Homology of arbitrary coefficients (algebraic) and Bockstein's spectrum sequence (algebraic)

(Now we had know the adjoint property of Hom and $\otimes$, then we'll see how this adjoint pair can act on the duality in homology and cohomology, and introduce $0 \to G_0 \to G_1 \to G_2 \to 0$ inducing the Bockstein's homomorphism and Bockstein's spectral sequence

$$\begin{pmatrix} \text{homology} \leftrightarrow \text{Hom} \\ \updownarrow \qquad\quad \updownarrow \\ \text{cohomology} \leftrightarrow \otimes \end{pmatrix}$$

I'll do all things on Homological algebra, the topological left to next speaker)

Def 1 (The p-th homology group of $\mathscr{C}$ with coeffiencents in G)
$\mathscr{C} \otimes G \overset{def}{=} \{C_p \otimes G, \partial_p \otimes 1_G\}$, and $H_p(\mathscr{C}; G) \overset{def}{=} H_p(\mathscr{C} \otimes G)$

Rk. Consider G as a complex $\cdots \to 0 \to G \to 0 \to 0 \to \cdots$, then $\mathscr{C} \otimes G$ is just the tensor of complexes)

Def 2 (Augmentation and reduced homology)
Let $C_{-1} \otimes G = \mathbb{Z} \otimes G = G$, with $C_0 \otimes G \xrightarrow{\varepsilon \otimes 1_G} G$
Then $\tilde{H}_p(\mathscr{C}; G) = \begin{cases} H_p(\mathscr{C}; G) ; p \geq 1 \\ \dfrac{\operatorname{Ker}(\partial_1 \otimes 1_G)}{\operatorname{Im}(\varepsilon \otimes 1_G)} ; p = 0 \end{cases}$

Prop 3. $\tilde{H}_0(\mathscr{C}; G) \oplus G \cong H_0(\mathscr{C}, G)$

Pf. $0 \to \operatorname{Ker}\varepsilon \to C_0 \to \mathbb{Z} \to 0 \Longrightarrow \operatorname{Ker}(\partial_1 \otimes 1_G) \cong \operatorname{Im}(\partial_1 \otimes 1_G) \oplus \dfrac{\operatorname{Ker}(\partial_1 \otimes 1_G)}{\operatorname{Im}(\varepsilon \otimes 1_G)}$

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker}\varepsilon \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \operatorname{Ker}\varepsilon \otimes G \arrow[r] & C_0 \otimes G \arrow[r] & G \arrow[r] & 0 \\ \cdots \arrow[r] & C_1 \otimes G \arrow[ur, "\partial_1 \otimes 1_G"] \arrow[u] & & & \end{tikzcd}$$

$\Longrightarrow H_0(\mathscr{C}; G) \cong \tilde{H}_0(\mathscr{C}; G) \oplus G$ ☒

(Rk. We using spilt $\Rightarrow$ pure exact here)

Def 4 (The induced chain map, chain homotopy)

$$\begin{tikzcd} \cdots \arrow[r] & C_p \arrow[r, "\partial_p"] \arrow[d, "\varphi_p"] \arrow[dl, "D"] & C_{p-1} \arrow[r] \arrow[d, "\varphi_{p-1}"] & \cdots \arrow[r] & C_0 \arrow[r] \arrow[d, "\varphi_0"] & \mathbb{Z} \arrow[d] \\ \cdots \arrow[r] & D_p \arrow[r] & D_{p-1} \arrow[r] & \cdots \arrow[r] & D_0 \arrow[r] & \mathbb{Z} \\ \cdots \arrow[r] & C_p \otimes G \arrow[r] \arrow[d, "\varphi_p \otimes 1_G"] & C_{p-1} \otimes G \arrow[r] \arrow[d, "\varphi_{p-1} \otimes 1_G"] & \cdots \arrow[r] & C_0 \otimes G \arrow[d, "\varphi_0 \otimes 1_G"] & \\ \cdots \arrow[r] & D_p \otimes G \arrow[r] & D_{p-1} \otimes G \arrow[r] & \cdots \arrow[r] & D_0 \otimes G & \end{tikzcd}$$

① Thus $\varphi \otimes 1_G$ is the induced chain map (verify commute!) and in particular,

② When φ_0 preserves augmentation, $\varphi_0 \otimes 1_G$ also (verify commute!)

③ if $\varphi \overset{D}{\simeq} \psi$
$\Longrightarrow \varphi \otimes 1_G \overset{D \otimes 1_G}{\simeq} \psi \otimes 1_G$ We only verify this

Pf. $\varphi \otimes 1_G - \psi \otimes 1_G = (\varphi - \psi) \otimes 1_G \overset{\text{dim}p}{=} (\partial_{p+1} D_p + D_{p-1} \partial_p) \otimes 1_G = (\partial_{p+1} \otimes 1_G)(D_p \otimes 1_G) + (D_{p-1} \otimes 1_G)(\partial_p \otimes 1_G)$ ☒

Coro 5 (Homological level) ① $\varphi \simeq \psi \Longrightarrow (\varphi \otimes 1_G)_* = (\psi \otimes 1_G)_*$
② φ is a chain equivalence, $\varphi \otimes 1_G$ also

Pf. Both trivial, we verify ② $\begin{cases} \varphi \circ \psi_1 \overset{D_1}{\simeq} \mathrm{Id} \\ \psi_2 \circ \varphi \overset{D_2}{\simeq} \mathrm{Id} \end{cases} \Longrightarrow \begin{cases} (\varphi \otimes 1_G) \circ (\psi_1 \otimes 1_G) \overset{D_1 \otimes 1_G}{\simeq} (\mathrm{Id} \otimes 1_G) \\ (\psi_2 \otimes 1_G) \circ (\varphi \otimes 1_G) \overset{D_2 \otimes 1_G}{\simeq} (\mathrm{Id} \otimes 1_G) \end{cases}$ ☒

Thm 6 (Long exact sequence)
$0 \to \mathscr{C} \otimes G \to \mathscr{D} \otimes G \to \mathscr{E} \otimes G \to 0$ inducing long exact sequence by Zig-Zag $\cdots \to H_p(\mathscr{C}; G) \to H_p(\mathscr{D}; G) \to H_p(\mathscr{E}, G) \xrightarrow{(\partial \otimes 1_G)_*} H_{p-1}(\mathscr{C}; G) \to \cdots$

in particulars $0 \to \mathscr{C} \to \mathscr{D} \to \mathscr{E} \to 0$ spilt / pure exact in each dimension ☒

Ref. J. H. PALMIERI. A user's guide to the Bockstein spectral sequence

Its dual analogue.

