

Rules for integrands of the form $(a + b x + c x^2)^p (d + e x + f x^2)^q (A + B x + C x^2)$

1. $\int (a + b x + c x^2)^p (d + e x + f x^2)^q (A + B x + C x^2) dx$ when $cd - af == 0 \wedge bd - ae == 0$

1: $\int (a + b x + c x^2)^p (d + e x + f x^2)^q (A + B x + C x^2) dx$ when $cd - af == 0 \wedge bd - ae == 0 \wedge (p \in \mathbb{Z} \vee \frac{c}{f} > 0)$

Derivation: Algebraic simplification

Basis: If $cd - af == 0 \wedge bd - ae == 0 \wedge (p \in \mathbb{Z} \vee \frac{c}{f} > 0)$, then $(a + b x + c x^2)^p == (\frac{c}{f})^p (d + e x + f x^2)^p$

Rule 1.2.1.7.1.1: If $cd - af == 0 \wedge bd - ae == 0 \wedge (p \in \mathbb{Z} \vee \frac{c}{f} > 0)$, then

$$\int (a + b x + c x^2)^p (d + e x + f x^2)^q (A + B x + C x^2) dx \rightarrow \left(\frac{c}{f}\right)^p \int (d + e x + f x^2)^{p+q} (A + B x + C x^2) dx$$

Program code:

```
Int[(a_+b_.*x_+c_.*x_^2)^p_.*(d_+e_.*x_+f_.*x_^2)^q_.*(A_+B_.*x_+C_.*x_^2),x_Symbol] :=
(c/f)^p*Int[(d+e*x+f*x^2)^(p+q)*(A+B*x+C*x^2),x] /;
FreeQ[{a,b,c,d,e,f,A,B,C,p,q},x] && EqQ[c*d-a*f,0] && EqQ[b*d-a*e,0] && (IntegerQ[p] || GtQ[c/f,0]) &&
(Not[IntegerQ[q]] || LeafCount[d+e*x+f*x^2]≤LeafCount[a+b*x+c*x^2])
```

```
Int[(a_+b_.*x_+c_.*x_^2)^p_.*(d_+e_.*x_+f_.*x_^2)^q_.*(A_+C_.*x_^2),x_Symbol] :=
(c/f)^p*Int[(d+e*x+f*x^2)^(p+q)*(A+C*x^2),x] /;
FreeQ[{a,b,c,d,e,f,A,C,p,q},x] && EqQ[c*d-a*f,0] && EqQ[b*d-a*e,0] && (IntegerQ[p] || GtQ[c/f,0]) &&
(Not[IntegerQ[q]] || LeafCount[d+e*x+f*x^2]≤LeafCount[a+b*x+c*x^2])
```

2: $\int (a+bx+cx^2)^p (d+ex+fx^2)^q (A+Bx+Cx^2) dx$ when $cd - af = 0 \wedge bd - ae = 0 \wedge p \notin \mathbb{Z} \wedge q \notin \mathbb{Z} \wedge \frac{c}{f} \neq 0$

Derivation: Piecewise constant extraction

Basis: If $cd - af = 0 \wedge bd - ae = 0$, then $\partial_x \frac{(a+bx+cx^2)^p}{(d+ex+fx^2)^p} = 0$

Basis: If $cd - af = 0 \wedge bd - ae = 0$, then $\frac{(a+bx+cx^2)^p}{(d+ex+fx^2)^p} = \frac{a^{\text{IntPart}[p]} (a+bx+cx^2)^{\text{FracPart}[p]}}{d^{\text{IntPart}[p]} (d+ex+fx^2)^{\text{FracPart}[p]}}$

Rule 1.2.1.7.1.2: If $cd - af = 0 \wedge bd - ae = 0 \wedge p \notin \mathbb{Z} \wedge q \notin \mathbb{Z} \wedge \frac{c}{f} \neq 0$, then

$$\int (a+bx+cx^2)^p (d+ex+fx^2)^q (A+Bx+Cx^2) dx \rightarrow \frac{a^{\text{IntPart}[p]} (a+bx+cx^2)^{\text{FracPart}[p]}}{d^{\text{IntPart}[p]} (d+ex+fx^2)^{\text{FracPart}[p]}} \int (d+ex+fx^2)^{p+q} (A+Bx+Cx^2) dx$$

Program code:

```
Int[(a+b_.**x+c_.**x^2)^p_.*(d+e_.**x+f_.**x^2)^q_.*(A_.+B_.**x+C_.**x^2),x_Symbol] :=
  a^IntPart[p]*(a+b*x+c*x^2)^FracPart[p]/(d^IntPart[p]*(d+e*x+f*x^2)^FracPart[p])*Int[(d+e*x+f*x^2)^(p+q)*(A+B*x+C*x^2),x] /;
FreeQ[{a,b,c,d,e,f,A,B,C,p,q},x] && EqQ[c*d-a*f,0] && EqQ[b*d-a*e,0] && Not[IntegerQ[p]] && Not[IntegerQ[q]] && Not[GtQ[c/f,0]]
```

```
Int[(a+b_.**x+c_.**x^2)^p_.*(d+e_.**x+f_.**x^2)^q_.*(A_.+C_.**x^2),x_Symbol] :=
  a^IntPart[p]*(a+b*x+c*x^2)^FracPart[p]/(d^IntPart[p]*(d+e*x+f*x^2)^FracPart[p])*Int[(d+e*x+f*x^2)^(p+q)*(A+C*x^2),x] /;
FreeQ[{a,b,c,d,e,f,A,C,p,q},x] && EqQ[c*d-a*f,0] && EqQ[b*d-a*e,0] && Not[IntegerQ[p]] && Not[IntegerQ[q]] && Not[GtQ[c/f,0]]
```

2: $\int (a+bx+cx^2)^p (d+ex+fx^2)^q (A+Bx+Cx^2) dx$ when $b^2 - 4ac = 0$

Derivation: Piecewise constant extraction

Basis: If $b^2 - 4ac = 0$, then $\partial_x \frac{(a+bx+cx^2)^p}{(b+2cx)^{2p}} = 0$

Basis: If $b^2 - 4ac = 0$, then $\frac{(a+bx+cx^2)^p}{(b+2cx)^{2p}} = \frac{(a+bx+cx^2)^{\text{FracPart}[p]}}{(4c)^{\text{IntPart}[p]} (b+2cx)^{2\text{FracPart}[p]}}$

Rule 1.2.1.7.2: If $b^2 - 4ac = 0$, then

$$\int (a+bx+cx^2)^p (d+ex+fx^2)^q (A+Bx+Cx^2) dx \rightarrow \frac{(a+bx+cx^2)^{\text{FracPart}[p]}}{(4c)^{\text{IntPart}[p]} (b+2cx)^{2\text{FracPart}[p]}} \int (b+2cx)^{2p} (d+ex+fx^2)^q (A+Bx+Cx^2) dx$$

