

Rules for integrands of the form $(a + b x^n)^p (c + d x^n)^q (e + f x^n)^r$

when $b c - a d \neq 0 \wedge b e - a f \neq 0 \wedge d e - c f \neq 0$

0: $\int (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$ when $(p \mid q \mid r) \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule 1.1.3.5.1: If $(p \mid q \mid r) \in \mathbb{Z}^+$, then

$$\int (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow \int \text{ExpandIntegrand}[(a + b x^n)^p (c + d x^n)^q (e + f x^n)^r, x] dx$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_.*(e_+f_.*x_^n_)^r_,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*x^n)^p*(c+d*x^n)^q*(e+f*x^n)^r,x],x] /;
FreeQ[{a,b,c,d,e,f,n},x] && IGtQ[p,0] && IGtQ[q,0] && IGtQ[r,0]
```

$$1. \int (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$$

$$1: \int \frac{e + f x^n}{(a + b x^n) (c + d x^n)} dx$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{e+fz}{(a+bz)(c+dz)} = \frac{be-af}{(bca-d)(a+bz)} - \frac{de-cf}{(bca-d)(c+dz)}$$

Rule 1.1.3.5.1.1:

$$\int \frac{e + f x^n}{(a + b x^n) (c + d x^n)} dx \rightarrow \frac{be - af}{bc - ad} \int \frac{1}{a + b x^n} dx - \frac{de - cf}{bc - ad} \int \frac{1}{c + d x^n} dx$$

Program code:

```
Int[(e+f_.**x_^n_)/((a+b_.**x_^n_)*(c+d_.**x_^n_)),x_Symbol] :=
  (b*e-a*f)/(b*c-a*d)*Int[1/(a+b*x^n),x] -
  (d*e-c*f)/(b*c-a*d)*Int[1/(c+d*x^n),x] /;
FreeQ[{a,b,c,d,e,f,n},x]
```

2:
$$\int \frac{e + f x^n}{(a + b x^n) \sqrt{c + d x^n}} dx$$

Derivation: Algebraic expansion

Basis: $\frac{e+fx}{a+bz} == \frac{f}{b} + \frac{be-af}{b(a+bz)}$

Rule 1.1.3.5.1.2:

$$\int \frac{e + f x^n}{(a + b x^n) \sqrt{c + d x^n}} dx \rightarrow \frac{f}{b} \int \frac{1}{\sqrt{c + d x^n}} dx + \frac{be - af}{b} \int \frac{1}{(a + b x^n) \sqrt{c + d x^n}} dx$$

Program code:

```
Int[(e_+f_.*x_^n_)/((a_+b_.*x_^n_)*Sqrt[c_+d_.*x_^n_]),x_Symbol] :=
  f/b*Int[1/Sqrt[c+d*x^n],x] +
  (b*e-a*f)/b*Int[1/((a+b*x^n)*Sqrt[c+d*x^n]),x] /;
FreeQ[{a,b,c,d,e,f,n},x]
```

3:
$$\int \frac{e + f x^n}{\sqrt{a + b x^n} \sqrt{c + d x^n}} dx$$

Derivation: Algebraic expansion

Basis:
$$\frac{e+fx}{\sqrt{a+bz}} = \frac{f\sqrt{a+bz}}{b} + \frac{be-af}{b\sqrt{a+bz}}$$

Rule 1.1.3.5.1.3:

$$\int \frac{e + f x^n}{\sqrt{a + b x^n} \sqrt{c + d x^n}} dx \rightarrow \frac{f}{b} \int \frac{\sqrt{a + b x^n}}{\sqrt{c + d x^n}} dx + \frac{be - af}{b} \int \frac{1}{\sqrt{a + b x^n} \sqrt{c + d x^n}} dx$$

Program code:

```
Int[(e+f_.**x_^n_)/(Sqrt[a+_b_.**x_^n_]*Sqrt[c+_d_.**x_^n_]),x_Symbol] :=
  f/b*Int[Sqrt[a+b*x^n]/Sqrt[c+d*x^n],x] +
  (b*e-a*f)/b*Int[1/(Sqrt[a+b*x^n]*Sqrt[c+d*x^n]),x] /;
FreeQ[{a,b,c,d,e,f,n},x] &&
  Not[EqQ[n,2] && (PosQ[b/a] && PosQ[d/c] || NegQ[b/a] && (PosQ[d/c] || GtQ[a,0] && (Not[GtQ[c,0]] || SimplersqrtQ[-b/a,-d/c])))]
```

4.
$$\int (a + b x^n)^p (c + d x^n)^q (e + f x^n) dx \text{ when } p < -1$$

1:
$$\int \frac{e + f x^2}{\sqrt{a + b x^2} (c + d x^2)^{3/2}} dx \text{ when } \frac{b}{a} > 0 \wedge \frac{d}{c} > 0$$

Derivation: Algebraic expansion

■ Basis:
$$\frac{e+fx^2}{\sqrt{a+bx^2} (c+dx^2)^{3/2}} = \frac{be-af}{(bc-ad)\sqrt{a+bx^2}\sqrt{c+dx^2}} - \frac{(de-cf)\sqrt{a+bx^2}}{(bc-ad)(c+dx^2)^{3/2}}$$

Rule 1.1.3.5.1.4.1: If $\frac{b}{a} > 0 \wedge \frac{d}{c} > 0$, then

$$\int \frac{e + f x^2}{\sqrt{a + b x^2} (c + d x^2)^{3/2}} dx \rightarrow \frac{b e - a f}{b c - a d} \int \frac{1}{\sqrt{a + b x^2} \sqrt{c + d x^2}} dx - \frac{d e - c f}{b c - a d} \int \frac{\sqrt{a + b x^2}}{(c + d x^2)^{3/2}} dx$$

Program code:

```
Int[(e_+f_.*x_^2)/(Sqrt[a_+b_.*x_^2]*(c_+d_.*x_^2)^(3/2)),x_Symbol] :=
  (b*e-a*f)/(b*c-a*d)*Int[1/(Sqrt[a+b*x^2]*Sqrt[c+d*x^2]),x] -
  (d*e-c*f)/(b*c-a*d)*Int[Sqrt[a+b*x^2]/(c+d*x^2)^(3/2),x] /;
FreeQ[{a,b,c,d,e,f},x] && PosQ[b/a] && PosQ[d/c]
```

2: $\int (a + b x^n)^p (c + d x^n)^q (e + f x^n) dx$ when $p < -1 \wedge q > 0$

Derivation: Binomial product recurrence 1 with $p = 0$

Rule 1.1.3.5.1.4.2: If $p < -1 \wedge q > 0$, then

$$\int (a + b x^n)^p (c + d x^n)^q (e + f x^n) dx \rightarrow$$

$$- \frac{(b e - a f) x (a + b x^n)^{p+1} (c + d x^n)^q}{a b n (p + 1)} +$$

$$\frac{1}{a b n (p + 1)} \int (a + b x^n)^{p+1} (c + d x^n)^{q-1} (c (b e n (p + 1) + b e - a f) + d (b e n (p + 1) + (b e - a f) (n q + 1)) x^n) dx$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_.*(e_+f_.*x_^n_),x_Symbol] :=
  -(b*e-a*f)*x*(a+b*x^n)^(p+1)*(c+d*x^n)^q/(a*b*n*(p+1)) +
  1/(a*b*n*(p+1))*
  Int[(a+b*x^n)^(p+1)*(c+d*x^n)^(q-1)*Simp[c*(b*e*n*(p+1)+b*e-a*f)+d*(b*e*n*(p+1)+(b*e-a*f)*(n*q+1))*x^n,x] /;
FreeQ[{a,b,c,d,e,f,n},x] && LtQ[p,-1] && GtQ[q,0]
```

$$\mathbf{3:} \int (a+bx^n)^p (c+dx^n)^q (e+fx^n) dx \text{ when } p < -1$$

Derivation: Binomial product recurrence 2a with $p = 0$

Rule 1.1.3.5.1.4.3: If $p < -1$, then

$$\int (a+bx^n)^p (c+dx^n)^q (e+fx^n) dx \rightarrow -\frac{(be-af)x(a+bx^n)^{p+1}(c+dx^n)^{q+1}}{an(bc-ad)(p+1)} + \frac{1}{an(bc-ad)(p+1)} \int (a+bx^n)^{p+1}(c+dx^n)^q (c(be-af) + en(bc-ad)(p+1) + d(be-af)(n(p+q+2)+1)x^n) dx$$

