E/F, $E = \Omega_f$ is the splitting field of $f(X) \in F[X]$ monic irreducible over F.
$\alpha_i \in E$ is one root of f, prove that:
① The multiplicity of each α_i (root of f) is $[F(\alpha_i):F]_i$
② $f(X) = [(X-\alpha_1)\cdots(X-\alpha_n)]^m$; $m = [F(\alpha_1):F]_i$, $n = [F(\alpha_1):F]_s$ over E

Pf. ① $\boxed{\text{Step 1}}$ $m(\alpha_1) = [F(\alpha_1):F]_i$

For $\operatorname{char} F = 0$, $m(\alpha_1) = 1$ by irreducible and F perfect $\Rightarrow F(\alpha_1)/F$ seperable
$\Rightarrow [F(\alpha_1):F]_i = [F(\alpha_1):F_{sep}] = 1$;

For $\operatorname{char} F = p$,

<u>Claim.</u> $f(X) = g(X^{p^r})$, $\exists\, r$, s.t. g is seperable over F, ~~and every root of $f(X)$ is root of $g(X)$ irreducible~~ ~~thus every root $X \mapsto X^{p^r}$ has multiplicity of ... and ...~~

<u>Pf of Claim</u> (<u>Prop 4.6, GTM 167, Morandi</u>)

$\exists\, r$ maximal, s.t. $f(X) \in F[X^{p^r}]$ (due to $r=0$ exist; $p^r \leq \deg f$ finite)
$\Rightarrow f(X) = g(X^{p^r})$
$\Rightarrow g(X) \notin F[X^p]$ by maximality $\Rightarrow g$ seperable
and g irreducible is clear ▨

We partion f to $\underbrace{X \mapsto X^{p^r}}_{\substack{\text{purely inseperable} \\ [F(\alpha_1):F]_i}} \underbrace{\mapsto g(X^{p^r})}_{\substack{\text{seperable} \\ [F(\alpha_1):F]_s}}$

then $m(\alpha_1)$
$= p^r m(\alpha_1^{p^r} \text{ in } g(Y))$
$= p^r \cdot 1 = [F(\alpha_1):F]_i$ ▨

$\boxed{\text{Step 2}}$ $m(\alpha_1) = m(\alpha_i)$, $\forall i$

Denote $m(\alpha_1) = m$
$\Rightarrow f(X) = (X-\alpha_1)^m g(X)$ over E

We have a field isomorphism over F: $\varphi: E \to E$, $\alpha_1 \mapsto \alpha_i$
$\Rightarrow$ it induce a $\varphi: E[X] \to E[X]$, $(X-\alpha_1) \mapsto (X-\alpha_i)$
$\Rightarrow f(X) = \varphi(f(X)) = (X-\alpha_i)^m \varphi(g(X))$
$\Rightarrow m(\alpha_i) = m$ ▨

Then by the Claim, directly
$[F(\alpha_1):F]_s = \deg(g)$
and $[F(\alpha_1):F]_i = p^r$ ▨ $= m(\alpha_1)$

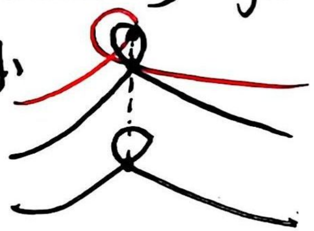
② $f(X) = \prod(X-\alpha_i)$ by splitting and monic and the multiplicity $\Rightarrow f(X) = (\prod(X-\alpha_i))^m$ by ①
$\deg f = [F(\alpha_1):F] = [F(\alpha_1):F]_s \cdot [F(\alpha_1):F]_i$
$\Rightarrow n = [F(\alpha_1):F]_s$ ▨

<u>Coro.</u> The ramification $(f)\mathcal{O}_X = (X-\alpha_1)^m \cdots (X-\alpha_n)^m$ is
$(f) \trianglelefteq \mathcal{O}_X$ prime by $f \in F[X]$ irreducible
$(X-\alpha_i) \trianglelefteq \mathcal{O}_X$ prime (maximal)

Picture

When ~~$n=1$~~ $\Rightarrow$ ~~totally~~ ramified

E.g.,



is unramified 2-covering and even étale ([Ha] III, §10, E
↳ For concre
constructi

Hausdorff measure's basic properties & application in geomertic analysis.

First, recall the definition of Hausdorff measure during the course, and study two things ① It's Borel regular ② Why it can reflect the measure then using the Hausdorff measure to do variation and studying minimal submanifold.

Idea described here (Due to may have no enough time)

①(Plateau problem) Finding surfaces S "filling γ" ($\partial S = \gamma$) s.t. Area(S) is minimal. In general, it gives γ closed boundary condition and a variation problem in a general manifold;

③ Then naturally: When varifold turns out to be manifold? This kind of problems are called the regularity; (Allard...)

④ Applying it to minimal surface theory.

If having enough time, I'll introduce some definitions in ② & applications in ④.

② Recall the classical analysis (PDE) tells us: a sequence of smooth solutions $\xrightarrow{\lim}$ may not smooth continuous... continuous... The solution to the "noncompleteness" is Weak solution (distribution). Such expanding category of objects. Here also: a sequence of smooth manifold $\longrightarrow$ may not smooth $\Longrightarrow$ adding the varifold (Current generalizes distribution);

Rk. Such idea is delighted after varifold $V = \underline{v}(M, \theta)$ is defined by purely abstract way of Hausdorff dimension. Thus our key of studying is using a measure to replace (abstract) a manifold. However, only left measure means that we can only study minimal surfaces' existence & regularity (as PDE) with the properties derived by convergence.

