Only possible one path by critical of Lagrangian $\rightsquigarrow$ All possible by Feymann's path integral

What's Quantum mechanics for Mathematicians?

1. particles $P(t) \in \mathbb{R}^n \times \mathbb{R}^1_t \rightsquigarrow$ wave function $\psi \in L^2(\mathbb{R}^n \times \mathbb{R}^1_t) = \mathcal{H}$.

oberserables $x(p)$. $\quad$ oberserables $\hat{x}(\psi) = \langle \hat{x} | \psi \rangle$

(Q) $\quad v(p) \rightsquigarrow \quad \hat{p}(\psi) = \langle \hat{p} | \psi \rangle$

$E(p)$.. $\quad \hat{H}(\psi) = \langle \hat{H} | \psi \rangle$

changes w.r.t. time t = changes w.r.t. time t $\quad \psi = e^{-i\hat{H}t} \cdot \psi(0)$

commutative $\rightsquigarrow$ noncommutative $\quad$ infinitesimal generator is $\hat{H}$

2. Axioms in Quantum Theory of Schrödinger's picture

$*$ Schrödinger equation $i\hbar \frac{\partial}{\partial t} | \psi \rangle = \hat{H} | \psi \rangle$ restrict on t, $\psi \in \mathcal{H}$

$\psi(t) = U(t)\psi(0)$, $\{U(t)\}$ strongly continuous unitary operator semi-group $t \in \mathbb{R}$

- State $\mathcal{H}$ Hilbert space $/ \mathbb{C}$.
- Observables $\hat{A} \in \mathcal{B}(\mathcal{H})$ Hermitian (self-adjoint as may unbounded), denoted $A(\mathcal{H})$; $\sigma_e(A) \subset \sigma(A)$ is the eigenvalues
- Measurement $A(\mathcal{H}) \times \mathcal{H} \xrightarrow{M} \mathrm{Hom}(\mathbb{R}, [0,1])$

$$(\hat{A}, \psi) \longmapsto \left(f_{\hat{A},\psi} : x \mapsto \begin{cases} 0 ; & x \notin \sigma_e(\hat{A}) \\ |\Sigma \langle \varphi_x | \psi \rangle|^2 ; & x \in \sigma_e(\hat{A}) \end{cases} \right)$$

$\hat{A}\varphi_x = x\varphi_x$ eigenvectors

sends the measure conclusion x

[illegible]

to its probability, i.e. $\int f_{A,\psi}(x)\,dx = 1$ always holds.

Recover probability by expectation & distribution $\langle A\psi, \psi \rangle \in \mathbb{R}$ & $\langle A^2\psi, \psi \rangle$

~~In particular, when $\hat{A} = \mathrm{Id}_{\mathbb{R}}$, then we recover Schrödinger's equation~~

~~$f_{Id,\psi}: x \mapsto \begin{cases} 0 ; x \neq 1 \\ 1 ; x = 1 \end{cases}$, when we measure ψ itself~~

Thus give $\langle A\psi, \psi \rangle \in \mathbb{R}$ is enough! $\quad$ thus due to A self-ad $\Rightarrow A^2$ also self-ad

Q. Three "•" are abstract in $A(\mathcal{H})$ and $\mathcal{H}$ level without any time

can we absorb "$*$" into them, by take $\hat{A}_t$ instead $\psi(t)$?

This's another picture due to Heisenberg $\rightsquigarrow$ Haag–Kastler axiom (By C^*-algebra A replace $\mathcal{B}(\mathcal{H})$) of AQFT (Algebraic)

If we stay at Schrödinger's picture, we should describe the dynamic w.r.t. time evolution of states: $\forall [t_1, t_2] \subset \mathbb{R}$,

$U([t_1,t_2]) : \mathcal{H}_{t_1} \cong \mathcal{H} \longrightarrow \mathcal{H} \cong \mathcal{H}_{t_2}$ as $(*)$, s.t. $U([t,t]) = \mathrm{Id}$

Such rule $\rightsquigarrow$ category level! $\quad U([t_1,t_2]) \circ U([t_2,t_3]) = U([t_1,t_3])$

日月光华 旦复旦兮

Date

FQFT (Functorial)

Thus Schrödinger's picture ⇝ (Atiyah-Segal axiom of TQFT) dual

Heisenberg's picture ⇝ Haag-Kastler axiom of AQFT

Such duality still mysterious, only special case formulated.

