

## Rules for integrands of the form $(dx)^m (ax^q + bx^n + cx^{2n-q})^p$

**1:**  $\int x^m (ax^n + bx^n + cx^n)^p dx$

Rule:

$$\int x^m (ax^n + bx^n + cx^n)^p dx \rightarrow \int x^m ((a+b+c)x^n)^p dx$$

Program code:

```
Int[x^m_.*(a_.*x^q_.+b_.*x^n_.+c_.*x^r_.)^p_,x_Symbol] :=
  Int[x^m*((a+b+c)*x^n)^p,x] /;
FreeQ[{a,b,c,m,n,p},x] && EqQ[q,n] && EqQ[r,n]
```

**2:**  $\int x^m (ax^q + bx^n + cx^{2n-q})^p dx$  when  $p \in \mathbb{Z}$

Rule: If  $p \in \mathbb{Z}$ , then

$$\int x^m (ax^q + bx^n + cx^{2n-q})^p dx \rightarrow \int x^{m+pq} (a + bx^{n-q} + cx^{2(n-q)})^p dx$$

Program code:

```
Int[x^m_.*(a_.*x^q_.+b_.*x^n_.+c_.*x^r_.)^p_,x_Symbol] :=
  Int[x^(m+pq)*(a+b*x^(n-q)+c*x^(2*(n-q)))^p,x] /;
FreeQ[{a,b,c,m,n,q},x] && EqQ[r,2*n-q] && IntegerQ[p] && PosQ[n-q]
```

3.  $\int \frac{x^m}{\sqrt{a x^q + b x^n + c x^{2n-q}}} dx$  when  $q < n \wedge b^2 - 4ac \neq 0$

1:  $\int \frac{x^m}{\sqrt{a x^q + b x^n + c x^{2n-q}}} dx$  when  $q < n \wedge b^2 - 4ac \neq 0 \wedge m = \frac{q}{2} - 1$

Derivation: Integration by substitution

Basis: If  $m = \frac{q}{2} - 1$ , then  $\frac{x^m}{\sqrt{a x^q + b x^n + c x^{2n-q}}} = -\frac{2}{n-q} \text{Subst} \left[ \frac{1}{4a-x^2}, x, \frac{x^{m+1}(2a+b x^{n-q})}{\sqrt{a x^q + b x^n + c x^{2n-q}}} \right] \partial_x \frac{x^{m+1}(2a+b x^{n-q})}{\sqrt{a x^q + b x^n + c x^{2n-q}}}$

Rule: If  $q < n \wedge b^2 - 4ac \neq 0 \wedge m = \frac{q}{2} - 1$ , then

$$\int \frac{x^m}{\sqrt{a x^q + b x^n + c x^{2n-q}}} dx \rightarrow -\frac{2}{n-q} \text{Subst} \left[ \int \frac{1}{4a-x^2} dx, x, \frac{x^{m+1}(2a+b x^{n-q})}{\sqrt{a x^q + b x^n + c x^{2n-q}}} \right]$$

Program code:

```
Int[x_^m_/Sqrt[a_*x^q_.+b_*x^n_.+c_*x^r_.],x_Symbol] :=
  -2/(n-q)*Subst[Int[1/(4*a-x^2),x],x,x^(m+1)*(2*a+b*x^(n-q))/Sqrt[a*x^q+b*x^n+c*x^r]] /;
FreeQ[{a,b,c,m,n,q,r},x] && EqQ[r,2*n-q] && PosQ[n-q] && NeQ[b^2-4*a*c,0] && EqQ[m,q/2-1]
```

2:  $\int \frac{x^m}{\sqrt{a x^q + b x^n + c x^{2n-q}}} dx$  when  $q < n$

Derivation: Piecewise constant extraction

Basis:  $\partial_x \frac{x^{q/2} \sqrt{a+b x^{n-q}+c x^{2(n-q)}}}{\sqrt{a x^q + b x^n + c x^{2n-q}}} = 0$

Rule: If  $q < n$ , then

$$\int \frac{x^m}{\sqrt{ax^q + bx^n + cx^{2n-q}}} dx \rightarrow \frac{x^{q/2} \sqrt{ax^{n-q} + bx^{2(n-q)} + cx^{2(n-q)}}}{\sqrt{ax^q + bx^n + cx^{2n-q}}} \int \frac{x^{m-q/2}}{\sqrt{a + bx^{n-q} + cx^{2(n-q)}}} dx$$