Program code:

```
Int[(a+b_.**x_^n_)^p_*(c+d_.**x_^n_)^q_*(e+f_.**x_^n_),x_Symbol] :=
  -(b*e-a*f)**x*(a+b*x^n)^(p+1)*(c+d*x^n)^(q+1)/(a*n*(b*c-a*d)*(p+1)) +
  1/(a*n*(b*c-a*d)*(p+1))*
  Int[(a+b*x^n)^(p+1)*(c+d*x^n)^q*Simp[c*(b*e-a*f)+e*n*(b*c-a*d)*(p+1)+d*(b*e-a*f)*(n*(p+q+2)+1)*x^n,x],x] /;
FreeQ[{a,b,c,d,e,f,n,q},x] && LtQ[p,-1]
```

$$\mathbf{5:} \int (a+bx^n)^p (c+dx^n)^q (e+fx^n) dx \text{ when } q > 0 \wedge n(p+q+1)+1 \neq 0$$

Derivation: Binomial product recurrence 3a with $p = 0$

Rule 1.1.3.5.1.5: If $q > 0 \wedge n(p+q+1)+1 \neq 0$, then

$$\int (a+bx^n)^p (c+dx^n)^q (e+fx^n) dx \rightarrow \frac{fx(a+bx^n)^{p+1}(c+dx^n)^q}{b(n(p+q+1)+1)} +$$

$$\frac{1}{b(n(p+q+1)+1)} \int (a+bx^n)^p (c+dx^n)^{q-1} (c(b e - a f + b e n(p+q+1)) + (d(b e - a f) + f n q (b c - a d) + b d e n(p+q+1)) x^n) dx$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_.*(e_+f_.*x_^n_),x_Symbol] :=
  f*x*(a+b*x^n)^(p+1)*(c+d*x^n)^q/(b*(n*(p+q+1)+1)) +
  1/(b*(n*(p+q+1)+1))*
  Int[(a+b*x^n)^p*(c+d*x^n)^(q-1)*Simp[c*(b*e-a*f+b*e*n*(p+q+1))+(d*(b*e-a*f)+f*n*q*(b*c-a*d)+b*d*e*n*(p+q+1))*x^n,x] /;
FreeQ[{a,b,c,d,e,f,n,p},x] && GtQ[q,0] && NeQ[n*(p+q+1)+1,0]
```

$$6. \int \frac{(a+bx^n)^p (e+fx^n)}{c+dx^n} dx$$

$$1: \int \frac{e+fx^4}{(a+bx^4)^{3/4} (c+dx^4)} dx$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{e+fx}{(a+bz)^{3/4} (c+dz)} == \frac{b e - a f}{(b c - a d) (a+bz)^{3/4}} - \frac{(d e - c f) (a+bz)^{1/4}}{(b c - a d) (c+dz)}$$

Rule 1.1.3.5.1.6.1:

$$\int \frac{e+fx^4}{(a+bx^4)^{3/4} (c+dx^4)} dx \rightarrow \frac{b e - a f}{b c - a d} \int \frac{1}{(a+bx^4)^{3/4}} dx - \frac{d e - c f}{b c - a d} \int \frac{(a+bx^4)^{1/4}}{c+dx^4} dx$$

Program code:

```
Int[(e_+f_.*x_^4)/((a_+b_.*x_^4)^(3/4)*(c_+d_.*x_^4)),x_Symbol] :=
  (b*e-a*f)/(b*c-a*d)*Int[1/(a+b*x^4)^(3/4),x] - (d*e-c*f)/(b*c-a*d)*Int[(a+b*x^4)^(1/4)/(c+d*x^4),x] /;
FreeQ[{a,b,c,d,e,f},x]
```

$$2: \int \frac{(a+bx^n)^p (e+fx^n)}{c+dx^n} dx$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{e+fx}{c+dz} = \frac{f}{d} + \frac{d e - c f}{d (c+dz)}$$

Rule 1.1.3.5.1.6.2:

$$\int \frac{(a+bx^n)^p (e+fx^n)}{c+dx^n} dx \rightarrow \frac{f}{d} \int (a+bx^n)^p dx + \frac{d e - c f}{d} \int \frac{(a+bx^n)^p}{c+dx^n} dx$$

Program code:

```
Int[(a+b_.**x_^n_)^p_.*(e+f_.**x_^n_)/(c+d_.**x_^n_),x_Symbol] :=
  f/d*Int[(a+b*x^n)^p,x] + (d*e-c*f)/d*Int[(a+b*x^n)^p/(c+d*x^n),x] /;
FreeQ[{a,b,c,d,e,f,p,n},x]
```

$$7: \int (a+bx^n)^p (c+dx^n)^q (e+fx^n) dx$$

Derivation: Algebraic expansion

Rule 1.1.3.5.1.7:

$$\int (a+bx^n)^p (c+dx^n)^q (e+fx^n) dx \rightarrow e \int (a+bx^n)^p (c+dx^n)^q dx + f \int x^n (a+bx^n)^p (c+dx^n)^q dx$$

Program code:

```
Int[(a+b_.**x_^n_)^p_.*(c+d_.**x_^n_)^q_.*(e+f_.**x_^n_),x_Symbol] :=
  e*Int[(a+b*x^n)^p*(c+d*x^n)^q,x] + f*Int[x^n*(a+b*x^n)^p*(c+d*x^n)^q,x] /;
FreeQ[{a,b,c,d,e,f,n,p,q},x]
```


2. $\int (a+bx^n)^p (c+dx^n)^q (e+fx^n)^r dx$ when $p \in \mathbb{Z}^-$

1. $\int \frac{(c+dx^2)^q (e+fx^2)^r}{a+bx^2} dx$

1: $\int \frac{1}{(a+bx^2)(c+dx^2)\sqrt{e+fx^2}} dx$

Derivation: Algebraic expansion

Basis: $\frac{1}{(a+bz)(c+dz)} = \frac{b}{(bca-d)(a+bz)} - \frac{d}{(bca-d)(c+dz)}$

Rule 1.1.3.5.2.1.1:

$$\int \frac{1}{(a+bx^2)(c+dx^2)\sqrt{e+fx^2}} dx \rightarrow \frac{b}{bca-d} \int \frac{1}{(a+bx^2)\sqrt{e+fx^2}} dx - \frac{d}{bca-d} \int \frac{1}{(c+dx^2)\sqrt{e+fx^2}} dx$$

Program code:

```
Int[1/((a+b_.**x_^2)*(c+d_.**x_^2)*Sqrt[e+f_.**x_^2]),x_Symbol] :=
  b/(b*c-a*d)*Int[1/((a+b*x^2)*Sqrt[e+f*x^2]),x] -
  d/(b*c-a*d)*Int[1/((c+d*x^2)*Sqrt[e+f*x^2]),x] /;
FreeQ[{a,b,c,d,e,f},x]
```

```
Int[1/(x^2*(c+d_.**x_^2)*Sqrt[e+f_.**x_^2]),x_Symbol] :=
  1/c*Int[1/(x^2*Sqrt[e+f*x^2]),x] -
  d/c*Int[1/((c+d*x^2)*Sqrt[e+f*x^2]),x] /;
FreeQ[{c,d,e,f},x] && NeQ[d*e-c*f,0]
```

$$2: \int \frac{\sqrt{c+dx^2} \sqrt{e+fx^2}}{a+bx^2} dx$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{\sqrt{c+dz}}{a+bz} = \frac{d}{b\sqrt{c+dz}} + \frac{bc-ad}{b(a+bz)\sqrt{c+dz}}$$

Rule 1.1.3.5.2.1.2:

$$\int \frac{\sqrt{c+dx^2} \sqrt{e+fx^2}}{a+bx^2} dx \rightarrow \frac{d}{b} \int \frac{\sqrt{e+fx^2}}{\sqrt{c+dx^2}} dx + \frac{bc-ad}{b} \int \frac{\sqrt{e+fx^2}}{(a+bx^2)\sqrt{c+dx^2}} dx$$

Program code:

```
Int[Sqrt[c+_.*x_^2]*Sqrt[e+_.*x_^2]/(a+_.*x_^2),x_Symbol] :=
  d/b*Int[Sqrt[e+f*x^2]/Sqrt[c+d*x^2],x] + (b*c-a*d)/b*Int[Sqrt[e+f*x^2]/((a+b*x^2)*Sqrt[c+d*x^2]),x] /;
FreeQ[{a,b,c,d,e,f},x] && GtQ[d/c,0] && GtQ[f/e,0] && Not[SimplerSqrtQ[d/c,f/e]]
```

