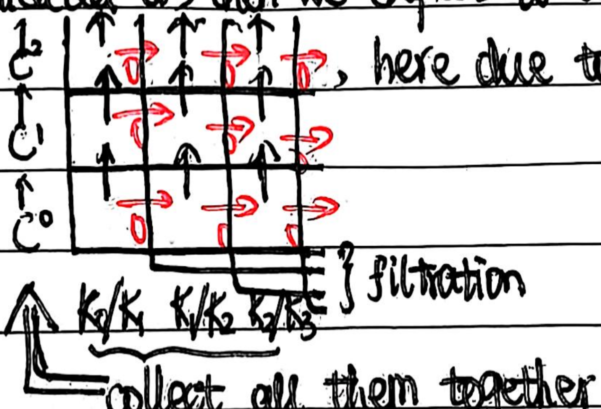
Spectral Sequences and its applications

We'll understand the SS (of double cpx), esp. Grothendieck's SS degenerates at E_2-page, then many examples will be given with full generality. In particular, we'll emphasize the algebraic de Rham cohomology and Hodge theory in an abstract level (omit analysis). We'll also not give full details of construction of SS via exact couples $\Rightarrow$ filtrated cpx $\Rightarrow$ bicpx

• Idea/motivation of SS: this is two steps in my mind: ① First, if $(C^\bullet, d)$ has quite complicated d, then we expect to "decompose" it

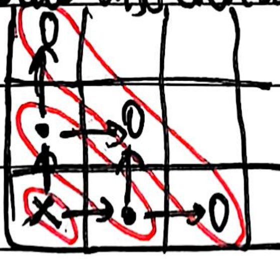
54:92 into different "levels"



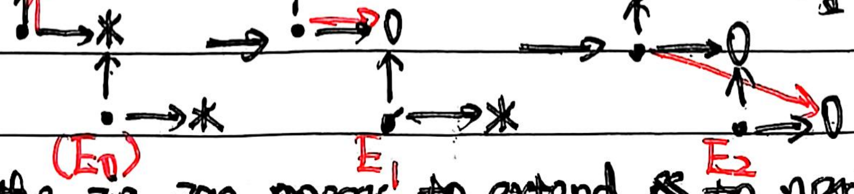
, here due to the quotient/filtrated structure one can view it bicpx via horizontal arrows D,

Hence it's equivalent to the total cohomology $D = d_h + (-1)^p d_v = (-1)^p d_h$

② Then we compute the total cohomology. the complexity is that the total differential has different domain & target mixed together:



When we consider the kernel of $2 \to 3$ in left picture, we can annilate the element in 3 step-by-steps: through the filtrat



note this is just the zig-zag process to extend SS to next page, similiarly for the image from $1 \to 2$, ~~this is~~ they just differ by their concrete (algebraic) equations

Hence when all 0, the final consequence ~~is~~ converge to total cohomology!

Only issue is extension of $Gr(K_p)$ to $K_\bullet$, this is hopeless to be generally solved

Vakil may draw some picture to describle in his own languages interestingly, anyway I don't read it

It's said that Leray do not explain the motivation of SS, and we even don't know how "spectral" named. Fu said that such a boring tool can only be developed in jail www.

Usually for use we mainly focus on computing total cohomology, all our SS refer to cohomological one and concentrated in the first quadrant, i.e. in form as before.

