

Rules for integrands of the form $(a + b x^2 + c x^4)^p$

1. $\int (a + b x^2 + c x^4)^p dx$ when $b^2 - 4 a c == 0$

x: $\int (a + b x^2 + c x^4)^p dx$ when $b^2 - 4 a c == 0 \wedge p \in \mathbb{Z}$

Derivation: Algebraic simplification

Basis: If $b^2 - 4 a c == 0$, then $a + b z + c z^2 == \frac{1}{c} \left(\frac{b}{2} + c z \right)^2$

Rule 1.2.2.1.1.1: If $b^2 - 4 a c == 0 \wedge p \in \mathbb{Z}$, then

$$\int (a + b x^2 + c x^4)^p dx \rightarrow \frac{1}{c^p} \int \left(\frac{b}{2} + c x^2 \right)^{2p} dx$$

Program code:

```
(* Int[(a_+b_.*x_^2+c_.*x_^4)^p_,x_Symbol] :=
  1/c^p*Int[(b/2+c*x^2)^(2*p),x] /;
FreeQ[{a,b,c,p},x] && EqQ[b^2-4*a*c,0] && IntegerQ[p] *)
```

2. $\int (a + bx^2 + cx^4)^p dx$ when $b^2 - 4ac = 0 \wedge p \notin \mathbb{Z}$

x: $\int \frac{1}{(a + bx^2 + cx^4)^{5/4}} dx$ when $b^2 - 4ac = 0$

Derivation: Square trinomial recurrence $2c$ with $m + 4(p + 1) + 1 = 0$

Rule 1.2.2.1.1.2.1: If $b^2 - 4ac = 0$, then

$$\int \frac{1}{(a + bx^2 + cx^4)^{5/4}} dx \rightarrow \frac{2x}{3a(a + bx^2 + cx^4)^{1/4}} + \frac{x(2a + bx^2)}{6a(a + bx^2 + cx^4)^{5/4}}$$

Program code:

```
(* Int[1/(a+b*x^2+c*x^4)^(5/4),x_Symbol] :=
  2*x/(3*a*(a+b*x^2+c*x^4)^(1/4)) + x*(2*a+b*x^2)/(6*a*(a+b*x^2+c*x^4)^(5/4)) /;
FreeQ[{a,b,c},x] && EqQ[b^2-4*a*c,0] *)
```

2: $\int (a + bx^2 + cx^4)^p dx$ when $b^2 - 4ac = 0 \wedge p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: If $b^2 - 4ac = 0$, then $\partial_x \frac{(a+bx^2+cx^4)^p}{(b+2cx^2)^{2p}} = 0$

Note: If $b^2 - 4ac = 0$, then $a + bz + cz^2 = \frac{1}{4c} (b + 2cz)^2$

Rule 1.2.2.1.1.2.2: If $b^2 - 4ac = 0 \wedge p \notin \mathbb{Z}$, then

$$\int (a + bx^2 + cx^4)^p dx \rightarrow \frac{(a + bx^2 + cx^4)^p}{(b + 2cx^2)^{2p}} \int (b + 2cx^2)^{2p} dx$$

Program code:

```
Int[(a+b_.**x_^2+c_.**x_^4)^p_,x_Symbol] :=
  (a+b*x^2+c*x^4)^p/(b+2*c*x^2)^(2*p)*Int[(b+2*c*x^2)^(2*p),x] /;
FreeQ[{a,b,c,p},x] && EqQ[b^2-4*a*c,0] && IntegerQ[p-1/2]
```

```
Int[(a+b_.**x_^2+c_.**x_^4)^p_,x_Symbol] :=
  a^IntPart[p]*(a+b*x^2+c*x^4)^FracPart[p]/(1+2*c*x^2/b)^(2*FracPart[p])*Int[(1+2*c*x^2/b)^(2*p),x] /;
FreeQ[{a,b,c,p},x] && EqQ[b^2-4*a*c,0] && Not[IntegerQ[2*p]]
```

$$2. \int (a + bx^2 + cx^4)^p dx \text{ when } b^2 - 4ac \neq 0 \wedge p > 0$$

$$1: \int (a + bx^2 + cx^4)^p dx \text{ when } b^2 - 4ac \neq 0 \wedge p \in \mathbb{Z}^+$$

