

Rules for integrands of the form $(e x)^m (a + b x^n)^p (c + d x^n)^q$

0. $\int (e x)^m (b x^n)^p (c + d x^n)^q dx$

1. $\int (e x)^m (b x^n)^p (c + d x^n)^q dx$ when $m \in \mathbb{Z} \vee e > 0$

1: $\int (e x)^m (b x^n)^p (c + d x^n)^q dx$ when $(m \in \mathbb{Z} \vee e > 0) \wedge \frac{m+1}{n} \in \mathbb{Z}$

Derivation: Algebraic expansion and integration by substitution

Basis: If $\frac{m+1}{n} \in \mathbb{Z}$, then $x^m (b x^n)^p = \frac{1}{b^{\frac{m+1}{n}-1}} x^{n-1} (b x^n)^{p+\frac{m+1}{n}-1}$

Basis: $x^{n-1} F[x^n] = \frac{1}{n} \text{Subst}[F[x], x, x^n] \partial_x x^n$

Rule 1.1.3.4.0.1.1: If $(m \in \mathbb{Z} \vee e > 0) \wedge \frac{m+1}{n} \in \mathbb{Z}$, then

$$\int (e x)^m (b x^n)^p (c + d x^n)^q dx \rightarrow \frac{e^m}{n b^{\frac{m+1}{n}-1}} \text{Subst}\left[\int (b x)^{p+\frac{m+1}{n}-1} (c + d x)^q dx, x, x^n\right]$$

Program code:

```
Int[(e_.**x_)^m_.*(b_.**x_^n_)^p_*(c_+d_.**x_^n_)^q_,x_Symbol] :=
  e^m/(n*b^(Simplify[(m+1)/n]-1))*Subst[Int[(b*x)^(p+Simplify[(m+1)/n]-1)*(c+d*x)^q,x],x,x^n] /;
FreeQ[{b,c,d,e,m,n,p,q},x] && (IntegerQ[m] || GtQ[e,0]) && IntegerQ[Simplify[(m+1)/n]]
```

$$\mathbf{2:} \int (e x)^m (b x^n)^p (c + d x^n)^q dx \text{ when } (m \in \mathbb{Z} \vee e > 0) \wedge \frac{m+1}{n} \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{(b x^n)^p}{x^{n p}} == 0$$

Rule 1.1.3.4.0.1.2: If $(m \in \mathbb{Z} \vee e > 0) \wedge \frac{m+1}{n} \notin \mathbb{Z}$, then

$$\int (e x)^m (b x^n)^p (c + d x^n)^q dx \rightarrow \frac{e^m b^{\text{IntPart}[p]} (b x^n)^{\text{FracPart}[p]}}{x^{n \text{FracPart}[p]}} \int x^{m+n p} (c + d x^n)^q dx$$

Program code:

```
Int[(e_.**x_)^m_.*(b_.**x_^n_)^p_*(c_+d_.**x_^n_)^q_,x_Symbol] :=
  e^m*b^IntPart[p]*(b*x^n)^FracPart[p]/x^(n*FracPart[p])*Int[x^(m+n*p)*(c+d*x^n)^q,x] /;
FreeQ[{b,c,d,e,m,n,p,q},x] && (IntegerQ[m] || GtQ[e,0]) && Not[IntegerQ[Simplify[(m+1)/n]]]
```

2: $\int (e x)^m (b x^n)^p (c + d x^n)^q dx$ when $m \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(e x)^m}{x^m} == 0$

Rule 1.1.3.4.0.2: If $m \notin \mathbb{Z}$, then

$$\int (e x)^m (b x^n)^p (c + d x^n)^q dx \rightarrow \frac{e^{\text{IntPart}[m]} (e x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (b x^n)^p (c + d x^n)^q dx$$

Program code:

```
Int[(e*x_)^m*(b_*x_^n_)^p*(c_+d_*x_^n_)^q_,x_Symbol] :=
  e^IntPart[m]*(e*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(b*x^n)^p*(c+d*x^n)^q,x] /;
FreeQ[{b,c,d,e,m,n,p,q},x] && Not[IntegerQ[m]]
```

$$\text{E1. } \int \frac{x^m}{(a+b x^2)^{1/4} (c+d x^2)} dx \text{ when } b c - 2 a d == 0 \wedge m \in \mathbb{Z} \wedge (a > 0 \vee \frac{m}{2} \in \mathbb{Z})$$

$$\text{1: } \int \frac{x}{(a+b x^2)^{1/4} (c+d x^2)} dx \text{ when } b c - 2 a d == 0 \wedge a > 0$$

Note: The result is real and continuous when the integrand is, and substitution $u \rightarrow x^2$ results in 2 inverse trig and 2 log terms.

Rule 1.1.3.4.E1.1: If $b c - 2 a d == 0 \wedge a > 0$, then

$$\int \frac{x}{(a+b x^2)^{1/4} (c+d x^2)} dx \rightarrow -\frac{1}{\sqrt{2} a^{1/4} d} \text{ArcTan}\left[\frac{\sqrt{a} - \sqrt{a+b x^2}}{\sqrt{2} a^{1/4} (a+b x^2)^{1/4}}\right] - \frac{1}{\sqrt{2} a^{1/4} d} \text{ArcTanh}\left[\frac{\sqrt{a} + \sqrt{a+b x^2}}{\sqrt{2} a^{1/4} (a+b x^2)^{1/4}}\right]$$

Program code:

```
Int[x_/((a+b_.**x_^2)^(1/4)*(c+d_.**x_^2)),x_Symbol] :=
  -1/(Sqrt[2]*Rt[a,4]*d)*ArcTan[(Rt[a,4]^2-Sqrt[a+b*x^2])/(Sqrt[2]*Rt[a,4]*(a+b*x^2)^(1/4))] -
  1/(Sqrt[2]*Rt[a,4]*d)*ArcTanh[(Rt[a,4]^2+Sqrt[a+b*x^2])/(Sqrt[2]*Rt[a,4]*(a+b*x^2)^(1/4))] /;
FreeQ[{a,b,c,d},x] && EqQ[b*c-2*a*d,0] && PosQ[a]
```

2: $\int \frac{x^m}{(a+b x^2)^{1/4} (c+d x^2)} dx$ when $b c - 2 a d == 0 \wedge m \in \mathbb{Z} \wedge (a > 0 \vee \frac{m}{2} \in \mathbb{Z})$

Rule 1.1.3.4.E1.2: If $b c - 2 a d == 0 \wedge m \in \mathbb{Z} \wedge (a > 0 \vee \frac{m}{2} \in \mathbb{Z})$, then

$$\int \frac{x^m}{(a+b x^2)^{1/4} (c+d x^2)} dx \rightarrow \int \text{ExpandIntegrand}\left[\frac{x^m}{(a+b x^2)^{1/4} (c+d x^2)}, x\right] dx$$

Program code:

```
Int[x_^m_/((a_+b_.*x_^2)^(1/4)*(c_+d_.*x_^2)),x_Symbol] :=
  Int[ExpandIntegrand[x^m/((a+b*x^2)^(1/4)*(c+d*x^2)),x],x] /;
FreeQ[{a,b,c,d},x] && EqQ[b*c-2*a*d,0] && IntegerQ[m] && (PosQ[a] || IntegerQ[m/2])
```

$$\text{E2. } \int \frac{x^m}{(a+b x^2)^{3/4} (c+d x^2)} dx \text{ when } b c - 2 a d == 0 \wedge m \in \mathbb{Z} \wedge (a > 0 \vee \frac{m}{2} \in \mathbb{Z})$$

$$1. \int \frac{x^2}{(a+b x^2)^{3/4} (c+d x^2)} dx \text{ when } b c - 2 a d == 0$$

$$\text{1: } \int \frac{x^2}{(a+b x^2)^{3/4} (c+d x^2)} dx \text{ when } b c - 2 a d == 0 \wedge \frac{b^2}{a} > 0$$

Reference: Eneström index number E688 in The Euler Archive

Rule 1.1.3.4.E2.1.1: If $b c - 2 a d == 0 \wedge \frac{b^2}{a} > 0$, then

$$\int \frac{x^2}{(a+b x^2)^{3/4} (c+d x^2)} dx \rightarrow -\frac{b}{a d \left(\frac{b^2}{a}\right)^{3/4}} \text{ArcTan}\left[\frac{b + \sqrt{\frac{b^2}{a}} \sqrt{a+b x^2}}{\left(\frac{b^2}{a}\right)^{3/4} x (a+b x^2)^{1/4}}\right] + \frac{b}{a d \left(\frac{b^2}{a}\right)^{3/4}} \text{ArcTanh}\left[\frac{b - \sqrt{\frac{b^2}{a}} \sqrt{a+b x^2}}{\left(\frac{b^2}{a}\right)^{3/4} x (a+b x^2)^{1/4}}\right]$$

Program code:

```
Int[x^2/((a+b_.**x^2)^(3/4)*(c+d_.**x^2)),x_Symbol] :=
  -b/(a*d*Rt[b^2/a,4]^3)*ArcTan[(b+Rt[b^2/a,4]^2*Sqrt[a+b*x^2])/(Rt[b^2/a,4]^3*x*(a+b*x^2)^(1/4))] +
  b/(a*d*Rt[b^2/a,4]^3)*ArcTanh[(b-Rt[b^2/a,4]^2*Sqrt[a+b*x^2])/(Rt[b^2/a,4]^3*x*(a+b*x^2)^(1/4))] /;
FreeQ[{a,b,c,d},x] && EqQ[b*c-2*a*d,0] && PosQ[b^2/a]
```

$$2: \int \frac{x^2}{(a+b x^2)^{3/4} (c+d x^2)} dx \text{ when } b c - 2 a d = 0 \wedge \frac{b^2}{a} \neq 0$$

Reference: Eneström index number E688 in The Euler Archive

Derivation: Integration by substitution

Basis: If $b c - 2 a d = 0$, then $\frac{x^2}{(a+b x^2)^{3/4} (c+d x^2)} = \frac{2b}{d} \text{Subst}\left[\frac{x^2}{4a+b^2 x^4}, x, \frac{x}{(a+b x^2)^{1/4}}\right] \partial_x \frac{x}{(a+b x^2)^{1/4}}$

Rule 1.1.3.4.E2.1.2: If $b c - 2 a d = 0 \wedge \frac{b^2}{a} \neq 0$, then

$$\int \frac{x^2}{(a+b x^2)^{3/4} (c+d x^2)} dx \rightarrow \frac{2b}{d} \text{Subst}\left[\int \frac{x^2}{4a+b^2 x^4} dx, x, \frac{x}{(a+b x^2)^{1/4}}\right]$$

$$\rightarrow -\frac{b}{\sqrt{2} a d \left(-\frac{b^2}{a}\right)^{3/4}} \text{ArcTan}\left[\frac{\left(-\frac{b^2}{a}\right)^{1/4} x}{\sqrt{2} (a+b x^2)^{1/4}}\right] + \frac{b}{\sqrt{2} a d \left(-\frac{b^2}{a}\right)^{3/4}} \text{ArcTanh}\left[\frac{\left(-\frac{b^2}{a}\right)^{1/4} x}{\sqrt{2} (a+b x^2)^{1/4}}\right]$$

Program code:

```
Int[x_^2/((a_+b_.*x_^2)^(3/4)*(c_+d_.*x_^2)),x_Symbol] :=
  -b/(Sqrt[2]*a*d*Rt[-b^2/a,4]^3)*ArcTan[(Rt[-b^2/a,4]*x)/(Sqrt[2]*(a+b*x^2)^(1/4))] +
  b/(Sqrt[2]*a*d*Rt[-b^2/a,4]^3)*ArcTanh[(Rt[-b^2/a,4]*x)/(Sqrt[2]*(a+b*x^2)^(1/4))] /;
FreeQ[{a,b,c,d},x] && EqQ[b*c-2*a*d,0] && NegQ[b^2/a]
```

2: $\int \frac{x^m}{(a+b x^2)^{3/4} (c+d x^2)} dx$ when $b c - 2 a d \neq 0 \wedge m \in \mathbb{Z} \wedge (a > 0 \vee \frac{m}{2} \in \mathbb{Z})$

Rule 1.1.3.4.E2.2: If $b c - 2 a d \neq 0 \wedge m \in \mathbb{Z}$, then

$$\int \frac{x^m}{(a+b x^2)^{3/4} (c+d x^2)} dx \rightarrow \int \text{ExpandIntegrand}\left[\frac{x^m}{(a+b x^2)^{3/4} (c+d x^2)}, x\right] dx$$

Program code:

```
Int[x_^m_/((a_+b_.*x_^2)^(3/4)*(c_+d_.*x_^2)),x_Symbol] :=
  Int[ExpandIntegrand[x^m/((a+b*x^2)^(3/4)*(c+d*x^2)),x],x] /;
FreeQ[{a,b,c,d},x] && EqQ[b*c-2*a*d,0] && IntegerQ[m] && (PosQ[a] || IntegerQ[m/2])
```

1: $\int x^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge m - n + 1 \neq 0$

Derivation: Integration by substitution

Basis: $x^{n-1} F[x^n] = \frac{1}{n} \text{Subst}[F[x], x, x^n] \partial_x x^n$

Rule 1.1.3.4.1: If $b c - a d \neq 0 \wedge m - n + 1 \neq 0$, then

$$\int x^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \frac{1}{n} \text{Subst}\left[\int (a+b x)^p (c+d x)^q dx, x, x^n\right]$$

Program code:

```
Int[x_^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  1/n*Subst[Int[(a+b*x)^p*(c+d*x)^q,x],x,x^n] /;
FreeQ[{a,b,c,d,m,n,p,q},x] && NeQ[b*c-a*d,0] && EqQ[m-n+1,0]
```


2: $\int x^m (a + b x^n)^p (c + d x^n)^q dx$ when $b c - a d \neq 0 \wedge (p \mid q) \in \mathbb{Z} \wedge n < 0$

Derivation: Algebraic expansion

Basis: If $p \in \mathbb{Z}$, then $(a + b x^n)^p = x^{n p} (b + a x^{-n})^p$

Rule 1.1.3.4.2: If $b c - a d \neq 0 \wedge (p \mid q) \in \mathbb{Z} \wedge n < 0$, then

$$\int x^m (a + b x^n)^p (c + d x^n)^q dx \rightarrow \int x^{m+n(p+q)} (b + a x^{-n})^p (d + c x^{-n})^q dx$$

Program code:

```
Int[x_^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  Int[x^(m+n*(p+q))*(b+a*x^(-n))^p*(d+c*x^(-n))^q,x] /;
FreeQ[{a,b,c,d,m,n},x] && NeQ[b*c-a*d,0] && IntegersQ[p,q] && NegQ[n]
```

$$3. \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge \frac{m+1}{n} \in \mathbb{Z}$$

$$1: \int x^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge \frac{m+1}{n} \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If $\frac{m+1}{n} \in \mathbb{Z}$, then $x^m F[x^n] = \frac{1}{n} \text{Subst}[x^{\frac{m+1}{n}-1} F[x], x, x^n] \partial_x x^n$

Note: If $n \in \mathbb{Z} \wedge \frac{m+1}{n} \in \mathbb{Z}$, then $m \in \mathbb{Z}$, and $(e x)^m$ automatically evaluates to $e^m x^m$.