(Almost) All SS are in form of three types:

① E_0-started = Two usual cohomology = Bicpx case

$E_0^{p,q}=K^{p,q}$, $E_1^{p,q}=H^p(K^{\cdot,q})$ or $H^q(K^{p,\cdot})$,

$E_2^{p,q}=H^qH^p(K^{\cdot\cdot})$ or $H^pH^q(K^{\cdot\cdot}) \Rightarrow \mathrm{Tot}^{p+q}(K^{\cdot\cdot}) = E_\infty$;

② E_1-started = One usual cohomology + one resolution = Cpx + functor case

$E_1^{p,q} = R^qF(K^p)$

$E_2^{p,q} = H^pR^qF(K^\cdot)$ or $R^qF\,H^p(K^\cdot)$ (view $H^p(K^\cdot)$ cpx with $d=0$)

$\Rightarrow \mathbb{R}^{p+q}F(K^\cdot)$ hypercohomology via CE resolution $=E_\infty$;

③ E_2-started = Two resolution = Object + two functors

$E_2^{p,q} = (R^qG\circ R^pF)(K)$ or $R^p((R^qG)\circ F)(K) \Rightarrow R^{p+q}(G\circ F)(K) = E_\infty$;

$0\to K\to A^0\to A^1\to\cdots$ resolution

$\boxed{0\to F(K)\to F(A^0)\to F(A^1)\to\cdots}$ $\Longrightarrow$ take $\boxed{\text{cohomology to } R^pF}$ Path I

take ②

CE resolute it, composite take cohomology to R^qG Path II

$0\to F(K)\to B^0\to B^1\to\cdots$ resolution

$\boxed{0\to GF(K)\to G(B^0)\to G(B^1)\to\cdots}$ $\Longrightarrow$ take $\boxed{\text{cohomology to } (R^qG)\circ F}$ Path III

take ②

CE resolute it $\Longrightarrow$ take cohomology to $R^p(R^qG)F$. Path II'

Thus Path I $=(R^qG\circ R^pF)(K)$

Path II = Path II' $=R^{p+q}(G\circ F)(K)$ ☐

Path III $=R^p((R^qG)\circ F)(K)$

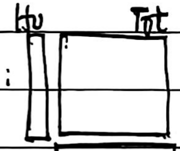
The last one ③ is called Grothendieck SS, ③ can be viewed as special case of ②, ② can be viewed as special case of ① as we're simply choosing an acyclic resolution then $R^nF(K)=H^n(F(A^\cdot))$ degenerate the usual cohomology of cpxes.

What about the converse? Precisely, "contract" cpx into one object and realize it as a resolution of it?

I found this question and tried myself, but finding it's hard (or impossible) to realize. In Tohoku's paper, Grothendieck defined so-called resolvent functors: given $F:C_1\to C_2$, $\mathbb{F}: C_1\to \mathrm{Comp}(C_2)$ is called its resolvent functor if $R^nF(K)=H^n(\mathbb{F}(K))$, i.e. $\mathbb{F}(K)=F(A^\cdot)$ is a process of "choosing a resolution". Then my question is just how to deduce one F from $\mathbb{F}$ conversely, but it's still hopeless...

Hence next we give various examples of ①, ② and ③, focus on how to "factor" complicated total cohomology by taking two cohomology seperately.

① • Comparison thm type:



if the rows exact $\Rightarrow$ Tot $=H_v$

if the columns exact $\Rightarrow$ Tot $=H_h$

For example, Čech-to-de Rham bicpx of a good cover has all exact by Poincaré lemma, hence $H_h=H_v$; same holds for E_1-degenerate (and symmetry between two types);

• Frölicher spectral sequence are Dolbeault double cpx $(\partial,\bar\partial)$

$E_1^{p,q}=H^q(X;\Omega^p)$ takes cohomology w.r.t. $\bar\partial$ (kill $\bar\partial$-part and this turns into algebraic, i.e. in algebraic level we can't extend to E_0 but adding $\bar\partial$ (choose a CE resolution) we can)

$E_1^{p,q} \Rightarrow E_\infty = H^{p+q}_{dR}(X;\mathbb{C})$

$E_0^{p,q}$

Compact Kähler case it's E_1-degenerate $\Rightarrow H^n_{dR}(X;\mathbb{C}) = \bigoplus_{p+q=n} H^q(X;\Omega^p) = \bigoplus_{p+q=n} H^{pq}$ is Hodge decomposition, and it admits Hodge

