

**Rules for integrands of the form $(g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r$
when $b c - a d \neq 0 \wedge b e - a f \neq 0 \wedge d e - c f \neq 0$**

0. $\int (g x)^m (b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$

1. $\int (g x)^m (b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$ when $m \in \mathbb{Z} \vee g > 0$

1: $\int (g x)^m (b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$ when $(m \in \mathbb{Z} \vee g > 0) \wedge \frac{m+1}{n} \in \mathbb{Z}$

Derivation: Integration by substitution

Basis: If $\frac{m+1}{n} \in \mathbb{Z}$, then $x^m (b x^n)^p = \frac{1}{b^{\frac{m+1}{n}-1}} x^{n-1} (b x^n)^{p+\frac{m+1}{n}-1}$

Basis: $x^{n-1} F[x^n] = \frac{1}{n} \text{Subst}[F[x], x, x^n] \partial_x x^n$

Rule 1.1.3.6.0.1.1: If $(m \in \mathbb{Z} \vee g > 0) \wedge \frac{m+1}{n} \in \mathbb{Z}$, then

$$\int (g x)^m (b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow \frac{g^m}{n b^{\frac{m+1}{n}-1}} \text{Subst}\left[\int (b x)^{p+\frac{m+1}{n}-1} (c + d x)^q (e + f x)^r dx, x, x^n\right]$$

Program code:

```
Int[(g_.**x_)^m_.*(b_.**x_^n_)^p_.*(c_+d_.**x_^n_)^q_.*(e_+f_.**x_^n_)^r_.,x_Symbol] :=
  g^m/(n*b^(Simplify[(m+1)/n]-1))*Subst[Int[(b*x)^(p+Simplify[(m+1)/n]-1)*(c+d*x)^q*(e+f*x)^r,x],x,x^n] /;
  FreeQ[{b,c,d,e,f,g,m,n,p,q,r},x] && (IntegerQ[m] || GtQ[g,0]) && IntegerQ[Simplify[(m+1)/n]]
```

2: $\int (g x)^m (b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$ when $(m \in \mathbb{Z} \vee g > 0) \wedge \frac{m+1}{n} \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(b x^n)^p}{x^{n p}} = 0$

Rule 1.1.3.6.0.1.2: If $(m \in \mathbb{Z} \vee g > 0) \wedge \frac{m+1}{n} \notin \mathbb{Z}$, then

$$\int (g x)^m (b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow \frac{g^m b^{\text{IntPart}[p]} (b x^n)^{\text{FracPart}[p]}}{x^{n \text{FracPart}[p]}} \int x^{m+n p} (c + d x^n)^q (e + f x^n)^r dx$$

Program code:

```
Int[(g_*x_)^m_*(b_*x_^n_)^p_*(c_+d_*x_^n_)^q_*(e_+f_*x_^n_)^r_,x_Symbol] :=
  g^m*b^IntPart[p]*(b*x^n)^FracPart[p]/x^(n*FracPart[p])*Int[x^(m+n*p)*(c+d*x^n)^q*(e+f*x^n)^r,x] /;
FreeQ[{b,c,d,e,f,g,m,n,p,q,r},x] && (IntegerQ[m] || GtQ[g,0]) && Not[IntegerQ[Simplify[(m+1)/n]]]
```

2: $\int (g x)^m (b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$ when $m \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(g x)^m}{x^m} == 0$

Rule 1.1.3.6.0.2: If $m \notin \mathbb{Z}$, then

$$\int (g x)^m (b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow \frac{g^{\text{IntPart}[m]} (g x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$$

Program code:

```
Int[(g_*x_)^m_*(b_*x_^n_)^p_*(c_+d_*x_^n_)^q_*(e_+f_*x_^n_)^r_,x_Symbol] :=
  g^IntPart[m]*(g*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(b*x^n)^p*(c+d*x^n)^q*(e+f*x^n)^r,x] /;
FreeQ[{b,c,d,e,f,g,m,n,p,q,r},x] && Not[IntegerQ[m]]
```

1: $\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$ when $p + 2 \in \mathbb{Z}^+ \wedge q \in \mathbb{Z}^+ \wedge r \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule 1.1.3.6.1: If $p + 2 \in \mathbb{Z}^+ \wedge q \in \mathbb{Z}^+ \wedge r \in \mathbb{Z}^+$, then

$$\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow \int \text{ExpandIntegrand}[(g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r, x] dx$$

Program code:

```
Int[(g_.**x_)^m_.*(a_+b_.**x_^n_)^p_.*(c_+d_.**x_^n_)^q_.*(e_+f_.**x_^n_)^r_,x_Symbol] :=
  Int[ExpandIntegrand[(g*x)^m*(a+b*x^n)^p*(c+d*x^n)^q*(e+f*x^n)^r,x],x] /;
FreeQ[{a,b,c,d,e,f,g,m,n},x] && IGtQ[p,-2] && IGtQ[q,0] && IGtQ[r,0]
```

2: $\int x^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$ when $m - n + 1 = 0$

Derivation: Integration by substitution

Basis: $x^{n-1} F[x^n] = \frac{1}{n} \text{Subst}[F[x], x, x^n] \partial_x x^n$

Rule 1.1.3.6.2: If $m - n + 1 = 0$, then

$$\int x^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow \frac{1}{n} \text{Subst}\left[\int (a + b x)^p (c + d x)^q (e + f x)^r dx, x, x^n\right]$$

Program code:

```
Int[x_^m_.*(a_+b_.**x_^n_)^p_.*(c_+d_.**x_^n_)^q_.*(e_+f_.**x_^n_)^r_,x_Symbol] :=
  1/n*Subst[Int[(a+b*x)^p*(c+d*x)^q*(e+f*x)^r,x],x,x^n] /;
FreeQ[{a,b,c,d,e,f,m,n,p,q,r},x] && EqQ[m-n+1,0]
```

3: $\int x^m (a+b x^n)^p (c+d x^n)^q (e+f x^n)^r dx$ when $(p \mid q \mid r) \in \mathbb{Z} \wedge n < 0$

Derivation: Algebraic expansion

Basis: If $(p \mid q \mid r) \in \mathbb{Z}$, then

$$(a+b x^n)^p (c+d x^n)^q (e+f x^n)^r = x^{n(p+q+r)} (b+a x^{-n})^p (d+c x^{-n})^q (f+e x^{-n})^r$$

Rule 1.1.3.6.3: If $(p \mid q \mid r) \in \mathbb{Z} \wedge n < 0$, then

$$\int x^m (a+b x^n)^p (c+d x^n)^q (e+f x^n)^r dx \rightarrow \int x^{m+n(p+q+r)} (b+a x^{-n})^p (d+c x^{-n})^q (f+e x^{-n})^r dx$$

Program code:

```
Int[x_^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_.*(e_+f_.*x_^n_)^r_,x_Symbol] :=
  Int[x^(m+n*(p+q+r))*(b+a*x^(-n))^p*(d+c*x^(-n))^q*(f+e*x^(-n))^r,x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && IntegersQ[p,q,r] && NegQ[n]
```

4. $\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$ when $\frac{m+1}{n} \in \mathbb{Z}$

1: $\int x^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$ when $\frac{m+1}{n} \in \mathbb{Z}$

Derivation: Integration by substitution

Basis: If $\frac{m+1}{n} \in \mathbb{Z}$, then $x^m F[x^n] = \frac{1}{n} \text{Subst}[x^{\frac{m+1}{n}-1} F[x], x, x^n] \partial_x x^n$

Note: If $n \in \mathbb{Z} \wedge \frac{m+1}{n} \in \mathbb{Z}$, then $m \in \mathbb{Z}$, and $(e x)^m$ automatically evaluates to $e^m x^m$.

