• Categorification. Given numerical invariant / polynomial invariant e. $\rightsquigarrow$ (Co)homology, s.t. $e \in H^{top}_{Hodge}(X)$. 元 By a grading structure

(Khovanov homology — categorification → Jones polynomial. Eg. Euler number — categorification → homology)

Lecture 1

Topological Quantum Field Theory and Khovanov Homology — Yin Tian

(Categorification is due to Atiyah's original work (bi) on TQFT)

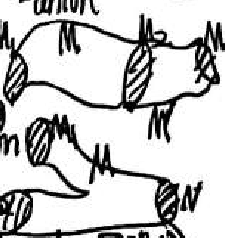
(called the vector space the phase space)

Def 1 (TQFT). A n-dim TQFT is a functor $F: n\text{-Cob} \to \boxed{\mathrm{Vect}_k}$ symmetric & monoidal: disjoint union $F \to \otimes$

(One may prefer notation of $(n+1)$) n-dim cobordism category

$\mathrm{Hom}(M_1^{n-1}, M_2^{n-1}) = \{M^n \mid \partial M^n = \overline{M_1^{n-1}} \amalg M_2^{n-1}\}$ is the horizonal categorification of cobordism

(Here $\overline{M_1}$ mean the $\boxed{\text{orientation}}$ reversed.)



(It corresponding to the complex conjugate on the $F(M^{n-1})$ a vector space when $k = \mathbb{C}$; In general, it gives an isomorphism $F(M^{n-1})^* \cong F(\overline{M^{n-1}})$)

① is the composition; ② is the monoidal structure.

*Eg. reverse the orientation

Eg 1. 1-dim TQFT is Quiver generated by

objects: $\bullet+$ $\bullet-$

morphism: identities; $\mathrm{Hom}(\emptyset, +-) \cong \mathrm{Hom}(\emptyset, -+)$; $\mathrm{Hom}(+-, \emptyset) \cong \mathrm{Hom}(-+, \emptyset)$

The isomorphism is interchange the $\otimes$, by F is symmetric, it has a topological interperation:

(Twist map); these generators satisfying relations: ① "… $\cong$ …" & ② "… $\cong$ …"

Rk. Note that the topological trivial $\Leftrightarrow$ identity; this is the TQFT distincts from QFT, it only depends on the topological structure, not the "time" in physics: … $\cong$ … adding one "time" not change anything.

Applying F to ① with $\square - (\cdot = \triangle$ … id

$\Rightarrow F(+) \to F(+) \otimes F(-) \otimes F(+) \to F(+)$, via $F(+) \otimes k$ and $k \otimes F(+)$. Here we used relation ①

$\Rightarrow \boxed{F(+) = F(-)^*}$ (By $F(+) \otimes F(-) \cong k \cong F(-) \otimes F(+)$; $x \otimes y \mapsto \langle x, y \rangle$) thus F determined by $F(+)$

Consider $\bigcirc \in \mathrm{Hom}(\emptyset, \emptyset)$, $F(\bigcirc): F(\emptyset) \to F(+-) \to F(\emptyset)$

$k \to F(+) \otimes F(+)^* \to k$

$\boxed{1 \mapsto \sum \varepsilon_i \otimes \varepsilon_i^*} \mapsto n$

$\Rightarrow F(\bigcirc) = \dim_k F(+)$, $\mathrm{tr}(F(\to)) = \mathrm{tr}(\mathrm{id}_{F(+)})$

Let $1 \mapsto \sum a_{ij} \varepsilon_i \otimes \varepsilon_j^*$, we determine a_{ij} by $\varepsilon_i = F(\ldots)(\varepsilon_i) = \sum a_{ij} \varepsilon_j \Rightarrow a_{ij} = \delta_{ij}$ ☒

Braid knot

$\boxed{\text{Compactification}} \to$ to $\bigcirc$ as the simplest one: using braid construct knot invariant

Called the planar trick: The 2-dim surfaces come from 7-kinds by

① Using twist, we can write as disjoint union of connected components $\Rightarrow$ assume connected.

② … normal form … ☒

Or by Morse theory (critical of height function): ind $= 0$ or 2: … or … maximal point; ind $= 1$: … or … saddle point ☒

The relations are just this, i.e. for isotopic two surfaces we can use relations identify them.

Conversely, the Geometric interpretation of Frobenius algebras is called the PRO (is a monoidal functor category)

Algebraically: $\mathbb{O} = (\iota: k \to F(S^1))$; $\mathbb{O} = (\varepsilon: F(S^1) \to k)$, $A = F(S^1)$; $= (m: A \otimes A \to A)$; $= (\Delta: A \to A \otimes A)$, $= (\mathrm{Flip})$

And relations correspond to their names well. Frobenius relation gives compatibility of product m & coproduct Δ

⑧ A is a commutative Frobenius algebra & F, the 2D TQFT is determined by $A = F(S^1)$ (② is hard to prove)

Eg 2. 2-dim TQFT. $1 \mapsto \sum \varepsilon_i \otimes \ldots$

[Kock. Frobenius algebra and 2-TQFT]

[Witten. Quantum field theory and Jones polynomial]

Generated by:

Objects: $\bigcirc$;

Morphisms: id, …, …, …, …, twist (abstract); pair of pants

Relations: ① "Locality" (change of places); ② "identity … $\cong$ …"; ③ "… $\cong$ …" sewing the disk; ④ "associativity"; ⑤ "co-associativity"; ⑥ "… $\cong$ …" communivity; ⑦ "… $\cong$ …" co-communivity; ⑧ Frobenius relation; ⑨ Twists … $\cong$ …

Issues (Example) … (normal form) $\cong$ … (not normal)

thus "the multiplication & co-… are dual"

Def2. (Frobenius algebra) A is finite-dimensional $\mathbb{k}$-algebra, with associative non-degenerator $\rho: A\otimes A \to \mathbb{k}$ ($\leftrightarrow$ ⊃⊃)

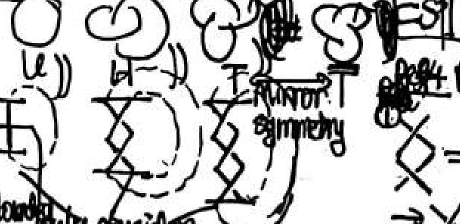
E.g.3. ① $Mat_n(\mathbb{k})$ is Frobenius algebra with $\varepsilon = tr: M_n(\mathbb{k}) \to \mathbb{k}$; ② Group algebra $\mathbb{k}[G]$ is Frobenius algebra;

③ $H^*(X;\mathbb{C})$ is Frobenius algebra, ρ given by $\rho([x],[y]) = \int \langle [x],[y] \rangle$ or $H^{even}(X;\mathbb{C})$ as odd part is skew-symmetric,

Taking $X = \mathbb{CP}^1 \Rightarrow H^*(X;\mathbb{k}) = \mathbb{k}\langle 1, x\rangle = \mathbb{k}[x]/(x^2) =: R$, R is the simplest Frobenius algebra, it has operations $\begin{cases} i: \mathbb{k} \to R, 1 \mapsto 1; \\ \varepsilon: R \to \mathbb{k}, 1 \mapsto 0, x \mapsto 1; \\ m: R\otimes R \to R; \Delta: R \to R\otimes R \\ 1\otimes 1 \mapsto 1 \qquad 1 \mapsto 1\otimes x + x\otimes 1 \\ x\otimes x \mapsto 0 \\ 1\otimes x = x\otimes 1 \mapsto x \qquad x \mapsto x\otimes x \end{cases}$

Conclusion $\boxed{\begin{array}{l} \{1\text{-TQFT}\} \cong \mathrm{Vect}_{\mathbb{k}} \text{ as sets} \\ \{2\text{-TQFT}\} \cong \{\text{communicative Frobenius algebra over } \mathbb{k}\} \end{array}}$ 3D, it not holds such a simple way as the 3-manifold is complex.

Restrict to knots (One can "recover" all 3-manifolds by surgery) and construct knot invariants. Jones polynomial (1980) does this, at least distinct T and $\bar{T}$



T $\xrightarrow{\text{mirror symmetry}}$ $\bar{T}$

Def4. B_n = {braids of n-strands} ① [braid] = || ② [braid] = [braid] $\Rightarrow$ X = X^{-1}; $s_1 s_2 s_1 = s_2 s_1 s_2$ ③ $s_i s_j = s_j s_i$ the braid relation if $|i-j| \geq 2$; Quantum Yang–Baxter equation;

Failed one: Alexander polynomial (can't distinct T & $\bar{T}$)

Def(Seifert) Seifert surface of knot L is an oriented surface F, bounded by L. It's constructed by Seifert resolution ↖↗ & ↖↗ ⇝)(

E.g.4.

L Seifert resolution Filling inside, Gluing back w.r.t the to knot orientation F

the Seifert surface allows us to define the ∞-covering space X_∞, and it turns out the Alexander polynomial $\Delta(L) \in H^1(X_\infty)$, $\Delta(L) = \det(tA - A^T)$, A is the link matrix of $H^1(F)$.

Both X_∞ and $\Delta(L)$ are knot invariants.

E.g.5

$A = \begin{pmatrix} \frac{p+q}{2} & \frac{q+1}{2} \\ \frac{q-1}{2} & \frac{q+r}{2} \end{pmatrix}$ & $\Delta = \frac{1}{4}(pq+qr+rp)(t-1)^2 + \frac{1}{4}(t+1)^2$

The categorification of Alexander is knot Floer homology

Combinational $\begin{cases} \Delta(\bigcirc) = 1 \\ \Delta(\nearrow\!\!\!\nwarrow) - \Delta(\nwarrow\!\!\!\nearrow) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta()(\,) \quad \cdots (*) \end{cases}$

only consider this part $\xrightarrow{\text{compactification}}$ $\boxed{\text{knots}}$

$\uparrow$ algebraization

$\boxed{\text{Braid group}}$ by Artin (generator is four kinds of ... it's the fundamental group

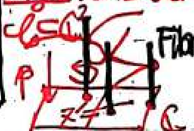
E.g.A $M_{0,n,1}$ is the moduli of $g=0$, marked n points, 1 boundary Riemannian surfaces, then $M_{0,n,1}(\mathbb{C}) = \{D^2 - \{p_1, \cdots, p_n\}\}$'s singleton thus the moduli stack $M_{0,n,1}$ is its mapping class group

$\Rightarrow M_{0,n,1} \cong B_n$ Braid group

We have $\pi_1(D^2 - \{p_1, \cdots, p_n\}) \overset{\text{Van Kampen}}{=} \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n} = \mathbb{F}_n$, we have mapping class group $\curvearrowright \pi_1$

$\Rightarrow \alpha: B_n \to \mathrm{Aut}(\mathbb{F}_n)$ the Artin representation, it's faithful

E.g.B. Consider algebraic curve over $\mathbb{C}$



$C_0 \subseteq \mathbb{C}^2$ Fibres p Z $\mathbb{C}$

$\Rightarrow \pi_1(\mathbb{C} - Z) \curvearrowright \pi_1(F_z)$, $z \in \boxed{Z}$ fibre as a monodromy: $\pi_1(\mathbb{C} - Z) \cong \mathbb{F}^S$ $\Downarrow$ Singularity $\#Z = S > 0$

Projection p is fibration $\to \mathrm{Aut}(\mathbb{F}_n)$

$\Rightarrow \mathbb{F}^S \to \mathrm{Aut}(\mathbb{F}_n)$ Artin representation

$\Rightarrow \pi_1(\mathbb{C}^2 - (C_0 \cup \mathcal{L}_0)) = \mathbb{F}^S \ltimes \mathbb{F}_n$, $\mathcal{L}_0 = \bigcup F_z$ fibres of lines

Rk. B_n is the deformation quantumization of S_n (or A_n);

Jones polynomial modifies (*) into $t^{-1}J(\nearrow\!\!\!\nwarrow) - t\,J(\nwarrow\!\!\!\nearrow) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})J()(\,)$ (*'), using (*) & (*') we can compute by resolution in all knot points. Similiarly, we have the Skein module

Def5. (Skein module) $S(M^3) = \mathbb{k}\{\text{links} \subset M\} / \langle \nearrow\!\!\!\nwarrow = q \nwarrow\!\!\!\nearrow + q^{-1} \rangle \langle \rangle$ has addition formal and multiplication $\amalg$; q is fixed number, $S(M^3)$ over $\mathbb{Z}[q, q^{-1}]$

$S(F^2) := S(F^2 \times I^1)$ called the Skein algebra.

E.g.6. $S(S^1 \times D) = \mathbb{Z}[q, q^{-1}][X]$. Q. Is it always finitely generated? √

Construction of Jones polynomial $J(t) \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ satisfy (*')

Thm1. ① $\exists (b,n) \in B_n \times \mathbb{Z}$, $\hat{b}$ (the closure of b) realises as link L (Alexander); ② (b,n) & (c,m) represent same link L ($\hat{b} = \hat{c} = L$) $\Leftrightarrow$ either $(b,n) \sim (gbg^{-1}, m)$ or $(b,n) \sim (bs_n^{\pm 1}, n+1)$ or their composite, called the Markov moves of type 1 & 2. (Markov)

E.g.A comes from the fact that $B_n = \pi_1(\mathrm{UConf}_n(\mathbb{C}))$ (in fact, $\mathrm{UConf}_n(\mathbb{C}) = K(B_n, 1)$) $\Rightarrow$ the flat bundle over $\mathrm{UConf}_n(\mathbb{C})$'s monodromy gives representation of B_n. Here configuration space $= \mathbb{C}^n - \Delta$ and unorder one, $\mathrm{UConf}_n(\mathbb{C}) \cong \mathrm{Conf}_n(\mathbb{C}) / S_n$

Fulton's $\mathbb{C}[n]$ is compactification $\mathrm{Conf}_n(\mathbb{C})$

Pf. Easy ☑ We need construct an invariant (at least) invariant under Markov move, then $J(L) = J(\hat{b}) = f(b)$

Def6. (Jone's polynomial) Set A_n a finite-dimensional von Neumann algebra generated by $1, e_1 \cdots e_{n-1}$ with relations (i) $e_i^2 = e_i$; (ii) $e_i e_j = e_j e_i$; (iii) $e_i e_{i+1} e_i = \frac{t}{(1+t)^2} e_i$; $t \in \mathbb{C}$; equipped with a trace map $tr: A_n \to \mathbb{C}$, ① $tr_n(1) = 1$; ② $tr_n(ab) = tr_n(ba)$; ③ $tr_n(a e_{n-1}) = \frac{t}{(1+t)^2} tr_{n-1}(a)$; then $\pi_t: B_n \to A_n$ is well-defined map satisfies the quatum Yang–Mills equation

then we set $J(L)(t) = \left(-\frac{t+1}{\sqrt{t}}\right)^{n-1} tr(\pi_t(\boxed{b})) \in \mathbb{C}$ $\boxed{\begin{cases} s_i \mapsto \sqrt{t}(t e_i - (1 - e_i)) \\ s_i^{-1} \mapsto (\sqrt{t})^{-1}(t^{-1} e_i - (1 - e_i)) \end{cases}}$ $\cdots$ (☆)

this not depend on b, but only depend on $\hat{b} = L$. (We denote it as $V(L)$, V_L then)

E.g.7. $V(\bigcirc)(t) = 1$, $V(H)(t) = -\sqrt{t}(t^2 + 1)$, $V(T)(t) = -t^4 + t^3 + t$

Prop1. ① $V(\bar{L})(t) = V(L)(t^{-1})$, $\bar{L}$ is the mirror of L ($\times \rightsquigarrow \times$)

② the relation of oriented skein: (*')

Pf. All comes from (☆) ☑ Rk. $V(T)(t)$ is totally determined by (*')

E.g.8. Using ② to T, we can unknot one to Hopf link $\Rightarrow tV(T)(t) - \frac{1}{t}V(\bigcirc)(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V(H)(t)$ $\Rightarrow V(T)(t) = -t^4 + t^3 + t$ also can be computed.

Rk. It's generalised to HOMFLY-PT polynomial $P(L)(t)$, s.t. $aP(\nearrow\!\!\!\nwarrow) - a^{-1}P(\nwarrow\!\!\!\nearrow) = bP()(\,)$ (H, O, ..., T is eight people)

Def7. (Kauffman's approach to Jone's polynomial) Unoriented skein relation: $\times$, $\asymp$, $|| \subseteq$ plane, the planar projection of L into a link diagram ① two link diagram represent same L iff either ∂ $\leftrightarrow$ —, or ⊃⊂ $\leftrightarrow$ ||, or [crossing] $\leftrightarrow$ [crossing], or [crossing] $\leftrightarrow$ [crossing], or their composite, called the Reidemeister moves. ② The Kauffman bracket $\langle K \rangle \in \mathbb{Z}[A, B, d]$ s.t. ① $\langle \bigcirc \rangle = 1$

Symplectic Khovanov homology $\overset{?}{=}$ Khovanov homology (done in char 0 by Abouzaid & Smith)
We have a spectral sequence, E^2 are symplectic Khovanov homology converges Floer homology
$Kh_{symp}(K) = HF^{*}(L_{\wp_n}, (\beta_K \times id) L_{\wp_n})$

(ii) $\langle O \cup K \rangle = d \langle K \rangle$; (iii) $\langle \times \rangle = A \langle \asymp \rangle + B \langle)(\rangle$ with $AB = 1$ & $d = -(A^2 + B^2) \Rightarrow \langle K \rangle \in \mathbb{Z}[A, A^{-1}]$, we define a Laurent polynomial of A by $f[K] = (-A)^{-3w(K)} \langle K \rangle$, where $w(K)$ the writhe of K is $\nearrow\!\!\!\nwarrow$ $+1$ & $\nwarrow\!\!\!\nearrow$ -1

$\Rightarrow \boxed{V(K)(t) = f[K](t^{-\frac{1}{4}})}$ (omitted)

By $\langle \times \rangle = \langle)(\rangle \Rightarrow \square \langle \cdot \rangle + \Delta \langle \cdot \rangle + d O \langle \cdot \rangle = 0$
$\langle)(\rangle + \square \langle \cdot \rangle + \Delta \langle \cdot \rangle + O \langle \cdot \rangle$ solution of the linear equations.

Return back to TQFT, we have the following

Quantization of the Jones invariance under Kirby moves $L_1 \leftrightarrow L_2$

WRT invariant (Witten–Reshetikhin–Turaev's invariant) Jone's polynomial
two approaches: Quantum group: $\boxed{U_q sl_2}$ (Kirby, ...) From links $\Rightarrow$ tangles, From $S^3 \Rightarrow M^3$
Affine Lie algebra: $\hat{sl}_2$

Σ_1^2, $M \xrightarrow{F}$, $V(\Sigma_1) \supset V_2$, $V(\Sigma_2) \supset V_4$, Σ_2^2

From the categorification of Jones polynomial we can "detect" some of 4D-TQFT.

TQFT relates the categorification by the lifting 2D-TQFT $\rightsquigarrow$ 3D-TQFT or general $nD \rightsquigarrow (n+1)D$
(but complex: 4D-TQFT; $H(M^4) \in$ "2-category" -- even for 4D-TQFT, it's still open.)
$F(\Sigma) \in \mathcal{C} \rightsquigarrow F(M) \in \boxed{Vect}$ categorification

Construction of Khovanov homology (Using the unoriented here, others are all possible) $Kh(L) \in Vect_{\mathbb{K}}$, L has planar projection P

$\boxed{Step 1}$ For planar curve, unknot, i.e. $O\ O\ O \cdots$;
Set $Kh(O) = \mathbb{Z}[X]/(X^2) =: R$ (Recall the Eg.3.③)
$\Rightarrow Kh(O^n) = R^n$ (tensor)

$\boxed{Step 2}$ 2D-TQFTs
$Kh(\text{cobordism}) := (m: R \otimes R \to R)$, others similiar; here cobordism $\subset S^3 \times [0,1]$

$\boxed{Step 3}$ Dealing knot case $Kh(\times)$
Set ① $Kh(8) := (Kh(\genfrac{}{}{0pt}{}{O}{O}) \xrightarrow{m} Kh(O)) = Kh(\text{cobordism})$
The idea comes from $\boxed{\langle \times \rangle = \underbrace{\langle \asymp \rangle A}_{0\text{-resolution}} + \underbrace{\langle)(\rangle A^{-1}}_{1\text{-resolution}}}$
and note $8 \cong O \Rightarrow Kh(O) \cong Kh(\text{cobordism})$
$\Rightarrow (R \otimes R \xrightarrow{m} R) \cong (R)$ as chain complex quasi-iso;
thus $8 = \genfrac{}{}{0pt}{}{O}{O} A + O A^{-1}$ sending $0 \to 1$
② $Kh(\cdot)$ is also a chain complex ($m \circ \Delta = 0$):
$$\begin{tikzcd} & Kh(\cdot)_{(0,1)} \arrow[dr, "m"] & \\ Kh(\cdot)_{(0,0)} \arrow[ur, "\Delta"] \arrow[dr, "\Delta"] & & Kh(\cdot)_{(1,1)} \\ & Kh(O\ O)_{(1,0)} \arrow[ur, "m"] & \end{tikzcd}$$
$Kh(\cdot) \cong Kh(O)$

And generalising ①&② to all knots, $Kh(P)$ is a communicative diagram of Δ & m, equivalent to a chain complex, s.t. $Kh(-)$ (Phylosophy is that: Δ adding holes & m minus holes) is invariant under topology, locally determined by $Kh(\times) = (Kh(\asymp) \xrightarrow{m} Kh()())$, it's the categorification of ⓢ

$\boxed{Step 4}$ Graded structure on R induces Graded structure of Kh (omitted).

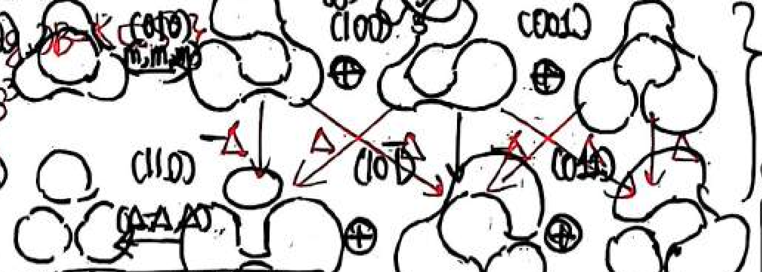
Thm2. (Khovanov) $Kh(L)$ is an invariant of isotopy of $L \subset S^3$, and $\chi_q(Kh(L)) \doteq V(L)(t)(-\sqrt{t} - \frac{1}{\sqrt{t}})$ after change of variables

(It's more powerful than Jone's polynomial in distinguishing knots: Jone's polynomial distinct unknot or not is open, but Khovanov homology does ($V(L)(t) = V(O)(t)$, is $L \simeq O$?)

Conjecture1. (Milnor), solved by Khovanov by Gauge, and later by Khovanov homology) $g_4(T_{p,q}) = \frac{(p-1)(q-1)}{2}$

...g the slice genus / 4-ball genus
$g_4(K) := \min\{g \mid g(\Sigma), \partial \Sigma = K\}$

Eg.8. (Trefoil's Khovanov homology) Later we denote resolutions (abc)



$A \otimes A \to A \oplus A \oplus A \to (A \otimes A) \oplus (A \otimes A) \oplus (A \otimes A) \to A \otimes A \otimes A$
algebraization: chain complex (hard to compute)
we decompose to chain complexes in each grade

deg(j) \ i	0	1	2	3	
-2	$\langle x \otimes x \rangle$	0	0	0	$\mathbb{Z}$
0	$\langle x \otimes 1, 1 \otimes x \rangle$	$\langle x \rangle$	$\langle x \otimes x \rangle$	$\langle x \otimes x \otimes x \rangle$	$\mathbb{Z}$
2	$\langle 1 \otimes 1 \rangle$	$\langle 1 \rangle$	$\langle 1 \otimes x, x \otimes 1 \rangle$	$\langle 1 \otimes x \otimes x, \ldots, x \otimes x \otimes 1 \rangle$	$\mathbb{Z}$
4	0	0	$\langle 1 \otimes 1 \rangle$	$\langle 1 \otimes 1 \otimes x, \ldots \rangle$	$\mathbb{Z}/2\mathbb{Z}$
6	0	0	0	$\langle 1 \otimes 1 \otimes 1 \rangle$	$\mathbb{Z}$

Summary

j \ i	0	1	2	3
6	0	0	0	$\mathbb{Z}$
4	0	0	0	$\mathbb{Z}/2\mathbb{Z}$
2	0	0	$\mathbb{Z}$	0
0	$\mathbb{Z}$	0	0	0
-2	$\mathbb{Z}$	0	0	0

Lee's homology, $A_{Lee} := \mathbb{Z}[X]/(X^2 - 1)$
$\Delta: 1 \mapsto 1 \otimes x + x \otimes 1$
$x \mapsto x \otimes x + 1 \otimes 1$, thus
$\begin{cases} a = \frac{1+x}{2} \\ b = \frac{1-x}{2} \end{cases} \Rightarrow m(a,a) = a,\ m(b,b) = 0,\ m(a,b) = 0,\ \Delta(a) = a \otimes a,\ \Delta(b) = b \otimes b.$

One can used spectral sequence to give such diagram directly [Dee. A End of the Khovanov homology]

We show its application to Milnor conjecture ($T_{p,q}$ is the K into the T^2 (toric knot), and into $\mathbb{R}^3$ $\Rightarrow$ with slope $\frac{p}{q} \in \mathbb{Q}$)

the s-invariant $s(K) = 2g(S) - 2\chi = 2g(S) \geq |s(K)| \Rightarrow g(S) = s(K)$ ($s(K) = g(T_{p,q})$ given by [illegible]'s polynomial and lead terms, S is the Seifert resolution of K)

• Milnor's original (Here may something wrong)
idea is the $x^p + y^q = 1$ cut the ball into the (p,q)-torus knot $T_{p,q}$
When the knot is algebraic curve $g(T_{p,q}) = \frac{(p-1)(q-1)}{2}$ by algebraic way easily
$g(T_{p,q}) \leq \frac{(p-1)(q-1)}{2}$ can't take inequality? Milnor conjectured not
First by Gauge theory by Kronheimer & Mrowka, and here the easy proof is by Rasmussen later

Relation with AG via Mirror symmetry (Fock–Goncharov)
$A(\Sigma) \leftrightarrow Om$ → X tropical points degeneration
$X \leftrightarrow Ot$ → X tropical points (mirror)
Both Teichmüller X is coordinate ring of X

• Cluster algebra
Set over $\mathbb{k} = \mathbb{Z}$ (classical) or $\mathbb{Z}[q, q^{-1}]$ (Quantum)
For every 4-gon Σ with $a_1, \dots, a_5$ & a_1' are two triangulation Δ, Δ'; $\Delta \leftrightarrow \Delta'$ are flips

The skein $Sk(\Sigma) = \mathbb{k}[a_1, a_1', a_2, a_3, a_4, a_5]/\sim$
skein relation is $\times = \asymp +)($ (classical case without coefficient)
i.e. $a_1 a_1' = a_3 a_5 + a_2 a_4$

Now set $\Sigma = (S, M)$, M are mark points on S (finite), S compact & oriented, γ on S {end at M (arcs/open string); no ends (loops/closed string)}
$D = \sqcup \gamma$ is a diagram with isotopy class $[D]$ $\Rightarrow$ $Sk(\Sigma) := \bigoplus \mathbb{k}[D]/\sim$, multiply is union
and relations are ① $\times = \asymp +)($; ② $O = -2$; ③ $O = 0$ ($p \in M$) ; ④ $[D] = 2$ ($p \in M$)
(comes from hyperbolic metric, one can compute the length when $p \in M$ is infinity: (punctured))

Eg.A. (Annulus) by triangulation looks like the Dehn twist.
Notes that $\mathbb{k}$-basis of $Sk(\Sigma)$ is determined by every family of triangulation, each triangulation gives basis, called the cluster monomial
One can replace γ_1 by $\gamma_1 \circ \gamma_1 \circ \cdots \circ \gamma_1$ to rotation many times.
thus computed that $Sk(\Sigma) = \mathrm{Span}\, \mathbb{Z}[b_1, b_2] \{ \gamma_k^{\mathrm{up}} \gamma_{k+1}^{\mathrm{down}}, L^{(a)} \mid a \in \mathbb{N} \}$
when Σ is 4-gon, but Σ general, cluster monomial can't be a basis!
↳ "real roots" = cluster monomial
"imaginary roots" = loop elements L^N } Basis.
In the meaning of root system of Lie algebra