Program code:

```
Int[x^m_/Sqrt[a_*x^q_+b_*x^n_+c_*x^(2*n-q_)],x_Symbol] :=
  x^(q/2)*Sqrt[a+b*x^(n-q)+c*x^(2*(n-q))]/Sqrt[a*x^q+b*x^n+c*x^(2*n-q)]*
  Int[x^(m-q/2)/Sqrt[a+b*x^(n-q)+c*x^(2*(n-q))],x] /;
FreeQ[{a,b,c,m,n,q},x] && EqQ[r,2*n-q] && PosQ[n-q] && (EqQ[m,1] && EqQ[n,3] && EqQ[q,2] ||
(EqQ[m+1/2] || EqQ[m,3/2] || EqQ[m,1/2] || EqQ[m,5/2]) && EqQ[n,3] && EqQ[q,1])
```

4:  $\int \frac{x^{\frac{3(n-1)}{2}}}{(ax^{n-1} + bx^n + cx^{n+1})^{3/2}} dx$  when  $b^2 - 4ac \neq 0$

Rule: If  $b^2 - 4ac \neq 0$ , then

$$\int \frac{x^{\frac{3(n-1)}{2}}}{(ax^{n-1} + bx^n + cx^{n+1})^{3/2}} dx \rightarrow -\frac{2x^{\frac{n-1}{2}}(b + 2cx)}{(b^2 - 4ac)\sqrt{ax^{n-1} + bx^n + cx^{n+1}}}$$

Program code:

```
Int[x^m_/((a_*x^q_+b_*x^n_+c_*x^(2*n-q_))^(3/2)),x_Symbol] :=
  -2*x^((n-1)/2)*(b+2*c*x)/((b^2-4*a*c)*Sqrt[a*x^(n-1)+b*x^n+c*x^(n+1)]) /;
FreeQ[{a,b,c,n},x] && EqQ[m,3*(n-1)/2] && EqQ[q,n-1] && EqQ[r,n+1] && NeQ[b^2-4*a*c,0]
```

5:  $\int \frac{x^{\frac{3n-1}{2}}}{(ax^{n-1} + bx^n + cx^{n+1})^{3/2}} dx$  when  $b^2 - 4ac \neq 0$

Rule: If  $b^2 - 4ac \neq 0$ , then

$$\int \frac{x^{\frac{3n-1}{2}}}{(a x^{n-1} + b x^n + c x^{n+1})^{3/2}} dx \rightarrow \frac{x^{\frac{n-1}{2}} (4 a + 2 b x)}{(b^2 - 4 a c) \sqrt{a x^{n-1} + b x^n + c x^{n+1}}}$$

Program code:

```
Int[x_^m_./(a_.*x_^q_.+b_.*x_^n_.+c_.*x_^r_.)^(3/2),x_Symbol] :=
  x^((n-1)/2)*(4*a+2*b*x)/((b^2-4*a*c)*Sqrt[a*x^(n-1)+b*x^n+c*x^(n+1)]) /;
FreeQ[{a,b,c,n},x] && EqQ[m,(3*n-1)/2] && EqQ[q,n-1] && EqQ[r,n+1] && NeQ[b^2-4*a*c,0]
```

6:  $\int x^m (a x^{n-1} + b x^n + c x^{n+1})^p dx$  when  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4 a c \neq 0 \wedge n \in \mathbb{Z}^+ \wedge m + p (n - 1) - 1 = 0$

Derivation: Generalized trinomial recurrence 3a with  $A = 0, B = 1, q = n - 1$  and  $m + p (n - 1) - 1 = 0$

Rule: If  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4 a c \neq 0 \wedge n \in \mathbb{Z}^+ \wedge m + p (n - 1) = 1$ , then

$$\int x^m (a x^{n-1} + b x^n + c x^{n+1})^p dx \rightarrow \frac{x^{m-n} (a x^{n-1} + b x^n + c x^{n+1})^{p+1}}{2 c (p + 1)} - \frac{b}{2 c} \int x^{m-1} (a x^{n-1} + b x^n + c x^{n+1})^p dx$$

