

Rules for integrands of the form $P[x]^p$

0: $\int u P[x]^p dx$ when $p \notin \mathbb{Z} \wedge P[x] = x^m Q[x]$

Derivation: Piecewise constant extraction

Basis: If $P[x] = x^m Q[x]$, then $\partial_x \frac{P[x]^p}{x^{mp} Q[x]^p} = 0$

Rule: If $p \notin \mathbb{Z} \wedge P[x] = x^m Q[x]$, then

$$\int u P[x]^p dx \rightarrow \frac{P[x]^{\text{FracPart}[p]}}{x^{m \text{FracPart}[p]} Q[x]^{\text{FracPart}[p]}} \int u x^{mp} Q[x]^p dx$$

Program code:

```
Int[u_.*P_^p_,x_Symbol] :=
  With[{m=MinimumMonomialExponent[P,x]},
    P^FracPart[p]/(x^(m*FracPart[p])*Distrib[1/x^m,P]^FracPart[p])*Int[u*x^(m*p)*Distrib[1/x^m,P]^p,x] /;
    FreeQ[p,x] && Not[IntegerQ[p]] && SumQ[P] && EveryQ[Function[BinoialQ[#,x],P] && Not[PolyQ[P,x,2]]
```

$$1. \int P[x]^p dx \text{ when } P[x] = P1[x] P2[x] \dots$$

$$1: \int P[x^2]^p dx \text{ when } p \in \mathbb{Z}^- \wedge P[x] = P1[x] P2[x] \dots$$

Derivation: Algebraic simplification

Note: This rule assumes host CAS distributes integer powers over products.

Rule: If $p \in \mathbb{Z}^- \wedge P[x] = P1[x] P2[x] \dots$, then

$$\int P[x^2]^p dx \rightarrow \int \text{ExpandIntegrand}[P1[x^2]^p P2[x^2]^p \dots, x] dx$$

Program code:

```
Int[P^p_, x_Symbol] :=
  With[{u=Factor[ReplaceAll[P, x→Sqrt[x]]]},
    Int[ExpandIntegrand[ReplaceAll[u, x→x^2]^p, x], x] /;
    Not[SumQ[NonfreeFactors[u, x]]] /;
    PolyQ[P, x^2] && ILtQ[p, 0]
```

2: $\int P[x]^p dx$ when $p \in \mathbb{Z}^- \wedge P[x] = P_1[x] P_2[x] \dots$

Derivation: Algebraic simplification

Note: This rule assumes host CAS distributes integer powers over products.

Rule: If $p \in \mathbb{Z}^- \wedge P[x] = P_1[x] P_2[x] \dots$, then

$$\int P[x]^p dx \rightarrow \int P_1[x]^p P_2[x]^p \dots dx$$

Program code:

```
Int[P_^p_,x_Symbol] :=
  With[{u=Factor[P]},
    Int[ExpandIntegrand[u^p,x],x] /;
    Not[SumQ[NonfreeFactors[u,x]]] /;
    PolyQ[P,x] && ILtQ[p,0]
```

2: $\int P[x]^p dx$ when $p \in \mathbb{Z} \wedge P[x] = P_1[x] P_2[x] \dots$

Derivation: Algebraic simplification

Note: This rule assumes host CAS distributes integer powers over products.

Rule: If $p \in \mathbb{Z} \wedge P[x] = P_1[x] P_2[x] \dots$, then

$$\int P[x]^p dx \rightarrow \int P_1[x]^p P_2[x]^p \dots dx$$

Program code:

```
Int[P^p_, x_Symbol] :=
  With[{u=Factor[P]},
    Int[u^p, x] /;
    Not[SumQ[NonfreeFactors[u, x]]] /;
    PolyQ[P, x] && IntegerQ[p]
```

$$\mathbf{x}: \int P_n[x]^p dx \text{ when } P_n[x] = Q_{n1}[x]^q R_{n2}[x]^r \dots \wedge p \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction

Basis: If $P_n[x] = Q_{n1}[x]^q R_{n2}[x]^r \dots$, then $\partial_x \frac{P_n[x]^p}{Q_{n1}[x]^{pq} R_{n2}[x]^{pr} \dots} = 0$

Rule: If $P_n[x] = Q_{n1}[x]^q R_{n2}[x]^r \dots \wedge p \notin \mathbb{Z}$, then

$$\int P_n[x]^p dx \rightarrow \frac{P_n[x]^p}{Q_{n1}[x]^{pq} R_{n2}[x]^{pr} \dots} \int Q_{n1}[x]^{pq} R_{n2}[x]^{pr} \dots dx$$

Program code:

```
(* Int[Pn^p,x_Symbol] :=
  With[{u=Factor[Pn]},
    Pn^p/DistributeDegree[u,p]*Int[DistributeDegree[u,p],x] /;
    Not[SumQ[u]] /;
    PolyQ[Pn,x] && Not[IntegerQ[p]] *)
```

2. $\int P[x]^p dx$ when $p \in \mathbb{Z}^+$

1: $\int (a + bx + cx^2 + dx^3)^p dx$ when $p \in \mathbb{Z}^+ \wedge c^2 - 3bd = 0$

Derivation: Integration by substitution

Basis: If $c^2 - 3bd = 0$, then $(a + bx + cx^2 + dx^3)^p = \frac{1}{3^p} \text{Subst} \left[\left(\frac{3ac - b^2}{c} + \frac{c^2 x^3}{b} \right)^p, x, \frac{c}{3d} + x \right] \partial_x \left(\frac{c}{3d} + x \right)$

Rule: If $p \in \mathbb{Z}^+ \wedge c^2 - 3bd = 0$, then

$$\int (a + bx + cx^2 + dx^3)^p dx \rightarrow \frac{1}{3^p} \text{Subst} \left[\int \left(\frac{3ac - b^2}{c} + \frac{c^2 x^3}{b} \right)^p dx, x, \frac{c}{3d} + x \right]$$

Program code:

```
Int[(a_.+b_.*x_+c_.*x_^2+d_.*x_^3)^p_,x_Symbol] :=
  1/3^p*Subst[Int[Simp[(3*a*c-b^2)/c+c^2*x^3/b,x]^p,x],x,c/(3*d)+x] /;
FreeQ[{a,b,c,d},x] && IGtQ[p,0] && EqQ[c^2-3*b*d,0]
```

2: $\int P[x]^p dx$ when $p \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $p \in \mathbb{Z}^+$, then

$$\int P[x]^p dx \rightarrow \int \text{ExpandToSum}[P[x]^p, x] dx$$

Program code:

```
Int[P_^p_, x_Symbol] :=
  Int[ExpandToSum[P^p, x], x] /;
  PolyQ[P, x] && IGtQ[p, 0]
```

3: $\int P[x]^p dx$ when $p \in \mathbb{Z} \wedge P[x] = (a + bx + cx^2)(d + ex + fx^2) \dots$

Derivation: Algebraic expansion

Rule: If $p \in \mathbb{Z} \wedge P[x] = (a + bx + cx^2)(d + ex + fx^2) \dots$, then

$$\int P[x]^p dx \rightarrow \int \text{ExpandIntegrand}[P[x]^p, x] dx$$

Program code:

```
Int[P_^p_, x_Symbol] :=
  Int[ExpandIntegrand[P^p, x], x] /;
  PolyQ[P, x] && IntegerQ[p] && QuadraticProductQ[Factor[P], x]
```

$$4. \int (a + bx + cx^2 + dx^3)^p dx$$

$$1. \int (a + bx + dx^3)^p dx$$

$$1. \int (a + bx + dx^3)^p dx \text{ when } 4b^3 + 27a^2d = 0$$

$$1: \int (a + bx + dx^3)^p dx \text{ when } 4b^3 + 27a^2d = 0 \wedge p \in \mathbb{Z}$$