PART I Basic theory

Recall. $\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A) = \lim_{\delta\to 0} \inf_{\substack{A \subset \bigcup C_j \\ \operatorname{diam}(C_j) \leq \delta}} \sum \underbrace{\frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}}_{\text{denoted } \alpha(s)} \left(\frac{\operatorname{diam}(C_j)}{2}\right)^s$ using small "balls" covering A to measure the s-dimensional Hausdorff measure. ($\alpha(s)$ ↳ the volume of standard s-ball)

"然而，对于一些读者来说这样的定义就仍然略显粗糙" (Lou) And for here $A \subset \mathbb{R}^n$, we generalise it into a general mertic space X. (One can directly redefine $A \subset X$ above), see [Folland], and using the following way, Borel regular is easier.

Df 1.1. (Carathéodory's construction) Given datum $(X, \mathcal{F}, \zeta)$, $\mathcal{F} \subset 2^X$, $\zeta: 2^X \to \overline{\mathbb{R}}$, restrict to $\zeta|_{\mathcal{F}}$ is non-negative (may $=\infty$) $\Longrightarrow$ then we construct a family a pretimary measure μ_δ, and $\mu = \lim_{\delta\to 0} \mu_\delta$ a final measure. ($0 < \delta \leq \infty$)

Construction $\forall 0 < \delta \leq \infty$, $\forall A \subset X$, $\mu_\delta(A) := \inf_{\substack{A \subset \bigcup C_j \\ \operatorname{diam}(C_j) \leq \delta \\ C_j \in \mathcal{F}}} \sum \zeta(C_j)$, then μ_δ is monotonely decreasing $\Longrightarrow \lim_{\delta \to 0} \mu_\delta(A)$ exists $\Longrightarrow \mu: A \longmapsto \lim_{\delta\to 0} \mu_\delta(A)$

One easily see (existence) $\mu_\delta(A \cup B) \geq \mu_\delta(A) + \mu_\delta(B)$ if $d(A, B) > \delta > 0 \Longrightarrow \delta \to 0$, $\mu(A \cup B) \geq \mu(A) + \mu(B)$ this triangle inequality implies all open sets are μ-measurable (not always μ_δ!) (triangle inequality reversed). μ does a measure, and taking $\mathcal{F} = \mathcal{B}_X$ the Borel sets $\Longrightarrow \mu$ is Borel regular this case.

Eg 1.2. (s-dimensional spherical measure) Taking $\mathcal{F} = \{\text{closed balls} \subset X\}$, $\zeta(S) = \frac{\Gamma(\frac{1}{2})^s}{\Gamma(\frac{s}{2}+1)} \left(\frac{\operatorname{diam}(S)}{2}\right)^s$ we call the result of Carathéodory construction to be s-dimensional spherical measure

Eg 1.3. Taking $\mathcal{F} = 2^X - \{\emptyset\}$, $\zeta(S)$ above, result the desired Hausdorff measure (almost tatological here, but if one involving the exterior products to construct more ζ in $\mathbb{R}^n$ or M, it's more convenient, see [Federer, GMT])

Eg 1.4. ① $\dim\{pt\} = 0$ & $\mathcal{H}^0$ is counting measure. By $\alpha(0) = 1 \Longrightarrow \mathcal{H}^0(\{pt\}) = 1$ by definition ▨

② $\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$. It'll be taught by next speaker omitted ▨

③ $\mathcal{H}^s = 0$ on $\mathbb{R}^n$ for all $s > n$, i.e. $\dim_H(\mathbb{R}^n) = n$

Notice that s can be fractional or even more strange. (Why we call "❄" objects the fractal ($\dim_H(A) > \dim(A)$) ↳ chain of closed/open sets) Later I'll explain this when discussing the Hausdorff dimension.

Here $\mathcal{H}^s([0,1]^n) = \lim_{m\to\infty} \mathcal{H}^s_{\frac{\sqrt{n}}{m}}([0,1]^n) = \lim_{m\to\infty} \inf (\cdots)$ We skip it due to later we'll compute a more complex example (Cantor set)

$\leq \lim_{m\to\infty} \sum_{j \leq m^n} \alpha(s) \left(\frac{\sqrt{n}}{m}\right)^s = \lim_{n\to\infty} \alpha(s) n^{\frac{s}{2}} m^{n-s}$ (partion into small cubes)

$= \begin{cases} 0; & n > s \\ \infty; & n < s \\ \alpha(s) n^{\frac{s}{2}}; & n = s \end{cases}$ ▨

Prop 1.6. Set L an affine transform $x \longmapsto ax' + b$ (x' is rotation of x) $\Longrightarrow \mathcal{H}^s(L(A)) = a^s \mathcal{H}^s(A)$

Pf. Trivial. By taking covering C_j under L, ball scaling $a^s \Longrightarrow \mathcal{H}^s$ $a(A)$ ▨

Thm 1.4. μ is Borel regular measure.

