Fredholm operator

Thm. X Banach, $A \in \mathcal{K}(X)$
then either ① $\forall x \in X$, $\exists ! y \in X$, s.t. $y = x + Ay$
or ② $\exists y \in Y$, $y = Ay$
(Fredholm Alternative)

Pf. See Lax & $\boxed{}$ Coro. $\operatorname{Ind}(I+A)=0$

Ex. $\{\varphi_n\} \subset H$. $\varphi_n \xrightarrow{\omega} 0$, $A \in \mathcal{B}(H)$
and $\|\varphi_n\|=1$ $\|A\varphi_n\| \to 0 \Rightarrow A$ not Fredholm operator

Recall, Fredholm operator has some form $I + A$ ($A \in \mathcal{K}(X)$), but $\operatorname{Ind}(I+A)=0$ ($\operatorname{Ind} T = \dim \ker T - \operatorname{codim} \operatorname{Ran} T$)
and $\operatorname{Ind}(F) \neq 0$ is later constructed

We'll give a description of the Atiyah-Singer index theorem, and give a proof by noncommutative Geometry way (Connes) in this course.

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Thm. (Atkinson) Equivalent:
① T is Fredholm ($\dim \ker T$, $\dim \operatorname{coker} T < \infty$) (or $\dim \frac{Y}{\operatorname{Ran} T} < \infty$)
② T has psedoinverse
($\exists S$, s.t. $ST = I_X + K$, $TS = I_Y + L$, K, L compact)
③ T has psedoinverse with K, L finite rank

Pf. Done in FA $\boxed{}$

The stability of index

Prepare. $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{matrix} X_1 \\ X_2 \end{matrix}$ over $\begin{matrix} Y_1 & Y_2 \end{matrix}$; $X = X_1 \oplus X_2$ closed $\downarrow T$ $Y = Y_1 \oplus Y_2$ closed

when D invertible:
$\Rightarrow \begin{pmatrix} I_{Y_1} & -BD^{-1} \\ 0 & I_{Y_2} \end{pmatrix} T \begin{pmatrix} I_{X_1} & 0 \\ -D^{-1}C & I_{X_2} \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix}$

thus T Fredholm $\Leftrightarrow A - BD^{-1}C$ Fredholm
and $\operatorname{Ind} T = \operatorname{Ind}(A - BD^{-1}C)$ [due to for $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = A$, A Fredholm $\Leftrightarrow A_1, A_2$ Fredholm and $\operatorname{ind} A = \operatorname{ind} A_1 + \operatorname{ind} A_2$]

Coro. ① T Fredholm $\Leftrightarrow S$ Fredholm
② $\operatorname{Ind} T + \operatorname{Ind} S = \operatorname{Ind} TS$ (For Atkinson's choice of S)

Proof ② When proving ① $\Rightarrow$ ③, $ST = I_X - P$ is P, Q projection map,
$TS = I_Y - Q$ and $\operatorname{Ran} P = \ker T$, $\ker(I-Q) = \operatorname{Ran} T$'s complement space
$\Rightarrow \operatorname{ind} T = \dim(\ker(I-P)) - \dim(\ker(I-Q))$
and $\operatorname{ind} S = \dim(\ker(I-Q)) - \dim(\ker(I-P))$
$= -\operatorname{ind} T$ $\boxed{}$

Thm. (Dieudonné) $\|T' - T\| < \varepsilon$, T Fredholm
$\Rightarrow T'$ Fredholm and $\operatorname{ind} T' = \operatorname{ind} T$ ($\operatorname{ind}^{-1}(n)$ is open in $\mathcal{B}(X,Y)$)

Pf. Let $X_1 = \ker T$, $Y_2 = \operatorname{Ran} T \Rightarrow T = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$ and D invertible (Here is by injective + surjective + X_2, Y_2 complete)
$T' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \Rightarrow \|D' - D\| < \varepsilon$, Claim. D' also invertible (it's nontrivial for $X_2 \neq Y_2$ here) We construct $(D')^{-1}$ from D & D'
By the Claim, and if D' invertible, $\dim X_1$, $\dim Y_1 < \infty \Rightarrow T'$ Fredholm and $\operatorname{ind} T' = \dim X_1 - \dim Y_1 = \operatorname{ind} T$
Pf of Claim (Algebraic). $D' = D + (D' - D) \Rightarrow D^{-1}D' = I_{X_2} + \boxed{D^{-1}(D'-D)}$, it suffices to show $\|D^{-1}(D'-D)\| < 1$ $\boxed{}$

Coro. $\operatorname{ind} T = C$ for every pathwise-components / $\gamma: [0,1] \to \{$Fredholm operators$\} \Rightarrow \operatorname{Ind}(\gamma(0)) = \operatorname{Ind}(\gamma(1))$

Prop. $\operatorname{Ind}(T+K) = \operatorname{Ind} T$, T Fredholm, K compact.
Pf. Let $\gamma(t) = T + tK$ $\boxed{}$

Thm. $\operatorname{Ind}(TS) = \operatorname{Ind} T + \operatorname{Ind} S$; $X \xrightarrow{S} Y \xrightarrow{T} W$
Hint. The "+" occurs when the form TS ... $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, thus;

Observation. TS also Fredholm by $S'T'$ the psedo-inverse (S', T' seperately the psedo-inverse of S, T)

Pf. Consider $\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$: $X \oplus Y \to Y \oplus W$
and $\begin{pmatrix} 0 & \text{invertible} \\ \text{invertible} & 0 \end{pmatrix}$: $X \oplus Y \to Y \oplus W$

change into by $\begin{pmatrix} I_Y & 0 \\ T & \varepsilon I_W \end{pmatrix} \begin{pmatrix} S & \boxed{\varepsilon I_Y} \\ 0 & T \end{pmatrix} \begin{pmatrix} \varepsilon I_X & 0 \\ -S & I_Y \end{pmatrix} = \begin{pmatrix} 0 & I_Y \\ TS & 0 \end{pmatrix}$ $(\to \varepsilon)$

invertible $\Rightarrow$ not change both $\dim \ker$, $\dim \operatorname{coker}$ thus ind also

We adding εI_Y not change

$\Rightarrow \operatorname{ind}\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} = \operatorname{ind}\begin{pmatrix} 0 & \text{invertible} \\ TS & 0 \end{pmatrix}$
$\operatorname{ind} T + \operatorname{ind} S = \operatorname{ind} TS$ $\boxed{}$

Eg. Atiyah-Singer for S^1 (Toeplitz operators)
$\operatorname{ind} T_f = -W_f$

Something about differential Geometry (Almost nothing amazing)

(Connection depends on coordinate choice but curvature not.)

Differential operator $d: f \mapsto \frac{df}{dx}$
$C^\infty \to C^\infty \Rightarrow$ not complete
extend C^∞ to bigger complete space ① by integral by part: $\forall g \in C_c^\infty(\mathbb{R})$
$\int_{\mathbb{R}} f'(x) g(x)\, dx = -\int_{\mathbb{R}} f(x) g'(x)\, dx$ the weak derivative in distribution theory
$\langle f', g \rangle = -\langle f, g' \rangle \Rightarrow$ by Riesz, we recover the f' by $\langle \square, g \rangle = \langle -f, g' \rangle$, let it replace the definition
② by Fourier transform $\mathcal{F}: L^1 \to L^\infty$
extend to $L^2 \to L^\infty$ by Plancherel formula $\|\mathcal{F}f\|_2 = \|f\|_2$ $f \mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$
and $C_c^\infty \subset L^2 \cap L^1 \subset L^2$ $\Rightarrow L^1 \cap L^2$ dense in L^2 $(\mathcal{F}^4 = I \Rightarrow \sigma(\mathcal{F}) = \{\pm i, \pm 1\})$

E.g. $\mathcal{F}(f')(\xi) = \frac{1}{\sqrt{2\pi}}\int f'(x)e^{-i\xi x}dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} (f(x)e^{-i\xi x})' + i\xi f(x)e^{-i\xi x}dx = (i\xi)(\mathcal{F}f)\xi \Rightarrow$ we recover f' by $f' = \mathcal{F}^{-1}(i\xi\mathcal{F}f)$ as its definition. Page 2

Both way ①② allow us to define $\frac{d}{dx}$ in a bigger space: ① to ensure the well-defined of $f(x)e^{-i\xi x} \xrightarrow{x\to\infty} 0$, we have the Schwartz space $S(\mathbb{R}) = \{f: \mathbb{R}\to\mathbb{C} \mid \forall n, \lim_{|x|\to\infty}|x|^n f(x) = 0\}$, then well-defined; ② It's still not enough! $f \in L^2 \mapsto \mathcal{F}^{-1}(i\xi(\mathcal{F}f)) \notin L^2$

we need $\xi(\mathcal{F}f) \in L^2$, thus we have the Sobolev space (1st order) $H^1(\mathbb{R}) = \{f \in L^2 \mid (1+|\xi|)(\mathcal{F}f) \in L^2\}$

$\Rightarrow$ then more generally, and $H^k(\mathbb{R}) = \{f \in L^2 \mid (1+|\xi|^k)(\mathcal{F}f) \in L^2\}$

$f \mapsto \mathcal{F}^{-1}P_k(\xi)\mathcal{F}f$, $f \in H^k(\mathbb{R})$ and $\deg P_k = k$ polynomial

"$= P_k(-i\frac{d}{dx})f$, C" means define, and equal when $f \in C^k(\mathbb{R})$)

Rk. We prefer ~~a~~ our ~~Hilbert~~ space to then the spectrum $\sigma(T) \subset \mathbb{R}_{\geq 0}$ (by self-adjoint, $\sigma(T) \subset \mathbb{R}$ is true) it allow us to $\sqrt{T}$, and let $T = \Delta$, $\sqrt{\Delta}$ is dirac operator, it's what we need!

Formally, the psedo-differential operators $a(x,D)$ for "a proper function $a(x,\xi)$"

$(a(x,D)f)(x) := \int_{\mathbb{R}} e^{ix\xi}a(x,\xi)\left(\int f(t)e^{-it\xi}dt\right)d\xi \cdot C$ (some const)

the $a(x,\xi)$ called the symbol of $a(x,D)$

(later we may explain what $a(x,\xi)$ is)

The conjudagte operator of $A = a(x,D): S(\mathbb{R}^n) \to S(\mathbb{R}^n)$

$\langle a(D)f, g\rangle_{L^2(\mathbb{R}^n_x)} = \langle \mathcal{F}^{-1}(a(\xi)\mathcal{F}f), \mathcal{F}^{-1}\mathcal{F}g\rangle \overset{\text{isometry}}{=} \langle a(\xi)\mathcal{F}f, \mathcal{F}g\rangle_{L^2(\mathbb{R}^n,\xi)}$

$= \langle \mathcal{F}f, \overline{a(\xi)}\mathcal{F}g\rangle_{L^2(\mathbb{R}^n,\xi)} = \langle f, \overline{a}(D)g\rangle_{L^2(\mathbb{R}^n,x)}$

$\Rightarrow a(D)^* = \overline{a}(D)$

The "psedo" is due to $a(x,D)$ depends on x, if only $a(D)$ it's a usual differential operator.

$\boxed{b(D)\circ a(D)}f) = \mathcal{F}^{-1}(b(\xi)\mathcal{F}(a(D)f)) = \mathcal{F}^{-1}(\boxed{b(\xi)a(\xi)}\mathcal{F}f)$

$\Rightarrow b(D)\circ a(D) = (ba)(D)$ the multiplication

In general $a(x,D)f(x) = \mathcal{F}^{-1}a(x,\xi)\mathcal{F}f(x) = \frac{1}{(2\pi)^n}\int f(y)\left(\int e^{i(x-y)\xi}a(x,\xi)d\xi\right)dy \Rightarrow a(x,D)f = \int K(x,\xi)f(\xi)d\xi$

Consider $a(x,\xi)$, s.t. every integral meaningful (sufficiently) $K(x,\xi)$ is the Kernel function

and the kernel function $K(x,\xi)^* = \overline{K(\xi,x)}$, once the $a(x,D)^*$ exists.

Symbol class.

For $m \in \mathbb{R}$, $S^m = S^m(\mathbb{R}^n\times\mathbb{R}^n) = \{a \in C^\infty(\mathbb{R}^n\times\mathbb{R}^n) \mid \forall\alpha,\beta\in\mathbb{N}^n, |\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}\}$

the symbol of order m, $\cdots \subset S^{-1} \subset S^0 \subset \cdots \subset S^n \subset \cdots$

(Alternatively, $C_{\alpha,\beta}(1+|\xi|)^{m-C'_{\alpha,\beta}|\beta|-C''_{\alpha,\beta}|\alpha|}$ is also a definition (Hörmander) But not equivalent)

For general $(a_1(x,D)\circ a_2(x,D)f)(x) = (2\pi)^{-n}\int e^{ix\xi}a_1(x,\xi)\mathcal{F}(a_2 f)\ldots$

$= (2\pi)^{-2n}\int e^{ix\xi - iy\xi + iy\eta - iz\eta}\, a_1(x,\xi)a_2(y,\eta)f(z)\,d\xi\,dz\,dy$

$= (2\pi)^{-n}\int e^{i(x-y)(\eta-\xi)}a_1(x,\xi)a_2(y,\eta)\,f\,d\xi\,d\eta\,dy$

$\Rightarrow a_1(x,D)\circ a_2(x,D) = a_1(x,\xi)\circ a_2(x,\eta)$ the composite of functions

E.g. ① Differential symbol $a(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$, each $a_\alpha(x) \in C^\infty(\mathbb{R}^n)$ has $\partial_x^\beta a_\alpha(x)$ bounded, $\forall \beta \geq 0$

② $\forall \lambda > 0$, $a(\lambda\xi) = \lambda^m a(\xi)$ and $a(\xi) \in C^\infty(\mathbb{R}^n - 0)$, and $\chi \in C_c^\infty$, $\chi(U_0) \equiv 1$, $\exists U_0 \in N_0 \Rightarrow \tilde{a}(\xi) = (1-\chi(\xi))a(\xi) \in S^m(\mathbb{R}^n\times\mathbb{R}^n)$

③ $a(x,\xi) = e^{ix\xi}$ isn't a symbol of $S^\infty = \bigcap_{m\in\mathbb{R}} S^m \Rightarrow$ isn't a symbol of any order

Prop. ① $a \in S^m \Rightarrow \partial_x^\alpha\partial_\xi^\beta a \in S^{m-|\beta|}$; ② $S = \bigcup S^m$ is graded algebra, i.e. $S^{m_1}S^{m_2} \subset S^{m_1+m_2}$ (each S^m is vector space: by adding/scaling $C_{\alpha,\beta}$)

Lemma. $a_1 \cdots a_k \in S^0$, $\forall F \in C^\infty(\mathbb{C}^k) \Rightarrow F(a_1\cdots a_k) \in S^0$ the functional calculus

The asymptotic expansion: $a_j \in S^{m_j}$, $m_j \to -\infty$, then $\boxed{a = \sum a_j} \Leftrightarrow a - \sum_{j<k} a_j \in S^{m_k+1}$, $\forall k \geq 0$ (A formal notation)

Thm. $a_1 \in S^{m_1}$ and $a_2 \in S^{m_2}$, then $b = a_1 \# a_2 \in S^{m_1+m_2}$ $\Rightarrow$ $B = A_1\circ A_2$ their corresponding operator

$b(x,\xi) = (2\pi)^{-n}\int e^{-i(x-y)(\xi-\eta)}a_1(x,\eta)a_2(y,\xi)dyd\eta$ has asymptotic expansion: $b \simeq \sum \frac{1}{\alpha!}\partial_\xi^\alpha a_1 \cdot D_x^\alpha a_2$ $(\alpha! = \alpha_1!\cdots\alpha_n!)$

Coro. (we denote $A = a(x,D)$) $a_1 \in S^{m_1}$, $a_2 \in S^{m_2} \Rightarrow [A_1, A_2]$ is (m_1+m_2-1)-order operator, with symbol $b = \frac{1}{i}\{a_1, a_2\}$ (mod $S^{m_1+m_2-2}$)

$\{a_1, a_2\} = \sum_j \frac{\partial a_1}{\partial \xi_j}\frac{\partial a_2}{\partial x_j} - \frac{\partial a_1}{\partial x_j}\frac{\partial a_2}{\partial \xi_j}$ the Poission bracket)

Pf. Prop, Coro are trivial, Thm had done by before computation, Lemma left as Exercise ☐

The local-global correspondence: local: differential (Newton-Leibniz ...) Global: integral, Fourier transformation

Now we define $a(x,D)$ rigorous, but consider $a(x,D): S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ well-defined?

Pf. $a(x,D) = \mathcal{F}^{-1}a(x,\xi)\mathcal{F}$, Note. ① $\mathcal{F}, \mathcal{F}^{-1}$ preserve $S(\mathbb{R}^n)$ ② $a(x,\xi)$ is a polynomially increasing. ☐

Thm. $a \in S^0 \Rightarrow A$ is bounded on $L^2 \to L^2$

Pf. $A = \mathcal{F}^{-1}a\mathcal{F}$, due to $a \in S^0 \Rightarrow a$ bounded, $\Rightarrow a: L^2 \to L^2$ multiply $\Rightarrow A: L^2 \to L^2$;

Then we prove the boundness: $\|Af\|_2 \leq \sqrt{M}\|f\|_2 \Leftrightarrow \langle (MI - A^*A)f, f\rangle_2 \geq 0$

$\Rightarrow MI - A^*A + R = C^*C$, $R \in S^{-1}$, Write as C^*C ($C = \sqrt{M}I - A$)

Observation. C has symbol $c(x,\xi) = (M - |a(x,\xi)|^2)^{\frac{1}{2}}$. First, $c \in S^0$; using the Lemma. $y = x^{\frac{1}{2}}$ smooth on $\{|x| > \frac{M}{2}\} \Rightarrow M - |a|^2 = M - a\cdot\overline{a} \in S^0$

we only to estimate R then, given $r \in S^{-k}$ $(k \geq 1)$ let $M \geq 2\sup|a(x,\xi)|^2$

$\|Rf\|_2^2 = \langle R^*Rf, f\rangle_2 \leq \|R^*R\|\|f\|^2 \Rightarrow (R^*R$ bounded $\Rightarrow R$ bounded)

thus we reduce R to R^*R, and $\overline{r}\#r$ is symbol of R^*R, s.t. $\overline{r}\#r \in S^{-2k}$ ... repeat!

It suffices to prove: for $k >> 0$, S^{-k} has all corresponding operator bounded.

Then the Kernel function $K(x,y)=(2\pi)^{-n}\int e^{i(x-y)\xi}a(x,\xi)d\xi$, $a\in S^{-(n+1)} \Rightarrow |a|\le C(1+|\xi|)^{-(n+1)} \Rightarrow |K(x,y)|\le(2\pi)^{-n}\int|a|d\xi$ Page 3

Then $(x_i-y_i)K(x,y)$ is the Kernel function of $Op(i\frac{\partial a}{\partial \xi_i}(x,\xi)) \in S^{-(n+2)} \Rightarrow (1+|x-y|^{n+1})|K(x,y)|\le C'$ a constant

We need to prove K is continuous by dominated by a polynomial of (x,y)

$\Rightarrow \underline{\int|K(x,y)|dx\le C'';\ \int|K(x,y)|dy\le C''}$ (*) (denote the corresponding operator)

Claim. Any K kernel satisfy (*) is a bounded operator: $|(Tf)(x)|^2=|\int K(x,y)f(y)dy|^2\le(\int|K(x,y)||f(y)|dy)^2 \overset{\text{Cauchy}}{\le}(\int|K(x,y)|dy)(\int|K(x,y)||f(y)|^2dy)$
$\le C\int|K||f|^2dy) \to \|Tf\|_2^2=\int|Tf|^2dx\le C\iint|K(x,y)||f(y)|^2dy\,dx\le C(\int|K(x,y)|dx)\int|f|^2dy\le C^2\|f\|_2^2$

Let $T=A$ ▨

The pseudo-differential operator on manifold
- Q_1: the pullback of differential $\le C'$ operators by different coordinates
- Q_2: differential operator $\boxed{\mathbb{R}^n \to U\subset\mathbb{R}^n \text{ locally}}$ (Extend by zero)
  (Recall, we need hole $\mathbb{R}^n$ for Q_1).

Summary. Kernel function $\underset{①}{\overset{②}{\rightleftarrows}}$ symbol function

* ⑤ ③ ④ pseudo-differential operator (* means only holds for differential operator $P=\sum A^\alpha(x)D^\alpha$)

① $K(x,y)=\frac{1}{(2\pi)^n}\int e^{i(x-y)\xi}a(x,\xi)d\xi$; ② $a(x,\xi)=\sum A^\alpha(x)\xi^\alpha$
③④ by definition (Fourier transform); ⑤ Fourier inversion

Prop. $a\in S^m$, TFAE
(1) $\exists b\in S^m$, $a(x,D)b(x,D)-Id\in Op(S^{-\infty})$
(2) $\exists b\in S^m$, $b(x,D)a(x,D)-Id\in Op(S^{-\infty})$
(3) $\exists C>0$, $|\xi|\ge C \Rightarrow |a(x,\xi)|\ge C|\xi|^m$

(Such Op has a role of order likes $O(n)$)

then b unique up to (mod $S^{-\infty}$).

We call such $a(x,D)$ is $\boxed{\text{elliptic}}$

($m=2$, it's the usual elliptic order PDE in $\mathbb{R}^n$; it has its Fourier transformation a elliptic equation)

Pf. b', $b''\in S^m$, s.t. $AB'-Id\in Op(S^{-\infty})$, $B''A-Id\in Op(S^{-\infty})$
$B''-B'=B''(Id-AB')+(B''A-Id)B'\in Op(S^{-\infty})$
$\Rightarrow$ unique;
(Recall Op: $a(x,\xi)\mapsto a(x,D)$) and $BA-Id=(B'-B'')A+(B''A-Id)\in Op(S^{-\infty})$
Similarly, $AB''-Id\in Op(S^{-\infty})$, thus (i)$\Leftrightarrow$(ii);
(i)/(ii) $\Rightarrow$ (iii)
$a(x,\xi)b(x,\xi)-1\in S^{-\infty}\subset S^{-1} \Rightarrow |a(x,\xi)b(x,\xi)-1|\le C_\alpha(1+|\xi|)^{-1}$
$|\xi|\to\infty$, $\frac{1}{2}\le|a(x,\xi)b(x,\xi)|\le C|a(x,\xi)||\xi|^m$
$\Rightarrow \frac{1}{2C}|\xi|^m\le|a(x,\xi)|$

(iii) $\Rightarrow$ (i)/(ii) (nontrivial)

Construction. $b(x,\xi)=(1+|\xi|^2)^{-\frac{m}{2}}F\left(\frac{a(x,\xi)}{(1+|\xi|^2)^{\frac{m}{2}}}\right)$
$F\in H(\mathbb{C})$, s.t. $F(z)=\frac{1}{z}$ for $|z|$ large

Observation. $a(x,\xi)\in S^m$, $(1+|\xi|^2)^{\frac{m}{2}}\in S^m \Rightarrow \frac{a(x,\xi)}{(1+|\xi|^2)^{\frac{m}{2}}}\in S^0$
$\Rightarrow F\left(\frac{a(x,\xi)}{(1+|\xi|^2)^{\frac{m}{2}}}\right)\in S^0 \Rightarrow b\in S^{-m}$

$ab-1=\frac{a}{(1+|\xi|^2)^{\frac{m}{2}}}F\left(\frac{a}{(1+|\xi|^2)^{\frac{m}{2}}}\right)-1$, by (iii), supp$(ab-1)$ compact (bounded)
as it has bound $\frac{C}{2}$: $\frac{a}{(1+|\xi|^2)^{\frac{m}{2}}}\ge\frac{C|\xi|^m}{(1+|\xi|^2)^{\frac{m}{2}}}\ge\frac{C}{2}$, we let $|z|\ge\frac{C}{2}\Rightarrow F(z)=\frac{1}{z}$

then $Op(a)Op(b)=Op(a\#b)$, and expansion $a\#b$
$\Rightarrow AB-Id=R\in S^{-1}$, let $b_k(x,D)=b(x,D)r(x,D)^k\in Op(S^{m-k})$
and the expansion gives $b'\sim\sum b_k$, s.t. $AB'=A(B'-\sum_{j<k}B_j)+AB(\sum_{j<k}R^j)$
$=(Id-R^k)+Op(S^{-k})=Id+Op(S^{-k})$
let $k\to\infty$ ▨

Rk. Here $-R$ is the lower order parts ($a\#b=ab+r$.)

Since $a\in S^m$, $(a(x,D)f)(x)=\int K(x,y)f(y)dy \Rightarrow K\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n-\underset{\text{diagonal}}{\Delta(\mathbb{R}^n)})$

E.g. Laplace equation $\Delta u=f$, $u,f\in C^\infty(\mathbb{R}^n)$
$R: L^\infty(\mathbb{R}^n)\rightleftarrows L^\infty(B_r)$: $\sim$, consider equation
$v\mapsto v|_{B_r}$ $\qquad v+RK\tilde{v}=f$ $\sim$ (*)
$\left.\begin{matrix}v, x\in B_r\\ 0, x\notin B_r\end{matrix}\right\}=\tilde{v}\leftarrow v$ $\qquad \Rightarrow \|RK\tilde{v}\|_\infty\le\sup_{B_r}|K(x,y)||v|$
$K: L^\infty(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)$ $\quad r\to0\Rightarrow(f-RK)$ is a contraction
$Ku(x)=\int K(x,y)u(y)dy$ $\quad$ operator, let $u=B\tilde{v}$
$\Rightarrow Au=AB\tilde{v}=\tilde{v}+K\tilde{v}$
$\Rightarrow ARu=v+RK\tilde{v}=f$, $A=\Delta$
then Ru is the local solution of $\Delta u=f$ ▨

In manifold. $A: C_0^\infty(M)\to C^\infty(M)$, denoted all m-order to be $\Psi^m(M)$, each coordinate has $a(x,D)$ the local symbol

Prop. $X:\Omega\to\Omega'$ is C^∞-diffeomorphism, $a\in S^m$, the kernel of a has compact support in $\Omega\times\Omega$, then: The coordinate change of symbol
① $a'(X(x),\eta)=e^{-iX(x)\eta}(a(x,D)e^{iX(x)\eta})$ for D fixed
it defines $a(y,\eta)\in S^m$ (after a zero extension outside Ω');
② $a'(x,D)$'s Kernel function has compact support in $\Omega'\times\Omega'$;
③ $a(x,D)(u\circ X)=(a'(x,D)u)\circ X$; $a'(X(x),\eta)\sim\sum\frac{1}{\alpha!}\partial_\xi^\alpha a(x,X'(\eta))D_y^\alpha(e^{i\rho_x(y)\eta})|_{y=x}$ where $\rho_x(y)=X(y)-X(x)-X'(x)(y-x)$, for all proper

$\Longrightarrow$ We can well-define pseudo-differential operator in M, not depend on coordinates

Pf. ① $a(x,D)e^{ix\xi}=e^{ix\xi}a(x,\xi)$ by $\exists\hat{u}\in C_0^\infty$, s.t. $(2\pi)^{-n}\int\hat{u}(\xi)d\xi=u(0)=1$ the kernel $\Rightarrow a(x,D)\underset{e^{ix\xi}}{\underline{u(\varepsilon x)e^{ix\xi}}}=e^{ix\xi}(2\pi)^{-n}\int e^{ix\xi}a(x,\xi+\varepsilon z)\hat{u}(z)dz$, let $\varepsilon\to0 \Rightarrow a(x,D)e^{ix\xi}=e^{ix\xi}a(x,\xi)$

$\exists\varphi\in C_0^\infty(\Omega)$, s.t. $\forall(x,y)$ near to supp$K$ (Kernel), $\varphi(x)=\varphi(y)=1 \Rightarrow a'(X(x),\eta)=\varphi(x)e^{-iX(x)\eta}(a(x,D)(\varphi(x)e^{iX(x)\eta}))$ $a(x,\xi)$

By ① we can prove ③: ①$\Rightarrow$③ holds for $u=e^{iy\eta}$, then by a Fourier inversion $\boxed{\in S^m}$ (By some harmonic techniques: oscillating integral, omitted)
$\Rightarrow$ holds for all u. ② is obvious ▨

Rk. The idea of proving ① is $e^{iX(x)\eta}$ is wave function can't use Fourier $\Rightarrow$ we modify it by multiply $\varphi(x)$ compact support)

Def. $A: C_0^\infty(M)\to\boxed{C^\infty(M)}$ is pseudo-differential operator, if for $\psi: V\to U\subset\mathbb{R}^n$, $\tilde{A}: C_0^\infty(U)\to C^\infty(U)$, $u\mapsto A(u\circ\psi)\circ\psi^{-1}$ is m-order pseudo-differential operator in $\mathbb{R}^m$ (extend by zero) $\Rightarrow \varphi A\psi\in Op(S^m)$)

(We can't expect $C_0^\infty(M)$ due to apply Fourier; local $\Rightarrow$ Global is must)

Def. A proper support of $A: C_0^\infty(\Omega)\to C_0^\infty(\Omega)$ is $\forall K\subset\Omega$, $K'\subset\Omega$ compact, ① if supp$u\subset K$ $\Rightarrow$ supp$Au\subset K'$ ② $Au|_{K'}\equiv0 \Rightarrow u|_K\equiv0$ (cont $K\to$ cont K')

Observation. ① $a \in S^m$, $a = a_m + S^{m-1}$, for $a(\chi(x), y) \sim \sum \frac{1}{\alpha!} \partial_\xi^\alpha a(x, \chi(y)) D_y^\alpha (e^{i\varphi(x,y)\eta})|_{y=x}$ has the highest part $a_m(\chi(x), y)$

(Here the decomposition is an m-deg homogenous, i.e. $|\partial_x^\alpha \partial_\xi^\beta a| \le C_{\alpha\beta}(1+|\xi|)^{m-|\beta|}$ and $|\partial_x^\alpha a| \le C_{\alpha,0}(1+|\xi|)^m$, $\ne C_{\alpha\beta}(1+|\xi|)^{m-1}$; $\in S^m \setminus S^{m-1}$) $\Rightarrow$ m-homogenous is invariant under χ a coordinate change

② Proper supported operator sends $C_0(\Omega)$ to $C_0(\Omega)$

Coro. ① $A_1, A_2 \in \Psi^{m_1}(M), \Psi^{m_2}(M)$ proper support $\Rightarrow A_1 A_2 \in \Psi^{m_1+m_2}(M)$ proper support

② The principal symbol of $A_1 A_2$ is $a_{m_1} a_{m_2}$; ③ $[A_1, A_2]$'s principal symbol is $\frac{1}{i}\{a_{m_1}, a_{m_2}\}$ poisson bracket.

(② not holds for symbol in general by expansion)

(It has lower order $m-1$; by ②, the m-order is 0)

Coro. ① $\chi: \Omega \to \Omega'$, $a_m(\chi(x), \eta) = a_m(x, {}^t\chi'(x)\eta)$

it induces $\chi^*: T_x^*\Omega \to T_{\chi(x)}^*\Omega'$

(i.e. the grading structure turns into the cotangent bundle)

② Let $\Omega = \Omega'$, then $a_m(\chi(x), \eta) = a_m(x, {}^t\chi'(x)\eta)$

Such a_m invariant, we call it the principal symbol of $A \in \Psi^m(M)$, (i.e. we omit the all lower part)

$\Downarrow$

Principal symbol is homogenous function on cotangent space/bundle (which transform in coordinate change)

E.g. $f(\lambda x) = \lambda^m f(x)$ (we only concern $|x| > C$, thus f not smooth at 0 doesn't matter) $\Rightarrow$ it's bad!

Recall. $\operatorname{grad} f = df$, $\operatorname{div} X = \operatorname{tr}(Y \mapsto \nabla_Y X) = \sum_i \langle \nabla_{Y_i} X, Y_i \rangle$ in manifold M, and $\Delta f = \operatorname{div} \circ \operatorname{grad}(f)$ by taking $\{Y_i\}$ orthogonal

Rk. $\operatorname{div}(X) = \sum \langle \nabla_{Y_i} X, Y_i \rangle$ depends on $\{Y_i\}$

"Δf" = "$\sum Y_i(Y_i f)$" depends on $\{Y_i\}$, indeed we have

$\Delta f = \operatorname{tr}(Y \mapsto \nabla_Y \operatorname{grad} f)$
$= \sum \langle \nabla_{X_i} \operatorname{grad} f, X_i \rangle$
$= \sum (X_j \langle \operatorname{grad} f, X_j \rangle - \langle \operatorname{grad} f, \nabla_{X_j} X_j \rangle)$
$= \sum X_j(df(X_j)) - \sum \langle \operatorname{grad} f, \nabla_{X_j} X_j \rangle$
$= \sum X_j(X_j f) - \langle \operatorname{grad} f, \nabla_{X_j} X_j \rangle$ not flat case

$A: C_c^\infty(M) \to C^\infty(M)$, $\boxed{\sqrt{-\Delta}}$ is what operator? (Dirac operator)

(Due to $\Delta = \sum \frac{\partial^2}{\partial x_j^2}$ has symbol $a(x, \xi) = -\sum \xi_j^2$)

$(b(\xi))^2 = |\xi|^2$, b polynomial on (ξ_i) is impossible!

E.g. ① $n=2$, $b(x,y) = x^2 + y^2 \Rightarrow b(x,y) = \lambda x + \mu y$

$\Rightarrow \begin{cases} \lambda^2 = \mu^2 = 1 \\ \lambda\mu + \mu\lambda = 0 \end{cases}$ no solution $\Rightarrow$ but $\boxed{\text{using matrix}}$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, then let $p(f, g) = 2i \begin{pmatrix} \frac{\partial g}{\partial \bar z} \\ \frac{\partial f}{\partial z} \end{pmatrix}$ i.e. $p = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial y}$

thus we add a varible "x" (in symbol) $\Rightarrow b(x, \xi)$ is the original idea of pseudo-differential operator $\Rightarrow p^2 = -\Delta \Rightarrow p = \sqrt{-\Delta}$

② $n=3$, $\begin{cases} \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = 1 & \text{replace by } -1 \Rightarrow \text{quartic number } \mathbb{H} \subset \mathbb{C} \oplus \mathbb{C}, \text{ i.e. } \mathbb{H} \to M_2(\mathbb{C}) \\ \gamma_i\gamma_j + \gamma_j\gamma_i = 0 & \text{and we need adding a "z"} \end{cases}$

$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $i \mapsto \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$; $j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; $k \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

Thus we embedding $C_c^\infty(M)$ to a proper vector bundle (higher rank), and $\sqrt{-\Delta}$ exist in this nature

Let $D = \sum \gamma_j \nabla_{X_j}$ and $D^2 = \Delta \Rightarrow D^2 = \sum_{i,j} \gamma_j(\nabla_{X_j} \gamma_i \nabla_{X_i} + \gamma_i \nabla_{X_j} \nabla_{X_i})$; compare $\sum_j X_j(X_j f) - \sum \langle \operatorname{grad} f, \nabla_{X_j} X_j \rangle$ of $-\Delta$

$= \sum_j (\gamma_j^2 \nabla_{X_j} \nabla_{X_j} + \sum_{i<j} (\gamma_j\gamma_i \nabla_{X_j} \nabla_{X_i} + \gamma_i\gamma_j \nabla_{X_i} \nabla_{X_j})) + \sum_i X_j(\nabla_{X_j} \gamma_i) \nabla_{X_i}) \Rightarrow \boxed{\begin{cases} \gamma_j^2 = 1 \\ \gamma_i\gamma_j + \gamma_j\gamma_i = 0 \end{cases}}$

($\gamma_j^2 = 1$; $\gamma_j\gamma_i \nabla_{X_j}\nabla_{X_i} + \gamma_i\gamma_j \nabla_{X_i}\nabla_{X_j} = 0$ if $\gamma_j\gamma_i + \gamma_i\gamma_j = 0$ $\Rightarrow \nabla_{X_j}\nabla_{X_i} - \nabla_{X_i}\nabla_{X_j}$ is the $R(X_i, X_j) + \nabla_{[X_i, X_j]}$ the curvature part.)