Rule 1.1.3.4.3.1: If $b c - a d \neq 0 \wedge \frac{m+1}{n} \in \mathbb{Z}$, then

$$\int x^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \frac{1}{n} \text{Subst}\left[\int x^{\frac{m+1}{n}-1} (a+b x)^p (c+d x)^q dx, x, x^n\right]$$

Program code:

```
Int[x_^m.*(a+b_.*x_^n_)^p.*(c+d_.*x_^n_)^q_,x_Symbol] :=
  1/n*Subst[Int[x^(Simplify[(m+1)/n]-1)*(a+b*x)^p*(c+d*x)^q,x],x,x^n] /;
FreeQ[{a,b,c,d,m,n,p,q},x] && NeQ[b*c-a*d,0] && IntegerQ[Simplify[(m+1)/n]]
```

2: $\int (e x)^m (a + b x^n)^p (c + d x^n)^q dx$ when $b c - a d \neq 0 \wedge \frac{m+1}{n} \in \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(e x)^m}{x^m} == 0$

Basis: $\frac{(e x)^m}{x^m} == \frac{e^{\text{IntPart}[m]} (e x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}}$

Rule 1.1.3.4.3.2: If $b c - a d \neq 0 \wedge \frac{m+1}{n} \in \mathbb{Z}$, then

$$\int (e x)^m (a + b x^n)^p (c + d x^n)^q dx \rightarrow \frac{e^{\text{IntPart}[m]} (e x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (a + b x^n)^p (c + d x^n)^q dx$$

Program code:

```
Int[(e*x_)^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  e^IntPart[m]*(e*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a+b*x^n)^p*(c+d*x^n)^q,x] /;
FreeQ[{a,b,c,d,e,m,n,p,q},x] && NeQ[b*c-a*d,0] && IntegerQ[Simplify[(m+1)/n]]
```

4: $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge (p \mid q) \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule 1.1.3.4.4: If $b c - a d \neq 0 \wedge (p \mid q) \in \mathbb{Z}^+$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \int \text{ExpandIntegrand}[(e x)^m (a+b x^n)^p (c+d x^n)^q, x] dx$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  Int[ExpandIntegrand[(e*x)^m*(a+b*x^n)^p*(c+d*x^n)^q,x],x] /;
  FreeQ[{a,b,c,d,e,m,n},x] && NeQ[b*c-a*d,0] && IGtQ[p,0] && IGtQ[q,0]
```

5. $\int (e x)^m (a+b x^n)^p (c+d x^n) dx$ when $b c - a d \neq 0$

1: $\int (e x)^m (a+b x^n)^p (c+d x^n) dx$ when $b c - a d \neq 0 \wedge a d (m+1) - b c (m+n(p+1)+1) = 0 \wedge m \neq -1$

Derivation: Trinomial recurrence 2b with $c = 0$ and $a d (m+1) - b c (m+n(p+1)+1) = 0$

Rule 1.1.3.4.5.1: If $b c - a d \neq 0 \wedge a d (m+1) - b c (m+n(p+1)+1) = 0 \wedge m \neq -1$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n) dx \rightarrow \frac{c (e x)^{m+1} (a+b x^n)^{p+1}}{a e (m+1)}$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_),x_Symbol] :=
  c*(e*x)^(m+1)*(a+b*x^n)^(p+1)/(a*e*(m+1)) /;
  FreeQ[{a,b,c,d,e,m,n,p},x] && NeQ[b*c-a*d,0] && EqQ[a*d*(m+1)-b*c*(m+n*(p+1)+1),0] && NeQ[m,-1]
```

```

Int[(e_.**x_)^m_.*(a1_+b1_.*x_^non2_.)^p_.*(a2_+b2_.*x_^non2_.)^p_.*(c_+d_.*x_^n_),x_Symbol] :=
  c*(e*x)^(m+1)*(a1+b1*x^(n/2))^(p+1)*(a2+b2*x^(n/2))^(p+1)/(a1*a2*e*(m+1)) /;
FreeQ[{a1,b1,a2,b2,c,d,e,m,n,p},x] && EqQ[non2,n/2] && EqQ[a2*b1+a1*b2,0] && EqQ[a1*a2*d*(m+1)-b1*b2*c*(m+n*(p+1)+1),0] && NeQ[m,-1]

```

2. $\int (e x)^m (a+b x^n)^p (c+d x^n) dx$ when $b c - a d \neq 0 \wedge m+n(p+1)+1=0$

1: $\int (e x)^m (a+b x^n)^p (c+d x^n) dx$ when $b c - a d \neq 0 \wedge m+n(p+1)+1=0 \wedge (n \in \mathbb{Z} \vee e > 0) \wedge (n > 0 \wedge m < -1 \vee n < 0 \wedge m+n > -1)$

Derivation: Trinomial recurrence 3b with $c = 0$

Rule 1.1.3.4.5.2.1: If $b c - a d \neq 0 \wedge (n \in \mathbb{Z} \vee e > 0) \wedge (n > 0 \wedge m < -1 \vee n < 0 \wedge m+n > -1)$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n) dx \rightarrow \frac{c (e x)^{m+1} (a+b x^n)^{p+1}}{a e (m+1)} + \frac{d}{e^n} \int (e x)^{m+n} (a+b x^n)^p dx$$

Program code:

```

Int[(e_.**x_)^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_),x_Symbol] :=
  c*(e*x)^(m+1)*(a+b*x^n)^(p+1)/(a*e*(m+1)) + d/e^n*Int[(e*x)^(m+n)*(a+b*x^n)^p,x] /;
FreeQ[{a,b,c,d,e,m,n,p},x] && NeQ[b*c-a*d,0] && EqQ[m+n*(p+1)+1,0] && (IntegerQ[n] || GtQ[e,0]) &&
  (GtQ[n,0] && LtQ[m,-1] || LtQ[n,0] && GtQ[m+n,-1])

```

2: $\int (e x)^m (a+b x^n)^p (c+d x^n) dx$ when $b c - a d \neq 0 \wedge m+n(p+1)+1=0 \wedge m \neq -1$

Derivation: Trinomial recurrence 2b with $c = 0$

Rule 1.1.3.4.5.2.2: If $b c - a d \neq 0 \wedge m+n(p+1)+1=0 \wedge m \neq -1$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n) dx \rightarrow \frac{(b c - a d) (e x)^{m+1} (a+b x^n)^{p+1}}{a b e (m+1)} + \frac{d}{b} \int (e x)^m (a+b x^n)^{p+1} dx$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_),x_Symbol] :=
  (b*c-a*d)*(e*x)^(m+1)*(a+b*x^n)^(p+1)/(a*b*e*(m+1)) + d/b*Int[(e*x)^m*(a+b*x^n)^(p+1),x] /;
FreeQ[{a,b,c,d,e,m,n,p},x] && NeQ[b*c-a*d,0] && EqQ[m+n*(p+1)+1,0] && NeQ[m,-1]
```

3: $\int (e x)^m (a+b x^n)^p (c+d x^n) dx$ when $b c - a d \neq 0 \wedge (n \in \mathbb{Z} \vee e > 0) \wedge (n > 0 \wedge m < -1 \vee n < 0 \wedge m+n > -1)$

Derivation: Trinomial recurrence 3b with $c = 0$

Rule 1.1.3.4.5.3: If $b c - a d \neq 0 \wedge (n \in \mathbb{Z} \vee e > 0) \wedge (n > 0 \wedge m < -1 \vee n < 0 \wedge m+n > -1)$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n) dx \rightarrow \frac{c (e x)^{m+1} (a+b x^n)^{p+1}}{a e (m+1)} + \frac{a d (m+1) - b c (m+n(p+1)+1)}{a e^n (m+1)} \int (e x)^{m+n} (a+b x^n)^p dx$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_),x_Symbol] :=
  c*(e*x)^(m+1)*(a+b*x^n)^(p+1)/(a*e*(m+1)) +
  (a*d*(m+1)-b*c*(m+n*(p+1)+1))/(a*e^n*(m+1))*Int[(e*x)^(m+n)*(a+b*x^n)^p,x] /;
FreeQ[{a,b,c,d,e,p},x] && NeQ[b*c-a*d,0] && (IntegerQ[n] || GtQ[e,0]) &&
  (GtQ[n,0] && LtQ[m,-1] || LtQ[n,0] && GtQ[m+n,-1]) && Not[ILtQ[p,-1]]
```

```

Int[(e_.**x_)^m_.*(a1_+b1_.*x_^non2_.)^p_.*(a2_+b2_.*x_^non2_.)^p_.*(c_+d_.*x_^n_),x_Symbol] :=
  c*(e*x)^(m+1)*(a1+b1*x^(n/2))^(p+1)*(a2+b2*x^(n/2))^(p+1)/(a1*a2*e*(m+1)) +
  (a1*a2*d*(m+1)-b1*b2*c*(m+n*(p+1)+1))/(a1*a2*e^n*(m+1))*Int[(e*x)^(m+n)*(a1+b1*x^(n/2))^p*(a2+b2*x^(n/2))^p,x] /;
FreeQ[{a1,b1,a2,b2,c,d,e,p},x] && EqQ[non2,n/2] && EqQ[a2*b1+a1*b2,0] && (IntegerQ[n] || GtQ[e,0]) &&
  (GtQ[n,0] && LtQ[m,-1] || LtQ[n,0] && GtQ[m+n,-1]) && Not[ILtQ[p,-1]]

```

4. $\int (e x)^m (a+b x^n)^p (c+d x^n) dx$ when $b c - a d \neq 0 \wedge p < -1$

1. $\int x^m (a+b x^2)^p (c+d x^2) dx$ when $b c - a d \neq 0 \wedge p < -1 \wedge \frac{m}{2} \in \mathbb{Z} \wedge (p \in \mathbb{Z} \vee m+2p+1 = 0)$

1: $\int x^m (a+b x^2)^p (c+d x^2) dx$ when $b c - a d \neq 0 \wedge p < -1 \wedge \frac{m}{2} \in \mathbb{Z}^+ \wedge (p \in \mathbb{Z} \vee m+2p+1 = 0)$

Derivation: ???

Note: If $\frac{m}{2} \in \mathbb{Z}^+$, $b^{m/2} x^{m-2} (c+d x^2) - (-a)^{m/2-1} (b c - a d)$ is divisible by $a+b x^2$.

Note: The degree of the polynomial in the resulting integrand is m .

Note: This rule should be generalized for integrands of the form $x^m (a+b x^n)^p (c+d x^n)$.

Rule 1.1.3.4.5.4.1.1: If $b c - a d \neq 0 \wedge p < -1 \wedge \frac{m}{2} \in \mathbb{Z}^+ \wedge (p \in \mathbb{Z} \vee m+2p+1 = 0)$, then

$$\int x^m (a+b x^2)^p (c+d x^2) dx \rightarrow \frac{(-a)^{m/2-1} (b c - a d) x (a+b x^2)^{p+1}}{2 b^{m/2+1} (p+1)} + \frac{1}{2 b^{m/2+1} (p+1)} \int (a+b x^2)^{p+1} \left(\frac{2 b (p+1) x^2 (b^{m/2} x^{m-2} (c+d x^2) - (-a)^{m/2-1} (b c - a d))}{a+b x^2} - (-a)^{m/2-1} (b c - a d) \right) dx$$

Program code:

```
Int[x_^m*(a_+b_.*x_^2)^p_*(c_+d_.*x_^2),x_Symbol] :=
  (-a)^(m/2-1)*(b*c-a*d)*x*(a+b*x^2)^(p+1)/(2*b^(m/2+1)*(p+1)) +
  1/(2*b^(m/2+1)*(p+1))*Int[(a+b*x^2)^(p+1)*
    ExpandToSum[2*b*(p+1)*x^2*Together[(b^(m/2)*x^(m-2)*(c+d*x^2)-(-a)^(m/2-1)*(b*c-a*d))/(a+b*x^2)]-(-a)^(m/2-1)*(b*c-a*d),x],x] /;
FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && LtQ[p,-1] && IGtQ[m/2,0] && (IntegerQ[p] || EqQ[m+2*p+1,0])
```


$$\mathbf{2:} \int x^m (a+b x^2)^p (c+d x^2) dx \text{ when } b c - a d \neq 0 \wedge p < -1 \wedge \frac{m}{2} \in \mathbb{Z}^- \wedge (p \in \mathbb{Z} \vee m+2p+1 = 0)$$

Derivation: ???

Note: If $\frac{m}{2} \in \mathbb{Z}^-$, $b^{m/2} (c+d x^2) - (-a)^{m/2-1} (b c - a d) x^{-m+2}$ is divisible by $a+b x^2$.

Note: The degree of the polynomial in the resulting integrand is $-m$.

Note: This rule should be generalized for integrands of the form $x^m (a+b x^n)^p (c+d x^n)$.

Rule 1.1.3.4.5.4.1.2: If $b c - a d \neq 0 \wedge p < -1 \wedge \frac{m}{2} \in \mathbb{Z}^- \wedge (p \in \mathbb{Z} \vee m+2p+1 = 0)$, then

$$\int x^m (a+b x^2)^p (c+d x^2) dx \rightarrow \frac{(-a)^{m/2-1} (b c - a d) x (a+b x^2)^{p+1}}{2 b^{m/2+1} (p+1)} + \frac{1}{2 b^{m/2+1} (p+1)} \int x^m (a+b x^2)^{p+1} \left(\frac{2 b (p+1) (b^{m/2} (c+d x^2) - (-a)^{m/2-1} (b c - a d) x^{-m+2})}{a+b x^2} - (-a)^{m/2-1} (b c - a d) x^{-m} \right) dx$$

Program code:

```
Int[x_^m*(a+b_*x_^2)^p*(c+d_*x_^2),x_Symbol] :=
  (-a)^(m/2-1)*(b*c-a*d)*x*(a+b*x^2)^(p+1)/(2*b^(m/2+1)*(p+1)) +
  1/(2*b^(m/2+1)*(p+1))*Int[x^m*(a+b*x^2)^(p+1)*
    ExpandToSum[2*b*(p+1)*Together[(b^(m/2)*(c+d*x^2)-(-a)^(m/2-1)*(b*c-a*d)*x^(-m+2))/(a+b*x^2)]-
    (-a)^(m/2-1)*(b*c-a*d)*x^(-m),x],x] /;
FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && LtQ[p,-1] && ILtQ[m/2,0] && (IntegerQ[p] || EqQ[m+2*p+1,0])
```

2: $\int (e x)^m (a+b x^n)^p (c+d x^n) dx$ when $b c - a d \neq 0 \wedge p < -1$

Derivation: Trinomial recurrence 2b with $c = 0$

Rule 1.1.3.4.5.4.2: If $b c - a d \neq 0 \wedge p < -1$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n) dx \rightarrow -\frac{(b c - a d) (e x)^{m+1} (a+b x^n)^{p+1}}{a b e n (p+1)} - \frac{a d (m+1) - b c (m+n (p+1) + 1)}{a b n (p+1)} \int (e x)^m (a+b x^n)^{p+1} dx$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.**x_^n_)^p_.*(c_+d_.**x_^n_),x_Symbol] :=
  -(b*c-a*d)*(e*x)^(m+1)*(a+b*x^n)^(p+1)/(a*b*e*n*(p+1)) -
  (a*d*(m+1)-b*c*(m+n*(p+1)+1))/(a*b*n*(p+1))*Int[(e*x)^m*(a+b*x^n)^(p+1),x] /;
FreeQ[{a,b,c,d,e,m,n},x] && NeQ[b*c-a*d,0] && LtQ[p,-1] &&
  (Not[IntegerQ[p+1/2]] && NeQ[p,-5/4] || Not[RationalQ[m]] || IGtQ[n,0] && ILtQ[p+1/2,0] && LeQ[-1,m,-n*(p+1)])
```