Pf. ① μ is measure: suffices check countable additivity: $\mu_\delta(\bigcup A_j) \leq \sum_{i,j} \zeta(C_{ij})$ taking inf with sum over $\{A_j \subset \bigcup C_{ij},\ \operatorname{diam}(C_{ij}) \leq \delta\} \Longrightarrow \mu_\delta(\bigcup A_j) \leq \sum_j \mu_\delta(A_j)$

Then let $\delta \to 0$ & $\Rightarrow \bigcup A_j \subset \bigcup C_{ij}$ $\Longrightarrow \mu_\delta(\bigcup A_j) \leq \sum_j \mu_\delta(A_j) \leq \sum_j \mu(A_j)$ $\Longrightarrow \mu(\bigcup A_j) \leq \sum_{j} \mu(A_j)$ ▨

② Taking $\mathcal{F} = \mathcal{B}_X \Longrightarrow \mu$ is Borel ▨ By the triangle inequality & Carathéodory's criterion

(Recall. It says that Borel $\Longleftrightarrow \mu(A \cup B) = \mu(A) + \mu(B)$ and $A \cap B = \emptyset$ ↳ $d(A, B) > 0$)

③ Regularity. For $\forall A \subset X$, $\exists B \in \mathcal{B}_X$, $A \subset B$ and $\mu(A) = \mu(B)$.

$\forall j \geq 1$, taking a family $\{C_{ij}\}_{i \geq 1}$, $\operatorname{diam}(C_{ij}) < \frac{1}{j}$ closed $A \subset \bigcup_{i \geq 1} C_{ij}$ & $\sum_i \zeta(C_{ij}) \leq \mu_{\frac{1}{j}}(A) + \frac{1}{j}$ it's always possible by $\mu_{\frac{1}{j}}(A)$'s definition is inf $\Longrightarrow$ taking $B = \bigcap_j \bigcup_i C_{ij}$, done ▨ (let $j \to \infty$)

To describle Hausdorff dimension, we need

Prop 1.7. Set $A \subset X$ and $0 \leq s < t < \infty$

① $\mathcal{H}^s(A) < \infty \Longrightarrow \mathcal{H}^t(A) = 0$

② $\mathcal{H}^t(A) > 0 \Longrightarrow \mathcal{H}^s(A) = \infty$

Pf. ① $\Longleftrightarrow$ ②, only prove ①: Taking $\{C_j\}$ covering A, $\operatorname{diam} C_j \leq \delta$ and $\sum \zeta^s(C_j) \leq \mathcal{H}^s_\delta(A) + 1 \leq \mathcal{H}^s(A) + 1$

$\Longrightarrow \mathcal{H}^t_\delta(A) \leq \sum \zeta^t(C_j) = \frac{\alpha(t)}{\alpha(s)} 2^{s-t} \sum \zeta^s(C_j) (\operatorname{diam} C_j)^{t-s} \leq \frac{\alpha(t)}{\alpha(s)} (\delta/2)^{t-s} (\mathcal{H}^s(A) + 1) \xrightarrow{\delta \to 0} 0$ ▨

By Prop 1.7 thus we define the Hausdorff dimension $\dim_H(A) = \inf\{0 \le s < \infty \mid \mathcal{H}^s(A) = 0\}$ for $A \subset X$

Eg. 1.8 ① $A \subset \mathbb{R}^1$ to be the Cantor set (it does a fractal), $\dim_H A = \log 2/\log 3 > 0 = \dim A$, this examples shows ② $\dim_H C = 2 > \dim C = 1$, C the Peano curve of filling triangle (also space-filling curves holds this) thus they're fractal (Omit the proof here); $\dim_H A \notin \mathbb{Q}$ is possible;

③ Further properties of Hausdorff dimension is preferred by person studying the fractal, I prefer studying objects with "regularity". Set M^n to a n-dimensional manifold. $\dim_H(M) \ge n$, due to $\dim_H$ isn't a topological invariant (see ①: $[0,1] \to C$ Peano; ② Rigorously $(\mathbb{R}, d(-_1, -_2) = \sqrt{|-_1 - -_2|})$ has $\dim_H = 2$.)

$\dim_H(A) \ge \dim A$

$\dim_H(A \times B) \ge \dim_H(A) + \dim_H(B)$

But it's a diffeomorphic invariant due to only bi-Lipschitz suffices preserve $\dim_H$ ([Evans, Thm 2.8 (i)]) $\Rightarrow \dim_H M^n = n$ by it's gluing morphisms & charts are all smooth ☑

We sketch the $\dim_H(A) = \log 2/\log 3$ at last:

Pf. $\mathcal{H}^d(A) = \lim_{\delta \to 0} \inf_{A \subset \bigcup C_j} \sum \alpha(d)\left(\frac{\operatorname{diam} C_j}{2}\right)^d \le \lim \sum_j \alpha(d)\left(\frac{\operatorname{diam} C_j}{2}\right)^d$ (C_j are 2^k-intervals length $\frac{1}{3^k}$ in k^{th} step)

$= \frac{\alpha(d)}{2^d} \lim_{k\to\infty} 2^k \frac{\operatorname{diam} C_j}{3^{kd}} \le \frac{\alpha(d)}{2^d} \lim_{k\to\infty} e^{\log 2 \cdot k - \log 3 \cdot kd}$

$= \begin{cases} \frac{\alpha(d)}{2^d} ; & d = \log 2/\log 3 \\ \infty ; & d < \log 2/\log 3 \\ 0 ; & d > \log 2/\log 3 \end{cases} \Rightarrow \dim_H(A) \le \log 2/\log 3$