The Clifford algebra. Given $Q: V \times V \to \mathbb{C}$ a quardic form

A Clifford algebra over $\mathbb{C}$ is an associative algebra over $\mathbb{C}$, $C(Q)$, s.t.

① $j: V \to C(Q)$ linear, s.t. $(j(v))^2 = Q(v, v) \cdot 1$;

② (Universal property) It's initial object for all (A, j) satisfy the above.

(It contains most information)

Equivalently, let $V^{\otimes} = \bigoplus_{n \ge 0} V^{\otimes n}$ graded, then $C(Q) = V^{\otimes} / \{v_1 \otimes v_2 + v_2 \otimes v_1 = -2Q(v_1, v_2)\}$

E.g. $Q = 0$, $C(Q) = \Lambda V$

E.g. $V = T_p M$, $Q = g$ the Riemannian metric

We skip the algebraic facts and set the following:

① (M^n, g) complete; ② S is vector bundle on M, as a ① representation of Clifford algebra; i.e. $\rho: C(g_p) \to \operatorname{End}(S_p)$, $\forall p \in M$, (S_p, ρ) is representation of $C(g_p)$

② And a Hermittian inner product (form) $(,)_p$ in S_p, s.t. $(v \cdot \psi, v \cdot \varphi) = (\psi, \varphi)$, $|v| = 1$

Globally $(\psi, \varphi) = \int (\psi_p, \varphi_p) d\mathrm{vol}_p$ if $\psi, \varphi \in \Gamma_c(S)$ compact supported!

③ A connection ∇^A on S i.e. action $C(g) \times S \to S$, $(C(g_p), S(p)) \mapsto (v_p \cdot s_p)_p$.

$\nabla^A: \Gamma(S) \to \Gamma(T^*M \otimes S)$

s.t. ① $\nabla_X^A (Y \cdot \psi) = (\nabla_X Y)\psi + Y(\nabla_X \psi)$

② $X \cdot (\psi, \varphi)_p = (\nabla_X^A \psi, \varphi)_p + (\psi, \nabla_X^A \varphi)_p$

We call such ∇^A is Hermite connection

Def. Such S is spinor bundle (The existence of S we omitted, it's not need in proving index)

Let $\Delta \psi = \sum \nabla_{e_i}^A \nabla_{e_i}^A \psi - \sum \operatorname{div}(e_i) \nabla_{e_i}^A \psi$

$D_A \psi = \sum e_i \cdot \nabla_{e_i}^A \psi$ is action, for $\psi \in \Gamma(S)$

$\boxed{D_A^2 = \text{"}\Delta\text{"}}$ by and + curvature parts

$\begin{cases} e_i \cdot e_j + e_j \cdot e_i = 0 & \text{as they orthgonal} \\ e_i^2 = 1 & \text{is obvious} \end{cases}$

Prop. ① $D_A(f\psi) = \operatorname{grad} f \cdot \psi + f D_A \psi$; ② $(D_A \psi, \varphi) = (\psi, D_A \varphi)$

Pf. ① $D_A(f\psi) = \sum e_i \cdot \nabla_{e_i}^A (f\psi) = \sum e_i (df(e_i) \cdot \psi + f \nabla_{e_i}^A \psi) = (\sum df(e_i) \cdot e_i) \cdot \psi + f D_A \psi = \operatorname{grad} f \cdot \psi + f D_A \psi$ ☑

② Locally. $(D_A \psi, \varphi)_p = \sum (e_i \cdot \nabla_{e_i}^A \psi, \varphi)_p = \sum (e_i \cdot e_i \cdot \nabla_{e_i}^A \psi, e_i \cdot \varphi)_p$

$= -\sum (\nabla_{e_i}^A \psi, e_i \cdot \varphi)_p = -\sum (e_i (\psi, e_i \cdot \varphi)_p - (\psi, \nabla_{e_i}(e_i \cdot \varphi))_p) = \boxed{\sum e_i (\psi, e_i \cdot \varphi)_p + \sum (\psi, \nabla_{e_i} e_i \cdot \varphi)_p} + (\psi, D_A \varphi)_p$

$\Rightarrow \nabla_{e_i} e_i \cdot \varphi + e_i \cdot \nabla_{e_i} \varphi$. A $\operatorname{div}(\cdot) \Rightarrow$ integral 0 ☑

Completion $(\Gamma_c(S), (,))$ to $L^2(S)$ (metric)

but D_A can't defined globally $L^2(S)$; it's unbounded, to deal with these operators, we have generally: $A: H \to H$

Def. A is closable iff $\overline{(\operatorname{Graph} A)} = \operatorname{Graph} \bar A$, $\exists \bar A$ called the closure of A (later almost all operators we need is closable)

For A unbounded, we need its domain of definition $D(A)$. By $\bar A x = \lim A x_n$, we have $D(\bar A) = \{x \in H \mid \exists (x_n) \subset D(A), x_n \to x, Ax_n$

Consider $\langle Ax, y\rangle = \langle x, A^*y\rangle$, $x \in D(A)$, what about y? Thus we have $D(A^*) = \{y \in H \mid \exists z, \forall x \in D(A), \langle Ax, y\rangle = \langle x, z\rangle\}$

Def. if $\forall x, y \in D(A)$, $\langle Ax, y\rangle = \langle x, Ay\rangle$, then A is symmetric operator (not self-adjoint!) (formal self-adjoint)
$\Rightarrow D(A) \subset D(A^*)$ and $A^*|_{D(A)} = A$, i.e. $A \subset A^*$ (Notation $A \subset B$ means $D(A) \subset D(B)$ and $B|_{D(A)} = A$)

(Now $D_A: L^2(S) \dashrightarrow L^2(S)$ domain of definition dense and unbounded)

Def. A self-adjoint $\Leftrightarrow A = A^* \Leftrightarrow A \subset A^*$ & $A^* \subset A$. Def. $\bar{A} = A^*$ is the essentially self-adjoint.
Observation. self-adjoint $\Rightarrow$ essentially self-adjoint by $A \subset \boxed{\bar{A} = A^{**}} \subset A^*$ ▨ (Von Neumann)

Claim. ① $D_A: L^2(S) \dashrightarrow L^2(S)$ is essentially self-adjoint;
② $D(D_A) \subset H^1(S)$ the sobolev space, $D_A: H^1(S) \to L^2(S)$ is Fredholm, we'll prove ② at first, ① later.

(Rk. $H^1(S) = W^{1,2}(S)$ the sobolev space with norm L^2)

② Set M compact, $H^1(S) = \{\psi \in L^2(S) \mid D_A\psi \in L^2(S)\}$. the Sobolev embedding $i: H^1(S) \hookrightarrow L^2(S)$
the compact embedding thm tells us $B_{H^1}(S) = \{\psi \in H^1(S) \mid \|\psi\|_{L^2}^2 + \|D_A\psi\|_{L^2}^2 \le 1\}$ is compact (By A-A thm. $\|\psi\|_{L^2}^2 \le 1 \Rightarrow$ uniformly bounded, $\|D_A\psi\|_{L^2} \le 1 \Rightarrow$ equicontinuous $\Rightarrow$ compact ▨)
Consider $\boxed{\operatorname{Ker} D_A}$, $K^1 = \{\psi \in H^1(S) \mid D_A\psi = 0, \|\psi\|_{L^2}^2 + \|D_A\psi\|_{L^2}^2 \le 1\}$, $K^0 = \{\psi \in H^1(S) \mid D_A\psi = 0, \|\psi\|_{L^2}^2 = 1\}$
$\Rightarrow K^1 = K^0 \subset \operatorname{Ker} D_A$, $K^1 = \operatorname{Ker} D_A \cap B_{H^1}(S)$ compact $\Rightarrow \dim \operatorname{Ker} D_A < \infty$
Consider $\boxed{\operatorname{Coker} D_A}$ as the $L^2 \ominus D_A(H^1)$, consider $(D_A\varphi, \psi) = 0 \Rightarrow \psi \in D(D_A^*)$

Rk. Weak derivative (D_A in H^1) bounded $\not\Rightarrow$ equicontinuous in general, but after a proof that D_A is elliptic regular we can do this.

by ① (D_A essentially self-adjoint) $\Rightarrow D_A\psi = 0 \Rightarrow D_A(H^1)^\perp = \operatorname{Ker} D_A$ is finite dimensional
Thus it suffices to show [illegible] is closed in $L^2(S)$ ($\operatorname{Coker} D_A$) Coro. $\operatorname{Ind} D_A = 0$
By Sobolev embedding $D_A(H^1) = \operatorname{Range}(D_A)$ (How can we modify D_A to make Index meaningful?)
$\Rightarrow \exists C > 0$, s.t. $\|\psi\|_{L^2} C \le \|D_A\psi\|_{L^2} \Rightarrow \{D_A\psi_n\}$ Cauchy implies $\{\psi_n\}$ Cauchy $\Rightarrow$ closed. ▨

The principal symbol of D_A: take coordinate of T_x^*M to be $(\xi_1 \cdots \xi_n) = \xi$
then $\nabla_{e_i} = \frac{\partial}{\partial x_i}$ locally coordinate, and $D_i = \frac{1}{i}\frac{\partial}{\partial x_i} \leftrightarrow \xi_i \Leftrightarrow \frac{\partial}{\partial x_i} \leftrightarrow i\xi_i \Leftrightarrow \nabla^A_{e_i} \leftrightarrow i\xi_i \Leftrightarrow \boxed{D_A \Leftrightarrow i\xi \ \& \ D_A^2 \Leftrightarrow -|\xi|^2}$

① (Wolf) Give norm in $D(D^*)$ (D is Dirac operator/domain): $\psi \in D(D^*)$, $N(\psi) = (\|\psi\|_{L^2}^2 + \|D^*\psi\|_{L^2}^2)^{\frac{1}{2}}$ the following three Lemmas gives out ▨
Lemma1. If $\overline{\Gamma(S)} = D(D^*)$ in the norm topology induced by $N \Rightarrow D$ essentially self adjoint. ($N \ge L^2$, and $\overline{\Gamma(S)}_N \subset \overline{\Gamma(S)}_{L^2} = D(D^*)$. the condition tells us two norm's same)
Pf. We known D is symmetry, i.e. $D \subset D^* \Rightarrow \bar{D} \subset D^* \Rightarrow D(\bar{D}) \subset D(D^*)$, we prove $D(D^*) \subset D(\bar{D})$
$\forall \psi \in D(D^*)$, $\exists (\psi_n) \subset \Gamma(S)$, s.t. $\psi_n \xrightarrow{N} \psi \Rightarrow \begin{cases}\psi_n \xrightarrow{L^2} \psi \\ D^*\psi_n \xrightarrow{L^2} D^*\psi\end{cases}$ due to $\Gamma(S) \subset D(D) \subset D(D^*)$ $\Rightarrow D^*\psi_n = D\psi_n \xrightarrow{L^2} D^*\psi \Rightarrow (\psi, D^*\varphi) = \lim(\psi_n, D^*\varphi) \Rightarrow \psi \in D(\bar{D})$ & $D^*\psi = \bar{D}\psi$ ▨
$\Gamma(S) \subset D(D) \subset D(D^*)$

Lemma2. $\overline{\Gamma_c(S)}_N = D_c(D^*) = \{\psi \in D(D^*) \mid \operatorname{supp}\psi \text{ compact}\}$
Pf. First by portion of unity subordinate to coordinate charts, one can view $\psi \in D_c(D^*)$ as (ψ_j) in open sets of $\mathbb{R}^n$, smoothing it again (We had smoothing it first as $\psi_j = f_j\psi$, f_j is the portion of unity $\sum f_i = 1$) $\Rightarrow \psi_j * h_\varepsilon \xrightarrow[\varepsilon \to 0]{L^2(S)} \psi_j$ and $(\psi_j * h_\varepsilon)_j \in \Gamma_c(S)$ when turn back to M, and $D^*(\psi_j * h_\varepsilon) \xrightarrow[\varepsilon \to 0]{L^2} D^*(\psi_j)$, let $\varepsilon = \frac{1}{n}$, $\psi_{j,n} = \psi_j * h_{\frac{1}{n}} \Rightarrow N(\psi_j - \psi_{j,n}) \xrightarrow[n \to \infty]{} 0$ ▨
Recall. (Smoothing function) $f(x) = e^{-\frac{1}{1-|x|^2}}$ & $f_\varepsilon(x) = \frac{1}{\varepsilon}f(\frac{x}{\varepsilon})$ integral 1, $\varepsilon \to 0$, it's distribution δ is unit of convolution
the portion of unity is by $f_\varepsilon(x) * g$, g is [illegible] For dimension $= n$, $h_\varepsilon(x) = \frac{1}{\varepsilon^n}f(\frac{x}{\varepsilon})$
Lemma3. $\overline{(D_c(D^*))}_N = D(D^*)$.
Pf. Fix $x_0 \in M$, define $\rho(x) = d(x_0, x)$, $|\rho(x_1) - \rho(x_2)| \le d(x_1, x_2) \Rightarrow$ 1-Lip $\Rightarrow$ a.e. differentiable $\Rightarrow |\operatorname{grad}\rho| \le 1$ a.e. (Here we only need a.e. not apply geodesic)
Let $B_{x_0}(r) = \{x \in M \mid \rho(x) \le r\}$ geodesic ball, $B_{x_0}(r)$ compact, $\alpha =$ [illegible], and $b_r(x) = \alpha(\frac{\rho(x)}{r})$ as 1-Lip $\Rightarrow |\operatorname{grad} b_r|^2 \le \frac{1}{r^2}|\alpha'(\frac{\rho}{r})|^2 < \frac{C}{r^2}$, ~~[illegible]~~ we estimate the grad due to D's composite ruleing does it.
$\psi \in D(D^*)$, $\psi_r = b_r\psi \in D_c(D^*)$, $D^*(\psi_r) = (\operatorname{grad} b_r)\psi + b_r D^*\psi \Rightarrow \|D^*(\psi_r - \psi)\|_{L^2}^2 = \|(\operatorname{grad} b_r)\psi + (b_r - 1)D^*\psi\|_{L^2}^2 \le 2(\|(\operatorname{grad} b_r)\psi\|_{L^2}^2 + \|(b_r - 1)D^*\psi\|_{L^2}^2)$
$= 2\left(\int_{B_{x_0}(2r) - B_{x_0}(r)} \|(\operatorname{grad} b_r)\psi\|^2 + \int_{M - B_{x_0}(r)} \|(b_r - 1)D^*\psi\|^2\right) \le 2\left(\frac{C}{r^2}\int\|\psi\|^2 + \int\|D^*\psi\|^2\right) \Rightarrow \|\psi_r - \psi\|_N^2 \le 2\left(\frac{C}{r^2}\int\|\psi\|^2 + \int(\|\psi\|^2 + \|D^*\psi\|^2)\right)$
$\le 2\int_{M - B_{x_0}(r)} \left(\frac{2C}{r^2} + 1\right)\|\psi\|^2 + 2\|D^*\psi\|^2 \xrightarrow[r \to \infty]{} 0$ ▨ (Here we need M is complete!)
We complete the proof of ① ▨

Prop. $\lambda \notin \sigma(\bar{D} = D^*)$, $(D - \lambda I)^{-1}: L^2(S) \to L^2(S)$ is compact.
Pf. $D - \lambda: D(D) \to L^2(S)$, $\lambda \notin \sigma(\bar{D} = D^* = D) \Rightarrow (D - \lambda)^{-1}: L^2(S) \to L^2(S)$ bounded
Claim. $(D - \lambda)^{-1}$ factor through $H^1(S)$ $\Rightarrow$ by compact sobolev embedding $\Rightarrow (D - \lambda)^{-1}$ compact
i.e. $R((D - \lambda)^{-1}) \subset H^1(S)$

$\boxed{\text{Schrödinger–Lichnerowicz formula: } D^2\psi = \Delta(\psi) + \frac{K}{4}\psi + \frac{1}{2}dA(\psi);\ K \text{ is the numerical curvature, } dA \text{ is curvature form.}}$

$\Rightarrow \|D\psi\|_{L^2}^2 = (\Delta\psi, \psi) + \int\frac{K}{4}\langle\psi, \psi\rangle d\mathrm{vol} + \int\frac{1}{2}\langle dA\psi, \psi\rangle d\mathrm{vol} = \|\nabla\psi\|^2 + \int\frac{K}{4}\langle\psi, \psi\rangle + \boxed{\int\frac{1}{2}\langle dA\psi, \psi\rangle}$ (dA bounded due to M compact)
$= \|\psi\|_{H^1}^2 - \|\psi\|_{L^2}^2$
$\Rightarrow \|\psi\|_{H^1}^2 + \left(\frac{K_{min}}{4} - C - 1\right)\|\psi\|_{L^2}^2 \le \|D\psi\|_{L^2}^2 \le \|\psi\|_{H^1}^2 + \left(\frac{K_{max}}{4} + C - 1\right)\|\psi\|_{L^2}^2$
$\Rightarrow \exists C$, s.t. $\left|\int\frac{1}{2}\langle dA\psi, \psi\rangle\right| \le C\|\psi\|_{L^2}^2 \le \|\psi\|_{L^2}^2 C$
thus $\|(D - \lambda)^{-1}\varphi\|_{H^1}^2 = \|(D - \lambda)^{-1}(D - \lambda)\psi\|_{H^1}^2 \le \|(D - \lambda)\psi\|_{L^2}^2 + \underbrace{(C + 1 + \lambda^2 - \frac{K_{min}}{4})}_{=:C'}\|\psi\|_{L^2}^2 = \|\varphi\|_{L^2}^2 + C'\|(D - \lambda)^{-1}\varphi\|_{L^2}^2$ (bounded in L^2)
$\le (1 + C'C'')\|\varphi\|_{L^2}^2 < \infty \Rightarrow (D - \lambda)^{-1}\psi \in H^1(S)$ ▨
Due to $\operatorname{Ind}(D) = 0$ now, we need some modification. We study more precise in $Cl(\Omega)$ [illegible]

Then we split the spinor bundle $S=S^+\oplus S^-$, s.t. Class $\cdot S^+\subset S^-$, Cl odd $S^-\subset S^+$ (this implies Cl even $\cdot S^+\subset S^+$, Cl even $\cdot S^-\subset S^-$) Page 6

then $D=\begin{pmatrix}0 & D_1\\ D_0 & 0\end{pmatrix}$ $\begin{matrix}S^+\\ S^-\end{matrix}$, $D_0:\Gamma(S^+)\to\Gamma(S^-)$ and $D_0^*=D_1\Rightarrow$ The ~~right~~ left hand (analytic index) in Atiyah–Singer index thm
$D_1:\Gamma(S^-)\to\Gamma(S^+)$ (By D self-adjoint) is $\mathrm{Ind}(D_0)$ ($\mathrm{Ind}(D_0)\neq 0$ in general)

Now I claim: $\mathrm{Ind}(D)$ only depend on its principal symbol (D is Fredholm differential operator)

Pf. By $\sigma(x,D):H^s\to H^{s-m}$, the expansion $\sigma=\underbrace{\sigma_m}_{\text{principal}}+\sigma_{m-1}+\cdots$, then $(\sigma-\sigma_m)(x,D)$ is compact as $\Rightarrow H^s\to H^{s-(m-1)}\xrightarrow[\text{compact embedding}]{\text{Sobolev}}H^{s-m}\Rightarrow$ compact

then by compact distribution ▨

Consider the principal symbol of D_0 (denoted σ_1) locally in a trivialize neiberhood of S^+ & S^-

denoted V_p, $\forall x\in V_p$, $s(x)=\begin{pmatrix}s^1(x)\\ \vdots\\ s^k(x)\end{pmatrix}$, $D_0 s(x)=\begin{pmatrix}s^1(x)\\ \vdots\\ s^l(x)\end{pmatrix}\Rightarrow D_0=\left(\sum_j a_j^{(l\times k)}\partial_j\right)_{l\times k}$, $a_j^{(l\times k)}$ for some functions $\Rightarrow\sigma_1(x,\xi)=(i\sum a_j\xi_j)$

Conclusion. If D_0 is elliptic $\Rightarrow(i\sum a_j\xi_j)$ matrix invertible $\Rightarrow l=k\Rightarrow\mathrm{rank}\,S^+=\mathrm{rank}\,S^-$

$\sigma_1:\Gamma(T^*M)\to\Gamma(GL(S^+,S^-))=\mathrm{Hom}(S^+,S^-)$ using $\pi^*S^+\xrightarrow{\sigma_1}$, we can denote

$(x,\xi)\mapsto(\sigma_1(x,\xi)=(i\sum a_j\xi_j):S^+\to S^-)$ ▨ $T^*M\xrightarrow{\pi}M$ D_0 as triple $[\pi^*S^+,\pi^*S^-,\sigma_1]$: a datum over T^*M

Reformulation of classical

Or by Clifford algebra

$\sigma(D_A)=\begin{pmatrix}0 & \sigma_1(D_A)\\ \sigma_0(D_A) & 0\end{pmatrix}$ s.t.

$\sigma(D_A)^2=(\sum\xi_i^2)\,\mathrm{I}_{n\times n}$

$\Rightarrow\sigma(D_A)^{\dagger}$ invertible

$\Rightarrow\sigma(D_A)$ invertible ▨

$E\longleftrightarrow\Gamma(E)$ or $\Gamma(E)$; We have ① (Gelfand–Naimark, 1943) $M\longleftrightarrow C(M)$ ($\mathrm{Max}(C(M))\cong M$), M compact $\longleftrightarrow C(M)$ unity

② (Serre–Swan, 1956) {Vector bundle over M} $\longleftrightarrow$ {f.g. projective module over C}

$M\longleftrightarrow C^\infty(M)$ or $C(M)$ Pf of ② $\Longrightarrow$ Using partion of unity in M on $\{U_i\}$, U_i trivialize E, $s\in\Gamma(E)$, $s\mapsto\begin{pmatrix}f_i s_i\\ \vdots\end{pmatrix}$

compact. $\mapsto\begin{pmatrix}s_{i1}(x)\\ \vdots\\ s_{in}(x)\end{pmatrix}\in C(M,\mathbb{R}^{n\times k})$ by their n components: $\Gamma(E)\xrightarrow{s_i}\{f\in M\to\mathbb{R},\ \mathrm{supp}\,f\subset\mathrm{supp}\,f_i\ldots\}$
$=C(M)^{n\times k}$

We accept then Thus we define $P:C(M)^{n\times k}\to C(M)^{n\times k}$ as $P=\begin{pmatrix}\boxed{P_{V_1}\ \ P_{n1}} & & \\ & \ddots & \\ & & \boxed{P_{V_k}\ \ P_{nk}}\end{pmatrix}$ by $V_i=\{f\in C(M),\ \mathrm{supp}\,f\subset\mathrm{supp}\ldots\}$ and L^2 projection to V_i ▨

① D_A is elliptic, with elliptic regularity in manifold

② $(D_A-\lambda)^{-1}$ compact for $\lambda\in\rho(D_A)$ ($D=D_A$)

$(\Longleftarrow)$ Omitted ▨

(compact resolvent) Cor. $\Gamma(E)=\Gamma^\infty(E)\otimes_{C^\infty(M)}C(M)$, thus we ablise the notation (as both $\Gamma(E)$ (by dual)

Serre–Swan gives another way describe the principal symbol: $D:L^2(S)\to L^2(S)$ and D extending A

$\Rightarrow\ker D,\ \ker D_1\subset\Gamma(S)$ ~~[struck out]~~

$\Gamma(E)\cong N\cdot(C(M))^n$ by enlage both n & m, $D=\sum e_i\cdot\nabla_{e_i}$

What is the meaning of $\sigma_1(\xi_1\cdots\xi_n)$ invertible?

$\Gamma(S)\cong N'\cdot(C(M))^m\Rightarrow$ set $n=m$ $\Rightarrow\sigma_1=\sum A_i(i\xi_i)=i\sum A_i\xi_i$ constant matrix i.e. over C

$\Longleftrightarrow(i\sum a_j\xi_j)$ invertible this coincides upper approach ▨ Then by elliptic, when $(\xi_i)\neq 0$, σ_1 invertible

Summary. The principal symbol is a matrix-value function on T^*M, out of zero section invertible iff elliptic

$E_1\xrightarrow{D}E_2\longleftrightarrow\sigma:T^*M_0\to\mathrm{Mat}(\pi^*E_1,\pi^*E_2)$ $E_1\xrightarrow{D}E_2$ In particular, for

D elliptic $\longleftrightarrow\sigma_0:(T^*M)_0\to GL(\pi^*E_1,\pi^*E_2)$ π^*E_1 $D_{A,0}=[\pi^*S^+,\pi^*S^-,\sigma_1]$
(section map)

$\Rightarrow D=[\pi^*E_1\to\pi^*E_2,\sigma]$ (sometimes denoted $\Gamma[\Gamma(\pi^*E_1),\Gamma(\pi^*E_2),\sigma]$) T^*M (Due to σ_1 determines $\mathrm{Ind}(D_{A,0})$ it's a good notation

Using T^*M's vector bundle (differential form) to give $\mathrm{Ind}(D_{A,0})\in\mathbb{R}$ only way is intergal!

Topological K-theory (Grothendieck (AR), Atiyah (C, easier) Singer, Bott (Periodicity) ...

K^0 group: Set X manifold compact (However it fails, we modify as following (Thus naturally we come into righthand's topologic index)

$\mathrm{Vect}(X)/\cong$ with $V_1\cong V_2\Leftrightarrow\exists f:V_1\to V_2$ invertible & fibres isomorphic as in A_6 expected to be a group

$\Rightarrow[V_1]+[V_2]:=[V_1\oplus V_2]\Rightarrow(\mathrm{Vect}(X)/\cong,+)$ (equivalent to $0\to V_1\to V\to V_2\to 0$, $V\sim V_1\oplus V_2$ quotiental)

Furthermore, $V_1\cong_s V_2$ stably isomorphic iff $V\oplus(X\times\mathbb{C}^{k_1})\sim V_2\oplus(X\times\mathbb{C}^{k_2})$ ~~[struck out]~~

$\mathrm{Vect}(X)/\cong_s$ is an Abelian semi-group $\Rightarrow$ Def. $K^0(X)$ = group generated by such semi-group

$\Rightarrow$ Algebraically, elements $\in K^0(X)$ is of the form $[V_1]-[V_2]$, here $[V_1]-[V_2]=[V_1']-[V_2']$ iff $\exists k_1,k_2:V_1\oplus V_2'\oplus X\times\mathbb{C}^{k_1}\sim V_1'\oplus V_2\oplus X\times\mathbb{C}^{k_2}$

Rk. $K^0:\mathrm{Top}\to\mathrm{Ab}$ is a generalised cohomology functor) Eg. $K^0(pt)=\mathbb{Z}$

When X noncompact, we define $K^0(X)$ as the compactification at pt, $K^0(X)=K^0(X^+)$ ~~[struck out]~~

Eg. Return back to $[\pi^*E_1,\pi^*E_2,\sigma]=D$ $X^+=X\sqcup\{pt\}$ Extending V to V^+ over X^+: Able to extension

we claim these two can be extending T^*M (x,ξ) M compact (x,ξ) M (T^*M not compact $\Rightarrow$ can't use Serre–Swan) $\Longleftrightarrow\exists K\subset X$, s.t. $V|_{X-K}\cong(X-K)\times\mathbb{C}^n$

to compactification $(T^*M)^+$

For this, we need: Atiyah–Bott–Shapiro construction : Case I Given $Y\subset X$ closed, V_1,V_2 over X (By the definition of the compactification

$V_1|_Y\cong V_2|_Y\Rightarrow$ an element $\in K^0(X/Y)$

Pf. Copy X_0, Y_0, X_1, Y_1, X_1, $Y_0=Y_1$ $[V_1,V_2,\sigma]$ gives V over $Z\Rightarrow 0\to K^0(Z/X_1)\to K^0(Z)\to K^0(X)\to 0$ (thin, it's split $\Rightarrow$ exact

called Z collapse $X_1\Rightarrow Z/X_1\cong X_0/Y=X/Y$ $K^0(X/Y)$ $[V]\mapsto 0$ then done by lifting ▨
lift (exact)

Case II ...

Let's put the K-theory later. Return back to $[\pi^*S^+, \pi^*S^-, \sigma = \sigma(D_A, \alpha)]$

Set $B_1(T^*M) = \{(x,\eta) \mid \|\eta\| \le 1\}$, $S_1(T^*M) = \partial B_1(T^*M) \Rightarrow B_1(T^*M) \cong T^*M$, thus reduce noncompact to compact, i.e.

construct a bundle according to $(T^*M)^+$ (one point compactification) $\Leftrightarrow$ trivial in infinity $\Leftrightarrow$ construct a bundle on $B_1(T^*M)$ trivial in neiberhood of $S_1(T^*M)$

We hope to gluing π^*S^+ & π^*S^- together to construct, by datum of translation functions

T^*M Observation ① Translation function of $\pi^*S^+ \simeq$ — of π^*S^- (homotopy by their same base space)

$\Rightarrow$ ② (translation function of $\pi^*S^-)^{-1} \cdot$ (translation function of $\pi^*S^+) \simeq$ Identity — denoted (φ_{ij}).

One asks that ① $(\varphi_{ij}) = \{\psi_{ij} \cdot \varphi_{ij}\}$ does cocycle? (Obviously if one replace "$\cdot$" by "$\circ$" (composition) it's not a cocycle

② Otherwise how to modify this?

(Let view $\varphi_i(V_i \cap V_j) \to \varphi_j(V_i \cap V_j)$)

but matrix function

$\Downarrow$

View a matrix function

$T_{ij}: \varphi_i(V_i \cap V_j) \to M_n(\mathbb{C})$

makes sense

$\Rightarrow T_{ij} \cdot (T_{ij})^{-1}$ does translation function

Expectation: (φ_{ij}) does a translation function (as trivial bundle) $\Leftrightarrow \varphi_{ij} \simeq Id \Leftrightarrow S^+ \cong S^-$

$\boxed{\text{Gluing } \pi^*S^+ \text{ \& } \pi^*S^- \text{ to a locally trivial bundle at } S_1(T^*M)}$

$\Rightarrow$ Gluing to $(T^*M)^+$'s vector bundle

$\Rightarrow \boxed{[\pi^*S^+, \pi^*S^-, \sigma] \in K^0(T^*M)}$

(Thus we glue this without ABS construction case II)

Setting. $D_A = \begin{array}{cc} \begin{pmatrix} 0 & D \\ D_A & 0 \end{pmatrix} & \begin{array}{c} L^2(S^+) \\ L^2(S^-) \end{array} \\ L^2(S^+)\ L^2(S^-) & \end{array}$

Projection operator $\Leftrightarrow$ closed subspace

Compact projection operator $\Leftrightarrow$ f.d. closed subspace

$\operatorname{tr} P_V = \dim V$

Functional calculus for unbounded operators:

$f(x) = \frac{x}{x+i}$ bounded continuous on $\mathbb{R} \supseteq \sigma(D_A)$

(Using $p_n \xrightarrow{\|\cdot\|_\infty} f$ in $\sigma(T)$, $\Rightarrow p_n(T) \xrightarrow{\|\cdot\|} f(T)$)

What about D_A unbounded? when bounded

Giving two projection operators $P_0: L^2(S) \to \mathrm{Graph}(D_-)$, $P_1: L^2(S) \to L^2(S^-)$

$P_0 - P_1 = \begin{pmatrix} (D_-D_+ + 1)^{-1} & D_-(D_+D_- + 1)^{-1} \\ D_A(D_A^2 + 1)^{-1} & (D_+D_- + 1)^{-1} \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(D_A^2 + 1)^{-1} + D_A(D_A^2+1)^{-1}$

$\Rightarrow P_0 - P_1$ (diagonal part) (antidiagonal part)

corresponds to finite dimensional compact closed subspace

$\boxed{D_A(D_A + iI)^{-1}}$ bounded, $\boxed{(D_A - iI)^{-1}}$ compact

$\forall \lambda \in \rho(D_A)$, $(D_A - \lambda I)^{-1}$ compact, D_A essentially self adjoint $\Rightarrow \sigma(D_A) \subset \mathbb{R}$

$\operatorname{tr}(P_0 - P_1) \stackrel{?}{=} \operatorname{Ind}(D_{A,0})$

Thus on analytic side, one has a number easy to deal with (index isn't good to deal) (commutation)

On the topological index side, we hope to give out a number also.

Idea. $M \times M \times I \overset{\text{Unit}}{\dashrightarrow} TM$

$(x, y_t, t) \dashrightarrow$ "$\lim_{t \to 0} \frac{y_t - x}{t}$"

Consider the Lie groupoid

$G = (M \times M \times (0,1]) \cup (TM)$

$G^{(0)} = M \times [0,1]$ embedded into G, called the tangent groupoid

$G \overset{s}{\underset{t}{\rightrightarrows}} G^{(0)} \hookrightarrow G$

$(x,y,\varepsilon) \overset{s}{\mapsto} (x,\varepsilon) \mapsto (x,x,\varepsilon)$, $\overset{t}{\mapsto} (y,\varepsilon) \mapsto (y,y,\varepsilon)$ if $\varepsilon > 0$

$(x,v) \overset{s}{\underset{t}{\mapsto}} (x,0) \mapsto (x,0)$: if $\varepsilon = 0$

$\Rightarrow$ operations $(x,y,\varepsilon) \circ (y,z,\varepsilon) = (x,z,\varepsilon)$

$(x,v) \circ (x,w) = (x, v+w)$

and $(x_n, y_n, \varepsilon_n) \to (x, X)$ with $X = \lim \frac{y_n - x_n}{\varepsilon_n}$

$\varepsilon \to 0$, degenerate to infinitesimal deformation

Generally Groupoid is $(G, G^{(0)}, s, t)$ satisfy some properties of s & t. (Generality for integral)

$G^{(0)} \subset G$, $G \overset{s}{\underset{t}{\rightrightarrows}} G^{(0)}$

$\Rightarrow G^{(2)} \subset G \times G$

$G^{(2)} = \{(\gamma_1, \gamma_2) \mid s(\gamma_2) = t(\gamma_1)\}$

and multiplication $\circ$

(i) $s(\gamma_1 \circ \gamma_2) = s(\gamma_1)$, $t(\gamma_1 \circ \gamma_2) = t(\gamma_2)$

(ii) $s(x) = t(x) = x \in G^{(0)}$

(iii) $\gamma \circ t(\gamma) = \gamma$; $s(\gamma) \circ \gamma = \gamma$

(iv) $(\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)$

(v) Inverse γ^{-1} exists two sided $\gamma \circ \gamma^{-1} = s(\gamma)$, $\gamma^{-1} \circ \gamma = t(\gamma)$

E.g. ① Given equivalence class

$\mathcal{R} \subset X \times X \Rightarrow (\mathcal{R}, \Delta, s = p_1, t = p_2)$

is groupoid;

② $(G, \{e\}, s, t)$ group is groupoid;

③ Given $G \curvearrowright X$, $X \rtimes G$ is groupoid

by $X \rtimes G = (X \times G, X \times \{e\}, id, \alpha)$

Let $G^x = s^{-1}(\{x\}) = \{x\}$

$G_x = t^{-1}(\{x\}) =$ (stabilizer)

G_x

Smooth groupoid (Ehresmann)

G, $G^{(0)}$ smooth & s, t are submersion & $G^{(0)} \hookrightarrow G$ smooth & $G^{(2)} \to G$ smooth

Rk. Such language is easy to proceed after G constructed (all are natural/functorial). Only hard part is construct G is analytic, others are all algebraic.