① Filtration $F^pH^n(X;\mathbb{C}) = \mathrm{Im}(H^n(X;\Omega_X^{\geq p}) \to H^n(X;\Omega_X^{\cdot}))$
$= \bigoplus_{p'\geq p} H^{p'q} = H^n(X;\mathbb{C}) \cap K^p$

② Start at E_1-page in $\bigoplus_{p+q=n} H^{pq} \cap K^p$

algebraic case above $\Rightarrow$ Hodge-to-de Rham SS

Hence we define algebraic de Rham cohomology $H^n_{dR}(X) = \mathbb{R}^n\Gamma(X;\Omega_X^{\cdot}) = \mathbb{H}^n(X;\Omega_X^{\cdot})$. Note that when it coincides classical de Rham cohomology when our topological is fine enough, s.t. $0 \to \mathbb{C} \to \Omega_X^0 \to \Omega_X^1 \to \cdots$ is acyclic resolution, the $\mathbb{R}^n$ degenerate to R^n. But in Zariski topology it doesn't, in algebraic case, we need some infinitesimal ~~topology~~ formally

① • Leray-Serre SS: $E_0^{pq} = C^p(\mathcal{U}; \Omega^q(F))$ for fibration $F \to X \to B$ and $\mathcal{U}$ is good cover of B, $C^p(\mathcal{U}, \square)$ is the Čech cohomology

$E_0^{pq} = C^p(\mathcal{U}; \Omega^q(F)) \Rightarrow E_2^{pq} = H^p(B) \otimes H^q(F) \Rightarrow E_\infty = H^{p+q}(X)$

(We know fibration fit's homotopy group via LES better, but here we can also use fibration to compute homology.)

Two applications of Leray-Serre:

(A) (Unstable) Homotopy group of spheres: $\pi_p(S^q) = 0$ for $p < q$ and $\pi_q(S^q) = \mathbb{Z}$ by definition

But $\pi_p(S^q)$ for $p > q$ is open problem in general, our strategy is compute $H_p(K_p(\pi_p(S^q), p), \mathbb{Z}) = \pi_p(S^q)$ by find a "good" fibration on $K(\pi_p(S^q), p)$: for example, ~~there~~ is $\Omega X \to PX \to X$ … path space, contractible, hence $H_\cdot(PX) = \mathbb{Z}$

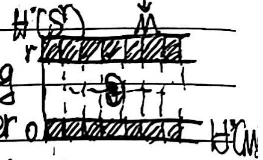
$\Omega K(A,p) = K(A, p-1)$ lower but this LS not converges at all!

Hence it's always computable, if $K(A,p)$ is clearly constructed.

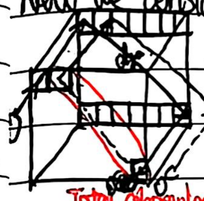
(For stable case, we have Adam's SS, which I know nothing)

(B) Gysin sequence: here we consider sphere bundle $S^r \to S(E) \to M$, rank $E = r+1$, $S(E) \subset E$ is its sphere bundle.

We have E_2-page has only two rows nonvanishing



Now we consider both E_2 & E_∞ same r-pages together,



form a "rotation", writing precisely: $\cdots \to H(S(E)) \to H(M) \xrightarrow{d_r} H(M) \to \cdots$

Here total cohomology factors into direct sum of two $\Rightarrow H(S(E)) = H_1 \oplus H_2$

$\Rightarrow d_2 = d_3 = \cdots = d_{r-1} = 0$, $d_r \neq 0$, $d_{r+1} = \cdots = 0$ $\Rightarrow E_\infty$ also two rows but needn't to be se…

and $H(M) \to H(S(E)) \to H(M)$ is $H(M) \to H_1 \subset H(S(E)) \twoheadrightarrow H_2 \to H(M)$

hence exact $\Rightarrow$ it's exact.