L^3 (Bangle) obvious a basis element replaced by Braid replaced by $\frac{1}{|\sigma|} \sum_\sigma$ a mean value called the band
to be the minimal self-intersection called bracelet ⇒ also a basis ($T_N(z + z^{-1}) = z^N + z^{-N}$ ⇒ $T_N(\mathrm{tr} M) = \mathrm{tr}(M^N)$ First & Second T_N Chebyshev polynomial)
(classical case not care or, thus not knots, quantum case it's knot)

Def B. (Cluster algebra combinatorial) the dual graph of triangulation Δ is Q, with orientation on Δ ⇒ Q comes to be a quiver, this quiver determine an algebra $A(Q)$. Phylosophically speaking, it measures the positivity of matrix on the Lie theory and quantum theory
$Sk(\Sigma)$ is Cluster algebra by upper construction, the interior part of triangulation (e.g. γ_1 & γ_2 or a_1 & a_1')'s dual subquiver Q
the generic basis $\mathbb{L} := \{$generic values of character of $\vec{Q}$-rep$\}$ the components of moduli of $\vec{Q}$-reps admits open dense subset s.t. character constant. ("many" $A(\vec{Q})$ admits $\mathbb{L}$, especially in geometric cases)

$\Sigma \xrightarrow{\text{triangulation}} \Delta \xrightarrow{\text{dual}} \vec{Q} \xrightarrow{\text{rep}}$ representation (can be viewed a categorification)
Called seed S_Δ
$Sk(\Sigma) = A(\vec{Q})$
|| [Gross–Hacking–Keel–Kontsevich] (using mirror symmetry)
$\boxed{\mathrm{span}\{\text{bracelets}\} = \mathrm{span}\{\theta\text{-functions}\}}$ (comes from abelian variety)
[Mandel–Qin, arxiv 2301.1101] $\boxed{\text{Main thm}}$

θ-function $\vec{Q} = M_{\mathbb{R}} - \square$ (scatter diagram)
$\forall m \in M$, θ-function is $\theta_{m,\vec{Q}} := x^m \sum_n c_n x^n$ with $c_n \in \mathbb{N}$
and $c_0 = 1$, these (c_n) comes from $\square$
Eg.C. (Tropical curves) $\theta_{f_1, \vec{Q}} = \sum$ broken lines = tropical curve

Observation (Teichmüller theory for $SL(2, \mathbb{k})$ or $PGL(2, \mathbb{k})$)
$\forall M \in SL(2, \mathbb{k})$, taking $T_N(\mathrm{tr} M) = \mathrm{tr}(M^N)$, this Chebyshev polynomial corresponds to bracelets $\mathrm{brac}^{(N)}(L)$ by view M the monodromy of $SL(2, \mathbb{R})$-local system.
($X_i = a^{v_i}$, v_i is representation, x_i generate X)
$\omega(e_i, e_j) := \#(i \to j) - \#(j \to i)$

[GHKK] Cluster monomial
$\boxed{\text{Main thm}}$ θ-functions — loop elements
and Eg.A. tells using ($d = f_1 - f_2$) $\vec{Q}$ closed to infinite Dehn twists — Bracelets ||
Chebyshev polynomial by computation
$\boxed{\theta_{d, \vec{Q}} = T_N(\theta_{d, \vec{Q}})}$
↳ this is Annulus' case
(general we reduce to one loop by dividing, not too hard, case ① & ②)
① Annulus; ② not annulus but can do

Wall and wall-crossing
$\vec{Q}$ gives lattice $\bigoplus \mathbb{Z} e_i = N$ and dual $\bigoplus \mathbb{Z} f_i = M$, for 4-gon $|\vec{Q}_0| = 4$ vertices
ω quadratic form on $\bigoplus \mathbb{Z} e_i$, scatter diagram $\square$ on $M_{\mathbb{R}} = M \otimes \mathbb{R}$ is a collection of walls (d, P_{f_d}), d is codim 1 polyhedral cone, s.t.
$d \pm f_3 = d \pm f_4 = d$, primitive normal vector n_d, the wall-cross vector P_{f_d} measure "what cost" when wall-crossing
f_d is a series $f_d \in \mathbb{Z}[[x_1, x_2]]$, constant 1,
$P_{f_d}(x^m) = x^m f_d^{\langle m, n_d \rangle}$, $m \in M$
The wall-crossing process has no monodromy (↺ not change P_{f_d})

$f_2 \to e_1$ ($\mathbb{R} f_1 \oplus \mathbb{R} f_2 \cap S_\Delta$)
e_2, P_{1+x_2}
$e_1 + 2e_2$, $P_{1+x_1 x_2^2}$
$-f_1$
$2e_1 + 3e_2$, $P_{1+x_1^2 x_2^3}$
limit $e_1 + e_2$, $P_{(1 + \sum \dots)}$
$3e_1 + 4e_2$
$-f_2$
$2e_1 + e_2$

⟷ every chamber ⟷ triangulation ⟺ Dehn twist
topological / algebraic ⟷ t-structure and Bridgeland stability

Lecture 2

Higgs bundles and minimal surfaces in ~~non-compact~~ symmetric spaces (Qiongling Li 2)

XIJIAO HOTEL BEIJING

Abelian Hodge correspondence

$S_{g\geq 2}$ a surface with $g\geq 2$, giving it a complex structure J (All assume compact)

$H^1(S,\mathbb{R})\cong H^0((S,J),K)$ the Holomorphic 1-form (or replace $\mathbb{R}$ by S^1) called Abelian differential

↳ replaced by non-Abelian? ⟹ replace K to Higgs bundle

Non-Abelian Hodge correspondence

Def 1 (Higgs bundle) $X=(S,J)$ as upper, (E,θ) over X is Higgs bundle: E is holomorphic forms over X, $\theta\in H^0(X,\mathrm{End}(E)\otimes K)$, K is the holomorphic cotangent bundle (i.e. $\bar\partial_E\theta=0$)

called the Higgs field

Def 2 $SL(n,\mathbb{C})$-Higgs bundle: $\det E=\theta$, $\mathrm{tr}\,\theta=0$

$GL(n,\mathbb{R})$-Higgs bundle: $\exists Q: E\times E\to\theta$ nondegenerated holomorphic quadratic form (We'll see why it gives a $\mathbb{R}$-structure later)

$SL(n,\mathbb{R})$-Higgs bundle: $Q(\theta s,t)=Q(s,\theta t)$ both of two

Eg 1 Line bundle is Higgs bundle, θ is canonically chosen

K is line bundle

Eg 2 $K^{\frac12}$ is line bundle $((K^{\frac12})^2=K)$ set $E=K^{\frac12}\oplus K^{-\frac12}$, $\theta=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, $a,d\in H^0(X,K)$, $c\in H^0(X,\mathcal{O})$, $b\in H^0(X,$ $\mathrm{Hom}(K^{-\frac12},K^{\frac12})\otimes K)\cong H^0(X,K^2)$

$c_1(K)\Rightarrow$ mod 2 $\Rightarrow\exists K^{\frac12}$, s.t. $K^{\frac12}\otimes K^{\frac12}=K$

When it's $SL(n,\mathbb{C})\Longleftrightarrow a+d=0$

$GL(n,\mathbb{R})\Longleftrightarrow Q=\begin{pmatrix}0&1\\1&0\end{pmatrix}, a=d$

$SL(n,\mathbb{R})\Longleftrightarrow a=d=0$, $\theta=\begin{pmatrix}0&q\\1&0\end{pmatrix}$, $q\in H^0(X,K^2)$

Rk. Besides the definition of degree in Algebraic Geometry, here $\deg E:=\int c_1(E)$, it can be computed by $\int c_1(E)=\int\frac{i}{2\pi}\mathrm{tr}(F(D))=\int\frac{i}{2\pi}\Lambda(\mathrm{tr}(F(D)))\omega$ (contraction)

Def 3 (Stability) If any θ-invariant holomorphic subbundle $F\subset E$, $\frac{\deg(F)}{\mathrm{rank}(F)}<\frac{\deg(E)}{\mathrm{rank}(E)}$ ($\leq$, semistable), polystable if $(E,\theta)=\oplus(E_i,\theta_i)$, $\frac{\deg(E_i)}{\mathrm{rank}(E_i)}$ is constant on i

E. $\mathrm{Pic}(X)$ by $\theta\to 2\theta$ sending

Eg 1' Line bundle has no nontrivial subbundle ⟹ stable

Eg 2' $(K^{\frac12}\oplus K^{-\frac12},\begin{pmatrix}0&0\\1&0\end{pmatrix})$ has only $K^{-\frac12}$ proper θ-invariant subbundle, $\deg K=2g-2\Rightarrow\deg K^{-\frac12}=-g+1$ $\Rightarrow\frac{1-g}{1}<0$ stable ☒

$[K]\mapsto c_1$ ⟹ $c_1(L)\Rightarrow$ $\geq 2a$, $a=[K]$

(Curvature) $F(\nabla^H):\nabla^H\circ\nabla^H:\Omega^k\to\Omega^{k+2}$

Def 4 A Hermitian metric H on (E,θ) is harmonic if $\boxed{F(\nabla^H)}+[\theta,\theta^*]=0$ (the Hitchin equation)

Here ∇^H is Chern connection compatible with H, θ^* defined by H-adjoint: $H(\theta s,t)=H(s,\theta^*t)$

⟹ $D=\nabla^H+\theta+\theta^*$ is flat ⟹ we have holonomy $\mathrm{Hol}(D):\pi_1(X)\to GL(n,\mathbb{C})$

But the holomorphic structure unique (2)

Thm 1 (Hitchin, Simpson) X admits a harmonic metric (uniquely) ⟺ (E,θ) is polystable (stable) of deg 0

Here $\nabla^H\in\Omega^{1,1}$, $\theta\in\Omega^{1,0}$, $\theta^*\in\Omega^{0,1}$ ⟹ $D\in\Omega^{1,1}$

is given by square root of the holomorphic section (non vanishing)

Eg 1 $H=1$ on line bundle

$(\hat h=h(\frac{\partial}{\partial z},\frac{\partial}{\partial\bar z}))$

Eg 2 $H=\begin{pmatrix}h^{-\frac12}&0\\0&h^{\frac12}\end{pmatrix}$ on $(K^{\frac12}\oplus K^{-\frac12},\begin{pmatrix}0&0\\1&0\end{pmatrix})$, h is Hermitian on K, $\partial\bar\partial\log\hat h=\hat h\Rightarrow h$ is constant curvature -4

H is harmonic ⟺ X admits a conformal hyperbolic metric

$K_h=-\frac{2\partial\bar\partial\log\hat h}{\hat h}=-2\boxed{2}=-4$ ($\mathrm{tr}\,(h^{-\frac12}\ h^{\frac12})$)

2^{2g} choices of lines

$\nabla^H=\partial g\begin{pmatrix}\partial\log\hat h&\\&-\partial\log\hat h\end{pmatrix}$, $\theta=\begin{pmatrix}0&0\\1&0\end{pmatrix}\Rightarrow\theta^*=\begin{pmatrix}0&\hat h\\0&0\end{pmatrix}\Rightarrow[\theta,\theta^*]=\begin{pmatrix}\hat h&0\\0&-\hat h\end{pmatrix}dz\wedge d\bar z$

square at this top has two choices; Eg 2 is $SL(2,\mathbb{R})$-compatible) ⟹ 2^{2g} ($2g$ loops)

When $\theta=\begin{pmatrix}0&a\\1&0\end{pmatrix}\Rightarrow\theta^*=\begin{pmatrix}0&\hat h\\\overline{a}&0\end{pmatrix}$) Q. H must diagonal (or isometric to diagonal)?

Def 4' H is compatible with $SL(n,\mathbb{C})$-structure if $\det(H)=1$.

$GL(n,\mathbb{R})$-structure if $\Phi_Q:E\rightleftarrows E^*$ is $\mathbb{C}$-linear isomorphism

$SL(n,\mathbb{R})$-structure if both.

$\Phi_H:E\to E^*$ is conjugate isomorphism.

⟹ $\boxed{\tau=\Phi_H^{-1}\circ\Phi_Q}:E\to E$ is real-structure on E ($\tau^2=\mathrm{id}$)

that's why it's called $GL(n,\mathbb{R})$

Rk. When $GL(n,\mathbb{R})$-compatible, the holonomy $\mathrm{Hol}(D):\pi_1(X)\to GL(n,\mathbb{R})$ factor through real

Algebraic $L\Longleftrightarrow\mathrm{Hom}(\pi_1(X),\mathbb{Z}_2)$ and $\#\mathrm{Hom}(\pi_1(X),\mathbb{Z}_2)=2^{2g}$ ☒

Harmonic map & minimal surface

$(M,g),(N,h)$ Riemannian, $\rho:\pi_1(M)\to\mathrm{Isom}(N)$ (isometry) a representation. $\tilde M$ is universal covering of M

$f:\tilde M\to N$ is ρ-equivariant ($f(\gamma.x)=\rho(\gamma).f(x)$), $df:T\tilde M\to f^*TN$, $\in C^\infty(\tilde M,T^*\tilde M\otimes f^*TN)$

With the ∇ on f^*TN the Levi-Civita connection induced by $N \Rightarrow d_\nabla : C^\infty(M, T^*M \otimes f^*TN) \longrightarrow C^\infty(M, \wedge^2 T^*M \otimes f^*TN)$

Def5. (Energy density) $\boxed{e(f) = \frac{1}{2}\|df\|^2_{g,h}}$ $\pi_1(M)$-invariant; energy $E(f) = \int_M e(f)\, dvol_g$ Page2

f is harmonic map if it's a critical point of $E(f)$ with respect to ρ-equivariant deformation

$\left(\frac{1}{2}\|df\|^2 := \frac{1}{2}(\|\partial_x f\|^2 + \|\partial_y f\|^2)/g_0 = h\langle \Psi_H, \Psi_H \rangle\right)$ thus H is harmonic metric can also defined

Prop1. f harmonic $\Leftrightarrow * d_\nabla * df = 0$

Pf. Here is surface, thus $*dx = dy$ & $*dy = -dx$, $\frac{d}{dt}E(f_t) = \frac{1}{2}\int_M \frac{d}{dt}\langle df, df\rangle dvol_g = \int_M \langle \nabla_{\partial_t} df, df\rangle dvol_g = \int_M \langle d_\nabla \frac{\partial f}{\partial t}, df\rangle$

$= \int_M d\langle \frac{\partial f}{\partial t} \wedge *df\rangle_h - \int_M \langle \frac{\partial f}{\partial t}, d_\nabla(*df)\rangle_h dvol_g = \int_M \langle \frac{\partial f}{\partial t}, d_\nabla(*df)\rangle_h dvol_g + 0 \;\big|_{t=0} = 0 \Rightarrow * d_\nabla * df = 0$ ▨

integral by part

Thm2. (Corlette, Donaldson) $\rho : \pi_1(M) \longrightarrow G$ a semi-simple/reductive group.

$\Rightarrow \exists$ a ρ-equivariant harmonic map $f : \tilde{M} \longrightarrow Sym(G)$ (G/K quotient defined by Killing form)

Eg3. $G = SL(n,\mathbb{C})$ $Sym(G) = SL(n,\mathbb{C})/U(n) \ni A \in$ Hermitian matrices, $\langle X, Y\rangle_A = tr(A^{-1}XA^{-1}Y)$ by compute tangent space

Prop2. f harmonic $\Leftrightarrow (d_\nabla)^{0,1}(\partial f) = 0 \Leftrightarrow (d_\nabla)^{0,1}(df)^{1,0} \Leftrightarrow \partial f$ is holomorphic (note that harmonic map thus not depend on metric)

Pf. $*dz = -idz$, $*d\bar{z} = id\bar{z}$, $d_\nabla * df = 0 \Leftrightarrow d_\nabla(-i\partial f + i\bar\partial f) = 0 \Leftrightarrow d_\nabla(-\partial f + \bar\partial f) = 0$

$\Rightarrow d_\nabla df = d_\nabla(\partial f + \bar\partial f) = 0 \Leftrightarrow d_\nabla(\partial f) = 0$ ▨

Def6. f conformal if $Hopf(f) := h(\partial f, \partial f) \in C^\infty((T^*M)^{1,0} \otimes (T^*M)^{1,0})$ vanishing (Hopf differential)

by Prop2 $\Rightarrow Hopf(f)$ is holomorphic if f is harmonic map.

Def7. A branched conformal immersion $f : \tilde{X} \hookrightarrow N$ is minimal if $f(X)$ is $H = 0$

Prop3. f minimal $\Leftrightarrow f$ harmonic in upper setting

Pf. Choosing frame $\{\partial x, \partial y\} \Rightarrow$ minimal $\Leftrightarrow \mathrm{II}(\partial x, \partial x) + \mathrm{II}(\partial y, \partial y) = 0 \Leftrightarrow \mathrm{II}(\partial z, \partial\bar{z}) = 0 \Leftrightarrow \boxed{(\nabla_{\partial_{\bar z}} \partial z)^N = 0 \Leftrightarrow \nabla_{\partial_{\bar z}} \partial z = 0}$ $\Leftrightarrow$ harmonic

conformal $\Rightarrow (\nabla_{\partial_{\bar z}} \partial z)^T = 0$ is left as Exercise ▨

Lemma5. (E, D, H) is vector bundle, connection, Hermitian $\Rightarrow \exists!\; D = \nabla^H + \Psi^H$ is unitary part and Hermitian part.

Pf. $H(s, \Psi^H t) = \frac{1}{2}\{H(Ds, t) + H(s, Dt) - d(H(s,t))\}$ (*) ▨ (Choosing $\Psi^H \in \Omega^1$, s.t. (*) holds $\Rightarrow \nabla^H = D - \Psi^H$, self-adjoint)

Prop4. TAFE ① Higgs bundle $(E, \bar\partial_E, \theta)$ + harmonic metric H; ② Flat vector bundle (E, D) + harmonic metric H

③ Flat vector bundle (E, D) + harmonic metric H; ④ $(\rho, f) \in Hom(\pi_1(X), SL(n,\mathbb{C})) \times Hom(\tilde{X}, SL(n,\mathbb{C})/U(n))$

Pf. Write their equations: ① $F(\nabla^H) + [\theta, \theta^*] = 0$; ② $d_{\nabla^H} * \Psi_H = 0$ (Using Lemma); ③ $(d_{\nabla^H})^{0,1}(\Psi_H)^{1,0} = 0$; ④ $(d_{\nabla^{LC}})^{0,1}(\partial f) = 0$

① $\Leftrightarrow$ ② $\Leftrightarrow$ ③ is done; ③ $\Rightarrow$ ④ $f(x)(V, W) := H_x(P^x_{x_0}(V), P^x_{x_0}(W))$, the Parallel transport by Levi-Civita

thus we have such map f; and $\Psi_H = -\frac{1}{2}f^{-1}df$, $\theta = (\Psi_H)^{1,0} = -\frac{1}{2}f^{-1}\partial f$, $\theta^* = (\Psi_H)^{0,1} = -\frac{1}{2}f^{-1}\bar\partial f \Rightarrow$

$Hopf(f) = tr(f^{-1}\partial f f^{-1}\partial f) = 4tr(\theta^2)$, $f^*g_{G} = 4(tr(\theta^2) + tr(\theta\theta^*) + tr(\theta^*)^2)$ we can write all things by (θ, θ^*) ▨

Non-Abelian Hodge correspondence: $\{\rho : \pi_1(S) \rightarrow G$ reductive$\}/\sim$ conjugate $\longleftrightarrow$ $\{(E, \theta)$ polystable G-Higgs bundle$\}/\sim$ gauge $=: \mathcal{M}_{Higgs}(G)$

Here the gauge equivalence is $(E_1, \theta_1) \sim (E_2, \theta_2)$ if

$$\begin{tikzcd} E_1 \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & E_2 \arrow[d] \\ E_1 \otimes K \arrow[r] & E_2 \otimes K \end{tikzcd}$$

(local conjugate)

$\Updownarrow$ Corlette, Donaldson $\qquad$ Hitchin, Simpson $\Updownarrow$

$\{(\rho, f) \mid f : \tilde{X} \rightarrow Sym(G)$ is ρ-equi, harmonic$\}$ $\xrightarrow[\text{algebraic}]{\text{obvious}}$ $\{(E, \theta, H) \mid H$ harmonic, compatible with $G\}$

$(\rho, f) \sim (g\rho g^{-1}, gf)$

We can relate this with $\{$reductive, flat G-connection$\}/\sim$ gauge $=: \mathcal{M}_{deRham}(G)$

Eg4. We know harmonic map $\Leftrightarrow$ Higgs bundle $\qquad$ Symmetric space ($\mathbb{R}^n$) by embedding $\tilde{X} \hookrightarrow \mathbb{R}^n$, $\sum w_i^2 = 0$ is the simplest Higgs bundle

$\Updownarrow$ conformal $\qquad \Updownarrow tr(\theta^2) = 0$

minimal surface $\Leftrightarrow \boxed{tr(\theta^2)\text{'s Higgs bundle}}$

By conformal $\Leftrightarrow tr((\partial f)^2) = 0 \Leftrightarrow tr(\theta^2) = 0$ ▨

Hitchin fibration. $\mathcal{M}_{Higgs}(SL(n,\mathbb{C})) \xrightarrow{f} \bigoplus_{2 \le i \le n} H^0(X, K^i) =: \mathcal{B}$ the Hitchin base, $\dim \mathcal{B} = \frac{1}{2}\dim \mathcal{M}_{Higgs}(SL(n,\mathbb{C}))$

We have $\boxed{\mathcal{M}_{Higgs}(SL(n,\mathbb{C})) \cong \mathcal{M}_{deRham}(SL(n,\mathbb{C}))}$ is the isomorphism of coarse moduli

any homogeneous polynomial of θ is also possible, and they're equivalent

$[(E, \theta)] \longmapsto (tr(\theta^2), tr(\theta^3), \dots, tr(\theta^n))$ integrable system

Generic fibre: torus

It's Hitchin section $s(q_2 \cdots q_n) \mapsto (K^{\frac{n-1}{2}} \oplus \cdots \oplus K^{\frac{1-n}{2}}, \theta = \begin{pmatrix} 0 & q_2 & \cdots & q_n \\ 1 & \ddots & & q_{n-1} \\ & \ddots & \ddots & q_2 \\ 0 & & 1 & 0 \end{pmatrix})$

when we take $f(E, \theta) = char(\theta)$ (characteristic polynomial, graded) $\Rightarrow \tilde\theta = \begin{pmatrix} 0 & & 0 & q_n \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots & q_2 \\ 0 & & 1 & 0 \end{pmatrix}$, and $\theta \sim \tilde\theta$ by gauge

such s gives a $SL(n,\mathbb{R})$-Higgs bundle, Ex1.2 shows stability

$\Rightarrow D$ has holonomy in $SL(n,\mathbb{R})$, and similarly $\mathcal{M}_{Higgs}(SL(n,\mathbb{R})) \subset \mathcal{M}_{Higgs}(SL(n,\mathbb{C}))$ have same dimension, and f is proper

map $\Rightarrow Im(s)$ compact, and s local diffeomorphism $\Rightarrow \boxed{Im(s)} \subset \mathcal{M}_{Higgs}(SL(n,\mathbb{R}))$ a connected component.

$Im(s)$ is called the Hitchin component, it's a higher Teichmüller space: when $n=2$, it's just the Teichmüller space.

We study its representation theory, comparing with Teichmüller space.

Setting. Γ is a finite generated hyperbolic group with generator set $G \subset \Gamma$, define norms $d(1, \gamma)$ = word length of γ with respect to G

& $\|\gamma\| = \inf_{\eta \in \Gamma} d(1, \eta\gamma\eta^{-1})$ on $\gamma \in \Gamma$;

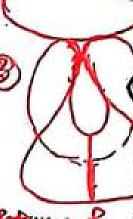
M is metric space, $\gamma \in Isom(M)$, we define the translation length (/displacement) of γ: $\ell_M(\gamma) = \inf_{x \in M} d(x, \gamma x)$

Eg6. When M is Cayley graph of $\Gamma \Rightarrow \ell_M(\gamma) = \|\gamma\|$

Return back to $SL(n,\mathbb{R})$, $M = X_d = SL(d,\mathbb{R})/SO(d)$ with $\langle X, Y\rangle_A = tr(A^{-1}XA^{-1}Y)$, $g \in SL(d,\mathbb{R})$, $g\cdot I = g^T g$, $d(I, gI) = 2\sqrt{\sum \log^2 \sigma_i(g)}$
where $\sigma_i(g)$ are singular values, we have the KAK decomposition $g = KDL$, $D = \begin{pmatrix} \sigma_1(g) & & 0 \\ & \ddots & \\ 0 & & \sigma_d(g)\end{pmatrix}$ ordered by decreasing
$\Rightarrow gg^T = KD^2K^{-1} \Rightarrow \rho_M(g) = 2\sqrt{\sum |\log \lambda_i(g)|^2}$, where $\lambda_i(g)$ is eigenvalue of g ordered by decreasing.
Def 8. $f: Y \to Z$ between metric spaces called (K, C)-quasi-isometric embedding if $\frac{1}{K}d_Y(a,b) - C \leq d_Z(f(a), f(b)) \leq K d_Y(a,b) + C$, if $\exists (K, C)$ we call it simply quasi-isometric embedding.
Def 9. $\rho: \Gamma \to Isom(X)$ is well-displacing if $\exists J, B > 0$, s.t. $\ell_X(\rho(\gamma)) \geq J\|\gamma\| - B$
We have the orbit map $\tau_\rho: \Gamma \to X$ then, $x_0 \in X$ fixed
Prop 5. (Delzant-Guichard-Labourie-Mozes) τ_ρ is quasi-isometric embedding $\Leftrightarrow$ ρ is well-displacing ($\gamma \mapsto \rho(\gamma)(x_0)$)
These are good in representation: $\frac{1}{K}d(1,\gamma) - C \leq d_X(\tau_\rho(id), \tau_\rho(\gamma)) \leq K d(1,\gamma) + C \Rightarrow \rho$ is discrete and almost faithful rep
further Γ torsion-free $\Rightarrow \rho$ faithful. Anosov is more better. (E.g. It's open but quasi-isometric embedding not)
Def 10. (P_k-Anosov representation singular value) (Original defined by dynamics by Labourie) $\rho: \Gamma \to SL(d,\mathbb{R})$
TAFE ① $\exists D, L \geq 0$ s.t. $\log\left(\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))}\right) \geq D\cdot d(1,\gamma) - L$, $\forall \gamma \in \Gamma$; ② $\exists J, B > 0$, s.t. $\log\frac{\lambda_k(\rho(\gamma))}{\lambda_{k+1}(\rho(\gamma))} \geq J\|\gamma\| - B$, $\forall \gamma \in \Gamma$
Prop 6. P_k-Anosov
$\Rightarrow$ Both ① quasi-isometric embedding & ② well-displacing.
Pf. Upper bound of (★) is true without Anosov, below bound by:
① $d_{X_d}(\tau_\rho(id), \tau_\rho(\gamma)) = 2\sqrt{\sum \log(\sigma_i(\rho(\gamma)))^2} \geq \log\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geq J d(1,\gamma) - B$ ▨ ② use ② similarly ▨
Eg. 6. $\rho: \mathbb{F}_2 \to SL(4,\mathbb{R})$ defined by $\rho_1 \oplus \rho_2$
(Guichard) $\rho_1: \mathbb{F}_2 \to SL(2,\mathbb{R})$, P_1-Anosov
$\rho_2: \mathbb{F}_2 \to SL(2,\mathbb{R})$, $\rho_2(a) = \rho_1(a)$ and $\rho_2(b) = id$ isn't P_1-Anosov $\Big\}$ $\Rightarrow$ ① ρ not P_1-Anosov as $\sigma_1(\rho(a)) = \sigma_2(\rho(a)) = \sigma_1(\rho_1(a))$ occurs twice
② ρ also not P_2-Anosov due to singular value is $\boxed{1, 1}$, ★, occurs twice in ρ_2
③ ρ is quasi-isometric embedding due to $d_{X_4}(\tau_\rho(id), \tau_\rho(\gamma)) \geq \sigma_1(\rho(\gamma)) \geq \sigma_1(\rho_1(\gamma)) \geq J d(1,\gamma) - B$ ▨
Eg. 7. $\rho_1: \Gamma \to SL(m,\mathbb{R})$, $\rho_2: \Gamma \to SL(n,\mathbb{R}) \Rightarrow \rho_1 \oplus \rho_2: \Gamma \to SL(m+n,\mathbb{R})$ is also P_k-Anosov if both ρ_1 & ρ_2 are
~~then by openness, there is a neiberhood of representation in $SL(m+n,\mathbb{R})$ not~~ comes from direct sum.
Considering $u: X \hookrightarrow \mathbb{R}^3$ conformal minimal branched immersion, $\partial u = (du)^{1,0} = (w_1, w_2, w_3)$, $\sum w_i^2 = 0$ by conformal
$\Rightarrow (w_1 - iw_2)(w_1 + iw_2) + w_3^2 = 0$, assume $w_3 \not\equiv 0 \Rightarrow \begin{cases} g = \frac{w_3}{w_1 - iw_2} \text{ meromorphic} \\ \eta = w_1 - iw_2 \text{ holomorphic}\end{cases} \Rightarrow \begin{cases} w_1 = \frac{1}{2}(1-g^2)\eta \\ w_2 = \frac{i}{2}(1+g^2)\eta \\ w_3 = g\eta\end{cases}$
$u(p) = u(p_0) + Re\int_{p_0}^{p}(\frac{1}{2}(1-g^2)\eta, \frac{i}{2}(1+g^2)\eta, g\eta)$ is the
Enneper-Weierstrass representation of minimal surfaces $\subset \mathbb{R}^3$. Here we hope to generalise such correspondence to an equivariant analogue.

