(SN)

Date

Descent theory in Algebraic Geometry

~~Without~~ large amount of proof of various descents, I only give out its ideas and uses in dealing topology outside Zariski's world.

Q. Gluing [redacted] problem in Zariski case? Intersection $\Leftrightarrow$ Cartesian product

$$\begin{tikzcd} U\cap V \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ U \arrow[r, hook] & U\cup V = X \end{tikzcd}$$

$U \hookrightarrow X$ is Zariski open subset $\Leftrightarrow$ $U \to X$ is open map if we do have a topology

Thus we can [redacted] revise our topology/covering as some special maps $U \to X$ satisfy some [redacted] other properties: étale, smooth [redacted], fpqc (fidèlement plate et quasi-compacte / faithfully flat and quasi-compact), fppf (fidèlement plate de présentation finie / faithfully flat and finite presentation), syntomic (flat, [redacted] l.c.i.), ... (All of them stable under base change)

☆ [redacted] I'm not familiar with when we should use this or that, not ask me! I'm only familiar with Euclidean topology [redacted]

I do know some examples using different of them, but don't know why.)

We [redacted] denote such property (P) [redacted] defining "open-sets" of X $\Rightarrow$

$$\begin{tikzcd} U\times_X V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

due to (P) stable under base change, $U\times_X V \to V$ or U also "open". same manner for sheaves...

Now our question, for example, is:

(*) Gluing $f_1: Y_1 \to U$, $f_2: Y_2 \to V$ to $f: Y \to X$, what condition needed for the pair (f_1, f_2)?

(Y is not given and need to be constructed)

$$\begin{tikzcd} & U\times_X Y_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y_2 \arrow[d] \\ \times Y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U\times_X V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d, "(P)"] \\ Y_1 \arrow[r] & U \arrow[r, "(P)"'] & X \end{tikzcd}$$

We'll first do Zariski case, i.e. $U, V \subset X$, then to general case, and we'll see where the gluing condition comes from.

Can we glue $Y = Y_1 \amalg Y_2$ directly (even in Zariski case)?

and $f: y \mapsto \begin{cases} f_1(y); & y \in Y_1 \\ f_2(y); & y \in Y_2 \end{cases} \in U \cup V = X$ without any condition on (f_1, f_2).

i.e. $f: Y = Y_1 \amalg Y_2 \to U \amalg V \to U \cup V = X$ not injective

what properties are broken?

My first idea is let $X = \{$line with double origins$\}$ with $U = V = \mathbb{A}^1$

$f_i : \mathbb{A}^1 \to \mathbb{A}^1$ is both identity, then $f: \mathbb{A}^1 \amalg \mathbb{A}^1 \to X$

gluing all except origin

such a map loses all good properties of $f_i = \mathrm{Id}$

$Y_i = Y \times_X U_i$

⇒ Generally we have $\{Y_i \to U_i\} \Leftrightarrow \{Y \to X\}$ for (U_i) covers X

without any gluing condition $Y = \coprod Y_i$

for arbitary topology defines $Cov(X)$ and $Cov(X) = \{\mathcal{U}\}$

Our key is $Y_1 \amalg Y_2 \to U \cup V$ $\quad \{U_i\} = \mathcal{U} \in Cov(X)$

$Y_1 \ni y_1 \mapsto f_1(y_1) \in U \cap V \subset U \subset U \amalg V$

$Y_2 \ni y_2 \mapsto f_2(y_2) \in U \cap V \subset V \subset U \amalg V$

when $f_1(y_1) = f_2(y_2) \in U \cap V$, then $f_1(y_1) \in U \amalg V \ni f_2(y_2)$ refers to different pts if we wite $U \amalg V = U \times \{0\} \cup V \times \{1\}$, then $(f_1(y_1), 0) \neq (f_2(y_2), 1)$

they send via $U \amalg V \to U \cup V$ to same pt when naive case

But what happens when $U \amalg V \to U \cup V$

$(x,0) \mapsto \varphi_U(x) \in U$; $\varphi_U : U \to U$

$(x,1) \mapsto \varphi_V(x) \in V$; $\varphi_V : V \to V$ • We don't need isomorphic or other conditions

then we need $\boxed{\varphi_V|_{U\cap V} = \varphi_U|_{U \cap V}}$ (1) is gluing condition

and for U, V and W, it's easy to see the cocycle condition holds as let $g_{UV} = \varphi_U|_{U\cap V}(\varphi_V|_{U\cap V})^{-1} = Id$ by (1) (For sheaves case $\varphi_U : \mathcal{E}|_U \to \mathcal{O}_U^{\oplus r}$)