Program code:

```
Int[x_^m_.*(a_.*x_^q_.+b_.*x_^n_.+c_.*x_^r_.)^p_,x_Symbol] :=
  x^(m-n)*(a*x^(n-1)+b*x^n+c*x^(n+1))^(p+1)/(2*c*(p+1)) -
  b/(2*c)*Int[x^(m-1)*(a*x^(n-1)+b*x^n+c*x^(n+1))^p,x] /;
FreeQ[{a,b,c},x] && EqQ[r,2*n-q] && PosQ[n-q] && Not[IntegerQ[p]] && NeQ[b^2-4*a*c,0] && IGtQ[n,0] &&
RationalQ[m,p,q] && EqQ[m+p*(n-1)-1,0]
```

7.  $\int x^m (ax^q + bx^n + cx^{2n-q})^p dx$  when  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p > 0$

1:  $\int x^m (ax^q + bx^n + cx^{2n-q})^p dx$  when  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p > 0 \wedge m + pq + 1 = n - q$

Derivation: Generalized trinomial recurrence 1b with  $A = 0$ ,  $B = 1$  and  $m + pq + 1 = 0$

Rule: If  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p > 0 \wedge m + pq + 1 = n - q$ , then

$$\int x^m (ax^q + bx^n + cx^{2n-q})^p dx \rightarrow \frac{x^{m-n+q+1} (b + 2cx^{n-q}) (ax^q + bx^n + cx^{2n-q})^p}{2c(n-q)(2p+1)} - \frac{p(b^2 - 4ac)}{2c(2p+1)} \int x^{m+q} (ax^q + bx^n + cx^{2n-q})^{p-1} dx$$

Program code:

```
Int[x_^m_.*(a_.*x_^q_.+b_.*x_^n_.+c_.*x_^r_.)^p_,x_Symbol] :=
  x^(m-n+q+1)*(b+2*c*x^(n-q))*(a*x^q+b*x^n+c*x^(2*n-q))^p/(2*c*(n-q)*(2*p+1)) -
  p*(b^2-4*a*c)/(2*c*(2*p+1))*Int[x^(m+q)*(a*x^q+b*x^n+c*x^(2*n-q))^(p-1),x] /;
FreeQ[{a,b,c},x] && EqQ[r,2*n-q] && PosQ[n-q] && Not[IntegerQ[p]] && NeQ[b^2-4*a*c,0] && IGtQ[n,0] && GtQ[p,0] &&
RationalQ[m,q] && EqQ[m+pq+1,n-q]
```

2:

$\int x^m (ax^q + bx^n + cx^{2n-q})^p dx$  when  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p > 0 \wedge m + pq + 1 > n - q \wedge m + p(2n - q) + 1 \neq 0 \wedge m + pq + (n - q)(2p - 1) + 1 \neq 0$

Derivation: Generalized trinomial recurrence 1b with  $A = 0$ ,  $B = 1$  and  $m = m - n + q$

Rule: If  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p > 0 \wedge$  , then

$$m + pq + 1 > n - q \wedge m + p(2n - q) + 1 \neq 0 \wedge m + pq + (n - q)(2p - 1) + 1 \neq 0$$

$$\int x^m (ax^q + bx^n + cx^{2n-q})^p dx \rightarrow \left( (x^{m-n+q+1} (b(n-q)p + c(m+pq+(n-q)(2p-1)+1)x^{n-q}) (ax^q + bx^n + cx^{2n-q})^p) / (c(m+p(2n-q)+1)(m+pq+(n-q)(2p-1)+1)) \right) +$$

$$\int x^{m-(n-2q)} \left( -ab(m+pq-n+q+1) + (2ac(m+pq+(n-q)(2p-1)+1) - b^2(m+pq+(n-q)(p-1)+1)) x^{n-q} \right) (ax^q + bx^n + cx^{2n-q})^{p-1} dx$$

Program code:

```
Int[x^m_.*(a_.x^q_.+b_.x^n_.+c_.x^r_.)^p_,x_Symbol] :=
  x^(m-n+q+1)*(b*(n-q)*p+c*(m+pq+(n-q)*(2*p-1)+1)*x^(n-q))*(a*x^q+b*x^n+c*x^(2*n-q))^p/(c*(m+pq+(n-q)*(2*p-1)+1)*(m+pq+(n-q)*(2*p-1)+1))
  (n-q)*p/(c*(m+pq+(n-q)*(2*p-1)+1)*(m+pq+(n-q)*(2*p-1)+1))*
  Int[x^(m-(n-2*q))*
    Simp[-a*b*(m+pq-n+q+1)+(2*a*c*(m+pq+(n-q)*(2*p-1)+1)-b^2*(m+pq+(n-q)*(p-1)+1))*x^(n-q),x]*
    (a*x^q+b*x^n+c*x^(2*n-q))^(p-1),x] /;
FreeQ[{a,b,c},x] && EqQ[r,2*n-q] && PosQ[n-q] && Not[IntegerQ[p]] && NeQ[b^2-4*a*c,0] && IGtQ[n,0] && GtQ[p,0] &&
RationalQ[m,q] && GtQ[m+pq+1,n-q] && NeQ[m+pq+(n-q)*(2*p-1)+1,0] && NeQ[m+pq+(n-q)*(2*p-1)+1,0]
```