Dirac's picture = mixed them together, both ψ_t and $\hat{A}_t$.

QM to QFT? (Or M to FT classically?)

Mechanics are for Hilbert space (quantum) or $\mathbb{R}^n$ (classical), while field theory it's a "Hilbert bundle" (quantum) or vector bund (classical) i.e. a relative analogue of mechanics:

Fix M, $\forall p \in M$ we have $\phi(p,t) \in F_p$ is a field, i.e. section of a specific bundle, finite-dimensional;

but for mechanics, some particles' system $p \in \mathbb{R}^{2n}$ the configuration space, discrete $\mathbb{R}^{2n}$ not continuous function ϕ

Lift to quantum: particle p $\xrightarrow{\text{quantization}}$ wave function ψ; field φ $\xrightarrow{\text{quantization}}$ quantum field $\hat{\Phi}$ is operator

Eg. $\frac{1}{i}[-,-] \rightsquigarrow [-,-]$

$\sum a_i(\omega)\psi_i \mapsto \sum \hat{a}_i(\omega)\psi_i = \hat{A}$ quantization (Γ^*M, ω)

Operator is section of an infinite dimensional bundle, each fibres are Hilbert spaces!

(Due to the lack of standard book of TQFT & AQFT, later I may write something on it as functors.)

Observables of mechanics are: classical particle system is $f(p)$ function; quantum is operator $\hat{A}(\psi)$

What're "observables of fields"? It's almost based on relative idea and simply fix at a point. Thus nothing interesting.

Start at QFT, all interesting mathematical physical theory occurs:

Gauge theory $\xleftarrow{\text{AdS/CFT}}$ Gravitation (Our string refers to super one, with both Bosonian and Fermanion)

∩ ∩ Type IIB

QFT $\xleftarrow{\text{AdS/CFT}}$ String theory

(CFT side) (AdS side)

~~Quantization in math & Physics,~~

Particle $\frac{1}{2}$-spin representation, Spin, and orientibility of moduli space

What a standard "Quantum Mechanic" course (by Physic department) need to contain?

- Schrödinger equation & How to solve it (Perturbation/Variational/WKB)
- Formalism (I had reviewed it) ← "
- Spin, Bosonian and Fermion → add realistic effect, it's QFT indeed
- Quantization

Path integral ⇄ reuse Lagrangian

All of them should be taken into Geometrist's eyes:

even computational method to PDEs can be lifted into homological perturbation and derived method. (as a type of Taylor expansion)

This note is a review of the Quantum Mechanic course in this semester, but it's predictable that it's useless for final test www

$\boxed{\text{PART I}}$ Angular momentum

Represention of $SO(3)$:

- It's cpt Lie group, thus all representation are finite-dimensional;
- It's semi-simple Lie group, thus all representation factor into irreducible.
- All irreducible representation of $SO(3)$ are tensor product of spin-$\frac{1}{2}$ rep.

The last step is due to passing $SU(2) \xrightarrow{2:1} SO(3)$, and $SU(2)$-rep are generated by restriction to $U(1) \underset{l_j}{\hookrightarrow} SU(2)$, embedding $(l_j)_{j=1,2,3}$ are given by exponent of generators of $su(2) \cong so(3) = \{A^T + A = 0\}$

Precisely, let $F_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ span $so(3) \cong su(2)$, then

$L^+ = i\hbar\lambda(i\exp(F_2) + \exp(F_1))$, $L^- = i\hbar\lambda(\exp(F_1) - i\exp(F_2))$, $L_3 = i\hbar\lambda\exp(F_3)$ $\in SU(2)$, and their action to V, $\dim V = 2l+1$, $\boxed{l \in \frac{N}{2}}$, basis $(v_m)_{-2l \le m \le 2l}$

$L^+ v_j = \begin{cases} j(2l+1-j)v_{j-1}, & j > -l \\ 0, & j = -l \end{cases}$ $L^- v_j = \begin{cases} v_{j+1}, & j < l \\ 0, & j = l \end{cases}$ shifting & $L_3 v_j = (l-j)v_j$

and for $SO(3)$, $\dim V = 2l+1$ $\boxed{l \in N}$ and basis $(v_m)_{-l \le m \le l}$

$L^+ v_j = j(2l+1+j)v_{j-1}$, $j > -l$, $L^- v_j = v_{j+1}$, $j < l$, $L_3 v_j = (l-j)v_j$

Thus one can see, although in Physics spin is totally new in quantum case, it's more easy to compute in Math.