$\Rightarrow g_{UV} \circ g_{VW} = g_{UW}$ is trivial

Thus we modify (1) to a more admissible case

$\varphi_V|_{U\cap V} \circ g_{UV} = \varphi_U|_{U \cap V}$, $g_{UV} : U \cap V \to U \cap V$ is isomorphism (2')

and satisfy $\boxed{g_{UV} = g_{VU}^{-1} \ \& \ g_{UV} \circ g_{VW} = g_{UW}}$ (2)

(2) is desired! We had seen it in mfds, vector bundles, sheaves, and here gluing maps... (φ_U) are all isomorphism

thus (2') ⇔ (2), general (2) weaker than (2')

Compare them, sheaves generalize the former two ($\mathcal{O}_X$ & locally free) and gluing map generalize $\mathcal{O}_X$ case into relative case.

Sheaf is a functor/presheave $\mathcal{C}^{op} \to Set$ $\boxed{\text{compatible}}$ with a topology on $\mathcal{C}$, what we do for gluing maps is generalise $\mathcal{C}^{op} - Sch^{op}$ to the morphism category (2-cat) $Mor^{op} \to Set$

Under this change, we generalize the compatibility condition of sheaf to a higher level, this is what descent theory doing; the basic idea is still gluing, but for AG use, our descent next will all not far away cocycle condition, which goes against Grothendieck's original motivation.

PART 1

~~Def 1~~ (Descent datum)

① A descent datum $\{(\mathcal{F}_i,\varphi_{ij})\}$ w.r.t. $\mathfrak{U}=\{f_i: S_i\to S\}\in \mathrm{Cov}(S)$

[QCoh] isomorphism and quasi-coherent sheaf $\mathcal{F}_i$ on S_i

s.t. (a) $\varphi_{ij}: pr_i^*\mathcal{F}_i \to pr_j^*\mathcal{F}_j$; $S_i\times_S S_j \overset{pr_i}{\underset{pr_j}{\rightrightarrows}} S_i, S_j$

(b)
$$\begin{tikzcd} pr_i^*\mathcal{F}_i \arrow[rr, "pr_{ik}^*\varphi_{ik}"] \arrow[dr, "pr_{ij}^*\varphi_{ij}"'] & \arrow[d, phantom, "\circlearrowleft"] & pr_k^*\mathcal{F}_k \\ & pr_j^*\mathcal{F}_j \arrow[ur, "pr_{jk}^*\varphi_{jk}"'] & \end{tikzcd}$$
; $S_i\times_S S_j\times_S S_k \to S_i\times_S S_j,\ S_j\times_S S_k,\ S_k\times_S S_i$ $\to S_i, S_j, S_k$

called descent condition/cocycle condition;

Morphism $\{(\mathcal{F}_i,\varphi_{ij})\}\to\{(\mathcal{F}_i',\varphi_{ij}')\}$ is given by $\psi_i:\mathcal{F}_i\to\mathcal{F}_i'$

s.t.
$$\begin{tikzcd} pr_i^*\mathcal{F}_i \arrow[r, "\varphi_{ij}"] \arrow[d, "pr_i^*\psi_i"'] & pr_j^*\mathcal{F}_j \arrow[d, "pr_j^*\psi_j"] \\ pr_i^*\mathcal{F}_i' \arrow[r, "\varphi_{ij}'"] & pr_j^*\mathcal{F}_j' \end{tikzcd}$$
(compatible with these "translation functions")

"Descent Datum" not "Doredeno Daisuki"

Thus we have a category $DD^{QCoh}(\mathfrak{U})$

algebraic

② Similarly in algebraic level,

a descent datum $\{(M_i,\varphi_{ij})\}$ w.r.t. $\mathfrak{U}=\{f_i:R\to R_i\}\in\mathrm{Cop}(R)$ and f.g. module M_i over R_i

s.t. (a) ✓ (b) ✓; morphism ✓ (*)

But here we have a more precise statement (**) usually used, giving $R\to A$ we have $A\rightrightarrows A\otimes_R A \Rrightarrow A\otimes_R A\otimes_R A$, denoted $A^{(n)}$ for n-fold. descent datum (N,φ) is A-module N and $\varphi: N\otimes_R A\to A\otimes_R N$.