$\boxed{\text{Holomorphic datum } (g,\eta), \text{ s.t. } Re\oint_\gamma(-,-,-) = 0 \text{ for } \gamma \text{ closed}} \xleftrightarrow{\text{(★) Real period} = 0} \boxed{X \subset \mathbb{R}^3 \text{ minimal surfaces conformal}}$

Recall that:
$\boxed{\text{Higgs bundle } \oplus(\mathcal{O}_X, w_i, 1) \text{ on } X} \xleftrightarrow[\text{Hodge polystable}]{\text{Non Abelian}} \boxed{X \hookrightarrow \mathbb{R}^3 \text{ conformal minimal surfaces}}$

① 
③ 

Pictures of minimal surface $\subset \mathbb{R}^3$

Eg. 8. $u: \mathbb{C}^* \hookrightarrow \mathbb{R}^3$
① $\begin{cases} w_1 = -\frac{1}{2}(1-w^2)\frac{dw}{w^2} \\ w_2 = \frac{i}{2}(1+w^2)\frac{dw}{w^2} \\ w_3 = \frac{dw}{w}\end{cases}$ $\Rightarrow$ catenoid

② It's conjugate surface $\begin{cases} w_1' = iw_1 \\ w_2' = iw_2 \\ w_3' = iw_3 = i\frac{dw}{w}\end{cases}$ not satisfy (★)

② 

due to it's the imagine part of $\log w$ $\Rightarrow$ lifting $\mathbb{C}^*$ to $\mathbb{C}$ by e^z $\Rightarrow$ Helicoid

③ $\mathbb{C} \hookrightarrow \mathbb{R}^3$ with $g = z$, $\eta = dz$ $\Rightarrow$ Enneper's surface

④ $T^2 - \{p_1, p_2, p_3\} \hookrightarrow \mathbb{R}^3$ with $g = \frac{1}{\wp'(z)}$, $\eta = \wp(z)dz$ $\Rightarrow$ Costa surface

Weierstrass elliptic function

④ 

they're the four formest minimal surfaces, 1744–1982

And for other embeddings
Eg. 9. $u: X \to \mathbb{H}^2 \times \mathbb{R}$ conformal minimal
$\Leftrightarrow (L \oplus L^{-1}, \begin{pmatrix} 0 & \beta \\ \alpha & 0\end{pmatrix}, \begin{pmatrix} h & 0 \\ 0 & h^{-1}\end{pmatrix}) \oplus (\mathcal{O}_X, w, 1)$
$tr(\Phi^2) = 0 \Leftrightarrow \alpha\beta + w^2 = 0$
Eg. 10. $u: X \to \mathbb{H}^2 \times \mathbb{H}^2$ conformal minimal
$\Leftrightarrow (L \oplus L^{-1}, \begin{pmatrix} 0 & \beta \\ \alpha & 0\end{pmatrix}, \begin{pmatrix} h & 0 \\ 0 & h^{-1}\end{pmatrix})^{\oplus 2}$
$tr(\Phi^2) = 0 \Leftrightarrow \alpha^2\beta^2 = 0$
then return back to Anosov representation:

Uniformization thm: elliptic, parabolic, hyperbolic } Riemann surfaces

• Fuchian representation: hyperbolic Riemannian S, $\rho: \pi_1(S) \to PSL(2,\mathbb{R})$, $\Gamma \subset PSL(2,\mathbb{R})$ Fuchian subgroup

Def 11. (Teichmüller space) $S_{g\geq 2}$ closed orientible, a marked hyperbolic structure on S is $(X, f: S \to X)$, X hyperbolic and f homeomorphism s.t. $S \cong \Gamma\backslash\mathbb{H}^2$, Riemannian surface
$\Rightarrow \mathcal{T}(S) := \{\text{marked hyperbolic structure on } S\}/\text{isotopy} = \{\rho: \pi_1(S) \to PSL(2,\mathbb{R}) \text{ Fuchian representation (discrete \& faithful)}\}/\text{conjugation}$
Pf. For equality second, set $X = \mathbb{H}^2/\Gamma$, Γ discrete in $PSL(2,\mathbb{R}) \Rightarrow \rho = f_*$ is isomorphism
Conversely, set $X_\rho := \mathbb{H}^2/\rho(\pi_1(S))$ and $S \simeq X_\rho$ homotopical equivalent $\xrightarrow[\text{Bother}]{\text{Nielsen}} \exists f: S \to X_\rho$ homeomorphic ▨
hyperbolic

We next show that ρ is Anosov by ① ρ is convex cocompact & ② convex cocompact "$\Longleftrightarrow$" P_1-Anosov

① $\rho: \Gamma \longrightarrow O_0(n,1)$ (defined by acting on $\mathbb{H}^n$ by $\begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$) is convex cocompact if orbit map $\tau_\rho: \Gamma \longrightarrow \mathbb{H}^n$ is quasi-isomorphic embedding

Prop 7. ρ Fuchsian and convex cocompact $\Longleftrightarrow \exists\, \Omega \subset \mathbb{H}^n$ convex, s.t. $\rho(\Gamma) \curvearrowright \Omega$ preserves and acts cocompactly on Ω

(Omitted) Note that $n=2 \Longrightarrow PSL(2,\mathbb{R}) = SO_0(2,1) \Longrightarrow$ Fuchsian is convex cocompact.

② Prop 8. View $O_0(n,1) \hookrightarrow SL(n+1,\mathbb{R})$, then convex cocompact $\Longleftrightarrow P_1$-Anosov

Pf. Embedding $\mathbb{H}^n \subset \mathbb{R}^{n+1}$ by $\{x \in \mathbb{R}^{n+1} \mid -(\sum_i^n x_i^2) + x_{n+1}^2 = 1\} \Longrightarrow d_{\mathbb{H}^n}(e_{n+1}, Te_{n+1}) = \log \sigma_1(T) = \log \frac{\sigma_1(T)}{\sigma_2(T)}$, similiarly $\cdots \Longrightarrow \sigma_1(T) \geq \sigma_2(T) = \cdots = \sigma_n(T) \geq \sigma_{n+1}(T)$ ▨

Return back to Hitchin representation $\rho: \pi_1(S) \longrightarrow SL(n,\mathbb{R})$, it NonAbelian Hodge to Hitchin section $\left(K^{\frac{n-1}{2}} \oplus \cdots \oplus K^{\frac{1-n}{2}}, \begin{pmatrix} & & & \alpha \\ 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{pmatrix}\right)$

$= Sym^{n-1}\left(K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}, \begin{pmatrix} 0 & \delta_2 \\ 1 & 0 \end{pmatrix}\right)$, "$Sym^{n-1}$" corresponds to $\tau_n: SL(2,\mathbb{R}) \hookrightarrow SL(n,\mathbb{R}) \Longrightarrow \rho = \underline{\tau_n \circ \rho_0\ \text{Fuchsian}}$, the right hand is an equivalent definition of Hitchin representation.

Prop 9. ρ_0 Fuchsian $\Longrightarrow \tau_n \circ \rho_0$ is P_k-Anosov for $\forall\, 1 \leq k \leq \frac{n}{2}$

Pf. By SDT decomposition / KAK decomposition $\Longrightarrow$ reduce to diagonal matrix with eigenvalue/singular value $\Longrightarrow \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \xrightarrow{\tau_n} \begin{pmatrix} \lambda^{n-1} & & \\ & \ddots & \\ & & \lambda^{1-n} \end{pmatrix}$

thus their below bound are holds ▨

ρ_0 without Fuchsian $\Longrightarrow \rho$ is still P_k-Anosov ② Hitchin representation is closed condition & Anosov is open condition $\Longrightarrow$ Hole component is Hitchin & Anosov (For other components, may also Anosov)

Eg 11. We have group $\mathrm{Anosov}_k(\Gamma, SL(n,\mathbb{R})) := \mathrm{Hom}^{P_k\text{-Anosov}}(\Gamma, SL(n,\mathbb{R}))/SL(n,\mathbb{R})$ is not reductive, but Hausdorff, we take $\mathrm{Anosov}_k^{red}(\Gamma, SL(n,\mathbb{R}))$

Usually $Out(\Gamma) = Aut(\Gamma)/Inn(\Gamma) \curvearrowright \mathrm{Anosov}_k^{red}(\Gamma, SL(n,\mathbb{R}))$ by $\rho \mapsto \rho \circ \varphi^{-1}$, $\varphi \in Out(\Gamma)$

Thm 3. (Labourie) This action is proper & discontinuous

Def 12. (Energy) $\rho: \pi_1(S) \longrightarrow Isom(N)$, we define the energy functional $E_\rho: \mathcal{T}(S) \longrightarrow \mathbb{R}^N$

Thm 4. (Labourie) If $\rho: \pi_1(S) \longrightarrow Isom(N)$, N is Cartan-Hadamard, ρ is well-displacing $\Longrightarrow E_\rho$ is proper i.e. achieve minium, $\exists\, J$ on S $\quad X = (S,J) \longmapsto E_\rho(X) := \inf\{\text{Energy of } f: \tilde{X} \to N \mid f \text{ is } \rho\text{-equivariant}\}$

Corollary 1. $\rho: \pi_1(S) \longrightarrow SL(d,\mathbb{R})$ reductive & well-displacing $\Longrightarrow \exists\, J$, and f is ρ-equivariant conformal harmonic $f: \tilde{X} \hookrightarrow X_d$ minimal

Conjecture (Labourie) ρ Hitchin. $\exists!$ ρ-equivariant minimal surface $\subset X_d$ (Recall X_d symmetric)

Answer. $n \geq 3$, not unique, $n \geq 4$, not exists ▨ (Recently by Sagman & Smillie)

Homework 1 for the minicourse "Higgs bundles and minimal surfaces in non-compact symmetric spaces"

Qiongling Li

June 2024

Problem 1. *Let M be a compact Riemann surface of genus at least 2 and $a \in H^0(M, K^2)$ a quadratic differential.*

- *Show the Higgs bundle*

$$E = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}, \quad \theta = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$$

is stable.

- *Show the harmonic metric H on the Higgs bundle satisfying $\det(H) = 1$ satisfies $H = diag(h^{-\frac{1}{2}}, h^{\frac{1}{2}})$.*
- *Show the associated flat connection*

$$D = \nabla^H + \theta + \theta^{*H}$$

has holonomy in a copy of $SL(2,\mathbb{R})$.

- *The 2-tensor*

$$\hat{h} = a + (h + \frac{a\bar{a}}{h}) + \bar{a} \in \Omega^0(M, S^2T^* \otimes \mathbb{C})$$

is a Riemannian metric on M,

the Gaussian curvature of $\hat{h}$ is -4,

any metric of constant curvature -4 on M is isometric to a metric of this form for some $a \in H^0(M, K^2)$ up to a diffeomorphism homotopic to the identity.

Recall that Teichmuller space T is the space of equivalence classes of metrics of constant curvature -4 on M, modulo the action of the group of diffeomorphisms of M homotopic to the identity.

- *As a corollary, show that Teichmuller space T is homeomorphic to $\mathbb{R}^{6g-6}$.*

(Reference: N.J.Hitchin, Self-duality equations on a Riemann surface. Section 11)

• (Last one) Using hyperbolic structure g_0 on M and other hyperbolic g_0, $\exists!\ \varphi: (M, g_0) \to (M, g)$ critical of energy (Eells-Sampson)

Quadratic differential $a \Longrightarrow$ Higgs field θ ☑

Hyperbolic structure, splitting

• (Teichmuller thm) $\dim_{\mathbb{C}} H^0(M, K^2) \overset{\text{Riemann-Roch}}{=} 3g-3 \Rightarrow \dim_{\mathbb{R}} H^0(M,K^2) = 6g-6$ ☑

(Note that $\theta: K^{\frac12} \oplus K^{-\frac12}$ sends to $K^{-\frac12} \oplus K^{\frac32}$, thus we need reverse)

• $\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \boxed{\begin{pmatrix} ay \\ x \end{pmatrix}}$

$x = ay$, $y = x$

$\Rightarrow \boxed{y = ay}$ and x arbitrary

$a \neq 1 \Rightarrow$ no nontrivial subbundle $\Rightarrow$ stable

$a = 1 \Rightarrow$ only $K^{-\frac12} \subsetneq K^{\frac12} \oplus K^{-\frac12}$

$\Rightarrow \mu(K^{-\frac12}) = \frac{-g-0}{1} = 1-g < 0 \Rightarrow$ stable ☑

• Conjugate action on Higgs field θ

$\sigma_t = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_t^* \theta := \sigma_t^{-1} \theta \sigma_t = \begin{pmatrix} 0 & \frac{a}{t} \\ t & 0 \end{pmatrix}$

$\Rightarrow$ the slope of $\sigma_t^* \theta$ & θ are up to a positive constant

$\Rightarrow$ stability same $\Rightarrow$ suffices check $\sigma_t^* \theta$, and also $\frac{1}{t}\sigma_t^* \theta = \begin{pmatrix} 0 & \frac{1}{t^2}a \\ 1 & 0 \end{pmatrix}$

And (slope) stability is open $\Rightarrow$ let $t \to 0$, we reduce $a \neq 0$ case to $a = 0$ case ☑

(open condition under Sobolev norm on θ and dense)

• $\sigma_{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_{-1}^* \theta = -\theta$

$\sigma_{-1}^* H = \overline{\sigma_{-1}^t} H \sigma_{-1}$ harmonic

& $(\sigma_{-1}^* \theta)^{*(\sigma_{-1}^* H)} = \sigma_{-1}^{-1} \theta^{*H} \sigma_{-1}$

$\Rightarrow F_{\nabla^{\sigma^* H}} + [\sigma_{-1}^* \theta, (\sigma_{-1}^* \theta)^{*(\sigma_{-1}^* H)}] = 0$

under new $\sigma_{-1}^* H$ metric also holds Hitchin equation

$\Rightarrow \sigma_{-1}^* H = H$, by uniqueness of Corlette-Donaldson solving equations

$\begin{cases} \det H = 1 \\ \sigma_{-1}^* H = H \end{cases}$, $H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ ☑

• By Corlette-Donaldson, $D = \nabla^H + \theta + \theta^{*H}$ is reductive flat $SL(n, \mathbb{C})$-connection and D is irreducible $\Rightarrow$ uniqueness of $\det(H) = 1$ ☑

• Done in Lecture ☑

• h is Hermitian metric on K^{-1}, a is $(2,0)$-form $\Rightarrow \hat{h}$ is symmetric bilinear form obviously ☑ By writing as $\hat{h} = h(dz + \frac{\bar{a} d\bar{z}}{h})(d\bar{z} + \frac{a dz}{h})$

• $K_{\hat{h}} = -\frac{2\partial\bar{\partial}\log h}{h}$

$\hat{h} = h(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})$

$\bar{\partial}(h^{-1}\partial h) = \begin{pmatrix} \partial_{\bar{z}}\partial_z \log h^{-\frac12} & 0 \\ 0 & -\partial_{\bar{z}}\partial_z \log h^{-\frac12} \end{pmatrix} dz \wedge d\bar{z}$

and $\theta = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$, $\theta^* = \begin{pmatrix} 0 & h \\ \bar{a} & 0 \end{pmatrix}$

$\Rightarrow [\theta, \theta^*] = \begin{pmatrix} a\bar{a} - h & 0 \\ 0 & h - a\bar{a} \end{pmatrix} dz \wedge d\bar{z}$

$\Rightarrow F(\nabla^H) = -[\theta, \theta^*]$ by Hitchin equation

$\Rightarrow \partial_{\bar{z}}\partial_z \log h^{\frac12} + \boxed{h - a\bar{a}} = 0$

$\Rightarrow K_{\hat{h}} = -2 \cdot 2 = -4$ ☑

• By minimizing energy $E = \int \|d\varphi\|^2$, $\varphi: (M, g_0) \to (M, g)$ under a path of such harmonic diffeomorphic

$\hookrightarrow$ $(2,0)$ part of $\varphi^* g$ is holomorphic under g_0; $\varphi^* g = a + b + \bar{a}$, $a \in H^0(M, K^2)$ ☑

① $\hat{h}$ is real ($\overline{\hat{h}} = \hat{h}$)

$\hat{h} = (\hat{h})^{2,0} + (\hat{h})^{1,1} + (\hat{h})^{0,2} = a + (h + \frac{a\bar{a}}{h}) + \bar{a}$

$\Rightarrow \overline{\hat{h}} = \hat{h}$;

② $\hat{h}$ is nondegenerate ($\hat{h}(v,v) > 0, v \neq 0$)

(Teichmuller's theorem)

$g = 0$, obvious $\neq 0$; $g \neq 0$, it's a quadratic function $\neq 0$ with $\Delta < 0$

③ Lemma (Numerical)

• $(h + \frac{a\bar{a}}{h})^2 - 4a\bar{a} > 0 \Leftrightarrow \|a\|^2 < 1$

• $\|a\|^2 < 1$ (very hard, numerical $\to$ PDE $\to$ Strong maximal principle)

② + ③ $\Rightarrow$ non-degenerate ☑

Locally $\hat{h} = h(dz + \frac{\bar{a}}{h} d\bar{z})(d\bar{z} + \frac{a}{h} dz) = h u\bar{u}$, $\hat{\omega} = u \wedge \bar{u}$ the Kähler form

$\nabla u = u \otimes \sigma$, σ some 1-form (∇ is Levi-Civita), assume $\sigma^{0,1} \neq 0$

$\Rightarrow du(X, Y) = \nabla_X u(Y) - \nabla_Y u(X) = (\sigma \wedge u)(X, Y)$, i.e. $du = \sigma \wedge u$ ...(1)

& $\hat{h}(u, u) = \frac{1}{h} \Rightarrow -\frac{dh}{h} = \sigma + \bar{\sigma}$ ...(2)

$\Rightarrow -\frac{\partial}{\partial z}\frac{h}{h} = \frac{a}{h}\sigma^{1,0} - \sigma^{0,1}$ ...(3) $\Rightarrow \sigma^{0,1} + \sigma^{1,0} = -\frac{1}{h}\frac{\partial h}{\partial z}$ ...(4)

$\Rightarrow \frac{a}{h}\bar{\sigma}^{0,1} = -\sigma^{0,1}$ ...(5) $\Rightarrow \|a/h\| = 1 \Rightarrow K^2 \cong \mathbb{C}$

But $g(M) \geq 2$, $K^2 \cong \mathbb{C}$ impossible

$\Rightarrow \sigma^{0,1} = 0$, i.e. $\sigma^{1,0} dz$

$\Rightarrow \hat{F} = \bar{\partial}\sigma = \bar{\partial}\sigma^{1,0} dz = \bar{\partial}(-\frac{1}{h}\frac{\partial h}{\partial z}) dz = -F_{K^{-1}}$

Hitchin equation: $2(1 - \frac{a\bar{a}}{h^2}) h\, dz \wedge d\bar{z}$

$\Rightarrow \hat{F} = 2\hat{\omega}$ $\Rightarrow -4$ ☑

扫描全能王 創建

Taking all $g_i = 0$
stable is similiar and easy (Or $(E_n, \theta) = \mathrm{Sym}^{n-1}(E_2, \theta)$)
In general, taking $\frac{1}{t^n} t^* \theta$ $\boxed{//}$
($t \to \infty$ to reduce to $g_i = 0$ case)
[Hitchin, Lie groups and Teichmüller space, Section 3]

Problem 2. *Let M be a compact Riemann surface of genus at least 2 and $q_i \in H^0(M, K^i)$ a holomorphic $i-$differential for $i = 2, \cdots, n$. Show the Higgs bundle in the Hitchin section:*

$$E = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{\frac{3-n}{2}} \oplus K^{\frac{1-n}{2}}, \quad \theta = \begin{pmatrix} 0 & q_2 & q_3 & q_4 & \cdots & q_n \\ 1 & 0 & q_2 & q_3 & \ddots & \vdots \\ & 1 & 0 & q_2 & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & q_3 \\ & & & \ddots & \ddots & q_2 \\ & & & & 1 & 0 \end{pmatrix}$$

is stable.

[redacted]

Notice that here $E = K^{\frac{1}{2}} + K^{-\frac{1}{2}}$ is the not splitting case
splitting $\Leftrightarrow$ $\partial_E = \begin{pmatrix} \partial_{K^{\frac{1}{2}}} & 0 \\ 0 & \partial_{K^{-\frac{1}{2}}} \end{pmatrix}$
thus when acting, we can't directly applying the matrix multiplication

Problem 3. *Let M be a compact Riemann surface of genus at least 2 and L a holomorphic line bundle of negative degree. Let $\beta \in \Omega^{0,1}(M, K^{-1})$. Let*

$$E = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}, \partial_E = diag(\partial_{K^{\frac{1}{2}}}, \partial_{K^{-\frac{1}{2}}}) + \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix}, \theta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

"+"

- *Show (E, θ) is a stable $SL(2, \mathbb{C})-$Higgs bundle.*
- *The associated harmonic map is an equivariant minimal immersion of $\widetilde{M}$ into $\mathbb{H}^3$.*
- *Show that any equivariant conformal minimal immersion of $\widetilde{M}$ into $\mathbb{H}^3$ arises from this way.*
- *The moduli space of equivariant conformal minimal immersion of $\widetilde{M}$ into $\mathbb{H}^3$ is defined as $\{(\rho, f)\}/\sim$, where $\rho : \pi_1(M) \to SL(2, \mathbb{C})$ and $f : \widetilde{M} \to \mathbb{H}^3$ the equivariant conformal minimal immersion and the equivalence means $(\rho, f) \sim (g\rho g^{-1}, g \cdot f)$.*
  Show that the moduli space of equivariant conformal minimal immersion of $\widetilde{M}$ into $\mathbb{H}^3$ is parametrized by the vector space $H^1(M, K^{-1})$.

Homework 2 for the minicourse "Higgs bundles and minimal surfaces in non-compact symmetric spaces"

Qiongling Li

June 2024

<u>Problem1</u> (<u>Delzant-Guichard-Labourie-Mozes</u>) [1]: Canary's lecture note on Anosov representations.

Problem 1. *Show that for a hyperbolic group Γ and a homomorphism $\rho : \Gamma \to Isom(X)$, the orbit map τ_ρ is a quasi-isometric embedding if and only if ρ is well-displacing.*

(Hint: (1) First show

$$l_X(g) \geq \lim_{n\to\infty} \frac{d(x_0, g^n \cdot x_0)}{n},$$

for $g \in Isom(X)$. And use the fact for Γ hyperbolic, there exists K such that for all $\gamma \in \Gamma$,

$$\left| ||\gamma|| - \lim_{n\to\infty} \frac{d(1, \gamma^n)}{n} \right| \leq K.$$

(2) Use the following fact: The result of Delzant-Guichard-Labourie-Mozes says for a hyperbolic group Γ, there exist $\alpha, \beta \in \Gamma$ and $K > 0$ so that

$$d(1, \gamma) \leq 2\max\{||\gamma||, ||\gamma\alpha||, ||\gamma\beta||\} + K,$$

*for all $\gamma \in \Gamma$. They call such property as the **U-property**.)*

(1) $l_X(g) := \inf ||\eta g \eta^{-1}||_S$, S the generator of Γ and $||\cdot||_S$ is the word length

Taking generators $g_1 \cdots g_n = g^1 \cdots g^n$

$\Rightarrow$ $\boxed{U\text{-property}}$ implies that.

$||\gamma|| \leq A \lim_{n\to\infty} \frac{d(1,\gamma^n)}{n} + B$

$\Rightarrow \left| ||\gamma|| - \lim_{n\to\infty} \frac{d(1,\gamma^n)}{n} \right| < \infty \Rightarrow \exists K$ dominate;

And the length is dominate on the paths on Cayley Graph ▨

(2) Taking generator $\Gamma_0 \trianglelefteq \Gamma$ and applying (1) $\Rightarrow \Gamma_0$ is U-property $\Rightarrow \Gamma$ also.

Last step by <u>Prop4.2.1</u>. ▨

↳ in [DGLM. Well-displacing rep. and orbit maps]

<u>Problem2</u>.

Problem 2. *1. Construct an example $\rho : \Gamma \to SL(4, \mathbb{R})$ is P_1-Anosov but not P_2-Anosov.*

2. Construct an example $\rho : \Gamma \to SL(4, \mathbb{R})$ is P_2-Anosov but not P_1-Anosov.

(Hint: Use $\rho \oplus \rho$ and $\rho \oplus T$ where $\rho : \Gamma \to SL(2, \mathbb{R})$ is P_1-Anosov and T is a trivial representation.)

(1) $\rho : \mathbb{F}_2 \to SL(2,\mathbb{R})$ by $\rho(a) = id$ and $\rho(b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, it's P_1-Anosov;

and $0 : \mathbb{F}_2 \to SL(2,\mathbb{R})$ sending to $0 \in SL(2,\mathbb{R})$

$\Rightarrow \sigma_1(\rho(a)) \neq \sigma_1(\rho(b))$, and $\sigma_1(\rho(a)) = 0$

$\Rightarrow P_1$-Anosov;

But P_2-Anosov is $\boxed{1, 1}, 0$ ↳ by ρ

$\Rightarrow \rho \oplus 0$ not P_2-Anosov ▨

(In fact, one summand not P_k-Anosov $\not\Rightarrow \rho$ is also not P_k-Anosov)

(2) $\rho \oplus \rho$ is P_2-Anosov but not P_1-Anosov ▨

<u>Problem3</u>.

Problem 3. *(Milnor-Svarc Lemma) If a group Γ acts properly discontinuously and cocompactly by isometries on a complete Riemannian manifold X (or more generally on a proper geodesic metric space), then the orbit map $\Gamma \to X$ is a quasi-isometry.*

We say that $f : X \to Y$ is a quasi-isometry if f is a $(K, C)-$quasi-isometric

$\tau_\rho : \Gamma \to X$ is a $(\max\{K_0, \frac{K_1}{r}\}, r)$-quasi-isometry where

$\begin{cases} d(\alpha(x_0), \beta(x_0)) \leq K_0 d_S(\alpha, \beta) \\ d_S(id, \gamma) \leq K_1 d_\Gamma(id, \gamma) \Leftrightarrow d_S(1, t) \leq K_1 \end{cases}$

$\Rightarrow \begin{cases} \frac{1}{K_0} d(\alpha(x_0), \beta(x_0)) \leq d(\alpha, \beta) \\ \frac{r}{K_1} d_S(\alpha, \beta) - r \leq d(\alpha(x_0), \beta(x_0)) \end{cases}$ ▨

emebdding and if $y \in Y$, there exists $x \in X$ such that $d(f(x), y) \leq C$, i.e., f is a quasi-isometric emebdding which is coarsely surjective.

(For the proof, check Reference[1] Lemma 1.1.)

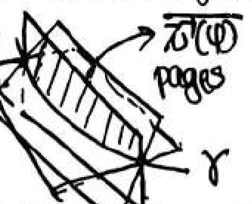
One may apply to $\rho : \pi_1(S) \to PSL(2, \mathbb{R})$ Fuchsian.