Derivation: Algebraic expansion

Rule 1.2.2.1.2.1: If $b^2 - 4ac \neq 0 \wedge p \in \mathbb{Z}^+$, then

$$\int (a + bx^2 + cx^4)^p dx \rightarrow \int \text{ExpandIntegrand}[(a + bx^2 + cx^4)^p, x] dx$$

Program code:

```
Int[(a_+b_.*x_^2+c_.*x_^4)^p_,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*x^2+c*x^4)^p,x],x] /;
FreeQ[{a,b,c},x] && NeQ[b^2-4*a*c,0] && IGtQ[p,0]
```

$$2: \int (a + bx^2 + cx^4)^p dx \text{ when } b^2 - 4ac \neq 0 \wedge p > 0$$

Derivation: Trinomial recurrence 1b with $m = 0$, $A = 1$ and $B = 0$

Rule 1.2.2.1.2.2: If $b^2 - 4ac \neq 0 \wedge p > 0$, then

$$\int (a + bx^2 + cx^4)^p dx \rightarrow \frac{x (a + bx^2 + cx^4)^p}{4p+1} + \frac{2p}{4p+1} \int (2a + bx^2) (a + bx^2 + cx^4)^{p-1} dx$$

Program code:

```
Int[(a_+b_.*x_^2+c_.*x_^4)^p_,x_Symbol] :=
  x*(a+b*x^2+c*x^4)^p/(4*p+1) +
  2*p/(4*p+1)*Int[(2*a+b*x^2)*(a+b*x^2+c*x^4)^(p-1),x] /;
FreeQ[{a,b,c},x] && NeQ[b^2-4*a*c,0] && GtQ[p,0] && IntegerQ[2*p]
```

3: $\int (a + bx^2 + cx^4)^p dx$ when $b^2 - 4ac \neq 0 \wedge p < -1$

Reference: G&R 2.161.5

Derivation: Trinomial recurrence 2b with $m = 0$, $A = 1$ and $B = 0$

Note: G&R 2.161.4 is a special case of G&R 2.161.5.

Rule 1.2.2.1.3: If $b^2 - 4ac \neq 0 \wedge p < -1$, then

$$\int (a + bx^2 + cx^4)^p dx \rightarrow -\frac{x(b^2 - 2ac + bcx^2)(a + bx^2 + cx^4)^{p+1}}{2a(p+1)(b^2 - 4ac)} + \frac{1}{2a(p+1)(b^2 - 4ac)} \int (b^2 - 2ac + 2(p+1)(b^2 - 4ac) + bc(4p+7)x^2)(a + bx^2 + cx^4)^{p+1} dx$$

Program code:

```
Int[(a_+b_.**x_^2+c_.**x_^4)^p_,x_Symbol] :=
  -x*(b^2-2*a*c+b*c*x^2)*(a+b*x^2+c*x^4)^(p+1)/(2*a*(p+1)*(b^2-4*a*c)) +
  1/(2*a*(p+1)*(b^2-4*a*c))*Int[(b^2-2*a*c+2*(p+1)*(b^2-4*a*c)+b*c*(4*p+7)*x^2)*(a+b*x^2+c*x^4)^(p+1),x] /;
FreeQ[{a,b,c},x] && NeQ[b^2-4*a*c,0] && LtQ[p,-1] && IntegerQ[2*p]
```

4. $\int \frac{1}{a + bx^2 + cx^4} dx$ when $b^2 - 4ac \neq 0$

1: $\int \frac{1}{a + bx^2 + cx^4} dx$ when $b^2 - 4ac \neq 0 \wedge b^2 - 4ac > 0$

Reference: G&R 2.161.1a

Derivation: Algebraic expansion

Basis: Let $q \rightarrow \sqrt{b^2 - 4ac}$, then $\frac{1}{a+bz+cz^2} = \frac{c}{q} \frac{1}{\frac{b}{2} - \frac{q}{2} + cz} - \frac{c}{q} \frac{1}{\frac{b}{2} + \frac{q}{2} + cz}$

■

Rule 1.2.2.1.4.1: If $b^2 - 4ac \neq 0$, let $q \rightarrow \sqrt{b^2 - 4ac}$, then

$$\int \frac{1}{a+bx^2+cx^4} dx \rightarrow \frac{c}{q} \int \frac{1}{\frac{b}{2} - \frac{q}{2} + cx^2} dx - \frac{c}{q} \int \frac{1}{\frac{b}{2} + \frac{q}{2} + cx^2} dx$$