$$\begin{tikzcd} N\otimes_R A\otimes_R A \arrow[rr, "\varphi_{02}"] \arrow[dr, "\varphi_{01}=(\varphi,\mathrm{id})"'] & \arrow[d, phantom, "\circlearrowleft"] & A\otimes_R A\otimes_R N \\ & A\otimes_R N\otimes_R A \arrow[ur, "(\mathrm{id},\varphi)=\varphi_{12}"'] & \end{tikzcd}$$

$\varphi_{02}: n\otimes 1\otimes 1 \mapsto \sum a_i\otimes 1\otimes n_i$

where $\varphi: n\otimes 1\mapsto \sum a_i\otimes n_i$

It's natural but need more word to say due to $\otimes$ not commutative, here $\times$ is commutative, thus easier.

Due to my restricted knowledge of simplicial method in algebra, only introduction is given next, see rigorous [SP, Descent §3]

Here, our φ_{02}, or general φ_{ij} is something subtle: due to we need to relate (*) with (**), it's not enough to give $R\to A$ to recover a "covering", thus our (φ_{ij}) should be complicated enough to carry enough datum.

Our method is, somehow, an algebraic simplicial approximation: we use $(A/R)_\bullet$, a cosimplicial algebra as R in (*)

$(A/R)_\bullet(\epsilon_j)$, as R_i covering (of $(A/R)_\bullet$) in (*)

N_{ij} over $(A/R)_\bullet(\epsilon_j)$, as module in (*)

gluing ⇝ $N_\bullet$, a cosimplicial $(A/R)_\bullet$-module as gluing result

• take $A=\prod R_i$, or dually in geometric case $S'=\coprod S_i$

Hence (*) "⟺" (**) thus we take the form (**) more. (⟸ is trivial)

Rk. "Descent/Descente" means originally "下降", here we can see why "下降" is related with "gluing":

($R\to A$: from R to A is simply pushforward/pullback, but A to R is descent. Later [SGA I] also, but more abstract.)

I'm not sure at the stage of Grothendieck writing [FGA], the simplicial method was exerted or not, but anyway is not surprise that without it Grothendieck can also see its core and give it such as name "Descente" accurately (not simply use "Gluing")

Later, except from gluing, we'll also consider properties such as $QCoh(S)\overset{equi}{\simeq} QCoh(S'/S)$, descent from $S'\to S$ to S directly & $Sch/S \xrightarrow{ff} DD^{Sch}(S'/S)$

Here we denote $DD^{QCoh}(S'\to S)=DD^{QCoh}(S'/S)$, so is later notation $DD^{Sch}(S'/S)$ for only one cover and given by $S'\times_S S'\times_S S' \Rrightarrow S'\times_S S' \rightrightarrows S'$

③ Morphisms/Schemes $\{(V_i,\varphi_{ij})\}$ w.r.t. $\mathfrak{U}=\{f_i:S_i\to S\}\in\mathrm{Cov}(S)$ and $\{g_i:V_i\to S_i\}$

s.t. (a) $\varphi_{ij}: V_i\times_S S_j \to S_i\times_S V_j$ isomorphism; (b)
$$\begin{tikzcd} V_i\times_S S_j\times_S S_k \arrow[rr, "pr_{ik}^*\varphi_{ik}"] \arrow[dr, "pr_{ij}^*\varphi_{ij}"'] & \arrow[d, phantom, "\circlearrowleft"] & S_i\times_S S_j\times_S V_k \\ & S_i\times_S V_j\times_S S_k \arrow[ur, "pr_{jk}^*\varphi_{jk}"'] & \end{tikzcd}$$

$(pr_i^*V_i = V_i\times_S(S_i\times_S S_j) = (V_i\times_S S_i)\times_S S_j = V_i\times_S S_j)$

Morphism is similiarly defined, we have a category $DD^{Sch}(\mathfrak{U})$

PART II After give various descents from gluing problems, we are eager to ask two things:

Q1. Can descent data glued? i.e. category equivalence $DD^{QCoh}(\mathfrak{U}) \simeq QCoh(S)$ & $DD^{Sch}(\mathfrak{U}) \simeq Sch/S$?

Q2. Choice of cover? Very aggressively, can we expect $DD^{QCoh}(S'/S) \simeq QCoh(S)$ & $DD^{Sch}(S'/S) \simeq Sch/S$, i.e. the "simplicial approximation" works well?