Other side set $\frac{1}{3^{k+1}}$-length done ☑

Exercises 1 ① Check $(\mathbb{R}, d(-_1, -_2) = \sqrt{|-_1 - -_2|})$ is of $\dim_H = 2$;

② For $A' \cong A$ homeomorphism $\Rightarrow \inf \dim_H(A') \ge \dim A$;

③ $\dim_H(A \times B) \ge \dim_H(A) + \dim_H(B)$; $A' \cong A$

④ φ is Lipschitz, then $\dim_H(\varphi(A)) \ge \dim_H(A)$;

⑤ (Rademacher's thm) $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz $\Rightarrow$ differentiable a.e. For $\varphi: M \to N$ holds still;

⑥ Compute $\dim_H(A)$, A generalised Cantor sets.

PART III Geometric applications. Time is tight, we only sketch the definition and start computation.

Def. 2.1. (Differentiable submanifolds are rectifiable manifolds), are rectifiable varifolds) Def 2.2. ① De Rham currents

A varifold on M^n Riemannian is datum (M, μ), μ is Radon on $M \times Gr_n(\mathbb{R}^m)$, where an isometry $M^n \subset \mathbb{R}^m$ is possible. (Here $Gr_n(\mathbb{R}^m)$ can be viewed as space of differential forms) Or I = the fibre bundle over M with fibre $Gr_n(\mathbb{R}^m)$ and μ on I compact supported. ② It's rectifiable if there is a rectifiable set $S \subset M$ & density function θ over S & θ locally integrable $\in L^1_{loc}(S, \mathcal{H}^k)$ then ①②③ inducing desired Radon measure $\mu(A) = \int_{\pi^{-1}(S, A)} \theta(p)\, d\mathcal{H}^k(p)$

locally rectifiable = oriented + Rectifiable varifold.

A (De Rham) current is $T \in (\Omega^k_c(M))^\vee$ and satisfy $\{\omega_j\} \subset \Omega^k_c(M)$, $\operatorname{supp} \omega_j$ bounded uniformly and $|\omega_j| \to 0 \Rightarrow T(\omega_j) \to 0$

Due to these two are quite abstract, we give some examples. (We denote $\underline{v}(S, \theta)$) As distribution, denote space of current $\mathcal{D}_k(M)$ equipped with weak-* topology

Eg 2.3 ① (Stationary varifolds) $V = \underline{v}(S, \theta)$ is stationary iff $\forall X \in \Gamma_c(TM)$ gives flow (φ_t), $\frac{d}{dt}\Big|_{t=0} (\varphi_t)_* \|V\| = 0$ (When $\theta = 1$ & is small → curvature estimate (Allard's Regularity))

where $\langle \|V\|, \varphi \rangle = \int_{S \times M} \varphi(x)\theta(x)\, d\mathcal{H}^n$, $\forall \varphi \in C_c(M)$ determines $\|V\|$ the mass.

② $\int (a\, dx \wedge dy + b\, dy \wedge dz + c\, dx \wedge dz) = \int_{\mathbb{R}^2} \varphi(x, y, 0)\, dx\, dy$ is a 2-current;

② $M^n \subset \mathbb{R}^{n+1}$ with normal vector field $\nu(p)$, $M(t) = \operatorname{Im}(F_t: (x,t) \mapsto x + t f(x)\nu(x))$, family (M_t) on M is rectifiable varifold

Rk. (De Rham cohomology) M^n rectifiable, oriented, denote $[\![M]\!](\omega) = \int_M \omega \xrightarrow{\text{Stokes}} [\![\partial M]\!](\omega) = [\![M]\!](d\omega)$, here we need $\partial M \ldots$ all also rectifiable. Think it as pair $\langle \partial M, \omega \rangle = \langle M, d\omega \rangle \Rightarrow \partial: \mathcal{D}_{n+1} \to \mathcal{D}_n$ determines a homology theory

For further homological integration, see [Federer Chap 4.] $\quad T \mapsto \partial T: \omega \mapsto T(d\omega)$)

Computation of some variation problems

Thm 2.4. (Minimalizing current and minimal surface) ① Using GMT's theory, one can determine existence of minimalizing current (minimalize the mass above) ② By precisely computation as Eg. 2.3 ② we have variation formula $\begin{cases} \frac{d}{dt}\big|_{t=0} \operatorname{vol}(M_t) = -\int_M Hf \operatorname{vol}(M) \text{ (f chosen above)} \\ \left(\frac{d}{dt}\right)^2\big|_{t=0} \operatorname{vol}(M_t) = \int_M (|\nabla f|^2 - (|A|^2 + Ric(\nu,\nu)) f^2)\, d\operatorname{vol}(M) \end{cases}$

Notation. $V = \underline{v}(S, \theta)$ as usual $T = \underline{\tau}(S, \theta, \xi)$ the current with multiplicity ξ over V (usually we assume $\xi \in \mathbb{Z}$)

Thm 2.5 (Poincaré inequality) This type is essential tool in GA

① $\fint_{B_r(p)} |f - \bar{f}| \le Cr \fint_{B_r(p)} |\nabla f|$ classically in Riemannian Geometry;

② $\mu\{x \in B_r(p) \mid f(x) > 0\} \le (1-\alpha)\omega_n r^n$, $e^{Mr} \le 1+\alpha$, $\exists \alpha \in (0,1)$ and $\mu(B_r(p)) \le Mr^n \Rightarrow$ upper holds for general varifold

Pf. Omitted ① as area-minimalising not prepared.

