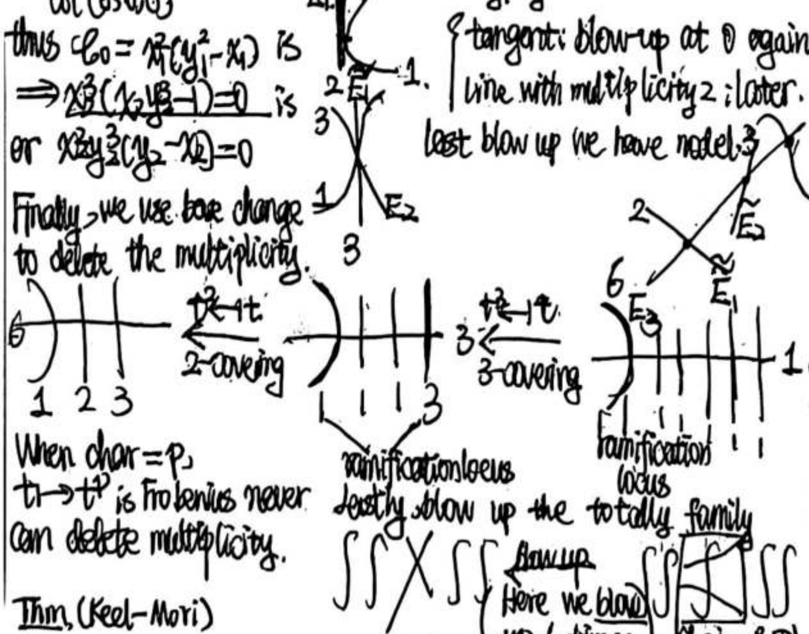
tangent: blow-up at 0 again; line with multiplicity 2: later.

$\Rightarrow x_2^3(x_2 y_2^3 - 1) = 0$ is

or $x_2 y_2^3 (y_2 - x_2) = 0$

last blow up we have model:

Finally, we use base change to delete the multiplicity.



When char $= p$, $t \mapsto t^p$ is Frobenius never can delete multiplicity.

Lastly blow up the totally family

Here we blow up 6 times. chain of $\mathbb{P}^1$ (due to ADE singularity) is rational

Step 4 (Continue)

Define {Special point 1: Rational tale $\Rightarrow C^2 = -1$; Special points 2: Rational bridge $\Rightarrow C^2 = -2$}

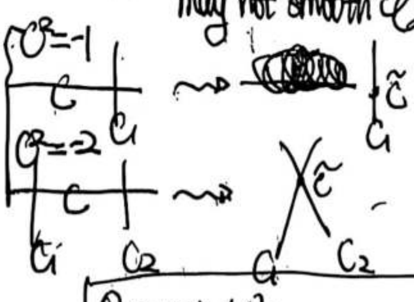
$C^2 = -1 \Leftrightarrow \exists$ contraction makes C to one point.

Contraction is $X \to X'$

$\mathrm{Proj}(\oplus H^0(X, L)) \to \mathrm{Proj}(\oplus H^0(X, L \otimes C^{\ldots}))$

$C \to pt$

$C^2 = -2 \Rightarrow \exists$ contraction makes C to one point, but may not smooth $\mathcal{C}$, but still nodal.



Thus $\overline{\mathcal{M}}_g$ is proper of finite type, smooth.

Thm. (Keel-Mori)

$\overline{\mathcal{M}}_g$ admits a proper coarse moduli $\overline{M}_g \in \mathrm{Sch}$, and coarse moduli map $\overline{\mathcal{M}}_g \to \overline{M}_g$

But $\overline{M}_g$ always not smooth

Our next task. Thm. $\overline{M}_g$ is projective scheme (Positivity of CM line bundle) Good moduli space.

Rk. By quotient, $\mathcal{M}_g$ & $\overline{\mathcal{M}}_g$ has smooth presentation naturally; but DM-stack needs étale presentation (although we can slice from smooth one)

We call $\mathcal{M}_g^n$ with level structure

$\mathcal{M}_g^n \xrightarrow[\text{étale}]{F} \mathcal{M}_g$, $\mathcal{M}_g^n = \{(C, \varphi) \in \mathcal{M}_g \times \mathrm{Aut}(H^1(C, \mathbb{Z}/n\mathbb{Z}))\}$ and $\mathcal{M}_g^n$ is fine moduli ($n \ge 3$)

n larger $\Rightarrow$ restriction more $\Rightarrow$ automorphism $\ldots$ finite $\Rightarrow F$ is finite, then take representation $\Rightarrow$ we have an étale presentation ☐

Projectivity. Def (Coarse moduli) $\mathcal{X} \to Sch$ algebraic stack. the coarse moduli of $\mathcal{X}$ is a morphism $\pi: \mathcal{X} \to X$ algebraic space (almost all time, it's a scheme)

s.t. ① $|\pi|: |\mathcal{X}| \to |X|$ homeomorphism, i.e. $\mathcal{X}(\bar{k}) \cong X(\bar{k})$ as set; ② π is universal (smart): $\mathcal{X} \to Y$, $\exists!$ for Y algebraic space

(This implies it's uniqueness $X = \mathcal{X}(\bar{k})$)

Thm (Keel-Mori). $\mathcal{X} \to Sch/S$ seperated DM-stack finite type, S Noetherian, admits $\pi: \mathcal{X} \to X$ coarse moduli space s.t.

① X seperated over S and $\boxed{\pi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X}$; $\rightsquigarrow$ Rk. In stack, the fibre disconnected; Zariski main thm is modified

② $|\pi|: |\mathcal{X}| \to |X|$ is proper homeomorphism;

③ Flat algebraic space base change $X' \to X$ to this $\Rightarrow \mathcal{X} \times_X X' \to X'$ is also coarse moduli space.

Rk. Existence of coarse moduli space is étale local, thus only check the quotient stack $[U/G_x]$, and this case is trivial: $[U/G_x](\bar{k}) = [U(\bar{k})/G_x(\bar{k})] = U(\bar{k})/G_x(\bar{k})$ ▨

Thm. $\overline{\mathcal{M}}_g$ admits a projective coarse moduli space $\overline{\mathcal{M}}_g \to \overline{M}_g$

Pf. (Mumford's way) We use the GIT, $\boxed{G \text{ reductive}}$, Y projective over $\bar{k}$, $G \curvearrowright Y$ properly, $Y = \mathrm{Proj} \bigoplus H^0(Y, \mathcal{L}_0^{\otimes n})$, $\mathcal{L}_0$ is a G-linearization (i.e. $Y \hookrightarrow |m\mathcal{L}_0|$, $G \mapsto PGL$ linear representation) (must!)

then $[Y^{ss}/G] \to Y/\!/_{\mathcal{L}_0} G = \mathrm{Proj} \bigoplus H^0(Y, \mathcal{L}_0^{\otimes m})^G$ is coarse moduli space/good quotient.

The proof is by: ① Reduce to affine $Y = \mathrm{Spec}\,R$, $[Y/G] \to Y/\!/G = \mathrm{Spec}(R^G)$

② Check R^G or $\bigoplus H^0(Y, \mathcal{L}_0^{\otimes n})^G$ finite generated (hard!).