Def 1' (The p-th cohomology of $\mathscr{C}$ with coeffeicent in G)
$\operatorname{Hom}(\mathscr{C}, G) \overset{def}{=} \{\operatorname{Hom}(C_p, G), \operatorname{Hom}(\partial_p, 1_G) = \partial_p^* = \delta_p\}$, $H^p(\mathscr{C}; G) \overset{def}{=} H_p(\operatorname{Hom}(\mathscr{C}, G))$

Def 2' (Augmentation and reduced cohomology)
Let $\operatorname{Hom}(C_{-1}, G) = \operatorname{Hom}(\mathbb{Z}, G) = G$ with $\operatorname{Hom}(C_0, G) \xleftarrow{\operatorname{Hom}(\varepsilon, 1_G)} G$
Then $\tilde{H}^p(\mathscr{C}; G) = \begin{cases} H^p(\mathscr{C}; G) ; p \geq 1 \\ \dfrac{\operatorname{Ker}(\delta_1)}{\operatorname{Im}(\operatorname{Hom}(\varepsilon, 1_G))} ; p = 0 \end{cases}$

Prop 3' (Pf same)
$\tilde{H}^0(\mathscr{C}; G) \oplus G \cong H^0(\mathscr{C}, G)$

Def 4' (The induced chain map, chain homotopy)
For the graph, only differs in contravariant, the consequence is completely the same. We prove that: ③' if $\varphi \overset{D}{\simeq} \psi$
$\Longrightarrow \operatorname{Hom}(\varphi, 1_G) \overset{\operatorname{Hom}(D, 1_G)}{\simeq} \operatorname{Hom}(\psi, 1_G)$

Pf. $\operatorname{Hom}(\varphi, 1_G) = \varphi^*$ (i.e. $\varphi^*(f) = f \circ \varphi$)
$\Longrightarrow \varphi^* - \psi^* = (\varphi - \psi)^* \overset{\text{dim}p}{=} (\partial_{p+1} D_p + D_{p-1} \partial_p)^*$
$= (\partial_{p+1})^* (D_p)^* + (D_{p-1})^* (\partial_p)^*$ ☒

Coro 5' (Cohomological level) (Pf is same)
①' $\varphi \simeq \psi \Longrightarrow (\varphi^*)_* = (\psi^*)_*$
②' φ is a chain equalence, φ^* also

Thm 6' (Long exact sequence)
$0 \to \operatorname{Hom}(\mathscr{E}, G) \to \operatorname{Hom}(\mathscr{D}, G) \to \operatorname{Hom}(\mathscr{C}, G) \to 0$ inducing long exact sequence by Zig-Zag
$\cdots \leftarrow H^p(\mathscr{C}, G) \leftarrow H^p(\mathscr{D}, G) \leftarrow H^p(\mathscr{E}, G) \leftarrow \cdots$

in particular, $0 \to \mathscr{C} \to \mathscr{D} \to \mathscr{E} \to 0$ spilt, $\mathscr{E}$ free

Then such three is a exact couple, thus it can induce the Bockstein spectral sequence in each dimension

Notice: for $0 \to \mathscr{C} \xrightarrow{\theta} \mathscr{C} \to \mathscr{B} = \operatorname{coker}\theta \to 0$, it still holds, thus can generalise

Another way is filter $\mathscr{C}$ by $F^s(\mathscr{C}) = \operatorname{Im}(\theta^s)$, then
$\cdots \subseteq F^2\mathscr{C} \subseteq F^1\mathscr{C} \subseteq F^0\mathscr{C} = \mathscr{C}$, it's also making sense

Rk. For the convergence, there are still something special

Now we come to our main task: Thm 51.1 and its dual analogue: Thm 45.5 (Thm 46.2 isn't necessarily, and we might teach it later)

Both of them tell us the homology isomorphism is preserved in Hom and $\otimes$ in chain level

Thm 7. $\mathcal{C}$ and $\mathcal{D}$ free. $\varphi: \mathcal{C} \to \mathcal{D}$ inducing homology isomorphisms in all dimensions, so does $\varphi \otimes 1_G$

Pf. $\boxed{\text{Step 1}}$ We first do for a short free exact sequence (by universal coefficient, it's trivial. But universal coefficient depends on this thm to prove)

$0 \to \mathcal{C}_1 \xrightarrow{f} \mathcal{C}_2 \to \mathcal{C}_3 \to 0$

Claim, if f induce homology isomorphisms all dimensions, so does $f \otimes 1_G$

Pf. It suffices to prove $H_p(\mathcal{C}_3; G) = 0$ (This part is wrong ① depend on universal coefficient ② Not using exactness)

$(0 \to H_p(\mathcal{C}_1; G) \xrightarrow{f \otimes 1_G} H_p(\mathcal{C}_2; G) \to H_p(\mathcal{C}_3; G))$

$\Rightarrow H_p(\mathcal{C}_1; G) \cong H_p(\mathcal{C}_2; G)$.

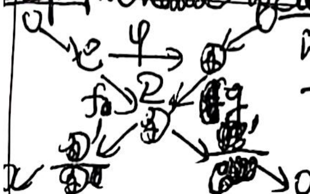
But we had known $H_p(\mathcal{C}_3) = 0$

$\Rightarrow H_p(\mathcal{C}_3; G) \cong (H_{p-1}(\mathcal{C}_3) * G) \oplus (H_p(\mathcal{C}_3) \otimes G)$

$= 0$ by universal coefficient (Thm 55.1) $\blacksquare$

RK. Here $*$ not the free product but torsion product, a more usual notion is $\mathrm{Tor}(H_{p-1}(\mathcal{C}_3), G) = H_{p-1}(\mathcal{C}_3) * G$

$\boxed{\text{Step 2}}$ We reduce to Step 1's case, we hope a graph (Only need commute up to a homology, i.e. $g \circ \varphi \simeq f$)



if f, g both $\Rightarrow \varphi$ also

Thus left is finding such $\mathcal{D}'$, this is Lemma 45.4

Using so-called "pullback of what" / "algebraic mapping cylinder" (We omit its Geometry motivation here, to be more algebraic)

Coro 8. $\mathcal{C}, \mathcal{D}$ free, $H_p(\mathcal{C}) \cong H_p(\mathcal{D})$

$\Rightarrow H_p(\mathcal{C}, G) \cong H_p(\mathcal{D}, G)$

Coro 8'. $\cdots \Rightarrow H^p(\mathcal{C}, G) \cong H^p(\mathcal{D}, G)$

RK. Quite trivial in the view of universal coefficient.

