Date

Le lingua marche à travers la number theory in geometry

PART II

I'd like to explain CFT from classical number theory first.

Recall the conclusion of CFT:

Setting $\operatorname{Spec}\mathbb{Z}$: (0), 2, 3, 5, 7

$\Uparrow \mathcal{O}_{\mathbb{Q}}$

more trivial

Residue: $\mathbb{Q}$, $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$, $\mathbb{F}_7$ — Global fields

Completion ⇓ ⇓ completion

of Residue: $\mathbb{R}$, $\mathbb{Q}_2$, $\mathbb{Q}_3$, $\mathbb{Q}_5$, $\mathbb{Q}_7$ — Local fields

infinite place/Archimedean, finite places/non-Archimedean

We call number fields are $K/\mathbb{Q}$ finite global (Over $\mathbb{F}_p$ trivial)

$K/\mathbb{Q}_p$ finite local

function fields $K/\mathbb{F}_q$ transcendental global $\overset{\text{1-dim}}{\Leftrightarrow}$ $K/\mathbb{F}_q(t)$ finite

$\hat{K}/\mathbb{F}_q$ transcendental local $\Leftrightarrow$ $K/\mathbb{F}_q((t))$ finite

Rk. ① $K/\mathbb{R}$ is something easy and we not concerned here

over $\mathbb{R}$ or $\mathbb{C}$ something come to be "trivial" / abelian extension, Galois rep

and p-adic becomes necessarily;

② By Gelfand–Mazur, Archimedean local field is $\mathbb{R}$ or $\mathbb{C}$,

thus the non-Archimedean local field / non-Archimedean places of global field is more interesting in number theory;

③ In geometric setting we prefer $/\mathbb{R}$ (or $\mathbb{C}$) basically, but for some reasons / algebraic (mod p) to $\mathbb{F}_p$, and specialize to $\mathbb{Q}$

analytic for convergence to non-Archimedean

we acquire these base field in number theory for technical reasons: to replace trivial/impossible cases in $\mathbb{R}$ or $\mathbb{C}$

• On the other hand, for number theory we need global ones to recover classical conclusions glued by local, esp. non-Archimedean fields (local-global results)

Hence, we have sufficient reason to consider p-adic fields.

It's more delicate for arithmetic than topologies such as Zariski, thus sometimes we acquire étale topology, dominate more functions

The study of number theory and geometry are "almost same fine": the algebra of function fields $K(X)$ $\leftrightarrow$ the topology of X

(adding algebraic structure $\leftrightarrow$) adding geometries

the arithmetic of $K(X)$

without additional structure,

For example, apply the strong approximation thm to $K(C)$ dominates pole & zero of functions on C, we can recover many basic facts on C with algebraic structure $/\mathbb{C}$

Thm 1. (CFT) ① (Global CFT) L/K $\boxed{\text{abelian}}$

(and $q=p^r$)

$\mathbb{F}_q(t)$ is $\mathbb{F}_q((t))$ instead $\mathbb{Q}_p(t)$)

$\Rightarrow 1 \to C_L \xrightarrow{N_{L/K}} C_K \xrightarrow{\Phi_{L/K}} \operatorname{Gal}(L/K) \to 1$

$\Rightarrow \operatorname{Gal}(L/K) \cong C_K/N_{L/K}(C_L)$ dense img

C_K is the idele class group $= \mathbb{A}_K^\times/K^\times = \mathbb{I}_K/K^\times$

and • It's natural for $L'/L/K$

$H = N_{L/K}(C_L)$

• $\{H \leq \mathbb{A}_K^\times/K^\times \text{ finite index}\} \leftrightarrow \{L/K \text{ finite abelian}\}$

L is called "class field" of H

② (Global CFT, maximal) K^{ab}/K maximal abelian in $\bar{K}$

on $\operatorname{Spec}(\mathbb{Z})$ above $\mathbb{Q}$, $\mathbb{Q}_p$'s

$\operatorname{Gal}(K^{ab}/K) =: \pi_1(K)^{ab}$ or $\operatorname{Gal}(K)^{ab}$

$\Rightarrow$ all $\Phi_{L/K}$ factor through $\pi_1(K)^{ab}$;

③ (Local CFT) Same statement with L/K local ($1 \to C_K^0 \to C_K \to \operatorname{Gal}(K)$ of profinite grps)

and compatibility:

$$\begin{array}{ccc} K_v^\times & \longrightarrow & \pi_1(K_v)^{ab} \\ \downarrow & ? & \downarrow \\ C_K & \longrightarrow & \pi_1(K)^{ab} \end{array}$$