(Using G (noncommutative) to study M is idea of noncommutative geometry by Connes)

G discrete $\Rightarrow G^x$ discrete, a, b functions on G

$\Rightarrow (a * b)(\gamma) = \sum_{\gamma_1 \circ \gamma_2 = \gamma} a(\gamma_1) b(\gamma_2)$ (discrete) convolution

Generalize $C_c^\infty(G, \Omega^{\frac{1}{2}})$, we can define convolution as $(a * b)(\gamma) = \int a(\gamma_1) b(\gamma_2)$ as compact supported

$= \int a(\gamma_1) b(\gamma_1^{-1}\gamma)$, $\gamma_1 \circ \gamma_2 = \gamma$, $\gamma_1 \in G^{s(\gamma)}$

$\Rightarrow C_c^\infty(G, \Omega^{\frac{1}{2}})$ has convolution.

Now all set G smooth groupoid

Def. $\Omega^{\frac{1}{2}}$ over G is a vector bundle.

with fibre $\Omega^{\frac{1}{2}}_\gamma = \{\rho: \Lambda^k T_\gamma(G^x) \otimes \Lambda^k T_\gamma(G_y) \to \mathbb{C} \mid \rho(\lambda v) = \sqrt{|\lambda|}\rho(v), \lambda \in \mathbb{R}\}$

$(x = s(\gamma)$ & $y = t(\gamma))$, $k = \dim G^x = \dim G_y = \dim G - \dim G^{(0)}$. We call ρ is $\frac{1}{2}$-density

(supp $a \times$ supp $b) \cap (\gamma_1 \circ \gamma_2 = \gamma)$ compact

$\hookrightarrow$ It makes sense as $a(\gamma_1) b(\gamma_2)$ on $G^{s(\gamma)} \times G_{t(\gamma)}$, $\gamma_1 \circ \gamma_2 = \gamma$

and ρ as $\Lambda^k T_\gamma(G^x) \to (\Lambda^k T_\gamma(G_y))^*$

then $\gamma_1 \circ \gamma_2$ is composite of "linear" maps thus ... of T_γ is different

$a(\gamma_1) b(\gamma_2)$ is $\Lambda^k T_{\gamma_1}(G^{s(\gamma)}) \otimes \Lambda^k T_{\gamma}(G_{t(\gamma)}) \to \mathbb{C}$

(Not rigorous ①: two $\frac{1}{2}$-density is 1-density i.e. almost "linear" $\Rightarrow$ how to turn back to $\frac{1}{2}$-density?

$\Rightarrow$ It's $\sqrt{|\lambda|}$, thus need modify the coefficient and $\gamma_1 \circ \gamma_2 \neq \gamma$

Solution. Set $\Lambda^k T_{\gamma_1}(G^{s(\gamma)})$ & $\Lambda^k T_{\gamma_2}(G_{t(\gamma)})$ all 1

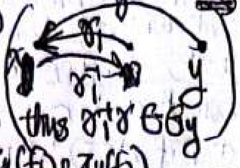
$\Rightarrow \rho(\lambda(x \otimes y)) = a(\lambda_1(x \otimes 1)) b(\lambda_2(1 \otimes y))$

$\sqrt{|\lambda_1|}\sqrt{|\lambda_2|}\rho(x \otimes y) = \sqrt{|\lambda|}\rho(x \otimes y)$

For C^*-algebra, we have involution for $\Gamma_c^\infty(G,\Omega^{\frac12})$ by $f^*: \gamma \mapsto \overline{f(\gamma^{-1})} \in \mathbb{C}$

For $\xi \in L^2(G_y)$, $f \in \Gamma_c^\infty(G,\Omega^{\frac12})$, define $\pi_y(f)(\xi) \in L^2(G_y)$

by $\pi_y(f)(\xi): \gamma \mapsto \int_{G^x} f(\gamma_1)\xi(\gamma_1^{-1}\gamma)$



thus $\gamma_1^{-1}\gamma \in G_y$

this gives rise to a $\Gamma_c^\infty(G,\Omega^{\frac12})$ — C^* algebra representation

i.e. $\Gamma_c^\infty(G,\Omega^{\frac12}) \xrightarrow{\pi_y} B(L^2(G_y))$

$f \longmapsto (\xi \mapsto \pi_y(f)(\xi)) = \pi_y(f)$

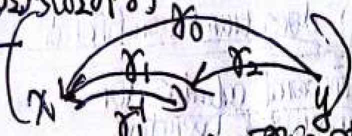
Check. ① $\pi_y(f_1 * f_2) = \pi_y(f_1)\circ\pi_y(f_2)$ ② $\pi_y(f^*) = (\pi_y(f))^*$

① $\pi_y(f_1*f_2)(\xi)(\gamma) = \int (f_1*f_2)(\gamma_1\gamma_2)\,\xi(\gamma_2^{-1}\gamma_1^{-1}\gamma)$

$\pi_y(f_1)\circ\pi_y(f_2)(\xi)(\gamma) = \int_{G^x} f_1(\gamma_1)(\pi_y(f_2))(\xi)(\gamma_1^{-1}\gamma)) = \int_{G^x} f_1(\gamma_1)\int_{G^{r(\gamma_1^{-1}\gamma)}} f_2(\gamma_2)\xi(\gamma_2^{-1}\gamma_1^{-1}\gamma)$

$(G^{r(\gamma_1^{-1}\gamma)} = G^{r(\gamma_1^{-1})} = G^{s(\gamma_1)}) = \int_{G^x}\int_{G^{s(\gamma_1)}} f_1(\gamma_1) f_2(\gamma_2)\xi(\gamma_2^{-1}\gamma_1^{-1}\gamma)$

and $(f_1*f_2)(\gamma_1\gamma_2) = \int_{G^{s(\gamma_1)}} f_1(\gamma_1)f_2(\gamma_2)$ ▨



⇒ is a priori: as we must need this representation to give $\Gamma_c^\infty(G,\Omega^{\frac12})$ does C^*-algebra: i.e. norm $\|f\| = \sup_{y\in G^{(0)}}\|\pi_y(f)\|_{(\text{operator norm})} \Rightarrow \|f^*f\| = 1$

thus we denote $C_r^*(G)$ as the closure

the reduced C^*-algebra of groupoid

Another norm makes $\Gamma_c^\infty(G,\Omega^{\frac12})$ into C^*-algebra comes from $C_r^*(G)$ is

$\|f\|_{\max} = \sup\{\|\pi(f)\|_{\text{operator norm}}, \pi \text{ is any } C^*\text{-algebra rep } (\pi(f^*) = (\pi(f))^*)\}$

$\geq \|f\|$ (nonempty by $C_r^*(G)$)

$\Rightarrow \overline{C_c^\infty(G,\Omega^{\frac12})}^{\|\cdot\|_{\max}}$ is C^*-algebra, denoted $C^*(G)$, the full C^*-algebra of groupoid

E.g. M compact, ($M\times M$, Δ, s,t natural) the

$\Rightarrow (f*g)(x,z) = \int f(x,y)\,g(y,z)\,d(x,y)$

$(G_y \cong M)$

$= \int_M f(x,y)\,g(y,z)\,dy$

$\Rightarrow (\pi_y(f)\xi)(x,y) = \int_M f(x,z)\,\xi(z,y)\,dz$

Observation. $\pi_y(f)$ is compact operator due (by finite rank approximate, comparing the Fredholm integral operator

the kernel f is continuous in compact subsets.

Conversely, all compact operators come from this.

by M compact, $\Rightarrow \Gamma_c^\infty(G,\Omega^{\frac12}) \cong C(M\times M) \cong K$

Furthermore, they isomorphic as C^*-algebra:

$\boxed{C_r^*(M\times M) \cong C^*(M\times M) \cong K}$

③ Return back to tangent groupoid $\mathbb{T}M = G$

E.g. $G = (M\times M\times(0,1]) \sqcup (TM) = G_1 \sqcup G_2$

G_1 open and G_2 closed

$\Rightarrow \Gamma_c^\infty(G,\Omega^{\frac12})$ "$=$" $\Gamma_c^\infty(G_1,\Omega^{\frac12}) \oplus$ "$\Gamma_c^\infty(G_2,\Omega^{\frac12})$

indeed we have (at least)

$$\begin{array}{ccccccccc} 0 \to & C^*(G_1) & \to & C^*(G) & \to & C^*(G_2) & \to 0 \\ & \cong & & \| & & \cong & \\ 0 \to & C_0(0,1]\otimes K & \to & C^*(G) & \to & \boxed{C^*(TM)} & \to 0 \end{array}$$

How can we relate $(x,X_1)\circ(x,X_2) = (x, X_1+X_2)$ with convolution on TM (= usual convolution in Fourier transformation)

$\Rightarrow C^*(T_xM) \cong C_0(T_x^*M)$ by Fourier transform "$\mapsto$" and Fourier inversion "$\leftarrow$"

$\Rightarrow 0 \to C_0(0,1]\otimes K \xrightarrow{R} C^*(G) \xrightarrow{\pi} C_0(T^*M) \to 0$

Summary ① On the one hand, we have $[\pi^*S^+, \pi^*S^-, \sigma] \in K^0(T^*M)$

② And Serre-Swan generalised to:

~~Compact~~ vector bundle over compact $\longleftrightarrow$ $C_0(M)$'s projection matrix, $P \in \mathrm{End}(C_0(M))$

$\rightsquigarrow$ vector bundle over locally compact $\longrightarrow$ $T \in K_0(C_0(M))$

Here setting is T^*M

thus $[\pi^*S^+, \pi^*S^-, \sigma] \in K^0(T^*M)$

$\updownarrow$

vector bundle trivial in T^*M outside compact

$\updownarrow$

$K_0(C_0(T^*M))$'s element!

④ by taking approximation of finite rank operators; (projective limit)

$\mathbb{C} \to M_2(\mathbb{C}) \to \cdots \to K$

and $K_0(M_n(\mathbb{C})) = \mathbb{Z} \Rightarrow K_0(K) = \mathbb{Z}$ ▨

$\boxed{K_0(C_0(T^*M)) \cong K^0(T^*M)}$

$K_0(C^*(M\times M\times\{1\})) \xleftarrow{\rho_*} \boxed{K_0(C^*(G)) \xrightarrow{\pi_*} K_0(C_0(T^*M))}$

induced by $\rho: M\times M\times\{1\} \hookrightarrow G$.

it's isomorphism, take its inverse to composite

$K_0(C^*(M\times M\times\{1\})) \cong K_0(K) \cong \mathbb{Z}$

$K^0(T^*M) \| \xrightarrow{\boxed{\mathrm{ind}_a}} \mathbb{Z}$ the analytic index

thus we defined the Ind

Pf of A-S thm. By immersion (may not Riemannian) (to preserve TM) $j: M \to \mathbb{R}^N \Rightarrow h: \mathbb{T}M \to \mathbb{R}^N$ is a homomorphism, then we have an action

$(x,y,\varepsilon) \mapsto \frac{j(x)-j(y)}{\varepsilon}$

$(x,X) \mapsto dj_x(X)$

$\mathbb{T}M \curvearrowright \mathbb{R}^N$ by $\mathbb{T}M \times \mathbb{R}^N \xrightarrow{(h,\mathrm{id})} \mathbb{R}^N\times\mathbb{R}^N \xrightarrow{+} \mathbb{R}^N$

it's free & proper

Observation $B\mathbb{T}M = (\mathbb{T}M^{(0)}\times\mathbb{R}^N)/\sim \;\cong \mathbb{R}^N\times(0,1] \sqcup NM$ (the normal bundle)

And the classifying space $BG = BG_1 \sqcup BG_2$

$(x,X)\sim(y,Y) \Leftrightarrow x,y$ have same stratification; $\exists\,\gamma\in\mathbb{T}M$, $s(\gamma)=x$, $t(\gamma)=y$ and $X = Y + h(\gamma)$

(Recall $\mathbb{T}M^{(0)} = M\times[0,1] = M\times(0,1]\sqcup M$)

And by stratification preserved

$\Rightarrow BG_1 \cong \mathbb{R}^N\times(0,1]$, $BG_2 \cong NM$

We can write out precisely:

$\alpha: \mathbb{R}^N\times(0,1] \to BG_1$

$(X,\varepsilon) \mapsto ((x,\varepsilon),X)$

and as an infinitesimal BG_2, we define convergence $(X_\varepsilon,\varepsilon) \to (x,Y)$

We have a new groupoid $\mathbb{T}M_h = \mathbb{T}M\times\mathbb{R}^N$

with $\mathbb{T}M_h^{(0)} = \mathbb{T}M^{(0)}\times\mathbb{R}^N$

$s(\gamma,X) = (s(\gamma),X)$ $\quad (\gamma_1,X_1)\circ(\gamma_2,X_2)$

$t(\gamma,X) = (t(\gamma), X+h(\gamma)) \quad = (\gamma_1\circ\gamma_2, X_2)$

$\Rightarrow \mathbb{T}M_h = G_h = G_{1h}\sqcup G_{2h}$

BG_1 is (properly) contractible $\Rightarrow$ $K^0(BG_2) \cong K^0(BG)$

$\Rightarrow$ ... $K^0(BG) \to K^0(\mathbb{R}^N) \cong \mathbb{Z}$

⑤ On the other hand

$K^0(T^*M) \cong K_0(C^*(G_2)) \cong K_0(C^*(G_{2,h})) \cong K^0(BG_2)$

thus by this way, we have composite

$\Phi: K^0(T^*M) \to \mathbb{Z}$, then $\boxed{\Phi = \mathrm{Ind}_a\,(A\text{-}S)}$

⑥ Thom Isomorphism $K^0(T^*M) \cong K^0(NM)$ (τ)

$\Phi = \beta\circ\psi\circ\tau$ is topological ▨

(Taking N even, by Bott periodicity, β)

第一讲 Fredholm 算子

2/26

§1 Fredholm Alternative

1900年, Erik Ivar Fredholm (1866–1927, 瑞典数学家) 研究了这样的问题. (The classical "Fredholm operator")

设 $K: [a,b]\times[a,b] \to \mathbb{C}$ 是一个连续函数, 考虑积分方程

$$x(t) + \int_a^b K(t,s)x(s)ds = f(t)$$

左边关于未知函数 x 是线性的.

(1) $x \overset{T_K}{\longmapsto} x + \int K(t,s)x(s)ds$ 是一个从 $L^2([a,b])$ 到 $L^2([a,b])$ 的有界线性算子 (It's just a domination of integral.)

(2) 这还是一个紧算子 (习题).

Our setting is quite general: only Banach spaces (not Hilbert) and operators between different spaces

Idea. Using finite rank operators approximate the T_K. Precisely,

① Taking $K_n = \sum_{\lambda\in P_n} a_i \chi_{[x_i, x_{i+1}]}$

$\Rightarrow T_{K_n}$ corresponding a matrix $\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$ diagonal $\Rightarrow$ finite rank.

② $K_n \xrightarrow{\|\cdot\|_2} K$

$\Rightarrow T_{K_n} \xrightarrow[\text{norm}]{\text{operator}} T_K$

$\|(T_{K_n} - T_K)(x)\|_{L^2}$

$= \|\int (K_n - K)(t,s)x(s)ds\|_{L^2}$

$\le \underline{\|\int (K_n-K)(t,s)x(s)ds\|_{L^2}} \xrightarrow[\text{set } \|x\|_{L^2}=1]{} 0$

$\Rightarrow \|T_{K_n} - T_K\| \to 0$ by definition of operator norm ▨

定理 (Fredholm Alternative): 设 X 是一个 Banach 空间, A 是 X 上的一个紧算子. 则下述两种情形:

(1) $\forall x\in X$, $\exists!\, y\in X$, $y = x + Ay$ $\Leftrightarrow$ $(I-A)$ invertible

(2) $\exists\, y\neq 0$, $y = Ay$ $\Leftrightarrow$ $\mathrm{Ker}(I-A)\neq\{0\}$

有且只有一种会发生.

thus $\mathrm{Ker}(I-A)=\{0\} \Leftrightarrow (I-A)$ invertible, the "$\Leftarrow$" is trivial.

$\Rightarrow$ **引理1.** 若 A 是紧算子, 且 $\ker(I-A)=\{0\}$, 则存在 $\varepsilon>0$, 使得 $\|(I-A)x\| > \varepsilon\|x\|$ $(\forall x\in X)$

证明: 若不然, 则有 $x_n\in X$, $\|x_n\|=1$, $\|(I-A)x_n\| \le \frac{1}{n}$

即 $(I-A)x_n \longrightarrow 0$

A 紧 $\Rightarrow \exists\, n_j$, $Ax_{n_j} \to x_\infty$, 则此时

$$x_{n_j} = (I-A)x_{n_j} + Ax_{n_j} \longrightarrow x_\infty \Rightarrow \|x_\infty\| = 1$$

$$\Rightarrow \|(I-A)x_\infty\| = \lim_j \|(I-A)x_{n_j}\| = 0$$

$$\Rightarrow 0 \neq x_\infty \in \ker(I-A),\quad \text{矛盾}$$

注. 这说明 $I-A: X \to \mathrm{Ran}(I-A)$ 是可逆的, then it suffices to prove the surjectivity of $(I-A)$

我们知道

Riesz引理：设 Y 是 X 的一个闭线性子空间，对于任意 $\varepsilon>0$，存在 $x\in X$，$\|x\|=1$ 使得 $d(x,Y)\geq 1-\varepsilon$.

引理2. 设 $C\in B(X)$，若 $\ker(C)=\{0\}$，且 $\mathrm{Ran}\, C\neq X$，则对于任意 n，$\mathrm{Ran}(C^{n+1})\subsetneq \mathrm{Ran}(C^n)$

证明：首先，$\mathrm{Ran}(C^{n+1})\subseteq \mathrm{Ran}(C^n)$

其次，取 $u\in X\setminus \mathrm{Ran}\, C$，$u\neq 0$，$C^n u\in \mathrm{Ran}\, C^n$

若 $C^n u = C^{n+1}w$，因为 C 是单射，所以 $C^{n-1}u = C^n w$，

依次类推，可得矛盾.

Such argument of $\cdots\subsetneq \mathrm{Ran}(C^{n+1})\subsetneq \mathrm{Ran}(C^n)\subsetneq\cdots$ stable (or kernel) is standard using Riesz Lemma.

性质1. 设 A 是紧算子，$\ker(I-A)=\{0\}$，则 $\mathrm{Ran}(I-A)=X$.

证明：记 $C=I-A$，设 $\mathrm{Ran}(C)\neq X$. 由引理2，我们有 X 的一列闭子空间 $Y_n=\mathrm{Ran}(C^n)$

$$\cdots\subsetneq Y_{n+1}\subsetneq Y_n\subsetneq Y_{n-1}\subsetneq\cdots$$

$\Rightarrow \exists\, y_n\in \mathrm{Ran}(C^n)$，$\|y_n\|=1$，$d(y_n, \mathrm{Ran}(C^m))\geq \frac{1}{2}$ $\forall m>n$ （在Riesz引理中取 $\varepsilon=\frac{1}{2}$）

而 $Ay_n = y_n - Cy_n$

当 $m>n$ 时，（因为 $[A, C^m]=0$）$Ay_m\in \mathrm{Ran}(C^m)\subset \mathrm{Ran}(C^{n+1})$

于是 $Ay_n - Ay_m = y_n - Cy_n - Ay_m \in y_n - \mathrm{Ran}(C^{n+1})$

$\Rightarrow \|Ay_n - Ay_m\|\geq\frac{1}{2}$ $(\forall n\neq m)$ 矛盾

这样，我们证明了：如果定理中的情形(2)不发生，那么情形(1)必然发生

$(\Longleftarrow)$ 如何证明情形(2)发生时，情形(1)必然不会发生？

By (2) $\Rightarrow \exists\, y\neq 0\in X$, $(I-A)y=0=(I-A)0$, thus for $0\in X$, the corresponding $y\neq 0$ isn't uniq

$\Rightarrow$ contradict to (1) $\boxed{/}$

§2. Fredholm 算子.

在把有限维空间(上的线性变换)的性质"搬"到无限维空间上去的时候:

(1) 如果我们考虑 $\dim \mathrm{Ran}T < +\infty$, 就有了有限秩算子的概念.
(进而可以考虑紧算子)

(2) 如果我们(对于 $T: E \to F$)考虑的是 $\dim\ker T + \dim \mathrm{Ran}T = \dim E$
其等价形式是 $\dim\ker T - (\dim F - \dim \mathrm{Ran}T) = \dim E - \dim F$.
当 $\dim E, \dim \mathrm{Ran}T, \dim F = +\infty$ 时, 该式还可能是有意义的.

定义 设 X, Y 是 Banach 空间, $T \in B(X,Y)$. 称 T 是一个 Fredholm 算子
若 $\dim\ker T < +\infty$ 且 $\mathrm{Ran}T$ 有有限维补空间 ~~(或者说~~ (Such description makes a kernel more subtle)
(或者说 $\dim\{l \in Y^*, l(\mathrm{Ran}T) = 0\} < +\infty$)
称 $\mathrm{ind}\,T = \dim\ker T - \mathrm{codim}\,\mathrm{Ran}T$ 为 T 的(Fredholm)指标.

(课上没讲) 定理. 在上述定义中, $\mathrm{Ran}T$ 一定是闭的.

证明. 设 Y 的有限维(从而一定是闭的)子空间 Z 满足 $Y = \mathrm{Ran}T \oplus Z$.
记 $\tilde{X} = X \oplus Z$, 定义 $\tilde{T}: \tilde{X} \to Y$, $\tilde{T}(x, z) = Tx + z$.
(不妨取 $\|(x,z)\|_{\tilde{X}} = (\|x\|_X^2 + \|z\|_Z^2)^{1/2}$) 显然 $\tilde{T}$ 连续, $\mathrm{Ran}\tilde{T} = Y$.
由开映照定理, $\tilde{T}(\{\tilde{x}, \|\tilde{x}\|_{\tilde{X}} < 1\})$ 包含 Y 中以原点为球心, 半径为 ε 的一个球. 即
$$\forall y \in Y, \exists \tilde{x} \in \tilde{X}, \|\tilde{x}\| \le \varepsilon^{-1}\|y\|, \quad \tilde{T}\tilde{x} = y.$$
特别地, 当 $y \in \mathrm{Ran}T$ 时, 这个 $\tilde{x}$ 在 X 中 (即 z 分量为0)
设 $y_\infty \in \overline{\mathrm{Ran}T}$, 则存在一列 $x_n \in X$, $Tx_n \to y_\infty$ 且 $\|Tx_{n+1} - Tx_n\| \le 2^{-n}$
由上式, 存在 $x_n^\# \in X$, 使得 $Tx_n^\# = Tx_n$ 且 $\|x_{n+1}^\# - x_n^\#\| \le 2^{-n} \cdot \varepsilon^{-1}$.
所以 $\{x_n^\#\}$ 是 Cauchy 列, 从而有极限 $x_\infty^\#$, 于是
$y_\infty = \lim Tx_n^\# = Tx_\infty^\#$, $y_\infty \in \mathrm{Ran}T$, 所以 $\mathrm{Ran}T$ 是闭的.

Some texts adding "the closed range" condition into the definition of Fredholm
i.e. three conditions
① Closed image;
② $\dim\ker T < \infty$;
③ $\dim\mathrm{coker}\,T < \infty$;
Three in fact has 2-to-3 property!

Recall. (Closed range thm)
T^* the transpose of T
TAFE ① $\mathrm{Ran}(T)$ closed
② $R(T^*)$ closed;
③ $\mathrm{Ran}(T) = \ker(T^*)^\perp$
④ $\mathrm{Ran}(T^*) = \ker(T)^\perp$

Now due to $\mathrm{codim}\,\mathrm{Ran}(T) = \dim\mathrm{coker}(T) = \dim\ker(T^*) < \infty$
$\Longrightarrow \ker(T^*)$ finite-dimensional,
thus holds ③ $\Longleftrightarrow$ ① ▨

例. $l^2(\mathbb{N})$ 上的左、右平移.

注. Fredholm 在积分方程中"看到"的形如 $I+A$ (A 紧) 的算子, 都是 Fredholm 算子, 但指标都是 0. 到了 1921 年, Fritz Noether (1866-19 Emy Noether 的弟弟) 才造出了指标不为 0 的例子.

Notice that here $X\neq Y$ making the pseudo inverses' formulas complexer, which isn't outlined in Functional analysis.

定义. 对于 $T\in B(X,Y)$, 我们称 $S\in B(Y,X)$ 是 T 的一个拟逆 (pseudo inverse), 若存在 $K\in\mathcal{K}(X)$, $L\in\mathcal{K}(Y)$, 使得

$$(*)\qquad ST=I_X+K,\qquad TS=I_Y+L.$$

性质 2. 若 $S_1T=I_X+K$, $TS_2=I_Y+L$ 则 $TS_1-I_Y\in\mathcal{K}(Y)$

证明. $S_1TS_2=S_1+S_1L=S_2+KS_2$

$\Rightarrow\quad S_2-S_1=S_1L-KS_2\quad\in\mathcal{K}(Y,X)$

$\Rightarrow\quad TS_1-I_Y=TS_2+T(S_1-S_2)-I_Y=L+T(S_1-S_2)\in\mathcal{K}(Y).$

定理 (Atkinson) 对于 $T\in B(X,Y)$, 下面三条等价.

(1) T 是 Fredholm 算子

(2) T 有拟逆.

(3) 上述 $(*)$ 式中的 K 和 L 可以取有限秩算子

证明: (1) $\Rightarrow$ (3) $\overset{\text{显然}}{\Longrightarrow}$ (2) $\Rightarrow$ (1)

(1) $\Rightarrow$ (3). 取有限维子空间 $\ker T$ 的一个补空间 Z (即对于 $x\in X$ 有唯一分解 $x=x_1+x_2$, $x_1\in\ker T$, $x_2\in Z$). 定义 $Px=x_1$. 则 $P^2=P$, $\mathrm{Ran}(I-P)=Z$. $T|_Z: Z\to\mathrm{Ran}\,T$ 连续双射. 因为 $\mathrm{Ran}\,T$ 闭, 从而逆映射定理告诉我们存在逆映射 $\tilde{S}:\mathrm{Ran}\,T\to Z$ 取 $\mathrm{Ran}\,T$ 的一个补空间 W, Q 是 Y 向 W 的投影 $\mathrm{Ran}(I-Q)=(\mathrm{Ran}$

定义 $S: Y\to X$, 可知 $ST=I-P$

$\eta\mapsto\tilde{S}(I-Q)\eta$ $\qquad TS=I-Q$

(2) ⇒ (1). 由 $ST=I_X+K$ 可知，若 $Tx=0$ 则 $(I+K)x=0$.

这样 $\ker T \subset \ker(I+K) = \sigma(K)$ 中 -1 的重数 $<+\infty$.

同样，$TS=I_Y+L$，$\mathrm{Ran}\, T \supset \mathrm{Ran}(I+L)$，于是

$$l\in Y^*,\ l(\mathrm{Ran}\, T)=0 \Rightarrow l(\mathrm{Ran}(I+L))=0$$
$$\Rightarrow \forall y\in Y,\ l(y)=-l(Ly)=-(L^T l)(y)$$
$$\Rightarrow l \in \ker(I_{Y^*}+L^T)$$ 有限维的.

$\Rightarrow \operatorname{coker} T \subset \ker(I_{Y^*}+L^T) < \infty$ ▨

Recall Riesz-Schauder thm mainly tells us that: ① $\sigma(T)$ contains $\{0\}\cup \boxed{\sigma_p(T)}$ ⇒ discrete; (infinite-dimensional) ② $\sigma(T^*)=\sigma(T)$; dim ker$(\lambda I-T)$=dim ker$(\bar{\lambda}I-T^*)$ … Here it means to prove the ① Note that: the key of original proof of ① is the Riesz Lemma. [Precisely: (i) Infinite dimensional ball not compact; (ii) Using $Y_{n+1}\subsetneq Y_n \subsetneq Y_{n-1}$ … in Range or Kernel.]

习题.

1. 如何从§1中陈述的结论推导紧算子谱的 Riesz-Schauder 定理.

2. (a) 设 $f\in L^2([-1,1])$，$f\geq 0$. 证明，存在 $g\in L^2([-1,1])$，
$g(x)\geq f(x)$ (a.e.) 且 $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=0$

(b) 由此证明 $\{xf,\ f\in L^2([-1,1])\}$ 的补空间不是有限维的.

3. 设 $\mathcal{H}$ 是 Hilbert 空间，$A\in B(\mathcal{H})$ 且存在 $\{\varphi_n\}_{n=1}^{\infty}\in\mathcal{H}$，
$\varphi_n \xrightarrow{w} 0$ (即 $\forall y\in\mathcal{H}$, $(\varphi_n,y)\longrightarrow 0$)，$\|\varphi_n\|=1$，$\|A\varphi_n\|\to 0$.
证明，A 不是 Fredholm 算子.

4. 证明：$x\longmapsto x(t)+\int_a^b K(t,s)x(s)ds$ 是个紧算子. $L^2([a,b])\to L^2([a,b])$
(~~Done in the first page~~)

Exercises:

1. $\forall \lambda\in\sigma(T)$, $\lambda I-A=\lambda(I-A/\lambda)$ (Here when $\lambda\neq 0$), applying Fredholm alternative to $I-A/\lambda$ (A/λ also compact)
⇒ once $\mathrm{Ran}(\lambda I-A)$ not surjective, then $\mathrm{Ker}(\lambda I-A)$ not injective ⇒ $\lambda\in\sigma_p(T)$ ▨

2. (a) Taking $g(x)=\frac{f(x)}{|x|^{\frac14}}\geq f(x)$ as $|x|^{\frac14}\leq 1$ and $\|g\|_2\leq\|f\|_2\|\frac{1}{|x|^{\frac14}}\|_2<\infty$
⇒ $g\in L^2([-1,1])$ and $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}|x|^{\frac14}=0$ ▨

(b) By taking $h_\varepsilon=\frac{|x|^\varepsilon}{f}\in L^2([-1,1])$, where $\varepsilon<\frac12$, then $h_\varepsilon\perp xf$ as $x|x|^\varepsilon$ is odd function but h_ε can't be spanned by finite L^2-functions ▨

3. Setting $\overline{\langle A^m\varphi_n\rangle}\subset\mathcal{H}$ a seperable Hilbert subspace of $\mathcal{H}$, then we replace $\mathcal{H}$ by it, without change φ_n's properties
when $\dim\overline{\langle A^m\varphi_n\rangle}<\infty$, $\varphi_n\xrightarrow{\omega}0 \Leftrightarrow \varphi_n\to 0$ ⇒ contradiction ⇒ Assume $\mathcal{H}$ is an infinite-dimensional seperable Hilbert space ⇒ $\mathcal{H}\cong l^2(\mathbb{Z})$.
And set $\mathcal{H}=\mathrm{Ker}\,A\oplus\mathcal{H}_1=\mathcal{H}_2\oplus\mathrm{Im}\,A$ ⇒ $A=\begin{array}{cc|c} & \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} & \begin{array}{c}\ker A\\ \mathcal{H}_1\end{array} \\ & \begin{array}{cc} \mathcal{H}_2 & \mathrm{Ran}(A)\end{array} & \end{array}$ with D invertible (inverse operator thm)
$\varphi_n=(\cdots a_0^n, a_1^n\cdots)$, when A is Fredholm, the matrix is finite 0
⇒ impossible to give out $\varphi_n=(\cdots 0,1,0\cdots)$ with $\|A\varphi_n\|\to 0$ ⇒ A not Fredholm ▨
(Only when 0's ~~imp~~ infinite dimensional, we can give out φ_n by taking it's basis).

第一讲(续) 3/5

在Atkinson定理的证明中, $ST=I_X-P\ (P^2=P)$, $TS=I_Y-Q\ (Q^2=Q)$

此时, $\mathrm{ind}\,T=\dim(\ker(I_X-P))-\dim(\ker(I_Y-Q))$

一般地, 若一个Fredholm算子 T 和它的一个拟逆 S 满足

$$\begin{cases} ST=I_X+K & \ker(I_X+K)=\ker(I_X+K)^2 \\ TS=I_Y+L & \ker(I_Y+L)=\ker(I_Y+L)^2 \end{cases}$$

则 $\mathrm{ind}\,T=\dim\ker(I_X+K)-\dim\ker(I_Y+L)$

进而可得 $\mathrm{ind}\,S=-\mathrm{ind}\,T$. [证明与本课程主题关联较小,略]

And T Fredholm $\Leftrightarrow$ S (its pseudo-inverse) also Fredholm // When setting on right holds, we prove the assertion by observing that: $\mathrm{Ran}(K)=\mathrm{Ker}(T)$, $\mathrm{Ker}(I+L)=\mathrm{coker}(T)$ $\Rightarrow \mathrm{Ind}(T)=\dim\ker(T)-\dim\mathrm{coker}(T)$ $=\dim\mathrm{Ran}(K)-\dim\ker(I+L)$ $=\dim\ker(I+K)-\dim\ker(I+L)$. One can also use $\mathrm{Ind}\,S+\mathrm{Ind}\,T=\mathrm{Ind}\,ST$ $=\mathrm{Ind}(I_X+K)=\mathrm{Ind}(I)=0$ as a priori.

§3 Fredholm算子(的指标)在(微小)扰动下的稳定性

我们先对于 $X\cong X_1\oplus X_2$, $Y\cong Y_1\oplus Y_2$, 考虑 $T: X\to Y$

$$T=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$$

We consider the partitioned matrix for later application to Dirac operator $\begin{pmatrix} 0 & D_0 \\ D_1 & 0\end{pmatrix}=D$, with

性质. (1) 若 $B=0$, $C=0$ (它们作用于不同的空间上, 因此我们不写 $B=C=0$)

则 T Fredholm $\Leftrightarrow$ A 和 D 都是Fredholm的, $\mathrm{Ind}\,D=\mathrm{Ind}(D_0)-\mathrm{Ind}(D_1)=0$

且此时 $\mathrm{ind}\,T=\mathrm{ind}\,A+\mathrm{ind}\,D$

thus must use D_0 replace itself.

(2) 设 D 可逆, 则

$$\underbrace{\begin{pmatrix} I_{Y_1} & -BD^{-1} \\ 0 & I_{Y_2}\end{pmatrix}}_{\text{可逆}}\begin{pmatrix} A & B \\ C & D\end{pmatrix}\underbrace{\begin{pmatrix} I_{X_1} & 0 \\ -D^{-1}C & I_{X_2}\end{pmatrix}}_{\text{可逆}}=\begin{pmatrix} A-BD^{-1}C & 0 \\ 0 & D\end{pmatrix}$$

故 T Fredholm $\Leftrightarrow$ $A-BD^{-1}C$ Fredholm

且此时 $\mathrm{ind}\,T=\mathrm{ind}(A-BD^{-1}C)$ $\Longrightarrow$ taking $A=D=0$, then $\mathrm{Ind}\,T=0$

(3) 设 D 可逆, $\dim X_1<\infty$, $\dim Y_1<\infty$, 则 T Fredholm

$$\mathrm{ind}\,T=\dim X_1-\dim Y_1$$

(因为有限维空间之间的映射都是Fredholm的)

Stability of index

- Norm invariance

$\Downarrow$

- Path invariance (homotopy invariance)

$\Downarrow$

- Compact invariance

J. Dieudonné. Sur les Homomorphismes d'Espaces Normés, Bull. Sci. Math 1943, 72-84

Combining with Sobolev compact embedding, we can find our key step later.

定理 (Dieudonné, 1943) 设 $T: X \to Y$ Fredholm, 则存在 $\varepsilon > 0$, 使得 $\forall T'$, $\|T' - T\| < \varepsilon \Rightarrow T'$ Fredholm, 且 $\operatorname{ind} T' = \operatorname{ind} T$.

