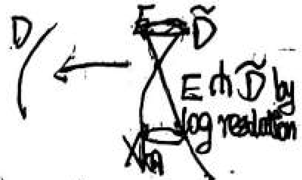
We'll use analytic (Milnor fibres), arithmetic (p-adic), topological (mixed Hodge module) to study hypersurface singularity (Ruijie Yang) **Page 1**

Lecture 1

Def 1. $X_m = \{[z_1, \dots, z_n] \in \mathbb{P}^{n-1}_{\mathbb{C}} \mid f_m(z) = 0\}$, $0 \in D$ is ordinary singularity if X_m is smooth, $D = V(f) \subset \mathbb{A}^n_{\mathbb{C}}$, m is the multiplicity of $0 \in D$

When ordinary case, the log resolution of $(\mathbb{C}^n, D)$ is given by blowing up $0 \in D \subset \mathbb{C}^n$ (Exercise)



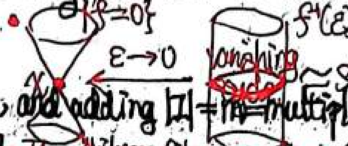
Eg 2. ① Cone is ordinary; ② Cusp isn't ordinary $\prec \longleftarrow$ not normal simple crossing E $\tilde{D}$

In fact, every ordinary singularity locally is the cone over $X_m \subset \mathbb{P}^{n-1}_{\mathbb{C}}$, thus it's totally determined by the lowest term, thus it's Morse-theoretic.

Prop 3. Any (hypersurface) singularity can be deformed to ordinary (even smooth by adding linear term distribution, but useless here), it's called Morsification.

Pf. **Step 1** By hypersurface cutting, assume $0 \in D$ is isolated;

Step 2 $D = V(f)$, $f = \sum C_I g^I$, g^I basis of $\mathbb{C}[z_1 \dots z_n]$ parameterized by $(C_I)_I \in \mathbb{C}^N$, and adding $|I| = m$ = multiplicity terms to distribute $\Rightarrow \exists\, 0 \in U \subset \mathbb{C}^N$, s.t. $\boxed{f_t} = f + C_I g^I$ is ordinary, $\forall t \in U$ ☐



• The Milnor fibres are just $(f_t)_{t \neq 0}$ nearby fibre at t via the Milnor fibration $S^{2n-1}_{\varepsilon} - D \to S^1$, $z \mapsto \frac{f(z)}{|f(z)|}$

Consider $\varphi_{f,c} : \mathbb{C}^n - D \to \mathbb{R}$, $z \mapsto \frac{1}{|f(z)|^c}$ and $C_0(f) := \sup\{c \in \mathbb{R}_{>0} \mid \varphi_{f,c}$ is locally integrable at $0\}$ as when $c \ll 1$ it's locally integrable and $c \gg 1$ not.

Eg 4. ① $f = z^2 - w^2$ node $\Rightarrow C_0(f) = 1$; ② $f = w^2 - z^3$ cusp $\Rightarrow C_0(f) = \frac{5}{6} < 1$, usually $C_0 \searrow$, the singularity is more complex.

Thm 5. $C_0(f) \le \min\{1, \frac{n}{m}\}$, m is multiplicity, n is number of variables; when $0 \in D$ is ordinary (in meaning of ordinary) $\Rightarrow$ equal, the converse not holds due to the $f = \sum z_i^{m_i} \Rightarrow C_0(f) = \min\{1, \sum \frac{1}{m_i}\}$ is equal by choosing proper (m_i), but not ordinary. (It's not concrete as we need!)

• ① x smooth $\Rightarrow F_x \simeq pt$
② (Milnor) x isolated singularity $\Rightarrow \dim H^i(F_x, \mathbb{C}) = \begin{cases} \dots \end{cases}$
③ F_x is Stein $\Rightarrow H^i = 0$, $i \ge n+1$

Rk. General computation of integral for $C_0(f)$ isn't worked well generically: we need transform into monomial by changing of variables but it's only existence by Hironaka and have no algorithm to compute generally.

• Hence take $\mathrm{supp}(\varphi) \subset B_\varepsilon$ then Milnor fibre

Consider extend $c \in \mathbb{R}_{>0}$ to $\mathbb{C}$: set Archimedean zeta function $Z_{f,A}(s) = \int |f|^{2s} \varphi \, dz \wedge d\bar{z}$ holomorphic for $\varphi \in C^\infty_c(\mathbb{C}^n)$ on right half plane $(\mathrm{Re}(s) > 0)$

• Archimedean ζ is associated to $(\mathbb{C}^n, D)$ here over $\mathbb{Z}_p$ localise. (The D-module $D[s] f^s$ ...) completion of $\mathbb{Q}$ ... to p-adic case, and here Deligne school method comes in

Thm 6. (Bernstein–Sato) $\forall f \in \mathbb{C}[z_1 \dots z_n]$, $\exists\, b(s) \in \mathbb{C}[s]$, s.t. $\exists\, P \in \mathbb{C}[z_1 \dots z_n]\langle \partial_1 \dots \partial_n \rangle[s]$ the differential operator $P = \sum P_i s^i$, P_i linear and $\underline{P f^{s+1} = b(s) f^s}$ (*), $I = \{b \mid (*)\} \subseteq \mathbb{C}[s]$ ideal, $\mathbb{C}[s]$ is PID $\Rightarrow I = \langle b_f(s) \rangle$, $b_f(s)$ called the Bernstein–Sato polynomial, which chosen to be unique monic.

• $\int |f|^{2s} \omega \wedge \bar{\omega} = \int |t|^{2s} dt\, d\bar{t} \left(\int \frac{\omega}{df} \wedge \overline{\frac{\omega}{df}} \right)$, $|t| \le \varepsilon$, $f^{-1}(t) \cap B_\varepsilon$ Milnor fibre

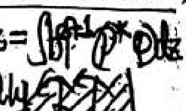
Pf. By $\partial_i f^{s+1} = (s+1) f^s$, then linear ☐ (A rigorous proof of existence relies on the holonomic D-module)

Eg 7. ① $b_f(s) = s+1 \iff D$ smooth, and $b_f(s)$ is more complex can recover the singularity.

② It has algorithm in Macaulay 2 code, thus can be compute generally by computer.

③ **Facts.** A. $(s+1) \mid b_f(s)$; B. $(s + C_0(f)) \mid b_f(s)$; C. $-C_0(f)$ is the largest root of b_f (Kollár); D. All roots rational (Kashiwara)

Thus we can extend $\{\mathrm{Re}(s) > 0\}$ to $\{\mathrm{Re}(s) > -1\}$ meromorphic $b_f(s) Z_{f,A}(s) = \int b_f(s) |f|^{2s} \varphi \, dz = \int (P f^{s+1}) \varphi \, dz = \int |f|^{s+1} P^* \varphi \, dz$ is holomorphic for $\mathrm{Re}(s) > -1 \Rightarrow Z_{f,A}(s)$ meromorphic with $\{$poles of $Z_{f,A}\} \subset V(b_f)$ and extend periodly



thus $\{$poles of $Z_{f,A}\} \subset \{$zeros of $b_f - l \mid l \in \mathbb{Z}_{\ge 0}\}$ (this idea origins Gelfand) (Similiar as Γ-function) thus we can extend zeta function by b_f-function

Thm 8. (Kollár, Lichtin) $\min\{\tilde{\alpha}_f, 1\} = C_0(f) = \mathrm{lct}(f)$ (log canonical threshold of (f) principal divisor (X, D)), $\tilde{\alpha}_f$ is the minimal exponent of $\tilde{b}_f = b_f / (s+1)$

• Kashiwara's Rationality thm

Conjecture 9. (Loeser, 1985) $\tilde{\alpha}_f$ is largest nontrivial pole of $Z_{f,A}(s)$

wild (lct) mild singularity

$\pi^* D = \tilde{D} + \sum a_i E_i$, $K_{Y/X} = \sum b_i E_i$ $\Big\} \Rightarrow \tilde{\alpha}_f \ge \min \frac{b_i + 1}{a_i}$ (except $-1, -2, \dots$) $= \mathrm{lct}(f)$ (Mustață, Popa, [?])

① $\mathrm{Jac}(f) \subseteq V(f)$; (By replace $V(f)$ by its open neighborhood)

② $\mathcal{N} := \mathcal{H}^0_{\mathcal{J}} \mathcal{O}_{X \times \mathbb{C}}$ ... (D ...) symplectic geometry, Hodge theory
③ $\partial_t \curvearrowright \mathcal{N} \xrightarrow{t} \mathcal{N} \to \frac{\mathcal{N}}{t\mathcal{N}} \to 0$
$\mathrm{Ch}(\mathcal{N}) = W_f \cap \Lambda$, $\Lambda \subset T^*X$ Lagrangian give $\mathcal{N}/t\mathcal{N}$ holonomic

Mustață–Popa conjectured that it takes min at ...

Thm 10 (Malgrange, 1983) $\{$eigenvalue of monodromy action T on $H^i(F_x, \mathbb{C})\} = \{e^{2\pi i \alpha} \mid b_f(\alpha) = 0\}$ (Bernstein–Beilinson–Deligne school) ⇒ $b_f(s)$ and induction step on $H^i(F_x)$ ...

Borel's old theorem had known eigenvalue is of form $e^{2\pi i \alpha}$, $\exists \alpha$ via Dehn twist on $H^i(F_x, \mathbb{Q})$

Nearby & vanishing cycle functor (It's a sheaf-theoretic invariant by Grothendieck and Deligne's well $\iff$ V-filtration)

Consider fibres over $\mathbb{C}^* \subset \mathbb{C}$: $\tilde{X} \xrightarrow{k} X \longleftarrow X_0 = \{f = 0\}$, then $\underline{\mathbb{C}}_X$ on X local system = local constant sheaf

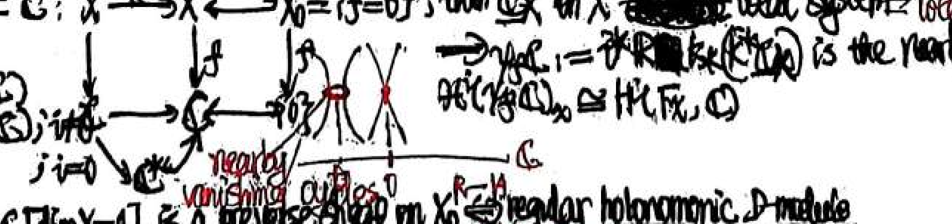
$\Rightarrow \psi_f \mathbb{C} := i^* R k_* (k^* \underline{\mathbb{C}}_X)$ is the nearby cycle

$\mathcal{H}^i(\psi_f \mathbb{C})_x \cong H^i(F_x, \mathbb{C})$

then vanishing cycle is

$\varphi_f \mathbb{C} := \mathrm{Cone}(i^* \mathbb{C} \to \psi_f \mathbb{C})$

$\Rightarrow \mathcal{H}^i(\varphi_f \mathbb{C})_x = \begin{cases} \tilde{H}^i(F_x, \mathbb{C}) & i \neq 0 \\ 0 & i = 0 \end{cases}$



Thm 11 (Gabber) ${}^p\psi_f \mathbb{C} := \psi_f \mathbb{C}[\dim X - 1]$ is a perverse sheaf on $X_0 \overset{R-H}{\iff}$ regular holonomic D-module correspondence

$\Rightarrow T \curvearrowright {}^p\psi_f \mathbb{C} = \bigoplus {}^p\psi_{f,\lambda} \mathbb{C}$ (monodromy action's eigenvalue in nearby cycle)

$f \in \mathbb{Z}[x_1 \cdots x_n]$ and modulo p^{i}, let $M_i := \#\{x \in (\frac{\mathbb{Z}}{p^i\mathbb{Z}})^n \mid f(x) \equiv 0 \bmod p^i\}$, then its generating function $g(T) := \sum_i \frac{M_i}{\#(\mathbb{Z}/p^i\mathbb{Z})^n} T^i$ [Igusa]

$= \sum \frac{M_i}{p^{in}} T^i$

Eg.11 (Hensel's lemma) $x \in \mathbb{Z}^n$, $f(x) \equiv 0 \bmod p$, $\exists j$, $\frac{\partial f}{\partial x_j} \not\equiv 0 \bmod p \Rightarrow \#\{y \in (\frac{\mathbb{Z}}{p^i\mathbb{Z}})^n \mid f(y) \equiv 0 \bmod p^i, x \equiv y \bmod p\} = p^{(i-1)(n-1)}$

$\Rightarrow g(T) = \frac{M_1}{p^n} \frac{p}{p-T}$

Eg.12. ① $f(x,y) = xy$; $b_f(s) = (s+1)^2$; $g(T) = \frac{p^2-T}{(p-T)^2}$ } the relation between b_f and g is given by monodromy conjecture

② $f = y^2 - x^3$; $b_f(s) = (s+1)(s+\frac{5}{6})(s+\frac{7}{6})$; $g(T) = \frac{p^4 + (p-p^3)T^5 - T^6}{(p-T)(p^5-T^6)}$

Thm 13. (Denef, Sharfarevich, Igusa) g is rational function

Pf. Consider $Z_{f,p}(s) = \int_{\mathbb{Z}_p^n} |f(x)|_p^s \cdot |dx|$, dx is the Haar measure on $\mathbb{Q}_p$, $\mathbb{Z}_p \subset \mathbb{Q}_p$ is the unit ball, i.e. $\mu(\mathbb{Z}_p) = 1$ (p-adic integral)

$\Rightarrow Z_{f,p}(s) = \sum_i \int_{\{\mathrm{ord} f(x) = i\}} p^{-is} |dx| = \sum p^{-is} \int_{\mathrm{ord} f(x) = i} |dx| = \sum p^{-is} [\mu(\{\mathrm{ord} f(x) \geq i\}) - \mu(\{\mathrm{ord} f(x) \geq i+1\})] = \sum p^{-is} (\frac{M_i}{p^{in}} - \frac{M_{i+1}}{p^{(i+1)n}}) = \frac{(p^{-s}-1) g(p^{-s}) + 1}{p^{-s}}$

$\Rightarrow g(T) = \frac{T Z_{f,p}(s)|_{p^{-s}=T} - 1}{T-1}$, thus it's rational (with integral coefficient) ▨ thus we study $Z_{f,p}(s)$ enough

By Hironaka's consequence, on $X = \mathbb{Q}_p^n$ we also have log resolution $\pi: \tilde{X} \to X$, $\pi^* D = \sum a_i E_i$ ($E_0 = \tilde{D}$, $a_0 = 1$), $K_{\tilde{X}/X} = \sum b_i E_i$

$\Rightarrow Z_{f,p}(s) = \frac{Q(p^{-s})}{\prod(1 - p^{-(a_i s + b_i + 1)})}$, Q is a (complex) polynomial $\Rightarrow$ poles of $Z_{f,p}$ is of form $s_0 = -\frac{b_j+1}{a_j} + \frac{2\pi i m}{(\log p)(b_j+1)}$, $\exists j \in I$, $m \in \mathbb{Z}$

i.e. $\{\mathrm{Re}\, s_0 \mid \text{pole}\} \subset \{-\frac{b_j+1}{a_j}\}$

Conjecture 14. (Strong monodromy conjecture, Igusa, 1988) $\{\mathrm{Re}\, s_0 \mid \text{pole}\} \subset \{\alpha \mid b_f(\alpha) = 0\}$ for "most" prime p

Eg.15. (cusp case) $f(x,y) = x^2 - y^3 \Rightarrow \{-1, -\frac{5}{6}, -\frac{7}{6}\} \supsetneq \{-1, -\frac{5}{6}\}$

Coro 16. (Weak monodromy conjecture) $\{\mathrm{Re}\, s_0 \mid \text{pole}\} \subset \{\alpha \mid e^{2\pi i \alpha}$ is eigenvalue of T monodromy action$\}$ Pf. Thm 9 ▨

Motivic integration: Replace $\mathbb{Z}_p$ by others

$\mathcal{L}_d(X) = \mathrm{Hom}(\mathrm{Spec} \frac{\mathbb{C}[t]}{(t^{d+1})}, X)$ is the d-order jet/deformation of X and $\mathcal{L}(X) = \varprojlim \mathcal{L}_d(X)$ $\mathrm{Hom}(\mathrm{Spec}\, \mathbb{C}[[t]], X) =: X_\infty \to X_m \to X$

• arc space / arc scheme

Integrand is $\mathrm{ord}_t(f \circ \gamma)$, $\gamma \in \mathcal{L}_d(X)$, and the constant has $X_k = \{\gamma \in \mathcal{L}_d(X) \mid \mathrm{ord}_t(f \circ \gamma) = k\}$ can be used to compute as sum

measure μ is motivic measure • Originally Kontsevich expect μ makes $\mathbb{L}$ invertible; it's ring valued, and behaves as in Grothendieck

$\Rightarrow$ the motivic zeta function $Z_{f,\text{motivic}}(s) = \int \mathbb{L}^{-\mathrm{ord}_t f \cdot s} d\mu$, $\boxed{\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}]} \in K_0(\mathrm{Var}/\mathbb{C})$ the Grothendieck group (ring) ring's operations

$= \frac{1}{\mathbb{L}^n} \sum [X_d] \mathbb{L}^{-d(s)} \in K_0(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-1}]$ $\mu(X \sqcup Y) = \mu(X) + \mu(Y)$, $Z \subset X$ closed (This is the idea of motivic, via $\mathbb{A}^1_{\mathbb{C}}$ (or $\mathbb{P}^1_{\mathbb{C}}$) as variable...)

We can use $Z_{f,\text{motivic}}$ to recover $Z_{f,p}$ by and $\mu(X \times Y) = \mu(X) \cdot \mu(Y)$. Eg. $A = \mathbb{Z}$ and $\#k < \infty \Rightarrow \#X(k) < \infty$

$\Rightarrow \mu(X) = \#X(k)$ is counting measure

Thm 17. (Denef, Loeser) $Z_{f,\text{motivic}} = \frac{1}{\mathbb{L}^n} \sum_{J \subset I} [E_J^\circ] \prod_{j \in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_j s + (b_j+1)} - 1}$ via the log resolution, E_j has index set I, $J \subset I$

taking its motivic characteristic we have $\Rightarrow E_J^\circ = \bigcap_{j \in J} E_j \setminus \bigcup_{i \notin J} E_i$

It's something subtle and omitted here

$Z_{f,\text{top}}(s) = \sum_{J \subset I} \chi(E_J^\circ) \prod_{j \in J} \frac{1}{a_j s + (b_j+1)} = \lim_{p \to 1} Z_{f,p}(s)$ (How this prime $p \to 1$?) if we admit it well-defined, we can use

L'Hôpital's rule and Grothendieck trace formula alone) ▨ related with the ring spectra

Precisely, $\lim_{p \to 1} Z_{f,p}(s) = \lim_{p \to 1} \frac{1}{p^n} \sum_{J \subset I} \# E_J^\circ(\mathbb{F}_p) \cdot \prod \frac{p-1}{p^{(a_j s + b_j + 1)} - 1} = \sum_{J \subset I} \boxed{\lim_{p \to 1} \# E_J^\circ(\mathbb{F}_p)} \prod \frac{1}{a_j s + b_j + 1} = \sum_{J \subset I} \lim_{p \to 1} \sum (-1)^i \mathrm{tr}(F_r^p \curvearrowright H^i_{et}) \prod \frac{1}{a_j s + b_j + 1}$

Igusa

$p \to 1$, the $F_r^p \to \mathrm{Id} \Rightarrow \# E_J^\circ(\mathbb{F}_p) \to \chi_{\ell\text{-adic}}(E_J^\circ) \overset{\text{etale comparison}}{=} \chi(E_J^\circ)$ $= \sum_{J \subset I} \chi(E_J^\circ) \prod \frac{1}{a_j s + b_j + 1}$

motivic case all things are expected to be generalized: one can show $-\mathrm{lct}(f)$ --- $\mathrm{lct}(f)$ is largest pole of $Z_{f,\text{motivic}}(s)$

Conjecture 18. (Monodromy conjecture) $\{\text{poles}\} \subset V(b_f)$

Conjecture 19. (Mustață–Popa, Loeser) ① $\lambda_f = \frac{b_i+1}{a_i}$, $\exists i$; ② $-\lambda_f$ is largest pole of $Z_{f,A}$ and $Z_{f,\text{motivic}}$.

Note that poles of $Z_{f,\text{motivic}}$ are all form $\frac{b_i+1}{a_i}$, thus ② $\Rightarrow$ ① ▨

D-module. $\mathcal{D}_X$ = sheaf of differential operator on X, $\mathcal{D}_X(X) = \mathbb{C}[z_1 \cdots z_n]\langle \partial_1 \cdots \partial_n \rangle / \begin{matrix}[\partial_i, \partial_j] = 0 \\ [\partial_i, z_j] = \delta_{ij}\end{matrix} =: D_X$

left D_X module $M \Leftrightarrow$ left $\mathcal{O}_X$-module M + flat connection $\nabla: T_X \otimes M \to M \Leftrightarrow d: M \to \Omega^1_X \otimes M$

Eg.20 $\mathcal{O}_X$, $\mathcal{O}_X(*\mathrm{div}(f))$ are D_X-module;

Eg.21. The de Rham complex is $[M \to \Omega^1_X \otimes_{\mathcal{O}_X} M \to \Omega^2_X \otimes M \to \cdots \to \Omega^n_X \otimes M][n]$, the Poincaré Lemma reformulated as $DR(\mathcal{O}_X) \overset{q.i.s}{\simeq} \mathbb{C}_X[n]$

$=: DR(\mathcal{O}_X)$ when $X = \mathbb{C}^n$ trivial

Eg.22. $DR(\mathcal{O}_X(*f)) = Rj_* \mathbb{C}_U[n]$, $U = X - \mathrm{div}(f) \xrightarrow{j} X$

V-filtration (D-module analogue of nearby cycle) (And vanishing cycle is $\mathrm{gr}^\alpha_V\, i_+ \mathcal{O}_X$)

Thm 23. (Riemann-Hilbert, Kashiwara–Mebkhout) We have category equivalence $\{$regular holonomic D_X-modules$\} \underset{\sim}{\xrightarrow{DR}} \{$perverse sheaves$\} \subset \{$constructible sheaves$\}$

To depict V-filtration, consider the graph embedding $i: X \to X \times \mathbb{C}_t$ ($t: X \to \mathbb{C}$), for

M is D-module, $i_+ M \cong M \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$, where i_+ is the direct image of i • due to X not smooth, can't define $\psi_f \mathbb{C}[n]$

Thm 24. (Kashiwara–Malgrange) M regular holonomic, eigenvalues of ψ_f ... replace it by $\mathbb{C}[n]$

(quasi-unipotent monodromy) $\Rightarrow \exists!$ filtration on $i_+ M$, $V^\bullet i_+ M$, s.t. ① (multiplier ideal) ... define V-filtration ... $V^\alpha \subset V^\beta$, $\alpha > \beta$, $V^\alpha = \bigcup_{\beta > \alpha} V^\beta = V^{\alpha + \epsilon}$

$\forall\, \epsilon \ll 1$; $\{\alpha \in \mathbb{Q} \mid V^\alpha / V^{>\alpha} \neq 0\}$ is discrete set ② (monodromy) The monodromy $S = -\partial_t t$, $V^\alpha i_+ M$ is coherent $\quad V^\beta = \bigcap_{\beta > \alpha} V^\beta$

$D_X\langle s, t \rangle$-module; ③ $V^\alpha \underset{\partial_t}{\overset{t}{\rightleftarrows}} V^{\alpha+1}$, ④ $S + \alpha = -\partial_t t + \alpha$ is nilpotent on $\mathrm{gr}^\alpha_V i_+ M = V^\alpha / V^{>\alpha}$ • multiplier ideal also has this ...