Program code:

```
Int[(a+b_.**x+c_.**x^2)^p_.*(d_.+e_.**x+f_.**x^2)^q_.*(A_.+B_.**x+C_.**x^2),x_Symbol] :=
  (a+b*x+c*x^2)^FracPart[p]/((4*c)^IntPart[p]*(b+2*c*x)^(2*FracPart[p]))*Int[(b+2*c*x)^(2*p)*(d+e*x+f*x^2)^q*(A+B*x+C*x^2),x] /;
FreeQ[{a,b,c,d,e,f,A,B,C,p,q},x] && EqQ[b^2-4*a*c,0]
```

```
Int[(a+b_.**x+c_.**x^2)^p_.*(d_.+e_.**x+f_.**x^2)^q_.*(A_.+C_.**x^2),x_Symbol] :=
  (a+b*x+c*x^2)^FracPart[p]/((4*c)^IntPart[p]*(b+2*c*x)^(2*FracPart[p]))*Int[(b+2*c*x)^(2*p)*(d+e*x+f*x^2)^q*(A+C*x^2),x] /;
FreeQ[{a,b,c,d,e,f,A,C,p,q},x] && EqQ[b^2-4*a*c,0]
```

```
Int[(a+b_.**x+c_.**x^2)^p_.*(d_.+f_.**x^2)^q_.*(A_.+B_.**x+C_.**x^2),x_Symbol] :=
  (a+b*x+c*x^2)^FracPart[p]/((4*c)^IntPart[p]*(b+2*c*x)^(2*FracPart[p]))*Int[(b+2*c*x)^(2*p)*(d+f*x^2)^q*(A+B*x+C*x^2),x] /;
FreeQ[{a,b,c,d,f,A,B,C,p,q},x] && EqQ[b^2-4*a*c,0]
```

```
Int[(a+b_.**x+c_.**x^2)^p_.*(d_.+f_.**x^2)^q_.*(A_.+C_.**x^2),x_Symbol] :=
  (a+b*x+c*x^2)^FracPart[p]/((4*c)^IntPart[p]*(b+2*c*x)^(2*FracPart[p]))*Int[(b+2*c*x)^(2*p)*(d+f*x^2)^q*(A+C*x^2),x] /;
FreeQ[{a,b,c,d,f,A,C,p,q},x] && EqQ[b^2-4*a*c,0]
```

$$4. \int (a+bx+cx^2)^p (d+ex+fx^2)^q (A+Bx+Cx^2) dx \text{ when } b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0 \wedge p < -1$$

$$1: \int (a+bx+cx^2)^p (d+ex+fx^2)^q (A+Bx+Cx^2) dx \text{ when } b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0 \wedge p < -1 \wedge q > 0$$

Derivation: Nondegenerate biquadratic recurrence 1

Rule 1.2.1.7.4.1: If $b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0 \wedge p < -1 \wedge q > 0$, then

$$\begin{aligned} & \int (a+bx+cx^2)^p (d+ex+fx^2)^q (A+Bx+Cx^2) dx \rightarrow \\ & \left(\left((Abc - 2aBc + a^2C - (c(bB - 2Ac) - C(b^2 - 2ac))x) (a+bx+cx^2)^{p+1} (d+ex+fx^2)^q \right) / (c(b^2 - 4ac)(p+1)) \right) - \\ & \frac{1}{c(b^2 - 4ac)(p+1)} \int (a+bx+cx^2)^{p+1} (d+ex+fx^2)^{q-1} \cdot \\ & \left((eq(Abc - 2aBc + a^2C) - d(c(bB - 2Ac)(2p+3) + C(2ac - b^2(p+2))) + \right. \\ & \left. (2fq(Abc - 2aBc + a^2C) - e(c(bB - 2Ac)(2p+q+3) + C(2ac(q+1) - b^2(p+q+2))))x - \right. \\ & \left. f(c(bB - 2Ac)(2p+2q+3) + C(2ac(2q+1) - b^2(p+2q+2)))x^2 \right) dx \end{aligned}$$

Program code:

```
Int[(a+b*x+c*x^2)^p*(d+e*x+f*x^2)^q*(A+B*x+C*x^2),x_Symbol] :=
(A*b*c-2*a*B*c+a^2*C-(c*(b*B-2*A*c)-C*(b^2-2*a*c))*x)*(a+b*x+c*x^2)^(p+1)*(d+e*x+f*x^2)^q/(c*(b^2-4*a*c)*(p+1)) -
1/(c*(b^2-4*a*c)*(p+1))*
Int[(a+b*x+c*x^2)^(p+1)*(d+e*x+f*x^2)^(q-1)*
Simp[e*q*(A*b*c-2*a*B*c+a^2*C)-d*(c*(b*B-2*A*c)*(2*p+3)+C*(2*a*c-b^2*(p+2)))+(
2*f*q*(A*b*c-2*a*B*c+a^2*C)-e*(c*(b*B-2*A*c)*(2*p+q+3)+C*(2*a*c*(q+1)-b^2*(p+q+2)))]*x-
f*(c*(b*B-2*A*c)*(2*p+2*q+3)+C*(2*a*c*(2*q+1)-b^2*(p+2*q+2)))*x^2,x],x] /;
FreeQ[{a,b,c,d,e,f,A,B,C},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && LtQ[p,-1] && GtQ[q,0] && Not[IGtQ[q,0]]
```

```

Int[ (a_+b_.*x_+c_.*x_^2)^p_*(d_+e_.*x_+f_.*x_^2)^q_*(A_+C_.*x_^2), x_Symbol] :=
  (A*b*c+a*b*c+(2*A*c^2+C*(b^2-2*a*c))*x)*(a+b*x+c*x^2)^(p+1)*(d+e*x+f*x^2)^q/(c*(b^2-4*a*c)*(p+1)) -
  1/(c*(b^2-4*a*c)*(p+1))*
  Int[ (a+b*x+c*x^2)^(p+1)*(d+e*x+f*x^2)^(q-1)*
    Simp[A*c*(2*c*d*(2*p+3)+b*e*q)-C*(2*a*c*d-b^2*d*(p+2)-a*b*e*q)+
      (C*(2*a*b*f*q-2*a*c*e*(q+1)+b^2*e*(p+q+2))+2*A*c*(b*f*q+c*e*(2*p+q+3)))*x-
      f*(-2*A*c^2*(2*p+2*q+3)+C*(2*a*c*(2*q+1)-b^2*(p+2*q+2)))*x^2,x], x] /;
FreeQ[{a,b,c,d,e,f,A,C},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && LtQ[p,-1] && GtQ[q,0] && Not[IGtQ[q,0]]

```

```

Int[ (a_+c_.*x_^2)^p_*(d_+e_.*x_+f_.*x_^2)^q_*(A_+B_.*x_+C_.*x_^2), x_Symbol] :=
  (a*B-(A*c-a*C)*x)*(a+c*x^2)^(p+1)*(d+e*x+f*x^2)^q/(2*a*c*(p+1)) -
  2/((-4*a*c)*(p+1))*
  Int[ (a+c*x^2)^(p+1)*(d+e*x+f*x^2)^(q-1)*
    Simp[A*c*d*(2*p+3)-a*(C*d+B*e*q)+(A*c*e*(2*p+q+3)-a*(2*B*f*q+C*e*(q+1)))*x-f*(a*C*(2*q+1)-A*c*(2*p+2*q+3))*x^2,x], x] /;
FreeQ[{a,c,d,e,f,A,B,C},x] && NeQ[e^2-4*d*f,0] && LtQ[p,-1] && GtQ[q,0] && Not[IGtQ[q,0]]