Lecture 3

Intersection theory of punctured holomorphic curves and planar open books. Zhenyi Zhou

Wendl (2010) Strongly fillable contact manifolds and J-holomorphic foliation
Wendl Lectures on 3-manifold, holomorphic curves and intersection theory

Def 1. (Open book) (B,π) is open book on M^3 a closed orientied



$\pi^{-1}(\varphi)$ pages

s.t. $\forall \gamma \subset B$, γ called a binding. (due to Siefring analytically)

γ: $\exists$ a neibourhood of γ, $U \supset \gamma$, $U \cong S^1\times D^2$, $\pi|_{S^1\times D^2-(0)}: S^1\times D^2-(0)\to S^1$, $(\psi, r, \varphi)\mapsto\varphi$

$S^1 \to R^3$, $S^1\to S^3$: A non-connected "knot" (including) "linked together" & "knotted together"

$B\subset M$ is orientied $\boxed{\text{link}}$, $\pi: M-B\to S^1$ is fibration. $\boxed{\text{knots}}\subset\boxed{\text{links}}$, and for B is knot $\Rightarrow$ π is unique

Eg 1. Consider the "degenerate" case, $M=S^2 \Longrightarrow$ then "$\times S^1$" gives an example:

Bindings, pages, B_1, B_2

Rk. Such M^3 always exists a open book

$M=S^2\times S^1$ & $B=B_1\times S^1 \amalg B_2\times S^1$ (not connected)
pages $\pi^{-1}(\varphi)=S^1\times(0,1)$

Eg 2. $M=S^3\subset\mathbb{C}^2$, $B=\{|z_1|=0\}\Rightarrow\pi^{-1}(\theta)=\{(\sqrt{1-r^2}e^{i\theta}, re^{i\varphi})\}=D^2$, $\pi=\frac{z_1}{|z_1|}$

$B=\{|z_1|=0\}\amalg\{|z_2|=0\}$ $\Rightarrow\pi^{-1}(\theta)=\{(\sqrt{1-r^2}e^{i\theta}, re^{i\theta})\}\cong S\times(0,1)$, $\pi=\frac{z_1z_2}{|z_1z_2|}$ (B called Hopf link)

Def 2. (Contact manifold) A co-orientied contact structure ξ on Y^3 is:
① $\xi\subset TY$ is codim $=1$ subbundle;
② (Contact form) $\exists\alpha\in\Omega^1(Y)$, s.t. $\xi=\ker\alpha$;
③ Totally nonintegrable: $\alpha\wedge d\alpha\neq0$;

Eg 3. $\xi=\ker\alpha$, $\alpha=x_1dy_1-y_1dx_1+x_2dy_2-y_2dx_2$ is standard contact form on S^3;

Def 3. (Giroux contact form) $\alpha\Leftrightarrow$ contact structure ξ supported by open book (B,π)?
① $\alpha|_{TB}>0$; ② $d\alpha|_{\pi^{-1}(\varphi)}>0$

Exercise ① α contact form, $f>0\Longrightarrow f\alpha$ also contact form. $\ker f\alpha=\ker f\cup\ker\alpha=\ker\alpha=\xi$ ☑
② Check that Eg.2's two examples and Eg.3's standard contact form are compatible in sense of Def 3.

Thm. Every open book (on such M^3) supports unique $[\xi_0]$ up to isotopy (Thurston–Winkelnkemper)
Open book is fully determined by its page P and its monodromy: given by vector field V on P, s.t. near ∂P, $V=\partial_\varphi\Longrightarrow$ diffeomorphism $\psi: P\to P$ and $\psi|_{\partial P}=id$, well-defined up to isotopy.
thus we have $\boxed{\text{Pages}\Longleftrightarrow\text{Open book}\Longleftrightarrow\text{Contact structure}}$ (Slogan) $\Longleftarrow$ is done. Next is "$\Longrightarrow$"

Def 4. (Abstract open book) (P,ψ), P^2 has boundary & $\psi\in \mathrm{Diff}_c(P)$
$\Longrightarrow$ we can construct $OB(P,\psi):=\partial P\times D^2\amalg P_\psi$, $P_\psi:=[0,1]\times P/(1,x)\sim(0,\psi(x))$ by monodromy
$B=\partial P\times\{(0,0)\}\subset OB(P,\psi)$ and $\pi: OB(P,\psi)-B\to S^1\Rightarrow(B,\pi)$ is open book on $OB(P,\psi)$

Def 5. (Thurston–Winkelnkempler construction) Fix $\lambda\in\Omega^1(P)$, $d\lambda>0$. $\lambda|_{O(\partial P)}=\rho d\varphi$, $O(\partial P)=\coprod(1-\varepsilon,1]\times[0,D]$ and admit that $\pi_0\mathrm{Diff}_c(P)=\pi_0\mathrm{Symp}_c(P,d\lambda)\Rightarrow\psi\in\mathrm{Symp}_c(P,d\lambda)$ $d\varphi$
$\psi^*d\lambda=d\lambda\Longrightarrow\psi^*\lambda=\lambda+\eta$, $\eta\in\Omega^1(P)$, $d\eta=0$, $\eta=0$ near ∂P
$\alpha=(\lambda|_{\partial P}+k\rho^2dt)\amalg(\lambda+b(t)\eta+kdt)$, $b(t)$ is the Bump function, $k\gg1$, α is Giroux contact
it's unique under isotopy

Introduction to J-holomorphic curves

Def 6. $u:(\Sigma,j)\to(M,J)$, $du\circ j=J\circ du\Leftrightarrow\bar\partial_J u=0$ — Locally $\mathbb{C}^n$ (linearization in PDE)

Thm 7. (Carleman similarity principle) "J-holomorphic curve (Cauchy–Riemannian equation) is closed to holomorphic curve in complex manifold" (here only almost complex) Precisely writting, $p>2$ fixed, $J(z)\in L^p(B_\varepsilon(0),\mathrm{End}(\mathbb{R}^{2n}))$
$J\in W^{1,p}(B_\varepsilon,\mathrm{End}(\mathbb{R}^{2n}))$, $J(z)=id$, $\xi u\in W^{1,p}(B_\varepsilon,\mathbb{R}^{2n})$ is a solution of $\boxed{\frac{\partial u}{\partial z}+J(t)\frac{\partial u}{\partial t}+C(z)u(z)=0}$ (*)

$\Rightarrow \exists \delta \in (0,\varepsilon)$, $\Phi \in W^{1,p}(B_\delta, \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n,\mathbb{R}^{2n}))$, holomorphic map $v: B_\delta(0) \to \mathbb{C}^n$, s.t. $\Phi(z)$ invertible and $u(z)=\Phi(z)v(z)$ $\boxed{\text{Page 2}}$

Pf. $\boxed{\text{Step 1}}$ Assume $J=\begin{pmatrix}0&-1\\1&0\end{pmatrix}=i$ (by a linear algebra);

$\boxed{\text{Step 2}}$ We can take $C\in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$; $\boxed{\text{Step 3}}$ $W^{1,p}(\mathbb{CP}^1,\mathbb{C}^{n\times n}) \cong L^p(\mathbb{CP}^1,\mathbb{C}^n)\oplus \mathbb{C}^{n\times n}$, thus $(\bar\partial u)^{0,1}$ its preimage gives the solution of equation $\underbrace{\boxed{\frac{\partial\Phi}{\partial s}+J_0\frac{\partial\Phi}{\partial t}}}_{=(\bar\partial\Phi)^{0,1}} = C\Phi \in L^p$

$u \mapsto (\bar\partial u)^{0,1}$

and let $u(z)=\Phi(z)v(z)$ in (*) $\Rightarrow \Phi(z)\left(\frac{\partial v(z)}{\partial s}+i\frac{\partial v(z)}{\partial t}\right)+\underbrace{\left(\frac{\partial\Phi}{\partial s}+i\frac{\partial\Phi}{\partial t}+C\Phi\right)}_{=0}v(z)=0$ ▨ (By $C\in L^\infty$ and $\Phi\in W^{1,p}$) done

Cor1. If J integrable $\Rightarrow$ J-holomorphic curve is holomorphic curve

Thm3. (Unique Continuation) For C^∞, $u,v: B_\varepsilon(0)\to\mathbb{C}^n$, $\bar\partial_{J_0}(u)=\bar\partial_{J_0}(v)=0$, $\forall k$, $u-v=O(r^k)$ when $r\to\infty \Rightarrow u=v$.

Pf. $\omega=u-v$, its vanishing locus $A=\{w=0\}\subset B_\varepsilon(0)$ is open and closed ▨

Def7. (Multi-covered curve) $\Sigma \xrightarrow{u} M$, $\exists \varphi: \Sigma\to\Sigma'$ holomorphic (branched cover) with $\deg\varphi\geq 2$, s.t. $\Sigma \xrightarrow{u} M$, $\Sigma\xrightarrow{\varphi}\Sigma'\xrightarrow{u'} M$

not multi-covered curve called simple curve

Def8. (Somewhere injective) $\Sigma\xrightarrow{u}M$ $\exists p\in\Sigma$, $du(z)\neq 0$, $\#(u^{-1}(u(z)))=1\Rightarrow z$ called injective point $\Rightarrow \Sigma$ is somewhere injective

Prop1. TFAE ① Simple; ② Somewhere injective; ③ $\#\{$non injective points$\}$ and $\#\{$self-intersected points$\}<\infty$

due to ③, we can consider the intersection theory. $(\Sigma_1\cdot\Sigma_2)=\sum_{u(p)=v(q)}\delta(p,q)$ locally counting multiplicity

Thm4. (Intersection of positivity) (M^4,J), $u_i:\Sigma_i\to M$ simple $(i=0,1)\Rightarrow(\Sigma_0\cdot\Sigma_1)\geq 0$

Thm5. (Adjunction formula) $(\Sigma\cdot\Sigma)-\chi(\Sigma)\leq[\Sigma]\cdot[\Sigma]-c_1(\Sigma)$ and "$=$" holds when $u=\Sigma\to M$ is immersion.

intersection number as homology class.

Def9. (Planar open book) If P is planar $(g(P)=0)$

A contact structure is planar if it's supported by planar open book

Def10. Giving contact form α, then the Reeb vector field R is $\alpha(R)=1$ & $i_R(d\alpha)=0$, and its flow called the Reeb flow

Eg.4 $(S^3,\alpha_{standard})$ has Reeb flow $(e^{it}z_1, e^{it}z_2)$

It's transverse flow (monodromy) $(e^{if(|z_2|)t}z_1, e^{if(|z_1|)t}z_2)$, where f is bump function

$\Rightarrow$ the monodromy $(r^{2k}e^{if(|r|^2)\theta}, \sqrt{1-r^2}^{\,k}e^{if(r)\theta})$ called the positive Dehn twist

$(s,t)\mapsto(se^{i2\pi t},t)$

(In general $\boxed{\text{Dehn twist}}$ is defined for curves embedding)

Exercise. (Lens space) $L(k,k-1):=S^3/\mathbb{Z}_k$ by action $\zeta\cdot(z_1,z_2)=(\zeta z_1,\zeta z_2)$, $\zeta=e^{\frac{2\pi i}{k}}\in\mathbb{Z}_k$

Show that $L(k,k-1)=OB([0,1]\times S^1,\tau^k)$, τ is the positive Dehn twist

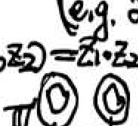
Def11. (Lefschetz fibration) A bordered Lefschetz fibration is $\pi: W^4\to\mathbb{D}^2$, finite critical points $W_{crit}\subset W^0$ (0-dim cells), s.t.

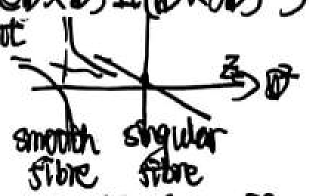
① $\partial W=\partial_h W\amalg\partial_v W$, $\pi^{-1}(\partial\mathbb{D}^2)=\partial_v W$ and $\pi|_{\partial_v W}\to\partial\mathbb{D}^2$ is a fibre bundle (it's horizontal boundary & vertical boundary e.g. $\partial(\mathbb{D}^2\times\mathbb{D}^2)=(\partial\mathbb{D}^2\times\mathbb{D}^2)\amalg(\mathbb{D}^2\times\partial\mathbb{D}^2)$)

② $\pi|_{\partial_h W}\to\mathbb{D}^2$ is smooth trivial fibre bundle;

③ local on each critical point p, local chart $\{z_1,z_2\}$, s.t. $\pi(z_1,z_2)=z_1\cdot z_2+$ constant

④ (Bordered) $\pi^{-1}(v)$ is connected, nonempty boundary in $\partial_h W$





$\Rightarrow$ Giving page are smooth fibres (thus contact)

monodromy are product of positive Dehn twist $\tau^k\Rightarrow\partial W$ inhert an open book from π

Eg.5. $W\subset\mathbb{C}^3_{1,2,3}\xrightarrow{\pi}\mathbb{C}$ smooth affine, $\pi|_W$ has finite critical points $\Rightarrow\exists R$ large s.t. $W\cap B^4(R)\times B^2(r)\xrightarrow{\pi}B^2(r)$ with $\mathrm{Crit}(\pi)\subsetneq\partial B^4(R)\times B^2(r)\Rightarrow\pi$ is Lefschetz fibration and $\partial W=OB([0,1]\times S^1,\tau^k)=L(k,k-1)$ when we set $W=(x^2+y^2+w^k=1)$ is A_{k-1}-singularity

Exercise. We have a distribution of contact structure $(x^2+y^2+w^k=1)\underset{contact}{\cong}(x^2+y^2+w^k=0)\underset{algebraic}{\cong}\mathbb{C}^2/\mathbb{Z}_k$

conclude that $S^3/\mathbb{Z}_k\cong\mathbb{C}^2/\mathbb{Z}_k$

Problem of symplectic filling: even for 3-dim nonoriented compact contact, we have an conterexample.

Symplectic filling is fundamental on symplectic cobordism

Def12. (Symplectic filling) (W^4,ω) is a strong symplectic filling of (Y^3,ξ) if ① $\partial W=Y$; ② near ∂W, $\exists$ a vector field X, X transverse ∂W pointout (i.e. $L_X\omega=\omega$) ③ $\ker(i_X\omega)=\xi$;

A Liouville filling if $\omega=d(i_X\omega)$, $i_X\omega$ is Liouville form (② $\Rightarrow$ Liouville filling by Cartan formula)

A Weinstein filling if X is an exhausting Morse function; $\nabla\varphi=X$, φ is Morse function

Rk. ω is compatible with gradient of the Lefschetz fibration π if ① $\pi^{-1}(v)$ symplectic; ② local at each critical point p ... as ③ in Def11. ③ ∂W is One can assume ∂W is level set

the Thurston's construction (all kinds)

Thm5. (Wendl) (W,ω) is a symplectic filling of a planar open book, then W admittes a compatible bordered Lefschetz fibration supporting OB. If (W,ω) is minimal (Meaning of algebraic geometry: no intersection -1 curves)

$\Rightarrow$ ① The fibres of π have no closed component (irreducible)

② ω is unique up to deformation, fix the π

Compare the MMP of surface: One can always $\boxed{\text{blowing-down}}$ to minimal $\Updownarrow$ $\boxed{\text{Lefschetz fibration}}$

Cor2. A planar contact structure is fillable (all kinds)

$\Leftrightarrow$ Monodromy can be factored to positive Dehn twist

Pf. Only "$\Leftarrow$", it's by construction of a Lefschetz fibration, this is by imitate Def5's construction ▨

Cor3. Classification of minimal filling up to symplectic deformation $\Leftrightarrow$ factorization of monodromy by positive Dehn twists

Eg.6 (Overtwisted structure) $\partial B([0,1]\times S^1, \xi) \cong (S^3, \xi_{ot})$

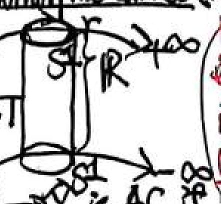
Eg.7. $(S^3, \xi_{standard})$, $(S^2\times S^1, \xi_{standard})$, $(L(k,k-1), \xi_{standard})$ has the following unique filling:
D^4 , $[0,1]\times S^1\times D^2$, A_{k-1}-singularity

Eg.8. For not minimal (genus large), fillings are distinct from each others:
$L(2,1)\cong \mathbb{RP}^3$ has $\{$ Liouville filling: A_1-singularity $= T^*S^2$
Strong filling: $(\mathbb{CP}^2 - ...) \longrightarrow \mathbb{CP}^2$

Def 13. (Symplectization) (Y, λ) is contact, $\hat{Y} := (Y\times \mathbb{R}_t, d(e^t\lambda))$ is symplectic 4-manifold, with an almost complex structure J compatible with λ: ① J is $\mathbb{R}_t$-translation invariant; ② $J\partial_t = R$ (Reeb vector field); ③ J preserves $\xi = \ker\lambda$; ④ $d\lambda(-, J-)$ is positive definite & symmetric (i.e. Riemannian metric)
Then we can deal the J-holomorphic curves, but on $\hat{Y}$, the closed J-holomorphic curve is constant by
$\frac{1}{2}\int |du|^2 dvol = \int u^*(d(e^t\lambda)) = 0 \Rightarrow$ we must puncture it, this is our setting.

Intersection theory on punctured holomorphic curves (no planar needed)

Eg.9. $\mathbb{R}^1\times S^1 \longrightarrow \hat{Y}$
$(s,t) \longmapsto (Ts, \gamma(Tt))$
γ is the Reeb orbit of λ of period T
$T(\gamma)$ called the twisted cylinder



Weinstein conjecture says that every Reeb field on contact (compact) manifold, it has a periodic orbit. (not unique!)
Up to isotopy it's solved by open books; 3-dim proven by Taubes

Def 14. (Asymptotically cylinder) $u: \dot{\Sigma} \to \hat{Y}$ is AC if near each puncture, $\exists$ a cylinder coordinate $(0,\infty)\times S^1 / (-\infty,0)\times S^1$
s.t. $u|_{Z_\pm} = \exp_{T(\gamma)} h(s,t)$, $|\partial^k h(s,t)| \xrightarrow{s\to\infty} 0$ (and ~~the~~ Hofer energy $<\infty$ and ~~Reeb orbit is nondegenerate~~)

We need construct a Lefschetz fibration: Step 1 a foliation on $\hat{Y}$

each pages are holomorphic curves by choosing good J & perturbations

Step 2 Symplectic filling W, $\hat{W} := W \sqcup Y\times[0,\infty)$ for proving the Wendl theorem
∂_t is Liouville vector field, we extend the foliation to W:

In fact here it's not a foliation on contact structure, but we'll discuss at last: it can be distribute to a stable Hamiltonian and it turns out to be a true foliation.

Observation. A general leaf of (push-forward) the foliation is positively asymptotic to all Bindings
Consider the moduli $\mathcal{M}_{0,B}(W) = \{ u: \dot{\Sigma} \to W \mid J$-holomorphic, positively asymptotic to all Bindings $\}$ ($\Leftarrow$ holomorphic)
Key: $\mathcal{M}_{0,B}$ is punctured disk $\Rightarrow \overline{\mathcal{M}_{0,B}} = D^2 \Rightarrow \partial\mathcal{M}_{0,B} = S^1 \cong \{$generic leaves of $\hat{Y}\}$
then we can pushforward by universal family $\mathcal{U} \xrightarrow{evaluation} W$ this gives foliation on W ☒
$\downarrow$
$\mathcal{M}_{0,B} \longrightarrow D$

For the definition of intersection numbers, we should note that we need adding hidden datum hidden by punctured; this method of counting is completely analytic

(This is our next discussion on the intersection theory)

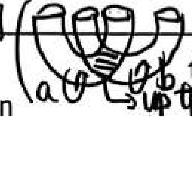
The lecture was lost in the following analytical details

Eg.10. (Analytic) $f: M\to\mathbb{R}$ a Morse function ($\nabla df(p): T_xM\otimes T_xM \to \mathbb{R}$ is nondegenerate quadratic form at critical point), $\dot{X}(s) = \nabla f(X(s))$ gradient flow, $\lim_{s\to\pm\infty} X(s) = p_\pm$, $\exists h_\pm(s)\in T_{p_\pm}(M)$ near ∞. s.t. $X(s) = \exp_{p_\pm} h_\pm(s)$, $h_\pm(s) = e^{\lambda_\pm s}(v_\pm + r_\pm(s))$, where $\lambda_\pm$: $\mp$-eigenvalence of Hess $f(p_\pm)$. (not identically $\equiv 0$)
This is finite-dim: $v_\pm$: eigenvector
we then have: $r_\pm$: bump function with $|r_\pm(s)| \xrightarrow{s\to\infty} 0$

$u = (f, v): Z_\pm \to \hat{Y}$ holomorphic $\Leftrightarrow \begin{cases} \partial_s f - \lambda(\partial_t v) = 0 \\ \partial_t f + \lambda(\partial_s v) = 0 \end{cases} \Leftrightarrow v^*\lambda\circ j = df$
$\mathcal{A}(\gamma) = \int \gamma^*\lambda$, and $\pi_\xi \partial_s v + J\pi_\xi \partial_t v = 0 \Leftrightarrow \partial_s v = \nabla\mathcal{A}(v(s))$, $\mathcal{A}$ is the contact action functional
$\frac{d}{dt}\mathcal{A}(\gamma_t) = \int d\lambda(\dot{\gamma}, \partial_t\gamma)$, thus $\{$critical points$\} \longleftrightarrow \{$representation of Reeb orbits $S^1 \to Y\}$
$\langle \eta_1, \eta_2 \rangle = \int d\lambda(\eta_1, J\eta_2)$ is L^2-product on $\eta_1, \eta_2 \in \Gamma(\gamma^*\xi) \Rightarrow L^2(\gamma^*\xi)$, on this (infinite-dim space) we have
$\mathrm{Hess}(\mathcal{A}(\gamma))\eta = -\nabla_\eta(J\pi_\xi \gamma') - J\pi_\xi\nabla_t\eta$ on Levi-Civita connection $\Rightarrow \mathrm{Hess}(\mathcal{A}(v)) = -J\pi_\xi[R, \eta]$

Thm 6. (Analogous to Eg.10) Hess $\mathcal{A}(\gamma)$ is unbounded self-adjoint operator, eigenvector nowhere zero
replace X in Eg.10. by u J-holomorphic curve, we have the same conclusion (two curves $h_u - h_v = e^{\lambda s}(v_\pm(t) + r(s,t))$)
These analytic datum gives the intersection number in ∞ = after permutation the hidden datum to ∞

[redacted] Thm 4. Two [redacted] curves u, v, not identical image $\Rightarrow \#\{(p,q) \mid u(p) = v(q)\} < \infty$ (Compare Prop 1, it's punctured case)
Pf. By unique continuation ☒ taking ab times universal covering $\Rightarrow u = v \xRightarrow{unique} u \equiv v$
$\Rightarrow$ contradiction continuation

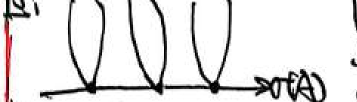


up to a period

Thm 7. $\exists S: S^1 \to End(\mathbb{R}^2)$, $A: C^\infty(\xi^*, \mathbb{R}^2) \to C^\infty(\xi, \mathbb{R}^2)$, $A = -(J_0 \frac{d}{dt} + S)$ self-adjoint. Choosing trivialisation τ of $\gamma^*\xi$, with $(J, d\lambda) \Rightarrow$ as symplectic space $\Rightarrow$ Hess$(A(\gamma))$, then we define wind: $\sigma(A) \to \mathbb{Z}$ is well-defined & monotone & acheieve $\forall n \in \mathbb{Z}$ twice.

the Asymptotic operator

Pf.



Well-defined: V_1, V_2 are two eigen vectors $\lambda \mapsto V_\lambda \mapsto$ wind(V_λ)

wind(V_1) − wind(V_2) = wind$(V_1 \overline{V_2})$ = 0

by $V_1\overline{V_2} = 0$ near $\mathbb{R}$

Thus

Due to if not monotone, we have , at t', wind(t') not well-defined

thus well-defined $\Rightarrow$ monotone //

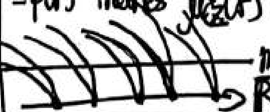
Def 15 (Conley-Zehnder index) Using $\sigma(A)$

$\mu_{CZ}^\tau(\gamma) = 2\alpha_-^\tau(\gamma) - p(\gamma) = \alpha_+^\tau + \alpha_-^\tau$

Here $\alpha_\pm^\tau = \min/\max\{\text{wind}^\tau(\alpha) \mid \alpha \in \sigma(A) \cap (0, \infty)\}$

$\Rightarrow p(\gamma) = \alpha_+^\tau - \alpha_-^\tau = 0$ or 1 not depending on τ

"$-p(\gamma)$" makes $\mu_{CZ}^\tau(\gamma)$ left:



$\Rightarrow \mu_{CZ}^\tau = \mu - \frac{1}{2}$

morse index μ ~~everything?~~

This from Morse index, and the "V" in picture we see it value is twice ☑

Page 4

two approaches to C-Z index

① Using axiomatic (i), (ii), (iii), (iv)

② Using spectrum of Asymptotic operator

Thm 8. (Siefring) $\boxed{\exists \text{ intersection theory}}$ for AC curves (punctured) by $u * v \in \mathbb{Z}$, s.t.