```
Int[Sqrt[c+_.*x_^2]*Sqrt[e+_.*x_^2]/(a+_.*x_^2),x_Symbol] :=
  d/b*Int[Sqrt[e+f*x^2]/Sqrt[c+d*x^2],x] + (b*c-a*d)/b*Int[Sqrt[e+f*x^2]/((a+b*x^2)*Sqrt[c+d*x^2]),x] /;
FreeQ[{a,b,c,d,e,f},x] && Not[SimplerSqrtQ[-f/e,-d/c]]
```

$$3. \int \frac{1}{(a+bx^2)\sqrt{c+dx^2}\sqrt{e+fx^2}} dx$$

$$1: \int \frac{1}{(a+bx^2)\sqrt{c+dx^2}\sqrt{e+fx^2}} dx \text{ when } \frac{d}{c} > 0 \wedge \frac{f}{e} > 0$$

Derivation: Algebraic expansion

Basis: $\frac{1}{(a+bx^2)\sqrt{e+fx^2}} = -\frac{f}{(be-af)\sqrt{e+fx^2}} + \frac{b\sqrt{e+fx^2}}{(be-af)(a+bx^2)}$

Rule 1.1.3.5.2.1.3.1: If $\frac{d}{c} > 0 \wedge \frac{f}{e} > 0$, then

$$\int \frac{1}{(a+bx^2)\sqrt{c+dx^2}\sqrt{e+fx^2}} dx \rightarrow -\frac{f}{be-af} \int \frac{1}{\sqrt{c+dx^2}\sqrt{e+fx^2}} dx + \frac{b}{be-af} \int \frac{\sqrt{e+fx^2}}{(a+bx^2)\sqrt{c+dx^2}} dx$$

Program code:

```
Int[1/((a+b_.**x_^2)*Sqrt[c+d_.**x_^2]*Sqrt[e+f_.**x_^2]),x_Symbol] :=
  -f/(b*e-a*f)*Int[1/(Sqrt[c+d*x^2]*Sqrt[e+f*x^2]),x] +
  b/(b*e-a*f)*Int[Sqrt[e+f*x^2]/((a+b*x^2)*Sqrt[c+d*x^2]),x] /;
FreeQ[{a,b,c,d,e,f},x] && GtQ[d/c,0] && GtQ[f/e,0] && Not[SimplerSqrtQ[d/c,f/e]]
```

2. $\int \frac{1}{(a+bx^2)\sqrt{c+dx^2}\sqrt{e+fx^2}} dx$ when $\frac{d}{c} \neq 0$

1: $\int \frac{1}{(a+bx^2)\sqrt{c+dx^2}\sqrt{e+fx^2}} dx$ when $\frac{d}{c} \neq 0 \wedge c > 0 \wedge e > 0$

Rule 1.1.3.5.2.1.3.2.1: If $\frac{d}{c} \neq 0 \wedge c > 0 \wedge e > 0$, then

$$\int \frac{1}{(a+bx^2)\sqrt{c+dx^2}\sqrt{e+fx^2}} dx \rightarrow \frac{1}{a\sqrt{c}\sqrt{e}\sqrt{-\frac{d}{c}}} \text{EllipticPi}\left[\frac{bc}{ad}, \text{ArcSin}\left[\sqrt{-\frac{d}{c}}x\right], \frac{cf}{de}\right]$$

Program code:

```
Int[1/((a+b_.**x_^2)*Sqrt[c+d_.**x_^2]*Sqrt[e+f_.**x_^2]),x_Symbol] :=
  1/(a*Sqrt[c]*Sqrt[e]*Rt[-d/c,2])*EllipticPi[b*c/(a*d), ArcSin[Rt[-d/c,2]*x], c*f/(d*e)] /;
FreeQ[{a,b,c,d,e,f},x] && Not[GtQ[d/c,0]] && GtQ[c,0] && GtQ[e,0] && Not[Not[GtQ[f/e,0]]] && SimplerSqrtQ[-f/e,-d/c]]
```

$$\text{2: } \int \frac{1}{(a+bx^2) \sqrt{c+dx^2} \sqrt{e+fx^2}} dx \text{ when } \frac{d}{c} \neq 0 \wedge c \neq 0$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{\sqrt{1+\frac{d}{c}x^2}}{\sqrt{c+dx^2}} == 0$$

Rule 1.1.3.5.2.1.3.2.2: If $\frac{d}{c} \neq 0 \wedge c \neq 0$, then

$$\int \frac{1}{(a+bx^2) \sqrt{c+dx^2} \sqrt{e+fx^2}} dx \rightarrow \frac{\sqrt{1+\frac{d}{c}x^2}}{\sqrt{c+dx^2}} \int \frac{1}{(a+bx^2) \sqrt{1+\frac{d}{c}x^2} \sqrt{e+fx^2}} dx$$

Program code:

```
Int[1/((a+b.*x.^2)*Sqrt[c+d.*x.^2]*Sqrt[e+f.*x.^2]),x_Symbol] :=
  Sqrt[1+d/c*x^2]/Sqrt[c+d*x^2]*Int[1/((a+b*x^2)*Sqrt[1+d/c*x^2]*Sqrt[e+f*x^2]),x] /;
FreeQ[{a,b,c,d,e,f},x] && Not[GtQ[c,0]]
```

$$4. \int \frac{\sqrt{c+dx^2}}{(a+bx^2) \sqrt{e+fx^2}} dx$$

$$\text{1: } \int \frac{\sqrt{c+dx^2}}{(a+bx^2) \sqrt{e+fx^2}} dx \text{ when } \frac{d}{c} > 0$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{\sqrt{c+dx^2} \sqrt{\frac{c(e+fx^2)}{e(c+dx^2)}}}{\sqrt{e+fx^2}} == 0$$

Rule 1.1.3.5.2.1.4.1: If $\frac{d}{c} > 0$, then

$$\int \frac{\sqrt{c+dx^2}}{(a+bx^2) \sqrt{e+fx^2}} dx \rightarrow \frac{c \sqrt{e+fx^2}}{e \sqrt{c+dx^2} \sqrt{\frac{c(e+fx^2)}{e(c+dx^2)}}}$$

$$\int \frac{1}{(a+bx^2) \sqrt{\frac{c(e+fx^2)}{e(c+dx^2)}}} dx \rightarrow \frac{c \sqrt{e+fx^2}}{a e \sqrt{\frac{d}{c}} \sqrt{c+dx^2} \sqrt{\frac{c(e+fx^2)}{e(c+dx^2)}}} \text{EllipticPi}\left[1 - \frac{bc}{ad}, \text{ArcTan}\left[\sqrt{\frac{d}{c}} x\right], 1 - \frac{cf}{de}\right]$$

$$\int \frac{\sqrt{c+dx^2}}{(a+bx^2) \sqrt{e+fx^2}} dx \rightarrow \frac{\sqrt{c+dx^2} \sqrt{\frac{c(e+fx^2)}{e(c+dx^2)}}}{\sqrt{e+fx^2}} \int \frac{1}{(a+bx^2) \sqrt{\frac{c(e+fx^2)}{e(c+dx^2)}}} dx \rightarrow \frac{\sqrt{c+dx^2} \sqrt{\frac{c(e+fx^2)}{e(c+dx^2)}}}{a \sqrt{\frac{d}{c}} \sqrt{e+fx^2}} \text{EllipticPi}\left[1 - \frac{bc}{ad}, \text{ArcTan}\left[\sqrt{\frac{d}{c}} x\right], 1 - \frac{cf}{de}\right]$$

Program code:

```
Int[Sqrt[c_+d_.*x_^2]/((a_+b_.*x_^2)*Sqrt[e_+f_.*x_^2]),x_Symbol] :=
  c*Sqrt[e+f*x^2]/(a*e*Rt[d/c,2]*Sqrt[c+d*x^2]*Sqrt[c*(e+f*x^2)/(e*(c+d*x^2))]) *
  EllipticPi[1-b*c/(a*d),ArcTan[Rt[d/c,2]*x],1-c*f/(d*e)] /;
FreeQ[{a,b,c,d,e,f},x] && PosQ[d/c]
```

```
(* Int[Sqrt[c_+d_.*x_^2]/((a_+b_.*x_^2)*Sqrt[e_+f_.*x_^2]),x_Symbol] :=
  Sqrt[c+d*x^2]*Sqrt[c*(e+f*x^2)/(e*(c+d*x^2))]/(a*Rt[d/c,2]*Sqrt[e+f*x^2]) *
  EllipticPi[1-b*c/(a*d),ArcTan[Rt[d/c,2]*x],1-c*f/(d*e)] /;
FreeQ[{a,b,c,d,e,f},x] && PosQ[d/c] *)
```

$$2: \int \frac{\sqrt{c+dx^2}}{(a+bx^2)\sqrt{e+fx^2}} dx \text{ when } \frac{d}{c} \neq 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{\sqrt{c+dx^2}}{a+bx^2} = \frac{d}{b\sqrt{c+dx^2}} + \frac{bc-ad}{b(a+bx^2)\sqrt{c+dx^2}}$$

Rule 1.1.3.5.2.1.4.2: If $\frac{d}{c} \neq 0$, then

$$\int \frac{\sqrt{c+dx^2}}{(a+bx^2)\sqrt{e+fx^2}} dx \rightarrow \frac{d}{b} \int \frac{1}{\sqrt{c+dx^2}\sqrt{e+fx^2}} dx + \frac{bc-ad}{b} \int \frac{1}{(a+bx^2)\sqrt{c+dx^2}\sqrt{e+fx^2}} dx$$

Program code:

```
Int[Sqrt[c+d.*x_^2]/((a+b.*x_^2)*Sqrt[e+f.*x_^2]),x_Symbol] :=
  d/b*Int[1/(Sqrt[c+d*x^2]*Sqrt[e+f*x^2]),x] +
  (b*c-a*d)/b*Int[1/((a+b*x^2)*Sqrt[c+d*x^2]*Sqrt[e+f*x^2]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NegQ[d/c]
```

$$5. \int \frac{(c+dx^2)^q (e+fx^2)^r}{a+bx^2} dx \text{ when } q > 0$$

$$1: \int \frac{\sqrt{e+fx^2}}{(a+bx^2)(c+dx^2)^{3/2}} dx \text{ when } \frac{d}{c} > 0 \wedge \frac{f}{e} > 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{1}{(a+bx^2)(c+dx^2)^{3/2}} = \frac{b}{(b^2c-ad)(a+bx^2)\sqrt{c+dx^2}} - \frac{d}{(b^2c-ad)(c+dx^2)^{3/2}}$$

Rule 1.1.3.5.2.1.5.1: If $\frac{d}{c} > 0 \wedge \frac{f}{e} > 0$, then

$$\int \frac{\sqrt{e+fx^2}}{(a+bx^2)(c+dx^2)^{3/2}} dx \rightarrow \frac{b}{b^2c-ad} \int \frac{\sqrt{e+fx^2}}{(a+bx^2)\sqrt{c+dx^2}} dx - \frac{d}{b^2c-ad} \int \frac{\sqrt{e+fx^2}}{(c+dx^2)^{3/2}} dx$$

Program code:

```
Int[Sqrt[e+f*x^2]/((a+b*x^2)*(c+d*x^2)^(3/2)),x_Symbol] :=
  b/(b*c-a*d)*Int[Sqrt[e+f*x^2]/((a+b*x^2)*Sqrt[c+d*x^2]),x] -
  d/(b*c-a*d)*Int[Sqrt[e+f*x^2]/(c+d*x^2)^(3/2),x] /;
FreeQ[{a,b,c,d,e,f},x] && PosQ[d/c] && PosQ[f/e]
```

$$2: \int \frac{(e+fx^2)^{3/2}}{(a+bx^2)(c+dx^2)^{3/2}} dx \text{ when } \frac{d}{c} > 0 \wedge \frac{f}{e} > 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{e+fx^2}{(a+bx^2)(c+dx^2)} = \frac{be-af}{(b^2c-ad)(a+bx^2)} - \frac{de-cf}{(b^2c-ad)(c+dx^2)}$$

Rule 1.1.3.5.2.1.5.2: If $\frac{d}{c} > 0 \wedge \frac{f}{e} > 0$, then

$$\int \frac{(e+fx^2)^{3/2}}{(a+bx^2)(c+dx^2)^{3/2}} dx \rightarrow \frac{be-af}{bc-ad} \int \frac{\sqrt{e+fx^2}}{(a+bx^2)\sqrt{c+dx^2}} dx - \frac{de-cf}{bc-ad} \int \frac{\sqrt{e+fx^2}}{(c+dx^2)^{3/2}} dx$$

Program code:

```
Int[(e+f.*x^2)^(3/2)/((a+b.*x^2)*(c+d.*x^2)^(3/2)),x_Symbol] :=
  (b*e-a*f)/(b*c-a*d)*Int[Sqrt[e+f*x^2]/((a+b*x^2)*Sqrt[c+d*x^2]),x] -
  (d*e-c*f)/(b*c-a*d)*Int[Sqrt[e+f*x^2]/(c+d*x^2)^(3/2),x] /;
FreeQ[{a,b,c,d,e,f},x] && PosQ[d/c] && PosQ[f/e]
```

$$\mathbf{3:} \int \frac{(c+dx^2)^{3/2} \sqrt{e+fx^2}}{a+bx^2} dx \text{ when } \frac{d}{c} > 0 \wedge \frac{f}{e} > 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{(c+dx^2)^{3/2}}{a+bx^2} = \frac{(bc-ad)^2}{b^2(a+bx^2)\sqrt{c+dx^2}} + \frac{d(2bc-ad+bdx^2)}{b^2\sqrt{c+dx^2}}$$

Rule 1.1.3.5.2.1.5.3: If $\frac{d}{c} > 0 \wedge \frac{f}{e} > 0$, then

$$\int \frac{(c+dx^2)^{3/2} \sqrt{e+fx^2}}{a+bx^2} dx \rightarrow \frac{(bc-ad)^2}{b^2} \int \frac{\sqrt{e+fx^2}}{(a+bx^2)\sqrt{c+dx^2}} dx + \frac{d}{b^2} \int \frac{(2bc-ad+bdx^2) \sqrt{e+fx^2}}{\sqrt{c+dx^2}} dx$$

Program code:

```
Int[(c+d.*x^2)^(3/2)*Sqrt[e+f.*x^2]/(a+b.*x^2),x_Symbol] :=
  (b*c-a*d)^2/b^2*Int[Sqrt[e+f*x^2]/((a+b*x^2)*Sqrt[c+d*x^2]),x] +
  d/b^2*Int[(2*b*c-a*d+b*d*x^2)*Sqrt[e+f*x^2]/Sqrt[c+d*x^2],x] /;
FreeQ[{a,b,c,d,e,f},x] && PosQ[d/c] && PosQ[f/e]
```


$$4: \int \frac{(c+dx^2)^q (e+fx^2)^r}{a+bx^2} dx \text{ when } q < -1 \wedge r > 1$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{(c+dx^2)^q (e+fx^2)^r}{a+bx^2} = \frac{b(b e - a f) (c+dx^2)^{q+2}}{(b c - a d)^2 (a+bx^2)} - \frac{(c+dx^2)^q (2 b c d e - a d^2 e - b c^2 f + d^2 (b e - a f) x^2)}{(b c - a d)^2}$$

Rule 1.1.3.5.2.1.5.4: If $q < -1 \wedge r > 1$, then

$$\int \frac{(c+dx^2)^q (e+fx^2)^r}{a+bx^2} dx \rightarrow \frac{b(b e - a f)}{(b c - a d)^2} \int \frac{(c+dx^2)^{q+2} (e+fx^2)^{r-1}}{a+bx^2} dx - \frac{1}{(b c - a d)^2} \int (c+dx^2)^q (e+fx^2)^{r-1} (2 b c d e - a d^2 e - b c^2 f + d^2 (b e - a f) x^2) dx$$

Program code:

```
Int[(c+d_.**x_^2)^q_*(e+f_.**x_^2)^r_/(a+b_.**x_^2),x_Symbol] :=
  b*(b*e-a*f)/(b*c-a*d)^2*Int[(c+d*x^2)^(q+2)*(e+f*x^2)^(r-1)/(a+b*x^2),x] -
  1/(b*c-a*d)^2*Int[(c+d*x^2)^q*(e+f*x^2)^(r-1)*(2*b*c*d*e-a*d^2*e-b*c^2*f+d^2*(b*e-a*f)*x^2),x] /;
FreeQ[{a,b,c,d,e,f},x] && LtQ[q,-1] && GtQ[r,1]
```

5: $\int \frac{(c+dx^2)^q (e+fx^2)^r}{a+bx^2} dx$ when $q > 1$

Derivation: Algebraic expansion

Basis: $c+dx = \frac{d(a+bz)}{b} + \frac{bc-ad}{b}$

Rule 1.1.3.5.2.1.5.5: If $q > 1$, then

$$\int \frac{(c+dx^2)^q (e+fx^2)^r}{a+bx^2} dx \rightarrow \frac{d}{b} \int (c+dx^2)^{q-1} (e+fx^2)^r dx + \frac{bc-ad}{b} \int \frac{(c+dx^2)^{q-1} (e+fx^2)^r}{a+bx^2} dx$$

Program code:

```
Int[(c_+d_.*x_^2)^q_*(e_+f_.*x_^2)^r_/(a_+b_.*x_^2),x_Symbol] :=
  d/b*Int[(c+d*x^2)^(q-1)*(e+f*x^2)^r,x] +
  (b*c-a*d)/b*Int[(c+d*x^2)^(q-1)*(e+f*x^2)^r/(a+b*x^2),x] /;
FreeQ[{a,b,c,d,e,f,r},x] && GtQ[q,1]
```

$$6. \int \frac{(c+dx^2)^q (e+fx^2)^r}{a+bx^2} dx \text{ when } q \leq -1$$

$$1: \int \frac{(c+dx^2)^q (e+fx^2)^r}{a+bx^2} dx \text{ when } q < -1$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{(c+dx^2)^q}{a+bx^2} = \frac{b^2 (c+dx^2)^{q+2}}{(b^2c-ad)^2 (a+bx^2)} - \frac{d(2bc-ad+bdx^2)(c+dx^2)^q}{(b^2c-ad)^2}$$

Rule 1.1.3.5.2.1.6.1: If $q < -1$, then

$$\int \frac{(c+dx^2)^q (e+fx^2)^r}{a+bx^2} dx \rightarrow \frac{b^2}{(b^2c-ad)^2} \int \frac{(c+dx^2)^{q+2} (e+fx^2)^r}{a+bx^2} dx - \frac{d}{(b^2c-ad)^2} \int (c+dx^2)^q (e+fx^2)^r (2bc-ad+bdx^2) dx$$

Program code:

```
Int[(c+d.*x^2)^q*(e+f.*x^2)^r/(a+b.*x^2),x_Symbol] :=
  b^2/(b*c-a*d)^2*Int[(c+d*x^2)^(q+2)*(e+f*x^2)^r/(a+b*x^2),x] -
  d/(b*c-a*d)^2*Int[(c+d*x^2)^q*(e+f*x^2)^r*(2*b*c-a*d+b*d*x^2),x] /;
FreeQ[{a,b,c,d,e,f,r},x] && LtQ[q,-1]
```

2: $\int \frac{(c+dx^2)^q (e+fx^2)^r}{a+bx^2} dx$ when $q \leq -1$

Derivation: Algebraic expansion

Basis: $1 = -\frac{d(a+bx^2)}{bc-ad} + \frac{b(c+dx^2)}{bc-ad}$

Rule 1.1.3.5.2.1.6.2: If $q \leq -1$, then

$$\int \frac{(c+dx^2)^q (e+fx^2)^r}{a+bx^2} dx \rightarrow -\frac{d}{bc-ad} \int (c+dx^2)^q (e+fx^2)^r dx + \frac{b}{bc-ad} \int \frac{(c+dx^2)^{q+1} (e+fx^2)^r}{a+bx^2} dx$$

Program code:

```
Int[(c_+d_.*x_^2)^q_*(e_+f_.*x_^2)^r_/(a_+b_.*x_^2),x_Symbol] :=
  -d/(b*c-a*d)*Int[(c+d*x^2)^q*(e+f*x^2)^r,x] +
  b/(b*c-a*d)*Int[(c+d*x^2)^(q+1)*(e+f*x^2)^r/(a+b*x^2),x] /;
FreeQ[{a,b,c,d,e,f,r},x] && LeQ[q,-1]
```

$$2. \int \frac{(c+dx^2)^p (e+fx^2)^p}{(a+bx^2)^2} dx \text{ when } -1 \leq q < 0 \wedge -1 \leq r < 0$$

$$1: \int \frac{\sqrt{c+dx^2} \sqrt{e+fx^2}}{(a+bx^2)^2} dx$$

Rule 1.1.3.5.2.2.1:

$$\int \frac{\sqrt{c+dx^2} \sqrt{e+fx^2}}{(a+bx^2)^2} dx \rightarrow \frac{x \sqrt{c+dx^2} \sqrt{e+fx^2}}{2a(a+bx^2)} + \frac{df}{2ab^2} \int \frac{a-bx^2}{\sqrt{c+dx^2} \sqrt{e+fx^2}} dx + \frac{b^2ce-a^2df}{2ab^2} \int \frac{1}{(a+bx^2) \sqrt{c+dx^2} \sqrt{e+fx^2}} dx$$

Program code:

```
Int[Sqrt[c+d.*x.^2]*Sqrt[e+f.*x.^2]/(a+b.*x.^2)^2,x_Symbol] :=
  x*Sqrt[c+d*x^2]*Sqrt[e+f*x^2]/(2*a*(a+b*x^2)) +
  d*f/(2*a*b^2)*Int[(a-b*x^2)/(Sqrt[c+d*x^2]*Sqrt[e+f*x^2]),x] +
  (b^2*c*e-a^2*d*f)/(2*a*b^2)*Int[1/((a+b*x^2)*Sqrt[c+d*x^2]*Sqrt[e+f*x^2]),x] /;
FreeQ[{a,b,c,d,e,f},x]
```

$$2: \int \frac{1}{(a+bx^2)^2 \sqrt{c+dx^2} \sqrt{e+fx^2}} dx$$

Rule 1.1.3.5.2.2.2:

$$\int \frac{1}{(a+bx^2)^2 \sqrt{c+dx^2} \sqrt{e+fx^2}} dx \rightarrow \frac{b^2x \sqrt{c+dx^2} \sqrt{e+fx^2}}{2a(bc-ad)(be-af)(a+bx^2)} - \frac{df}{2a(bc-ad)(be-af)} \int \frac{a+bx^2}{\sqrt{c+dx^2} \sqrt{e+fx^2}} dx +$$

$$\frac{b^2 c e + 3 a^2 d f - 2 a b (d e + c f)}{2 a (b c - a d) (b e - a f)} \int \frac{1}{(a + b x^2) \sqrt{c + d x^2} \sqrt{e + f x^2}} dx$$

Program code:

```
Int[1/((a_+b_.*x_^2)^2*Sqrt[c_+d_.*x_^2]*Sqrt[e_+f_.*x_^2]),x_Symbol] :=
  b^2*x*Sqrt[c+d*x^2]*Sqrt[e+f*x^2]/(2*a*(b*c-a*d)*(b*e-a*f)*(a+b*x^2)) -
  d*f/(2*a*(b*c-a*d)*(b*e-a*f))*Int[(a+b*x^2)/(Sqrt[c+d*x^2]*Sqrt[e+f*x^2]),x] +
  (b^2*c*e+3*a^2*d*f-2*a*b*(d*e+c*f))/(2*a*(b*c-a*d)*(b*e-a*f))*Int[1/((a+b*x^2)*Sqrt[c+d*x^2]*Sqrt[e+f*x^2]),x] /;
FreeQ[{a,b,c,d,e,f},x]
```

3: $\int (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$ when $p \in \mathbb{Z}^- \wedge q > 0$

Derivation: Algebraic expansion

Basis: $c + d z = \frac{d(a+bz)}{b} + \frac{b c - a d}{b}$

Rule 1.1.3.5.2.4: If $p \in \mathbb{Z}^- \wedge q > 0$, then

$$\int (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow \frac{d}{b} \int (a + b x^n)^{p+1} (c + d x^n)^{q-1} (e + f x^n)^r dx + \frac{b c - a d}{b} \int (a + b x^n)^p (c + d x^n)^{q-1} (e + f x^n)^r dx$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_*(e_+f_.*x_^n_)^r_,x_Symbol] :=
  d/b*Int[(a+b*x^n)^(p+1)*(c+d*x^n)^(q-1)*(e+f*x^n)^r,x] +
  (b*c-a*d)/b*Int[(a+b*x^n)^p*(c+d*x^n)^(q-1)*(e+f*x^n)^r,x] /;
FreeQ[{a,b,c,d,e,f,n,r},x] && ILtQ[p,0] && GtQ[q,0]
```

4: $\int (a+bx^n)^p (c+dx^n)^q (e+fx^n)^r dx$ when $p \in \mathbb{Z}^- \wedge q \leq -1$