For ②, only compute first variation: $\frac{d}{dt}\operatorname{vol}(M_t) = \frac{d}{dt}\int_M \sqrt{\det(g_{ij}(t))}\, dx$

Pf. ① Later Exercises, ② [Simon, Lecture notes on GMT, Thm 6.6] Essentially the same way as ① ☑

$= \frac{1}{2}\int_M g^{ij}(t) \frac{\partial}{\partial t} g_{ij}(t) \sqrt{\det(g_{ij}(t))}\, dx = \int_{M_t} \operatorname{div}_{M_t}(F_t) \sqrt{\det(g_{ij}(t))}\, d\operatorname{vol}(M_t)$

$\overset{t=0}{=} \int_M \operatorname{div}_M(f(t)\nu(t)) \sqrt{\det g_{ij}(t)}\, d\operatorname{vol}(M_t) = \int_M f \operatorname{div}_M(\nu(t)) \sqrt{\det g_{ij}}\, d\operatorname{vol}(M)$

$= \int_M -Hf\, d\operatorname{vol}(M)$

※ by $\frac{\partial}{\partial t}(g_{ij}(t)) = \overline{\nabla}_{F_t}\langle F_i, F_j \rangle = \langle \overline{\nabla}_{F_i} F_t, F_j \rangle + \langle F_i, \overline{\nabla}_{F_j} F_t \rangle$

$\Rightarrow 2\operatorname{div}_{M_t}(F_t) = g^{ij}(t) \frac{\partial}{\partial t}(g_{ij}(t)) = \frac{\partial}{\partial t} \log \det(g_{ij}(t))$ ☑

(Cheeger–Colding) M^n, $Ric \ge (n-1)\Lambda > -\infty$, $A_1, A_2 \subset B_R(p) \subset M$ $\forall y_i \in A_i$, $\exists W$ s.t. $\bigcup_{y_1, y_2} \gamma_{y_1 y_2} \subset W \Rightarrow \forall f \in L^1(M)$, $\iint \int_0^{d(y_1,y_2)} f(\gamma_{y_1y_2}(s))\, ds\, dy_1\, dy_2 \le C R \int_W f (\operatorname{vol}(A_1) + \operatorname{vol}(A_2))$

Exercises 2 ① Determine when (M_t) is stationary varifold; ② Compute second variation formula; ③ Check the Eilenberg–Steenrod axioms for Rk.; ④ Prove the Segment inequality in metric Geometry; ⑤ Using Segment inequality prove Thm 2.5 ①, Check ① holds ②'s condition; ⑥* Modify Segment inequality & proving Thm 2.5 ②.

Analytic Langlands. Etingof–Frenkel–Kazhdan.

X smooth algebraic variety over $\mathbb{F}_q$

Slogan. Langlands theory on $\mathbb{F}_q(X)$ = Harmonic analysis on $Bun_G(X)(\mathbb{F}_q)$

Here for "Harmonic analysis", we need $L^2(Bun_G(X)(\mathbb{F}_q))$ w.r.t. measure $\mu(E)=\frac{1}{|Aut E|}$ (all complex-valued function)

Consider the eigenfunction of Hecke operators (point spectrum) + Continuous spectrum $\longleftrightarrow$ Rep of Galois group into $G^\vee$

Rk. ① Replace $\mathbb{F}_q$ by any local fields F $\begin{cases} \mathbb{Q}_p, \mathbb{F}_q((t)); \\ \mathbb{R}, \mathbb{C} \end{cases}$ (*) we prefer this case next;

Hecke operator on $L^2 Bun_G(X)(F)$ denoted $H_{x,\lambda}$, it commutes Hitchin differential operators when $\mathbb{R}/\mathbb{C}$, thus the spectrum of $H_{x,\lambda} \longleftrightarrow$ spectrum of Hitchin operator

We'll then ① Reduce to stable bundle; ② Investigate Hecke operators; ③ State and prove the thm. in details.

② The moduli stack $Bun_G(X)(F)$ we should only consider measure-non-zero part: we can delete degenerated bundles as they're measure zero, so is unstable one $\Rightarrow Bun_G^s(X)(F)$ is stable one, and it's open dense, with a smooth mfd structure $\subset Bun_G(X)(F)$

• Analogue with arimetric Langlands/geometric one? One side all automorphic rep & Galois side consists Galois action on $G^\vee$, and the analogue is the action spectrum of Hecke on $G^\vee$

• On this view $\{L^2(Bun_G(X)(F))\}$, we write ① (Arithmetic) $\{$Hecke eigenfunctions$\}$ = A $G^\vee(\mathbb{C})$-local system on $X = \{\pi_1^{et}(X) \to G^\vee(\mathbb{C})\}$; ② (Geometric) $\{$Hecke eigenfunctions$\}$ = $\{G$-operator on X | topological restrictions$\}$

Setting. $G = PGL_2$, X smooth proj curve over F local field, consider $Bun_G(X, t_1 \cdots t_N)$, (t_i) marked $\subset X(F)$ is the ramification locus. The marked pts gives the moduli a parabolic structure:

Def1. (Bundle with parabolic structure) E is principal PGL_2-bundle on X, at each marked pts t_i, $\mathbb{P}E_{t_i} \cong \mathbb{P}^1$, and a (quasi)-parabolic structure is a choice of $s_i \in \mathbb{P}E_i$, each $i \le N$

This can be viewed as a must for defining stability in marked pts nature: $\mu(E) := \frac{1}{2}\deg E + \frac{N}{4}$ is natural, but when consider $L \subset E$ a line bundle, we need consider the local intersection of L and $S = (s_i)$ at each fibres over t_i

$\mu(L) := \deg L + \frac{N_L}{2}$; $N_L := \#\{i \mid s_i = L_{t_i}\}$;

then E stable $\iff \mu(L) < \mu(E)$ in marked setting.

