

Rules for integrands of the form $(d + e x)^m \sinh[a + b x + c x^2]^n$

1. $\int \sinh[a + b x + c x^2]^n dx$

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Derivation: Algebraic expansion

Basis: $\sinh[z] == \frac{e^z}{2} - \frac{e^{-z}}{2}$

Rule:

$$\int \sinh[a + b x + c x^2] dx \rightarrow \frac{1}{2} \int e^{a+b x+c x^2} dx - \frac{1}{2} \int e^{-a-b x-c x^2} dx$$

Program code:

```
Int[Sinh[a_.+b_.*x+c_.*x^2],x_Symbol] :=
  1/2*Int[E^(a+b*x+c*x^2),x] - 1/2*Int[E^(-a-b*x-c*x^2),x] /;
FreeQ[{a,b,c},x]
```

```
Int[Cosh[a_.+b_.*x+c_.*x^2],x_Symbol] :=
  1/2*Int[E^(a+b*x+c*x^2),x] + 1/2*Int[E^(-a-b*x-c*x^2),x] /;
FreeQ[{a,b,c},x]
```

2: $\int \sinh[a + b x + c x^2]^n dx$ when $n \in \mathbb{Z} \wedge n > 1$

Derivation: Algebraic expansion

Rule: If $n \in \mathbb{Z} \wedge n > 1$, then

$$\int \sinh[a + b x + c x^2]^n dx \rightarrow \int \text{TrigReduce}[\sinh[a + b x + c x^2]^n] dx$$

Program code:

```
Int[Sinh[a_.+b_.*x_.+c_.*x_^2]^n_,x_Symbol] :=
  Int[ExpandTrigReduce[Sinh[a+b*x+c*x^2]^n,x],x] /;
FreeQ[{a,b,c},x] && IGtQ[n,1]
```

```
Int[Cosh[a_.+b_.*x_.+c_.*x_^2]^n_,x_Symbol] :=
  Int[ExpandTrigReduce[Cosh[a+b*x+c*x^2]^n,x],x] /;
FreeQ[{a,b,c},x] && IGtQ[n,1]
```

3: $\int \sinh[v]^n dx$ when $n \in \mathbb{Z}^+ \wedge v = a + b x + c x^2$

Derivation: Algebraic normalization

Rule: If $n \in \mathbb{Z}^+ \wedge v = a + b x + c x^2$, then

$$\int \sinh[v]^n dx \rightarrow \int \sinh[a + b x + c x^2]^n dx$$

Program code:

```
Int[Sinh[v_]^n_., x_Symbol] :=
  Int[Sinh[ExpandToSum[v, x]]^n, x] /;
IGtQ[n, 0] && QuadraticQ[v, x] && Not[QuadraticMatchQ[v, x]]
```

```
Int[Cosh[v_]^n_., x_Symbol] :=
  Int[Cosh[ExpandToSum[v, x]]^n, x] /;
IGtQ[n, 0] && QuadraticQ[v, x] && Not[QuadraticMatchQ[v, x]]
```

$$2. \int (d+e x)^m \sinh[a+b x+c x^2]^n dx$$

$$1. \int (d+e x)^m \sinh[a+b x+c x^2] dx$$

$$1. \int (d+e x)^m \sinh[a+b x+c x^2] dx \text{ when } m > 0$$

$$1. \int (d+e x) \sinh[a+b x+c x^2] dx$$

$$\text{1: } \int (d+e x) \sinh[a+b x+c x^2] dx \text{ when } b e - 2 c d = 0$$

Rule: If $b e - 2 c d = 0$, then

$$\int (d+e x) \sinh[a+b x+c x^2] dx \rightarrow \frac{e \cosh[a+b x+c x^2]}{2 c}$$

Program code:

```
Int[(d_+e_.x_)*Sinh[a_+b_.x_+c_.x_^2],x_Symbol] :=
  e*Cosh[a+b*x+c*x^2]/(2*c) /;
FreeQ[{a,b,c,d,e},x] && EqQ[b*e-2*c*d,0]
```

```
Int[(d_+e_.x_)*Cosh[a_+b_.x_+c_.x_^2],x_Symbol] :=
  e*Sinh[a+b*x+c*x^2]/(2*c) /;
FreeQ[{a,b,c,d,e},x] && EqQ[b*e-2*c*d,0]
```