Thus via raising & lowering operators, we done, and the composite of raising/lowering operators $\Leftrightarrow$ tensoring with Spin-$\frac{1}{2}$ rep.

Note that $SO(3) \xleftarrow[1:2]{} SU(2)$ ($SO(3) \cong SU(2)/\{-I_2, I_2\}$)

$2l+1 \Leftrightarrow 4l+1$ $\quad$ $4l+1$ ⋯ the number of choice of m

$\boxed{l \in \mathbb{N}} \xleftrightarrow{1:2} \boxed{l \in \frac{\mathbb{N}}{2}}$ ⋯ quantum number/eigenvalue / basis of rep

classical angular momentum $\xleftarrow{1:2}$ spin momentum (double cover $\Leftrightarrow$ $\frac{\text{eigenvalue}}{2}$)

* And this is due to abstract representation, in physics, $l \in \frac{\mathbb{N}}{2}$ not allowed in classical setting, thus we take $2l+1$ and throw away some non-physical dimension of V. This's why double cover $\Leftrightarrow$ classical angular to spin

We then generalize such notation $Spin(3) = SU(2) \xrightarrow{2:1} SO(3)$ into $Spin(n) \xrightarrow{2:1} SO(n)$

Rk. Note that we're all discussing $\boxed{\text{Fermion}}$, before and future. For Boson, a parallel story is $Mp(2d) \xrightarrow{2:1} Sp(2d)$, which I don't know.

- We know the angular momentum and Spin angular momentum are determined by (canonical) (anti) commutation relation (in representation level):

$[\hat{J}_1, \hat{J}_2] = \hat{J}_3$, $[\hat{J}_2, \hat{J}_3] = \hat{J}_1$, $[\hat{J}_3, \hat{J}_1] = \hat{J}_2$ (or $\hat{S}_i$). Our next question is relation Spin with Clifford algebra relations $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$, note that the later one are Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$su(2) = \langle i\sigma_1, i\sigma_2, i\sigma_3 \rangle \cong \{A \in M_3(\mathbb{R}) \mid A + A^T = 0\}$

$\sigma_i \longleftrightarrow F_i$ before $\quad$ (Ex. Work out this iso precisely)

due to double cover doesn't change the tangent space = Lie algebra thus $spin(n) \cong so(n)$, and we focus the relation under the isomorphism

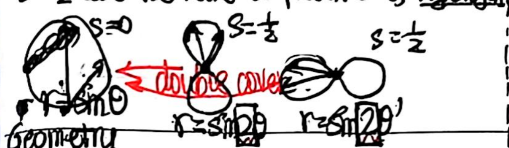
$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij} \longleftrightarrow [F_1, F_2] = F_3$ ⋯ or $[L_3, L^+] = L_+$ } oscillator / interpretation

easier to generalize $\quad$ harder $\longleftarrow$ $[L_3, L^-] = -L_-$ $\quad$ / quantum bit

$[L^+, L^-] = 2L_3$

and gives a natural canonical representation of $spin(n)$ by Clifford algebra

$s = \frac{1}{2}$ case we have a picture of hydrogen



Geometry example of $SU(2)$-representation:

- $s = 0$ scalar field, componet 1
- $s = \frac{1}{2}$ spinor field, componet 2 $\langle \uparrow | \downarrow$
- $s \geq 1$ vector field, No. component ≥ 3

Fermion $s = \frac{1}{2}, \frac{3}{2}$ ⋯ $\Leftrightarrow$ Anti-symmetric

Boson $s = 0, 1$ ⋯ $\Leftrightarrow$ Symmetric

in QFT: particles $\Leftrightarrow$ excited states of field via creation from vacuum/ground state

- Hopf fibrations/Dirac's magnetic monopole

We'll see spin/quantum effect occurs due to Dirac (in 1931) explain the "magnetic monopole" as quantization of electric monopole.