**3:**  $\int x^m (ax^q + bx^n + cx^{2n-q})^p dx$  when  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p > 0 \wedge m + pq + 1 < -(n - q) \wedge m + pq + 1 \neq 0$

Derivation: Generalized trinomial recurrence 1a with  $A = 1$  and  $B = 0$

Rule: If  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p > 0 \wedge m + pq + 1 \leq -(n - q) + 1 \wedge m + pq + 1 \neq 0$ , then

$$\int x^m (ax^q + bx^n + cx^{2n-q})^p dx \rightarrow \frac{x^{m+1} (ax^q + bx^n + cx^{2n-q})^p}{m + pq + 1} - \frac{(n - q)p}{m + pq + 1} \int x^{m+n} (b + 2cx^{n-q}) (ax^q + bx^n + cx^{2n-q})^{p-1} dx$$

Program code:

```
Int[x^m_.*(a_.x^q_.+b_.x^n_.+c_.x^r_.)^p_,x_Symbol] :=
  x^(m+1)*(a*x^q+b*x^n+c*x^(2*n-q))^p/(m+pq+1) -
  (n-q)*p/(m+pq+1)*Int[x^(m+n)*(b+2*c*x^(n-q))*(a*x^q+b*x^n+c*x^(2*n-q))^(p-1),x] /;
FreeQ[{a,b,c},x] && EqQ[r,2*n-q] && PosQ[n-q] && Not[IntegerQ[p]] && NeQ[b^2-4*a*c,0] && IGtQ[n,0] && GtQ[p,0] &&
RationalQ[m,q] && LeQ[m+pq+1,-(n-q)+1] && NeQ[m+pq+1,0]
```

**4:**  $\int x^m (a x^q + b x^n + c x^{2n-q})^p dx$  when  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p > 0 \wedge m + pq + 1 > -(n - q) \wedge m + p(2n - q) + 1 \neq 0$

Derivation: Generalized trinomial recurrence 1a with  $A = 0$ ,  $B = 1$  and  $m = m - n$

Derivation: Generalized trinomial recurrence 1b with  $A = 1$  and  $B = 0$

Rule: If  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p > 0 \wedge m + pq + 1 > -(n - q) \wedge m + p(2n - q) + 1 \neq 0$ , then

$$\int x^m (a x^q + b x^n + c x^{2n-q})^p dx \rightarrow \frac{x^{m+1} (a x^q + b x^n + c x^{2n-q})^p}{m + p(2n - q) + 1} + \frac{(n - q)p}{m + p(2n - q) + 1} \int x^{m+q} (2a + b x^{n-q}) (a x^q + b x^n + c x^{2n-q})^{p-1} dx$$

Program code:

```
Int[x_^m.*(a_.*x_^q_.+b_.*x_^n_.+c_.*x_^r_.)^p_,x_Symbol] :=
  x^(m+1)*(a*x^q+b*x^n+c*x^(2*n-q))^p/(m+p*(2*n-q)+1) +
  (n-q)*p/(m+p*(2*n-q)+1)*Int[x^(m+q)*(2*a+b*x^(n-q))*(a*x^q+b*x^n+c*x^(2*n-q))^(p-1),x] /;
FreeQ[{a,b,c},x] && EqQ[r,2*n-q] && PosQ[n-q] && Not[IntegerQ[p]] && NeQ[b^2-4*a*c,0] && IGtQ[n,0] && GtQ[p,0] &&
RationalQ[m,q] && GtQ[m+p*q+1,-(n-q)] && NeQ[m+p*(2*n-q)+1,0]
```

8.  $\int x^m (a x^q + b x^n + c x^{2n-q})^p dx$  when  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1$

1:  $\int x^m (a x^q + b x^n + c x^{2n-q})^p dx$  when  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge m + pq + 1 = -(n - q)(2p + 3)$

Derivation: Generalized trinomial recurrence 2b with  $A = 1, B = 0$  and  $m + pq + 1 = -(n - q)(2p + 3)$