```
Int[(e_.**x_)^m_.*(a1_+b1_.**x_^non2_)^p_.*(a2_+b2_.**x_^non2_)^p_.*(c_+d_.**x_^n_),x_Symbol] :=
  -(b1*b2*c-a1*a2*d)*(e*x)^(m+1)*(a1+b1*x^(n/2))^(p+1)*(a2+b2*x^(n/2))^(p+1)/(a1*a2*b1*b2*e*n*(p+1)) -
  (a1*a2*d*(m+1)-b1*b2*c*(m+n*(p+1)+1))/(a1*a2*b1*b2*n*(p+1))*Int[(e*x)^m*(a1+b1*x^(n/2))^(p+1)*(a2+b2*x^(n/2))^(p+1),x] /;
FreeQ[{a1,b1,a2,b2,c,d,e,m,n},x] && EqQ[non2,n/2] && EqQ[a2*b1+a1*b2,0] && LtQ[p,-1] &&
  (Not[IntegerQ[p+1/2]] && NeQ[p,-5/4] || Not[RationalQ[m]] || IGtQ[n,0] && ILtQ[p+1/2,0] && LeQ[-1,m,-n*(p+1)])
```

5: $\int (e x)^m (a+b x^n)^p (c+d x^n) dx$ when $b c - a d \neq 0 \wedge m+n(p+1)+1 \neq 0$

Derivation: Trinomial recurrence 2b with $c = 0$ composed with binomial recurrence 1b

Rule 1.1.3.4.5.5: If $b c - a d \neq 0 \wedge m+n(p+1)+1 \neq 0$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n) dx \rightarrow \frac{d (e x)^{m+1} (a+b x^n)^{p+1}}{b e (m+n(p+1)+1)} - \frac{a d (m+1) - b c (m+n(p+1)+1)}{b (m+n(p+1)+1)} \int (e x)^m (a+b x^n)^p dx$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.**x_^n_)^p_.*(c_+d_.**x_^n_),x_Symbol] :=
  d*(e*x)^(m+1)*(a+b*x^n)^(p+1)/(b*e*(m+n*(p+1)+1)) -
  (a*d*(m+1)-b*c*(m+n*(p+1)+1))/(b*(m+n*(p+1)+1))*Int[(e*x)^m*(a+b*x^n)^p,x] /;
FreeQ[{a,b,c,d,e,m,n,p},x] && NeQ[b*c-a*d,0] && NeQ[m+n*(p+1)+1,0]
```

```
Int[(e_.**x_)^m_.*(a1_+b1_.**x_^non2_)^p_.*(a2_+b2_.**x_^non2_)^p_.*(c_+d_.**x_^n_),x_Symbol] :=
  d*(e*x)^(m+1)*(a1+b1*x^(n/2))^(p+1)*(a2+b2*x^(n/2))^(p+1)/(b1*b2*e*(m+n*(p+1)+1)) -
  (a1*a2*d*(m+1)-b1*b2*c*(m+n*(p+1)+1))/(b1*b2*(m+n*(p+1)+1))*Int[(e*x)^m*(a1+b1*x^(n/2))^p*(a2+b2*x^(n/2))^p,x] /;
FreeQ[{a1,b1,a2,b2,c,d,e,m,n,p},x] && EqQ[non2,n/2] && EqQ[a2*b1+a1*b2,0] && NeQ[m+n*(p+1)+1,0]
```

6. $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge n \in \mathbb{Z}$

1. $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+$

0: $\int \frac{(e x)^m (a+b x^n)^p}{c+d x^n} dx$ when $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule 1.1.3.4.6.1.0: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p \in \mathbb{Z}^+$, then

$$\int \frac{(e x)^m (a+b x^n)^p}{c+d x^n} dx \rightarrow \int \text{ExpandIntegrand}\left[\frac{(e x)^m (a+b x^n)^p}{c+d x^n}, x\right] dx$$

Program code:

```
Int[(e_.*x_)^m_.*(a_+b_.*x_^n_)^p_/(c_+d_.*x_^n_),x_Symbol] :=
  Int[ExpandIntegrand[(e*x)^m*(a+b*x^n)^p/(c+d*x^n),x],x] /;
  FreeQ[{a,b,c,d,e,m},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && IGtQ[p,0] && (IntegerQ[m] || IGtQ[2*(m+1),0] || Not[RationalQ[m]])
```

$$1. \int (e x)^m (a+b x^n)^p (c+d x^n)^2 dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+$$

$$1: \int (e x)^m (a+b x^n)^p (c+d x^n)^2 dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge m < -1 \wedge n > 0$$

Derivation: ?

Rule 1.1.3.4.6.1.1.1: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge m < -1 \wedge n > 0$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^2 dx \rightarrow \frac{c^2 (e x)^{m+1} (a+b x^n)^{p+1}}{a e (m+1)} - \frac{1}{a e^n (m+1)} \int (e x)^{m+n} (a+b x^n)^p (b c^2 n (p+1) + c (b c - 2 a d) (m+1) - a (m+1) d^2 x^n) dx$$

Program code:

```
Int[(e_.*x_)^m_.*(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^2,x_Symbol] :=
  c^2*(e*x)^(m+1)*(a+b*x^n)^(p+1)/(a*e*(m+1)) -
  1/(a*e^n*(m+1))*Int[(e*x)^(m+n)*(a+b*x^n)^p*Simp[b*c^2*n*(p+1)+c*(b*c-2*a*d)*(m+1)-a*(m+1)*d^2*x^n,x],x] /;
  FreeQ[{a,b,c,d,e,p},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && LtQ[m,-1] && GtQ[n,0]
```

2: $\int (e x)^m (a+b x^n)^p (c+d x^n)^2 dx$ when $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1$

Derivation: ?

Rule 1.1.3.4.6.1.1.2: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^2 dx \rightarrow -\frac{(b c - a d)^2 (e x)^{m+1} (a+b x^n)^{p+1}}{a b^2 e n (p+1)} + \frac{1}{a b^2 n (p+1)} \int (e x)^m (a+b x^n)^{p+1} ((b c - a d)^2 (m+1) + b^2 c^2 n (p+1) + a b d^2 n (p+1) x^n) dx$$

Program code:

```
Int[(e.*x_)^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^2,x_Symbol] :=
  -(b*c-a*d)^2*(e*x)^(m+1)*(a+b*x^n)^(p+1)/(a*b^2*e*n*(p+1)) +
  1/(a*b^2*n*(p+1))*Int[(e*x)^m*(a+b*x^n)^(p+1)*Simp[(b*c-a*d)^2*(m+1)+b^2*c^2*n*(p+1)+a*b*d^2*n*(p+1)*x^n,x],x] /;
FreeQ[{a,b,c,d,e,m,n},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && LtQ[p,-1]
```

$$\mathbf{3:} \int (e x)^m (a+b x^n)^p (c+d x^n)^2 dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge m+n(p+2)+1 \neq 0$$

Derivation: ?

Rule 1.1.3.4.6.1.1.3: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge m+n(p+2)+1 \neq 0$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^2 dx \rightarrow \frac{d^2 (e x)^{m+n+1} (a+b x^n)^{p+1}}{b e^{n+1} (m+n(p+2)+1)} + \frac{1}{b (m+n(p+2)+1)} \int (e x)^m (a+b x^n)^p (b c^2 (m+n(p+2)+1) + d ((2 b c - a d) (m+n+1) + 2 b c n (p+1)) x^n) dx$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.**x_^n_)^p_*(c_+d_.**x_^n_)^2,x_Symbol] :=
  d^2*(e*x)^(m+n+1)*(a+b*x^n)^(p+1)/(b*e^(n+1)*(m+n*(p+2)+1)) +
  1/(b*(m+n*(p+2)+1))*Int[(e*x)^m*(a+b*x^n)^p*Simp[b*c^2*(m+n*(p+2)+1)+d*((2*b*c-a*d)*(m+n+1)+2*b*c*n*(p+1))*x^n,x],x] /;
FreeQ[{a,b,c,d,e,m,n,p},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && NeQ[m+n*(p+2)+1,0]
```

$$\mathbf{2:} \int x^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge m \in \mathbb{Z} \wedge \text{GCD}[m+1, n] \neq 1$$

Derivation: Integration by substitution

Basis: If $n \in \mathbb{Z} \wedge m \in \mathbb{Z}$, let $k = \text{GCD}[m+1, n]$, then $x^m F[x^n] = \frac{1}{k} \text{Subst}[x^{\frac{m+1}{k}-1} F[x^{n/k}], x, x^k] \partial_x x^k$

Rule 1.1.3.4.6.1.2: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}$, let $k = \text{GCD}[m+1, n]$, if $k \neq 1$, then

$$\int x^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \frac{1}{k} \text{Subst}\left[\int x^{\frac{m+1}{k}-1} (a+b x^{n/k})^p (c+d x^{n/k})^q dx, x, x^k\right]$$

Program code:

```
Int[x_^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  With[{k=GCD[m+1,n]},
    1/k*Subst[Int[x^((m+1)/k-1)*(a+b*x^(n/k))^p*(c+d*x^(n/k))^q,x],x,x^k] /;
    k!=1] /;
FreeQ[{a,b,c,d,p,q},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && IntegerQ[m]
```

$$\mathbf{3:} \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge m \in \mathbb{F}$$

Derivation: Integration by substitution

Basis: If $k \in \mathbb{Z}^+$, then $(e x)^m F[x] = \frac{k}{e} \text{Subst}\left[x^{k(m+1)-1} F\left[\frac{x^k}{e}\right], x, (e x)^{1/k}\right] \partial_x (e x)^{1/k}$

Rule 1.1.3.4.6.1.3: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge m \in \mathbb{F}$, let $k = \text{Denominator}[m]$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \frac{k}{e} \text{Subst}\left[\int x^{k(m+1)-1} \left(a + \frac{b x^{k n}}{e^n}\right)^p \left(c + \frac{d x^{k n}}{e^n}\right)^q dx, x, (e x)^{1/k}\right]$$

Program code:

```
Int[(e_.**x_)^m_*(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  With[{k=Denominator[m]},
    k/e*Subst[Int[x^(k*(m+1)-1)*(a+b*x^(k*n))/e^n]^p*(c+d*x^(k*n)/e^n)^q,x],x,(e*x)^(1/k)] /;
  FreeQ[{a,b,c,d,e,p,q},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && FractionQ[m] && IntegerQ[p]
```

$$4. \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1$$

$$1. \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge q > 0$$

$$\mathbf{1:} \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge q > 0 \wedge m - n + 1 > 0$$

Derivation: Binomial product recurrence 1 with $A = 0$, $B = 1$ and $m = m - n$

Derivation: Binomial product recurrence 3a with $A = c$, $B = d$ and $q = q - 1$

Rule 1.1.3.4.6.1.4.1.1: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge q > 0 \wedge m - n + 1 > 0$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow$$

$$\frac{e^{n-1} (e x)^{m-n+1} (a+b x^n)^{p+1} (c+d x^n)^q}{b n (p+1)} - \frac{\tilde{e}^n}{b n (p+1)} \int (e x)^{m-n} (a+b x^n)^{p+1} (c+d x^n)^{q-1} (c(m-n+1) + d(m+n(q-1)+1) x^n) dx$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.**x_^n_)^p_*(c_+d_.**x_^n_)^q_,x_Symbol] :=
  e^(n-1)*(e*x)^(m-n+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^q/(b*n*(p+1)) -
  e^n/(b*n*(p+1))*Int[(e*x)^(m-n)*(a+b*x^n)^(p+1)*(c+d*x^n)^(q-1)*Simp[c*(m-n+1)+d*(m+n*(q-1)+1)*x^n,x],x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && LtQ[p,-1] && GtQ[q,0] && GtQ[m-n+1,0] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

2: $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge q > 1$

Derivation: Binomial product recurrence 1 with $A = c$, $B = d$ and $q = q - 1$

Rule 1.1.3.4.6.1.4.1.2: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge q > 1$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \frac{- (c b - a d) (e x)^{m+1} (a+b x^n)^{p+1} (c+d x^n)^{q-1}}{a b e n (p+1)} + \frac{1}{a b n (p+1)} \cdot \int (e x)^m (a+b x^n)^{p+1} (c+d x^n)^{q-2} (c (c b n (p+1) + (c b - a d) (m+1)) + d (c b n (p+1) + (c b - a d) (m+n(q-1)+1)) x^n) dx$$

Programcode:

```
Int[(e_.**x_)^m_.*(a_+b_.**x_^n_)^p_*(c_+d_.**x_^n_)^q_,x_Symbol] :=
  -(c*b-a*d)*(e*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(q-1)/(a*b*e*n*(p+1)) +
  1/(a*b*n*(p+1))*Int[(e*x)^m*(a+b*x^n)^(p+1)*(c+d*x^n)^(q-2)*
  Simp[c*(c*b*n*(p+1)+(c*b-a*d)*(m+1))+d*(c*b*n*(p+1)+(c*b-a*d)*(m+n*(q-1)+1))*x^n,x],x] /;
FreeQ[{a,b,c,d,e,m},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && LtQ[p,-1] && GtQ[q,1] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

$$\mathbf{3:} \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge 0 < q < 1$$

Derivation: Binomial product recurrence 1 with $A = 1$ and $B = 0$

Derivation: Binomial product recurrence 3b with $A = c$, $B = d$ and $q = q - 1$

Rule 1.1.3.4.6.1.4.1.3: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge 0 < q < 1$, then

$$\begin{aligned} & \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \\ & - \frac{(e x)^{m+1} (a+b x^n)^{p+1} (c+d x^n)^q}{a e n (p+1)} + \\ & \frac{1}{a n (p+1)} \int (e x)^m (a+b x^n)^{p+1} (c+d x^n)^{q-1} (c (m+n (p+1)+1) + d (m+n (p+q+1)+1) x^n) dx \end{aligned}$$

Program code:

```
Int[(e_.**x_)^m_.*(a+_b_.**x_^n_)^p_*(c+_d_.**x_^n_)^q_,x_Symbol] :=
  -(e*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^q/(a*e*n*(p+1)) +
  1/(a*n*(p+1))*Int[(e*x)^m*(a+b*x^n)^(p+1)*(c+d*x^n)^(q-1)*Simp[c*(m+n*(p+1)+1)+d*(m+n*(p+q+1)+1)*x^n,x],x] /;
FreeQ[{a,b,c,d,e,m},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && LtQ[p,-1] && LtQ[0,q,1] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

$$\mathbf{2.} \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge m - n + 1 > 0$$

$$\mathbf{1:} \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge m - n + 1 > n$$

Derivation: Binomial product recurrence 3a with $A = 0$, $B = 1$ and $m = m - n$

Rule 1.1.3.4.6.1.4.2.1: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge m - n + 1 > n$, then

$$\begin{aligned} & \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \\ & - \frac{a e^{2n-1} (e x)^{m-2n+1} (a+b x^n)^{p+1} (c+d x^n)^{q+1}}{b n (b c - a d) (p+1)} + \end{aligned}$$

$$\frac{e^{2n}}{b n (b c - a d) (p+1)} \int (e x)^{m-2n} (a+b x^n)^{p+1} (c+d x^n)^q (a c (m-2n+1) + (a d (m-n+nq+1) + b c n (p+1)) x^n) dx$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  -a*e^(2*n-1)*(e*x)^(m-2*n+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(q+1)/(b*n*(b*c-a*d)*(p+1)) +
  e^(2*n)/(b*n*(b*c-a*d)*(p+1))*Int[(e*x)^(m-2*n)*(a+b*x^n)^(p+1)*(c+d*x^n)^q*
  Simp[a*c*(m-2*n+1)+(a*d*(m-n+n*q+1)+b*c*n*(p+1))*x^n,x],x] /;
FreeQ[{a,b,c,d,e,q},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && LtQ[p,-1] && GtQ[m-n+1,n] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

2: $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge n \geq m - n + 1 > 0$

Derivation: Binomial product recurrence 3a with $A = 1$ and $B = 0$

Derivation: Binomial product recurrence 3b with $A = 0$, $B = 1$ and $m = m - n$

Rule 1.1.3.4.6.1.4.2.2: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge n \geq m - n + 1 > 0$, then

$$\frac{\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \frac{e^{n-1} (e x)^{m-n+1} (a+b x^n)^{p+1} (c+d x^n)^{q+1}}{n (b c - a d) (p+1)} - \frac{e^n}{n (b c - a d) (p+1)} \int (e x)^{m-n} (a+b x^n)^{p+1} (c+d x^n)^q (c (m-n+1) + d (m+n (p+q+1) + 1) x^n) dx$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  e^(n-1)*(e*x)^(m-n+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(q+1)/(n*(b*c-a*d)*(p+1)) -
  e^n/(n*(b*c-a*d)*(p+1))*Int[(e*x)^(m-n)*(a+b*x^n)^(p+1)*(c+d*x^n)^q*Simp[c*(m-n+1)+d*(m+n*(p+q+1)+1)*x^n,x],x] /;
FreeQ[{a,b,c,d,e,q},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && LtQ[p,-1] && GeQ[n,m-n+1] && GtQ[m-n+1,0] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

$$\mathbf{3:} \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1$$

Derivation: Binomial product recurrence 3b with $A = 1$ and $B = 0$

Rule 1.1.3.4.6.1.4.3: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1$, then

$$\begin{aligned} & \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \\ & - \frac{b (e x)^{m+1} (a+b x^n)^{p+1} (c+d x^n)^{q+1}}{a e n (b c - a d) (p+1)} + \\ & \frac{1}{a n (b c - a d) (p+1)} \int (e x)^m (a+b x^n)^{p+1} (c+d x^n)^q (c b (m+1) + n (b c - a d) (p+1) + d b (m+n (p+q+2)+1) x^n) dx \end{aligned}$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.**x_^n_)^p_*(c_+d_.**x_^n_)^q_,x_Symbol] :=
  -b*(e*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(q+1)/(a*e*n*(b*c-a*d)*(p+1)) +
  1/(a*n*(b*c-a*d)*(p+1))*
  Int[(e*x)^m*(a+b*x^n)^(p+1)*(c+d*x^n)^q*Simp[c*b*(m+1)+n*(b*c-a*d)*(p+1)+d*b*(m+n*(p+q+2)+1)*x^n,x],x] /;
FreeQ[{a,b,c,d,e,m,q},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && LtQ[p,-1] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

$$\mathbf{5.} \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge q > 0$$

$$\mathbf{1.} \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge q > 0 \wedge m < -1$$

$$\mathbf{1:} \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge q > 0 \wedge m < -1 \wedge p > 0$$

Derivation: Binomial product recurrence 2a with $A = a$, $B = b$ and $p = p - 1$

Rule 1.1.3.4.6.1.5.1.1: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge q > 0 \wedge m < -1 \wedge p > 0$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow$$

$$\frac{(e x)^{m+1} (a+b x^n)^p (c+d x^n)^q}{e (m+1)} - \frac{n}{e^n (m+1)} \int (e x)^{m+n} (a+b x^n)^{p-1} (c+d x^n)^{q-1} (b c p + a d q + b d (p+q) x^n) dx$$

Program code:

```
Int[(e_.**x_)^m_*(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  (e*x)^(m+1)*(a+b*x^n)^p*(c+d*x^n)^q/(e*(m+1)) -
  n/(e^n*(m+1))*Int[(e*x)^(m+n)*(a+b*x^n)^(p-1)*(c+d*x^n)^(q-1)*Simp[b*c*p+a*d*q+b*d*(p+q)*x^n,x],x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && GtQ[q,0] && LtQ[m,-1] && GtQ[p,0] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

2: $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge q > 1 \wedge m < -1$

Derivation: Binomial product recurrence 2a with $A = c$, $B = d$ and $q = q - 1$

Rule 1.1.3.4.6.1.5.1.2: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge q > 1 \wedge m < -1$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \frac{c (e x)^{m+1} (a+b x^n)^{p+1} (c+d x^n)^{q-1}}{a e (m+1)} - \frac{1}{a e^n (m+1)} \cdot \int (e x)^{m+n} (a+b x^n)^p (c+d x^n)^{q-2} (c (c b - a d) (m+1) + c n (b c (p+1) + a d (q-1)) + d ((c b - a d) (m+1) + c b n (p+q)) x^n) dx$$

Program code:

```
Int[(e_.**x_)^m_*(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  c*(e*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(q-1)/(a*e*(m+1)) -
  1/(a*e^n*(m+1))*Int[(e*x)^(m+n)*(a+b*x^n)^p*(c+d*x^n)^(q-2)*
  Simp[c*(c*b-a*d)*(m+1)+c*n*(b*c*(p+1)+a*d*(q-1))+d*((c*b-a*d)*(m+1)+c*b*n*(p+q))*x^n,x],x] /;
FreeQ[{a,b,c,d,e,p},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && GtQ[q,1] && LtQ[m,-1] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

$$\mathbf{3:} \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge 0 < q < 1 \wedge m < -1$$

Derivation: Binomial product recurrence 2a with $A = 1$ and $B = 0$

Derivation: Binomial product recurrence 4b with $A = c$, $B = d$ and $q = q - 1$

Rule 1.1.3.4.6.1.5.1.3: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge 0 < q < 1 \wedge m < -1$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \frac{(e x)^{m+1} (a+b x^n)^{p+1} (c+d x^n)^q}{a e (m+1)} - \frac{1}{a e^n (m+1)} \int (e x)^{m+n} (a+b x^n)^p (c+d x^n)^{q-1} (c b (m+1) + n (b c (p+1) + a d q) + d (b (m+1) + b n (p+q+1)) x^n) dx$$

Program code:

```
Int[(e_.**x_)^m_*(a_+b_.**x_^n_)^p_*(c_+d_.**x_^n_)^q_,x_Symbol] :=
  (e*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^q/(a*e*(m+1)) -
  1/(a*e^n*(m+1))*Int[(e*x)^(m+n)*(a+b*x^n)^p*(c+d*x^n)^(q-1)*
    Simp[c*b*(m+1)+n*(b*c*(p+1)+a*d*q)+d*(b*(m+1)+b*n*(p+q+1))*x^n,x],x] /;
FreeQ[{a,b,c,d,e,p},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && LtQ[0,q,1] && LtQ[m,-1] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

$$\mathbf{2:} \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge q > 0 \wedge p > 0$$

Derivation: Binomial product recurrence 2b with $A = a$, $B = b$ and $p = p - 1$

Rule 1.1.3.4.6.1.5.2: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge q > 0 \wedge p > 0$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \frac{(e x)^{m+1} (a+b x^n)^p (c+d x^n)^q}{e (m+n (p+q)+1)} +$$

$$\frac{n}{m+n(p+q)+1} \int (e x)^m (a+b x^n)^{p-1} (c+d x^n)^{q-1} (a c (p+q) + (q (b c - a d) + a d (p+q)) x^n) dx$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  (e*x)^(m+1)*(a+b*x^n)^p*(c+d*x^n)^q/(e*(m+n*(p+q)+1)) +
  n/(m+n*(p+q)+1)*Int[(e*x)^m*(a+b*x^n)^(p-1)*(c+d*x^n)^(q-1)*Simp[a*c*(p+q)+(q*(b*c-a*d)+a*d*(p+q))*x^n,x] /;
FreeQ[{a,b,c,d,e,m},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && GtQ[q,0] && GtQ[p,0] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

3: $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge q > 1$

Derivation: Binomial product recurrence 2b with $A = c$, $B = d$ and $q = q - 1$

Rule 1.1.3.4.6.1.5.3: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge q > 1$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \frac{d (e x)^{m+1} (a+b x^n)^{p+1} (c+d x^n)^{q-1}}{b e (m+n(p+q)+1)} + \frac{1}{b (m+n(p+q)+1)} \int (e x)^m (a+b x^n)^p (c+d x^n)^{q-2} \cdot (c((b c - a d)(m+1) + c b n(p+q)) + (d(c b - a d)(m+1) + d n(q-1)(b c - a d) + c b d n(p+q)) x^n) dx$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  d*(e*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(q-1)/(b*e*(m+n*(p+q)+1)) +
  1/(b*(m+n*(p+q)+1))*Int[(e*x)^m*(a+b*x^n)^p*(c+d*x^n)^(q-2)*
  Simp[c*((c*b-a*d)*(m+1)+c*b*n*(p+q))+(d*(c*b-a*d)*(m+1)+d*n*(q-1)*(b*c-a*d)+c*b*d*n*(p+q))*x^n,x] /;
FreeQ[{a,b,c,d,e,m,p},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && GtQ[q,1] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

$$\text{4: } \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge q > 0 \wedge m - n + 1 > 0$$

Derivation: Binomial product recurrence 2b with $A = 0$, $B = 1$ and $m = m - n$

Derivation: Binomial product recurrence 4a with $A = c$, $B = d$ and $q = q - 1$

Rule 1.1.3.4.6.1.5.4: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge q > 0 \wedge m - n + 1 > 0$, then

$$\begin{aligned} & \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \\ & \frac{e^{n-1} (e x)^{m-n+1} (a+b x^n)^{p+1} (c+d x^n)^q}{b (m+n (p+q) + 1)} - \\ & \frac{e^n}{b (m+n (p+q) + 1)} \int (e x)^{m-n} (a+b x^n)^p (c+d x^n)^{q-1} (a c (m-n+1) + (a d (m-n+1) - n q (b c - a d)) x^n) dx \end{aligned}$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.**x_^n_)^p_*(c_+d_.**x_^n_)^q_,x_Symbol] :=
  e^(n-1)*(e*x)^(m-n+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^q/(b*(m+n*(p+q)+1)) -
  e^n/(b*(m+n*(p+q)+1))*
  Int[(e*x)^(m-n)*(a+b*x^n)^p*(c+d*x^n)^(q-1)*Simp[a*c*(m-n+1)+(a*d*(m-n+1)-n*q*(b*c-a*d))*x^n,x],x] /;
FreeQ[{a,b,c,d,e,p},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && GtQ[q,0] && GtQ[m-n+1,0] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

$$\text{6: } \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge m - n + 1 > n$$

Derivation: Binomial product recurrence 4a with $A = 0$, $B = 1$ and $m = m - n$

Rule 1.1.3.4.6.1.6: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge m - n + 1 > n$, then

$$\begin{aligned} & \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \\ & \frac{e^{2n-1} (e x)^{m-2n+1} (a+b x^n)^{p+1} (c+d x^n)^{q+1}}{b d (m+n (p+q) + 1)} - \frac{e^{2n}}{b d (m+n (p+q) + 1)} \int (e x)^{m-n} (a+b x^n)^p (c+d x^n)^q dx \end{aligned}$$

$$\int (e x)^{m-2 n} (a+b x^n)^p (c+d x^n)^q \left(a c (m-2 n+1) + (a d (m+n (q-1)+1) + b c (m+n (p-1)+1)) x^n \right) dx$$

Program code:

```
Int[(e_.**x_)^m_.*(a+_b_.**x_^n_)^p_*(c+_d_.**x_^n_)^q_,x_Symbol] :=
  e^(2*n-1)*(e*x)^(m-2*n+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(q+1)/(b*d*(m+n*(p+q)+1)) -
  e^(2*n)/(b*d*(m+n*(p+q)+1))*
  Int[(e*x)^(m-2*n)*(a+b*x^n)^p*(c+d*x^n)^q*Simp[a*c*(m-2*n+1)+(a*d*(m+n*(q-1)+1)+b*c*(m+n*(p-1)+1))*x^n,x] /;
FreeQ[{a,b,c,d,e,p,q},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && GtQ[m-n+1,n] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

7: $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge m < -1$

Derivation: Binomial product recurrence 4b with $A = 1$ and $B = 0$

Rule 1.1.3.4.6.1.7: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge m < -1$, then

$$\frac{\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx}{a c e^{n(m+1)}} \rightarrow \frac{(e x)^{m+1} (a+b x^n)^{p+1} (c+d x^n)^{q+1}}{a c e^{n(m+1)}} - \frac{1}{a c e^{n(m+1)}} \int (e x)^{m+n} (a+b x^n)^p (c+d x^n)^q ((b c + a d)(m+n+1) + n(b c p + a d q) + b d(m+n(p+q+2)+1) x^n) dx$$

Program code:

```
Int[(e_.**x_)^m_.*(a+_b_.**x_^n_)^p_*(c+_d_.**x_^n_)^q_,x_Symbol] :=
  (e*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(q+1)/(a*c*e^(m+1)) -
  1/(a*c*e^n*(m+1))*
  Int[(e*x)^(m+n)*(a+b*x^n)^p*(c+d*x^n)^q*Simp[(b*c+a*d)*(m+n+1)+n*(b*c*p+a*d*q)+b*d*(m+n*(p+q+2)+1))*x^n,x] /;
FreeQ[{a,b,c,d,e,p,q},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && LtQ[m,-1] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

$$8. \int \frac{(e x)^m (c + d x^n)^q}{a + b x^n} dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+$$

$$1. \int \frac{(e x)^m}{(a + b x^n) (c + d x^n)} dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+$$

$$\text{1: } \int \frac{(e x)^m}{(a + b x^n) (c + d x^n)} dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge n \leq m \leq 2 n - 1$$

Derivation: Algebraic expansion

Basis: If $n \in \mathbb{Z}$, then $\frac{(e x)^m}{(a+b x^n) (c+d x^n)} = -\frac{a e^n (e x)^{m-n}}{(b c-a d) (a+b x^n)} + \frac{c e^n (e x)^{m-n}}{(b c-a d) (c+d x^n)}$

Rule 1.1.3.4.6.1.8.1.1: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge n \leq m \leq 2 n - 1$, then

$$\int \frac{(e x)^m}{(a + b x^n) (c + d x^n)} dx \rightarrow -\frac{a e^n}{b c - a d} \int \frac{(e x)^{m-n}}{a + b x^n} dx + \frac{c e^n}{b c - a d} \int \frac{(e x)^{m-n}}{c + d x^n} dx$$

Program code:

```
Int[(e_.**x_)^m_/((a_+b_.**x_^n_)*(c_+d_.**x_^n_)),x_Symbol] :=
  -a*e^n/(b*c-a*d)*Int[(e*x)^(m-n)/(a+b*x^n),x] + c*e^n/(b*c-a*d)*Int[(e*x)^(m-n)/(c+d*x^n),x] /;
FreeQ[{a,b,c,d,e,m},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && LeQ[n,m,2*n-1]
```

$$\mathbf{2:} \int \frac{(e x)^m}{(a+b x^n)(c+d x^n)} dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{1}{(a+b x)(c+d x)} = \frac{b}{(b c - a d)(a+b x)} - \frac{d}{(b c - a d)(c+d x)}$$

Rule 1.1.3.4.6.1.8.1.2: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+$, then

$$\int \frac{(e x)^m}{(a+b x^n)(c+d x^n)} dx \rightarrow \frac{b}{b c - a d} \int \frac{(e x)^m}{a+b x^n} dx - \frac{d}{b c - a d} \int \frac{(e x)^m}{c+d x^n} dx$$

Program code:

```
Int[(e_.**x_)^m_/((a_+b_.**x_^n_)*(c_+d_.**x_^n_)),x_Symbol] :=
  b/(b*c-a*d)*Int[(e*x)^m/(a+b*x^n),x] - d/(b*c-a*d)*Int[(e*x)^m/(c+d*x^n),x] /;
FreeQ[{a,b,c,d,e,m},x] && NeQ[b*c-a*d,0] && IGtQ[n,0]
```

$$\text{2: } \int \frac{(e x)^m (c + d x^n)^q}{a + b x^n} dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge n \leq m \leq 2 n - 1$$

Derivation: Algebraic expansion

Basis: If $n \in \mathbb{Z}$, then $\frac{1}{a+b x^n} = \frac{e^n (e x)^{-n}}{b} - \frac{a e^n (e x)^{-n}}{b (a+b x^n)}$

Rule 1.1.3.4.6.1.8.2: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^+ \wedge n \leq m \leq 2 n - 1$, then

$$\int \frac{(e x)^m (c + d x^n)^q}{a + b x^n} dx \rightarrow \frac{e^n}{b} \int (e x)^{m-n} (c + d x^n)^q dx - \frac{a e^n}{b} \int \frac{(e x)^{m-n} (c + d x^n)^q}{a + b x^n} dx$$