```

```

Int[ (a_+c_.*x_^2)^p_*(d_+e_.*x_+f_.*x_^2)^q_*(A_+C_.*x_^2), x_Symbol] :=
  -(A*c-a*C)*x*(a+c*x^2)^(p+1)*(d+e*x+f*x^2)^q/(2*a*c*(p+1)) +
  2/(4*a*c*(p+1))*
  Int[ (a+c*x^2)^(p+1)*(d+e*x+f*x^2)^(q-1)*
    Simp[A*c*d*(2*p+3)-a*C*d+(A*c*e*(2*p+q+3)-a*C*e*(q+1))*x-f*(a*C*(2*q+1)-A*c*(2*p+2*q+3))*x^2,x], x] /;
FreeQ[{a,c,d,e,f,A,C},x] && NeQ[e^2-4*d*f,0] && LtQ[p,-1] && GtQ[q,0] && Not[IGtQ[q,0]]

```

```

Int[ (a_+b_.*x_+c_.*x_^2)^p_*(d_+f_.*x_^2)^q_*(A_+B_.*x_+C_.*x_^2), x_Symbol] :=
  (A*b*c-2*a*B*c+a*b*C-(c*(b*B-2*A*c)-C*(b^2-2*a*c))*x)*(a+b*x+c*x^2)^(p+1)*(d+f*x^2)^q/(c*(b^2-4*a*c)*(p+1)) -
  1/(c*(b^2-4*a*c)*(p+1))*
  Int[ (a+b*x+c*x^2)^(p+1)*(d+f*x^2)^(q-1)*
    Simp[-d*(c*(b*B-2*A*c)*(2*p+3)+C*(2*a*c-b^2*(p+2)))+
      (2*f*q*(A*b*c-2*a*B*c+a*b*C))*x-
      f*(c*(b*B-2*A*c)*(2*p+2*q+3)+C*(2*a*c*(2*q+1)-b^2*(p+2*q+2)))*x^2,x], x] /;
FreeQ[{a,b,c,d,f,A,B,C},x] && NeQ[b^2-4*a*c,0] && LtQ[p,-1] && GtQ[q,0] && Not[IGtQ[q,0]]

```

```

Int[(a_+b_.*x_+c_.*x_^2)^p_*(d_+f_.*x_^2)^q_*(A_+C_.*x_^2),x_Symbol] :=
  (A*b*c+a*b*c+(2*A*c^2+C*(b^2-2*a*c))*x)*(a+b*x+c*x^2)^(p+1)*(d+f*x^2)^q/(c*(b^2-4*a*c)*(p+1)) -
  1/(c*(b^2-4*a*c)*(p+1))*
  Int[(a+b*x+c*x^2)^(p+1)*(d+f*x^2)^(q-1)*
    Simp[A*c*(2*c*d*(2*p+3))-C*(2*a*c*d-b^2*d*(p+2))+
      (C*(2*a*b*f*q)+2*A*c*(b*f*q))*x-
      f*(-2*A*c^2*(2*p+2*q+3)+C*(2*a*c*(2*q+1)-b^2*(p+2*q+2)))*x^2,x],x] /;
FreeQ[{a,b,c,d,f,A,C},x] && NeQ[b^2-4*a*c,0] && LtQ[p,-1] && GtQ[q,0] && Not[IGtQ[q,0]]

```

$$\mathbf{2:} \int (a+bx+cx^2)^p (d+ex+fx^2)^q (A+Bx+Cx^2) dx \text{ when } b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge p < -1 \wedge q \neq 0 \wedge (cd-af)^2 - (bd-ae)(ce-bf) \neq 0$$

Derivation: Nondegenerate biquadratic recurrence 3

Rule 1.2.1.7.4.2: If

$b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge p < -1 \wedge q \neq 0 \wedge (cd-af)^2 - (bd-ae)(ce-bf) \neq 0$, then

$$\begin{aligned} & \int (a+bx+cx^2)^p (d+ex+fx^2)^q (A+Bx+Cx^2) dx \rightarrow \\ & \frac{(a+bx+cx^2)^{p+1} (d+ex+fx^2)^{q+1}}{(b^2-4ac) ((cd-af)^2 - (bd-ae)(ce-bf)) (p+1)} \cdot \\ & \frac{((Ac-aC)(2ace-b(cd+af)) + (Ab-aB)(2c^2d+b^2f-c(be+2af)) + \\ & c(A(2c^2d+b^2f-c(be+2af)) - B(bcd-2ace+abf) + C(b^2d-abe-2a(cd-af)))x) +}{1} \int (a+bx+cx^2)^{p+1} (d+ex+fx^2)^q \cdot \\ & ((bB-2Ac-2aC)((cd-af)^2 - (bd-ae)(ce-bf)) (p+1) + \\ & (b^2(Cd+Af) - b(Bcd+Ace+aCe+abf) + 2(Ac(cd-af) - a(cCd-Bce-aCf))) (af(p+1) - cd(p+2)) - \\ & e((Ac-aC)(2ace-b(cd+af)) + (Ab-aB)(2c^2d+b^2f-c(be+2af))) (p+q+2) - \\ & (2f((Ac-aC)(2ace-b(cd+af)) + (Ab-aB)(2c^2d+b^2f-c(be+2af))) (p+q+2) - \\ & (b^2(Cd+Af) - b(Bcd+Ace+aCe+abf) + 2(Ac(cd-af) - a(cCd-Bce-aCf))) (bf(p+1) - ce(2p+q+4)))x - \\ & cf(b^2(Cd+Af) - b(Bcd+Ace+aCe+abf) + 2(Ac(cd-af) - a(cCd-Bce-aCf))) (2p+2q+5)x^2) dx \end{aligned}$$

Program code:

```

Int[(a+b*x+c*x^2)^p*(d+e*x+f*x^2)^q*(A+B*x+C*x^2),x_Symbol] :=
(a+b*x+c*x^2)^(p+1)*(d+e*x+f*x^2)^(q+1)/((b^2-4*a*c)*((c*d-a*f)^2-(b*d-a*e)*(c*e-b*f))*(p+1)*
((A*c-a*c)*(2*a*c*e-b*(c*d+a*f))+(A*b-a*B)*(2*c^2*d+b^2*f-c*(b*e+2*a*f))+
c*(A*(2*c^2*d+b^2*f-c*(b*e+2*a*f))-B*(b*c*d-2*a*c*e+a*b*f)+C*(b^2*d-a*b*e-2*a*(c*d-a*f)))*x) +
1/((b^2-4*a*c)*((c*d-a*f)^2-(b*d-a*e)*(c*e-b*f))*(p+1))*
Int[(a+b*x+c*x^2)^(p+1)*(d+e*x+f*x^2)^q*
Simp[(b*B-2*A*c-2*a*C)*((c*d-a*f)^2-(b*d-a*e)*(c*e-b*f))*(p+1)+
(b^2*(C*d+A*f)-b*(B*c*d+A*c*e+a*C*e+a*B*f)+2*(A*c*(c*d-a*f)-a*(c*C*d-B*c*e-a*C*f)))*(a*f*(p+1)-c*d*(p+2))-
e*((A*c-a*c)*(2*a*c*e-b*(c*d+a*f))+(A*b-a*B)*(2*c^2*d+b^2*f-c*(b*e+2*a*f)))*(p+q+2)-
(2*f*((A*c-a*c)*(2*a*c*e-b*(c*d+a*f))+(A*b-a*B)*(2*c^2*d+b^2*f-c*(b*e+2*a*f)))*(p+q+2)-
(b^2*(C*d+A*f)-b*(B*c*d+A*c*e+a*C*e+a*B*f)+2*(A*c*(c*d-a*f)-a*(c*C*d-B*c*e-a*C*f)))*
(b*f*(p+1)-c*e*(2*p+q+4)))*x-
c*f*(b^2*(C*d+A*f)-b*(B*c*d+A*c*e+a*C*e+a*B*f)+2*(A*c*(c*d-a*f)-a*(c*C*d-B*c*e-a*C*f)))*(2*p+2*q+5)*x^2,x],x]/;
FreeQ[{a,b,c,d,e,f,A,B,C,q},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && LtQ[p,-1] &&
NeQ[(c*d-a*f)^2-(b*d-a*e)*(c*e-b*f),0] && Not[Not[IntegerQ[p]] && ILtQ[q,-1]] && Not[IGtQ[q,0]]