③ Then trivial ▨

Then we expect to use it: $U = \mathrm{Hilb}(\mathbb{P}^N)^{ss}$ and $\overline{\mathcal{M}}_g = [U/PGL]$

where $C \hookrightarrow |mK_C| = \mathbb{P}^N$ (Hilb scheme carries $\mathcal{O}(1)$ naturally linearization) $\rightsquigarrow$

(Gieseker) $m \geq 5$ case the upper holds ▨

However GIT's way has disadvantage: in higher dimension it's useless: (except for moduli of sheaves)

① Y^{ss} hard to depict; ② May not a quotient stack.

(In higher dimension, Donaldson's works in differential Geometry is best)

Pf. (Kollar's way) Studying the Chern-Mumford line bundle: show $\overline{M}_g$ admits an ample line bundle ($\overline{M}_g$ is proper by $\overline{\mathcal{M}}_g$ is). Which line bundle we choose?

Lemma. (Numerical criterion of ampleness) X proper algebraic space over $\bar{k}$, $\mathcal{L} \in \mathrm{Pic}(X)$

① $\mathcal{L}$ ample; ② $\mathcal{L}$ nef. and $\forall Z \subset X$ closed $\mathcal{L}|_Z$ big; ③ $\mathcal{L}$ strictly nef, $\mathcal{L}^{\otimes m}|_Z$ effective for some $m > 0$ (only compute the intersection number)

Rk. ① Alternatively, using Chow variety

$\overline{\mathcal{M}}_g = [V/PGL]$, $V = \mathrm{Chow}(\mathbb{P}^N)^{ss}$

weaken to $m \geq 4$

(Chow variety also $\mathcal{O}(1)$-linearization)

② Furthermore all of these are reducing to the $\boxed{\text{stability condition}}$, in general

$\overline{M}_g = \{$space of polystable elements$\}$

semistable + closed orbit

Stable $\Leftrightarrow$ stack (fine moduli)

polystable $\Leftrightarrow$ coarse moduli

Pf of Kollar (Continue) Consider $\mathrm{Pic}(\overline{M}_g)$, we have $\{\boxed{\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q} \cong \mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}}$, $\mathrm{rank}(\mathrm{Pic}(\overline{M}_g)) = 1 + \lfloor \frac{g}{2} \rfloor$

Let $f: \mathcal{C}_g \to \overline{\mathcal{M}}_g$ the universal family.

then $V_m = f_*(\omega_{\mathcal{C}_g/\overline{\mathcal{M}}_g}^{\otimes m})$, take $\det(V_m) = c_1(V_m) \in \mathrm{Pic}(\overline{\mathcal{M}}_g) \Rightarrow$ it gives a line bundle on $\overline{M}_g$ (descends). I claim. It's ample for m large

(↳ $\det V_m = \pi^* H$ after a power $V_m^{\otimes n}$, n large)

By projection formula

$\pi: \overline{\mathcal{M}}_g \to \overline{M}_g$ $\pi_*\pi^*(\mathcal{L}_0) = \pi_*\mathcal{O}_{\overline{\mathcal{M}}_g} \otimes \mathcal{L}_0 \overset{\text{Keel-Mori}}{=} \mathcal{O}_{\overline{M}_g} \otimes \mathcal{L}_0 = \mathcal{L}_0$ ▨

$\pi^*\pi_*\mathcal{L}_0 = \mathcal{L}_0$

We use the criteria to show ampleness: check ②: $Z \subset \overline{M}_g$, $\mathcal{C} \xrightarrow{f} Z \Rightarrow \det(V_m)|_Z = \det(f_*(\omega_{\mathcal{C}/Z}^{\otimes m}))$, only check for it. (③ is hard to use due to strictly nef)

Now we define $\lambda = c_1(f_*(\omega_{\mathcal{C}_g/\overline{\mathcal{M}}_g}^{\otimes m}))$ for simplicity and $\lambda|_Z$

$\boxed{\text{Step 1}}$ λ is nef, $\lambda \cdot$ curve ≥ 0, denote the curve $B \Longrightarrow$

{ ① Assume B is smooth and $g(B) \geq 2$: by blowup & branch covering $\xrightarrow{g} B' \to B$ smooth, $\lambda|_{B'}$ nef $\Rightarrow \lambda|_B$ nef ▨

②
$$\begin{tikzcd} X \arrow[r] \arrow[d, "\varphi"] \arrow[dr, phantom, "\square"] & \mathcal{C}_g \arrow[d, "f"] \\ B \arrow[r] & \overline{\mathcal{M}}_g \end{tikzcd}$$
X is minimal. By (1) curve exists in the fibration $X \to B$ $\Rightarrow$ the horizonal curve genus 0 can't dominate $g(B) \geq 2 \Rightarrow$ contradiction ▨

$X \to B$ then X is proper surface $\Leftrightarrow$ projective surface

③ Reduce to char $k = p \neq 0$ (spread out and specialization)

By: Let B over k, then B is over a $\mathbb{Z}$-algebra R finitely generated

$$\begin{tikzcd} B \arrow[r] \arrow[d] & B' \arrow[d] \\ \mathrm{Spec}\,k \arrow[r] & \mathrm{Spec}\,R \end{tikzcd}$$

X can be spread out to be $\mathcal{X}$ (integral model)

$\mathcal{X} \to \boxed{\mathrm{Spec}\,R}$ higher dimension

Then over $R \Rightarrow$ modulo p (i.e. $(p) \subseteq \mathbb{Z}$ and mod $(p)R$) $\Rightarrow$ special fibre $X_p \to \mathrm{Spec}\,R/(p) \Rightarrow$ it's a base change and nef is stable ▨

(mixed characteristic)

$\boxed{\text{Step 2}}$ Kollar's criterion: nef + little big $\Rightarrow$ big

Then by the numerical criterion $\Rightarrow \lambda|_B$ nef for all curves, λ big

$\Leftrightarrow \lambda$ ample ▨

$\boxed{\text{Step 3}}$ (Viehweg, Kollar) $\mathrm{Sym}^d \mathcal{E} \twoheadrightarrow Q$ locally free ($\dim = N$)

if $\mathcal{E}$ nef and classifying map $\alpha: X \to [\mathrm{Gr}(q, N)/PGL(N)]$ quasi-finite $\Rightarrow \mathcal{E}$ ample.

Setting $\mathcal{E} = f_*(\omega_{\mathcal{C}_g/\overline{\mathcal{M}}_g}^{\otimes 3})$, $\lambda = c_1(\mathcal{E})$ and $\varphi_d: \mathrm{Sym}^d \mathcal{E} \twoheadrightarrow$

$Q = f_*(\omega_{\mathcal{C}_g/\overline{\mathcal{M}}_g}^{\otimes 3d})$ d large, s.t. { φ_d surjective, determined by degree d-equations } Fibres of $f: \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ is

(A Finite condition)

Rk. (Positivity) $\mathcal{L}$ ample $\Leftrightarrow \varphi^*\mathcal{L}$ ample: finite; $\mathcal{L}$ nef $\Leftrightarrow \varphi^*\mathcal{L}$ nef: proper

$\mathcal{L}$ big $\Leftrightarrow \varphi^*\mathcal{L}$ big: generically finite

Then we prove $\mathscr{C} \to \mathrm{Fr}: \mathcal{Y}/PGL(n)$ has finite fibres (checked locally)

$\#$ Fibre $= \#$ (deg of equations defining this curve) $\leq \infty$

The classifying map sends two points $C_1, C_2 \in \mathscr{C}$ (sections of $\omega^{\otimes k}_{C/B}$ is local coordinate in embedding to $\mathbb{P}^N$) $\boxed{}$

to same point $\Leftrightarrow F_{C_1} \cong F_{C_2}$, Fibres in $X \to \mathscr{C}$ (as $g(C) \geq 2$ $\Rightarrow$ isomorphic $C' \cong C$ are finite)