证明: 我们记 $X_1 = \operatorname{Ker} T$, $Y_2 = \operatorname{Ran} T$, X_2 和 Y_1 分别是它们的闭的补空间, 这样 $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$. $T = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$

D 是 X_2 到 Y_2 的双射, 由逆映照定理, D 可逆.

记 $T' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ 我们有 $T'\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \begin{pmatrix} B'x_2 \\ D'x_2 \end{pmatrix}$

因为我们处理的是 Banach 空间, 所以我们做不到 $\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\| \geq \left\|\begin{pmatrix} 0 \\ y \end{pmatrix}\right\|$.

我们有 $(T' - T)\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \begin{pmatrix} B'x_2 \\ (D' - D)x_2 \end{pmatrix}$

~~注意到 $B': X_2 \to Y_1$, $\dim Y_1 < +\infty$, 我们考虑 $\operatorname{Ker} B' =: X_{22}$~~

~~取它的一个(在 X_2 中)闭的补空间 X_{21}, 即 $X_2 = X_{21} \oplus X_{22}$~~

~~由 D 是可逆双射, 定义 $Y_{21} = DX_{21}$, $Y_{22} = DX_{22}$, 则 $Y_2 = Y_{21} \oplus Y_{22}$~~

~~我们记 $D = \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix}$~~

在 Dieudonné 的原文中, 他也没注意到这个问题.

因为 Y_1 中的单位球面是紧的

因为 Y_1 是有限维的, Y_2 是闭的, 所以 Y_1 中的单位球面到 Y_2 的距离有正的下(确)界. 即 $\left\|\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\| \geq k\left\|\begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right\|$

从而 $\left\|\begin{pmatrix} 0 \\ y_2 \end{pmatrix}\right\| \leq \left\|\begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right\| + \left\|\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\| \leq \left(\frac{1}{k} + 1\right)\left\|\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\|$

所以 $\|D' - D\| \leq (1 + \frac{1}{k})\|T' - T\|$, 这样当 $\|T' - T\|$ 很小时, $\|D' - D\|$ 也很小 (小到 $D' = (D' - D) + D$ 可逆)

这时利用上一条性质的(3), 即可得证.

感谢王炲宇同学指出这里的问题.

在$(B(X,Y),\|\cdot\|)$中，

It's useful to write as $\pi_0(\mathcal{F}(X,Y)) \xrightarrow{ind} \mathbb{Z}$ due to the later topological (homotopy) theory advise us to view $\mathcal{F}(X,Y)$ as the topological space we focus!!

推论 ind 在$\underbrace{\{\text{Fredholm算子}\}}_{=:\mathcal{F}(X,Y)}$的每一个(道路)连通分支上是常数。

是一个开集$\subset B(X,Y)$; Open by $\forall T$, the neiberhood $B_T(\varepsilon) \subset \mathcal{F}(X,Y)$

推论 设$\gamma:[0,1]\to B(X,Y)$是关于范数拓扑连续的道路，$\gamma(t)$ Fredholm ($\forall t\in[0,1]$)，则 $ind\,\gamma(0) = ind\,\gamma(1)$.

推论 设 T Fredholm，K 紧 则 $T+K$ Fredholm 且 $ind(T+K) = ind\,T$.

注：此时对$\|K\|$无限制。Pf. We can take the path $(T+tK)_{t\in[0,1]}$; here we need K compact to ensure $T+tK\in\mathcal{F}(X,Y)$ ▨

The compact invariance is most important: Sobolev embedding $\Rightarrow$ lower terms of symbol are compact $\Rightarrow$ $\boxed{\text{index only depends on the principal symbol}}$

§4 Fredholm指标的可加性

我们设$T:X\to Y$，$S:Y\to Z$ Fredholm

则 $\begin{matrix}T\\ \oplus\\ S\end{matrix} : \begin{matrix}X\\ \oplus\\ Y\end{matrix} \longrightarrow \begin{matrix}Y\\ \oplus\\ Z\end{matrix}$ 也是Fredholm算子，且 $ind(T\oplus S) = ind\,T + ind\,S$

这个算子可以写成2x2矩阵的形式 $\begin{pmatrix}T & 0\\ 0 & S\end{pmatrix}$

而 $\begin{pmatrix}0 & \text{可逆}\\ ST & 0\end{pmatrix}$ 会是一个指标为$ind(ST)$的Fredholm算子

<u>如果ST是一个Fredholm算子的话</u>

$\ker(ST) = \underbrace{T^{-1}}_{\text{原像}}(\ker S) = \ker T \oplus \underbrace{T^{-1}}_{}(\ker S\cap Ran\,T)$

T视为$\ker T$的补空间到$Ran\,T$的双射

$Ran(ST) = S(Ran\,T)$ 它的一个补空间是

$Ran\,S$的补空间 $\oplus$ S ($Ran\,T$的补 $\cap$ $\ker S$的补)

定理 $ind\,S + ind\,T = ind\,ST$.

思路：把上述两个2x2矩阵(可加上适当小扰动)通过<u>可逆矩阵</u>联系起来，就能证明它们的指标相同。

方法：凑.

$$\begin{matrix} Y \\ \oplus \\ Z \end{matrix} \rightarrow \begin{matrix} Y \\ \oplus \\ Z \end{matrix} \qquad \begin{matrix} X \\ \oplus \\ Y \end{matrix} \rightarrow \begin{matrix} X \\ \oplus \\ Y \end{matrix}$$

$$\begin{pmatrix} \varepsilon I_Y & 0 \\ S & \varepsilon I_Z \end{pmatrix}\begin{pmatrix} \cancel{S}T & -\varepsilon I_Y \\ 0 & S \end{pmatrix}\begin{pmatrix} \varepsilon I_X & 0 \\ T & I_Y \end{pmatrix} = \varepsilon\begin{pmatrix} & \varepsilon I_Y \\ ST & \end{pmatrix}$$

Both left and right matrixes are invertible when ε small $\Rightarrow$ not change index.

This example will provide a most trivial example for Atiyah-Singer index thm.

§5 例: S^1 上的 Toeplitz 算子 $\Rightarrow$ ind ST = ind RHS = ind LHS = ind S + ind T ▨

考虑 Hilbert 空间 $H = L^2(S^1, \frac{d\theta}{2\pi})$, 对于 $f \in L^\infty(S^1, \frac{d\theta}{2\pi})$

定义 $M_f: H \to H$, $g \mapsto f\cdot g$ 那么自然有 $\|M_f g\|_{L^2} \leq \|f\|_{L^\infty}\|g\|_{L^2}$

In this case, we have 由 Fourier 级数理论, H 有一组自然的标准正交基 $\{e^{in\theta}\}_{n=-\infty}^{\infty}$

SES: 设内积

$0 \to \mathbb{K} \to$ ~~紧算子~~ $\to 0$

$$\langle e^{in\theta}, M_f(e^{im\theta})\rangle = \int_{S^1} f(e^{i\theta})e^{i(m-n)\theta}\frac{d\theta}{2\pi} = a_{n-m}$$

compact operators 线性 Toeplitz algebra is the degenerate case of 设 $f(e^{i\theta}) \sim \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$

$\langle T\varphi + K\rangle$ forget K 从而在 L^2 中, 当我们把每个元素视为一个双边无限的数组时,

$0 \to \mathbb{K} \to \mathcal{T}^*(H) \to C(T^*M)$ used in A-S M_f 的 "Fourier 系数" 总是能"读"出来的.

L^2 中的等号

pseudo-differential operator for principal symbol. (Here $\sigma(T\varphi+K) = \varphi$ is the principal symbol of $T\varphi+K$, and the symbol of $T\varphi$).

H 我们考虑 $L^2(S^1, \frac{d\theta}{2\pi})$ 的子空间 $H^2(S^1, \frac{d\theta}{2\pi}) = \{g = \sum b_k e^{ik\theta}, \ b_k = 0 \ \forall k<0\}$

forget G.H.Hardy lower terms ~~[illegible]~~ 记 $P_+: L^2 \to H^2$ 为正交投影

Such spaces H^2 are called Hardy space. We'll not use it later as it abuses notation of Sobolev spaces.

我们定义 ① $T_f = P_+ \circ M_f: H^2 \subset L^2 \to H^2$ called T_f the Toeplitz operators

One can find ② $T_{fg} \overset{?}{=} T_f T_g$ The Fredholm properties of T_f? ③ Index theories: Ind T_{fg} = Ind T_f + Ind T_g?

此时, 关于 H^2 的标准正交基 $\{e^{im\theta}, m \geq 0\}$, T_f 可以表达为无限矩阵

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & \\ a_1 & a_0 & a_{-1} & \\ a_2 & a_1 & a_0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

(这样的矩阵叫 Toeplitz 矩阵)

(Operator is Toeplitz $\Longleftrightarrow$ the matrix is of constant diagonal)

问题: 对怎样的 $f \in C(S^1)$, T_f 是 Fredholm 的?

记 $P_- = (I - P_+)$, $\underbrace{\text{容易看到 } P_- M_{f_m} P_+ \text{ 是有限秩算子}}_{\text{对于形如 } f_m(e^{i\theta}) = e^{im\theta} \text{ 的单项式}}$

① 如何证明?

从而对于三角多项式 f, $P_- M_f P_+$ 也是有限秩的.

By ① $f_m: z \mapsto z^m$ then M_{f_m} corresponds to the translation operator in L^2; with the "cutting" of P_+ and P_-, it turns out to be $(0 \cdots 0, a_0, a_1 \cdots) \xrightarrow{P_-} (0 \cdots 0, a_0, a_1 \cdots a_m, 0 \cdots)$ $\xleftarrow{M_{f_m}}$ $-m$ 0

由 $\|M_f - M_g\| \leq \|f-g\|_{L^\infty}$, 由 Weierstrass 第二逼近定理② 对任意 $f \in C(S^1)$, $P_- M_f P_+$ 紧

∴ this result is of finite rank $\leq m$ ▨

② Proof of Weierstrass's second approximation thm

Using Stone-Weierstrass, consider $\mathcal{A} = \{f \mid f(e^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}\} \subset L^2(S^1, \frac{d\theta}{2\pi})$ subalgebra

It seperates points as $e^{i\theta}$ seperates points; it vanishes nowhere as 1 does; However, it's not self-adjoint! Thus we set $\mathcal{A}' = \{f \mid f(e^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}\}$ then $1 \notin \mathcal{A}'$! Setting $\mathcal{A} = \langle \mathcal{A}' \cup \{1\}\rangle$ is the right method ▨

注意到 $M_f M_g = M_{fg}$，$T_f = P_+ M_f P_+$ 有

$$T_f T_g - T_{fg} = P_+ M_f P_+ M_g P_+ - P_+ M_f M_g P_+ = -P_+ M_f P_- M_g P_+$$

对任意连续的 f, g 都是紧算子。

thus $\mathrm{Ind}(T_f T_g) = \mathrm{Ind}(T_f) + \mathrm{Ind}(T_g) = \mathrm{Ind}(T_{fg})$

设 $f \in C(S^1)$，$f(e^{i\theta}) \neq 0\ (\forall\theta)$，则 $g = \frac{1}{f} \in C(S^1)$，

T_g 是 T_f 的一个拟逆，从而 T_f 是 Fredholm 算子

对这样的一个 f，~~注意~~ $f(S^1)$ 是 $\mathbb{C} \setminus \{0\}$ 中的一条闭曲线 Γ_f，从而一定同伦等价于 $f_m(S^1)$ $f_m(e^{i\theta}) = e^{im\theta}$，$m$ 是 Γ_f 关于原点的绕数 (winding number)，由 Fredholm 指标的同伦不变性，可得 $\mathrm{ind}(T_f) = \mathrm{ind}(T_{f_m}) = -m$。

习题.

1. 对于 §5 中的 $f \in C(S^1)$，$f(e^{i\theta}) \neq 0\ (\forall\theta)$，求 $\sigma(T_f)$

2. 若 $f \in C(S^1)$，$f(1) = 0$，(T_f is Fredholm $\Leftrightarrow$ f nowhere vanishing, the "$\Leftarrow$" is clear in upper argument, now this exercise proves "$\Rightarrow$" part)

(a) 求 g_n，$\|g_n\|_2 = 1$ 且 $\|f g_n\|_2 \xrightarrow{n\to\infty} 0$

(b) 对于 $k_n \uparrow +\infty$，找到适当的 g_n，使得 $\|P_+ z^{k_n} g_n\|_2 \longrightarrow 1$ 并由此得到 h_n，$\|h_n\|_2 = 1$，$h_n \in \mathrm{Ran}(P_+)$，且 $h_n \xrightarrow{w.} 0$，$\|T_f h_n\|_2 \to 0$

(c) 由此证明 T_f 不是 Fredholm 算子（提示：利用上次的习题 3）

3. 设 P, Q 是 Hilbert 空间上的投影算子. The difference of two projection operators is the key method to simplify the computing of analytic index: using $P_0 - P_1$ is finite rank $\Rightarrow$ computing its trace

(a) 设 $P-Q$ 是一个紧算子，若 $\ker(QP)$ 是有限维的，证明 $\mathrm{Ran}(I-P)$ 是有限维的（从而 $\mathrm{Ran}(I-Q)$ 也是）

(b) 若 $\mathrm{Ran}(I-P)$ 和 $\mathrm{Ran}(I-Q)$ 都是有限维的，证明 QP 是 Fredholm 算子，并且 $\mathrm{ind}(QP) = 0$（提示：$Q - QP$ 紧）

3.(a) $\mathrm{Ran}(I-P) = \ker P \subset \ker(QP)$

$\forall x = (I-P)y$

$Px = (P-P^2)y = 0$

(even not need $P-Q$ compact!)

(b) $\mathrm{Ran}(I-P) = \mathrm{coker}(P)$ $= \ker(P) \Rightarrow P, Q$ are both Fredholm operator with index 0 $\Rightarrow PQ$ also index 0

~~3.(a) In the proof of Atkinson thm's (b)⇒(a) From a $\tilde P$ with Ran($\tilde P$) f.d. we can construct $\tilde S T = I - \tilde P$, $T\tilde S = I - \tilde Q$~~

Exercises. 1. Given any nowhere vanishing function $f \in C(S^1)$, then $T_f T_{\frac{1}{f}} - T_1$, $T_{\frac{1}{f}} T_f - T_1$ are both compact, and $T_1 =$ Identity operator $\Rightarrow \mathrm{Ind}\, T_f + \mathrm{Ind}\, T_{\frac{1}{f}} = 0$

I claim, $\sigma(T_f) = \{z \in \mathbb{C} \mid \text{"the winding number of } f(S^1) = \Gamma_f \text{ around } z\text{"} \neq 0 \text{ or } z \in \Gamma_f\}$ (the winding number $= W_{\Gamma_f}(z)$)

~~this not depending on whether $f \neq 0$ or not~~, it's a theorem of Krein

(and we can determine $\sigma_c, \sigma_r, \sigma_p$ precisely)

2. (b)$\Rightarrow$(c) is trivial ($W_{\Gamma}(z) > 0$ Γ_f $W_{\Gamma}(z) < 0$) ∎ Only do (a) & (b);

(a) This can always be done by set $f_n \to f$ with $f_n = \sum a_n z^{n}$, then set $g_n \perp f_n$ ∎ (and normalised)

(b) g_n can be always found by taking G_n, s.t. $\|P_+ G_n\| \longrightarrow 1 \Rightarrow g_n = G_n / z^{k_n}$ then take $\tilde h_n = P_+ z^{k_n} g_n \Rightarrow \tilde h_n \xrightarrow{\omega} 0$ as $k_n \to \infty \Rightarrow$ set $h_n = \tilde h_n / \|\tilde h_n\|$, and $\|T_f h_n\| = \|P_+ f h_n\|_2 \leq \sum |a_n| \|P_+ z^{n} h_n\|$

due to $f(1) = \sum |a_n| = 0 \Rightarrow$ RHS $\leq$ [illegible] $\to 0$ ∎

第二讲 微分几何初步 3/12

用微积分的工具讨论几何(对象的)性质，比较早的有 Gaspar Monge 他写过一本叫做《分析学在几何中的应用》(Application de l'Analyse à la Géometrie) 的书 (有1850年 Liouville 作注的版本 在网上可以找到)

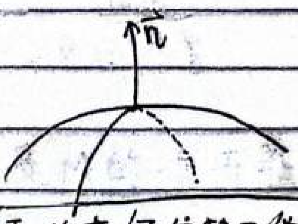
一般来说，内蕴(intrinsic)几何的发端是 Gauss 的工作《Disquisitiones generales circa ~~surf~~ ~~sp~~ superficies curvas》(1827) (General Investigations of Curved Surfaces) 其中的一个重要结果(在原书中是一个推论) 日后被冠以"(Gauss的)绝妙定理" (Theorema Egregium) 之名：Gauss 曲率只依赖于第一基本形式(=度量)

Egregium (latin) = remarkable (en)

Gauss 只有晚年的著作是用德语写的，之前基本上都是用拉丁文写的。

Gauss 曲率 = 主曲率之积

平面曲线在一点处的曲率半径 = 该点处的密切圆半径



密切直线 = 切线 ← 纯几何定义，$\boxed{\text{不必有/不依赖于坐标系}}$

密切圆 = 和原曲线有相同二阶 Taylor 展开的圆 ←

一个不显然的事实：两个主方向互相垂直

One can prove this by the eigenvalue of the Gauss map i.e. apply the spectral thm to the $DG: T_pM \to T_pS^n$ $\overset{\shortparallel}{S}$ the shape operator, then orthogonality by the spectral thm //

Rk. $(\mathrm{Hess} f)(X,Y) = \langle S(X), Y \rangle$ thus also can determine the principal curvature by eigenvalue of Hess f, if Σ^2 is a graph of

例. 两个主曲率相同的点称为脐点 (embilic point)

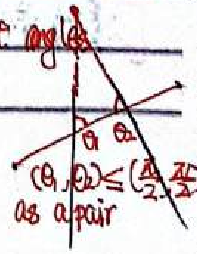
在椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ $(a>b>c>0)$ 上，恰有四个脐点

(1826年2月)

差不多同时，Lobachevsky 给出了非欧几何的第一种表述

[查一下《几何原本》中的第五公设是如何表述的？]

(Parallel postulate) If a line segment intersects two straight lines forming two interior angles on the same side that are less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum less than two right angles.



1854年6月10日，B. Riemann 在哥廷根大学作了题为《关于几何学基础的假设》的就职演说。

Riemann 认为，在十九世纪中期关于各种(非欧)几何学的困难，实际上来自于当时的几何学家没有把拓扑性质与度量性质区分开来。

geometry 来自古埃及的土地测量

度量是什么？

"传统"的作法：曲线长度 = 折线长度的极限

有了微积分之后，设 $\gamma: [0,1] \to \mathbb{R}^d$ $(d \geq 2)$ 连续可微，则

$$L_\gamma = \int_0^1 \sqrt{\sum_{i=1}^{d} \left(\frac{d\gamma_i}{dt}\right)^2}\, dt$$

Riemann 的作法：

我们考虑的对象是"局部'等同'于欧氏空间的(几何)对象"

定义：设 M 是一个 Hausdorff 空间，$\{V_i\}$ 是 M 的一个开覆盖，$U_i \subset \mathbb{R}^n$ 是开集，$\varphi_i: V_i \to U_i$ 是同胚

称这样的一个 M 是(拓扑)流形 (manifold)，称 (V_i, φ_i) 是一个坐标卡 (chart，也称坐标邻域)，$\{(V_i, \varphi_i)\}$ 为一个坐标图册 (atlas)

如果存在 $r \in \mathbb{N}$，使得对于任意 i, j，$\varphi_j \circ \varphi_i^{-1}: U_i \to U_j$ 是一个 C^r 微分同胚，那么称 M 是一个 C^r 微分流形。

在一个流形 M 上，设 V 是一个开集，$f: V \to \mathbb{R}$ 或 $\mathbb{C}$。可以讨论 $f \circ \varphi_i^{-1}: \varphi_i(V \cap V_i) \to \mathbb{R}$ 或 $\mathbb{C}$ 的光滑性

固定 $p \in M$，我们记 $C_p^\infty = \{f$ 是 p 的某个开邻域上的光滑函数$\}$

此处讲课时可能有误

$p\in$

设 $f_1: V_1 \to \mathbb{R}$, $f_2: V_2 \to \mathbb{R}$, 则 f_1+f_2, $f_1 \cdot f_2$ 在 $V_1 \cap V_2$ 上有良定义,

~~[illegible]~~.

在 C_p^∞ 上可以定义一个等价关系 $\sim$: $f \sim g$ 当且仅当存在 p 的一个开邻域 W, 使得 $f|_W = g|_W$.

f 在 C_p^∞ 中关于 $\sim$ 的等价类记为 $[f]$, 称为 M 在 p 点处的一个 C^∞-函数芽 (germ)

定义: $\mathcal{F}_p = C_p^\infty / \sim$,

$\mathcal{F}_p$ 有线性空间结构:
$$[f]+[g] := [f+g]$$
$$\alpha[f] = [\alpha f]$$

设 γ 是 M 上过 p 点的一段曲线: $\gamma: (-\delta, \delta) \to M$, C^∞, $\gamma(0)=p$.

所有这样的曲线的集合, 记为 Γ_p.

对于 $\gamma \in \Gamma_p$, $[f] \in \mathcal{F}_p$, $\langle\langle \gamma, [f] \rangle\rangle := \left.\dfrac{d(f\circ\gamma)}{dt}\right|_{t=0}$ 是一个只依赖于 γ 和 $[f]$ 的确定的数.

且这个配对 (pairing) 对于 $[f]$ 是线性的:
$$\langle\langle \gamma, [f]+[g] \rangle\rangle = \langle\langle \gamma, [f] \rangle\rangle + \langle\langle \gamma, [g] \rangle\rangle$$
$$\langle\langle \gamma, \alpha[f] \rangle\rangle = \alpha \langle\langle \gamma, [f] \rangle\rangle$$

我们记 $\mathcal{H}_p = \{ [f] \in \mathcal{F}_p \mid \langle\langle \gamma, [f] \rangle\rangle = 0, \ \forall \gamma \in \Gamma_p \}$

性质1. $\mathcal{H}_p$ 恰好是在 p 点关于局部坐标的偏导数都是 0 的光滑函数的芽所构成的线性空间.

定义. 我们称商空间 $\mathcal{F}_p/\mathcal{H}_p$ 为 M 在 p 点的余切空间(cotangent space), 记作 T_p^*. 它的元素是函数芽 $[f]$ (它本身是等价类) 的 $\mathcal{H}_p$-等价类, 称为余切向量, 记为 $(df)_p$.

T_p^* 有从 $\mathcal{F}_p$ 诱导的线性空间结构.

定理. 设 $f_1,\cdots,f_k\in C_p^\infty$, $F(y_1,\cdots,y_k)$ 是点 $(f_1(p),f_2(p),\cdots,f_k(p))\in\mathbb{R}^k$ 的邻域内的光滑函数, 则 $f=F(f_1,\cdots,f_k)\in C_p^\infty$

且 $$(df)_p=\sum_{j=1}^k\left(\frac{\partial F}{\partial y_j}\right)_{(f_1(p),\cdots,f_k(p))}(df_j)_p$$

证明. 要证明的是, 对于任意 $\gamma\in\Gamma_p$ 有

$$\langle\langle\gamma,[f]\rangle\rangle=\langle\langle\gamma,\sum_{j=1}^k\left(\frac{\partial F}{\partial y_j}\right)_{(f_1(p),\cdots,f_k(p))}\cdot[f_j]\rangle\rangle$$

$$\langle\langle\gamma,[f]\rangle\rangle=\frac{d}{dt}\Big|_{t=0}(f\circ\gamma)=\frac{d}{dt}\Big|_{t=0}F(f_1\circ\gamma(t),\cdots,f_k\circ\gamma(t))=\sum_{j=1}^k\left(\frac{\partial F}{\partial y_j}\right)_{(f_1(p),\cdots,f_k(p))}\frac{d}{dt}\Big|_{t=0}(f_j\circ\gamma(t))$$

这说明 $[f]-\sum_{j=1}^k\left(\frac{\partial F}{\partial y_j}\right)_{(f_1(p),\cdots,f_k(p))}\cdot[f_j]\in\mathcal{H}_p$, 即得结论. □

推论1 对于 $f,g\in C_p^\infty$, $a\in\mathbb{R}$, 有

$$d(f+g)_p=(df)_p+(dg)_p$$
$$d(af)_p=a\cdot(df)_p$$
$$d(fg)_p=f(p)\cdot(dg)_p+g(p)\cdot(df)_p$$

推论2. $\dim T_p^*=n$.

证明. 取一个 (V,φ), $p\in V$, 将 $\varphi(V)\subset\mathbb{R}^n$ 中的坐标函数 $f_k(x_1,\cdots,x_n)=x_k$ 与 φ 复合可得到局部坐标函数 $u_k=f_k\circ\varphi$

证明 $\{(du_k)_{k=1}^n\}$ 构成 T_p^* 的一组基即可.

T_p^* 上的线性函数：注意到配对 $\langle\langle \gamma, [f] \rangle\rangle$ 关于第二个变量是线性的，并根据 $\mathcal{H}_p$ 的定义 可知上述配对诱导了一个 T_p^* 上的线性映射 $(df)_p \longmapsto \langle\langle \gamma, (df)_p \rangle\rangle$ 所以我们有个映射 $\Gamma_p \to (T_p^*)^*$

进而可在 Γ_p 上定义等价关系：

$$\gamma_1 \sim \gamma_2 \iff \forall (df)_p \in T_p^*, \ \langle\langle \gamma_1, (df)_p \rangle\rangle = \langle\langle \gamma_2, (df)_p \rangle\rangle$$

$\{[\gamma]\} \longrightarrow (T_p^*)^*$ 是单射，利用局部坐标可以证明也是满射。

从而我们得到了 M在p点的切空间 T_p。

我们用 $\langle [\gamma], (df)_p \rangle$ 来表示 T_p 和 T_p^* 之间的配对/对偶

如果我们的 M 是某个 $\mathbb{R}^m$ 的（嵌入）子流形，那么

$\gamma_1 \sim \gamma_2 \iff \frac{d\gamma_1}{dt}\Big|_{t=0} = \frac{d\gamma_2}{dt}\Big|_{t=0} \in \mathbb{R}^m$ 即 γ_1 和 γ_2 在 $t=0$ 处有相同的切向量

定义 设 f 是定义在 p 点附近的 C^∞ 函数，$(df)_p \in T_p^*$ 也称为 f 在 p 点的微分 (differential)

定义 设 $X \in T_p$，$f \in C_p^\infty$，记 $Xf = \langle X, (df)_p \rangle$

称为函数 f 沿切向量 X 的方向导数

性质：
$$X(\alpha f + \beta g) = \alpha Xf + \beta Xg \qquad X \in T_p,\ f, g \in C_p^\infty$$
$$X(fg) = f(p)Xg + g(p)Xf \qquad \alpha, \beta \in \mathbb{R}$$

例 考虑 $SO(3,\mathbb{R}) = \{A \in M_3(\mathbb{R}),\ A^TA = I,\ \det A = 1\}$

$I_3 \in SO(3,\mathbb{R})$，$T_{I_3} = \{B \in M_3(\mathbb{R}),\ B^T + B = 0\}$ $\dim T_{I_3} = 3$

回到Riemann，他是怎么定义曲线的长度的？

设 $\gamma:[0,1]\longrightarrow M$，Riemann认为/假定这条曲线的长度应该具有形式 $L_\gamma=\int_0^1 f(\gamma(t),X(t))\,dt$，这里 $X(t_0)=\frac{d\gamma}{dt}\Big|_{t=t_0}\in T_{\gamma(t_0)}$

为简单起见，他取 $f(\gamma(t),X(t))=\sqrt{g_{\gamma(t)}(X(t),X(t))}$

其中 $g_{\gamma(t)}$ 是 $T_{\gamma(t)}$ 上的一个（正定）二次型

1918年Finsler在他的博士论文里考虑了更一般的情形.

$$f(\gamma(t),X(t))=\langle Hess(F_{\gamma(t)})X(t),\ X(t)\rangle^{1/2}.$$

讨论：什么是"几何"？

Descartes --- Monge --- Riemann -- 通过(局部)坐标计算得到，但又不依赖于坐标系选取的性质是"几何"性质

Klein --- Lie -- 在适当变换群作用下保持不变的性质是"几何"性质

(3) $\phi_{p,q}:V\to V$ is rotation operator $B(0,1)^FV$

F only left the triangle inequality

this is by ① by linearity, only do $|x|=|y|=1$ case

② $x+y$ and F preserves rotation $\Longrightarrow F(x+y)\leq F(x)+F(y)$ ▨

习题：

Exercises

1. (1) 正定阵一定可以对角化.

(1) Positive definite matrix is defined to be symmetric and $z^TMz>0$

Any symmetric matrix is diagonalisable by spectral thm ▨

(2) 设B是n维向量空间中原点O的一个有界邻域. 证明，在所有包含B的椭球中，存在唯一一个体积达到最小值的.

(3) 设 $F:V\longrightarrow\mathbb{R}$ 满足：(a) 连续；(b) $\forall v\neq 0$, $F(v)>0$；(c) $F(\lambda v)=|\lambda|F(v)$

设对于任意 $p,q\in\{v\in V,\ F(v)=1\}$，都存在线性变换 $\phi_{p,q}:V\to V$，满足 $\begin{cases}\phi_{p,q}(p)=q\\ F(\phi(v))=F(v),\ \forall v\in V\end{cases}$. 证明，F是由某个正定内积决定的范数.

(2) $U\ni 0$ bounded: $\forall x,y\in U$, $\|x-y\|<\infty$ set U closed, thus compact

Set $\mathcal{B}_0=\{(r_1,\dots,r_n)\in\mathbb{R}^n_{>0}\mid\exists x_0\in\mathbb{R}^n, s.t.\ U\subset\overline{B}\}\neq\emptyset$ (as bounded)

$\mathcal{B}_0$ is all compact. We know $Vol(\overline{B})=Cr_1\cdots r_n$, thus take $r=\inf_{(r_1,\dots,r_n)\in\mathcal{B}_0}(r_1\cdots r_n)$

I claim. the inf is achievable

Set $\overline{B}(x_0;r_1\dots r_n)$ be any elliptic ball $x_0\in U$. Set $\overline{B}_m=\overline{B}(x_m;r_1^m,\dots,r_n^m)$, s.t. $r_1^m\cdots r_n^m\longrightarrow r$ and $r_1^m\cdots r_n^m\leq r+\frac{1}{m}$

$\|x_m-y\|\leq(\sum(r_j^m)^2)^{\frac12}\leq Cr_1^m\cdots r_n^m\leq C(r+\frac1m)\leq C(r+1)$

$\boxed{\Rightarrow \text{converge subsequence } (x_{m_k})}\xrightarrow{k\to\infty}x_0$ is the desired limit achieve minimal ▨

replaced by weak convergence (I think it holds for reflective Banach spaces)

第二讲 微分几何初步(续) 3/19

注. 若 $M\subseteq\mathbb{R}^d$, 则 T_pM 与 p 点处 M (在$\mathbb{R}^d$中)的切向量全体构成的线性空间是一致的.

进一步的问题: 为什么力学中"力的三要素"包含了"力的作用点"?
因为 加速度/力向量是"生活"在切空间中的.
(严格地说, 是速度向量在切空间中. 而加速度向量是对速度再求一次导数)

这就引出了(至少)两个进一步的问题:

1° 不同点处的切空间中的向量如何"等同"起来?
在$\mathbb{R}^3$中, $T_{p_1}\mathbb{R}^3$ 和 $T_{p_2}\mathbb{R}^3$ 可以自然地"等同", 这就是原始的(naive)"平行"概念. 在《微分几何》课程中, 我们看到过平行移动的概念. 这其实就是把不同点处的切空间等同起来的意思.

2° "平行移动等同于联络", 那么联络(connection)是什么?

定义 在 $TM=\coprod_{p\in M}T_pM$ 上定义拓扑, 使得对于任意 $p\in M$, 存在一个开邻域 V_p, $\coprod_{q\in V_p}(T_qM)\cong V_p\times\mathbb{R}^n$. 这一性质称为"局部平凡性"

若 $V_{p_1}\cap V_{p_2}\neq\varnothing$, 则 $\varphi_1: V_{p_1}\to U_1\subseteq\mathbb{R}^n$, $\varphi_2: V_{p_2}\to U_2\subseteq\mathbb{R}^n$ 诱导了 $T_{\varphi_1(q)}\mathbb{R}^n$ 到 $T_{\varphi_2(q)}\mathbb{R}^n$ 的一个线性映射, 当 q 变化时, 我们可以从中"读"出空间的弯曲(程度).

称 $X: M\longrightarrow TM$, $p\longmapsto X_p\in T_pM$ 为 M 上的一个切向量场

我们可以定义 $(Xf)(p):=\langle X,(df)_p\rangle$, 容易得到

1. $X(fg)(p)=f(p)(Xg)(p)+(Xf)(p)\,g(p)$
2. $X(\alpha f+\beta g)(p)=\alpha(Xf)(p)+\beta(Xg)(p)$

接下来我们计算：

$$Y(X(fg)) = (Yf)\cdot(Xg) + f\cdot(Y(Xg)) + \cancel{(Yg)\cdot(Xf)}\ (Xf)(Yg) + (Y(Xf))\cdot g$$

$$X(Y(fg)) = (Xf)\cdot(Yg) + f\cdot(X(Yg)) + (Yf)\cdot(Xg) + (X(Yf))\cdot g$$

从而 $[X,Y] := XY - YX$ 满足：$[X,Y](\alpha f+\beta g) = \alpha[X,Y]f + \beta[X,Y]g$

且 $[X,Y](fg) = f([X,Y]g) + ([X,Y]f)g$）于是 $[X,Y](p) \in (T_p^*)^* \cong T_p$,

这样从两个向量场出发，可以用一种双线性，反对称的方式得到一个新的向量场，这个 $[X,Y]$ 称为 X 和 Y 的 Lie 括号（Sophus Lie）

The useful Koszul formula

$\langle\nabla_X Y, Z\rangle = X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle + \langle[X,Y],Z\rangle - \langle[Y,Z],X\rangle - \langle[Z,X],Y\rangle$

联络的(较)现代定义是 Koszul 在20世纪40年代给出的：

设 X, Y 是 C^∞ 向量场，f 是 C^∞ 函数，记号 $\nabla_X Y$ 是一个切向量场满足：

1. 关于 X 线性　　2. 关于 Y 线性

（1+3：关于 X 是 $C^\infty(M)$-线性的.）

3. $\nabla_{fX} Y = f\nabla_X Y$　　4. $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$

以曲面为例，容易看到，联络本身的表达形式严重依赖于坐标的选取

这样，我们可以对"流形上的微分算子"有一个初步的体会.

设 $p\in M$，它的一个邻域 $V_p \xrightarrow{\varphi} U_{\varphi(p)} \subseteq \mathbb{R}^n$ 那么（过 $U_{\varphi(p)}$中的点 且）平行于各条坐标轴的开线段就可以"拉回"到 M 上（更确切地说是 V_p 上），成为一些光滑曲线，于是就有了 n 个切向量场 X_i $(i=1,\cdots,n)$

那么，$\sum_{i=1}^n X_i^2$ 是否就可以叫做流形上（p 点附近）的 Laplace 算子了呢？

问题1：尽管在 V_p 的每一点 q 处，~~线性~~ $\{X_i(q)\}$ 都是 T_qM 的一组基

但（尤其在流形 M 上有度量时）它们并非两两垂直的.

