Quantum mechanics recalled: • axioms: ~ & ~ & ··
• spectral decomposition
• Stone - Von Neumann
• Weyl's quantization

Path integral formalism: ···, then quantization in math.

Rk. One uses $v \in \mathcal{H}$ as pure states, which we called it states in general usually, and all states are all trace class operators; pure $\iff$ project to 1-dimensional $\langle v \rangle \subset \mathcal{H}$. No essential difference as $P_W = \sum P_{v_i}$ viewed as linear combination of eigenvalues.

It means the measurement of observable A on state P is $\langle A | P \rangle := \int_{\mathbb{R}} \lambda \, d\mu_A^P$ depend heavily on P's spectral decomposition.

• This is due to some physical reason about density matrix? $\boxed{\sum a_i P_{v_i} \neq P_{\sum a_i v_i}}$ (infinite sum) gives mix states

Recall of spectral decomposition:

Two versions $\begin{cases} \text{operator } A = \int_{\sigma(A)} \lambda \, dP_A(\lambda) \\ \text{space } \mathcal{H} \cong \int_{\sigma(A)}^{\oplus} \mathcal{H}_\lambda \, d\mu(\lambda), \text{ given } A \curvearrowright \mathcal{H} \end{cases}$

operator form is familiar: functional calculus $f(A) = \int_{\sigma(A)} f(\lambda) \, dP_A(\lambda)$

truncation by χ_E: $P_A(E) = \chi_E(A)$ as $\langle \psi | P_A(E) | \psi \rangle$ = measurement probability of A at ψ, restricted in E

Space form is a decomposition into irreducible representation, we define $\int^{\oplus}$, direct integral, occurring in conformal field theory and its Lie group representation theory (unitary rep of A)

Def. $\int_X^{\oplus} \mathcal{H}_\lambda \, d\mu(\lambda) := \left\{ s: \lambda \mapsto s(\lambda) \in \mathcal{H}_\lambda \,\middle|\, \begin{array}{l} \lambda \mapsto \langle e_i(\lambda) | s(\lambda) \rangle_{\mathcal{H}_\lambda} \text{ measurable} \\ \text{and } \int_X \langle s(\lambda) | s(\lambda) \rangle_{\mathcal{H}_\lambda} d\mu(\lambda) < \infty \end{array} \right\}$, μ is measure on X, valued in $\mathbb{R}$, $\{e_i(\lambda)\}^{\infty}$ is orthonormal basis of $\mathcal{H}_\lambda$ but allowing some of them to be 0, $\forall \lambda \in X$; $\lambda \mapsto \langle e_i(\lambda) | e_i(\lambda) \rangle_{\mathcal{H}_\lambda}$; $\dim \mathcal{H}_\lambda = \infty$ is allowed, but $\lambda \mapsto \dim \mathcal{H}_\lambda$ is measurable function acquired.

Thm (Space form of spectral decomposition)

$\mathcal{H} \cong \int_{\sigma(A)}^{\oplus} \mathcal{H}_\lambda \, d\mu(\lambda)$ is a unitary isomorphism $U: \mathcal{H} \to \boxdot$ is measurable

s.t. $UAU^{-1}: \boxdot \to \boxdot$ is $(UAU^{-1}(s))(\lambda) = \lambda \cdot s(\lambda)$ (eigenvalue / diagonalize)

• "Equivalence" of two forms: the idea is eigenvector spaces $\leftrightarrow$ operator truncated at the eigenvalue

Let $V_E = \mathrm{Ran}(P_A(E)) \subset \mathcal{H}$. It corresponds to $\tilde{V}_E \subset \boxdot$, $\tilde{V}_E = \{ s \mid \mathrm{supp}\, s \subset E \}$

Thus let projection $\boxdot \twoheadrightarrow \tilde{V}_E$ denoted as $\tilde{P}_A(E)$, then $U^{-1} \tilde{P}_A(E) U = P_A(E)$! ☑

Rk (Important!) Such an "equivalence" is not true, especially in A unbounded. space form is strictly stronger and useful in practice)

Axiomatically, we expect a quantization of $(M, \{-,-\}, H)$ classical system is $(Q_\hbar: C^\infty(M) \to \mathcal{A}, \mathcal{A} \curvearrowright \mathcal{H})$

s.t. • (Classical limit) $\lim_{\hbar \to 0} \frac{1}{2} Q_\hbar^{-1}(Q_\hbar(f) \circ Q_\hbar(g) + Q_\hbar(g) \circ Q_\hbar(f)) = fg = gf$

• (Commutator) $\lim_{\hbar \to 0} Q_\hbar^{-1}\left(\frac{i}{\hbar}[Q_\hbar(f), Q_\hbar(g)]\right) = \{f, g\}$

asymptotic expansion $\Rightarrow$ deformation quantization $Q_\hbar^{-1}(Q_\hbar(f) \circ Q_\hbar(g)) \sim fg - \frac{i\hbar}{2}\{f, g\} + O(\hbar^2)$

No details given as no general detailed framework works for all quantizations, we can only say what we expect/need, not say what is it precisely.

We call this condition semi-classical

E.g. $Q_\hbar(xp) = Q_\hbar(px) = ?$ • Pseudo-differential operator quantization (via symbol) $Q(xp) = x \frac{\partial}{\partial x}(-)$ or $Q(xp) = \frac{\partial}{\partial x}(x -)$

• Weyl quantization $Q_{Weyl}(xp) = \left(x \frac{\partial}{\partial x} + \frac{\partial}{\partial x} x\right) \frac{1}{2}$

• Wick quantization (in geometric quantization) $Q_{Wick}((x + \sqrt{-1}\partial p)(x - \sqrt{-1}\partial p)) = \left(x - \sqrt{-1}\partial \frac{\partial}{\partial x}\right)\left(x + \sqrt{-1}\partial \frac{\partial}{\partial x}\right)$

note that it depends on $\sqrt{-1} = i$ the cpx structure (almost)

or $Q_{anti\text{-}Wick}(\cdots) = \left(x + \sqrt{-1}\partial \frac{\partial}{\partial x}\right)\left(x - \sqrt{-1}\partial \frac{\partial}{\partial x}\right)$

Rk. (Weyl quantization) $Q_{Weyl}(x^i p^j) = \frac{1}{(i+j)!} \sum_{\sigma \in S_{i+j}} (\cdots)$ all commutation terms $\Rightarrow Q_{Weyl}((ax+bp)^k) = \left(ax + b\frac{\partial}{\partial x}\right)^k \Rightarrow Q_{Weyl}(e^{ax+bp}) = e^{ax + b\frac{\partial}{\partial x}}$

$\Rightarrow f(x,p) = \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^n \mathbb{R}^n} \hat{f}(a,b) e^{i(ax+bp)} \, da\, db \Rightarrow Q_{Weyl}(f) = \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^n \mathbb{R}^n} \hat{f}(a,b) e^{i(ax + b\frac{\partial}{\partial x})} \, da\, db \Rightarrow \boxed{\text{locality}}$

Fourier transform

Thus we can recover Weyl quantization from its value on $(x,p)^{\text{polynomial}} \in \mathbb{R}^n \times \mathbb{R}^n$

BCH asymptotic expansion above axioms.

Rk. Later, path-integral formalism can "unify" these quantization by modifying the order of paths $\leftrightarrow$ the order of commutation

Weyl quantization recalled: $Q_{Weyl}(f)(\varphi) = \int_{\mathbb{R}^n} K_f(x,y) \varphi(y) \, dy$

$K_f(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}\left(a, \frac{y-x}{2}\right) e^{\frac{ia(x+y)}{2}} \, da$ is the kernel function of $Q_{Weyl}(f)$

Now for some reason (index thm), $= \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} f\left(\frac{x+y}{2}, p\right) e^{-\frac{i(y-x)p}{\hbar}} \, dp$ (position $\xrightarrow{\text{Fourier Transform}}$ momentum)

One asks $\mathrm{Tr}(Q_{Weyl}(f)) = \int_{\mathbb{R}^n} K_f(x,x) \, dx = ?$? $= \int_{\mathbb{R}^n} K_f(x,x) \, dx = \frac{1}{\hbar^n} \iint_{\mathbb{R}^n \mathbb{R}^n} e^{iax} \hat{f}(a,0) \, da\, dx = \frac{1}{\hbar^n} \hat{f}(0,0)$

($Q_{Weyl}(f)$ is trace-class operator) as upper integral finite)

$Q_{Weyl}(f) \circ Q_{Weyl}(g) = Q_{Weyl}(f *_\hbar g)$, $*_\hbar$ the Moyal product $(f *_\hbar g)(x,p) = \iiiint_{(\mathbb{R}^n)^4} e^{-i\hbar(a_1 b_2 - b_1 a_2)/2} = \frac{1}{(2\pi)^n} \int \hat{f}(x,p) \, da \, dp$ $\hat{f}(a_1, b_1) \hat{g}(a_2, b_2) \, da_1 \, db_1 \, da_2 \, db_2$ $e^{i(a_1 + a_2)x + i(b_1 + b_2)p}$

$f * g \sim fg - \frac{i\hbar}{2}\{f, g\} + O(\hbar^2)$

Those f, g are polynomials ⇒ we do prove them, when $f,g\in C^\infty(\mathbb{R}^n_x\times\mathbb{R}^n_p)$, we can write it formally:
⇒ we also proven for Schwarz function as Fourier transform works