Rule 1.1.3.6.4.1: If $\frac{m+1}{n} \in \mathbb{Z}$, then

$$\int x^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow \frac{1}{n} \text{Subst}\left[\int x^{\frac{m+1}{n}-1} (a + b x)^p (c + d x)^q (e + f x)^r dx, x, x^n\right]$$

Program code:

```
Int[x_^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_.*(e_+f_.*x_^n_)^r_.,x_Symbol] :=
  1/n*Subst[Int[x^(Simplify[(m+1)/n]-1)*(a+b*x)^p*(c+d*x)^q*(e+f*x)^r,x],x,x^n] /;
FreeQ[{a,b,c,d,e,f,m,n,p,q,r},x] && IntegerQ[Simplify[(m+1)/n]]
```

2: $\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$ when $\frac{m+1}{n} \in \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(g x)^m}{x^m} = 0$

Basis: $\frac{(g x)^m}{x^m} = \frac{g^{\text{IntPart}[m]} (g x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}}$

Rule 1.1.3.6.4.2: If $\frac{m+1}{n} \in \mathbb{Z}$, then

$$\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow \frac{g^{\text{IntPart}[m]} (g x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$$

Program code:

```
Int[(g*x_)^m_.*(a+b_*x_^n_)^p_.*(c+d_*x_^n_)^q_.*(e+f_*x_^n_)^r_,x_Symbol] :=
  g^IntPart[m]*(g*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a+b*x^n)^p*(c+d*x^n)^q*(e+f*x^n)^r,x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p,q,r},x] && IntegerQ[Simplify[(m+1)/n]]
```

$$5. \int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \text{ when } n \in \mathbb{Z}$$

$$1. \int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \text{ when } n \in \mathbb{Z}^+$$

$$1: \int x^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \text{ when } n \in \mathbb{Z}^+ \wedge m \in \mathbb{Z} \wedge \text{GCD}[m+1, n] \neq 1$$

Derivation: Integration by substitution

Basis: If $n \in \mathbb{Z} \wedge m \in \mathbb{Z}$, let $k = \text{GCD}[m+1, n]$, then $x^m F[x^n] = \frac{1}{k} \text{Subst}[x^{\frac{m+1}{k}-1} F[x^{n/k}], x, x^k] \partial_x x^k$

Rule 1.1.3.6.5.1.1: If $n \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}$, let $k = \text{GCD}[m+1, n]$, if $k \neq 1$, then

$$\int x^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow \frac{1}{k} \text{Subst}\left[\int x^{\frac{m+1}{k}-1} (a + b x^{n/k})^p (c + d x^{n/k})^q (e + f x^{n/k})^r dx, x, x^k\right]$$

Program code:

```
Int[x_^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_.*(e_+f_.*x_^n_)^r_,x_Symbol] :=
  With[{k=GCD[m+1,n]},
    1/k*Subst[Int[x^((m+1)/k-1)*(a+b*x^(n/k))^p*(c+d*x^(n/k))^q*(e+f*x^(n/k))^r,x],x,x^k] /;
    k!=1] /;
FreeQ[{a,b,c,d,e,f,p,q,r},x] && IGtQ[n,0] && IntegerQ[m]
```

$$\mathbf{2:} \int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \text{ when } n \in \mathbb{Z}^+ \wedge m \in \mathbb{F}$$

Derivation: Integration by substitution

Basis: If $k \in \mathbb{Z}^+$, then $(g x)^m F[x] = \frac{k}{g} \text{Subst}\left[x^{k(m+1)-1} F\left[\frac{x^k}{g}\right], x, (g x)^{1/k}\right] \partial_x (g x)^{1/k}$

Rule 1.1.3.6.5.1.2: If $n \in \mathbb{Z}^+ \wedge m \in \mathbb{F}$, let $k = \text{Denominator}[m]$, then

$$\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow \frac{k}{g} \text{Subst}\left[\int x^{k(m+1)-1} \left(a + \frac{b x^{kn}}{g^n}\right)^p \left(c + \frac{d x^{kn}}{g^n}\right)^q \left(e + \frac{f x^{kn}}{g^n}\right)^r dx, x, (g x)^{1/k}\right]$$

Program code:

```
Int[(g_.*x_)^m_*(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_*(e_+f_.*x_^n_)^r_,x_Symbol] :=
  With[{k=Denominator[m]},
    k/g*Subst[Int[x^(k*(m+1)-1)*(a+b*x^(k*n)/g^n)^p*(c+d*x^(k*n)/g^n)^q*(e+f*x^(k*n)/g^n)^r,x],x,(g*x)^(1/k)]] /;
  FreeQ[{a,b,c,d,e,f,g,p,q,r},x] && IGtQ[n,0] && FractionQ[m]
```

$$\mathbf{3.} \int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \text{ when } n \in \mathbb{Z}^+$$

$$\mathbf{1.} \int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \text{ when } n \in \mathbb{Z}^+ \wedge p < -1$$

$$\mathbf{1:} \int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \text{ when } n \in \mathbb{Z}^+ \wedge p < -1 \wedge q > 0$$

Derivation: Binomial product recurrence 1

Rule 1.1.3.6.5.1.3.1.1: If $n \in \mathbb{Z}^+ \wedge p < -1 \wedge q > 0$, then

$$\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow$$

$$-\frac{(b e - a f) (g x)^{m+1} (a + b x^n)^{p+1} (c + d x^n)^q}{a b g n (p+1)} + \frac{1}{a b n (p+1)}.$$

$$\int (g x)^m (a+b x^n)^{p+1} (c+d x^n)^{q-1} \left(c (b e n (p+1) + (b e - a f) (m+1)) + d (b e n (p+1) + (b e - a f) (m+n q+1)) x^n \right) dx$$