Thm 25. (Kashiwara) $DR(\mathrm{gr}^\alpha_V(i_+ M)) \cong \psi_{f,\lambda} DR(M)$, $\forall\, 0 < \alpha \leq 1 \Rightarrow \bigoplus DR(\mathrm{gr}^\alpha_V(i_+ M)) = \psi_f DR(M)$ $\mathcal{J}(\alpha D)$ resolution of singularity

Eg.26. $f = x_1^{m_1} \cdots x_n^{m_n} \Rightarrow V^\alpha(i_+ \mathcal{O}_X) = D_X(\prod x_i^{\lceil \alpha m_i - 1 \rceil} \otimes 1)$, thus all f, the V-filtration and nearby cycle are computable via

Prop 27 (Malgrange) $\{\beta \mid gr_V^\beta (\iota_+ \mathcal{O}_X) \neq 0\} = \{\beta \mid b_f(\beta) = 0\} \bmod \mathbb{Z}$, it advices that we can view V-filtration by two ways: (local) b-function (Saito), and the Hodge module (Davis)

composite of

Consider $M[s]f^s$, as $D_X\langle s,t\rangle$-module, as $\mathcal{O}_X$-module $M[s]f^s \cong M \otimes \mathbb{C}[\partial_t] \cong \iota_+ M$, and the two $\mathcal{O}_X$-module isomorphism is in fact $D_X\langle s,t\rangle$-module isomorphism

Thm 28. (Mustata–Popa) $M[s]f^s \xrightarrow{\cong} \iota_+ M$ given by $m s^k f^s \mapsto (-\partial_t t)^k (m \otimes 1)$

$\frac{m}{f} Q_k(s) f^s \leftarrow m \otimes \partial_t^k$, $Q_k(x) = \prod_{0 \le i \le k-1} (x+i)$

Def 29. local b-function for $u \in M[s]f^s$

$b_u(s)$ satisfy $P \cdot (tu) = b_u(s) u$

Eg 30. When $u = f^s$, $b_u(s) = b_f(s)$ globally

Thm 31. (Kashiwara–Saito) $V^\alpha \iota_+ M = \{u \in \iota_+ M \cong M[s]f^s \mid$ all roots of $b_u \le -\alpha\}$; $b_u(s)$ has only rational roots

Eg 32. $X = M_n(\mathbb{C}) \xrightarrow{f = \det} \mathbb{C}$, then $G = GL_n \times GL_n \curvearrowright X$ reductively $\Rightarrow$ decomposition $\mathbb{C}[X, \frac{1}{f}] = \bigoplus W_p$, p is the dominated weight $p = (p_1, \dots, p_n) \in \mathbb{Z}^n$ and $p_1 \ge p_2 \ge \dots \ge p_n$, $W_p = S_p \mathbb{C}^n \otimes S_p \mathbb{C}^n$, S_p is the Schur functor at p

Thus ① $u \in W_p \subset M$, $b_u(s) = \prod (s + 1 + p_i + n - i) \Rightarrow b_f(s) = \prod (s+i)$ by taking $p = 0$ ($\det(\partial_{ij}) f^{s+1} = \prod (s+i) f^s$ the Cayley's formula); ② we have $(\mathbb{C}[s], G)$-decomposition $V^\alpha \iota_+ M = \bigoplus (P_{p,\alpha}(s)) \otimes W_p \cdot f^s$, $P_{p,\alpha}(s) := \prod [s + \lambda_i]_{\lceil \alpha - \lambda_i \rceil}$, It's the second the representation argument of multiplicity-free space

$\le \mathbb{C}[s]$ ideal $\qquad [s+a]_k := \begin{cases} 1; & k \le 0 \\ \prod (s+a+i); & k \ge 1 \end{cases}$

Multiplier ideal. $\lambda \in \mathbb{Q}_{>0}$

$\mathcal{J}(f^\lambda)$ is defined $U \mapsto \{g \in \mathcal{O}_U \mid \frac{|g|^2}{|f|^{2\lambda}}$ locally integrable$\}$ • the analytic multiplier ideal

$\alpha(f) = \sup\{\lambda \mid \mathcal{J}(f^\lambda) = \mathcal{O}_X\}$ • $\mathcal{J}_{an}$ & $\mathcal{J}$ is related by $\mathcal{J}_{an}(D)$ $\quad \mathcal{J}_{an}(D)$, and the plurisubharmonic function and f semi-invariant

• the rep V decompose into irreducible each one time, V is called multiplicity free

Thm 33. (Budur–Saito) $\mathcal{J}(f^\lambda) = (V^{\lambda} \iota_+ \mathcal{O}_X) \cap (\mathcal{O}_X \otimes 1) = \mathcal{J}(f^{\lambda - \epsilon})$, $\mathcal{O}_X \otimes 1 = F_0 \iota_+ \mathcal{O}_X$ (Filtration $F_\bullet$) [illegible]

Hodge theory $V^\alpha \iota_+ \mathcal{O}_X = V^\alpha (\iota_+ M) \cap (\iota_+ \mathcal{O}_X)$, $j: U = X - D = X - f^{-1}(0) \hookrightarrow X$, $\frac{\mathcal{O}_U[s]f^s}{(s+\alpha)^m}$ is a mixed Hodge module $\Leftrightarrow$ singular analogue of VHS

$\partial_t \frac{\mathcal{O}_U[s]f^s}{(s+\alpha)^m}$ also mixed Hodge $\Rightarrow F_p j_* \left(\frac{\mathcal{O}_U[s]f^s}{(s+\alpha)^m}\right)$ is $\mathcal{O}_X$-module

• Variation of Hodge structure $\boxed{\text{variation}}$ of mixed Hodge structure (mixed Hodge conn) (in cohomology / Geometrically) singular analogue of connection

and we can take $\varinjlim_m F_p j_* \left(\frac{\mathcal{O}_U[s]f^s}{(s+\alpha)^m}\right)$

$\mathcal{J}(\varphi)(U) = \{g \mid \frac{|g|^2}{e^{\varphi}}$ loc$\}$ is ideal, $\mathcal{O}$-module of Hodge

Thm 34. (Davis, 2024) $V^\alpha \iota_+ \mathcal{O}_X f^s \cong \bigcup \varinjlim F_p j_* \left(\frac{\mathcal{O}_U[s]f^s}{(s+\alpha)^m}\right)$ (ideal is

geometric representation-theoretic) refers theory, the minimal. $\tilde{\alpha} = \min_{k \in \mathbb{N}} \{k + \alpha \mid gr_k^F V^\alpha \iota_+ \mathcal{O}_X \neq gr_k^F [illegible]\}$ $\alpha \in (0,1]$

to additional filtration

Coro 35. $\exists \delta$, s.t. $0 \to V^\alpha \iota_+ M \to$ [illegible] $\xrightarrow{s = -\alpha} M f^{-\alpha} \to 0$

• Recall multiplier ideal is for Kawamata–Viehweg vanishing & Nadel vanishing

$\mathcal{J}(D) = \mu_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}/X} - \lfloor \mu^* D \rfloor) \subseteq \mathcal{O}_X$ ②

Observation: $\pi_X \leftarrow$ higher multiplier ideals, log resolution of (X, D) (Schnell, 2023) $\quad \iota_+ M \xrightarrow{s = -\alpha} \frac{M[s]f^s}{(s+\alpha)}$ Hodge ideal (Mustata–Popa, 2016, 2018) $\quad \cong M[s]f^s \xrightarrow{s = -\alpha}$

$gr_0^F V^\alpha \iota_+ \mathcal{O}_X = \mathcal{J}(f^{\alpha})$

$\trianglelefteq gr_0^F$ is ideal,

and for $k > 0$, $gr_k^F V^\alpha \iota_+ \mathcal{O}_X =: \mathcal{J}^k(f^\alpha) \trianglelefteq gr_k^F$ higher multiplier ideals

Here $F_\bullet \iota_+ \mathcal{O}_X$ is by $F_k \iota_+ \mathcal{O}_X := \sum \mathcal{O}_X \otimes \partial_t^i$ and thus $gr_k^F V^\alpha \iota_+ \mathcal{O}_X = \frac{F_k \cap V^\alpha}{F_{k-1} \cap V^\alpha} \hookrightarrow \frac{F_k}{F_{k-1}} = gr_k^F$

Coro 38. We can partial solving the Mustata–Popa conjecture: $\exists i, l \in \mathbb{Z}$ s.t.

$\tilde{\alpha}_f = \frac{b_i + 1}{a_i} + \frac{l}{a_i}$ (using interpretation of Coro 36.)

• Lichtin's thm tell use that roots of $b_f(s)$ are this form $-\frac{b_i + 1 + l}{a_i}$, it's not a new consequence!

We can compute $gr_k^F V^\alpha$ by log resolution:

Thm 37. (Davis) $\pi: \tilde{X} \to X$ w.r.t. $D = (f)$

then $gr_\bullet^F V^{>\alpha} = \bigoplus gr_k^F V^{>\alpha}$ is $\mathrm{Sym}\, T_X$-module

$\Rightarrow gr_\bullet^F V^{>\alpha} = R\pi_* (\omega_{\tilde{X}/X} \otimes gr_\bullet^F V^{>\alpha} \otimes_{\mathrm{Sym}} \mathrm{Sym}(\pi^* T_X))$

taking $k = 0 \Rightarrow \mathcal{J}(f^\alpha) = R\pi_* (\mathcal{J}(\tilde{f}^\alpha) \otimes \omega)$

(Rk. $F^\bullet$ is decreasing, $F_\bullet$ is increasing) ⊗ [illegible]

mixed Hodge module (MHM)

[Deligne, 1974, ICM] heuristic dictionary

ℓ-adic cohomology on varieties over $\mathbb{F}_p$ $\longleftrightarrow$ Hodge theory on complex mfds

$\cup \qquad \cup$

mixed ℓ-adic complex $\xleftrightarrow{[BBDG]}$ M.H.M (Saito)

Vector bundles $=$ $\mathcal{O}$-module for smooth

{ singular • L^2-cohomology is "completion of de Rham cohomology"

$\mathcal{O}$-module a generalisation;

Now replace vector bundles by $R^k f_* \mathbb{C}$, a generalisation is filtered D-module, (i) & (ii) generalises well but the MHM, and ($R^k f_*$) Hodge module needs D-modules; extending Saito's proof used forms.

① Classical Hodge (Hodge decomposition) $H^k(X; \mathbb{C}) = \bigoplus H^{p,q} \Leftrightarrow \exists$ Hodge filtration $F^\bullet$ s.t. $H^k = F^p \oplus \overline{F^{k+1-p}}$

Adding $F^\bullet$ to $H^k(X; \mathbb{C})$ is Hodge structure (of weight k) ($p + q = k$)

② Modern Hodge. Relative & Variation Hodge structure

$(\Rightarrow H^{p,q} = F^p \cap \overline{F^q}$, $F^p = \bigoplus_{i \ge p} H^{i, k-i})$

$X \xrightarrow{f} B$ fibres X_b $\qquad$ [Griffiths, 1960s]

$R^k f_* \mathbb{C}$ the higher direct image

$(R^k f_* \mathbb{C})_b \cong H^k(X_b; \mathbb{C})$

$\Rightarrow R^k f_* \mathbb{C}$ consist Hodge structure (of weight k)

③ (more) Modern Hodge. Singular f. $R^k f_* \mathbb{C}$ from to B's singular fibres and recording degeneration of Hodge

(A) VHS near singularity $\Delta^n \setminus (\Delta^*)^n$

(B) HS on L^2-cohomology on $C_0 = C \setminus \{p_1, \dots, p_n\}$, C is smooth projective curve

(C) Resolution of singularity

singular fibre has no pure Hodge, how pure degenerate to mixed is what important

Def 40. (VHS) Variation of Hodge structure (of weight k) on B smooth is $(V_{\mathbb{Q}}, \mathcal{O}, \nabla, F^\bullet \mathcal{O})$

① $V_{\mathbb{Q}}$ is $\mathbb{Q}$-local system (i.e. local constant)

② $\mathcal{O}$ holomorphic vector bundle on B;

③ $\nabla: \mathcal{O} \to \mathcal{O} \otimes \Omega_B^1$ flat connection on $\mathcal{O}$;

④ $F^\bullet \mathcal{O}$ consists subholomorphic vector bundle

(Rk. In classical $H^{p,q}$ not holomorphic, but F^p are.)

s.t. (i) $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \cong \ker \nabla \subset \mathcal{O}$

$(\Rightarrow V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_B \cong \mathcal{O})$

$\boxed{\text{(ii) (Griffith's transversality) } \nabla(F^p \mathcal{O}) \subset \Omega_B^1 \otimes F^{p-1} \mathcal{O}}$

(iii) $\forall b \in B$, $(V_b, F^\bullet(V_b))$ is Hodge structure (of weight k)

$(\Rightarrow V_{\mathbb{Q}} / V_{\mathbb{C}} \Leftrightarrow V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ underlies a variation of HS)

Def 41. (Hodge module) The category of Hodge module of weight w on X is $HM(X, w) \Rightarrow HM_{\ge d}(X, w) := \bigcup \bigoplus_{Z \subset X \text{ closed}} HM_Z(X, w)$, $\dim X \ge 1$

category of HS weight w, $X = pt$

$HM_Z(X, w)$ the Hodge module supported by Z, $M \in HM_Z(X, w)$ is $(M, F_\bullet M, K_{\mathbb{Q}})$

(Rk. M generalises $(\mathcal{O}, \nabla)$, $K_{\mathbb{Q}}$ generalises $V_{\mathbb{Q}}$, $F_\bullet M$ is $F^\bullet \mathcal{O}$)

Where $\mathcal{M}$ is <u>regular holonomic D_X-module</u>, $F_\bullet\mathcal{M}$ is coherent $\mathcal{O}$-module filtration, $K_{\mathbb{Q}}$ is $\mathbb{Q}$-perverse sheaf s.t. (i) $K_{\mathbb{Q}}$ (restrict to an open set, it's still $(\mathcal{O},\nabla)$) $\otimes_{\mathbb{Q}} \mathbb{C} \underset{qis}{\simeq} DR(\mathcal{M})$; (ii) $F_\bullet\mathcal{M}$ is good filtration: $F_iD_X \cdot F_k\mathcal{M} \subset F_{k+i}\mathcal{M}$, F_iD_X = sheaf of differential operator order $\leq i$; $k \gg 0$ $F_iD_X \cdot F_k\mathcal{M} = F_{k+i}\mathcal{M}$.

(iii) (Difficult due to restrict to pt/fibre is hard for D-modules, and it's equivalent to the classical Griffith transversality) selection for this is V-filtration) $\forall\, U \subset X$ open, $\forall f: U \to \mathbb{C}$ holomorphic $f|_{Z \cap U} \neq 0$, we restrict to $V(f)$ single pt. Again using the graph $\iota: U \to U \times \mathbb{C}$ to replace $\mathcal{M}$: $gr^{\alpha}_V \iota_+(\mathcal{M}|_U) = V^{\alpha}\iota_+ / V^{>\alpha}\iota_+(\mathcal{M}|_U)$

$\Rightarrow \operatorname{supp} gr^{\alpha}_V \iota_+(\mathcal{M}|_U) \subset V(f)$, and we put a filtration on $gr^{\alpha}_V \iota_+(\mathcal{M}|_U)$ by $W(N)$: the monodromy weight filtration, $N := s\partial_s - \alpha$

N act nilpotently on $gr^{\alpha}_V \iota_+(\mathcal{M}|_U) \Rightarrow \exists!\, W(N)$, s.t. $\begin{cases} N: W(N)_l \to W(N)_{l-2} \\ N^l: gr^{W(N)}_l \iota_+(\mathcal{M}|_U) \to gr^{W(N)}_{-l}\iota_+(\mathcal{M}|_U) \text{ isomorphism} \end{cases}$ (Jordan standard form) Under upper settings, we need

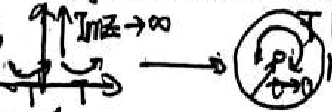
(A) $\bigoplus_{0<\alpha\leq 1} gr^{W(N)}_l gr^{\alpha}_V \iota_+(\mathcal{M}, F_\bullet\mathcal{M})|_U \in HM_{\leq d-1}(X_0 \cap f^{-1}(0))$ (nearby cycle)

(B) $F_k \iota_+\mathcal{M} = \bigoplus F_{k-i}\mathcal{M} \otimes \partial_t^i$ & $F_p V^{\alpha} \xrightarrow{t} F_p V^{\alpha+1}$; $\alpha > 0$; $F_p gr^{\alpha}_V \xrightarrow{\partial_t} F_{p+1} gr^{\alpha-1}_V$; $\alpha < 1$.

Return back to Saito's proof of (A), i.e. Schmid's work on local degeneration on disk $\Delta^* \hookrightarrow \Delta$, namely, $\lim (\mathcal{O}, F^\bullet\mathcal{O})_t = ?$ (This will tell us why V-filtration occurs)

Step 1 lift the limit to universal covering $p: \mathbb{H} \to \Delta^*$, $z \mapsto e^{2\pi i z} = t$. let $\mathbb{V} = \ker\nabla$, $p^*\mathbb{V}$ trivial, i.e. $p^*\nabla$ injective

$\Rightarrow (F^\bullet\mathcal{O})_{p} = F_z V \Rightarrow$ we left to consider $\lim\limits_{\operatorname{Im} z \to \infty} F_z V = ?$ (Here we can see its advantage is centered in the only one fibre) thus we have canonical fibre (all $\cong$) $V = \Gamma(\mathbb{H}, p^*\mathcal{O})^{p^*\nabla}$, with monodromy $T: V \to V$ and $F_{z+1}V = T F_z V$ thus $\operatorname{Im} z \to \infty$ is $z \mapsto z+1$ well-defined as $\operatorname{Re} z$ can be transformed invariantly



However, it's still may not exist

Step 2 (Existence of limiting Hodge structure) First $T = T_s \cdot T_u$, semisimple × unipotent, $T_u = e^{2\pi i N}$, N nilpotent

Thm 42 (Schmid) $F_{\lim} = \lim\limits_{\operatorname{Im} z \to \infty} e^{-2\pi i z N} \cdot F_z$ exists and induce limiting Hodge structure $F_{\lim}$, not pure. $T_s F_{\lim} = F_{\lim}$ is T_s-invariant.

With eigenspace decomposition $V = \bigoplus E_\lambda(T_s)$, then $W(N)$ is monodromy on $E_\lambda(T_s)$

Thm 43 (Schmid) $(\mathcal{O}, F^\bullet\mathcal{O})$ polarized VHS (weight w) $\Rightarrow (E_\lambda(T_s), W(N)_{\bullet - w}, F_{\lim})$ is MHS (weight w) (Hodge–Riemann condition)

Step 3 (V-filtration) We can't extend directly $j_*\mathcal{O}$ as it's too large.

<u>Deligne's meromorphic extension</u> $\tilde{\mathcal{O}}^{\alpha} := \sum \mathcal{O}_\Delta \cdot \tilde{s}_i$, $\{v_i\}$ basis of V, $\tilde{s}_i: \mathbb{H} \to V$, $z \mapsto e^{2\pi i z R_\alpha} e^{-2\pi i z N} v_i$, $R_\alpha = \begin{pmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_n \end{pmatrix}$ are proper chosen "log of T_s", s.t. $\beta_i \in [\alpha, \alpha+1)$

Why V-filtration occurs? $\tilde{\mathcal{O}} = \bigcup \tilde{\mathcal{O}}^{\alpha}$ desired

(1) $\tilde{\mathcal{O}}$ is D_Δ-mod; (2) $\tilde{\mathcal{O}}^{\alpha+1}$ is V-filtration (3) $\tilde{\mathcal{O}}$ is regular

D-module structure is by $\partial_t \tilde{\mathcal{O}} = \nabla_{\frac{\partial}{\partial t}} \tilde{\mathcal{O}}$, $t\tilde{\mathcal{O}}^{\alpha} = \tilde{\mathcal{O}}^{\alpha+1}$, $\tilde{\mathcal{O}}^{\beta} \subset \tilde{\mathcal{O}}^{\alpha}$ for $\beta \geq \alpha$ (3) $\nabla: \tilde{\mathcal{O}} \to \mathcal{O}_\Delta\left(\frac{dt}{t}\right) \otimes \tilde{\mathcal{O}}$ makes regular.

($\tilde{\mathcal{O}}$ is the most base and motivation example of regular holonomic D-module)

Cor 44. $V = \tilde{\mathcal{O}}^{\alpha} / t\tilde{\mathcal{O}}^{\alpha} = \tilde{\mathcal{O}}^{\alpha} / \tilde{\mathcal{O}}^{\alpha+1} \to \tilde{\mathcal{O}}^{\alpha} / \tilde{\mathcal{O}}^{>\alpha} = gr^{\alpha}\tilde{\mathcal{O}}$, and induce isomorphism $E_{e^{-2\pi i\alpha}}(T_s) \xrightarrow{\simeq} gr^{\alpha}\tilde{\mathcal{O}}$

<u>Application</u>. $\mathcal{M}_g \to \{$principally polarized abelian variety$\}$ Q. (Riemann–Schottky problem) the image of this map (It's answered by analytic/nonconstructive methods)

Thm 44. $C \mapsto (\operatorname{Jac}(C), \Theta)$ (Θ (theta)-diviser is ample and used good moduli)

$\operatorname{Jac}(C) \cong \operatorname{Pic}^{g-1}(C)$ (Abel–Jacobi)
$\cup \qquad \cup$
$\Theta \cong \{[L] \mid h^0(C, L) \geq 1\}$
$\cup \qquad \cup$
$\operatorname{Sing}_m\Theta \cong \{[L] \mid h^0(C, L) \geq m\}$
(multiplicity $\geq m$ singularities)

<u>Thm 45.</u> (Martens, Mumford) $\dim \operatorname{Sing}_m(\Theta) \leq g - 2m + 1$, $\forall m \geq 2$, if $\exists m$, s.t. equality holds $\Leftrightarrow \forall m$, equality holds $\Leftrightarrow C$ is hyperelliptic curve

<u>Thm 46.</u> $(\operatorname{Jac}(C), \Theta)$ is indecomposable ppav (principal polarised abelian variety) $\Leftrightarrow \Theta$ irreducible

<u>Conjecture 47</u> (Casalaina-Martin, 2008) (A, Θ) indecomposable ppav $\Rightarrow$

(1) $\dim \operatorname{Sing}_m\Theta \leq \dim A - 2m + 1$, $\forall m \geq 2$;

(2) $\exists m$ equality holds $\Leftrightarrow (A, \Theta) \in \{(\operatorname{Jac}(C), \Theta), C \text{ hyperelliptic}\} \amalg \{(IJ(X_3), \Theta)\}$

(IJ is <u>intermediate Jacobian</u>, X_3 is smooth cubic 3-fold)

(Rk. $\dim IJ(X_3) = 5 \Rightarrow \dim \operatorname{Sing}_3\Theta = 0$ (unique pt mult $= 3$) $= 5 - 2 \times 3 + 1$, thus does equal)

It's proven for (1), (2) holds for $\dim A \leq 5$

<u>Conjecture 48.</u> (Grushevsky) $\operatorname{mult}_x \Theta \leq \frac{\dim A + 1}{2}$ (almost totally open)

Thm 49. (1) (Kollár, 1996) $\operatorname{mult}_x\Theta \leq \dim A$; (2) (Ein–Lazarsfeld) $\operatorname{mult}_x\Theta \leq \dim A - 1$

Pf of (1). (Others like (48) are all its improvements)

$\dim \operatorname{Sing}_m\Theta \leq \dim A - m \Leftarrow (A, \Theta)$ <u>log canonical</u> $\Leftrightarrow \mathcal{J}(A, (1-\varepsilon)\Theta) = \mathcal{O}_A$, $\forall\, 0 < \varepsilon < 1$

Otherwise, $\exists \varepsilon > 0$, s.t. $\mathcal{J} \neq \mathcal{O}_A \Rightarrow 0 \to \mathcal{J} \to \mathcal{O}_A \to \mathcal{O}_Z \to 0$, Z = vanishing locus of $\mathcal{J}$

$H^i(A, \mathcal{J} \otimes \mathcal{O}(\Theta)) = 0$ by Nadel vanishing

(By $H^i(A, \mathcal{J} \otimes \mathcal{O}(\Theta)) = H^i(A, \mathcal{J}((1-\varepsilon)\Theta) \otimes (\mathcal{O}(\Theta) \otimes \omega_A))$, $\omega_A = \mathcal{O}_A$ by Abelian; $\Theta - (1-\varepsilon)\Theta = \varepsilon\Theta$ ample)

$\Rightarrow 0 \to \mathcal{J}(\Theta) \to \mathcal{O}_A(\Theta) \to \mathcal{O}_Z(\Theta) \to 0$'s LHS is $0 \to H^0(\mathcal{J}(\Theta)) \to H^0(A, \mathcal{O}(\Theta)) \to H^0(A, \mathcal{O}_Z(\Theta)) \to$ [illegible]

$\Rightarrow H^0(A, \mathcal{O}_Z(\Theta)) = 0$, but it's impossible using additive structure on A Abelian, $a \in A$ general, s.t. $(\Theta + a) \cap Z \neq \emptyset$

$\Rightarrow \dim H^0(A, \mathcal{O}_Z(\Theta)) = H^0(A, \mathcal{O}_Z(\Theta)) \neq 0$ ☐ (Rk. Such approximate is already best, to sharpen, only way is refinement of the multiplier ideals

Our next task is Hodge ideals and higher multiplier ideal to prove

(1) (Mustaţă–Popa, 2016) Conj 48 holds if $\dim \operatorname{Sing}\Theta = 0$ (Hodge ideal)

(2) (Schnell–Yang, 2023) Conj 47 holds if dimension of <u>centre of minimal exponent</u> $= 1$

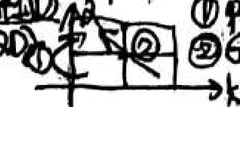
(One can almost say its just C desired)

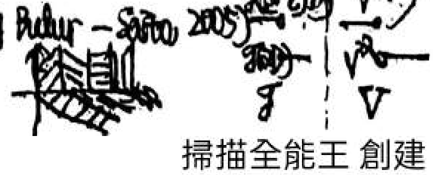
<u>Recall</u> that higher multiplier ideal is by intersecting with V-filtration, $\mathcal{J}_k(\alpha D) \subset \mathcal{O}_X$; $\mathcal{J}_0 = \mathcal{J}$ (by Budur–Saito 2005)

$(k, \alpha) := \frac{\mathcal{J}_k((\alpha+1)D)}{\mathcal{J}_k(\alpha D)}$

(1) periodicity $\mathcal{J}_k((\alpha+1)D) = \mathcal{J}_k(\alpha D) \otimes \mathcal{O}(-D)$

(2) Griffiths transversality is $\mathcal{J}_k(\alpha D) = \mathcal{J}_{k+1}((\alpha+1)D)$. Thus stratified





To imitate the proof of Kollár, we need a Nadel-type vanishing: $B-2\Theta$ ample $\Rightarrow H^q(A, \mathcal{J}_k(2\Theta)\otimes(B+k\Theta))=0$, $q>0$ Page 11

Thm 50. (Schnell–Yang) $Y=$ center of minimal exponent of (A,Θ) (When Θ symmetric, it's connected; otherwise take a component). (Continue lecture 1)

$\dim Y=1 \Rightarrow$ ① Y smooth projective hyperelliptic curve (and reduced, normal, rational singularity...)

② $\dim \operatorname{Sing}_m\Theta \le \dim A-2m+1$; $\forall m\ge 2$; ③ Equality holds for $\exists m \Rightarrow$ $A=\operatorname{Jac}(Y)$

or exotic; $g(Y)=2m$, $\dim A=2m-1$, $\exists x\in\Theta$, $\operatorname{mult}_x\Theta=m$.