问题2. $[X_i, X_j] = 0$ 么？

Problems. 1. One conterexample is the polar coordinate $(dr)^2 + r^2(d\theta)^2$

In fact the $\nabla_{X_i}X_j = \Gamma_{ij}^k X_k$, i.e. X_j can "change" in direct of X_i when Γ_{ij}^k, thus we have no reason to give $X_i(p)\perp X_j(p)$, even in single point //

2. $[X_i, X_j] = \underbrace{\nabla_{X_i}X_j - \nabla_{X_j}X_i}_{\text{torsion-free}} = (\Gamma_{ij}^k - \Gamma_{ji}^k)X_k = 0$ for this we call the holonomic basis //

(It's cyclic argument, revise : $[X_i, X_j] = [\varphi^* \frac{\partial}{\partial x_i}, \varphi^* \frac{\partial}{\partial x_j}] = \varphi^*[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$)

第二讲 (拟)微分算子

我们知道 $\frac{d}{dx}: C_c^1(\mathbb{R}) \to C_c^0(\mathbb{R})$ 或 $C_c^1((0,1)) \to C_c^0((0,1))$ 在我们把 C^1 视为 C^0 的子空间时，可以给出典型的无界算子的例子。

与此同时，在泛函分析中我们学到的工具大多数是对完备赋范线性空间来说的。所以我们要对微分算子的定义方式做适当的处理。

注意：课上这里写了一个求导记号

方法1 (分布/广义函数) 对适当的光滑函数类 ($\mathcal{D} = C_0^\infty(\mathbb{R})$ 或 $S = S(\mathbb{R}) = \{f, \forall n,k, \lim_{|x|\to\infty} f^{(k)}(x)(1+|x|)^n = 0\}$)，考虑其上的连续线性泛函 g，把 $g(f)$ 写成 $\langle f, g\rangle$，并定义

C_0^∞ is a Fréchet space

g' 为满足 $\langle f, g'\rangle = -\langle f', g\rangle$ $\forall f$ 的唯一的连续线性泛函。 integral by part

这牵涉到：① 在光滑函数类上取什么样的拓扑？

② g 全体构成的空间中包含怎样的函数类？

标准参考文献 L. Hörmander. The Analysis of Linear Partial Differential Operators I.

② it's little vague, But I prefer to find

$S_0 \in \mathcal{D}(U)$ Dirac function can be related with Dirac operator $D^2 = \Delta I$ by $\Delta\varphi = S_0$, $\varphi = \frac{C_n}{\|x\|^{n-2}}$ for a proper C_n

方法2 (与方法1不是互斥的，而是"叠加"的，Fourier 变换) $\Rightarrow D^2(\varphi I) = S_0 I$

我们知道"Fourier 变换把求导运算转化为乘上一个多项式"。

这句话的意思是：在所有记号都有意义的情况下，

$$\widehat{f'}(\xi) = * \int_{-\infty}^{\infty} e^{-i\xi x} f'(x)\,dx = * \int \left[(e^{-i\xi x} f(x))' + i\xi e^{-i\xi x} f(x) \right] dx$$

$$\left(\text{对 } f \in \mathcal{D} \text{ 或 } S,\ \lim_{x\to\pm\infty} e^{-i\xi x} f(x) = 0 \right) = i\xi \hat{f}(\xi)$$

即 (记 $D = \frac{1}{i}\frac{d}{dx}$) $\widehat{Df} = \xi \hat{f}$

Laurent Schwartz (1915–2003)

① Topology is determined by the compact convergence with all $\|\partial^\alpha \varphi_n\| \to \|\partial^\alpha f\|$ (where underlying $\mathbb{R}^n \supset U$). It induces weak*-topology on distribution spaces $\mathcal{D}(U)$

We generalising the differential operators to pseudo-differential operators by Fourier transform: this coincides in case of differential operator by the 1:1 correspondence:

① Differential operator $P = \sum A_i(x)\partial_i$;

② Symbol $a(p) = \sum A_i(x) x_i$;

③ The Kernel function of operator P;

where ② $\Rightarrow$ ① by Fourier transform and we extend this from polynomial to general Schwarz function.

现在，我们还需要~~做~~作一个理论上的准备，定义 L^2 上的 Fourier 变换.

1° $f \longmapsto \int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx$ 定义了一个从 L^1 到 L^∞ 的变换

(可以证明不是满射)

2° $f \longmapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx$ 定义了一个从 $L^1 \cap L^2 \to L^2$ 的~~映射~~ 等距映射

(Planchrel / Parseval)

3° 所以 2° 中的映射可以连续延拓为 L^2 到 L^2 的等距映射.

By $C_c^\infty \subset L^1 \cap L^2 \subset L^2 \Rightarrow L^1 \cap L^2$ dense in L^2

记 $\mathcal{F}: L^2 \to L^2$ 为 L^2 上的 Fourier 变换

4° 可以证明这是一个满射，从而是一个酉算子.

5° $\mathcal{F}^4 = Id$，从而 $\sigma(\mathcal{F}) \subseteq \{\pm 1, \pm i\}$，容易举例说明这 4 个数都是特征值

6° ~~记~~ $Df = \mathcal{F}^{-1}(\boxed{\xi}\, \mathcal{F}f(\xi))$ $\qquad D = \frac{1}{i} \boxed{\frac{d}{dx}}$

使这个表达式(只涉及积分)有意义的函数 f 比使得左边有意义(C^1)的函数要多一些

6° using Fourier transform replace derivative to expand differentiable functions, and replace ξ (algebraic datum) to other functions on ξ can change D (analytic datum)

Q: 什么时候 $\xi \cdot (\mathcal{F}f)(\xi) \in L^2$?

定义：一阶 Sobolev 空间 $H^1(\mathbb{R}) = \{ f \in L^2,\ (1+|\xi|^2)\hat{f} \in L^2 \}$

类似可定义 $H^k(\mathbb{R}) = \{ f \in L^2,\ (1+|\xi|)^k \hat{f} \in L^2 \}$

其中 $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx$.

只要我们讨论的是全空间 $\mathbb{R}$(或 $\mathbb{R}^n$)，那么各种定义都是等价的.

对于 $f \in H^k$，和一个 k 次多项式 P_k，可以考虑 $f \longmapsto \mathcal{F}^{-1}(P_k(\xi)\,\mathcal{F}f(\xi))$

我们知道，当 f 足够光滑时，这等于 $P_k(D)f$.

之所以把 D 写成这样，是因为 $\int_{\mathbb{R}} (Df)g = \int_{\mathbb{R}} f(Dg)$.

一般地，对于适当的函数 $a(x,\xi)$，我们定义

$$(\boxed{a(x,D)} f)(x) = \frac{1}{2\pi} \int e^{ix\xi}\, \boxed{a(x,\xi)} \left(\int_{\mathbb{R}} e^{-is\xi} f(s)\,ds \right) d\xi \quad \cdots (*)$$

It also depends on x, not only ξ

此时 $a(x,D)$ 称为是个拟微分算子(pseudo-differential operator)

函数 $a(x,\xi)$ 称为这个算子的象征(symbol)

M.Riesz — F.Riesz

L.Gårding 拟微分算子理论的标准参考书是

Hörmander

L. Hörmander. The Analysis of Partial Differential Operators III

(这套书总共四卷，一和三使用的频率比二和四高得多)

习题:

1. 在一个(2维)曲面上，你能否(只)用微分同胚到平面的办法定义出一个 Laplace 算子么?

2. (1) 证明，$\mathcal{F}: L^2 \to L^2$ 是满射

(2) 证明：$\{\pm 1, \pm i\}$ 都是 $\mathcal{F}$ 的特征值 (For 1, the Gaussian integral $\widehat{e^{-\pi a|x|^2}} = a^{-\frac{n}{2}} e^{-\frac{\pi}{a}|\xi|^2}$

(3) 证明，Fourier 变换把速降函数变成速降函数

3. 证明，在 S 的定义式中，"$\lim = 0$" 可替换为 "有界"

Exercises. 1. One has the first basic form on $S^2 \subset \mathbb{R}^3$, thus one can do this by give Riemannian metric for all (M^n, g): $\Delta: f \mapsto \operatorname{div}(\operatorname{grad} f)$ with $\operatorname{div}: X \mapsto \sum_i E_i(X_i)$ and $\operatorname{grad}: f \mapsto E_i(f)E_i$ by given a geodesic frame, then writting E_i as ∂_i in local cooridnates $\Rightarrow \Delta: f \mapsto \frac{1}{\sqrt{\det(g_{ij})}} \sum \frac{\partial}{\partial x_j}\left(g^{ij}\sqrt{\det(g_{ij})}\frac{\partial}{\partial x_i} f\right)$ is defined locally to $\mathbb{R}^n$. For $n=2$, $\Delta f = \frac{1}{\sqrt{\det(I)}} \sum \frac{\partial}{\partial x_j}\left(I^{-1}\sqrt{\det(I)}\frac{\partial}{\partial x_i} f\right)$ ▨

Rk. One can extend to L^p $(1 \le p \le 2)$'s Fourier transform by Riesz–Thorin thm. (For $p>2$ need distribution) but only L^2 has equality, others are the Hausdorff–Young inequality $\|\hat{f}\|_{p'} \le \|f\|_p$.

2. (1) One has the Plancherel equality $\|\hat{f}\|_2 = \|f\|_2$ isometry $\Rightarrow$ surjective ✗ (I'm sorry it's wrong even in Hilbert space: by translation in $l^2(\mathbb{Z})$.) Using Fourier inversion formula $\mathcal{F}^{-1}: L^2 \to L^2$ with $\mathcal{F}^{-1} = \mathcal{F}R = R\mathcal{F}$ with $R: f \mapsto f(-\square)$ flip operator, thus $\mathcal{F}$ is isomorphism in L^2 ▨

(2) $\mathcal{F}^4 = Id \Rightarrow \{\pm 1, \pm i\} \supseteq \sigma(\mathcal{F})$, then we can construct to eigenfunction by form $P(x)e^{-\pi|x|^2}$ with $p(x)$ is a polynomial. Precisely: ① $\mathcal{F}(e^{-\pi|x|^2}) = e^{-\pi|\xi|^2}$; ② $\mathcal{F}(-2\pi i x f) = \frac{d}{d\xi}(\mathcal{F}(f)) \Rightarrow \mathcal{F}(P(-2\pi i x)e^{-\pi|x|^2}) = P\left(\frac{d}{d\xi}\right)e^{-\pi|\xi|^2}$

③ Then set $\lambda P(-2\pi i x)e^{-\pi|x|^2} = P\left(\frac{d}{dx}\right)e^{-\pi|x|^2} \Rightarrow \begin{cases} \lambda = -1: f(x) = P(-2\pi i x)e^{-\pi|x|^2} = (\text{some cubic}) \cdot e^{-\pi|x|^2} \\ \lambda = i: f(x) = P(-2\pi i x)e^{-\pi|x|^2} = -2\pi i x e^{-\pi|x|^2} \\ \lambda = -i: f(x) = P(-2\pi i x)e^{-\pi|x|^2} = (\text{Some quadratic}) \cdot e^{-\pi|x|^2} \end{cases}$ ▨

(3) $\lim\limits_{|\xi|\to\infty} (\mathcal{F}f)^{(k)}(\xi)(1+|\xi|^n) = \lim\limits_{|\xi|\to\infty} \left(\int e^{-2\pi i \xi x} f(x)\,dx\right)^{(k)}(1+|\xi|^n) \overset{\text{Lebesgue}}{=} \lim\limits_{|\xi|\to\infty}\left(\int \left(e^{-2\pi i \xi x} f(x)\right)^{(k)} dx\right)(1+|\xi|^n) \overset{(1+|\xi|^n)}{=} \lim\limits_{|\xi|\to\infty}\left(\int \left(\frac{d}{dx}\right)^{(n)} e^{-2\pi i \xi x} \mathcal{F}(x)\,dx\right)(1+|\xi|^n) \xrightarrow[\text{by part}]{\text{Integral}} 0$ ▨

Rk. For $\mathbb{R}^n$ in general one can replace $\|x\|^m$ by $x^{m'}$, $m \in \mathbb{N}$ and $m' \in \mathbb{N}^n$, then they're also equivalent

3. "$\lim \overset{DCT}{=} 0$" $\Rightarrow$ bounded is obvious; Conversely, $\forall k, n$, $f^{(k)}(x)(1+|x|^n)$ bounded (Riemann–Lebesgue Lemma)

i.e. $\sup\limits_{x\in\mathbb{R}} |f^{(k)}(x)(1+|x|^n)| < M < \infty$

$\Rightarrow f^{(k)}(x)(1+|x|^n) < M, \forall x \in \mathbb{R} \Rightarrow f^{(k)}(x)(1+|x|^{n+1}) < \frac{M}{|x|}$

$\Rightarrow \lim\limits_{|x|\to\infty} f^{(k)}(x)(1+|x|^{n+1}) = 0$

thus $\forall k, n$, replace n by $(n+1)$, done ▨

第二讲 (拟)微分算子(续) 3/26

象征 $\longleftrightarrow$ 算子

$a(x,\xi)$ $\qquad$ $Op(a)=a(x,D)$

我们知道,一般来说,两个算子是不交换的,而两个函数的(逐点)乘法是交换的。我们希望我们考虑的算子集合在复合运算下具有某种封闭性。

Thus we expect the order of commutator $\leq$ the order of operators' product

注意: 象征的定义不是唯一的!

定义. 设 $m\in\mathbb{R}$, 我们定义 This is motivated by: $a(x,\xi)$ respect to ξ is closed to polynomial

$$S^m=S^m(\mathbb{R}^n\times\mathbb{R}^n)=\left\{a\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n),\ \forall\alpha,\beta,\ \left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right|\leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}\right\}$$

$\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{N}^n$

$\xi=(\xi_1,\cdots,\xi_n)$

$\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$

$|\alpha|=\alpha_1+\cdots+\alpha_n$

S^m 的元素称为一个 m 阶象征,并记 $S^{-\infty}=\bigcap_m S^m$ Simlliar to polynomial. It gives a grading structure

例 (1) 设 $a(x,\xi)=\sum_{|\alpha|\leq m}a_\alpha(x)\xi^\alpha$, 其中 $a_\alpha\in C^\infty(\mathbb{R}^n)$ 且每个 a_α 及其任意阶导数都是有界的,那么 $a\in S^m$. 我们称 a 是一个微分象征.

$\cdots\subset S^{m-n}\subset S^{m-n+1}\subset\cdots\subset S^m$

thus $S^{-\infty}=\lim_{m\to-\infty}S^m$

(2) 设 $a(\xi)$ 是一个 m 阶(正)齐次函数(即 $\forall\lambda>0$, $a(\lambda\xi)=\lambda^m a(\xi)$)且当 $\xi\neq0$ 时,$a(\xi)$ 是 C^∞ 的. 若 $\chi\in C_c^\infty$ 且在原点附近 $\chi=1$ 那么 $\tilde{a}(\xi)=(1-\chi(\xi))a(\xi)$ 是一个 m 阶象征.

(3) 若 $\varphi\in S$, 则 $\varphi(\xi)\in S^m$ (只依赖于 ξ, 在 x 方向是常数)

(4) $a(x,\xi)=e^{ix\xi}$ 不是一个象征.

基本性质.

1) $a\in S^m\Longrightarrow\partial_x^\alpha\partial_\xi^\beta a\in S^{m-|\beta|}$

(1) $\forall\alpha',\beta'$, $\left|\partial_x^{\alpha'}\partial_\xi^{\beta'}(\partial_x^\alpha\partial_\xi^\beta a)(x)\right|=\left|\partial_x^{\alpha'+\alpha}\partial_\xi^{\beta'+\beta}a(x)\right|\leq C_{\alpha',\beta'}(1+|\xi|)^{m-|\beta'+\beta|}\leq C_{\alpha',\beta'}(1+|\xi|)^{(m-|\beta|)-|\beta'|}$

$\Rightarrow\partial_x^\alpha\partial_\xi^\beta a(x)\in S^{m-|\beta|}$ ▨

2) $a\in S^{m_1}$, $b\in S^{m_2}\Rightarrow ab\in S^{m_1+m_2}$

(2) $\left|\partial_x^\alpha\partial_\xi^\beta(ab)\right|$
$=\left|(\partial_x^\alpha\partial_\xi^\beta a)b+\cdots+a(\partial_x^\alpha\partial_\xi^\beta b)\right|$
$\leq\left|(\partial_x^\alpha\partial_\xi^\beta a)\right||b|+\cdots+|a|\left|\partial_x^\alpha\partial_\xi^\beta b\right|$
$\leq C_{\alpha\beta}(1+|\xi|)^{m_1-\beta}C_{0,0}(1+|\xi|)^{m_2}+\cdots$
$+C_{\alpha\beta}'(1+|\xi|)^{m_2-\beta}C_{0,0}(1+|\xi|)^{m_1}$
$\leq\tilde{C}_{\alpha,\beta}(1+|\xi|)^{m_1+m_2-\beta}$ ▨

3) 引理. 若 $a_1,\cdots,a_K\in S^0$, $F\in C^\infty(\mathbb{C}^k)$, 则 $F(a_0,\cdots,a_k)\in S^0$ (Functional Calculus)

证明. 首先,我们不妨设 a_j 都是实值函数(为什么"不妨"?) 且 $F\in C^\infty(\mathbb{R}^k)$, 于是

$$\frac{\partial F}{\partial x_j}=\sum_{s=1}^k\frac{\partial F}{\partial a_s}\partial_{x_j}a_s,\quad\frac{\partial F}{\partial\xi_j}=\sum_{s=1}^k\frac{\partial F}{\partial a_s}\partial_{\xi_j}a_s$$

我们对 $|\alpha|+|\beta|\leq p$ 进行归纳,证明 S^m 定义中的估计成立.

- $p=0$ 显然,每个 a_j 都满足 $|a_j(x,\xi)|\leq C_{0,0,j}$, 而光滑(连续就够了)函数 F 在有界集 $\prod_{j=1}^k[-C_{0,0,j},C_{0,0,j}]$ 上有界.

· $p \Rightarrow p+1$：对 $\frac{\partial f}{\partial \xi}$ 用归纳假设，对上式用 Leibniz 公式.

$S^m \xrightarrow{Op} \Psi^m(M)$ (Here $M=\mathbb{R}^n$)

$a \longmapsto Op(a)$ is pseudo-differential operator of order m.

extending into grading

$S^* \xrightarrow{Op} \Psi^*(M)$

We expect a "inverse" at order 0 (principal symbol)

At last, $0 \to K \to \Psi^0(M) \to S^0 \to C^\infty(M)$

the reconstruction of $Op(a)=a(x,D)$ is by the (*) at first, later use the kernel function.

性质. 如果 $a\in S^m$, $f\in S$, 那么

$$(Op(a)f)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}a(x,\xi)\Big(\underbrace{\int_{\mathbb{R}^n} e^{-iy\xi}f(y)dy}_{\hat{f}}\Big)d\xi$$

定义了 S 中的一个函数，且 $(a,f)\longmapsto Op(a)f$ 是连续的.

证明. $f\in S \Rightarrow \hat{f}\in S$ 而由 $a\in S^m$ 可知

$$|(Op(a)f)(x)| \le \underbrace{*}_{\text{(Some constants)}}\ \underbrace{\sup\left(a(x,\xi)(1+|\xi|)^{-m}\right)}_{\text{由}S^m\text{的定义}\Rightarrow\text{有界}}\int_{\mathbb{R}^n}\underbrace{(1+|\xi|)^m|\hat{f}(\xi)|}_{\in S^m(\mathbb{R}^n)}d\xi$$

由此得到 $Op(a)f$ 是有界函数，且由 a 关于 x 的连续性可得到 $Op(a)f$ 连续

为说明 $Op(a)f\in S$，我们要建立下述两个有用的关系式

$$\boxed{[Op(a),D_j] = i\,Op(\partial_{x_j}a)\ ;\ [Op(a),x_j] = -i\,Op(\partial_{\xi_j}a)}$$

其中 x_j 表示在 x 点从 $f(x)$ 变为 $x_jf(x)$ 的乘积算子. Here "[,]" means the commutor of two operators

它们可以用分部积分很快得到：

$$(Op(a)D_jf)(x) = *\int e^{ix\xi}\underbrace{\widehat{D_jf}(\xi)}_{a(x,\xi)}d\xi = *\int e^{ix\xi}a(x,\xi)\xi_j\hat{f}(\xi)d\xi$$

而

$$D_j(Op(a)f)(x) = -i\left[*\int e^{ix\xi}\cdot i\xi_j\,a(x,\xi)\hat{f}(\xi)d\xi + Op(\partial_{x_j}a)f(x)\right]$$

相减即得第一个式子. 由这两个关系式可以推出：

$\boxed{x^\alpha D^\beta(Op(a)f)\text{ 是一些形如 } Op(\partial_\xi^{\alpha'}\partial_x^{\beta'}a)(x^{\alpha''}D^{\beta''}f)\ (\text{其中 } \alpha'+\alpha''=\alpha,\ \beta'+\beta''=\beta) \text{ 的项的线性组合}}$，从而是有界的，因此 $Op(a)f\in S$.

要说明 $(a,f)\longmapsto Op(a)f$ 连续，我们需要先定义 S^m 和 S 中的拓扑.

· 在 S 上，$\forall\alpha,\beta\in\mathbb{N}^n$，$\sup|x^\alpha\partial_x^\beta f|$ 定义了一个半范数，这一族半范数决定了 S 上的常用拓扑 (By last Exercise 3, we had check it $<\infty$)

Maurice Fréchet

· 在 S^m 上，$|a|^m_{\alpha,\beta} = \sup_{(x,\xi)}\left\{(1+|\xi|)^{-(m-|\beta|)}\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right|\right\}$ 是一个半范数，这一族半范数定义的拓扑下，S^m 是一个 Fréchet 空间

上述第二个红框中的结论即说明 $(a,f)\longmapsto Op(a)f$ 是连续的.

设 $a\in S^m(\mathbb{R}^n\times\mathbb{R}^n)$

性质 $a\longmapsto Op(a)$ 是单射.

证明 我们假设对于所有的 $f\in S$, $x\in\mathbb{R}^n$, 有

$$\int e^{ix\xi}a(x,\xi)\hat{f}(\xi)\,d\xi=0$$

固定 x, 首先可以注意到函数 $b(\xi)=\dfrac{a(x,\xi)}{(1+|\xi|^2)^{\frac{m}{2}+\frac{n}{4}+\frac{1}{2}}}\in L^2(\mathbb{R}^n)$

$\left[|a|\le * (1+|\xi|)^m \Rightarrow |b|\le \dfrac{*}{(1+|\xi|^2)^{\frac{n}{4}+\frac{1}{2}}}\Rightarrow |b|^2\le\dfrac{*}{(1+|\xi|^2)^{\frac{n}{2}+1}}\text{ 可积}\right]$

且 b 对于任意形如 $v(\xi)=e^{-ix\xi}(1+|\xi|^2)^{\frac{m}{2}+\frac{n}{4}+\frac{1}{2}}\overline{\hat{f}(\xi)}$ 的函数正交.

当 f 取遍 $S(\mathbb{R}^n)$ 时, v 也取遍 $S(\mathbb{R}^n)$ (为什么?), 这样由 S 在 L^2 中的稠密性即可得到 $b=0$.

算子的核函数 (kernel function)

Recall our (*) at first

$a(x,D)f(x)=\left(\frac{1}{2\pi}\right)^n\int e^{ix\xi}\boxed{a(x,\xi)}\left(\int e^{-is\xi}f(s)ds\right)d\xi$

$=(2\pi)^{-n}\int f(s)\left(\int e^{i(x-s)\xi}a(x,\xi)d\xi\right)ds$

先假设 $a\in S^{-\infty}$, 对于 $f\in S$, 有

$$\begin{aligned}(Op(a)f)(x)&=\int e^{ix\xi}a(x,\xi)\hat{f}(\xi)d\xi\\&=(2\pi)^{-n}\int e^{ix\xi}a(x,\xi)\left(\int e^{-is\xi}f(s)ds\right)d\xi\\&=(2\pi)^{-n}\int f(s)\left(\int e^{i(x-s)\xi}a(x,\xi)d\xi\right)ds\end{aligned}$$

于是 $Op(a)$ 的核函数由

$$K(x,y)=(2\pi)^{-n}\int e^{i(x-y)\xi}a(x,\xi)d\xi \quad \text{给出}$$

~~当~~ $\Rightarrow K(x,x-y)=(2\pi)^{-n}\int e^{iy\xi}a(x,\xi)d\xi \xrightarrow[\text{inversion}]{\text{Fourier}} a(x,\xi)=\int K(x,x-y)e^{-iy\xi}dy$

算子的共轭算子

thus we can recover $a(x,\xi)$ from $K(x,y)$

对于 $A:S\to S$, 我们希望找到 $A^*:S\to S$ 满足对于任意 $u,v\in S$

取 L^2 中内积

$$(Au,v)=(u,A^*v)$$

我们称 A^* 是 A 的共轭算子.

例 设 $a(D)$ 是一个常系数拟微分算子 (即象征 a 不依赖于 x)

对于 $u,v\in S$, 我们有

$$(a(D)u,v)=(2\pi)^{-n}(a\hat{u},\hat{v})=(2\pi)^{-n}(\hat{u},\bar{a}\hat{v})=(u,\bar{a}(D)v)$$

这样 $a(D)^*=\bar{a}(D)$.

如果 A^* 存在，那么它的核函数 K^* 满足

$$\int (Au)(x)\overline{v(x)}dx = (Au, v) = (u, A^*v) = \int u(y)\overline{(A^*v)(y)}dy$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\int\int K(x,y)u(y)dy\,\overline{v(x)}dx \qquad\qquad \int u(y)\overline{\left(\int K^*(y,x)v(x)dx\right)}dy$$

从而 $K^*(y,x) = \overline{K(x,y)}$

即 $K^*(x,y) = \overline{K(y,x)} = (2\pi)^{-n}\int e^{i(x-y)\xi}\,\overline{a(y,\xi)}\,d\xi$

而

$$a^*(x,\xi) = \int K^*(x,x-y)e^{-iy\xi}dy = (2\pi)^{-n}\int e^{iy(\eta-\xi)}\,\overline{a}(x-y,\eta)\,dy\,d\eta$$

$$= (2\pi)^{-n}\int e^{-iy\eta}\,\overline{a}(x-y,\xi-\eta)\,dy\,d\eta$$

象征的渐近展开 (asymptotic expansion)

实际应用中，常取 $m_j = m-j$ 或 $m_j = m-\frac{j}{2}$

对于一个递减数列 $m_j \searrow -\infty$ $(j\in\mathbb{N})$，我们考虑 $a_j \in S^{m_j}$，我们希望给和式 $\sum_j a_j$ 某种意义（一般来说这不是一个收敛的函数项级数）

我们约定：

$$a \sim \sum a_j \iff \forall k\geq 0 \quad a-\sum_{j=0}^{k} a_j \in S^{m_{k+1}}$$

我们有如下的

Borel 引理. 设 (b_j) 是一个复数列，则存在一个光滑函数 $f\in C^\infty(\mathbb{R})$，满足 $\forall j$, $f^{(j)}(0) = b_j$，即当 $x\to 0$ 时，$f(x) \sim \sum b_j \frac{x^j}{j!}$

证明. 我们取一个 C^∞ 函数 χ，满足 $\chi(x) = \begin{cases} 1 & \text{当 } |x|\leq 1 \text{ 时} \\ 0 & \text{当 } |x|\geq 2 \text{ 时} \end{cases}$

$\binom{k}{l} = C_k^l$

我们要选取适当的正数列 $\lambda_j \nearrow +\infty$，使得 $f(x) = \sum_{j=0}^{\infty} b_j \frac{x^j}{j!}\chi(\lambda_j x)$ 满足要求

首先，f 是处处良定义的.

其次，对于整数 k 和 $j\geq k$，第 j 项的 k 阶导数 $f_j^{(k)}(x) = \sum_{0\leq l\leq k}\binom{k}{l} b_j \frac{x^{j-l}}{(j-l)!}\chi^{(k-l)}(\lambda_j x)\lambda_j^{k-l}$

因为 $\lambda_j x$ 在 χ 及其各阶导数的支集上都是有界的，故存在常数 C_k，满足

$$|f_j^{(k)}(x)| \leq C_k |b_j| \lambda_j^{k-j}\frac{1}{(j-k)!}$$

现在取 $\lambda_j \geq 1+|b_j|$，那么对于任意 $x\in\mathbb{R}$，$\sum_j |f_j^{(k)}(x)|$ 是一致收敛的.

由此保证 $f\in C^{\infty}$ 且可逐项求导，于是 $f^{(k)}(0)=b_k\ (\forall k)$.

习题.

1. 写出一个满足 Borel 引理证明要求的 χ.

2. 设 $a\in S^0(\mathbb{R}^n\times\mathbb{R}^n)$，记 $a_\varepsilon(x,\xi)=a(x,\varepsilon\xi)$

(1) 证明，对任意重指标 α,β，有 $|a_\varepsilon-a_0|^m_{\alpha,\beta}\le C_{\alpha,\beta,m}\,\varepsilon^m$.

(2) 对于 $m>0$，证明，当 $\varepsilon\to 0$ 时，在 S^m 中，$a_\varepsilon\to a_0$.

(3) 在 S^0 中，$a_\varepsilon\xrightarrow{\varepsilon\to 0}a_0$ 成立么？

3. 设 $a_j\in S^{m_j}$，$m_j\searrow-\infty$，

(1) 证明，可以选取一个适当的函数 χ 和一列适当的参数 ε_j，使得

$$a=\sum\left(1-\chi(\varepsilon_j\xi)\right)a_j\in C^{\infty}$$

(2) 证明，$a-\sum_{j=0}^{k}a_j\in S^{m_{k+1}}$

(3) 证明 $\operatorname{supp}a\subset\bigcup\operatorname{supp}a_j$

定义 称一个象征 $a\in S^m$ 为古典的，若 $a\sim\sum_j a_j$，其中 a_j 对于 $|\xi|\ge 1$ 是一个齐次函数，即对于 $|\xi|\ge1,\ \lambda\ge1$，$a_j(x,\lambda\xi)=\lambda^{m-j}a_j(x,\xi)$

齐次：$m-j$ 阶

Exercises.

1. value, r, 1 2: the bump function

Set $\chi_1(x)=\begin{cases}e^{-\frac{1}{x}}, & x>0\\ 0; & x\le 0\end{cases}$

(Only do $\mathbb{R}$ case, otherwise one can replace x by $|x|^2$)

then $\chi_2(x)=\dfrac{\chi_1(x)}{\chi_1(x)+\chi_1(1-x)}$ is 1, 0 1, then lost

$\chi_{abcd}(x)=\chi_2\!\left(\frac{x-a}{b-a}\right)\chi_2\!\left(\frac{d-x}{d-c}\right)$ is 1, a b c d

$\Rightarrow \chi(x)=\chi_{-2-112}(x)$

$=\chi_2(x+2)\,\chi_2(2-x)$

$=\dfrac{\chi_1(x+2)}{\chi_1(x+2)+\chi_1(-x-1)}\cdot\dfrac{\chi_1(2-x)}{\chi_1(2-x)+\chi_1(x-1)}$

$=\begin{cases}1; & |x|<1\\ 0; & |x|>2.\\ \square; & \text{otherwise}\end{cases}$ ▨

2. (1) $|a_\varepsilon-a_0|^m_{\alpha,\beta}=\sup_{(x,\xi)}(1+|\xi|)^{-(m-|\beta|)}\left|\partial_x^\alpha\partial_\xi^\beta(a_\varepsilon-a_0)\right|\le\sup_{(x,\xi)}(1+|\xi|)^{-(m-|\beta|)}\left(|\partial_x^\alpha\partial_\xi^\beta(a_\varepsilon)|-|\partial_x^\alpha\partial_\xi^\beta(a_0)|\right)$

$\le\sup_{(x,\xi)}\left(|\partial_x^\alpha\partial_\xi^\beta(a_\varepsilon)|(1+|\xi|)^{-(m-|\beta|)}-|\partial_x^\alpha\partial_\xi^\beta(a_0)|(1+|\xi|)^{-(m-|\beta|)}\right|$ $\left(\partial_{\varepsilon\xi}=\frac{1}{\varepsilon}\partial_\xi\right)$

$\le C_{\alpha,\beta}(1+|\xi|)^{-m-|\beta|}(1+|\xi|)^{-|\beta|}-|\partial_x^\alpha\partial_{\varepsilon\xi}^\beta(a_0)|(1+|\xi|)^{-m-|\beta|}\varepsilon^{m+|\beta|}|\le C_{\alpha,\beta}(1+|\xi|)^{-m}-|C_{\alpha,\beta}(1+|\xi|)^{-m-|\beta|}(1+|\varepsilon\xi|)^{-|\beta|}\varepsilon^{m+|\beta|}$

$=C_{\alpha,\beta}(1+|\xi|)^{-m}\varepsilon^m=C_{\alpha,\beta,m}\varepsilon^m$ ▨

(2) $|a_\varepsilon-a_0|^m_{\alpha,\beta}\le C_{\alpha,\beta,m}\varepsilon^m\xrightarrow{\varepsilon\to0}0\Rightarrow a_\varepsilon-a_0\in S^{m+1}\Rightarrow a_\varepsilon\xrightarrow{\varepsilon\to0}a_0$ in S^m ▨

(3) Taking $n=1$, and choice $a(x,\xi)\in S^0(\mathbb{R}^2)$, s.t. $|\partial_x^\alpha\partial_\xi^\beta a|=C_{\alpha,\beta}(1+|\xi|)^{-\beta}$ $(\alpha,\beta\in\mathbb{N})$, i.e. a polynomial, then all inequality in (1) takes equality $\longrightarrow|a_\varepsilon-a_0|^0_{\alpha,\beta}=C_{\alpha,\beta}\varepsilon^0=C_{\alpha,\beta}>0\Rightarrow$ no ▨

3. (1) All $a_j(x,\xi)$, $\chi(\varepsilon_j\xi)$ are smooth, thus the destruction is the infinite sum. Consider $\forall(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$, $a(x,\xi)$ can be defined by finite sum by distinct supports of χ choosen in (1) and ε_j choosed property ▨

(3) this choice directly gives $\operatorname{supp}a\subset\bigcup\operatorname{supp}a_j$ ▨

(2) $a-\sum a_j=\boxed{\sum_{j\ge k+1}a_j}-\left(\sum_{j\le k}\chi(\varepsilon_j\xi)a_j\right)$

$\in S^{m_{k+1}}$ by definition

$\chi(\varepsilon_j\xi)a_j\in S^{-\infty}\subset S$ by $\varepsilon_j\ge\varepsilon_k>0$ (decreasing) ▨

第三节（拟）微分算子（续2） 4/2.

现在我们来看算子的复合.

设 A_1, A_2 是两个拟微分算子，对 $f\in\mathcal{S}$，我们有

$$(A_1A_2f)(x) = * \int e^{ix\xi} a_1(x,\xi)\,\widehat{(A_2f)}(\xi)\,d\xi$$

$$= *' \int e^{ix\xi} a_1(x,\xi)\left(\int e^{-iy(\xi-\eta)} a_2(y,\eta)\hat f(\eta)\,dy\,d\eta\right) d\xi$$

$$= *' \int e^{iy\eta + i\xi(x-y)} a_1(x,\xi)\,a_2(y,\eta)\,\hat f(\eta)\,dy\,d\eta\,d\xi$$

于是 $B = A_1A_2$ 的象征

$$b(x,\xi) = *' \int e^{-i(x-y)(\xi-\eta)} a_1(x,\eta)\,a_2(y,\xi)\,dy\,d\eta$$

我们承认如下定理.

1) $\#: S^{m_1}\times S^{m_2} \to S^{m_1+m_2}$

2) $a_1\#a_2 - a_2\#a_1 \in S^{m_1+m_2-1}$

3) $(a_1\#a_2 - a_2\#a_1) - \frac{1}{i}\{a_1,a_2\} \in S^{m_1+m_2-2}$

定理 若 $a_1\in S^{m_1}$，$a_2\in S^{m_2}$，我们有 $Op(a_1)\,Op(a_2) = Op(b)$，其中 $b = a_1\# a_2 \in S^{m_1+m_2}$ 由上述积分给出，且 $b \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha a_1\, D_x^\alpha a_2$.