Rule: If  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge m + pq + 1 = -(n - q)(2p + 3)$ , then

$$\int x^m (a x^q + b x^n + c x^{2n-q})^p dx \rightarrow -\frac{x^{m-q+1} (b^2 - 2ac + bc x^{n-q}) (a x^q + b x^n + c x^{2n-q})^{p+1}}{a(n-q)(p+1)(b^2 - 4ac)} + \frac{2ac - b^2(p+2)}{a(p+1)(b^2 - 4ac)} \int x^{m-q} (a x^q + b x^n + c x^{2n-q})^{p+1} dx$$

Program code:

```
Int[x_^m_.*(a_.x_^q_.+b_.x_^n_.+c_.x_^r_.)^p_,x_Symbol] :=
  -x^(m-q+1)*(b^2-2*a*c+b*c*x^(n-q))*(a*x^q+b*x^n+c*x^(2*n-q))^(p+1)/(a*(n-q)*(p+1)*(b^2-4*a*c)) +
  (2*a*c-b^2*(p+2))/(a*(p+1)*(b^2-4*a*c))*
  Int[x^(m-q)*(a*x^q+b*x^n+c*x^(2*n-q))^(p+1),x] /;
FreeQ[{a,b,c},x] && EqQ[r,2*n-q] && PosQ[n-q] && Not[IntegerQ[p]] && NeQ[b^2-4*a*c,0] && IGtQ[n,0] && LtQ[p,-1] &&
RationalQ[m,p,q] && EqQ[m+pq+1,-(n-q)*(2*p+3)]
```

2:  $\int x^m (a x^q + b x^n + c x^{2n-q})^p dx$  when  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge m + pq + 1 > 2(n - q)$

Derivation: Generalized trinomial recurrence 2a with  $A = 0, B = 1$  and  $m = m - n + q$

Rule: If  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge m + pq + 1 > 2(n - q)$ , then

$$\int x^m (a x^q + b x^n + c x^{2n-q})^p dx \rightarrow -\frac{x^{m-2n+q+1} (2a + b x^{n-q}) (a x^q + b x^n + c x^{2n-q})^{p+1}}{(n-q)(p+1)(b^2 - 4ac)} +$$



$$\frac{1}{(n-q)(p+1)(b^2-4ac)} \int x^{m-2n+q} (2a(m+pq-2(n-q)+1) + b(m+pq+(n-q)(2p+1)+1)x^{n-q}) (ax^q + bx^n + cx^{2n-q})^{p+1} dx$$

## Program code:

```
Int[x_^m_.*(a_.*x_^q_.+b_.*x_^n_.+c_.*x_^r_.)^p_,x_Symbol] :=
  -x^(m-2*n+q+1)*(2*a+b*x^(n-q))*(a*x^q+b*x^n+c*x^(2*n-q))^(p+1)/((n-q)*(p+1)*(b^2-4*a*c)) +
  1/((n-q)*(p+1)*(b^2-4*a*c))*
  Int[x^(m-2*n+q)*(2*a*(m+p*q-2*(n-q)+1)+b*(m+p*q+(n-q)*(2*p+1)+1)*x^(n-q))*(a*x^q+b*x^n+c*x^(2*n-q))^(p+1),x] /;
FreeQ[{a,b,c},x] && EqQ[r,2*n-q] && PosQ[n-q] && Not[IntegerQ[p]] && NeQ[b^2-4*a*c,0] && IGtQ[n,0] && LtQ[p,-1] &&
RationalQ[m,q] && GtQ[m+p*q+1,2*(n-q)]
```

**3:**  $\int x^m (a x^q + b x^n + c x^{2n-q})^p dx$  when  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge m + pq + 1 < n - q$

Derivation: Generalized trinomial recurrence 2b with  $A = 1$  and  $B = 0$

Rule: If  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge m + pq + 1 < n - q$ , then

$$\int x^m (a x^q + b x^n + c x^{2n-q})^p dx \rightarrow -\frac{x^{m-q+1} (b^2 - 2ac + bc x^{n-q}) (a x^q + b x^n + c x^{2n-q})^{p+1}}{a (n-q) (p+1) (b^2 - 4ac)} + \frac{1}{a (n-q) (p+1) (b^2 - 4ac)} \int x^{m-q} (b^2 (m + pq + (n-q) (p+1) + 1) - 2ac (m + pq + 2(n-q) (p+1) + 1) + bc (m + pq + (n-q) (2p+3) + 1) x^{n-q}) (a x^q + b x^n + c x^{2n-q})^{p+1} dx$$