Both are positive with some conditions, we'll prove the following main thms next

Thm1. In Zariski topology & $\mathfrak{U}$ is effective $\Rightarrow$ $QCoh(S) \longrightarrow DD^{QCoh}(\mathfrak{U})$ is fully faithful (given by pullback through $S_i \to S$ each)

Here effective descent datum $\{(\mathcal{F}_i, \varphi_{ij})\} \cong \{(\mathcal{F}|_{S_i}, \varphi_{ij}^{can})\}$ $\exists\, \mathcal{F}$ on S globally, φ_{ij}^{can} induced by pullback called canonical descent w.r.t. $\mathcal{F} \in QCoh(S)$

Pf. Is there any possibility to fail? Who can tell me the obstruction? ▨

Thm2. ① We have equivalence of category $DD^{Sch}(\mathfrak{U}) \xrightarrow{\simeq} DD^{Sch}(S'/S)$;

② Giving two covering $\mathfrak{U}$, $\mathfrak{V} \in Cov(S)$, $\mathfrak{U} = (U_i)_I$, $\mathfrak{V} = (V_j)_J$, $\alpha: I \to J$

If $\forall j \in J$, $\{g_i : U_i \to V_j\}_{\alpha(i)=j} \in Cov(V_j)^{fpqc}$ $\Rightarrow$ pullback $DD^{Sch}(\mathfrak{V}) \longrightarrow DD^{Sch}(\mathfrak{U})$ is fully faithful;

(Define pullback is complicated by natural, omitted here)

② gives the descent datum will stay stable under change of cover;

③ With a proper topology, $DD^{Sch}(\mathfrak{U}) \simeq Sch/S$ at least for Zariski case it's right.

Pf. ③ for Zariski topology is the discussion at first ▨

① $S' = \coprod S_i$ and $V' = \coprod V_i$, $(\varphi : V' \times_S S' \longrightarrow V' \times S')$

$= (\coprod \varphi_{ij} : \coprod_{i,j} V_i \times_S S_j \longrightarrow \coprod_{i,j} S_i \times_S V_j)$ ▨

② Hardest, see [SP, Descent, §35], our strategy is using ① to reduce to only one case ▨ with $S' \to S$

PART III

Descent properties part will be primary, it says the property of of S', or objects of $QCoh(S')$, Sch/S', $Mor(Sch/S')$ descend to objects of $QCoh(S)$, Sch/S, $Mor(Sch/S)$.

First we define the flat, faithfully flat, smooth, étale maps

Def2. ① $S \xrightarrow{f} S'$ is flat at $s \in S$ if $\mathcal{O}_{S,s} \longleftarrow \mathcal{O}_{S', f(s)}$ is flat

f is flat if f is flat at $\forall s \in S$;

② Same for faithfully flat; ($\Leftrightarrow$ surjective & flat)

(Recall our algebraic/local definition is $A \to B$ is flat if $- \otimes_A B$ is exact, and faithfully flat if $- \otimes_A B$ is faithful & exact)

③ f is smooth at $s \in S$ if f is flat at $s \in S$ and $\Omega_{S/S', s}$ generated by $\leq \dim_s(S_{f(s)})$ elements, as $\mathcal{O}_{S,s}$-module ($S_{f(s)}$ means the fibre $f(s) \longleftarrow$ local dimension)

And f is smooth if f is smooth at $\forall s \in S$;

Rk. flat can't give dimension of fibres' conclusion without finiteness condition such as locally Noetherian, but smooth can (only inequality)

④ f is étale if f is smooth and unramified

$\Leftrightarrow$ f is flat and unramified

$\Leftrightarrow$ f is smooth and relative dim 0

(Here relative dim give by $\dim_s(S_{f(s)}) = 0$)

Later we refer to a cover/topology, it'll be one of the Zariski, fpqc, fppf, étale, smooth, syntomic, $\mathcal{U}=\{S_i\to S\} \in Cov(S)$

Left has stronger property & topology is coarser;
Right has weaker property & topology is finer.

and a property of map $X_i\to S_i$, saying P.
P should satisfy following general conditions:

① The descent datum $\{X_i,\varphi_{ij}\}$ is effective: $\exists X\to S \in Cov(S)$, $\{X_i,\varphi_{ij}\}\cong\{X|_{S_i},\varphi_{ij}^{can}\}$;

② P stable under base change;

③ If $(Y_k\to V_k)$ all hold P, so is $\coprod Y_k\to\coprod V_k$;

④ $\forall (S'\to S)\in Cov(S)$ in a given topology, $\forall (X',\varphi)$ descent datum $\in DD^{sch}(S'/S)$ satisfy $X'\to S'$ hold $P \Rightarrow (X,\varphi)$ effective.