Precisely $|2^{n}(I/\Delta)| \leq |\mathrm{Stab}(\Delta)| < \infty$ $\boxed{}$

Def. $\delta_d = \{ \text{curve } d,\ g-d \}$, $d = 0, 1, \dots [\frac{g}{2}]$, for δ_0 it's $\{\alpha\}$ (multiplicity 2)

$\in \mathrm{Pic}(\overline{M}_g)_{\mathbb{Q}} \qquad = \mathbb{Q}\langle \lambda, \delta_i \rangle$

Thm. ① $\mathrm{Pic}(\overline{M}_g)_{\mathbb{Q}} = \mathrm{Span}\{\lambda, \delta_0 \cdots \delta_{[\frac{g}{2}]}\}$ (Recall $\lambda = c_1(\cdots)$)

② (Strengthen Kollar) $\lambda - k\sum \delta_i$ is ample if $k \ll 1$ (later proven)

We proven $m \geq 2$, when $m = 1$, λ_1 is nef & big but not ample

$\Rightarrow M_g \subset \overline{M}_g$ quasi-projective

Thm. $M_g \subset \overline{M}_g$ quasi-projective Pf. $\mathscr{C}_g \xrightarrow{\pi} M_g$, E, $R^i\pi_*(\mathbb{Q})$ or $R^i\pi_*(\mathbb{C})$

Then in analytic way, using Kodaira embedding $M_g \hookrightarrow \mathrm{Proj}(\bigoplus_m H^0(M_g, \lambda^{\otimes m}))$

$\Rightarrow$ Quasi-projective $\boxed{}$ $\quad = \overline{M}_g^{BB}$ (Arithmetic compactification by Hodge bundle)

$H^i(C, \mathbb{Q})$ or $H^i(C, \mathbb{C}) = \mathbb{C}^{2g}$; $i = 1$ (0 or 2 trivial) $= \mathbb{C}^g \oplus \mathbb{C}^g = H^0(C, \Omega_C) \oplus \overline{H^0(C, \Omega_C)}$

$\mathrm{Ab}(\pi_1(C)) \otimes \mathbb{Q} = \mathbb{Q}^{2g}$, $i = 1$ — (topological) — analytic — GAGA — algebraic (note not analytic coherent $\Rightarrow$ decompose into coherent to apply GAGA)

$H^0(C, \Omega_C) = \mathbb{C}^g$ the Hodge bundle

$\pi_*(\Omega_{\mathscr{C}_g/M_g}) =: E$ over M_g with fibres over C to be $H^0(C, \Omega_C)$ and $\lambda = c_1(E)$

Here λ is nef & big and not ample, it turns out $\lambda = \lambda_1$

④ Now we reduce to $g(B) \geq 2$, char $= p$, X minimal (minimal by no rational tales)

Then we start the proof: Otherwise $\mathcal{E} = \pi_*(\omega^{\otimes}_{X/B})$ not nef

$\Rightarrow \exists$ line bundle $\mathcal{E} \to \mathcal{L}$, $\mathcal{L}$ ample, if $\mathcal{L}$ has large degree

(ample) $\Rightarrow \mathcal{L} = \omega_C \otimes D$, D very ample

$\Rightarrow H^1(\mathscr{C}, \omega_C^{\vee}(D) \otimes \mathcal{E}) \neq 0 \Rightarrow H^0(\omega_X \otimes \omega^{\otimes}(-D)) \neq 0$ contradiction $\boxed{}$

(Using Leray SS to compute)

We need char $= p$ for this

The absolute & relative Frobenius

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d] \arrow[rr, bend left, "F"] & \mathscr{C}^{(1)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathscr{C} \arrow[d] \\ & \operatorname{Spec} k \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

relative Frobenius; $\mathscr{C} \xrightarrow{F_r} \mathscr{C}$ composite the absolute Frobenius: endomorphism with $\deg(F_r) = p^n > 1$ not isomorphism.

Frobenius for k, char $k = p$

By compositing, one has $(\deg F_r^n)$ enough large, not changing the base but $\mathcal{L}$ fold large $\boxed{}$

$\overline{M}_g^{BB}$ smaller than $\overline{M}_g$ in boundary ($\overline{M}_g^{BB} - M_g$ even can't form divisor), we have ① $\overline{M}_g^{BB} \to \overline{M}_g$ birational ② $\mathrm{Pic}(M_g) \underset{\mathbb{Q}}{\cong} \mathrm{Pic}(\overline{M}_g) \cong \mathbb{Q}\langle\lambda\rangle$

③ is open for AG (only topologically proven)

※ We concern $\mathrm{Pic}(\overline{M}_g)$ due to it determines all projective Geometry) One compute $\dim \delta_i = \dim M_i + \dim M_{g-i} + \underline{1+1}$

$= 3i - 3 + 3(g-i) - 3 + 2 = 3g - 4$ (moving intersection through two curves respectively)

$\Rightarrow$ codimension 1 $\boxed{}$

We study $\overline{M}_g$ more precisely { ① Canonical class; ② Ample divisors; ③ Birational model.

① $T_{[C]} = \pi_*(\Omega_C, \mathcal{O}_C)$

Smooth $H^1(C, \Omega_C^{\vee} \otimes \mathcal{O}_C) = H^1(C, T_C)$; and cotangent $T^{\vee}_{[C]} \overset{\text{Serre}}{=} H^0(C, \Omega_C \otimes \omega_C) \Rightarrow$ we define cotangent sheaf

$\Rightarrow \Omega$ locally free

To compute $\pi_*(\Omega \otimes \omega)$ we use a base change through presentation $\pi_*(\Omega_{\mathscr{C}_g/M_g} \otimes \omega_{\mathscr{C}_g/M_g})$

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathscr{C}_g \arrow[d] \\ B \arrow[r] & M_g \end{tikzcd}$$

$\Rightarrow \pi_*(\Omega_{X/B} \otimes \omega_{X/B})$, due to we concern c_1, let's assume $\dim B = 1$

Grothendieck–Riemann–Roch. $\pi: X \to Y$ proper, X, Y smooth $\Rightarrow \mathrm{ch}(\pi_!(E)) \mathrm{td}(Y) = \pi_*(\mathrm{ch}(E)\mathrm{td}(X))$, $\forall E \in \mathrm{Coh}(X)$

$\Rightarrow \mathrm{ch}(\pi_*(\Omega_{X/B} \otimes \omega_{X/B})) = \pi_*(\mathrm{ch}(\Omega^{2}_{X/B}) \mathrm{td}(X/B))$ here $\omega_{X/B} \neq \Omega_{X/B}$ here $\omega_{X/B}$ dualising, X may not smooth

$\mathrm{ch}(\pi_!(E)) = \pi_*(\mathrm{ch}(E)\mathrm{td}(X/Y))$; usually $\pi_! = \pi_*$ without higher direct image. $(\mathrm{td}(X) = \mathrm{td}(T_X)$, $\mathrm{td}(X/Y) = \mathrm{td}(T_{X/Y}))$

$= \pi_*\left((1 + 2\lambda + \frac{(2\lambda)^2}{2} + 2\eta)\right)\left(1 - \frac{\lambda}{2} + \frac{\lambda^2 + \eta}{12}\right)$, $\lambda = c_1(\Omega_{X/B})$, $\eta = c_2(\Omega_{X/B})$

$= \pi_*(\lambda + 2\lambda^2 - 2\eta - \lambda^2 + \frac{\lambda^2 + \eta}{12})$, $\eta = c_2(\Omega_{X/B})$ (We only concern λ^2 and η) $= \pi_*(\frac{13}{12}\lambda^2 - \frac{11}{12}\eta)$ × $\Rightarrow$ it's not number we want!)