Program code:

```
Int[x_^m_.*(a_.*x_^q_+b_.*x_^n_+c_.*x_^r_.)^p_,x_Symbol] :=
  -x^(m-q+1)*(b^2-2*a*c+b*c*x^(n-q))*(a*x^q+b*x^n+c*x^(2*n-q))^(p+1)/(a*(n-q)*(p+1)*(b^2-4*a*c)) +
  1/(a*(n-q)*(p+1)*(b^2-4*a*c))*
  Int[x^(m-q)*
    (b^2*(m+pq+(n-q)*(p+1)+1)-2*a*c*(m+pq+2*(n-q)*(p+1)+1)+b*c*(m+pq+(n-q)*(2*p+3)+1)*x^(n-q))*
    (a*x^q+b*x^n+c*x^(2*n-q))^(p+1),x] /;
FreeQ[{a,b,c},x] && EqQ[r,2*n-q] && PosQ[n-q] && Not[IntegerQ[p]] && NeQ[b^2-4*a*c,0] && IGtQ[n,0] && LtQ[p,-1] &&
RationalQ[m,q] && LtQ[m+pq+1,n-q]
```

**4:**  $\int x^m (a x^q + b x^n + c x^{2n-q})^p dx$  when  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge n - q < m + pq + 1 < 2(n - q)$

Derivation: Generalized trinomial recurrence 2a with  $A = 1$  and  $B = 0$

Derivation: Generalized trinomial recurrence 2b with  $A = 0$ ,  $B = 1$  and  $m = m - n$

Rule: If  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge n - q < m + pq + 1 < 2(n - q)$ , then

$$\int x^m (a x^q + b x^n + c x^{2n-q})^p dx \rightarrow \frac{x^{m-n+1} (b + 2 c x^{n-q}) (a x^q + b x^n + c x^{2n-q})^{p+1}}{(n-q) (p+1) (b^2 - 4 a c)} - \frac{1}{(n-q) (p+1) (b^2 - 4 a c)} \int x^{m-n} (b (m+p q - n + q + 1) + 2 c (m+p q + 2 (n-q) (p+1) + 1) x^{n-q}) (a x^q + b x^n + c x^{2n-q})^{p+1} dx$$

Program code:

```
Int[x_^m_.*(a_.x_^q_.+b_.x_^n_.+c_.x_^r_.)^p_,x_Symbol] :=
  x^(m-n+1)*(b+2*c*x^(n-q))*(a*x^q+b*x^n+c*x^(2*n-q))^(p+1)/((n-q)*(p+1)*(b^2-4*a*c)) -
  1/((n-q)*(p+1)*(b^2-4*a*c))*
  Int[x^(m-n)*(b*(m+p*q-n+q+1)+2*c*(m+p*q+2*(n-q)*(p+1)+1)*x^(n-q))*(a*x^q+b*x^n+c*x^(2*n-q))^(p+1),x] /;
FreeQ[{a,b,c},x] && EqQ[r,2*n-q] && PosQ[n-q] && Not[IntegerQ[p]] && NeQ[b^2-4*a*c,0] && IGtQ[n,0] && LtQ[p,-1] &&
RationalQ[m,q] && LtQ[n-q,m+p*q+1,2*(n-q)]
```

9.  $\int x^m (a x^q + b x^n + c x^{2n-q})^p dx$  when  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4 a c \neq 0 \wedge n \in \mathbb{Z}^+ \wedge -1 \leq p < 0$

1:  $\int x^m (a x^q + b x^n + c x^{2n-q})^p dx$  when  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4 a c \neq 0 \wedge n \in \mathbb{Z}^+ \wedge -1 \leq p < 0 \wedge m + p q + 1 = 2 (n - q)$

Derivation: Generalized trinomial recurrence 3a with  $A = 0, B = 1$  and  $m = (-p q + 2 (n - q) - 1) - n + q$

Rule: If  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4 a c \neq 0 \wedge n \in \mathbb{Z}^+ \wedge -1 \leq p < 0 \wedge m + p q + 1 = 2 (n - q)$ , then

$$\int x^m (a x^q + b x^n + c x^{2n-q})^p dx \rightarrow \frac{x^{m-2n+q+1} (a x^q + b x^n + c x^{2n-q})^{p+1}}{2 c (n-q) (p+1)} - \frac{b}{2 c} \int x^{m-n+q} (a x^q + b x^n + c x^{2n-q})^p dx$$