$f \underset{h}{*} g = e^{-\frac{i\hbar}{2}(\frac{\partial}{\partial x}\otimes\frac{\partial}{\partial p} - \frac{\partial}{\partial p}\otimes\frac{\partial}{\partial x})}(f\otimes g)$

We can use Weyl quatization to prove Stone-Von Neumann thm: (self-adjoint case)

Thm (Stone-Von Neumann) $A_1,\dots,A_n,B_1,\dots,B_n\in B(H)$ satisfying canonical commutation relation (CCR): $[A_i,A_j]=0=[B_i,B_j]$, $[A_i,B_j]=\delta_{ij}(-i\hbar)$

then $H=\oplus V_{ij}$, s.t. ② ∃ unitary $U_{ij}: V_{ij}\to L^2(\mathbb{R}^n)$ isometry

① V_{ij} is invariant under $e^{itA_i}, e^{itB_j}, \forall t$ s.t. $U_{ij}e^{itA_i}U_{ij}^{-1}=e^{itX_i}$, $U_{ij}e^{itB_j}U_{ij}^{-1}=e^{itP_j}$

⇔ $[e^{itA_i}, e^{itA_j}]=0=[e^{itB_i},e^{itB_j}]$

In particular, when A_i, B_i are irreducible rep ⇒ $H\cong L^2(\mathbb{R}^n)$ (X_i and P_j are always irreducible on L^2)

Pf. By Weyl Quantization, set $\tilde{Q}: S(\mathbb{R}^{2n})\to B(H)$ satisfying axioms of Weyl Quantization $f\mapsto \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}}\hat f(a,b)e^{i(aA+bB)}da\,db$

unitary case: $e^{it_1A_i}e^{it_2B_j} = e^{it_2B_j}e^{it_1A_i}e^{-it_1\delta_{ij}(it_2)}$

(Recall these axioms only depend on CCR, instead X and P theirselves! We denote Weyl Quantization by X and P theirselves Q)

Our proof is using $\tilde{Q}$ imitate Q from "the ground state" of an "abstract harmonic osillator":

Let $f_0(x,p)=2^n e^{-\frac{|x|^2}{\sigma}}e^{-\frac{\sigma|p|^2}{\hbar^2}}$ with $\varphi_0(x)=(\pi\sigma)^{-\frac{n}{4}}e^{-\frac{|x|^2}{2\sigma}}\in L^2(\mathbb{R}^n)$ ground state ⇒ $Q(f_0)=P_{\varphi_0}$ projection to ground state (same as computation for QM's harmonic oscillator) $=|\varphi_0\rangle\langle\varphi_0|$

Now $\tilde Q(f_0)e^{i(aA+bB)}\tilde Q(f_0) = e^{-\frac{\hbar^2|b|^2}{4\sigma}}e^{-\frac{\sigma|a|^2}{4}}\tilde Q(f_0)$ via axioms (complicated computation, but easy) ⋯ (*)

• $H=\oplus V_{ij}$ $\begin{cases}\langle e^{i(a_1A_i+b_1B_j)}\varphi_{ij} \mid e^{i(a_2A_{i'}+b_2B_{j'})}\varphi_{i'j'}\rangle=0 \text{ (acting by } \tilde Q(f_0) \text{ and move to one side, then use (*) to simplify)}\\ \text{No other components: } H=(\oplus V_{ij})\oplus V' \text{ impossible, due to irreducibility}\end{cases}$

• Each V_{ij}, apply $\tilde Q = Q$ up to unitary operator by (*) ☒

Path integral. The propagator is the integral kernel of unitary $e^{-itH/\hbar}$ (involution operator), and we rewrite the propagator into integral over path/space configuration

$H=H_0+V$ is hard to compute kernel for addition of operators as $e^{H}=e^{H_0}\cdot e^{V}$ is multiplication we know nothing

But we have Lie-Kato-Trotter's product formula: $e^{i(A+B)}=\lim_{n\to\infty}e^{\frac{i}{n}A}\circ e^{\frac{i}{n}B}$ (A, B adjoint) and we know kernel of composition is multiplication

Now for Quantizations, the path integral over $DpDq$, $DqDp$ ⋯ in configuration space of $\mathbb{R}^{2n}$, but when $\mathbb{R}^{2n}$ is replaced by manifolds, no global charts, the measure on the configuration space over M matters

Their propagator is also given by integral kernel: (we had computed Weyl's, others similar)

{ Weyl: symplectic; Wick/anti-Wick: cpx/Kähler; pq/qp: no good geometric other than $\mathbb{R}^{2n}$'s analysis }

$K_{Weyl}(q|q')=\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}f(p,\frac{q+q'}{2})e^{\frac{i}{\hbar}p(q-q')}dp$;

$K_{pq}(q|q')=\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}f(p,q')e^{\frac{i}{\hbar}p(q-q')}dp$; $K_{qp}(q|q')=\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}f(p,q)e^{\frac{i}{\hbar}p(q-q')}dp$

$K_{Wick}(q|q')=\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}f(p,\frac{q-\sqrt{-1}q'}{2})e^{\frac{i}{\hbar}p(q-q')}dp$, $K_{anti\text{-}Wick}(q|q')=\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}f(p,\frac{q+\sqrt{-1}q'}{2})e^{\frac{i}{\hbar}p(q-q')}dp$ ☒

E.g. We show how to compute path integral over infinite-dim configuration space, into a finite-dim well-defined integral (Gaussian-type), or further a concrete function:

The (rough) idea is, first write path integral as limit of piecewise integral by (physical) definition, then compute the $(n-1)$-integral, and then take $\lim_{n\to\infty}$ formally.

(Gaussian type always from the Lagrangians, in various cases ⇒ can be computed concretely)

Consider harmonic oscillator. $K(q',t'|q,t)=\lim_{n\to\infty}\left(\frac{m}{2\pi\hbar i\Delta t}\right)^{\frac{n}{2}}\underbrace{\int\cdots\int}_{n-1}\exp\Big(\frac{i}{\hbar}\underbrace{\sum_j\frac{m}{2}\big(\frac{q_{j+1}-q_j}{\Delta t}\big)^2-\frac{m\omega^2q_j^2}{2}}_{\text{A quadric form in Gaussian type }(A\vec q,\vec q)+\text{linear terms}}\Big)\Delta t\cdot dq_1\cdots dq_{n-1}$

$\Delta t=(t'-t)/n$

A is the $(n-1)$ matrix $\frac{1}{\sqrt{\det A}} \sim = \lim_{n\to\infty}\boxed{\text{constants}}\cdot\boxed{\text{Gaussian type}}\cdot\boxed{\text{linear terms}}$

$n\to\infty$: $\frac{1}{\sqrt{\det "A"}}\sim = e^{-(\frac{\pi i}{4}+\frac{\pi i\nu}{2})}\cdot\sqrt{\frac{m\omega}{2\pi\hbar|\sin\omega T|}}\cdot e^{\frac{im\omega}{2\hbar(\sin\omega T)}[(q^2+q'^2)(\cos\omega T)-2qq']}$

Now our question arises: what is the "A" after we take continuous limit $n\to\infty$? It's an operator but how to define its determinant?

First we find by comparing result above, the $\frac{1}{\sqrt{\det "A"}}\sim\sqrt{\frac{\omega}{|\sin\omega T|}}$

Rk. ① det"A" is related with index thm, by $\frac{1}{\sqrt{\det "A"}}=\lim_{n\to\infty}\frac{1}{\sqrt{\det A_{n-1}}}=\lim_{n\to\infty}\frac{1}{\sqrt{N\frac{\sin n\theta}{\sin\theta}}}$, $\theta\sim\omega\Delta t$ especially the harmonic oscillator case; as above limit

② We use "regularization method" to define determinant (via path integral): $=\lim_{n\to\infty}\sqrt{\frac{\omega\Delta t}{|\sin\omega T|}}=\boxed{\sqrt{\frac{\omega}{|\sin\omega T|}}}(\lim\Delta t)$ cancelled by other terms

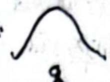
③ $tr = det'$, if we have trace class operators, we can find its exponential to exhibit a well-defined determinant, this is mathematical reason of why we can use path integral to find determinant (via zeta function). The two methods are equivalent: $\det A \overset{③}{=} e^{-\frac{dZ_A}{ds}|_{s=0}}$, $Z_A(s) = \sum \frac{1}{\lambda_k^s}$, λ_k is eigenvalues of A, Z_A extend to 0

④ What is the quadratic part $\langle Aq, q\rangle$, or what is the meaning of matrix A_{n-1}?

$\overset{④}{=} \left(\int_H e^{-\pi\langle q, Aq\rangle} Dq\right)^{-2}$, $A \curvearrowright H$ infinite-dimensional path integral is a continuous limit of Gaussian integral;

It's the Laplacian of a graph, the graph is the piecewise linear approximation to the path of a path; for example, a free particle and the final limit operator is Laplacian, is the integral over loops, and A is Laplacian of loop.

q_1, q_0, $q_{n-1} = q'$



q

E.g. ① $A_\omega = -\left(\frac{d}{dt}\right)^2 - \omega^2 \curvearrowright L^2[t, t']$, $A_{i\omega} = -\left(\frac{d}{dt}\right)^2 + \omega^2 \curvearrowright L^2[t, t'] \Rightarrow \det A_\omega = \frac{2\sin(\omega T)}{\omega T}$, $\det A_{i\omega} = \frac{2\sinh(\omega T)}{\omega T}$

② $A_\omega = -\left(\frac{d}{dt}\right)^2 + v(x)\frac{d}{dt} + u(x) \curvearrowright L^2[0, T]$, $\det(A_\omega - \lambda I) = e^{-\frac{1}{2}\int_0^T u(x)dx}\left(y_1(T,\lambda) + y_2'(T,\lambda) - 1 - e^{\int_0^T v(x)dx}\right)$ by ODE tricks (highly technical!)