Pf. (A,Θ) indecomposable $\underset{\text{Ein-Laz}}{\Longrightarrow}$ Θ has rational singularity $\underset{\text{Saito}}{\Longleftarrow}$ $\tilde{\alpha}_\Theta>1$

$\tilde{\alpha}_\Theta \le \min\limits_{m\ge 2} \dfrac{\operatorname{codim}_A \operatorname{Sing}_m\Theta}{m} \Longrightarrow \operatorname{codim}_A\operatorname{Sing}_m\Theta \ge m+1$

Otherwise, $\exists m\ge 2$, $\dim \operatorname{Sing}_m\Theta \ge \dim A-2m+1 \Rightarrow \operatorname{codim}_A\operatorname{Sing}_m\Theta \le 2m-2 \Rightarrow 1<\tilde{\alpha}_\Theta \le \frac{2m-2}{m}<2$, however it's impossible.

Consider the higher multiplier ideal attached to $\tilde{\alpha}_\Theta=\min\{k+\alpha \mid \mathcal{J}_k(\alpha\Theta)\ne\mathcal{O}_A\}$ $k\in\mathbb{Z}_{\ge0}$, $\alpha\in[0,1)$

Eg 51 $\tilde{\alpha}_\Theta$ =
- ≤ 2, $\forall C$
- $=2$, C Brill–Noether general
- $=\frac{3}{2}$, C hyperelliptic
- $=\frac{5}{3}$, $A=\prod(X_3)=\operatorname{Jac}(C)$

$\mathcal{J}_1=\mathcal{J}_1(\beta\Theta)$, $\tilde{\alpha}_\Theta=1+\beta$, $0<\beta<1$ satisfy

$0\to\mathcal{J}_1\to\mathcal{O}_A\to\mathcal{O}_Z\to 0$, Z reduced, $\mathcal{O}_Z=F_1\operatorname{gr}_1^V$ is lowest piece of a mixed Hodge module.

$Z=\{x\in\Theta \mid \tilde{\alpha}_{\Theta,x}=\tilde{\alpha}_\Theta\}$. $0\to\mathcal{J}_1\to\mathcal{O}_A\to\mathcal{O}_Z\to 0$ $(*)$ Then using Nadel vanishing upper to $(**)$

$\Longrightarrow H^q(A,\mathcal{O}(2\Theta))\twoheadrightarrow H^q(Y,\mathcal{O}_Y(2\Theta))$

Rk. Can't apply to $(*)$ due to

$(*)\Leftrightarrow 0\to\mathcal{J}_1\to\tilde{\mathcal{J}}_1\to W^{\cdot}\operatorname{gr}_1^F\operatorname{gr}_V^\beta \iota_+\mathcal{O}_A\to 0$

Then by Saito's vanishing $H^1(Y,\mathcal{O}_Y(2\Theta)\otimes\rho)=0$, $\rho\in\operatorname{Pic}^0(Y)$ $\square$

Nadel vanishing isn't enough.

$\operatorname{gr}_1^F\operatorname{gr}_V^{\beta}\iota_+\mathcal{O}_A$

$\operatorname{gr}^W\operatorname{gr}_1^F\operatorname{gr}_V^\beta\iota_+\mathcal{O}_A$

A refinement is an extra vanishing as generalisation of Kodaira vanishing

② ① $\Rightarrow$ $A \xrightarrow{2:1} |H^0(A,2\Theta)|$

$Y \xrightarrow[f]{2:1} |H^0(Y,2\Theta|_Y)| =: \mathbb{P}^N$

$\mathcal{O}_Y \to 0$ $(**)$

(Saito) M is D-twisted MHM $\Rightarrow H^i(X,\operatorname{gr}_k^F DR(M)\otimes\mathcal{O}(B))=0$ if $B+2D$ ample

$N=h^0(Y,\mathcal{O}_Y(2\Theta))-1=\chi(Y,\mathcal{O}_Y(2\Theta))-1 \overset{RR}{=} \deg_Y(2\Theta)-g(Y)$

$=2(\Theta\cdot Y)-g(Y)$

and note that $\operatorname{gr}^W\operatorname{gr}_1^F\operatorname{gr}_V^\beta\iota_+\mathcal{O}_A$ is $-\beta(\Theta)$-twisted $\square$

and $\deg(\mathcal{O}_Y(2\Theta))\ge 2g(Y)-1 \Rightarrow (\Theta\cdot Y)\ge g(Y) \Rightarrow (\Theta\cdot Y)=g(Y) \Rightarrow \deg(\operatorname{Im} f)=N \Rightarrow \operatorname{Im} f\subset\mathbb{P}^N$ rational normal curve $\Rightarrow Y$ hyperellipti[c]

Du Bois singularities $\boxed{\text{Lecture 4}}$ (Wanchun Shen)

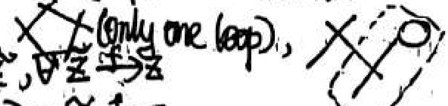
Not change locally under smoothing (= deformation), such as Hodge numbers ...

[Steenbrink, Kollár, Kovács, Saito] Rational singularities $\subset$ Du Bois singularities

By other works of Ishii, Kollár and Kovács

ADE is the smallest type, compare that:

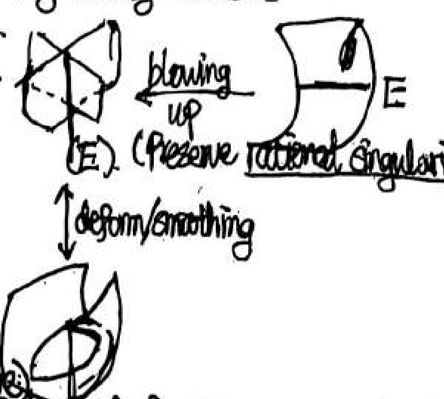
① Du Bois singularity $\Leftrightarrow$ Dual graph more than ADE (only one loop),



② Rational singularity $\Leftrightarrow \mathcal{O}_Z \overset{q.i.s}{\to} Rf_*\mathcal{O}_{\tilde{Z}}$, $\forall \tilde{Z} \xrightarrow{f} Z$ $\Leftrightarrow \exists \tilde{Z} \xrightarrow{1} Z$.

ADE $\Leftrightarrow R^{k}f_*\mathcal{O}_{\tilde{Z}} = 0$ (elliptic curve)

E.g. (Whitney's umbrella)



Def1. (Du Bois singularity, Steenbrink, 1983) $\mathcal{O}_Z \overset{q.i.s}{\to} \underline{\Omega}_Z^0$ the Du Bois complex (hard to define)

Thm2. (Saito) ① $\mathcal{O}_X(*Z) := \bigcup \mathcal{O}_X(kZ)$ is a mixed Hodge module; • It's $(M, F_\bullet, \mathcal{F})$ (M is filtered D-module, $F_\bullet$ is its Hodge filtration,

② Du Bois singular $\Leftrightarrow \tilde{\alpha} \geq 1 \Leftrightarrow F_0 = P_0$ the leading terms of Hodge filtration and Pole order filtration

$\Leftrightarrow$ "A condition on V-filtration"; Rational singular $\Leftrightarrow \tilde{\alpha} > 1$

$\mathcal{F}$ is the preverse sheaf, s.t. the Riemann–Hilbert functor (de Rham functor)

DR: Hol-D-Mod $\longrightarrow$ Perverse

$(M, F_\bullet) \longmapsto \mathcal{F}$

$\underline{\Omega}_X^k$ is represented by a complex, s.t.

① $\underline{\Omega}_X^k = 0$ for $k<0$ or $k > \dim X$; ② cohomologically supported by degree $0 \cdots \dim X$

Idea. A singular differential?

• (Saito) $(M, F_\bullet, \mathcal{F})$ has Hodge structure. It's "mixed" due to it's various functors, denoted $\underline{\Omega}_X^k$

① Reflexive differential; $j: X_{sing} \hookrightarrow X$ closed and $i: X_{sm} \longrightarrow X$ open then $j_*\Omega^k_{X_{sm}}$ is well-defined for X, if X is normal; (It's Du Bois complex when toric ...) (when X is Gorenstein...)

③ Voh-differential $= \mathcal{H}^0(\underline{\Omega}_X^k)$; $\underline{\Omega}_X^k$ generalises both them, and it's good, Hodge-theoritically: we have generalisation of degeneration

$E_1^{pq} = \mathbb{H}^q(X, \underline{\Omega}_X^p) \Rightarrow H^{p+q}(X, \mathbb{C})$; $\mathbb{H}^q(X, \underline{\Omega}_X^p \otimes \mathcal{L}) = 0$ (Kodaira vanishing)

• h-differential is sheafification of Kähler differential w.r.t. h-topology (Voevodsky) $\Omega_h(X) = \underline{\Omega}_h(X)$

Construction (Similiar to Deligne's mixed Hodge, we take a sequence smooth to "approximate" singular and "glue" $\Omega_{X_i}^k$ to $\underline{\Omega}_X^k$)

$\boxed{\text{Step1}}$ A hyperresolution of X is a cohomologically descent semisimplicial sequence of smooth X_i ($\cdots \Rrightarrow X_1 \rightrightarrows X_0 \to X = X_\bullet \to X$

$\boxed{\text{Step2}}$ First we push forward all $\Omega_{X_i}^k$ to X, then taking the M-V sequence of hyperresolution. $\mathbb{Z}_X \to R\varepsilon_{X_\bullet *}\mathbb{Z}_{X_\bullet}$ is q.i.s (Simplicical has inverse arrows, each) $(\varepsilon_i)_{i\leq j}$ s.t. $\varepsilon_j \varepsilon_{j+1,i} = \varepsilon_j \varepsilon_{j,i+1}$

where $X_\bullet$ ... keep track on their space and induce the pullback of functions in sheaf-theoretic

Cubical hyperresolution Y 0-cubical, $Y_1 \to Y_0$ 1-cubical; $Y_{10} \to Y_{00}$ 2-cubical; $Y_{11} \to Y_{01}$; $Y_{100} \to Y_{000}$ 3-cubical ...

Same as the
$$\begin{array}{ccc} E & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ P & \longrightarrow & X \end{array}$$
the 2-cubical of 0-cubical X (replace X by $E \to P$)

If $\tilde{X}(E, P)$ is singular, the 1 cubical $(E \to P)$ can have "a 2-cubical of 1-cubical"

thus we have a 2-cubical, repeat this until not singular ▨

And the hyperresolution is given by diagonal.

Prop3 ① $\forall k, \exists\, \underline{\Omega}_X^k \to \Omega_X^k$, (E.g. the exceptional divisor of A_2-singularity, which is singular) when X is smooth $\Rightarrow$ isomorphisms

② $\underline{\Omega}_X^k|_U = \Omega_U^k$; ③ $E_1^{pq} = \mathbb{H}^q(X, \underline{\Omega}_X^p) \Rightarrow H^{p+q}(X; \mathbb{C})$; ④ $\underline{\Omega}_Y^k \to Rf_*\underline{\Omega}_X^k$ induced by $f: X \to Y$

Pf. ① trivial; ② $\underline{\Omega}_X^k|_U = \Omega_{U_i}^k$, $\forall i$, then $i \to \infty$; ③ $\mathbb{H}^q(X_i, \Omega_{X_i}^p) \Rightarrow H^{p+q}(X_i, \mathbb{C})$; ④ $\underline{\Omega}_X^k = R\varepsilon_{X_\bullet *}\Omega_{X_\bullet}^k$.

Prop4 ① Consider the category of Du Bois complexes, we have: ① Du Bois triangle $\underline{\Omega}_X^k \to Rf_*\underline{\Omega}_{\tilde{X}}^k \oplus \underline{\Omega}_Z^k \to Rf_*\underline{\Omega}_E^k \xrightarrow{+1}$

② Steenbrink triangle

Blowing up (X, Z), $Z \subset X$ closed

$\tilde{X} \xrightarrow{f} X$, $E = f^{-1}(Z)_{red}$ the exceptional is SNC

$\Rightarrow Rf_*\Omega_{\tilde{X}}^k(\log E)(-E) \to \underline{\Omega}_X^k$, ③ $R\mathcal{H}om_{\mathcal{O}_X}(\underline{\Omega}_X^0, \omega_X^\bullet) \hookrightarrow R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X^\bullet)$ injective, thus rational $\Rightarrow$ Du Bois

$$\begin{array}{ccc} H^{p+q}(X_i, \mathbb{C}) & \xrightarrow{d=0} & H^{p+q+1}(X_i, \mathbb{C}) \\ \Downarrow & & \Downarrow \\ H^{p+q}(X, \underline{\Omega}_X^p) & \xrightarrow{d=0} & H^{p+q}(X, \underline{\Omega}_X^{p+1}) \\ \Downarrow & & \Downarrow \\ H^{p+q}(X, \mathbb{C}) & \xrightarrow{d=0} & H^{p+q+1}(X, \mathbb{C}) \end{array}$$
▨

$Rf_*R\varepsilon_{X_\bullet *} \Omega^k \cong R\varepsilon_{X_\bullet *} Rf_* \Omega^k$. $Rf_*\underline{\Omega}_X^k$ ▨

Pf. ① Note that $Rf_*\underline{\Omega}_{\tilde{X}}^k \oplus \underline{\Omega}_Z^k \cong Rf_*\underline{\Omega}_E^k$, $Rf_*\underline{\Omega}_E^k \cong Rf_*\Omega_{E_\bullet}^k$ via hyperresolution ▨

② E is SNC $\Rightarrow Rf_*\underline{\Omega}_E^k = Rf_*(\Omega_{\tilde{X}}^k / \Omega_{\tilde{X}}^k(\log E)(-E))$ thus use ① cancel $Rf_*\underline{\Omega}_E^k$ by axiom of triangle category ▨

③ Rational $\Rightarrow \mathcal{O}_X \to \underline{\Omega}_X^0 \to Rf_*\mathcal{O}_{\tilde{X}}$ is q.i.s, take dual $\Rightarrow$ surjective $\Rightarrow$ q.i.s $\Rightarrow$ Du Bois ▨

Our next prove the Thm2 ②, with ① (MHM structure) admitted. Thus consider $\mathcal{O}_X(*Z)$ ① Pole order filtration $P_k(\mathcal{O}_X(*Z)) = \mathcal{O}_X((k+1)Z)$

② Hodge filtration $F_k \subset P_k$ (the difference of F_k and P_k reflects the datum of singularity;

③ V-filtration [See Lecture 1]

Pf. Du Bois $\Leftrightarrow \mathcal{O}_Z \to \underline{\Omega}_Z^0$ q.i.s $\overset{dual}{\Leftrightarrow} R\mathcal{H}om(\underline{\Omega}_Z^0, \omega_Z^\bullet) \longrightarrow R\mathcal{H}om(\mathcal{O}_Z, \omega_Z^\bullet)$ q.i.s

$\overset{①}{\Leftrightarrow} (Gr^F_{-n}DR)(\mathcal{O}_X(*Z)/\mathcal{O}_X) \to (Gr^F_{-n}DR)(\mathcal{O}_X(*Z)/\mathcal{O}_X)$ q.i.s $\Leftrightarrow F_0(\mathcal{O}_X(*Z)/\mathcal{O}_X) = P_0(\mathcal{O}_X(*Z)/\mathcal{O}_X)$

$\Leftrightarrow F_0 = P_0 \overset{②}{\Leftrightarrow} P_0 \subset F_0 \Leftrightarrow \mathcal{O}_X \frac{1}{f} \subseteq F_0 \Leftrightarrow \frac{1}{f} \in F_0 \overset{③}{\Leftrightarrow} \frac{1}{f} \otimes 1 \in F_0(M[\partial_t]) \overset{④}{\Leftrightarrow} \frac{1}{f} \otimes 1$

$\in V^0(M[\partial_t]) \overset{⑤}{\Leftrightarrow} 1 \otimes 1 \in V^1(M[\partial_t]) \overset{⑥}{\Leftrightarrow} \tilde{\alpha} \geq 1$. ② $F_0 \subset P_0$ by

Pf. $F_k = P_k \otimes I_k$, I_k is the Hodge ideal

① $Gr^F_{-n}DR)(\mathcal{O}_X(*Z)/\mathcal{O}_X) \cong R\mathcal{H}om(\underline{\Omega}_Z^0, \omega_Z^\bullet)$ ③ [Saito's thm in $\boxed{\text{Lecture 1}}$] $F_k(\mathcal{O}_X(*Z)) \hookrightarrow j_*(F_k(\mathcal{O}_X(*Z)))$ by Mustață–Popa $j: V \hookrightarrow X$, $V \cap Z$ smooth to give isomorphism ▨

by direct computation: $Gr^F DR(f_+ \ldots)[-\dim Z]$

$Rf_*(Gr^F_{-n}DR)$ the compatibility to $\mathcal{O}_X(*Z) = f_+\mathcal{O}_X(*E)$, $\mathcal{O}_X(*E)$ has trivial Hodge filtration ▨

③ $F_p(M[\ast\partial_t]) = \oplus(F_{p-i}M \otimes \partial_t^i)$ ④ $F_0(i_+M) = V^0(i_+M)$ via $F_0(i_+M) \xrightarrow[\text{formula}]{\text{Saito}} \sum_i \partial_t^i V^0(i_+M) \cap j_\ast j^\ast F_{-i}(i_+M) = V^0(i_+M)$ $\boxtimes$ $\boxed{\text{Page2}}$

by the quasi-unipotent condition $\boxtimes$

⑤ The multiplication $t: V^0(i_+M) \to V^1(i_+M)$ is bijective due to i_+M has no subobject supported on Z $\boxtimes$

<u>Higer singularities</u>

Def 5. X is k-Du Bois if $\forall\, i \le k$, $\Omega_X^i \to \underline{\Omega}_X^i$ qis; X is k-rational if $\forall\, i \le k$, $\underline{\Omega}_X^i \to \mathbb{D}(\underline{\Omega}_X^{n-i})$ qis

Rk. 0-rational = rational holds, but needs a proof (See exercises in Day3)

Thm. (Mustață-Popa, Saito) ① k-Du Bois $\Leftrightarrow \tilde{\alpha} \ge k+1$; k-Rational $\Leftrightarrow \tilde{\alpha} > k+1$;

② For not hypersurface but lci with codim j, then k-Du Bois $\Leftrightarrow \tilde{\alpha} \ge k+j$; k-Rational $\Leftrightarrow \tilde{\alpha} > k+j$.

Rk (i) The speak found a method for (some) lci cases, but due to it has no interpretations via V-filtration (which holds for lci cases well), it's not widely accepted.

(ii) We not expect ② holds for non-lci cases, with (co)torsions (See exercise, it's works by the speaker).

Day 1: Introduction to rational and Du Bois singularities.

Plan:
- Motivation and definition of Bernstein-Sato polynomial, and computations in simple examples;
- Rational and Du Bois singularities, and their relation to singularities from birational geometry;
- Roots of the Bernstein-Sato polynomial and invariants of singularities.

1. The Gamma function is defined as

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx, \quad \Re(s) > 0.$$

(a) Derive the functional equation for the Gamma function, and conclude that it can be analytically continued to a meromorphic function on the entire complex plane with poles at $0, -1, -2, \cdots$. ✓

(b) What is the key identity that was used in the derivation?

$\partial_x x^{s+1} = (s+1) x^s$

$P f^{s+1} = b(s) f^s$

2. Work out the Bernstein-Sato polynomial $b_f(s)$ in the following examples:

(a) $f = x$

$\frac{\partial}{\partial x} x^{s+1} = b_f(s) x^s$

$b_f(s) = s+1$

(b) $f = x^2 + y^2$

$\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right)(x^2+y^2)^{s+1} = b_f(s)(x^2+y^2)^s$

$b_f(s) = (s+1)^2$

(c) $f = x^2 + y^3$

$(s+1)(s+\frac{5}{6})(s+\frac{7}{6}) = b_f(s)$ and P is similiar.

The Macaulay2 code for computing the Bernstein-Sato polynomial of $f = x^2 + y^3$ (You can use the M2 interface online at https://www.unimelb-macaulay2.cloud.edu.au):

```
loadPackage "Dmodules";
R = QQ[x,y];
f = x^2+y^3;
b = factorBFunction globalBFunction f
```

$f(x)$	$b_f(s)$	timing data by Kan on S-4/20
$x^3 - y^2$	$(s+1)(s+\frac{5}{6})(s+\frac{7}{6})$	0.2s
$(x^3 - y^2)^2$	$(s+1)(s+\frac{1}{12})(s+\frac{5}{12})(s+\frac{1}{2})(s+\frac{7}{12})(s+\frac{11}{12})$	0.7s
$x^5 - y^2$	$(s+1)(s+\frac{7}{10})(s+\frac{9}{10})(s+\frac{11}{10})(s+\frac{13}{10})$	0.2s
$x^5 + y^5$	$(s+1)^2(s+\frac{2}{5})(s+\frac{3}{5})(s+\frac{4}{5})(s+\frac{6}{5})(s+\frac{7}{5})(s+\frac{8}{5})$	0.8s
$x^5 + y^5 + x^2y^2$	$(s+1)^2(s+\frac{2}{5})(s+\frac{3}{5})(s+\frac{4}{5})(s+\frac{6}{5})(s+\frac{7}{5})$	180s
$x^3y + y^3 + z^3$	$(s+1)(s+\frac{35}{18})(s+\frac{31}{18})(s+\frac{29}{18})$ $\times(s+\frac{3}{2})(s+\frac{25}{18})(s+\frac{23}{18})(s+\frac{19}{18})$	5s

Figure 1: Oaku's algorithm, 1996

when $f \neq \mathbb{C}$

3. (a) Show that $b_f(-1) = 0$, so $(s+1) \mid b_f(-1)$. Let $\tilde{b}_f(s) := b_f(s)/(s+1)$ be the *reduced Bernstein-Sato polynomial.*

$\mathcal{P} f^{s+1} = b_f(s) f^s$

$\mathcal{P} f^0 = b_f(-1) f^{-1}$

$\mathcal{P} 1 = b_f(-1) f^{-1} = 0$

$\Rightarrow b_f(-1) = 0 \Rightarrow (s+1) \mid b_f(-1)$

(b) The *minimal exponent* of f is the negative of the greatest root of $\tilde{b}_f(s)$. Compute the minimal exponent in the following examples:

i. $f = x$

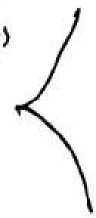
$\tilde{\alpha} = \infty$

ii. $f = x^2 + y^2$

$\tilde{\alpha} = 1$

iii. $f = x^2 + y^3$

$\tilde{\alpha} = \frac{5}{6} = c_0(f)$



(c) For each of the examples above, draw a picture of the zero set of f (over the complex numbers $\mathbb{C}$). What do you observe?

4. A variety Z has *rational singularities* if the natural map $\mathcal{O}_Z \to \mathbf{R}f_*\mathcal{O}_{\tilde{Z}}$ is a quasi-isomorphism, for some (any) resolution of singularities $f : \tilde{Z} \to Z$.

(a) Let $Z = (x^2 + y^3 = 0) \subset \mathbb{A}^2$ be the cuspidal cubic. Show that Z does not have rational singularities.

$\mathcal{O}_Z = \mathbb{C}[t^2, t^3]$, $\mathbb{A}^1 \xrightarrow{f} Z$ parameterizes

$\mathcal{O}_{\mathbb{A}^1} = \mathbb{C}[t]$ $\quad t \mapsto (t^3, t^2)$

but $f^*\mathcal{O}_Z \not\subset \mathcal{O}_{\mathbb{A}^1}$ ▨

(b) Surfaces in $\mathbb{A}^3$ with rational singularities are classified by Artin: they are either smooth, or have ADE singularities:

- $A_n : x^2 + y^2 + z^{n+1} = 0,\ n \geq 1$
- $D_n : x^2 + y(z^2 + y^{n-2}) = 0,\ n \geq 4$
- $E_6 : x^2 + y^3 + z^4 = 0$
- $E_7 : x^2 + y(y^2 + z^3) = 0$
- $E_8 : x^2 + y^3 + z^5 = 0$ $\exists$

• This can be proven by representation-theortic method

weighted homogeneous $\Leftrightarrow$ weights (w_i) has $\sum_i w_i j_i$ is same for all $\alpha = (j_1 \cdots j_n)$

Compute the minimal exponents of these polynomials.

For quasi-homogenous function we can compute via weight: $\tilde{\alpha}(A_n) = \frac{1}{2} + \frac{1}{2} + \frac{1}{n+1}$, $\tilde{\alpha}(D_n) = \frac{1}{2} + \frac{1}{n-1} + \frac{n-2}{2(n-1)}$, $\tilde{\alpha}(E_6) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$, $\tilde{\alpha}(E_7) = \frac{1}{2} + \frac{1}{3} + \frac{2}{9}$, $\tilde{\alpha}(E_8) = \frac{1}{2} + \frac{1}{3} + \frac{1}{5}$

($\tilde{\alpha}(f) =$ weight(x) + weight(y) + weight(z)) ▨

5. A complex algebraic variety Z is said to have *Du Bois singularities* if the natural map $\mathcal{O}_Z \to \underline{\Omega}^0_Z$ is a quasi-isomorphism. Here, $\underline{\Omega}^0_Z$ is the Du Bois complex that we will discuss more tomorrow.