Program code:

```
Int[1/(a_+b_.*x_^2+c_.*x_^4),x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    c/q*Int[1/(b/2-q/2+c*x^2),x] - c/q*Int[1/(b/2+q/2+c*x^2),x]] /;
  FreeQ[{a,b,c},x] && NeQ[b^2-4*a*c,0] && PosQ[b^2-4*a*c]
```

2: $\int \frac{1}{a+bx^2+cx^4} dx$ when $b^2 - 4ac \neq 0 \wedge b^2 - 4ac \not> 0$

Derivation: Algebraic expansion

Basis: If $q \rightarrow \sqrt{\frac{a}{c}}$ and $r \rightarrow \sqrt{2q - \frac{b}{c}}$, then $\frac{1}{a+bz^2+cz^4} = \frac{r-z}{2cqr(q-rz+z^2)} + \frac{r+z}{2cqr(q+rz+z^2)}$

Note: If $(a \mid b \mid c) \in \mathbb{R} \wedge b^2 - 4ac < 0$, then $\frac{a}{c} > 0$ and $2\sqrt{\frac{a}{c}} - \frac{b}{c} > 0$.

Rule 1.2.2.1.4.2: If $b^2 - 4ac \neq 0 \wedge b^2 - 4ac \not> 0$, let $q \rightarrow \sqrt{\frac{a}{c}}$ and $r \rightarrow \sqrt{2q - \frac{b}{c}}$, then

$$\int \frac{1}{a+bx^2+cx^4} dx \rightarrow \frac{1}{2cqr} \int \frac{r-x}{q-rx+x^2} dx + \frac{1}{2cqr} \int \frac{r+x}{q+rx+x^2} dx$$

Program code:

```
Int[1/(a+b_.*x_^2+c_.*x_^4),x_Symbol] :=
  With[{q=Rt[a/c,2]},
    With[{r=Rt[2*q-b/c,2]},
      1/(2*c*q*r)*Int[(r-x)/(q-r*x+x^2),x] + 1/(2*c*q*r)*Int[(r+x)/(q+r*x+x^2),x]] /;
  FreeQ[{a,b,c},x] && NeQ[b^2-4*a*c,0] && NegQ[b^2-4*a*c]
```

5. $\int \frac{1}{\sqrt{a+bx^2+cx^4}} dx$ when $b^2 - 4ac \neq 0$

1. $\int \frac{1}{\sqrt{a+bx^2+cx^4}} dx$ when $b^2 - 4ac > 0$

1: $\int \frac{1}{\sqrt{a+bx^2+cx^4}} dx$ when $b^2 - 4ac > 0 \wedge c < 0$

Derivation: Algebraic expansion

Basis: If $b^2 - 4ac > 0 \wedge c < 0$, let $q \rightarrow \sqrt{b^2 - 4ac}$, then

$$\sqrt{a + bx^2 + cx^4} = \frac{1}{2\sqrt{-c}} \sqrt{b + q + 2cx^2} \sqrt{-b + q - 2cx^2}$$

■ Rule 1.2.2.1.5.1.1: If $b^2 - 4ac > 0 \wedge c < 0$, let $q \rightarrow \sqrt{b^2 - 4ac}$, then

$$\int \frac{1}{\sqrt{a + bx^2 + cx^4}} dx \rightarrow 2\sqrt{-c} \int \frac{1}{\sqrt{b + q + 2cx^2} \sqrt{-b + q - 2cx^2}} dx$$

Program code:

```
Int[1/Sqrt[a+_.*x_^2+c_.*x_^4],x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    2*Sqrt[-c]*Int[1/(Sqrt[b+q+2*c*x^2]*Sqrt[-b+q-2*c*x^2]),x]] /;
  FreeQ[{a,b,c},x] && GtQ[b^2-4*a*c,0] && LtQ[c,0]
```


$$2. \int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \text{ when } b^2 - 4ac > 0 \wedge c \neq 0$$

$$1: \int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \text{ when } b^2 - 4ac > 0 \wedge \frac{c}{a} > 0 \wedge \frac{b}{a} < 0$$

Reference: G&R 3.165.2

Derivation: Piecewise constant extraction

$$\text{Basis: Let } q = \left(\frac{c}{a}\right)^{1/4}, \text{ then } \partial_x \frac{(1+q^2 x^2) \sqrt{\frac{(a+bx^2+cx^4)}{a(1+q^2 x^2)^2}}}{\sqrt{a+bx^2+cx^4}} = 0$$