$= \pi_*(\frac{13}{12}\lambda^2 - \frac{11}{12}[\mathrm{Sing}(B)]) = \boxed{13\lambda - 2\delta}$

Modify: $\Omega_{X/B} = \mathcal{I}_{\mathrm{Sing}(B)} \otimes \omega_{X/B}$

Some points $\Rightarrow$ only $\eta = c_2$ need change.

$\Rightarrow \mathrm{ch}_2(\Omega_{X/B}) = [\mathrm{Sing}(B)] \cdot \lambda^2$

$c_2(\Omega_{X/B})$

Denote $\lambda = c_1(\pi_*(\omega_{X/B})) \Rightarrow$ desired number]

$[\mathrm{Sing}(B)] = \{$Singularity of fibres in $X \to B\}$

$\Rightarrow \pi_*([\mathrm{Sing}(B)]) = 2\delta = 2\,\mathrm{Sing}(B)$ (as usual, due to node multiplicity 2)

[We omit the proof of $\pi_*(\frac{\lambda^2 + [\mathrm{Sing}(B)]}{12}) = \lambda$ (pure birational Geometry)]

② $D = a\lambda + \sum b_i \delta_i$ ample? When $b_i \neq b_j$ it's hard, but for $D = a\lambda - b\delta$ it's done by ①: Thm. (Kodamata–Harris) $\lambda - \mu\delta$ ample $\Leftrightarrow \mu < \frac{1}{11}$ (and $\mu > 0$)

Coro. $\overline{M}_g = \mathrm{Proj}(\bigoplus H^0(\overline{M}_g, (\lambda - \mu\delta)^{\otimes m}))$ when $\mu < \frac{1}{11}$

Pf. Mainly by next thm

Thm. $\pi: X \to B$ ($\dim B = 1$) family polarized $\subset \mathbb{P}^n$ bounded, $E = \pi_*(\mathcal{O}(1))$ locally free

$\Rightarrow (n+1) c_1(\mathcal{O}(1))^{\dim X} \geq \dim X\, c_1(\mathcal{O}(1))^{\dim X - 1} c_1(E)$ (components same) (When X family of curves, it turns back to CM line bundle in Kollar's projectivity)

③ $\mu = \frac{1}{11}$ case, $\overline{M}_g^{\mathrm{cusp}} = \mathrm{Proj}\bigoplus H^0(\overline{M}_g, (\lambda - \frac{1}{11}\delta)^{\otimes m})$, boundary $= \{X_{a-i} \cup \cdots\}$

All of these are quotient (by $C \hookrightarrow |\omega_C^{\otimes 4}|$ or larger) stack $\Rightarrow$ D-M stacks. μ larger ↗ and then tacnode … occurs
(called $\overline{M}_g^{\,\text{cusp}}$ ~~pseudo-stable curve~~) Various compactifications (adding boundary) occurs $\overline{M}_g$, $\overline{M}_g^{BB}$, $\overline{M}_g^{\,\text{cusp}}$, K-compactification, Hodge-compactification
all these compactification are $\overline{M}_g$ best.

Two ways: $[U/G]$ not "good" $\longrightarrow$ Kollár's way: $[U/G]$'s coarse moduli, turn back to classical birational Geometry.
Classical GIT start)
$\mathcal{M}(\mathbb{K})$'s Geometry, Good moduli way

DM can be Artinian, generalising quotient $\Rightarrow$ Modern GIT
done by classical

E.g. $\mathcal{M}$ = moduli of Fano varieties (anticanonically
[KSPV] done the moduli of canonically polarized variety
e.g. $\mathcal{M}_g$ (general type).

polarized $\Leftrightarrow -K_X$ ample) $\Rightarrow |\mathrm{Aut}(X)| \overset{\text{can be}}{=} \infty$

E.g. $X = \mathrm{Bl}_{pt}(\mathbb{P}^2)$, $\mathrm{Aut}(X) \subseteq PGL(3)$ fix this $\{pt\}$
$\Rightarrow \mathrm{Aut} = PGL(3)$ / stablizer infinite
and X is Fano with volume $K_X^2 = 8$
Let $\mathcal{M}_{K_X^2=8}$ isn't DM

E.g. Let $\mathcal{M}$ = {moduli of K-semi-stable Fano varieties with some numerical datum}
(Xu, Jarod …) $\mathcal{M}$ admits a good moduli $\mathcal{M} \to M$ algebraic space (existence is proven in 2021, now M's properties are not well understanded)

Def. $\pi: \mathcal{M} \to M$ is good moduli iff
(1) M algebraic space, quasi-compact & quasi-connected
(2) π Affine.
(3) $\pi_*: \mathrm{Coh}(\mathcal{M}) \to \mathrm{Coh}(M)$ is exact functor (all over $\mathbb{K}$)

Rk. (Quasi-)coherent by pullback to $\forall$ presentation all

Thm. $\pi: \mathcal{M} \to M$ Good, then
① π is universally closed & surjective;
② $\mathcal{Z}_1, \mathcal{Z}_2 \subset \mathcal{M}$ closed substack $\Rightarrow \pi(\mathcal{Z}_1 \cap \mathcal{Z}_2) = \pi(\mathcal{Z}_1) \cap \pi(\mathcal{Z}_2)$
$\Rightarrow |\mathcal{M}| \xrightarrow{1:1} |M|$; (orbit intersection)
③ $\mathcal{M}$ noetherian, π universal for morphism to algebraic spaces
i.e. $\mathcal{M} \to M$, $\mathcal{M} \to M'$, $M \overset{\exists!}{\to} M'$ (categorical quotient), all these are generalisation

Only ② needs (3) and hardest.
Pf. $\mathrm{Coh}(M) \underset{\pi^{-1}}{\overset{\pi_*}{\rightleftarrows}} \mathrm{Coh}(\mathcal{M})$ exact by (3)
$\mathcal{I}_{\mathcal{Z}_1}$, $\mathcal{I}_{\mathcal{Z}_2}$, $\mathcal{I}_{\mathcal{Z}_1 \cap \mathcal{Z}_2}$

Preserve surjectivity $\longleftarrow \mathcal{I}_{\mathcal{Z}_1} \oplus \mathcal{I}_{\mathcal{Z}_2} \twoheadrightarrow \mathcal{I}_{\mathcal{Z}_1 \cap \mathcal{Z}_2}$
$\Updownarrow$ $\Updownarrow$
$\pi(\mathcal{Z}_1) \cap \pi(\mathcal{Z}_2) = \pi(\mathcal{Z}_1 \cap \mathcal{Z}_2) = \mathcal{Z}_1 \cap \mathcal{Z}_2$ ▨

Rk. Although good generalises coarse,
Coarse $\not\Rightarrow$ good, and affine GIT is good $\Leftrightarrow$ G reductive.

Classical GIT. ① Affine GIT. Good quotient is G-equivariant map $U \xrightarrow{\pi} Y$ iff (1) Affine morphism
(2) $\mathcal{O}_Y \cong (\pi_*\mathcal{O}_U)^G$.
then $G \curvearrowright X$ proper & algebraic, $X = \mathrm{Spec}\,A \Rightarrow X$ admits a good quotient $X \to X/\!/G = Y = \mathrm{Spec}(A^G)$ (linear reductive $\Rightarrow A^G$ f.g.)