Program code:

```
Int[(e_.**x_)^m_*(c_+d_.*x_^n_)^q_/ (a_+b_.*x_^n_), x_Symbol] :=
  e^n/b*Int[(e*x)^(m-n)*(c+d*x^n)^q, x] - a*e^n/b*Int[(e*x)^(m-n)*(c+d*x^n)^q/(a+b*x^n), x] /;
FreeQ[{a,b,c,d,e,m,q}, x] && NeQ[b*c-a*d, 0] && IGtQ[n, 0] && LeQ[n, m, 2*n-1] && IntBinomialQ[a,b,c,d,e,m,n,-1,q,x]
```

$$\begin{aligned}
3. \int \frac{x (a+b x^3)^q}{c+d x^3} dx & \text{ when } b c - a d \neq 0 \wedge q^2 = \frac{1}{4} \wedge (b c - 4 a d = 0 \vee b c + 8 a d = 0 \vee b^2 c^2 - 20 a b c d - 8 a^2 d^2 = 0) \\
1. \int \frac{x}{(c+d x^3) \sqrt{a+b x^3}} dx & \text{ when } b c - a d \neq 0 \wedge (b c - 4 a d = 0 \vee b c + 8 a d = 0 \vee b^2 c^2 - 20 a b c d - 8 a^2 d^2 = 0) \\
1. \int \frac{x}{(a+b x^3) \sqrt{c+d x^3}} dx & \text{ when } b c - a d \neq 0 \wedge 4 b c - a d = 0 \\
1: \int \frac{x}{(a+b x^3) \sqrt{c+d x^3}} dx & \text{ when } b c - a d \neq 0 \wedge 4 b c - a d = 0 \wedge c > 0
\end{aligned}$$

Reference: Goursat pseudo-elliptic integral

Contributed by Martin Welz on 24 January 2018 via sci.math.symbolic

Derivation: Algebraic expansion

Basis: If $4 b c - a d = 0 \wedge c > 0$, let $q \rightarrow \left(\frac{d}{c}\right)^{1/3}$, then

$$\frac{x}{(a+b x^3) \sqrt{c+d x^3}} = -\frac{q}{6 \times 2^{2/3} b x \sqrt{c+d x^3}} + \frac{d q x^2}{2^{5/3} b (4 c+d x^3) \sqrt{c+d x^3}} - \frac{q^2 (2^{2/3} - 2 q x)}{12 b (2 + 2^{1/3} q x) \sqrt{c+d x^3}} + \frac{q (2^{4/3} + 3 q^2 x^2 - 2^{1/3} q^3 x^3)}{6 \times 2^{2/3} b x (2^{4/3} - 2^{2/3} q x + q^2 x^2) \sqrt{c+d x^3}}$$

Rule 1.1.3.4.6.1.8.3.1.1.1: If $b c - a d \neq 0 \wedge 4 b c - a d = 0 \wedge c > 0$, let $q \rightarrow \left(\frac{d}{c}\right)^{1/3}$, then

$$\begin{aligned}
& \int \frac{x}{(a+b x^3) \sqrt{c+d x^3}} dx \rightarrow \\
& - \int \frac{q}{6 \times 2^{2/3} b x \sqrt{c+d x^3}} dx + \int \frac{d q x^2}{2^{5/3} b (4 c+d x^3) \sqrt{c+d x^3}} dx - \\
& \int \frac{q^2 (2^{2/3} - 2 q x)}{12 b (2 + 2^{1/3} q x) \sqrt{c+d x^3}} dx + \int \frac{q (2^{4/3} + 3 q^2 x^2 - 2^{1/3} q^3 x^3)}{6 \times 2^{2/3} b x (2^{4/3} - 2^{2/3} q x + q^2 x^2) \sqrt{c+d x^3}} dx \rightarrow \\
& \frac{q \operatorname{ArcTanh}\left[\frac{\sqrt{c+d x^3}}{\sqrt{c}}\right]}{9 \times 2^{2/3} b \sqrt{c}} + \frac{q \operatorname{ArcTan}\left[\frac{\sqrt{c+d x^3}}{\sqrt{3} \sqrt{c}}\right]}{3 \times 2^{2/3} \sqrt{3} b \sqrt{c}} - \frac{q \operatorname{ArcTan}\left[\frac{\sqrt{3} \sqrt{c} (1+2^{1/3} q x)}{\sqrt{c+d x^3}}\right]}{3 \times 2^{2/3} \sqrt{3} b \sqrt{c}} - \frac{q \operatorname{ArcTanh}\left[\frac{\sqrt{c} (1-2^{1/3} q x)}{\sqrt{c+d x^3}}\right]}{3 \times 2^{2/3} b \sqrt{c}}
\end{aligned}$$

Program code:

```

Int[x_/((a_+b_.*x_^3)*Sqrt[c_+d_.*x_^3]),x_Symbol] :=
  With[{q=Rt[d/c,3]},
    q*ArcTanh[Sqrt[c+d*x^3]/Rt[c,2]]/(9*2^(2/3)*b*Rt[c,2]) +
    q*ArcTan[Sqrt[c+d*x^3]/(Sqrt[3]*Rt[c,2])]/(3*2^(2/3)*Sqrt[3]*b*Rt[c,2]) -
    q*ArcTan[Sqrt[3]*Rt[c,2]*(1+2^(1/3)*q*x)/Sqrt[c+d*x^3]]/(3*2^(2/3)*Sqrt[3]*b*Rt[c,2]) -
    q*ArcTanh[Rt[c,2]*(1-2^(1/3)*q*x)/Sqrt[c+d*x^3]]/(3*2^(2/3)*b*Rt[c,2]) /;
  FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && EqQ[4*b*c-a*d,0] && PosQ[c]

```

$$2: \int \frac{x}{(a+b x^3) \sqrt{c+d x^3}} dx \text{ when } b c - a d \neq 0 \wedge 4 b c - a d = 0 \wedge c \neq 0$$

Reference: Goursat pseudo-elliptic integral

Derivation: Algebraic expansion

Basis: If $4 b c - a d = 0 \wedge c > 0$, let $q \rightarrow \left(\frac{d}{c}\right)^{1/3}$, then

$$\frac{x}{(a+b x^3) \sqrt{c+d x^3}} = -\frac{q}{6 \times 2^{2/3} b x \sqrt{c+d x^3}} + \frac{d q x^2}{2^{5/3} b (4 c+d x^3) \sqrt{c+d x^3}} - \frac{q^2 (2^{2/3} - 2 q x)}{12 b (2 + 2^{1/3} q x) \sqrt{c+d x^3}} + \frac{q (2^{4/3} + 3 q^2 x^2 - 2^{1/3} q^3 x^3)}{6 \times 2^{2/3} b x (2^{4/3} - 2^{2/3} q x + q^2 x^2) \sqrt{c+d x^3}}$$

Rule 1.1.3.4.6.1.8.3.1.1.2: If $b c - a d \neq 0 \wedge 4 b c - a d = 0 \wedge c \neq 0$, let $q \rightarrow \left(\frac{d}{c}\right)^{1/3}$, then

$$\begin{aligned} & \int \frac{x}{(a+b x^3) \sqrt{c+d x^3}} dx \rightarrow \\ & - \int \frac{q}{6 \times 2^{2/3} b x \sqrt{c+d x^3}} dx + \int \frac{d q x^2}{2^{5/3} b (4 c+d x^3) \sqrt{c+d x^3}} dx - \int \frac{q^2 (2^{2/3} - 2 q x)}{12 b (2 + 2^{1/3} q x) \sqrt{c+d x^3}} dx + \int \frac{q (2^{4/3} + 3 q^2 x^2 - 2^{1/3} q^3 x^3)}{6 \times 2^{2/3} b x (2^{4/3} - 2^{2/3} q x + q^2 x^2) \sqrt{c+d x^3}} dx \\ & \rightarrow -\frac{q \operatorname{ArcTan}\left[\frac{\sqrt{c+d x^3}}{\sqrt{-c}}\right]}{9 \times 2^{2/3} b \sqrt{-c}} - \frac{q \operatorname{ArcTanh}\left[\frac{\sqrt{c+d x^3}}{\sqrt{3} \sqrt{-c}}\right]}{3 \times 2^{2/3} \sqrt{3} b \sqrt{-c}} - \frac{q \operatorname{ArcTanh}\left[\frac{\sqrt{3} \sqrt{-c} (1+2^{1/3} q x)}{\sqrt{c+d x^3}}\right]}{3 \times 2^{2/3} \sqrt{3} b \sqrt{-c}} - \frac{q \operatorname{ArcTan}\left[\frac{\sqrt{-c} (1-2^{1/3} q x)}{\sqrt{c+d x^3}}\right]}{3 \times 2^{2/3} b \sqrt{-c}} \end{aligned}$$

Program code:

```
Int[x_/((a+b_.*x^3)*Sqrt[c_+d_.*x^3]),x_Symbol] :=
  With[{q=Rt[d/c,3]},
    -q*ArcTan[Sqrt[c+d*x^3]/Rt[-c,2]]/(9*2^(2/3)*b*Rt[-c,2]) -
    q*ArcTanh[Sqrt[c+d*x^3]/(Sqrt[3]*Rt[-c,2])]/(3*2^(2/3)*Sqrt[3]*b*Rt[-c,2]) -
    q*ArcTanh[Sqrt[3]*Rt[-c,2]*(1+2^(1/3)*q*x)/Sqrt[c+d*x^3]]/(3*2^(2/3)*Sqrt[3]*b*Rt[-c,2]) -
    q*ArcTan[Rt[-c,2]*(1-2^(1/3)*q*x)/Sqrt[c+d*x^3]]/(3*2^(2/3)*b*Rt[-c,2]] /;
    FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && EqQ[4*b*c-a*d,0] && NegQ[c]
```

$$\mathbf{2:} \int \frac{x}{(a+b x^3) \sqrt{c+d x^3}} dx \text{ when } b c - a d \neq 0 \wedge 8 b c + a d = 0$$

Reference: Goursat pseudo-elliptic integral

Contributed by Martin Welz on 22 January 2018 via sci.math.symbolic

Derivation: Algebraic expansion

Basis: If $8 b c + a d = 0$, let $q \rightarrow \left(\frac{d}{c}\right)^{1/3}$, then $\frac{x}{a+b x^3} = \frac{d q x^2}{4 b (8 c - d x^3)} - \frac{q^2 (1+q x)}{12 b (2-q x)} + \frac{2 c q^2 - 2 d x - d q x^2}{12 b c (4+2 q x + q^2 x^2)}$

Rule 1.1.3.4.6.1.8.3.1.2: If $b c - a d \neq 0 \wedge 8 b c + a d = 0$, let $q \rightarrow \left(\frac{d}{c}\right)^{1/3}$, then

$$\int \frac{x}{(a+b x^3) \sqrt{c+d x^3}} dx \rightarrow \frac{d q}{4 b} \int \frac{x^2}{(8 c - d x^3) \sqrt{c+d x^3}} dx - \frac{q^2}{12 b} \int \frac{1+q x}{(2-q x) \sqrt{c+d x^3}} dx + \frac{1}{12 b c} \int \frac{2 c q^2 - 2 d x - d q x^2}{(4+2 q x + q^2 x^2) \sqrt{c+d x^3}} dx$$

Program code:

```
Int[x_/((a+_.*x^3)*Sqrt[c+_.*x^3]),x_Symbol] :=
  With[{q=Rt[d/c,3]},
    d*q/(4*b)*Int[x^2/((8*c-d*x^3)*Sqrt[c+d*x^3]),x] -
    q^2/(12*b)*Int[(1+q*x)/((2-q*x)*Sqrt[c+d*x^3]),x] +
    1/(12*b*c)*Int[(2*c*q^2-2*d*x-d*q*x^2)/((4+2*q*x+q^2*x^2)*Sqrt[c+d*x^3]),x]] /;
  FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && EqQ[8*b*c+a*d,0]
```

$$\mathbf{3.} \int \frac{x}{(c+d x^3) \sqrt{a+b x^3}} dx \text{ when } b c - a d \neq 0 \wedge b^2 c^2 - 20 a b c d - 8 a^2 d^2 = 0$$

$$\mathbf{1:} \int \frac{x}{(c+d x^3) \sqrt{a+b x^3}} dx \text{ when } b c - a d \neq 0 \wedge b^2 c^2 - 20 a b c d - 8 a^2 d^2 = 0 \wedge a > 0$$

Reference: Goursat pseudo-elliptic integral

Note: If $b^2 c^2 - 20 a b c d - 8 a^2 d^2 = (b c - 10 a d + 6 \sqrt{3} a d) (b c - 10 a d - 6 \sqrt{3} a d) = 0$, then $\frac{b c - 10 a d}{6 a d}$ should simplify to $\sqrt{3}$ or $-\sqrt{3}$.

Rule 1.1.3.4.6.1.8.3.1.3.1: If $b c - a d \neq 0 \wedge b^2 c^2 - 20 a b c d - 8 a^2 d^2 = 0 \wedge a > 0$, let $q \rightarrow \left(\frac{b}{a}\right)^{1/3}$ and $r \rightarrow \frac{b c - 10 a d}{6 a d}$, then

$$\int \frac{x}{(c + d x^3) \sqrt{a + b x^3}} dx \rightarrow$$

$$-\frac{q(2-r) \operatorname{ArcTan}\left[\frac{(1-r) \sqrt{a+b x^3}}{\sqrt{2} \sqrt{a} r^{3/2}}\right]}{3 \sqrt{2} \sqrt{a} d r^{3/2}} - \frac{q(2-r) \operatorname{ArcTan}\left[\frac{\sqrt{a} \sqrt{r} (1+r) (1+q x)}{\sqrt{2} \sqrt{a+b x^3}}\right]}{2 \sqrt{2} \sqrt{a} d r^{3/2}} - \frac{q(2-r) \operatorname{ArcTanh}\left[\frac{\sqrt{a} (1-r) \sqrt{r} (1+q x)}{\sqrt{2} \sqrt{a+b x^3}}\right]}{6 \sqrt{2} \sqrt{a} d \sqrt{r}} - \frac{q(2-r) \operatorname{ArcTanh}\left[\frac{\sqrt{a} \sqrt{r} (1+r-2 q x)}{\sqrt{2} \sqrt{a+b x^3}}\right]}{3 \sqrt{2} \sqrt{a} d \sqrt{r}}$$

Program code:

```
Int[x_/((c_+d_.x^3)*Sqrt[a_+b_.x^3]),x_Symbol] :=
  With[{q=Rt[b/a,3],r=Simplify[(b*c-10*a*d)/(6*a*d)]},
    -q*(2-r)*ArcTan[(1-r)*Sqrt[a+b*x^3]/(Sqrt[2]*Rt[a,2]*r^(3/2))]/(3*Sqrt[2]*Rt[a,2]*d*r^(3/2)) -
    q*(2-r)*ArcTan[Rt[a,2]*Sqrt[r]*(1+r)*(1+q*x)/(Sqrt[2]*Sqrt[a+b*x^3])]/(2*Sqrt[2]*Rt[a,2]*d*r^(3/2)) -
    q*(2-r)*ArcTanh[Rt[a,2]*(1-r)*Sqrt[r]*(1+q*x)/(Sqrt[2]*Sqrt[a+b*x^3])]/(6*Sqrt[2]*Rt[a,2]*d*Sqrt[r]) -
    q*(2-r)*ArcTanh[Rt[a,2]*Sqrt[r]*(1+r-2*q*x)/(Sqrt[2]*Sqrt[a+b*x^3])]/(3*Sqrt[2]*Rt[a,2]*d*Sqrt[r]) /;
    FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && EqQ[b^2*c^2-20*a*b*c*d-8*a^2*d^2,0] && PosQ[a]
```

$$2: \int \frac{x}{(c + d x^3) \sqrt{a + b x^3}} dx \text{ when } b c - a d \neq 0 \wedge b^2 c^2 - 20 a b c d - 8 a^2 d^2 = 0 \wedge a \neq 0$$

Reference: Goursat pseudo-elliptic integral

Note: If $b^2 c^2 - 20 a b c d - 8 a^2 d^2 = (b c - 10 a d + 6 \sqrt{3} a d) (b c - 10 a d - 6 \sqrt{3} a d) = 0$, then $\frac{b c - 10 a d}{6 a d}$ should simplify to $\sqrt{3}$ or $-\sqrt{3}$.