① $u*v$ is homotopic invariant;

② $\mathrm{Im}\, u \neq \mathrm{Im}\, v \Rightarrow u * v = (u \cdot v) + \boxed{l_\infty(u,v)}$ infinite intersection

③ $u * v = 0$, $\mathrm{Im}\, u \neq \mathrm{Im}\, v \Rightarrow \mathrm{Im}\, u \cap \mathrm{Im}\, v = \phi$

Pf. Define $l_\infty(u,v)$ by trivialization $\tau \Rightarrow (u \cdot_\tau v) := (u \cdot v^\tau)$, v^τ is pullback near $\pm\infty$ in direction of τ, V^τ approximate to $\exp_{v(s,t)} \eta(t)$, wind$^\tau(\eta) = 0$. $l_\infty^\tau(u,v) := (u \cdot_\tau v) - (u \cdot v)$ (only left ∞ datum), and the pullback of v^τ makes their Reeb orbits coincide

"γ_z" = "γ_ζ" ($\mathrm{Im}\,\gamma_z = \mathrm{Im}\,\gamma_\zeta$) $\mathrm{Im}\,\gamma_z = \mathrm{Im}\,\gamma_\zeta$ Observation ① + ② $\Rightarrow$ ③

$\{\gamma_z = \gamma_\zeta$ (multiplicity k) , here $k^2 l_\infty^\tau = $ "l_∞^τ in next case"

$\gamma_z \neq \gamma_\zeta$ (multiplicity $k_z \neq k_\zeta$), set $l_\infty(u,v) := \sum_{(z,\zeta)} l_\infty(u,v,z,\zeta) := \sum_{(z,\zeta)} \left(l_\infty^\tau(u,v,z,\zeta) - \min(-k_\zeta \alpha_\mp^\tau(\gamma^{k_z}), k_z \alpha_\mp^\tau(\gamma^{k_\zeta}))\right)$ ☑

Both dependent on τ but the minus not

Coro 5. $u_1, u_2 \in M_{OB} \Rightarrow u_1 * u_2 = 0$

$\Rightarrow u_1 \cap u_2 = \phi$ Pf. By construction of foliation ☑ (*)

Summary. Open book $\Rightarrow$ it's J-holomorphic must punctured $\Rightarrow$ Intersection theory of punctured J-holomorphic curve (AC)

We explain (*): [Wendl, Open book decompositions and stable Hamiltonian structures]

(X is Reeb field)

Def 15. (Stable Hamiltonian structure) M^3 oriented closed, $\mathcal{H} = (\xi, X, \omega)$ ① $\xi \subset TM$, $TM/\xi = \mathbb{R}$; ② $X \in \Gamma(TM)$ and $X \pitchfork \xi$ ③* $\xi = \ker \lambda$, λ is integrable (key difference with contact structure, it's softer); ④* $\lambda \wedge d\lambda - k\,\mathrm{vol}_M$, ⑤ $\omega \in \Omega^2(M)$, $\omega|_\xi$ is symplectic

(③*, ④* isn't necessary)

Prop 1. $OB(P, D) = OP \times D \amalg P_\psi$, $P_\psi \xrightarrow{\pi} S^1$ fibrebundle, the vertical distribution $\ker \pi_*$ on P_ψ can be extend to M as a cofoliation ξ_0, s.t. C^0 permutation of ξ_0 to ξ_ε supported on the open book, compatible with $\mathcal{H}_0 = (\xi_0, X_0, \omega_0)$, $\mathcal{H}_\varepsilon = (\xi_\varepsilon, X_\varepsilon, \omega_\varepsilon)$

(On open book, we can distribute contacts to stable Hamiltonian, and foliation the latter one)

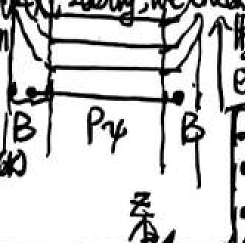
Eg 11. $(\Sigma, \sigma) \to M$, σ is symplectic form on Σ curve $\Rightarrow \omega = \sigma + dH \wedge \alpha_t = \sigma + dH \wedge \pi_* (X)$ ($\omega|_\Sigma = \sigma$) $\Rightarrow \mathcal{H} = (\ker \pi_*, X, \omega)$

(Hamiltonian on S^1)

Pf. $\lambda_0 = f(\rho)\,d\theta + g(\rho)\,d\varphi$, here θ is S^1 and (ρ, φ) is D^2, and we directly compute f, g by conditions we need to achieve $\Rightarrow$ construct $\lambda_0 \Rightarrow \xi_0$ by theorem of Giroux $\Rightarrow \exists \xi_\varepsilon$ contact, with Reeb vector field X_ε from $X_0 \to X$. Lastly, we check the compatibility of ξ_0 with $\mathcal{H}_0$: imitate the Eg. 11. by $\exists (P, \sigma) \hookrightarrow P_\psi$ a J_0-holomorphic curve, and then

$\Rightarrow \frac{\partial u}{\partial s} + J_0(u) \frac{\partial u}{\partial t} = 0$

$\Rightarrow \begin{cases} a_s = f\theta_t + g\varphi_t \\ a_t = -f\theta_s - g\varphi_s \\ \rho_s = \frac{1}{\rho D}(f'\theta_t - g'\varphi_t) \\ \rho_t = -\frac{1}{\rho D}(f'\theta_s + g'\varphi_s) \end{cases}$ ~ (*)



Its construction of foliation near B: this is solving of PDEs: $X_0 = \left(\frac{g'}{D}\right)\partial_\theta - \left(\frac{f'}{D}\right)\partial_\varphi$

(*) is still a PDEs, we change it into ODEs by $\rho_s = \theta_t$, $\rho_t = \theta_s$ ☑

Legendrian invariants (Same Legendrian type ⇍ Same knot type), we distinct by $\pi_{xz}(K)$

• Thurston-Bennequin $tb(K) = \text{writhe}(\pi_{xz}(K)) - \frac{1}{2}\#\{\text{cusps in } \pi_{xz}(K)\}$

• Rotation: $rot(K) := \frac{1}{2}\{\#\{\downarrow \text{cusps in } \pi_{xz}(K)\} - \#\{\uparrow \text{cusps in } \pi_{xz}(K)\}\}$

$tb(K) + |rot(K)| \leq -\chi(\Sigma)$, Σ is the Seifert surface of K ($\uparrow$ & $\downarrow$ are slopes)

• Legendrian knots & Lagragian filling

$\mathbb{R}^3$, $\xi = \ker(\alpha = dz - y\,dx)$ contact

Def A. (Legendrian knot) $K \subset \mathbb{R}^3$, s.t. $TK \subset \xi$

Eg B. (Milnor (5,2), $m(5,2)$)

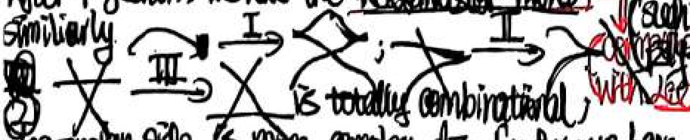
$m(5,2)$ two Legendrian types have same tb & rot, but no classical way to identify / distinct them

Front projection π_{xz}

Lagragian projection, π_{xy}

Note that $y = \frac{dz}{dx} \Rightarrow$ we can recover K from $\pi_{xz}(K)$

After projection, we have the Reidemeister moves similiarly



is totally combinational,

Lagrangian side is more complex, $dz = \int y\,dx$, we have the Ng's resolution relates ① to ②

Morden way: Chekanov-Eliashberg DGA F: Leg $\to$ DGA functor

such $\mathcal{Z}$ is called given by $\mathbb{Z}_2$-graded algebra $\mathbb{Z}_2\langle$Reeb chords on $K\rangle$ by $\succ$ and $\times$, ∂ by Maslov [illegible], upper has grade (degree) and counting

Legendrian link cobordism = 0 and morphism are cobordism index

compare to zag with Lecture 1, Def 7

Reeb vector field X / Reeb chord Λ : integral curve of X, ends on Λ

Reeb orbit : needs more condition to define.

the J-holomorphic curves — Reeb chords of K

We have the invariance (1-1 correspondence)

$\partial a := \Sigma^{\#} \ldots b_1 b_2$, sum over the moduli (After compactification) of J-holo curves transverse to Reeb chords

Thus by the structure, the Maslov index down 1, thus the degree also down 1

Legendrian isotopy class $[K] \longleftrightarrow [F(K)] \hookrightarrow$ J-holo curves $\xRightarrow{\text{homology}}$ Legendrian contact homology (hard to compute)

$\Rightarrow$ augmentation : $A(K) \xrightarrow{\varepsilon} \mathbb{Z}_2$ — the stable tame isotopy class

But we still can read some information from it

due to $\mathbb{Z}_2$ is small, this is very easy to computation, and one can view ε as representation.

• Note that it's clear that Legendrian contact homology can't distinguish unknot by Chekanov-Eliashberg original

(The speaker even don't know it)

① $\#\{\varepsilon$ is augmentation$\} < \infty$ is invariant under homotopy

it's invariant can distinct two Milnor-(5,2).

Legendrian contact homology in $\mathbb{R}^3$, Etnyre & Ng

② And linearisation $LCH^{\varepsilon}_{*}(K) := H_*(\mathbb{Z}_2\langle \text{Reeb chords}\rangle, \partial^{\varepsilon})$ has differential easy to compute also:

has $\partial a = b_1 b_2 \in \mathbb{Z}(b) \ldots + b_1 b_3 \in \ldots + b_2 b_3 \in \ldots$

Lagrangian filling $\mathbb{R}^4$

$(\mathbb{R}^3, \xi)$, Legendrian Λ

L is called the (exact) Lagrangian filling of Λ if it's cylindrical over Λ on ends. (ELF)

Contactification $\mathbb{R}^4$ to $\mathbb{R}^5$ — $L \subset \mathbb{R}^4$ is exact Lagrangian

$\mathbb{R}^4$ has symplectic form $d(e^t\alpha)$ (as the symplectification of $(\mathbb{R}^3, \xi)$)

$(\mathbb{R}_t \times \mathbb{R}^3, d(e^t\alpha)) \xleftarrow{f} (\mathbb{R}_t \times \mathbb{R}^3 \times \mathbb{R}_w, dw + e^t\alpha)$

$L \longrightarrow \tilde{L}$ thus we have $\{$Legendrian knots$\} \longleftrightarrow \{$Legendrian 3-folds $\subset \mathbb{R}^5\}$

$p \mapsto (p, f(p))$

$\mathrm{Aug}(\Lambda, \mathbb{Z}_2)$ (one two augmentation category)

The rigidity of ELF • $tb(\Lambda) = -\chi(L) = 2g_4 - 1$ (g_4 the slicing genus) (※) in "good" cases

{ objects $(F(\Lambda) \xrightarrow{\varepsilon} \mathbb{Z}_2$; morphism $\mathrm{Hom}(\varepsilon_1, \varepsilon_2) := \mathbb{Z}_2\langle \text{Reeb chords}\rangle$

• (m,n)-torus link if $(m,n) \geq (3,6)$ or $(4,4)$ (dictionary order) up to symmetry $\Rightarrow$ infinite many ELFs (By the Gao Hong Hua and its coauthors)

$(2,n), (3,3), (3,4), (3,5)$ are all finite.

$\mathrm{Aug}(\Lambda, \mathbb{Z}_2)$ objects : — ; morphism $\mathrm{Hom}(\varepsilon_1, \varepsilon_2) := \mathbb{Z}_2\langle$ Reeb chords$\rangle$

• The moduli of (infinite) ELFs has the Cluster structure $A(\vec{\alpha})$, $\vec{\alpha}$ can also be described

We have the correspondence of ELF and augmentation

And Fukaya category $\longleftrightarrow$ Augmentation (Speaker let me to ask Gao...)

Both A_∞, with multiplication with different

(※) "equivalent" definition (※※)

Fukaya category $\longrightarrow$ Augmentation category

Microlocal sheaves on $\mathbb{R}^3 \longrightarrow$ Microlocal sheaves on $\mathbb{R}^3$ & focus on Λ

• Ekholm's work concerns the

The Ekholm-Lekili (also A_∞)'s work gives the duality of Legendrain and Lagrangian via Fukaya category

The inverse not holds by (※): Seidel's isomorphism $LCH^{\varepsilon}_{*}(\Lambda) \cong H^{1-*}(L)$ the categorification of (※)

$\Rightarrow tb(\Lambda) = -\chi(L)$ recover (※) by taking Euler number

However, we can turn into "good" case by a permutation (e.g. disk case D.)

however it holds only for "good" case, thus contradiction $\Rightarrow$ not 1-1 correspondence.

The $\tilde{L}$ Legendrian is determined by $L \subset M$ Lagrangian submanifold, the setting is different with here ($M \neq \mathbb{R}^{2n}$), and Fukaya category is partially wrapped

Thm. (Pan-Rutherford) $\forall$ augmentation $\varepsilon : F(\Lambda) \to \mathbb{Z}_2$, can be induced by some (L, α) is ELF.

The construction of L is based on locally like $m(5,2)$

When L is immersed, it generalises the Seidel isomorphism $LCH^{\varepsilon}_{*}(\Sigma) \cong H^{1-*}(L) \cong LCH^{\varepsilon}_{*}(\Lambda)$, then

$\#$ double pts of $L \geq \frac{1}{2}(P^{\varepsilon}_{\Lambda}(1) - 1)$, $P^{\varepsilon}_{\Lambda}(t) = t + 2g_4(\Lambda)$

(※) is still some arguments in symplectic topology in general, for definition of Fukaya category, this is by Nadler & Tamarkin define by sheaf method. But in here $\mathbb{R}^{2n}$ case, this holds obviously

• Concordance of fibred knots, Heegaard Floer

(※※) (It's another work of Gao) Augmentation $\Longrightarrow$ Augmentation polynomial

Two knots $K_0, K_1 \subset S^3$ are smoothly concordant if

where $-S^3$ means reverse embedded orientation. annulus

Lagrangian filling of Λ $\subset$ Filling $\Updownarrow$ Sheaves through microlocalization

$\Updownarrow$

Microlocal sheaves microlocalization on Λ

A-polynomial (mirror symmetry) Augmentation variety $\Downarrow$ vanishing

$\Downarrow$

A-variety — closed

$\Longleftrightarrow K_0 \# \overline{K_1}$ (mirror) is boundary of disk $D^2 \subset B^4$

We call $K_0 \# \overline{K_1}$ slice

K is ribbon if radical function $r : B^4 \to [0,1]$ has no local maximal

Conjecture A (Slice-Ribbon) Slice knot $\Rightarrow$ Ribbon knot, The Heegaard Floer τ-invariant $\tau \leq g_4 \leq g$ of K.

Definition B. (Fibred knot) If $S^3 - K \to S^1$ is fibration $\Leftrightarrow K$ is the binding of open book.

Thm C. (Hedden, 2005) K supports "tight" contact structure $\Rightarrow \tau = g_4 = g$.

Here our contact structure is tight means not overtwisted, $\exists$ overtwisted disk D^2 given by $D^2 \cap \xi$, $\xi = \ker\beta = \frac{\partial}{\partial z}$, $r\tan(\frac{\partial}{\partial z} \frac{\partial}{\partial \theta})$ locally

it's not well-defined until Prop D.

Prop D. (Eliashberg, 1989) Homotopy class of 2-planes $\leftrightarrow$ isotopy class of overtwisted contact structure

Prop E. (Giroux, 2020) isotopy class of contact structure $\leftrightarrow$ open book / positive stabilization (No complete proof written down) (Σ, φ) open book

Prop F. (Baker, Etnyre, VanHorn-Morris, 2011) tight $\Leftarrow$ weakly symplectic fillable $\Leftarrow$ strong symplectic fillable $\Leftarrow$ stein $\Leftarrow$ Dehn(Σ)

(note the statement we use the change of Giroux) Rk. Our Wend's thm says planar $\Rightarrow$ last three equal.

Def G. (Right-veering; Honda-Kazez-Matić, 2005) $h: \Sigma \to \Sigma$, $h|_{\partial\Sigma} = id$, if $(h(\alpha), \alpha)$ determine orientation of Σ at x where $\alpha \subset \Sigma$ start at x

Thm H. (HKM, 2005) (M, ξ) tight $\Leftrightarrow$ all open books are right-veering. Pf. Using convex surface.

Heegaard decomposition of M is $M = H_1 \coprod H_2$ two Heegaard bodies given by $(\Sigma, h, K) \Rightarrow H_1 = \Sigma \times [0, \frac{1}{2}]$, $H_2 = \Sigma \times [\frac{1}{2}, 1]$, $\partial H_1 = \Sigma_{\frac{1}{2}} \coprod -\Sigma_0$, $\partial H_2 = -\Sigma_{\frac{1}{2}} \coprod \Sigma_1$ glued by $g: \partial H_2 \to \partial H_1 = id \coprod h$ (K is convexity condition, used to avoid over-twist) $\Rightarrow M = H_1 \bigcup_g H_2$

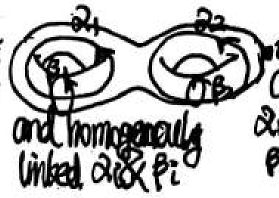
Prop I. After positive-stabilization, we always make open book to be right-veering.

Knot Floer homology Doubly pointed Heegaard diagram $(\Sigma_g, \alpha, \beta, w, z)$ where $\alpha = \{\alpha_1, \cdots \alpha_g\}$, $\beta = \{\beta_1, \cdots \beta_g\}$ simple closed curves on Σ_g and homogeneously linked α_i & β_i, $\alpha_i \cap \alpha_j = \emptyset$, $\beta_i \cap \beta_j = \emptyset$

and $w, z \in \Sigma_g$ are pointed "given α, β to H_α, H_β"

Concrete construction is change the Morse index

In our setting, the Heegaard diagram comes from



$g = 2$, $\alpha = (\alpha_1, \alpha_2) = (a_1 \cup a_1, \beta\ \ a_2 \cup a_2)$

$\beta = (\beta_1, \beta_2) = (b_1 \cup h(b_1), b_2 \cup h(b_2))$

where a_1, a_2, b_1, b_2 is graphed in picture, it's more clear written in $T^2 \# T^2$ as

K is fibred knot. $\Rightarrow$ we can define the Heegaard Floer homology

two Heegaard bodies $Sym^g(\Sigma)$ has $T_\alpha: \alpha_1 \times \cdots \times \alpha_g \to Sym^g(\Sigma)$, $T_\beta: \beta_1 \times \cdots \times \beta_g \to Sym^g(\Sigma)$

$\Rightarrow u: D^2 \to Sym^g(\Sigma)$ J-holomorphic curve, $x, y \in T_\alpha \cap T_\beta \Rightarrow$ with $\pi_2(x, y)$ = relative homology class of D^2

$\forall \varphi \in \pi_2(x, y)$, $\mathcal{M}(\varphi) = \{$J-holomorphic representative of $\varphi\}$

$\mu(\varphi)$ = Maslov index is $\dim \mathcal{M}(\varphi)$, by Gromov compactness $\Rightarrow \widehat{\mathcal{M}}(\varphi) = \mathcal{M}(\varphi)/\mathbb{R}$ is finite, $\#\widehat{\mathcal{M}}(\varphi)$ is used next as differential of "homology" via graded by Maslov grading $M: T_\alpha \cap T_\beta \to \mathbb{Z}$, s.t. $M(x) - M(y) = \mu(\varphi) - 2n_w(\varphi) = \mu(\varphi) - 2\#(\varphi \cap \{w\} \times Sym^{g-1}(\Sigma))$ & Alexander grading $A: T_\alpha \cap T_\beta \to \mathbb{Z}$, s.t. $A(x) - A(y) = n_z(\varphi) - n_w(\varphi) = \Delta$

chain level: $CFK^\infty(S^3, K)$ is free $\mathbb{F}_2[u, u^{-1}]$-module generated by $x \in T_\alpha \cap T_\beta \Leftrightarrow \mathbb{F}_2$-vector space spanned by $[x, i, j]$ with $x \in T_\alpha \cap T_\beta$ as well. Then counting u gives differential $\partial^\infty [x, i, j] = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\varphi \in \pi_2(x,y), \mu(\varphi)=1} \#\widehat{\mathcal{M}}(\varphi) [y, i - n_w(\varphi), j - n_z(\varphi)]$ (mod 2)

$\Rightarrow$ bicomplex $\mathbb{Z} \oplus \mathbb{Z}$ filtered & this bicomplex is homotopical type is a knot invariant

taking the filtration and homology $\Rightarrow H_*(\partial^\infty) = HF^\infty(S^3) = \mathbb{F}_2[u, u^{-1}]$

② taking homology for $i = 0 \Rightarrow H_*(i=0) = \widehat{CF}(S^3)$; ③ $H_*(i = 0, \partial) = \widehat{HF}(S^3)$, with $\partial x = \sum_{y \in T_\alpha \cap T_\beta, n_w(\varphi)=0} \sum \#\widehat{\mathcal{M}}(\varphi) y$;

All of them are not depend on K; ④ (Heegaard Floer homology group) $H_*(\widehat{\partial})$, with $\partial x = \sum_{y \in T_\alpha \cap T_\beta, n_w(\varphi)=0, n_z(\varphi)=0} \sum \#\widehat{\mathcal{M}}(\varphi) y$ $= \widehat{HFK}(K)$

$\Rightarrow$ τ-invariant defined $\tau(K) := \min\{m \mid H(i=0, j \leq m) \to \widehat{HF}(S^3)$ nontrivial$\}$ in j-axis, and $v_t(K) := \min\{s \mid H_*(s \leq \frac{t}{2} i + (1 - \frac{t}{2}) j) \to HF^\infty(S^3)$ nontrivial$\}$, $t \in [0, 2]$ for vary the "slope" of filtration set $\Upsilon_K(t) := -2v_t(K)$ is a piecewise-linear function is a knot invariant, its slope correspond to the $\Delta(t)$ Alexander grading

Prop J. ① $\Upsilon_K(t)$ & $\tau(K)$ are concordant invariant; ② $\tau(\bar{K}) = -\tau(K)$, $\Upsilon_{\bar{K}}(t) = -\Upsilon_K(t)$; ③ $\Upsilon_K(t) = \Upsilon_K(2 - t)$; ④ $|\Upsilon_K(t)| \leq g(K)$

Prop K. ① $\widehat{HFK}(K) = \bigoplus_{-g \leq j \leq g} \widehat{HFK}(K, j)$, with ① $\widehat{HFK}(K, j) \cong \widehat{HFK}(K, -j)$; ② $\widehat{HFK}(K, j) = 0$ $|j| > g$ also 0; ③ rank $\widehat{HFK}(K, g) = 1$ iff K is fibred.

Thm L. (2020)

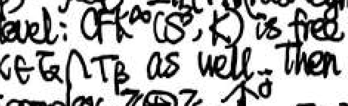
If $\exists t \in [0, 1]$, s.t. $\Upsilon_K(t) = g \Rightarrow \varphi$ is right-veering; (converse proven by Baldwin-Sek-Ni, 2023)

If K_1, K_2 ribbon concordant and $\boxed{\exists t \in [0, 1], \Upsilon_{K_1}(t) = g(K_1)}$ and $\Upsilon_{K_2} \leq g(K_2) \Rightarrow K_1 = K_2$ tight

(if $t_1, t_2 \in [0, 1]$, then (ribbon) concordant make it $t_1 = t_2$)

• Convex surface.

$\Sigma \subset (M^3, \xi)$, $x \in \Sigma$ is singular if $\xi_x = T_x M$, denote $l_x = \xi_x \cap T_x M$, the characteristic foliation Σ_ξ with each leaf



E.g. A. leaves ∂D^2 dividing set ∂D^2 are singular locus; leaves; 0 is singular; overtwisted D^2

Def B. $\Sigma \subset M$ proper embedded, it's convex if $\exists$ contact vector field v ($\mathcal{L}_v \alpha = g\alpha$, g is function), s.t. $v \pitchfork \Sigma$

boundary

Prop C. ① C^∞-generic closed embedded surface Σ is convex (after permutation); ② If with Legendrian, C^0-generic (after permutation)

The dividing set $\Gamma_\Sigma := \{x \in \Sigma \mid v(x) \in \xi_x\}$ satisfy ① Γ_Σ is 1-mfd; ② $\Gamma_\Sigma \pitchfork \Sigma_\xi$; ③ Γ_Σ not depend on v; ④ $\Sigma - \Gamma_\Sigma = R_+(\Gamma_\Sigma) \cup R_-(\Gamma_\Sigma)$

⑤ $\Gamma_\Sigma \neq \emptyset$; ⑥ compact $\partial \Gamma_\Sigma = \emptyset$, $\partial \Gamma_\Sigma \subset \partial \Sigma$

E.g. $D \subset (\mathbb{R}^3, \xi)$

leaves — dividing set

Thm E. (Giroux's criterion) $\begin{cases} \Sigma \neq S^2 \text{ has tight neiberhood} \Leftrightarrow \Gamma_\Sigma \text{ has no homotopically 0 dividing curves} \\ \Sigma = S^2 \text{ has tight neiber hood} \Leftrightarrow \#\pi_0(\Gamma_\Sigma) = 1 \text{ (i.e. component)} \end{cases}$

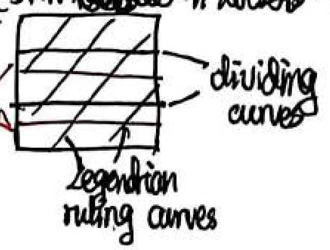
Thm F. (Eliashberg)

$\exists!$ tight contact structure on B^3 with boundary fixed contact structure, with $\#\pi_0(\Gamma_{\partial B}) = 1$.

We can see for S^2, things almost done, next consider torus T^2

$L \subset S^3$ Legendrian knot, $tb(L) = n$. it's standard neiberhood $N(L)$

s.t. (1) $T^2 \cong \partial N(L)$; (2) $\#\pi_0(\Gamma_{T^2}) = 2$; (3) slope$(\Gamma_{T^2}) = n$



slope $= \frac{q}{p}$ is $q\lambda + q\mu$ homologous to L itself i.e.

Thm G. ① (Kanda, Makar–Limanov) μ Solid torus $S^1 \times D^2$ and (1)(2)(3) boundary conditions $\Rightarrow \exists!$ tight contact structure on $S^1 \times D^2$ up to isotopy relative to boundary;

② (Thurston–Bennequin inequality) $tb(L) + |rot(L)| \le 2g(\Sigma) - 1$.

$\Rightarrow \overline{tb}(K) := \max\{tb(L) \mid L \text{ Legendrian of } K\}$ and $\overline{tb}(K) \le 2g(K) - 1$

The contact width $w(K) := \sup\{\text{slope}(\Gamma_{\partial S}) \mid S \text{ is a contact solid torus in of type } K, \partial S \text{ convex}, \Gamma_{\partial S}\}$ +1

(Etnyre–Honda)

Eg. H. For $T_{p,q}$, $(p,q)=1$ & $pq<0 \Rightarrow w(T_{p,q}) = \overline{tb}(T_{p,q}) = pq$ (by Lidman–Sivek)

Thm I. (Etnyre–Li–Tosun) K is L-space knot, $\overline{tb}(K) = 2g(K) - 1 \Rightarrow w(K) = \overline{tb}(K)$. Rk. It's conjectured that L-space knot $\Rightarrow \overline{tb}(K) = 2g - 1$

Def J. L-space knot if a surgery $S^3_r(K)$, $r>0$ is L-space, i.e. rank $\widehat{HF}(S^3_r(K)) = \# H_1(S^3_r(K))$

"Pf": Symplectic filling of L-space knot is negative definite.. ▨

Thm K. ([ELT]) ① $\frac{q}{p} < w(K)$, Knot Lagrangian slice $\Rightarrow \omega(K_{p,q}) = \overline{tb}(K_{p,q})$. $K_{p,q}$ is the cable knot, s.t. $p\lambda + q\mu$ cable on $\partial N(K)$, $p \neq 0$ and $(p,q)=1$.

② or $\frac{q}{p} \le \min\{w(K), -\frac{1}{2}\}$, K is Lagrangian slice

Def L. Lagrangian slice as K if K has Legendrian representative L. filled by Lagrangian disk (B^4, ω_{stand})

Coro M. K is L-space, $\overline{tb}(K) = 2g(K) - 1 \Rightarrow \omega(K_{p,q}) = \overline{tb}(K_{p,q})$

We know $\omega(K_{p,q}) \le pq$, is $\overline{tb}(K_{p,q}) \le pq$? It's disproved by Yasui, he showed $\exists K_{n,-1}$ s.t. $\overline{tb}(K_{n,-1}) > n$

we call $\overline{tb}(K_{p,q}) > pq$ is LLC property of K.

Prop N. (McCullough) K is LLC $\Rightarrow \exists$ a contact solid torus of type K, virtually overtwisted (tight but it's finite cover overtwisted, otherwise we call universally tight)

and Chakraborty – Etnyre – Min determine such virtually overtwisted neiberhood precisely.

Thm O. (Question in [CEM], Answered by [ELT]) LLC, K then

① K admits a Legendrian representative which bounds disk in (B^4, ω); ② $(p,q) = (n,-1)$

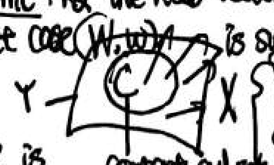
The proof depends on the concrete virtually overtwisted neiberhood in [CEM]

• Dynamics, nonlinear Maslov indexs via Floer cones. Here dynamics $\Leftrightarrow$ continuous is (Φ_t) Hamiltonian flow $\Leftrightarrow$ isotopy

Denote $Cont(Y,\xi)$ and $Cont_0(Y,\xi)$ the contactomorphism and identical component. $Cont(Y,\xi)$ with $\mathbb{R}$-action.

Reeb dynamic. R_α the Reeb vector field uniquely determined by $\begin{cases} \text{orthogonal}: d\alpha(R_\alpha, -) = 0 \\ \text{normalisation}: \alpha(R_\alpha) = 1 \end{cases}$, and R_α its flow

Non-compact case (W,ω) is symplectic



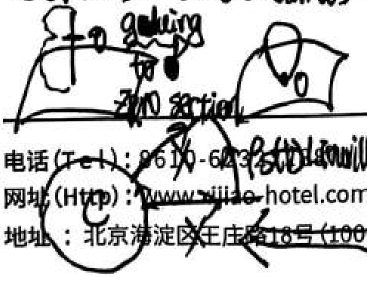
$X \begin{cases} \text{inside } C: \text{convex (transversal)} \\ \text{outside } C: \text{Liouville vector field} \end{cases}$ $\begin{cases} \text{denote its flow } \varphi_t \Rightarrow U = \{x \in W \mid \varphi_t(x) \to \infty\} \\ \Rightarrow \mathbb{R} \curvearrowright U \text{ by } s \mapsto \varphi_s(x) \end{cases}$ $\subset C$

Prop A. then $U/\mathbb{R}$ is turned out a compact contact manifold.