Program code:

```
Int[(g_.**x_)^m_.*(a+_b_.**x_^n_)^p_*(c+_d_.**x_^n_)^q_.*(e+_f_.**x_^n_),x_Symbol] :=
- (b*e-a*f)*(g*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^q/(a*b*g*n*(p+1)) +
1/(a*b*n*(p+1))*Int[(g*x)^m*(a+b*x^n)^(p+1)*(c+d*x^n)^(q-1)*
Simp[c*(b*e*n*(p+1)+(b*e-a*f)*(m+1))+d*(b*e*n*(p+1)+(b*e-a*f)*(m+n*q+1))*x^n,x],x] /;
FreeQ[{a,b,c,d,e,f,g,m},x] && IGtQ[n,0] && LtQ[p,-1] && GtQ[q,0] && Not[EqQ[q,1] && SimplerQ[b*c-a*d,b*e-a*f]]
```

2: $\int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n) dx$ when $n \in \mathbb{Z}^+ \wedge p < -1 \wedge m-n+1 > 0$

Derivation: Binomial product recurrence 3a

Rule 1.1.3.6.5.1.3.1.2: If $n \in \mathbb{Z}^+ \wedge p < -1 \wedge m-n+1 > 0$, then

$$\int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n) dx \rightarrow \frac{g^{n-1} (b e - a f) (g x)^{m-n+1} (a+b x^n)^{p+1} (c+d x^n)^{q+1}}{b n (b c - a d) (p+1)} - \frac{g^n}{b n (b c - a d) (p+1)} \cdot \int (g x)^{m-n} (a+b x^n)^{p+1} (c+d x^n)^q \left(c (b e - a f) (m-n+1) + (d (b e - a f) (m+n q+1) - b n (c f - d e) (p+1)) x^n \right) dx$$

Program code:

```
Int[(g_.**x_)^m_.*(a+_b_.**x_^n_)^p_*(c+_d_.**x_^n_)^q_.*(e+_f_.**x_^n_),x_Symbol] :=
g^(n-1)*(b*e-a*f)*(g*x)^(m-n+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(q+1)/(b*n*(b*c-a*d)*(p+1)) -
g^n/(b*n*(b*c-a*d)*(p+1))*Int[(g*x)^(m-n)*(a+b*x^n)^(p+1)*(c+d*x^n)^q*
Simp[c*(b*e-a*f)*(m-n+1)+(d*(b*e-a*f)*(m+n*q+1)-b*n*(c*f-d*e)*(p+1))*x^n,x],x] /;
FreeQ[{a,b,c,d,e,f,g,q},x] && IGtQ[n,0] && LtQ[p,-1] && GtQ[m-n+1,0]
```

$$\mathbf{3:} \int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n) dx \text{ when } n \in \mathbb{Z}^+ \wedge p < -1$$

Derivation: Binomial product recurrence 3b

Rule 1.1.3.6.5.1.3.1.3: If $n \in \mathbb{Z}^+ \wedge p < -1$, then

$$\int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n) dx \rightarrow$$

$$-\frac{(b e - a f) (g x)^{m+1} (a+b x^n)^{p+1} (c+d x^n)^{q+1}}{a g n (b c - a d) (p+1)} + \frac{1}{a n (b c - a d) (p+1)} \cdot$$

$$\int (g x)^m (a+b x^n)^{p+1} (c+d x^n)^q (c (b e - a f) (m+1) + e n (b c - a d) (p+1) + d (b e - a f) (m+n(p+q+2)+1) x^n) dx$$

Program code:

```
Int[(g_.**x_)^m_.*(a_+b_.**x_^n_)^p_*(c_+d_.**x_^n_)^q_*(e_+f_.**x_^n_),x_Symbol] :=
  -(b*e-a*f)*(g*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(q+1)/(a*g*n*(b*c-a*d)*(p+1)) +
  1/(a*n*(b*c-a*d)*(p+1))*Int[(g*x)^m*(a+b*x^n)^(p+1)*(c+d*x^n)^q*
    Simp[c*(b*e-a*f)*(m+1)+e*n*(b*c-a*d)*(p+1)+d*(b*e-a*f)*(m+n*(p+q+2)+1)*x^n,x],x] /;
FreeQ[{a,b,c,d,e,f,g,m,q},x] && IGtQ[n,0] && LtQ[p,-1]
```

$$2. \int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n) dx \text{ when } n \in \mathbb{Z}^+ \wedge q > 0$$

$$\mathbf{1:} \int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n) dx \text{ when } n \in \mathbb{Z}^+ \wedge q > 0 \wedge m < -1$$

Derivation: Binomial product recurrence 2a

Rule 1.1.3.6.5.1.3.2.1: If $n \in \mathbb{Z}^+ \wedge q > 0 \wedge m < -1$, then

$$\int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n) dx \rightarrow$$

$$\frac{e (g x)^{m+1} (a+b x^n)^{p+1} (c+d x^n)^q}{a g (m+1)} - \frac{1}{a g^n (m+1)} \cdot$$

$$\int (g x)^{m+n} (a+b x^n)^p (c+d x^n)^{q-1} (c (b e - a f) (m+1) + e n (b c (p+1) + a d q) + d ((b e - a f) (m+1) + b e n (p+q+1)) x^n) dx$$

Program code:

```
Int[(g_.**x_)^m_*(a_+b_.**x_^n_)^p_.*(c_+d_.**x_^n_)^q_.*(e_+f_.**x_^n_),x_Symbol] :=
  e*(g*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^q/(a*g*(m+1)) -
  1/(a*g^n*(m+1))*Int[(g*x)^(m+n)*(a+b*x^n)^p*(c+d*x^n)^(q-1)*
    Simp[c*(b*e-a*f)*(m+1)+e*n*(b*c*(p+1)+a*d*q)+d*((b*e-a*f)*(m+1)+b*e*n*(p+q+1))*x^n,x],x] /;
FreeQ[{a,b,c,d,e,f,g,p},x] && IGtQ[n,0] && GtQ[q,0] && LtQ[m,-1] && Not[EqQ[q,1] && SimplerQ[e+f*x^n,c+d*x^n]]
```

2: $\int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n) dx$ when $n \in \mathbb{Z}^+ \wedge q > 0$

Derivation: Binomial product recurrence 2b

Rule 1.1.3.6.5.1.3.2.2: If $n \in \mathbb{Z}^+ \wedge q > 0$, then

$$\int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n) dx \rightarrow \frac{f (g x)^{m+1} (a+b x^n)^{p+1} (c+d x^n)^q}{b g (m+n (p+q+1)+1)} + \frac{1}{b (m+n (p+q+1)+1)} \cdot \int (g x)^m (a+b x^n)^p (c+d x^n)^{q-1} (c ((b e - a f) (m+1) + b e n (p+q+1)) + (d (b e - a f) (m+1) + f n q (b c - a d) + b e d n (p+q+1)) x^n) dx$$