Pf. ③ by Tate cohomology (or other methods), the glue to ① $\Rightarrow$ ② ▨

Here $\mathbb{A}_K = \prod' K_v$, what's its spec looks like? It's hard

$= \varinjlim_{S \text{ finite}} \left(\prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_{K_v}\right)$

but we have $C_K \to Cl(\mathcal{O}_K)$ by product all components

K has
r_1 real embedding & r_2 cpx embedding

$C_K^0 \subset C_K$ identical component, $C_K^0 \cong \mathbb{R}_{>0} \times \underbrace{(S^1)^{r_2} \times (A_\mathbb{Q}/\mathbb{Q})^{r_1+r_2-1}}_{(C_K^1)}$

~~has cpt & connect kernel.~~ norm

to relate with geometry, and thus C_K is finer than the Picard group (Jacobian) in geometry.

Ψ_K is called the Artin (reciprocity) map, the motivation of CFT is the problem of reciprocity:

Consider the splitting problem mod p:

$f(x) \equiv (x-a_1)\cdots(x-a_n) \pmod p$, $a_i \in \mathbb{F}_p$

$\Leftrightarrow$ (Kummer) $\mathbb{Q}(\alpha)/\mathbb{Q} =: L/K$, $p\mathcal{O}_L = \mathfrak{P}_1^{e_1}\cdots\mathfrak{P}_k^{e_k}$ has $e_1 = e_2 = \cdots = e_k$ split

Ramification $\Leftrightarrow$ Frob $= 1 \in$ Gal

$\mathrm{Gal}(\mathcal{O}_L/\mathfrak{P}/\mathcal{O}_K/p) \hookrightarrow \mathrm{Gal}(L/K)$

$\cong$

$D(\mathfrak{P}|p)$

Now we reduce the problem to determine whether a Frobenius element $\in \mathrm{Gal}(L/K)$ is 1. The structure of $\mathrm{Gal}(L/K) \cong C_K/N_{L/K}(C_L)$ gives the full answer of the question, when $\boxed{\mathrm{Gal}(L/K) \text{ Abelian}}$

Rk. The symmetry of $\mathrm{Gal}(L/K)$ in non-Abelian case is much complicated and we consider Galois rep $\mathrm{Gal}(L/K) \to GL_n(\mathbb{C})$, some cases the $\mathbb{C}$-rep is too easy and we consider $GL_n(\mathbb{Q}_p)$, then these cases we comes to Langlands ~~program, and~~ CFT became a degenerated story: one dimensional $\mathrm{Gal}(L/K) \cong J_K/J_L$

J_K & J_L are generalize Jacobian of some curves and no Galois-rep taken participate in.

We have a 1-dim rep $\Leftrightarrow$ character

$\rho: \mathrm{Gal}(K) \to \overline{\mathbb{Q}}_l^\times$ (for ...) $\mathbb{Q}_L$

$A_\mathbb{Q}/\mathbb{Q} \cong \mathbb{R} \times \hat{\mathbb{Z}}/\mathbb{Z} \cong \varprojlim \mathbb{R}/n\mathbb{Z} \cong \varprojlim_{x \to x^n} S^1$

Q. Why $\Psi_{L/K}$ is called reciprocity map?

Consider the example of degree 2, the splitting problem is nothing more than famous quadric reciprocity law:

x^2+ax+b split in $\mathbb{F}_p$ $\Leftrightarrow$ $(a^2-4b$ not square$)$, $p \nmid a^2-4b$ splitting behavior is determined by the congruence class of $p \pmod{a^2-4b}$

The second statement says why it's so $\Leftarrow \left(\frac{p}{\frac{a^2-4b}{4}}\right)\left(\frac{\frac{a^2-4b}{4}}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{\frac{a^2-4b}{4}-1}{2}}$ classical

named: the splitting behavior of prime ideal p on L is determined by the congruence class of p (mod $N_{L/K}(C_L)$) in C_K.

ART II We found it's hard to study the geometry of C_K, A_K, I_K.. but it ~~satisfies some~~ shares some properties of ideal class group / Picard group /..., then we have the generalized Jacobian; (fractional ideals)

And for the finer study for arithmetic use, etale topo is natural, and etale cohomology takes place of Tate cohom

Setting. $K = k(X)$ is global function field of X/k smooth projective & $\boxed{\cdots}$

$\boxed{\cdots}$ is usually conditions filling to break of $\bar{k} \neq k$, ~~for even~~ as here we not consider Grothendieck's etale fundamental grps.