Here we need W is ideal bounded by Y

Eg. B. ① $(W,\omega) = (\mathbb{C}^n, \omega_{stand}) \Rightarrow U/\mathbb{R} \cong S^{2n-1}$, $X = \frac{1}{2}\sum x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}$

② $(W,\omega) = (T^*Q, \omega_{stand}) \Rightarrow U/\mathbb{R} \cong S^*Q$ the unit spherical bundle, $U = T^*Q - \Gamma(Q_0)$ the zero section



We can extension the isotopy from ideal boundary Y to W

Prop C. We have correspondenc

$\left\{\begin{array}{l}\text{Hamiltonian isotopy } (\Phi_t) \text{ on} \\ (W,\omega), \text{ outside some compact} \\ \text{set is Liouville flow-equivalence} \end{array}\right\} \Big/ \sim \text{agree in a compact set} \longleftrightarrow \left\{\begin{array}{l}\text{Contact isotopy } (\varphi_t) \\ \text{on the ideal boundary} \\ (Y,\xi)\end{array}\right\}$

(Φ_t) Liouville

For dynamics, we consider the fixed points of a ρ_s-equivariant Hamiltonian dynamics (ρ_s is the Liouville flow) (Φ_t) is Hamiltonian isotopy, ρ_s-equi outsider $K \subset W \Rightarrow$ its fixed points has two types { isolated. I / "line". II. We prefer this type (isolated & compact is easy)

by the equivariant $\Phi_1 \circ \rho_s(x) = \rho_s \circ \Phi_1(x) = \rho_s(x)$ not depend on (Φ_t)

$Fix_{type\,II}(\Phi_1) \subset U$, thus $Disc(\varphi_t) := Fix_{II}(\Phi_1)/\mathbb{R}$ called the discriminant points of φ_1

Eg.D. $\varphi = \varphi_1$ fixed, then $(\varphi_t = R_2^{st} \circ \varphi)_t$, $s \in \mathbb{R}$ fixed
$\Rightarrow Disc(\varphi)$ is called the translated point with time shift.
It's the period orbits in the Weinstein conjecture.

Flexibility v.s. Rigidity

Thm E. (S^{2n-1}, ξ_{stand}). $\exists\, \varphi = \varphi_1$ contactomorphism, not admit any translated points. $(n \geq 2)$

(Rk. It's rigidity, and in fact one shown $\exists$ infinite many φ admit translated points flexibly)

Thus $\Sigma \subset Cont(Y, \xi)$ is all contactomorphism admit at least one discriminant point. Eg. G.

Eg. F. (Linear) $(W, \omega) = (\mathbb{C}^n, \omega_{std})$, $A \in Sp(n)$ is ρ_s-equi, $\rho_s(z) = e^{\frac{s}{2}} z$ (rotation)
$\Rightarrow Fix_I(A) = \{0\}$; $Fix_{II}(A) = \{e^{\frac{s}{2}} z \mid z \in Ker(A - I)\} \Rightarrow \Sigma = \{Ax = x \mid A \in Sp(2n)$ with eigen value $1\}$

more flexible

$\Sigma \subset Cont(S^{2n-1}, \xi)$

more rigid

$\Sigma \subset Cont(\mathbb{R}P^{2n-1}, \xi)$ are graph of the flow acting on the Σ

Due to we study the Fixed pts, thus we use Floer theory naturally: $HF(\Phi_t)$ and generalised to the filtered Morse homology $\{HF^{<\lambda}(\Phi_t)\}_{\lambda \in \mathbb{R}}$ with $\tau_{\lambda_1\lambda_2} : HF^{<\lambda_1}(\Phi_t) \to HF^{<\lambda_2}(\Phi_t)$, $\lambda_1 \leq \lambda_2$ is called a persisted module. $V = (\{V_\lambda\}_{\lambda \in \mathbb{R}}, \tau_{\lambda_1\lambda_2} : V_{\lambda_1} \to V_{\lambda_2})$ is indeed a "quiver", can be decomposed similar as $\mathbb{Z}$-graded case.

persistence module $\Leftarrow$ the presistent homological analogue of Morse theory, thus one can view taking filtration of space W

$(\hookrightarrow \; ||||||||| \to \mathbb{R}$.

Prop G. $V = \bigoplus \mathbb{K}_{[a_i, b_i)}$ unique up to ordering of $([a_i, b_i))$'s

V as $|||||||||||$ a multi-set of interval.

Such convention is by [Polterovich, 2017]

$\emptyset = W_0 \subset W_1 \subset \cdots \subset W_n = W$
$\Rightarrow H_p(W_i) \to H_p(W_j)$'s image is called persistent homology, and extend from $p \in \mathbb{Z}$ to $\mathbb{R}$

Rk. $\mathbb{K}_{[a,b)}$ looks like
$0 \;\; [||||||||) \;\; 0$
it's a called interval-type persisted module $\mathbb{K}_{[a,b)}$

And by the presisted Floer homology group, its mapping cone is call the Floer cone homology.

Appendix.

④ (D-contracting) $\forall B(x,r), B(x,r)\cap\gamma=\emptyset \Rightarrow \operatorname{diam}\pi_\gamma(B(x,r))<D$

All the 5 equivalent definition of δ-hyperbolicity is hard to prove (planar Geometry tech)

Lecture 4

Groups acting on Hyperbolic spaces

Weng Yuan Yang

Slogan. Gromov's Hyperbolic Geometry $\longleftrightarrow$ Tree's Geometry

Why? Consider $\mathbb{D}\subset\mathbb{R}^2$ the Poincaré disk embedded in $\mathbb{R}^2$, and small disk tangent in Euclid sense.

Area of the small disk $\le Ce^{R}$ $\Rightarrow$ R can be small

Poincaré's disk model, triangles

③ (Tree) $\forall x,y,z$ $\exists$ triangle comparison map

② $\forall x,y,z, \exists o$, s.t. $d(o,x),d(o,y),d(o,z)<\delta$

Def1. (Hyperbolic space) (X,d) is geodesic metric space, $\delta\ge0$, X is δ-hyperbolic if any geodesic triangle Δ is δ-thin (i.e. any side of Δ is in δ-neiberhood of the other)

A δ-thin triangle Δ has δ-center is point $\in\cap(\delta$-neiberhood of all side); in hyperbolic it must exists.

s.t. $\forall u,v$, $d(\bar{u},\bar{v})-d(u,v)\le\delta$

Eg1. ① $\{pt\}$ is hyperbolic; ② Tree is hyperbolic; ③ M Riemannian, $\sec_M\le-1 \Rightarrow \tilde{M}$ is hyperbolic

Gromov ideas

Def2. (Geodesic) Geodesic is shortest path under metric d; ($\delta=0$, usually δ can be very large (as $e^{\square}$ order)) (λ,C)-quasi-geodesic ($\lambda\ge1$, $C\ge0$) α if $\forall\beta\subset\alpha$, $|\beta|\le\lambda d(\beta_-,\beta_+)+C$

④ A space holds Morse lemma.

$(l(\gamma)=\sup\sum d(x_i,x_{i+1}))$ with same endding points (can be ∞)

Lemma1. (Morse) α,β are two (λ,C)-quasi-geodesic in (X,d), δ-hyperbolic, $\exists D$ depending on λ,C,δ, s.t. $\alpha\subset B(\beta,D)$, $\beta\subset B(\alpha,D)$

⑤ Any geodesic is D-contracting (D fixed).

i.e. $d_H(\alpha,\beta)\le D<\infty$

Eg2. ① Connect two geodesices is $(1,C)$-quasi-geodesic

② it's a $(3,0)$-geodesic

③ In $\mathbb{R}^2$, let $h\to\infty$ $\Rightarrow$ Morse Lemma not holds in $\mathbb{R}^2$

$d(z_0,x)+d(x,p)\le l(\gamma|_{[p,z_0]})\le d(x,p)+d(p,z_0)\le 2d(p,z)+d(z,p)\le 3d(p,z)$ ☐

Pf of Morse Lemma. $\forall\gamma$ has same end points with α, $\exists C_0$, $|\gamma|>e^{\operatorname{Cd}(m,\gamma)}$ $\Rightarrow \exists C=\delta\log|\gamma|+1$, s.t. $\alpha\subset B(\gamma,C)$

(By $\log_2|\gamma|$ times partion on γ $\Rightarrow$ we can choose such C)

middle

- Taking γ is (λ,C)-quasi geodesic

$m\in\alpha$

$\Rightarrow$ we need determine $d(m,\gamma)<\infty$; assume $m\in\alpha$ has maximal $d(m,\gamma)$

$\beta\subset\gamma$ then has $\beta\cap B(m,R)=\emptyset$ due to triangle inequality

- Taking γ is β $\Rightarrow e^{C_0R}\le|\beta|\le 2R+\lambda R+C$ by β is (λ,C)-quasi-geodesic

$\Rightarrow e^{C_0R}<(2+6\lambda)R+C \Rightarrow R<\infty$ ☐

$\ge2R$, m, α, $2R$, $\hookrightarrow$ if $d(q,m)<2R$ take p simply

Def3. An action $\Phi: G\to \operatorname{Isom}(X,d)$ is called:

(i) Faithful if $\ker\Phi=1$; (ii) Effective if $\#\ker\Phi<\infty$; (iii) Free if $\operatorname{stab}_G(x)=1$;

(iv) Proper if $\forall K\subset X$ compact, $\#\{g\mid gK\cap K=\emptyset\}<\infty$; (v) Metrically proper if $\forall K\subset X$, $\operatorname{diam}(K)<\infty$ (bounded), $\#\{g\mid gK\cap K\}<\infty$

(vi) Cocompact if $\exists K\subset X$ compact, s.t. $G.K=X$; (vii) Cobounded if $\exists K\subset X$, $\operatorname{diam}(K)<\infty$, s.t. $G.K=X$

Rk. A metric space may not locally compact $\Rightarrow$ no enough compact sets $\Rightarrow$ modify (iv) & (vi) to (v) & (vii)

Def4. (Cayley graph) $\operatorname{Cay}(G,S)$ = vertices: $G\ni g$ is point $\bullet g$; $S\subset G$ is the generator set $S=S^{-1}$; edges: $G\times S\ni(g,s)$ $g \longrightarrow gs$, $g=gs s^{-1}$. Equipped the metric called word metric $d_S(g,h):=\min\{n\mid g^{-1}h=s_1\cdots s_n, s_j\in S\}$

word $\longleftrightarrow$ path & relation $\longleftrightarrow$ loop

Observation: ① $\operatorname{Cay}(G,S)$ connected ($g=s_1\cdots s_n$ connected e) ② $G\curvearrowright\operatorname{Cay}(G,S)$ is a faithful isometric action ③ $\deg(\operatorname{Cay}(G,S))=|S|$

Cay (Heisenberg) looks structure have: $t=aba^{-1}b^{-1}$

Eg3. $\mathbb{Z}$, $\{\pm1\}$ & $(\mathbb{Z},\{\pm2,\pm3\})$ $\longrightarrow$ & $\cdots$ this motivate us to define quasi-isometry

Exercise. Draw the Cayley graph ① Heissenberg group $H=\langle a,b,t\mid[a,t]=[b,t]=1$ and $[a,b]=t\rangle$; ② $\langle a,t\mid tat^{-1}=a^2\rangle$, and See that its Cayley graph correspends to a hyperbolic plane

the lines identify two vertex

Lemma2. (Milnor-Švarc) $G\curvearrowright X$ is proper and cocompact, X is proper (closed ball $\Rightarrow$ compact) geodesic space ($\Leftrightarrow$ local compact & complete)

$\Rightarrow$ ① G is finitely generated by $S\subset G$, $\#S<\infty$ (can be replaced by cobounded) and $S=S^{-1}$

② Fix $o\in X$, $\exists\lambda\ge1$, $C\ge0$, s.t. $\operatorname{Cay}(G,S)\longrightarrow X$ is a (λ,C)-quasi-isometry, $g\mapsto g.o$

$ab=ba$; Embed in $\mathbb{R}^3$ is the four-dim volume growth

Coro. ① $\operatorname{Cay}(G,S_1)\simeq\operatorname{Cay}(G,S_2)$ quasi-isometry ② $H\le G$ has finite index, finite generated, and $\operatorname{Cay}(H,S)\simeq\operatorname{Cay}(G,S)$ are quasi-isometry

$\begin{cases}\frac{1}{\lambda}d(g_1,g_2)-C\le d(g_1o,g_2o)\le\lambda d(g_1,g_2)+C \cdots(1)\ (\lambda,C)\text{-quasi-embedding}\\ \exists R, B(G.o,R)=X\cdots(2)\end{cases}$

Pf. ① trivial; ② By $H\curvearrowright\operatorname{Cay}(G,S)$, and one check proper (easy) & cocompact to apply Milnor-Švarc lemma.

QI-graph the quaternion

Pf of Lemma 2. Set $R:=\mathrm{Diam}(K)<\infty \Rightarrow B(G.0,R)=X$ --- (2) holds;
and $\forall g$, we partion the path $0\to g$ by $0=x_0, x_1, \dots, x_j, x_{j+1}, \dots, x_n$, $\forall j$, $\exists g_j$, s.t. $d(x_j, g_j)\leq R$.
writing $g=g_n=g_0(g_0^{-1}g_1)(\dots)(g_{n-1}^{-1}g_n)$ and taking $g_n=g$, g_0
$\Rightarrow g\in\langle S\rangle$, then we prove the (1) for quasi-isometry $\in S$ $\Leftarrow$ $d(0,g_j^{-1}g_{j+1}0)=d(g_j, g_{j+1})<2R+1$.
Upperbound: taking $\lambda=\max\{d(0,s0)\mid s\in S\}$ and $C=0$, later we let C bigger
Lower bound: $d(e,g)\leq n<d(0,g.0)+1$
thus it's a $(\max\{d(0,s.0)\mid s\in S\},1)$-quasi-isometry ☒

Hyperbolic group $\subset$ Relative Hyperbolic group $\subset$ Acylindrically Hyperbolic group
(Uniform Lattice) (Lattice) (Mapping class group)

Def 5. (Hyperbolic) G is Gromov hyperbolic if $G\curvearrowright(X,d)$ Geometrically (proper & cocompact), where (X,d) is δ-hyperbolic
$\Leftrightarrow \forall S$, $\mathrm{Cay}(G,S)$ is Gromov hyperbolic

Rk. ① The equivalence is by Milnor–Svarc Lemma and Gromov-hyperbolicity is invariant under quasi-isometry on geodesic spaces (this is by the Morse Lemma, pullback the δ-neiberhood of image of quasi-embedding; ② cocompact it essential due any finite generated group acting on (pt,d) is proper.

Prop 1. ① Hyperbolic is finite presented, when it's torsion-free, it acts on contractible space;
② $\mathrm{Cay}(G_1,S_1)\cong\mathrm{Cay}(G_2,S_2)$, where it's quasi-isometry $\Rightarrow G_1\cong G_2$; ③ Given two hyperbolic group, we can determine it's isomorphic or not;
Given two words, we can determine it's conjugate, ~~or~~ equal or not. Pf. Omitted ☒

Thm 1. (Gromov) Generic finite generated group is hyperbolic. Here the generic means that $G=\langle S\mid R\rangle$, $\#S=k$, $\#R=l$
then $\dfrac{\#\{G \text{ hyperbolic}\}}{\#\{G\mid \#S=k,\#R=l\ \&\ \forall r\in R, l(r)\leq n\}}\xrightarrow{n\to\infty}1$

Eg 4. $\mathbb{Z}^2$ isn't hyperbolic: $\mathbb{Z}^2\curvearrowright\mathbb{R}^2$ isn't proper.

Conjecture 1. ① Is there $H\leq G$, $\forall G$ hyperbolic, H finite index, s.t. H is torsion-free? (Virtually torsion-free)
② $\partial X:=\{\text{geodesic}\}/\sim$, $\alpha\sim\beta\Leftrightarrow\exists R$, s.t. $B(\alpha,R)\supset\beta$, $B(\beta,R)\supset\alpha$
the Gromov boundary with the fellow-travel topology, $[\alpha_n]\in\partial X$, $\downarrow n\to\infty$, $[\alpha_0]\in\partial X$ fellow time; then ∂X compact if X is proper.
∂: Met $\to$ Top is a functor preserve quasi-isometry to homeomorphism and quasi-embedding to embedding
$\bar{X}:=X\sqcup\partial X$ compactification, s.t. $X\subset\bar{X}$ dense and open
∂X, $[\gamma]$ Topology neiberhood $\{[\eta]\mid d(\gamma(n),\eta(n))<C\}$, $\mathcal{B}=\{B([\gamma],k,C)\}$ (n fixed)
∂: Grp $\xrightarrow{\partial\mathrm{Cay}}$ Top, we have $\partial G:=\partial\mathrm{Cay}(G,S)$. and $G\curvearrowright\partial G$ is a homeomorphic action.
Then if $\partial G=S^2$, G is virtually uniform lattice in $\mathbb{H}^3$? (Cannon) It's shown S^n ($n\geq3$) not holds. called the Kleinian group

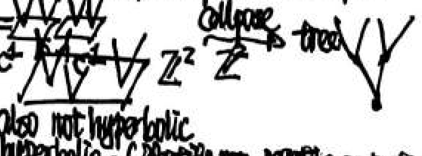
Eg 5. ① ∂F_2 = Cantor set; ② $\partial\pi_1(\Sigma_g)=S^1$, and a theorem by Tukia/Gabai conversely if $\partial G=S^1\Rightarrow G$ is virtually Σ_g ($g\geq2$)
(Virtually means that finite index $H\leq G$, H satisfy ...) topological basis of virtual metric
Exercise. If ∂G = Cantor set $\Rightarrow G$ is virtually F_2
It's induced by visual metric d_α, $\dfrac{a^{-d(x,y)_0}}{K(\lambda,S)}\leq d_\alpha(x,y)\leq\lambda\, a^{-d(x,y)_0}$
• Visualibity of Gromov boundary: $\forall x\neq y\in X$, $\exists$ geodesic joint x & y: ① ② ③
• then we can define $d(x,y)_0 < \infty$, and $d_0=\lambda^{-d(x,y)_0}$ not metric, using λ-tripod, projection to γ_j and $\gamma_j\to\gamma_0$ dense
A-A Lemma ☒

Rk. ① S^n is important due to for manifold, all $\partial X=S^n$, and ② the Cannon conjecture is part of Thurston Geometrization theorem (proven by Perelman) (2007)
the fractal X can be not sphere;

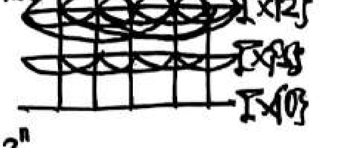
Relative hyperbolic group (Bowditch)
Set M^n has finite volume hyperbolic manifold, the thick-thin decomposition of M gives thick (compact) $\neq$ the $G\curvearrowright M$'s Margulis-thin, for curvature negative, via combinational horoball.
its universal covering and $G\curvearrowright\mathbb{H}^n-\bigsqcup_{\text{all horoballs}}B_i$ is cocompact, then this action (in particular $n=2$), it occurs part of $\mathbb{Z}^2$ $\Rightarrow \pi_1(M)$ isn't hyperbolic, but it still relatively hyperbolic

At first, there're horo boundary $\cong\mathbb{R}^{n-1}$, horosphere $\cong$ hyperplane, glue interior $\subset\mathbb{H}^n$, horoballs
and $\pi_1(M)\curvearrowright\mathbb{H}^n$ gives many horoballs gB_1, $g'B_1$, ..., gB_2, gB_3, ..., denote $G=\pi_1(M)$

Idea. Adding horoballs to turn into hyperbolic/collapse $\mathbb{Z}^2$-like part.
Eg 6. $G=\mathbb{Z}^2*\mathbb{Z}\Rightarrow\mathrm{Cay}(G,S)=$ $S=\{a^{\pm},b^{\pm},c^{\pm}\}$ $\mathbb{Z}^2$, $\mathbb{Z}^2$ collapse to tree



we need use some additional edge on $\mathbb{Z}^2$ to "make" it into hyperbolic: also not hyperbolic (identify non-negative part to pt)
Eg 6'. $=$ $\mathbb{Z}^2$ this "fills" the $\mathbb{Z}^2$ to a combinational horoball $\mathcal{H}(\mathbb{Z}^2)$, Boundary $=\mathbb{Z}^2$, and can be viewed as $\mathbb{H}^3$ hole
then $\Rightarrow$ hyperbolic

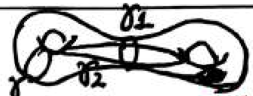


$\Gamma\times\{2\}$, $\Gamma\times\{1\}$, $\Gamma\times\{0\}$

Def 6. (Combinational horoball) Γ locally finite length, $\mathcal{H}(\Gamma)$ is a graph with vertex $\Gamma\times\mathbb{Z}_{\geq0}\ni(v,n)$, edge $\begin{cases}(v,n)-(v,n+1)\\(v,n)-(v',n)\text{ if } d(v,v')\leq 2^n\end{cases}$
Then $\mathcal{H}(\Gamma)$ is hyperbolic with $\partial\mathcal{H}(\Gamma)=\{pt\}$ (by 2^m are quasi-geodesic, all of them are R-thin triangle as there are 2^n length connecting by definition)
And $\mathrm{Cay}(G,S)\curvearrowright\mathcal{H}(\mathrm{Cay}(G,S))$ is isometry and proper

Def 7. (Relative hyperbolic group, informal!) $\mathrm{Cay}(G,S)\to X:=\mathrm{Cay}(G,S)\bigsqcup\mathcal{H}(g,\mathbb{Z}^2)$ is δ-hyperbolic and G is relative hyperbolic.
$X=X_{core}\sqcup X_{cusp}$, $G\curvearrowright X_{core}$ geometrically, and $G\curvearrowright X_{cusp}$ result all g the quotient X_{cusp}/G is degenerate to "tree" $\Leftarrow$ finite union of rays.

Acylindrically Hyperbolic group We only introduce case of Mapping class group, useful in lower-dim topology ($\mathrm{Mod}(\Sigma_g)=\pi_0(\mathrm{Homeo}(\Sigma_g))$)
$\mathrm{Mod}(\Sigma_g)\curvearrowright\mathrm{Teich}(\Sigma_g)$ (Teichmüller space of Σ_g) is geodesic, proper, but not hyperbolic:
Thm 2. (Minsky) $\mathrm{Teich}(\Sigma_g)=T^{\geq\varepsilon_0}\sqcup T^{<\varepsilon_0}$ the thick-thin decomposition ($\geq\varepsilon_0$ means injective radius), $\mathrm{Mod}(\Sigma_g)\curvearrowright T^{\geq\varepsilon_0}$ geometrically, and $T^{<\varepsilon}$ is quasi-isometry to $\mathbb{T}(S_1\times\mathbb{T}_{S_2})\Rightarrow$ not hyperbolic ($\to\ \to\ S_1\times S_2$)

Again we identify T^{∞} into one point; consider $\{T^{\infty}(\gamma) \mid \gamma$ simple closed curve on $\Sigma_g\}$ 

Caratheodory's Lemma gives that $|\gamma_1| \to 0$ & $|\gamma_2| \to \infty \Rightarrow T^{\infty}(\gamma_1) \cap T^{\infty}(\gamma_2) \neq \emptyset$

such γ forms space. $\gamma_1 \cap \gamma_2 = \emptyset$ but γ can intersection both γ_1 and γ_2 in Teichmüller space. It looks like 

denoted $\mathcal{C}(\Sigma_g) \subset \mathrm{Teich}(\Sigma_g)$, and $\mathcal{C}(\Sigma_g)$ is hyperbolic by a theorem of Masur–Minsky

Thm3. (Masur–Minsky) $\mathcal{C}(\Sigma_g)$ is δ-hyperbolic and ① $\mathrm{Mod}(\Sigma_g) \curvearrowright \mathcal{C}(\Sigma_g)$ is $\boxed{\text{Lox}}$, $g \in \mathrm{Mod}(\Sigma_g)$ has coarse fixed point) loxodromic WPD

② $\forall \varepsilon, \exists R, N$, s.t. $\# \mathrm{Stab}(x \overset{\geq R}{\longleftrightarrow} y) \leq N$; $\mathrm{diam}(x), \mathrm{diam}(y) < \varepsilon$, this is why it's called Acylindrically (Bowditch)

Ref. Osin. Acylindrically hyperbolic groups. 2018 ICM] (Rk. Here Acylindrically is generalisation of relative due to: $\mathrm{Teich}(\Sigma_g)$ can't be filled by horoballs due to the intersections of "holes" are complex )

(We consider $G \curvearrowright \partial G$ and Cannon conjecture next)

- Convergence group action: $G \curvearrowright M$ compact is convergence if $G \curvearrowright \mathrm{Conf}(M,3)$ is proper ($\mathrm{Conf}(M,3) = M^3 - \Delta$), it has good dynamic property

Lemma A. $f: X \to X$ quasi-symmetric $\Rightarrow \partial f: \partial X \to \partial X$ quasi-conformal.
$\left(\frac{d(f(x),f(a))}{d(f(x),f(b))} < g(t) \Rightarrow \frac{d(x,a)}{d(x,b)} < t\right)$ (Same as complex analysis

$\Lambda G := \{$limit of orbits of G in $M\} \subset M$, one restrict $G \curvearrowright M$ to $G \curvearrowright \Lambda G$ is a minimal action.

Def A. $x \in \Lambda G$ is called conical point if $\exists g \in G$ hyperbolic element as an attractor ($\exists g_n, z, g_n z \to x$)

It's called $\begin{cases} \text{parabolic}: \mathrm{Stab}(x) \text{ is parabolic elements} \Rightarrow x \text{ also parabolic} \\ \text{hyperbolic}: \mathrm{Stab}(x) \curvearrowright M - \{x\} \text{ cocompact} \end{cases}$

$G \curvearrowright M$ is called geometric finite if $\Lambda G = \{\text{conical}\} \cup \{\text{parabolic}\}$

Eg A. When M is complete hyperbolic manifold with finite volume $\Rightarrow \pi_1(M) \curvearrowright M$ is geometric finite

Lemma B. $\{g_n\} \subset G$, $(a,b,c) \in \mathrm{Conf}(M,3)$, $g_n(a,b,c) \xrightarrow{n\to\infty} (x,y,z) \Rightarrow (x,y,z) \in \mathrm{Conf}(M,3)$

Pf. Trivial, otherwise $g_n(x,y,z) = (x,y,z)$, $\forall n$ ▨

Coro B.1. $g_n(a,b,c) \xrightarrow{n\to\infty} (x,x,z) \Rightarrow \forall d \neq a,b,c$, $g_n \cdot d \to x$ ▨

Coro B.2. $g \in G$ infinite order and $g_n := g^n$ $\Rightarrow g$ has ≤ 2 fix points $\begin{cases} \text{elliptic} & \text{all finite order} \\ \text{parabolic} & \text{one finite \& one infinite} \\ \text{hyperbolic} & \text{all infinite order} \end{cases}$

Eg B. For $M = \mathbb{H}^n$, there is most two fix points x, y $\exists g \in G$ hyperbolic element (all finite order of x & y) with $g^n x \to y$, in dynamic systems, x is called the repeller and attracter

Thm A. G hyperbolic/relative hyperbolic $\Rightarrow G \curvearrowright \partial G$ is convergence group action

- Thm C (Bowditch) $G \curvearrowright M$ convergence, $M = \{\text{conical points}\} \Rightarrow$ ① G is hyperbolic, ② $\partial G \xrightarrow{\cong} M$ is G-equi homeomorphism

Thm D. (Yaman) $G \curvearrowright M$ geometric finite $\Rightarrow$ ① G is relative hyperbolic $\boxed{\text{relative to } P \leq G}$ the maximal parabolic subgroup

② $\partial(G,P) \xrightarrow{\cong} M$ is G-equi homeomorphism.

(relative to $H \leq G$ if $\Gamma(G,H)$ is $\begin{cases} \delta\text{-hyperbolic} \\ \text{fine: edge belongs to finite cycle uniformly} \end{cases}$

$\Gamma(G,H)$ vertex are cosets and edge $g \to gt$ ($t \in H$), adding to $\Gamma(G)$)

- Other boundaries.

Def E. (End boundary) $\partial_e X := \varprojlim \pi_0(X - K)$ Def F. (Floyd boundary) $(\Gamma, 0)$ is graph with base point 0, $f: \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$, $\sum f(n) < \infty$

(for distinct we denote ∂_G Gromov boundary) $\partial_f G = \bar{G} - G$, $\bar{G}$ is compactification w.r.t $l(e) = f(d(0,e))$)

$\begin{cases} \# \partial_G G = 0 \text{ or } 2 \text{ or } \infty \\ \# \partial_e G = 0 \text{ or } 1 \text{ or } 2 \text{ or } \infty \\ \# \partial_f G = ? \text{ It's not well known now} \end{cases}$

Lecture 5

Symplectomorphism of 4-manifolds

Weiwei Wu

The Hamiltonian diffeomorphism is the Hamiltonian flow given by $-df = \omega(X_f, -)$, X_f the Hamiltonian field.