② Projective GIT (Mumford). $X = \mathrm{Proj}\,R$, $G \curvearrowright X$ "preserving grading" $\longrightarrow X \dashrightarrow^{\pi} X/\!/G = Y = \mathrm{Proj}\,R^G$
(depending on chosing an ample line bundle (linearization) due to different ways only coincide in $n \gg 0$ grading)
{Domain of π} $= X^{ss} \underset{\text{HM}}{=}$ Hilbert-Mumford criteria.

Def. $X^{ss} = \{x \in X \mid \exists s \in R^G, s(x) \neq 0\}$, $R = \bigoplus_{m \geq 0} H^0(X, L^{\otimes m})$
$X^{ss} \longrightarrow X/\!/G$. $\exists x' \in X^{ss}$
$x \longmapsto \overline{G.x}$ s.t. $\overline{G.x} = G.x'$, x' called polystable.
$x \longrightarrow x'$ is a limit by $\mathbb{G}_m \subset G$ $\Rightarrow$ stable.

E.g. $\mathcal{M}_{1,1} \subset [|\mathcal{O}_{\mathbb{P}^2}(3)| / PGL(3)] \Rightarrow M_{1,1} \subset |\mathcal{O}_{\mathbb{P}^2}(3)| /\!/ PGL(3)$
by $\mathcal{M}_{1,1} \subset [|\mathcal{O}_{\mathbb{P}^2}(3)| / PGL(3)] \supset [|\mathcal{O}_{\mathbb{P}^2}(3)|^{ss} / PGL(3)]$
$\downarrow$ $\downarrow$ GIT
$M_{1,1} \hookrightarrow |\mathcal{O}_{\mathbb{P}^2}(3)|^{ss} /\!/ PGL(3)$ $\Leftrightarrow$ every smooth cubic curve $\in$ semi-stable $\Leftrightarrow \exists\, s \in H^0(|\mathcal{O}_{\mathbb{P}^2}(3)|, \mathcal{O}(m))^G$
$s(E) \neq 0$, $E \subset \mathbb{P}^2$ the elliptic curve $\Leftrightarrow \exists$ hypersurface H, G-invariant and $H \not\ni [E]$

Claim. $\exists$ divisor (hypersurface): discriminant divisor Δ = {all singular} (does codim 1 by all the Jacobi $= 0$) $\Rightarrow \Delta \not\ni [E]$ ▨
Then claim. the inclusion $M_{1,1} \hookrightarrow |\mathcal{O}_{\mathbb{P}^2}(3)| /\!/ PGL(3) \cong \mathbb{P}^1$
$\quad\; \| \qquad\qquad\qquad \|$
$\quad\; \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$
Now $|\mathcal{O}_{\mathbb{P}^2}(3)| /\!/ PGL(3) = \mathrm{Proj} \bigoplus_{m \geq 0} H^0(|\mathcal{O}_{\mathbb{P}^2}(3)|, \mathcal{O}(m))^G$
thus we consider the j-invariant of cubic equation $y^2 = x^3 + ax + b$
$\Rightarrow$ invariant $f(a,b)$, $g(a,b)$ $\Rightarrow \mathbb{C}[f,g] = \mathbb{P}^1$
On the other hand, $M_{1,1} \overset{j}{\cong} \mathbb{A}^1$ ▨ $(E, 0) \mapsto j(E)$

Rk. $\mathbb{P}^1 - \mathbb{A}^1 = \mathbb{P}(|\mathcal{O}_{\mathbb{P}^2}(3)|^{ss}) - M_{1,1}$ = {nodal};
$|\mathcal{O}_{\mathbb{P}^2}(3)|^{ss}$ = {smooth or [illegible] or cusp nodal}
Polystable point is "the most singular"

Main thm.
$\mathcal{M}$ admits good moduli $\Leftrightarrow$ complete equivalent to Θ-complete + S-complete
(~~Core step of the upper~~ work by Xu & Jarod)

E.g. $G = \mathbb{G}_m$, $U = \mathbb{A}^n_k$ torus action, $\mathbb{K} = k$
$\Rightarrow [U/G] \to \mathrm{Spec}\,k$ is good moduli space (by $\mathrm{Spec}(k[x_1, \dots, x_n]^G) \cong \mathrm{Spec}\,k$)
One can see $[U - \{0\} / G] \to \mathbb{P}^{n-1}$ is coarse (by closed points in $\mathbb{A}^n$ $\Leftrightarrow$ closed orbits in action and taking $\lim \to 0$)
here good and coarse just complement.
$*$ "=" $*$ "+" $\bullet$ all not closed except $\Leftarrow$

Eg. $\mathbb{G}_m \curvearrowright \mathbb{A}^2 \Rightarrow (\mathbb{A}^2)^{\mathbb{G}_m} = \mathrm{Spec}(k[x,y])^{\mathbb{G}_m} = \mathrm{Spec}(k[xy])$
$t.(x,y) = (tx, t^{-1}y) \Rightarrow [\mathbb{A}^2/\mathbb{G}_m] \to \mathrm{Spec}\, k[xy]$ is good;

Thm. (Jarod Alper) $\mathcal{X} \to X$ good, $\mathcal{X}$ locally Noetherian $\Rightarrow \forall x \in \mathcal{X}$ closed, G_x stablizer $\boxed{\text{reductive}}$ ($G_x = \mathrm{Aut}(x)$)

Rk. Given $\mathcal{X}$ moduli meaning, then $\forall x \in \mathcal{X}$ object $\Rightarrow \mathrm{Aut}(x)$ reductive, it's very useful. Pf & App. Later ▨

Θ-complete is related with the Θ-stratification in GIT;
S-complete is related with the seperatness
$\Rightarrow$ One can think the main thm as a valuative critia

Eg.1 not Θ-complete; Eg.2 not S-complete, we only need delete these

Prop. $\mathcal{X}$ seperated DM-stack, finite type over B and tame ($|\mathrm{Aut}(x)|$ coprime char k) $\Rightarrow$ coarse moduli ($\exists$ by Keel-Mori) is good moduli
Thus char 0 + seperated at most cases, coarse $\Rightarrow$ good

Def. $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$ & $\Theta_R = \Theta \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec} R$.
Def. (Θ-completeness/reductive)
$\forall$ DVR R:

$$\begin{tikzcd} \Theta_R - \{0\} \arrow[r] \arrow[d] & \mathcal{X} \\ \Theta_R \arrow[ur, dashed, "\exists!\text{ extension}"'] & \end{tikzcd}$$

(also "equivalent" to H-M critia)

Def. $\overline{ST}_R := \left[\mathrm{Spec}\left(\frac{R[s,t]}{(st-\varpi)}\right)/\mathbb{G}_m\right]$
$\mathbb{G}_m \curvearrowright R$ trivial, $\mathbb{G}_m \curvearrowright R[s,t]$ by $\lambda.(s,t) = (\lambda s, \lambda^{-1} t)$, ϖ is uniformizer of R; DVR;
(Notation "$\overline{ST}_R$" is preferred by K-moduli)
Consider the open substack, $s \neq 0$
$(\overline{ST}_R)_s \cong [\mathrm{Spec}\, R[s, \frac{\varpi}{s}]/\mathbb{G}_m] = [\mathrm{Spec}(R[s]_s)/\mathbb{G}_m] = \mathrm{Spec} R$; $t \neq 0$, $(\overline{ST}_R)_t = \mathrm{Spec} R$.
$\Rightarrow \overline{ST}_R = \mathrm{Spec} R \cup \mathrm{Spec} R$, thus it's similiar to Eg.2, thus we need a similiar valuative critia for "seperatness"