(In fact, the latter one is motivated by the general GCFT mainly in history, said by Katz & Lang)

Rk. A more coherent statement in geometric setting is using sheaf (more specially, local system) to replace $k(X)$ later by Grothendieck's sheaf-to-function dictionary

Rk ① Thus the fine structure of idèle / Picard grp makes it possible to deal with ramification. At place x, the ramification by acting some related to x is Hecke operator $L(x)$ & Frob_x is Wilson operator

Thm 2 (CFT) $\{\text{Characters } K^\times\backslash \mathbb{A}_K^\times/\mathcal{O}_K^\times \to \overline{\mathbb{Q}}_\ell^\times\}$
$\cong \{\text{Characters } \mathrm{Gal}(\overline{K}/K) \to \overline{\mathbb{Q}}_\ell^\times\}$ $L(x) = \mathrm{Frob}_x L$

as sets (with additional conditions: send Frob to what?)

if $\overline{K}/K$ unramified, Abelian

(Ramified case we can have an analogue by dominate at places of ramification)

② Bi-coset $\leftrightarrow$ Picard grp given by $/\mathcal{O}_K^\times \Leftarrow$ quotient all trivializations

Pf. $K^\times\backslash \mathbb{A}_K^\times/\mathcal{O}_K^\times$ is dense image to $K^\times\backslash$ $\leftrightarrow$ quotient all values of scalars Gal$(\overline{K}/K)$, we extend it as $\overline{\mathbb{Q}}_\ell^\times$ is cpt

This result realize CFT as 1-dim (to $\overline{\mathbb{Q}}_\ell^\times$ rep / $\overline{\mathbb{Q}}_\ell$) Langlands,

its geometric story: $K = K(X) \Rightarrow K^\times\backslash \mathbb{A}_K^\times/\mathcal{O}_K^\times$
$(GL(1))$ $= \mathbb{G}_m(K)\backslash \mathbb{G}_m(\mathbb{A}_K)/\mathbb{G}_m(\mathcal{O}_K)$

parametrize all $\mathbb{G}_m$-principal bundle / X, which holds for general reductive algebraic grp $G(K)\backslash G(\mathbb{A}_K)/G(\mathcal{O}_K)$

On the other hand, the $= \mathrm{Bun}_G(X)$

fundamental grp rep $\pi_1(X) \to G^\vee$ ($G^\vee$ is something related to G)

$\Leftrightarrow$ Flat $G^\vee$-principal bundles / X

Gauge $\Leftrightarrow$ ~~local~~ $G^\vee$-local systems / X $\mathrm{Loc}_{G^\vee}(X)$

Hence we have next $\boxed{\text{picture}}$:

Character variety $\mathrm{Loc}_{G^\vee}(X)$ $\{\text{D-modules on } \mathrm{Bun}_G(X)\}/\sim$

$\updownarrow$ Gauge $\cup$ RH

$\{\pi_1(X) \to G^\vee\}/\sim$ $\xleftrightarrow[\text{Geometric Langlands}]{}$ $\{\square \text{ Sheaves on } \mathrm{Bun}_G(X)\}/\sim$

We call these special sheaves generalizing modular forms on Biquotient the Hecke eigensheaves.

They form a "basis" of D-modules / $\mathrm{Bun}_G(X)$.

日月光华 旦复旦兮

"Eigen" as the eigenvector of differential operator $\Leftrightarrow$ solution to a PDE hence identified as D-mod

Rk. The bi-coset $\Gamma\backslash G/H$ has $\Gamma \hookrightarrow G$ lattice & $H \hookrightarrow G$ maximal Lie subgrp, which refers to arithmetic local symmetric spaces with discrete & continuous symmetry $\Gamma\backslash\square$ $\square/H$ known to physists

Observation: skycraper sheaves k_{x_0} forms a "basis" of QCoh in same meaning, for moduli space $\mathrm{Loc}_{G^\vee}(X) \ni [\mathcal{E}]$, $k_{[\mathcal{E}]} \cong \mathcal{O}_{\mathcal{E}}$

then $\mathrm{QCoh}(\mathrm{Loc}_{G^\vee}(X)) \longleftrightarrow \text{D-mod}(\mathrm{Bun}_G(X)) \ni e$

sending skycraper sheaves to basis elements $\boxed{\frac{1}{i x_0}\frac{d}{dt}} e^{i x_0 t} = e^{i x_0 t}$

$S_{x_0} \longmapsto e^{(i x_0)t}$ Grothendieck's sheaf-to-function dictionary

like Fourier–Mukai transform locally constant

• For abelian case, the proof base on just FM transform!:

Fourier theory. Recall $L^2(G) \cong L^2(G^\vee)$ for Pontryagin duality and the Fourier–Mukai equivalence sending skycraper sheaves to locally constant sheaves length 1

The Langlands ~~duality~~ in 1-dim is about $\mathbb{G}_m(K)\backslash \mathbb{G}_m(\mathbb{A}_K)/\mathbb{G}_m(\mathcal{O}_K)$

but similiar as $\mathbb{Z} \hookrightarrow \mathbb{Q}$ lattice ($\mathbb{Z}^\vee = \mathbb{Z}\backslash\mathbb{Q}$)

$\Rightarrow (K^\times)^\vee = K^\times\backslash \mathbb{A}_K^\times$ (Fairly speaking, I don't know!)

The Langlands dual of G is closer? to Pontryagin dual, but here we refer to latter one: $G^\vee = \mathrm{Hom}(G, \mathbb{G}_m)$ in algebraic case (affine)

G	$G^\vee$	$= \mathrm{Hom}(\mathrm{Spec}A, \mathrm{Spec}\, k[x, x^{-1}])$
Character	point •	$= \mathrm{Spec}(\mathrm{Hom}(k[x,x^{-1}], A))$
function	function	$= \mathrm{Spec}(A^\times) = \mathrm{Spec}(\Gamma(\mathcal{O}_G))$
Representation	$\boxed{?}$ Bundle!	$= \mathrm{Spec}(k[G])$ ~~the~~ group algebra

$G \to \mathrm{End}(V) \Leftrightarrow V$ is $\Gamma(\mathcal{O}_G)$ module $\Leftrightarrow V$ is $k[G]$-module

$\Leftrightarrow \exists\ \mathcal{V} \to G^\vee$ vector bundle with fibre V

Generally $G \to H \Leftrightarrow H$-principal bundle $\to G^\vee$

As $G^{\vee\vee} \cong G$, we take the G as $\mathrm{Loc}_{G^\vee}(X)$, and we show by example; the result G-bundle is something "eigen"!

E.g. ① CFT thus is an equivalence between $\Bbbk[Cl(K)]$ and $\Bbbk[Loc_{\mathbb{G}_m}(\mathcal{O}_K)]$ (or their Spectrum), Unramified. Ramified case needs moving some pts. $\Bbbk$-Rep

② Let $G = \mathbb{A}^1 = \mathrm{Spec}(\Bbbk[T])$, $\check{G} = \mathbb{G}_a$, $G \curvearrowright V$ by operator $T: V \to V$.

$G = \langle T \rangle$

The vector bundle $/\mathbb{A}^1$ has each fibre at $\lambda \in \mathbb{A}^1$

$\Rightarrow$ we have diagram $T \mapsto \lambda$ $\Leftrightarrow$ maximal ideal $(T-\lambda)$ of $\Bbbk[T]$ ($\Bbbk = \bar{\Bbbk}$)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(T-\lambda) \arrow[r] & \Bbbk[T] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Bbbk \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(T-\lambda) \arrow[r] & V \arrow[r] & \frac{V}{\mathrm{Ker}(T-\lambda)} \arrow[r] & 0
\end{tikzcd}$$

the fibre diagram at $\lambda \in \mathbb{A}^1$

$\Rightarrow$ $\mathcal{V}_x = \mathrm{coker}(T-\lambda)$ is co-eigenspace of T

$\boxed{\text{Conclusion}}$: Langlands correspondence is a kind of Fourier transform, or one can see as it's motivated $\boxed{\text{and CFT}}$ by CFT and Fourier /representation theory both.

Rk. Gauge theory in GLP is also used to explain it as S-duality, omitted due to it's common, the reduction in UGD-case to MS from 4d TQFT to 2d TQFT I think the E-M duality is quite strange!

Rk (Ramification theory) When one deal with ramified case, we consider ramification divisor, thus ramification $\Leftrightarrow$ boundary divisor $\Leftrightarrow$ boundary condition on TQFT

$\star$ Rk. Commutative operator families are emphasised in algebraic studies as we can take the spectrum to recover its space easily:

[Hecke operator $H_x(\chi)$ on D_x, $\in \mathrm{Pic}$] [stacky issue — higher cobordism cat.]

on $Loc(X)$

Commutation $\Leftrightarrow$ Geometry $\subset$ Topological theory
GLP ramified case
Noncommutativity $\Leftrightarrow$ Time without time involution
(By Connes' thesis)

In unramified case, no eigenvectors & stack used!

Such relation to physics, esp. topological ones are often in geometric Langlands.