Rule 1.1.3.4.6.1.8.3.1.3.2: If $b c - a d \neq 0 \wedge b^2 c^2 - 20 a b c d - 8 a^2 d^2 = 0 \wedge a \neq 0$, let $q \rightarrow \left(\frac{b}{a}\right)^{1/3}$, and $r \rightarrow \frac{b c - 10 a d}{6 a d}$, then

$$\int \frac{x}{(c + d x^3) \sqrt{a + b x^3}} dx \rightarrow$$

$$\frac{q(2-r) \operatorname{ArcTanh}\left[\frac{(1-r) \sqrt{a+b x^3}}{\sqrt{2} \sqrt{-a} r^{3/2}}\right]}{3 \sqrt{2} \sqrt{-a} d r^{3/2}} - \frac{q(2-r) \operatorname{ArcTanh}\left[\frac{\sqrt{-a} \sqrt{r}(1+r)(1+q x)}{\sqrt{2} \sqrt{a+b x^3}}\right]}{2 \sqrt{2} \sqrt{-a} d r^{3/2}} - \frac{q(2-r) \operatorname{ArcTan}\left[\frac{\sqrt{-a}(1-r) \sqrt{r}(1+q x)}{\sqrt{2} \sqrt{a+b x^3}}\right]}{6 \sqrt{2} \sqrt{-a} d \sqrt{r}} - \frac{q(2-r) \operatorname{ArcTan}\left[\frac{\sqrt{-a} \sqrt{r}(1+r-2 q x)}{\sqrt{2} \sqrt{a+b x^3}}\right]}{3 \sqrt{2} \sqrt{-a} d \sqrt{r}}$$

Program code:

```
Int[x_/((c+d_.*x_^3)*Sqrt[a_+b_.*x_^3]),x_Symbol] :=
  With[{q=Rt[b/a,3],r=Simplify[(b*c-10*a*d)/(6*a*d)]},
    q*(2-r)*ArcTanh[(1-r)*Sqrt[a+b*x^3]/(Sqrt[2]*Rt[-a,2]*r^(3/2))]/(3*Sqrt[2]*Rt[-a,2]*d*r^(3/2)) -
    q*(2-r)*ArcTanh[Rt[-a,2]*Sqrt[r]*(1+r)*(1+q*x)/(Sqrt[2]*Sqrt[a+b*x^3])]/(2*Sqrt[2]*Rt[-a,2]*d*r^(3/2)) -
    q*(2-r)*ArcTan[Rt[-a,2]*(1-r)*Sqrt[r]*(1+q*x)/(Sqrt[2]*Sqrt[a+b*x^3])]/(6*Sqrt[2]*Rt[-a,2]*d*Sqrt[r]) -
    q*(2-r)*ArcTan[Rt[-a,2]*Sqrt[r]*(1+r-2*q*x)/(Sqrt[2]*Sqrt[a+b*x^3])]/(3*Sqrt[2]*Rt[-a,2]*d*Sqrt[r]) /;
    FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && EqQ[b^2*c^2-20*a*b*c*d-8*a^2*d^2,0] && NegQ[a]
```

$$2: \int \frac{x \sqrt{a+b x^3}}{c+d x^3} dx \text{ when } b c - a d \neq 0 \wedge (b c - 4 a d = 0 \vee b c + 8 a d = 0 \vee b^2 c^2 - 20 a b c d - 8 a^2 d^2 = 0)$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{\sqrt{a+b z}}{c+d z} = \frac{b}{d \sqrt{a+b z}} - \frac{b c - a d}{d (c+d z) \sqrt{a+b z}}$$

Rule 1.1.3.4.6.1.8.3.2: If $b c - a d \neq 0 \wedge (b c - 4 a d = 0 \vee b c + 8 a d = 0 \vee b^2 c^2 - 20 a b c d - 8 a^2 d^2 = 0)$, then

$$\int \frac{x \sqrt{a+b x^3}}{c+d x^3} dx \rightarrow \frac{b}{d} \int \frac{x}{\sqrt{a+b x^3}} dx - \frac{b c - a d}{d} \int \frac{x}{(c+d x^3) \sqrt{a+b x^3}} dx$$

Program code:

```
Int[x*Sqrt[a+b_.**x_^3]/(c+d_.**x_^3),x_Symbol] :=
  b/d*Int[x/Sqrt[a+b*x^3],x] - (b*c-a*d)/d*Int[x/((c+d*x^3)*Sqrt[a+b*x^3]),x] /;
FreeQ[{c,d,a,b},x] && NeQ[b*c-a*d,0] && (EqQ[b*c-4*a*d,0] || EqQ[b*c+8*a*d,0] || EqQ[b^2*c^2-20*a*b*c*d-8*a^2*d^2,0])
```

$$4. \int \frac{x^2 (c+d x^4)^q}{a+b x^4} dx \text{ when } b c - a d \neq 0 \wedge q^2 = \frac{1}{4}$$

$$1: \int \frac{x^2}{(a+b x^4) \sqrt{c+d x^4}} dx \text{ when } b c - a d \neq 0$$

Derivation: Algebraic expansion

$$\text{Basis: If } \frac{r}{s} = \left(-\frac{a}{b}\right)^{1/2}, \text{ then } \frac{x^2}{a+b x^4} = \frac{s}{2 b (r+s x^2)} - \frac{s}{2 b (r-s x^2)}$$

Rule 1.1.3.4.6.1.8.4.1: If $b c - a d \neq 0$, let $\frac{r}{s} = \left(-\frac{a}{b}\right)^{1/2}$, then

$$\int \frac{x^2}{(a+b x^4) \sqrt{c+d x^4}} dx \rightarrow \frac{s}{2b} \int \frac{1}{(r+s x^2) \sqrt{c+d x^4}} dx - \frac{s}{2b} \int \frac{1}{(r-s x^2) \sqrt{c+d x^4}} dx$$

Program code:

```
Int[x_^2/((a+b_.**x_^4)*Sqrt[c+d_.**x_^4]),x_Symbol] :=
  With[{r=Numerator[Rt[-a/b,2]], s=Denominator[Rt[-a/b,2]]},
    s/(2*b)*Int[1/((r+s*x^2)*Sqrt[c+d*x^4]),x] - s/(2*b)*Int[1/((r-s*x^2)*Sqrt[c+d*x^4]),x] /;
  FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0]
```

2: $\int \frac{x^2 \sqrt{c+d x^4}}{a+b x^4} dx$ when $b c - a d \neq 0$

Derivation: Algebraic expansion

Basis: $\frac{\sqrt{c+d z}}{a+b z} = \frac{d}{b \sqrt{c+d z}} + \frac{b c - a d}{b (a+b z) \sqrt{c+d z}}$

Rule 1.1.3.4.6.1.8.4.2: If $b c - a d \neq 0$, then

$$\int \frac{x^2 \sqrt{c+d x^4}}{a+b x^4} dx \rightarrow \frac{d}{b} \int \frac{x^2}{\sqrt{c+d x^4}} dx + \frac{b c - a d}{b} \int \frac{x^2}{(a+b x^4) \sqrt{c+d x^4}} dx$$

Program code:

```
Int[x_^2*Sqrt[c+d_.**x_^4]/(a+b_.**x_^4),x_Symbol] :=
  d/b*Int[x^2/Sqrt[c+d*x^4],x] + (b*c-a*d)/b*Int[x^2/((a+b*x^4)*Sqrt[c+d*x^4]),x] /;
  FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0]
```

9. $\int \frac{x^m}{\sqrt{a+b x^n} \sqrt{c+d x^n}} dx$ when $b c - a d \neq 0 \wedge (m|n) \in \mathbb{Z} \wedge 0 < m - n + 1 < n$

$$1: \int \frac{x^2}{\sqrt{a+b x^2} \sqrt{c+d x^2}} dx \text{ when } b c - a d \neq 0 \wedge \frac{b}{a} > 0 \wedge \frac{d}{c} > 0$$

Rule 1.1.3.4.6.1.9.1: If $b c - a d \neq 0 \wedge \frac{b}{a} > 0 \wedge \frac{d}{c} > 0$, then

$$\int \frac{x^2}{\sqrt{a+b x^2} \sqrt{c+d x^2}} dx \rightarrow \frac{x \sqrt{a+b x^2}}{b \sqrt{c+d x^2}} - \frac{c}{b} \int \frac{\sqrt{a+b x^2}}{(c+d x^2)^{3/2}} dx$$

Program code:

```
Int[x_^2/(Sqrt[a+b_*x_^2]*Sqrt[c+d_*x_^2]),x_Symbol] :=
  x*Sqrt[a+b*x^2]/(b*Sqrt[c+d*x^2]) - c/b*Int[Sqrt[a+b*x^2]/(c+d*x^2)^(3/2),x] /;
FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && PosQ[b/a] && PosQ[d/c] && Not[SimplerSqrtQ[b/a,d/c]]
```

$$2: \int \frac{x^n}{\sqrt{a+b x^n} \sqrt{c+d x^n}} dx \text{ when } b c - a d \neq 0 \wedge (n = 2 \vee n = 4)$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{z}{\sqrt{a+b z}} == \frac{\sqrt{a+b z}}{b} - \frac{a}{b \sqrt{a+b z}}$$

Rule 1.1.3.4.6.1.9.2: If $b c - a d \neq 0 \wedge (n = 2 \vee n = 4)$, then

$$\int \frac{x^n}{\sqrt{a+b x^n} \sqrt{c+d x^n}} dx \rightarrow \frac{1}{b} \int \frac{\sqrt{a+b x^n}}{\sqrt{c+d x^n}} dx - \frac{a}{b} \int \frac{1}{\sqrt{a+b x^n} \sqrt{c+d x^n}} dx$$

Program code:

```
Int[x_^n/(Sqrt[a+b_*x_^n]*Sqrt[c+d_*x_^n]),x_Symbol] :=
  1/b*Int[Sqrt[a+b*x^n]/Sqrt[c+d*x^n],x] - a/b*Int[1/(Sqrt[a+b*x^n]*Sqrt[c+d*x^n]),x] /;
FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && (EqQ[n,2] || EqQ[n,4]) && Not[EqQ[n,2] && SimplerSqrtQ[-b/a,-d/c]]
```

10: $\int x^m (a+b x^n)^p (c+d x^n)^q dx$ when $n \in \mathbb{Z}^+ \wedge \left(p + \frac{m+1}{n} \mid q\right) \in \mathbb{Z} \wedge -1 < p < 0$

Derivation: Integration by substitution

Basis: If $p + \frac{m+1}{n} \in \mathbb{Z}$, let $k = \text{Denominator}[p]$, then

$$x^m (a+b x^n)^p F[x^n] = \frac{k a^{p+\frac{m+1}{n}}}{n} \text{Subst}\left[\frac{x^{\frac{k(m+1)}{n}-1}}{(1-b x^k)^{p+\frac{m+1}{n}+1}} F\left[\frac{a x^k}{1-b x^k}\right], x, \frac{x^{n/k}}{(a+b x^n)^{1/k}}\right] \partial_x \frac{x^{n/k}}{(a+b x^n)^{1/k}}$$

Basis: If $\left(p + \frac{m+1}{n} \mid q\right) \in \mathbb{Z}$, let $k = \text{Denominator}[p]$, then

$$x^m (a+b x^n)^p (c+d x^n)^q = \frac{k a^{p+\frac{m+1}{n}}}{n} \text{Subst}\left[\frac{x^{\frac{k(m+1)}{n}-1} (c - (b c - a d) x^k)^q}{(1-b x^k)^{p+q+\frac{m+1}{n}+1}}, x, \frac{x^{n/k}}{(a+b x^n)^{1/k}}\right] \partial_x \frac{x^{n/k}}{(a+b x^n)^{1/k}}$$

Note: The exponents in the resulting integrand are integers.

Rule 1.1.3.4.6.1.10: If $n \in \mathbb{Z}^+ \wedge \left(p + \frac{m+1}{n} \mid q\right) \in \mathbb{Z} \wedge -1 < p < 0$, let $k = \text{Denominator}[p]$, then

$$\int x^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \frac{k a^{p+\frac{m+1}{n}}}{n} \text{Subst}\left[\int \frac{x^{\frac{k(m+1)}{n}-1} (c - (b c - a d) x^k)^q}{(1-b x^k)^{p+q+\frac{m+1}{n}+1}} dx, x, \frac{x^{n/k}}{(a+b x^n)^{1/k}}\right]$$

Program code:

```
Int[x_^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_.,x_Symbol] :=
  With[{k=Denominator[p]},
    k*a^(p+(m+1)/n)/n*
    Subst[Int[x^(k*(m+1)/n-1)*(c-(b*c-a*d)*x^k)^q/(1-b*x^k)^(p+q+(m+1)/n+1),x],x,x^(n/k)/(a+b*x^n)^(1/k)] /;
  FreeQ[{a,b,c,d},x] && IGtQ[n,0] && RationalQ[m,p] && IntegersQ[p+(m+1)/n,q] && LtQ[-1,p,0]
```

$$2. \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^-$$

$$1. \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^- \wedge m \in \mathbb{Q}$$

$$\text{1: } \int x^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^- \wedge m \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } F[x] = -\text{Subst}\left[\frac{F\left[\frac{x^{-1}}{x^2}\right]}{x^2}, x, \frac{1}{x}\right] \partial_x \frac{1}{x}$$

Rule 1.1.3.4.6.2.1.1: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^- \wedge m \in \mathbb{Z}$, then

$$\int x^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow -\text{Subst}\left[\int \frac{(a+b x^{-n})^p (c+d x^{-n})^q}{x^{m+2}} dx, x, \frac{1}{x}\right]$$

Program code:

```
Int[x_^m.*(a_+b_.*x_^n_)^p.*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  -Subst[Int[(a+b*x^(-n))^p*(c+d*x^(-n))^q/x^(m+2),x],x,1/x] /;
FreeQ[{a,b,c,d,p,q},x] && NeQ[b*c-a*d,0] && ILtQ[n,0] && IntegerQ[m]
```

2: $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge n \in \mathbb{Z}^- \wedge m \in \mathbb{F}$

Derivation: Integration by substitution

Basis: If $n \in \mathbb{Z} \wedge g > 1$, then $(e x)^m F[x^n] = -\frac{g}{e} \text{Subst}\left[\frac{F[e^{-n} x^{-g n}]}{x^{g(m+1)+1}}, x, \frac{1}{(e x)^{1/g}}\right] \partial_x \frac{1}{(e x)^{1/g}}$

Rule 1.1.3.4.6.2.1.2: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^- \wedge m \in \mathbb{F}$, let $g = \text{Denominator}[m]$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow -\frac{g}{e} \text{Subst}\left[\int \frac{(a+b e^{-n} x^{-g n})^p (c+d e^{-n} x^{-g n})^q}{x^{g(m+1)+1}} dx, x, \frac{1}{(e x)^{1/g}}\right]$$

Program code:

```
Int[(e_.*x_)^m_*(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  With[{g=Denominator[m]},
    -g/e*Subst[Int[(a+b*e^(-n)*x^(-g*n))^p*(c+d*e^(-n)*x^(-g*n))^q/x^(g*(m+1)+1),x],x,1/(e*x)^(1/g)] /;
  FreeQ[{a,b,c,d,e,p,q},x] && ILtQ[n,0] && FractionQ[m]
```


$$\mathbf{2:} \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{Z}^- \wedge m \notin \mathbb{Q}$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: } \partial_x \left((e x)^m (x^{-1})^m \right) = 0$$

$$\text{Basis: } F[x] = -\text{Subst} \left[\frac{F[x^{-1}]}{x^2}, x, \frac{1}{x} \right] \partial_x \frac{1}{x}$$

Rule 1.1.3.4.6.2.2: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^- \wedge m \notin \mathbb{Q}$, then

$$\begin{aligned} \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx &\rightarrow (e x)^m (x^{-1})^m \int \frac{(a+b x^n)^p (c+d x^n)^q}{(x^{-1})^m} dx \\ &\rightarrow - (e x)^m (x^{-1})^m \text{Subst} \left[\int \frac{(a+b x^{-n})^p (c+d x^{-n})^q}{x^{m+2}} dx, x, \frac{1}{x} \right] \end{aligned}$$