$S^1 \to S^3 \Leftrightarrow U(1) \to S^3$ is $U(1)$-principal bundle / $U(1)$-gauged theory on S^2

$\quad \downarrow \qquad\qquad \downarrow$ / electromagnetic theory

$\quad S^2 \subset \mathbb{R}^3 \quad S^2$

$S^2 \simeq \mathbb{R}^3 - \{0\}$, with our "magnetic monopole" exists in $\{0\}$

Fact. The associated vector bundle of $U(1)$-principal bundle is given by $S^2 \xrightarrow{F} M_2(\mathbb{C})$, $F(x_1, x_2, x_3) = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$, $F^2 = 1 \Rightarrow$ we construct idempotent element $e = \frac{1+F}{2} \in C(S^2, M_2(\mathbb{C})) \cong M_2(C(S^2))$, thus by Serre-Swan, idempotent of matrix algebra $\Leftrightarrow$ vector bundle

Hence we get a rank 2 real vector bundle, i.e. cpx line bundle. ∎

Rk. ① It's isomorphic to $\mathcal{O}(-1)$ on $\mathbb{P}^1_{\mathbb{C}}$. A more precise construction can be given, but here we only abstractly use SS correspondence and idempotent element, due to it generalize into quantum spheres;

② We omit gauge-theoretic computations on it due to I don't want to entre QFT now, the consequence in physics is the fact that: if such "magnetic monopole" exists, $\boxed{\text{all electric charges are quantized}}$, compatible with our physical intuition. (i.e. integral times of h)

Representation of $Spin(n)$:

- We have $spin(n) \xrightarrow{\exp} Spin(n)$ for f.d. representations

$\quad \| \, d\pi \qquad\qquad \pi \downarrow 2:1$

$so(n) \xrightarrow{\exp} SO(n)$

$gl(V) \xrightarrow{\exp} GL(V)$. $\boxed{\text{Lie's third thm}}$

Recall: $\{\pm I\} \to Spin(n) \to SO(n)$, $\circlearrowright GL(V)$; $SO(n)$-rep is $Spin(n)$-rep with both I_n and $-I_n$ send to identity $H \xrightarrow{id} H$ ∎

It's advised that they're generated by Clifford relations (Both Lie group & algebraic level)

Spin $\boxed{\ }$ classical $+$ $-$

$=$ Möbius band

$\boxed{\text{"Orientiblity"}}$

The spinor bundle $\boxed{\text{split into + and -}}$, indeed spinor, as section of spinor bundle, described by two components in such manner: one S^+, one $\hat{\Sigma}$.

I omit the general theory as one can directly look at the masterpiece [Spin Geometry], but give the example when $n=4$:

$\gamma_0 = \begin{pmatrix} 0 & I_{2\times2} \\ I_{2\times2} & 0 \end{pmatrix}$, $\gamma_1 = -i\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}$, $\gamma_2 = -i\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}$, $\gamma_3 = -i\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$, with relations $\gamma_0^2 = \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = 1$ and $-\frac{1}{2}\gamma_0\gamma_i = -\frac{i}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$, $-\frac{1}{2}\gamma_1\gamma_i = -\frac{i}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$, $-\frac{1}{2}\gamma_2\gamma_3 = -\frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$ for all $\gamma_i\gamma_j$ with $i<j$, thus we can cancell the σ_i at the first 2×2 block & the fourth 2×2 block:

$-\frac{1}{4}(\gamma_0\gamma_1 + \gamma_2\gamma_3) = -\frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix}$, $-\frac{1}{4}(\gamma_0\gamma_2 + \gamma_1\gamma_3) = -\frac{i}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix}$, $-\frac{1}{4}(\gamma_0\gamma_3 + \gamma_1\gamma_2) = -\frac{i}{2}\begin{pmatrix} \sigma_3 & \\ & 0 \end{pmatrix}$

and replaced by $- \cdots$

$\Rightarrow spin(4) \cong su(2) \oplus su(2)$ via this splitting generators

$\boxed{\text{PART II}}$ Orientibility of moduli space

We had seen spin structure is related with orientibility, now we make it more precise. In history, first Donaldson find it for his gauge-theortic moduli spaces in instanton theory, in his excellent paper win him a fields medal. By Atiyah-Floer's guildence, we're confirmed that same thing work for Lagrangian moduli spaces.