$\Rightarrow \widetilde{\det}\left(-\left(\frac{d}{dt}\right)^2 + \omega\frac{d}{dt}\right) = \frac{2T}{\omega}\sinh\left(\frac{\omega T}{2}\right)$

Where y_1, y_2 are the two fundamental solutions to $A_\omega y = \lambda y$ $(y_1(0,\lambda) = 1, y_1'(0,\lambda) = 0, y_2(0,\lambda) = 0, y_2'(0,\lambda) = 1)$

Note that here our $\widetilde{\det}(D) := \lim_{\lambda \to 0} \det(D - \lambda I)$, the limit of regularized determinant to $\lambda = 0$

Index theory

Atiyah gives an explanation to Witten's proof to index thm in mathematical language, $\mathrm{Ind}\, \not{D} = \int_X \hat{A}(TX)$ (via $\widetilde{\det}$; $\underbrace{\mathrm{Ind}\,\not{D}}_{\boxed{\text{analytic index}}}$, $\underbrace{\int_X \hat{A}(TX)}_{\text{topological index}}$; usual index thm $\cap$ Fedosov-Nest-Tsygan algebraic index thm)

• Setting: X^n orientible, structure group of TX reduce to $SO(n)$ after we fix metric. Here we take the notation the ∇ is covariant derivative on TX

and $R \in \Gamma(X, \Omega^2_X \otimes so(n))$ is 2-form on $Fr_{SO(n)} \times_{Ad} so(n)$

$\subset \Gamma(X, \mathrm{End}(TX) \otimes \Omega^2_X)$, or as characteristic class ($=$ sum of some Pontryagin classes)

• $\hat{A}$-genus of bundle

$\hat{A}(TX) := \det^{\frac{1}{2}}\left(\frac{R/2}{\sinh(\frac{R}{2})}\right) = \exp\left(tr\left(\frac{1}{2}\log\frac{R/2}{\sinh(\frac{R}{2})}\right)\right) \in \Omega^\bullet(X)$, R the curvature tensor, due to trace, $\mathrm{End}(TX) \to \mathbb{R}$ is an ordinary differential form

(For more general $E = P \times_{Ad} \mathfrak{g}$, similiar $\hat{A}$ defined)

$\subset$ algebraic index, replace pseudo-differential operators by deformation quantization of algebra

The motivation of $\hat{A}$-genus is a hard story, see [Hirzebruch, Manifolds and modular forms], they (Atiyah-Singer) first conjecture $\hat{A}$ genera by polynomial satisfy several algebraic properties, e.g. multiplicative, then determined various notation of "genus"

• We need consider the loop space with natural "symplectic" structure (infinite-dim) $M = Map(S^1, X)$, $S^1 \curvearrowright M$ by rotate domain S^1, induce V_{S^1} vector field, $dH = \omega(V_{S^1}, -)$, H associated Hamiltonian

• In finite-dim $(M^{2n}, \omega) \curvearrowleft S^1$, we have DH localization formula: if $Fix_{S^1}(M) = \coprod X_j$, α_{ij} are chern root of $N_{X_j/M}$. $S^1 \curvearrowright N_{X_j/M}$ can be decomposed into $\frac{1}{2}\mathrm{codim}(X_j, M)$ complex plane $\mathbb{C}$'s rotations, ($N_{X_j/M}$ is even rank/$\mathbb{R}$ $\Rightarrow$ cpx vector bundle) the "winding number" are m_{ij}

$\Rightarrow \int_M e^{-iH}\frac{\omega^n}{n!} = \sum_j \int_{X_j} \frac{e^{-tH}e^{\omega}}{\boxed{\prod_i (tm_{ij} - \alpha_{ij}\sqrt{-1})}} dx$ ("Pf")

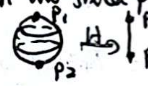
Q. How to generalize to $M = Map(S^1, X)$?

A. • The Liouville volume $\frac{\omega^n}{n!}$ replaced by some measure

• The $\boxed{\cdot\cdot}$ part $= \det^{-\frac{1}{2}}$ defined by regularized determinant above $= \frac{1}{Pff}$

In finite dim, $Pff(J)\, vol_g = \frac{\omega^n}{n!}$, in M, vol_g is placed by Wiener measure, thus if we have a Pff, all things done!

The proof of DH can by Morse theory: if fixed pt is isolated, then two fixed pt connected like $S^1 \curvearrowright S^2$:



p_1, p_2, H

then the left hand integral is reduce to p_1 and p_2 by Newton-Lebinz rule, generally, by Morse-Bott method ▨

Rk. $T_f M = \Gamma(S^1, f^*TX)$, it has natural (g, ω) by $g(v_1, v_2) = \int g(v_1(\theta), v_2(\theta))d\theta$, $\omega(v_1, v_2) = \int_{S^1} g(\boxed{\nabla_{f_*(\frac{\partial}{\partial\theta})}} v_1(\theta), v_2(\theta))d\theta$ (denoted ∇_f simply)

but ω is anti-symmetric, closed, degenerate iff $\exists$ geodesic loop f in X

but luckily ① here $\nabla_{f_*(\frac{\partial}{\partial\theta})}$ take the role of J, we'll compute its $Pff(\nabla_f)$ at $f \in M$; ② it not effects H in weaker sense $H = \pm\int_{S^1} |f_*(\frac{\partial}{\partial\theta})|^2 d\theta$

• Fact. $\widetilde{\det}(\nabla_f) = \det(1 - T_f)$, T_f is the holonomy matrix of X at loop f: $T_pX \to T_pX$

Pff: If X is odd dim, both 0, for the right hand, always $\exists$ direction invariant under T_f parallel transport (consider S^{2n+1}); if X is even dim, the holonomy Jordan block is $\begin{pmatrix} e^{2\pi\alpha_i} & & \\ & \ddots & \\ 0 & & \end{pmatrix} \cong T_f$, they are eigenvalues of ∇_f ($+2\pi i k$, $k \in \mathbb{Z}$)

• Recall in finite dim, orientible $\Leftrightarrow$ Pff is globally defined

$\Leftrightarrow \sqrt{\ }$ is globally defined $(Pff = \sqrt{\det}) \Leftrightarrow$ Reduce from $O(2n)$ to $SO(2n)$

$\Leftrightarrow Spin(2n)$

Here we just take it as the definition of "orientible" in infinite dim spaces $\Rightarrow$

$\begin{array}{ccc} Spin(2n) & \xrightarrow{\sqrt{\varphi}} & \mathbb{R} \\ \downarrow 2:1 & & \\ SO(2n) & \xrightarrow{\varphi} & \mathbb{R} \end{array}$

(here $\varphi = \det$ are character of $SO(2n)$-rep $\Rightarrow$ Pff are character of $Spin(2n)$-rep)

(Map(S^1, X) orientible called string, and righthand is Witten genus $\to$ enhanced version)

$(\widetilde{\det}(1 - T_f))(M) = \chi_M(S^+) - \chi_M(S^-)$: $Spin(2n) \to \mathbb{R}$

χ is character/trace of $Spin(2n) \curvearrowright S^+ \oplus S^-$ is spinor rep

Reminder: (Physical)
Regularize determinant value
$\leftrightarrow$ renormalization to form effective (f.d) theory;
Spin-rep (on Clifford algebra)
$\leftrightarrow$ Fermion;
Ind $\not{D}$ $\leftrightarrow$ partition function/...

Then we can prove index thm, ⓪ the right side is computation of characteristic class, written as curvature terms as our definition (not usual one yet), ... ① ...

The left side is which we need to work with, we need relate $\mathrm{Ind}(\not{D})$ with physical terms, and the renormalization gives $\sqrt{\det}$ term, this is by Spin geometry, same as classical pf by K-theory, precisely:

① DH localization on M infinite dim, " $\int_M e^{-tH}\left(\frac{\omega^\infty}{\infty!}\right)$ " $:= \int_M e^{-tH(f)} \frac{1}{\sqrt{\det(\cdots)}} d\mu(f)$, μ is Wiener measure on M (Riemannian) — physically, it absorbs momentum part to avoid infty or quotient by Gaussian integral

To apply

Now one need compute $\int_M e^{-tH}\frac{\omega^\infty}{\infty!}$ by DH: $= \int_M e^{-tH(f)}(\mathrm{Tr}\,S^+(T_f) - \mathrm{Tr}\,S^-(T_f))\,d\mu(f)$

DH localize to $X \subset M$ are those constant loops, i.e. X is just the target X

$\Rightarrow$ we localize to a f.d. effective theory

is the path-integral over M^∞, it's propagator

$= \mathrm{Tr}(e^{-2i\pi t\Delta^+}) - \mathrm{Tr}(e^{-2i\pi t\Delta^-})$ $\left(\Delta^+ = \not{D}^\dagger\not{D},\ \Delta^- = \not{D}\not{D}^\dagger,\ \Delta = \Delta^+ + \Delta^-,\ \not{D}: S^+ \to S^-,\ \not{D}^\dagger: S^- \to S^+\right)$

by each term as Heat kernel

$(N_X M)_x = T_x M / T_x X$, $\forall x \in X \subset M$

$= \Gamma(S^1, S^1 \times T_x X)$ / constant section

$= \Gamma^\perp(S^1, S^1 \times T_x X)$ $\xrightarrow{\text{Fourier expansion}}$ {without constant term series, value in $T_x X$}

spectral theory / McKean–Singer trick:

$= (\sum e^{-2i\pi t\lambda^+} + \dim\mathrm{Ker}\,\not{D}) - (\sum e^{-2i\pi t\lambda^-} + \dim\mathrm{Ker}\,\not{D}^\dagger)$, $\lambda^\pm$ are eigenvalue of $\Delta^\pm$

$= \dim\mathrm{Ker}\,\not{D} - \dim\mathrm{Ker}\,\not{D}^\dagger = \dim\mathrm{Ker}\,\not{D} - \dim\mathrm{coker}\,\not{D} = \mathrm{Ind}(\not{D})$ $\Rightarrow \lambda^+ = \lambda^-$

~~these steps is standard~~

thus it's "even-dimensional" although it's infinite, the Fourier coefficient of $>0 \leftrightarrow <0$, $\exists$ almost cpx structure, s.t. $N_X M$ is complex vector bundle $\Rightarrow$ Chern roots $C(N_X M) = \prod(1+\alpha_j)(1-\alpha_j) = \prod_i \prod_{k \neq 0}(k^2 + \alpha_i^2) = \prod_i \frac{\sinh(\alpha_i/2)}{(\alpha_i/2)}$ (nontrivial!) (k: winding number)

② LHS $= \int_X \frac{e^{-tH}e^{\omega}}{[\cdots]}$, RHS $= \int_X \sqrt{\det[\cdots]}$

$= \int_X \sqrt{\det[\cdots]}\, e^{-tH}e^{\omega}$ compare the two part in the integral carefully, they're same! ▨ (top part — only this contribute to integral)

Rk. ① This is a variant of heat kernel proof to Atiyah–Singer index thm, there are many proof to AS index thm, but in general only two kinds

Index thm:
- K-theory: pair of operator K-homology with topological K-cohomology, by index map
  - variants on K_0, e.g. operator algebra on tangent groupoid by Connes
  - variants on K^0
  - ...
- Heat kernel: using geometric analysis, compute $\mathrm{Tr}(e^{-2i\pi t\Delta^\pm})$
  - by supersymmetry
  - by stochastic method (Wiener measure = Brownian motion)
  - by propagator and regulized det, as above.

$\boxed{\text{Our proof's specialty is write } \mathrm{Tr}(e^{-2i\pi t\Delta^\pm}) \text{ as propagator, and compute it by localize to effective theory}}$, other steps are standard;

② The non-standard step? reduce $\int d\mu$ to Tr is nontrivial, one can view this as: integral $\int d\mu \rightsquigarrow$ supertrace

as here is e^{-tH} not e^{-itH}, the PDE is parabolic than hyperbolic, it tells us $e^{-t\Delta}$ send solution $u(0,-)$ to $u(t,-)$ this's same physical meaning as propagator from 0 to t!

③ Obviously, here is nonperturbative method of QFT. Historically, the measure " $\frac{\omega^\infty}{\infty!}$ " is hard to find, some mathematician need Wiener measure directly, but it's naive! The loop space's propagator has a trace term in perturbative QFT, so is the non-perturbative case (as result is same, the difference is computation approaches) (consider Feynman diagrams)

The choice of perturbative/non-perturbative is just depends on whether we can find measure or not!

We'll introduce ②, by supersymmetry:

Supertrace and Fermion system

A Boson system $H_B := \mathbb{C}[x_1 \cdots x_n] = \mathbb{C}\langle x_1 \cdots x_n\rangle / \langle x_i x_j = x_j x_i\rangle$ is infinite-dim $\subset L^2(\mathbb{C})$ (as linear space, not algebra) $\Rightarrow X_n = x_n \cdot$, $P_n = \frac{\partial}{\partial x_n}$

Adding SUSY, the Fermion system $H_F := \mathbb{C}\langle\bar\theta_1 \cdots \bar\theta_n\rangle / \langle\bar\theta_i\bar\theta_j = -\bar\theta_j\bar\theta_i,\ \bar\theta_i^2 = 0\rangle$ is finite-dim, as $\bar\theta_i^2 = 0 \Rightarrow a_n = \bar\theta_n \cdot$, $a_n^\dagger = \frac{\partial}{\partial\bar\theta_n}$

$\mathbb{C}\langle\theta_1 \cdots \theta_n, \bar\theta_1 \cdots \bar\theta_n\rangle / \langle\cdots\rangle$, involution/SUSY is the conjugation.

we set $\overline{f(\bar\theta)} := \overline{\sum c_j \bar\theta_{j_1} \cdots \bar\theta_{j_{n_j}}} = \sum \bar c_j \theta_{j_{n_j}} \cdots \theta_{j_1}$ interwined the order $\in \mathbb{C}\langle\theta_1 \cdots \theta_n\rangle / \langle\cdots\rangle$

then the inner product on H_F: $\langle f_1, f_2\rangle = \int_{Ber} f_1(\bar\theta)\overline{f_2(\bar\theta)}\, e^{-\bar\theta\theta}\, d\bar\theta\, d\theta$, $\bar\theta\theta = \sum \bar\theta_j \theta_j$

the Berizann integral is $\int_{Ber} f(\theta, \bar\theta)\, d\bar\theta\, d\theta := \frac{\partial}{\partial\theta_n} \cdots \frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\bar\theta_1} \cdots \frac{\partial}{\partial\bar\theta_n} f$ (Why "integral"?)

Now all things integral in H_F are good, the integral kernel/symbol are matrix than operators:

(there're some subtle convention of order, omitted), but $\int_{Ber} e^{-\bar\theta\theta}\, d\bar\theta\, d\theta = 1$ needed) $\Downarrow$ $\bar\theta_1 \cdots \bar\theta_n$ is orthogonal basis of H_F

Def. (Matrix symbol) $A: H_F \to H_F$, $\tilde a(\bar\theta, \theta) := \sum \langle A f_J, f_I\rangle f_I(\bar\theta)\overline{f_J(\bar\theta)} \in \mathbb{C}\langle\theta_1 \cdots \theta_n, \bar\theta_1 \cdots \bar\theta_n\rangle / \langle\cdots\rangle$

• (weak symbol) $\forall A$ can be written as $\sum K_{IJ}(a_I^\dagger a_J) \Rightarrow \tilde a(\bar\theta, \theta) := \sum_{I,J} K_{IJ}\bar\theta_I\theta_J$

They're different, to order in each terms, it's like the difference between Weyl Quantization v.s. pq Quantization

Rk. ① $\mathrm{End}_{\mathbb{C}}(H_F) = Cl_{2n}(\mathbb{C})$ generated by $a^\dagger$ and a, thus we can define the Wick symbol; ② Generally, we couple H_B and H_F and their operators;

Thm. (Quantization) • $(Af)(\bar\theta) = \int_{Ber} \tilde a(\bar\theta, \theta')f(\bar\theta')e^{-\bar\theta'\theta'}\, d\theta'\, d\bar\theta'$ • $\tilde a(\bar\theta, \theta) = e^{\bar\theta\theta} a(\bar\theta, \theta) \Rightarrow (Af)(\bar\theta) = \int_{Ber} e^{\bar\theta\theta'} a(\bar\theta, \theta')e^{-\bar\theta'\theta'}\, d\theta'\, d\bar\theta'$

It has properties: ① $A = A_1 \circ A_2 \Rightarrow \tilde a = \int_{Ber} \tilde a_1(\bar\theta, \theta')\tilde a_2(\bar\theta', \theta)e^{-\bar\theta'\theta'}\, d\theta'\, d\bar\theta'$

$\Rightarrow a = \int_{Ber} a_1(\bar\theta, \theta')a_2(\bar\theta', \theta)e^{-(\bar\theta - \bar\theta')(\theta - \theta')}\, d\theta'\, d\bar\theta'$ are convolution product (Gaussian integral)

② The trace and supertrace of $A: \mathcal{H}_F \to \mathcal{H}_F = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : \mathcal{H}_F^{even} \oplus \mathcal{H}_F^{odd} \to \mathcal{H}_F^{even} \oplus \mathcal{H}_F^{odd}$, $tr(A) = tr(A_{11}) + tr(A_{22})$

We have $tr A = \int_{Ber} \tilde{a}(\bar\theta, \theta) e^{-\bar\theta\theta} d\bar\theta d\theta$ $\quad tr_S(A) := tr(A_{11}) - tr(A_{22})$

$\boxed{tr_S A = \int_{Ber} a(\bar\theta, \theta) d\theta d\bar\theta} = \int_{Ber} \tilde{a}(\theta, \theta) e^{\bar\theta\theta} d\theta d\bar\theta$ conversely (ok)

Pf. The quantization is by definition; for ①, it's direct computation; for ②, $\int_{Ber} \tilde{a}(\theta, \theta) e^{-\bar\theta\theta} d\bar\theta d\theta = \sum_{I,J} \langle Af_I, f_J \rangle \int_{Ber} f_I(\theta) \overline{f_J(\theta)} e^{\theta\bar\theta} d\bar\theta d\theta$

$= \sum_{I,J} \langle Af_I, f_J \rangle \delta_{IJ}$

$= \sum_I \langle Af_I, f_I \rangle = tr(A)$;

similarly, $\int_{Ber} \tilde{a}(\bar\theta, \theta) e^{-\bar\theta\theta} d\theta d\bar\theta = \sum_I \langle Af_I, f_I \rangle (-1)^{|I|} = tr_S(A)$ ▨