2: $\int (d+e x) \sinh[a+b x+c x^2] dx$ when $b e - 2 c d \neq 0$

Rule: If $b e - 2 c d \neq 0$, then

$$\int (d+e x) \sinh[a+b x+c x^2] dx \rightarrow \frac{e \cosh[a+b x+c x^2]}{2 c} - \frac{b e - 2 c d}{2 c} \int \sinh[a+b x+c x^2] dx$$

Program code:

```
Int[(d_.+e_.**x_)*Sinh[a_.+b_.**x_+c_.**x_^2],x_Symbol] :=
  e*Cosh[a+b*x+c*x^2]/(2*c) -
  (b*e-2*c*d)/(2*c)*Int[Sinh[a+b*x+c*x^2],x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b*e-2*c*d,0]
```

```
Int[(d_.+e_.**x_)*Cosh[a_.+b_.**x_+c_.**x_^2],x_Symbol] :=
  e*Sinh[a+b*x+c*x^2]/(2*c) -
  (b*e-2*c*d)/(2*c)*Int[Cosh[a+b*x+c*x^2],x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b*e-2*c*d,0]
```

$$2. \int (d+e x)^m \sinh[a+b x+c x^2] dx \text{ when } m > 1$$

$$1: \int (d+e x)^m \sinh[a+b x+c x^2] dx \text{ when } m > 1 \wedge b e - 2 c d == 0$$

Rule: If $m > 1 \wedge b e - 2 c d == 0$, then

$$\int (d+e x)^m \sinh[a+b x+c x^2] dx \rightarrow \frac{e (d+e x)^{m-1} \cosh[a+b x+c x^2]}{2 c} + \frac{e^2 (m-1)}{2 c} \int (d+e x)^{m-2} \cosh[a+b x+c x^2] dx$$

Program code:

```
Int[(d_.+e_.**x_)^m_*Sinh[a_.+b_.**x_+c_.**x_^2],x_Symbol] :=
  e*(d+e*x)^(m-1)*Cosh[a+b*x+c*x^2]/(2*c) -
  e^2*(m-1)/(2*c)*Int[(d+e*x)^(m-2)*Cosh[a+b*x+c*x^2],x] /;
FreeQ[{a,b,c,d,e},x] && GtQ[m,1] && EqQ[b*e-2*c*d,0]
```

```
Int[(d_.+e_.**x_)^m_*Cosh[a_.+b_.**x_+c_.**x_^2],x_Symbol] :=
  e*(d+e*x)^(m-1)*Sinh[a+b*x+c*x^2]/(2*c) -
  e^2*(m-1)/(2*c)*Int[(d+e*x)^(m-2)*Sinh[a+b*x+c*x^2],x] /;
FreeQ[{a,b,c,d,e},x] && GtQ[m,1] && EqQ[b*e-2*c*d,0]
```

$$\mathbf{2:} \int (d+e x)^m \sinh[a+b x+c x^2] dx \text{ when } m > 1 \wedge b e - 2 c d \neq 0$$

Rule: If $m > 1 \wedge b e - 2 c d \neq 0$, then

$$\int (d+e x)^m \sinh[a+b x+c x^2] dx \rightarrow \frac{e (d+e x)^{m-1} \cosh[a+b x+c x^2]}{2 c} - \frac{b e - 2 c d}{2 c} \int (d+e x)^{m-1} \sinh[a+b x+c x^2] dx - \frac{e^2 (m-1)}{2 c} \int (d+e x)^{m-2} \cosh[a+b x+c x^2] dx$$