$Symp(X,\omega)$ is infinite-dimensional, rigidity (likes finite-dimensional) is less than $Isom(X)$ & $Aut(X)$ (holomorphic)

Thm (Gromov, 1985) notes the first rigidity of $Symp(X,\omega)$: $\boxed{Symp(S^2\times S^2, \omega\oplus\omega) \simeq SO(3)\times SO(3) \rtimes (\mathbb{Z}/2\mathbb{Z})}$ permutation of two S^2 section

Setting. $\mathcal{J}_\omega^c$, $\mathcal{J}_\omega^t$ to be compatible complex structure & tame one. J^l is C^l-smooth, $\mathcal{J}_\omega^l$ is C^{l}

$\mathcal{J}_\omega^c$ is contractible: view it as $\mathcal{J}_\omega \xrightarrow{\pi} X$ with fibre $J_\omega|_x \Rightarrow$ a fibre bundle contractible iff each fibres are ☒

We have the moduli space $\mathcal{M}(A,X;J) = \{u : S^2 \to X \mid \bar\partial_J u = 0, [u] = A \in H^2(X;\mathbb{Z})\}$, and compactification to a Banach manifold $\bar{\mathcal{M}}(A,X;J)$, $\mathcal{M}$ is constructed as moduli stack by: $\Lambda^{0,1}(S^2, u^*TX) = \mathcal{E}_u \dashrightarrow \mathcal{E}$ (universal family)

Adding all J together, we set $\mathcal{M}(A,X;\mathcal{J}^l) := \{(u,J) \mid \bar\partial_J u = 0, [u] = A\}$ and compactification to a bigger $\bar{\mathcal{M}}(A,X;\mathcal{J}^l) \xrightarrow[\text{projection } p]{\text{natural}} \mathcal{J}^l$ as a fibre bundle with each fibre at $J \in \mathcal{J}^l$ is the smaller one. $\mathcal{E} \xrightarrow{\pi} \mathcal{B} = \{u : S^2 \to X \mid \text{smooth}\}$ (choose it), $\bar\partial_J$

Linearization of $\bar\partial_J$ and p are denoted $D\bar\partial_J$ and Dp, they're Fredholm with the same index, and their surjectivity are equivalent.

$\Rightarrow \forall l$, $\mathcal{J}^{reg} \subset \mathcal{J}^l$ is second category (regular means that $D\bar\partial_J$ surjective), $J \in \mathcal{J}^{reg}$, $\dim \bar{\mathcal{M}}(A,X;J) = 2n + 2c_1(A) \ldots$

$J \in \mathcal{J}^{reg}$ is called transversal, they're generic (with some additional condition)

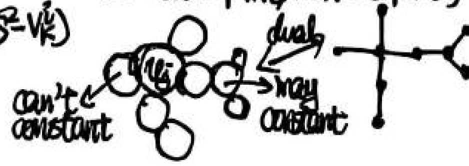
And $\bar{\mathcal{M}}(A,X;J_t)$ is the $p^{-1}(\{J_t\})$ also a manifold with boundary.

Eg.1 (Using intersection inequality) $u: S^2 \to S^2\times S^2$, $\bar\partial_J u = 0$, $\forall J$, $[u] = A \Rightarrow 2\cdot\delta(u)$ (singular term) $\geq c_1(A) - A^2 = 2 \Rightarrow \delta(u) = 0$

$\Rightarrow u$ is embedding (Chern does not depend on J, we can use standard one as $\mathbb{CP}^1$)

Gromov's compactness. Similar with algebraic geometry, we have stable curves (as Gromov–Hausdorff limit of a sequence of (u_t, J_t)) ~~stable~~

Def1 (Stable curves) ① Dual graph is a tree; ② $u = \{u_i\}$, $\sum [u_i] = u$, $\omega(u_i) = 0 \Rightarrow$ three intersection points/marked points; ③ $\exists t_j \to \infty$, $\forall i$, $\exists z_k^i \in V_k^i \subset S^2$, $k = 1 \cdots n_i$, $\gamma_i \in PSL(2,\mathbb{C})$, $\forall i$ $u_{t_j}(S^2 - V_k^i) \longrightarrow u_i(S^2 - V_k^i)$



Return back to Thm1. $(X,\omega) = (S^2\times S^2, \omega\oplus\omega)$, $A, B \in H_2(S^2\times S^2;\mathbb{Z})$ two generators

$\mathcal{M}_0(X,A;J) = \{u : \mathbb{P}^1 \to X \mid [u] = A, \bar\partial_J u = 0\}/PSL(2,\mathbb{C})$ & $\mathcal{M}_0(X,B;J)$

Lemma1. $\mathcal{M}_0(X,A;J) \cong S^2$ homeomorphism

Pf. Using intersection, we know that the $S^2 \subset S^2\times S^2$ are all "good" $|\,|\,|\,|$, i.e. $[S^2] = A \in \mathcal{M}_0(X,A;J_0)$, J_0 standard, and $\mathcal{M}_0(X,A;J_0)$ just consists these $[S^2] = A$ by unique continuous $\Rightarrow \mathcal{M}_0(X,A;J_0) \cong S^2$

Then for other J_t, taking path (contractibility) $J_0 \to J_1$, we have $\mathcal{M}_0(X,A;\{J_t\}) \longrightarrow [0,1]$ is fibre bundle

By intersection again, only ③ case is possible $\Rightarrow \mathcal{M}_0(X,A;J_0) = \mathcal{M}_0(X,A;J_t)$, $\forall t \in [0,1] \Rightarrow$ set-theoretic

Applying this to all A-curves $\Sigma \Rightarrow$ we have map $\pi : \mathcal{M}(X,A;J_1) \longrightarrow \Sigma$, by $(A.B) = 1$, positivity intersection $\Rightarrow$ well-defined π, and Gromov compactness $\Rightarrow \pi$ & π^{-1} are $\sigma \longmapsto \sigma \cap \Sigma$ both continuous ☒

Rk. Observe that here $\mathcal{J}_\omega = \mathcal{J}_\omega^{reg}$ by $\mathcal{J}_\omega$ has only one component by a stable argument, and adjunction formula $\Rightarrow A, B$ curves are embedding $\Rightarrow$ transverse $\Rightarrow \mathcal{J}_\omega^{reg} \supset \mathcal{J}_\omega$ ☒

Eg.2 (Gromov–Witten invariants) $\dim \mathcal{M}_0(X,A;J) = 2n + 2c_1(A) - \boxed{6}$, $2n = \dim X$

Cycles on X, $\beta_i \in H_*(X)$, A-curve through β_i ($n = 2$, it's $-\dim PSL(2,\mathbb{C})$) degenerate to point $p_i, \ldots$

$\Longleftrightarrow$ marked points $p_1 \cdots p_m \Rightarrow \dim \mathcal{M}_0(X,A;p_1 \cdots p_m;J) = \boxed{2n + 2c_1(A) - 2m - 6 = 0}$

$\Rightarrow \mathcal{M}_0(X,A;p_1 \cdots p_m;J)$ is discrete points and $\#\mathcal{M}_0(X,A;p_1 \cdots p_m;J) = GW(X,A,J,\beta_1 \cdots \beta_m)$ the Gromov–Witten invariant. Our proof upper is computation that $GW_0(S^2\times S^2) = 1$ (for A & B)

The stable curve w.r.t. to the [illegible] almost complex structure due to complex manifold = (strict) stable curve (A) lives ... although the stable ... in different sense G-H convergence Grothendieck topology ... here it's more used to stable maps, counting problem
$\Rightarrow$ finiteness of intersection
$\Rightarrow$ finiteness of symplectic energy
e.g. Gromov–Witten invariant on right.

Exercises. ① $X = \mathbb{CP}^2$, marked p_1, p_2 generic $\Rightarrow \dim \mathcal{M}_0(X,L;p_1,p_2;J) = 4 + 6 - 4 - 6 = 0$, compute it's G-W invariants $= 1$

② Replace ①'s L by $2L$, marked 5 points, compute it's G-W invariants $= 1$. (Imitate the arguement above). (one can also use $\mathcal{M}_0(X,\Delta;p_1 \cdots p_5;J)$ this case, Δ is the diagonal).

$Symp(X) \longrightarrow Aut_\omega^*(H_2(X),*) \subset Aut(H_2(X))$ preserves ω and intersection pair, then it's kernel $Symp_0(X) \subset Symp(X)$ is homological trivial.

Set $\mathcal{M} = \{(u,v,J) \mid \bar\partial_J u = \bar\partial_J v = 0, u_*[S^2] = A, v_*[S^2] = B, (p,q) \in u \cap v\}$, $G = PSL(2,\mathbb{C})$ now important: $Symp(X)$ identify as its subgroup $\Rightarrow G\times G \curvearrowright (u,v)$ naturally $\Rightarrow G\times G \curvearrowright \mathcal{M} \Rightarrow SO(3)\times SO(3) \hookrightarrow Symp_0(X) \longrightarrow Symp_0(X)/SO(3)\times SO(3)$

where τ is constructed by $(p', q') := \psi^{-1}(p,q)$, $\psi \in Symp_0(X)$

$\tau(\psi) = (\psi\circ\alpha, \psi\circ\beta, \psi_* J_0)$, $\alpha(x) = (x, q')$, $\beta(y) = (p', y)$

$0 \to G\times G \xrightarrow{\text{reparameterization}} \mathcal{M} \longrightarrow \mathcal{M}/G\times G \to 0$

By exactness, at last we use semi-direct product $\rtimes (\mathbb{Z}/2\mathbb{Z})$

Denote $\tau \circ \iota(SO(3)\times SO(3)) =: \mathcal{M}_0 \subset \mathcal{M}$

~~Exercises. $\mathcal{M}_0 \hookrightarrow \mathcal{M}$ is weak equivalence (i.e. $\pi_i(\mathcal{M}_0) \cong \pi_i(\mathcal{M})$), $\mathcal{M}_0 \cong SO(3)\times SO(3)$~~ ($\mathcal{M}_0 \to \mathcal{J}_\omega$ is a Serre fibration)

Prop 1. (McDuff–Salamon) $Symp(X)$ is cellular complex (infinite), thus weak equivalence $\Rightarrow$ homotopical equivalence

This $\mathcal{M}_0 \simeq \mathcal{M}$, if we can construct it's homotopical inverse of τ $\Rightarrow Symp_0(X) \simeq \mathcal{M} \simeq \mathcal{M}_0 \simeq SO(3)\times SO(3)$

Construction of inverse $\mathcal{F}: \mathcal{M} \to Symp_0(X)$, set $\varphi: S^2\times S^2 \to S^2\times S^2$, $(z_1,z_2) \mapsto \varphi(z_1,z_2)$. However, generally $\varphi \notin Symp(X)$ but $\varphi \in Diff(X)$

Using $\boxed{\text{Moser technique}}$ $(\omega, J) \mapsto \boxed{\psi}$

$[\omega_t]$ with $\frac{d}{dt}[\omega_t]=0 \Rightarrow \exists (\psi_t) \in Diff(X)\times I$, s.t. $\omega_t = \psi_t^* \omega_0$ (By flow X_t, $\frac{d}{dt}\omega_t = d\lambda_t$, $\lambda_t = \iota(X_t, -)$) (*)

then for φ, set $\omega_t = (1-t)\omega_0 + t\varphi^*\omega_0 \Rightarrow$ $\begin{cases} \omega_t \text{ is symplectic; the coordinate curves } (a,z) \,\&\, (z,b) \text{ are } J\text{-holomorphic} \\ \frac{d}{dt}[\omega_t]=0 \text{ trivial} \end{cases}$

$\boxed{J \in \mathcal{J}_{\omega_0} \Rightarrow J \in \mathcal{J}_{\varphi^*\omega_0}}$ $\Rightarrow (a,z)\,\&\,(z,b)$ are $\varphi^*(J)$-holomorphic $\Rightarrow \omega_t$ is symplectic as it $\subset$ a convex set

$\Rightarrow$ Moser can retract $\mathcal{M} \to Diff(X)$ to $Symp(X) \Rightarrow$ desired $\psi \in Symp_0(X)$, thus

$Symp_0(X) \simeq SO(3)\times SO(3)$ homotopical equivalence.

(contract is the λ_t, λ_t's space is contractible (Note that ω_t's space isn't contractible), thus well-defined)

At last adding "$\times \mathbb{Z}/2\mathbb{Z}$", we complete the proof of Thm 1 ▨

Prop 2. $Symp(S^2) \simeq SO(3)$ (proof easier & similiar)

Pf. $0 \to Diff_p^+(S^2) \to Diff^+(S^2) \to Conf^1(S^2) = S^2 \to 0$ ⋯(1)

(Fix p, stabilizer) $(\psi \mapsto \psi(p))$

and $Diff_0^+(S^2) \subset Diff_p^+(S^2)$ defined $:= \{\psi \in Diff_p^+(S^2) \mid \psi_* e_i = e_i$, e_i frame of $S^2\}$

$\Rightarrow 0 \to Diff_0^+(S^2) \to Diff_p^+(S^2) \to \mathcal{F}_p := \{\text{All frames positively oriented}\}$

(Smale) $Diff_0^+(S^2)$ contractible $(\psi \mapsto (\psi_* e_1, \psi_* e_2))$

$\Rightarrow Diff_p^+(S^2) \simeq \mathcal{F}_p \simeq SO(2)$ ⋯(2)

And return back to (1), $Diff_p^+(S^2) \hookrightarrow Diff^+(S^2) \to Conf(S^2)$

Constructing homotopical inverse of i, saying $\exists f$ is easy

$$\begin{array}{ccc} Diff_p^+(S^2) & \hookrightarrow & Diff^+(S^2) \\ & & \uparrow i \\ SO(2) & \overset{\iota}{\hookrightarrow} & SO(3) \end{array}$$

(Diff is very flexible)

$\Rightarrow Diff^+(S^2) \simeq SO(3)$, at last, using Moser's trick ▨

For general surface this argument holds due to here the ω the area = volume, taking $\varphi \in Diff^+(S^2)$, $\varphi^*\omega$; $S^2 \to \mathbb{R}$ is a fibre bundle with base contractible $\Rightarrow$ fibre bundle contractible $\Rightarrow$ all $\varphi \in Symp(S^2)$ ▨

Preserve volume $\not\Rightarrow$ Preserve area in higher dimensional, but contere example isn't easy to construct.

* Another application of Moser technique:

Thm 2. (Auroux) (X,ω) compact, (M_t) continuous symplectic isotopy $\Rightarrow \exists \Phi_t: X \to X$, s.t. $\Phi_0 = id$ and $\Phi_t(M_0) = M_t$

Pf. By Moser $\Rightarrow \exists (\rho_t: M_0 \to M_t)_t$ (It's proof is also Moser's trick)

By Weinstein symplectic neiborhood theorem $\Rightarrow \exists \psi_t: N_0 \to N_t$, each $M_t \subset N_t$; $\psi_t(M_0) = M_t$ extend ρ_t, we can further extend to $X \to X$

First we can allow ρ_t only diffeomorphic, then applying the Moser trick to "make" ρ_t into a symplectomorphism

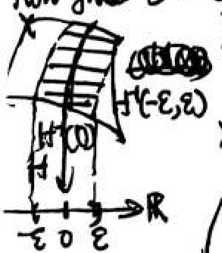
$\boxed{\text{Step 1}}$ Finding vector field ξ_t on X, s.t. $d(\iota_{\xi_t}\omega) = -\frac{d}{dt}(\rho_t^*\omega)$

$\boxed{\text{Step 2}}$ Then it's flow $\Psi_t: X \to X$, we can composite to $\Phi_t = \rho_t \circ \Psi_t$ is symplectomorphism ▨

Here details: ① ξ_t given by contact 1-form $\alpha_t = \iota_{\xi_t}\omega$ ② $\frac{d}{dt}(\Psi_t^*\rho_t^*\omega) = 0$ symplectic; ③ $\rho_t \circ \Psi_t(M_0) = M_t$. Need be checked

(Gromov told him this proof in his 20s)

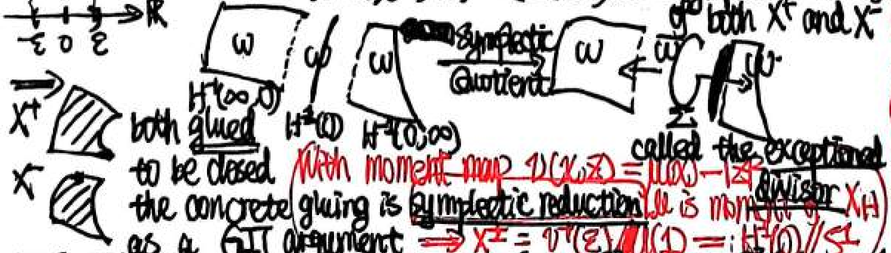
Symplectic cut $H: X \to \mathbb{R}$ Hamiltonian, $H^{-1}(-\varepsilon,\varepsilon) \cap \partial X = \emptyset$. X_H's Hamiltonian flow gives $S^1 \curvearrowright H^{-1}(-\varepsilon,\varepsilon)$, this action "cuts" X into $X^{\pm}$



X^+ $\begin{cases} (1)\ (H^{-1}(0,\infty), \omega) \\ (2)\ (H^{-1}(0)/S^1, \overline{\omega}): S^1 \curvearrowright H^{-1}(0) \text{ free} \Rightarrow H^{-1}(0)/S^1 \text{ also a manifold.} \end{cases}$

One can view $X^{\pm}$ as the compactification of their (1) and $\psi_0 = id$

X^- $\begin{cases} (1)\ (H^{-1}(-\infty,0), \omega) \\ (2)\ (H^{-1}(0)/S^1, \overline{\omega}) = (\Sigma, \overline{\omega}) \text{ is a symplectic submanifold of both } X^+ \text{ and } X^- \end{cases}$



both glued to be closed the concrete gluing is symplectic reduction as a GIT argument

called the exceptional divisor

With moment map $\mu(x,z) = \mu(x) - |z|^2$, μ is moment of X_H

$\Rightarrow X^{\pm} = \mu^{-1}(\varepsilon)/U(1) =: H^{-1}(0)//S^1$

Eg 3. ① $X = \mathbb{C}^2$ and $H = |z|^2 - 1 \Rightarrow H^{-1}(0) = S^3 \Rightarrow \Sigma = S^3/S^1 = \mathbb{CP}^1$ by Hopf fibration $\Rightarrow \begin{cases} X^+ = \mathbb{C}^2 \# Bl_0(\mathbb{CP}^2) \text{ (Blowing up to exceptional divisor } \Sigma\text{)} \\ X^- = \mathbb{CP}^2 \text{ (with infinite line } \Sigma \subset \mathbb{CP}^2\text{)} \end{cases}$

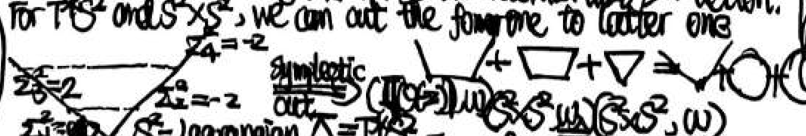
Rk. $H_k = |z|^2 - k \Rightarrow X^{\pm}$ not change topologically, but volume (area) differs, thus the "blowing-up" depends more the algebraic geometry, it blow-ups a ball

Generally $i: B^{2n}(k) \hookrightarrow X^{2n} \Rightarrow \partial(i(B^{2n}(k)))/S^1 = \mathbb{CP}^{n-1} =: Z_k$, if $\exists Z' \subset X$ (with (Weinstein) symplect neiborhood $U' \cong U_k$) $\Rightarrow$ replace Z' by $B^{2n}(k)$, called the symplectic blow-down, and we can glue back when $\dim = 4$, $S^2 \subset X$ symplectic embedding

② (T^*M, ω) Louville form $\lambda = p\,dq$ and $\omega = d\lambda = dp \wedge dq$, $H = |p|$ is Hamiltonian also S^1-action.

Consider $T^*S^2 \cong TS^2$ for example thus it's a S^1-action, vector of flow note depend on vector 2π time

For T^*S^2 and $S^2\times S^2$, we can cut the former one to latter one



$Z' \simeq S^2$, $\boxed{(Z', Z') = -1}$ enough (Castelnuovo theorem in AG)

Thm B (McDuff) $B^{2n} \hookrightarrow X^{2n}$ the embedding space is connected ($\exists$ Hamiltonian change any one to another)

Idea. $B_1 \Leftrightarrow B_2$. Cut both B_1 & B_2, and give $f: \Sigma_1 \to \Sigma_2$ symplectic of exceptional divisors and induction lemma ▨

Appendix (for (*))

General Moser's trick

Thm A. (Moser's stability thm) $\boxed{\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t = 0}$ admits solution $(X_t) \Rightarrow \exists (\psi_t)$, s.t. $\psi_t^*\omega_t = \omega_0$

Pf. $\frac{d}{dt}(\psi_t^*\omega_t) = \psi_t^*\left(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t\right) = 0$ $\Rightarrow$ taking ψ_t to be flow of X_t

$\Rightarrow \psi_t^*\omega_t \equiv C \Rightarrow \psi_t^*\omega_t = \psi_0^*\omega_0 = \omega_0$

Here (ω_t) only is differential forms!) ▨

Coro B. (Symplectic analogue) $\frac{d}{dt}\omega_t + d\sigma_t = 0$ admits solution $(\sigma_t) \Rightarrow \exists (\psi_t)$, s.t. ⋯

(Here ω_t is symplectic) ⋯(*)

Pf. $\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t = -d\sigma_t + \iota_{X_t}(d\omega_t) + d(\iota_{X_t}\omega_t)$ ($d\omega_t = 0$, Cartan's Magic formula)

$= d(-\sigma_t + \iota_{X_t}\omega_t)$

$0 = d(-\sigma_t + \iota_{X_t}\omega_t)$ is solvable?

Notice that $\begin{pmatrix} TM & \xrightarrow{\cong} & T^*M \\ X_t & \mapsto & \iota_{X_t}\omega_t \end{pmatrix} \Rightarrow \exists X_t$ ▨

Linear isotopy connecting makes sense almost everywhere, and this the only nontrivial condition is convexity (it's sometimes the only hard step)

Rk. For ellipsoid we believe it's also holds, but for polydisk, it's much harder.

Thm4. (McDuff-Opshtein) $\Sigma \subset X$ embedding, $\Sigma^2 \leq -1$, $a \in H_2(M;\mathbb{Z})$, (i) $GW(a) = 0$; (ii) $(a, E) \geq 0$ for $E \in \mathcal{E} - \{0\}$, $\mathcal{E} = \{E \in H_2 \mid E^2 = -1, (a, E) = 1, E$ can be represented as a embedding of $S^2\}$; (iii) $(a, \Sigma) \geq 0$

$\Rightarrow \exists$ generic $\mathcal{J}(\Sigma, a) \subset \mathcal{J}(\Sigma) := \{J \in \mathcal{J}_\omega \mid \Sigma$ is J-holomorphic$\}$, s.t. $J \in \mathcal{J}(\Sigma, a)$, a is represented as a embedding curve and $\mathcal{J}(\Sigma, a)$ connected.

Thm5. (Li-Liu) $GW(E) = 1$ for $\forall E \in \mathcal{E} \Rightarrow \forall J \in \mathcal{J}_\omega$, $\exists$ embedding J-curve represents E (here Li is Tian-Jun Li (not Jun Li) Liu is Ai-Ko Liu.) then we can glue S^2 to it to realise a blow-down.

Conjecture1. (Arnold's nearby Lagrangian) $\forall L \subset (T^*X, d\lambda = dp \wedge dq)$ is compact Lagrangian, $\lambda|_L = df_L$ exact Lagrangian $\Rightarrow \exists$ compact supported symplectomorphism Hamiltonian $\varphi_H : T^*X \to T^*X$, s.t. $\varphi_H(L) = \Gamma(0_X)$

It's only solved in $\mathbb{RP}^2$, S^2, T^2 cases, we show $\mathbb{RP}^2/S^2$ next.

Consider A, B-curves and symplect cut, in $S^2 \times S^2$ we have two symplectic form $\omega_0 = \omega_{stand} \oplus \omega_{stand} = A + B$ and $\omega_\lambda = A + (1+\lambda)B$

• When $\lambda \in (0,1)$, consider ω_λ instead of ω_0, the key difference is $\omega_\lambda \cdot (A - B) = \lambda > 0$, by intersection theory $(A \cdot A - B) < 0$ impossible $\Rightarrow A$ and $A - B$ can be both J-holomorphic, i.e. $\mathcal{J}(A) \cap \mathcal{J}(A-B) = \varphi$ in $\mathcal{J}_\omega$

Lemma2. $\mathcal{J}_\omega = \mathcal{J}(A) \sqcup \mathcal{J}(A-B)$; $\dim \mathcal{J}(A-B) = \dim \mathcal{J}_\omega - 2$

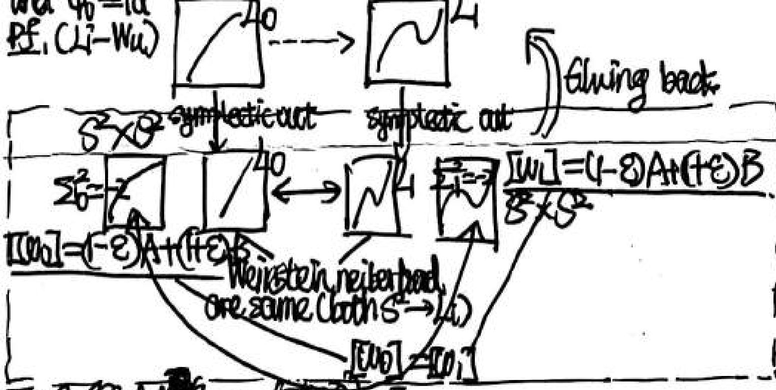
Pf. Latter one is Fredholm theory, we prove the former one: $GW(A) \neq 0 \Rightarrow A$ is represented as stable curve, $u = aA + bB$, by Adjunction formula $a(u) - [u]^2 \leq 2 \Rightarrow (a-1)(b-1) \geq 0$. By $\omega_\lambda(u) > 0 \Rightarrow a, b < 0$ is impossible, thus:

$\begin{cases} a, b \geq 1 \cdots (1) \\ a = 1, b \leq 0 \cdots (2) \\ a = 0, b = 1 \cdots (3) \end{cases}$ Using $\lambda < 1$, (2) sharpen to $\begin{cases} a = 1, b = -1 \cdots (2.1) \\ a = 1, b = 0 \cdots (2.2) \end{cases}$ thus $\begin{cases} A = A \text{ not bubble} \Leftrightarrow \mathcal{J}(A) \\ A = B + (A - B) \text{ bubble} \Leftrightarrow \mathcal{J}(A-B) \end{cases}$ ☒

Thm6. (Abreu-McDuff) $\mathcal{J}(A-B) \subset \mathcal{J}_\omega$ is connected codim2 Frechet manifold

Thm7. (Hind) Two Lagrangians $L_0, L_1 \subset S^2 \times S^2$, $[L_i] = A - B$ only class vanishing area $\Rightarrow \exists \varphi_t \in \mathrm{Symp}_0(S^2 \times S^2)$, s.t. $\varphi_1(L_0) = L_1$ and $\varphi_0 = id$

Pf. (Li-Wu)



Thus we got φ: $L_0 \longrightarrow L_1$, $\Sigma_1 \neq \varphi(\Sigma_0)$ sending L_0 to L_1, but not sending Σ_0 to Σ_1! Using symplectomorphism send $\Sigma_1 \to \varphi(\Sigma_0)$, this is by first diffeomorphism & Moser's trick, one can recheck the proof of Thm2.