Adding Fermion varibles, we also have path integral: give Hamiltonian $H: \mathcal{H}_F \to \mathcal{H}_F$

the propagator is the Wick symbol of U_t, $U_t = e^{-itH}$ involution operator,

denoted $u(\bar\theta, \theta; t)$, it can be written as path integral:

we have $U_t = \lim_{N\to\infty} \underbrace{U_{t/N} \circ U_{t/N} \circ \cdots \circ U_{t/N}}_{N \text{ times}}$, the we write its symbol as convolution product $u(\bar\theta, \theta; t) = \lim_{N\to\infty} \underbrace{\int_{Ber} \cdots \int_{Ber}}_{N \text{ times}} \exp(\boxed{\cdots})$

$\boxed{\cdots}$ contains two parts, one is symbol of $U_{t/N}$ has lived in exp as it's represented by symbol of H, another is the $e^{-\bar\theta\theta}$ part:

$\boxed{\cdots} = \{[\sum_j^N \bar\theta_j (\theta_{j-1} - \theta_j)] + \bar\theta(\theta_N - \theta)\} - (\sum -i H(\bar\theta_j, \theta_{j-1}) \Delta t)$, $H(\bar\theta, \theta)$ Wick symbol of H

take the limit $\Rightarrow u(\bar\theta, \theta; t) = \int e^{i\int_0^t (\bar\partial \dot\partial - H(\bar\partial, \partial)) ds + \bar\theta(\partial(t) - \theta)}$ "$d\mu$" $\Rightarrow tr_S(U_t)$

$d\mu(\partial) d\mu(\bar\partial)$

$\overset{(*)}{=} \int_{Ber} u(\bar\theta, \theta; t) d\theta d\bar\theta$ cancel Fermion varibles

$\left\{ \begin{array}{l} \partial \boxed{\text{path}}, \\ \partial(0) = \theta, \partial(t) = \bar\theta \end{array} \right.$ is desired propagator

What is it means? No real path here!

$= \int e^{i\int_0^t (\bar\partial\partial - H(\bar\partial, \partial)) dt} d\mu(\partial) d\mu(\bar\partial)$

intertwine integral order [P23]

(☆)

Return to index theory, $H = \Delta$

$Tr(e^{-2i\pi t \Delta^+}) - Tr(e^{-2i\pi t \Delta^-}) = tr_S(e^{-2i\pi t \Delta})$, we get a coupled theory on enlarged loop space, add Fermion varibles θ $= \int e^{-S(r, \theta)} \underset{\text{Boson, Wiener measure}}{d\mu(r)} \underset{\text{Fermion}}{d\mu(\theta)}$ $\overset{(☆)}{=}$ $\boxed{\text{cancel Fermion varibles}}$ $\int e^{-S(r)} \underbrace{d\mu(r) \sqrt{\det(\nabla_F)}}_{\text{Boson + Pffian, not Wiener measure}}$

i.e. the Pffiafian part is contributed by SUSY. done ▨

Liouville volume part

Rk. ① Why Berizmann integral is "integral"? For each mfd M we associate supermfd ΠTM as vector fields are anti-commute locally. $C^\infty(\Pi TM) = C^\infty(U) \otimes \mathbb{C}\langle \theta_1 .. \theta_n \rangle / \langle , \rangle$ is superalgebra; $\Omega^\bullet(M) = C^\infty(\Pi TM)$, we integral against $[M]$.

shift 1, $\mathbb{Z}/2\mathbb{Z}$-graded, shift commute to anti commute / parity change

$\Rightarrow \int_M \omega = \int \omega(x, \theta) dx d\theta$ is the Berizmann integral exactly, this is a noncommutative geometry idea, make $\int_{Ber}$ integral on algebra

$TM \longleftrightarrow$ Fermion ΠTM

$C^\infty(\Pi TM) \longrightarrow$ Fermion system (Generalize to more polyvector fields)

② (Equivariant cohomology) Follows the upper remark, let $S^1 \curvearrowright M$ generates vector field V, twist usual differential $D = d - \iota_V$

$\Rightarrow D^2 = -L_V$ by Cartan's magic formula, but restrict to $D: (\Omega^\bullet(M))^{S^1} \to (\Omega^\bullet(M)[1])^{S^1} \Rightarrow D^2 = 0$ well-defined $\Rightarrow$ we can define cohomology

$\omega \in \Omega^\bullet(M)$ is D-closed $\Leftrightarrow$ ω is D-exact on the S^1-fixed points of M, then we can do localization:

(Berline-Vergne) M cpt, 2n-dim, isolated fixed pts, $D\omega = 0$, top form (otherwise also holds, both sides $= 0$.)

with (ω, g) $\to$ compatible with S^1-action $\quad$ $\to$ D is supercharge operator

$\Rightarrow \int_M \omega = (2\pi)^n \sum_{\text{fixed pts}} \frac{\omega_x \, Pff(g_x)}{Pff(W_x)}$ $\quad$ Pf. Use Rk①, use Berizmann integral (Omitted) ▨

③ $Hom(\mathcal{O}(\Pi\mathbb{R}), C^\infty(\mathbb{R})) \in M$

$= \{0\}$ as all $\mathcal{O}(\Pi\mathbb{R})$ Fermion are nilpontent, but $C^\infty(\mathbb{R})$ none $\Rightarrow$ $Map(\mathbb{R}, \Pi\mathbb{R})$ has no $\mathbb{R}$-pts

but we can put geometric (?) stack structure on it, make it have $\mathbb{R}^{0|n}$-pts, i.e. $Hom(\mathcal{O}(\Pi\mathbb{R}), C^\infty(\mathbb{R}) \otimes \mathbb{R}[\theta_1 .. \theta_n]) \neq \{0\}$

using Grothendieck's 'functor of pts' idea! $\quad$ adds Fermion varibles

④ $Spin(n) \hookrightarrow Cl_n(\mathbb{C})$ $\curvearrowright \mathcal{H}_F = \mathcal{H}_F^+ \oplus \mathcal{H}_F^-$ two irreducible representations are left & right handed spinors

$\quad Spin(n) \quad Spin(n)$

The SUSY indeed add $\mathbb{Z}/2\mathbb{Z}$-graded theory by left-handed + right-handed

For odd-dimensional & $\mathbb{R}$-valued cases, the rep is even harder

the $\mathbb{Z}/2\mathbb{Z}$-graded $\Rightarrow$ differential operator $d_\pm$; $d_+ d_- = d_- d_+ = 0$ $\to$ supercharge operators

two kinds of Fermion varibles/operators

intertwine Boson and Fermion, grade 0, grade 1

thus this $\mathbb{Z}/2\mathbb{Z}$ comes from the double cover $SO(1,3) \overset{2:1}{\twoheadrightarrow} Spin(1,3)$

⑤ The spin-statistic thm is formulated as $\quad$ as apply the Fermion operator shift spin $\pm\frac{1}{2}$;

$Spin \curvearrowright \mathcal{H}_F \otimes \mathcal{H}_B$

spin $\longleftrightarrow$ statistic (CCR/ACCR)

Deformation Quantization v.s. Geometric Quantization

Recall:

Graded Abelion groups $\xrightarrow{+\text{multiplication}}$ Graded (associative) algebra $\xrightarrow{+\text{higher multiplication coherence in associative}}$ A_∞-algebra $\xrightarrow{\text{universal envelope}}$ L_∞-algebra

Graded Abelion groups $\xrightarrow{+\text{differential}}$ Differential graded Abelion group

$\to$ DG-algebra $\xrightarrow{\text{universal envelope}}$ DGLA $\xrightarrow{+\text{higher Lie bracket coherence in Jacobi identity}}$ L_∞-algebra

Differential graded Abelion group $=$ Chain complex of Abelion group

(differential δ_0, multiplication δ_1, higher $\delta_2, \delta_3, \cdots$)

This part is our setting for DQ

We'll not formulate all definition, but give something interesting:

• DGLA as differential graded coalgebra: set $C=\bigoplus \mathrm{Sym}^k \mathfrak{g}[1]$ with coderivation $\delta^n: C^i \to C^{i+n}$ shift [n]

then as ① $(\mathrm{Sym}^2\mathfrak{g}[1])_\bullet \cong (\Lambda^2\mathfrak{g})[2]$

$\downarrow \delta^1$ ⇠ recover the bracket

$\mathfrak{g}[2]$

($\mathfrak{g}[2]=\mathfrak{g}[1][1]$ by shift 1 in δ^1.)

this iso is strange, but it's by computing index $(-1)^{\boxed{\cdot}}$ coincides when interchange

• DGLA as degree 1 vector fields: take the dual of $\delta^1 \Rightarrow Q_1$, $[Q_1,Q_1]=0$ is desired vector fields on $\mathfrak{g}[1]$

② and we only use δ^1 (and δ^0 for d dual) as coderivation is determined by its projectors in target, $\delta^2=\delta^3=\cdots=0$ $\Downarrow$ $(\delta^1)^2=0$

(dually, derivation is determined by its generators in domain)

$\Rightarrow$ we have an equivalence of categories

$\{CDGA\} \overset{\simeq}{\Leftrightarrow} \{DGLA\}$ ☒

$\Rightarrow \{L_\infty\} \Leftrightarrow \{$higher DGLA$\}$ for non-vanishing higher terms

Here for DQ order 1 suffices to control deformation

E.g. Differential form are anti-symmetric 0-tensor $\Leftrightarrow$ symmetric 1-tensor