Program code:

```
Int[(d_.+e_.**x_)^m_*Sinh[a_.+b_.**x_+c_.**x_^2],x_Symbol] :=
  e*(d+e*x)^(m-1)*Cosh[a+b*x+c*x^2]/(2*c) -
  (b*e-2*c*d)/(2*c)*Int[(d+e*x)^(m-1)*Sinh[a+b*x+c*x^2],x] -
  e^2*(m-1)/(2*c)*Int[(d+e*x)^(m-2)*Cosh[a+b*x+c*x^2],x] /;
FreeQ[{a,b,c,d,e},x] && GtQ[m,1] && NeQ[b*e-2*c*d,0]
```

```
Int[(d_.+e_.**x_)^m_*Cosh[a_.+b_.**x_+c_.**x_^2],x_Symbol] :=
  e*(d+e*x)^(m-1)*Sinh[a+b*x+c*x^2]/(2*c) -
  (b*e-2*c*d)/(2*c)*Int[(d+e*x)^(m-1)*Cosh[a+b*x+c*x^2],x] -
  e^2*(m-1)/(2*c)*Int[(d+e*x)^(m-2)*Sinh[a+b*x+c*x^2],x] /;
FreeQ[{a,b,c,d,e},x] && GtQ[m,1] && NeQ[b*e-2*c*d,0]
```

2. $\int (d+e x)^m \sinh[a+b x+c x^2] dx$ when $m < -1$

1: $\int (d+e x)^m \sinh[a+b x+c x^2] dx$ when $m < -1 \wedge b e - 2 c d == 0$

Rule: If $m < -1 \wedge b e - 2 c d == 0$, then

$$\int (d+e x)^m \sinh[a+b x+c x^2] dx \rightarrow \frac{(d+e x)^{m+1} \sinh[a+b x+c x^2]}{e (m+1)} - \frac{2 c}{e^2 (m+1)} \int (d+e x)^{m+2} \cosh[a+b x+c x^2] dx$$

Program code:

```
Int[(d_.+e_.x_)^m_*Sinh[a_.+b_.x_+c_.x_^2],x_Symbol] :=
  (d+e*x)^(m+1)*Sinh[a+b*x+c*x^2]/(e*(m+1)) -
  2*c/(e^2*(m+1))*Int[(d+e*x)^(m+2)*Cosh[a+b*x+c*x^2],x] /;
FreeQ[{a,b,c,d,e},x] && LtQ[m,-1] && EqQ[b*e-2*c*d,0]
```

```
Int[(d_.+e_.x_)^m_*Cosh[a_.+b_.x_+c_.x_^2],x_Symbol] :=
  (d+e*x)^(m+1)*Cosh[a+b*x+c*x^2]/(e*(m+1)) -
  2*c/(e^2*(m+1))*Int[(d+e*x)^(m+2)*Sinh[a+b*x+c*x^2],x] /;
FreeQ[{a,b,c,d,e},x] && LtQ[m,-1] && EqQ[b*e-2*c*d,0]
```


2: $\int (d+e x)^m \sinh[a+b x+c x^2] dx$ when $m < -1 \wedge b e - 2 c d \neq 0$

Rule: If $m < -1 \wedge b e - 2 c d \neq 0$, then

$$\int (d+e x)^m \sinh[a+b x+c x^2] dx \rightarrow \frac{(d+e x)^{m+1} \sinh[a+b x+c x^2]}{e (m+1)} - \frac{b e - 2 c d}{e^2 (m+1)} \int (d+e x)^{m+1} \cosh[a+b x+c x^2] dx - \frac{2 c}{e^2 (m+1)} \int (d+e x)^{m+2} \cosh[a+b x+c x^2] dx$$