Program code:

```
Int[(e_.**x_)^m_*(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  -(e*x)^m*(x^(-1))^m*Subst[Int[(a+b*x^(-n))^p*(c+d*x^(-n))^q/x^(m+2),x],x,1/x] /;
FreeQ[{a,b,c,d,e,m,p,q},x] && NeQ[b*c-a*d,0] && ILtQ[n,0] && Not[RationalQ[m]]
```

$$7. \int (e x)^m (a + b x^n)^p (c + d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{F}$$

$$1: \int x^m (a + b x^n)^p (c + d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge n \in \mathbb{F}$$

Derivation: Integration by substitution

Basis: If $g \in \mathbb{Z}^+$, then $x^m F[x^n] = g \text{Subst}[x^{g(m+1)-1} F[x^{g n}], x, x^{1/g}] \partial_x x^{1/g}$

Rule 1.1.3.4.7.1: If $b c - a d \neq 0 \wedge n \in \mathbb{F}$, let $g = \text{Denominator}[n]$, then

$$\int x^m (a + b x^n)^p (c + d x^n)^q dx \rightarrow g \text{Subst}\left[\int x^{g(m+1)-1} (a + b x^{g n})^p (c + d x^{g n})^q dx, x, x^{1/g}\right]$$

Program code:

```
Int[x_^m.*(a+b.*x_^n_)^p.*(c+d.*x_^n_)^q_,x_Symbol] :=
  With[{g=Denominator[n]},
    g*Subst[Int[x^(g*(m+1)-1)*(a+b*x^(g*n))^p*(c+d*x^(g*n))^q,x],x,x^(1/g)]] /;
  FreeQ[{a,b,c,d,m,p,q},x] && NeQ[b*c-a*d,0] && FractionQ[n]
```

2: $\int (e x)^m (a + b x^n)^p (c + d x^n)^q dx$ when $b c - a d \neq 0 \wedge n \in \mathbb{F}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(e x)^m}{x^m} == 0$

Basis: $\frac{(e x)^m}{x^m} == \frac{e^{\text{IntPart}[m]} (e x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}}$

Rule 1.1.3.4.7.2: If $b c - a d \neq 0 \wedge n \in \mathbb{F}$, then

$$\int (e x)^m (a + b x^n)^p (c + d x^n)^q dx \rightarrow \frac{e^{\text{IntPart}[m]} (e x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (a + b x^n)^p (c + d x^n)^q dx$$

Program code:

```
Int[(e*x_)^m_*(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  e^IntPart[m]*(e*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a+b*x^n)^p*(c+d*x^n)^q,x] /;
FreeQ[{a,b,c,d,e,m,p,q},x] && NeQ[b*c-a*d,0] && FractionQ[n]
```

$$8. \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge \frac{n}{m+1} \in \mathbb{Z}$$

$$\text{X: } \int x^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge \frac{n}{m+1} \in \mathbb{Z}^- \wedge m \neq -1 \wedge -1 \leq p < 0 \wedge -1 \leq q < 0$$

Derivation: Integration by substitution

$$\text{Basis: If } \frac{n}{m+1} \in \mathbb{Z}, \text{ then } x^m F[x^n] = -\frac{1}{m+1} \frac{F\left[\left(x^{-(m+1)}\right)^{-\frac{n}{m+1}}\right]}{\left(x^{-(m+1)}\right)^2} \partial_x x^{-(m+1)}$$

Rule 1.1.3.4.8.x: If $b c - a d \neq 0 \wedge m \neq -1 \wedge \frac{n}{m+1} \in \mathbb{Z}^- \wedge -1 \leq p < 0 \wedge -1 \leq q < 0$, then

$$\int x^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow -\frac{1}{m+1} \text{Subst}\left[\int \frac{(a+b x^{-\frac{n}{m+1}})^p (c+d x^{-\frac{n}{m+1}})^q}{x^2} dx, x, x^{-(m+1)}\right]$$

Program code:

```
(* Int[x_^m.*(a_+b_.*x_^n_)^p.*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  -1/(m+1)*Subst[Int[(a+b*x^Simplify[-n/(m+1)])^p*(c+d*x^Simplify[-n/(m+1)])^q/x^2,x],x,x^(-(m+1))]/;
FreeQ[{a,b,c,d,m,n},x] && NeQ[b*c-a*d,0] && NeQ[m,-1] && ILtQ[Simplify[n/(m+1)+1],0] &&
GeQ[p,-1] && LtQ[p,0] && GeQ[q,-1] && LtQ[q,0] && Not[IntegerQ[n]] *)
```

$$\mathbf{1:} \int x^m (a + b x^n)^p (c + d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge \frac{n}{m+1} \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If $\frac{n}{m+1} \in \mathbb{Z}$, then $x^m F[x^n] = \frac{1}{m+1} \text{Subst}[F[x^{\frac{n}{m+1}}], x, x^{m+1}] \partial_x x^{m+1}$

Rule 1.1.3.4.8.1: If $b c - a d \neq 0 \wedge \frac{n}{m+1} \in \mathbb{Z}$, then

$$\int x^m (a + b x^n)^p (c + d x^n)^q dx \rightarrow \frac{1}{m+1} \text{Subst}\left[\int (a + b x^{\frac{n}{m+1}})^p (c + d x^{\frac{n}{m+1}})^q dx, x, x^{m+1}\right]$$

Program code:

```
Int[x_^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  1/(m+1)*Subst[Int[(a+b*x^Simplify[n/(m+1)])^p*(c+d*x^Simplify[n/(m+1)])^q,x],x,x^(m+1)] /;
FreeQ[{a,b,c,d,m,n,p,q},x] && NeQ[b*c-a*d,0] && IntegerQ[Simplify[n/(m+1)]] && Not[IntegerQ[n]]
```

$$\mathbf{2:} \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge \frac{n}{m+1} \in \mathbb{Z}$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{(e x)^m}{x^m} == 0$$

$$\text{Basis: } \frac{(e x)^m}{x^m} == \frac{e^{\text{IntPart}[m]} (e x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}}$$

Rule 1.1.3.4.8.2: If $b c - a d \neq 0 \wedge \frac{n}{m+1} \in \mathbb{Z}$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \frac{e^{\text{IntPart}[m]} (e x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (a+b x^n)^p (c+d x^n)^q dx$$

Program code:

```
Int[(e*x_)^m_.*(a+b_*x_^n_)^p_*(c+d_*x_^n_)^q_,x_Symbol] :=
  e^IntPart[m]*(e*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a+b*x^n)^p*(c+d*x^n)^q,x] /;
  FreeQ[{a,b,c,d,e,m,n,p,q},x] && NeQ[b*c-a*d,0] && IntegerQ[Simplify[n/(m+1)]] && Not[IntegerQ[n]]
```

$$9. \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge p < -1$$

$$1. \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge p < -1 \wedge q > 0$$

$$\mathbf{1:} \int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge p < -1 \wedge q > 1$$

Derivation: Binomial product recurrence 1 with $A = c$, $B = d$ and $q = q - 1$

Rule 1.1.3.4.9.1.1: If $b c - a d \neq 0 \wedge p < -1 \wedge q > 1$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \frac{-(b c - a d) (e x)^{m+1} (a+b x^n)^{p+1} (c+d x^n)^{q-1}}{a b e n (p+1)} + \frac{1}{a b n (p+1)} .$$

$$\int (e x)^m (a+b x^n)^{p+1} (c+d x^n)^{q-2} \left(c (c b n (p+1) + (c b - a d) (m+1)) + d (c b n (p+1) + (c b - a d) (m+n (q-1) + 1)) x^n \right) dx$$

Programcode:

```
Int[(e_.**x_)^m_.*(a+_b_.**x_^n_)^p_*(c+_d_.**x_^n_)^q_,x_Symbol] :=
  -(c*b-a*d)*(e*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(q-1)/(a*b*e*n*(p+1)) +
  1/(a*b*n*(p+1))*Int[(e*x)^m*(a+b*x^n)^(p+1)*(c+d*x^n)^(q-2)*
    Simp[c*(c*b*n*(p+1)+(c*b-a*d)*(m+1))+d*(c*b*n*(p+1)+(c*b-a*d)*(m+n*(q-1)+1))*x^n,x],x] /;
FreeQ[{a,b,c,d,e,m,n},x] && NeQ[b*c-a*d,0] && LtQ[p,-1] && GtQ[q,1] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

2: $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge p < -1 \wedge 0 < q < 1$

Derivation: Binomial product recurrence 1 with $A = 1$ and $B = 0$

Derivation: Binomial product recurrence 3b with $A = c$, $B = d$ and $q = q - 1$

Rule 1.1.3.4.9.1.2: If $b c - a d \neq 0 \wedge p < -1 \wedge 0 < q < 1$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow$$

$$- \frac{(e x)^{m+1} (a+b x^n)^{p+1} (c+d x^n)^q}{a e n (p+1)} +$$

$$\frac{1}{a n (p+1)} \int (e x)^m (a+b x^n)^{p+1} (c+d x^n)^{q-1} (c (m+n (p+1) + 1) + d (m+n (p+q+1) + 1) x^n) dx$$

Program code:

```
Int[(e_.**x_)^m_.*(a+_b_.**x_^n_)^p_*(c+_d_.**x_^n_)^q_,x_Symbol] :=
  -(e*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^q/(a*e*n*(p+1)) +
  1/(a*n*(p+1))*Int[(e*x)^m*(a+b*x^n)^(p+1)*(c+d*x^n)^(q-1)*Simp[c*(m+n*(p+1)+1)+d*(m+n*(p+q+1)+1))*x^n,x],x] /;
FreeQ[{a,b,c,d,e,m,n},x] && NeQ[b*c-a*d,0] && LtQ[p,-1] && LtQ[q,1] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

2: $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge p < -1$

Derivation: Binomial product recurrence 3b with $A = 1$ and $B = 0$

Rule 1.1.3.4.9.2: If $b c - a d \neq 0 \wedge p < -1$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow -\frac{b (e x)^{m+1} (a+b x^n)^{p+1} (c+d x^n)^{q+1}}{a e n (b c - a d) (p+1)} + \frac{1}{a n (b c - a d) (p+1)} \int (e x)^m (a+b x^n)^{p+1} (c+d x^n)^q (c b (m+1) + n (b c - a d) (p+1) + d b (m+n (p+q+2)+1) x^n) dx$$

Program code:

```
Int[(e.*x_)^m.*(a+b.*x_^n)^p.*(c+d.*x_^n)^q,x_Symbol] :=
  -b*(e*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(q+1)/(a*e*n*(b*c-a*d)*(p+1)) +
  1/(a*n*(b*c-a*d)*(p+1))*
  Int[(e*x)^m*(a+b*x^n)^(p+1)*(c+d*x^n)^q*
    Simp[c*b*(m+1)+n*(b*c-a*d)*(p+1)+d*b*(m+n*(p+q+2)+1)*x^n,x],x] /;
FreeQ[{a,b,c,d,e,m,n,q},x] && NeQ[b*c-a*d,0] && LtQ[p,-1] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

10. $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge q > 0$

1: $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge q > 0 \wedge p > 0$

Derivation: Binomial product recurrence 2b with $A = a$, $B = b$ and $p = p - 1$

Rule 1.1.3.4.10.1: If $b c - a d \neq 0 \wedge q > 0 \wedge p > 0$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \frac{(e x)^{m+1} (a+b x^n)^p (c+d x^n)^q}{e (m+n (p+q)+1)} +$$

$$\frac{n}{m+n(p+q)+1} \int (e x)^m (a+b x^n)^{p-1} (c+d x^n)^{q-1} (a c (p+q) + (q (b c - a d) + a d (p+q)) x^n) dx$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  (e*x)^(m+1)*(a+b*x^n)^p*(c+d*x^n)^q/(e*(m+n*(p+q)+1)) +
  n/(m+n*(p+q)+1)*Int[(e*x)^m*(a+b*x^n)^(p-1)*(c+d*x^n)^(q-1)*Simp[a*c*(p+q)+(q*(b*c-a*d)+a*d*(p+q))*x^n,x],x] /;
FreeQ[{a,b,c,d,e,m,n},x] && NeQ[b*c-a*d,0] && GtQ[q,0] && GtQ[p,0] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

2: $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge q > 1$

Derivation: Binomial product recurrence 2b with $A = c$, $B = d$ and $q = q - 1$

Rule 1.1.3.4.10.2: If $b c - a d \neq 0 \wedge q > 1$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \frac{d (e x)^{m+1} (a+b x^n)^{p+1} (c+d x^n)^{q-1}}{b e (m+n(p+q)+1)} + \frac{1}{b (m+n(p+q)+1)} \int (e x)^m (a+b x^n)^p (c+d x^n)^{q-2} dx$$

$$(c((b c - a d)(m+1) + c b n(p+q)) + (d(c b - a d)(m+1) + d n(q-1)(b c - a d) + c b d n(p+q)) x^n) dx$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  d*(e*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(q-1)/(b*e*(m+n*(p+q)+1)) +
  1/(b*(m+n*(p+q)+1))*Int[(e*x)^m*(a+b*x^n)^p*(c+d*x^n)^(q-2)*
  Simp[c*((c*b-a*d)*(m+1)+c*b*n*(p+q))+(d*(c*b-a*d)*(m+1)+d*n*(q-1)*(b*c-a*d)+c*b*d*n*(p+q))*x^n,x],x] /;
FreeQ[{a,b,c,d,e,m,n,p},x] && NeQ[b*c-a*d,0] && GtQ[q,1] && IntBinomialQ[a,b,c,d,e,m,n,p,q,x]
```

11. $\int \frac{(e x)^m}{(a+b x^n) (c+d x^n)} dx$ when $b c - a d \neq 0$

$$1: \int \frac{x^m}{(a+b x^n)(c+d x^n)} dx \text{ when } b c - a d \neq 0 \wedge (m = n \vee m = 2n - 1)$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{x^m}{(a+b x^n)(c+d x^n)} = -\frac{a x^{m-n}}{(b c - a d)(a+b x^n)} + \frac{c x^{m-n}}{(b c - a d)(c+d x^n)}$$

Rule 1.1.3.4.11.1: If $b c - a d \neq 0 \wedge (m = n \vee m = 2n - 1)$, then

$$\int \frac{x^m}{(a+b x^n)(c+d x^n)} dx \rightarrow -\frac{a}{b c - a d} \int \frac{x^{m-n}}{a+b x^n} dx + \frac{c}{b c - a d} \int \frac{x^{m-n}}{c+d x^n} dx$$

Program code:

```
Int[x_^m_/((a_+b_.x_^n_)*(c_+d_.x_^n_)),x_Symbol] :=
  -a/(b*c-a*d)*Int[x^(m-n)/(a+b*x^n),x] + c/(b*c-a*d)*Int[x^(m-n)/(c+d*x^n),x] /;
FreeQ[{a,b,c,d,m,n},x] && NeQ[b*c-a*d,0] && (EqQ[m,n] || EqQ[m,2*n-1])
```

$$2: \int \frac{(e x)^m}{(a+b x^n)(c+d x^n)} dx \text{ when } b c - a d \neq 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{1}{(a+b z)(c+d z)} = \frac{b}{(b c - a d)(a+b z)} - \frac{d}{(b c - a d)(c+d z)}$$

Rule 1.1.3.4.11.2: If $b c - a d \neq 0$, then

$$\int \frac{(e x)^m}{(a+b x^n)(c+d x^n)} dx \rightarrow \frac{b}{b c - a d} \int \frac{(e x)^m}{a+b x^n} dx - \frac{d}{b c - a d} \int \frac{(e x)^m}{c+d x^n} dx$$

Program code:

```
Int[(e_.x_)^m_/((a_+b_.x_^n_)*(c_+d_.x_^n_)),x_Symbol] :=
  b/(b*c-a*d)*Int[(e*x)^m/(a+b*x^n),x] - d/(b*c-a*d)*Int[(e*x)^m/(c+d*x^n),x] /;
FreeQ[{a,b,c,d,e,n,m},x] && NeQ[b*c-a*d,0]
```