In $\mathbb{A}^2$, it's easy to say only node is Du Bois, cusp not Du Bois $\Rightarrow \alpha = 1$

Steenbrink classified all normal Du Bois surfaces in $\mathbb{A}^3$. They can be smooth, $\boxed{\text{ADE}}$ $\tilde{\alpha} > 1$ or has $\boxed{\text{simple elliptic or cusp singularities}}$ $\tilde{\alpha} = 1$. Here are some examples:

- $\tilde{E}_6 : x^3 + y^3 + z^3 + xyz$
- $\tilde{E}_7 : x^2 + y^4 + z^4 + xyz$
- $\tilde{E}_8 : x^2 + y^3 + z^6 + xyz$
- $T_{237} : x^2 + y^3 + z^7 + xyz$

Conjecture. Du Bois $\Leftrightarrow \tilde{\alpha} \geq 1$
$\cup$
Rational $\Leftrightarrow \tilde{\alpha} > 1$

It's thm of Saito. ▨

Compute the minimal exponent in each case. Compare your answers with question 5, and try to make some conjectures.

Exercises

1. (Examples of Bernstein-Sato polynomial) Compute the Bernstein-Sato polynomial $b_f(s)$ and the minimal exponent $\tilde{\alpha}_f$ for the following polynomials f. It is highly recommended that you use computer algebras such as Macaulay2 or SINGULAR to find patterns.

   (a) $f = x_1^2 + x_2^2 + \cdots + x_n^2$; $(s+1)^2$ and $\frac{n}{2}$

   (b) $f = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$; $(s+1)^{\sum a_i}$ and $\sum \frac{1}{a_i}$

   (c) $f = \det(x_{ij})_{1 \le i,j \le n}$. $(s+1)^n$ and n

- All of the three needn't computer.

2. (Log canonical threshold and minimal exponent) Given a divisor $D \subset X$ on a smooth variety X, Hironaka's work shows that one can find a log resolution of (X, D). That is, a proper birational map $f : \tilde{X} \to X$ such that $(f^* D)_{\mathrm{red}}$ has simple normal crossings support. Let $\tilde{D}$ be the strict transform of D, and E_i the exceptional divisors of f. We define the numbers a_i, b_i by

$$K_{Y/X} = \sum_i b_i E_i$$

$$f^* D = \tilde{D} + \sum_i a_i E_i$$

   Then the *log canonical threshold* of (X, D) is

$$\mathrm{lct}(X, D) = \min_i \frac{b_i + 1}{a_i}.$$

   In the following examples, compute $\mathrm{lct}(X, D)$ and $\tilde{\alpha}_D$, and verify that $\mathrm{lct}(X, D) = \min\{1, \tilde{\alpha}_D\}$.

   (a) D = cone over a degree d hypersurface in $\mathbb{P}^n$, $X = \mathbb{A}^{n+1}$;

   (b) $D = (x^2 - y^3 = 0)$, $X = \mathbb{A}^2$.

3. (Existence of the Bernstein-Sato polynomial) Let $f \in \mathbb{C}[x_1, x_2 \cdots, x_n]$ be a polynomial. Let $X = \mathbb{A}^n$ and $Z = (f = 0) \subset X$ the hypersurface defined by f. Consider

$$\mathcal{M}_f := \mathcal{O}_X(*Z)[s] f^s.$$

- $\mathcal{O}_X(*Z) := \bigcup_{k \ge 0} \mathcal{O}_X(kZ)$, in affine case it's D-module A_f

(Saito) it has additional Hodge structure as a mixed Hodge module. (MHM)

(a) Set $X_s = X \times_{\mathrm{Spec}\,\mathbb{C}} \mathrm{Spec}\,\mathbb{C}[s]$ and $U_s = U \times_{\mathbb{C}} \mathbb{C}(s) = (X - Z) \times_{\mathbb{C}} \mathbb{C}(s)$ then $\mathcal{M} = \mathcal{O}_{U_s} \cdot f^s$ is naturally $\mathcal{D}_{U_s}$-module with $\mathcal{D}_{U_s} = \mathcal{D}_X[s]$ and $D(uf^s) = (D(u) + su\frac{D(f)}{f}) f^s$

   (a) Show that $\mathcal{M}_f$ is a $\mathcal{D}_X[s]$-module extending the natural $\mathcal{O}_X[s]$-module structure, and that $D \in \mathcal{T}_X$ acts by the "expected" rule

$$D(uf^s) = D(u) f^s + s \frac{D(f)}{f} u f^s, \quad u \in \mathcal{O}_X(*Z).$$

It has ① Hodge filtration $F_\bullet$ ② Pole order filtration $\mathcal{O}_X(\cdot Z)$ $= P_\bullet$; ③ V-filtration $V_\bullet$

(When smoothing, $F_\bullet = P_\bullet = V_\bullet$)

$F_0 = P_0 \Leftrightarrow$ Condition on $V_\bullet \Leftrightarrow \tilde{\alpha}_f \ge 1$ $\Leftrightarrow$ Du Bois

(If you are familiar with the theory of D-modules, $\mathcal{M}_f$ is isomorphic to the pushforward of $\mathcal{O}_X(*Z)$ under the graph embedding $i : X \to X \times \mathbb{C}$, $x \mapsto (x, f(x))$.)

(b) Let $\mathcal{N}_f := \mathcal{D}_X[s] \cdot f^s$ be the submodule of $\mathcal{M}_f$ generated by f^s. Define an action

$$t : \mathcal{N}_f \to \mathcal{N}_f \quad P(s) \cdot f^s \mapsto P(s+1) f^{s+1}.$$

Verify that $ts = (s+1)t$, so that the natural s-action on $\mathcal{N}_f$ induces an action on the quotient $\mathcal{N}_f / t\mathcal{N}_f$.

(c) Show that the image of t is $\mathcal{D}_X[s] f \cdot f^s$, hence conclude that the Bernstein-Sato polynomial $b_f(s)$ is the minimal polynomial the s-action.

(d) It is a theorem of Kashiwara that if $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, then the endomorphism ring $\mathcal{E}nd_{\mathcal{D}_X}(\mathcal{M})_x$ is a finite-dimensional $\mathbb{C}$-vector space for all $x \in X$. Use this theorem, as well as the fact that $\mathcal{N}_f / t\mathcal{N}_f$ is holonomic, to prove the existence of the Bernstein-Sato polynomial $b_f(s)$.

(b) $\mathcal{N}_f = D_X[s] f^s = D_{X,s} f^s$. We in order to show $\mathcal{N}_f$ is holonomic by
$\mathcal{N}' \subset \mathcal{N}$ is the maximal holonomic submodule $\Rightarrow \mathcal{N}'|_{U_S} = \mathcal{N}|_{U_S}$
$\Rightarrow 0 \to \mathcal{N}' \to \mathcal{N} \to Q \to 0$, $\operatorname{supp} Q \subset V(f) \times_{\mathbb{C}} \mathbb{C}(s)$
thus $D_{X,s} f^k f^s \subset \mathcal{N}'$, $D_{X,s} f^k f^s$ holonomic by maximal, $\exists k$
$\Rightarrow D_{X,s} f^s \xrightarrow{\cong} D_{X,s} f^k f^s$
$P(s) f^s \mapsto P(s+k) f^{k+s}$, thus $\exists k$, s.t. t^k is isomorphism ▨

(c) then filtration $\cdots \subset t\mathcal{N}_f \subset t\mathcal{N}_f \subset \mathcal{N}_f$, $\mathcal{N}_f$ holonomic $\Rightarrow$ it's finite length
$\Rightarrow \exists k'$, s.t. $f^k f^s \in D_{X,s} f^{k+1} f^s = t^{k+1} \mathcal{N}_f \Rightarrow f^s = Q(s) f f^s$
let $b(s) = \frac{P(s)}{Q(s)}$ by clearing denominators ▨

(d) Done ▨

Day 2: Introduction to the Du Bois complexes

Plan:
- Motivation and definition of the Du Bois complexes, and computations in simple examples;
- A better understanding of Du Bois singularities;
- Rational singularities are Du Bois.

1. (a) Let $X = (xy = 0) \subset \mathbb{A}^2$ be the union of two lines. Describe the Kähler differential Ω^1_X, and find a torsion element in it.

$R = \mathbb{k}[x,y]/(xy) \Rightarrow \Omega^1_X = Rdx \oplus Rdy/(xdy+ydx)$

Note that $(x+y)(xdy) = (x+y)(-ydx) = 0 \Rightarrow x+y$ torsion $\boxtimes$

(b) Let $Y = (x^2 + y^3 = 0) \subset \mathbb{A}^2$ be a cusp. Describe the Kähler differential Ω^1_X, and find a torsion element in it.

$R = \mathbb{k}[x,y]/(x^2+y^3) \Rightarrow \Omega^1_X = Rdx \oplus Rdy/(2xdx+3y^2dy)$

Note that $y(3ydx-2xdy) = 3y^2dx - 2xydy = 0 \Rightarrow y$ torsion $\boxtimes$

2. (a) Compute the Du Bois complexes of the node X and the cusp Y. Conclude that X has Du Bois singularities, and Y doesn't.

$X \leftarrow \mathbb{P}^1 \amalg p \leftleftarrows r \amalg s$

P exceptional divisor, also one of singular

$P = E$

Only one step, P needn't more hyperresolution:

$\underline{\Omega}^0_X = [\sigma_* \mathcal{O}_{\tilde{X}} \oplus \mathcal{O}_p \to \mathcal{O}_E]$

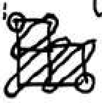
$= \sigma_* \mathcal{O}_{\mathbb{P}^1} \neq \mathcal{O}_X$

$\Rightarrow$ not Du Bois ▨

Q. Is it depending on resolution?

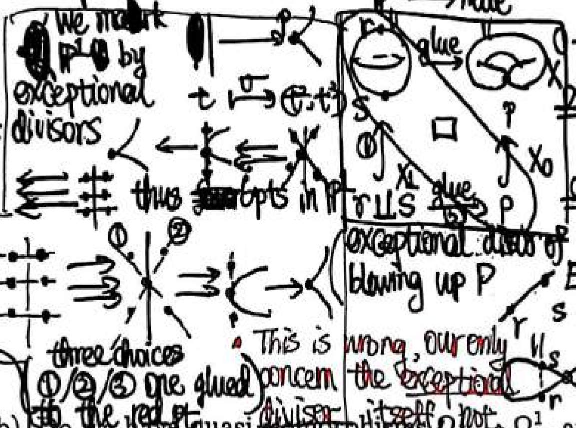
It's origin by Deligne and solved by Du Bois.

One can see that if E and P are "more complex" (E have singularity) we need more such cubes:



① Node is by normalisation $\tilde{X} \xrightarrow{\sigma} X$

② Cusp also $\mathbb{P}^1 \xrightarrow{\sigma} X$



This is wrong, our only concern the exceptional divisor itself, not its intersection behavior. thus here E—P is still smooth.

thus the MV sequence of hyperresolution is

$0 \to H^0(X) \to H^0(\mathbb{P}^1) \oplus H^0(p) \to H^0(r \amalg s) \to H^1(X) \to H^1(\mathbb{P}^1) \to 0$

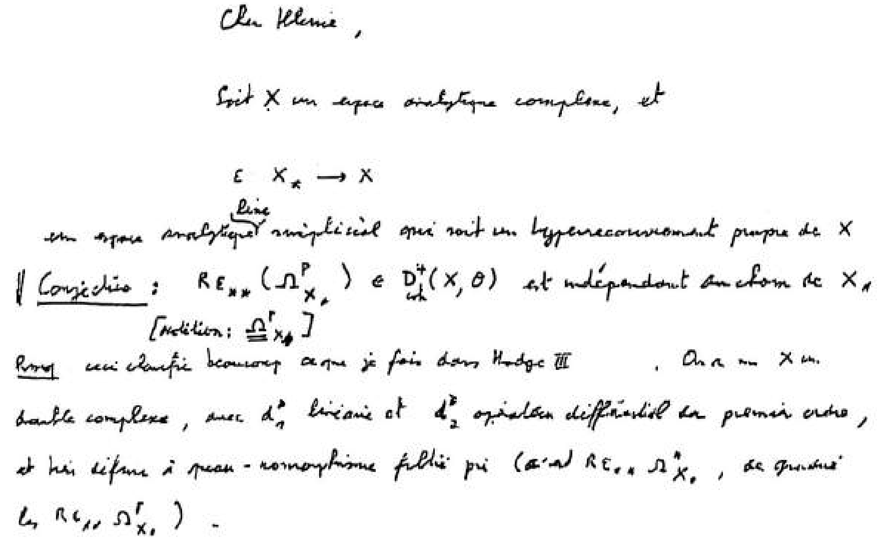
$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to 0$

$\underline{\Omega}^0_X = [\sigma_* \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_p \to \mathcal{O}_{r \amalg s}]$ (corresponding pullback of functions)

$= \mathcal{O}_X \Rightarrow$ node is Du Bois ▨

$\underline{\Omega}^1_X = \sigma_* \Omega^1_{\mathbb{P}^1} \neq \Omega^1_X$, we call node is not 1-Du Bois ▨

(b) ~~Do we have quasi-isomorphisms~~ $\Omega^1_X \to \underline{\Omega}^1_X$ and $\Omega^1_Y \to \underline{\Omega}^1_Y$?


Cher Illusie,

Soit X un espace analytique complexe, et

$\varepsilon\ X_* \to X$

un espace analytique (lisse) simplicial qui soit un hyperrecouvrement propre de X

Conjecture : $R\varepsilon_{**}(\Omega^p_{X_*}) \in D^+_{coh}(X, \mathcal{O})$ est indépendant du choix de X_*

[notation: $\underline{\Omega}^p_{X_*}$]

Preuve ceci clarifie beaucoup ce que je fais dans Hodge III. On a sur X_* un double complexe, avec d_1^* linéaire et d_2^* opérateur différentiel du premier ordre, et lui définir à quasi-isomorphisme filtré près ($R\varepsilon_{**}\Omega^*_{X_*}$, de gradués les $R\varepsilon_{**}\Omega^p_{X_*}$).

Figure 1: Deligne's 1973 letter to Illusie

3. Let $E = E_1 \cup E_2 \cup E_3 \subset \mathbb{A}^3$ be the union of 3 coordinate planes in $\mathbb{A}^3$. Find a hyperresolution of E.

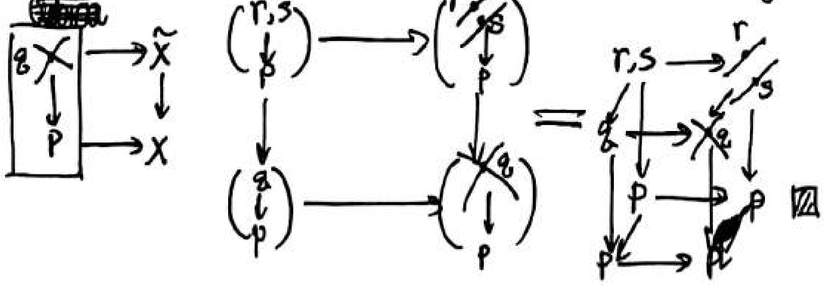
$E_1\cap E_2\cap E_3 \rightrightarrows E_1\cap E_2 \sqcup E_2\cap E_3 \sqcup E_3\cap E_1 \rightrightarrows E_1\sqcup E_2\sqcup E_3 \to E$

(Can you generalize this example to any simple normal crossings divisor?)

$\Leftrightarrow 0\to \mathbb{Z}_E \to \mathbb{Z}_{E_1\sqcup E_2\sqcup E_3} \to \bigoplus \mathbb{Z}_{E_i\cap E_j} \to \mathbb{Z}_{E_1\cap E_2\cap E_3} \to 0$ ∎

4. Let $X = (x^2 + y^2 + z^3 = 0) \subset \mathbb{A}^3$ be a surface with A_2 singularities. Find a cubical hyperresolution of X, and write down the associated hyperresolution.

By Guillén–Navarro Aznar–Pascual Gainza–Puerta (Similarly idea Carlson developed polyhedral resolution)



5. Use the hyperresolutions in questions 3 and 4 to compute the Du Bois complexes in the following examples:

(a) $E = E_1 \cup E_2 \cup E_3$ is the union of 3 coordinate axes in $\mathbb{A}^3$.
(This arguement can be generalized to show $\underline{\Omega}_E^k = \Omega_X^k / \Omega_X^k(\log E)(-E)$ for any simple normal crossing divisor $E \subset X$ in a smooth variety X.)

(b) $X = (x^2 + y^2 + z^3 = 0) \subset \mathbb{A}^3$.

6. Properties of the Du Bois complexes:

All these properties holds for smooth case, and approximation works well.

(a) There is a natural map $\Omega^k_X \to \underline{\Omega}^k_X$ for each k. All these maps are isomorphisms when X is smooth.

(b) For an open subvariety $U \subset X$, we have $\underline{\Omega}^k_X|_U = \underline{\Omega}^k_U$.

(c) $\underline{\Omega}^\bullet_X$ is quasi-isomorphic to the constant sheaf $\mathbb{C}$. Thus, there is an analog of the Hodge-to-de Rham spectral sequence

$$E_1 = \mathbb{H}^q(X, \underline{\Omega}^p_X) \implies H^{p+q}(X, \mathbb{C}).$$

It degenerates at E_1 if X is proper.

$$\begin{array}{ccc} H^q(X_i, \Omega^p_i) & \xrightarrow{d=0} & H^q(X_i, \Omega^{p+1}_i) \\ \Downarrow & & \Downarrow \\ H^{p+q}(X_i, \mathbb{C}) & \xrightarrow{d=0} & H^{p+q+1}(X_i, \mathbb{C}) \\ \Downarrow & & \Downarrow \\ \mathbb{H}^{q}(X, \underline{\Omega}^p_X) & \xrightarrow{d=0} & \mathbb{H}^{q}(X, \underline{\Omega}^{p+1}_X) \end{array} \quad \boxed{/\!/}$$

(d) (Functoriality) Any map $f : X \to Y$ induces map on their Du Bois complexes $\underline{\Omega}^k_Y \to \mathbf{R}f_*\underline{\Omega}^k_X$.

$$\begin{array}{l} \underline{\Omega}^k_Y = \mathbb{R}\varepsilon_{*\bullet}\Omega^k_{Y_\bullet} \\ \quad\downarrow \\ \mathbb{R}\varepsilon_{*\bullet}\mathbb{R}f_{\bullet *}\Omega^k_{X_\bullet} = \mathbb{R}f_*\mathbb{R}\varepsilon_{*\bullet}\Omega^k_{X_\bullet} \end{array} \quad \boxed{/\!/}$$

Computation hyperesolution (Du Bois' triangle) Let $f : Y \to X$ be a proper map that is isomorphism away from
$\Rightarrow \underline{\Omega}_X^k$ isn't easy via bicomplex$X$, and let $E = f^{-1}(Z)_{\mathrm{red}}$. Then for each k, there is an exact triangle
(hyperhomology of Cartan-Eilenberg resolution)
we use the triangle to compute usually.

$$\begin{tikzcd}
\underline{\Omega}_X^k \arrow[r] \arrow[d, equal, "k"] & \mathbf{R}f_*\underline{\Omega}_Y^k \oplus \underline{\Omega}_Z^k \arrow[r] \arrow[d, equal, "k"] & \mathbf{R}f_*\underline{\Omega}_E^k \arrow[r, "+1"] \arrow[d, equal] & {} \\
\underline{\Omega}_X^k \arrow[r] & \mathbf{R}\varepsilon_* \underline{\Omega}_{X_0}^k \arrow[r] & \mathbf{R}f_*\underline{\Omega}_{X_1}^k \arrow[r, "+1"] & {}
\end{tikzcd} \quad \boxed{/\!/}$$

$$\begin{tikzcd}
E = f^{-1}(Z)_{\mathrm{red}} \arrow[r] \arrow[d] & Y \arrow[d] \\
Z \arrow[r] & X
\end{tikzcd}$$

$=X_1$, $=X_0$

Both E, Z are smooth here

(f) (Steenbrink's triangle) Let Z be a closed subvariety of X. Let $f : \tilde{X} \to X$ be a resolution of singularities which is an isomorphism outside Z, and $E := f^{-1}(Z)_{\mathrm{red}}$ is a simple normal crossing divisor. Then for each k, there is an exact triangle

Note that SNC $\Rightarrow \mathbf{R}f_*\underline{\Omega}_E^k = \mathbf{R}f_*[\Omega_{\tilde{X}}^k / \Omega_{\tilde{X}}^k(\log E)(-E)]$

and apply it into (e)
$$\mathbf{R}f_*\Omega^p_{\tilde{X}}(\log E)(-E) \to \underline{\Omega}^p_X \to \underline{\Omega}^p_Z \xrightarrow{+1}.$$

$\Rightarrow [\underline{\Omega}_X^k \to \mathbf{R}f_*\Omega_{\tilde{X}}^k \oplus \underline{\Omega}_Z^k \to \mathbf{R}f_*[\Omega_{\tilde{X}}^k / \Omega_{\tilde{X}}^k(\log E)(-E)] \xrightarrow{+1}]$

Check this when X has an isolated singular point $Z = \{p\}$.

$\cong [\underline{\Omega}_X^k \to \underline{\Omega}_Z^k \xrightarrow{+1} \mathbf{R}f_*\Omega_{\tilde{X}}^k(\log E)(-E) \to]$ $\boxed{/\!/}$

It's originally proven (g) (Kovács-Schwede) The natural map
by them hard via Hodge, a
new proof given recently:

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\underline{\Omega}^0_X, \omega^\bullet_X) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \omega^\bullet_X)$$

$\mathbf{R}\mathcal{H}om(\underline{\Omega}_X^0, \omega_X^\bullet) = F_{-n}\mathbf{R}\Gamma_X(\mathcal{O}_Y)$ ($Y \supset X$ smoothly) is injective on cohomology.
the $-n$ piece of Hodge filtration of $\mathbf{R}\Gamma_X\mathcal{O}_Y$
Use this theorem to show that rational singularities are Du Bois.

$$\begin{tikzcd}
\mathbf{R}\mathcal{H}om(\underline{\Omega}_X^0, \omega_X^\bullet) & \mathbf{R}\mathcal{H}om(\mathcal{O}_X, \omega_X^\bullet) \arrow[l] \arrow[d] \\
& \mathbf{R}\Gamma_X\mathcal{O}_Y \arrow[lu, "\text{injective}"]
\end{tikzcd}$$

thus it must injective
(cohomological level) $\boxed{/\!/}$

Rational $\Rightarrow$
$\mathcal{O}_X \to \underline{\Omega}_X^0 \to \mathbf{R}f_*\mathcal{O}_{\tilde{X}}$ is qis
"dual" $\mathbf{R}\mathcal{H}om(\mathbf{R}f_*\mathcal{O}_{\tilde{X}}, \omega_X^\bullet)$
$\to \mathbf{R}\mathcal{H}om(\underline{\Omega}_X^0, \omega_X^\bullet) \to \mathbf{R}\mathcal{H}om(\mathcal{O}_X, \omega_X^\bullet)$
is qis
$\Rightarrow \mathbf{R}\mathcal{H}om(\underline{\Omega}_X^0, \omega_X^\bullet) \to \mathbf{R}\mathcal{H}om(\mathcal{O}_X, \omega_X^\bullet)$
surjective cohomologically.
by Kovács-Schwede $\Rightarrow$ isomorphism
$\Rightarrow$ Du Bois $\boxed{/\!/}$

• Reflexive differential form sheaf $\Omega^{[p]}_{X_{sm}}$ defined on $X_{sm} \subset X$, with (by MMP) $X_{sing} = \mathrm{lc}, \mathrm{c}, \mathrm{t}$

$\tilde{X} \xrightarrow{\pi} X$ log resolution of klt pair (X, D)

$\Rightarrow \pi_* \Omega^p_{\tilde{X}}$ is reflexive sheaf $=: \Omega^{[p]}_X$

Exercises

1. (Description of $\underline{\Omega}^n_X$) Let X be a variety of dimension n. Show that

$$\underline{\Omega}^n_X = f_* \omega_{\tilde{X}},$$

where $f : \tilde{X} \to X$ is a resolution of singularities of X. On the other hand, it is non-trivial fact that

$\mathcal{O}_X \to \mathcal{O}_{X^{sn}}$ is final object in $\mathcal{O}$-module categories

(1) $\mathcal{H}^0 \underline{\Omega}^0_X = \mathcal{O}_{X^{sn}}$, where X^{sn} is the seminormalization of X.

$\mathcal{O}_{X^{sn}} = i_* \mathcal{O}_{X^{sn}}$, I don't know

(2) If X has rational singularities, then $\mathcal{H}^0 \underline{\Omega}^k_X = \Omega^{[k]}_X$ for every k, where $\Omega^{[k]}_X$ is the reflexive differentials on X.

Rational $\Rightarrow$ Du Bois $\Rightarrow \mathcal{H}^0 \underline{\Omega}^k_X = \mathcal{H}^0 i_* \Omega^k_{X_{sn}} = i_* \Omega^k_{X_{sn}} = \Omega^{[k]}_X$ ▨

• Similar as normalisation, it's locally algebraic:

① Seminormal means that ~~[illegible]~~ worst is cusp, i.e. $\forall x, y \in A$, $x^3 = y^2 \Rightarrow \exists!\, a \in A$, s.t. $x = a^2$, $y = a^3$;

② Universal homeomorphism & iso in residue maps $\{$ has initial object $(X^{sn} \to X)$ is the seminormalisation

2. (Du Bois complexes for curves)

   (a) Let X be a variety with isolated singularities, such that its normalization is smooth. Show that $\underline{\Omega}^k_X$ is concentrated in degree zero.

   Due to the hyperresolution of 2-elements $\{E_i\} \to \tilde{X}$, E_i are not connected each other $\Rightarrow$ smooth $\Rightarrow$ only 1-term in hyperresolution ▨

   $\{P_i\} \to X$, $\tilde{X} \xrightarrow{\sigma} X$

   (b) Describe the Du Bois complexes for curves. (*Hint.* The fact in the previous exercise might be helpful.)