Thm 7'. $\cdots$, so does φ^* (i.e. all cohomology dimensions)

Pf. $\boxed{\text{Step 1'}}$ (Lemma 45.3) $0 \to \mathcal{C}_1 \xrightarrow{f} \mathcal{C}_2 \to \mathcal{C}_3 \to 0$

Claim. $\cdots$, so does f^* (i.e. all cohomology dimensions)

$H^p(\mathcal{C}_3; G) \cong \mathrm{Hom}(H_p(\mathcal{C}_3), G) \oplus \mathrm{Ext}(H_{p-1}(\mathcal{C}_3), G)$

$= 0$ $\blacksquare$ (Thm 53.1) ($H_p(\frac{\mathcal{D}'}{\mathcal{C}}) = 0$)

$\boxed{\text{Step 2}}$ Same as Thm 7's $\boxed{\text{Step 2}}$ $\blacksquare$ (We then prove f or g induce g_* isomorphism)

$\boxed{\text{Pf of Step 2}}$: $\mathcal{D}'$ is direct: for $\frac{\mathcal{D}'}{\mathcal{D}}, \frac{\mathcal{D}'}{\mathcal{C}}$ free

$\mathcal{D}'_p$ containing $C_p \oplus D_p$ as a direct summand $c_p \mapsto (0, 0, c_p)$

and to commute up to a homotopy, we need $C_p \to D_{p+1}$

thus D'_p containing C_{p-1} as a direct summand (Motivation algebraic)

thus we set $D'_p = C_p \oplus D_p \oplus C_{p-1}$, bound any

$\partial'_p(c_p, d_p, c_{p-1}) = (\partial c_p, \partial d_p, 0) + (-c_{p-1}, \varphi(c_{p-1}), \partial c_{p-1})$

(Such the second also construct directly force it commute)

Then $\partial'^2 = 0$ $\frac{\mathcal{D}'}{\mathcal{D}}, \frac{\mathcal{D}'}{\mathcal{C}}$ free, $j \circ \varphi \simeq i$

(From the construction directly)

$\boxed{\text{Claim}}$ $(j)_*$ is isomorphism in homology level

$\Rightarrow f = g \circ \varphi$, $f_* = (g \circ \varphi)_* = g_* \circ \varphi_*$ also, then by (or $\varphi_* \circ g_*$) step 1 $\blacksquare$

Pf the Claim similiar to Step 1, it suffices to prove

$H_p(\frac{\mathcal{D}'}{\mathcal{D}}) = 0$, by $C_p(\frac{\mathcal{D}'}{\mathcal{D}}) = \frac{D'_p}{D_p} = C_p \oplus C_{p-1}$

if $(c_p, c_{p-1}) \in \ker \partial'$, i.e. $(\partial c_p - c_{p-1}, -\partial c_{p-1}) = 0$

$\Rightarrow \partial c_p = c_{p-1} \Rightarrow (c_p, c_{p-1}) = (c_p, \partial c_p) = -\partial'(0, c_p)$

$\Rightarrow \ker \partial' = \mathrm{Im}\,\partial' \Rightarrow H_p(\mathcal{D}') = 0$ $\blacksquare$ $\in \mathrm{Im}\,\partial'$

Then we focus on when we changing these G, i.e. given a chain of G, we'll introducing Bockstein's homomorphism and Bockstein's spectral sequence next (Construction: Bicomplex, exact couple, filtered, graded)

Coro 9 and 9'. Apply the Zig-zag to $0 \to \mathcal{C} \otimes G \to \mathcal{C} \otimes G' \to \mathcal{C} \otimes G'' \to 0$

and $0 \to \mathrm{Hom}(\mathcal{C}, G) \to \mathrm{Hom}(\mathcal{C}, G') \to \mathrm{Hom}(\mathcal{C}, G'') \to 0$

$\beta_*: H_p(\mathcal{C}; G'') \to H_{p-1}(\mathcal{C}; G)$

and $\beta^*: H^p(\mathcal{C}; G'') \to H^{p+1}(\mathcal{C}; G)$ are Bockstein homomorphism

Lemma 69.2 tells us their duality: It commutes with the Poincare duality up to a sign)

RK. $0 \to \mathcal{C} \otimes G \to \mathcal{C} \otimes G' \to \mathcal{C} \otimes G'' \to 0$ is just the tensor, thus the previous observation is natural and generise to spectral sequence of two sequence

RK. The motivation of Bockstein is arthmetic, the first our concern is $0 \to \mathbb{Z}_m \xrightarrow{m} \mathbb{Z}_{m^2} \to \mathbb{Z}_m \to 0$ or $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}_p \to 0$

or $0 \to \mathcal{C} \xrightarrow{p} \mathcal{C} \to \mathcal{C} \otimes \mathbb{Z}_p \to 0$ for $\mathcal{C}$ free (Tensor the second one); for topology, we can recover $\mathbb{Z}$-coeffiencent from $\mathbb{Z}_p$-coeffiencent, this is nontrivial, $\mathbb{Z}_p$-coeffiencent carries less information than $\mathbb{Z}$: we can always do by universal coeffiencent.

As the discussion upper, given $0 \to \mathcal{C} \xrightarrow{p} \mathcal{C} \to \mathcal{C} \otimes \mathbb{Z}_p \to 0$ with its long exact sequence, then any three countinou objects have two in common (graded by suspension $\Sigma \mathcal{C}$)

Pf of Step 1 (rewised) $0 \to \mathcal{C}_1 \to \mathcal{C}_2 \to \mathcal{C}_3 \to 0$

We showed that $H_p(\mathcal{C}_3, G) = 0 \Leftrightarrow H_p(\mathcal{C}_1, G) \cong H_p(\mathcal{C}_2, G)$

$H_p(\mathcal{C}_3) = 0 \Leftrightarrow H_p(\mathcal{C}_1) \cong H_p(\mathcal{C}_2)$.

Pf. $(\mathcal{C}_3)_p = B_p \oplus U_p$

(B_p is the boundary (and by $H_p(\mathcal{C}_3) = 0 \Leftrightarrow B_p = Z_p$)

$(\mathcal{C}_3)_p$ is the chain

U_p is by standard basis theorm: $(\mathcal{C}_3)_p = Z_p \oplus U_p$ ($Z_p = B_p$)

and $\partial_p(U_p)$ is isomorphic $(\partial_p|_{U_p})$

$\Rightarrow (\mathcal{C}_3)_p \otimes G = (B_p \otimes G) \oplus (U_p \otimes G)$

and $\partial_p \otimes 1_G (B_p \otimes G)$

$\partial B_p \otimes 1_G G = 0 \otimes G = 0$; $(\partial_p \otimes 1_G)(U_p \otimes G) = \partial_p(U_p) \otimes G$ isomorphism

$\Rightarrow B_p \otimes G = \tilde{Z}_p$, $U_p \otimes G = \tilde{U}_p$

$(\mathcal{C}_3 \otimes G)_p = \tilde{Z}_p \oplus \tilde{U}_p = \tilde{B}_p \oplus \tilde{U}_p$

$\Rightarrow \tilde{Z}_p = \tilde{B}_p \Rightarrow H_p(\mathcal{C}_3; G) = 0$ $\blacksquare$

(For its dual analogue, only note that

$\mathrm{Hom}((\mathcal{C}_3)_p, G) \cong \mathrm{Hom}(B_p, G) \oplus \mathrm{Hom}(U_p, G)$

$\cong \tilde{Z}_p \oplus \tilde{U}_p \cong \tilde{B}_p \oplus \tilde{U}_p$ $\blacksquare$

On the homology theory of Grassmannian 周文夫 22302130126

Abstract. The Grassmannian is central to the study of Schubert calculus, K-theory, Here we'll mainly to do the Schubert cell decomposition and compute the homology by CW-complex; And then we will deal with the cohomology especially computing its cohomology ring; at last, we skeleeth another computation of the (co)homology theory of it, and introduce some applications and open problems. We summarize the course by study, compute it.