Denote these coarse moduli by Bun^s and Bun^{ss}, it's easy to see that $Bun^s \subset Bun^{ss}$ is an open set, and with quasi-proj & proj variety structure respectively. (Both smooth) Observation. N odd $\Rightarrow$ s equivalent to ss, i.e. $Bun^s = Bun^{ss}$.

Consider $\mathcal{H} := L^2(Bun^s)$: due to no natural measure given on Bun^s, we define it as half-density, thus it's Hilbert.

The Hecke operators: recall the older geometric Langlands, their Hecke operator is a sum over all Hecke modifications, but now we must integrate.

• Half-density is $\lambda: \mathcal{G} \to \mathbb{C}$ on groupoid generally, with $\lambda(eA) = |\det A|^{\frac{1}{2}}\lambda(e)$, thus it's free w.r.t any Haar measure choice

Def2. (Hecke modification) E' is the Hecke modification of E at $x \in X$ if $\exists s \in \mathbb{P}E_x$, s.t. $0 \to E \to E' \to \delta_s \to 0$ (skyscraper) $\Rightarrow$ the (global) section of E' more than E: adding a first order pole at x, with residue in s.

We denote $Z = \{(E, E', x, s)\}$, it's called Hecke correspondence (as incidence correspondence), and Z_x is the fiber of p_3 at x.

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{p_1} & \downarrow^{p_2} & \searrow^{p_3} & \\ Bun^s & & Bun^s & & X \end{array}$$

we abuse notation to p_1 & p_2 to Z_x

Thm1. (Beilinson–Drinfeld) $\exists$ canonical nonvanishing section $g \in \Gamma(Z, p_1^* K_{Bun^s}^{-1} \otimes p_2^* K_{Bun^s} \otimes \omega^2 \otimes p_3^* K_X^{-1})$

Def3. (Hecke operator) $H_x : \mathcal{H} \to \mathcal{H}$ is defined as also operator of $-\frac{1}{2}$-density, w.r.t. $x \in X$

$$\psi \mapsto H_x\psi : E \longmapsto \int_{(E, E', s) \in Z_x} \psi(E')|g|^{\frac{1}{2}}$$

Rk. In general H_x is well-defined is still open, we only know the case when restrict to $Bun^{vs} \subset Bun^s$ open, deleting all nontrivial nilpotent Higgs fields: take $\mathrm{supp}\,\psi \subset Bun^{vs}$ compact subset $\Rightarrow (H_x\psi)(E)$ converges to Bun^s, $\forall E \in Bun^s \Rightarrow H_x$ densely defined in Bun^{vs}, the well-defined is conjectured below

Conj1. ① H_x extends to a compact self-adjoint operator on $\mathcal{H}$; ② $[H_x, H_y] = 0$; ③ $\cap \ker H_x = 0$

By ①②③ $\Rightarrow \boxed{\mathcal{H} = \bigoplus_{\Lambda \text{ (discrete)}} \mathcal{H}_\Lambda}$ the decomposition of eigenspaces. (By theory of spectrum of compact operator, $\Lambda = (\lambda(x))$, $x \in X(F)$)

Thm2. For $X \cong \mathbb{P}^1$, the conjecture holds.

Note that $N = 1, 2$ are trivial, N = odd also easy to deal as $Bun^{ss} = Bun^s$, we only consider $N \ge 4$

We denote Hecke modification at t_i by $S_i : Bun^s \to Bun^s$, due to $\begin{cases} Bun^s \text{ has two components with degree odd \& even} \\ \text{Hecke modification decrease the degree by 1} \end{cases}$

$\Rightarrow Bun_0^s \overset{S_i}{\underset{S_i}{\rightleftarrows}} Bun_1^s$ and $S_i^2 = id$, $S_iS_j = S_jS_i$ (action by $(\mathbb{Z}/2\mathbb{Z})^N$) induces isomorphisms $Bun_0^s \cong Bun_1^s$ by (S_i), and these (S_i) also gives the parabolic structure of E at each (t_i) respectively.