$dx\, dy \underset{=}{=} (-1)^{\deg(dx)\deg(dy)+1} dy\, dx$ 1-tensor, $= dy\,dx$

$\Omega^\bullet(\mathbb{R}^n)=\mathrm{Sym}\,\mathcal{O}(\mathbb{R}^n\oplus\mathbb{R}^n[1])$

$\boxed{\text{Shift by 1 changes symmetry}}$

$dx\wedge dy=-dy\wedge dx$ 0-tensor

E.g. ① $(\mathfrak{g}=\Omega^{0,\bullet}(X,TX),\bar\partial,[-,-])$ is deformation of X's cpx structure, {the MC form / solution to MC equations} = deformation space by K-S map $=H^1$

② $(\mathfrak{g}_E=\Omega^{0,\bullet}(X,\mathrm{End}(E)),\bar\partial_E,[-,-])$ ($\bar\partial_E$: cpx connection of E or End(E)) is deformation of E's cpx structure

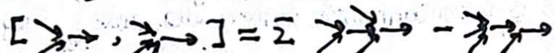
Rk. (A) General DGLA, the MC form are also deformation space $=H^1$;

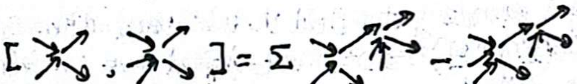
(B) In AG, it's moduli of X and moduli of E, fixing X;

(C) Here we can couple ①, ② by $\Gamma(TM) \overset{\rho}{\curvearrowright} \mathrm{End}(E)$ and the semi-direct product $\mathfrak{g}\rtimes_\rho \mathfrak{g}_E$ controls total deformation of X & E;

(D) $\Omega^{1,0}$ is holomorphic, it respects cpx structure, thus when we deform cpx structure, we use $\Omega^{0,\bullet}$ than $\Omega^{1,0}$!

③ Polyvector field $(\mathfrak{g}=\Gamma(M,\bigoplus\Lambda^k TM)=\bigoplus\Gamma(M,\Lambda^k TM),\ 0,\ [-,-])$, the $[-,-]$ is by generalizing commutator of usual vector fields: 0 is usual dual of d in $\Omega^\bullet$





is polyvector field: multivariable in target

④ Hochschild chain complex $(\mathfrak{g}_H=\bigoplus HC^n=\bigoplus \mathrm{Hom}_{\mathbb{C}}(A^{\otimes n},A),\ d_\mu,\ [-,-])$, $\mu: A\otimes A\to A$ is the multiplication of algebra $A/\mathbb{C}$

$d_\mu f(a_0\otimes\cdots\otimes a_n)=\boxed{\cdots}$ see Hochschild chain, omitted.

same as ③ as elements $A^{\otimes n}\to A$ viewed as $A, A, A \to A$ $\Rightarrow$ thus, ③ ⊂ ④

called the Gerstenhaber structure

μ associative $\Leftrightarrow d_\mu^2=0$

⑤ We quantize ④ into $(\mathfrak{g}_H[[\hbar]],\ d_{\mu_\hbar},\ [-,-]_G)$

$\mu_\hbar$ linearly extend $A[[\hbar]]\otimes A[[\hbar]]\to A[[\hbar]]$

$[-,-]_G$ also linearly extended $[-,-]_G$

$\mu_\hbar=\mu_0+\hbar\mu_1+\hbar^2\mu_2+\cdots$, $\mu_0=\mu$, $\mu_i\ (i\geq 1)$ composite with $[-,-]_G$; $\mu_\hbar$ associative

$\Updownarrow$

$[\mu_\hbar,\mu_\hbar]_G=0$ quantum one

$\Updownarrow$

$\mu_\hbar=\mu+\Theta_\hbar$, $d_\mu\Theta_\hbar+\frac{1}{2}[\Theta_\hbar,\Theta_\hbar]_G=0$ satisfy MC equation (classical one, also classical one)

$\Theta_\hbar\in\mathfrak{g}_H^2[[\hbar]]$, classical operators act on it by acting on coefficients of $\hbar^k$

We'll back to it in DQ next

Rk. (D) Now it's time to couple all things together:

①, ②, ③ are something orthogonal, and ⑤ can quantize ③

$H^0(X,\Lambda^i TX)+H^j(X,TX) \overset{③}{\Rightarrow} H^j(X,\Lambda^i TX)$

$\overset{②}{\Rightarrow} H^j(X,\Lambda^i\,\mathrm{End}(E))$

$\overset{⑤}{\Rightarrow} H^j(X,\Lambda^i\,\mathrm{End}(E))[[\hbar]]$

"orthogonal": enhance algebraic structure v.s. enhance geometric structure

Our question: Quantize $\mu_\hbar$ exists for polyvector fields?

(Kontsevich) Given Possion structure $\pi\in HC^2(A,A)$, $\exists\,\Theta_\hbar\in\mathfrak{g}_H^2[[\hbar]]$ s.t. ① $\Theta_\hbar$ satisfy MC equation; ② $\mu_1(f,g)=\frac{1}{2}\{f,g\}$ classical limit

(thus let A be ring of functions, it recovers many quantizations before: pq, qp, Weyl, Wick...)

One may want to solve two equations ①② order by order, but it's technically impossible: as we can see, it needs π doesn't from equations $+[[\hbar]]$

(☆) It's nontrivial issue is: we construct Hochschild chain from polyvector field, but it's not DGLA morphism, it has many quantum correlations lives in higher orders, as L_∞-morphism $\Rightarrow$ MC equation (Formality thm)

Condition$(\mathfrak{g}[1], Q)$ $Q \curvearrowright \mathcal{O}(\mathfrak{g}[1]) \simeq \mathrm{Sym}\,\mathfrak{g}^\vee[1]$ derivation $\leftrightarrow$ degree 1 ($\deg 1 \in$) $\mathfrak{g}[-1] \otimes \mathfrak{g}[1]$ $\cong \mathfrak{g}^\vee[-1] \otimes \mathfrak{g}[1]$

$\Updownarrow \simeq$ $\Rightarrow$ $(\deg 1)\mathrm{Sym}^2\mathfrak{g}^\vee[-1]\otimes\mathfrak{g}[1]$ $\cong \Lambda^2\mathfrak{g}^\vee[-2]\otimes\mathfrak{g}[1]$

$(\mathrm{Sym}\,\mathfrak{g}[1], D) \Rightarrow \delta: \mathfrak{g} \to \mathfrak{g}$ degree 1 $(\deg 1)\mathrm{Sym}^3\mathfrak{g}^\vee[-1]\otimes\mathfrak{g}[1]$ $\cong \Lambda^3\mathfrak{g}^\vee[-3]\otimes\mathfrak{g}[1]$

$[-,-]: \Lambda^2\mathfrak{g} \to \mathfrak{g}$ degree 0

$[-,-,-]: \Lambda^3\mathfrak{g} \to \mathfrak{g}$ degree -1 $\cong$

Observation.

We extend MC equation to L_∞ by

$\delta + [-,-] + [-,-,-] + \cdots = 0$

But by degree reason, only $\deg x = 1$, $\delta x + [x,x] + \cdots$ has degree $1+1 = 1+1+0 = 1+1+1-1 = \cdots = 2$

otherwise they not at same degree $\Rightarrow$ all $0 \Rightarrow x = 0$

Prop.

x is solution to MC equation $\Leftrightarrow$ $Q_{\mathfrak{g}}x = 0$ i.e. x is fixed by vector field Q

$\Rightarrow \phi: (\mathfrak{h}[1], Q_{\mathfrak{h}}) \to (\mathfrak{g}[1], Q_{\mathfrak{g}})$

ϕ is L_∞-algebra homomorphism $\Leftrightarrow$ ϕ respect Q-action on $\mathcal{O}(\mathfrak{g}[1])$ & $\mathcal{O}(\mathfrak{h}[1])$ $\Leftrightarrow$ $\phi \in \mathcal{O}(\mathfrak{g}[1]) \otimes \mathfrak{h}[1]$ is L_∞-algebra

$\Leftrightarrow$ ϕ send $Q_{\mathfrak{g}}$-fixed points to $Q_{\mathfrak{h}}$-fixed points (inherited from $\mathfrak{h}[1]$)

ϕ satisfy MC equation

Setting. $\mathfrak{g} = T_{poly}(\mathbb{R}^n)[[\hbar]]$, $\mathfrak{h} = D_{poly}(\mathbb{R}^n)$ is the poly-differential operators

We're using the map $\mathfrak{g} \to \mathfrak{h}$ to dominate the quantized $\mathfrak{g}$

$= \bigoplus_{i=-1} \mathrm{Diff}(C^\infty(\mathbb{R}^n)^{\otimes(i+1)}, C^\infty(\mathbb{R}^n))[[\hbar]] \subset HC^\bullet$, thus it's DGLA by Hochschild chain structure

by multiplication & derivative on $\mathbb{R}^n$

Although both $\mathfrak{g}$ & $\mathfrak{h}$ are DGLA, they have no DGLA map, but L_∞-morphism. the obstruction of DGLA morphism / higher quantum corrections lies in the Possion structure $\hbar\pi \in \mathfrak{g}^1$, $\frac{1}{2}[\hbar\pi, \hbar\pi] = 0$

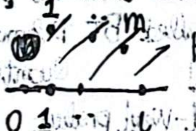
Rk. For general Possion mfds than $\mathbb{R}^n$, use some descent method to glue (nontrivial).

