Rules for integrands of the form $(a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x])^n$

1: $\int (a + b \, \text{Sin}[e + f \, x]) \, (c + d \, \text{Sin}[e + f \, x]) \, dx$ when $b \, c - a \, d \neq 0$

Derivation: Algebraic expansion

Basis: $(a + b \, z) \, (c + d \, z) == \frac{1}{2} \, (2 \, a \, c + b \, d) + (b \, c + a \, d) \, z - \frac{1}{2} \, b \, d \, \left(1 - 2 \, z^2\right)$

Rule: If $b \, c - a \, d \neq 0$, then

$$\int (a + b \, \text{Sin}[e + f \, x]) \, (c + d \, \text{Sin}[e + f \, x]) \, dx \rightarrow \frac{(2 \, a \, c + b \, d) \, x}{2} - \frac{(b \, c + a \, d) \, \text{Cos}[e + f \, x]}{f} - \frac{b \, d \, \text{Cos}[e + f \, x] \, \text{Sin}[e + f \, x]}{2 \, f}$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])*(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  (2*a*c+b*d)*x/2 - (b*c+a*d)*Cos[e+f*x]/f - b*d*Cos[e+f*x]*Sin[e+f*x]/(2*f) /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0]
```

2: $\displaystyle\int \frac{a + b\,\text{Sin}[e + f\,x]}{c + d\,\text{Sin}[e + f\,x]}\,dx$ when $b\,c - a\,d \neq 0$

Reference: G&R 2.551.2

Derivation: Algebraic expansion

Basis: $\dfrac{a+b\,z}{c+d\,z} \;==\; \dfrac{b}{d} - \dfrac{b\,c - a\,d}{d\,(c+d\,z)}$

Rule: If $b\,c - a\,d \neq 0$, then

$$\int \frac{a + b\,\text{Sin}[e + f\,x]}{c + d\,\text{Sin}[e + f\,x]}\,dx \;\longrightarrow\; \frac{b\,x}{d} - \frac{b\,c - a\,d}{d}\int \frac{1}{c + d\,\text{Sin}[e + f\,x]}\,dx$$

Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])/(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  b*x/d - (b*c-a*d)/d*Int[1/(c+d*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0]
```

3. $\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x])^n \, dx$ when $b \, c + a \, d == 0 \; \wedge \; a^2 - b^2 == 0$

   1: $\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x])^n \, dx$ when $b \, c + a \, d == 0 \; \wedge \; a^2 - b^2 == 0 \; \wedge \; m \in \mathbb{Z}$

Derivation: Algebraic simplification

Basis: If $b \, c + a \, d == 0 \; \wedge \; a^2 - b^2 == 0$, then $(a + b \, \text{Sin}[z]) \; (c + d \, \text{Sin}[z]) == a \, c \, \text{Cos}[z]^2$

Rule: If $b \, c + a \, d == 0 \; \wedge \; a^2 - b^2 == 0 \; \wedge \; m \in \mathbb{Z}$, then

$$\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x])^n \, dx \;\longrightarrow\; a^m \, c^m \int \text{Cos}[e + f \, x]^{2m} \, (c + d \, \text{Sin}[e + f \, x])^{n-m} \, dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_.*(c_+d_.*sin[e_.+f_.*x_])^n_.,x_Symbol] :=
  a^m*c^m*Int[Cos[e+f*x]^(2*m)*(c+d*Sin[e+f*x])^(n-m),x] /;
FreeQ[{a,b,c,d,e,f,n},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && IntegerQ[m] && Not[IntegerQ[n] && (LtQ[m,0] && GtQ[n,0] || LtQ[0,n,m
```

2. $\int (a + b\,\mathrm{Sin}[e + f\,x])^m\,(c + d\,\mathrm{Sin}[e + f\,x])^n\,dx$ when $b\,c + a\,d == 0 \,\wedge\, a^2 - b^2 == 0 \,\wedge\, m \notin \mathbb{Z} \,\wedge\, n \notin \mathbb{Z}$

   1. $\int (a + b\,\mathrm{Sin}[e + f\,x])^m\,(c + d\,\mathrm{Sin}[e + f\,x])^n\,dx$ when $b\,c + a\,d == 0 \,\wedge\, a^2 - b^2 == 0 \,\wedge\, m + \frac{1}{2} \in \mathbb{Z}^+$

      1. $\int \sqrt{a + b\,\mathrm{Sin}[e + f\,x]}\,(c + d\,\mathrm{Sin}[e + f\,x])^n\,dx$ when $b\,c + a\,d == 0 \,\wedge\, a^2 - b^2 == 0$

        1: $\int \dfrac{\sqrt{a + b\,\mathrm{Sin}[e + f\,x]}}{\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}}\,dx$ when $b\,c + a\,d == 0 \,\wedge\, a^2 - b^2 == 0$

Derivation: Piecewise constant extraction

Basis: If $b\,c + a\,d == 0 \,\wedge\, a^2 - b^2 == 0$, then $\partial_x\,\dfrac{\mathrm{Cos}[e + f\,x]}{\sqrt{a + b\,\mathrm{Sin}[e + f\,x]}\,\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}} == 0$

Rule: If $b\,c + a\,d == 0 \,\wedge\, a^2 - b^2 == 0$, then

$$\int \frac{\sqrt{a + b\,\mathrm{Sin}[e + f\,x]}}{\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}}\,dx \;\longrightarrow\; \frac{a\,c\,\mathrm{Cos}[e + f\,x]}{\sqrt{a + b\,\mathrm{Sin}[e + f\,x]}\,\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}}\int \frac{\mathrm{Cos}[e + f\,x]}{c + d\,\mathrm{Sin}[e + f\,x]}\,dx$$

Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/Sqrt[c_+d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  a*c*Cos[e+f*x]/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]])*Int[Cos[e+f*x]/(c+d*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0]
```

        2: $\int \sqrt{a + b\,\mathrm{Sin}[e + f\,x]}\,(c + d\,\mathrm{Sin}[e + f\,x])^n\,dx$ when $b\,c + a\,d == 0 \,\wedge\, a^2 - b^2 == 0 \,\wedge\, n \ne -\frac{1}{2}$