推论 在定理的条件下 $[A_1, A_2] \equiv A_1A_2 - A_2A_1$ 是一个 m_1+m_2-1 阶算子，且其象征 $b = \frac{1}{i}\{a_1, a_2\} \mod S^{m_1+m_2-2}$

这里 Poisson 括号 $\{f,g\} = \sum \frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}$

推论的证明. 算子 A_1A_2 的象征 $b_1 \sim \sum \frac{1}{\alpha!}\partial_\xi^\alpha a_1\, D_x^\alpha a_2$

A_2A_1 $\quad b_2 \sim \sum\frac{1}{\alpha!}\partial_\xi^\alpha a_2\, D_x^\alpha a_1$

这样，对于 $b = b_1 - b_2$ 就有

$$b = \sum_j \left(\frac{\partial a_1}{\partial \xi_j} D_j a_2 - \frac{\partial a_2}{\partial \xi_j} D_j a_1\right) \mod S^{m_1+m_2-2}$$

现在，我们来看拟微分算子在函数空间上的作用.

It's a hard thm

定理. 如果 $a\in S^0$，那么 $a(x,D)$ 是 L^2 上的有界线性算子

Idea of proof

注意. $a\in S^0 \Rightarrow a$ 有界 $\Rightarrow |(Af)(x)| \le K\cdot\|\hat f\|_{L^1}$

$B = M - A^*A = C^*C$

证明. 思路是 $\|Af\|_{L^2}^2 = (Af, Af) = (A^*Af, f)$，不等式 $\|Af\|_{L^2}^2 \le M\|f\|_{L^2}^2$

(modulo S^{-m}, m large)

也可以写作 $(Bf, f) \ge 0$ 其中 $B = M - A^*A$ 是一个 0 阶自伴算子.

Using kernel function

为了证明对于充分大的 M，算子 B 满足 $(Bf, f)\ge 0$，最简单的办法

to dominate their

是把 B 写成 C^*C 的形式.

difference($M-A^*A-C^*C$)

a) 我们取 $M \geq 2\sup|a(x,\xi)|^2$，并取 $c(x,\xi)=(M-|a(x,\xi)|^2)^{1/2}$
根据上次课的第一个引理，我们知道 $c\in S^0$，而前一页的定理告诉我们 $C^*C=M-A^*A+R$，$r\in S^{-1}$，由此 $\|Af\|_{L^2}^2\leq M\|f\|_{L^2}^2+(Rf,f)$

b) 现在来控制"误差" (Rf,f) 的上界。我们假设 $r\in S^{-k}$ $(k\geq 1)$，那么因为 $\|Rf\|_{L^2}^2=(R^*Rf,f)$，只要 R^*R 在 L^2 上连续，那么 R 也是，且 $\|R\|_{L^2\to L^2}\leq\|R^*R\|_{L^2\to L^2}^{1/2}$。但我们知道 $r^*\#r\in S^{-2k}$

反复使用这个逻辑，我们只需证明，存在某个充分大的 k，使得对于任意 $r\in S^{-k}$，$Op(r)$ 在 L^2 上连续

c) 我们在 $\mathbb{R}^n$ 中考虑问题，从而可以取 $k=n+1$。此时 $r(x,D)$ 的核函数 $K(x,y)$ 满足 $|K(x,y)|\leq\boxed{*}\int|r(x,\xi)|d\xi$

不仅如此，$(x_j-y_j)K(x,y)$ 是 $i\frac{\partial r}{\partial\xi_j}(x,D)$ 的核函数，是比 r "更好"的函数。重复 $(n+1)$ 次后（取多种可能的 $i_1,i_2\cdots i_{n+1}$）我们得到 $(1+|x-y|^{n+1})|K(x,y)|\leq$ 常数，特别地，

$$\int|K(x,y)|dy\leq\text{常数},\quad\int|K(x,y)|dx\leq\text{常数}\qquad(*)$$

d) 我们有

$$|Rf(x)|^2\leq\int|K(x,y)||f(y)|^2dy\int|K(x,y)|dy$$
$$\leq C\int|K(x,y)||f(y)|^2dy$$

因此 $\int|Rf(x)|^2dx\leq C\int|f(y)|^2dy\int|K(x,y)|dx\leq C^2\int|f(y)|^2dy$

即：核函数满足 $(*)$ 的算子是连续的。

一般地，对于 $s\in\mathbb{R}$，我们 Here it's particular case: usual $W^{s,p}$ can't be discripted by

定义 Sobolev 空间 $H^s(\mathbb{R}^n)=\{\int(1+|\xi|^2)^s|\hat{f}(\xi)|^2d\xi<+\infty\}$ Fourier transform

注意：$(1+D^2)^{s/2}$ 是从 H^s 到 L^2 的等距映射。only $p=2$ it's Hilbert $\Rightarrow W^{s,2}=H^s$

相应的范数记为 $\|f\|_s$： One can extend to $W^{s,p}=\{f\in L^p|\mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}(f)]\in L^p\}$

$$\|f\|_s^2=(2\pi)^{-n}\int(1+|\xi|^2)^s|\hat{f}(\xi)|^2d\xi$$

$\|(1+D^2)^{\frac{s}{2}}(f)\|_{L^2}^2$
$=\int((1+D^2)^{\frac{s}{2}}(f))^2dm$
$=\int(1+D^2)^s(\hat{f})^2dm\,(\frac{1}{2\pi})^n$
$=\|f\|_s^2$ ☒

我们马上有

定理 如果 $a\in S^m$，那么 $a(x,D)$ 把 H^s 映到 H^{s-m}。

Thm is direct due to the differential operator dengenerates to multiplication $(2\pi i\xi)$ in Fourier transform, thus order of $(1+|\xi|^2)$ $+m\Rightarrow s+m$ ☒

(iv) 微分算子 → 微分方程 → 求解

我们有

性质 设 $a\in S^m$，则以下两条等价

① 存在 $b\in S^{-m}$，$a(x,D)b(x,D)-Id\in Op(S^{-\infty})$

② 存在 $b\in S^{-m}$，$b(x,D)a(x,D)-Id\in Op(S^{-\infty})$

由它们可以推出：

③ 存在某个 $C>0$ 使得当 $|\xi|\geq C$ 时，$|a(x,\xi)|\geq C|\xi|^m$

反之，如果③成立，那么存在 $b\in S^{-m}$ 满足①和②，并且所有满足①或②的 b' 在 mod $S^{-\infty}$ 的意义下都和 b 相等

定义 满足③的象征称为椭圆象征，相应的算子称为椭圆算子

例：$\mathbb{R}^n$ 上的 Laplace 算子 $\Delta=-\sum\partial_{x_i}^2$ 的象征是 $\sum\xi_i^2=|\xi|^2$

证明. 若 $b',b''\in S^{-m}$ 满足 $AB'-Id\in Op(S^{-\infty})$，$B''A-Id\in Op(S^{-\infty})$，则

$$B''-B'=B''(Id-AB')+(B''A-Id)B'\in Op(S^{-\infty})$$

这样就得到 $B'A-Id\in Op(S^{-\infty})$，$AB''-Id\in Op(S^{-\infty})$

在另一方面，①或②推出 $a(x,\xi)b(x,\xi)-1\in S^{-1}$，且当 $|\xi|$ 充分大时

$$1/2\leq|a(x,\xi)|\cdot|b(x,\xi)|\leq C\cdot|a(x,\xi)|\cdot|\xi|^{-m}$$，即得③

$b\in S^{-m}\Rightarrow|b|\leq C\cdot(1+|\xi|)^{-m}$

反之，如果 a 是椭圆的，我们定义 $b=(1+|\xi|^2)^{-\frac{m}{2}}F\left(a(1+|\xi|^2)^{-m/2}\right)$

其中 F 是定义在 $\mathbb{C}$ 上的光滑函数，且当 z 充分大时，$F(z)=\frac{1}{z}$.

所以 $b\in S^{-m+0}=S^{-m}$，且对于某个具有紧支集的函数 χ，成立 $ab=1+\chi(\xi)$。这样，根据算子复合的象征渐近展开公式，就有

$$a(x,D)b(x,D)=Id-r(x,D)\qquad r\in S^{-1}$$

现在对 $k\geq0$，定义 $b_k(x,D)=b(x,D)r(x,D)^k\in Op(S^{-m-k})$，

取 $b'\sim\sum_{k\geq0}b_k\in S^{-m}$，即可得到

$$AB' = A\Big(B' - \sum_{j<k} B_j\Big) + AB\sum_{j<k} R^j$$
$$= (\mathrm{Id} - R)\sum_{j<k} R^j + O_p(S^{-k})$$
$$= \mathrm{Id} - R^k + O_p(S^{-k}) = \mathrm{Id} + O_p(S^{-k})$$ 即得①.

类似地可构造满足②的 B''，从而命题得证. □

注意，存在B使得 $BA - \mathrm{Id} \in O_p(S^{-\infty})$ 并不说明A是单射，但只能说明 $\ker A$ 由 C^∞ 函数构成；更一般地，$Af \in C^\infty \Rightarrow f \in C^\infty$.

例. 我们尝试在原点附近局部地解调和方程 $\Delta f = g \in C^\infty(\mathbb{R}^n)$

设 B_r 是以原点为中心、r为半径的球. 设算子S满足 $\Delta S - \mathrm{Id} \in O_p(S^{-\infty})$

如果解 $f = Su$, 那么存在一个核函数光滑的算子K, 使得 $u + Ku = g$.

我们规定：对于 $u \in L^\infty(B_r)$, $\tilde{u} = \begin{cases} u & 若\ x \in B_r \\ 0 & 若\ x \notin B_r \end{cases}$，规定 Rh 为函数h在B_r上的限制, 我们在 $L^\infty(B_r)$ 中求解 ~~...~~ 方程

$$u + RK\tilde{u} = f \iff u = f - RK\tilde{u}$$

设K的核函数为 $k(x,y)$，则

$$\|RK\tilde{u}\|_{L^\infty(B_r)} \leq \sup_{|x|<r,\,|y|<r} \Big|\int k(x,y)\,\tilde{u}(y)\,dy\Big| \leq \sup_{\substack{|x|<r\\|y|<r}} |k(x,y)|\cdot\|u\|_{L^\infty}\cdot r^n$$

因此当r充分小的时候，$\big\|(f - RK\tilde{u}_1) - (f - RK\tilde{u}_2)\big\|_{L^\infty(B_r)} \leq \alpha\|u_1 - u_2\|_{L^\infty(B_r)}$ $(\alpha<1)$

从而由"压缩映照必有不动点"

我们可以找到满足 $u_0 = f - RK\tilde{u}_0$ 的函数 u_0，取 $f_0 = S\tilde{u}_0$,

则 $\Delta f_0 = \Delta S\tilde{u}_0 = \tilde{u}_0 + K\tilde{u}_0$; $\Delta(Rf_0) = u_0 + RK\tilde{u}_0 = f$.

从而 Rf_0 就是要找的局部解. 我们可以证明它是光滑的

(因为 $u_0 = f + C^\infty$)

On all other sets (open sets $\subset \mathbb{R}^n$ or manifolds), we reduce to $\mathbb{R}^n$ by ① Extend by zero outside $\Omega \subset \mathbb{R}^n$ "smoothly"; ② Locally diffeomorphism.

Q. Depending on the choice of bump functions? Depending on local cooridnates?

接下来，为了（定义）流形上的微分算子，我们要先考虑（$\mathbb{R}^n$中）一个开集上的算子。首先，我们有

性质. 设 $a\in S^m$，并设 K 是 $a(x,D)$ 的核函数，那么对于 $x\neq y$，K 是 C^∞ 的.

证明. 对 $x\neq y$，取 $\chi,\psi\in C_c^\infty$，使得 $\begin{cases}\text{在}x\text{附近}\ \chi=1\\ \text{在}y\text{附近}\ \psi=1\\ \operatorname{supp}\chi\cap\operatorname{supp}\psi=\emptyset\end{cases}$

考虑算子 $\chi\,Op(a)\,\psi$，根据复合算子的象征渐近展开定理，~~象征~~ $\chi a\psi\sim 0$，从而是 $S^{-\infty}$ 中的，所以该算子的核函数 $\tilde{K}=\chi(x)K(x,y)\psi(y)\in C^\infty$，所以当 $x\neq y$ 时，$K(x,y)\in C^\infty$.

定义. 对于开集 $\Omega\subset\mathbb{R}^n$，定义

$$S^m_{loc}(\Omega\times\mathbb{R}^n)=\{a\in C^\infty(\Omega\times\mathbb{R}^n),\ \forall\varphi\in C_c^\infty(\Omega),\ \varphi a\in S^m(\mathbb{R}^n\times\mathbb{R}^n)\}$$

The proof of the prop is left as Exercise.

性质 设连续线性算子 $A: C_c^\infty(\Omega)\to C^\infty(\Omega)$ 满足：对任意的 $\varphi,\psi\in C_c^\infty(\Omega)$，$\varphi A\psi\in Op(S^m)$，那么存在 $a'\in S^m_{loc}(\Omega\times\mathbb{R}^n)$，使得 $A=a'(x,D)+R$，其中 R 是一个核函数在 $C^\infty(\Omega\times\Omega)$ 中的算子. 这里象征 a' 在 $\mathrm{mod}\,S^{-\infty}_{loc}(\Omega\times\mathbb{R}^n)$ 的意义下是唯一的.

在上述性质所描述的（情况）下，我们称 A 是 Ω 上的拟微分算子，并把 a' 在 $S^m_{loc}/S^{-\infty}_{loc}$ 中的等价类称为 A 的象征.

定义 连续线性算子 $A: C_c^\infty(\Omega)\to C^\infty(\Omega)$ 被称作具有恰当支撑的，如果对任意紧子集 $K\subset\Omega$，存在紧子集 $K'\subset\Omega$ 满足

$\operatorname{supp} f\subset K\Rightarrow \operatorname{supp} Af\subset K'$，且 在 K' 上 $f=0\Rightarrow$ 在 K 上 $Af=0$

性质. 设算子 $A=a(x,D)$ 的象征 $a\in S^m_{loc}(\Omega\times\mathbb{R}^n)$. 存在一个具有 $C^\infty(\Omega\times\mathbb{R}^n)$ 的核函数的算子 R，使得 $A+R$ 是恰当支撑的.

Exercises

习题.

1. 证明 若 $a_1(x,D)$, $a_2(x,D)$ 是两个微分算子, 则对于 $b(x,D)=a_1(x,D)a_2(x,D)$, 我们有 $b(x,\xi)=\sum_{\alpha\ge0}\frac{1}{\alpha!}\partial_\xi^\alpha a_1 D_x^\alpha a_2$ (注意和式是有限的)

2. 在0阶算子的 L^2 有界性定理的证明中, c)部分的第一个不等式右边以*代替的常数可取为多少?

3. 我们知道, L^2 上的有界线性算子全体 $B(L^2)$ 有自然的Banach空间结构. a) 验证 $S^0 \xrightarrow{Op} B(L^2)$, $a\mapsto a(x,D)$ 是连续的. 它的像是闭的么?

b) 设 $a\in S^0$, 证明 $[0,1]\to B(L^2)$, $\varepsilon\mapsto a(x,\varepsilon D)$ 一般来说不是连续的.

4. 对于自然数 s, 证明 $H^s=\{u\in L^2,\ D^\alpha u\in L^2,\ |\alpha|\le s\}$

5. 设 $\Omega\subset\mathbb{R}^n$ 是开集, $(U_i)_{i\in I}$ 是一族相对紧开子集, $\Omega\subset\bigcup_{i\in I}U_i$. 我们要证明存在一族函数 $(\varphi_i)_{i\in I}$, 满足

① $\forall i$, $\varphi_i\in C_0^\infty(U_i)$, $0\le\varphi_i\le1$;

② 对 Ω 的任意紧子集 K, $\{i,\ \operatorname{supp}\varphi_i\cap K\ne\emptyset\}$ 是有限集

③ 在 Ω 中, $\sum\varphi_i\equiv1$

a) 证明, 我们可以假设 $I=\mathbb{N}$.

b) 假设对于任意 $K\subset\Omega$ 紧, $\{i,\ U_i\cap K\ne\emptyset\}$ 有限, 证明存在一族开集 (V_i) 使得 $\forall i$, $\overline{V_i}\subset U_i$, 且 $\Omega=\bigcup_{i\in I}V_i$

取 $\psi_i\in C_c^\infty(U_i)$, $\psi_i\ge0$, $\psi_i|_{V_i}>0$, 则 $\varphi_i=\frac{\psi_i}{\sum_j\psi_j}$ 即为所求

c) 对一般情况进行证明.

6. 对 5.b) 中的 φ_i, 和上页第一个框中的 A, 记 $A_{jk}=\varphi_j A\varphi_k\in Op(S^m)$

(1) 证明, $a'=\sum_{\operatorname{supp}\varphi_j\cap\operatorname{supp}\varphi_k\ne\emptyset} a_{jk}\in S^m_{loc}(\Omega\times\mathbb{R}^n)$

(2) 设 A 的核函数为 $K(x,y)$, 证明 $\sum_{\operatorname{supp}\varphi_j\cap\operatorname{supp}\varphi_k=\emptyset} A_{jk}$ 的核函数是 $\sum\varphi_j(x)K(x,y)\varphi_k(y)$

(3) 由此证明 ~~[illegible]~~ 该条性质.

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1. $Op(b)=Op(a_1)Op(a_2)$
$\Rightarrow b=a_1\# a_2 \pmod{S^{m_1+m_2-N}}$
$\Rightarrow b\sim\sum\frac{1}{\alpha!}(\partial_\xi^\alpha a_1)(D_x^\alpha a_2)$
By a_1 & a_2 are differential operators (polynomial on ξ)
$\Rightarrow b-\sum\frac{1}{\alpha!}(\partial_\xi^\alpha a_1)(D_x^\alpha a_2)\in S^{m_1+m_2-N}=\{0\}$
$\Rightarrow b=\sum\frac{1}{\alpha!}(\partial_\xi^\alpha a_1)(D_x^\alpha a_2)$ ☒

2. $|K(x,y)|$ the kernel of $r(x,\xi)$
$=|(\frac{1}{2\pi})^n\int e^{i(x-y)\xi}r(x,\xi)d\xi|$
$\le(\frac{1}{2\pi})^n|\int e^{i(x-y)\xi}d\xi|\,|\int r(x,\xi)d\xi|$
$\le|(\frac{1}{2\pi})^n|\int e^{i(x-y)\xi}d\xi|\int|r(x,\xi)|d\xi$
"*"
$\Rightarrow *\le(\frac{1}{2\pi})^n(\frac{2}{x-y})^n$
thus one can take $*=\frac{1}{(\pi(x-y))^n}$ ☒

3.(a) $\|Op\|^2=\sup_{\|a\|_{S^0}=1}\|a(x,D)\|^2_{L^2}$
$=\sup\|a(x,D)f\|^2_{L^2}$, $\|f\|_{L^2}=1$
$=\sup\int|a(x,D)f|^2$, $\int f^2=1$
$=\sup\int|a(x,D)f|^2$, $\int f^2=1$
$\le\sup\int$ [illegible]
$\Rightarrow$ continuous ☒
A sequence $a_i(x,D)\to T\in B(L^2)$
then I claim $T=Op(\lim a_i(x,\xi))\Rightarrow$ closed
due to for S^0, one can construct $a_i(x,\xi)$ from $a_i(x,D)$
(b) By Exercise 2(3) previous, it not continuous at 0 ☒

4. $D^\alpha u\in L^2\iff(1+|\xi|^2)^{\frac{|\alpha|}{2}}\hat u\in L^2$ ($\forall|\alpha|\le s$)
Parseval
$D^\alpha u\in L^2\iff\xi^\alpha\hat u\in L^2$ ☒
with norm $\|f\|_s^2=\sum\|\xi^\alpha f\|^2$

5.(a) By $\Omega\subset\mathbb{R}^n$ is G_δ, every covering has countable refinement (Lindelöf) ☒
(The existence of partition of unity for $\Omega\subset\mathbb{R}^n$)
(b) $\forall K$ compact and $(U_i\cap K)_{finite}$ covers $K\Rightarrow(U_i)$ is locally finite
(i.e. cases of $K=pt$, ... pt applied) $V_i\subset\overline{V_i}\subset U_i$ always exists ☒ (Idea. Taking $U_i\to U_i$)
(c) to reduce to (b), one show that (U_i) is locally finite covering. this is by the topological fact that open subset of para compact ($\mathbb{R}^n$) is also para compact ☒
(Here φ_j, φ_k means to be the multiplicative operator)

6.(1) $A_{jk}(f)=\varphi_j A(\varphi_k f)$ supported in $\operatorname{supp}\varphi_j\cap$ ... compact
$\in C_c^\infty(\Omega\times\Omega)$
$=\varphi_j a(x,D)\varphi_k f$
$=\begin{cases}0, & \operatorname{supp}\varphi_j\cap\operatorname{supp}\varphi_k=\emptyset\\ \varphi_j\varphi_k A(f)\end{cases}$
$\Rightarrow A=\sum_{\operatorname{supp}\varphi_j\cap\operatorname{supp}\varphi_k\ne\emptyset}A_{jk}\pmod{S^{-\infty}}$
as $\sum\varphi_j\varphi_k=1$
$\Rightarrow a'=\sum a_{jk}$ ☒
(2) $\sum A_{jk}(f)(x)=\sum(\frac{1}{2\pi})^n\int e^{ix\xi}$ ... $=\sum\varphi_j(x)K(x,y)\varphi_k(y)$
(3) modulo $S^{-\infty}$, the A is recovered by its kernel $\Rightarrow A=\sum A_{jk}+R$
$=a'(x,D)+R$ ☒

Using the definition 4/9

"force" all local 流形上的算子. 设 M 是一个光滑 (C^∞) 流形, $A: C_c^\infty(M) \to C^\infty(M)$

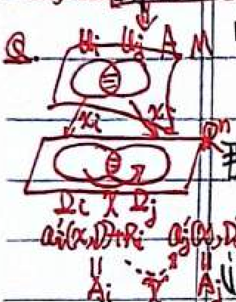
coordinates and all 是一个连续线性算子

extension $U \subset \mathbb{R}^n$ 称 A 是一个拟微分算子, 如果它在每一个坐标卡 (同胚到 $\mathbb{R}^n$ 中的开集) 上

to ensure it's well 都是 (这个开集上的) 拟微分算子

-defined 坐标变换

Q.



性质 设 $\chi: \Omega \to \Omega'$ 是 $\mathbb{R}^n$ 中两个开集之间的一个光滑微分同胚

我们假设对于 $a \in S^m$, 算子 $a(x,D)$ 的核函数在 $\Omega \times \Omega$ 中具有紧支集

那么 (i) 由 $a'(\chi(x),\eta) = e^{-i\chi(x)\eta} a(x,D) e^{i\chi(x)\eta}$ 定义的函数 $a'(y,\eta)$

$(a'(y,\eta) = 0$ 若 $y \notin \Omega')$ 是一个 S^m 类象征

(ii) $a'(y,D)$ 的核函数在 $\Omega' \times \Omega'$ 中具有紧支集.

Gluing all $a_i(x,\xi)$ in 证明. 首先, 如果 $a \in S^m$, 那么 $a(x,D)e^{ix\xi} = e^{ix\xi} a(x,\xi)$. 这是因为

$\Omega_i \overset{\chi_i}{\cong} U_i$ into $a(x,\xi)$ 当 $\hat{u} \in C_c^\infty$ 时, $a(x,D)u(\varepsilon x)e^{ix\xi} = e^{ix\xi}(2\pi)^{-n}\int e^{i\varepsilon x\zeta} a(x,\xi+\varepsilon\zeta)\hat{u}(\zeta)\,d\zeta$

on manifold: invariant 如果还有 $(2\pi)^{-n}\int \hat{u}(\zeta)\,d\zeta = u(0) = 1$, 那么 $\operatorname{supp}\hat{u}$ 紧 $\Rightarrow$ 在 $\operatorname{supp}\hat{u}$ 上,

multiply the approximate unity to make convergen

under χ? $a_j(x,D)+R_j = a_i'(\chi(x),D)+R_i$ $e^{i\varepsilon x\zeta} \Rightarrow 1$, $a(x,\xi+\varepsilon\zeta) \Rightarrow a(x,\xi)$ 从而结论成立

用广义函数的语言, 而对于 $v \in S$, 容易看到 $\lim\limits_{\varepsilon\to 0}\int u(\varepsilon x)e^{ix\xi}v(x)\,dx = \int e^{ix\xi}v(x)\,dx$

"$u(\varepsilon x)e^{ix\xi} \to e^{ix\xi}$ $\Leftarrow \lim\limits_{\varepsilon\to 0}\langle u(\varepsilon x)e^{ix\xi}, v\rangle = \langle e^{ix\xi}, v\rangle, \forall v \in S$

在 S' 中成立"

和上面的讨论

现在, 假设我们能够证明 (i), 那么 a' 的定义式表明, 当 $u(x) = e^{ix\eta}$ 时,

$$a(x,D)(u\circ\chi) = (a'(x,D)u)\circ\chi \quad 且 \ (记\ \rho_x(y) = \chi(y) - \chi(x) - \chi'(x)(y-x))$$

$$a'(\chi(x),\eta) \sim \sum \frac{1}{\alpha!}\partial_\xi^\alpha a(x, {}^t\chi'(x)\eta)\, D_y^\alpha(e^{i\rho_x(y)\eta})\big|_{y=x}$$

最后来证明 (i), 如果 $\varphi \in C_0^\infty(\Omega)$ 对于 a 的核函数的支集的一个邻域中的

(x,y) 满足 $\varphi(x) = \varphi(y) = 1$, 那么

$$a'(\chi(x),\eta) = \varphi(x)\, e^{-i\chi(x)\eta} a(x,D)(\varphi(x) e^{i\chi(x)\eta})$$

记 $\lambda = 1+|\eta|$, 习题 2(5) 给出了一个依赖于 λ, $a'(\chi(x),\eta)$ 及其关于 x 和 η 的各阶导数的渐近展开式, 容易看到这个展开式对于参数 $\alpha = \frac{\eta}{\lambda}$ 是一致的,

由此可知, a' 是一个关于 (y,η) 的象征.

View $C_c^\infty(M)$ and 这样我们可以给出流形上的拟微分算子的精确定义

$C^\infty(M)$ as sections 定义 算子 $A: C_c^\infty(M) \longrightarrow C^\infty(M)$ 被称作一个 m 阶拟微分算子，

we can generalise 如果对于任意局部坐标系 $\varphi: V \longrightarrow U \subset \mathbb{R}^n$，相应的 $\tilde{A}: C_c^\infty(U) \to C^\infty(U)$

to pseudo-differential 是 U 上的 m 阶拟微分算子，即对于任意 $\psi, \eta \in C_c^\infty(U)$ $u \mapsto [A(u\circ\varphi)]\circ\varphi^{-1}$

operator from $\Gamma(M,\mathcal{O}_M)$ $\psi\tilde{A}\eta \in Op(S^m)$，记作 $A \in \Psi^m(M)$

(or $\Gamma^\infty(M,\mathcal{O}_M)$) to other

bundles $\Gamma(M,E)$ 在上述性质中，如果把 a 写成 $a = a_m + S^{m-1}$ 的形式，其中 a_m 是 m 次齐次的，

(Due to later we 那么，对 a' 也成立同样的结论，且 $a'_m(\chi(x),\eta) = a_m(x, {}^t\chi'(x)\eta)$

see the Dirac operator 此时我们称 a_m 是 A 的主象征

doesn't act on rank 1) 在流形上，如果 $A \in \Psi^m(M)$ 在每个坐标图的表述式中都有一个主象征，那么

thus $a(x,\xi)$ the 容易看到这其实是 $\boxed{T^*M \text{上的一个}(m\text{次齐次})\text{函数}}$ 在不同的局部坐标中的表述式，

principal symbol 我们称这个函数是 A 的主象征。 From the Prop (i), one can see $a'(y,\eta) = e^{\ldots}a(x,D)\ldots$

where $\chi: U_i \longrightarrow U_j$ preserves in $U_i \cap U_j \ni x$, $a'(y,\eta) = a(\chi(x),\ldots) = a(x,\ldots)$

can be a matrix 我们无法确定 A 的 $m-1$ 阶象征，所以对于流形上的算子，我们只有

(lower terms change in coordinate changing)

-value function 定理 (a) 如果 $A_i \in \Psi^{m_i}$ $(i=1,2)$ 是恰当支撑且具有主象征 a_i $(i=1,2)$，

那么 $A = A_1A_2 \in \Psi^{m_1+m_2}(M)$ 也是恰当支撑的，且具有主象征 a_1a_2

(b) 交换子 $[A_1, A_2]$ 具有主象征 $\frac{1}{i}\{a_1, a_2\}$

证，(a)(b) reduce to $\Omega \subset \mathbb{R}^n$ easily, and then (b) is obvious, consider (a) only:

and K_2, $f=0$ 引理 $\forall K_2 \subset \Omega$, $\exists K_2' \subset \Omega$, s.t. $\mathrm{supp} f \subset K_2 \Rightarrow \mathrm{supp}\, Af \subset K_2'$, $\exists K_1' \subset \Omega$, $K_1 := K_2' \Rightarrow \mathrm{supp}(A\ldots) \subset K_1'$ 意义

$\Rightarrow K_2$, $A_2f = 0$ 1. 振荡积分：我们要给出一大类形如 $\int_{\mathbb{R}^N} e^{i\varphi(x)}a(x)\,d^n x$ 的积分的合理

$\Rightarrow K_1$, $A_2f=0$ (1) 设 $K \subset\subset \mathbb{R}^N$ (即"K是$\mathbb{R}^N$中的一个紧子集")，设 $\varphi \in C^\infty(\mathbb{R}^N)$ 满足：在 K 中任意

$\Rightarrow K_1$, $(A_1A_2)f = 0$ 一点 x 处，$|\varphi\text{的梯度}| \geq C_0 > 0$，那么对于任意 $a \in C_c^\infty(K)$ 和任意 $k \in \mathbb{N}$ $\lambda \geq 1$

$\nabla\varphi \neq 0$ nonvanishing we call φ is non-stationary, and we'll

thus A_1A_2 is proper 成立 $\lambda^k \left|\int e^{i\lambda\varphi(x)}a(x)dx\right| \leq C_{k+1}(\varphi)\,C(c_0,K)\sup_{|\alpha|\leq k}|\partial^\alpha a|$ estimate by the principle of nonstationary phase

supported. Then this shows that $\forall \varphi,\psi$, $\varphi(A_1A_2)\psi$'s symbol

Some times one write $I_\varphi(a)(x) = \int e^{i\varphi(x,\xi)}a(x,\xi)d\xi$ more general

preserves highest order term (2) 对于 $\rho \in (-\infty, 1]$，$m \in \mathbb{R}$，我们记

$\Rightarrow a = a_1a_2$ principal value $A_\rho^m(\mathbb{R}^N) = \{a \in C^\infty(\mathbb{R}^N), \forall x \in \mathbb{R}^N, \alpha \in \mathbb{N}^N, |\partial^\alpha a(x)| \leq C_\alpha(1+|x|)^{m-\rho|\alpha|}\}$ Our question of estimate is the stationary case

证明，当 $m < -N$ 时，对于 $a \in A_\rho^m$，积分 $I_\varphi(a) = \int_{\mathbb{R}^N} e^{i\varphi(x)}a(x)d^n x$ 是有意义的

其中 $\varphi \in C^\infty(\mathbb{R}^N\setminus\{0\})$ 是一个 d 次齐次函数 $(d>0)$.

(5) 对于任意 $\alpha,\beta\in\mathbb{N}^n$，证明，$(2\pi)^{-n}\int e^{-iy\cdot\eta}y^\alpha\eta^\beta d^ny\,d^n\eta=(-i)^{|\alpha|}\cdot\alpha!\cdot\delta_{\alpha,\beta}$. $\delta_{\alpha,\beta}=\begin{cases}1 & 若\alpha=\beta\\ 0 & 若\alpha\neq\beta.\end{cases}$

(3) 找两个函数 $\chi_0,\chi\in C_c^\infty(\mathbb{R}^N)$，使得 $\operatorname{supp}\chi_0\subset\{|x|\le 1\}$，$\operatorname{supp}\chi\subset\{\frac{1}{2}\le|x|\le 2\}$

且 对于任意 $x\in\mathbb{R}^N$，$1=\chi_0(x)+\sum_{p=0}^{\infty}\chi(2^{-p}\cdot x)$

(4) 利用(1)，证明，若 φ 满足 $(\nabla\varphi)(x)\neq 0$ $(\forall x\in\mathbb{R}^N\setminus\{0\})$，那么 对于任意 $a\in A_p^m$，级数 $\underbrace{\int_{\mathbb{R}^N}e^{i\varphi}\chi_0 a+\sum_{p=0}^{\infty}\int e^{i\varphi(x)}\chi(2^{-p}x)a(x)d^nx}_{(2)中的}$ 都是收敛的，且对于 $m<-N$，就等于 $I_\varphi(a)$，从而可以作为 $I_\varphi(a)$ 的定义.

2. 设 $\psi\in C^\infty(\mathbb{R}^n)$ 是一个实值函数，且对于任意 $x\in\mathbb{R}^n$，$(\nabla\psi)(x)\neq 0$，并设 $u\in C_c^\infty(\mathbb{R}^n)$，$a\in S^m(\mathbb{R}^n\times\mathbb{R}^n)$. 记 $I(x,\lambda)=e^{-i\lambda\psi(x)}a(x,D)(ue^{i\lambda\psi})(x)$ 对 $\lambda\ge 1$.

(1) 记 $\Phi(x,\eta,\lambda)=\int e^{-iz\cdot\eta}e^{i\lambda r_x(x+z)}u(x+z)d^nz$ (其中 $r_x(y)=\psi(y)-\psi(x)-d\psi(x)(y-x)$)

(利用适当的变量代换) 证明：$I(x,\lambda)=(2\pi)^{-n}\int a(x,\eta+\lambda d\psi(x))\Phi(x,\eta,\lambda)\,d\eta$

$\begin{cases}z\mapsto x+z\\ y\mapsto y+\lambda d\psi(x)\end{cases}$

(2) 证明，对于充分大的 C，在 $\frac{\lambda}{C}\le|\eta+\lambda d\psi(x)|\le C\lambda$ 之外，对任意 $k\in\mathbb{N}$，存在 C_k，使得 $|\Phi(x,\eta,\lambda)|\le C_k(1+|\eta|+\lambda)^{-k}$

(3) 把 $I(x,\lambda)$ 写成 $I_1(x,\lambda)+I_2(x,\lambda)$，其中

$$I_2(x,\lambda)=(2\pi)^{-n}\iint e^{-iz\cdot\eta}e^{i\lambda r_x(x+z)}u(x+z)\chi\left(\frac{\eta}{\lambda}+d\psi(x)\right)a(x,\eta+\lambda d\psi(x))\,dz\,d\eta$$

这里的 $\chi\in C_c^\infty(\mathbb{R}^n)$，满足 $\chi(\zeta)=\begin{cases}0 & 当 |\zeta|<\frac{1}{2C}时\\ 1 & 当 \frac{1}{C}\le|\zeta|\le C时.\end{cases}$ 证明 I_1 关于 λ 是速降的.