Program code:

```
Int[x_^m_.*(a_.x_^q_.+b_.x_^n_.+c_.x_^r_.)^p_,x_Symbol] :=
  x^(m-2*n+q+1)*(a*x^q+b*x^n+c*x^(2*n-q))^(p+1)/(2*c*(n-q)*(p+1)) -
  b/(2*c)*Int[x^(m-n+q)*(a*x^q+b*x^n+c*x^(2*n-q))^p,x] /;
FreeQ[{a,b,c},x] && EqQ[r,2*n-q] && PosQ[n-q] && Not[IntegerQ[p]] && NeQ[b^2-4*a*c,0] && IGtQ[n,0] && GeQ[p,-1] && LtQ[p,0] &&
RationalQ[m,q] && EqQ[m+p*q+1,2*(n-q)]
```

**2:**  $\int x^m (ax^q + bx^n + cx^{2n-q})^p dx$  when  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge -1 \leq p < 0 \wedge m + pq + 1 = -2(n-q)(p+1)$

Derivation: Generalized trinomial recurrence 3b with  $A = 1, B = 0$  and  $m + pq + 1 = -2(n-q)(p+1)$

Rule: If

$q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge m + pq + 1 \neq 0 \wedge n \in \mathbb{Z}^+ \wedge -1 \leq p < 0 \wedge m + pq + 1 = -2(n-q)(p+1)$ , then

$$\int x^m (ax^q + bx^n + cx^{2n-q})^p dx \rightarrow -\frac{x^{m-q+1} (ax^q + bx^n + cx^{2n-q})^{p+1}}{2a(n-q)(p+1)} - \frac{b}{2a} \int x^{m+n-q} (ax^q + bx^n + cx^{2n-q})^p dx$$

Program code:

```
Int[x_^m_.*(a_.*x_^q_+b_.*x_^n_+c_.*x_^r_.)^p_,x_Symbol] :=
  -x^(m-q+1)*(a*x^q+b*x^n+c*x^(2*n-q))^(p+1)/(2*a*(n-q)*(p+1)) -
  b/(2*a)*Int[x^(m+n-q)*(a*x^q+b*x^n+c*x^(2*n-q))^p,x] /;
FreeQ[{a,b,c},x] && EqQ[r,2*n-q] && PosQ[n-q] && Not[IntegerQ[p]] && NeQ[b^2-4*a*c,0] && IGtQ[n,0] && GeQ[p,-1] && LtQ[p,0] &&
RationalQ[m,q] && EqQ[m+pq+1,-2*(n-q)*(p+1)]
```

**3:**  $\int x^m (ax^q + bx^n + cx^{2n-q})^p dx$  when  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge -1 \leq p < 0 \wedge m + pq + 1 > 2(n-q)$

Derivation: Generalized trinomial recurrence 3a with  $A = 0, B = 1$  and  $m = m - n + q$

Note: If  $-1 \leq p < 0$  and  $m + pq + 1 > 2(n-q)$ , then  $m + pq + 2(n-q)p + 1 \neq 0$ .

Rule: If  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge -1 \leq p < 0 \wedge m + pq + 1 > 2(n-q)$ , then

$$\int x^m (ax^q + bx^n + cx^{2n-q})^p dx \rightarrow \frac{x^{m-2n+q+1} (ax^q + bx^n + cx^{2n-q})^{p+1}}{c(m+pq+2(n-q)p+1)} -$$

$$\frac{1}{c(m+pq+2(n-q)p+1)} \int x^{m-2(n-q)} (a(m+pq-2(n-q)+1) + b(m+pq+(n-q)(p-1)+1)x^{n-q}) (ax^q + bx^n + cx^{2n-q})^p dx$$

Program code:

```
Int[x^m.*(a_.*x^q_.+b_.*x^n_.+c_.*x^r_.)^p_,x_Symbol] :=
  x^(m-2*n+q+1)*(a*x^q+b*x^n+c*x^(2*n-q))^(p+1)/(c*(m+p*q+2*(n-q)*p+1)) -
  1/(c*(m+p*q+2*(n-q)*p+1))*
  Int[x^(m-2*(n-q))*(a*(m+p*q-2*(n-q)+1)+b*(m+p*q+(n-q)*(p-1)+1)*x^(n-q))*(a*x^q+b*x^n+c*x^(2*n-q))^p,x] /;
FreeQ[{a,b,c},x] && EqQ[r,2*n-q] && PosQ[n-q] && Not[IntegerQ[p]] && NeQ[b^2-4*a*c,0] && IGtQ[n,0] && GeQ[p,-1] && LtQ[p,0] &&
RationalQ[m,q] && GtQ[m+p*q+1,2*(n-q)]
```