• First we recall what is spin structure on mfds:

Def1. A spin mfd is pair $(M, g, E, P_{spin}, \varphi)$

(M,g) is Riemannian, E is oriented rank n vector bundle, P_{spin} is Spin(n)-principal bundle $\downarrow E$

$\Rightarrow$ $\exists$ SO(n)-principal bundle $P_{SO}(E)$: the associated/orthonormal frame bundle

① $P_{spin} \xrightarrow{\varphi} P_{SO}(E)$; ② φ is 2-cover; ③ φ is Spin(n)-equivariant $\searrow M \swarrow$

$\boxed{\text{Obstruction}}$ $\Rightarrow$ We call E spinor bundle & sections are spinor field

(M,g,E) orientible $\Leftrightarrow$ $w_1(E) \in H^1(M; \mathbb{Z}/2\mathbb{Z})$ vanishes

(M,g,E) spin/allow spin structure $\Leftrightarrow$ $w_2(E) \in H^2(M; \mathbb{Z}/2\mathbb{Z})$ vanishes

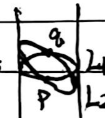
shows the spin condition is 2-analogue of orientibility, thus not strange to correspond to orientibility of "2-geometric object"/moduli stack

• Without menention E, we admit $E = TM$

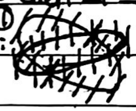
furthermore, spin too

• Prove of orientibility for Gauge-theortic moduli space by Donaldson is via SW classes as above, recently (2017) Joyce also directly prove via the orientation bundle on moduli space. Some essential techniques can be found in [Spin Geometry] and [D], as the tangent space are some elliptic differential operators.

• Lagrangian case: consider T^*S^1 and Lagrangian we prefer:

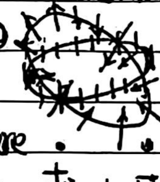


Then the spin structure on each $L_i \cong S^1$ are two types, trivial and Möbius band: ①



two type $\not\uparrow$ and $\not\downarrow$ determines orientation of $\mathcal{M}(p,q) = \cdot \quad \cdot$

②



are $\overset{+}{\cdot} \quad \overset{+}{\cdot}$ and $\overset{-}{\cdot} \quad \overset{-}{\cdot}$

are $\overset{+}{\cdot} \quad \overset{-}{\cdot}$ and $\overset{-}{\cdot} \quad \overset{+}{\cdot}$, thus these spin structures do lift to orientation of moduli space, in similiar way as monodromy: two differential paths $\Leftrightarrow$ two different points $\in$ moduli space

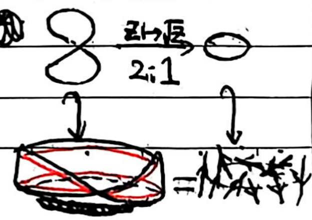
$\boxed{\text{Q. Can we rewrite these terms into infinitsimal neighbourhood, in analogue of Alexander-Spanier cohomology?}}$

Details see [S].

$\boxed{\text{PART III}}$ Infinite-dimensional representation and Geometric quantization

"Quantization" is such a philosophy, in both mathematics and physics, from f.d. to infinite-dimensional (Hilbert space formalism) & adding hidden higher terms to noncommutative (path-integral formalism/perturbation case). The latter one tells us to view it as a $\boxed{\text{deformation}}$ and lift to derived laguage, it's deformation quantization, which's a word slightly tautological. "日月光华 旦复旦兮" Some physical terms, e.g. second quantization or "regular quantization" are not taken into my considerate.