$E \in Bun_0^s$ is generally trivial, i.e. $E \cong \mathcal{O} \oplus \mathcal{O}$ (rank 2), thus we can write out all (t_i) and (s_i) this case:

Pf of Thm2. $t_1 = 0$; $t_2 \cdots t_{N-1}$; $t_N = \infty$; $s_1 = (1, 0)$, $s_2 = (1, y_1) \cdots s_{N-1} = (1, y_{N-1})$, $s_N = (0, 1)$ $\Rightarrow Bun_0^s \simeq \mathbb{P}^{N-2}$ a birational equivalence by $E_y \leftrightarrow y = (y_1 \cdots y_{N-1})$ $\Rightarrow$ the Hecke modification $HM_{x,s}(E_y) = E_z$, $z = (z_j)$, each $z_j = \frac{xy_j - st_j}{y_j - s}$

$\mathcal{H} = \{\psi : (y_2 \cdots y_{N-1}) \mapsto \psi(y_2 \cdots y_{N-1}) \mid \psi(\lambda y) = \|\lambda\|^{-\frac{N-2}{2}}\psi(y)\}$ is the $-\frac{N-2}{2}$-densities $\Rightarrow (H_x\psi)(y) = \int_F \psi\left(\frac{t_1 - x}{s - y_1} \cdots \frac{t_{N-1} - x}{s - y_{N-1}}\right) \left\| \frac{ds}{\prod (s - y_j)} \right\|^{\frac{1}{2}}$ (a subtle thing is that $\|\cdot\|$ is defined differently in $\mathbb{R}/\mathbb{C}$: $\|a\| = \begin{cases} |a|, & a \in \mathbb{R} \\ |a|^2, & a \in \mathbb{C} \end{cases}$)

Thm3. Over $\mathbb{C}$, $\sigma(H_x)$ = $\{$operators T on $\mathcal{H}$ | T has real monodromy & unipotent singularity at $(t_i)\}$ (For $\mathbb{R}$, similar consequence also given)

To consider monodromy $\in SL_2(\mathbb{R})$, we need consider quantum Hitchin operators/Hamiltonians on $Bun^s(X)$

First we set $H_{x,\lambda}$, where λ is dominant coweights of G (upper is $PGL_2(\mathbb{C})$), root system Λ_+: only takes $\lambda = 1$ and $H_{x,\lambda} = H_x$

$\mathcal{A} \hookrightarrow \mathcal{D}(Bun_G(X)) \to \mathcal{D}(Bun_G(X))$ unbounded normal differential operators, and $\bar{\mathcal{A}}$ its closure. They're call quantum Hitchin operators.

Take $p \in X$ with $\Delta \in N(p)$, $\mathcal{O} = \mathbb{C}[\Delta] \cong \mathbb{C}[[t]]$ formal neiberhood, $K = \mathrm{Frac}\,\mathcal{O} = \mathbb{C}((t)) \Longrightarrow Bun_G(X) = G(X \setminus \{p\}) \backslash G(K) / G(\mathcal{O})$ is a double quotient (covering X by $\{X - \{p\}, \Delta\}$), each G-bundle $\Longleftrightarrow$ translation function $g \in G(\mathbb{C}((t)))$ on Δ^* (punctured).

<u>Def 4.</u> (Quantum Hitchin system) Taking the Feigin–Frenkel center of $U(\hat{\mathfrak{g}})$ ($U(\mathfrak{g})$ is the universal evelope algebra of the associated Lie algebra of G), then it's a 2-sided invariant differential operator on $G(K) \Longrightarrow$ descending to the double quotient $Bun_G(X)$ (and the descend also differential by some nontrivial facts)

• Simply the set of all double cosets and the orbits of $H \backslash (G/K)$ & $(H \backslash G)/K$ has 1-1 correspondence

<u>E.g.</u> $G = SL_2(\mathbb{C}) \Longrightarrow$ the quantum Hitchin operator are the <u>Sugawara operators</u>

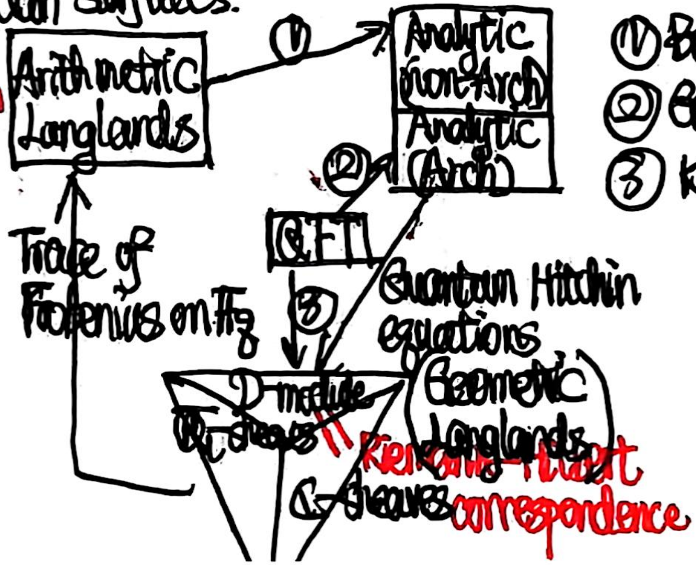
Hecke operators also formulated via the double quotient: set $Gr_G := G(K)/G(\mathcal{O})$ the affine Grassmannian, $G(\mathcal{O}) \curvearrowright Gr_G$ with orbits $(Gr_G^\lambda)_{\lambda \in \Lambda^+}$, then the Hecke operator are convolution with δ_λ: $H_{x,\lambda}\psi = \psi * \delta_\lambda$, δ_λ is the distribution of orbit Gr_G^λ, thus $\psi * \delta_\lambda$ is an integral on $L^2(Bun_G(X))$ (well-defined is nontrivial)