4:  $\int x^m (ax^q + bx^n + cx^{2n-q})^p dx$  when  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge -1 \leq p < 0 \wedge m + pq + 1 < 0$

Derivation: Generalized trinomial recurrence 3b with  $A = 1$  and  $B = 0$

Rule: If  $q < n \wedge p \notin \mathbb{Z} \wedge b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge -1 \leq p < 0 \wedge m + pq + 1 < 0$ , then

$$\frac{\int x^m (ax^q + bx^n + cx^{2n-q})^p dx}{x^{m-q+1} (ax^q + bx^n + cx^{2n-q})^{p+1}} = \frac{1}{a(m+pq+1)} \int x^{m+n-q} (b(m+pq+(n-q)(p+1)+1) + c(m+pq+2(n-q)(p+1)+1)x^{n-q}) (ax^q + bx^n + cx^{2n-q})^p dx$$

Program code:

```
Int[x^m.*(a_.*x^q_.+b_.*x^n_.+c_.*x^r_.)^p_,x_Symbol] :=
  x^(m-q+1)*(a*x^q+b*x^n+c*x^(2*n-q))^(p+1)/(a*(m+p*q+1)) -
  1/(a*(m+p*q+1))*
  Int[x^(m+n-q)*(b*(m+p*q+(n-q)*(p+1)+1)+c*(m+p*q+2*(n-q)*(p+1)+1)*x^(n-q))*(a*x^q+b*x^n+c*x^(2*n-q))^p,x] /;
FreeQ[{a,b,c},x] && EqQ[r,2*n-q] && PosQ[n-q] && Not[IntegerQ[p]] && NeQ[b^2-4*a*c,0] && IGtQ[n,0] && GeQ[p,-1] && LtQ[p,0] &&
RationalQ[m,q] && LtQ[m+p*q+1,0]
```

**10:**  $\int x^m (ax^q + bx^n + cx^{2n-q})^p dx$  when  $p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis:  $\partial_x \frac{(ax^q + bx^n + cx^{2n-q})^p}{x^{pq} (a + bx^{n-q} + cx^{2(n-q)})^p} = 0$

Rule: If  $p \notin \mathbb{Z}$ , then

$$\int x^m (ax^q + bx^n + cx^{2n-q})^p dx \rightarrow \frac{(ax^q + bx^n + cx^{2n-q})^p}{x^{pq} (a + bx^{n-q} + cx^{2(n-q)})^p} \int x^{m+pq} (a + bx^{n-q} + cx^{2(n-q)})^p dx$$

Program code:

```
Int[x^m.*(a.*x^q_.+b.*x^n_.+c.*x^r_.)^p_,x_Symbol] :=
  (a*x^q+b*x^n+c*x^(2*n-q))^p/(x^(p*q)*(a+b*x^(n-q)+c*x^(2*(n-q)))^p)*
  Int[x^(m+p*q)*(a+b*x^(n-q)+c*x^(2*(n-q)))^p,x] /;
FreeQ[{a,b,c,m,n,p,q},x] && EqQ[r,2*n-q] && Not[IntegerQ[p]] && PosQ[n-q]
```

**S:**  $\int u^m (a u^q + b u^n + c u^{2n-q})^p dx$  when  $u = d + e x$

Derivation: Integration by substitution

Rule: If  $u = d + e x$ , then

$$\int u^m (a u^q + b u^n + c u^{2n-q})^p dx \rightarrow \frac{1}{e} \text{Subst} \left[ \int x^m (a x^q + b x^n + c x^{2n-q})^p dx, x, u \right]$$

Program code:

```
Int[u^m_.*(a_.*u^q_.+b_.*u^n_.+c_.*u^(2*n-q).)^p_,x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[x^m*(a*x^q+b*x^n+c*x^(2*n-q))^p,x],x,u] /;
FreeQ[{a,b,c,m,n,p,q},x] && EqQ[r,2*n-q] && LinearQ[u,x] && NeQ[u,x]
```