$\Rightarrow$ In S^1-case, the double cover is just $S^1 \to S^1$



First we have a natural unitary (infinite-dimensional) $SO(3)$-rep by $\pi: SO(3) \longrightarrow U(\mathcal{H})$, $\mathcal{H} = L^2(\mathbb{R}^3)$ the translation, produce a representation via rotation ~~tautologically~~.
$R \longmapsto (\pi(R): \mathcal{H} \to \mathcal{H},\ \psi \mapsto \psi R^{-1})$

and then $so(3) \xrightarrow{\exp} SO(3) \xrightarrow{\pi} U(\mathcal{H})$

This isn't canonical start point of quantization, but it do produce the Hilbert space of quantum states we expect in physics, it looks a little more complicated than next Eg.1, indeed this's a baby version of geometric quantization, not deformation quantization next, we'll focus on the Hilbert space of state directly instead of the algebra of operators/observables (Indeed, this's predicted as a duality).

(Recall something we're familiar with)

Eg.1 ① $\{H_1, H_2\} = \omega(X_{H_1}, X_{H_2})$, H_1 and H_2 are Hamiltonians, X_{H_1} and X_{H_2} are Hamiltonian vector fields $\iota(X_{H_i})\omega = dH_i$

$\Gamma(M, \mathcal{O}_M) \longleftrightarrow \Gamma(M, TM)$ is Lie algebra homomorphism
$H \longleftrightarrow v_H$ (choose a proper H)

② Comoment map $\mu^*: \mathfrak{g} \to \Gamma(M, \mathcal{O}_M)$ is G-equi $\Rightarrow$ Lie algebra homomorphism

③ Principal symbol map $\sigma_p: \mathrm{Gr}\, \mathcal{D}(M) \longrightarrow C^\infty(T^*M)$ is Lie algebra homomorphism, here we take associated graded algebra due to $\mathcal{D}^m(M)/\mathcal{D}^{m-1}(M)$ behaves better (isomorphism) under taking symbol and from $[-,-] \mapsto \{-,-\}$

All of the three are isomorphism of Lie algebra, up to constant, we call the inverse map the (pre)quantization, which is a process shared by both algebraic/deformation & geometric. (Note that, in ③, it's nothing more than we done in QM, up to $\hbar$)

Then ② is special case of ② by definition of μ^*:
$\mathfrak{g} \xrightarrow{da} \boxed{\Gamma(M, TM) \longrightarrow \Gamma(M, \mathcal{O}_M)}$, thus just take the action as multiplication.
$G \xrightarrow{a} \Gamma(M, \mathcal{O}_M)$

(Also related closely with Duistermaat–Heckman localisation)

This is a way of ② "forget G", another way is "forget (M,ω)" by take $\xi \in \mathfrak{g}^*$, then let $M = \mathcal{O}(\xi)$ the orbit under adjoint action $G \curvearrowright \mathfrak{g}$, then $\mu^* =$ inclusion of orbit. it will have a natural ω

In this case, the representation is linked with (geometric) quantization: (Kirillov's character formula) $\dim \mathcal{O}(\xi) = 2d$
$\{\mathcal{O} \subset \mathfrak{g}^*\} \longleftrightarrow \{\text{rep } \pi_{\mathcal{O}}: G \to GL(V)\}$ and
"Twist" of character of $\pi_{\mathcal{O}} = \widehat{\left(\frac{\omega}{2\pi}\right)^d}$, where the FT is the meaning of FT of measure, as ω^d is top form/volume form.

Such "twist" and FT is the step beyond prequantization, here we do have geometric quantization after this "twist".

Let $G = SU(2) = Spin(3)$ now, the phase space of spin is Bloch sphere S^2 (In quantum case, phase space may not cotangent bundle!), and the S^2 just equip the symplectic form coincide above. Prequantization $(S^2, C^\infty(S^2), \{-,-\})$ is easy, then we note that S^2 doesn't have global (Darboux) chart, thus then we'll take local charts $\longleftrightarrow$ decompose into irreducible rep by Kirillov's correspondence above, here the "twist" is just computation of local charts. All these fields comm

(Borel–Weil–Bott) We generalize above correspondence to general G, this is a start of GRT, leading to Springer theory

G semi-simple, $G_{\mathbb{C}}$ is the complexification, $P \le G_{\mathbb{C}}$ parabolic subgroup thus also $\mathbb{C}$-algebraic group (e.g. flag varieties)

$\Rightarrow \{G \text{ f.d. irreducible representation}\} \longleftrightarrow \Gamma(G_{\mathbb{C}}/P, \mathcal{O}^{an}_{G_{\mathbb{C}}/P})$

thus $SU(2)$-rep $\Longleftrightarrow \Gamma(S^2, \mathcal{O}^{an}_{S^2})$ $\quad G_{\mathbb{C}}/P \simeq G/T$ as Lie group

Here we view S^2 as complexified, i.e. We view its tangent bundle complexified $TS^2 \oplus iTS^2$, then its tangent spaces are naturally symplectic

we consider G/T, T maximal tori due to T not change the quantum state
In $SU(2)$, it's $SU(2)/U(1) = S^2$

No.