Notice: ① All homology has coeffienent $\mathbb{Z}$ for simplicity, we'll not concern its relation with de Rham or Dolbeault cohomology
② We admit that: Grassmannian is a manifold, thus homology manifold, it's also a variety by Plücker embedding/relation
③ Some details, especially these details on combinations, we'll omit them
④ Theorems as Leray-Hirsch, Poincare duality.. ; technique as Leray-Serre spectral sequence, Chern class ... will be used without preparations
⑤ It's sad that this thesis will not contain some related topics, such as the classifying space, enumerative geometry ——
⑥ The Grassmannian we denote $Gr(n,m)$ here, and usually we let it over $\mathbb{C}$, sometimes $\mathbb{R}$ or others ($m=\infty$ is admitted)

Def 1. (Grassmannian) For $n\le m\in\mathbb{Z}_{\ge1}$, The Grassmannian $Gr(n,V)=\{\Lambda\subset V \text{ as subspace} \mid \dim\Lambda=n\}$, where $\dim V=m$
$Gr(n,m)$ over $\mathbb{K}$ is defined to be $Gr(n,\mathbb{K}^m)$; $Gr(n,\infty)\overset{def}{=}\bigcup_m Gr(n,m)=\varinjlim_m Gr(n,m)$
By Notice②, we have it's a manifold, and variety

PART I The CW-structure and Homology for the grassmannian

In this part, we make it clear for the CW-structure of $G(n,m)$, but we find it hard to give $H_pG(n,m)$ in general, thus we only compute $G(2,4)$ in particular, explictly: the general case of coeffiecent G still open now, the boundary map is almost a pure combinatoric problem

Def 2. (Schubert symbol) We need to encode the elements $\Lambda\in G(n,m)$ by some tuples
Let $\Lambda=\mathrm{span}\{v_1\cdots v_n\}$, thus Λ can be represented as a $(n\times m)$-matrix A by linear algebra, we can make it equivalent a row-reduced matrix, then, we define the Schubert symbol $\lambda(A)=(\lambda_1\cdots\lambda_n)\in\mathbb{Z}^n$ a index, where λ_i represents the column coordinate of "1" in the ith row vector. $\lambda(\Lambda)=\lambda(A)$ doesn't depend on A by our linear algebra, denoted as λ for convience

Example 3. $\Lambda\subset G(3,7)$ a plane has the martix row-reduced as $\begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0\\ * & * & 0 & 1 & 0 & 0 & 0\\ * & * & 0 & 0 & * & 1 & 0\end{bmatrix}$ then $\lambda(A)=(3,4,6)$

Def 4. (Schubert cell) $\Omega_\lambda^\circ=\{\Lambda\in Gr(n,m)\mid\lambda(\Lambda)=\lambda\}$ is the Schubert cell
Showing $\{\Omega_\lambda^\circ\}_\lambda$ form a celluar decomposition isn't a trivial thing

Lemma 5. $\Omega_\lambda^\circ\approx\mathbb{D}^{\sum_{i\le n}(\lambda_i-i)}$ the open disk of dimension $\sum_{i\le n}(\lambda_i-i)$, when we consider it over $\mathbb{R}$ or $\mathbb{C}$

Pf. $\dim\Omega_\lambda^\circ$ = The number of "$*$" in a generic palane $\Lambda\in\Omega_\lambda^\circ=(\lambda_1-1)+\cdots+(\lambda_n-n)$
And homeomphism just mapping these "$*$"s ☒ (A skeleeth, the details are contained in Thm6)
Then it's natural to consider $(\Omega_\lambda^\circ)^c$ closed: $(\Omega_\lambda^\circ)^c=\{\Lambda\in Gr(n,m)\mid\forall t\le n, \dim(\Lambda\cap F_{\lambda_t})=t, \dim(\Lambda\cap F_i)<t \text{ for } i<\lambda_t\}^c$, where F_λ is the λth flag of $\mathbb{R}^m$ or $\mathbb{C}^m$
$=\{\Lambda\in Gr(n,m)\mid\forall t\le n: \dim(V\cap F_{\lambda_t})\ge t\}$ is Zariski-closed, it's just the Schubert variety
Next is our main consequence

Thm 6. $\{\Omega_\lambda^\circ\}_\lambda$ gives a CW-structure on $G(n,m)$, let us assume it over $\mathbb{C}$ here

Pf. We deal with A, choose A such that it has all row vectors orthonormal, such form is uniquely corresponding to each Λ, by our linear algebra; we denote it as $\widetilde{\Lambda}$, Ignoring the last zeros after the λ_ith coordinate of ith row, then every row is an element in a closed hemisphere $H_i\subset S^{2(\lambda_i-1)}\subset\mathbb{C}^{\lambda_i}\subset\mathbb{C}^m$: $H_i=\{x\in S^{2(\lambda_i-1)}\mid x_{\lambda_i}\ge0\}$ closed upper hemisphere

Example 7. The same as Example 3, $\widetilde{\Lambda}=\begin{bmatrix} * & * & 1 & & & \\ * & * & 0 & 1 & & \\ * & * & 0 & 0 & * & 1\end{bmatrix}$ and $v_1=(*,*,1)\in H_1\subset S^2$
$v_2=(*,*,0,1)\in H_2\subset S^3$
$v_3=(*,*,0,0,*,1)\in H_3\subset S^5$

We define $E_\lambda=\{(v_1\cdots v_n)\in(S^{2\lambda_m-1})^n\mid v_1\cdots v_n \text{ are orthonormal}, v_i\in H_i\}$
I claim: ① $\Omega_\lambda^\circ\approx E_\lambda^\circ$, ② $E_\lambda\approx H_1^{(0)}\times H_1^{(1)}\times\cdots\times H_1^{(n-1)}$, ③ $H_1^{(i)}\approx D^{(\lambda_2-1)}$; where $H_1^{(i)}\subset S^{\lambda_1^{(i)}-1}$, $\lambda^{(i)}=(\lambda_{i+1}-i,\cdots,\lambda_n-i)$
Then complete the proof, next we prove the claim ①②③