   Induction on number of isolated singularities. ▨

3. (Du Bois complexes for SNC divisors) Generalize the argument in question 5(a) to show for any simple normal crossing divisor $E \subset X$ in some smooth variety X, we have

$$\underline{\Omega}^k_E = \Omega^k_X / \Omega^k_X(\log E)(-E).$$

Taking $X \xrightarrow{id} X$ is log resolution of (X, E) with exceptional divisor $E = E$ into Steenbrink's triangle ▨

4. (Schwede's criterion) Let X be a variety embedded in some smooth variety Y. Let $f : \tilde{Y} \to Y$ be a log resolution the pair (Y, X), with exceptional divisor $E = f^{-1}(X)_{red}$. Using Steenbrink's triangle, show that

$\mathbf{R}f_* \Omega^p_{\tilde{Y}}(\log E)(-E) \to \underline{\Omega}^p_Y \to \underline{\Omega}^p_X \xrightarrow{+1}$

$$\mathbf{R} f_* \mathcal{O}_E = \underline{\Omega}^0_X.$$

Due to here Y is smooth, so is $\tilde{Y}$

$\Rightarrow \mathbf{R}f_* \mathcal{O}_{\tilde{Y}}(-E) \to 0$ by cancel $\underline{\Omega}^p_Y \overset{qis}{\cong} \mathcal{O}_Y \cong f_* \mathcal{O}_{\tilde{Y}} \cong \mathcal{O}_Y$

$\Rightarrow \underline{\Omega}^0_X \cong \mathbf{R}f_* \mathcal{O}_E$ ▨

Thus, X has ~~Du Bois singularities~~ if and only if the natural map $\mathcal{O}_X \to \mathbf{R} f_* \mathcal{O}_E$ is a quasi-isomorphism.

5. (Support of cohomology sheaves) Show that $\mathcal{H}^q \underline{\Omega}^p_X = 0$ for $q < 0$ and $q > \dim X$[1]. More generally,

$$\operatorname{codim} \operatorname{supp} \mathcal{H}^q \underline{\Omega}^p_X \geq q$$

for all p, q.

[1] Actually, Steenbrink's vanishing theorem states that $\mathcal{H}^q \underline{\Omega}^p_X = 0$ for $p + q > \dim X$.

Day 3: Survey of higher singularities

Frame of proof
Du Bois $\Longleftrightarrow$ $\tilde{\alpha}_Z \geq 1$
$\Updownarrow$ (R, 8.2) $\quad \Updownarrow$ Lecture 1, Schnell
$P_0 = F_0$ $\overset{(Q4)}{\Longleftrightarrow}$ conditions on V-filtration
(GB: $F_k \subset P_k$)

> Plan:
> - Idea for proving Saito's theorem that DB $\iff$ $\tilde{\alpha} \geq 1$;
> - Generalizations leading to the notions of higher singularities.

- Generalizations.
① X is k-Du Bois if $\forall p \leq k$, $\underline{\Omega}^p_Z \to \underline{\Omega}^p_Z$ are q.i.s
② X is k-rational if $\forall p \leq k$ $\underline{\Omega}^p_Z \to \mathbb{D}(\underline{\Omega}^{n-p}_Z)$ are q.i.s.
③ By exercise, 0-rational = rational
④ Thm (Mustață–Popa, Saito...)
k-Du Bois $\Longleftrightarrow$ $\tilde{\alpha} \geq k+1$
k-Rational $\Longleftrightarrow$ $\tilde{\alpha} > k+1$
⑤ Higher B-S polynomial ✓
Higher V-filtration ✓
⑥ With these higher analogues, ④ holds true for lci than hypersurface.
⑦ Non-lci case: (i) $\underline{\Omega}_Z$ is bad
(ii) k-Du Bois + nonlci is still unknown

1. Let M be a left D-module with a filtration F. Define

$$\mathrm{Gr}^F_p \mathrm{DR}(M) := [\mathrm{Gr}^F_p M \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathrm{Gr}^F_{p+1} M \to \cdots \to \omega_X \otimes_{\mathcal{O}_X} \mathrm{Gr}^F_{p+n} M].$$

Let $Z = (f = 0) \subset X$ be a reduced hypersurface, and let $f : \tilde{X} \to X$ be a log resolution of (X, Z) with exceptional divisor E. Then

(a) For $M = \mathcal{O}_X$ with the trivial Hodge filtration $F_k\mathcal{O}_X = \begin{cases} \mathcal{O}_X, & k \geq 0 \\ 0, & \text{otherwise} \end{cases}$

show that $\mathrm{Gr}^F_{-n} \mathrm{DR}(\mathcal{O}_X) = \omega_X$.

$$\mathrm{Gr}^F_k \mathcal{O}_X = \begin{cases} \mathcal{O}_X ; & k = 0 \\ 0 ; & k \neq 0 \end{cases}$$
$$\Rightarrow \mathrm{Gr}^F_{-n} \mathrm{DR}(\mathcal{O}_X) = [0 \to 0 \to \cdots \to \omega_X] = \omega_X \quad \square$$

E is better than Z, we expect to compute $F_\cdot \mathcal{O}_{\tilde{X}}(*E)$ first and then pushforward to $F_\cdot \mathcal{O}_X(*Z)$

(b) For $M = \mathcal{O}_{\tilde{X}}(*E)$ with the Hodge filtration given by

$$F_k\mathcal{O}_{\tilde{X}}(*E) = \underbrace{(F_k\mathcal{D}_X)}_{= \text{differential operator order} \leq k} \cdot \mathcal{O}_{\tilde{X}}(E),$$

show that $\mathrm{Gr}^F_{-n} \mathrm{DR}(\mathcal{O}_{\tilde{X}}(*E)) = \omega_{\tilde{X}}(E)$.

$$\mathrm{Gr}^F_k \mathcal{O}_{\tilde{X}}(*E) = \begin{cases} F_0\mathcal{O}_{\tilde{X}}(*E) ; & k = 0 \\ 0 ; & k \neq 0 \end{cases} = \begin{cases} \mathcal{O}_{\tilde{X}}(E) ; & k = 0 \\ 0 ; & k \neq 0 \end{cases}$$
$$\Rightarrow \mathrm{Gr}^F_{-n} \mathrm{DR}(\mathcal{O}_{\tilde{X}}(*E)) = [0 \to 0 \to \cdots \to \omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(E)]$$
$$= \omega_{\tilde{X}}(E) \quad \square$$

(c) It is a fact that $\mathcal{O}_X(*Z)$ is the pushforward of $\mathcal{O}_{\tilde{X}}(*E)$[1]. By compatibility of the pushforward and Gr DR, we have

$$\mathrm{Gr}^F_{-n} \mathrm{DR}(\mathcal{O}_X(*Z)) = \mathbf{R}f_*\omega_{\tilde{X}}(E).$$

By compatibility $(\mathrm{Gr}^F_{-n} \mathrm{DR})(f_+) = (\mathbf{R}f_*)(\mathrm{Gr}^F_{-n} \mathrm{DR})$ $\square$

[1] In the category of mixed Hodge modules.

(d) Applying $\mathrm{Gr}^F_{-n}\,\mathrm{DR}$ to the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(*Z) \to \mathcal{O}_X(*Z)/\mathcal{O}_X \to 0$$

gives an exact triangle. Compare this with Steenbrink's triangle for the Du Bois complex to conclude

$$\mathrm{Gr}^F_{-n}\,\mathrm{DR}(\mathcal{O}_X(*Z)/\mathcal{O}_X) \simeq \mathbf{R}\mathcal{H}om(\underline{\Omega}^0_Z, \omega^\bullet_Z)[-\dim Z].$$

$$\begin{tikzcd}
(\mathrm{Gr}^F_{-n}\mathrm{DR})(\mathcal{O}_X(*Z)/\mathcal{O}_X) \arrow[r, "+1"] & (\mathrm{Gr}^F_{-n}\mathrm{DR})(\mathcal{O}_X(*Z)) \arrow[r] \arrow[d, equal] & (\mathrm{Gr}^F_{-n}\mathrm{DR})(\mathcal{O}_X) \arrow[d, equal] \\
\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\underline{\Omega}^0_Z, \omega^\bullet_Z)[-\dim X] \arrow[u, "+1"] \arrow[r] & \omega_X \arrow[r] & (\mathbf{R}f_*)\omega_{\tilde{X}}(E) \\
\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\underline{\Omega}^0_Z, \omega^\bullet_Z) \arrow[u, "+\dim X"] \arrow[r] & \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \omega^\bullet_X) \arrow[u, "+\dim X"] \arrow[r] & \mathbf{R}\mathcal{H}om(\mathbf{R}f_*\mathcal{O}_{\tilde{X}}(E), \omega^\bullet_X) \arrow[u, "+\dim X"]
\end{tikzcd}$$

-- dual of Steenbrink ☑

$\Rightarrow (\mathrm{Gr}^F_{-n}\mathrm{DR})(\mathcal{O}_X(*Z)/\mathcal{O}_X) \cong \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\underline{\Omega}^0_Z, \omega^\bullet_Z)[-\dim X+1] = \cdots [-\dim Z]$

(e) Let

$$P_k\mathcal{O}_X(*Z) = \begin{cases} \mathcal{O}_X((k+1)Z), & k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

be the pole-order filtration on $\dot{Z}$. It induces a natural filtration on $\mathcal{O}_X(*Z)/\mathcal{O}_X$. Show that $\mathrm{Gr}^P_{-n}\,\mathrm{DR}(\mathcal{O}_X(*Z)/\mathcal{O}_X) = \omega_Z$.

Repeat (a)-(d) by replace $F_k\mathcal{O}_{\tilde{X}}$ by $P_k\mathcal{O}_X$

$\Rightarrow (\mathrm{Gr}^P_{-n}\mathrm{DR})(\mathcal{O}_X(*Z)) = (\mathbf{R}f_*)(\mathrm{Gr}^P_{-n}\mathrm{DR})(\mathcal{O}_{\tilde{X}}(*E))$

$\mathrm{Gr}^P_k(\mathcal{O}_X) = \begin{cases} \mathcal{O}_X \ ; k=0 \\ 0 \ ; \text{otherwise} \end{cases} \Rightarrow (\mathrm{Gr}^P_{-n}\mathrm{DR})(\mathcal{O}_X) = \omega_X$

$\mathrm{Gr}^P_k(\mathcal{O}_{\tilde{X}}(*E)) = \begin{cases} \mathcal{O}_X((k+1)Z)/\mathcal{O}_X(kZ) \ ; k \geq 0 \\ 0 \ ; \text{otherwise} \end{cases} \Rightarrow (\mathrm{Gr}^P_{-n}\mathrm{DR})(\mathcal{O}_{\tilde{X}}(*E)) = \omega_{\tilde{X}}$ and note that

F_k needs smooth $\Rightarrow$ only on $(\tilde{X}, E)$, but P_k not $\Rightarrow$ directly on (X, Z)

$\Rightarrow$ By triangle $\Rightarrow (\mathrm{Gr}^P_{-n}\mathrm{DR})(\mathcal{O}_X(*Z)/\mathcal{O}_X) = \omega_Z$ ☑

2. Let $Z = (f = 0) \subset X$ be a reduced hypersurface, embedded in a smooth variety X. Use part (d) and (e) in the previous question to show that

$$P_0\mathcal{O}_X(*Z) = F_0\mathcal{O}_X(*Z) \iff Z \text{ has Du Bois singularities.}$$

Du Bois $\iff \mathcal{O}_Z \to \underline{\Omega}^0_Z$ qis $\overset{\text{dual}}{\iff} R\mathcal{H}om(\underline{\Omega}^0_Z, \omega^\bullet_Z) \to R\mathcal{H}om(\mathcal{O}_Z, \omega^\bullet_Z)$ qis

$\underset{(d)(e)}{\iff} (\mathrm{Gr}^F_{-n}\mathrm{DR})(\mathcal{O}_X(*Z)/\mathcal{O}_X) \to (\mathrm{Gr}^F_{-n}\mathrm{DR})(\mathcal{O}_X(*Z)/\mathcal{O}_Z)$ qis $\iff F_0(\mathcal{O}_X(*Z)/\mathcal{O}_X) = P_0(\mathcal{O}_X(*Z)/\mathcal{O}_X)$

$\iff F_0 = P_0$ ∎

Saito's formula: If (M, F, V) is a sufficiently nice[2] D-module with filtrations V and F, then

$$F_pM = \sum_{i\geq 0}(\partial_t^i V^0 M \cap \underline{j_*j^*F_{p-i}M}) = \begin{cases} F_{p-i}\mathcal{O}_U ; & i \le p \\ 0 ; & \text{otherwise} \end{cases}$$

(if take $M = \mathcal{O}_X(*Z)$ & $j : X - Z \hookrightarrow X$)

3. (a) Suppose $Z = (f = 0) \subset X$ is a smooth hypersurface. Then the V-filtration on $\mathcal{O}_X(*Z)$ is given by

$$V^k\mathcal{O}_X(*Z) = \mathcal{O}_X \cdot t^{k-1}, \quad k \in \mathbb{Z}.$$

Use Saito's formula to show $F_k\mathcal{O}_X(*Z) = P_k\mathcal{O}_X(*Z)$[3].

$F_k\mathcal{O}_X(*Z) \overset{\text{Saito}}{=} \sum_{0\le i\le p} \partial_t^i(V^0M) = \sum_{0\le i\le p} \partial_t^i(\mathcal{O}_X \tfrac{1}{t}) = \mathcal{O}_X\{\tfrac{1}{t}, \tfrac{1}{t^2}, \cdots, \tfrac{1}{t^{p+1}}\} = \mathcal{O}_X((p+1)Z) = P_k\mathcal{O}_X(*Z)$ ∎

(b) Show that $F_k\mathcal{O}_X(*Z) \subset P_k\mathcal{O}_X(*Z)$ for any reduced hypersurface $Z = (f = 0) \subset X$.

$F_p(\mathcal{O}_X(*Z)) \hookrightarrow j_*(F_p\mathcal{O}_X(*Z)|_V) \cong j_*(P_p\mathcal{O}_X(*Z)|_V) = P_p\mathcal{O}_X(*Z)$

where $V \subset X$, codim$(V, X) \geq 2$, $V \cap Z$ is smooth ∎

[2] This means: (M, F) is a filtered D-module that is quasi-unipotent along Z (here the V-filtration is taken along Z and indexed by t), such that the vanishing cycle map $\mathrm{Gr}\,t : \mathrm{Gr}^0_V M \to \mathrm{Gr}^1_V M$ is injective and strict with respect to F. The quasi-unipotent condition is satisfied by any mixed Hodge module. The condition on $\mathrm{Gr}\,t$ is satisfied for the objects $\mathcal{O}_X(*Z), i_+\mathcal{O}_X(*Z)$ that we are interested in.

[3] The Hodge filtration in 1(b) can be computed in a similar fashion.

4. Let $Z = (f = 0) \subset X$ be a reduced hypersurface. Let $M = \mathcal{O}_X(*Z)$, and

$$i_+M = M \otimes_{\mathbb{C}} \mathbb{C}[\partial_t] =: M[\partial_t]$$

be the pushforward of M by the graph embedding.

Sabbah's thm

(a) If you remember, i_+M is the holonomic D-module in Handout 1, Exercise 2. There is an s action on i_+M whose minimal polynomial is the Bernstein-Sato polynomial $b_f(s)$, and whose eigenvalues (i.e. roots of $b_f(s)$) are up to integer shifts the indices where the V-filtrations changes. This perspective allows one to show

$$1 \otimes 1 \in V^1(M[\partial_t]) \iff \text{all roots of } b_f(s) \le -1.$$

(b) Use Saito's formula to show $F_0(i_+M) = V^0(i_+M)$.

Note that $j_*j^*F_{-i}(i_+M) = \begin{cases} 0, & i \neq 0 \\ F_0(i_+M) & i = 0 \end{cases}$

thus $F_0(i_+M) = \partial_t^0 V^0(i_+M) \cap F_0(i_+M)$
$= V^0(i_+M)$ $\square$

(c) There are two final facts we need:

- $t : V^0(i_+M) \to V^1(i_+M)$ is bijective (this is related to the fact that i_+M has no subobject supported on Z);
- $F_p(M[\partial_t]) = \oplus_i(F_{p-i}M \otimes \partial_t^i)$ (this follows from the quasi-unipotent condition)

(d) Combining results in question 3 and 4, show that

$$P_0\mathcal{O}_X(*Z) = F_0\mathcal{O}_X(*Z) \iff \tilde{\alpha} \ge 1.$$

$P_0 = F_0 \overset{(3.3)}{\iff} P_0 \subset F_0 \iff \mathcal{O}_X \frac{1}{f} \subset F_0 \iff \frac{1}{f} \in F_0 \overset{(d.2)}{\iff} \frac{1}{f} \otimes 1 \in F_0(M[\partial_t]) \overset{\text{Saito (b)}}{\iff} \frac{1}{f} \otimes 1 \in V^0(M[\partial_t])$
$\overset{(c.1)}{\iff} 1 \otimes 1 \in V^1(M[\partial_t]) \overset{(a)}{\iff}$ all roots of $b_f(s) \le -1 \iff \tilde{\alpha} \ge 1$ $\square$

Together with question 2, this completes the proof of Saito's theorem.

Exercises.

• $\mathcal{H}$ is the ker/im of complex of sheaves, it's still sheaf, and Z is the support.

1. (Injectivity theorem in the lci case) Let $Z \subset X$ be a local complete intersection of codimension r. Consider the local cohomology module $\mathcal{H}^r_Z\mathcal{O}_X$.

   (a) Generalizing the argument in 1(d), show that

   $$\mathrm{Gr}^F_{k-n}\,\mathrm{DR}(\mathcal{H}^r_Z\mathcal{O}_X) = \mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}(\Omega^k_Z, \omega^\bullet_Z)[k - \dim Z].$$

   (b) Define the *Ext filtration* on $\mathcal{H}^r_Z\mathcal{O}_X$ by

   $$E_k\mathcal{H}^r_Z\mathcal{O}_X = \mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^{[illegible]}, \mathcal{O}_X).$$

   by Steenbrink

   • It's derived and many notations are not familiar, omitted

   Show that

   $$\mathrm{Gr}^E_k\,\mathcal{H}^r_Z\mathcal{O}_X = \mathrm{Sym}^k\,\mathcal{N}_{Z/X} \otimes \omega_Z \otimes \omega_X^{-1}.$$

   (c) Show that

   $$\mathrm{Gr}^E_{k-n}\,\mathrm{DR}(\mathcal{H}^r_Z\mathcal{O}_X) = \mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}(\mathbb{L}^k_Z, \omega^\bullet_Z)[k - \dim Z],$$

   where $\mathbb{L}^k_Z$ is the k-th derived wedge product of the cotangent complex of Z.

   (d) Deduce that local complete intersections with k-rational singularities are k-Du Bois.

2. (0-rational = rational) Verify that a complex algebraic variety Z has rational singularities if and only if it has 0-rational singularities in the sense of Friedman-Laza.

• The last two exercises are the recent work by the speaker, I omit it

3. (Kähler differentials in the non-lci case) Let Z_d be the cone over the d-th Veronese embedding of $\mathbb{P}^r$.

   (a) Show that for $d \geq 3$, the Kähler differential $\Omega^1_{Z_d}$ has torsion, as follows: Let $\mathcal{I}_d$ be the ideal sheaf defining Z_d in its embedding. It is a fact that $\Omega^1_{Z_d}$ has torsion if $H^1(\mathcal{I}_d^2(2)) \neq 0$. Show that this is indeed the case when $d \geq 3$.

   (b) Show that for $d \geq 2$, the Kähler differential $\Omega^1_{Z_d}$ has cotorsion, following the steps below:

   - Show that $Z_d = \mathbb{A}^{r+1}/\mu_d$, where μ_d is the cyclic group of order d;
   - Show that the embedding dimension of Z_d is $e = \binom{r+d}{d}$;
   - Show that the minimal number of generators of $(\Omega^1_{\mathbb{A}^{r+1}})^{\mu_r}$ is $m = (r + 1)\binom{d+r-1}{d-1}$;

- It is a fact that if $e < m$, then Z_d has cotorsion. Show that this is the case for $d \geq 2$.

• This work generalises (part) to non-lci case. But we don't except all things holds in non-lci. In cone & toric case the speaker doesn't have a V-filtration, thus no much credit for this work

4. (Higher singularities for cones and toric varieties)

   (a) Let X be a toric variety, and $f : Y \to X$ a toric resolution of singularities with exceptional divisor E. Show that

$$X \text{ is simplicial} \iff R^1 f_* \Omega^1_Y(\log E) = 0.$$

   (*Hint.* Let $D = D_1 + D_2 + \cdots + D_k + E$ be the torus-invariant divisor of Y. Use the fact that X has rational singularities to show the residue sequence

$$0 \to \Omega^1_Y(\log E) \to \Omega^1_Y(\log D) \cong \mathcal{O}_Y^{\oplus n} \xrightarrow{\text{Res}} \oplus_i \mathcal{O}_{D_i} \to 0$$

   is a f_*-acyclic resolution of $\Omega^1_Y(\log E)$, and use it to compute $R^1 f_* \Omega^1_Y(\log E)$.) In particular, non-simplicial toric varieties do not have pre-1-rational singularities (although they do have pre-k-Du Bois singularities for all k).

   (b) Let $X \subset \mathbb{P}^N$ be a smooth projective variety, and

$$Z = \operatorname{Spec}\Big(\bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(m))\Big)$$

   be the affine cone over X. Then, one can show that the Du Bois complexes of Z are given by the following formulas:

$$\Gamma(Z, \mathcal{H}^0 \underline{\Omega}^0_Z) \simeq \mathbb{C} \oplus \bigoplus_{m \geq 1} H^0(X, \mathcal{O}_X(m)),$$

$$\Gamma(Z, \mathcal{H}^i \underline{\Omega}^0_Z) \simeq \bigoplus_{m \geq 1} H^i(X, \mathcal{O}_X(m)) \text{ for } i \geq 1,$$

   and for $k \geq 1$,

$$\Gamma(Z, \mathcal{H}^i \underline{\Omega}^k_Z) \cong \bigoplus_{m \geq 1} H^i(X, \Omega^k_X(m)) \oplus \bigoplus_{m \geq 1} H^i(X, \Omega^{k-1}_X(m)).$$

   Use these to give criteria for Z to have pre-k-Du Bois singularities.

Hodge theory in combinational geometry (Botong Wu) $\boxed{\text{Lecture 3}}$

Read–Heron–Rota–Welsh conjecture

Dowling–Wilson conjecture

Lorentzian polynomials

Conjecture 1. (Read, proven by Huh, 2012) $P_G(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n$ ($a_0 = 1$, and (a_k) turns into 0 for some $k \le n$) is the chromatic polynomial $\Rightarrow |a_k|$ is log-concave sequence, i.e. $|a_k|^2 \ge |a_{k-1}||a_{k+1}|$

Facts. ① $\deg P_G = \# G_0$ (vertex number); ② $P_G(t) | t$; ③ (a_n) is alternative sequence, and no 0 in middle (One can see this from Read easily, but it can be computed directly); ④ $P_G \in \mathbb{Z}[t]$ (This is by induction on the deleting $G \setminus e$ and contracting G/e: $P_G(t) = P_{G \setminus e}(t) - P_{G/e}(t)$ (delete e / contract G/e))

a_n

• Recall chromatic polynomial is $P_G: t \mapsto$ the number of coloring G using t colors, nearby vertex not same color

E.g. $P_G = t(t-1)((t-2)^2 + (t-1))$

t, $t-1$, $t-2/t-1$, $t-2/1$

Conjecture 2. (Dowling–Wilson) $V = k^d$, $E \subset V$ finitely spanned, $\mathcal{F} := \{\text{all subspace of } V \text{ generated by subsets of } E\}$, $\mathcal{F}_k = \{W \in \mathcal{F} \mid \dim W = k\}$, then $|\mathcal{F}_k| \le |\mathcal{F}_{d-k}|$, $\forall k \le \frac{d}{2}$ (d is maximal $\ne 0$)

Eg 3. $d = 3$, its thm of de Bruijn–Erdos: in $\mathbb{P}^2_k$ n pts not same line, then they connect at least n lines

But note that it can't inductively proven: in char 0 ; but in char 2 ($\mathbb{P}^2_{\mathbb{F}_2}$) : 7 lines = 7 pts

9 lines

In general choosing of elements $\in E \Rightarrow \# \mathcal{F}_k = \begin{cases} \binom{n}{k} & k < d \\ 1 & k = d \\ 0 & k > d \end{cases}$

Rk. The proof of 1 is by Hodge–Riemann relation / Hodge index; The proof of 2 is by Hard Lefschetz (easier)

Pf of Conj 2. ① We construct an algebraic variety corresponds to $\mathcal{F} = \bigoplus \mathcal{F}_k$ (in meaning graded isomorphism)

$k = \mathbb{C}$, choosing of basis determines that $F = (e_1, \dots, e_n)$: $V^* \to \mathbb{A}^n_{\mathbb{C}} \subset (\mathbb{P}^1_{\mathbb{C}})^n$, and then $Y = \overline{F(V^*)}$ desired complete variety

② $F(V^*) \curvearrowleft V$ and $\{F(V^*)\text{-orbits}\} \longleftrightarrow \mathcal{F}$; thus Y has stratification by affine spaces $\Rightarrow$ cellular structure

$\Rightarrow \dim H^{2k}(Y, \mathbb{Q}) = \begin{cases} \# \mathcal{F}_k & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$, $\mathcal{F} \to \mathrm{Stab}(\mathbb{C})^\perp$ with $\# \{\ge k \text{ cells}\} \longleftrightarrow \# \mathcal{F}_k$, only $k =$ even cells exist

Then by Poincare duality / Hard Lefschetz $\Rightarrow$ proven for Y smooth case (even equal)

③ A singular analogue of cohomology: intersection cohomology, we expect also a Poincare duality / Hard Lefschetz [By BBDG], but only over $\mathbb{C}$

• Idea: take constant sheaf on X_{sm}, and $\exists !$ extension to X makes it perverse, then taking derived section $\Rightarrow IH^*(X)$

$IH^*(Y)$ is graded $H^*(Y, \mathbb{Q})$-module $\Rightarrow H^*(Y, \mathbb{Q}) \hookrightarrow IH^*(Y, \mathbb{Q})$, then we have MHM $\ker(H^k(Y, \mathbb{Q}) \to IH^k(Y, \mathbb{Q})) = W_{k-1} H^k(Y, \mathbb{Q})$ the weighted filtration; note that when k even, $k-1$ is odd $\Rightarrow Y = \mathbb{C}^d \amalg \mathbb{C}^{d-1} \amalg \mathbb{C}^{d-1} \amalg \cdots \amalg pt$ (by $\# \mathcal{F}_k$ cells) $\Rightarrow$

$H^{2k}(Y, \mathbb{Q}) = \bigoplus H_c^{2k}(\mathbb{C}^?, \mathbb{Q}) \Rightarrow W_{2k-1} H^{2k}(Y, \mathbb{Q}) = 0 \Rightarrow H^{2k}(Y, \mathbb{Q}) \hookrightarrow IH^{2k}(Y, \mathbb{Q})$; by [BBDG], $\boxed{IH^{2k}(Y, \mathbb{Q}) \xrightarrow[A^{d-2k}]{\cong} IH^{2d-2k}(Y, \mathbb{Q})}$

$\Rightarrow$ inequality holds; Rk. $\mathbb{C}$ done $\Rightarrow$ char 0 done is easy

(• Very hard, using Deligne–Weil) $H^{2k}(Y, \mathbb{Q}) \xrightarrow{A^{d-2k}} H^{2d-2k}(Y, \mathbb{Q})$

④ Char $k > 0$ using l-adic cohomology ▨

Rk. l-adic cohomology has no Hodge–Riemann relation, but others all have.