Derivation: Algebraic expansion

Basis: If $4b^3 + 27a^2d = 0$, then $a + bx + dx^3 = \frac{1}{3^3 a^2} (3a - bx)(3a + 2bx)^2$

Rule: If $4b^3 + 27a^2d = 0 \wedge p \in \mathbb{Z}$, then

$$\int (a + bx + dx^3)^p dx \rightarrow \frac{1}{3^{3p} a^{2p}} \int (3a - bx)^p (3a + 2bx)^{2p} dx$$

Program code:

```
Int[(a_.+b_.**x_+d_.**x_^3)^p_,x_Symbol] :=
  1/(3^(3*p)*a^(2*p))*Int[(3*a-b*x)^p*(3*a+2*b*x)^(2*p),x] /;
FreeQ[{a,b,d},x] && EqQ[4*b^3+27*a^2*d,0] && IntegerQ[p]
```

$$2: \int (a + bx + dx^3)^p dx \text{ when } 4b^3 + 27a^2d = 0 \wedge p \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction

Basis: If $4b^3 + 27a^2d = 0$, then $\partial_x \frac{(a+bx+dx^3)^p}{(3a-bx)^p (3a+2bx)^{2p}} = 0$

Rule: If $4b^3 + 27a^2d = 0 \wedge p \notin \mathbb{Z}$, then

$$\int (a + bx + dx^3)^p dx \rightarrow \frac{(a + bx + dx^3)^p}{(3a - bx)^p (3a + 2bx)^{2p}} \int (3a - bx)^p (3a + 2bx)^{2p} dx$$

Program code:

```
Int[(a_.+b_.**x_+d_.**x_^3)^p_,x_Symbol] :=
  (a+b*x+d*x^3)^p/((3*a-b*x)^p*(3*a+2*b*x)^(2*p))*Int[(3*a-b*x)^p*(3*a+2*b*x)^(2*p),x] /;
FreeQ[{a,b,d,p},x] && EqQ[4*b^3+27*a^2*d,0] && Not[IntegerQ[p]]
```

2. $\int (a + bx + dx^3)^p dx$ when $4b^3 + 27a^2d \neq 0$

1: $\int (a + bx + dx^3)^p dx$ when $4b^3 + 27a^2d \neq 0 \wedge p \in \mathbb{Z}$

Derivation: Algebraic expansion

Basis: If $r \rightarrow \left(-9ad^2 + \sqrt{3}d\sqrt{4b^3d + 27a^2d^2}\right)^{1/3}$, then $a + bx + dx^3 = \frac{2b^3d}{3r^3} - \frac{r^3}{18d^2} + bx + dx^3$

Basis:

$$\frac{2b^3d}{3r^3} - \frac{r^3}{18d^2} + bx + dx^3 = \frac{1}{d^2} \left(\frac{18^{1/3}bd}{3r} - \frac{r}{18^{1/3}} + dx \right) \left(\frac{bd}{3} + \frac{12^{1/3}b^2d^2}{3r^2} + \frac{r^2}{3 \times 12^{1/3}} - d \left(\frac{2^{1/3}bd}{3^{1/3}r} - \frac{r}{18^{1/3}} \right) x + d^2x^2 \right)$$

Rule: If $4b^3 + 27a^2d \neq 0 \wedge p \in \mathbb{Z}$, let $r \rightarrow \left(-9ad^2 + \sqrt{3}d\sqrt{4b^3d + 27a^2d^2}\right)^{1/3}$, then

$$\int (a + bx + dx^3)^p dx \rightarrow \frac{1}{d^{2p}} \int \left(\frac{18^{1/3}bd}{3r} - \frac{r}{18^{1/3}} + dx \right)^p \left(\frac{bd}{3} + \frac{12^{1/3}b^2d^2}{3r^2} + \frac{r^2}{3 \times 12^{1/3}} - d \left(\frac{2^{1/3}bd}{3^{1/3}r} - \frac{r}{18^{1/3}} \right) x + d^2x^2 \right)^p dx$$