$\Rightarrow$ finally we have $\varphi : X_0 \to X_1$ after gluing then we extend φ to φ_t, s.t. $\varphi_1 = \varphi$ by Moser's trick ☒

Thm8. (Siedel) $\mathrm{Symp}_c(T^*S^2) \simeq \mathbb{Z}$

Pf. (Similiar to Gromov's $S^2 \times S^2$) Consider fix $\Delta \subset S^2 \times S^2$, denote $\mathcal{S} = \mathrm{Symp}(S^2 \times S^2, \Delta) \simeq \{id, \iota : (x,y) \mapsto (y,x)\}$

Similiar to Gromov, $\exists \psi : (S^2 \times S^2, \omega_0) \to (S^2 \times S^2, \omega_0)$, but here we need restrict to Δ is 0, thus we need a relative Moser, anyway it always holds. $\psi(z_1, z_2)$, then $\alpha : \mathcal{S} \to \mathcal{J}_\Delta$ and $\beta : \mathcal{J}_\Delta \to \mathcal{S}$ are constructed to be homotopical equivalence by pullback & pushforward (z_1, z_2) via ψ, same as before $\Rightarrow \mathcal{J}_\Delta \simeq \mathcal{S} \simeq$ two pts

Modify $\mathcal{S}$ to $\mathcal{S}'$, $\mathcal{S}'$ fix an infinitesimal neiberhood of $\Delta \simeq \mathrm{Symp}_c(S^2 \times S^2 - \Delta) \simeq \mathrm{Symp}_c(T^*S^2)$ desired $(S^2 \times S^2 - \Delta = \mathrm{Conf}(S^2, 2))$ and $\mathcal{S}' \to \mathcal{S}$ is fibre bundle, each fibre is "can't rotation" (Δ), denoted $\mathrm{Map}(S^2, SL(2,\mathbb{R})) \simeq S^1$

$\Rightarrow 0 \to \mathcal{S}' \to \mathcal{S} \to \mathrm{Map} \to 0$, the homotopy sequence of fibre bundle is $0 \to \pi_1(S^1) \xrightarrow{i} \pi_0(S^1) \xrightarrow{\delta} \pi_0(\mathcal{S}) \to 0$

Claim. Only case is $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$. We show that:

$\exists \varphi \in \pi_0(\mathcal{S}')$, $\delta(\varphi) \neq 0$ and $\varphi^2 \in \mathrm{Im}(i)$. Such element is Lagrangian Dehn twist $\mathbb{Z} \longrightarrow \square \longrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$

It's generator of $\mathrm{Symp}_c(T^*S^2)$

and can be generalised to S^n, DT

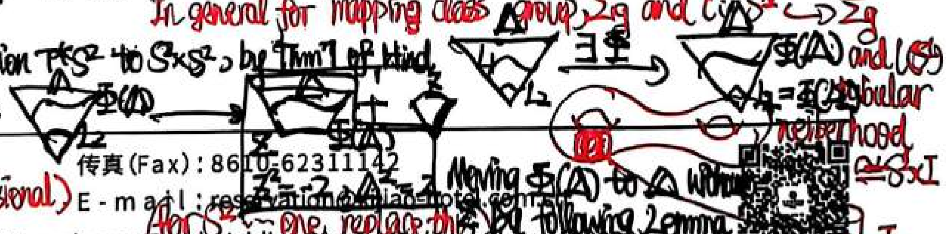
Here $\varphi = \tau_{S^2}$ ☒

$\infty > H$ thus τ_{S^2} is compact supported

Precisely, the definition of Dehn twist as here for T^*S^2 is defined by $(s, t) \mapsto (e^{2\pi i s t}, t)$ well (as the lecture on open books). In general for mapping class group Σ_g and $\iota : S^1 \hookrightarrow \Sigma_g$

Pf of Conjecture 1 in S^2. ① Compactification T^*S^2 to $S^2 \times S^2$, by Thm7 of Hind.

② Move back $\Phi(\Delta)$ to Δ; first cut off L_2



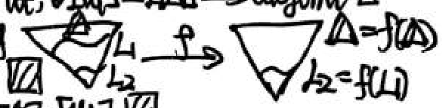
Dickerish twist τ_Δ generates $\mathrm{Map}(\Sigma_g)$ ($3g-1$-dimensional) $\Sigma / \simeq$ isotopy

(to $\iota : S^1 \to M$)

电话(Tel)：8610-62322288 传真(Fax)：8610-62311142
网址(http)：www.xijiao-hotel.com.cn E-mail：reservation@xijiao-hotel.com.cn
地址：北京海淀区王庄路18号(100083) / No.18 Wangzhuang Road, Haidian District, Beijing, China (100083)

<u>Lemma 3</u>. $p_1^1 \cdots p_5^1 \in \Delta$ and $p_1^2 \cdots p_5^2 \in \Delta$, and J_1, J_2 make (Δ, Z), $(\varphi(\Delta), Z)$ holomorphic $\Rightarrow \exists (J_t, p_i^t)$ connecting J_1 & J_2, s.t. Z is J_t-holomorphic and for p_i^t, $\exists !$ J_t-holomorphic curve u_t, $[u_t] = [\Delta] \Rightarrow$ disjoint Z

② Gluing. Now $\exists f \in \mathrm{Symp}_c(T^*S^2)$, s.t. $f(L_1) = L_2$ after gluing



and by $\mathrm{Symp}_c(T^*S^2) \simeq \mathbb{Z}_{\tau_{S^2}} \Rightarrow f \sim \tau_{S^2}^k$ and $\tau_{S^2}(O_{S^2}) = O_{S^2}$ ▨

<u>Pf of Lemma 3</u>. By <u>Mcduff–Opshtein</u> i.e. <u>Thm 4 & Thm 5</u>, fix $a = [\Delta] = [u_t]$ ▨

<u>Thm 9</u>. (B^4, ω_{stand}) has $\mathrm{Symp}_c(B^4)$ contractible. [Even's thesis]

Pf. (Similiar to <u>Siedel's thm 8</u>) $0 \to \mathrm{Stab}_0(C_1 \cup C_2) \to \mathrm{Symp}_0(S^2 \times S^2) \to \mathcal{C}(S^2 \times S^2) \to 0$ is a $\boxed{\text{fibration}}$, $\mathcal{C}(S^2 \times S^2) = \{C_1 \cup C_2 \mid u, v$ J-holo, $[u] = A$, $[v] = B$, $u(S^2) = C_1$, $v(S^2) = C_2$, $C_1 \pitchfork C_2$ in a pt $\}$

$\simeq$ "replace $C_1 \pitchfork C_2$ by $C_1 \perp_\omega C_2$" $\simeq \{(p, J) \mid p \in S^2 \times S^2, J \in \mathcal{J}_\omega\} \simeq S^2 \times S^2$ as $\mathcal{J}_\omega$ contractible

A-curve C_1, p_0, C_2 $\xrightarrow{u, v}$ C_1, p

$C_1 \cup C_2 \longleftarrow (p, J)$

$C_1 \cup C_2 \longmapsto (C_1 \cap C_2, J)$

B-curve

$$\begin{array}{ccccccccc} \Rightarrow 0 \to & \boxed{\mathrm{Stab}_0(C_1 \cup C_2)} & \to & \mathrm{Symp}_0(S^2 \times S^2) & \to & \mathcal{C} & \to 0 \\ & \| & & \| & & \| & \\ 0 \to & S^1 \times S^1 & \to & SO(3) \times SO(3) & \to & S^2 \times S^2 & \to 0 \end{array}$$

Such $0 \to S^1 \times S^1 \to SO(3) \times SO(3) \to S^2 \times S^2 \to 0$ is fibration by a consequence in 2002. ⋯ (1)

By rotation of curve C_1 and C_2

$\mathrm{Aut}(C_1 \cup C_2) = S^1 \times S^1 \Rightarrow \mathrm{Stab}_0 \to \mathrm{Aut}(C_1 \cup C_2)$ by restriction is a homotopical equivalence

$C_1 \pitchfork C_2$ (C_1, C_2, p_0, C_1, p_1, C_2, p_0)

$\Rightarrow 0 \to \mathrm{Stab}_1(C_1 \cup C_2) \to \mathrm{Stab}_0(C_1 \cup C_2) \to \mathrm{Aut}(C_1 \cup C_2) \to 0$ ⋯ (2) and $0 \to \mathrm{Stab}_2(C_1 \cup C_2) \to \mathrm{Stab}_1(C_1 \cup C_2) \to (S^1 \times S^1) \to 0$

Applying (contractible) the sequence of homotopy sequence to (1) $\Rightarrow$ (2) $\Rightarrow$ (3)

$\mathrm{Stab}_2 \cong \mathrm{Symp}_c(S^2 \times S^2 - (C_1 \cup C_2)) = \mathrm{Symp}_c(B^4)$ ⋯ (3)

$\Rightarrow \mathrm{Symp}_c(B^4) \simeq pt$ ▨

We explain the <u>J-inflaction</u> (occured in <u>Thm 3 by Mcduff</u>)

$X = \Sigma \times S^2$, $\omega_\mu = \mu \sigma_\Sigma + \sigma_{S^2}$, σ is the area element s.t. $\int \sigma = 1$.

$\Rightarrow G_\mu := \mathrm{Symp}(X, \omega_\mu) \cap \mathrm{Diff}_0(X)$ and $S_\mu := \{\omega \mid \omega \sim \omega_\mu$ isotopy$\} \Rightarrow G_\mu \to \mathrm{Diff}_0(X) \to S_\mu \to 0$

$\psi \mapsto \psi \omega_\mu = (\psi^{-1})^* \omega_\mu$

<u>Thm 10</u>. (Mcduff) $S_\mu \simeq \mathcal{J}_{\omega_\mu}$

Pf. $S_\mu \times \mathcal{J}_{\omega_\mu}$ (incidence correspondence)

$$S_\mu \xleftarrow{p_1} S_\mu \times \mathcal{J}_{\omega_\mu} \xrightarrow{p_2} \mathcal{J}_{\omega_\mu}$$

$\mathcal{J}_{\omega_\mu}$ contractible $\Rightarrow S_\mu \times \mathcal{J}_{\omega_\mu}$ contractible

p_1 has contractible fibre $\Rightarrow S_\mu \simeq \mathcal{J}_{\omega_\mu}$ also contractible ▨

(Donaldson's conjecture that "<u>tame-to-compatible</u>")

<u>tame</u> $\Rightarrow$ called "<u>tame-to-tame</u>"

$\nearrow$ this called "<u>compatible-to-tame</u>"

<u>Lemma 4</u>. (Inflation Lemma; Mcduff, Anjos, Li–Li–Pinsonnault) (X^4, ω_0), J <u>compatible</u> with ω_0, $\exists$ J-holomorphic curve Z with $\boxed{(Z, Z) \geq 0}$

$\Rightarrow \exists (\omega_\lambda)_{\lambda \geq 0}$ are symplectic forms, all tame J, $[\omega_\lambda] = [\omega_0] + \lambda Z$ ($PD[Z]$ is inverse of $[PD(Z)] = Z$) (When $(Z, Z) < 0$, it has a bound for $\lambda < \frac{\omega(Z)}{(Z, Z)}$)

In general standard textbooks such as Mcduff–Salamon.. one usually adding more additions on Z. • normal bundle ν_Z is J-invariant

• $b_2^+ = 1$

Pf. "It's quite hard to prove, <u>the preparation lemma</u>. $J \in \mathcal{J}_\omega^t$, we can isotope ω to ω', s.t. • $J \in \mathcal{J}_{\omega'}(X)$

• ν_Z trivial

It's a "tame-to-tame" consequence and locally "tame-to-compatible" $J|_Z \in \mathcal{J}_{\omega'}^c(Z)$

• $(Z, Z) \geq 0$

The proof of Lemma is technical and cost more arguments, omitted.

• $(Z, Z) < 0$

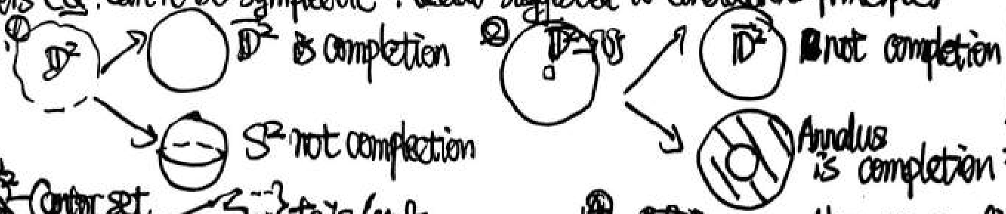
We turn it into linear algebra by setting $\langle v, w \rangle_J = \frac{1}{2}(\omega(v, Jw) + \omega(w, Jv))$ and the preparation Lemma is used in represent the Thom class of Z by a better one (thus replaced by ω') ▨

Next are two talks non-related with Lectures

• Compactification of manifolds (low dimensional topology and homotopy theory)

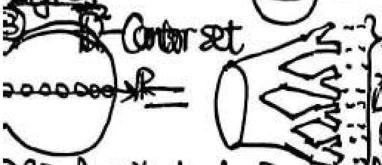
Def A. (Completable) M is completable if $\exists \hat{M}$ compact, boundary $\partial\hat{M} \neq \emptyset$, $\exists C \subset \partial\hat{M}$, s.t. $\hat{M} - C \cong M$ diffeomorphic or others (Q. Can it be symplectic? Gao suggested to consider h-principle)

E.g. B. ① $\mathbb{D}^2$ → $\bar{\mathbb{D}}^2$ is completion; → S^2 not completion. ② $\mathbb{D}^2 - \{0\}$ → $\bar{\mathbb{D}}^2$ not completion; → Annulus is completion $=$



Observation. Completion not change the topological of M?

E.g. C. ③ $\mathbb{D}^2$ − Cantor set, $\mathbb{R}$; tails/ends infinite, this method similiar to ② isn't hold! ④ $\mathbb{D}^2$ − Cantor also



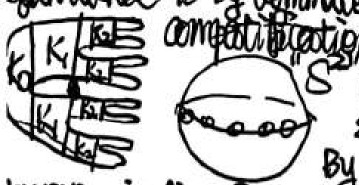
Here we can directly "see" $\bar{\mathbb{D}}^2$ is it's completion (filling Cantor set back directly)

In general, it holds for M^2, $\partial M \neq \emptyset$ $\Rightarrow$ $M^2 - E$, $E \subset \partial M$ a (complex enough) set then $M^2 - E$ completable

Def D. A neiberhood of an end in noncompact W is a subspace $U \subset W$ containing a component of $W - K$. Or equivalently infinite tails.

An end of W is an equivalence class of sequence of neiberhood of end $W \supset U_1 \supset U_2 \supset \cdots$ s.t. $\cap U_i = \emptyset$, equivalence is by dominate each other ($U_i \subset V_j \subset U_k$, $\forall j, \exists i, k$)

However, it's this "S^2" does S^2? It's the work of Kerékjártó in 1923 and Richard in 1963, and more generally



compactification 1 pt the ④ to S^2 with boundary's topology defined by sequence (similiar to Gromov's boundary). By Jordan's thm, it's topological sphere (Jordan–Schöenflis)

Thm E. (Gu–Guilbault, 2020) M^2 connected ($\partial M^2 = \emptyset$ or $\neq \emptyset$) completable $\Leftrightarrow$ $H_1(M^2)$ is finite generated. (locally ② & ④)

Pf. Locally as ②&④, and using classification of 2-manifold ☐

However, in dim ≥ 3, Jordan–Schöenflis not holds due to the Alexander horned sphere

Rk. $A \amalg_{S} B = \mathbb{R}^3$ the Alexander sphere & its complement Alexander ball.

(Bin) $AB \amalg_{\partial AB} AB = S^3$ is used in 4-Poincaré conjecture's proof.

Thm F. (Stalling, 1962) A contractible open n-manifold homeomorphic to $\mathbb{R}^n$ iff simply connected at ∞ (simply connected in ends) for $n \geq 5$

E.g. G. (Whitehead manifold) $M^3 = S^3 - \cap T_i$. (T_i are solid torus linked) is Whitehead manifold, it's contractible but not $\simeq \mathbb{R}^3$ ($\pi_1^\infty(M) \neq 0$). Intersecting this is $(\mathbb{R}^3 - M^3) \times \mathbb{R} \simeq \mathbb{R}^4$.

Rk. Thm F is Poincaré conjecture, we know it's right for $\forall n$.)

Thm H. (Tucker, 1974; Husch–Price, 1970; Kakimizu, 1987). M^3 completable iff each component of each clean neiberhood of ∞ ($\partial N \cap \partial M = \emptyset$) are π_1 is finite generated

Prop I. ① (Browder–Levine–Livesay, 1965) M is 1-ended, simply connected at ∞, completable $\Leftrightarrow$ $H_*(M)$ is finite generated

② (Siebenmann, 1965) M^n completable $\Leftrightarrow$ (i) pro-π_1-stable at each end. (ii) inward tame $\Leftrightarrow$ finitely dominated: $\simeq$ a finite polyhedron (iii) Wall obstruction $= 0$ (hard) when M^n compact/$\partial M \neq \emptyset$

Explain (i): Let $U_0 \supset V_0 \supset U_1 \supset V_1 \supset \cdots$ $\Rightarrow$ ladder $V_0 \leftarrow V_1 \leftarrow V_2 \leftarrow \cdots$ over $U_0 \leftarrow U_1 \leftarrow U_2 \cdots$ and $\pi_1(V_0) \leftarrow \pi_1(V_1) \leftarrow \cdots$, $\pi_1(U_0) \leftarrow \pi_1(U_1) \leftarrow \cdots$ we define the proisomorphic (pro-trivial if to $1 \leftarrow 1 \leftarrow$)

pro-stable if it is proisomorphic to $\pi_1(U) \leftarrow \pi_1(U) \leftarrow \cdots$. $\pi_1^\infty(M) = 1$ is slightly weaker than simply connected ∞

Thm J. (Wang, 2023) Open + complete positive scalar curvature + $\pi_1^\infty = 1 \Rightarrow M \simeq \mathbb{R}^3$

Conjecture K. (Yau) Open + complete positive scalar curvature $\Longrightarrow M \simeq \mathbb{R}^3$

Idea by Gu. Fox–Artin ray r, and neiberhood $N(r)$ open, $M := \mathbb{R}^3 - \mathrm{Int}(N) \not\simeq \mathbb{R}^3$ ... M completable and $\partial M \simeq \mathbb{R}^2$. But if $\mathrm{Double}_{\partial M}(M) := M \cup_{\partial M} M$ (completable and) complete positive scalar curvature $\Rightarrow$ Yau's conjecture not holds // ($\pi_1^\infty(\mathrm{Double}_{\partial M}(M))$ f.g. is obvious) (If complete PSC holds, it's helpful for completability)

Pf of Prop I. Core: h-cobordism: W with $\pi_1(M) = \pi_1(N) = 1$, $M \times [0,1] \simeq N \times [0,1] \simeq W$; then we use s-cobordism: adding $\pi_1 = 1$ to h-cobordism ☐; then it's purely AT.

applied to a collar end which has a multiplicative collar neiberhood structure. $\partial M \times [0, 1)$

Thm L. (Gu–Guilbault, 2020) M^n, $n \neq 4$ completable $\Leftrightarrow$ (i) pro-π_1 stable (ii) inward tame (iii) [illegible] (iv) $\mathrm{Wh}(\pi_1) = 0$

• $\mathbb{Z}_2$ harmonic 1-forms : connections in topology and geometry (Gauge, calibrated Geometry and representation theory) Page 31

(Taubes, 2012) $P \longleftarrow SU(2)$ M closed orientible

Set M is dim 2 or 3, $\downarrow$ a $SU(2)$-principal bundle with flat $SL(2,\mathbb{C})$-connection (A,φ) with A unitary connection on P and $\varphi \in \Omega^1(su(2))$, s.t. $\begin{cases} F_A - \varphi\wedge\varphi = 0 \\ d_A\varphi = 0, d_A * \varphi = 0 \end{cases}$ ... (*)

• $\exists\, \varphi \in \Omega(T^*M)$ s.t. $d\varphi = 0$

By decomposition $sl(2,\mathbb{C}) \cong su(2) \oplus su(2)$

Usually it's a φ-submanifold, $\varphi|_\Sigma = \lambda\, vol_\Sigma$, $\lambda \le 1$

$A \longmapsto A + i\varphi \Rightarrow F_A = 0$

The moduli $\mathcal{M} := \{\text{solution to } (*)\}/\sim \;\cong \{\rho : \pi_1(M) \to SL_2(\mathbb{C})\}/\sim$ isn't compact, our care about it's compactification via Seiberg-Witten due to it has good compactification

and $\Sigma \subset TM$ some times coupled

$\boxed{\mathcal{M}_C = \mathcal{M} \cap \{L^2 \le C\} \text{ is compact}}$ ① (A_n, φ_n) with $\|\varphi_n\|_{L^2} \xrightarrow{n\to\infty} \infty \Rightarrow (A_n, \frac{\varphi_n}{\|\varphi_n\|_{L^2}}) \xrightarrow{\text{it's } \infty}$ $\mathbb{Z}_2$-harmonic 1-form over $M-Z$, Z is divisor (dim 2)

② $(A_n, \varphi_n) \xrightarrow{n\to\infty}$ 2-spinor SW $\Rightarrow$ $\mathbb{Z}_2$-harmonic spinor over $M-Z$ $\Rightarrow$ ①② are just desired compactification

Def A. $\mathbb{Z}_2$-harmonic 1-form is $(Z, \mathcal{I}, v)$ s.t. (i) Z divisor; (ii) $\mathcal{I}|_{M-Z}$ is flat $\mathbb{R}$-bundle (with -1 monodromy) along loops surrounding Z

(iii) $v \in \Gamma(T^*M \otimes \mathcal{I})$ $dv = d*v = 0$; (iv) $|v| \to 0$ along $Z \Leftrightarrow |v|$ bounded

For spinor, replace T^*M by $\mathcal{S}$

Eg. B ① $\{0\}$, Mobius bundle, $Re(\sqrt{z}dz)$ on $\mathbb{C}$ is $\mathbb{Z}$-valued; ② $(|q^{-1}(0)|, \mathcal{I}, Re(\sqrt{q}))$ on Σ^2, $q = zdz\otimes dz \in H^0(K_\Sigma^2)$

③ We have Hitchin-Kobashyi $\{\rho : \pi_1(M) \to SL(2,\mathbb{C})\}/\sim \longleftrightarrow \{\text{stable Higgs bundles}\}$ $\mathcal{I}$ is the ideal sheaf of $|q^{-1}(0)|$, $\#|q^{-1}(0)| = 4g-4 < \infty$

$(\mathcal{E},\varphi) \overset{\mathcal{H}}{\longmapsto} \det\varphi \in H^0(K_\Sigma^2)$ the Hitchin map $\Rightarrow \mathcal{H}^{-1}(\|q\|_{L^2} \le C) = \mathcal{M}_C$ thus compact by $\mathcal{H}$ is proper map. and by $\varphi_\infty \in SH^0(K_\Sigma^2)$ we can complete the proof of (*) in dim $= 2$ ▨

From ①②, we see it's multi-valued, thus we call it multi-valued harmonic 1-form sometimes

In general statement (*) is proven by Morgan-Shalen. But the existence is still mystery even in S^3.

Rk. ① It's expected that it can develop $SL(2,\mathbb{C})$-gauge and $SU(2)$-gauge, and generalising SW theory

② The multivalue can describe the minimal submanifold with multiplicity — $\mathbb{Z}_2$-harmonic form is a tangent cone model.

③ D.Wang. Doan-Rezchikov's work on Floer theory; ④ Positive mass problem singular for non-spin.

④ When g (Riemannian) varies (g_t), the $\mathbb{Z}_2$-harmonic varys solved by Donaldson in 2019.

⑤ $(\rho_n : \pi_1(M) \to SL_2(\mathbb{C})) \xrightarrow[\text{abelianize}]{t\to\infty} (\rho_\infty : \pi_1(M-Z) \to U(1)\oplus U(1))$ using pass to limit & upper bound estimate for ρ_∞, Markovic disprove the Labourie conjecture. When ρ_∞ is empty space $\Rightarrow$ $\boxed{\text{character variety compact}}$

• Here we can understand it as the moduli in nonabelian Hodge correspondence with complex variety structure, given by rep of π_1

Natural question is classifying $\mathbb{Z}_2$-harmonic form on $\mathbb{R}^3/S^2$. $\boxed{\text{For } S^2}$, we mark $P \subset S^2$, $\#P = 2n$ marked, for general P Taubes showed it's impossible to find solution the eigenequation of $\boxed{\Delta f = \lambda f}$, $\boxed{\Delta : \Gamma(\mathcal{I}_P) \to \Gamma(\mathcal{I}_P)}$ Laplacian. And Taubes-Wu ask about whether $\exists$ configuation of P.

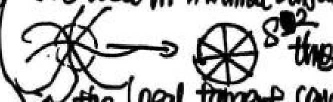
$\Delta = -\star d\star d$ on $S^2 - P$, something

Lemma C. (Taubes-Wu's expansion) $f \in \Gamma(\mathcal{I}_P)$, local on $p \in P$ we have $f(z,\bar z) = \sum (1 + \mathcal{U}_n(r))(a_n z^{n+\frac12} + b_n \bar z^{n+\frac12})$ and vanishing order $n(a_n, b_n)$ the first $\ne 0$, f is critical if $n_p(f)$ (vanishing order) ≥ 1, and we need find a critical configuation

$\mathcal{I}_P := P \times_{\mathbb{Z}_2} \mathbb{R}$ with metric by $\mathbb{Z}_2 \curvearrowright \mathbb{R}$

with $P \to S^2 - P$ is $\mathbb{Z}/2\mathbb{Z}$-principal bundle, and we have

It's local in minimal surface



the local tangent cone model

$\{\mathbb{Z}_2\text{-principal bundle on } S^2 - P\}$ $\longleftrightarrow Hom(H_1(S^2 - P; \mathbb{Z}), \mathbb{Z}_2)$

Prop D. (Taubes-Wu, 2020) $\mathcal{I}_P$ has critical section if P is antipodal. (Variational)

Def E. Compactification configuation space $\mathcal{C}_{2n}$ to $\overline{\mathcal{C}_{2n}} = \mathcal{C}_{2n} \cup \cdots \cup \mathcal{C}_2 \cup \mathcal{C}_0$ with topology to boundary is $\mathbb{Z}_2$-degeneration ($\longrightarrow \bullet$ and $\longrightarrow \emptyset$)

Hence $\mathcal{I}_P$ is Mobius bundle locally on P i.e. 2-valued as Eg. B①

$\Rightarrow$ extend bundle $\mathbb{RP}^1 \to \mathcal{C}_{2n}$ to $\overline{\mathbb{RP}^1} \to \overline{\mathcal{C}_{2n}}$ • this is basic idea of building bridge of

Now from spectral geometry, take $P = \Sigma_P - P$ is the conical singular surface, equipped deck transformation e.

$G \subset O(3) \curvearrowright S^2$, $\tilde{G}$ a group extension of $G \curvearrowright \Sigma_P$, and it's equivalent to group rep of eigenspace $\tilde G \to GL(L^2(\Sigma_P))$

In antipodal case, it corresponds to the D_{2m}-representation eigenvalue of $\Delta \longleftrightarrow$ classifying $\mathbb{Z}_2$-form, it's the generic philosophy of spectral geometry

Thm F. (He-Cheng, 2024) Critical configuation $\Longleftrightarrow$ Containing D_{2m}-subrep and can decompose into irreducible.

By varying $P(t)$ configuation the representation defomation in this path, $P(t) \subset \mathcal{C}_{2n}$, due to $\mathcal{C}_{2n}$ isn't good we consider $t \to \infty$ case. But here it's good due to $t\to\infty$, $P(t)$ is antipodal case. (Observation: Here the path is $Spec(\Delta_{P(t)})$)

$\mathcal{Z}_P^t = H_+^t \oplus H_-^t \oplus \boxed{W^t}$ = all D_{2m}-subrep, and $H_+^t \oplus H_-^t$ is the $\mathcal{U}$-symmetry and $\mathcal{U}$-anti-symmetry decomposition

Thm G. (He-Cheng 2024) $n \ge 1$, infinite configuations critical in $\mathcal{C}_{2n}$. Pf. Drawing the spectral flow, $n \nearrow$ the flow ▨

Coro H. (He-Cheng 2024) Generic configuation of $\mathcal{C}_{2n}$ are noncritical (Small permutation of $P(t) \subset \mathcal{C}_{2n}$ not preserve critical)

Facts on character variety (viewed the moduli in nonAbelian Hodge). $\mathcal{M}_B = Hom(\pi_1(X), G)/$Adjoint rep

① $\mathcal{M}_{Higgs}$ is the cotangent bundle of character varizety, it's hyperKähler

② (Riemann-Hilbert). $\mathcal{M} \longleftrightarrow$ holonomic D_X-module ③ (Hitchin-Kobayashi) ... we know.

④ (Betti Geometric Langlands) $\mathcal{M} = Loc Sys_G^{Betti}$ the Betti moduli of local systems $=$ $Map(X, BG)$. In good cases, it's symplectic.