Def. (S-complete) $\forall$ DVR R,

$$\begin{tikzcd} \overline{ST}_R - \{0\} \arrow[r] \arrow[d] & \mathcal{X} \\ \overline{ST}_R \arrow[ur, dashed, "\exists!\text{ extension}"'] & \end{tikzcd}$$

Rk. ① Affine diagonal $\Rightarrow \mathrm{Aut}(x)$ is affine group scheme thus this condition $\forall x \in \mathcal{X}(S)$ is natural
② When the case of Prop upper holds good = coarse the DM $\Rightarrow$ stabilizer has dim $= 0$, classical GIT tells us $U^{us} = \emptyset \Rightarrow$ Good quotient (GIT) = Geometric quotient
Consider moduli of coherent sheaves $\mathcal{C}oh(X)$ (or $\mathcal{Q}co(X)$, $\mathcal{B}un(X)$ ...)
then $\mathcal{C}oh(X) \to Coh(X)$ admittes a coarse moduli $Coh(X)$ consist all of S-equivalence classes: before we can only do this artifically but not natural, in order to have $\mathcal{C}oh(X)(k) = Coh(X)(k)$; but such S-equivalence class using Good moduli theory makes it natural (However when $\mathcal{C}oh(X)$ to $D^b(X)$, using this way even can't give it moduli meaning! (No GIT))

Rk ③. In Eg.1, we used the criteria following: $U \to [U/G]$ π is good moduli

$$\begin{tikzcd} U \arrow[r] \arrow[dr, "\tilde{\pi}"'] & {[U/G]} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

$\Leftrightarrow \tilde{\pi}$ is good quotient $\Rightarrow \tilde{\pi}$ is affine
thus not affine $\Rightarrow$ not admittes Good moduli ▨

Eg.1 $G = \mathbb{G}_m \curvearrowright U = \mathbb{A}^2$ or $\mathbb{P}^1$ $(\mathbb{A}^2)^*$ closed orbits only $(1,0)$ & $(0,1)$
$t.(x,y) = (tx, y)$
all other points (x,y) has $\overline{\mathbb{G}_m.(x,y)} \ni (1,0), (0,1)$ not closed
$\Rightarrow [U/G] \to$ affine "orbit space" $\to \mathrm{Spec}\, k$
covering: $U \xrightarrow{\text{affine}} \mathbb{P}^1$ (or $\mathbb{A}^{2*}$) projective (or not affine) $\Rightarrow$ contradiction $\Rightarrow$ such "orbit space" not exist
$\Rightarrow$ Good moduli not exist ▨
(If one want to find such an example, taking a not affine one to quotient to affine, esp. one point)

Eg.2 Over $\mathbb{C}$, $G = \mathbb{Z}/2\mathbb{Z} \curvearrowright U = \mathbb{A}^1 \cup_{\mathbb{A}^1 - \{0\}} \mathbb{A}^1$ = affine line doubled origin.
$0.x = 0$
$1.x = \begin{cases} x, & x \neq 0 \\ \text{another } 0, & x = 0 \end{cases}$
Thus $\Rightarrow [U/G] = \mathbb{A}^1$
Rk. Although not seperated not holds the Keel-Mori, but it's coarse here
Claim. $[U/G] \to \mathbb{A}^1$ is coarse but not good
When U has higher dim $\Rightarrow$ not coarse. coarse $\not\Rightarrow$ good.

Θ comes from GIT by: $G \curvearrowright U \subset \mathbb{P}^n$ by linearization $PGL \curvearrowright \mathbb{P}^n$ induced
(Recall. $U^{ss} = \{x \in U \mid \exists s \in H^0(L)^G, s(x) \neq 0\} = \{x \in U \mid \overline{G.x} \not\ni 0\}$
(0 is the affine cone's origin; $U \subset \mathbb{P}^n \subset \mathbb{A}^{n+1}$) $U^s = \{x \in U \mid G.x$ closed & finite stabilizer$\}$)
Using Hilbert-Mumford we reduce $G.x$ into $\mathbb{G}_m.x$, which is numerical.
thus $\forall\, \mathbb{G}_m \hookrightarrow G$, and taking limit, we extend $\mathbb{G}_m \curvearrowright U^s$ into $\mathbb{A}^1 \curvearrowright U$
$\Rightarrow \exists\, \mathbb{A}^1 \to U \Rightarrow \Theta \to [U/\mathbb{G}_m]$, thus we can use Θ to detect the GIT quotient! One can think it's "equivalent" to H-M criteria
By the upper example we have Cartesian diagram

$$\begin{tikzcd} \overline{ST}_R \arrow[r] \arrow[d] & {[\mathrm{Spec}(k[x,y])/\mathbb{G}_m]} \arrow[d] \\ \mathrm{Spec} R \arrow[r] & \mathrm{Spec}(k[xy]) \end{tikzcd}$$

induced by $xy \mapsto \varpi$ ▨

For proving the main thm, the core step is: Lemma. G reductive $\Rightarrow [\mathrm{Spec} A/G]$ is Θ-complete & S-complete (Locally)
($\Leftarrow$ affine diagonal)

Pf. Using BG, and $G \overset{\text{reductive}}{\hookrightarrow} GL_n$, thus consider BGL_n classifying vector bundles $\Rightarrow \Theta_R - \{0\} \to BGL_n \Leftrightarrow$ extending vector bundle $\mathcal{E}$ over $\Theta_R - \{0\}$ to Θ_R; Now $\dim \Theta_R = 2$ & Θ_R is smooth $\Rightarrow$ normal
$\Rightarrow$ extension holds as a coherent sheaf, torsion-freely to $\tilde{\mathcal{E}}$
$\Rightarrow \tilde{\mathcal{E}} = (\tilde{\mathcal{E}})^{\vee\vee} \Rightarrow$ locally-free ▨
(Double dual of torsion-free is locally free)

$\mathcal{C}oh(X) \rightsquigarrow D^b(X)$
slope stability, Bridgeland stability

Rk ④. Θ-completeness fixed
S-completeness $\Leftrightarrow X$ is seperated
But there is no relation with the seperateness of $\mathcal{X}$: Later examples shows: Nonseperate $\mathcal{X}$ can admittes seperated good moduli;

Prop. $\pi: \mathcal{X} \to X$ good, $\mathcal{X}$ Noetherian, finity type, affine diagonal.
then (1) $\mathcal{X}$ is Θ-complete; (2) X seperated $\Leftrightarrow$ S-complete

Pf. (1) Description of Θ_R:

$$\begin{tikzcd} & \mathrm{Spec} R \arrow[dl, "x \neq 0"'] & \\ \Theta_R & \lozenge & B\mathbb{G}_m k \arrow[ul] \arrow[dl] \\ & \mathrm{Spec} k \times \Theta \arrow[ul, "\varpi \neq 0"] & \end{tikzcd}$$