(4*) 假设 $r\in C^\infty(\mathbb{R}^n)$，$dr(0)=0$. 设 $b\in C^\infty(\mathbb{R}^n_z\times\mathbb{R}^n_\eta\times[1,+\infty))$ 满足：存在某个 m，使得 $\forall\alpha,\beta$，$|\partial_z^\alpha\partial_\eta^\beta b(z,\eta,\lambda)|\le C_{\alpha\beta}\lambda^{m-|\beta|}$，我们还假设 b 相对于 η,λ 具有一致的 z 支集. 证明，$J(\lambda):=(2\pi)^{-n}\iint e^{-iz\cdot\eta}e^{i\lambda r(z)}b(z,\eta,\lambda)\,dz\,d\eta$ 在 $\lambda\to+\infty$ 时 紧 具有渐近展开 $J(\lambda)\sim\sum_{\alpha\ge 0}\frac{1}{\alpha!}\partial_z^\alpha(e^{i\lambda r(z)}D_\eta^\alpha b(z,0,\lambda))\Big|_{z=0}$，且其中 α 阶项可以被 $C_{|\alpha|}\lambda^{m-|\alpha|/2}$ 控制.

$\forall(\eta,\lambda)\in\mathbb{R}^n\times[1,\infty)$
$b_{\eta,\lambda}$ has same support
$\operatorname{supp}(b_{\eta,\lambda})$ compact

(5) 证明，若在(3)中记 $b(x,z,\eta,\lambda)=u(x+z)a(x,\eta+\lambda d\psi(x))\chi\left(\frac{\eta}{\lambda}+d\psi(x)\right)$，那么可以利用(4)得到：当 $\lambda\to+\infty$ 时，$I(x,\lambda)$ 具有关于 x 局部一致的渐近展式

$$I(x,\lambda)\sim\sum_{\alpha\ge 0}\frac{1}{\alpha!}\partial_y^\alpha(e^{i\lambda r_x(y)}u(y))\Big|_{y=x}D_\xi^\alpha a(x,\lambda,d\psi(x))$$

Hanc marginis exiguitas non caperet.

I move these two Exercisely to last page.

下面我们考虑流形上的一个基本的算子：Laplace算子。

这里我们考虑 $\Delta: C_c^\infty(M) \to C^\infty(M)$，这个算子应该具有的性质是，当我们用任意的局部坐标系把流形M上的函数"搬"到 $\mathbb{R}^n$ 中的一个开集上去的时候，这个算子的作用和（的一个"局部"(=开子集)）$\mathbb{R}^n$ 上的 Laplace算子是一致的。

向量的模长 用 $|\cdot|$ 或 $\|\cdot\|$ 表示均可。

这样，按上面的讨论，Δ 的(主)象征就应该是 $\sigma(\xi) = -\|\xi\|^2$ (principal symbol preserved under diffeomorphism)

我们也可以有更为"直观"的办法来定义 Δ.

1) 在经典的多元微积分中，有三个基本运算

梯度：从数量(值函数)得到向量(值函数) $f \mapsto \nabla f$

散度：从向量 得到数量 $X \mapsto \nabla\cdot X$

旋度：从向量 得到向量 $X \mapsto \nabla\times X$

$$\Delta f = \mathrm{div}(\mathrm{grad} f) = \nabla^2 f$$

2) 在一个黎曼流形上，记度量 $g=(g_{ij})$ $(g^{ij}) = (g_{ij})^{-1}$ (这也是一个正定阵)

函数 f 的梯度 $\mathrm{grad} f$ 自然要满足：对于任意向量场 Y，$\langle \mathrm{grad} f, Y\rangle = df(Y) = Y(f)$

这可以作为梯度的定义。在局部坐标系 $x_1,\cdots,x_n$ 下，$\mathrm{grad} f = \sum_i\sum_j g^{ij}\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i}$

向量场X的散度可定义为：

$$(\mathrm{div} X)(p) = \mathrm{trace}(Y \mapsto \nabla_Y X) \quad Y\in M_p$$
$$= \sum_{i=1}^n \langle \nabla_{Y_i} X, Y_i\rangle \quad Y_1,\cdots,Y_n\in M_p \text{ 标准正交}$$

Γ_{jk}^i 是Christoffel符号，而在一个局部坐标系里，$X=\sum a_i\frac{\partial}{\partial x_i} \Rightarrow \mathrm{div}\, X = \sum_{i=1}^n\left(\frac{\partial a_i}{\partial x_i} + \sum_{j=1}^n a_j\Gamma_{ji}^i\right)$

$\nabla_{\partial_j}\partial_k = \sum \Gamma_{jk}^i\partial_i$
$\Rightarrow \Gamma_{jk}^i = \langle \nabla_{\partial_j}\partial_k, \partial_i\rangle$
where $\partial_i = \frac{\partial}{\partial x_i}$ for simplicity.

这样，$\Delta f = \mathrm{div}(\mathrm{grad} f)$

$$= \sum_{i=1}^n\left[\frac{\partial}{\partial x_i}\left(\sum_{j=1}^n g^{ij}\frac{\partial f}{\partial x_j}\right) + \sum_{j=1}^n \left(\sum_k g^{jk}\frac{\partial f}{\partial x_k}\right)\Gamma_{ji}^i\right]$$

$$\overset{(*)}{=} \sum_{i,j=1}^n g^{ij}\left(\frac{\partial^2 f}{\partial x_i\partial x_j} - \sum_{k=1}^n\frac{\partial f}{\partial x_k}\Gamma_{ij}^k\right)$$

若我们取p点处的法坐标系，$\Gamma_{ij}^k(p)=0$，$g_{ij}(p)=\delta_{ij}$，则 $(\Delta f)(p) = \sum_{i=1}^n\frac{\partial^2 f}{\partial x_i^2}(p)$

换个写法就是：对于在p点标准正交的向量场 $X_1,\cdots,X_n$

$$(\Delta f)(p) = (\mathrm{div}(\mathrm{grad} f))(p) = \mathrm{trace}(X\longmapsto \nabla_X \mathrm{grad} f) \quad X\in M_p$$
$$= \sum_{i=1}^n\langle \nabla_{X_i(p)}\mathrm{grad} f, X_i(p)\rangle = \sum_{i=1}^n X_i(p)\langle \mathrm{grad} f, X_i\rangle - \sum_{i=1}^n\langle \mathrm{grad} f(p), \nabla_{X_i(p)}X_i\rangle$$

$$= \sum_{i=1}^{n} X_i(p)(df(X_i)) - \sum_{i=1}^{n} \langle (\operatorname{grad} f)(p), \nabla_{X_i(p)} X_i \rangle$$

$$= \sum_{i=1}^{n} (X_i X_i f)(p) - \sum_{i=1}^{n} \langle (\operatorname{grad} f)(p), \nabla_{X_i(p)} X_i \rangle$$

所以，当 $X_1, \cdots, X_n$ 在 p 点处标准正交 且 $(\nabla_{X_i} X_i)(p) = 0$ 时，

$$(\Delta f)(p) = \sum_{i=1}^{n} (X_i X_i f)(p)$$

注：1. 最早考虑曲面上的 Δ 的人是 Beltrami，所以上述算子亦称 Laplace-Beltrami 算子

2. 我们为了把算子"开方"，需要考虑的算子是正的，因此在定义 Δ 时经常会加一个"$-$"号 (Laplace-Hodge operator $\Delta(\text{Hodge}) = -\Delta(\text{Beltrami})$)

$(\varphi, d^*\psi) = (\varphi, (-1)^{np+n+1}(*d*)\psi) = (-1)^{np+n+1}\int \varphi \wedge *(*d*)\psi = (-1)^{np+n+1}\int \langle \varphi, (*d*)\psi \rangle dvol \overset{\text{Stokes}}{=} \int d\varphi \wedge (*\psi)$

$= (d\varphi, \psi)$ ▨

习题：

1. 验证 $\operatorname{div}(fX) = Xf + f \operatorname{div} X = df(X) + f \cdot \operatorname{div} X$

2. (1) 写出 Christoffel 符号的定义 (Written in last Page 26)

(2) 证明上页中的 (*) 式

最后，如果我们要找到 Laplace 算子的"平方根" Dirac 算子 D，假设具有下述形式：$D = \sum_{i=1}^{n} a_i \nabla_{e_i}$ (e_i 是在 p 点处标准正交的一组向量场)

a_i 是待定的数对象，那么形式计算

$$D^2 = \sum_{i=1}^{n} a_i \nabla_{e_i}\left(\sum_{j=1}^{n} a_j \nabla_{e_j}\right)$$

$$= \underbrace{\sum_{i=1}^{n} a_i a_i \nabla_{e_i}\nabla_{e_i}}_{\Delta \;\Leftarrow\; \sum a_i^2 = 1} + \boxed{\sum_{i<j} (a_i a_j + a_j a_i)\nabla_{e_i}\nabla_{e_j}} + \sum_{i<j} a_j a_i (\nabla_{e_j}\nabla_{e_i} - \nabla_{e_i}\nabla_{e_j})$$

$\sum(a_i a_j + a_j a_i) = 0$

$$+ \underbrace{\sum_{i=1}^{n} a_i \sum_{j=1}^{n} (\nabla_{e_i} a_j) \nabla_{e_j}}$$

symmetric terms (respect to i, j)

在微分几何中，对于 Levi-Civita 联络，$\nabla_{e_j}\nabla_{e_i} - \nabla_{e_i}\nabla_{e_j} "=" R(e_i, e_j) - \nabla_{[e_i, e_j]}$ 会是一个关于 i, j 有对称性的量，

因此 $\boxed{}$ 中 $\nabla_{e_i}\nabla_{e_j}$ 的"系数" $a_i a_j + a_j a_i$ 要等于 0。

The Laplace-"Hodge" name comes from:

$\Omega(X) \cong \Lambda^*(T^*M)$

$D \downarrow \uparrow \hat{D} \qquad \downarrow d \uparrow d^*$

$\Omega(X) \cong \Lambda^*(T^*M)$

d^* the adjoint of d

$\Rightarrow d^* = (-1)^{np+n+1} * d *$

p is dim of $*$ acting the $\Lambda^p(T^*M)$ ($\boxed{\text{Hodge}}$ $*$ operator)

$D \cong d + d^*$; Dirac

$\hat{D} \cong (-1)^p(d - d^*)$, Dirac (right)

$\Rightarrow \Delta = D\hat{D} = \hat{D}D$ is the $\boxed{\text{Hodge}}$ analogue

Exercises.

1. $\operatorname{div}(fX) = \sum \langle \nabla_{E_i} fX, E_i \rangle$

$= \sum \langle f \nabla_{E_i} Y + (E_i f) X, E_i \rangle$

$= f \sum \langle \nabla_{E_i} Y, E_i \rangle + \sum \langle (E_i f) X, E_i \rangle$

$= f \operatorname{div} X + (\operatorname{grad} f)(X)$

$= f \operatorname{div} X + df(X)$ ▨

2. (2) (*) $=$

$\sum_i \frac{\partial}{\partial x_i}\left(\sum_j g^{ij}\frac{\partial f}{\partial x_j}\right) + \sum_k g^{ik}\frac{\partial f}{\partial x_k}\Gamma^i_{ji} = \sum_{i,j}\left(\frac{\partial}{\partial x_i}\left(g^{ij}\frac{\partial f}{\partial x_j}\right) - \sum_k g^{jk}\frac{\partial f}{\partial x_k}\Gamma^i_{ij}\right) = \sum_{i,j} g^{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{k \neq i}\Gamma^k_{ji} g^{ij}\frac{\partial f}{\partial x_k} - \sum_k g^{jk}\frac{\partial f}{\partial x_k}\Gamma^i_{ij}$

$= \sum_{i,j} g^{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_k \Gamma^k_{ij} g^{ij}\frac{\partial f}{\partial x_k}$

$= \sum_{i,j} g^{ij}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_k \Gamma^k_{ij}\frac{\partial f}{\partial x_k}\right)$ ▨

(By $\frac{\partial}{\partial x_i} g(-,-) = g(\boxed{\nabla_{\frac{\partial}{\partial x_i}} -}, -) + g(-, \boxed{\nabla_{\frac{\partial}{\partial x_i}} -}) + \cdots$ writing $\square$ by Christoffel, writing in local $g^{ij} = g_{ij}$ case)

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Clifford 代数 (ℝ或ℂ上的)

1. 张量代数：设 V 是一个（有限维）线性空间，$\{e_i\}$ 是一组基，$V\otimes V$ 是由 $\{e_i\otimes e_j\}$ 生成的线性空间。

暂时把它理解成一个记号

对于 $u,v\in V$，设 $u=\sum a_i e_i$，$v=\sum b_j e_j$，那么 $u\otimes v=\sum_{i,j} a_i b_j\, e_i\otimes e_j$

类似定义 $V^{\otimes n}$，并规定 $T(V)=\bigoplus_n V^{\otimes n}$ （$n=0$ 时 $V^{\otimes 0}:=$ 基域ℝ或ℂ）

2. 设 Q 是 V 上的一个二次型，

记 $I(Q)=$ 所有形如 $v\otimes v-Q(v)$ $(v\in V)$ 生成的双边理想

$=$ 所有形如 $v\otimes w+w\otimes v-2Q(v,w)$ $(v,w\in V)$ 生成的双边理想

也可以记为 $Cl(Q)$ 我们定义 Clifford 代数 $C(Q):=T(V)/I(Q)$

Such decomposition into $T(V)=\left(\bigoplus_{k=0}^{\infty}V^{\otimes 2k}\right)\oplus\left(\bigoplus_{k=1}^{\infty}V^{\otimes(2k+1)}\right)=:T^0V\oplus T^1V$

odd and even action on Spinor bundles

induces $S=S^+\oplus S^-$ decomposition

(representation).

是 $T(V)$ 的子代数，且 $I(Q)$ 是这个子代数的理想。

记 $C_0(Q)=\left(\bigoplus_{k=1}^{\infty}V^{\otimes 2k}\right)/I(Q)$，我们有分解 $C(Q)=C_0(Q)\oplus C_1(Q)$

我们以 $\cdot$ 表示 Clifford 代数中的乘法。

习题：

1. 证 $C_0(Q)\cdot C_0(Q)\subset C_0(Q)$，$C_0(Q)\cdot C_1(Q)\subset C_1(Q)$，$C_1(Q)\cdot C_1(Q)\subset C_0(Q)$

分次代数的分次张量积

2. (1) 设 Q_i 是 V_i 上的二次型 $(i=1,2)$，$Q_1\oplus Q_2$ 是 $V_1\oplus V_2$ 上的二次型

我们考虑 $C(Q_1)\hat{\otimes}C(Q_2)=\left(C_0(Q_1)\otimes C_0(Q_2)\oplus C_1(Q_1)\otimes C_1(Q_2)\right)$

Exercises 1. $\cdot: TV\times TV\to TV$ holds the property by counting $\oplus\left(C_0(Q_1)\otimes C_1(Q_2)\oplus C_1(Q_1)\otimes C_0(Q_2)\right)$

$T^0V\cdot T^0V\subset T^0V$

$T^0V\cdot T^1V\subset T^1V$

$T^1V\cdot T^1V\subset T^0V$

passing to quotient

对于 $a\in C(Q_1)$，$a_i\in C_i(Q_1)$，$b\in C(Q_2)$，$b_j\in C_j(Q_2)$

grading pieces

$(a\otimes b_j)\cdot(a_i\otimes b)=(-1)^{ij}(aa_i)\otimes(b_jb)$

证明：$C(Q_1\oplus Q_2)\cong C(Q_1)\hat{\otimes}C(Q_2)$ also ▨

(2) 由此证明，当 $\dim V=n$ 时 $\dim C(Q)=2^n$

2.(1) $C(Q_1\oplus Q_2)\cong C_0(Q_1\oplus Q_2)\oplus C_1(Q_1\oplus Q_2)$ $\quad\left(C_0(Q_1)\otimes C_0(Q_2)\oplus C_1(Q_1)\otimes C_1(Q_2)\right)\oplus\left(C_0(Q_1)\otimes C_1(Q_2)\oplus C_1(Q_1)\otimes C_0(Q_2)\right)$

(2) Run induction $\quad\xleftarrow{id}\sum(a_0\otimes b_0+a_1\otimes b_1)\;+\sum(a_0\otimes b_1+a_1\otimes b_0)$

$\dim V=1$, $\dim C(Q)\leq 2$ $\quad\sum(C_0+C_1)\xrightarrow{id}$ each C_0, C_1 is spanned by their $Q=Q_1\oplus Q_2\Rightarrow I(Q)=I(Q_1)\oplus I(Q_2)$

$\cong\;\Rightarrow$ via modulo only Q_1/Q_2, $C_0\xrightarrow{P_1}a_0$ or a_1, $\xrightarrow{P_2}b_0$ or b_1 ▨

$\dim V=n$, $\dim C(Q)=\dim C(Q_1\oplus Q_2)=\dim C(Q_1)\hat{\otimes}C(Q_2)$

$=\dim C(Q_1)\cdot\dim C(Q_2)=2^{n-1}\cdot 2=2^n$, where $Q_1\otimes Q_2=Q$ has Q_2 is dim1 subspace. ▨

Dirac 算子:

设 (M^n, g) 是一个 n 维黎曼流形，$\{e_i\}$ 是一族局部标准正交基向量场，并假设存在 M 上的一个 Hermite 向量丛 S，使得 Clifford 代数 $C(g) = T(T_pM)/I(g)$ 在 S_p 上作用，且当 $v \in T_pM$ 满足 $g(v,v) = 1$ 时，v 在 S_p 上的作用是保距的。

用 • 表示 Clifford 代数的作用

我们进一步假设存在丛 S 上的联络 ∇^A：对于任意切向量场 X, Y 和任意 $\psi_1, \psi_2 \in \Gamma(S)$：

$$\nabla^A_Y (X \cdot \psi) = (\nabla_Y X) \cdot \psi + X \cdot \nabla^A_Y \psi$$

Here $(-,-)$ defined to be $\int \langle -,- \rangle dvol$

$$(\nabla^A_X \psi_1, \psi_2) + (\psi_1, \nabla^A_X \psi_2) = X(\psi_1, \psi_2)$$

By $Cl^2(V) = spin(V)$ as Lie algebra, its corresponding Lie group Spin(V) (exponential image); then generalise this construction from vector space V to bundle TM. spin(V) ↷ Spin V gives Cl(V) ↷ Spin V, thus Cl(g) ↷ Spinor bundle ▨

在上述假设下，我们定义 $\psi \in \Gamma(S)$，$\{e_i\}$ 是局部正交标准向量场

(1) 旋量丛 S 上的 Laplace 算子：$\Delta_A \psi = -\sum_{i=1}^n \nabla^A_{e_i} \nabla^A_{e_i} \psi - \sum_{i=1}^n \mathrm{div}(e_i) \nabla^A_{e_i} \psi$

(2) Dirac 算子：$D_A \psi = \sum_{i=1}^n e_i \cdot \nabla^A_{e_i} \psi$ (By definition of Clifford algebra, D_A^2 does cancel the second term)

The spinor bundle is $S = P \times_{spin} \mathcal{S}$ the associated bundle of the Spin-principal bundle. $\mathcal{S}$ is the spinor space with $Cl(V) \cong End(\mathcal{S})$

我们马上看到，

$$D_A(f\psi) = \sum_{i=1}^n e_i \cdot \nabla^A_{e_i}(f\psi) = \sum_{i=1}^n e_i \cdot \{df(e_i)\psi + f \nabla^A_{e_i} \psi\}$$

$$= \mathrm{grad}(f) \cdot \psi + f D_A \psi$$

而 $(D_A\psi_1, \psi_2) = \sum_{i=1}^n (e_i \cdot \nabla^A_{e_i} \psi_1, \psi_2) = -\sum_{i=1}^n (\nabla^A_{e_i} \psi_1, e_i \cdot \psi_2)$

$$= -\sum_{i=1}^n \left[e_i (\psi_1, e_i \cdot \psi_2) - (\psi_1, (\nabla_{e_i} e_i) \cdot \psi_2) - (\psi_1, e_i \cdot \nabla^A_{e_i} \psi_2) \right]$$

$$= -\sum_{i=1}^n e_i (\psi_1, e_i \cdot \psi_2) - \sum_{i=1}^n \mathrm{div}(e_i)(\psi_1, e_i \cdot \psi_2) - (\psi_1, D_A \psi_2)$$

这里用到了 $\sum_{i=1}^n \nabla_{e_i} e_i = \sum_{i=1}^n \sum_{j=1}^n (\nabla_{e_i} e_i, e_j) e_j = -\sum_{i=1}^n \sum_{j=1}^n (e_i, \nabla_{e_i} e_j) e_j$

$$= -\sum_{j=1}^n \sum_{i=1}^n (e_j, \nabla_{e_j} e_i) e_i = -\sum_{i=1}^n (\mathrm{div}\, e_i) e_i$$

Exercises 1. For example, the Dirac operator $D_A: \Gamma(S) \to \Gamma(S)$, $\psi \mapsto \sum_i e_i \cdot \nabla^A_{e_i} \psi$ can be identified as $\Gamma(S) \xrightarrow{\nabla} S \otimes T^*M \cong S \otimes TM \to S \otimes Cl(TM) \xrightarrow{\text{Clifford action}} S$ (taking section, D_A), and the Laplacian is similiar ▨

习题

1. 证明，S 上的 Laplace 算子与 Dirac 算子的定义和 $\{e_i\}$ 的选取无关。
2. 对于给定的 $\psi_1, \psi_2 \in \Gamma(S)$，定义切向量场 $V \in \Gamma(T^*M)$ 如下：对任意 $W \in \Gamma(T^*M)$，
$$\langle V, W \rangle_x = -\langle \psi_1, W \cdot \psi_2 \rangle_x \quad \forall x \in M.$$
证明 V 存在唯一。定义 $(\mathrm{div}\, V)_x = \sum_i \langle \nabla_{e_i} V, e_i \rangle_x$，证明，div V 与 $\{e_i\}$ 的选取无关。

2. $(V, W)_x = -(\psi_1, W \cdot \psi_2)_x$ is determined $\Rightarrow$ V(x) is well-defined for $\forall x \in M \Rightarrow$ V also ▨

$\mathrm{div}(V) = \sum (\nabla_{e_i} V, e_i) = \sum \int \langle \nabla_{e_i} V, e_i \rangle dvol = \int \sum \langle \nabla_{e_i} V, e_i \rangle dvol = \int tr(\nabla V) dvol = \int \underbrace{\mathrm{div}(V)}_{\text{(older one)}} dvol$ ▨

利用上一页习题2中的定义，我们马上看到

$(\psi_1, D_A\psi_2) - (D_A\psi_1, \psi_2) = -\sum_{i=1}^{n} e_i(\psi_1, e_i\cdot\psi_2) + (\psi_1, (\nabla_{e_i}e_i)\cdot\psi_2)$

$\Rightarrow \int (\psi_1, D_A\psi_2) - (D_A\psi_1, \psi_2)\, dvol$ $\sum_{i=1} e_i(V, e_i) - (V, \nabla_{e_i}e_i) = \sum(\nabla_{e_i}V, e_i) = \mathrm{div}\, V$

$= \int \mathrm{div}(V)\, dvol = 0$ $\boxed{}$

现在我们假设 流形M 是紧致无边的，此时 $\int_M \mathrm{div} V\, dvol = 0$ (为什么?)

By Stokes formula $\int_M \mathrm{div}\, V\, dvol = \int_{\partial M} V\, dvol = 0$

这样我们就证明了：

性质 (1) 在紧致无边的黎曼流形M上，(如果存在) S 和 D_A，对于 $\psi_1, \psi_2 \in \Gamma(S)$

Here $V \in \Gamma(T^*M)$, T^*M also no boundary $\boxed{}$

成立 $\int_M (D_A\psi_1, \psi_2) = \int_M (\psi_1, D_A\psi_2)$

(2) 去掉 M 紧致的条件，加上"ψ_1, ψ_2 有紧支集"的条件，上式仍成立。

接下来，我们比较 D_A^2 和 Δ_A (One need defining analogues of ∇^A and R^S in S instead of T^*M)

$$D_A^2\psi - \Delta_A\psi = \sum_{i,j} e_i\cdot\nabla^A_{e_i}(e_j\cdot\nabla^A_{e_j}\psi) + \sum_i \nabla^A_{e_i}\nabla^A_{e_i}\psi + \sum_i \mathrm{div}(e_i)\nabla^A_{e_i}\psi$$

$$= \sum_{i,j} e_i\cdot\left[(\nabla_{e_i}e_j)\cdot\nabla^A_{e_j}\psi + e_j\cdot\nabla^A_{e_i}\nabla^A_{e_j}\psi\right] + \sum_i \nabla^A_{e_i}\nabla^A_{e_i}\psi + \sum_i \mathrm{div}(e_i)\nabla^A_{e_i}\psi$$

$$= \sum_{i,j,k}(\nabla_{e_i}e_j, e_k)\, e_ie_k\cdot\nabla^A_{e_j}\psi + \sum_{i,j} e_ie_j\cdot\nabla^A_{e_i}\nabla^A_{e_j}\psi + \sum_i \nabla^A_{e_i}\nabla^A_{e_i}\psi + \sum_i \mathrm{div}(e_i)\nabla^A_{e_i}\psi$$

$$= \sum_j\sum_{i\neq k}(\nabla_{e_i}e_j, e_k)\, e_ie_k\cdot\nabla^A_{e_j}\psi + \sum_j\sum_i \underbrace{(\nabla_{e_i}e_j, e_i)}_{=\mathrm{div}(e_j)}\underbrace{e_ie_i}_{=-1}\cdot\nabla^A_{e_j}\psi$$

$$+ \sum_{i\neq j} e_ie_j\cdot\nabla^A_{e_i}\nabla^A_{e_j}\psi + \underbrace{\sum_i \underbrace{e_ie_i}_{=-1}\cdot\nabla^A_{e_i}\nabla^A_{e_i}\psi + \sum_i \nabla^A_{e_i}\nabla^A_{e_i}\psi}_{=0} + \sum_i \mathrm{div}(e_i)\nabla^A_{e_i}\psi \quad (=0)$$

$$= \sum_j\sum_{i\neq k}(\nabla_{e_i}e_j, e_k)\, e_ie_k\cdot\nabla^A_{e_j}\psi + \sum_{i\neq j} e_ie_j\cdot\nabla^A_{e_i}\nabla^A_{e_j}\psi$$

$e_ie_k + e_ke_i = 0$

$$= \sum_j\sum_{i\neq k} -(e_j, \nabla_{e_i}e_k)\, e_ie_k\cdot\nabla^A_{e_j}\psi + \sum_{i\neq j} e_ie_j\cdot\nabla^A_{e_i}\nabla^A_{e_j}\psi$$

$$= \sum_j\sum_{i<k}(e_j, \nabla_{e_k}e_i - \nabla_{e_i}e_k)\, e_ie_k\cdot\nabla^A_{e_j}\psi + \sum_{i<j} e_ie_j\cdot(\nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{e_j}\nabla^A_{e_i})\psi$$

$$= \sum_{i<k} e_ie_k\cdot\sum_j(e_j, [e_k, e_i])\nabla^A_{e_j}\psi + \sum_{i<j} e_ie_j\cdot(\nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{e_j}\nabla^A_{e_i})\psi$$

$$= \sum_{i<k} e_ie_k\cdot(-\nabla^A_{[e_i,e_k]}\psi) + \sum_{i<j} e_ie_j\cdot(\nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{e_j}\nabla^A_{e_i})\psi$$

R^S: 曲率张量 $= \sum_{i,j} e_ie_j\cdot(\nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{e_j}\nabla^A_{e_i} - \nabla^A_{[e_i,e_j]})\psi = \frac12\sum_{i,j}\overbrace{e_ie_j\cdot}R^S(e_i,e_j)\psi \Rightarrow (D_A^2-\Delta_A)$

$= \frac12\sum(e_ie_j)\cdot R^S(e_i,e_j)$

("S" means that on 由 ∇^A 的性质, bundle S, respect to the metric $\langle,\rangle$ and ∇^A)

$\nabla^A_{Y_1}\nabla^A_{Y_2}(X\cdot\psi) = \nabla^A_{Y_1}((\nabla_{Y_2}X)\cdot\psi + X\cdot\nabla^A_{Y_2}\psi)$ One the calculate $R^S(e_i,e_j)$ precisely.

$= (\nabla_{Y_1}\nabla_{Y_2}X)\cdot\psi + (\nabla_{Y_2}X)\cdot\nabla^A_{Y_1}\psi + (\nabla_{Y_1}X)\cdot\nabla^A_{Y_2}\psi + X\cdot\nabla^A_{Y_1}\nabla^A_{Y_2}\psi$

从而有 $R^S(e_i,e_j)(e_j\cdot\psi) = (R(e_i,e_j)e_j)\cdot\psi + e_j\cdot R^S(e_i,e_j)\psi$

于是 $\sum_{i,j} e_i\cdot R^S(e_i,e_j)(e_j\cdot\psi) = \sum_{i,j} e_i\cdot(R(e_i,e_j)e_j)\cdot\psi + \sum_{i,j} e_ie_j\cdot R^S(e_i,e_j)\psi$

① $\| \quad\quad \| $ ②

$\sum_{i,j} R^S(e_i,e_j)(e_ie_j\cdot\psi) - \sum_{i,j}(R(e_i,e_j)e_i)\cdot(e_j\cdot\psi)$

$\sum_{i,j} R(e_i,e_j)e_i\cdot(e_j\cdot\psi) = -\frac{scal}{2}\psi$

by scal $= tr(Ric) = \sum_{i,j}\langle R(e_i,e_j)e_j, e_i\rangle$

$= \sum_{i,j} R(e_i,e_j)$ 就是 (1,1) 张量

① 是一个与$\{e_i\}$选取无关(为什么?)的量 作用在ψ上，得到一个$\Gamma(S)$的元素.

② 除了最后一步作用在ψ上以外，都是 Clifford 代数中的运算，(算算看)可以得到

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这恰好是一个数量(即 Clifford 代数中的0阶元)($-\frac{K}{2}$) (K = M上的数量曲率)

所以我们得到 Lichnérowicz (-Weitzenböck 或 -Schrödinger) 公式:

$$D_A^2\psi - \Delta_A\psi = \frac{K}{4}\psi + \boxed{\tfrac12 dA}(\psi)$$ 当成一个记号 (A is the connection 1-form and dA is thus 2-form it's due to $dA+[A,A]=0$ the Maurer-Cartan's form ☒)

Another analogue is that $D_A^*D_A\psi = \nabla_A^*\nabla_A\psi + \frac{scal}{4}\psi + \frac12\langle A^+,\psi\rangle$ (A^+ is self-dual part of connection)

这里的重点是从一堆关于丛S的运算中"剥离"出了一个只和TM有关的项 (利用下一页上定义的记号)

thus one has the Lichnérowicz-Laplacian $\Delta_L = \nabla^*\nabla + cRic$, c some constant; when $c=1$, it's Hodge-Laplacian.

当M是一个紧黎曼流形时，我们有:

$$\|D_A\psi\|^2_{L^2} = (D_A^2\psi,\psi)_{L^2} = (\sum_i\nabla^A_{e_i}\nabla^A_{e_i}\psi,\psi) + \int_M\frac{K}{4}(\psi,\psi) + \int_M(\tfrac12 dA(\psi),\psi)$$

(integral) $$= \|\psi\|^2_{H^1} - \|\psi\|^2_{L^2} + \int_M\frac{R}{4}|\psi|^2 + \int_M(\tfrac12 dA(\psi),\psi)$$

而最后一项中，把 $\frac12 dA$ 视为 $S\to S$ 的线性映射 是有界的，于是存在常数c，使得 $-c|\psi|^2 \le (\frac12 dA(\psi),\psi) \le c|\psi|^2$

于是我们得到: ($K_{min/max} = \min/\max_{p\in M}\{K(p)\}$)

$$\|\psi\|^2_{H^1} + (\frac{K_{min}}{4} - c - 1)\|\psi\|^2_{L^2} \le \|D_A\psi\|^2_{L^2} \le \|\psi\|^2_{H^1} + (\frac{K_{max}}{4} + c - 1)\|\psi\|^2_{L^2}$$

(Such domination $\Rightarrow D_A$ has closed range)

When $\dim M = 4$, $D_A\psi = 0$ and $A^+ = -\tau(\psi,\psi)$ (*) where $\tau: \Gamma(S^+)\times\Gamma(S^+)\to\Lambda^+\otimes\mathbb{C}$ is the trace-free part of endomorphism $\alpha\mapsto\langle\alpha,\psi\rangle\psi$

$0 = (D_A^*D_A\psi,\psi)_{L^2} = (\nabla_A^*\nabla_A\psi + \frac12 A^+\psi + \frac{scal}{4}\psi,\psi)$

$= \int(|\nabla_A\psi|^2 + \frac12(A^+\psi,\psi) + \frac{scal}{4}|\psi|^2)$

$= \int(|\nabla_A\psi|^2 + \frac14|\psi|^4 + \frac{scal}{4}|\psi|^2)$ the Selberg-Witten equation ☒

为讨论 Dirac 算子的分析性质，我们需要确定几个基本的空间：

$\Gamma(S) = \{S$ 的光滑截面$\}$，$L^2(S) = \Gamma(S)$ 在 $\|\cdot\|_{L^2}$ 范数下的完备化

$H^1(S) = \Gamma(S)$ 在 $\|\cdot\|_{H^1}$ 范数下的完备化

这里对于 $\psi \in \Gamma(S)$，$\|\psi\|_{L^2}^2 = \int_M (\psi(x), \psi(x))\, dvol_x$

而 $\|\psi\|_{H^1}^2 = \|\psi\|_{L^2}^2 + \int_M \sum_{i=1}^n (\nabla^A_{e_i}\psi, \nabla^A_{e_i}\psi)\, dvol$

我们来证明（先说）对 $\psi \in H^1$，$\|D_A\psi\|_{L^2} \leq \sum_{i=1}^n \|\nabla^A_{e_i}\psi\|_{L^2} \leq \sqrt{n}\, \|\psi\|_{H^1}$

性质 设 (M,g) 是一个紧黎曼流形，则 Dirac 算子 $D_A : H^1(S) \to L^2(S)$ 是一个指标为 0 的 Fredholm 算子

证明. 我们要说明 $\ker(D_A)$ 与 $L^2/\mathrm{Ran}(D_A)$ 都是有限维空间且维数相同.

在 $\ker(D_A)$ 中，我们考虑集合 $K_0 = \{\psi \in H^1(S) : D_A(\psi) = 0,\ \|\psi\|_{L^2} \leq 1\}$

和 $K_1 = \{\psi \in H^1(S) ; D_A(\psi) = 0,\ \|\psi\|_{H^1}^2 = \|\psi\|_{L^2}^2 + \|D_A\psi\|_{L^2}^2 \leq 1\}$.

自然有 $K_0 = K_1$. 同时考虑到（我们暂时承认）$H^1(S) \to L^2(S)$ 是

By Rellich compact embedding theorem, we'll prove it by analysis at last.

紧算子，所以 $K_0 = K_1$ 是 $L^2(S)$ 中的紧集. 这样，$\ker D_A$ 中的球在 L^2 范数下是紧的，从而 $\ker D_A$ 是有限维的. (Riesz)

现在我们来确定 $D_A(H^1)$ 在 L^2 中的正交补，这个子空间中的一个元素 φ 满足：对于任意 $\psi \in H^1(S)$，$(D_A\psi, \varphi)_{L^2} = 0$.

我们取一个支集包含于一个坐标卡的 ψ，把上述等式搬到欧氏空间上，我们得到 $\mathbb{R}^n$ 上的一个椭圆微分算子 D，和相应的 $\tilde{\varphi} \in L^2(\mathbb{R}^n)$，满足对于任意 $\psi \in C_c^\infty(\mathbb{R}^n)$，$(D\psi, \tilde{\varphi})_{L^2} = 0$. 于是由椭圆正则性定理可知 $\tilde{\varphi}$ 光滑.

这里的 $\tilde{\varphi}$ 和 ψ 都是向量值的

从而 $\varphi \in \Gamma(S) \subset H^1$. 由前面证明的性质可知 $(\psi, D\varphi)_{L^2} = 0\ \forall \psi \in H^1$.

于是因为 H^1（包含 $\Gamma(S)$）在 L^2 中稠密）$D\varphi = 0$. 这样就有 $(D_A(H^1))^\perp = \ker D_A$.