Derivation: Doubly degenerate sine recurrence 1a with p → 0

Rule: If $b\,c + a\,d == 0 \,\wedge\, a^2 - b^2 == 0 \,\wedge\, n \ne -\frac{1}{2}$, then

$$\int \sqrt{a + b \, Sin[e + f \, x]} \, (c + d \, Sin[e + f \, x])^n \, dx \longrightarrow -\frac{2 \, b \, Cos[e + f \, x] \, (c + d \, Sin[e + f \, x])^n}{f \, (2 \, n + 1) \, \sqrt{a + b \, Sin[e + f \, x]}}$$

Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -2*b*Cos[e+f*x]*(c+d*Sin[e+f*x])^n/(f*(2*n+1)*Sqrt[a+b*Sin[e+f*x]]) /;
FreeQ[{a,b,c,d,e,f,n},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && NeQ[n,-1/2]
```

2. $\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^n \, dx$ when $b \, c + a \, d == 0 \; \wedge \; a^2 - b^2 == 0 \; \wedge \; m - \frac{1}{2} \in \mathbb{Z}^+$

  1: $\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^n \, dx$ when $b \, c + a \, d == 0 \; \wedge \; a^2 - b^2 == 0 \; \wedge \; m - \frac{1}{2} \in \mathbb{Z}^+ \; \wedge \; n < -1$

Derivation: Doubly degenerate sine recurrence 1a with $p \rightarrow 0$

Rule: If $b \, c + a \, d == 0 \; \wedge \; a^2 - b^2 == 0 \; \wedge \; m - \frac{1}{2} \in \mathbb{Z}^+ \; \wedge \; n < -1$, then

$$\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^n \, dx \longrightarrow$$

$$-\frac{2 \, b \, Cos[e + f \, x] \, (a + b \, Sin[e + f \, x])^{m-1} \, (c + d \, Sin[e + f \, x])^n}{f \, (2 \, n + 1)} - \frac{b \, (2 \, m - 1)}{d \, (2 \, n + 1)} \int (a + b \, Sin[e + f \, x])^{m-1} \, (c + d \, Sin[e + f \, x])^{n+1} \, dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -2*b*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m-1)*(c+d*Sin[e+f*x])^n/(f*(2*n+1)) -
  b*(2*m-1)/(d*(2*n+1))*Int[(a+b*Sin[e+f*x])^(m-1)*(c+d*Sin[e+f*x])^(n+1),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && IGtQ[m-1/2,0] && LtQ[n,-1] && Not[ILtQ[m+n,0] && GtQ[2*m+n+1,0]]
```

2: $\int \left(a + b\, Sin[e + f\, x]\right)^m \left(c + d\, Sin[e + f\, x]\right)^n dx$ when $b\, c + a\, d == 0 \wedge a^2 - b^2 == 0 \wedge m - \frac{1}{2} \in \mathbb{Z}^+ \wedge n \nless -1$

Derivation: Doubly degenerate sine recurrence 1b with p → 0

Rule: If $b\, c + a\, d == 0 \wedge a^2 - b^2 == 0 \wedge m - \frac{1}{2} \in \mathbb{Z}^+ \wedge n \nless -1$, then

$$\int \left(a + b\, Sin[e + f\, x]\right)^m \left(c + d\, Sin[e + f\, x]\right)^n dx \longrightarrow$$

$$-\frac{b\, Cos[e + f\, x]\, \left(a + b\, Sin[e + f\, x]\right)^{m-1} \left(c + d\, Sin[e + f\, x]\right)^n}{f\, (m + n)} + \frac{a\, (2\, m - 1)}{m + n} \int \left(a + b\, Sin[e + f\, x]\right)^{m-1} \left(c + d\, Sin[e + f\, x]\right)^n dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -b*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m-1)*(c+d*Sin[e+f*x])^n/(f*(m+n)) +
  a*(2*m-1)/(m+n)*Int[(a+b*Sin[e+f*x])^(m-1)*(c+d*Sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,n},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && IGtQ[m-1/2,0] && Not[LtQ[n,-1]] &&
  Not[IGtQ[n-1/2,0] && LtQ[n,m]] && Not[ILtQ[m+n,0] && GtQ[2*m+n+1,0]]
```

2. $\int \left(a + b\, Sin[e + f\, x]\right)^m \left(c + d\, Sin[e + f\, x]\right)^n dx$ when $b\, c + a\, d == 0 \wedge a^2 - b^2 == 0 \wedge m + n \in \mathbb{Z}^-$

 1. $\int \left(a + b\, Sin[e + f\, x]\right)^m \left(c + d\, Sin[e + f\, x]\right)^n dx$ when $b\, c + a\, d == 0 \wedge a^2 - b^2 == 0 \wedge m + n + 1 == 0$

 1: $\int \dfrac{1}{\sqrt{a + b\, Sin[e + f\, x]}\, \sqrt{c + d\, Sin[e + f\, x]}} dx$ when $b\, c + a\, d == 0 \wedge a^2 - b^2 == 0$

Derivation: Piecewise constant extraction

Basis: If $b\, c + a\, d == 0 \wedge a^2 - b^2 == 0$, then $\partial_x \dfrac{Cos[e + f\, x]}{\sqrt{a + b\, Sin[e + f\, x]}\, \sqrt{c + d\, Sin[e + f\, x]}} == 0$

Rule: If $b\, c + a\, d == 0 \wedge a^2 - b^2 == 0$, then

$$\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx \;\longrightarrow\; \frac{\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{c+d\,\text{Sin}[e+f\,x]}} \int \frac{1}{\text{Cos}[e+f\,x]}\,dx$$

Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[c_+d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  Cos[e+f*x]/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]])*Int[1/Cos[e+f*x],x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0]
```

2: $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c+a\,d == 0 \wedge a^2-b^2 == 0 \wedge m+n+1 == 0 \wedge m \neq -\frac{1}{2}$