Derivation: Algebraic expansion

Basis: $1 = -\frac{d(a+bx^n)}{b(c+dx^n)} + \frac{b(c+dx^n)}{b(c+dx^n)}$

Rule 1.1.3.5.2.5: If $p \in \mathbb{Z}^- \wedge q \leq -1$, then

$$\int (a+bx^n)^p (c+dx^n)^q (e+fx^n)^r dx \rightarrow \frac{b}{b(c+dx^n)} \int (a+bx^n)^p (c+dx^n)^{q+1} (e+fx^n)^r dx - \frac{d}{b(c+dx^n)} \int (a+bx^n)^{p+1} (c+dx^n)^q (e+fx^n)^r dx$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_*(e_+f_.*x_^n_)^r_,x_Symbol] :=
  b/(b*c-a*d)*Int[(a+b*x^n)^p*(c+d*x^n)^(q+1)*(e+f*x^n)^r,x] -
  d/(b*c-a*d)*Int[(a+b*x^n)^(p+1)*(c+d*x^n)^q*(e+f*x^n)^r,x] /;
FreeQ[{a,b,c,d,e,f,n,q},x] && ILtQ[p,0] && LeQ[q,-1]
```

$$3. \int (a+bx^2)^p (c+dx^2)^q (e+fx^2)^r dx$$

$$1: \int \frac{1}{\sqrt{a+bx^2} \sqrt{c+dx^2} \sqrt{e+fx^2}} dx$$

Derivation: Piecewise constant extraction and integration by substitution

$$\blacksquare \text{ Basis: } \partial_x \frac{\sqrt{c+dx^2} \sqrt{\frac{a(e+fx^2)}{e(a+bx^2)}}}{\sqrt{e+fx^2} \sqrt{\frac{a(c+dx^2)}{c(a+bx^2)}}} = 0$$

$$- \text{ Basis: } \frac{1}{(a+bx^2)^{3/2} \sqrt{\frac{a(c+dx^2)}{c(a+bx^2)}} \sqrt{\frac{a(e+fx^2)}{e(a+bx^2)}}} = \frac{1}{a} \text{ Subst} \left[\frac{1}{\sqrt{1 - \frac{(bca-d)x^2}{c}}} \sqrt{1 - \frac{(be-af)x^2}{e}}}, x, \frac{x}{\sqrt{a+bx^2}} \right] \partial_x \frac{x}{\sqrt{a+bx^2}}$$

Rule 1.1.3.5.2.3.1:

$$\begin{aligned} \int \frac{1}{\sqrt{a+bx^2} \sqrt{c+dx^2} \sqrt{e+fx^2}} dx &\rightarrow \frac{a \sqrt{c+dx^2} \sqrt{\frac{a(e+fx^2)}{e(a+bx^2)}}}{c \sqrt{e+fx^2} \sqrt{\frac{a(c+dx^2)}{c(a+bx^2)}}} \int \frac{1}{(a+bx^2)^{3/2} \sqrt{\frac{a(c+dx^2)}{c(a+bx^2)}} \sqrt{\frac{a(e+fx^2)}{e(a+bx^2)}}} dx \\ &\rightarrow \frac{\sqrt{c+dx^2} \sqrt{\frac{a(e+fx^2)}{e(a+bx^2)}}}{c \sqrt{e+fx^2} \sqrt{\frac{a(c+dx^2)}{c(a+bx^2)}}} \text{Subst} \left[\int \frac{1}{\sqrt{1 - \frac{(bca-d)x^2}{c}}} \sqrt{1 - \frac{(be-af)x^2}{e}}} dx, x, \frac{x}{\sqrt{a+bx^2}} \right] \end{aligned}$$

Program code:

```
Int[1/(Sqrt[a+b.*x^2]*Sqrt[c+d.*x^2]*Sqrt[e+f.*x^2]),x_Symbol] :=
  Sqrt[c+d*x^2]*Sqrt[a*(e+f*x^2)/(e*(a+b*x^2))]/(c*Sqrt[e+f*x^2]*Sqrt[a*(c+d*x^2)/(c*(a+b*x^2))])*
  Subst[Int[1/(Sqrt[1-(b*c-a*d)*x^2/c]*Sqrt[1-(b*e-a*f)*x^2/e]),x],x,x/Sqrt[a+b*x^2]] /;
FreeQ[{a,b,c,d,e,f},x]
```


$$2: \int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2} \sqrt{e+fx^2}} dx$$

Derivation: Piecewise constant extraction and integration by substitution

$$\blacksquare \text{ Basis: } \partial_x \frac{\sqrt{c+dx^2} \sqrt{\frac{a(e+fx^2)}{e(a+bx^2)}}}{\sqrt{e+fx^2} \sqrt{\frac{a(c+dx^2)}{c(a+bx^2)}}} = 0$$

$$- \text{ Basis: } \frac{1}{\sqrt{a+bx^2} \sqrt{\frac{a(c+dx^2)}{c(a+bx^2)}} \sqrt{\frac{a(e+fx^2)}{e(a+bx^2)}}} = \text{Subst} \left[\frac{1}{(1-bx^2) \sqrt{1-\frac{(bca-d)x^2}{c}} \sqrt{1-\frac{(be-af)x^2}{e}}}, x, \frac{x}{\sqrt{a+bx^2}} \right] \partial_x \frac{x}{\sqrt{a+bx^2}}$$

Rule 1.1.3.5.2.3.2:

$$\begin{aligned} \int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2} \sqrt{e+fx^2}} dx &\rightarrow \frac{a\sqrt{c+dx^2} \sqrt{\frac{a(e+fx^2)}{e(a+bx^2)}}}{c\sqrt{e+fx^2} \sqrt{\frac{a(c+dx^2)}{c(a+bx^2)}}} \int \frac{1}{\sqrt{a+bx^2} \sqrt{\frac{a(c+dx^2)}{c(a+bx^2)}} \sqrt{\frac{a(e+fx^2)}{e(a+bx^2)}}} dx \\ &\rightarrow \frac{a\sqrt{c+dx^2} \sqrt{\frac{a(e+fx^2)}{e(a+bx^2)}}}{c\sqrt{e+fx^2} \sqrt{\frac{a(c+dx^2)}{c(a+bx^2)}}} \text{Subst} \left[\int \frac{1}{(1-bx^2) \sqrt{1-\frac{(bca-d)x^2}{c}} \sqrt{1-\frac{(be-af)x^2}{e}}} dx, x, \frac{x}{\sqrt{a+bx^2}} \right] \end{aligned}$$

Program code:

```
Int[Sqrt[a+b_.**x^2]/(Sqrt[c+d_.**x^2]*Sqrt[e+f_.**x^2]),x_Symbol] :=
  a*Sqrt[c+d*x^2]*Sqrt[a*(e+f*x^2)/(e*(a+b*x^2))]/(c*Sqrt[e+f*x^2]*Sqrt[a*(c+d*x^2)/(c*(a+b*x^2))]) *
  Subst[Int[1/((1-b*x^2)*Sqrt[1-(b*c-a*d)*x^2/c]*Sqrt[1-(b*e-a*f)*x^2/e]),x],x,x/Sqrt[a+b*x^2]] /;
FreeQ[{a,b,c,d,e,f},x]
```

$$3: \int \frac{\sqrt{c+dx^2}}{(a+bx^2)^{3/2} \sqrt{e+fx^2}} dx$$

Derivation: Piecewise constant extraction and integration by substitution

$$\blacksquare \text{ Basis: } \partial_x \frac{\sqrt{c+dx^2}}{\sqrt{e+fx^2}} \frac{\sqrt{\frac{a(e+fx^2)}{e(a+bx^2)}}}{\sqrt{\frac{a(c+dx^2)}{c(a+bx^2)}}} = 0$$

$$\blacksquare \text{ Basis: } \frac{\sqrt{\frac{a(c+dx^2)}{c(a+bx^2)}}}{(a+bx^2)^{3/2} \sqrt{\frac{a(e+fx^2)}{e(a+bx^2)}}} = \frac{1}{a} \text{Subst} \left[\frac{\sqrt{1 - \frac{(bca-d)x^2}{c}}}{\sqrt{1 - \frac{(be-af)x^2}{e}}}, x, \frac{x}{\sqrt{a+bx^2}} \right] \partial_x \frac{x}{\sqrt{a+bx^2}}$$

Rule 1.1.3.5.2.3.3:

$$\begin{aligned} \int \frac{\sqrt{c+dx^2}}{(a+bx^2)^{3/2} \sqrt{e+fx^2}} dx &\rightarrow \frac{\sqrt{c+dx^2}}{\sqrt{e+fx^2}} \frac{\sqrt{\frac{a(e+fx^2)}{e(a+bx^2)}}}{\sqrt{\frac{a(c+dx^2)}{c(a+bx^2)}}} \int \frac{\sqrt{\frac{a(c+dx^2)}{c(a+bx^2)}}}{(a+bx^2)^{3/2} \sqrt{\frac{a(e+fx^2)}{e(a+bx^2)}}} dx \rightarrow \\ &\frac{\sqrt{c+dx^2}}{a \sqrt{e+fx^2}} \frac{\sqrt{\frac{a(e+fx^2)}{e(a+bx^2)}}}{\sqrt{\frac{a(c+dx^2)}{c(a+bx^2)}}} \text{Subst} \left[\int \frac{\sqrt{1 - \frac{(bca-d)x^2}{c}}}{\sqrt{1 - \frac{(be-af)x^2}{e}}} dx, x, \frac{x}{\sqrt{a+bx^2}} \right] \end{aligned}$$

Program code:

```
Int[Sqrt[c+d.*x^2]/((a+b.*x^2)^(3/2)*Sqrt[e+f.*x^2]),x_Symbol] :=
  Sqrt[c+d*x^2]*Sqrt[a*(e+f*x^2)/(e*(a+b*x^2))]/(a*Sqrt[e+f*x^2]*Sqrt[a*(c+d*x^2)/(c*(a+b*x^2))]) *
  Subst[Int[Sqrt[1-(b*c-a*d)*x^2/c]/Sqrt[1-(b*e-a*f)*x^2/e],x],x,x/Sqrt[a+b*x^2] /;
FreeQ[{a,b,c,d,e,f},x]
```

$$4. \int \frac{\sqrt{a+bx^2} \sqrt{c+dx^2}}{\sqrt{e+fx^2}} dx$$

$$1: \int \frac{\sqrt{a+bx^2} \sqrt{c+dx^2}}{\sqrt{e+fx^2}} dx \text{ when } \frac{de-cf}{c} > 0$$

Rule 1.1.3.5.2.3.4.1: If $\frac{de-cf}{c} > 0$, then

$$\int \frac{\sqrt{a+bx^2} \sqrt{c+dx^2}}{\sqrt{e+fx^2}} dx \rightarrow$$

$$\frac{dx \sqrt{a+bx^2} \sqrt{e+fx^2}}{2f \sqrt{c+dx^2}} - \frac{c(de-cf)}{2f} \int \frac{\sqrt{a+bx^2}}{(c+dx^2)^{3/2} \sqrt{e+fx^2}} dx +$$

$$\frac{bc(de-cf)}{2df} \int \frac{1}{\sqrt{a+bx^2} \sqrt{c+dx^2} \sqrt{e+fx^2}} dx - \frac{bde-bcf-adf}{2df} \int \frac{\sqrt{c+dx^2}}{\sqrt{a+bx^2} \sqrt{e+fx^2}} dx$$

Program code:

```
Int[Sqrt[a+_.*x_^2]*Sqrt[c+_.*x_^2]/Sqrt[e+_.*x_^2],x_Symbol] :=
  d*x*Sqrt[a+b*x^2]*Sqrt[e+f*x^2]/(2*f*Sqrt[c+d*x^2]) -
  c*(d*e-c*f)/(2*f)*Int[Sqrt[a+b*x^2]/((c+d*x^2)^(3/2)*Sqrt[e+f*x^2]),x] +
  b*c*(d*e-c*f)/(2*d*f)*Int[1/(Sqrt[a+b*x^2]*Sqrt[c+d*x^2]*Sqrt[e+f*x^2]),x] -
  (b*d*e-b*c*f-a*d*f)/(2*d*f)*Int[Sqrt[c+d*x^2]/(Sqrt[a+b*x^2]*Sqrt[e+f*x^2]),x] /;
FreeQ[{a,b,c,d,e,f},x] && PosQ[(d*e-c*f)/c]
```

$$2: \int \frac{\sqrt{a+bx^2} \sqrt{c+dx^2}}{\sqrt{e+fx^2}} dx \text{ when } \frac{de-cf}{c} \neq 0$$

Rule 1.1.3.5.2.3.4.2: If $\frac{de-cf}{c} \neq 0$, then

$$\int \frac{\sqrt{a+bx^2} \sqrt{c+dx^2}}{\sqrt{e+fx^2}} dx \rightarrow$$

$$\frac{x \sqrt{a+bx^2} \sqrt{c+dx^2}}{2 \sqrt{e+fx^2}} + \frac{e(b e - a f)}{2 f} \int \frac{\sqrt{c+dx^2}}{\sqrt{a+bx^2} (e+fx^2)^{3/2}} dx +$$

$$\frac{(b e - a f)(d e - 2 c f)}{2 f^2} \int \frac{1}{\sqrt{a+bx^2} \sqrt{c+dx^2} \sqrt{e+fx^2}} dx - \frac{b d e - b c f - a d f}{2 f^2} \int \frac{\sqrt{e+fx^2}}{\sqrt{a+bx^2} \sqrt{c+dx^2}} dx$$

Program code:

```
Int[Sqrt[a_+b_.x_^2]*Sqrt[c_+d_.x_^2]/Sqrt[e_+f_.x_^2],x_Symbol] :=
  x*Sqrt[a+b*x^2]*Sqrt[c+d*x^2]/(2*Sqrt[e+f*x^2]) +
  e*(b*e-a*f)/(2*f)*Int[Sqrt[c+d*x^2]/(Sqrt[a+b*x^2]*(e+f*x^2)^(3/2)),x] +
  (b*e-a*f)*(d*e-2*c*f)/(2*f^2)*Int[1/(Sqrt[a+b*x^2]*Sqrt[c+d*x^2]*Sqrt[e+f*x^2]),x] -
  (b*d*e-b*c*f-a*d*f)/(2*f^2)*Int[Sqrt[e+f*x^2]/(Sqrt[a+b*x^2]*Sqrt[c+d*x^2]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NegQ[(d*e-c*f)/c]
```

5:
$$\int \frac{\sqrt{a+bx^2} \sqrt{c+dx^2}}{(e+fx^2)^{3/2}} dx$$

Derivation: Algebraic expansion

■ Basis:
$$\frac{\sqrt{a+bx^2}}{(e+fx^2)^{3/2}} = \frac{b}{f\sqrt{a+bx^2}\sqrt{e+fx^2}} - \frac{be-af}{f\sqrt{a+bx^2}(e+fx^2)^{3/2}}$$

Rule 1.1.3.5.2.3.5:

$$\int \frac{\sqrt{a+bx^2} \sqrt{c+dx^2}}{(e+fx^2)^{3/2}} dx \rightarrow \frac{b}{f} \int \frac{\sqrt{c+dx^2}}{\sqrt{a+bx^2}\sqrt{e+fx^2}} dx - \frac{be-af}{f} \int \frac{\sqrt{c+dx^2}}{\sqrt{a+bx^2}(e+fx^2)^{3/2}} dx$$

Program code:

```
Int[Sqrt[a+b_.**x_^2]*Sqrt[c+d_.**x_^2]/(e+f_.**x_^2)^(3/2),x_Symbol] :=
  b/f*Int[Sqrt[c+d*x^2]/(Sqrt[a+b*x^2]*Sqrt[e+f*x^2]),x] -
  (b*e-a*f)/f*Int[Sqrt[c+d*x^2]/(Sqrt[a+b*x^2]*(e+f*x^2)^(3/2)),x] /;
FreeQ[{a,b,c,d,e,f},x]
```

4: $\int (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$ when $n \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule 1.1.3.5.3: If $n \in \mathbb{Z}^+$, let $u = \text{ExpandIntegrand}[(a + b x^n)^p (c + d x^n)^q (e + f x^n)^r, x]$, if u is a sum, then

$$\int (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow \int u dx$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_*(e_+f_.*x_^n_)^r_,x_Symbol] :=
  With[{u=ExpandIntegrand[(a+b*x^n)^p*(c+d*x^n)^q*(e+f*x^n)^r,x]},
    Int[u,x] /;
    SumQ[u] /;
    FreeQ[{a,b,c,d,e,f,p,q,r},x] && IGtQ[n,0]
```

5: $\int (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$ when $n \in \mathbb{Z}^-$

Derivation: Integration by substitution

Basis: $F[x] \Rightarrow -\text{Subst}\left[\frac{F[x^{-1}]}{x^2}, x, \frac{1}{x}\right] \partial_x \frac{1}{x}$

Rule 1.1.3.5.4: If $n \in \mathbb{Z}^-$, then

$$\int (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow -\text{Subst}\left[\int \frac{(a + b x^{-n})^p (c + d x^{-n})^q (e + f x^{-n})^r}{x^2} dx, x, \frac{1}{x}\right]$$

Program code:

```
Int[(a+_.*x_^n_)^p_*(c+_.*x_^n_)^q_*(e+_.*x_^n_)^r_,x_Symbol] :=
  -Subst[Int[(a+b*x^(-n))^p*(c+d*x^(-n))^q*(e+f*x^(-n))^r/x^2,x],x,1/x] /;
FreeQ[{a,b,c,d,e,f,p,q,r},x] && !LtQ[n,0]
```

U: $\int (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$

Rule 1.1.3.5.X:

$$\int (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow \int (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$$

Program code:

```
Int[(a+_.*x_^n_)^p_.*(c+_.*x_^n_)^q_.*(e+_.*x_^n_)^r_,x_Symbol] :=
  Unintegrable[(a+b*x^n)^p*(c+d*x^n)^q*(e+f*x^n)^r,x] /;
FreeQ[{a,b,c,d,e,f,n,p,q,r},x]
```