Pf of the claim: ① $\Omega_\lambda^\circ\cong E_\lambda^\circ$ as set is obvious: we just change the place of tuples; to see the continuity we do by consider the topology of $Gr(n,m)$ as the quotient space of n-truples of orthonormal vectors in $\mathbb{C}^m$ //
② We have $E_\lambda\xrightarrow{P}H_1=H_1^{(0)}$ the projection $(v_1\cdots v_n)\mapsto v_1$, as our usual technique in Algebraic Geometry, we choose "a general fibre" $\pi^{-1}(0\cdots0,1)$, where $v_0=(0,\cdots,0,1)\in\mathbb{C}^{\lambda_1}$, let $\lambda^{(1)}=(\lambda_2-1,\cdots\lambda_n-1)$, then: $E_\lambda\approx H_1^{(0)}\times\pi^{-1}(v_0)\approx H_1^{(0)}\times E_{\lambda^{(1)}}$

By induction, and definition of $H_i^{(j)} \subset S^{\lambda_{i-1}}$, $x_i^{(j)} = (\lambda_{t+1}-i, \cdots \lambda_n - i)$ the hemisphere, then $E_\lambda \approx H_1^{(0)} \times H_1^{(1)} \times \cdots H_1^{(n-1)}$ //

③ (And note that the choice of fibre v_0 isn't matter: as in a hemisphere, we can by rotation, which is a homeomorphism)

③ Now directly deal with $D_\lambda = D_{\Sigma n}^{\sum(\lambda_i - i)}$, we have the characteristic map $\Phi_\lambda : D_\lambda \to Gr(n,m)$ to be the composition:

$D_\lambda \to H_1^{(0)} \times H_1^{(1)} \times \cdots \times H_1^{(n-1)} \xrightarrow{②} E_\lambda \xrightarrow{①} G(n,m)$, and $\Phi_\lambda|_{\mathring{D}_\lambda} : \mathring{D}_\lambda \to \mathring{H}_1^{(0)} \times \mathring{H}_1^{(1)} \times \cdots \times \mathring{H}_1^{(n-1)} \xrightarrow{②} \mathring{E}_\lambda \xrightarrow{①} G(n,m)$

$\Phi_\lambda|_{\partial D_\lambda} : \partial D_\lambda \to \partial H_1^{(0)} \times \cdots \times \partial H_1^{(n-1)} \longrightarrow G(n,m)$, where $\partial H_i = S^{i-2}$ decrease the dimension

And we show Φ_λ homeomorphism inductively: denote X^i as the i-skeleton

Attaching every $(i+1)$-cell $e_\lambda^i \in X^{i+1}$ via $\Phi_\lambda|_{\partial D_\lambda} : \partial D_\lambda \to X^i$, the resulting space $Y = \partial D_\lambda \sqcup_{\Phi_\lambda} X^i \approx X^i$ naturally

$\Rightarrow X^{i+1}$ also a CW-subcomplex // ☒ (For $m=\infty$, this way still makes sense)

Such an abstract proof only shows how to compute the $H(Gr(n,m))$ vaguely, indeed the computation of a concrete formula of $H_p(Gr(n,m))$ is still open problem, combinatorially. (Although for any Grassmannian, the algorithm is available)

Next we compute $Gr(2,4)$, the trivialest Grassmannian for example:

Example 8. $H_p(Gr(2,4)) = \begin{cases} 0 ; & p = 0,3, \geq 5 \\ \mathbb{Z}/2\mathbb{Z} ; & p = 1,2 \\ \mathbb{Z} ; & p = 4 \end{cases}$ it follows that $Gr(2,4)$ is orientible

Computation: By thm 6. $Gr(2,4)$ has a 0-cell, a 1-cell, two 2-cell, a 3-cell, a 4-cell, we describle them later

$\Rightarrow 0 \to \mathbb{Z} \xrightarrow{\partial_4} \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \longrightarrow 0$, the computation of ∂_p by attach cells into X^{p-1} as in the proof of Thm 6.

$p=1$: $H_1(Gr(2,4)) = \frac{\operatorname{Ker}\partial_1}{\operatorname{Im}\partial_2}$, I claim: $\operatorname{Ker}\partial_1 = \mathbb{Z}$, $\operatorname{Im}\partial_2 = 2\mathbb{Z}$ by $\begin{cases} \operatorname{Ker}\partial_1 = \mathbb{Z} \text{ is trivial, we attach 1-cell to 0-cell form } S^1 \\ \operatorname{Im}\partial_2 = 2\mathbb{Z} \text{ by } \begin{cases} \text{The first 2-cell consisting all lines in plane not meeting the 0-cell} \\ \text{The second 2-cell consisting all lines out of plane but meeting the 0-cell} \end{cases} \\ \Rightarrow \text{Their boundary are same, adding to be a 2-copy} \end{cases}$

$\Rightarrow H_1(Gr(2,4)) = \frac{\mathbb{Z}}{2\mathbb{Z}}$ //

$p=2$: $H_2(Gr(2,4)) = \frac{\operatorname{Ker}\partial_2}{\operatorname{Im}\partial_3}$. To attach 3-cell to the X^2, especially the two disjoint 2-cells, we need divide the 3-cell into two disjoint 2-cells, with the antipodal identified in the equator:

, thus the two 2-cells now are joined by a 1-cell, namely the equator, after the identification; these "new" two 2-cells, 1-cell coincide with X^2, thus the generator is just single, and order 2 by same computation in $p=2 \Rightarrow H_2(Gr(2,4)) = \frac{\mathbb{Z}}{2\mathbb{Z}}$ //

$p=3$: $H_3(Gr(2,4)) = \frac{\operatorname{Ker}\partial_3}{\operatorname{Im}\partial_4} = 0$ is trivial: by the attach process of 3-cell, the boundary, i.e. $\operatorname{Ker}\partial_3$ is just two-times of the generator of $H_2(Gr(2,4))$, homological to $0 \Rightarrow \operatorname{Ker}\partial_3 = \operatorname{Im}\partial_4$ //

$p=4$: $H_4(Gr(2,4)) = \operatorname{Ker}\partial_4 = \mathbb{Z}$ also trivial; we had shown that the boundary homological to $0 \Rightarrow \partial_4 = 0$ // ☒

PART II The cohomology ring of Grassmannian

R9. Why we not dealing the usual cohomology group here? When we compute $H^*(Gr(n,m))$ or $H^*(Gr(n,\infty))$, we apply to technique of vector bundle and Gysin sequence, which not makes sense when computing cohomology groups; and the cellular cohomology only differs with cellular homology up to $\operatorname{Hom}(\square, G)$ chain-level, thus nothing to discuss about; and $H^p(Gr(n,m))$ also can be computed by De Rham-Čech double complex, which we omit it here, but at all, its concrete formula is still open; also be careful when apply Poincare duality: not all Grassmannian orientible, the trivialest example is $\mathbb{RP}^2$