Derivation: Doubly degenerate sine recurrence 1c with $n \to -m-1, \; p \to 0$

Rule: If $b\,c+a\,d == 0 \wedge a^2-b^2 == 0 \wedge m+n+1 == 0 \wedge m \neq -\frac{1}{2}$, then

$$\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx \;\longrightarrow\; \frac{b\,\text{Cos}[e+f\,x]\,(a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n}{a\,f\,(2\,m+1)}$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  b*Cos[e+f*x]*(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^n/(a*f*(2*m+1)) /;
FreeQ[{a,b,c,d,e,f,m,n},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && EqQ[m+n+1,0] && NeQ[m,-1/2]
```

2: $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c+a\,d == 0 \wedge a^2-b^2 == 0 \wedge m+n+1 \in \mathbb{Z}^- \wedge m \neq -\frac{1}{2}$

Derivation: Doubly degenerate sine recurrence 1c with $p \to 0$

Rule: If $b\,c+a\,d == 0 \wedge a^2-b^2 == 0 \wedge m+n+1 \in \mathbb{Z}^- \wedge m \neq -\frac{1}{2}$, then

$$\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx \;\longrightarrow$$

$$\frac{b \cos[e + f\,x]\,(a + b \sin[e + f\,x])^m\,(c + d \sin[e + f\,x])^n}{a\,f\,(2\,m + 1)} + \frac{m + n + 1}{a\,(2\,m + 1)} \int (a + b \sin[e + f\,x])^{m+1}\,(c + d \sin[e + f\,x])^n\,dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  b*Cos[e+f*x]*(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^n/(a*f*(2*m+1)) +
  (m+n+1)/(a*(2*m+1))*Int[(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && ILtQ[Simplify[m+n+1],0] && NeQ[m,-1/2] &&
  (SumSimplerQ[m,1] || Not[SumSimplerQ[n,1]])
```

3: $\int (a + b \sin[e + f\,x])^m\,(c + d \sin[e + f\,x])^n\,dx$ when $b\,c + a\,d == 0 \;\wedge\; a^2 - b^2 == 0 \;\wedge\; m < -1$

Derivation: Doubly degenerate sine recurrence 1c with p → 0

Rule: If $b\,c + a\,d == 0 \;\wedge\; a^2 - b^2 == 0 \;\wedge\; m < -1$, then

$$\int (a + b \sin[e + f\,x])^m\,(c + d \sin[e + f\,x])^n\,dx \;\longrightarrow$$

$$\frac{b \cos[e + f\,x]\,(a + b \sin[e + f\,x])^m\,(c + d \sin[e + f\,x])^n}{a\,f\,(2\,m + 1)} + \frac{m + n + 1}{a\,(2\,m + 1)} \int (a + b \sin[e + f\,x])^{m+1}\,(c + d \sin[e + f\,x])^n\,dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  b*Cos[e+f*x]*(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^n/(a*f*(2*m+1)) +
  (m+n+1)/(a*(2*m+1))*Int[(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,n},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && LtQ[m,-1] && Not[LtQ[m,n,-1]] && IntegersQ[2*m,2*n]
```

4: $\int \left(a + b\,\text{Sin}[e + f\,x]\right)^m \left(c + d\,\text{Sin}[e + f\,x]\right)^n \,dx$ when $b\,c + a\,d == 0 \,\wedge\, a^2 - b^2 == 0 \,\wedge\, m \notin \mathbb{Z} \,\wedge\, n \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: If $b\,c + a\,d == 0 \,\wedge\, a^2 - b^2 == 0$, then $\partial_x \dfrac{\left(a + b\,\text{Sin}[e+f\,x]\right)^m \left(c + d\,\text{Sin}[e+f\,x]\right)^m}{\text{Cos}[e+f\,x]^{2\,m}} == 0$

Rule: If $b\,c + a\,d == 0 \,\wedge\, a^2 - b^2 == 0 \,\wedge\, m \notin \mathbb{Z} \,\wedge\, n \notin \mathbb{Z}$, then

$$\int \left(a + b\,\text{Sin}[e + f\,x]\right)^m \left(c + d\,\text{Sin}[e + f\,x]\right)^n \,dx \;\longrightarrow$$

$$\left(\left(a^{\text{IntPart}[m]}\, c^{\text{IntPart}[m]} \left(a + b\,\text{Sin}[e + f\,x]\right)^{\text{FracPart}[m]} \left(c + d\,\text{Sin}[e + f\,x]\right)^{\text{FracPart}[m]}\right) \big/ \text{Cos}[e + f\,x]^{2\,\text{FracPart}[m]}\right) \int \text{Cos}[e + f\,x]^{2\,m} \left(c + d\,\text{Sin}[e + f\,x]\right)^{n-m} \,dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  a^IntPart[m]*c^IntPart[m]*(a+b*Sin[e+f*x])^FracPart[m]*(c+d*Sin[e+f*x])^FracPart[m]/Cos[e+f*x]^(2*FracPart[m])*
    Int[Cos[e+f*x]^(2*m)*(c+d*Sin[e+f*x])^(n-m),x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && (FractionQ[m] || Not[FractionQ[n]])
```

4: $\displaystyle\int \frac{\left(a + b\, \text{Sin}\left[e + f\, x\right]\right)^2}{c + d\, \text{Sin}\left[e + f\, x\right]}\, dx$ when $\mathbf{b\, c - a\, d \neq 0}$

Derivation: Algebraic expansion

Basis: $\displaystyle\frac{(a+b\,z)^2}{c+d\,z} == \frac{b^2\,z}{d} + \frac{a^2\,d - b\,(b\,c - 2\,a\,d)\,z}{d\,(c+d\,z)}$

Rule: If $b\, c - a\, d \neq 0$, then

$$\int \frac{\left(a + b\, \text{Sin}\left[e + f\, x\right]\right)^2}{c + d\, \text{Sin}\left[e + f\, x\right]}\, dx \;\longrightarrow\; -\frac{b^2\, \text{Cos}\left[e + f\, x\right]}{d\, f} + \frac{1}{d}\int \frac{a^2\, d - b\,(b\, c - 2\, a\, d)\, \text{Sin}\left[e + f\, x\right]}{c + d\, \text{Sin}\left[e + f\, x\right]}\, dx$$

Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^2/(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -b^2*Cos[e+f*x]/(d*f) + 1/d*Int[Simp[a^2*d-b*(b*c-2*a*d)*Sin[e+f*x],x]/(c+d*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0]
```

5: $\displaystyle\int \frac{1}{(a + b\,\text{Sin}[e + f\,x])\,(c + d\,\text{Sin}[e + f\,x])}\,dx$ when $b\,c - a\,d \neq 0$

Derivation: Algebraic expansion

Basis: $\displaystyle\frac{1}{(a + b\,z)\,(c + d\,z)} == \frac{b}{(b\,c - a\,d)\,(a + b\,z)} - \frac{d}{(b\,c - a\,d)\,(c + d\,z)}$

Rule: If $b\,c - a\,d \neq 0$, then

$$\int \frac{1}{(a + b\,\text{Sin}[e + f\,x])\,(c + d\,\text{Sin}[e + f\,x])}\,dx \longrightarrow \frac{b}{b\,c - a\,d}\int \frac{1}{a + b\,\text{Sin}[e + f\,x]}\,dx - \frac{d}{b\,c - a\,d}\int \frac{1}{c + d\,\text{Sin}[e + f\,x]}\,dx$$

Program code:

```
Int[1/((a_.+b_.*sin[e_.+f_.*x_])*(c_.+d_.*sin[e_.+f_.*x_])),x_Symbol] :=
  b/(b*c-a*d)*Int[1/(a+b*Sin[e+f*x]),x] - d/(b*c-a*d)*Int[1/(c+d*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0]
```

6. $\int (a + b\, Sin[e + f\, x])^m\ (c + d\, Sin[e + f\, x])\ dx$ when $b\, c - a\, d \neq 0$

   1: $\int (b\, Sin[e + f\, x])^m\ (c + d\, Sin[e + f\, x])\ dx$

   ## Derivation: Algebraic expansion

   ## Rule: If $b\, c - a\, d \neq 0\ \wedge\ a^2 - b^2 \neq 0$, then

   $$\int (b\, Sin[e + f\, x])^m\ (c + d\, Sin[e + f\, x])\ dx\ \longrightarrow\ c \int (b\, Sin[e + f\, x])^m\ dx + \frac{d}{b} \int (b\, Sin[e + f\, x])^{m+1}\ dx$$

   ## Program code:

```
Int[(b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  c*Int[(b*Sin[e+f*x])^m,x] + d/b*Int[(b*Sin[e+f*x])^(m+1),x] /;
FreeQ[{b,c,d,e,f,m},x]
```

2. $\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x]) \, dx$ when $b \, c - a \, d \neq 0 \land a^2 - b^2 == 0$

   1: $\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x]) \, dx$ when $b \, c - a \, d \neq 0 \land a^2 - b^2 == 0 \land a \, d \, m + b \, c \, (m + 1) == 0$

Derivation: Singly degenerate sine recurrence 2a with $A \to -\dfrac{a \, d \, m}{b \, (m+1)}$, $B \to d$, $n \to 0$, $p \to 0$

Derivation: Singly degenerate sine recurrence 2c with $A \to -\dfrac{a \, d \, m}{b \, (m+1)}$, $B \to d$, $n \to 0$, $p \to 0$

Note: If $a^2 - b^2 == 0 \land a \, d \, m + b \, c \, (m + 1) == 0$, then $m + 1 \neq 0$.

Rule: If $b \, c - a \, d \neq 0 \land a^2 - b^2 == 0 \land a \, d \, m + b \, c \, (m + 1) == 0$, then

$$\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x]) \, dx \; \longrightarrow \; -\frac{d \, \text{Cos}[e + f \, x] \, (a + b \, \text{Sin}[e + f \, x])^m}{f \, (m + 1)}$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -d*Cos[e+f*x]*(a+b*Sin[e+f*x])^m/(f*(m+1)) /;
FreeQ[{a,b,c,d,e,f,m},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && EqQ[a*d*m+b*c*(m+1),0]
```

2: $\int (a + b\,\text{Sin}[e + f\,x])^m\,(c + d\,\text{Sin}[e + f\,x])\,dx$ when $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge m < -\frac{1}{2}$

Derivation: Singly degenerate sine recurrence 2a with $A \to c$, $B \to d$, $n \to 0$, $p \to 0$

Rule: If $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge m < -\frac{1}{2}$, then

$$\int (a + b\,\text{Sin}[e + f\,x])^m\,(c + d\,\text{Sin}[e + f\,x])\,dx \;\longrightarrow$$

$$\frac{(b\,c - a\,d)\,\text{Cos}[e + f\,x]\,(a + b\,\text{Sin}[e + f\,x])^m}{a\,f\,(2\,m + 1)} + \frac{a\,d\,m + b\,c\,(m + 1)}{a\,b\,(2\,m + 1)} \int (a + b\,\text{Sin}[e + f\,x])^{m+1}\,dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  (b*c-a*d)*Cos[e+f*x]*(a+b*Sin[e+f*x])^m/(a*f*(2*m+1)) +
  (a*d*m+b*c*(m+1))/(a*b*(2*m+1))*Int[(a+b*Sin[e+f*x])^(m+1),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && LtQ[m,-1/2]
```

3: $\int (a + b\, Sin[e + f\, x])^m\, (c + d\, Sin[e + f\, x])\, dx$ when $b\, c - a\, d \neq 0 \wedge a^2 - b^2 == 0 \wedge m \not< -\frac{1}{2}$

Derivation: Singly degenerate sine recurrence 2c with $A \to c$, $B \to d$, $n \to 0$, $p \to 0$

Rule: If $b\, c - a\, d \neq 0 \wedge a^2 - b^2 == 0 \wedge m \not< -\frac{1}{2}$, then

$$\int (a + b\, Sin[e + f\, x])^m\, (c + d\, Sin[e + f\, x])\, dx \longrightarrow$$

$$-\frac{d\, Cos[e + f\, x]\, (a + b\, Sin[e + f\, x])^m}{f\, (m + 1)} + \frac{a\, d\, m + b\, c\, (m + 1)}{b\, (m + 1)} \int (a + b\, Sin[e + f\, x])^m\, dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -d*Cos[e+f*x]*(a+b*Sin[e+f*x])^m/(f*(m+1)) +
  (a*d*m+b*c*(m+1))/(b*(m+1))*Int[(a+b*Sin[e+f*x])^m,x] /;
FreeQ[{a,b,c,d,e,f,m},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && Not[LtQ[m,-1/2]]
```

3. $\int (a + b\, Sin[e + f\, x])^m\, (c + d\, Sin[e + f\, x])\, dx$ when $b\, c - a\, d \neq 0 \wedge a^2 - b^2 \neq 0$

  1. $\int (a + b\, Sin[e + f\, x])^m\, (c + d\, Sin[e + f\, x])\, dx$ when $b\, c - a\, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge 2\, m \in \mathbb{Z}$

    1: $\int \frac{c + d\, Sin[e + f\, x]}{\sqrt{a + b\, Sin[e + f\, x]}}\, dx$ when $b\, c - a\, d \neq 0 \wedge a^2 - b^2 \neq 0$