Conjecture 4. $(\# \mathcal{F}_n)_n$ is log-concave sequence. (solved in $d \le 4$ case ($d \le 3$ is trivial))

Rk. Same as Read, the core is finding an intersection interpretation of $\# \mathcal{F}_n$: historical people already know that a_n can be written as Betti numbers, but only intersection number makes sense in proof.

Pf of Conj 1. ① Hyperplane arrangement $V = k^d$, $\mathcal{A} = \{H_1, \dots, H_n\}$ is hyperplane arrangement. $\mathcal{A}$ essential if $\bigcap H_i = \{0\}$; we can "make" into essential by $\mathcal{A}' = \{H_1/\bigcap H_i, \dots, H_n/\bigcap H_i\}$; the characteristic polynomial is $\chi_{\mathcal{A}}(t) = \sum_{S \subset \{1, \dots, n\}} (-1)^S t^{\dim H_S}$, $H_S = \bigcap_{i \in S \subset \{1, \dots, n\}} H_i$

② Realize P_G as $\chi_{\mathcal{A}}$ for some $\mathcal{A}$. ③ Prove for $\chi_{\mathcal{A}}$ in general.

We need correspond $\mathcal{A}$ to a matroid, $E \subset V$ still

Def 5. (Matroid) TAFAE. (A) $(E, \mathcal{I})$, $\mathcal{I} \subset 2^E$ independent set, s.t. ① $\emptyset \in \mathcal{I}$ ② $I_1 \subset I_2 \in \mathcal{I} \Rightarrow I_1 \in \mathcal{I}$; ③ $|I_2| > |I_1|$, $I_1, I_2 \in \mathcal{I} \Rightarrow \exists e \in I_2$ s.t. $e \cup I_1 \in \mathcal{I}$;

(B) (E, rank), $\mathrm{rank}: 2^E \to \mathbb{Z}_{\ge 0}$, s.t. ① $\mathrm{rank}(S) \le \# S$; ② $S \subset T \Rightarrow \mathrm{rank}(S) \le \mathrm{rank}(T)$; ③ (submodular) $\mathrm{rank}(S) + \mathrm{rank}(T) \ge \mathrm{rank}(S \cap T) + \mathrm{rank}(S \cup T)$

(C) $(E, \mathcal{F})$, $\mathcal{F} \subset 2^E$ the flats, s.t. ① $\mathcal{F} \ne \emptyset$; ② $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$; ③ $\forall F \subset E$, $\forall e \in E - F$, $\exists ! F'$, s.t. $e \in F'$ and $F' \supset F$ minimal contain

(D) $(E, \mathcal{F})$, $\mathcal{F} = \{F \mid \mathrm{Span}(F) \cap E = F\}$.

Rk. Note that with all rank, independent classical, it's just same as linear algebra, thus we can prove the equivalence by

$\mathrm{rank} \to \mathcal{I} = \{I \subset E \mid \mathrm{rank}(I) = |I|\}$

$\mathcal{I} \to \mathrm{rank}$; $S \mapsto \max\{|I| \mid I \subset S, I \in \mathcal{I}\}$

$\mathrm{rank} \to \mathcal{F} = \{F \subset E \mid \mathrm{rank}(F \cup \{e\}) > \mathrm{rank}(F), \forall e \in E - F\}$

$\mathcal{F} \to \mathrm{rank}$; $S \mapsto \max\{|F| \mid F \subset S, F \in \mathcal{F}\}$ ▨

② Using classical definition $\forall E \subset V$, $\exists$ a matroid structure on it but finite

Eg 6. $E = \{(1,0,0), (0,1,0), (0,0,1), (1,1,0)\} \stackrel{\text{def}}{=} \{1, 2, 3, 4\}$

its flats are 1234, 124, 13, 23, 34, $\emptyset$ $\Leftrightarrow$ 1, 2, 3, 4 in $\mathbb{P}^2$

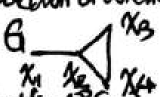
Conjecture 2' (Dowling–Wilson conjecture of matroids)

The characteristic polynomial of matroid $\chi_M(t) = \sum (-1)^{\#S} t^{\operatorname{rank}(M) - \operatorname{rank}(S)}$, this is preserved under the correspondence:
$\mathcal{A} = \{H_1 \cdots H_n\} \longleftrightarrow M_{\mathcal{A}} = (E, \operatorname{rank})$, $E = \{1 \ldots n\}$, $S \subseteq E$ is the combinational information of $\mathcal{A}$ & rank: $S \mapsto \operatorname{codim}(\bigcap_{i \in S} H_i)$
We have $\chi_{\mathcal{A}} = \chi_{M_{\mathcal{A}}}$ after reduce $\mathcal{A}$ to essential, and $\chi_{\mathcal{A}_G} = P_G$ by the induction formula under deletion & contraction.
($\mathcal{A}/H_i = \mathcal{A}$ on $k^d / \bigcap H_i$) Here is: $\mathcal{A} - \{H_n\} = \{H_1 \cdots H_{n-1}\}$
$\mathcal{A}/\{H_n\} = \{H_1 \cap H_n, \cdots H_{n-1} \cap H_n\}$

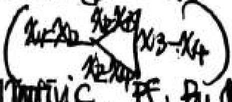
<u>Summary</u>

Graph —①→ Matroid ③ is done
② ↓ ↑ ③ ① by $(E, \mathcal{I}) = (\{\text{edges}\}, \{\text{edges} \not\supset \text{any cycles}\})$
Hyperplane — Configuration ② by vertices $x_1 \cdots x_n$, edges connecting x_i & $x_j \Rightarrow (x_i - x_j) \in \mathcal{A}_G$.
arrangement of vectors then $\{(x_1 - x_2) \cdots (x_{n-1} - x_n)\} = \mathcal{A}_G$
④ $\longmapsto$ vector f (normal vector)

Eg.7. G



$P_G(t) = t(t-1)(t-2)^2$ up to t.
$\chi_{\mathcal{A}_G}(t) = (t-1)^2(t-2)$
$\rightsquigarrow P_G(t) = t^{\#\pi_0(G)} \chi_{\mathcal{A}_G}(t)$



Prop 8. $U = k^d - \{H_1 \cup \cdots \cup H_n\} \Rightarrow$ in $K_0(\mathrm{Var}_k)$, $[U] = \chi_{\mathcal{A}}(\mathbb{L})$, $\mathbb{L} = [\mathbb{A}^1]$ motivic. Pf. By de Morgan ▨
Cor 9. $k = \mathbb{F}_q \Rightarrow \#\{\mathbb{F}_{q^r}\text{-rational pts} \in U\} = \chi_{\mathcal{A}}(q^r)$
Cor 10. $k = \mathbb{C} \Rightarrow \chi_{\mathcal{A}}(t) = \sum (-1)^k \dim H^k(U, \mathbb{Q}) t^{d-k} = \sum (-1)^{d-k} \dim H^{d-k}(U, \mathbb{Q}) t^k$ (thus does "characteristic" polynomial via Betti numbers)
Thm 11. (Huh, Katz) $\mathcal{A}$ essential $\Rightarrow \overline{\chi}_{\mathcal{A}}(t) = \chi_{\mathcal{A}}(t)/(t-1)$ (When loopless, it's divisible) is log-concave (thus so is $\chi_{\mathcal{A}}(t)$, complete the proof of Read conjecture)

Key construction (intersection) $\mathcal{A} = \{H_1 \cdots H_n\}$, $H_i = V(l_i)$, construct
Two nef classes $\alpha = f^* g^*([H]) \in H^2(X, \mathbb{Q})$
$\beta = g^*([H]) \in H^2(X, \mathbb{Q})$

$$
X \xrightarrow{\pi} X' \to \mathbb{P}^{n-1}, \quad X' \to \mathbb{P}^{n-1}
$$

$\mathbb{P}^{d-1} \dashrightarrow [l_1 \cdots l_n] \to \mathbb{P}^{n-1} \to [\frac{1}{l_1(x)} \cdots \frac{1}{l_n(x)}]$

$U = \mathbb{P}^{d-1} - \bigcup \overline{H}_i$, take $X' = \overline{(f, g)(U)} \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ and $X \xrightarrow{\pi} X'$ resolution (Here over $\mathbb{C}$, thus by Hironaka)

then by Huh, $\overline{\chi}_{\mathcal{A}}(t) = t^{d-1} - b_1 t^{d-2} + \cdots + (-1)^{d-1} b_{d-1}$, with $b_k = \deg_X(\alpha^k \beta^{d-k-1})$ intersection number (∗)

① (∗) $\Rightarrow$ log-concave

(Q (By Zhi Jiang & Chen Jiang) can we directly use intersection on X' singular? (Yes))

Thm 12. (Khovanskii–Teissier inequality) X smooth projective, $\dim X = d$, A, B nef $\in H^2(X, \mathbb{Q}) \Rightarrow \deg(A^m B^{d-m})$ is log-concave
Pf. Assume A, B ample (by ample approximate nef) and suffices to prove $\deg(AB)^2 \geq \deg(A^2)\deg(B^2)$ (surface case)
Then $\deg(A \cdot B) =: \langle A, B \rangle$ is non-degenerated quadratic form
Let $A = \lambda B + C$, by Hodge index thm, $\langle\,,\rangle$ have eigenvalue $(+, -, \cdots, -) \Rightarrow \langle C, C \rangle \leq 0$
$\langle A, B \rangle^2 = \lambda^2 \langle B, B \rangle^2$ as $\langle B, C \rangle = 0$; $\langle A, A \rangle \langle B, B \rangle = \lambda^2 \langle B, B \rangle^2 + \underline{\langle C, C \rangle \langle B, B \rangle}_{\leq 0 \text{ by } \langle B, B \rangle > 0 \text{ ample}}$ ▨
② Why (∗) holds (Betti = Intersection) Set $\operatorname{grad}(f): \mathbb{P}^{d-1} \dashrightarrow \mathbb{P}^{d-1}$
Thm 13. (Morse, Dimca–Papadima) $p \longmapsto [\frac{\partial f}{\partial x_1}(p) \cdots \frac{\partial f}{\partial x_d}(p)]$ $\overline{U} = \mathbb{P}^{d-1} \setminus \bigcup \overline{H}_i = \mathbb{P}^{d-1} \setminus V(f)$
For a general $\mathbb{P}^{d-2} \subset \mathbb{P}^{d-1}$, homotopical equivalence $\overline{U} \simeq (\overline{U} \cap \mathbb{P}^{d-2}) \cup \{\deg \operatorname{grad}(f)\ \text{cells}\}$ $f = l_1 \cdots l_n$, $\overline{H}_i = V(l_i)$
$\deg(\operatorname{grad}(f))$ = intersection = Betti by $\begin{cases} \dim H^k(\overline{U}, \mathbb{Q}) = \dim H^k(\overline{U} \cap \mathbb{P}^{d-2}, \mathbb{Q}),\ k \leq d-2 \Rightarrow H^{d-1}(\overline{U}, \mathbb{Q}) = \deg \operatorname{grad} f \\ \text{Note that } \frac{\partial f}{\partial x_i} = \frac{\partial (l_1 \cdots l_n)}{\partial x_i} = \langle [\frac{\partial l_1}{\partial x_i}, \cdots \frac{\partial l_n}{\partial x_i}], [l_2 \cdots l_n, \cdots, l_1 \cdots l_{n-1}] \rangle = \langle [\frac{\partial l_1}{\partial x_i} \cdots \frac{\partial l_n}{\partial x_i}], [\frac{1}{l_1} \cdots \frac{1}{l_n}] \rangle \\ \text{thus we factor } \operatorname{grad} f: \mathbb{P}^{d-1} \longrightarrow \mathbb{P}^{d-1} \Rightarrow \deg(\beta^{d-1}) = H^{d-1}(\overline{U}, \mathbb{Q}) = \deg(\operatorname{grad} f) ▨ \end{cases}$

Rk. We thus proven Thm 11 in realisible case, in not realisible case, we replace $H^*(X, \mathbb{Q})$ by $CH^*(M)$, M the matroid.

$p \mapsto [\frac{1}{l_1(p)} \cdots \frac{1}{l_n(p)}] \in \mathbb{P}^{n-1} \to \mathbb{P}^{d-1}$ projection

Thm 14. (Adiprasito–Huh–Katz) $CH^*(M)$ has Hodge theory, such as Poincaré duality, hard Lefschetz, Hodge–Riemann relation (generalise Hodge index)
(Realisible means comes from geometry, in non-realisible case, we can't give M an algebraic variety, thus Hodge isn't trivial thing
Pf of <u>non-realisible Dowling–Wilson</u>. (Without $H^*(Y) \longleftrightarrow IH^*(Y)$, what's its analogue?)
$H^*(M)$ (graded Möbius algebra of M) $\longleftrightarrow IH^*(M)$ (Intersection cohomology of matroid) has Hodge theory, then are all same ▨
<u>Lorentzian polynomial</u> Consider Y smooth projective, $D_1 \cdots D_n$ nef classes, $\dim Y = d$. Consider $\deg(t_1 D_1 + \cdots + t_n D_n)^d$
Prop 15. ① $(E, \mathcal{I})$ is matroid, $E = \{1, \cdots n\}$ and $\mathcal{I} = \{I \mid \prod_{i \in I} D_i \neq 0\}$; ② Every D_i is base point free, then $(E, \mathcal{I})$ is realisible over $\mathbb{C}$; (Rk. Without base pts free, it's also open now)
Pf. Must use Lorentzian polynomial ▨ (Due to time, no definition given in Lecture)

Stability of klt (Ziquan Zhuang) $\boxed{\text{Lecture 4}}$

[Ref. ArXiv.1801.07112 & ArXiv.2307.10525]

Kähler-Einstein metric is Kähler & Einstein metric (compatible with J & $Ric(g)=\lambda g$), classifying by Einstein constant:

$\lambda>0$	$\lambda=0$	$\lambda<0$
Fano	Calabi Yau	Canonical polarized

Facts. $\lambda>0$ ① this cpt complex mfd turns out to be algebraic, i.e. anti-canonical ample (via Kodaira embedding)

② (Donaldson–Sun) its Gromov–Hausdorff limit also $\boxed{\text{algebraic}}$, but may singular (Still Fano with limit metric, after replacing by a subsequence)

Rk. (Singular KE metric) $\omega|_{X_{sm}}$ is KE and in singular part, we have same volume form: $vol(\omega|_{X_{sing}}) = \ldots$ the anti-canonical volume • Recall such volume is due to $L^n = C(L)^n$, $C(L)^n$ is volume form

(ii) the tangent cone (metric tangent cone by G-H limit of scaling $\lambda\omega$) are all f.L. affine varieties with Ricci-flat Kähler cone metric. i.e. local metric written as cone

Q1. Now taking metric cone to $\lambda\omega\to\infty$ and Ricci flat ($Ric\equiv 0$), not depending on constant. Can we interpret this by pure algebraic geometry? (i) + (ii) $ds^2 = dr^2 + r^2 ds_1^2$, ds_1^2 is the sphere metric

Thm 1 (Yau-Tian-Donaldson conjecture) solved by
Smooth: Berman, Chen-Donaldson-Sun, Tian
Singular: Berman-Boucksom-Jonsson, Li-Liu-Xu-Z

$\exists$ KE metric $\Leftrightarrow$ K-polystable

Thm 2. (K-moduli thm, many people) The coarse moduli of KE Fano is proper. (ii) is more related with singularity:

Conjecture 3. (Donaldson–Sun) $x_1\in(X_1,\omega_1)$, $x_2\in(X_2,\omega_2)$, $\widehat{\mathcal{O}}_{X_1,x_1}\cong\widehat{\mathcal{O}}_{X_2,x_2}\Rightarrow$ their metric tangent cone $\cong$ (only depends algebraically)

• Symplectically viewing it, can depict it as Reeb cone → Reeb vector in normal bundle

Q2. How can we give an algebraic construction of metric tangent cone?

Our main thm is

Thm 4. (Stable degeneration, "algebraic Ricci flow") $\forall$ klt singularity $x\in X$ ($\supseteq$ all singularity occur in ②), $\exists$ volume preserving degeneration of $x\in X$ to $\exists!$ $y\in Y$, s.t. $y\in Y$ admits a Ricci-flat Kähler cone metric.

Replacement of metric is valuation (sometimes we use DVR as replacement of ball) & Divisor $\Leftrightarrow$ valuation can record information of metric/volume)

Def 5. (Valuation of singularity) $x\in X$ singularity, then the valuation is $v: \mathbb{C}(X)^*\longrightarrow\mathbb{R}$, $v(f)>0$, $f\in\mathcal{O}_{X,x}$

Eg 6. Take v to be the multiplicity of zero of f at $x\in X$. The Lelong number formula is $mult_x(f)=\liminf_{z\to x}\frac{\log|f(z)|}{\log|z-x|}$ gives an analytic interpretation of valuation, where the righthand $|z-x|=d(x,z)$ depending on the KE metric, thus we extend this definition to general; However, $\liminf_{z\to x}\frac{\log|f(z)|}{\log d(z,x)} =: v_\omega(f)$ isn't (in AG) valuation due to the liminf may turn to ∞? Another consequence of Donaldson-Sun told us no:

Thm 7. (Donaldson-Sun) (X,x) is limit (G-H) of KE Fano, then ① It does valuation; ② metric tangent cone of $x\in X$ can be realised as 2-step degeneration {K-semistable degeneration; K-polystable degeneration} ③ The first step "induced" by v_ω.

Construction of metric tangent cone

• Idea from differential (Mabuchi): the Mabuchi energy functional M: {Kähler metric} $\longrightarrow\mathbb{R}$, minimal is KE-metric;
Chi Li & Chenyang Xu's normalized volume: $\widehat{vol}$: {valuations} $\longrightarrow\mathbb{R}$, minimal is v_0 (proven by Li & Xu for G-H limit case, and we expect prove for klt cases), is it in general klt, not exist a minimal?)
① Using minimal to give v_ω, v_ω $\exists!$ up to scalaring; ② $gr_\omega R$ is f.g; ③ then v_ω induce the K-semistable degeneration by $gr_\omega R = \bigoplus \mathfrak{a}_\lambda/\mathfrak{a}_{>\lambda}$ ($\mathfrak{a}_\lambda=\{f\in\mathcal{O}_{X,x}\mid v(f)\geq\lambda\}$) is graded $\mathbb{C}$-algebra and f.g, $x_0\in X_0 = Spec(gr_\omega R)$ is K-semistable Fano cone; ④ X_0 $\exists!$ K-polystable degeneration $y\in Y$, admits Ricci flat Kähler cone metric.
This ①–④ is the Stable degeneration conjecture, all operation algebraic.

Eg. ① We expect in $(\mathbb{A}^n,0)$, it's mult$_0$ ② quotient singularities $(\mathbb{A}^n/G,0)$ ($\#G<\infty$), its descend of mult.

Rk. Here Spec$(gr_\omega R)$ similar with usual tangent cone Spec$(gr_{m_x}\mathcal{O}_{X,x})$

klt singularity (analytic)

Def 8. $x\in X$ is klt if $\exists r\in\mathbb{Z}_{>0}$, rK_X is Cartier (X singular, K_X is only Weil) and local generator σ, $(\sigma\wedge\bar\sigma)^{\frac{1}{r}}$ top volume form is locally L^2-integrable: $\int_{X\ni x}(\sigma\wedge\bar\sigma)^{\frac{1}{r}}<\infty$

Eg 9. ① $\mathbb{A}^n/G\ni 0$ is klt as $0\in\mathbb{A}^n$ is klt ② Generally, X is klt, X/G is still klt if $K_{X/G}$ is $\mathbb{Q}$-factorial | $\mathbb{A}^n/G$ is $\mathbb{Q}$-factorial (All G is $\#G<\infty$)

Recall that the algebraic definition is $Y\xrightarrow{f}X$ resolution (Hironaka), $K_Y=f^*K_X+\sum a_iE_i$, a_i then is order of zero/pole of $(\sigma\wedge\bar\sigma)^{\frac{1}{r}}$

By calculus, the order >-1 to make it locally integrable, that's why we define klt as $a_i>-1$, $\forall i$

Eg 10. ① Consider the affine cone over Fano mfd V, $X=C(V,-K_V)=Spec(\bigoplus H^0(V,-mK_V))\ni x$ is the vertex, the only singularity. It's klt: $Y=Bl_xX=Tot(K_V)\longrightarrow K_Y+E=f^*K_X+(1+a)E$, restrict to E $\Rightarrow a=0\Rightarrow 1>-1$, is klt (Restrict to normal bundle)

② Generally, V projective normal variety, L ample, $X=C(V,L)\Rightarrow x\in X$ vertex is klt $\Leftrightarrow$ V is Fano and $L=k(-K_V)$, $\exists k\in\mathbb{Z}_{>0}$

Normalized volume We record the order of pole/zero along E again: denote $A_X(E)=1+a_i$ the log discrepancy, generalize to valuation $\Rightarrow\widehat{vol}(v):=A_X(v)^n vol(v)$, where $vol(v):=\limsup\frac{\dim(\mathcal{O}_{X,x}/\mathfrak{a}_\lambda)}{\lambda^n/n!}$ (Rk. $mult_xX:=\lim_{k\to\infty}\frac{\dim(\ldots)}{(k^n/n!)}$) (Here local is $E_1+\cdots+E_n$)

Eg 11. ① Let $\alpha\in\mathbb{R}^n$, $v_\alpha(f)=v_\alpha(\sum c_\beta x^\beta)=\min\{\langle\alpha,\beta\rangle \mid c_\beta\neq 0\}$, let $\alpha=(1,\ldots,1)\Rightarrow v_1=mult_0$ in $\mathbb{A}^n$ $\Rightarrow vol(v_\alpha)=\frac{1}{\alpha_1\cdots\alpha_n}$, it's called monomial valuation. then we allow ($x_1,\ldots,x_n$) ...

② Quasi-monomial is locally monomial valuation; every valuation is limit (point-wise) of quasi-monomial valuation is ...