Program code:

```
Int[(a_.+b_.**x_+d_.**x_^3)^p_,x_Symbol] :=
  With[{r=Rt[-9*a*d^2+Sqrt[3]*d*Sqrt[4*b^3*d+27*a^2*d^2],3]},
  1/d^(2*p)*Int[Simp[18^(1/3)*b*d/(3*r)-r/18^(1/3)+d*x,x]^p*
    Simp[b*d/3+12^(1/3)*b^2*d^2/(3*r^2)+r^2/(3*12^(1/3))-d*(2^(1/3)*b*d/(3^(1/3)*r)-r/18^(1/3))*x+d^2*x^2,x]^p,x] /;
FreeQ[{a,b,d},x] && NeQ[4*b^3+27*a^2*d,0] && IntegerQ[p]
```

2: $\int (a + bx + dx^3)^p dx$ when $4b^3 + 27a^2d \neq 0 \wedge p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: If $r \rightarrow (-9ad^2 + \sqrt{3}d\sqrt{4b^3d + 27a^2d^2})^{1/3}$, then

$$\partial_x \left((a + bx + dx^3)^p / \left(\left(\frac{18^{1/3}bd}{3r} - \frac{r}{18^{1/3}} + dx \right)^p \left(\frac{bd}{3} + \frac{12^{1/3}b^2d^2}{3r^2} + \frac{r^2}{3 \times 12^{1/3}} - d \left(\frac{2^{1/3}bd}{3^{1/3}r} - \frac{r}{18^{1/3}} \right) x + d^2x^2 \right)^p \right) \right) = 0$$

Rule: If $4b^3 + 27a^2d \neq 0 \wedge p \notin \mathbb{Z}$, let $r \rightarrow (-9ad^2 + \sqrt{3}d\sqrt{4b^3d + 27a^2d^2})^{1/3}$, then

$$\int (a + bx + dx^3)^p dx \rightarrow \left((a + bx + dx^3)^p / \left(\left(\frac{18^{1/3}bd}{3r} - \frac{r}{18^{1/3}} + dx \right)^p \left(\frac{bd}{3} + \frac{12^{1/3}b^2d^2}{3r^2} + \frac{r^2}{3 \times 12^{1/3}} - d \left(\frac{2^{1/3}bd}{3^{1/3}r} - \frac{r}{18^{1/3}} \right) x + d^2x^2 \right)^p \right) \right) \cdot \int \left(\frac{18^{1/3}bd}{3r} - \frac{r}{18^{1/3}} + dx \right)^p \left(\frac{bd}{3} + \frac{12^{1/3}b^2d^2}{3r^2} + \frac{r^2}{3 \times 12^{1/3}} - d \left(\frac{2^{1/3}bd}{3^{1/3}r} - \frac{r}{18^{1/3}} \right) x + d^2x^2 \right)^p dx$$

Program code:

```
Int[(a_.+b_.*x_+d_.*x_^3)^p_,x_Symbol] :=
  With[{r=Rt[-9*a*d^2+Sqrt[3]*d*Sqrt[4*b^3*d+27*a^2*d^2],3]}],
  (a+b*x+d*x^3)^p/
  (Simp[18^(1/3)*b*d/(3*r)-r/18^(1/3)+d*x,x]^p*
   Simp[b*d/3+12^(1/3)*b^2*d^2/(3*r^2)+r^2/(3*12^(1/3))-d*(2^(1/3)*b*d/(3^(1/3)*r)-r/18^(1/3))*x+d^2*x^2,x]^p)*
  Int[Simp[18^(1/3)*b*d/(3*r)-r/18^(1/3)+d*x,x]^p*
   Simp[b*d/3+12^(1/3)*b^2*d^2/(3*r^2)+r^2/(3*12^(1/3))-d*(2^(1/3)*b*d/(3^(1/3)*r)-r/18^(1/3))*x+d^2*x^2,x]^p,x] /;
  FreeQ[{a,b,d,p},x] && NeQ[4*b^3+27*a^2*d,0] && Not[IntegerQ[p]]
```