Derivation: Algebraic expansion

Basis: $c + d\, z == \frac{b\, c - a\, d}{b} + \frac{d}{b}\, (a + b\, z)$

Rule: If $b\, c - a\, d \neq 0 \wedge a^2 - b^2 \neq 0$, then

$$\int \frac{c + d \, \text{Sin}[e + f \, x]}{\sqrt{a + b \, \text{Sin}[e + f \, x]}} \, dx \;\longrightarrow\; \frac{b \, c - a \, d}{b} \int \frac{1}{\sqrt{a + b \, \text{Sin}[e + f \, x]}} \, dx + \frac{d}{b} \int \sqrt{a + b \, \text{Sin}[e + f \, x]} \; dx$$

Program code:

```
Int[(c_.+d_.*sin[e_.+f_.*x_])/Sqrt[a_+b_.*sin[e_.+f_.*x_]],x_Symbol] :=
  (b*c-a*d)/b*Int[1/Sqrt[a+b*Sin[e+f*x]],x] + d/b*Int[Sqrt[a+b*Sin[e+f*x]],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0]
```

2: $\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x]) \; dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge m > 0 \wedge 2 \, m \in \mathbb{Z}$

Reference: G&R 2.551.1 inverted

Derivation: Nondegenerate sine recurrence 1b with $A \to a \, c$, $B \to b \, c + a \, d$, $C \to b \, d$, $m \to 0$, $n \to n - 1$, $p \to 0$

Rule: If $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge m > 0 \wedge 2 \, m \in \mathbb{Z}$, then

$$\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x]) \; dx \;\longrightarrow$$

$$-\frac{d \, \text{Cos}[e + f \, x] \, (a + b \, \text{Sin}[e + f \, x])^m}{f \, (m + 1)} + \frac{1}{m + 1} \int (a + b \, \text{Sin}[e + f \, x])^{m-1} \, \big(b \, d \, m + a \, c \, (m + 1) + (a \, d \, m + b \, c \, (m + 1)) \, \text{Sin}[e + f \, x] \big) \; dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -d*Cos[e+f*x]*(a+b*Sin[e+f*x])^m/(f*(m+1)) +
   1/(m+1)*Int[(a+b*Sin[e+f*x])^(m-1)*Simp[b*d*m+a*c*(m+1)+(a*d*m+b*c*(m+1))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && GtQ[m,0] && IntegerQ[2*m]
```

3: $\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x]) \, dx$ when $b \, c - a \, d \ne 0 \,\wedge\, a^2 - b^2 \ne 0 \,\wedge\, m < -1 \,\wedge\, 2 \, m \in \mathbb{Z}$

Reference: G&R 2.551.1

Derivation: Nondegenerate sine recurrence 1a with $A \to c$, $B \to d$, $C \to 0$, $n \to 0$, $p \to 0$

Rule: If $b \, c - a \, d \ne 0 \,\wedge\, a^2 - b^2 \ne 0 \,\wedge\, m < -1 \,\wedge\, 2 \, m \in \mathbb{Z}$, then

$$\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x]) \, dx \longrightarrow$$

$$-\frac{(b \, c - a \, d) \, Cos[e + f \, x] \, (a + b \, Sin[e + f \, x])^{m+1}}{f \, (m + 1) \, (a^2 - b^2)} + \frac{1}{(m + 1) \, (a^2 - b^2)} \int (a + b \, Sin[e + f \, x])^{m+1} \, ((a \, c - b \, d) \, (m + 1) - (b \, c - a \, d) \, (m + 2) \, Sin[e + f \, x]) \, dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -(b*c-a*d)*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+1)/(f*(m+1)*(a^2-b^2)) +
  1/((m+1)*(a^2-b^2))*Int[(a+b*Sin[e+f*x])^(m+1)*Simp[(a*c-b*d)*(m+1)-(b*c-a*d)*(m+2)*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && LtQ[m,-1] && IntegerQ[2*m]
```

2. $\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x]) \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge 2 \, m \notin \mathbb{Z}$

   1: $\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x]) \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge 2 \, m \notin \mathbb{Z} \wedge c^2 - d^2 == 0$

Derivation: Piecewise constant extraction and integration by substitution

Basis: $\partial_x \dfrac{\text{Cos}[e + f \, x]}{\sqrt{1 + \text{Sin}[e + f \, x]} \, \sqrt{1 - \text{Sin}[e + f \, x]}} == 0$

Basis: $\text{Cos}[e + f \, x] == \dfrac{1}{f} \, \partial_x \, \text{Sin}[e + f \, x]$

Rule: If $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge 2 \, m \notin \mathbb{Z} \wedge c^2 - d^2 == 0$, then

$$\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x]) \, dx \longrightarrow$$

$$\dfrac{c \, \text{Cos}[e + f \, x]}{\sqrt{1 + \text{Sin}[e + f \, x]} \, \sqrt{1 - \text{Sin}[e + f \, x]}} \int \dfrac{\text{Cos}[e + f \, x] \, (a + b \, \text{Sin}[e + f \, x])^m \, \sqrt{1 + \frac{d}{c} \, \text{Sin}[e + f \, x]}}{\sqrt{1 - \frac{d}{c} \, \text{Sin}[e + f \, x]}} \, dx \longrightarrow$$

$$\dfrac{c \, \text{Cos}[e + f \, x]}{f \, \sqrt{1 + \text{Sin}[e + f \, x]} \, \sqrt{1 - \text{Sin}[e + f \, x]}} \, \text{Subst}\left[\int \dfrac{(a + b \, x)^m \, \sqrt{1 + \frac{d}{c} \, x}}{\sqrt{1 - \frac{d}{c} \, x}} \, dx, \, x, \, \text{Sin}[e + f \, x] \right]$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  c*Cos[e+f*x]/(f*Sqrt[1+Sin[e+f*x]]*Sqrt[1-Sin[e+f*x]])*Subst[Int[(a+b*x)^m*Sqrt[1+d/c*x]/Sqrt[1-d/c*x],x],x,Sin[e+f*x]] /;
FreeQ[{a,b,c,d,e,f,m},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && Not[IntegerQ[2*m]] && EqQ[c^2-d^2,0]
```

2: $\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x]) \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge 2 \, m \notin \mathbb{Z} \wedge c^2 - d^2 \neq 0$

Derivation: Algebraic expansion

Basis: $c + d \, z == \frac{b \, c - a \, d}{b} + \frac{d}{b} \, (a + b \, z)$

Rule: If $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0$, then

$$\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x]) \, dx \; \rightarrow \; \frac{b \, c - a \, d}{b} \int (a + b \, \text{Sin}[e + f \, x])^m \, dx + \frac{d}{b} \int (a + b \, \text{Sin}[e + f \, x])^{m+1} \, dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  (b*c-a*d)/b*Int[(a+b*Sin[e+f*x])^m,x] + d/b*Int[(a+b*Sin[e+f*x])^(m+1),x] /;
FreeQ[{a,b,c,d,e,f,m},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0]
```