```

```

Int[(a+b*x+c*x^2)^p*(d+e*x+f*x^2)^q*(A+C*x^2),x_Symbol] :=
(a+b*x+c*x^2)^(p+1)*(d+e*x+f*x^2)^(q+1)/((b^2-4*a*c)*((c*d-a*f)^2-(b*d-a*e)*(c*e-b*f))*(p+1)*
((A*c-a*c)*(2*a*c*e-b*(c*d+a*f))+(A*b)*(2*c^2*d+b^2*f-c*(b*e+2*a*f))+
c*(A*(2*c^2*d+b^2*f-c*(b*e+2*a*f))+C*(b^2*d-a*b*e-2*a*(c*d-a*f)))*x) +
1/((b^2-4*a*c)*((c*d-a*f)^2-(b*d-a*e)*(c*e-b*f))*(p+1))*
Int[(a+b*x+c*x^2)^(p+1)*(d+e*x+f*x^2)^q*
Simp[(-2*A*c-2*a*C)*((c*d-a*f)^2-(b*d-a*e)*(c*e-b*f))*(p+1)+
(b^2*(C*d+A*f)-b*(A*c*e+a*C*e)+2*(A*c*(c*d-a*f)-a*(c*C*d-a*C*f)))*(a*f*(p+1)-c*d*(p+2))-
e*((A*c-a*c)*(2*a*c*e-b*(c*d+a*f))+(A*b)*(2*c^2*d+b^2*f-c*(b*e+2*a*f)))*(p+q+2)-
(2*f*((A*c-a*c)*(2*a*c*e-b*(c*d+a*f))+(A*b)*(2*c^2*d+b^2*f-c*(b*e+2*a*f)))*(p+q+2)-
(b^2*(C*d+A*f)-b*(A*c*e+a*C*e)+2*(A*c*(c*d-a*f)-a*(c*C*d-a*C*f)))*
(b*f*(p+1)-c*e*(2*p+q+4)))*x-
c*f*(b^2*(C*d+A*f)-b*(A*c*e+a*C*e)+2*(A*c*(c*d-a*f)-a*(c*C*d-a*C*f)))*(2*p+2*q+5)*x^2,x],x]/;
FreeQ[{a,b,c,d,e,f,A,C,q},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && LtQ[p,-1] &&
NeQ[(c*d-a*f)^2-(b*d-a*e)*(c*e-b*f),0] && Not[Not[IntegerQ[p]] && ILtQ[q,-1]] && Not[IGtQ[q,0]]

```



```

Int[(a+c_.**x_^2)^p_*(d_+e_.**x_+f_.**x_^2)^q_*(A_.+B_.**x_+C_.**x_^2),x_Symbol] :=
(a+c*x^2)^(p+1)*(d+e*x+f*x^2)^(q+1)/((-4*a*c)*(a*c*e^2+(c*d-a*f)^2)*(p+1))*
((A*c-a*c)*(2*a*c*e)+(-a*B)*(2*c^2*d-c*(2*a*f))+
c*(A*(2*c^2*d-c*(2*a*f))-B*(-2*a*c*e)+C*(-2*a*(c*d-a*f)))*x) +
1/((-4*a*c)*(a*c*e^2+(c*d-a*f)^2)*(p+1))*
Int[(a+c*x^2)^(p+1)*(d+e*x+f*x^2)^q*
Simp[(-2*A*c-2*a*c)*((c*d-a*f)^2-(-a*e)*(c*e))*(p+1)+
(2*(A*c*(c*d-a*f)-a*(c*C*d-B*c*e-a*C*f)))*(a*f*(p+1)-c*d*(p+2))-
e*((A*c-a*c)*(2*a*c*e)+(-a*B)*(2*c^2*d-c*(2*a*f)))*(p+q+2)-
(2*f*((A*c-a*c)*(2*a*c*e)+(-a*B)*(2*c^2*d-c*(2*a*f)))*(p+q+2)-
(2*(A*c*(c*d-a*f)-a*(c*C*d-B*c*e-a*C*f)))*
(-c*e*(2*p+q+4)))*x-
c*f*(2*(A*c*(c*d-a*f)-a*(c*C*d-B*c*e-a*C*f)))*(2*p+2*q+5)*x^2,x],x]/;
FreeQ[{a,c,d,e,f,A,B,C,q},x] && NeQ[e^2-4*d*f,0] && LtQ[p,-1] && NeQ[a*c*e^2+(c*d-a*f)^2,0] && Not[Not[IntegerQ[p]] && ILtQ[q,-1]] &&

```

```

Int[(a+c_.**x_^2)^p_*(d_+e_.**x_+f_.**x_^2)^q_*(A_.+C_.**x_^2),x_Symbol] :=
(a+c*x^2)^(p+1)*(d+e*x+f*x^2)^(q+1)/((-4*a*c)*(a*c*e^2+(c*d-a*f)^2)*(p+1))*
((A*c-a*c)*(2*a*c*e)+c*(A*(2*c^2*d-c*(2*a*f))+C*(-2*a*(c*d-a*f)))*x) +
1/((-4*a*c)*(a*c*e^2+(c*d-a*f)^2)*(p+1))*
Int[(a+c*x^2)^(p+1)*(d+e*x+f*x^2)^q*
Simp[(-2*A*c-2*a*c)*((c*d-a*f)^2-(-a*e)*(c*e))*(p+1)+
(2*(A*c*(c*d-a*f)-a*(c*C*d-a*C*f)))*(a*f*(p+1)-c*d*(p+2))-
e*((A*c-a*c)*(2*a*c*e))*(p+q+2)-
(2*f*((A*c-a*c)*(2*a*c*e))*(p+q+2)-(2*(A*c*(c*d-a*f)-a*(c*C*d-a*C*f)))*(-c*e*(2*p+q+4)))*x-
c*f*(2*(A*c*(c*d-a*f)-a*(c*C*d-a*C*f)))*(2*p+2*q+5)*x^2,x],x]/;
FreeQ[{a,c,d,e,f,A,C,q},x] && NeQ[e^2-4*d*f,0] && LtQ[p,-1] && NeQ[a*c*e^2+(c*d-a*f)^2,0] && Not[Not[IntegerQ[p]] && ILtQ[q,-1]] && No