2: $\int (a + bx + cx^2 + dx^3)^p dx$

Derivation: Integration by substitution

Rule:

$$\int (a + bx + cx^2 + dx^3)^p dx \rightarrow \text{Subst} \left[\int \left(\frac{2c^3 - 9bcd + 27ad^2}{27d^2} - \frac{(c^2 - 3bd)x}{3d} + dx^3 \right)^p dx, x, x + \frac{c}{3d} \right]$$

Program code:

```
Int[P3^p_, x_Symbol] :=
  With[{a=Coeff[P3, x, 0], b=Coeff[P3, x, 1], c=Coeff[P3, x, 2], d=Coeff[P3, x, 3]},
    Subst[Int[Simp[(2*c^3-9*b*c*d+27*a*d^2)/(27*d^2) - (c^2-3*b*d)*x/(3*d) + d*x^3, x]^p, x], x, x+c/(3*d)] /;
    NeQ[c, 0] /;
    FreeQ[p, x] && PolyQ[P3, x, 3]
```

$$5. \int (a + bx + cx^2 + dx^3 + ex^4)^p dx$$

$$1: \int (a + bx + cx^2 + dx^3 + ex^4)^p dx \text{ when } p \in \mathbb{Z}^- \wedge a \neq 0 \wedge c = \frac{b^2}{a} \wedge d = \frac{b^3}{a^2} \wedge e = \frac{b^4}{a^3}$$

Derivation: Algebraic simplification

$$\text{Basis: If } a \neq 0 \wedge c = \frac{b^2}{a} \wedge d = \frac{b^3}{a^2} \wedge e = \frac{b^4}{a^3}, \text{ then } a + bx + cx^2 + dx^3 + ex^4 = \frac{a^5 - b^5 x^5}{a^3 (a - bx)}$$

$$\text{Rule: If } p \in \mathbb{Z}^- \wedge a \neq 0 \wedge c = \frac{b^2}{a} \wedge d = \frac{b^3}{a^2} \wedge e = \frac{b^4}{a^3}, \text{ then}$$

$$\int (a + bx + cx^2 + dx^3 + ex^4)^p dx \rightarrow \frac{1}{a^{3p}} \int \text{ExpandIntegrand}\left[\frac{(a - bx)^{-p}}{(a^5 - b^5 x^5)^{-p}}, x\right] dx$$

Program code:

```
Int[P4^p_, x_Symbol] :=
  With[{a=Coeff[P4,x,0],b=Coeff[P4,x,1],c=Coeff[P4,x,2],d=Coeff[P4,x,3],e=Coeff[P4,x,4]},
    1/a^(3*p)*Int[ExpandIntegrand[(a-b*x)^(-p)/(a^5-b^5*x^5)^(-p),x],x] /;
    NeQ[a,0] && EqQ[c,b^2/a] && EqQ[d,b^3/a^2] && EqQ[e,b^4/a^3] /;
    FreeQ[p,x] && PolyQ[P4,x,4] && ILtQ[p,0]
```

$$2: \int (a + bx + cx^2 + dx^3 + ex^4)^p dx \text{ when } b^3 - 4abc + 8a^2d = 0 \wedge 2p \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: If } b^3 - 4abc + 8a^2d = 0, \text{ then}$$

$$(a + bx + cx^2 + dx^3 + ex^4)^p = -16a^2 \text{ Subst}\left[\frac{1}{(b-4ax)^2} \left(\frac{1}{(b-4ax)^4} a (-3b^4 + 16ab^2c - 64a^2bd + 256a^3e - 32a^2(3b^2 - 8ac)x^2 + 256a^4x^4)\right)^p, \frac{b}{4a} + \frac{1}{x}\right] \partial_x \left(\frac{b}{4a} + \frac{1}{x}\right)$$

Note: The substitution transforms a dense quartic polynomial into a symmetric quartic trinomial over the 4th power of a linear.