$\mathrm{Spec} R \times B\mathbb{G}_m$: $x = 0$; $\varpi = 0$

R with 0 is $\mathrm{Spec}\, k$.
$\Rightarrow \Theta_R - \{0\} = \mathrm{Spec} R \coprod_{\mathrm{Spec} k} \Theta_k$
Using $\mathcal{X} \to X$ good, Φ factor through $\mathcal{X}$, called $\tilde{\Phi}$

then

$$\begin{tikzcd} \mathrm{Spec} R \arrow[r] & \Theta_R - \{0\} \arrow[r, "\Phi"] \arrow[d] & \mathcal{X} \arrow[d] \\ & \Theta_R \arrow[d] \arrow[ur, dashed, "\Psi"] & X \\ & \mathrm{Spec} R \arrow[ur] & \end{tikzcd}$$

Good Moduli

Using Local structure thm, one can lift Φ to Ψ to complete the proof;

$[\operatorname{Spec} A/G_m] \xrightarrow[\text{affine}]{\text{étale}} \mathcal{X}$ & $\operatorname{Spec} R \to \operatorname{Spec} A^{G_m}$; affine étale

$\operatorname{Spec} A^{G_m} \xrightarrow[\text{affine}]{\text{étale}} X$; $\operatorname{Spec} R \to X$

By Local structure thm of Artinian stack

due to affine étale pullback to at least étale, étale over

DVR local is also DVR, (smooth/regular dim 1)

localisation $\Rightarrow$ R local DVR done

Thus S-complete $\Leftrightarrow$ Lifting f to desired map

This if separated $\Rightarrow$ by valuative critia done.

Conversely is harder:

Then separatness $\Rightarrow$ lifting to $\tilde{f} = \tilde{g}$

$\Rightarrow \overline{ST}_R \xrightarrow{(f,g)} \mathcal{X}$

Lifting

$\operatorname{Spec} R' \to \operatorname{Spec} R \rightrightarrows \mathcal{X} \to X$

$(f|_K = g|_K \Rightarrow f = g)$ $\hookrightarrow$ need to prove.

thus S-complete up to a base extension in Sch

at last we need $f \circ h = g \circ h \Rightarrow f = g$

i.e. h is an epi in Sch $\Leftrightarrow h: \operatorname{Spec} R' \to \operatorname{Spec} R$ given by $R \hookrightarrow R'$ injective

This is obvious, done ☒

By separatness $\tilde{f} = \tilde{g}$ $\exists!$

thus locally, we lift as: $\Theta_R \to [\operatorname{Spec} A/G_m]$, $\operatorname{Spec} R \to \operatorname{Spec} A^{G_m}$

then by the classical DVR done.

Composition will give the final lifting globally ☒

Lifting is algebraic up to a ring extension

$\operatorname{Spec} R \to \mathcal{U} \to \mathcal{X}$, $\operatorname{Spec} R_{K'}$, $\operatorname{Spec} R \to \operatorname{Spec} R \to X$

① taking $\mathcal{U} \to \mathcal{X}$ finite type presentation over $k = \bar{k}$;

② $\mathcal{U}$ has Geometric point $\Rightarrow \operatorname{Spec} K \to \mathcal{U}$;

③ Finite type $\Rightarrow$ finite equations, adding coefficients (finite) to K', extending K

④ $(R)_{K'}$ is dim 1 Noetherian normal $\Rightarrow$ DVR ☒

(2) First describe $\overline{ST}_R$: $\operatorname{Spec} R \to \overline{ST}_R$ ◇

$\Rightarrow \overline{ST}_R = \operatorname{Spec} R \coprod_{\operatorname{Spec} K} \operatorname{Spec} R$

the "affine line double origin"

Then S-completeness $\Leftarrow$ (f,g)

$\overline{ST}_R - \{0\} = \operatorname{Spec} R \coprod \operatorname{Spec} R \to \mathcal{X}$

$f = g$

Good moduli by S-completeness

(Q) Can $\mathcal{X}$ separated $\Rightarrow$ S-completeness this case?

Prop. $\mathcal{X}$ has affine quasi-finite diagonal, Θ-complete, then $\mathcal{X}$ S-complete $\Leftrightarrow \mathcal{X}$ separated

Pf. Omitted ☒

Eg. $[A/G_m]$, $\operatorname{Spec} k$, Good; $[\mathbb{A}^1 \amalg \mathbb{A}^1/\mathbb{Z}_2]$, $\mathbb{A}^1$, Coarse (Summary)

When Good coarse is almost same separated DM

Eg. $\Theta_R \to \mathcal{M}_g \Leftrightarrow \mathbb{A}^1 \to \Theta \to \mathcal{M}_g$

Given $\mathbb{A}^1$'s trivial G_m bundle $\Leftrightarrow \mathbb{A}^1 \times G_m \xrightarrow{\text{multi}} \mathbb{A}^1$

Using moduli meaning

$\Leftrightarrow \mathcal{C} \times G_m \to \mathbb{A}^1 \times G_m \to \mathbb{A}^1$; $\mathcal{C} \to \mathbb{A}^1 \to \Theta$

$\mathbb{A}^1 \to \Theta \to \mathcal{M}_g$

thus Θ-completeness $\Leftrightarrow$ filling a degeneration in $0 \in \mathbb{A}^1$'s fibre.

More generally, it holds for polarized varieties, $\hookrightarrow$ such problems are hard!

We have $\mathcal{M}_K^{ss}$ the K-semistable Fano varieties are Θ & S-complete

Bridgeland moduli space

In fact $\mathcal{M}_g^{all}$ is not $\mathcal{M}_g$

Conclusion

Step1 Construct moduli stack/functor

directly

as substack of a larger stack (easier to describe)

Step2 Algebraicity

Quotient stack.

Artin's axioms (deformation theory)

Step3 $\mathcal{M} \xrightarrow{\pi} M$

DM & separatness $\xRightarrow[\text{KM}]{\text{Keel}}$ coarse

Θ & S-completeness $\xRightarrow{\text{Jarod}}$ Good

Step4 Positivity $\Rightarrow$ projective M

This method also makes sense in arithmetic or Geometric representation theory.

Another method recently concerns stability, without known desired $\mathcal{M}$ at first, but want to classify: (Method of GIT's compactification)

Step1 Given enough large moduli stack $\mathcal{M}$

Step2 Given stability σ, $\mathcal{M}^{ss} \subset \mathcal{M}$ open substack & algebraic condition

check stability of $\mathcal{M}$ is open condition) (σ gives σ-stable / semi-stable)

Step3 Boundness (finite type)

Step4 Properness (Valuative Critia, hard)

$\Rightarrow \mathcal{M}^{ss} \to M^{ss}$

Step5 Positivity of CM line bundle (hard)

open $\Leftrightarrow$ stable; $\lim$; Proper $\Leftrightarrow$ semi-stable

Thm. $\mathcal{X}$ S-complete, $\forall x \in \mathcal{X}$ closed, then if G_x smooth affine $\Rightarrow G_x$ linear reductive.