Rule 1.2.2.1.5.1.2.1: If $b^2 - 4ac > 0 \wedge \frac{c}{a} > 0 \wedge \frac{b}{a} < 0$, let $q \rightarrow \left(\frac{c}{a}\right)^{1/4}$, then

$$\int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \rightarrow \frac{(1+q^2 x^2) \sqrt{\frac{(a+bx^2+cx^4)}{a(1+q^2 x^2)^2}}}{2q \sqrt{a+bx^2+cx^4}} \text{EllipticF}\left[2 \text{ArcTan}[qx], \frac{1}{2} - \frac{bq^2}{4c}\right]$$

Program code:

```
Int[1/Sqrt[a+_b_.**x_^2+c_.**x_^4],x_Symbol] :=
  With[{q=Rt[c/a,4]},
    (1+q^2*x^2)*Sqrt[(a+b*x^2+c*x^4)/(a*(1+q^2*x^2)^2)]/(2*q*Sqrt[a+b*x^2+c*x^4])*EllipticF[2*ArcTan[q*x],1/2-b*q^2/(4*c)]] /;
  FreeQ[{a,b,c},x] && GtQ[b^2-4*a*c,0] && GtQ[c/a,0] && LtQ[b/a,0]
```

2: $\int \frac{1}{\sqrt{a+bx^2+cx^4}} dx$ when $b^2 - 4ac > 0 \wedge a < 0 \wedge c > 0$

Reference: G&R 3.152.3+

Note: Not sure if the shorter rule is valid for all q .

■ Rule 1.2.2.1.5.1.2.2: If $b^2 - 4ac > 0 \wedge a < 0 \wedge c > 0$, let $q \rightarrow \sqrt{b^2 - 4ac}$, then

$$\int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \rightarrow \frac{\sqrt{\frac{2a+(b-q)x^2}{2a+(b+q)x^2}} \sqrt{\frac{2a+(b+q)x^2}{q}}}{2\sqrt{a+bx^2+cx^4} \sqrt{\frac{a}{2a+(b+q)x^2}}} \text{EllipticF}\left[\text{ArcSin}\left[\frac{x}{\sqrt{\frac{2a+(b+q)x^2}{2q}}}\right], \frac{b+q}{2q}\right]$$

$$\int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \rightarrow \frac{\sqrt{-2a-(b-q)x^2} \sqrt{\frac{2a+(b+q)x^2}{q}}}{2\sqrt{-a} \sqrt{a+bx^2+cx^4}} \text{EllipticF}\left[\text{ArcSin}\left[\frac{x}{\sqrt{\frac{2a+(b+q)x^2}{2q}}}\right], \frac{b+q}{2q}\right]$$

Program code:

```
Int[1/Sqrt[a_+b_.*x_^2+c_.*x_^4],x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    Sqrt[-2*a-(b-q)*x^2]*Sqrt[(2*a+(b+q)*x^2)/q]/(2*Sqrt[-a]*Sqrt[a+b*x^2+c*x^4])*
      EllipticF[ArcSin[x/Sqrt[(2*a+(b+q)*x^2)/(2*q)]],(b+q)/(2*q)] /;
    IntegerQ[q] /;
    FreeQ[{a,b,c},x] && GtQ[b^2-4*a*c,0] && LtQ[a,0] && GtQ[c,0]
```

```
Int[1/Sqrt[a_+b_.*x_^2+c_.*x_^4],x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    Sqrt[(2*a+(b-q)*x^2)/(2*a+(b+q)*x^2)]*Sqrt[(2*a+(b+q)*x^2)/q]/(2*Sqrt[a+b*x^2+c*x^4]*Sqrt[a/(2*a+(b+q)*x^2)])*
      EllipticF[ArcSin[x/Sqrt[(2*a+(b+q)*x^2)/(2*q)]],(b+q)/(2*q)] /;
    FreeQ[{a,b,c},x] && GtQ[b^2-4*a*c,0] && LtQ[a,0] && GtQ[c,0]
```

$$3. \int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \text{ when } b^2 - 4ac > 0 \wedge \frac{b+\sqrt{b^2-4ac}}{a} > 0$$

$$1: \int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \text{ when } b^2 - 4ac > 0 \wedge \frac{b+\sqrt{b^2-4ac}}{a} > 0$$