$\phi: \mathfrak{g} \to \mathfrak{h}$ is given by send polyvector fields $\pi_1 \cdots \pi_m$ to a differential operator on $f_0 \otimes \cdots \otimes f_l$

denote as $\mathcal{U}(\pi_1 \cdots \pi_n)(f_0 \otimes \cdots \otimes f_l) = \sum w_\Gamma P_\Gamma$, $w_\Gamma \in \mathbb{R}$ is weight of diagram Γ, $P_\Gamma \in C^\infty(\mathbb{R}^n)$

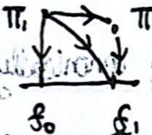
$w_\Gamma = \hbar^{|E(\Gamma)|} \int_{\mathrm{Conf}(\mathbb{H})} \bigwedge_{e \in E(\Gamma)} P_e$, P_e is Green function, is a 1-form, it takes value at start and end of edge $e \in E(\Gamma)$

$\mathrm{Conf}(\mathbb{H})$ is the configuration space of



not intersect $(m+l+1)$ points, modulo automorphisms (it's mfd with corners)

P_Γ is by: Γ give order of doing derivation



- connecting π_i and π_j $\Rightarrow$ composite in $\mathfrak{g}$
- connecting π_i and f_j $\Rightarrow$ doing derivation of f_j, π_i has many out arrows means which component of polyvector field

Rk. Where is our Feymann diagram live? We need equip $\mathbb{H}$ with hyperbolic metric to gain Green function, hyperbolic metric helps us converge the sum

Thm (Kontsevich's formality thm) ϕ above is L_∞-morphism.

Pf. Use integral by part, check MC equation. ☑ split integral order by order

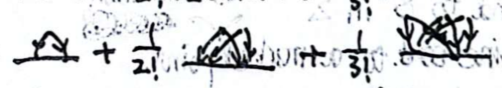
close $\Rightarrow [-,-]$ is T_{poly} & closed $\Rightarrow [-,-]$ in D_{poly}

boundary points of $\mathrm{Conf}(\mathbb{H})$ as two points get together

Eg. ① Let $\mathbb{R}^n = \mathbb{R}^{2d}$, the Possion bivector $=$ standard $\sum \partial_{x_i} \wedge \partial_{p_i}$ is constant

$\Rightarrow$ no composite in T_{poly} $\Rightarrow \phi(\hbar\pi) = \phi_1(\hbar\pi) + \frac{1}{2!}\phi_2(\hbar\pi, \hbar\pi) + \frac{1}{3!}\cdots$

② If $\mathfrak{g}$ is f.d usual Lie algebra

$\mathfrak{g} = \mathrm{span}\{x_i\} \Rightarrow \pi = \sum C_{ij}^k \partial_{x_i} \wedge \partial_{x_j}$, C_{ij}^k structure constant $=$  $\Rightarrow$ Weyl Quantization

$\Rightarrow (\mathfrak{g}[[\hbar]], *) \overset{iso}{\cong}$ to universal eveloping algebra (no internal arrows between π_1, π_2)

$U_\hbar(\mathfrak{g}) = \otimes\mathfrak{g} / (x \otimes y - y \otimes x = \hbar[x,y])$

and $\phi(\hbar\pi) =$  it's worser but only one, as π is lineear, only can compose once

Rk. Why we use Feymann diagram is clear: we can simplify $\mathcal{U}$ by combinational datum, but, ① Why we send Feymann diagram in $\mathbb{H}$ with hyperbolic metric?

② What's the physical model for $\mathbb{H}$, i.e. write $\mathcal{U}$ or ϕ as perturbative expansion of an interacted theory / $\mathbb{H}$?

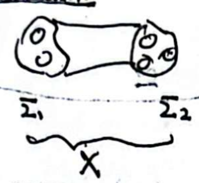
We should take $\mathbb{H}$ as base space-time • $f_0 \cdots f_l$ as observables (input at boundary) • $\pi_1 \cdots \pi_m$ as interaction terms (interaction inner edges)

$\Rightarrow$ the action $S = \int \mathrm{free} + \hbar\,\mathrm{I}\pi_1 + \cdots + \hbar\,\mathrm{I}\pi_m$

$\Rightarrow$ the path integral $\int_\cdot e^{-iS/\hbar} O_{S_0} \cdots O_{S_{k}}\, d\mu_1 = \langle 0 | O_{S_0} \cdots O_{S_1} | 0 \rangle$ i.e. $\mu_1 = \langle 0 | O_{(\cdot)} \cdots O_{(\cdot)} | 0 \rangle : T_{poly} \to D_{poly}$
perturbatively expand with $\hbar(I_{\pi_1} + \cdots + I_{\pi_n}) \Rightarrow$ serveral terms by act I_{π_i} to $O_{f_j} = \sum$ Feymann diagram in $\mathbb{H}$!
Why we use $\mathbb{H}$ hyperbolic model / $\mathbb{D}$ Poincaré disk model? This is due to its conformal structure / open string model, it has advantage that conformal $\Leftrightarrow$ rescaling-invariant $\Leftrightarrow$ no renormalization $\Leftrightarrow$ the sum of Feymann diagram converges $\Leftrightarrow$ algebraic

Geometric Quantization v.s. Deformation Quantization

Consider a $(2+1)$-theory (Chern-Simons)



For $(\Sigma, \tilde{J})$, the moduli of $U(1)$-bundle $\mathcal{M}_\Sigma$ with $\tilde{J}$ is Abelian variety with ample line bundle L.
$\mathcal{H}_{pre} = C^\infty(\mathcal{M}_\Sigma, L^{\otimes k})$ (for $L^{\otimes k}$ gives embedding to $\mathbb{P}^N$)
$\mathcal{H} = H^0_{\bar\partial}(\mathcal{M}_\Sigma, L^{\otimes k})$ $\hookrightarrow$ space of conformal blocks

$\Rightarrow$ QFT gives functor Z, s.t.
$Z(\boxminus) : \mathcal{H}_1 \to \mathcal{H}_2$, now let $\Sigma_1 = \Sigma_2 \Rightarrow Z(\boxminus) \curvearrowright \mathcal{H}$. $\tilde{J}$ isn't unique

Fact. ① Here $Z(\boxminus)$ is by the geometric quantization of $(\mathcal{M}_\Sigma, \tilde{J})$, $C^\infty(\mathcal{M}_\Sigma) \curvearrowright \mathcal{H}_{pre}$ geometric quantisation;
② Let $\hbar = \frac{1}{k}$, BT quantisation recover Moyal-Weyl product $\underset{\hbar}{*}$; $\rightsquigarrow C^\infty(\mathcal{M}_\Sigma)[[\hbar]] \curvearrowright \mathcal{H}$ Berezin-Toeplitz quantization;

※. ① When $\Sigma_1 = \Sigma_2$, $Z(\boxminus)$ is by some links in $X \supset L$ ② It's perturbative method, [WRT] has non-perturbative method $\mathrm{tr}(Z(L))$ = Jone's polynomial of L, by quantum groups;
③ Fact ② is surprising, as $\underset{\hbar}{*}$ not depend on choice of $\tilde{J}$, but geometric quantization depends heavily.

BT quantization on (M^{2n}, J, L) polarized Abelian variety

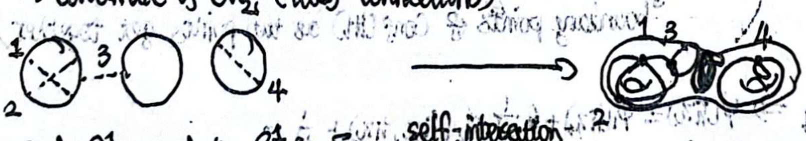
Setting: $(L, \langle\cdot,\cdot\rangle, \nabla)$ is Hermittian bundle with Chern connection $\nabla \Rightarrow H^0_{\bar\partial} = \mathrm{Ker}\,\bar\partial \subset C^\infty(M, L^{\otimes k})$
Fix $f \in C^\infty(M)$, the BT operator of f is: (just same form as classical Toeplitz) $\Rightarrow \exists$ projection $C^\infty \xrightarrow{\pi} H^0_{\bar\partial}$
$T_f : \mathcal{H} \to \mathcal{H}$ $\qquad \Rightarrow C^\infty(M) \curvearrowright \mathcal{H}$, we $\boxed{\text{quantize } f \text{ to } T_f}$ $(*)$
$\quad g \mapsto \pi(gf)$

Recall our Geometric Quantization is $P_f = -\frac{1}{k}\nabla^{(k)}_{X_f} + \sqrt{-1} f$ (∇_{X_f} is on L, extend to $L^{\otimes k}$ denoted as $\nabla^{(k)}_{X_f}$)
i.e. $P_f : \mathcal{H}_{pre} \to \mathcal{H}_{pre}$, $g \mapsto -\frac{1}{k}\nabla^{(k)}_{X_f} g + \sqrt{-1}(gf) \Rightarrow C^\infty(M) \curvearrowright \mathcal{H}_{pre}$
Thm. (Tuynman) $\pi \circ P_f = T_{f - \frac{1}{2k}\Delta f} \cdot \sqrt{-1}$