Our main task is compuation of ① $H^*(Gr(n,\infty)) = \mathbb{Z}[c_1(E^n) \cdots c_n(E^n)]$, where $c_i(E^n)$ is the ith-Chern class of the tautological bundle over $Gr(n,\infty)$; ② $H^*(Gr(n,m)) = \frac{\mathbb{Z}[c_1(E^n) \cdots c_n(E^n), c_1(Q) \cdots c_{m-n}(Q)]}{}$, where E^n still the tautological bundle, and Q defined to be $Gr(n,m) \times \frac{\mathbb{C}^m}{E^n}$; $c(E) \overset{\text{def}}{=} (c(E^n)\, c(Q) = 1)$ $1 + c_1(E) + \cdots + c_n(E)$ for the vector bundle E over some space X

Something interesting is

Conjecture 10. $H^*(Gr(n,m)) = \begin{cases} \frac{\mathbb{R}[p_1 \cdots p_{i-j}, \overline{p_1} \cdots \overline{p_j}]}{(p\overline{p} = 1)} \ (p_j \in H^{4j}(Gr(n,m), \mathbb{R}) \text{ the Pontryagin class}); & (n,m) = (2j, 2i) \\ & (n,m) = (2j, 2i+1) \\ & (n,m) = (2j+1, 2i+1) \\ \frac{\mathbb{R}[p_1 \cdots p_{i-j}, \overline{p_1} \cdots \overline{p_j}, \sigma_\omega]}{(p\overline{p} = 1, \sigma_\omega^2)} \ (\sigma_\omega \text{ is the Schubert class}); & (n,m) = (2j+1, 2i+2) \end{cases}$

We choose $\mathbb{R}$-coeffiencent here

As an imitation of $H^*(Gr(n,m))$, as $H^*(BO(n); \mathbb{R}) = \mathbb{R}[p_1 \cdots p_{[\frac{n}{2}]}]$ arises naturally

This is an example: Although we have universal coeffiencent theorem, derive G-coeffiencent from $\mathbb{Z}$-coeffiencent is still not trivial thing

Next we start to prove ①, we first do some preparations

Def 11. (Tautological bundle over (in)finite Grassmannian) $\pi: E_m^n \to Gr(n,m)$; $E_m^n = \{(\Lambda, v) \in Gr(n,m) \times \mathbb{C}^m \mid v \in \Lambda\}$

and $\pi: E^n \longrightarrow Gr(n,\infty) \underset{\text{denote}}{=} Gr(n)$; $E^n = \{(\Lambda, v) \in Gr(n) \times \mathbb{C}^\infty \mid v \in \Lambda\}$
$(\Lambda, v) \mapsto \Lambda$

it's easy to verify it's locally trivial :

Prop12. (E^n_m, π) is locally trivial for $m \le \infty$

Pf. For $m < \infty$, $\Lambda_0 \in Gr(n,m)$, $U_{[\Lambda_0]} = \{\Lambda \in Gr(n,m) \mid \Lambda \cap \Lambda_0^\perp = 0\} \in \mathcal{N}_{[\Lambda_0]}$

then $\varphi_{[\Lambda_0]}: \pi^{-1}(U_{[\Lambda_0]}) \longrightarrow U_{[\Lambda_0]} \times \Lambda_0$

$(\Lambda, v) \longmapsto (\Lambda, P_{[\Lambda_0]}(v))$, where $P_{[\Lambda_0]}: \Lambda \to \Lambda_0$ the projection
$v \mapsto P_{[\Lambda_0]}(v)$

$(\Lambda, \tilde{v} + D(\tilde{v})) \longleftarrow (\Lambda, \tilde{v})$

where $D: X \longrightarrow X^\perp$ is all continous
$\tilde{v} \longmapsto \tilde{v}^\perp$

For $m = \infty$, a convienent way is $Gr(n) = \varinjlim_m Gr(n,m)$

Here we use some way functional analytically : $\Lambda_0 \in Gr(n)$, $\exists N \ge n$: $\Lambda_0 \subset \mathbb{C}^N$; we now use $U_{[\Lambda_0]} = \{\Lambda \in Gr(n) \mid P_{[\Lambda_0]}: \Lambda \to \Lambda_0$ is surjective$\}$

then we prove it as same formula, but the continuity comes from restriction $\mathbb{C}^\infty$ to $\mathbb{C}^N$ ▨

Next, we can prove ① , without using flags, see the proof by flags , you can see Hacter for example

Lemma13. We have map $E_0^{n+1} \xrightarrow{f} Gr(n)$ and induced morphism f^*, $H^*(Gr(n)) \xrightarrow{\cong} H^*(E_0^{n+1})$ (where E_0 is defined to be the pullback of the non-zero sections quotient by the tautological bundle, i.e. $\pi: E \longrightarrow X$ then $E_0 = \pi_0^*(E/S)$, S the tautological subbundle of E^*: $S = \{(e, ze) \mid e \in E$ and $z \in \mathbb{C}\}$
$\cup$
$\pi_0 \overset{\text{denote}}{=} \pi|_{E^*}: E^* \to X$)

what's more, $H^*(Gr(n)) \xrightarrow{\pi_0^*} H^*(E_0^{n+1}) \xrightarrow{f^{*-1}} H^*(Gr(n-1))$ maps the Chern class $c_i(E^{n+1})$ to $c_i(E^n)$, $\forall i$, denote it as Φ

Pf. Step 1. Construction of f : $E_0^{n+1} \longrightarrow Gr(n)$

$(\Lambda, v) \in \mathbb{C}^\infty \times \Lambda^* \longmapsto \Lambda \cap v^\perp$, the orthogonal is respect to the Hermitian inner product as usual

$v^\perp$ is cutted by one equation, thus $\dim(\Lambda \cap v^\perp) = \dim \Lambda - 1 = n+1-1 = n$ well-defined

Consider $Gr(n,N)$ and $E_N^n \xrightarrow{\pi} Gr(n,N) \subset Gr$, to arise into chomological-level, we study the map f_N, factor it into a part of projection, another part of map preserving cohomological ring as follows : as we done in Gysin sequence

$$\begin{array}{ccc} \cdot & \uparrow f_N = f|_{E_{N,0}^{n+1}} & \uparrow f \\ & E_{N,0}^{n+1} & \subset E_0^{n+1} \end{array}$$

Let $D_0 = \{(\Lambda, v) \in Gr(n,N) \times \mathbb{C}^N \mid \Lambda \cap v^\perp = \Lambda\} \subset Gr(n,N) \times \mathbb{C}^N$, p has the structure of $\mathbb{C}^{N-n}$-bundle

$Gr(n,N) \xleftarrow{p} D_0 \xrightarrow{q} \mathbb{C}^N$

Then we have $E_{N,0}^{n+1} \longrightarrow D_0 \longrightarrow Gr(n,N)$ is precisely the f_N we want
$(\Lambda, v) \longmapsto (\Lambda \cap v^\perp, v) \mapsto \Lambda \cap v^\perp$

And $E_{N,0}^{n+1} \simeq D_0 \Longrightarrow H^*(E_{N,0}^{n+1}) \cong H^*(D_0)$. I claim : $H^*(D_0) \cong H^*(G(n,N))$ when N suffiencently large