Reference: G&R 3.152.1+

■ Rule 1.2.2.1.5.1.2.3.1: If $b^2 - 4ac > 0$, let $q \rightarrow \sqrt{b^2 - 4ac}$, if $\frac{b+q}{a} > 0$, then

$$\int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \rightarrow \frac{(2a + (b+q)x^2) \sqrt{\frac{2a+(b-q)x^2}{2a+(b+q)x^2}}}{2a \sqrt{\frac{b+q}{2a}} \sqrt{a+bx^2+cx^4}} \text{EllipticF}\left[\text{ArcTan}\left[\sqrt{\frac{b+q}{2a}} x\right], \frac{2q}{b+q}\right]$$

Program code:

```
Int[1/Sqrt[a+_.*x_^2+c_.*x_^4],x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    (2*a+(b+q)*x^2)*Sqrt[(2*a+(b-q)*x^2)/(2*a+(b+q)*x^2)]/(2*a*Rt[(b+q)/(2*a),2]*Sqrt[a+b*x^2+c*x^4])*
    EllipticF[ArcTan[Rt[(b+q)/(2*a),2]*x],2*q/(b+q)] /;
    PosQ[(b+q)/a] && Not[PosQ[(b-q)/a] && SimplersqrtQ[(b-q)/(2*a),(b+q)/(2*a)]] /;
    FreeQ[{a,b,c},x] && GtQ[b^2-4*a*c,0]
```

$$2: \int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \text{ when } b^2 - 4ac > 0 \wedge \frac{b-\sqrt{b^2-4ac}}{a} > 0$$

Reference: G&R 3.152.1-

■ Rule 1.2.2.1.5.1.2.3.2: If $b^2 - 4ac > 0$, let $q \rightarrow \sqrt{b^2 - 4ac}$, if $\frac{b-q}{a} > 0$ then

$$\int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \rightarrow \frac{(2a+(b-q)x^2) \sqrt{\frac{2a+(b+q)x^2}{2a+(b-q)x^2}}}{2a \sqrt{\frac{b-q}{2a}} \sqrt{a+bx^2+cx^4}} \text{EllipticF}\left[\text{ArcTan}\left[\sqrt{\frac{b-q}{2a}} x\right], -\frac{2q}{b-q}\right]$$

Program code:

```
Int[1/Sqrt[a+_.*x_^2+c_.*x_^4],x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    (2*a+(b-q)*x^2)*Sqrt[(2*a+(b+q)*x^2)/(2*a+(b-q)*x^2)]/(2*a*Rt[(b-q)/(2*a),2]*Sqrt[a+b*x^2+c*x^4])*
    EllipticF[ArcTan[Rt[(b-q)/(2*a),2]*x],-2*q/(b-q)] /;
  PosQ[(b-q)/a] /;
  FreeQ[{a,b,c},x] && GtQ[b^2-4*a*c,0]
```

$$4. \int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \text{ when } b^2 - 4ac > 0 \wedge \frac{b \pm \sqrt{b^2 - 4ac}}{a} \neq 0$$

$$1: \int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \text{ when } b^2 - 4ac > 0 \wedge \frac{b + \sqrt{b^2 - 4ac}}{a} \neq 0$$

Reference: G&R 3.152.7+

■ Rule 1.2.2.1.5.1.2.4.1: If $b^2 - 4ac > 0$, let $q \rightarrow \sqrt{b^2 - 4ac}$, if $\frac{b+q}{a} \neq 0$ then

$$\int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \rightarrow \frac{\sqrt{1 + \frac{(b+q)x^2}{2a}} \sqrt{1 + \frac{(b-q)x^2}{2a}}}{\sqrt{-\frac{b+q}{2a}} \sqrt{a+bx^2+cx^4}} \text{EllipticF}\left[\text{ArcSin}\left[\sqrt{-\frac{b+q}{2a}} x\right], \frac{b-q}{b+q}\right]$$

Program code:

```
Int[1/Sqrt[a_+b_.**x_^2+c_.**x_^4],x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    Sqrt[1+(b+q)*x^2/(2*a)]*Sqrt[1+(b-q)*x^2/(2*a)]/(Rt[-(b+q)/(2*a),2]*Sqrt[a+b*x^2+c*x^4])*
    EllipticF[ArcSin[Rt[-(b+q)/(2*a),2]*x],(b-q)/(b+q)] /;
    NegQ[(b+q)/a] && Not[NegQ[(b-q)/a] && SimplersqrtQ[-(b-q)/(2*a),-(b+q)/(2*a)]] /;
    FreeQ[{a,b,c},x] && GtQ[b^2-4*a*c,0]
```

$$2: \int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \text{ when } b^2 - 4ac > 0 \wedge \frac{b - \sqrt{b^2 - 4ac}}{a} \neq 0$$