```

```

Int[(a_+b_.*x_+c_.*x_^2)^p_*(d_+f_.*x_^2)^q_*(A_+B_.*x_+C_.*x_^2),x_Symbol] :=
(a+b*x+c*x^2)^(p+1)*(d+f*x^2)^(q+1)/((b^2-4*a*c)*(b^2*d*f+(c*d-a*f)^2)*(p+1))*
((A*c-a*c)*(b*(c*d+a*f))+(A*b-a*B)*(2*c^2*d+b^2*f-c*(2*a*f))+
c*(A*(2*c^2*d+b^2*f-c*(2*a*f))-B*(b*c*d+a*B*f)+C*(b^2*d-2*a*(c*d-a*f)))*x) +
1/((b^2-4*a*c)*(b^2*d*f+(c*d-a*f)^2)*(p+1))*
Int[(a+b*x+c*x^2)^(p+1)*(d+f*x^2)^q*
Simp[(b*B-2*A*c-2*a*c)*((c*d-a*f)^2-(b*d)*(-b*f))*(p+1)+
(b^2*(C*d+A*f)-b*(B*c*d+a*B*f)+2*(A*c*(c*d-a*f)-a*(c*C*d-a*C*f)))*(a*f*(p+1)-c*d*(p+2))-
(2*f*(A*c-a*c)*(b*(c*d+a*f))+(A*b-a*B)*(2*c^2*d+b^2*f-c*(2*a*f)))*(p+q+2)-
(b^2*(C*d+A*f)-b*(B*c*d+a*B*f)+2*(A*c*(c*d-a*f)-a*(c*C*d-a*C*f)))*
(b*f*(p+1)))*x-
c*f*(b^2*(C*d+A*f)-b*(B*c*d+a*B*f)+2*(A*c*(c*d-a*f)-a*(c*C*d-a*C*f)))*(2*p+2*q+5)*x^2,x],x]/;
FreeQ[{a,b,c,d,f,A,B,C,q},x] && NeQ[b^2-4*a*c,0] && LtQ[p,-1] && NeQ[b^2*d*f+(c*d-a*f)^2,0] && Not[Not[IntegerQ[p]] && ILtQ[q,-1]] &&

```

```

Int[(a_+b_.*x_+c_.*x_^2)^p_*(d_+f_.*x_^2)^q_*(A_+C_.*x_^2),x_Symbol] :=
(a+b*x+c*x^2)^(p+1)*(d+f*x^2)^(q+1)/((b^2-4*a*c)*(b^2*d*f+(c*d-a*f)^2)*(p+1))*
((A*c-a*c)*(b*(c*d+a*f))+(A*b)*(2*c^2*d+b^2*f-c*(2*a*f))+
c*(A*(2*c^2*d+b^2*f-c*(2*a*f))+C*(b^2*d-2*a*(c*d-a*f)))*x) +
1/((b^2-4*a*c)*(b^2*d*f+(c*d-a*f)^2)*(p+1))*
Int[(a+b*x+c*x^2)^(p+1)*(d+f*x^2)^q*
Simp[(-2*A*c-2*a*c)*((c*d-a*f)^2-(b*d)*(-b*f))*(p+1)+
(b^2*(C*d+A*f)+2*(A*c*(c*d-a*f)-a*(c*C*d-a*C*f)))*(a*f*(p+1)-c*d*(p+2))-
(2*f*(A*c-a*c)*(b*(c*d+a*f))+(A*b)*(2*c^2*d+b^2*f-c*(2*a*f)))*(p+q+2)-
(b^2*(C*d+A*f)+2*(A*c*(c*d-a*f)-a*(c*C*d-a*C*f)))*
(b*f*(p+1)))*x-
c*f*(b^2*(C*d+A*f)+2*(A*c*(c*d-a*f)-a*(c*C*d-a*C*f)))*(2*p+2*q+5)*x^2,x],x]/;
FreeQ[{a,b,c,d,f,A,C,q},x] && NeQ[b^2-4*a*c,0] && LtQ[p,-1] && NeQ[b^2*d*f+(c*d-a*f)^2,0] && Not[Not[IntegerQ[p]] && ILtQ[q,-1]] && No

```

5: $\int (a+bx+cx^2)^p (d+ex+fx^2)^q (A+Bx+Cx^2) dx$ when $b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0 \wedge p > 0 \wedge p+q+1 \neq 0 \wedge 2p+2q+3 \neq 0$

Derivation: Nondegenerate biquadratic recurrence 2

Rule 1.2.1.7.5: If $b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0 \wedge p > 0 \wedge p+q+1 \neq 0 \wedge 2p+2q+3 \neq 0$, then

$$\begin{aligned} & \int (a+bx+cx^2)^p (d+ex+fx^2)^q (A+Bx+Cx^2) dx \rightarrow \\ & \left(\left((Bcf(2p+2q+3) + C(bfp - ce(2p+q+2)) + 2cCf(p+q+1)x) (a+bx+cx^2)^p (d+ex+fx^2)^{q+1} \right) / (2cf^2(p+q+1)(2p+2q+3)) \right) - \\ & \quad \frac{1}{2cf^2(p+q+1)(2p+2q+3)} \int (a+bx+cx^2)^{p-1} (d+ex+fx^2)^q \cdot \\ & (p(bd - ae)(C(ce - bf)(q+1) - c(ce - Bf)(2p+2q+3)) + (p+q+1)(b^2Cdfp + ac(C(2df - e^2(2p+q+2)) + f(Be - 2Af)(2p+2q+3))) + \\ & \quad (2p(cd - af)(C(ce - bf)(q+1) - c(ce - Bf)(2p+2q+3)) + \\ & \quad (p+q+1)(Cefp(b^2 - 4ac) - bc(C(e^2 - 4df)(2p+q+2) + f(2Cd - Be + 2Af)(2p+2q+3)))) x + \\ & \quad (p(ce - bf)(C(ce - bf)(q+1) - c(ce - Bf)(2p+2q+3)) + (p+q+1) \\ & \quad (Cf^2p(b^2 - 4ac) - c^2(C(e^2 - 4df)(2p+q+2) + f(2Cd - Be + 2Af)(2p+2q+3)))) x^2) dx \end{aligned}$$