And by property of sphere bundle, we know $d_r = -\cup e$, $e \in H^{r+1}(M)$

$\Rightarrow$ Gysin sequence is $\cdots \xrightarrow{\pi^*} \boxed{H^\cdot(S(E)) \xrightarrow{\pi_*} H^\cdot(M)} \xrightarrow{\cup e} H^\cdot(M)[r+1] \xrightarrow{\pi^*} H^\cdot(S(E))[\ldots$

Gysin map

Another approach is by "SES" "$0 \to M \xhookrightarrow{s} E \to S(E) \to 0$" (indeed, the relative homology LES) $\Rightarrow \cdots \to H^\cdot(E, s(M)) \xrightarrow{+1} H^\cdot(E) \to H^\cdot(s(M)) \to H^\cdot(E, \ldots$

also Gysin sequence (Poincaré dual to upper one) $\to H^\cdot(S(E)) \xrightarrow{-r} H^\cdot(M) \xrightarrow[+(r+1)]{\cup e} H^\cdot(M) \to H^\cdot(S(E)\ldots$

(vertical maps: $\cong$, Thom $\cong$ $-(r+1)$, $\cong$, $\cong$)

$\boxed{\text{Slogan}}$: From SES induces LES, one can factor through a SS to induce LES

Two far-reaching generalization of Gysin sequence is

• Replace $S(E) \subset E$ by $E \twoheadrightarrow \mathbb{P}(E)$ in algebraic set. the Gysin homomorphism is fundamental in intersection theory, as $\cup e \Leftrightarrow$ intersect with zero section

• Conne's SS $\Rightarrow$ Conne's IBS LES is a noncommutative parallel story of above. from $0 \to C_\lambda \to C \to C/C_\lambda \to 0$ one can directly get

$\cdots \xrightarrow{I} HH^\cdot(C) \xrightarrow{+1} H^\cdot(C/C_\lambda) \to H^\cdot(C_\lambda) \to H^\cdot(C) \xrightarrow{+1} \cdots$

$\| \qquad$ "cobordism" $\cong -2 \qquad \| \qquad \|$

$\cdots \xrightarrow{I} HH^\cdot(A) \xrightarrow[B]{-1} HC^\cdot(A) \xrightarrow[S]{+2} HC^\cdot(A) \xrightarrow{I} HH^\cdot(A) \xrightarrow{B} \cdots$

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cyclic cohomology related with T-duality.

Partially generalizing Gysin in $r=1$, $U(1)$-gauged case, and in this case, $\pi^*\circ\pi_*: H^\bullet(S(E)) \xrightarrow{-1} H^\bullet(S(E))$ = differential

$I\circ B: HH^\bullet(A) \xrightarrow{-1} HH^\bullet(A)$ = Connes's NC differential B

$\pi_*\circ\pi^*: H^\bullet(M) \xrightarrow{-1} H^\bullet(M) = \cdots$

$B\circ I: HC^\bullet(A) \xrightarrow{-1} HC^\bullet(A) = \cdots$

$HH^\bullet$ always contains more redundant datum than $HC^\bullet$, one can view it as S^1-bundle over it (although open string in M, it has S^1-fibres closed:

② Replace LS SS with $E_1^{pq} = C^p(\mathfrak{U}, H^q(F))$ by generalized cohomology $E^q(F)$,
$E_1^{pq} = C^p(\mathfrak{U}, E^q(F)) \Rightarrow E_\infty = E^{p+q}(X)$ is Atiyah-Hirzebruch SS
it don't have E_0 page and writing it as derived functor in K-theory case is also subtle.

③ Understand LS as derived functor of global sections, we have $E_2^{pq} = (R^pf_*)\circ(R^qg_*)(\mathcal{F}) \Rightarrow E_\infty = R^{p+q}(f\circ g)_*(\mathcal{F})$ also called LS SS as special case of Grothendieck SS.