Rule: If $b^3 - 4abc + 8a^2d = 0 \wedge 2p \in \mathbb{Z}$, then

$$\int (a + bx + cx^2 + dx^3 + ex^4)^p dx \rightarrow -16a^2 \text{Subst} \left[\int \frac{1}{(b-4ax)^2} \left(\frac{1}{(b-4ax)^4} a (-3b^4 + 16ab^2c - 64a^2bd + 256a^3e - 32a^2(3b^2 - 8ac)x^2 + 256a^4x^4) \right)^p dx, x, \frac{b}{4a} + \frac{1}{x} \right]$$

Program code:

```
Int[P4^p_, x_Symbol] :=
  With[{a=Coeff[P4,x,0],b=Coeff[P4,x,1],c=Coeff[P4,x,2],d=Coeff[P4,x,3],e=Coeff[P4,x,4]},
    -16*a^2*Subst[
      Int[1/(b-4*a*x)^2*(a*(-3*b^4+16*a*b^2*c-64*a^2*b*d+256*a^3*e-32*a^2*(3*b^2-8*a*c)*x^2+256*a^4*x^4)/(b-4*a*x)^4)^p,x],
      x,b/(4*a)+1/x] /;
    NeQ[a,0] && NeQ[b,0] && EqQ[b^3-4*a*b*c+8*a^2*d,0]] /;
  FreeQ[p,x] && PolyQ[P4,x,4] && IntegerQ[2*p] && Not[IGtQ[p,0]]
```

6: $\int (a + bx^2 + cx^3 + dx^4 + ex^6)^p dx$ when $p \in \mathbb{Z}^- \wedge b^2 - 3ad = 0 \wedge b^3 - 27a^2e = 0$

Algebraic expansion

Basis: If $b^2 - 3ad = 0 \wedge b^3 - 27a^2e = 0$, then

$$a + bx^2 + cx^3 + dx^4 + ex^6 = \frac{1}{27a^2} (3a + 3a^{2/3}c^{1/3}x + bx^2) (3a - 3(-1)^{1/3}a^{2/3}c^{1/3}x + bx^2) (3a + 3(-1)^{2/3}a^{2/3}c^{1/3}x + bx^2)$$

Note: If $\frac{m+1}{2} \in \mathbb{Z}^+$, then $cx^m + (a + bx^2)^m = \prod_{k=1}^m (a + (-1)^k (1 - \frac{1}{a}) c^{\frac{1}{3}} x + bx^2)$

Rule: If $p \in \mathbb{Z}^- \wedge b^2 - 3ad = 0 \wedge b^3 - 27a^2e = 0$, then

$$\int (a + bx^2 + cx^3 + dx^4 + ex^6)^p dx \rightarrow \frac{1}{3^p a^{2p}} \int \text{ExpandIntegrand}[(3a + 3a^{2/3}c^{1/3}x + bx^2)^p (3a - 3(-1)^{1/3}a^{2/3}c^{1/3}x + bx^2)^p (3a + 3(-1)^{2/3}a^{2/3}c^{1/3}x + bx^2)^p, x] dx$$

Program code:

```
Int[Q6^p_, x_Symbol] :=
  With[{a=Coeff[Q6,x,0],b=Coeff[Q6,x,2],c=Coeff[Q6,x,3],d=Coeff[Q6,x,4],e=Coeff[Q6,x,6]},
    1/(3^(3*p))*a^(2*p)*Int[ExpandIntegrand[
      (3*a+3*Rt[a,3]^2*Rt[c,3]*x+b*x^2)^p*
      (3*a-3*(-1)^(1/3)*Rt[a,3]^2*Rt[c,3]*x+b*x^2)^p*
      (3*a+3*(-1)^(2/3)*Rt[a,3]^2*Rt[c,3]*x+b*x^2)^p,x],x] /;
    EqQ[b^2-3*a*d,0] && EqQ[b^3-27*a^2*e,0] /;
    ILtQ[p,0] && PolyQ[Q6,x,6] && EqQ[Coeff[Q6,x,1],0] && EqQ[Coeff[Q6,x,5],0] && RationalFunctionQ[u,x]
```