Reference: G&R 3.152.7-

■ Rule 1.2.2.1.5.1.2.4.2: If $b^2 - 4ac > 0$, let $q \rightarrow \sqrt{b^2 - 4ac}$, if $\frac{b-q}{a} \neq 0$ then

$$\int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \rightarrow \frac{\sqrt{1+\frac{(b-q)x^2}{2a}} \sqrt{1+\frac{(b+q)x^2}{2a}}}{\sqrt{-\frac{b-q}{2a}} \sqrt{a+bx^2+cx^4}} \text{EllipticF}\left[\text{ArcSin}\left[\sqrt{-\frac{b-q}{2a}} x\right], \frac{b+q}{b-q}\right]$$

Program code:

```
Int[1/Sqrt[a+_.*x_^2+c_.*x_^4],x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    Sqrt[1+(b-q)*x^2/(2*a)]*Sqrt[1+(b+q)*x^2/(2*a)]/(Rt[-(b-q)/(2*a),2]*Sqrt[a+b*x^2+c*x^4])*
    EllipticF[ArcSin[Rt[-(b-q)/(2*a),2]*x],(b+q)/(b-q)] /;
  NegQ[(b-q)/a] /;
  FreeQ[{a,b,c},x] && GtQ[b^2-4*a*c,0]
```

$$2. \int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \text{ when } b^2 - 4ac \neq 0$$

$$1: \int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \text{ when } b^2 - 4ac \neq 0 \wedge \frac{c}{a} > 0$$

Reference: G&R 3.165.2

Derivation: Piecewise constant extraction

$$\text{Basis: Let } q = \left(\frac{c}{a}\right)^{1/4}, \text{ then } \partial_x \frac{(1+q^2 x^2) \sqrt{\frac{(a+bx^2+cx^4)}{a(1+q^2 x^2)^2}}}{\sqrt{a+bx^2+cx^4}} = 0$$

Rule 1.2.2.1.5.2.1: If $b^2 - 4ac \neq 0 \wedge \frac{c}{a} > 0$, let $q \rightarrow \left(\frac{c}{a}\right)^{1/4}$, then

$$\int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \rightarrow \frac{(1+q^2 x^2) \sqrt{\frac{(a+bx^2+cx^4)}{a(1+q^2 x^2)^2}}}{2q \sqrt{a+bx^2+cx^4}} \text{EllipticF}\left[2 \text{ArcTan}[qx], \frac{1}{2} - \frac{bq^2}{4c}\right]$$

Program code:

```
Int[1/Sqrt[a+_b_.**x_^2+c_.**x_^4],x_Symbol] :=
  With[{q=Rt[c/a,4]},
    (1+q^2*x^2)*Sqrt[(a+b*x^2+c*x^4)/(a*(1+q^2*x^2)^2)]/(2*q*Sqrt[a+b*x^2+c*x^4])*EllipticF[2*ArcTan[q*x],1/2-b*q^2/(4*c)]] /;
  FreeQ[{a,b,c},x] && NeQ[b^2-4*a*c,0] && PosQ[c/a]
```

$$2: \int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \text{ when } b^2 - 4ac \neq 0 \wedge \frac{c}{a} < 0$$

Derivation: Piecewise constant extraction

Basis: If $q \rightarrow \sqrt{b^2 - 4ac}$, then $\partial_x \frac{\sqrt{1 + \frac{2cx^2}{b-q}} \sqrt{1 + \frac{2cx^2}{b+q}}}{\sqrt{a+bx^2+cx^4}} = 0$

■ Rule 1.2.2.1.5.2.2: If $b^2 - 4ac \neq 0 \wedge \frac{c}{a} \neq 0$, let $q \rightarrow \sqrt{b^2 - 4ac}$, then

$$\int \frac{1}{\sqrt{a+bx^2+cx^4}} dx \rightarrow \frac{\sqrt{1 + \frac{2cx^2}{b-q}} \sqrt{1 + \frac{2cx^2}{b+q}}}{\sqrt{a+bx^2+cx^4}} \int \frac{1}{\sqrt{1 + \frac{2cx^2}{b-q}} \sqrt{1 + \frac{2cx^2}{b+q}}} dx$$