12: $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge p+2 \in \mathbb{Z}^+ \wedge q+2 \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule 1.1.3.4.12: If $b c - a d \neq 0 \wedge p+2 \in \mathbb{Z}^+ \wedge q+2 \in \mathbb{Z}^+$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \int \text{ExpandIntegrand}[(e x)^m (a+b x^n)^p (c+d x^n)^q, x] dx$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  Int[ExpandIntegrand[(e*x)^m*(a+b*x^n)^p*(c+d*x^n)^q,x],x] /;
FreeQ[{a,b,c,d,e,m},x] && NeQ[b*c-a*d,0] && IGtQ[p,-2] && (IGtQ[q,-2] || EqQ[q,-3] && IntegerQ[(m-1)/2])
```

A. $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge m \neq -1 \wedge m \neq n-1$

1: $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge m \neq -1 \wedge m \neq n-1 \wedge (p \in \mathbb{Z} \vee a > 0) \wedge (q \in \mathbb{Z} \vee c > 0)$

Rule 1.1.3.4.A.1: If $b c - a d \neq 0 \wedge m \neq -1 \wedge m \neq n-1 \wedge (p \in \mathbb{Z} \vee a > 0) \wedge (q \in \mathbb{Z} \vee c > 0)$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \frac{a^p c^q (e x)^{m+1}}{e (m+1)} \text{AppellF1}\left[\frac{m+1}{n}, -p, -q, 1+\frac{m+1}{n}, -\frac{b x^n}{a}, -\frac{d x^n}{c}\right]$$

Program code:

```
Int[(e_.**x_)^m_.*(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  a^p*c^q*(e*x)^(m+1)/(e*(m+1))*AppellF1[(m+1)/n,-p,-q,1+(m+1)/n,-b*x^n/a,-d*x^n/c] /;
FreeQ[{a,b,c,d,e,m,n,p,q},x] && NeQ[b*c-a*d,0] && NeQ[m,-1] && NeQ[m,n-1] &&
  (IntegerQ[p] || GtQ[a,0]) && (IntegerQ[q] || GtQ[c,0])
```

2: $\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx$ when $b c - a d \neq 0 \wedge m \neq -1 \wedge m \neq n-1 \wedge \neg (p \in \mathbb{Z} \vee a > 0)$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(a+b x^n)^p}{\left(1+\frac{b x^n}{a}\right)^p} == 0$

Rule 1.1.3.4.A.2: If $b c - a d \neq 0 \wedge m \neq -1 \wedge m \neq n-1 \wedge \neg (p \in \mathbb{Z} \vee a > 0)$, then

$$\int (e x)^m (a+b x^n)^p (c+d x^n)^q dx \rightarrow \frac{a^{\text{IntPart}[p]} (a+b x^n)^{\text{FracPart}[p]}}{\left(1+\frac{b x^n}{a}\right)^{\text{FracPart}[p]}} \int (e x)^m \left(1+\frac{b x^n}{a}\right)^p (c+d x^n)^q dx$$

Program code:

```
Int[(e_.*x_)^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  a^IntPart[p]*(a+b*x^n)^FracPart[p]/(1+b*x^n/a)^FracPart[p]*Int[(e*x)^m*(1+b*x^n/a)^p*(c+d*x^n)^q,x] /;
FreeQ[{a,b,c,d,e,m,n,p,q},x] && NeQ[b*c-a*d,0] && NeQ[m,-1] && NeQ[m,n-1] && Not[IntegerQ[p] || GtQ[a,0]]
```

$$\text{S. } \int u^m (a + b v^n)^p (c + d v^n)^q dx \text{ when } v = e + f x \wedge u = g v$$

$$\text{1: } \int x^m (a + b v^n)^p (c + d v^n)^q dx \text{ when } v = e + f x \wedge m \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If $m \in \mathbb{Z}$, then $x^m F[e + f x] = \frac{1}{f^{m+1}} \text{Subst}[(x - e)^m F[x], x, e + f x] \partial_x (e + f x)$

Rule 1.1.3.4.S.1: If $v = e + f x \wedge m \in \mathbb{Z}$, then

$$\int x^m (a + b v^n)^p (c + d v^n)^q dx \rightarrow \frac{1}{f^{m+1}} \text{Subst}\left[\int (x - e)^m (a + b x^n)^p (c + d x^n)^q dx, x, v\right]$$

Program code:

```
Int[x_^m.*(a_.+b_.*v_^n_)^p.*(c_.+d_.*v_^n_)^q_,x_Symbol] :=
  1/Coefficient[v,x,1]^(m+1)*Subst[Int[SimplifyIntegrand[(x-Coefficient[v,x,0])^m*(a+b*x^n)^p*(c+d*x^n)^q,x],x],x,v] /;
FreeQ[{a,b,c,d,n,p,q},x] && LinearQ[v,x] && IntegerQ[m] && NeQ[v,x]
```

2: $\int u^m (a + b v^n)^p (c + d v^n)^q dx$ when $v = e + f x$ \wedge $u = g v$

Derivation: Integration by substitution and piecewise constant extraction

Basis: If $u = g v$, then $\partial_x \frac{u^m}{v^m} = 0$

Rule 1.1.3.4.S.2: If $v = e + f x$ \wedge $u = g v$, then

$$\int u^m (a + b v^n)^p (c + d v^n)^q dx \rightarrow \frac{u^m}{f v^m} \text{Subst}\left[\int x^m (a + b x^n)^p (c + d x^n)^q dx, x, v\right]$$

Program code:

```
Int[u_^m.*(a_.+b_.*v_^n_)^p_.*(c_.+d_.*v_^n_)^q_. ,x_Symbol] :=
  u^m/(Coefficient[v,x,1]*v^m)*Subst[Int[x^m*(a+b*x^n)^p*(c+d*x^n)^q,x],x,v] /;
FreeQ[{a,b,c,d,m,n,p,q},x] && LinearPairQ[u,v,x]
```

$$\text{N. } \int (e x)^m (a+b x^n)^p (c+d x^{-n})^q dx$$

$$1. \int x^m (a+b x^n)^p (c+d x^{-n})^q dx$$

$$\text{1: } \int x^m (a+b x^n)^p (c+d x^{-n})^q dx \text{ when } q \in \mathbb{Z}$$

Derivation: Algebraic normalization

Basis: If $q \in \mathbb{Z}$, then $(c+d x^{-n})^q = x^{-nq} (d+c x^n)^q$

Rule 1.1.3.4.N.1.1: If $q \in \mathbb{Z}$, then

$$\int x^m (a+b x^n)^p (c+d x^{-n})^q dx \rightarrow \int x^{m-nq} (a+b x^n)^p (d+c x^n)^q dx$$

Program code:

```
Int[x_^m_.*(a_+b_.*x_^n_.)^p_.*(c_+d_.*x_^mn_.)^q_,x_Symbol] :=
  Int[x^(m-n*q)*(a+b*x^n)^p*(d+c*x^n)^q,x] /;
FreeQ[{a,b,c,d,m,n,p},x] && EqQ[mn,-n] && IntegerQ[q] && (PosQ[n] || Not[IntegerQ[p]])
```

$$\mathbf{2:} \int x^m (a + b x^n)^p (c + d x^{-n})^q dx \text{ when } q \notin \mathbb{Z} \wedge p \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{x^n q (c+d x^{-n})^q}{(d+c x^n)^q} == 0$$

$$\text{Basis: } \frac{x^n q (c+d x^{-n})^q}{(d+c x^n)^q} == \frac{x^n \text{FracPart}[q] (c+d x^{-n})^{\text{FracPart}[q]}}{(d+c x^n)^{\text{FracPart}[q]}}$$

Rule 1.1.3.4.N.1.2: If $q \notin \mathbb{Z} \wedge p \notin \mathbb{Z}$, then

$$\int x^m (a + b x^n)^p (c + d x^{-n})^q dx \rightarrow \frac{x^n \text{FracPart}[q] (c + d x^{-n})^{\text{FracPart}[q]}}{(d + c x^n)^{\text{FracPart}[q]}} \int x^{m-n q} (a + b x^n)^p (d + c x^n)^q dx$$

Program code:

```
Int[x_^m.*(a_+b_.x_^n.)^p.*(c_+d_.x_^mn.)^q_,x_Symbol] :=
  x^(n*FracPart[q])*(c+d*x^(-n))^FracPart[q]/(d+c*x^n)^FracPart[q]*Int[x^(m-n*q)*(a+b*x^n)^p*(d+c*x^n)^q,x] /;
FreeQ[{a,b,c,d,m,n,p,q},x] && EqQ[mn,-n] && Not[IntegerQ[q]] && Not[IntegerQ[p]]
```


$$2: \int (e x)^m (a+b x^n)^p (c+d x^{-n})^q dx$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{(e x)^m}{x^m} == 0$$

$$\text{Basis: } \frac{(e x)^m}{x^m} == \frac{e^{\text{IntPart}[m]} (e x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}}$$

Rule 1.1.3.4.N.2:

$$\int (e x)^m (a+b x^n)^p (c+d x^{-n})^q dx \rightarrow \frac{e^{\text{IntPart}[m]} (e x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (a+b x^n)^p (c+d x^{-n})^q dx$$

Program code:

```
Int[(e*x_)^m_*(a_+b_*x_^n_)^p_*(c+d_*x_^mn_)^q_,x_Symbol] :=
  e^IntPart[m]*(e*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a+b*x^n)^p*(c+d*x^(-n))^q,x] /;
FreeQ[{a,b,c,d,e,m,n,p,q},x] && EqQ[mn,-n]
```

```
(* IntBinomialQ[a,b,c,d,e,m,n,p,q,x] returns True iff (e*x)^m*(a+b*x^n)^p*(c+d*x^n)^q is integrable wrt x in terms of non-Appell funct
IntBinomialQ[a_,b_,c_,d_,e_,m_,n_,p_,q_,x_Symbol] :=
  IntegersQ[p,q] || IGtQ[p,0] || IGtQ[q,0] ||
  EqQ[n,2] && (IntegersQ[m,2*p,2*q] || IntegersQ[2*m,p,2*q] || IntegersQ[2*m,2*p,q]) ||
  EqQ[n,4] && (IntegersQ[m,p,2*q] || IntegersQ[m,2*p,q]) ||
  EqQ[n,2] && IntegersQ[m/2,p+1/3,q] && (EqQ[b*c+3*a*d,0] || EqQ[b*c-9*a*d,0]) ||
  EqQ[n,2] && IntegersQ[m/2,q+1/3,p] && (EqQ[a*d+3*b*c,0] || EqQ[a*d-9*b*c,0]) ||
  EqQ[n,3] && IntegersQ[(m-1)/3,q,p-1/2] && (EqQ[b*c-4*a*d,0] || EqQ[b*c+8*a*d,0] || EqQ[b^2*c^2-20*a*b*c*d-8*a^2*d^2,0]) ||
  EqQ[n,3] && IntegersQ[(m-1)/3,p,q-1/2] && (EqQ[4*b*c-a*d,0] || EqQ[8*b*c+a*d,0] || EqQ[8*b^2*c^2+20*a*b*c*d-a^2*d^2,0])
```

Rules for integrands of the form $u \left(a_1 + b_1 x^{n/2} \right)^p \left(a_2 + b_2 x^{n/2} \right)^p F[x^n]$

1: $\int u \left(a_1 + b_1 x^{n/2} \right)^p \left(a_2 + b_2 x^{n/2} \right)^p F[x^n] dx$ when $a_2 b_1 + a_1 b_2 = 0 \wedge (p \in \mathbb{Z} \vee (a_1 > 0 \wedge a_2 > 0))$

Derivation: Algebraic simplification

Basis: If $a_2 b_1 + a_1 b_2 = 0 \wedge (p \in \mathbb{Z} \vee (a_1 > 0 \wedge a_2 > 0))$, then $(a_1 + b_1 x^{n/2})^p (a_2 + b_2 x^{n/2})^p = (a_1 a_2 + b_1 b_2 x^n)^p$

Rule: If $a_2 b_1 + a_1 b_2 = 0 \wedge (p \in \mathbb{Z} \vee (a_1 > 0 \wedge a_2 > 0))$, then

$$\int u \left(a_1 + b_1 x^{n/2} \right)^p \left(a_2 + b_2 x^{n/2} \right)^p F[x^n] dx \rightarrow \int u \left(a_1 a_2 + b_1 b_2 x^n \right)^p F[x^n] dx$$

Program code:

```
Int[u_.*(a1_+b1_.*x_^non2_.)^p_.*(a2_+b2_.*x_^non2_.)^p_.*(c_+d_.*x_^n_.)^q_,x_Symbol] :=
  Int[u*(a1*a2+b1*b2*x^n)^p*(c+d*x^n)^q,x] /;
FreeQ[{a1,b1,a2,b2,c,d,n,p,q},x] && EqQ[non2,n/2] && EqQ[a2*b1+a1*b2,0] && (IntegerQ[p] || GtQ[a1,0] && GtQ[a2,0])
```

```
Int[u_.*(a1_+b1_.*x_^non2_.)^p_.*(a2_+b2_.*x_^non2_.)^p_.*(c_+d_.*x_^n_.+e_.*x_^n2_.)^q_,x_Symbol] :=
  Int[u*(a1*a2+b1*b2*x^n)^p*(c+d*x^n+e*x^(2*n))^q,x] /;
FreeQ[{a1,b1,a2,b2,c,d,e,n,p,q},x] && EqQ[non2,n/2] && EqQ[n2,2*n] && EqQ[a2*b1+a1*b2,0] && (IntegerQ[p] || GtQ[a1,0] && GtQ[a2,0])
```

2: $\int u (a_1 + b_1 x^{n/2})^p (a_2 + b_2 x^{n/2})^p F[x^n] dx$ when $a_2 b_1 + a_1 b_2 = 0 \wedge \neg (p \in \mathbb{Z} \vee (a_1 > 0 \wedge a_2 > 0))$

Derivation: Piecewise constant extraction

Basis: If $a_2 b_1 + a_1 b_2 = 0$, then $a_x \frac{(a_1 + b_1 x^{n/2})^p (a_2 + b_2 x^{n/2})^p}{(a_1 a_2 + b_1 b_2 x^n)^p} = 0$

Rule: If $a_2 b_1 + a_1 b_2 = 0$, then

$$\int u (a_1 + b_1 x^{n/2})^p (a_2 + b_2 x^{n/2})^p F[x^n] dx \rightarrow \frac{(a_1 + b_1 x^{n/2})^{\text{FracPart}[p]} (a_2 + b_2 x^{n/2})^{\text{FracPart}[p]}}{(a_1 a_2 + b_1 b_2 x^n)^{\text{FracPart}[p]}} \int u (a_1 a_2 + b_1 b_2 x^n)^p F[x^n] dx$$

Program code:

```
Int[u_.*(a1_+b1_.*x_^non2_)^p_.*(a2_+b2_.*x_^non2_)^p_.*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  (a1+b1*x^(n/2))^FracPart[p]*(a2+b2*x^(n/2))^FracPart[p]/(a1*a2+b1*b2*x^n)^FracPart[p]*
  Int[u*(a1*a2+b1*b2*x^n)^p*(c+d*x^n)^q,x] /;
FreeQ[{a1,b1,a2,b2,c,d,n,p,q},x] && EqQ[non2,n/2] && EqQ[a2*b1+a1*b2,0] && Not[EqQ[n,2] && IGtQ[q,0]]
```

```
Int[u_.*(a1_+b1_.*x_^non2_)^p_.*(a2_+b2_.*x_^non2_)^p_.*(c_+d_.*x_^n_+e_.*x_^n2_)^q_,x_Symbol] :=
  (a1+b1*x^(n/2))^FracPart[p]*(a2+b2*x^(n/2))^FracPart[p]/(a1*a2+b1*b2*x^n)^FracPart[p]*
  Int[u*(a1*a2+b1*b2*x^n)^p*(c+d*x^n+e*x^(2*n))^q,x] /;
FreeQ[{a1,b1,a2,b2,c,d,e,n,p,q},x] && EqQ[non2,n/2] && EqQ[n2,2*n] && EqQ[a2*b1+a1*b2,0]
```