Later we'll describe elliptic regularity in Exercises.

最后来证明 $D_A(H^1)$ 是 $L^2(S)$ 中的闭子空间. 设序列 $D_A(\psi_n)$ 在 $L^2(S)$ 中收敛于 ψ，不失一般性我们可以假设 $\psi_n \perp \ker(D_A)$，而此时我们有 $\|D_A\psi_n\|_{L^2} \geq C\|\psi_n\|_{L^2}$（练习）这就是说 $\{\psi_n\}$ 在 L^2 中是一个 Cauchy 列，由 Lichnerowicz 公式的推论可知 $\{\psi_n\}$ 在 H^1 中也是 Cauchy 列，从而收敛到 ψ^*. 由 $D_A : H^1(S) \to L^2(S)$ 的连续性，可知

$$\psi = \lim_n D_A(\psi_n) = D_A(\lim_n \psi_n) \xlongequal{\text{连续}} D_A(\psi^*)，即\ \psi \in D_A(H^1). \qquad \square$$

It's standard to dealing such thing in manifold by reducing to $\mathbb{R}^n$ and using elliptic regularity $\Rightarrow$ smoothness $\Rightarrow$ return back to M^n

this is the elliptic regularity in manifold

An unbounded operator A may not defined in hole $X \xrightarrow{A} Y$ it's closed if Graph(A) ⊂ X⊕Y closed subspace ⟺ $A=\bar{A}$

Cor. A is closed and densely defined iff $A^{**}=A$ (by Von Neumann)

Such operator is normal if $A^*A=AA^*$ ⟺ $Dom(A)=Dom(A^*)$ have same norm.

Self adjoint ⟹ normal.

4/23 More operator theories (unbounded cases)

对于一个(复)Hilbert空间$\mathcal{H}$上的(无界)算子A，除了Dom A 和 Ran A 外，另一个与之关联的重要空间是 $Graph(A)=\{(x,Ax),x\in Dom A\}\subset H\times H$.

我们定义：

1. A的闭包 $\bar{A}$：$Graph(\bar{A})=\overline{Graph(A)}$，$Dom\bar{A}=\{x,\exists x_n\in Dom A, x_n\to x, Ax_n\to y\}$

2. A的伴随 A^*：$Dom A^*=\{x\in H, \exists y\in H, \forall z\in Dom A, \langle Az,x\rangle=\langle z,y\rangle\}$ $=\{x\in\mathcal{H}\mid y\mapsto\langle Ay,x\rangle \text{ is continuous}\}$

The existence of $\langle Ax,y\rangle=\langle x,Ay\rangle$ we use ~~Hahn-Banach first then~~ Riesz ▨ 此时 $A^*x=y$

3. 对称算子：$\forall x,y\in Dom A \quad \langle Ax,y\rangle=\langle x,Ay\rangle$

因此上次课证明的第一个性质就是说，在紧致无边流形M上，取 $Dom D_A=\Gamma(S)$ 那么D_A是对称算子

而对称算子A，$Dom A\subset Dom A^*$，且 $A^*|_{Dom A}=A$. 记为 $A\subset A^*$

自共轭算子 $A=A^*$

E.g. Symmetric operator $\not\Rightarrow$ self-adjoint (even for closed)

注意，由von Neumann定理 $\bar{A}=A^{**}$，于是自共轭算子满足 $\bar{A}=A$，从而是闭的。

本质自伴算子 (essentially self-adjoint) $\bar{A}=A^*$

$A=\frac{1}{i}\frac{d}{dx}: Dom(A)\subset L^2[0,\infty]\to L^2[0,\infty)$ is symmetric: $\int f'\bar{g}=\int f\bar{g'}$, but it's not self-adjoint for $Dom(A)=\{f\in L^2[0,\infty) \text{ absolutely continuous} \mid f'\in L^2 \text{ \& } f(0)=0\}$ ▨

以下我们考虑一个(可能非紧，但没有边界的)黎曼流形M，假设其上有旋量丛S和Dirac算子D_A. 此时 $L^2(S)$ 是 $\Gamma_c(S)$ 的闭包. $Dom D_A=\Gamma_c(S)$

Rk. In general $A=A_1+A_2$, A_1 any symmetric ~~$A_1=cA$, A_2~~ is self-adjoint (may $=0$)

我们来证明：D_A是本质自伴的，即 $\bar{D_A}=D_A^*$.

我们在 $Dom(D_A^*)$ 上定义一个范数 $N(\psi)=\sqrt{\|\psi\|_{L^2}^2+\|D_A^*\psi\|_{L^2}^2}$ the graph norm

引理1. 如果 $\Gamma_c(S)$ 在 $Dom D_A^*$ 中关于N范数是稠密的，那么D_A是本质自伴的.

证明. 我们要证明的是 $Dom D_A^*\subset Dom\bar{D_A}$. 对于 $\psi\in Dom D_A^*$，由假设条件，存在一列 $\psi_n\in\Gamma_c(S)$，$\lim_n N(\psi_n-\psi)=0$. 这蕴含了 $L^2(S)$ 中的 $\lim_n\psi_n=\psi$，和 $\lim_n D_A^*\psi_n=D_A^*\psi$. 注意到$\psi_n$是光滑有紧支集的，从而 $D_A^*(\psi_n)=D_A\psi_n$. 所以 $\{D_A\psi_n\}$ 在L^2中收敛，这就表明 $\psi\in Dom\bar{D_A}$. □

我们接着引进记号 $Dom_c D_A^*=\{\psi\in Dom D_A^*, supp\,\psi \text{ 紧}\}=Dom(D_A^*)\cap\Gamma_c(S)$

引理2. $\Gamma_c(S)$ 在 $Dom_c D_A^*$ 中关于N范数稠密.

证明. 取M的一个局部有限的坐标图册$\{(V_i,\varphi_i),i\in I\}$，使得$\bar{V_i}$是M中的紧

于是 设 f_i 是一个从属于 $\{V_i\}$ 的单位分解，$\mathrm{supp} f_i \subset V_i$

对于给定的 $\psi \in \mathrm{Dom}_c D_A^*$，只有有限个 $i \in I$ 使得 $\mathrm{supp} f_i \cap \mathrm{supp}\,\psi \neq \emptyset$.

记为 $i_1, \cdots, i_l \in I$ 再记 $\psi_j = f_{i_j}\psi$ $(1 \le j \le l)$，那么 $\psi = \psi_1 + \cdots + \psi_l$、具体维数)

ψ_j 可以看成 $\mathbb{R}^n$ 上有紧支集的向量值函数 (取值在 $\mathbb{R}^{2^{[n/2]}}$ 中，不过我们不会用到

考虑 $\mathbb{R}^n$ 上的光滑函数 $h(x) = \begin{cases} 0 & 若 |x| \ge 1 \\ e^{-\frac{1}{1-|x|^2}} & 若 |x| < 1 \end{cases}$ 并记 $h_\varepsilon(x) = \frac{1}{\varepsilon^n} h\left(\frac{x}{\varepsilon}\right)$

那么 $\psi_j * h_\varepsilon$ (每个坐标分量作卷积) 是 $\mathbb{R}^n$ 上的紧支集光滑函数，$\psi_j * h_\varepsilon \xrightarrow{L^2} \psi_j$

并且 $\psi_j \in L^2$，$D_A^* \psi_j * h_\varepsilon \to D_A^* \psi_j$，再回到流形 M 上，我们就有 $\Gamma_c(S)$

中的一列元素 $\psi_{j,1}, \psi_{j,2}, \cdots$ 满足 $\lim\limits_{k\to\infty} N(\psi_j - \psi_{j,k}) = 0$

现在定义 $\psi_k = \psi_{1,k} + \psi_{2,k} + \cdots + \psi_{l,k} \in \Gamma_c(S)$，$\lim\limits_{k\to\infty} N(\psi - \psi_k) = 0$ □

引理3 设 (M, g) 是一个完备黎曼流形，则 $\mathrm{Dom}_c D_A^*$ 在 $\mathrm{Dom}\, D_A^*$ 中关于 N 范数稠密

[illegible] 证明 固定 $m_0 \in M$，对于 $m \in M$，记 $\rho(m) = d(m_0, m)$，这样 $|\rho(m_1) - \rho(m_2)| \le d(m_1, m_2)$，

[illegible] 于是 ρ 是一个 1-Lipschitz 函数。实分析的知识告诉我们 ρ 几乎处处可微，$\mathrm{grad}(\rho)$

[illegible] 几乎处处存在 且 $\|\mathrm{grad}\,\rho\| \le 1$

[illegible] 现在考虑测地球 $B_r = \{m \in M, \rho(m) \le r\}$，根据 M 的完备性假设，B_r 是紧子集.

[illegible] 考虑一个 C^∞ 函数 $\alpha: \mathbb{R} \to [0,1]$，满足：$\alpha(t) \equiv 1$ $(t \in (-\infty, 1])$，$\alpha(t) \equiv 0$ $(t \in [2, +\infty))$

[illegible] 记 $K = \sup |\alpha'(t)|$.

[illegible] 定义 $b_r: M \to [0,1]$ 如下，$b_r(m) = \alpha\left(\frac{\rho(m)}{r}\right)$，则 b_r 在 B_r 上恒为1，$\mathrm{supp}(b_r) \subset B_{2r}$

[illegible] b_r 也是 Lipschitz 函数，$\|\mathrm{grad}(b_r)\|^2 = \frac{1}{r^2}\left|\alpha'\left(\frac{\rho}{r}\right)\right|^2 \|\mathrm{grad}(\rho)\|^2 \le \frac{K^2}{r^2}$

[illegible]

给定 $\psi \in \mathrm{Dom}_c(D_A^*)$，则 $\psi_r = b_r\psi \in \mathrm{Dom}\, D_A$，且 $D_A^*(\psi_r) = \mathrm{grad}(b_r)\cdot\psi + b_r D_A^*(\psi)$.

直接对 L^2-范数的下述估计：

$$\|D_A^*(\psi - \psi_r)\|_{L^2}^2 = \|(1-b_r) D_A^*\psi - \mathrm{grad}(b_r)\cdot\psi\|_{L^2}^2 \le \int_{M \setminus B_r} 2|D_A^*\psi|^2 + \frac{2K^2}{r^2}\int_M |\psi|^2$$

从而，对于 N-范数，我们就有

$$N^2(\psi - \psi_r) = \|\psi - \psi_r\|_{L^2}^2 + \|D_A^*(\psi - \psi_r)\|_{L^2}^2 \le \int_{M\setminus B_r} |\psi|^2 + \int_{M \setminus B_r} 2|D_A^*\psi|^2 + \frac{2K^2}{r^2}\int_M |\psi|^2$$

而 $\int_M |\psi|^2$ 和 $\int_M |D_A^*\psi|^2$ 都有限，利用控制收敛定理，于是 $\lim\limits_{r\to\infty} N^2(\psi - \psi_r) = 0$

综合上述三个引理，我们就得到

性质 如果完备黎曼流形上有旋量丛 S，那么 Dirac 算子 D_A 是 $L^2(S)$ 上的本质自伴算子

接下来我们讨论 Dirac 算子的谱。首先，来明确一些记号的意义：设 A 是 $\mathcal{H}$ 上的(无界)算子。

Point spectrum
$\sigma_p(A)=\{\lambda\in\mathbb{C},\ \ker(A-\lambda I)\neq\{0\}\}$，$\sigma_r(A)=\{\lambda\in\mathbb{C},\ \ker(A-\lambda I)=\{0\},\ \overline{\mathrm{Ran}(A-\lambda I)}\neq\mathcal{H}\}$

Residual spectrum
$\sigma_c(A)=\{\lambda\in\mathbb{C},\ \ker(A-\lambda I)=\{0\},\ \overline{\mathrm{Ran}(A-\lambda I)}=\mathcal{H},\ (A-\lambda I)^{-1}$ 无界$\}$

Continuous spectrum
$\mathbb{C}\setminus(\sigma_p(A)\cup\sigma_r(A)\cup\sigma_c(A))=\rho(A)=\{\lambda\in\mathbb{C},\ \overline{\mathrm{Ran}(A-\lambda I)}=\mathcal{H},\ (A-\lambda I)^{-1}\in B(\mathcal{H})\}$.

(Compression)s

One also have the essential spectrum

现在在紧黎曼流形 M 上，假设我们有 S 和 D_A，那么

性质 i) $\sigma_r(\overline{D_A})=\sigma_c(\overline{D_A})=\varnothing$ 从而 $\sigma(\overline{D_A})=\sigma_p(\overline{D_A})$

ii) $\sigma_p(\overline{D_A})=\sigma_p(D_A)$ $(=\sigma(D_A)=\sigma(\overline{D_A}))$ (Exercises)

证明. 首先，我们考虑 $\lambda\in\sigma_r(D_A)$，那么由定义，存在 $\varphi\in L^2(S)$，使得对于任意 $\psi\in\Gamma(S)$，$((D_A-\lambda I)\psi,\varphi)_{L^2}=0$.

事实上，$P=D_A-\lambda I$

取一个支集在一个坐标卡中的 ψ，把上式搬到欧氏空间中去，那么 $D_A-\lambda I$ 会对应到一个椭圆微分算子(为什么?) P，φ 会对应到一个 $\tilde{\varphi}\in L^2(\mathbb{R}^n)$，使得对于任意 $\tilde{\psi}\in C_c^\infty(\mathbb{R}^n)$，$(P\tilde{\psi},\tilde{\varphi})_{L^2}=0$. 利用欧氏空间上的椭圆正则性定理，$\tilde{\varphi}$ 是光滑的，这会对应到一个光滑的 φ，从而 $\varphi\in\mathrm{Dom}\,D_A$，于是

$\lambda\in\sigma_r(D_A)$ and D_A is elliptic operator (assume $M=\mathbb{R}^n$), i.e.

$(\psi,(D_A-\lambda I)\varphi)_{L^2}=0\quad\forall\psi\in\Gamma(S)\Rightarrow(D_A-\lambda I)\varphi=0\Rightarrow\lambda\in\sigma_p(D_A)$，矛盾.

$D_AT-\mathrm{Id}\in Op(S^{-\infty})$, $\exists T=b(x,D)$, $b\in S^{-m}\ni a(x,\xi)$, $D_A=a(x,D)$

then due to $\lambda\in\sigma_r(D_A)$

$(D_A-\lambda I)(\cdots)-\mathrm{Id}\in Op(S^{-\infty})$

"$\cdots$" is the reduced spectral resolvent $\Rightarrow(D_A-\lambda I)$ also elliptic. ▨

现在我们对于一般的(无界)算子 $A:\mathrm{Dom}A\to\mathcal{H}$ 定义近似谱 (approximation spectrum)

$\sigma_\alpha(A)=\{\lambda\in\mathbb{C},\ \exists x_n\in\mathrm{Dom}A,\ \|x_n\|=1,\ \|Ax_n-\lambda x_n\|\xrightarrow{n\to\infty}0\}$ or approximate point spectrum $\subset\sigma_p(D_A)$

我们来证明 $\sigma_c(D_A)=0$，即 $\sigma_\alpha(D_A)=\sigma_p(D_A)$. 设 $\lambda\in\sigma_\alpha(D_A)$，则存在 $\psi_n\in\Gamma(S)$，满足 $\|\psi\|_{L^2}=1$ 且 $\|D_A\psi_n-\lambda\psi_n\|_{L^2}\to0$ 利用 Lichnerowicz 公式，有

$$\frac{1}{2}\|D_A\psi-\lambda\psi\|_{L^2}^2\leqslant\|D_A\psi_n\|_{L^2}^2+\lambda^2\|\psi_n\|_{L^2}^2\in\sum_i\int|\nabla_{e_i}^A\psi_n|^2+\int_M\frac{K}{4}|\psi_n|^2+\frac{1}{2}\int_M(dA\cdot\psi,\psi)+\lambda^2\|\psi_n\|_{L^2}^2$$

由 M 紧，$\|\psi_n\|_{L^2}=1$ 以及 $\|D_A\psi_n-\lambda\psi_n\|_{L^2}\to0$，可知最右边的第一个积分(的表达式)是有界的，从而 $\{\|\psi_n\|_{H^1}\}$ 有界，利用 $H^1\to L^2$ 的紧性，可知 $\psi_n\xrightarrow{L^2}\psi_0\Rightarrow D_A\psi_0=\lambda\psi_0$.

Exercises. 1. Consider $Ker(A-\lambda I)$, $\overline{Ran(A-\lambda I)}$ and $Ker(\overline{A}-\lambda I)$, $\overline{Ran(\overline{A}-\lambda I)}$, $\overline{A}-\lambda I=\overline{A-\lambda I}$
the by definition of $\overline{A}$ when A is closable operator, $Ker(A-\lambda I)=\overline{Ker(A-\lambda I)}$, thus $Ker(\overline{A}-\lambda I)=\{0\} \Leftrightarrow Ker(A-\lambda I)=\{0\}$
and $\overline{Ran(A-\lambda I)}=\overline{Ran(\overline{A}-\lambda I)} \Longrightarrow \overline{A}$ not change the spectrum ▨

4. (i) is proven, (ii) directly by Exercise 1 ▨

Replace "~" parts, we have L^p-regularity on manifolds.

2. Otherwise $\exists \lambda \in \sigma_r(A)$ 习题. I assume $A: H \to H$ Hilbert spaces.

3. (Elliptic regularity) $M^n = \mathbb{R}^n$ and A is elliptic (pseudo-)differential operator on M with order m, smooth coefficients, u distribution on M s.t. $Au \in L^p$ ($p=2$ here) (i.e. $\exists v \in L^p$, s.t. $\langle u, A^*\varphi\rangle = \int v\varphi$, $\forall \varphi \in C^\infty$) $\Rightarrow u \in W^{m,p}$ ($=H^m$ here) and $\|u\|_{W^{m,p}} \le C(\|Au\|_{L^p} + \|u\|_{L^p})$, $\exists C \ge 0$ ▨

In general the proof is recently done, but for $p\geq 2$ even for manifold it's easy.

$(A-\lambda I)^{-1}: H \to H$ not densely-defined, $\overline{Dom(A-\lambda I)^{-1}} \subsetneq H$, we can decompose H: $H=\overline{Dom(A-\lambda I)^{-1}} \oplus H_2 =: H_1 \oplus H_2$, $\exists y \in H_2 \subset H$, $y \neq 0$, orthogonal to $Dom(A-\lambda I)^{-1}$, this contradict to $Ker(A-\lambda I)=0$ ▨

1. 证明：$\sigma(A)=\sigma(\overline{A})$，加上下标 p, r, c 也是如此。
2. 证明. 若 A 自伴，则 $\sigma_r(A)=\emptyset$.
3. 陈述欧氏空间上的椭圆正则性定理，并看一遍证明.
4. 补全上面性质的证明.

性质. 设 M 紧，则 $\overline{D_A}=D_A^*$，$Dom\,\overline{D_A}=H^1(S)$.

证明：设 $\psi \in Dom\,\overline{D_A}$，则存在 $\psi_n \in Dom\,D_A = \Gamma(S)$，使得 $\psi_n \xrightarrow{L^2} \psi$，且 $\{D_A(\psi_n)\}$ 在 L^2 中收敛. 此时，根据上次课中的不等式(*)（the domination of $\|D_A\psi\|^2$ by Lichnerowicz）可知 $\{\psi_n\}$ 在 $H^1(S)$ 中是一个 Cauchy 列，于是 ψ_n 在 $H^1(S)$ 中收敛到某个 ψ^*. 因为 $H^1(S) \hookrightarrow L^2(S)$ 是连续的 ($\|\psi\|_{H^1} \ge \|\psi\|_{L^2}$)，所以 $\psi^*=\psi$. 这样 $\psi \in H^1(S)$. 反过来的包含关系是平凡的. □

$(D_A-\lambda I)^{-1}$ is the resolvent of operator D_A

性质. 设 $\lambda \notin \sigma(\overline{D_A})$，则 $(D_A-\lambda I)^{-1}: L^2(S) \to L^2(S)$ 是紧算子.

证明. 我们把不等式(*)改写为

$$\|(D_A-\lambda I)^{-1}(D_A-\lambda I)\psi\|_{H^1}^2 \le \|(D_A-\lambda I)\psi\|_{L^2}^2 + \left(C+1+\lambda^2-\frac{K_{min}}{4}\right)\|\psi\|_{L^2}^2$$

记 $\varphi=(D_A-\lambda I)\psi \in Ran(D_A-\lambda I)$，我们就有 $\left(C=c+1+\lambda-\frac{K_{min}}{4}\right)$

$$\|(D_A-\lambda I)^{-1}\varphi\|_{H^1}^2 \le \|\varphi\|_{L^2}^2 + C\|(D_A-\lambda I)^{-1}\varphi\|_{L^2}^2$$

而根据假设 $(D_A-\lambda I)^{-1} \in B(L^2(S))$，所以存在常数 C^*，使得

$$\|(D_A-\lambda I)^{-1}\varphi\|_{H^1} \le C^*\|\varphi\|_{L^2}$$

这样 $Ran(D_A-\lambda I) \subset H^1(S)$. 由 $H^1(S) \hookrightarrow L^2(S)$（注意，我们还没证明过这一条）即得结论. □

Later it's used for $D_A(D_A^2+1)^{-1} = \underbrace{D_A(D_A+i)^{-1}}_{\text{bounded}}\underbrace{(D_A-i)^{-1}}_{\text{compact}}$
$\Rightarrow$ compact operator
$\Rightarrow$ not change index by adding it

Exercise. Leray-Schauder
$T: B \to B$ compact continuous operator
$D \subset B$ is nonempty convex, closed
$\Rightarrow T$ has fixed point $\in D$

性质. 存在 $L^2(S)$ 的一个由 D_A 的特征向量构成的标准正交基 $\{\psi_n\}$
$D_A\psi_n = \lambda_n\psi_n$, $\lim_{n\to\infty}|\lambda_n| = +\infty$.

证明. 我们对于 $(D_A-\lambda I)^{-1}$ ($\lambda \notin \sigma(\overline{D_A})$) 应用紧算子的 Leray-Schauder 定理即可.

习题. 你能把 Hilbert 空间上的 Leray-Schauder 定理叙述并证明么?

Pf. We can apply the Schauder's original fixed point theorem: set $D'=\overline{T(D)}$ also compact $\subset D \Rightarrow$ co(D') (convex hull) is ... bounded $\Rightarrow D''=\overline{co(D')}$ is compact convex ... $\Rightarrow$ Using Schauder to $T|_{D''}$ ▨

推论. 存在常数 $C>0$, 使得对于任意 $\psi\in H^1(S)\ominus \operatorname{Ker} D_A$ 都成立

$$|(D_A\psi,\psi)_{L^2}|\geq C\|\psi\|_{L^2}^2$$

习题:

1. 利用不等式 (*) 证明 $\|\psi\|_{H^1}$ 和 $\left(\|\psi\|_{L^2}^2+\|D_A\psi\|_{L^2}^2\right)^{1/2}$ 是等价范数.

于是, 我们可以定义 k-阶 Sobolev 空间 $H^k(S)$ 为 $\Gamma(S)$ 关于范数

$$\|\psi\|_{H^k}^2=\sum_{i=1}^{k}\|D_A^i\psi\|_{L^2}^2$$ 的完备化.

如果 $\psi\in\Gamma(S)$ 关于前述由 D_A 的特征向量构成的基 $\{\psi_n\}$ 的分解式为

$$\psi=\sum_{n=1}^{\infty}a_n\psi_n \qquad (D_A\psi_n=\lambda_n\psi_n)$$

那么 $$\|D_A^k\psi\|_{L^2}^2=\sum|a_n|^2\lambda_n^{2k}$$

最后, 我们来说明 $H^1(S)\longrightarrow L^2(S)$ 是一个紧算子, 即证明: $H^1(S)$ 中的任何一个有界点列可以选出在 $L^2(S)$ 中收敛的子列.

设 $\psi_n\in H^1(S)$, $\|\psi_n\|_{H^1(S)}\leq 1$. 由定义, 存在 $\varphi_n\in\Gamma(S)$, $\|\psi_n-\varphi_n\|_{H^1}\leq\frac{1}{n}$.
这样, 如果 $\{\varphi_n\}$ 的一个子列 $\{\varphi_{n_k}\}$ 在 $L^2(S)$ 中收敛, 那么相应的 $\{\psi_{n_k}\}$ 也在 $L^2(S)$ 中收敛.

Exercise 1. (*) $\|\psi\|_{H^1}^2+\left(\frac{K_{min}}{4}-C-1\right)\|\psi\|_{L^2}^2\leq\|D_A\psi\|_{L^2}^2\leq\|\psi\|_{H^1}^2+\left(\frac{K_{max}}{4}+C-1\right)\|\psi\|_{L^2}^2$

$\Longleftrightarrow \|\psi\|_{H^1}^2+C_1\|\psi\|_{L^2}^2\leq\|D_A\psi\|_{L^2}^2\leq\|\psi\|_{H^1}^2+C_2\|\psi\|_{L^2}^2$ $(C_1+2C\leq C_2,\ C\geq 0)$

then it turns a primary problem: $x,y,z\geq 0$, $x+C_1y\leq z\leq x+C_2y$, ~~[illegible]~~

then $\exists\, C_1', C_2'\geq 0$ s.t. $C_1'(y+z)\leq x\leq C_2'(y+z)$.

This is obvious: taking $C_1'=2\max\{1,-C_1\}$, $C_2'=\frac{1}{N}$ for N large $\boxtimes$

Exercises

1. (Oscillatory integral)

(1) $\boxed{\text{Step 1}}$ Reduce to dimension 1 case: in each direction $\frac{\partial}{\partial x_i}$ if we have domination uniformly, then by Fubini theorem;

$\boxed{\text{Step 2}}$ Denote as $I_\varphi(a)(\lambda)$, then we reduce to $I_\varphi(1)(\lambda)$ by $I_\varphi(a)(\lambda)=\frac{-1}{i\lambda}I_\varphi\left(\frac{d}{dx}\frac{a}{\varphi'}\right)(\lambda)$

this is by $\int e^{i\lambda\varphi(x)}a(x)\,dx=-\int a(x)\left(\int e^{i\lambda\varphi(s)}ds\right)'dx\left(\frac{1}{i\lambda}\right)$ by integration by part;

$\boxed{\text{Step 3}}$ (Van der Corput Lemma) Estimate $\int e^{i\lambda\varphi(x)}dx$ has order $O(\lambda^{-k})$, $\forall k$

this is by $I_\varphi(1)(\lambda)=\frac{1}{i\lambda\varphi'(x)}e^{i\lambda\varphi(x)}\Big|_{\text{boundary}}-\int\left(\frac{d}{dx}\frac{1}{i\lambda\varphi'(x)}\right)e^{i\lambda\varphi(x)}dx$ by integration by part $\cdots(*)$

$\Rightarrow |I_\varphi(1)(\lambda)|\leq\frac{1}{\lambda}\int\left|\frac{d}{dx}\frac{1}{\varphi'(x)}\right|dx\leq\frac{C}{\lambda}$. and $\forall k$, $|\varphi^{(k)}(x)|\geq C>0\Rightarrow$ repeating integration by part for $(*)$ ☒

(2) By (1), $\lambda^k|I_\varphi(a)(\lambda)|\leq C_{k,\varphi,K}\sup_{x\in K}(1+|x|)^{m-\rho|\alpha|}$ $<C_{k,\varphi,K}\sup_{x\in K}(1+|x|)^{-(N+\rho|\alpha|)}$ has $\left(\frac{1}{1+|x|}\right)^{N+\rho|\alpha|}$ order $\leq N+k$

thus when $k\to 0$, order $<N$

thus $\left(\frac{1}{1+|x|}\right)^{N+\rho|\alpha|}$ is still meaningful ☒

(3) $\exists\,\chi_0\in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp}\chi_0=\{|x|\leq 1\}$ and we have the covering $\{\{|x|\leq1\},\{\frac12\leq|x|\leq2\},\{1\leq|x|\leq4\},\{2\leq|x|\leq8\},\cdots\}=:\mathcal{U}$

we can assume $\chi_0=0$ and covering $\mathbb{R}^n-\{|x|\leq\frac12\}$ by later we set $\varphi_i=\frac{\chi_i}{\sum\chi_i}$ we can using $\varphi_i=\frac{\chi_i}{\sum\chi_i-\chi_0}$ to replace.

For the covering consisting "torus" , we can set a family φ_i with each $\operatorname{supp}\varphi_i\subset\{2^{i-2}\leq|x|\leq 2^i\}$

and $\sum\varphi_i=1$. and we can set

$\varphi_{i+1}(x)=\varphi_i\left(\frac{x}{2}\right)$ inductively: if it's possible for all $i\leq n$, $\sum\varphi_i(2^ix)+\sum\varphi_i(x)=1$, we modify φ_{n+1}, s.t. $\forall x\in\operatorname{supp}\varphi_n(x)\cap\operatorname{supp}\varphi_{n+1}(x)$, $\varphi_{n+1}(x)=\varphi_n\left(\frac{x}{2}\right)$, and otherwise we can always choose $\varphi_{n+1}(x)$, s.t. $\sum_{i\leq n}\varphi_i(2^ix)+\varphi_{n+1}(x)=-\sum_{i\geq n+1}\varphi_i(x)$ ☒

(4) $\sum_p\left(\int e^{i\lambda\varphi(x)}\chi(2^px)a(x)dx\right)$ has each term supported by finite (only nearby terms support intersected nonempty)

thus $\left|\sum_p\int e^{i\varphi(x)}\chi(2^px)a(x)dx\right|\leq 2\sum_p\int|e^{i\varphi(x)}||a(x)|dx\overset{(1)\text{and}(2)}{<}\infty$ done ☒

(5) By Fubini, reduce to $n=1$ case: due to $\alpha=\beta\Leftrightarrow$ all $\alpha_i=\beta_i\Leftrightarrow$ all $\delta_{\alpha_i\beta_i}=1$.

$\frac{1}{2\pi}\int_{\mathbb{R}^2}e^{-iy\eta}y^\alpha\eta^\beta dy\,d\eta\overset{\text{Fubini}}{=}\frac{1}{2\pi}\int_{\mathbb{R}}y^\alpha\left(\int e^{-iy\eta}\eta^\beta d\eta\right)dy\overset{\text{integral by part}}{=}\frac{1}{2\pi}\int_{\mathbb{R}}y^\alpha\frac{\beta}{iy}\left(\int e^{-iy\eta}\eta^{\beta-1}d\eta\right)dy=\cdots=\frac{1}{2\pi}\int_{\mathbb{R}}y^{\alpha-\beta}\frac{\beta!}{i^\beta}\left(\int e^{-iy\eta}d\eta\right)$

$=(-i)^\beta\cdot\beta!\cdot\delta_{\alpha\beta}=(-i)^\alpha\alpha!\cdot\delta_{\alpha\beta}$ ☒

2. (1) $\left(\frac{1}{2\pi}\right)^n\int a(x,\eta+\lambda d\psi(x))\Phi(x,\eta,\lambda)d\eta=\left(\frac{1}{2\pi}\right)^n\int a(x,\eta+\lambda d\psi(x))\left(\int e^{-iz\eta}e^{i\lambda[\psi(x+z)-\psi(x)-d\psi(x)(z)]}u(x+z)dz\right)d\eta$

$=e^{-i\lambda\psi(x)}\left(\frac{1}{2\pi}\right)^n\int a(x,\eta+\lambda d\psi(x))e^{-iz\eta+i\lambda(\psi(x+z)-d\psi(x)(z))}u(x+z)dz\,d\eta$

$=e^{-i\lambda\psi(x)}\left(\frac{1}{2\pi}\right)^n\int(ue^{i\lambda\psi})(x+z)\int e^{i(x+z-z)(\eta+\lambda d\psi(x))}a(x,\eta+\lambda d\psi(x))\,dz\,d\eta$

$=e^{-i\lambda\psi(x)}\left(\frac{1}{2\pi}\right)^n\int(ue^{i\lambda\psi})(x+z)\int e^{i(x+z-z)(\eta+\lambda d\psi(x))}a(x,\eta+\lambda d\psi(x))\,dz\,d(\eta+\lambda d\psi(x))$

$=e^{-i\lambda\psi(x)}[a(x,D)](ue^{i\lambda\psi})(x+z)=e^{-i\lambda\psi(x)}[a(x,D)(ue^{i\lambda\psi})](x)=I(x,\lambda)$ ☒

(2) $\left|\int e^{-iz\eta}e^{i\lambda(\psi(x+z)-\psi(x)-d\psi(x)(z))}u(x+z)dz\right|\leq\left|\int e^{-i|\eta+\lambda d\psi(x)|(z)+i\lambda(\psi(x+z)-\psi(x))}u(x+z)dz\right|\leq\left|\int e^{-i|1+\eta+\lambda d\psi(x)|(z)}u(x+z)dz\right|$

$\leq C_k|1+\eta+\lambda d\psi(x)|^{-k}$ by the Exercise 1(1) ☒

(3) $I_1(x,\lambda)=(I-I_2)(x,\lambda)=\left(\frac{1}{2\pi}\right)^n\iint e^{-iz\eta}e^{i\lambda\psi_x(x+z)}u(x+z)\left(1-\chi\left(\frac{\eta}{\lambda}+d\psi(x)\right)\right)a(x,\eta+\lambda d\psi(x))\,dz\,d\eta$

$\Rightarrow|I_1(x,\lambda)|=\left|\left(\frac{1}{2\pi}\right)^n\int a(x,\eta+\lambda d\psi(x))\Phi(x,\eta,\lambda)d\eta\right|$ $\quad=\begin{cases}1; & |\eta+\lambda d\psi(x)|\leq\frac{\lambda}{2C}\\ 0; & \frac{\lambda}{C}\leq|\eta+\lambda d\psi(x)|\leq C\lambda\end{cases}\Rightarrow$ we can apply (2) outside $[\frac{\lambda}{C},C\lambda]$

$\leq\left(\frac{1}{2\pi}\right)^n\int|a(x,\eta+\lambda d\psi(x))|C_k(1+\eta+\lambda d\psi(x))^{-k}d\eta\in S$ by $a\in S^m$ ☒

(4) and (5) We do it same time: set $b(z,\eta,\lambda)=b(x+z,\eta,\lambda)=u(x+z)a(x,\eta+\lambda d\psi(x))\chi\left(\frac{\eta}{\lambda}+d\psi(x)\right)$

We show that $I_2(x,\lambda)=D(u)(x,\lambda)$ is a pseudo-differential operator: by (3)

then $D=Op(I_2)$ and I_2 as the defines symbol of D has a natural asymptotic expansion that

$I_2(x,\lambda)=\sum\frac{1}{\alpha!}(\partial_\eta^\alpha\partial_z^\alpha)(a(x,\lambda d\psi(x))e^{i\lambda d\psi}u(y))+\text{"}\square\text{"}$ and "$\square$" $:=\int e^{ix\xi}(|\alpha|+1)\sum_{|\beta|=|\alpha|+1}^{|\alpha|+1}\frac{1}{\beta!}\int_0^1(\partial_y^\beta a)(x,\xi,t\xi+\eta)\xi^\beta(1-t)^{|\alpha|}dt$

$=\sum\frac{1}{\alpha!}\partial_y^\alpha(e^{i\lambda d\psi}u(y))\big|_{y=x}(\partial_\xi^\alpha a)(x,\lambda,d\psi(x))+\text{"}\square\text{"}$ $\quad\leq C_{\alpha\beta}(1+|\xi+\eta|)^{m-\rho|\beta|}(1+|\xi|)^{-k|\alpha|}$

$\underset{|\beta|=|\alpha|+1}{\leq}C_\alpha(1+|\xi|)^{m-\frac{|\alpha|}{2}}\leq C_\alpha\lambda^{m-\frac{\alpha}{2}}$ ☒

(This is the Thm 3.5 in "Spin Geometry")