Program code:

```
Int[(a+b*x+c*x^2)^p*(d+e*x+f*x^2)^q*(A+B*x+C*x^2),x_Symbol] :=
  (B*c*f*(2*p+2*q+3)+C*(b*f*p-c*e*(2*p+q+2))+2*c*C*f*(p+q+1)*x)*(a+b*x+c*x^2)^p*
  (d+e*x+f*x^2)^(q+1)/(2*c*f^2*(p+q+1)*(2*p+2*q+3)) -
  (1/(2*c*f^2*(p+q+1)*(2*p+2*q+3)))*
  Int[(a+b*x+c*x^2)^(p-1)*(d+e*x+f*x^2)^q*
  Simp[p*(b*d-a*e)*(C*(c*e-b*f)*(q+1)-c*(C*e-B*f)*(2*p+2*q+3))+
  (p+q+1)*(b^2*C*d*f*p+a*c*(C*(2*d*f-e^2*(2*p+q+2))+f*(B*e-2*A*f)*(2*p+2*q+3)))+
  (2*p*(c*d-a*f)*(C*(c*e-b*f)*(q+1)-c*(C*e-B*f)*(2*p+2*q+3))+
  (p+q+1)*(C*e*f*p*(b^2-4*a*c)-b*c*(C*(e^2-4*d*f)*(2*p+q+2)+f*(2*C*d-B*e+2*A*f)*(2*p+2*q+3)))]*x+
  (p*(c*e-b*f)*(C*(c*e-b*f)*(q+1)-c*(C*e-B*f)*(2*p+2*q+3))+
  (p+q+1)*(C*f^2*p*(b^2-4*a*c)-c^2*(C*(e^2-4*d*f)*(2*p+q+2)+f*(2*C*d-B*e+2*A*f)*(2*p+2*q+3)))]*x^2,x],x] /;
FreeQ[{a,b,c,d,e,f,A,B,C,q},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && GtQ[p,0] && NeQ[p+q+1,0] && NeQ[2*p+2*q+3,0] && Not[IGtQ[p,0]
```

```

Int[(a+b_.**x_+c_.**x_^2)^p_*(d_+e_.**x_+f_.**x_^2)^q_*(A_+C_.**x_^2),x_Symbol] :=
(C*(b*f*p-c*e*(2*p+q+2))+2*C*C*f*(p+q+1)*x)*(a+b*x+c*x^2)^p*
(d+e*x+f*x^2)^(q+1)/(2*c*f^2*(p+q+1)*(2*p+2*q+3)) -
(1/(2*c*f^2*(p+q+1)*(2*p+2*q+3)))*
Int[(a+b*x+c*x^2)^(p-1)*(d+e*x+f*x^2)^q*
Simp[p*(b*d-a*e)*(C*(c*e-b*f)*(q+1)-c*(C*e)*(2*p+2*q+3))+
(p+q+1)*(b^2*C*d*f*p+a*c*(C*(2*d*f-e^2*(2*p+q+2))+f*(-2*A*f)*(2*p+2*q+3)))+
(2*p*(c*d-a*f)*(C*(c*e-b*f)*(q+1)-c*(C*e)*(2*p+2*q+3))+
(p+q+1)*(C*e*f*p*(b^2-4*a*c)-b*c*(C*(e^2-4*d*f)*(2*p+q+2)+f*(2*C*d+2*A*f)*(2*p+2*q+3)))*x+
(p*(c*e-b*f)*(C*(c*e-b*f)*(q+1)-c*(C*e)*(2*p+2*q+3))+
(p+q+1)*(C*f^2*p*(b^2-4*a*c)-c^2*(C*(e^2-4*d*f)*(2*p+q+2)+f*(2*C*d+2*A*f)*(2*p+2*q+3)))*x^2,x],x] /;
FreeQ[{a,b,c,d,e,f,A,C,q},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && GtQ[p,0] && NeQ[p+q+1,0] && NeQ[2*p+2*q+3,0] && Not[IGtQ[p,0]]

```

```

Int[(a+c_.**x_^2)^p_*(d_+e_.**x_+f_.**x_^2)^q_*(A_+B_.**x_+C_.**x_^2),x_Symbol] :=
(B*c*f*(2*p+2*q+3)+C*(-c*e*(2*p+q+2))+2*C*C*f*(p+q+1)*x)*(a+c*x^2)^p*
(d+e*x+f*x^2)^(q+1)/(2*c*f^2*(p+q+1)*(2*p+2*q+3)) -
(1/(2*c*f^2*(p+q+1)*(2*p+2*q+3)))*
Int[(a+c*x^2)^(p-1)*(d+e*x+f*x^2)^q*
Simp[p*(-a*e)*(C*(c*e)*(q+1)-c*(C*e-B*f)*(2*p+2*q+3))+
(p+q+1)*(a*c*(C*(2*d*f-e^2*(2*p+q+2))+f*(B*e-2*A*f)*(2*p+2*q+3)))+
(2*p*(c*d-a*f)*(C*(c*e)*(q+1)-c*(C*e-B*f)*(2*p+2*q+3))+
(p+q+1)*(C*e*f*p*(-4*a*c)))*x+
(p*(c*e)*(C*(c*e)*(q+1)-c*(C*e-B*f)*(2*p+2*q+3))+
(p+q+1)*(C*f^2*p*(-4*a*c)-c^2*(C*(e^2-4*d*f)*(2*p+q+2)+f*(2*C*d-B*e+2*A*f)*(2*p+2*q+3)))*x^2,x],x] /;
FreeQ[{a,c,d,e,f,A,B,C,q},x] && NeQ[e^2-4*d*f,0] && GtQ[p,0] && NeQ[p+q+1,0] && NeQ[2*p+2*q+3,0] && Not[IGtQ[p,0]] && Not[IGtQ[q,0]]

```

```

Int[ (a_+c_.*x_^2)^p_*(d_+e_.*x_+f_.*x_^2)^q_*(A_+C_.*x_^2), x_Symbol] :=
  (C*(-c*e*(2*p+q+2))+2*c*C*f*(p+q+1)*x)*(a+c*x^2)^p*(d+e*x+f*x^2)^(q+1)/(2*c*f^2*(p+q+1)*(2*p+2*q+3)) -
  (1/(2*c*f^2*(p+q+1)*(2*p+2*q+3)))*
  Int[(a+c*x^2)^(p-1)*(d+e*x+f*x^2)^q*
    Simp[p*(-a*e)*(C*(c*e)*(q+1)-c*(C*e)*(2*p+2*q+3))+(p+q+1)*(a*c*(C*(2*d*f-e^2*(2*p+q+2))+f*(-2*A*f)*(2*p+2*q+3)))+(
      2*p*(c*d-a*f)*(C*(c*e)*(q+1)-c*(C*e)*(2*p+2*q+3))+(p+q+1)*(C*e*f*p*(-4*a*c)))*x+
      (p*(c*e)*(C*(c*e)*(q+1)-c*(C*e)*(2*p+2*q+3)))+(
        (p+q+1)*(C*f^2*p*(-4*a*c)-c^2*(C*(e^2-4*d*f)*(2*p+q+2)+f*(2*C*d+2*A*f)*(2*p+2*q+3)))]*x^2,x],x] /;
FreeQ[{a,c,d,e,f,A,C,q},x] && NeQ[e^2-4*d*f,0] && GtQ[p,0] && NeQ[p+q+1,0] && NeQ[2*p+2*q+3,0] && Not[IGtQ[p,0]] && Not[IGtQ[q,0]]

```

```

Int[ (a_+b_.*x_+c_.*x_^2)^p_*(d_+f_.*x_^2)^q_*(A_+B_.*x_+C_.*x_^2), x_Symbol] :=
  (B*c*f*(2*p+2*q+3)+C*(b*f*p)+2*c*C*f*(p+q+1)*x)*(a+b*x+c*x^2)^p*
  (d+f*x^2)^(q+1)/(2*c*f^2*(p+q+1)*(2*p+2*q+3)) -
  (1/(2*c*f^2*(p+q+1)*(2*p+2*q+3)))*
  Int[(a+b*x+c*x^2)^(p-1)*(d+f*x^2)^q*
    Simp[p*(b*d)*(C*(-b*f)*(q+1)-c*(-B*f)*(2*p+2*q+3))+(
      (p+q+1)*(b^2*C*d*f*p+a*c*(C*(2*d*f)+f*(-2*A*f)*(2*p+2*q+3)))+(
      2*p*(c*d-a*f)*(C*(-b*f)*(q+1)-c*(-B*f)*(2*p+2*q+3)))+(
      (p+q+1)*(-b*C*(C*(-4*d*f)*(2*p+q+2)+f*(2*C*d+2*A*f)*(2*p+2*q+3)))]*x+
      (p*(-b*f)*(C*(-b*f)*(q+1)-c*(-B*f)*(2*p+2*q+3)))+(
        (p+q+1)*(C*f^2*p*(b^2-4*a*c)-c^2*(C*(-4*d*f)*(2*p+q+2)+f*(2*C*d+2*A*f)*(2*p+2*q+3)))]*x^2,x],x] /;
FreeQ[{a,b,c,d,f,A,B,C,q},x] && NeQ[b^2-4*a*c,0] && GtQ[p,0] && NeQ[p+q+1,0] && NeQ[2*p+2*q+3,0] && Not[IGtQ[p,0]] && Not[IGtQ[q,0]]