Cor. X K-stable Fano $\Rightarrow$ Aut(X) reductive (this is earlier than differential Geometry)

To prove thm, we need ① $\Leftrightarrow$ ② gives the upper thm ☒

Thm. G affine smooth algebraic group TAFE ① Reductive; ② BG S-complete;

③ $\exists$ Cartan decomposition $G = K T K$

T the torus;

(i.e. $\forall$ DVR R, Frac$R = K$

$\lambda: G_m \to G$, $g = h_1 \lambda|_K h_2$)

Pf. ② $\Leftrightarrow$ ③

BG S-complete $\Leftrightarrow \overline{ST}_R \to BG$

$\Leftrightarrow$ Given two trivial G-bundle

$P_1 \to \operatorname{Spec} R$

$P_2 \to \operatorname{Spec} R$ gluing $\operatorname{Spec} K$ by g. - action

$P_1 = P_2$

Lemma. $\mathcal{X} \to \mathcal{Y}$ affine, $\mathcal{Y}$ S-complete $\Rightarrow \mathcal{X}$ S-complete

Eg (BG_a not S-complete)

$R = k[[x]]$, $K = k((x))$

$\forall g \in G_a(K)$, $g \neq h_1 h_2$

$\Leftrightarrow g = h_1 \lambda|_K h_2$ where $\lambda: G_m \to G$ comes from $BG_m \to BG$

$[0]/G_m$ (i.e. $P = P_g$)

then extend to this G-bundle over $\overline{ST}_R$ to 0

① $\Leftarrow$ ② $\Leftrightarrow$ Done

($\Leftarrow$) Otherwise, $R_u(G)$ the unipotent radical nontrivial

$\Rightarrow \exists G_a \trianglelefteq R_u(G)$ (by $G_a \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$)

$\Rightarrow BG_a \to BR_u G \to BG$

and $G/R_u G$, $R_u G/G_a$ are affine

X smooth projective variety

$\dim \operatorname{Aut}(X) = \dim H^0(X, T_X)$ may not finite type

Eg1. $\omega_X = \mathcal{O}_X$, Calabi-Yau

$\Rightarrow \dim \operatorname{Aut}(X) = 0$ but not finite type as G_a not admits G_m

Eg2. M_1 isn't finite Eg3. K-stable $\mathcal{M}$ isn't DM but S-complete

$M_{1,1}$ $|\operatorname{Aut}(X)| < \infty$

Eg4. $|\operatorname{Aut}(X)| < \infty$ if general type X

☒ It's also used in Hilbert-Mumford

$\Rightarrow BG_a \to BG$ is affine;

$\Rightarrow BG_a$ is S-complete by lemma, contradiction

G_a, ☒

$\begin{pmatrix} 1 & G_a^* \\ & 1 \end{pmatrix}$

$= R_u(G)$

扫描全能王 創建

From <u>E.g.1 – E.g.4</u>, the usual objects X in AG has bad $\mathrm{Aut}(X)$, thus <u>bad $\mathcal{M}$</u> parameters all classes of X in <u>Reascreahs</u>

($\hookrightarrow$ not admittes a scheme structure M, with $\mathcal{M} \to M$, even $\mathcal{M}$ not algebraic (not Artinian))

Our final task is giving several ways to modify $\mathcal{M}$ to $\mathcal{M}^{\sim}$ "good" which is almost used in all recent papers:

$\mathcal{M}^{\sim} = \{(X, \Sigma) \mid X \in \mathcal{M}, \Sigma$ is a natural condition on $X\} \xrightarrow[\text{functor}]{\text{Forgetful}} \mathcal{M}$

$\left\{\begin{array}{l} ① \ X \subset \mathbb{P}^N \Leftrightarrow (X, \mathcal{L}) \text{ polarisation; (Calabi-Yau not exists)} \\ \quad \mathbb{P}^N = |\mathcal{L}| \\ ② \ X \subset Y, \text{ with } Y \in A \text{ better moduli known;} \\ ③ \ D \subset X, \text{ with } D \in A \text{ better moduli known;} \\ ④ \ X \xrightarrow{f} Y \text{ or } Y \xrightarrow{f} X, \text{ with } f \text{ has some properties.} \end{array}\right.$

(② ③: E.g. taking Fano or general type ...)

④ is most general, but in form they're all used

<u>Eg1</u>. $\mathcal{M}_{1,1} \overset{①}{\underset{②}{=}} \{E \subset \mathbb{P}^2 \mid \cdots\} \overset{④}{=} \{E \to \mathbb{P}^1 \mid 2:1, \text{ branched in 4pts}\}$

<u>Eg2</u>. $\mathcal{H} = \{\text{Hyperelliptic curves}\} \overset{④}{=} \{X \to \mathbb{P}^1 \mid 2:1\}$ the Hurwitz space

<u>E.g.3</u>. (Best) $\mathcal{M}_H = \{\text{Hyperelliptic surfaces}\} \overset{④}{=} \{S \to \mathbb{P}^1 \times \mathbb{P}^1 \mid 2:1,$ branched over C, $\deg(C) = (4,4)$ (bidegree)$\}$ then $S \to \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$

$\overset{①}{=} \{(S, \pi^* \mathcal{O}_{\mathbb{P}^3}(1)^{\otimes 2})\} \overset{①}{=} \{S \subset \mathbb{P}^N \mid \cdots\} \overset{②}{=} \{S \subset Q \mid Q \text{ Fano threefold}, S \in |-K_Q|\}$, thus hyperelliptic surface $\Leftrightarrow$ deg (4,4) curve $\Leftrightarrow$ Fano threefold $\Leftrightarrow$ $\pi^*\mathcal{O}_{\mathbb{P}^3}(1)^{\otimes 2}$ correspondence beautifully through Moduli theory.

Done.

WRITING PROJECT OF MODULI THEOREY

ZHIYUAN LI

Please choose one of the following topics and write a paper including a short introduction, statement and the proof (at least 5 pages.

1. Castelnuovo-Mumford Regularity $(*)$. Introduce the regularity, prove the Castelnuovo-Mumford Regularity. Reference: [2, Chapter 5]

2. Representability of the Quot scheme $(*)$. Reference: [2, Chapter 5]

3. Chow variety $(***)$. Introduce the Chow variety and show it is a fine moduli space. Reference: [4, Chapter 1]

4. Coarse moduli space and Keel-Mori theorem $(**)$. Introduce and prove the Keel-Mori theorem. [1, Chapter 4]

5. Moduli space of elliptic curves $(*)$. Discuss the properties of moduli space of elliptic curves $\mathcal{M}_{1,1}$. Prove its coarse moduli space is $\mathbb{A}^1$. [1, Chapter 5]

6. Artin's approximation $(*)$. Introduce the Néron-Popescu desingularization theorem and G-ring. Show that the localization of finitely generated k-algebra or $\mathbb{Z}$-algebra is a G-ring. Prove the Artin approximation. [1, Appendix B]

7. Deformation of schemes $(**)$. Discuss the formal deformation theory of schemes. Prove that the first-order deformation of a smooth scheme is $\mathrm{H}^1(X, \mathcal{T}_X)$ and the obstruction space is in $\mathrm{H}^2(X, \mathcal{T}_X)$. [3, Section 2 and 6] If possibly, discuss the Grothendieck's existence theorem.

8. Deformation of coherent sheaves $(**)$. Prove that the first-order deformation of a coherent sheaf $\mathcal{F}$ over a variety is $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$ and the obstruction space is in $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$. [3, Section 5 and 10]

9. Moduli space of marked curves: $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ $(*)$. Introduction to moduli space of marked curves,such as definitions, why they are smooth separated DM stacks. [1, Chapter 5]

10. Irreducibility of $\mathcal{M}_g$ $(**)$. [1, Chapter 5]

References

[1] Jarod Alper. Stacks and moduli. *https://sites.math.washington.edu/ jarod/moduli.pdf.*
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[3] Robin Hartshorne. *Deformation theory*, volume 257. Springer, 2010.
[4] János Kollár. *Rational curves on algebraic varieties*, volume 32. Springer Science & Business Media, 2013.