Program code:

```
Int[(d_+e_.x_)^m_*Sinh[a_+b_.x_+c_.x_^2],x_Symbol] :=
  (d+e*x)^(m+1)*Sinh[a+b*x+c*x^2]/(e*(m+1)) -
  (b*e-2*c*d)/(e^2*(m+1))*Int[(d+e*x)^(m+1)*Cosh[a+b*x+c*x^2],x] -
  2*c/(e^2*(m+1))*Int[(d+e*x)^(m+2)*Cosh[a+b*x+c*x^2],x] /;
FreeQ[{a,b,c,d,e},x] && LtQ[m,-1] && NeQ[b*e-2*c*d,0]
```

```
Int[(d_+e_.x_)^m_*Cosh[a_+b_.x_+c_.x_^2],x_Symbol] :=
  (d+e*x)^(m+1)*Cosh[a+b*x+c*x^2]/(e*(m+1)) -
  (b*e-2*c*d)/(e^2*(m+1))*Int[(d+e*x)^(m+1)*Sinh[a+b*x+c*x^2],x] -
  2*c/(e^2*(m+1))*Int[(d+e*x)^(m+2)*Sinh[a+b*x+c*x^2],x] /;
FreeQ[{a,b,c,d,e},x] && LtQ[m,-1] && NeQ[b*e-2*c*d,0]
```

$$\mathbf{3:} \int (d+e x)^m \operatorname{Sinh}[a+b x+c x^2] \, dx$$

Rule:

$$\int (d+e x)^m \operatorname{Sinh}[a+b x+c x^2] \, dx \rightarrow \int (d+e x)^m \operatorname{Sinh}[a+b x+c x^2] \, dx$$

Program code:

```
Int[(d_+e_.x_)^m_.Sinh[a_+b_.x_+c_.x_^2],x_Symbol] :=
  Unintegrable[(d+e*x)^m*Sinh[a+b*x+c*x^2],x] /;
FreeQ[{a,b,c,d,e,m},x]
```

```
Int[(d_+e_.x_)^m_.Cosh[a_+b_.x_+c_.x_^2],x_Symbol] :=
  Unintegrable[(d+e*x)^m*Cosh[a+b*x+c*x^2],x] /;
FreeQ[{a,b,c,d,e,m},x]
```

$$\mathbf{2:} \int (d+e x)^m \operatorname{Sinh}[a+b x+c x^2]^n \, dx \text{ when } n \in \mathbb{Z} \wedge n > 1$$

Derivation: Algebraic expansion

Rule: If $n \in \mathbb{Z} \wedge n > 1$, then

$$\int (d+e x)^m \operatorname{Sinh}[a+b x+c x^2]^n \, dx \rightarrow \int (d+e x)^m \operatorname{TrigReduce}[\operatorname{Sinh}[a+b x+c x^2]^n] \, dx$$

Program code:

```
Int[(d_+e_.x_)^m_.Sinh[a_+b_.x_+c_.x_^2]^n_,x_Symbol] :=
  Int[ExpandTrigReduce[(d+e*x)^m,Sinh[a+b*x+c*x^2]^n,x],x] /;
FreeQ[{a,b,c,d,e,m},x] && IGtQ[n,1]
```

```
Int[(d_+e_.*x_)^m_.*Cosh[a_+b_.*x_+c_.*x_^2]^n_,x_Symbol] :=
  Int[ExpandTrigReduce[(d+e*x)^m,Cosh[a+b*x+c*x^2]^n,x],x] /;
FreeQ[{a,b,c,d,e,m},x] && IGtQ[n,1]
```

3: $\int u^m \sinh[v]^n dx$ when $n \in \mathbb{Z}^+ \wedge u = d + e x \wedge v = a + b x + c x^2$

Derivation: Algebraic normalization

Rule: If $n \in \mathbb{Z}^+ \wedge u = d + e x \wedge v = a + b x + c x^2$, then

$$\int u^m \sinh[v]^n dx \rightarrow \int (d + e x)^m \sinh[a + b x + c x^2]^n dx$$

Program code:

```
Int[u_^m_.*Sinh[v_]^n_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*Sinh[ExpandToSum[v,x]]^n,x] /;
FreeQ[m,x] && IGtQ[n,0] && LinearQ[u,x] && QuadraticQ[v,x] && Not[LinearMatchQ[u,x] && QuadraticMatchQ[v,x]]
```

```
Int[u_^m_.*Cosh[v_]^n_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*Cosh[ExpandToSum[v,x]]^n,x] /;
FreeQ[m,x] && IGtQ[n,0] && LinearQ[u,x] && QuadraticQ[v,x] && Not[LinearMatchQ[u,x] && QuadraticMatchQ[v,x]]
```