```

```

Int[(a+b*x+c*x^2)^p*(d+f*x^2)^q*(A+C*x^2),x_Symbol] :=
(C*(b*f*p)+2*C*C*f*(p+q+1)*x)*(a+b*x+c*x^2)^p*
(d+f*x^2)^(q+1)/(2*C*f^2*(p+q+1)*(2*p+2*q+3)) -
(1/(2*C*f^2*(p+q+1)*(2*p+2*q+3)))*
Int[(a+b*x+c*x^2)^(p-1)*(d+f*x^2)^q*
Simp[p*(b*d)*(C*(-b*f)*(q+1))+
(p+q+1)*(b^2*C*d*f*p+a*c*(C*(2*d*f)+f*(-2*A*f)*(2*p+2*q+3)))+
(2*p*(c*d-a*f)*(C*(-b*f)*(q+1))+
(p+q+1)*(-b*c*(C*(-4*d*f)*(2*p+q+2)+f*(2*C*d+2*A*f)*(2*p+2*q+3))))*x+
(p*(-b*f)*(C*(-b*f)*(q+1))+
(p+q+1)*(C*f^2*p*(b^2-4*a*c)-c^2*(C*(-4*d*f)*(2*p+q+2)+f*(2*C*d+2*A*f)*(2*p+2*q+3))))*x^2,x],x] /;
FreeQ[{a,b,c,d,f,A,C,q},x] && NeQ[b^2-4*a*c,0] && GtQ[p,0] && NeQ[p+q+1,0] && NeQ[2*p+2*q+3,0] && Not[IGtQ[p,0]] && Not[IGtQ[q,0]]

```

6: $\int \frac{A+Bx+Cx^2}{(a+bx+cx^2)(d+ex+fx^2)} dx$ when $b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge c^2d^2-bcde+ace^2+b^2df-2acdf-abef+a^2f^2 \neq 0$

Derivation: Algebraic expansion

Basis: Let $q \rightarrow c^2d^2-bcde+ace^2+b^2df-2acdf-abef+a^2f^2$, then $\frac{A+Bx+Cx^2}{(a+bx+cx^2)(d+ex+fx^2)} =$

$$\frac{1}{q(a+bx+cx^2)} (Ac^2d-acCd-Abce+aBce+Ab^2f-abBf-aAcf+a^2Cf+c(Bcd-bCd-Ace+aCe+Abf-aBf)x) +$$

$$\frac{1}{q(d+ex+fx^2)} (cCd^2-Bcde+Ace^2+bBdf-Acdf-aCdf-Abef+aAf^2-f(Bcd-bCd-Ace+aCe+Abf-aBf)x)$$

Rule 1.2.1.7.6: If $b^2-4ac \neq 0 \wedge e^2-4df \neq 0$, let

$q \rightarrow c^2d^2-bcde+ace^2+b^2df-2acdf-abef+a^2f^2$, if $q \neq 0$, then

$$\int \frac{A+Bx+Cx^2}{(a+bx+cx^2)(d+ex+fx^2)} dx \rightarrow$$

$$\frac{1}{q} \int \frac{1}{a+bx+cx^2} (Ac^2d-acCd-Abce+aBce+Ab^2f-abBf-aAcf+a^2Cf+c(Bcd-bCd-Ace+aCe+Abf-aBf)x) dx +$$

$$\frac{1}{q} \int \frac{1}{d+ex+fx^2} (c^2d^2 - Bcde + Ace^2 + bBdf - Acdf - aCdf - Abef + aAf^2 - f(Bcd - bCd - Ace + aCe + Abf - aBf)x) dx$$

Program code:

```
Int[(A_.+B_.xx_+C_.xx_^2)/( (a_+b_.xx_+c_.xx_^2)*(d_+e_.xx_+f_.xx_^2)),x_Symbol] :=
  With[{q=c^2*d^2-b*c*d*e+a*c*e^2+b^2*d*f-2*a*c*d*f-a*b*e*f+a^2*f^2},
    1/q*Int[(A*c^2*d-a*c*C*d-A*b*c*e+A*B*c*e+A*b^2*f-a*b*B*f-a*A*c*f+a^2*C*f+
      c*(B*c*d-b*C*d-A*c*e+a*C*e+A*b*f-a*B*f)*x)/(a+b*x+c*x^2),x] +
    1/q*Int[(c*C*d^2-B*c*d*e+A*c*e^2+b*B*d*f-A*c*d*f-a*C*d*f-A*b*e*f+a*A*f^2-
      f*(B*c*d-b*C*d-A*c*e+a*C*e+A*b*f-a*B*f)*x)/(d+e*x+f*x^2),x] /;
  NeQ[q,0] /;
  FreeQ[{a,b,c,d,e,f,A,B,C},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0]
```

```
Int[(A_.+C_.xx_^2)/( (a_+b_.xx_+c_.xx_^2)*(d_+e_.xx_+f_.xx_^2)),x_Symbol] :=
  With[{q=c^2*d^2-b*c*d*e+a*c*e^2+b^2*d*f-2*a*c*d*f-a*b*e*f+a^2*f^2},
    1/q*Int[(A*c^2*d-a*c*C*d-A*b*c*e+A*b^2*f-a*A*c*f+a^2*C*f+
      c*(-b*C*d-A*c*e+a*C*e+A*b*f)*x)/(a+b*x+c*x^2),x] +
    1/q*Int[(c*C*d^2+A*c*e^2-A*c*d*f-a*C*d*f-A*b*e*f+a*A*f^2-
      f*(-b*C*d-A*c*e+a*C*e+A*b*f)*x)/(d+e*x+f*x^2),x] /;
  NeQ[q,0] /;
  FreeQ[{a,b,c,d,e,f,A,C},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0]
```

```
Int[(A_.+B_.xx_+C_.xx_^2)/( (a_+b_.xx_+c_.xx_^2)*(d_+f_.xx_^2)),x_Symbol] :=
  With[{q=c^2*d^2+b^2*d*f-2*a*c*d*f+a^2*f^2},
    1/q*Int[(A*c^2*d-a*c*C*d+A*b^2*f-a*b*B*f-a*A*c*f+a^2*C*f+c*(B*c*d-b*C*d+A*b*f-a*B*f)*x)/(a+b*x+c*x^2),x] +
    1/q*Int[(c*C*d^2+b*B*d*f-A*c*d*f-a*C*d*f+a*A*f^2-f*(B*c*d-b*C*d+A*b*f-a*B*f)*x)/(d+f*x^2),x] /;
  NeQ[q,0] /;
  FreeQ[{a,b,c,d,f,A,B,C},x] && NeQ[b^2-4*a*c,0]
```

```

Int[(A_+C_.*x_^2)/((a_+b_.*x_+c_.*x_^2)*(d_+f_.*x_^2)),x_Symbol] :=
  With[{q=c^2*d^2+b^2*d*f-2*a*c*d*f+a^2*f^2},
    1/q*Int[(A*c^2*d-a*c*C*d+A*b^2*f-a*A*c*f+a^2*C*f+C*(-b*C*d+A*b*f)*x)/(a+b*x+c*x^2),x] +
    1/q*Int[(c*C*d^2-A*c*d*f-a*C*d*f+a*A*f^2-f*(-b*C*d+A*b*f)*x)/(d+f*x^2),x] /;
  NeQ[q,0] /;
  FreeQ[{a,b,c,d,f,A,C},x] && NeQ[b^2-4*a*c,0]