Program code:

```
Int[(g_.**x_)^m_*(a_+b_.**x_^n_)^p_.*(c_+d_.**x_^n_)^q_.*(e_+f_.**x_^n_),x_Symbol] :=
  f*(g*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^q/(b*g*(m+n*(p+q+1)+1)) +
  1/(b*(m+n*(p+q+1)+1))*Int[(g*x)^m*(a+b*x^n)^p*(c+d*x^n)^(q-1)*
    Simp[c*((b*e-a*f)*(m+1)+b*e*n*(p+q+1))+(d*(b*e-a*f)*(m+1)+f*n*q*(b*c-a*d)+b*e*d*n*(p+q+1))*x^n,x],x] /;
FreeQ[{a,b,c,d,e,f,g,m,p},x] && IGtQ[n,0] && GtQ[q,0] && Not[EqQ[q,1] && SimplerQ[e+f*x^n,c+d*x^n]]
```

3: $\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n) dx$ when $n \in \mathbb{Z}^+ \wedge m > n - 1$

Derivation: Binomial product recurrence 4a

Rule 1.1.3.6.5.1.3.3: If $n \in \mathbb{Z}^+ \wedge m > n - 1$, then

$$\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n) dx \rightarrow \frac{f g^{n-1} (g x)^{m-n+1} (a + b x^n)^{p+1} (c + d x^n)^{q+1}}{b d (m + n (p + q + 1) + 1)} - \frac{g^n}{b d (m + n (p + q + 1) + 1)} \cdot \int (g x)^{m-n} (a + b x^n)^p (c + d x^n)^q (a f c (m - n + 1) + (a f d (m + n q + 1) + b (f c (m + n p + 1) - e d (m + n (p + q + 1) + 1))) x^n) dx$$

Program code:

```
Int[(g_.**x_)^m_.*(a_+b_.**x_^n_)^p_.*(c_+d_.**x_^n_)^q_.*(e_+f_.**x_^n_),x_Symbol] :=
  f*g^(n-1)*(g*x)^(m-n+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(q+1)/(b*d*(m+n*(p+q+1)+1)) -
  g^n/(b*d*(m+n*(p+q+1)+1))*Int[(g*x)^(m-n)*(a+b*x^n)^p*(c+d*x^n)^q*
  Simp[a*f*c*(m-n+1)+(a*f*d*(m+n*q+1)+b*(f*c*(m+n*p+1)-e*d*(m+n*(p+q+1)+1))]*x^n,x],x] /;
FreeQ[{a,b,c,d,e,f,g,p,q},x] && IGtQ[n,0] && GtQ[m,n-1]
```

4: $\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$ when $n \in \mathbb{Z}^+ \wedge m < -1$

Derivation: Binomial product recurrence 4b

Rule 1.1.3.6.5.1.3.4: If $n \in \mathbb{Z}^+ \wedge m < -1$, then

$$\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow \frac{e (g x)^{m+1} (a + b x^n)^{p+1} (c + d x^n)^{q+1}}{a c g (m+1)} + \frac{1}{a c g^n (m+1)} \cdot \int (g x)^{m+n} (a + b x^n)^p (c + d x^n)^q (a f c (m+1) - e (b c + a d) (m+n+1) - e n (b c p + a d q) - b e d (m+n(p+q+2)+1) x^n) dx$$

Program code:

```
Int[(g_.**x_)^m_*(a_+b_.**x_^n_)^p_.*(c_+d_.**x_^n_)^q_.*(e_+f_.**x_^n_),x_Symbol] :=
  e*(g*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(q+1)/(a*c*g*(m+1)) +
  1/(a*c*g^n*(m+1))*Int[(g*x)^(m+n)*(a+b*x^n)^p*(c+d*x^n)^q*
  Simp[a*f*c*(m+1)-e*(b*c+a*d)*(m+n+1)-e*n*(b*c*p+a*d*q)-b*e*d*(m+n*(p+q+2)+1)*x^n,x],x] /;
FreeQ[{a,b,c,d,e,f,g,p,q},x] && IGtQ[n,0] && LtQ[m,-1]
```

$$\text{5: } \int \frac{(g x)^m (a + b x^n)^p (e + f x^n)}{c + d x^n} dx \text{ when } n \in \mathbb{Z}^+$$

Derivation: Algebraic expansion

Rule 1.1.3.6.5.1.3.5: If $n \in \mathbb{Z}^+$, then

$$\int \frac{(g x)^m (a + b x^n)^p (e + f x^n)}{c + d x^n} dx \rightarrow \int \text{ExpandIntegrand}\left[\frac{(g x)^m (a + b x^n)^p (e + f x^n)}{c + d x^n}, x\right] dx$$

Program code:

```
Int[(g_.**x_)^m_.*(a_+b_.**x_^n_)^p_.*(e_+f_.**x_^n_)/(c_+d_.**x_^n_),x_Symbol] :=
  Int[ExpandIntegrand[(g**x)^m*(a+b*x^n)^p*(e+f*x^n)/(c+d*x^n),x],x] /;
FreeQ[{a,b,c,d,e,f,g,m,p},x] && IGtQ[n,0]
```

$$\text{6: } \int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n) dx \text{ when } n \in \mathbb{Z}^+$$

Derivation: Algebraic expansion

Rule 1.1.3.6.5.1.3.6: If $n \in \mathbb{Z}^+$, then

$$\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n) dx \rightarrow$$

$$e \int (g x)^m (a + b x^n)^p (c + d x^n)^q dx + \frac{f}{e^n} \int (g x)^{m+n} (a + b x^n)^p (c + d x^n)^q dx$$

Program code:

```
Int[(g_.**x_)^m_.*(a_+b_.**x_^n_)^p_.*(c_+d_.**x_^n_)^q_.*(e_+f_.**x_^n_),x_Symbol] :=
  e*Int[(g**x)^m*(a+b*x^n)^p*(c+d*x^n)^q,x] +
  f/e^n*Int[(g**x)^(m+n)*(a+b*x^n)^p*(c+d*x^n)^q,x] /;
FreeQ[{a,b,c,d,e,f,g,m,p,q},x] && IGtQ[n,0]
```

4: $\int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n)^r dx$ when $n \in \mathbb{Z}^+ \wedge r \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule 1.1.3.6.5.1.4: If $n \in \mathbb{Z}^+ \wedge r \in \mathbb{Z}^+$, then

$$\int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n)^r dx \rightarrow e \int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n)^{r-1} dx + \frac{f}{e^n} \int (g x)^{m+n} (a+b x^n)^p (c+d x^n)^q (e+f x^n)^{r-1} dx$$