We the define $A_{X,x}: \mathrm{Val}_{X,x} \to \mathbb{R}_{\geq 0}$ by extending all quasi-monomials, $E=\sum E_i$ with weight $\alpha=(\alpha_1,\dots,\alpha_r) \Rightarrow A_{X,x}(v_\alpha)$ $\boxed{\text{Page 29}}$
$=\sum \alpha_i a_i = \sum \alpha_i A_{X,x}(E_i)$ (it's well-defined, i.e. not depending on resolution) $\Rightarrow A_{X,x}(v) = \sup A_{X,x}(v_\alpha)$ (may ∞) (Jonsson–Mustață)
Eg.12. For monomial valuation $A_{\mathbb{A}^n}(v_\alpha)=\sum \alpha_i \cdot 1 = |\alpha|$, thus $\widehat{\mathrm{vol}}(v_\alpha)=\frac{|\alpha|^n}{\alpha_1\cdots\alpha_n} \geq n^n$. minimal $\exists!$ $\alpha=(a,a,\dots,a)$ up to a-times to multiplicity.
It's the special case of stable degeneration, $\mathbb{A}^n/G$ similar.
Fact: the minimal of $\widehat{\mathrm{vol}}$ must be quasi-monomial valuation.
Def13. The local volume $\widehat{\mathrm{vol}}(x,X) := \inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}(v) = \inf \mathrm{vol}(v_\alpha)$
Prop14. ① $A_{X,x}(\lambda v)=\lambda A_{X,x}(v)$, $\mathrm{vol}(\lambda v)=\frac{1}{\lambda^n}\mathrm{vol}(v)$ $\Big\} \Rightarrow \widehat{\mathrm{vol}}(\lambda v)=\widehat{\mathrm{vol}}(v)$; ② (Li) $\widehat{\mathrm{vol}}(x,X)>0$ if $x\in X$ is klt. ③ (Blum, Liu) $\widehat{\mathrm{vol}}(x,X) \leq \widehat{\mathrm{vol}}(0,\mathbb{A}^n)=n^n$ (Liu, Xu) equal $\Leftrightarrow$ smooth
④ $\widehat{\mathrm{vol}}(-,X): X \to \mathbb{R}_{>0}$ is lower semi-continuous function (Zariski topology)
⑤ (Xu–Zhang) $G \curvearrowright X$ ($\#G \leq \infty$), v_0 minimizer must G-invariant (due to $G\times X \to X$ must have exist all minimizer) but $\exists!$ ▨
• Using ⑤ we can prove the Fact upper: using toric action $\mathbb{G}_m \curvearrowright \mathbb{A}^n$, $\mathbb{G}_m$-invariant $\Leftrightarrow$ monomial ▨
⑥ (Li, Liu, Xu) (Return to Eg.10 setting) $\widehat{\mathrm{vol}}(x,X)=\widehat{\mathrm{vol}}(\mathrm{ord}_E) \Leftrightarrow V$ is K-semistable; ord_E is valuation by counting order of zero/pole along exceptional divisor E.
If $L=-K_V$, $\widehat{\mathrm{vol}}(x,X)=(-K_V)^n$ anti-canonical volume • Without K-semistable, it still holds
Rk. In general, $\widehat{\mathrm{vol}}(x,X) \notin \mathbb{Q}$ (even for toric case) $\Rightarrow v \neq \lambda\,\mathrm{ord}_E$ for toric case can't be written from a log-resolution (called divisorial valuation)
Thm15. (Li, Xu) If minimizer is divisor $\Rightarrow E$ is K-semistable Kollár component
Def16. (Kollár component) If $\exists$ diagram $E \subset Y \xrightarrow{\text{birational}} x \in X$, E the unique exceptional divisor s.t. ① $-(K_Y+E)$ ample on X, ② (Y,E) is plt singularity.
Lemma18. (Xu) $\forall$ klt singularity, $\exists$ Kollár component.
Eg.17. Again E is only Weil, if $\exists k$ s.t. kE is Cartier, then
E is Kollár component $\Leftrightarrow E$ is unique exceptional & E is klt Fano. • plt, dlt, lc.. similiar as klt, occurs and first defined in log minimal model; plt $a_i>-1$, $\forall i$
However, the stable degeneration tells that $\widehat{\mathrm{vol}}(x,X)\in\overline{\mathbb{Q}}$ (algebraic numbers), and distribution of its value is also can say:
Thm19. (Xu, Zhang) $\mathrm{Vol}_n = \{\widehat{\mathrm{vol}}(x,X) \mid x\in X \text{ klt}, \dim X=n\} \subset \mathbb{R}$ discrete, but $B(0,\varepsilon)$ $\exists\,\varepsilon>0$. (Looks like the spectrum of compact operator)
plt: pure log terminal; dlt: divisorial log terminal; dlt slightly subtle; lc $a_i \geq -1$, $\forall i$; klt $a_i>-1$, $\forall i$; $\Sigma a_i<0$
Conjecture20. $\widehat{\mathrm{vol}}(x,X) \leq \widehat{\mathrm{vol}}$(ordinary double pt) for $x\in X$ singular (i.e. cusp is best, it's called the cusp conjecture)
Conjecture21. $\widehat{\mathrm{vol}}(x,X)\in\mathbb{Q} \Leftrightarrow$ minimizer divisorial
Eg.22. (Li, Xu) When GH limit of KE Fano case originally, $\frac{\widehat{\mathrm{vol}}(x,X)}{\widehat{\mathrm{vol}}(0,\mathbb{A}^n)}$ = volume density $=\lim_{r\to 0}\frac{\mathrm{Vol}(B(r))}{\mathrm{Vol}(B_0(r))}$
Pf of stable degeneration conjecture.
• In Birational Geometry, these dlt, klt, canonical, terminal; such dlt, klt, plt are all rational, lc
① Uniqueness of minimizer $\Leftrightarrow$ Uniqueness of Kähler–Einstein metric. By Bando–Mabuchi, the Mabuchi energy functional is geodesic convex functional) $\Rightarrow$ minimal is unique. by MMP.
Purely algebraic method (Blum–Liu–Qi) $\boxed{\text{Step 1}}$ "Geodesics" in $\mathrm{Val}_{X,x}$: it's impossible do in $\mathrm{Val}_{X,x}$ (too small), even can't find a one, extend to $\mathrm{Val}_{X,x} \subsetneq \mathrm{Filt}_{X,x}=\{(\mathfrak{a}_\lambda)_{\lambda\in\mathbb{R}} \mid \text{decreasing ideals}, \mathfrak{a}_\lambda\mathfrak{a}_\mu\subset\mathfrak{a}_{\lambda+\mu}\}$, the inclusions $\mathrm{Val}_{X,x} \hookrightarrow \mathrm{Filt}_{X,x}$, $(\mathfrak{a}_\lambda(v)=\{f\in\mathcal{O}_X \text{ and } v(f)\geq\lambda\})_{\lambda\in\mathbb{R}_{\geq 0}}$ easier to deal; independent with log resolution but lc not
then for $\forall v_0, v_1 \in \mathrm{Val}_{X,x} \Rightarrow v_t=(1-t)v_0+tv_1$ not valuation, but it still induced $(\mathfrak{a}_\lambda(v_t))\in\mathrm{Filt}_{X,x} \Rightarrow$ it's the desired geodesic, denoted $(\mathfrak{a}^{(t)}_\lambda)$
Extend $\widehat{\mathrm{vol}}$ to $\mathrm{Filt}_{X,x}$ also: $\widehat{\mathrm{vol}}(\mathfrak{a}_\bullet)=\mathrm{lct}(\mathfrak{a}_\bullet)^n \cdot \mathrm{mult}_x(\mathfrak{a}_\bullet)$ (Recall log canonical threshold)
(note the restrict back to $\mathrm{Val}_{X,x}$, mult_x is $\mathrm{vol}(v)$ but lct isn't equal to $A_{X,x}$ always) $\mathrm{Filt}_{X,x}$; $\mathrm{Val}_{X,x}$ seems "discrete" $\subset \mathrm{Filt}_{X,x}$
Thm23. ① (Xu–Zhang) $\mathrm{lct}(\mathfrak{a}^{(t)}_\bullet) \leq (1-t)\,\mathrm{lct}(\mathfrak{a}^{(0)}_\bullet)+t\,\mathrm{lct}(\mathfrak{a}^{(1)}_\bullet)$ • Here $\mathrm{lct}(\mathfrak{a})=\frac{A_{X,x}(E)}{\mathrm{ord}_E(\mathfrak{a})}$, thus good generalisation well; $\exists$ a metric makes it to be a true geodesic
② (Blum–Li–Qi) $\mathrm{mult}(\mathfrak{a}^{(t)}_\bullet)^{\frac{1}{n}} \leq (1-t)\,\mathrm{mult}(\mathfrak{a}^{(0)}_\bullet)^{\frac{1}{n}}+t\,\mathrm{mult}(\mathfrak{a}^{(1)}_\bullet)^{\frac{1}{n}}$, equality holds $\Leftrightarrow v_0=\lambda v_1$.
Pf. Birational geometry. Using multiplier ideal & subadditivity formula for ①; Okounkov body for ②: $\mathrm{mult}=\mathrm{vol}(\mathbb{R}^n\setminus\Delta)$, Δ a convex body to reduce to combinational & Brunn–Minkowski inequality (Minkowski lattices $\mathrm{Vol}((1-t)\Delta_0+t\Delta_1)^{\frac{1}{n}} \leq (1-t)\mathrm{Vol}(\Delta_0)^{\frac{1}{n}}+t\mathrm{Vol}(\Delta_1)^{\frac{1}{n}}$) ▨ • K-polystable degeneration (semi-stable one easier): locally $x\in\mathrm{Spec}\,R$ $\xrightarrow{\text{degeneration}}$ $x_0\in(X_0,\xi_0)$ K-semistable Fano cone
② Finite-generated. Q. What's original K-stable case for vector bundle? Schieder refinement thm ③ Existence (two-stable degeneration omitted)
Eg.24. $\mathrm{gr}_v R$ f.g. is nontrivial thing: $D=(xyz=x^3+y^3)\subset\mathbb{A}^3=X$ $\exists\, U\subset\mathbb{R}^2_{>0}$ s.t. $\{$ (Blum) $\}$ quasi-monomial (Xu)
$D\cap E$ is nodal cubic curve ∝ P. The common refinement of two Jordan–Hölder filtration.
$\Rightarrow$ 3 analytic branches at P. (Li–Wang–Xu) Two from one $\Rightarrow$ same refinement; [2] $\mathrm{gr}_v R$ is f.g.$\}$ $=U(\mathbb{Q})$ rational pts
Thm25. (Xu, Zhang) v quasi-monomial $E+D \subset Y = \mathrm{Bl}_0 X$ thus in general $\mathrm{gr}_v R$ isn't f.g.
Pf of Xu: Idea: $\widehat{\mathrm{vol}}(v_k)\to\widehat{\mathrm{vol}}(x,X)$ but limit not exists; let v_i be monomial on same (uniform) log resolution $\Rightarrow \exists$ uniform bound by Birkar's boundedness of complement thm
$\mathrm{gr}_v R$ is f.g. and $\mathrm{Spec}(\mathrm{gr}_v R)$ is klt $\Leftrightarrow v$ is Kollár valuation
Def26. ① (Kollár model) If $\exists$ diagram $E \subset Y \xrightarrow{\text{birational}} x\in X$, $E=f^{-1}(x)$, s.t. ① $-(K_Y+E)$ ample ② (Y,E) is dlt singularity (e.g. SNC)
② (Kollár valuation) $\exists$ Kollár model (E,Y), s.t. the valuation is quasi-monomial and monomial goes to (Y,E) • In original definition by Xu, it's qdlt (quotient of dlt), it's not essential (only technical)
s.t. $\exists\, v_1,\dots,v_r$ & $D_1,\dots,D_r$, $D_i\in|-NK_X|$ for N uniform, v_i monomial & $\exists$ uniform log resolution of family $\Rightarrow$ limit exists and quasi-monomial by
Pf of Thm25. Induction on rational rank of $v_0 := \mathrm{rank}_{\mathbb{Z}}(v_0(K^*))$
① $r(v_0)=1$, assume $v_0=\mathrm{ord}_E$ is divisorial $\Rightarrow$ we can turn $\mathrm{gr}_v R$ to [illegible], thus case use [illegible]
② $\Big\{$ Finite generate property preserves in geodesic; Such desired geodesic $\subset \mathrm{Val}_{X,x}$ $\quad v'$ (lower) $\curvearrowright v$ (lower), v_0 (higher rank) ▨
[illegible] invariance of plurigenera ▨
(Siu) $\mathcal{X}\to B\ni b \Rightarrow h^0(mK_{X_b})$ constant in B [illegible].

Partion of integers Lecture 8 (Chen Jiang)

Def1. (Restricted partion) Partion n into $\le d$ parts, each $(\lambda_i) \in \mathbb{Z}^d_{0\le \lambda_i \le k}$, $\sum \lambda_i = n \xleftrightarrow{1:1}$ Young diagram; denote $P(k,d,n)$ all such partions and $p(k,d,n) = \# P(k,d,n)$

Prop2 ① $P(k,d,n) = P(d,k,n)$; ② $P(k,d,n) = P(k,d,kd-n)$; ③ $p(k,d,n) = p(k,d-1,n-k) + p(k-1,d,n-d)$

Pf. ① Rotation the Young diagram; ② Complement the Young diagram; ③ Cancel the Young diagram ☒

Prop3. (Stanley) ① $p(k,d,n) = \dim_{\mathbb{Q}} H^{2n}(Gr(k,k+d),\mathbb{Q})$; ② log-concave $\begin{cases} p(k,d,n) \le p(k,d,n+1); & n \le \frac{kd}{2} - 1 \\ p(k,d,n) \ge p(k,d,n+1); & n \ge \frac{kd}{2} \end{cases}$

Rk. The partion also 1:1 corresponds to $(k^{e_k}, (k-1)^{e_{k-1}}, \ldots, 0^{e_0})$ with $\sum i e_i = n$ (Diophantine equation), then we can inductively construct partion by this way:

$P(k,d,n-1) \longrightarrow P(k,d,n)$

$(k^{e_k} \ldots 0^{e_0}) \longmapsto (k^{e_k} \ldots (i+1)^{e_{i+1}}, i^{e_i - 1}, \ldots, 0^{e_0}) =: \lambda(i)$

An important matrix then is $A_{k,d,n} \in \mathbb{Z}^{p(k,d,n-1)\times p(k,d,n)}$ with entry $(A_{k,d,n})_{\lambda\mu} = \begin{cases} e_i; & \mu = \lambda(i) \\ 0; & \text{otherwise} \end{cases}$

Eg.4. $P(4,3,5) = \{(410),(320),(311),(221)\}$

$P(4,3,6) = \{(420),(411),(330),(321),(222)\}$

$\cdots$ $P(4,3,7) = \{(430),(421),(331),(322)\}$

$$A_{436} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix} \quad A_{437} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Thm5 (Chen Jiang) $\operatorname{rank}(A_{k,d,n}) = \begin{cases} p(k,d,n-1); & n \le \frac{kd}{2} \\ p(k,d,n); & n \ge \frac{kd}{2} + 1 \end{cases}$

It turns out to relate Hard Lefschetz:

$\mathbb{R}^{p(k,d,0)} \xrightarrow{\cdot A_{k,d,1}} \mathbb{R}^{p(k,d,1)} \xrightarrow{\cdot A_{k,d,2}} \cdots \xrightarrow{\cdot A_{k,d,kd}} \mathbb{R}^{p(k,d,kd)}$

we can expect this chain is cohomology. As $\mathbb{R}^{p(k,d,n)} = H^{2n}(Gr(k,k+d);\mathbb{R})$, this cohomology given by an ample divisor $L \subset Gr(k,k+d)$ via representation theory, which is already know in 1974, Proter after replacing $A_{k,d,n}$ by $\tilde{A}_{k,d,n}$, which e_i replaced by all 1 and $[X] = \tilde{A}_{k,d,n}$

Idea of proof. Adding $\overset{X}{\underset{Y}{\rightleftarrows}} \mathbb{R}^{p(k,d,0)}$ as a sl_2-representation $(X,Y,H) = \mathrm{Sym}^d(W_k)$, W_k^Y is the $(k+1)$-dim sl_2-rep, s.t. $[X] = A_{k,d,n}$, thus we have all converse arrows ☒

Algebraic dynamical system is $f \in \mathrm{Aut}(X)$, $\dim X = d$

Entropy of f: ① (Topological) $h_{top}(f) := \sup\limits_{U \subset X} \{n \mid \bigcup f^i(U) = X\}$

② (Algebraic) $h_{alg}(f) = \max\limits_{0\le k\le d} \log(\rho_k(f))$, $\rho_k(f) = \rho(f^*: H^{k,k}(X) \to H^{k,k}(X))$

③ (Intersection) $\le h_{top}(f)$ (1974) $= \rho(f^*: H^{2k}(X) \to H^{2k}(X))$

$\rho_k(f) = \lim\limits_{n\to\infty} ((f^*)^n(H^k) \cdot H^{d-k})^{\frac{1}{n}}$, H is ample line bundle; $= \rho(f^*: N^k(X) \to N^k(X))$ (spectral radius)

Gromov [1982] $h_{top}(f) \le lov(f) \le h_{alg}(f)$, $lov(f)$ is the log-volume growth $= \limsup\limits_{n\to\infty} \frac{\log(\mathrm{vol}(\Gamma_n))}{n}$, $\mathrm{vol}(\Gamma_n) = \Gamma_n \cdot H^d = (\sum (f^{*})^i H)^d$

Γ_n is the graph of $(f^*, \ldots, (f^*)^{n-1}): X \to X \times \cdots \times X$

Thus all are same and denoted the entropy $h(f)$. Q. When $h(f) = 0$, Γ_n has no exponential growth, we then modify to consider the polynomial log-growth by (Cantat-Paris-Romaskevich) 2021 $plov(f) = \limsup\limits_{n\to\infty} \frac{\log \mathrm{vol}(\Gamma_n)}{\log(n)}$

↳ slow dynamic (when ample fixed, it's trivial, thus also Δ_n interesting)

$B = \bigoplus H^0(X, \Delta_m)$ in noncommunicative geometric, its Gelfand-Kirillov dimension $= plov(f)$

$h(f) = 0 \Leftrightarrow f^*|_{N^1(X)}$ is quasi-unipotent (all eigenvalues are roots of 1)

Estimate of $plov(f)$ $\begin{cases} d + k \le plov(f) \le k(d-1) + d, & f^*|_{N^1(X)} = I + N \text{ decomposition and } k = (\text{maximal size of Jordan blocks}) - 1 \\ d + 2k + 2 \le plov(f) \le k(d-1) + d - 2 & [LOZ] \\ plov(f) \le (\frac{k}{2} + 1)d & [\text{Cheng Jiang}] \end{cases}$

Combining with $k \le 2d - 2 \Rightarrow plov(f) \le d^2$ a conjective by [CPR]

By [LOZ], but only over $\mathbb{C}$; Cheng generalised it into any algebraically closed field.

Eg.6. Elliptic curve E

$X = E^d$, then $k = 2d - 2$

and $plov(f) = d^2$ by $f = \begin{pmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$

↳ proof via direct computation Δ_m and using $\dim B(\text{last})$, it turns out to have part "$\sum j e_j$", then relates with the partion of integers $A_{k,d,n}$, $D_{\mu} = 0$ to complete the pf ☒

Hodge theory and BBDG decomposition thm. Qizheng Yin $\boxed{\text{lecture}*}$ [Deligne, Hodge I–III]

Setting: Over $\mathbb{C}$, X nonsingular projective variety, X^{an} is associated analytic space

Hodge theory concerns that ① (Absolute) $H^*(X^{an},\mathbb{Q})$ (or $\mathbb{R},\mathbb{Z},\mathbb{Z}_p$)'s information? (Algebraic $\Rightarrow$ Topological)

Linear algebra: ② (Relative) $f: X \to Y$, from Y and fibres F, $H^*(X^{an},\mathbb{Q})$ information?

Def1. (Hodge structure) $\triangle$. Hodge theory answered these by giving additional structure on $H^*(X^{an},\mathbb{Q})$, as vector space/$\mathbb{Q}$

A pure $\mathbb{Q}$-Hodge structure of weight k on a $\mathbb{Q}$-vector space H (f.d.) is a decomposition $H_{\mathbb{C}} = H\otimes\mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$, Each are $\mathbb{C}$-vector space (Hodge decomposition), s.t. $\overline{H^{p,q}} = H^{q,p}$

Eg.2 (Tate Hodge structure) $\mathbb{Q}(n) = (H, H^{p,q}) = ((2\pi i)^n\mathbb{Q}, H_{\mathbb{C}} = H^{-n,-n})$, only weight $(-2n)$-Hodge exists on H as its 1-dimensional $(2\pi i)^n\mathbb{Q}\subset\mathbb{C}$

Eg.3 (Hodge decomposition) $H^k(X,\mathbb{Q})$ admits a HS of weight k, it's geometric (not all HS comes from geometry)

When X also compact $\Rightarrow \mathbb{Q}(-n) \cong H^{2n}(X,\mathbb{Q})$ as Hodge structure isomorphism (send $H^{p,q}$ to $H^{p,q}$, converse also)

$\left(\frac{1}{2\pi i}\right)^n \int_X \omega \longleftarrow [\omega]$ (de Rham) • It's the Tate twist, $H^{2n}(X,\mathbb{Q})$ is geometric side, $\mathbb{Q}(-n)$ is arithmetic side,

We have $H\otimes H'$ the $\mathbb{Q}(-n)\cong H^{2n}(X,\mathbb{Q})$ turns out to be a duality between geometric and arithmetic

$\mathrm{Hom}_{\mathbb{Q}}(H,H')$ with $wt(H\otimes H') = wt(H)+wt(H')$ and $wt(\mathrm{Hom}_{\mathbb{Q}}(H,H')) = wt(H')-wt(H)$, where not all Hodge structure come from them

Def1'. (Hodge structure via filtration) H, $H_{\mathbb{C}}$ admited a decreasing complex filtration $F^\bullet H_{\mathbb{C}}$, s.t. $H_{\mathbb{C}} = F^pH_{\mathbb{C}} \oplus \overline{F^{k-p+1}H_{\mathbb{C}}}$

Pf. $(H, H^{p,q}) \longleftrightarrow (H, F^pH_{\mathbb{C}})$ Idea. $H^{k,0}\ \cdots\ H^{p,q}\ \cdots\ H^{0,k}$, $\underbrace{\quad}_{F^pH_{\mathbb{C}}}$ $\underbrace{\quad}_{\overline{F^{k-p+1}H_{\mathbb{C}}}}$

$H^{p,q} \longmapsto F^pH_{\mathbb{C}} = \bigoplus_{k\geq p} H^{k,q}$

$H^{p,q} = F^pH_{\mathbb{C}} \cap \overline{F^qH_{\mathbb{C}}} \longleftarrow F^pH_{\mathbb{C}}$ ▨ and the conjugate inverse the red arrow

• p-adic Tate twist: replace $\mathbb{Q}$ by $\mathbb{Z}_p$, $A(1) = A\otimes\mathbb{Z}_p(1)$, replace $\mathbb{C}$ by K with char $K = p$, and $A(n) = A\otimes\mathbb{Z}_p(1)^{\otimes n}$, $\mathbb{Z}_p(1) = \varprojlim \mu_{p^n}$, the cyclotomic character

Def1''. (Hodge structure via representation theory)

A Deligne torus $\mathbb{S}$ is an algebraic group with value $\mathbb{S}(\mathbb{R}) = \left\{\begin{pmatrix} u & -v \\ v & u\end{pmatrix} \middle| u,v\in\mathbb{R}, u,v \text{ not all } 0\right\} \subset GL_2(\mathbb{R})$

$\mathbb{S}(\mathbb{R}) \cong \mathbb{C}^*$ ($u+vi$) $\Rightarrow \mathbb{S} \cong \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ (Res is restriction from $\mathbb{C}$ to $\mathbb{R}$), then a Hodge structure is H with a real representation $\varphi: \mathbb{S}(\mathbb{R}) \to GL(H_{\mathbb{R}})$, s.t. $\forall t$ (By weil) $\in \mathbb{R}^* \subset \mathbb{S}(\mathbb{R})$, $\varphi(t)$ has eigenvalue t^k ($\varphi(t) = t^k \mathrm{Id}$)

Pf. $(H, H^{p,q}) \longleftrightarrow (H,\varphi)$ ($\Leftarrow$) Define $H^{p,q}$ to be such a subspace $V\subset H_{\mathbb{C}}$, s.t. $\varphi(z,\bar z)|_V = z^p\bar z^q\cdot\mathrm{Id}$

Where $(z,\bar z)\in\mathbb{C}^*\times\mathbb{C}^* \cong \mathbb{S}(\mathbb{C})$ (due to here we extend H to $H_{\mathbb{C}}$, φ also complexified) $\Longleftrightarrow \varphi(z)|_V = z^p\bar z^q\cdot\mathrm{Id}$ for $z\in\mathbb{C}^*\cong\mathbb{S}(\mathbb{R})$

$(\Rightarrow)$ $\varphi(z) = z^p\bar z^q\cdot\mathrm{Id}$ in each level $H^{p,q}$ ▨ • Such representation-theoritic definition is easiest to generalise to $\boxed{\text{p-adic case}}$

Weil operator of $\mathbb{S}(\mathbb{R})$, $C := \varphi(i) \in GL(H_{\mathbb{R}})$, with $C|_{H^{p,q}} = i^{p-q}\cdot\mathrm{Id}$ is the Weil representation: a real operator due to

$\overline{C|_{H^{p,q}}} = \overline{i^{p-q}}\,\mathrm{Id} = i^{q-p}\,\mathrm{Id} = C|_{H^{q,p}}$ ▨ with a Galois representation (rep of absolute Gal grp) adding to H

Def4. (Polarization of Hodge structure) Adding a bilinear form $Q: H\times H \to \mathbb{Q}$, s.t. ① $Q(\alpha,\beta) = (-1)^kQ(\beta,\alpha)$, ② $Q(H^{p,q}, H^{p',q'}) = 0$ except $p=q'$ & $q=p'$, ③ Q restrict to Hermition form on $H^{p,q}$, denoted $\tilde{Q}: H_{p,q}\times H_{p,q} \to \mathbb{Q}$ is positive definite

$(\alpha,\beta) \longmapsto Q(C\alpha,\bar\beta)$

Eg.5 ① The category of all vector spaces with HS is Abelian category (with fixed weight k).

② Each morphism of $H \xrightarrow{f} H'$ must preserve the filtration (strict) i.e. $f(F^pH) = f(H_{\mathbb{C}})\cap F^pH'_{\mathbb{C}}$ • Useful to view these category as a geometric module due to Deligne's observation on period domain

③ The category of all polarized HS vector space is a semi-simple abelian category; later we'll see how polarization comes from geometry.