This is by Gysin sequence, when we have $p \le 2(N-n-1)$, then $H^p(D_0) \cong H^p(G(n,N))$, then let $N \to \infty$ here

$\Longrightarrow H^*(D_0) \cong H^*(G(n))$ $\Longrightarrow H^*(E_0^{n+1}) \cong H^*(G(n))$ //

Step2. Dealing the Chern class. First consider the top Chern class $c_{n+1}(E^{n+1})$ = The Euler charistric number $e \in H^*(Gr(n))$

and we have the graph

$$\begin{array}{ccccc} u & \longmapsto & j^*u & & H^*(Gr(n+1)) \\ \cdots \longrightarrow H^*(E^{n+1}, E_0^{n+1}) & \xrightarrow{j^*} & H^*(E^{n+1}) & \xrightarrow{i^*} & H^*(E_0^{n+1}) \longrightarrow \\ & & \pi^* e \;\nwarrow \pi^* & & \uparrow \pi_0^* \\ & & & & H^*(Gr(n)) \end{array}$$

(u the fundamental class $\in H^*(E^{n+1}, E_0^{n+1})$)

$j^*u = \pi^* e$ by definition
$\Longrightarrow \pi_0^* e = i^* \pi^* e = i^* j^* u = 0$
and $c_{n+1}(E^n) = 0$
$\Longrightarrow c_{n+1}(E^{n+1}) = c_{n+1}(E^n)$

Then in general, for $i \le n$, we lift the map $f : E_0^{n+1} \to G(n)$ to $\widetilde{E_0^{n+1}} = \{((\Lambda, v), [\omega]) \in E_0^{n+1} \times \Lambda\}$ induced by $E^{n+1} \supset E_0^{n+1}$

the equalence $\sim$ is defined to be $\omega \sim \omega' \Leftrightarrow \omega - \omega' = zv, \exists z \in \mathbb{C}$

i.e. complex line

$$\begin{array}{ccc} E^{n+1} & \supset & E_0^{n+1} \\ & \pi \searrow & \downarrow \pi_0 = \pi|_{E_0^{n+1}} \\ & & Gr(n) \end{array}$$

with $\widetilde{E_0^{n+1}} \xrightarrow{\tilde{f}} Gr(n)$; where $\omega \cap v^\perp \in [\omega]$ the unique $\omega_0 \in [\omega]$ such that $\omega_0 \perp v$

$(\Lambda, v, [\omega]) \longmapsto (\Lambda \cap v^\perp, \omega_0 \cap v^\perp)$

$\searrow (\Lambda \cap v^\perp, \omega \cap v^\perp) \nearrow$
$\cap$
E^{n+1} factor through

$$\begin{array}{ccc} \widetilde{E_0^{n+1}} & \longrightarrow & E^{n+1} \\ \downarrow & \searrow \tilde{f} & \downarrow \\ E_0^{n+1} & \xrightarrow{f} & Gr(n) \end{array}$$

Thus we have a covering or lifting of vector bundles: as a pullback graph $\Longrightarrow c_i(\widetilde{E^{n+1}}) = f^* c_i(E^n)$

and $c_i(\widetilde{E^{n+1}}) = \pi_0^* c_i(E^{n+1})$ by definition of Chern class $\Longrightarrow (f^*)^{-1} \pi_0^* c_i(E^{n+1}) = (f^*)^{-1} c_i(\widetilde{E^{n+1}}) = c_i(E^n)$ ▨

Now prove ① is direct by the upper Lemma

Thm14. ① $H^*(Gr(n)) \cong \mathbb{Z}[c_1(E^n), \cdots c_n(E^n)]$ freely generated, denoted by $\mathbb{Z}[c(E^n)]$ for conviencence ② $H^*(Gr(n,m)) \cong \dfrac{\mathbb{Z}[c(E^n), c(Q)]}{(c(E^n) \cdot c(Q) = 1)}$

Pf① Induction on n , assume the result holds for n

We have Gysin sequence $\to H^p(Gr(n+1)) \longrightarrow H^{p+2n+2}(Gr(n+1)) \longrightarrow H^{p+2n+2}(E_0^{n+1}) \to H^{p+1}(Gr(n+1)) \longrightarrow \cdots$

$$\begin{array}{ccccccc} & \| & & \| & & \| \text{ Lemma13} & \| \\ \cdots \to & H^p(Gr(n+1)) & \longrightarrow & H^{p+2n+2}(Gr(n+1)) & \xrightarrow{\Phi} & H^{p+2n+2}(G(n)) \to & H^{p+1}(Gr(n+1)) \to \cdots \end{array}$$

we break into short: $0 \to H^p(Gr(n+1)) \to H^{p+2n+2}(Gr(n+1)) \xrightarrow{\Phi} H^{p+2n+2}(Gr(n)) \to 0$ by the Lemma13 ,

For p sufficently small and $p\le p_0$, $H^p(Gr(n+1))=0 \Rightarrow H^{p+2n+2}(Gr(n+1)) \xrightarrow{\Phi} H^{p+2n+2}(Gr(n))$ isomorphism
$\Rightarrow \forall x\in H^{p+2n+2}(Gr(n+1))$, $\Phi(x)=$ a polynomial of $C_1(E^{n+1})\cdots C_n(E^{n+1})$ by induction hypothesis, thus we prove it for p sufficiently small. I claim: all p hold this, we do induction again, assuming that $(p+2n)$-dimensional had been done
If we prove the claim, the main induction completes as we done $(n+1)$-case
Pf of claim: we have $0\to H^p(Gr(n+1)) \xrightarrow{\cup C_{n+1}(E^{n+1})} H^{p+2n+2}(Gr(n+1)) \xrightarrow{\Phi} H^{p+2n+2}(Gr(n)) \to 0$
$\mathrm{Ker}\,\Phi \cong H^p(Gr(n+1))$, and $\mathrm{Ker}\,\Phi=\{x-$ "a polynomial of $C_1(E^{n+1})\cdots C_n(E^{n+1})$"$\}$
$\Rightarrow x-P(C_1(E^{n+1})\cdots C_n(E^{n+1}))=y\,C_{n+1}(E^{n+1})$ for $\exists y\in H^p(Gr(n+1))$
By the secondary induction hypothesis, $y=q(C_1(E^{n+1})\cdots C_{n+1}(E^{n+1}))\Rightarrow x$ is represented as a polynomial of $C_1(E^{n+1})\cdots C_{n+1}(E^{n+1})$ and uniqueness trivially holds ▨ (This way is the double-induction)
Next we prove ②, but we make a Remark to compare $H^*(Gr(n,m))$ and $H^*(Gr(n))$

(This only holds for $\mathbb{C}$)