Program code:

```
Int[(g_.**x_)^m_.*(a_+b_.**x_^n_)^p_.*(c_+d_.**x_^n_)^q_.*(e_+f_.**x_^n_)^r_. ,x_Symbol] :=
  e*Int[(g*x)^m*(a+b*x^n)^p*(c+d*x^n)^q*(e+f*x^n)^(r-1),x] +
  f/e^n*Int[(g*x)^(m+n)*(a+b*x^n)^p*(c+d*x^n)^q*(e+f*x^n)^(r-1),x] /;
FreeQ[{a,b,c,d,e,f,g,m,p,q},x] && IGtQ[n,0] && IGtQ[r,0]
```

$$2. \int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \text{ when } n \in \mathbb{Z}^-$$

$$1. \int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \text{ when } n \in \mathbb{Z}^- \wedge m \in \mathbb{Q}$$

$$\text{1: } \int x^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \text{ when } n \in \mathbb{Z}^- \wedge m \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } F[x] = -\text{Subst}\left[\frac{F\left[\frac{x^{-1}}{x^2}\right]}{x^2}, x, \frac{1}{x}\right] \partial_x \frac{1}{x}$$

Rule 1.1.3.6.5.2.1.1: If $n \in \mathbb{Z}^- \wedge m \in \mathbb{Z}$, then

$$\int x^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow -\text{Subst}\left[\int \frac{(a + b x^{-n})^p (c + d x^{-n})^q (e + f x^{-n})^r}{x^{m+2}} dx, x, \frac{1}{x}\right]$$

Program code:

```
Int[x^m_.*(a+b_.x^n_)^p_.*(c+d_.x^n_)^q_.*(e+f_.x^n_)^r_,x_Symbol] :=
  -Subst[Int[(a+b*x^(-n))^p*(c+d*x^(-n))^q*(e+f*x^(-n))^r/x^(m+2),x],x,1/x] /;
FreeQ[{a,b,c,d,e,f,p,q,r},x] && ILtQ[n,0] && IntegerQ[m]
```


$$\mathbf{2:} \int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \text{ when } n \in \mathbb{Z}^- \wedge m \in \mathbb{F}$$

Derivation: Integration by substitution

$$\text{Basis: If } n \in \mathbb{Z}^- \wedge k > 1, \text{ then } (g x)^m F[x^n] = -\frac{k}{g} \text{Subst}\left[\frac{F[g^{-n} x^{-kn}]}{x^{k(m+1)+1}}, x, \frac{1}{(g x)^{1/k}}\right] \partial_x \frac{1}{(g x)^{1/k}}$$

Rule 1.1.3.6.5.2.1.2: If $n \in \mathbb{Z}^- \wedge m \in \mathbb{F}$, let $k = \text{Denominator}[m]$, then

$$\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow -\frac{k}{g} \text{Subst}\left[\int \frac{(a + b g^{-n} x^{-kn})^p (c + d g^{-n} x^{-kn})^q (e + f g^{-n} x^{-kn})^r}{x^{k(m+1)+1}} dx, x, \frac{1}{(g x)^{1/k}}\right]$$

Program code:

```
Int[(g_.*x_)^m_*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_.*(e_+f_.*x_^n_)^r_,x_Symbol] :=
  With[{k=Denominator[m]},
    -k/g*Subst[Int[(a+b*g^(-n)*x^(-k*n))^p*(c+d*g^(-n)*x^(-k*n))^q*(e+f*g^(-n)*x^(-k*n))^r/x^(k*(m+1)+1),x],x,1/(g*x)^(1/k)]] /;
  FreeQ[{a,b,c,d,e,f,g,p,q,r},x] && ILtQ[n,0] && FractionQ[m]
```

$$\mathbf{2:} \int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \text{ when } n \in \mathbb{Z}^- \wedge m \notin \mathbb{Q}$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: } \partial_x \left((g x)^m (x^{-1})^m \right) = 0$$

$$\text{Basis: } F[x] = -\text{Subst}\left[\frac{F[x^{-1}]}{x^2}, x, \frac{1}{x}\right] \partial_x \frac{1}{x}$$

Rule 1.1.3.6.5.2.2: If $n \in \mathbb{Z}^- \wedge m \notin \mathbb{Q}$, then

$$\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow (g x)^m (x^{-1})^m \int \frac{(a + b x^n)^p (c + d x^n)^q (e + f x^n)^r}{(x^{-1})^m} dx$$

$$\rightarrow -(g x)^m (x^{-1})^m \text{Subst} \left[\int \frac{(a+b x^{-n})^p (c+d x^{-n})^q (e+f x^{-n})^r}{x^{m+2}} dx, x, \frac{1}{x} \right]$$

Program code:

```
Int[(g_.**x_)^m_*(a_+b_.**x_^n_)^p_*(c_+d_.**x_^n_)^q_*(e_+f_.**x_^n_)^r_,x_Symbol] :=
  -(g*x)^m*(x^(-1))^m*Subst[Int[(a+b*x^(-n))^p*(c+d*x^(-n))^q*(e+f*x^(-n))^r/x^(m+2),x],x,1/x] /;
FreeQ[{a,b,c,d,e,f,g,m,p,q,r},x] && ILtQ[n,0] && Not[RationalQ[m]]
```

6. $\int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n)^r dx$ when $n \in \mathbb{F}$

1: $\int x^m (a+b x^n)^p (c+d x^n)^q (e+f x^n)^r dx$ when $n \in \mathbb{F}$

Derivation: Integration by substitution

Basis: If $k \in \mathbb{Z}^+$, then $x^m F[x^n] = k \text{Subst}[x^{k(m+1)-1} F[x^{kn}], x, x^{1/k}] \partial_x x^{1/k}$

Rule 1.1.3.6.6.1: If $n \in \mathbb{F}$, let $k = \text{Denominator}[n]$, then

$$\int x^m (a+b x^n)^p (c+d x^n)^q (e+f x^n)^r dx \rightarrow k \text{Subst} \left[\int x^{k(m+1)-1} (a+b x^{kn})^p (c+d x^{kn})^q (e+f x^{kn})^r dx, x, x^{1/k} \right]$$

Program code:

```
Int[x_^m_*(a_+b_.**x_^n_)^p_*(c_+d_.**x_^n_)^q_*(e_+f_.**x_^n_)^r_,x_Symbol] :=
  With[{k=Denominator[n]},
    k*Subst[Int[x^(k*(m+1)-1)*(a+b*x^(k*n))^p*(c+d*x^(k*n))^q*(e+f*x^(k*n))^r,x],x,x^(1/k)]] /;
FreeQ[{a,b,c,d,e,f,m,p,q,r},x] && FractionQ[n]
```