Date

From classical mechanic to quantum mechanic is just change of notation, it's not usually works; Groenewold–Van Hove No-go thms show that prequantization not enough, and even not well-defined.

Motivated by this, we give the general Geometric Quantization process (details see [GTM 267, last chapter])

- Prequantization (change of notations)
- Polarization: choose $L_p \subset (T_pM \oplus iT_pM, \omega_{std})$ Lagrangian subspace, they form a distribution/foliation on the complexified phase space ("Polarization" = choose a direction, before we just choose the real part)
- Correction: due to we can only do locally, we'll compute locally. When gluing together, sections of a bundle need a covariant condition (line)

In conclusion:

$s \longleftrightarrow$ irreducible representation

$m \longleftrightarrow$ weight of this representation correspond to s. } spin-rep

(the concrete construction is spherical harmonic function) generalize

holomorphic sections $\longleftrightarrow$ irreducible representation (BWB) generalize

Krillov's GQ formalism.

(GQ commutes with reduction is conjectured by Guillemin–Sternberg)

PART IV Deformation Quantization

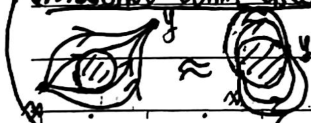
Above the correction can also be realized via path-integral formalism (on the complexified phase space) due to it's local by definition

- Recall of path-integral formalism in physics:

Quantum propagator $K(x,y)$, $x,y \in$ extended phase space

$K(x,y) = \int \square \exp(\frac{i}{\hbar} S(x,y))$, $\square$ = limit (add t) of a infinite product

Aharonov–Bohm effect tells us we can push x, y to boundary:



, thus $U: \psi \mapsto \int K(-,y)\psi(y)\,d\mu$ is FQFT functor from Cob cat, not Mfd cat. (integral kernel of)

(Here we omit how to make it into a measure make sense to have rigorous math foundation)

(In QFT, adding relativistic effect, the path-integral only changes $\mathcal{L}$, others all same)

Date

$S(x,y) = \int_x^y \mathcal{L}(q,\dot{q},t)\,dt$ the integral of Lagrangian, i.e. action functional and S depend on path if the extended phase space isn't simple connected

$\Rightarrow K(x,y) = \int \exp(\frac{i}{\hbar} S[q(t)])$ "$d\mu$" integral over all paths $q(t)$ with $(q(t_0); t_0) = x$ and $(q(t_1); t_1) = y$, i.e. $\frac{1}{\hbar} \to \infty$

Our classical limit is when $\hbar \to 0$, the oscillator integral in form of $\int e^{i\omega S} \square\, d\mu \xrightarrow{\omega \to \infty}$ High frequency part $\pm$ cancelled

$\xrightarrow{\omega \to \infty}$ Low frequency must $S' = 0$, is lefted

Rigorously, this's the method of stationary phase.

Thus in path integral formalism, the quantization is adding these terms back to classical part, and when $\hbar \to 0$ degenerate back.

(Recall) E.g. 2 $\sigma(A \circ B) = a \# b$, $\sigma_p(A \circ B) = ab$

$\approx ab + \{a,b\} + \cdots$ asymptotic expansion

This's the Weyl product $(f \underset{W}{*} g)(x,y) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x+u)g(y+v)\, e^{i(u_1v_2 - u_2v_1)}\,du\,dv$

$\underset{W}{*}: C^\infty(\mathbb{R}^2) \times C^\infty(\mathbb{R}^2) \longrightarrow C^\infty(\mathbb{R}^4)$

note that $\sim fg + \{f,g\} + \cdots$

- Integral representation is global, but asyptotic expansion is local (change one pt value not change integral)
- $f \underset{W}{*} g - g \underset{W}{*} f$

$= 2\{f,g\} + \cdots$ recover a $[-,-] = \{-,-\}$ as before discussion

- Later the Moyal given the semi-classical analogue.