7. $\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x])^n \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0$

    1: $\int (a + b \, \text{Sin}[e + f \, x])^m \, (d \, \text{Sin}[e + f \, x])^n \, dx$ when $a^2 - b^2 == 0 \wedge m \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $a^2 - b^2 == 0 \wedge m \in \mathbb{Z}^+$, then

$$\int (a + b \, \text{Sin}[e + f \, x])^m \, (d \, \text{Sin}[e + f \, x])^n \, dx \longrightarrow \int \text{ExpandTrig}\big[(a + b \, \text{Sin}[e + f \, x])^m \, (d \, \text{Sin}[e + f \, x])^n, x\big] \, dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_.*(d_.*sin[e_.+f_.*x_])^n_.,x_Symbol] :=
  Int[ExpandTrig[(a+b*sin[e+f*x])^m*(d*sin[e+f*x])^n,x],x] /;
FreeQ[{a,b,d,e,f,n},x] && EqQ[a^2-b^2,0] && IGtQ[m,0] && RationalQ[n]
```

2. $\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x])^2 \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0$

    1. $\int \text{Sin}[e + f \, x]^2 \, (a + b \, \text{Sin}[e + f \, x])^m \, dx$ when $a^2 - b^2 == 0$

        1: $\int \text{Sin}[e + f \, x]^2 \, (a + b \, \text{Sin}[e + f \, x])^m \, dx$ when $a^2 - b^2 == 0 \wedge m < -\frac{1}{2}$

Derivation: ???

Rule: If $a^2 - b^2 == 0 \wedge m < -\frac{1}{2}$, then

$$\int \text{Sin}[e + f \, x]^2 \, (a + b \, \text{Sin}[e + f \, x])^m \, dx \;\longrightarrow$$

$$\frac{b \, \text{Cos}[e + f \, x] \, (a + b \, \text{Sin}[e + f \, x])^m}{a \, f \, (2 \, m + 1)} - \frac{1}{a^2 \, (2 \, m + 1)} \int (a + b \, \text{Sin}[e + f \, x])^{m+1} \, (a \, m - b \, (2 \, m + 1) \, \text{Sin}[e + f \, x]) \, dx$$

Program code:

```
Int[sin[e_.+f_.*x_]^2*(a_+b_.*sin[e_.+f_.*x_])^m_,x_Symbol] :=
  b*Cos[e+f*x]*(a+b*Sin[e+f*x])^m/(a*f*(2*m+1)) -
  1/(a^2*(2*m+1))*Int[(a+b*Sin[e+f*x])^(m+1)*(a*m-b*(2*m+1)*Sin[e+f*x]),x] /;
FreeQ[{a,b,e,f},x] && EqQ[a^2-b^2,0] && LtQ[m,-1/2]
```

2: $\int Sin[e+f\,x]^2\,(a+b\,Sin[e+f\,x])^m\,dx$ when $a^2-b^2==0 \wedge m \nless -\frac{1}{2}$

Derivation: Nondegenerate sine recurrence 1b with $A \to a^2,\ B \to 2\,a\,b,\ C \to b^2,\ m \to 0,\ p \to 0$

Rule: If $a^2-b^2==0 \wedge m \nless -\frac{1}{2}$, then

$$\int Sin[e+f\,x]^2\,(a+b\,Sin[e+f\,x])^m\,dx \ \longrightarrow$$

$$-\frac{Cos[e+f\,x]\,(a+b\,Sin[e+f\,x])^{m+1}}{b\,f\,(m+2)} + \frac{1}{b\,(m+2)} \int (a+b\,Sin[e+f\,x])^m\,(b\,(m+1)-a\,Sin[e+f\,x])\,dx$$

Program code:

```
Int[sin[e_.+f_.*x_]^2*(a_+b_.*sin[e_.+f_.*x_])^m_,x_Symbol] :=
  -Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+1)/(b*f*(m+2)) +
  1/(b*(m+2))*Int[(a+b*Sin[e+f*x])^m*(b*(m+1)-a*Sin[e+f*x]),x] /;
FreeQ[{a,b,e,f,m},x] && EqQ[a^2-b^2,0] && Not[LtQ[m,-1/2]]
```

2. $\int \left(a + b \, \text{Sin}[e + f \, x]\right)^m \left(c + d \, \text{Sin}[e + f \, x]\right)^2 dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0$

  1: $\int \left(a + b \, \text{Sin}[e + f \, x]\right)^m \left(c + d \, \text{Sin}[e + f \, x]\right)^2 dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0 \wedge m < -1$

Derivation: Singly degenerate sine recurrence 2a with $A \rightarrow c$, $B \rightarrow d$, $n \rightarrow 1$, $p \rightarrow 0$

Rule: If $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0 \wedge m < -1$, then

$$\int \left(a + b \, \text{Sin}[e + f \, x]\right)^m \left(c + d \, \text{Sin}[e + f \, x]\right)^2 dx \longrightarrow$$

$$\frac{(b \, c - a \, d) \, \text{Cos}[e + f \, x] \, \left(a + b \, \text{Sin}[e + f \, x]\right)^m \left(c + d \, \text{Sin}[e + f \, x]\right)}{a \, f \, (2 \, m + 1)} +$$

$$\frac{1}{a \, b \, (2 \, m + 1)} \int \left(a + b \, \text{Sin}[e + f \, x]\right)^{m+1} \left(a \, c \, d \, (m - 1) + b \, \left(d^2 + c^2 \, (m + 1)\right) + d \, \left(a \, d \, (m - 1) + b \, c \, (m + 2)\right) \, \text{Sin}[e + f \, x]\right) dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^2,x_Symbol] :=
  (b*c-a*d)*Cos[e+f*x]*(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])/(a*f*(2*m+1)) +
  1/(a*b*(2*m+1))*Int[(a+b*Sin[e+f*x])^(m+1)*Simp[a*c*d*(m-1)+b*(d^2+c^2*(m+1))+d*(a*d*(m-1)+b*c*(m+2))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && LtQ[m,-1]
```

  2: $\int \left(a + b \, \text{Sin}[e + f \, x]\right)^m \left(c + d \, \text{Sin}[e + f \, x]\right)^2 dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0 \wedge m \not< -1$