2: $\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$ when $n \in \mathbb{F}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(g x)^m}{x^m} == 0$

Basis: $\frac{(g x)^m}{x^m} == \frac{g^{\text{IntPart}[m]} (g x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}}$

Rule 1.1.3.6.6.2: If $n \in \mathbb{F}$, then

$$\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow \frac{g^{\text{IntPart}[m]} (g x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$$

Program code:

```
Int[(g*x_)^m_*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_.*(e_+f_.*x_^n_)^r_.,x_Symbol] :=
  g^IntPart[m]*(g*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a+b*x^n)^p*(c+d*x^n)^q*(e+f*x^n)^r,x] /;
FreeQ[{a,b,c,d,e,f,g,m,p,q,r},x] && FractionQ[n]
```

$$7. \int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \text{ when } \frac{n}{m+1} \in \mathbb{Z}$$

$$1: \int x^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \text{ when } \frac{n}{m+1} \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If $\frac{n}{m+1} \in \mathbb{Z}$, then $x^m F[x^n] = \frac{1}{m+1} \text{Subst}[F[x^{\frac{n}{m+1}}], x, x^{m+1}] \partial_x x^{m+1}$

Rule 1.1.3.6.7.1: If $\frac{n}{m+1} \in \mathbb{Z}$, then

$$\int x^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow \frac{1}{m+1} \text{Subst}\left[\int (a + b x^{\frac{n}{m+1}})^p (c + d x^{\frac{n}{m+1}})^q (e + f x^{\frac{n}{m+1}})^r dx, x, x^{m+1}\right]$$

Program code:

```
Int[x^m.*(a+b.*x^n)^p.*(c+d.*x^n)^q.*(e+f.*x^n)^r.,x_Symbol] :=
  1/(m+1)*Subst[Int[(a+b*x^Simplify[n/(m+1)])^p*(c+d*x^Simplify[n/(m+1)])^q*(e+f*x^Simplify[n/(m+1)])^r,x],x,x^(m+1)] /;
FreeQ[{a,b,c,d,e,f,m,n,p,q,r},x] && IntegerQ[Simplify[n/(m+1)]]
```

2: $\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$ when $\frac{n}{m+1} \in \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(g x)^m}{x^m} = 0$

Basis: $\frac{(g x)^m}{x^m} = \frac{g^{\text{IntPart}[m]} (g x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}}$

Rule 1.1.3.6.7.2: If $\frac{n}{m+1} \in \mathbb{Z}$, then

$$\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow \frac{g^{\text{IntPart}[m]} (g x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$$

Program code:

```
Int[(g*x_)^m_.*(a+b_*x_^n_)^p_.*(c+d_*x_^n_)^q_.*(e+f_*x_^n_)^r_,x_Symbol] :=
  g^IntPart[m]*(g*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a+b*x^n)^p*(c+d*x^n)^q*(e+f*x^n)^r,x] /;
  FreeQ[{a,b,c,d,e,f,g,m,n,p,q,r},x] && IntegerQ[Simplify[n/(m+1)]]
```

8. $\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n) dx$

1. $\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n) dx$ when $p < -1$

1: $\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n) dx$ when $p < -1 \wedge q > 0$

Derivation: Binomial product recurrence 1

Rule 1.1.3.6.8.1.1: If $p < -1 \wedge q > 0$, then

$$\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n) dx \rightarrow -\frac{(b e - a f) (g x)^{m+1} (a + b x^n)^{p+1} (c + d x^n)^q}{a b g n (p+1)} + \frac{1}{a b n (p+1)}.$$

$$\int (g x)^m (a+b x^n)^{p+1} (c+d x^n)^{q-1} (c (b e n (p+1) + (b e - a f) (m+1)) + d (b e n (p+1) + (b e - a f) (m+n q+1)) x^n) dx$$

Program code:

```
Int[(g_.**x_)^m_.*(a+_b_.**x_^n_)^p_.*(c+_d_.**x_^n_)^q_.*(e+_f_.**x_^n_),x_Symbol] :=
- (b*e-a*f)*(g*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^q/(a*b*g*n*(p+1)) +
1/(a*b*n*(p+1))*Int[(g*x)^m*(a+b*x^n)^(p+1)*(c+d*x^n)^(q-1)*
Simp[c*(b*e*n*(p+1)+(b*e-a*f)*(m+1))+d*(b*e*n*(p+1)+(b*e-a*f)*(m+n*q+1))*x^n,x],x] /;
FreeQ[{a,b,c,d,e,f,g,m,n},x] && LtQ[p,-1] && GtQ[q,0] && Not[EqQ[q,1] && SimplerQ[b*c-a*d,b*e-a*f]]
```

2: $\int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n) dx$ when $p < -1$

Derivation: Binomial product recurrence 3b

Rule 1.1.3.6.8.1.2: If $p < -1$, then

$$\int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n) dx \rightarrow$$

$$-\frac{(b e - a f) (g x)^{m+1} (a+b x^n)^{p+1} (c+d x^n)^{q+1}}{a g n (b c - a d) (p+1)} + \frac{1}{a n (b c - a d) (p+1)} \cdot$$

$$\int (g x)^m (a+b x^n)^{p+1} (c+d x^n)^q (c (b e - a f) (m+1) + e n (b c - a d) (p+1) + d (b e - a f) (m+n (p+q+2)+1) x^n) dx$$

Program code:

```
Int[(g_.**x_)^m_.*(a+_b_.**x_^n_)^p_.*(c+_d_.**x_^n_)^q_.*(e+_f_.**x_^n_),x_Symbol] :=
- (b*e-a*f)*(g*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(q+1)/(a*g*n*(b*c-a*d)*(p+1)) +
1/(a*n*(b*c-a*d)*(p+1))*Int[(g*x)^m*(a+b*x^n)^(p+1)*(c+d*x^n)^q*
Simp[c*(b*e-a*f)*(m+1)+e*n*(b*c-a*d)*(p+1)+d*(b*e-a*f)*(m+n*(p+q+2)+1))*x^n,x],x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,q},x] && LtQ[p,-1]
```

2: $\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$ when $q > 0$

Derivation: Binomial product recurrence 2b

Rule 1.1.3.6.8.2: If $q > 0$, then

$$\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx \rightarrow$$

$$\frac{f (g x)^{m+1} (a + b x^n)^{p+1} (c + d x^n)^q}{b g (m + n (p + q + 1) + 1)} + \frac{1}{b (m + n (p + q + 1) + 1)} \cdot$$

$$\int (g x)^m (a + b x^n)^p (c + d x^n)^{q-1} \left(c \left((b e - a f) (m + 1) + b e n (p + q + 1) \right) + \left(d (b e - a f) (m + 1) + f n q (b c - a d) + b e d n (p + q + 1) \right) x^n \right) dx$$