We had known case in $PGL_2(\mathbb{C})$ with orbit $Gr_G^\lambda = Gr_G = \mathbb{P}^1$, they're all <u>unramified</u> cases except at the spectrum of Hitchin algebra $\mathcal{A}$ is determined by $\mathrm{Spec}\,\mathcal{A} = \{G^\vee\text{-operators on } X\}$ parametered by $G^\vee$-operators, then we reformulate Thm

<u>Conj 3.</u> Eigenvectors of Hecke & Hitchin operators are also parametrized by $G^\vee$-operators with real monodromy $\pi_1(X) \to G^\vee$ splitting on $G^\vee(\mathbb{R})$

<u>Rk.</u> When over $\mathbb{R}$, story holds via an involution $\tau: \Sigma \to \Sigma$ and $\mathbb{R}$ identified fixed locals, i.e. τ is $\mathbb{R}$-map relatively, then all story holds well in Automorphism of Riemannian surfaces.

• Quantization of usual Hitchin system / fibration
$T^*Bun_G \to \mathcal{B}$ the Hitchin base, inducing $\mathcal{O}_{\mathcal{B}} \longleftrightarrow \mathcal{O}_{T^*Bun_G}$ is a Lagrangian fibration, this is called Hitchin system.
Quantum case is $\mathcal{O}(\mathcal{B}) \longleftrightarrow \mathcal{D}(Bun_G)$ with operators and differential operators



① Braverman–Kazhdan–Polishchuk, 2006
② Gaiotto–Witten, 2021
③ Kapustin–Witten, 2021

Interesting Games: Eckmann-Hilton argument and Eilenberg-Mazur swindle

① Given S a set, and two bi-operations $\begin{cases} \oplus: S\times S \to S \text{ with unit } 0 \\ \otimes: S\times S\to S \text{ with unit } 1\end{cases}$ compatibly

$\Rightarrow$ the "structure" of these two operations are "essentially same";

$$(0\oplus a)\otimes(b\oplus 0) \to a\otimes b \to (a\oplus 0)\otimes(0\oplus b) \to (a\otimes 1)\oplus(1\otimes b) \to a\oplus b \to (1\otimes a)\oplus(b\otimes 1) \to (0\oplus b)\otimes(a\oplus 0) \to b\otimes a \to (b\oplus 0)\otimes(0\oplus a) \to (b\otimes 1)\oplus(1\otimes a) \to b\oplus a \to (1\otimes b)\oplus(a\otimes 1) \to (0\oplus a)\otimes(b\oplus 0)$$

commutes

Coro. $\pi_n(X), n\geq 2$ are communicative; (H-space) and if $\exists\, \mu: X\times X\to X$ and a unit $e\in X$, then $\pi_1(X)$

Pf. $S=\pi_n(X)$ $\begin{cases} \gamma_1\otimes\gamma_2 = \begin{cases}\gamma_1(2x,y\cdots); & 0\leq x\leq \frac{1}{2}\\ \gamma_2(2x-1,y\cdots); & \frac{1}{2}\leq x\leq 1\end{cases} \\ \gamma_1\oplus\gamma_2 = \begin{cases}\gamma_1(x,2y\cdots); & 0\leq y\leq\frac{1}{2}\\ \gamma_2(x,2y-1\cdots); & \frac{1}{2}\leq y\leq 1\end{cases}\end{cases}$

When $\exists\,\mu$, replace $\otimes$ by μ directly ☒

② Consider the convergence of the series in mathematical analysis: $1 = 1+(-1+1)+\cdots$

$= (1-1)+(1-1)+\cdots = 0$, replace $+$ by other operation as $\oplus$ or $\#$:

Coro1. (Cantor-Bernstein thm) ⋯ (all we know)

Pf. Set $A+B = A\sqcup B$, and equal by $\exists$ bijection $\Rightarrow X\hookrightarrow Y \Leftrightarrow \exists\, A\subset Y, \exists\, B\subset X$, s.t. $X = Y+A = (X+B)+A$

$\Rightarrow X = \Sigma(A+B)+Z$, $\exists\, Z$, and $Y = X+B = B+(A+B)+\cdots+Z = (B+A)+(B+A)+\cdots+Z = \Sigma(B+A)+Z = \Sigma(A+B)+Z = Y$ ☒

Coro2. M proj $\Leftrightarrow \exists\, F$ free, s.t. $M\oplus F = F$

Pf. $\exists\, N$, $M\oplus N = K$ free, set $F = \bigoplus K \Rightarrow M\oplus F = M\oplus(N\oplus M)\oplus\cdots = (M\oplus N)\oplus\cdots\oplus(M\oplus N) = F$ ☒

Coro3. If Abelian category $\mathcal{A}$ admits infinite direct sum $\Rightarrow K(\mathcal{A}) = 0$ (all equivalent)

Coro4. (Mazur swindle) $M\# N \simeq S^n$, then M, N both S^n (homeomorphism preserving orientation is "$\simeq$")

Pf. $\#$ has unit $S^n \Rightarrow (M\# N)\#\cdots = S^n - \{pt\}$

$\Rightarrow \underbrace{M\#(N\# M)\#\cdots}_{\boxed{S^n-\{pt\}}} = \underbrace{M\#(S^n-\{pt\})}_{\boxed{M-\{pt\}}}$, $\boxed{S^n-\{pt\} = M-\{pt\}} \Rightarrow S^n = M$ ☒