Derivation: Nondegenerate sine recurrence 1b with $A \rightarrow a^2$, $B \rightarrow 2 \, a \, b$, $C \rightarrow b^2$, $m \rightarrow 0$, $p \rightarrow 0$

Rule: If $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0 \wedge m \not< -1$, then

$$\int \left(a + b \, \text{Sin}[e + f \, x]\right)^m \left(c + d \, \text{Sin}[e + f \, x]\right)^2 dx \longrightarrow$$

$$-\frac{d^2 \, \text{Cos}[e + f \, x] \, \left(a + b \, \text{Sin}[e + f \, x]\right)^{m+1}}{b \, f \, (m + 2)} +$$

$$\frac{1}{b\,(m+2)}\int\left(a+b\,\text{Sin}\left[e+f\,x\right]\right)^{m}\left(b\,\left(d^{2}\,(m+1)+c^{2}\,(m+2)\right)-d\,\left(a\,d-2\,b\,c\,(m+2)\right)\,\text{Sin}\left[e+f\,x\right]\right)dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^2,x_Symbol] :=
  -d^2*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+1)/(b*f*(m+2)) +
  1/(b*(m+2))*Int[(a+b*Sin[e+f*x])^m*Simp[b*(d^2*(m+1)+c^2*(m+2))-d*(a*d-2*b*c*(m+2))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,m},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && Not[LtQ[m,-1]]
```

3. $\int\left(a+b\,\text{Sin}\left[e+f\,x\right]\right)^{m}\left(c+d\,\text{Sin}\left[e+f\,x\right]\right)^{n}dx$ when $b\,c-a\,d\neq0\,\wedge\,a^{2}-b^{2}==0\,\wedge\,c^{2}-d^{2}\neq0\,\wedge\,m>1$

    1: $\int\left(a+b\,\text{Sin}\left[e+f\,x\right]\right)^{m}\left(c+d\,\text{Sin}\left[e+f\,x\right]\right)^{n}dx$ when $b\,c-a\,d\neq0\,\wedge\,a^{2}-b^{2}==0\,\wedge\,c^{2}-d^{2}\neq0\,\wedge\,m>1\,\wedge\,n<-1$

Derivation: Singly degenerate sine recurrence 1a with A → a, B → b, m → m − 1, p → 0

Rule: If $b\,c-a\,d\neq0\,\wedge\,a^{2}-b^{2}==0\,\wedge\,c^{2}-d^{2}\neq0\,\wedge\,m>1\,\wedge\,n<-1$, then

$$\int\left(a+b\,\text{Sin}\left[e+f\,x\right]\right)^{m}\left(c+d\,\text{Sin}\left[e+f\,x\right]\right)^{n}dx\,\longrightarrow$$

$$-\left(\left(b^{2}\,(b\,c-a\,d)\,\text{Cos}\left[e+f\,x\right]\,\left(a+b\,\text{Sin}\left[e+f\,x\right]\right)^{m-2}\,\left(c+d\,\text{Sin}\left[e+f\,x\right]\right)^{n+1}\right)/\left(d\,f\,(n+1)\,(b\,c+a\,d)\right)\right)+$$

$$\frac{b^{2}}{d\,(n+1)\,(b\,c+a\,d)}\int\left(a+b\,\text{Sin}\left[e+f\,x\right]\right)^{m-2}\left(c+d\,\text{Sin}\left[e+f\,x\right]\right)^{n+1}\left(a\,c\,(m-2)-b\,d\,(m-2\,n-4)-\left(b\,c\,(m-1)-a\,d\,(m+2\,n+1)\right)\,\text{Sin}\left[e+f\,x\right]\right)dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -b^2*(b*c-a*d)*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m-2)*(c+d*Sin[e+f*x])^(n+1)/(d*f*(n+1)*(b*c+a*d)) +
  b^2/(d*(n+1)*(b*c+a*d))*Int[(a+b*Sin[e+f*x])^(m-2)*(c+d*Sin[e+f*x])^(n+1)*
    Simp[a*c*(m-2)-b*d*(m-2*n-4)-(b*c*(m-1)-a*d*(m+2*n+1))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[m,1] && LtQ[n,-1] &&
  (IntegersQ[2*m,2*n] || IntegerQ[m+1/2] || IntegerQ[m] && EqQ[c,0])
```

2: $\int \left(a + b \, \text{Sin}[e + f \, x]\right)^m \left(c + d \, \text{Sin}[e + f \, x]\right)^n \, dx$ when $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge m > 1 \wedge n \nleqslant -1$

Derivation: Singly degenerate sine recurrence 1b with $A \rightarrow a$, $B \rightarrow b$, $m \rightarrow m - 1$, $p \rightarrow 0$

Rule: If $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge m > 1 \wedge n \nleqslant -1$, then

$$\int \left(a + b \, \text{Sin}[e + f \, x]\right)^m \left(c + d \, \text{Sin}[e + f \, x]\right)^n \, dx \; \rightarrow$$

$$-\frac{b^2 \, \text{Cos}[e + f \, x] \, \left(a + b \, \text{Sin}[e + f \, x]\right)^{m-2} \left(c + d \, \text{Sin}[e + f \, x]\right)^{n+1}}{d \, f \, (m + n)} \; +$$

$$\frac{1}{d \, (m + n)} \int \left(a + b \, \text{Sin}[e + f \, x]\right)^{m-2} \left(c + d \, \text{Sin}[e + f \, x]\right)^n \cdot$$