Program code:

```
Int[(g_.**x_)^m_.*(a_+b_.**x_^n_)^p_.*(c_+d_.**x_^n_)^q_.*(e_+f_.**x_^n_),x_Symbol] :=
  f*(g*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^q/(b*g*(m+n*(p+q+1)+1)) +
  1/(b*(m+n*(p+q+1)+1))*Int[(g*x)^m*(a+b*x^n)^p*(c+d*x^n)^(q-1)*
    Simp[c*((b*e-a*f)*(m+1)+b*e*n*(p+q+1))+(d*(b*e-a*f)*(m+1)+f*n*q*(b*c-a*d)+b*e*d*n*(p+q+1))*x^n,x],x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p},x] && GtQ[q,0] && Not[EqQ[q,1] && SimplerQ[e+f*x^n,c+d*x^n]]
```

3: $\int \frac{(g x)^m (a + b x^n)^p (e + f x^n)}{c + d x^n} dx$ when $b c - a d \neq 0$

Derivation: Algebraic expansion

Rule 1.1.3.6.8.3: If $b c - a d \neq 0$, then

$$\int \frac{(g x)^m (a + b x^n)^p (e + f x^n)}{c + d x^n} dx \rightarrow \int \text{ExpandIntegrand}\left[\frac{(g x)^m (a + b x^n)^p (e + f x^n)}{c + d x^n}, x\right] dx$$

Program code:

```
Int[(g_.**x_)^m_.*(a_+b_.**x_^n_)^p_*(e_+f_.**x_^n_)/(c_+d_.**x_^n_),x_Symbol] :=
  Int[ExpandIntegrand[(g**x)^m*(a+b*x^n)^p*(e+f*x^n)/(c+d*x^n),x],x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p},x]
```

4: $\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n) dx$ when $b c - a d \neq 0$

Derivation: Algebraic expansion

Rule 1.1.3.6.8.4: If $b c - a d \neq 0$, then

$$\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n) dx \rightarrow e \int (g x)^m (a + b x^n)^p (c + d x^n)^q dx + \frac{f (g x)^m}{x^m} \int x^{m+n} (a + b x^n)^p (c + d x^n)^q dx$$

Program code:

```
Int[(g_.**x_)^m_.*(a_+b_.**x_^n_)^p_*(c_+d_.**x_^n_)^q_*(e_+f_.**x_^n_),x_Symbol] :=
  e*Int[(g**x)^m*(a+b*x^n)^p*(c+d*x^n)^q,x] +
  f*(g**x)^m/x^m*Int[x^(m+n)*(a+b*x^n)^p*(c+d*x^n)^q,x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p,q},x]
```


$$9. \int (g x)^m (a + b x^n)^p (c + d x^{-n})^q (e + f x^n)^r dx$$

$$1. \int x^m (a + b x^n)^p (c + d x^{-n})^q (e + f x^n)^r dx$$

$$\text{1: } \int x^m (a + b x^n)^p (c + d x^{-n})^q (e + f x^n)^r dx \text{ when } q \in \mathbb{Z}$$

Derivation: Algebraic normalization

Basis: If $q \in \mathbb{Z}$, then $(c + d x^{-n})^q = x^{-nq} (d + c x^n)^q$

Rule 1.1.3.6.9.1.1: If $q \in \mathbb{Z}$, then

$$\int x^m (a + b x^n)^p (c + d x^{-n})^q (e + f x^n)^r dx \rightarrow \int x^{m-nq} (a + b x^n)^p (d + c x^n)^q (e + f x^n)^r dx$$

Program code:

```
Int[x_^m_.*(a_+b_.*x_^n_.)^p_.*(c_+d_.*x_^mn_.)^q_.*(e_+f_.*x_^n_.)^r_,x_Symbol] :=
  Int[x^(m-n*q)*(a+b*x^n)^p*(d+c*x^n)^q*(e+f*x^n)^r,x] /;
FreeQ[{a,b,c,d,e,f,m,n,p,r},x] && EqQ[mn,-n] && IntegerQ[q]
```

$$\mathbf{2:} \int x^m (a + b x^n)^p (c + d x^{-n})^q (e + f x^n)^r dx \text{ when } p \in \mathbb{Z} \wedge r \in \mathbb{Z}$$

Derivation: Algebraic normalization

Basis: If $p \in \mathbb{Z}$, then $(a + b x^n)^p = x^{np} (b + a x^{-n})^p$

Rule 1.1.3.6.9.2: If $p \in \mathbb{Z} \wedge r \in \mathbb{Z}$, then

$$\int x^m (a + b x^n)^p (c + d x^{-n})^q (e + f x^n)^r dx \rightarrow \int x^{m+n(p+r)} (b + a x^{-n})^p (c + d x^{-n})^q (f + e x^{-n})^r dx$$

Program code:

```
Int[x_^m_.*(a_.+b_.*x_^n_.)^p_.*(c_+d_.*x_^mn_.)^q_.*(e_+f_.*x_^n_.)^r_,x_Symbol] :=
  Int[x^(m+n*(p+r))*(b+a*x^(-n))^p*(c+d*x^(-n))^q*(f+e*x^(-n))^r,x] /;
FreeQ[{a,b,c,d,e,f,m,n,q},x] && EqQ[mn,-n] && IntegerQ[p] && IntegerQ[r]
```

3: $\int x^m (a+b x^n)^p (c+d x^{-n})^q (e+f x^n)^r dx$ when $q \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{x^n q (c+d x^{-n})^q}{(d+c x^n)^q} == 0$

Basis: $\frac{x^n q (c+d x^{-n})^q}{(d+c x^n)^q} == \frac{x^n \text{FracPart}[q] (c+d x^{-n})^{\text{FracPart}[q]}}{(d+c x^n)^{\text{FracPart}[q]}}$

Rule 1.1.3.6.9.3: If $q \notin \mathbb{Z}$, then

$$\int x^m (a+b x^n)^p (c+d x^{-n})^q (e+f x^n)^r dx \rightarrow \frac{x^n \text{FracPart}[q] (c+d x^{-n})^{\text{FracPart}[q]}}{(d+c x^n)^{\text{FracPart}[q]}} \int x^{m-n q} (a+b x^n)^p (d+c x^n)^q (e+f x^n)^r dx$$

Program code:

```
Int[x^m.*(a_.+b_.*x^n_.)^p.*(c_.+d_.*x^mn_.)^q.*(e_.+f_.*x^n_.)^r_,x_Symbol] :=
  x^(n*FracPart[q])*(c+d*x^(-n))^FracPart[q]/(d+c*x^n)^FracPart[q]*Int[x^(m-n*q)*(a+b*x^n)^p*(d+c*x^n)^q*(e+f*x^n)^r,x] /;
FreeQ[{a,b,c,d,e,f,m,n,p,q,r},x] && EqQ[mn,-n] && Not[IntegerQ[q]]
```

$$2: \int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n)^r dx$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{(g x)^m}{x^m} == 0$$

$$\text{Basis: } \frac{(g x)^m}{x^m} == \frac{g^{\text{IntPart}[m]} (g x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}}$$