$(f \underset{W}{\overset{\hbar}{*}} g)(x,y) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x+u)g(y+v)\, e^{\frac{i}{\hbar}\boxed{(u_1v_2 - u_2v_1)}}\,du\,dv \sim fg + \hbar\{f,g\} + \frac{\hbar^2}{2!}\cdots$ → action functional

by stationary phase, $\lim_{\hbar \to 0} (f \underset{W}{\overset{\hbar}{*}} g) = fg$

Another example is small quantum cohomology with product defined by GW potential $= 2\beta + O(\hbar)$, thus is the quantization of original cohomology ring (classical intersection product; commutative)

• Note that all above discussions are pure algebraic, without concerning the states $\in$ some linear space, but directly in (operator) algebra level

• Only the first (classical term, commutes) & the second ($[-,-] \leftrightarrow \{-,-\}$ Lie algebra homomorphism/ canonical commutator relations) terms is still not enough, we do can inductively compute all higher order terms in the asymptotic expansion, classically we have WKB method... all are belong to perturbative theory of quantization, lifting into derived geometric setting ("Derived = Taylor expansion") and we call it deformation quantization (Derived)

Compare the following steps with WKB.

Def 1. A formal (reach $O(\hbar^{\infty})$ order) deformation quantization of M Poisson is put a star product (generalize Moyal's before) on $C^{\infty}(M)[[t]]$

$a * b = \sum B_n(a,b) t^n$, ① $B_0(a,b) = ab$, ② $B_1(a,b) - B_1(b,a) = \{a,b\}$, $B_i \in HC^2(C^{\infty}(M), C^{\infty}(M))$

(We can add $\hbar$ into this case, this's strict deformation quantization of C^*-alg and when $\hbar \to 0$, it's just formal deformation quantization)

• The (A) existence of $*$ is highly nontrivial, for Possion mfd this's proven in [K] by some "Feymann diagram"

• Further topics refer to [K] too, such as (B) Deligne's conjecture, (C) Kontsevich's formality thm $HH^{\bullet}(C^{\infty}(M), C^{\infty}(M)) \cong \bigoplus_k C^{\infty}(\Lambda^k TM)$ ...

cohomology-level $\leq$ chain-level

Here I only do last thing: Associativity $\Rightarrow$ Solving B_0, B_1, B_2 ... inductively

$\underbrace{B_0 \circ B_n}_{\text{3-cochain}} + B_1 \circ B_{n-1} + \cdots + B_n \circ B_0$ $\boxed{\text{Higher}}$ As $\boxed{L_\infty\text{-algebra}}$

$= 0 = \delta B_n$? (A_∞-algebra with $\times$ replaced by $[-,-]$)

$\in HH^3(A,A)$ ... $(*)$ By $(*)$ + (B_0 & B_1 given) $\mapsto$ All B_n ✓

Inductively, B_n holds $(*) \Leftrightarrow [B_n] = 0 \in HH^3(A,A)$

So $HH^3(A,A)$ is obstruction of associativity/ deformation quantization.

What's quantization? (Conclusion)

Commutative $\rightsquigarrow$ Noncommutative

Finite-dimensional $\rightsquigarrow$ Infinite-dimensional

$\boxed{\text{Geometric object/morphism}}$ $\rightsquigarrow$ $\boxed{\text{Function space/ operators}}$

One term $\rightsquigarrow$ Asymptotic expansion

$\boxed{\text{Sigma model}}$ $\rightsquigarrow$ $\boxed{\text{Sheaves/Bundles}}$ $\Downarrow$ lift via function = section of vector bun

Eg. GW $\rightsquigarrow$ DT, thus in [MNOP], they're up to a exponential.

They give rise to different "formalisms". BV is interesting, but omitted here.

$O(\hbar^0)$	$O(\hbar^1)$	...	$O(\hbar^{\infty})$
classical	semi-classical		formal