Thm3. If P holds these properties, then P descends for all covering under this topology $\Leftrightarrow$ P descend for single map of affines.

Pf. Given two covering $\mathcal{V}\to\mathcal{U}$, due to ②, P also holds for $\mathcal{V}$ by pullback descent datum;

- Hence we can reduce to single case by ① and ③ via we can find $\{X_i,\varphi_{ij}\}\cong\{X|_{S_i},\varphi_{ij}^{can}\}$ to pullback;
- For S affine, ④ will hold and done;
- For general S, a standard argument of gluing ▨

What the thm told us is: all covering descending properties usually seen can be reduced to single map of affine schemes! That's why [GW] all are single algebra map at first.

Now we'll list them and pick some to prove.

Prop4. ①
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\square"] & Y' \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ S' \arrow[rrr] & & & S \end{tikzcd}$$
$\boxed{S'\to S} \in Cov(S)$ in some topology τ

If f' satisfy P_τ then so is f;
(indeed equivalent as "$\Leftarrow$" is stable under base change)

② In particular, $\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$, if X' satisfy Q_τ (or $X'\to S'$), then so is X (or $X\to S$);

③ In particular in particular, $X'\to X$, if X' satisfy Q'_τ, then so is X;

④ $S'\xrightarrow{\varphi}S$ induce pullback functor $QCoh(S')\xrightarrow{\varphi^*}QCoh(S)$ preserve R_τ on Morphisms and R'_τ on Objects.

Let $\boxed{\tau=fpqc}$ topology as [GW] concerns and used often (as it's only slightly weaker than Zariski), then P_τ, Q_τ, Q'_τ, R_τ, R'_τ are listed:

P_τ = (locally) finite type, (locally) finite presentation, isomorphic, monomorphism, open immersion, closed immersion, quasi-cpt immersion, proper, affine, finite, quasi-finite;

Q_τ = (locally) finite type, (locally) finite presentation, isomorphic, mono & surj, universally open & closed & submersion, open & closed immersion, quasi-cpt immersion, separated, proper, finite, quasi-finite, affine, quasi-affine, flat, fppf, smooth, étale, syntomic, unramified;

Q'_τ = Quasi-cpt, (locally) Noetherian, integral, reduced, normal, regular;

R_τ = iso, inj & surj; R'_τ = finite type, finite presentation, flat, locally free, rank n; if $S'\to S$ both Noetherian, coherence also.

With finer topology, these properties should decrease in number, it's a subtle thing can't be all listed here. 参AG1の補

Pf. ① See [GW, Prop14.53]

②-④ See [Jarod Alper, Prop2.1.19, Prop2.1.22, Prop2.1.16] ▨

I pick only two to prove, most of them are just Hartshorne's exercise.

(a). If $(X'\hookrightarrow Y')=(X\times_S S'\to Y\times_S S')$ is closed immersion, so is $X\to Y$

(b) If $\varphi^*g_i\to\varphi^*g_j$ is isomorphism $\Rightarrow g_i\to g_j$ isomorphism

Lemma. Recall the scheme-theortic image of $X \xrightarrow{f} Y$ is the smallest $Z \subset Y$, closed subscheme

① such $f(X)$ exists

② Preserved under flat & qc base change

"Pf." ① $\mathcal{I} = \ker(\mathcal{O}_Y \to$ … otherwise $\exists\, \mathcal{I}' \subset \mathcal{I}$ largest

In [GW] he do this step, but it's useless ?

Pf of (a). Closed immersion is (Zariski) local property, thus assume $Y = \operatorname{Spec} A$,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & f(X) \arrow[ur] & \end{tikzcd}$$

$f(X)$ scheme-theortic image

$S' \to S$ is flat and quasi-c…

(scheme-theortic image preserved under base change)

$$\Rightarrow \begin{tikzcd} X' \arrow[rr, "f'"] \arrow[dr] & & Y' \\ & f'(X) \arrow[ur] & \end{tikzcd}$$

due to f' is closed immersion

$f'(X) = f(X) \times_S S' \Rightarrow X' \to f'(X)$ is isomorphism

descent to $X \to f(X)$ ▨

(b) Reduce to algebraic case. It says $A \to A'$ faithfully flat then $M \to N$ iso $\Leftrightarrow M \otimes_A A' \to N \otimes_A A'$ iso

This is just definition: $\Rightarrow$ is flat

$\Leftarrow$ is faithfully flat ▨