Now we need to explain $(*)$, i.e. we take $f_1, f_2 \in C^\infty(M)$, $T_{f_1} \circ T_{f_2} \sim \sum_l T_{c_l(f_1, f_2)} \left(\frac{1}{k}\right)^l$ asymptotic expansion resp. $\frac{1}{k}$
Let $f_1 \overset{BT}{\underset{\hbar}{*}} f_2 := \sum (-1)^l c_l(f_1, f_2)\hbar^l$ called BT Deformation Quantization for some c_l, $\forall l \geq 0$
$\left\{\begin{array}{l} c_0(f_1, f_2) = f_1 f_2 \text{ (need prove)} \\ c_1(f_1, f_2) = ? \\ \vdots \end{array}\right.$
Thm. $E = e^{-\frac{\hbar}{2}\Delta} : C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$
$\underbrace{E^{-1}\left(E(f_1) \overset{BT}{\underset{\hbar}{*}} E(f_2)\right)}_{\text{BT Quantization}} = \underbrace{f_1 \underset{\hbar}{*} f_2 \text{ the Moyal-Weyl product}}_{\text{Deformation Quantization}}$ if we take the polarization L the Θ-line bundle on the Abelian variety canonically

Geometric side: $\boxed{\Delta, T_f \cdots \text{all depend on } J}$ $=$ Algebraic side: $\boxed{\text{not depend on } J}$

※. Here our $f \in C^\infty(M)$ is chosen in CS theory by: $L \subset X$ a link, we can project to Σ, gives a Chord diagram, its holonomy function $h \in C^\infty(M) \Rightarrow Z(L) : \mathcal{H} \to \mathcal{H} \Leftarrow T_h : \mathcal{H} \to \mathcal{H}$
$\hookrightarrow$ coordinate of $\mathcal{M}_\Sigma$ ($U(1)$-connections)



each S^1 send to S^1 in Σ self-intersection
each dash line send to a marked point
(needn't to mark all self-intersection points)
$\uparrow$ projection (create self-intersections)

knots and links $L = \bigcup L_i$

Two Chord diagram of two torus links D_1, D_2, $[D_1], [D_2]$ has natural Possion bracket
$\{[D_1], [D_2]\} = \sum_{p \in D_1 \cap D_2} \varepsilon(p)[D_1 \underset{p}{\cup} D_2]$, $\varepsilon(p) = \left\{\begin{array}{l} +1 \\ -1 \end{array}\right.$ (perturbe to most traversed)
Higher C_l also have similiar combinatorics explanation. as $\{f_1, f_2\} \Leftarrow c_1(f_1, f_2)$ above

the Possion structure induce symplectic structure is the Goldman symplectic structure on $\mathcal{M}_\Sigma$
$\boxed{\text{the BT quantization of } f_{C_i} \text{ essentially is the quantization of Goldman symplectic structure}}$

② One might think let $\hbar \sim \frac{1}{k}$ is surprising, but we can take asympototic expansion by $\hbar$ as the same as the expansion by coupling constant $\lambda = \square\hbar$, and this is equivalent to the Large-N expansion $(\frac{1}{N})^i \sim \lambda^i$, where N describle the scale of symmetry group $SO(N)$, i.e. $k \sim N$ describle the symmetry of the Abelian variety X, then the comparsion between scale of automorphism group is more ($SO(N)$ v.s. $PSL(N)$ (embedded $(X, L) \hookrightarrow \mathbb{P}^N, \mathcal{O}(1)$)) reasonable; ↺ $PSL(N)$

phase space A after normalized, also projective, thus $PSL(N)$, too

<u>Fedosov Quantization & Algebraic index</u>

Given (M^{2n}, ω) symplectic, we had constructed $\underset{\hbar}{*}$ on $C^\infty(M)[[\hbar]]$ via deformation quantization locally on $\mathbb{R}^{2n}$, as symplectic $\subset$ Possion, we can use geometric data to construct alternatively, although they're not equal, they're equivalent in some sense by the discussion of the comparsion between BT quantization and deformation quantization above.

Consider the frame bundle $Sp_n \to Fr_{Sp_n} \downarrow M$, locally the Darboux chart $\omega_{M|_U} = dy^1 \wedge dy^2 + \cdots + dy^{2n-1} \wedge dy^{2n}$

We have symplectic connection ∇ s.t. $\nabla_X Y - \nabla_Y X = [X, Y]$ and $\nabla \omega = 0$ and its curvature $R^\nabla(X, Y)(Z) = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})(Z)$

set $W_n = \mathbb{C}[y^1 \ldots y^{2n}][[\hbar]] \Rightarrow \mathcal{W}_n = Fr_{Sp_n} \times_{Sp_n} W_n$ the associated vector bundle given by the action $Sp_n \curvearrowright \mathbb{C}[y^1 \ldots y^{2n}]$ extend to $[[\hbar]]$

thus $\mathcal{W}_n$ has fibre W_n, locally glued by the rule of cotangent vectors i.e. $\mathcal{W}_n \cong (\mathrm{Sym}\, T^*M)[[\hbar]]$ called Weyl bundle

$R^\nabla \in \Omega^2(\mathrm{End}\, TM) \overset{\omega}{\cong} \Omega^2(T^*M \otimes T^*M)$, ω induce a pair, pair induce $TM \overset{\omega}{\cong} T^*M$

i.e. $R^\nabla_\omega(X, Y, Z, W) = \omega(R^\nabla(X, Y)Z, W)$ (also same as Riemannian)

<u>Fact</u>. For Levi-Civita (Riemannian), ∇ $\exists!$, but for symplectic, it exists not unique. (Tensor computation)

The Fedosov Abelian connection on $\mathcal{W}_n$ can be extended from $T^*M \overset{\omega}{\cong} TM$, let ∇ be induced by a symplectic connection

Note that we have a fibrewise $\underset{\hbar}{*}$ on $\mathcal{W}_n$, we denote $[-,-]_{\underset{\hbar}{*}}$ the commutator $(X \underset{\hbar}{*} Y + Y \underset{\hbar}{*} X)\frac{1}{2}$

<u>Fact</u>. $\nabla^2 s = \frac{1}{\hbar}[-R^\nabla_\omega, s]_{\underset{\hbar}{*}}$ (Tensor computation)

Then our Fedosov Abelian connection is $D = \nabla + \frac{1}{\hbar}[\gamma, -]_{\underset{\hbar}{*}}$, fix $\gamma \in \Omega^1(M, \mathcal{W}_n)$, s.t. $D^2 = 0$ (γ is justification to make D flat)

$D^2 = \frac{1}{\hbar}[-R^\nabla_\omega + \nabla\gamma + \frac{1}{2\hbar}[\gamma, \gamma]_{\underset{\hbar}{*}}, -]_{\underset{\hbar}{*}} = 0 \Leftarrow (-R^\nabla_\omega + \nabla\gamma + \frac{1}{2\hbar}[\gamma, \gamma]_{\underset{\hbar}{*}}) \in \Omega^2(M, \mathbb{C}[[\hbar]])$ is "constant form" (in $\mathbb{C}[[\hbar]]$, thus $[-,-]$

<u>Fact</u> (Fedosov) $\forall \{\omega_k\} \subset \Omega^2(M, \mathbb{C}[[\hbar]])$ closed, $\exists\, \gamma$ s.t. $-R^\nabla + \nabla\gamma + \frac{1}{2\hbar}[\gamma, \gamma]_{\underset{\hbar}{*}} = \omega_\hbar := -\omega + \sum \omega_k \cdot \hbar^k$ (BV formalism, pertub

Rk. ① Not exists if we need to find holomorphic form over cpx variety;

② When flat $\mathbb{R}^{2n}$, $D = d + \delta$ simply;

We have $0 \to [\mathrm{Ker} D \xrightarrow[\sigma]{\cong} C^\infty(M)[[\hbar]]] \xrightarrow{D} C^\infty(M, \mathcal{W}_n) \xrightarrow{D} \Omega^1(M, \mathcal{W}_n) \xrightarrow{D} \Omega^2(M, \mathcal{W}_n) \xrightarrow{D} \cdots$ a resolution of $\mathrm{Ker} D \overset{\sigma}{\cong} C^\infty(M)[[\hbar]]$

(all $y_i = 0$)

then we have the Fedosov product $f \overset{F}{\underset{\hbar}{*}} g = \sigma(\sigma^{-1}(f) \underset{\hbar}{*} \sigma^{-1}(g))$, where $\underset{\hbar}{*}$ is the fibrewise Moyal-Weyl product

$\sigma^{-1}(f) = f + \partial_i f y^i + \cdots$ adding higher terms, to make $D(\sigma^{-1}(f)) = 0$ well-defined

it induces <u>σ is isomorphism</u>

Set $\mathcal{A}^\hbar = \mathrm{Ker} D \overset{\sigma}{\cong} C^\infty(M)[[\hbar]]$,

$\mathrm{Tr}: \mathcal{A}^\hbar \to \mathbb{C}((\hbar))$

s.t. • $\mathrm{Tr}(f) = \frac{1}{(2\pi\hbar)^n}(\int f \frac{\omega^n}{n!} + O(\hbar))$ leading term determined

• $\mathrm{Tr}(f \overset{F}{\underset{\hbar}{*}} g) = \mathrm{Tr}(g \overset{F}{\underset{\hbar}{*}} f)$

<u>Fact</u>. Then Tr is $\exists!$ determined;

Thm (Algebraic index) $\mathrm{Tr}(1) = \int e^{-\frac{\omega_\hbar}{\hbar}} \hat{A}(M)$ ▨

• Replace $C^\infty(M)[[\hbar]]$ by $C^\infty(M, \mathrm{End}(E))[[\hbar]]$ and $\mathcal{W}_n$ by $\mathcal{W}_n \otimes \mathrm{End}(E) \rightsquigarrow \mathrm{Tr}(1) = \int e^{-\frac{\omega_\hbar}{\hbar}} \mathrm{Ch}(E) \hat{A}(M)$ twisted by E ▨

• Next semester, the professor will introduce BV formalism, factorization algebra and higher algebra.