Program code:

```
Int[1/Sqrt[a_+b_.*x_^2+c_.*x_^4],x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    Sqrt[1+2*c*x^2/(b-q)]*Sqrt[1+2*c*x^2/(b+q)]/Sqrt[a+b*x^2+c*x^4]*
    Int[1/(Sqrt[1+2*c*x^2/(b-q)]*Sqrt[1+2*c*x^2/(b+q)]),x] /;
  FreeQ[{a,b,c},x] && NeQ[b^2-4*a*c,0] && NegQ[c/a]
```


6: $\int (a + bx^2 + cx^4)^p dx$ when $b^2 - 4ac \neq 0$

Derivation: Piecewise constant extraction

Basis: If $q \rightarrow \sqrt{b^2 - 4ac}$, then $\partial_x \frac{(a+bx^2+cx^4)^p}{\left(1 + \frac{2cx^2}{b+q}\right)^p \left(1 + \frac{2cx^2}{b-q}\right)^p} = 0$

■

Rule 1.2.2.1.6: If $b^2 - 4ac \neq 0$, let $q \rightarrow \sqrt{b^2 - 4ac}$, then

$$\int (a + bx^2 + cx^4)^p dx \rightarrow \frac{a^{\text{IntPart}[p]} (a + bx^2 + cx^4)^{\text{FracPart}[p]}}{\left(1 + \frac{2cx^2}{b+q}\right)^{\text{FracPart}[p]} \left(1 + \frac{2cx^2}{b-q}\right)^{\text{FracPart}[p]}} \int \left(1 + \frac{2cx^2}{b+q}\right)^p \left(1 + \frac{2cx^2}{b-q}\right)^p dx$$

Program code:

```
Int[(a+b_.**x_^2+c_.**x_^4)^p_,x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    a^IntPart[p]*(a+b*x^2+c*x^4)^FracPart[p]/((1+2*c*x^2/(b+q))^FracPart[p]*(1+2*c*x^2/(b-q))^FracPart[p])*
    Int[(1+2*c*x^2/(b+q))^p*(1+2*c*x^2/(b-q))^p,x] /;
    FreeQ[{a,b,c,p},x] && NeQ[b^2-4*a*c,0]
```

S: $\int (a + bx + cx^2 + dx^3 + ex^4)^p dx$ when $d^3 - 4cde + 8be^2 = 0 \wedge p \notin \{1, 2, 3\}$

Derivation: Integration by substitution

Basis: If $d^3 - 4cde + 8be^2 = 0$, then

$$(a + bx + cx^2 + dx^3 + ex^4)^p = \text{Subst} \left[\left(a + \frac{d^4}{256e^3} - \frac{bd}{8e} + \left(c - \frac{3d^2}{8e} \right) x^2 + ex^4 \right)^p, x, \frac{d}{4e} + x \right] \partial_x \left(\frac{d}{4e} + x \right)$$

Note: The substitution transforms a dense quartic polynomial into a symmetric quartic trinomial.

Rule: If $d^3 - 4cde + 8be^2 = 0 \wedge p \notin \{1, 2, 3\}$, then

$$\int (a + bx + cx^2 + dx^3 + ex^4)^p dx \rightarrow \text{Subst} \left[\int \left(a + \frac{d^4}{256e^3} - \frac{bd}{8e} + \left(c - \frac{3d^2}{8e} \right) x^2 + ex^4 \right)^p dx, x, \frac{d}{4e} + x \right]$$

Program code:

```
Int[P4^p_,x_Symbol] :=
  With[{a=Coeff[P4,x,0],b=Coeff[P4,x,1],c=Coeff[P4,x,2],d=Coeff[P4,x,3],e=Coeff[P4,x,4]},
    Subst[Int[SimplifyIntegrand[(a+d^4/(256*e^3)-b*d/(8*e)+(c-3*d^2/(8*e))*x^2+e*x^4)^p,x],x],x,d/(4*e)+x] /;
  EqQ[d^3-4*c*d*e+8*b*e^2,0] && NeQ[d,0] /;
  FreeQ[p,x] && PolyQ[P4,x,4] && NeQ[p,2] && NeQ[p,3]
```