

Rules for integrands of the form $\text{Trig}[d + e x]^m (a + b \cos[d + e x]^p + c \sin[d + e x]^q)^n$

1. $\int \sin[d + e x]^m (a + b \cos[d + e x]^p + c \sin[d + e x]^q)^n dx$ when $\frac{m}{2} \in \mathbb{Z} \wedge \frac{p}{2} \in \mathbb{Z} \wedge \frac{q}{2} \in \mathbb{Z} \wedge n \in \mathbb{Z}$

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Derivation: Integration by substitution

$$\text{Basis: } \cos[z]^2 == \frac{\cot[z]^2}{1 + \cot[z]^2}$$

$$\text{Basis: } \sin[z]^2 == \frac{1}{1 + \cot[z]^2}$$

$$\begin{aligned} \text{Basis: If } \frac{m}{2} \in \mathbb{Z}, \text{ then } \sin[d + e x]^m F[\cos[d + e x]^2, \sin[d + e x]^2] == \\ -\frac{1}{e} \text{Subst}\left[\frac{F\left[\frac{x^2}{1+x^2}, \frac{1}{1+x^2}\right]}{(1+x^2)^{m/2+1}}, x, \cot[d + e x]\right] \partial_x \cot[d + e x] \end{aligned}$$

Rule: If $\frac{m}{2} \in \mathbb{Z} \wedge \frac{p}{2} \in \mathbb{Z} \wedge \frac{q}{2} \in \mathbb{Z} \wedge n \in \mathbb{Z} \wedge 0 < p \leq q$, then

$$\int \sin[d + e x]^m (a + b \cos[d + e x]^p + c \sin[d + e x]^q)^n dx \rightarrow -\frac{1}{e} \text{Subst}\left[\int \frac{(c + b x^p (1+x^2)^{\frac{q}{2}-\frac{p}{2}} + a (1+x^2)^{q/2})^n}{(1+x^2)^{m/2+nq/2+1}} dx, x, \cot[d + e x]\right]$$

Program code:

```
Int[sin[d_+e_.x_]^m_*(a_+b_.cos[d_+e_.x_]^p_+c_.sin[d_+e_.x_]^q_)^n_,x_Symbol] :=
Module[{f=FreeFactors[Cot[d+e*x],x]},
-f/e*Subst[Int[ExpandToSum[c+b*(1+f^2*x^2)^(q/2-p/2)+a*(1+f^2*x^2)^(q/2),x]^n/(1+f^2*x^2)^(m/2+n*q/2+1),x],x,Cot[d+e*x]/f]] /;
FreeQ[{a,b,c,d,e},x] && IntegerQ[m/2] && IntegerQ[p/2] && IntegerQ[q/2] && IntegerQ[n] && GtQ[p,0] && LeQ[p,q]
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Int[cos[d_+e_.x_]^m_*(a_+b_.sin[d_+e_.x_]^p_+c_.cos[d_+e_.x_]^q_)^n_,x_Symbol] :=
Module[{f=FreeFactors[Tan[d+e*x],x]},
f/e*Subst[Int[ExpandToSum[c+b*(1+f^2*x^2)^(q/2-p/2)+a*(1+f^2*x^2)^(q/2),x]^n/(1+f^2*x^2)^(m/2+n*q/2+1),x],x,Tan[d+e*x]/f]] /;
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2: $\int \sin[d+e x]^m (a+b \cos[d+e x]^p+c \sin[d+e x]^q)^n dx$ when $\frac{m}{2} \in \mathbb{Z} \wedge \frac{p}{2} \in \mathbb{Z} \wedge \frac{q}{2} \in \mathbb{Z} \wedge n \in \mathbb{Z} \wedge 0 < q < p$

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$$- \frac{1}{e} \text{Subst}\left[\frac{F\left[\frac{x^2}{1+x^2}, \frac{1}{1+x^2}\right]}{(1+x^2)^{m/2+1}}, x, \cot[d+e x]\right] \partial_x \cot[d+e x]$$

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