S: $\int (a + b u^n)^p (c + d u^n)^q (e + f u^n)^r dx$ when $u = g + h x$

Derivation: Integration by substitution

Rule 1.1.3.5.S: If $u = g + h x$, then

$$\int (a + b u^n)^p (c + d u^n)^q (e + f u^n)^r dx \rightarrow \frac{1}{h} \text{Subst} \left[\int (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx, x, u \right]$$

Program code:

```
Int[(a_.+b_.*u_^n_)^p_.*(c_.+d_.*v_^n_)^q_.*(e_.+f_.*w_^n_)^r_. ,x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(a+b*x^n)^p*(c+d*x^n)^q*(e+f*x^n)^r,x],x,u] /;
FreeQ[{a,b,c,d,e,f,p,n,q,r},x] && EqQ[u,v] && EqQ[u,w] && LinearQ[u,x] && NeQ[u,x]
```


$$6. \int (a + b x^n)^p (c + d x^{-n})^q (e + f x^n)^r dx$$

$$1: \int (a + b x^n)^p (c + d x^{-n})^q (e + f x^n)^r dx \text{ when } q \in \mathbb{Z}$$

Derivation: Algebraic normalization

Basis: If $q \in \mathbb{Z}$, then $(c + d x^{-n})^q = \frac{(d + c x^n)^q}{x^{nq}}$

Rule 1.1.3.5.5.1: If $q \in \mathbb{Z}$, then

$$\int (a + b x^n)^p (c + d x^{-n})^q (e + f x^n)^r dx \rightarrow \int \frac{(a + b x^n)^p (d + c x^n)^q (e + f x^n)^r}{x^{nq}} dx$$

Program code:

```
Int[(a_.+b_.**x_^n_.)^p_.*(c_.+d_.**x_^mn_.)^q_.*(e_.+f_.**x_^n_.)^r_. ,x_Symbol] :=
  Int[(a+b*x^n)^p*(d+c*x^n)^q*(e+f*x^n)^r/x^(n*q),x] /;
FreeQ[{a,b,c,d,e,f,n,p,r},x] && EqQ[mn,-n] && IntegerQ[q]
```

2: $\int (a + b x^n)^p (c + d x^{-n})^q (e + f x^n)^r dx$ when $p \in \mathbb{Z} \wedge r \in \mathbb{Z}$

Derivation: Algebraic normalization

Basis: If $p \in \mathbb{Z}$, then $(a + b x^n)^p = x^{np} (b + a x^{-n})^p$

Rule 1.1.3.5.5.2: If $p \in \mathbb{Z} \wedge r \in \mathbb{Z}$, then

$$\int (a + b x^n)^p (c + d x^{-n})^q (e + f x^n)^r dx \rightarrow \int x^{n(p+r)} (b + a x^{-n})^p (c + d x^{-n})^q (f + e x^{-n})^r dx$$

Program code:

```
Int[(a_.+b_.**x_^n_.)^p_.*(c_.+d_.**x_^mn_.)^q_.*(e_.+f_.**x_^n_.)^r_.,x_Symbol] :=
  Int[x^(n*(p+r))* (b+a*x^(-n))^p*(c+d*x^(-n))^q*(f+e*x^(-n))^r,x] /;
FreeQ[{a,b,c,d,e,f,n,q},x] && EqQ[mn,-n] && IntegerQ[p] && IntegerQ[r]
```

3: $\int (a+bx^n)^p (c+dx^{-n})^q (e+fx^n)^r dx$ when $q \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{x^n q (c+dx^{-n})^q}{(d+cx^n)^q} = 0$

Basis: $\frac{x^n q (c+dx^{-n})^q}{(d+cx^n)^q} = \frac{x^n \text{FracPart}[q] (c+dx^{-n})^{\text{FracPart}[q]}}{(d+cx^n)^{\text{FracPart}[q]}}$

Rule 1.1.3.5.5.3: If $q \notin \mathbb{Z}$, then

$$\int (a+bx^n)^p (c+dx^{-n})^q (e+fx^n)^r dx \rightarrow \frac{x^n \text{FracPart}[q] (c+dx^{-n})^{\text{FracPart}[q]}}{(d+cx^n)^{\text{FracPart}[q]}} \int \frac{(a+bx^n)^p (d+cx^n)^q (e+fx^n)^r}{x^{nq}} dx$$

Program code:

```
Int[(a_.+b_.**x_^n_.)^p_.*(c_.+d_.**x_^mn_.)^q_.*(e_.+f_.**x_^n_.)^r_. ,x_Symbol] :=
  x^(n*FracPart[q])*(c+d*x^(-n))^FracPart[q]/(d+c*x^n)^FracPart[q]*Int[(a+b*x^n)^p*(d+c*x^n)^q*(e+f*x^n)^r/x^(n*q),x] /;
FreeQ[{a,b,c,d,e,f,n,p,q,r},x] && EqQ[mn,-n] && Not[IntegerQ[q]]
```

Rules for integrands of the form $(a + b x^n)^p (c + d x^n)^q (e_1 + f_1 x^{n/2})^r (e_2 + f_2 x^{n/2})^r$

1. $\int (a + b x^n)^p (c + d x^n)^q (e_1 + f_1 x^{n/2})^r (e_2 + f_2 x^{n/2})^r dx$ when $e_2 f_1 + e_1 f_2 = 0$

1: $\int (a + b x^n)^p (c + d x^n)^q (e_1 + f_1 x^{n/2})^r (e_2 + f_2 x^{n/2})^r dx$ when $e_2 f_1 + e_1 f_2 = 0 \wedge (r \in \mathbb{Z} \vee e_1 > 0 \wedge e_2 > 0)$

Derivation: Algebraic simplification

Basis: If $e_2 f_1 + e_1 f_2 = 0 \wedge (r \in \mathbb{Z} \vee e_1 > 0 \wedge e_2 > 0)$, then $(e_1 + f_1 x^{n/2})^r (e_2 + f_2 x^{n/2})^r = (e_1 e_2 + f_1 f_2 x^n)^r$

Rule: If $e_2 f_1 + e_1 f_2 = 0 \wedge (r \in \mathbb{Z} \vee e_1 > 0 \wedge e_2 > 0)$, then

$$\int (a + b x^n)^p (c + d x^n)^q (e_1 + f_1 x^{n/2})^r (e_2 + f_2 x^{n/2})^r dx \rightarrow \int (a + b x^n)^p (c + d x^n)^q (e_1 e_2 + f_1 f_2 x^n)^r dx$$

Program code:

```
Int[(a+b_.**x_^n_)^p_.*(c+d_.**x_^n_)^q_.*(e1+f1_.**x_^n2_)^r_.*(e2+f2_.**x_^n2_)^r_,x_Symbol] :=
  Int[(a+b*x^n)^p*(c+d*x^n)^q*(e1*e2+f1*f2*x^n)^r,x] /;
FreeQ[{a,b,c,d,e1,f1,e2,f2,n,p,q,r},x] && EqQ[n2,n/2] && EqQ[e2*f1+e1*f2,0] && (IntegerQ[r] || GtQ[e1,0] && GtQ[e2,0])
```

2: $\int (a + b x^n)^p (c + d x^n)^q (e_1 + f_1 x^{n/2})^r (e_2 + f_2 x^{n/2})^r dx$ when $e_2 f_1 + e_1 f_2 = 0$

Derivation: Piecewise constant extraction

■ Basis: If $e_2 f_1 + e_1 f_2 = 0$, then $\partial_x \frac{(e_1 + f_1 x^{n/2})^r (e_2 + f_2 x^{n/2})^r}{(e_1 e_2 + f_1 f_2 x^n)^r} = 0$

Rule: If $e_2 f_1 + e_1 f_2 = 0$, then

$$\int (a + b x^n)^p (c + d x^n)^q (e_1 + f_1 x^{n/2})^r (e_2 + f_2 x^{n/2})^r dx \rightarrow \frac{(e_1 + f_1 x^{n/2})^{\text{FracPart}[r]} (e_2 + f_2 x^{n/2})^{\text{FracPart}[r]}}{(e_1 e_2 + f_1 f_2 x^n)^{\text{FracPart}[r]}} \int (a + b x^n)^p (c + d x^n)^q (e_1 e_2 + f_1 f_2 x^n)^r dx$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_.*(e1_+f1_.*x_^n2_)^r_.*(e2_+f2_.*x_^n2_)^r_,x_Symbol] :=
  (e1+f1*x^(n/2))^FracPart[r]*(e2+f2*x^(n/2))^FracPart[r]/(e1*e2+f1*f2*x^n)^FracPart[r]*
  Int[(a+b*x^n)^p*(c+d*x^n)^q*(e1*e2+f1*f2*x^n)^r,x] /;
FreeQ[{a,b,c,d,e1,f1,e2,f2,n,p,q,r},x] && EqQ[n2,n/2] && EqQ[e2*f1+e1*f2,0]
```