```

7: $\int \frac{A+Bx+Cx^2}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx$ when $b^2-4ac \neq 0 \wedge e^2-4df \neq 0$

Derivation: Algebraic expansion

Basis: $\frac{A+Bx+Cx^2}{a+bx+cx^2} = \frac{C}{c} + \frac{Ac-aC+(Bc-bC)x}{c(a+bx+cx^2)}$

Rule 1.2.1.7.7: If $b^2-4ac \neq 0 \wedge e^2-4df \neq 0$, then

$$\int \frac{A+Bx+Cx^2}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \rightarrow \frac{C}{c} \int \frac{1}{\sqrt{d+ex+fx^2}} dx + \frac{1}{c} \int \frac{Ac-aC+(Bc-bC)x}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx$$

Program code:

```

Int[(A_+B_.*x_+C_.*x_^2)/((a_+b_.*x_+c_.*x_^2)*Sqrt[d_+e_.*x_+f_.*x_^2]),x_Symbol] :=
  C/c*Int[1/Sqrt[d+e*x+f*x^2],x] +
  1/c*Int[(A*c-a*C+(B*c-b*C)*x)/((a+b*x+c*x^2)*Sqrt[d+e*x+f*x^2]),x] /;
  FreeQ[{a,b,c,d,e,f,A,B,C},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0]

```

```

Int[(A_+C_.*x_^2)/((a_+b_.*x_+c_.*x_^2)*Sqrt[d_+e_.*x_+f_.*x_^2]),x_Symbol] :=
  C/c*Int[1/Sqrt[d+e*x+f*x^2],x] + 1/c*Int[(A*c-a*C-b*C*x)/((a+b*x+c*x^2)*Sqrt[d+e*x+f*x^2]),x] /;
  FreeQ[{a,b,c,d,e,f,A,C},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0]

```



```
Int[(A_+B_.*x_+C_.*x_^2)/(a_+c_.*x_^2)*Sqrt[d_+e_.*x_+f_.*x_^2]),x_Symbol] :=
  C/c*Int[1/Sqrt[d+e*x+f*x^2],x] + 1/c*Int[(A*c-a*C+B*c*x)/(a+c*x^2)*Sqrt[d+e*x+f*x^2]),x] /;
FreeQ[{a,c,d,e,f,A,B,C},x] && NeQ[e^2-4*d*f,0]
```

```
Int[(A_+C_.*x_^2)/(a_+c_.*x_^2)*Sqrt[d_+e_.*x_+f_.*x_^2]),x_Symbol] :=
  C/c*Int[1/Sqrt[d+e*x+f*x^2],x] + (A*c-a*C)/c*Int[1/(a+c*x^2)*Sqrt[d+e*x+f*x^2]),x] /;
FreeQ[{a,c,d,e,f,A,C},x] && NeQ[e^2-4*d*f,0]
```

```
Int[(A_+B_.*x_+C_.*x_^2)/(a_+b_.*x_+c_.*x_^2)*Sqrt[d_+f_.*x_^2]),x_Symbol] :=
  C/c*Int[1/Sqrt[d+f*x^2],x] + 1/c*Int[(A*c-a*C+(B*c-b*C)*x)/(a+b*x+c*x^2)*Sqrt[d+f*x^2]),x] /;
FreeQ[{a,b,c,d,f,A,B,C},x] && NeQ[b^2-4*a*c,0]
```

```
Int[(A_+C_.*x_^2)/(a_+b_.*x_+c_.*x_^2)*Sqrt[d_+f_.*x_^2]),x_Symbol] :=
  C/c*Int[1/Sqrt[d+f*x^2],x] + 1/c*Int[(A*c-a*C-b*C*x)/(a+b*x+c*x^2)*Sqrt[d+f*x^2]),x] /;
FreeQ[{a,b,c,d,f,A,C},x] && NeQ[b^2-4*a*c,0]
```

S: $\int (a+bu+cu^2)^p (d+eu+fu^2)^q (A+Bu+Cu^2) dx$ when $u = g+hx$

Derivation: Integration by substitution

Rule 1.2.1.7.S: If $u = g+hx$, then

$$\int (a+bu+cu^2)^p (d+eu+fu^2)^q (A+Bu+Cu^2) dx \rightarrow \frac{1}{h} \text{Subst} \left[\int (a+bu+cu^2)^p (d+eu+fu^2)^q (A+Bu+Cu^2) dx, x, u \right]$$

Program code:

```
Int[(a_.+b_.*u_.+c_.*u_^2)^p_.*(d_.+e_.*u_.+f_.*u_^2)^q_.*(A_.+B_.*u_.+C_.*u_^2),x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(a+b*x+c*x^2)^p*(d+e*x+f*x^2)^q*(A+B*x+C*x^2),x],x,u] /;
FreeQ[{a,b,c,d,e,f,A,B,C,p,q},x] && LinearQ[u,x] && NeQ[u,x]
```

```
Int[(a_.+b_.*u_.+c_.*u_^2)^p_.*(d_.+e_.*u_.+f_.*u_^2)^q_.*(A_.+B_.*u_),x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(a+b*x+c*x^2)^p*(d+e*x+f*x^2)^q*(A+B*x),x],x,u] /;
FreeQ[{a,b,c,d,e,f,A,B,C,p,q},x] && LinearQ[u,x] && NeQ[u,x]
```

```
Int[(a_.+b_.*u_.+c_.*u_^2)^p_.*(d_.+e_.*u_.+f_.*u_^2)^q_.*(A_.+C_.*u_^2),x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(a+b*x+c*x^2)^p*(d+e*x+f*x^2)^q*(A+C*x^2),x],x,u] /;
FreeQ[{a,b,c,d,e,f,A,C,p,q},x] && LinearQ[u,x] && NeQ[u,x]
```

```
Int[(a_.+c_.*u_^2)^p_.*(d_.+e_.*u_.+f_.*u_^2)^q_.*(A_.+B_.*u_.+C_.*u_^2),x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(a+c*x^2)^p*(d+e*x+f*x^2)^q*(A+B*x+C*x^2),x],x,u] /;
FreeQ[{a,c,d,e,f,A,B,C,p,q},x] && LinearQ[u,x] && NeQ[u,x]
```

```
Int[(a_.+c_.*u_^2)^p_.*(d_.+e_.*u_.+f_.*u_^2)^q_.*(A_.+B_.*u_),x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(a+c*x^2)^p*(d+e*x+f*x^2)^q*(A+B*x),x],x,u] /;
FreeQ[{a,c,d,e,f,A,B,C,p,q},x] && LinearQ[u,x] && NeQ[u,x]
```

```
Int[(a_.+c_.*u_^2)^p_.*(d_.+e_.*u_.+f_.*u_^2)^q_.*(A_.+C_.*u_^2),x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(a+c*x^2)^p*(d+e*x+f*x^2)^q*(A+C*x^2),x],x,u] /;
FreeQ[{a,c,d,e,f,A,C,p,q},x] && LinearQ[u,x] && NeQ[u,x]
```