Rk15. Can we do $H^*(Gr(n))=H^*(\varinjlim_{m\to\infty} G(n,m))=\varprojlim H^*(G(n,m))$? We can do as $H^*(Gr(n,m))$ is torsion-free, and $H^{odd}(Gr(n,m))=0$; we should notice that the class $c(E^n)$ and $c(Q)$ arise naturally as in Leray-Hirsch theorm; and the following proof must involve the flag: a standard way to study the Grassmannian, we'll apply Whitney product formula and splitting principle
Def16. (complete flag) $\pi:E\to X$, the flag $Fl(E)=\{(x,\Lambda_0\le\Lambda_1\le\cdots\le\Lambda_n)\mid \Lambda_n\le E_x,\ \dim\Lambda_i=i\}$
Rk ⑫ $\Psi: \dfrac{\mathbb{Z}[c(E^n),c(Q)]}{(c(E^n)c(Q)-1)} \longrightarrow H^*(Gr(n,m))$ Step 1: Ψ is injective the flag has a natural structure of bundle: $Fl(Q) \xrightarrow{\sigma} X$ (with $Q\xrightarrow{\pi} X$)
we do pullback $\sigma^*Q\to Q$, then we have a graph:

$$\begin{tikzcd} Fl(\sigma^*Q) \arrow[r] \arrow[d] & \sigma^*(Q) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Q \arrow[d, "\pi"] \\ Fl(Q) \arrow[r, "\sigma"] & Fl(Q) \arrow[r, "\sigma"] & X \end{tikzcd}$$

$Fl(\sigma^*Q) \,\|\, Fl(pt\times\mathbb{C}^m)$

Let us explain $Fl(\sigma^*Q)=Fl(pt\times\mathbb{C}^m)$: $\sigma^*Q=\{(\Lambda,\Lambda_0\le\cdots\le\Lambda_n,\omega)\mid \omega\in\mathbb{C}^m/\Lambda\}$
$\cong Fl(\mathbb{C}^m/V)\cong Fl(pt\times\mathbb{C}^m)$
Now we apply the spiltting formula to both sides of the equation $Fl(\sigma^*Q)\cong Fl(pt\times\mathbb{C}^m)$ in the cohomology-level:
$H^*(Fl(pt\times\mathbb{C}^m))\cong \dfrac{\mathbb{Z}[x_1\cdots x_m]}{(\prod_{i\le m}(1+x_i)-1)}$; $H^*(Fl(\sigma^*Q))=\dfrac{H^*(Fl(E^n))[y_1\cdots y_{m-n}]}{(\prod_{j\le m-n}(1+y_j)-c(Q))}$
$\Rightarrow$ They equal to $\dfrac{\mathbb{Z}[x_1\cdots x_m]}{(\prod_{i\le m-n}(1+x_i)-c(Q),\ \prod_{i\ge m-n+1}(1+x_i)-c(E^n))} \Leftrightarrow \prod_{i\le m-n}(1+x_i)\prod_{i\ge m-n+1}(1+x_i)=\prod_{i\le m}(1+x_i)=1 \Leftrightarrow c(E^n)c(Q)=1$ the only restriction
By replace notions, we find Ψ injective //
Step 2: Ψ is surjective. We run by induction on n, Check to proof of Thm14② the equation holds without gap for $2n-1\le m$, as the only using m is taking a large $N\to\infty$ the direct limit, and for $2n-1>m$, the Grassmanian considered as a orthogonal projection we have a homeomorphism $Gr(n,m)\overset{\varphi}{\approx} Gr(m-n,m)$ and $\varphi^*E^{m-n}=Q_n$, thus nothing to prove then
$\Lambda\mapsto\Lambda^\perp$, $\Lambda^\perp\leftarrow\!\!\mapsto\Lambda$ $\qquad \varphi^*Q_{m-n}=E^n$
Next, we check for $2n-1\le m$: consider Gysin sequence, again: $\cdots\to H^{p-2n}(Gr(n,m))\to H^p(Gr(n,m))\xrightarrow{\pi_0^*} H^p(E_0^n)\to H^{p-2n+1}(Gr(n,m))$
(Here we use $(p-2n,p)$ instead of $(p,p+2n)$ for later compuation)
I claim: $H^p(Q_{n-1,0})\cong H^p(E_0^n)$ (Here $Q_{n-1,0}$ also means that delete the zero section)
Then

$$\begin{tikzcd} \cdots \arrow[r] & H^{p-2n}(Gr(n-1,m)) \arrow[r] & H^p(Gr(n-1,m)) \arrow[r, "\pi_0^*"] & H^p(Q_{n-1,0}) \arrow[r] \arrow[d, equal] & H^{p-2n+1}(Gr(n,m)) \arrow[r] & \cdots \\ \cdots \arrow[r] & H^{p-2n}(Gr(n,m)) \arrow[r] & H^p(Gr(n,m)) \arrow[r, "\pi_0^*"] & H^p(E_0^n) \arrow[r] & H^{p-2n+1}(Gr(n,m)) \arrow[r] & \cdots \end{tikzcd}$$

This graph relates the induction hypothesis, the two Chern classes, we have these formulas naturally:
$\pi_0^*C_i(E^n)=C_i(E_0^n)$; $\pi_0^*C_j(Q_{n-1})=C_j(Q_{n-1,0})=(f^{-1})^*\pi_0^*C_j(Q_n)$
$=(f^{-1})^*(\pi_0)^*C_j(E^{n-1})$ Then all the same as the Pf of Thm14① //
Left is proving the claim: indeed $Q_{n-1,0}\approx E_0^n$ given by $f:E_0^n\to Q_{n-1,0}$ $\quad g:Q_{n-1,0}\to E_0^n$
$(\Lambda,v)\mapsto(\Lambda\cap v^\perp,v)$ $\quad (\Lambda,w)\mapsto(\Lambda\oplus w,w)$ ▨
RK17. $Gr(n,m)$ is a homogenous space, by Leray-Serre spectral sequence also makes sense

PART III. Application: coeffiencent $\mathbb{Z}/2\mathbb{Z}$

Thm18. $H^*(Gr(n),\mathbb{Z}/2\mathbb{Z})\cong\mathbb{Z}/2\mathbb{Z}[t_1\cdots t_n]$, when $Gr(n)$ over $\mathbb{C}$, these generators coincide the Chern class
Pf. Cubum autem in duos cubos, aut quadratoquadratum in duos cubos quadratoquadratos. Et médiévalgeneraliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.
Skeleton of the proof, the natural map $h_n:(\mathbb{RP}^\infty)^n\to Gr(n)$ as the classfying map, the cohomology level $h_n^*:H^*(Gr(n))\to H^*((\mathbb{RP}^\infty)^n)$ and $H^*((\mathbb{RP}^\infty)^n)\cong\frac{\mathbb{Z}}{2\mathbb{Z}}[x_1\cdots x_n]$ is done, we need to prove that ① h_n^* injective ② $\mathrm{Im}\,h_n^*$ generated by n symmetric functions ▨

Reference. Hacter; Charistristic class by Milnor