Rule 1.1.3.6.9.2:

$$\int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n)^r dx \rightarrow \frac{g^{\text{IntPart}[m]} (g x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (a+b x^n)^p (c+d x^n)^q (e+f x^n)^r dx$$

Program code:

```
Int[(g_*x_)^m_*(a_+b_*x_^n_)^p_*(c_+d_*x_^mn_)^q_*(e_+f_*x_^n_)^r_,x_Symbol] :=
  g^IntPart[m]*(g*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a+b*x^n)^p*(c+d*x^n)^q*(e+f*x^n)^r,x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p,q,r},x] && EqQ[mn,-n]
```

$$X: \int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n)^r dx$$

Rule 1.1.3.6.X:

$$\int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n)^r dx \rightarrow \int (g x)^m (a+b x^n)^p (c+d x^n)^q (e+f x^n)^r dx$$

Program code:

```
Int[(g_*x_)^m_*(a_+b_*x_^n_)^p_*(c_+d_*x_^n_)^q_*(e_+f_*x_^n_)^r_,x_Symbol] :=
  Unintegrable[(g*x)^m*(a+b*x^n)^p*(c+d*x^n)^q*(e+f*x^n)^r,x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p,q,r},x]
```

S: $\int u^m (a + b v^n)^p (c + d v^n)^q (e + f v^n)^r dx$ when $v = h + i x \wedge u = g v$

Derivation: Integration by substitution and piecewise constant extraction

Basis: If $u = g v$, then $\partial_x \frac{u^m}{v^m} = 0$

Rule 1.1.3.6.S: If $v = h + i x \wedge u = g v$, then

$$\int u^m (a + b v^n)^p (c + d v^n)^q (e + f v^n)^r dx \rightarrow \frac{u^m}{i v^m} \text{Subst}\left[\int x^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx, x, v\right]$$

Program code:

```
Int[u_^m.*(a_.+b_.*v_^n_)^p.*(c_.+d_.*v_^n_)^q.*(e_.+f_.*v_^n_)^r_,x_Symbol] :=
  u^m/(Coefficient[v,x,1]*v^m)*Subst[Int[x^m*(a+b*x^n)^p*(c+d*x^n)^q*(e+f*x^n)^r,x],x,v] /;
FreeQ[{a,b,c,d,e,f,m,n,p,q,r},x] && LinearPairQ[u,v,x]
```

Rules for integrands of the form $(g x)^m (a + b x^n)^p (c + d x^n)^q (e_1 + f_1 x^{n/2})^r (e_2 + f_2 x^{n/2})^r$

1. $\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e_1 + f_1 x^{n/2})^r (e_2 + f_2 x^{n/2})^r dx$ when $e_2 f_1 + e_1 f_2 = 0$

1: $\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e_1 + f_1 x^{n/2})^r (e_2 + f_2 x^{n/2})^r dx$ when $e_2 f_1 + e_1 f_2 = 0 \wedge (r \in \mathbb{Z} \vee e_1 > 0 \wedge e_2 > 0)$

Derivation: Algebraic simplification

Basis: If $e_2 f_1 + e_1 f_2 = 0 \wedge (r \in \mathbb{Z} \vee e_1 > 0 \wedge e_2 > 0)$, then $(e_1 + f_1 x^{n/2})^r (e_2 + f_2 x^{n/2})^r = (e_1 e_2 + f_1 f_2 x^n)^r$

Rule: If $e_2 f_1 + e_1 f_2 = 0 \wedge (r \in \mathbb{Z} \vee e_1 > 0 \wedge e_2 > 0)$, then

$$\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e_1 + f_1 x^{n/2})^r (e_2 + f_2 x^{n/2})^r dx \rightarrow \int (g x)^m (a + b x^n)^p (c + d x^n)^q (e_1 e_2 + f_1 f_2 x^n)^r dx$$

Program code:

```
Int[(g_*x_)^m_.*(a+b_*x_^n_)^p_.*(c+d_*x_^n_)^q_.*(e1+f1_*x_^n2_)^r_.*(e2+f2_*x_^n2_)^r_.,x_Symbol] :=
  Int[(g*x)^m*(a+b*x^n)^p*(c+d*x^n)^q*(e1*e2+f1*f2*x^n)^r,x] /;
FreeQ[{a,b,c,d,e1,f1,e2,f2,g,m,n,p,q,r},x] && EqQ[n2,n/2] && EqQ[e2*f1+e1*f2,0] && (IntegerQ[r] || GtQ[e1,0] && GtQ[e2,0])
```

2: $\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e_1 + f_1 x^{n/2})^r (e_2 + f_2 x^{n/2})^r dx$ when $e_2 f_1 + e_1 f_2 = 0$

Derivation: Piecewise constant extraction

■ Basis: If $e_2 f_1 + e_1 f_2 = 0$, then $\partial_x \frac{(e_1 + f_1 x^{n/2})^r (e_2 + f_2 x^{n/2})^r}{(e_1 e_2 + f_1 f_2 x^n)^r} = 0$

Rule: If $e_2 f_1 + e_1 f_2 = 0$, then

$$\int (g x)^m (a + b x^n)^p (c + d x^n)^q (e_1 + f_1 x^{n/2})^r (e_2 + f_2 x^{n/2})^r dx \rightarrow$$

$$\frac{(e_1 + f_1 x^{n/2})^{\text{FracPart}[r]} (e_2 + f_2 x^{n/2})^{\text{FracPart}[r]}}{(e_1 e_2 + f_1 f_2 x^n)^{\text{FracPart}[r]}} \int (g x)^m (a + b x^n)^p (c + d x^n)^q (e_1 e_2 + f_1 f_2 x^n)^r dx$$

Program code:

```
Int[(g_.*x_)^m_.*(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_.*(e1_+f1_.*x_^n2_)^r_.*(e2_+f2_.*x_^n2_)^r_,x_Symbol] :=
  (e1+f1*x^(n/2))^FracPart[r]*(e2+f2*x^(n/2))^FracPart[r]/(e1*e2+f1*f2*x^n)^FracPart[r]*
  Int[(g*x)^m*(a+b*x^n)^p*(c+d*x^n)^q*(e1*e2+f1*f2*x^n)^r,x] /;
FreeQ[{a,b,c,d,e1,f1,e2,f2,g,m,n,p,q,r},x] && EqQ[n2,n/2] && EqQ[e2*f1+e1*f2,0]
```