Thm6. (Classical Hodge decomposition) X nonsingular proj var/$\mathbb{C}$ (or compact Kähler) $\Rightarrow H^k(X,\mathbb{Q})$ admits a HS naturally

(i.e. $f^*: H^k(Y,\mathbb{Q}) \to H^k(X,\mathbb{Q})$ is map of HS), with $H^{p,q}(X) = H^q(X,\Omega_X^p)$ the Doubeault cohomology

Hodge diamond: $H^{n,n}$; $H^{n,0}$, $H^{0,n}$ Serre duality (of Ω_X sheaf); $H^{0,0}$; conjugate $\overline{H^{p,q}} = H^{q,p}$

• Mirror symmetry is pair of Calabi–Yau, s.t. $\begin{cases} H^{1,1}(X) = H^{1,n-1}(X') \\ H^{1,1}(X') = H^{1,n-1}(X)\end{cases}$ (not algebraic) $\Rightarrow$ the proof also not algebraic, must acquire a Kähler metric

$\Rightarrow$ X $\xrightarrow{45^\circ}$ X', $\xleftarrow{45^\circ}$ rotation of their Hodge diamond

• p-adic case not have good Kähler metric, how can we deal with it? ① No naive Hodge decomposition, but generalised by Tate-Hodge, proven by Faltings; ② Using rigid analytic variety replace "harmonic objects"

Q. How can we avoid the conjugate condition $\overline{H^{p,q}} = H^{q,p}$ in algebraic geometry?

① GAGA: we can allow holomorphic objects;

② Taking Hypercohomology of a holomorphic resolution

Pf. ① $H^k(X,\mathbb{C}) = H^k(X,\underline{\mathbb{C}}_X)$, with the fine resolution of $\underline{\mathbb{C}}_X$

$0 \to \underline{\mathbb{C}}_X \to \mathcal{A}_X^0 \xrightarrow{d} \mathcal{A}_X^1 \xrightarrow{d} \cdots$ (It's same as de Rham thm)

$H^k(X,\mathbb{C})$ differential forms' sheaves

$= H^k(X,\underline{\mathbb{C}}_X) = H^k_{dR}(X,\mathbb{C})$

② Repeat for Ω_X^p: $0 \to \Omega_X^p \to \mathcal{A}_X^{p,0} \xrightarrow{\bar\partial} \mathcal{A}_X^{p,1} \xrightarrow{\bar\partial} \cdots$

$\Rightarrow H^q(X,\Omega_X^p) = H^q_{Dol}(X,\Omega_X^p)$ the Dolbeault cohomology

③ (Hodge) Choose a Kähler metric $\Rightarrow$ Laplace Δ on $\mathcal{A}(X, \mathcal{A}_X^k)$, then use harmonic form to replace more:

$H^k_{dR}(X,\mathbb{C}) = \mathcal{H}^k$; for only Riemannian, $\Delta = \Delta_d$

$H^q_{Dol}(X,\Omega_X^p) = \mathcal{H}^{p,q}$; for only Hermittian, $\Delta = \Delta_{\bar\partial}$

④ Now use Kähler to relate Riemannian & Hermittian: $\boxed{\Delta_d = 2\Delta_{\bar\partial}}$

⑤ Harmonic decomposition $\mathcal{H}^k = \bigoplus \mathcal{H}^{p,q}$ ($\overline{\mathcal{H}^{p,q}} = \mathcal{H}^{q,p}$ trivial)

⑥ Not depend on choice of Kähler metric ▨

• From geometric case $X = \mathbb{G}_m = \mathrm{Spec}(\mathbb{C}[T,T^{-1}])$ is $H^1(X,\mathbb{C}) = \mathbb{C}\frac{dT}{T}$, thus Tate twist $\mathbb{C}(-1) = \frac{1}{2\pi i}\mathbb{C}$

$\Rightarrow \Omega_X^\bullet = [\mathcal{O} \to \Omega_X \to \Omega_X^2 \to \cdots \to \Omega_X^n \to 0]$ and $H^k(X,\mathbb{C}) = \mathbb{H}^k(X,\Omega_X^\bullet)$, and the spectral sequence'll give filtration

Naive/Stupid filtration $F^p\Omega_X^\bullet = \Omega_X^{\geq p}$ a cutting, the filtration gives the Hodge-to-de Rham spectral sequence (Frölicher) Page2
$E_1^{pq} = H^q(X, \Omega_X^p) \Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet) \cong H^{p+q}(X;\mathbb{C})$ with Hodge filtration $F^p H^{p+q}(X;\mathbb{C})$

Coro7. Hodge-to-de Rham is E_1-degenerate

Pf. E_1-degenerate $\Leftrightarrow$ $\mathrm{Gr}_F^p H^{p+q}(X;\mathbb{C}) = E_1^{p,q} = H^{p,q} \Leftarrow$ Hodge decomposition ☒

Rk. Hence we have (only) a part of pure algebraic information in Hodge decomposition.
Q. Is there an algebraic pf to Coro7? (Deligne–Illusie) proven it by (mod p)

Hard Lefschetz thm

The Kähler class $c_1(L) \in H^2(X;\mathbb{Q}) \cap H^{1,1}(X)$, L is ample line bundle
(It's a limited generalisation of complex geometry, the classical Kähler class)
contains more information than c_1: $c_1(L) \in H^2(X;\mathbb{Q}(1))$
We have natural operator $\mathcal{L}: H^k(X;\mathbb{Q}) \to H^{k+2}(X;\mathbb{Q})$
$[Y] \mapsto c_1(L) \cup [Y]$

Thm8. (Hard Lefschetz)
$\mathcal{L}^k: H^{n-k}(X;\mathbb{Q}) \to H^{n+k}(X;\mathbb{Q})$ is a isomorphism $\Leftrightarrow \exists \Lambda: H^n(X;\mathbb{Q}) \to H^{n-k}(X;\mathbb{Q})$. $\Lambda^k, \mathcal{L}^k$ are inverse
and $H = [\mathcal{L}, \Lambda]$, s.t. $H|_{H^k} = (k-1)\,\mathrm{id}$ (this is more useful form, as a $\mathfrak{sl}(2)$-rep $(\mathcal{L}, H, \Lambda) \leftrightarrow (\begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix})$)

Coro9. (Lefschetz decomposition)
$H^k(X;\mathbb{Q}) = \bigoplus \mathcal{L}^j H^{k-2j}_{prim}(X;\mathbb{Q})$, the primitive cohomology $H^k_{prim}(X;\mathbb{Q}) := \mathrm{Ker}(\mathcal{L}^{n-k+1})$

Pf. Pure $SL(2)$-rep theory ☒

Pf of thm8. (Original by Lefschetz it's Hard, using Lefschetz pencil, but Hodge is much easier even general to Kähler, but in Kähler must be $\mathbb{R}$-coeffienent) Omitted ☒

Hodge index and Hodge index thm

• Another type (algebraic) Hodge index on surface, concerning the self-intersection number in [Ha]

Def10. Consider $Q: H^k(X;\mathbb{Q}) \times H^k(X;\mathbb{Q}) \to \mathbb{Q}$ a quadratic form
• The "index" in topological always constructed by (co)homological classes via ① pairing to top class; ② Integral (as differential form)

Thm11. (Hodge index thm $= $ Hodge–Riemannian condition) $(\alpha, \beta) \longmapsto \int \mathcal{L}^{n-k}(\alpha \cup \beta) \cdot (-1)^{\frac{k(k-1)}{2}}$

The Lefschetz decomposition is orthogonal w.r.t. Q, and Q is the polarization on $H^k_{prim}(X;\mathbb{Q})$

Idea of Proof. Decompose into small pieces with positive/negative definite $Q|_W$ ☒

Rk. (Abelian variety) Slogan: Abelian variety $\Leftrightarrow$ HS with weight 1.
In fact we have ① $\left\{\begin{array}{l}\text{effective torsion-free}\\ \mathbb{Z}\text{-HS of weight 1}\end{array}\right\} \xleftrightarrow{1:1} \{\text{complex torus}\}$
and they're both categorical equivalence (effective is $H^{p,q} \neq 0$ $(p,q) \geq 0$ in Hodge decomposition)
② $\left\{\begin{array}{l}\text{effective torsion-free}\\ \text{polarized } \mathbb{Z}\text{-HS of wt 1}\end{array}\right\} \xleftrightarrow{1:1} \{\text{Abelian variety}\}$ via taking $H^1(-;\mathbb{Z})$

• It's due to Riemann, and not that then the HS are $\{(0,1),(1,0)\}$ type, the parameter space of both can be realised as $Sp_{2n}(\mathbb{Z}) - h_n$, $h_n =$ Seigel upper half plane $\cong Sp_{2n}(\mathbb{Z}) / K$ compact maximal

Hodge conjecture

Conj12. (Hodge) $Z \subset X$ closed subvariety of X non-singular proj, $\dim X = n$, with $[Z] \in H^{2k}(X;\mathbb{Q})$, $k = \mathrm{codim}(Z, X)$. (Rk. For arithmetic case one consider the Tate twist $H^{2k}(X, \mathbb{Q}(k))$) Taking resolution $\tilde{Z} \to Z \hookrightarrow X \Rightarrow H^0(\tilde{Z};\mathbb{Q}) \cong H_{2n-2k}(\tilde{Z};\mathbb{Q}) \xrightarrow{f_*} H_{2n-2k}(X;\mathbb{Q}) \cong H^{2k}(X;\mathbb{Q})$ via poincaré duality, it composites to $[\tilde{Z}] \mapsto [Z] \in H^{2k}(X;\mathbb{Q}) \cap H^{k,k}(X)$ by f_* preserve the Hodge structure
(Observe it's canonical, not depending on choice of resolution: taking a common resolution)
$\Rightarrow$ then we can $[Z] \in H^{2k}(X;\mathbb{Q}) \cap H^{k,k}(X) =: \mathrm{Hdg}^{2k}(X)$ the Hodge class.
Consider $\{\mathbb{Q}\text{-cycles}\} \longrightarrow \mathrm{Hdg}^*(X)$, it's the cycle class map $Z^*_{\mathbb{Q}}(X) \to \mathrm{Hdg}^*_{\mathbb{Q}}(X)$ $\boxed{\text{Hodge conjectured that it's surjective}}$
$Z \longmapsto [Z]$ the Hodge class
thus we can recover algebraic information via topological information, or algebraic information is almost all!

Rk. ① $\mathbb{Z}$-coeffienent is wrong {topological obstruction: Atiyah–Hirzebruch
② Kähler also wrong, it's wrong {algebraic obstruction: Kollár
originally by Hodge, disprover by Voisin
③ It's proven in $k=1$ by Lefschetz (1,1)-thm. thus $k = (n-1)$ also holds.
④ $n \geq 4$, even Abelian is unknown now.
• Lefschetz (1,1)-form $c_1: \mathrm{Pic}(X) \to H^2(X;\mathbb{Z}) \cap H^{1,1}(X) \Rightarrow n \leq 3$ solved.

Pf of Thm8 (A third pf) Inductive approach on $\dim X$: take $Z \subset X$ closed, we'll use the weak Lefschetz
(WL) $H^k(X;\mathbb{Q}) \cong H^k(Z;\mathbb{Q})$; $k < n-1$ for $Z^{n-1} \subset X^n$ is hyperplane section
$H^k(X;\mathbb{Q}) \hookrightarrow H^k(Z;\mathbb{Q})$; $k = n-1$
It's easy to prove by topological LES: $U := X - Z$ is affine $\Rightarrow \to H^k_c(U;\mathbb{Q}) \to H^k(X;\mathbb{Q}) \to H^k(Z;\mathbb{Q}) \to H^{k+1}_c(U;\mathbb{Q}) \to \cdots$ and Artin vanishing $H^k(U;\mathbb{Q}) = 0, \forall k > n \xRightarrow{\text{Poincaré}} H^k_c(U;\mathbb{Q}) = 0, \forall k < n$ ☒
then take $Z \in |L| \Longrightarrow H^{n-k}(X;\mathbb{Q}) \xrightarrow{i^*} H^{n-k}(Z;\mathbb{Q}) \xrightarrow{i_*} H^{n-k+2}(X;\mathbb{Q}) \Rightarrow$ HL holds for $k > 1$
but for $k = 1$ case
$H^{n-1}(X;\mathbb{Q}) \xrightarrow{i^*} H^{n-1}(Z;\mathbb{Q}) \xrightarrow{i_*} H^{n+1}(X;\mathbb{Q})$, it's isomorphism $\Leftrightarrow$ the intersection form on $H^{n-1}(Z;\mathbb{Q})$, restrict to $\mathrm{Im}(i^*)$ is nondegenerate.
$\mathcal{L}$ (by projection formula) $\mathcal{L}^1: H^{n-1}(X;\mathbb{Q}) \to H^{n+1}(X;\mathbb{Q})$ iso
We use the HR to give the intersection form is positive definite for Z
$\forall \alpha \in \mathrm{Ker}\,\mathcal{L} \subset H^{n-1}(X;\mathbb{Q}) \Rightarrow i^*\alpha \in \mathrm{Ker}\,\mathcal{L}|_Z \Rightarrow i^*\alpha \in H^{n-1}_{prim}(Z;\mathbb{Q})$, write $\alpha = \sum \alpha^{p,q}$, $\exists \alpha^{p,q} \neq 0$
$\Rightarrow 0 = \int \mathcal{L}(\alpha^{p,q} \cup \overline{\alpha^{p,q}}) = \int (i^*\alpha^{p,q}) \cup (\overline{i^*\alpha^{p,q}})$, contradiction ☒
• Intersection form is $Q_X: H^n(X;\mathbb{Q}) \times H^n(X;\mathbb{Q}) \to \mathbb{Q}$, $([\alpha], [\beta]) \mapsto \langle \alpha, \beta \rangle$. Here is Z and $\dim Z = n-1$; $\mathbb{Q}$ replaced by $\mathbb{Z}/\mathrm{Tor}$ sometimes, and $\partial M \neq \emptyset$ also OK by relative

Degeneration & Deformation of $\mathbb{Z}$HS

Variation of HS (Deformation) Def13. (VHS) S nonsingular, A VHS weight k on S is

① A local system H (local constant sheaf), $\mathbb{Q}$-local system H means that locally constant value $\mathbb{Q}^{\oplus N}$, f.d. $\Rightarrow N<\infty$ Page?
$\Rightarrow \mathcal{H}=H\otimes_{\mathbb{Q}}\mathcal{O}_S$ is holomorphic vector bundle;
② Flat connection $\nabla:\mathcal{H}\to\mathcal{H}\otimes\Omega^1_S$; ③ We have a decreasing sequence of holomorphic subbundles $F^\bullet\mathcal{H}$ called Hodge bundle;
writing in local $\sum f_i\sigma_i\mapsto\sum df_i\otimes\sigma_i$; ④ Restrict to each $s\in S$, $(H_s,(F^\bullet\mathcal{H})_{s})$ is a Hodge structure;
frame (σ_i) with constant $\mathbb{Q}$-value (f_i) ⑤ (Griffith's transversality) $\nabla(F^p\mathcal{H})\subset\Omega^1_S\otimes F^{p-1}\mathcal{H}$.
① & ② shows the variation is locally constant; ③ & ④ shows that it's a relative HS with base S; ⑤ is the geometrization
For ⑤, consider the geometric setting $f:X\to S$ projective, relative dimension $=:d$
then $H=R^kf_*\mathbb{Q}_X$ is $\mathbb{Q}$-local system topologically (i.e. f is topologically locally trivial) by thm of Ehresmann
$\Rightarrow H_s=H^k(X_s;\mathbb{Q})\Rightarrow\mathcal{H}=R^kf_*\mathbb{Q}_X\otimes\mathcal{O}_S=R^kf_*(f^{-1}\mathcal{O}_S)$, with $\mathcal{H}\xrightarrow{\nabla}\mathcal{H}\otimes\Omega^1_S$ ~~Gauss~~-Manin connection, and Hodge bundles given
by de Rham resolution $0\to f^{-1}\mathcal{O}_S\to\mathcal{O}_X\to\Omega^1_{X/S}\to\Omega^2_{X/S}\to\cdots\to\Omega^d_{X/S}\to0\Rightarrow R^kf_*(f^{-1}\mathcal{O}_S)=\mathbb{H}R^kf_*(\Omega^\bullet_{X/S})$, $\mathbb{H}R$ the Hyper
derived functor of f_* $\Rightarrow\mathcal{H}=\mathbb{H}R^kf_*(\Omega^\bullet_{X/S})\Rightarrow$ it admits Hodge subbundle as we have the silly filtration $F^p\Omega^\bullet_{X/S}=\Omega^{\geq p}_{X/S}$
$\Rightarrow F^p\mathcal{H}:=\mathrm{Im}(\mathbb{H}R^kf_*(F^p\Omega^\bullet_{X/S})\to\mathbb{H}R^kf_*(\Omega^\bullet_{X/S}))$, but now it's only coherent sheaf

Claim. $F^p\mathcal{H}$ is locally free. Pf. $F^p\mathcal{H}$ locally free $\underset{E_1\text{-converge}}{\Leftarrow} R^kf_*\Omega^k_{X/S}$ locally free $\underset{\text{Grauert thm}}{\Leftrightarrow} s\mapsto H^k(X_s,\Omega^k_{X_s})$ constant $\Leftarrow$ upper semi continuity, E_1-degeneration

Griffith transversity:
(Katz–Oda) ~~Gauss~~-Manin connection (defined topologically) $\Leftrightarrow$ (Algebraically) $\nabla:\mathcal{H}=\mathbb{H}R^kf_*(\Omega^\bullet_{X/S})\to\mathbb{H}R^{k+1}f_*(f^*\Omega^1_S\otimes\Omega^{\bullet-1}_{X/S})$ ☒
is induced by applying $\mathbb{H}R^k$ to $0\to f^*\Omega^1_S\otimes\Omega^{\bullet-1}_{X/S}\to\Omega^\bullet_X/f^*\Omega^2_S\wedge\Omega^{\bullet-2}_X\to\Omega^\bullet_{X/S}\to0$ projection formula $\mathcal{H}\otimes\Omega^1_S$
$\Rightarrow$ the Hodge bundle by $F^p\mathcal{H}=\mathbb{H}R^kf_*(F^p\Omega^\bullet_{X/S})\xrightarrow{\nabla}\mathbb{H}R^{k+1}f_*(f^*\Omega^1_S\otimes F^p\Omega^{\bullet-1}_{X/S})=F^{p-1}\mathcal{H}\otimes\Omega^1_S$ ☒

Degeneration of HS (Geometrically, it's smooth $\rightsquigarrow$ singular $\sim$) is the mixed: mix weight k varies.
Def 14. (MHS) On $\mathbb{Q}$-vector space H, we have two filtrations $\begin{cases}\text{weight filtration } W_\bullet H\text{, increasing, s.t. }\forall k, (\mathrm{Gr}^W_kH, F^\bullet\mathrm{Gr}^W_kH_{\mathbb{C}})\\ \text{Hodge filtration } F^\bullet H_{\mathbb{C}}\text{ decreasing} \quad\text{is pure HS of weight } k\end{cases}$
• One can understand MHS as $\oplus$(HS), but not direct sum but extension
• A mixed Hodge module is mixed (weight) + variation of HS (relative HS) + singular generalisation
Rk. ① Category of MHS is Abelian; ② Morphism of MHS (preserve two filtrations) $\Rightarrow$ strict respect to both filtration; (weight) (like $\mathbb{Z}(1)$) (perverse)
Thm 15. (Deligne, Hodge II & III) X arbitrary (Before pure HS $\Leftrightarrow$ smooth proj) variety $\Rightarrow H^l(X;\mathbb{Q})$ admits natural MHS.
and ① when X nonsingular but not proper, weight $\subset[l,2l]$
② when X proper but not nonsingular: weight $\subset[0,l]$
③ when X general, weight $\subset[0,2l]$
④ when X nonsingular and proper: pure weight l.
Eg 16. ① $X=\mathbb{C}^*\Rightarrow H^1(X;\mathbb{Q})=\mathbb{Q}(-1)$, weight 2 pure (winding number on $0\in\mathbb{C}^*$)
② $X=$ ⊖ $\simeq$ ∞ $\Rightarrow H^1(X;\mathbb{Q})=\mathbb{Q}(0)$, weight 0 pure
• All of these facts hold ∵ 1-1 correspondence between l-adic cohomology ... mixed HS
Eg 17. ① X nonsingular but not proper, $X\subset\bar{X}$ compactification $\Rightarrow W_lH^l(X;\mathbb{Q})=\mathrm{Im}(H^l(\bar{X};\mathbb{Q})\to H^l(X;\mathbb{Q}))$
② X proper but not nonsingular, $\pi:\tilde{X}\to X$ resolution $\Rightarrow W_{l-1}H^l(X;\mathbb{Q})=\mathrm{Ker}(H^l(X;\mathbb{Q})\to H^l(\tilde{X};\mathbb{Q}))$
① pure weight module ... n; ② Homomorphism compatible with Galois action $\Leftrightarrow$ morphism of MHS; ③ intersection cohomology generalise ... both singular and ...
Thm 18. (Deligne decomposition) $f:X\to S$ smooth & proj map, relative dimension d $\Rightarrow R^kf_*\mathbb{Q}_X$ is VHS, $\forall k$
then $H^l(X;\mathbb{Q})\cong\bigoplus_{k\leq 2d}H^{l-k}(S,R^kf_*\mathbb{Q}_X)\twoheadrightarrow H^0(S,R^lf_*\mathbb{Q}_X)$ the global section of local constant sheaf $R^lf_*\mathbb{Q}_X=H^l(F;\mathbb{Q})^{\pi_1(S)}$
Idea of Pf. $\mathbb{Z}(1)\leftrightarrow\mathbb{Z}(1)$ ($\mathbb{Z}(1)$ Tate module)
A global section of local constant sheaf = global section at each fibre, which coincide after "loop" (Monodromy $\pi_1(S)$-action)
Note that the restriction to fibre is false in general, due to local obstruction, it's not surjective;
thus $=H^l(F;\mathbb{Q})^{\pi_1(S)}$ where F is fibres of f, all equal ☒ is MHM ...
Hopf fibration $S^3\to S^2$ with fibre $S^1\cong\mathbb{P}^1$, thus where dim even not same $\Rightarrow H^1(S^3;\mathbb{Q})=0\not\twoheadrightarrow H^1(\mathbb{P}^1;\mathbb{Q})^{\pi_1(S^2)}$
l-adic cohomology ... It shows that we must use algebraicity in next proof:
we use Leray spectral sequence.
$E_2^{p,q}=H^p(S,R^qf_*\mathbb{Q}_X)\Rightarrow H^{p+q}(S,Rf_*\mathbb{Q}_X)=H^{p+q}(X;\mathbb{Q})$
E_2-degeneration $\Rightarrow E_2^{p,q}=\mathrm{Gr}^p_LH^{p+q}(X;\mathbb{Q})$ the Leray filtration $\Rightarrow H^l(X;\mathbb{Q})\cong\bigoplus_{p+q=l}H^p(S;R^qf_*\mathbb{Q}_X)$ complete the proof.
E_2-degeneration $\Leftrightarrow d_r:E_r^{p,q}\to E_r^{p+r,q-r+1}$ all vanishing for $r\geq2$, We prove it by induction on $r\geq2$
By Hard Lefschetz decomposition, $R^kf_*\mathbb{Q}_X=\bigoplus\eta^jR^{k-2j}_{\mathrm{prim}}f_*\mathbb{Q}_X$, where we take $\eta\in H^0(S,R^2f_*\mathbb{Q}_X)$ and identify it with
operator $\eta:R^kf_*\mathbb{Q}_X\to R^{k+2}f_*\mathbb{Q}_X$. Taking $q=d-k\Rightarrow$

$$\begin{array}{ccc}H^p(S,R^{d-k}_{\mathrm{prim}}f_*\mathbb{Q}_X) & \xrightarrow{d_r} & H^{p+r}(S,R^{d-k-r+1}f_*\mathbb{Q}_X)\\ {\scriptstyle 0\neq\eta^{k+1}}\downarrow & ? & \downarrow{\scriptstyle\eta^{k+1}}\\ H^p(S,R^{d+k+2}f_*\mathbb{Q}_X) & \xrightarrow{d_r} & H^{p+r}(S,R^{d+k-r+3}f_*\mathbb{Q}_X)\end{array}$$

Injective due to $k+1\leq k+r-1$ (By $r\geq2$) $\Rightarrow d_r=0$ upper one ☒

Thm 19. (Global invariant cycle thm) $f:X\to S$ smooth & proj, $X\subset\bar{X}$ compactification also smooth
$\Rightarrow H^l(\bar{X};\mathbb{Q})\twoheadrightarrow H^l(F,\mathbb{Q})^{\pi_1(S)}$
Pf. $H^l(\bar{X};\mathbb{Q})\to H^l(X;\mathbb{Q})\twoheadrightarrow H^l(F,\mathbb{Q})^{\pi_1(S)}$, due to fibre all smooth $\Rightarrow$ its HS is pure, by strict $\Rightarrow$ it comes from weight $\leq l$, but we know $H^l(X;\mathbb{Q})$ has weight $\subset[l,2l]\Rightarrow$ weight l, i.e. coming from $H^l(\bar{X};\mathbb{Q})$ also weight l purely $\Rightarrow$ surj ☒
Thm 20. (Deligne decomposition, strong type)
(i) (Decomposition) $Rf_*\mathbb{Q}_X\cong\bigoplus R^kf_*\mathbb{Q}_X[-k]$; (ii) (Relative Hard Lefschetz) $\eta^kR^{d-k}f_*\mathbb{Q}_X\cong R^{d+k}f_*\mathbb{Q}_X$;
(iii) (Semisimple) $R^kf_*\mathbb{Q}_X$ are all semisimple (Using positivity of quadratic form via Hodge index)
Eg 21. Consider singular case (still proj), $f:X\to Y$ with only X smooth, Y normal s.t. $H^l(Y;\mathbb{Q})$ not pure
by Zariski main thm $\Rightarrow f_*\mathbb{Q}_X=\mathbb{Q}_Y\Rightarrow H^l(Y;\mathbb{Q})=H^{l-0}(Y;R^0f_*\mathbb{Q}_X)$ mixed, not a summand of $H^l(X;\mathbb{Q})$ pure!
Thm 22 [BBDG]. Beilinson–Bernstein–Deligne–Gabber decomposition thm (Issue: sheaves/local system $\mathbb{Q}$ not enough for singular)
... on singular ... the BBDG next