$$\left(a \, b \, c \, (m - 2) + b^2 \, d \, (n + 1) + a^2 \, d \, (m + n) - b \, \left(b \, c \, (m - 1) - a \, d \, (3 \, m + 2 \, n - 2)\right) \, \text{Sin}[e + f \, x]\right) \, dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -b^2*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m-2)*(c+d*Sin[e+f*x])^(n+1)/(d*f*(m+n)) +
  1/(d*(m+n))*Int[(a+b*Sin[e+f*x])^(m-2)*(c+d*Sin[e+f*x])^n*
    Simp[a*b*c*(m-2)+b^2*d*(n+1)+a^2*d*(m+n)-b*(b*c*(m-1)-a*d*(3*m+2*n-2))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,n},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[m,1] && Not[LtQ[n,-1]] &&
  (IntegersQ[2*m,2*n] || IntegerQ[m+1/2] || IntegerQ[m] && EqQ[c,0])
```

4. $\int \left(a + b \, \text{Sin}[e + f \, x]\right)^m \left(c + d \, \text{Sin}[e + f \, x]\right)^n \, dx$ when $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge m < -1$

   1. $\int \left(a + b \, \text{Sin}[e + f \, x]\right)^m \left(c + d \, \text{Sin}[e + f \, x]\right)^n \, dx$ when $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge m < -1 \wedge n > 0$

      1: $\int \left(a + b \, \text{Sin}[e + f \, x]\right)^m \left(c + d \, \text{Sin}[e + f \, x]\right)^n \, dx$ when $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge m < -1 \wedge 0 < n < 1$

Derivation: Singly degenerate sine recurrence 2a with $A \rightarrow 1$, $B \rightarrow 0$, $p \rightarrow 0$

Rule: If $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge m < -1 \wedge 0 < n < 1$, then

$$\int \left(a + b \, \text{Sin}[e + f \, x]\right)^m \left(c + d \, \text{Sin}[e + f \, x]\right)^n \, dx \; \longrightarrow$$

$$\frac{b \, \text{Cos}[e + f \, x] \, \left(a + b \, \text{Sin}[e + f \, x]\right)^m \left(c + d \, \text{Sin}[e + f \, x]\right)^n}{a \, f \, (2 \, m + 1)} \, -$$

$$\frac{1}{a \, b \, (2 \, m + 1)} \int \left(a + b \, \text{Sin}[e + f \, x]\right)^{m+1} \left(c + d \, \text{Sin}[e + f \, x]\right)^{n-1} \left(a \, d \, n - b \, c \, (m + 1) - b \, d \, (m + n + 1) \, \text{Sin}[e + f \, x]\right) \, dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  b*Cos[e+f*x]*(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^n/(a*f*(2*m+1)) -
  1/(a*b*(2*m+1))*Int[(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^(n-1)*Simp[a*d*n-b*c*(m+1)-b*d*(m+n+1)*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[m,-1] && LtQ[0,n,1] &&
  (IntegersQ[2*m,2*n] || IntegerQ[m] && EqQ[c,0])
```

2: $\int \left(a + b\, \mathrm{Sin}[e + f\, x]\right)^m \left(c + d\, \mathrm{Sin}[e + f\, x]\right)^n dx$ when $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge m < -1 \wedge n > 1$

Derivation: Singly degenerate sine recurrence 2a with $A \to c$, $B \to d$, $n \to n - 1$, $p \to 0$

Rule: If $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge m < -1 \wedge n > 1$, then

$$\int \left(a + b\, \mathrm{Sin}[e + f\, x]\right)^m \left(c + d\, \mathrm{Sin}[e + f\, x]\right)^n dx \; \to$$

$$\frac{(b\,c - a\,d)\, \mathrm{Cos}[e + f\, x]\, \left(a + b\, \mathrm{Sin}[e + f\, x]\right)^m \left(c + d\, \mathrm{Sin}[e + f\, x]\right)^{n-1}}{a\,f\,(2\,m + 1)} +$$

$$\frac{1}{a\,b\,(2\,m+1)} \int \left(a + b\, \mathrm{Sin}[e + f\, x]\right)^{m+1} \left(c + d\, \mathrm{Sin}[e + f\, x]\right)^{n-2} \left(b\left(c^2\,(m+1) + d^2\,(n-1)\right) + a\,c\,d\,(m-n+1) + d\left(a\,d\,(m-n+1) + b\,c\,(m+n)\right) \mathrm{Sin}[e + f\, x]\right) dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  (b*c-a*d)*Cos[e+f*x]*(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^(n-1)/(a*f*(2*m+1)) +
  1/(a*b*(2*m+1))*Int[(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^(n-2)*
    Simp[b*(c^2*(m+1)+d^2*(n-1))+a*c*d*(m-n+1)+d*(a*d*(m-n+1)+b*c*(m+n))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[m,-1] && GtQ[n,1] &&
  (IntegersQ[2*m,2*n] || IntegerQ[m] && EqQ[c,0])
```

2: $\int \left(a + b\, \mathrm{Sin}[e + f\, x]\right)^m \left(c + d\, \mathrm{Sin}[e + f\, x]\right)^n dx$ when $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge m < -1 \wedge n \not> 0$

Derivation: Singly degenerate sine recurrence 2b with $A \to 1$, $B \to 0$, $p \to 0$

Rule: If $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge m < -1 \wedge n \not> 0$, then

$$\int \left(a + b\, \mathrm{Sin}[e + f\, x]\right)^m \left(c + d\, \mathrm{Sin}[e + f\, x]\right)^n dx \; \to$$

$$\frac{b^2\, \mathrm{Cos}[e + f\, x]\, \left(a + b\, \mathrm{Sin}[e + f\, x]\right)^m \left(c + d\, \mathrm{Sin}[e + f\, x]\right)^{n+1}}{a\,f\,(2\,m + 1)\,(b\,c - a\,d)} +$$

$$\frac{1}{a\,(2\,m+1)\,(b\,c-a\,d)}\int(a+b\,Sin[e+f\,x])^{m+1}\,(c+d\,Sin[e+f\,x])^{n}\,(b\,c\,(m+1)-a\,d\,(2\,m+n+2)+b\,d\,(m+n+2)\,Sin[e+f\,x])\,dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  b^2*Cos[e+f*x]*(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^(n+1)/(a*f*(2*m+1)*(b*c-a*d)) +
  1/(a*(2*m+1)*(b*c-a*d))*Int[(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^n*
    Simp[b*c*(m+1)-a*d*(2*m+n+2)+b*d*(m+n+2)*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,n},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[m,-1] && Not[GtQ[n,0]] &&
  (IntegersQ[2*m,2*n] || IntegerQ[m] && EqQ[c,0])
```

5. $\int\dfrac{(c+d\,Sin[e+f\,x])^{n}}{a+b\,Sin[e+f\,x]}\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq 0$

    1: $\int\dfrac{(c+d\,Sin[e+f\,x])^{n}}{a+b\,Sin[e+f\,x]}\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq 0\,\wedge\,n>1$

Derivation: Singly degenerate sine recurrence 2a with $A\to c$, $B\to d$, $m\to-1$, $n\to n-1$, $p\to 0$

Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq 0\,\wedge\,n>1$, then

$$\int\frac{(c+d\,Sin[e+f\,x])^{n}}{a+b\,Sin[e+f\,x]}\,dx\longrightarrow$$

$$-\frac{(b\,c-a\,d)\,Cos[e+f\,x]\,(c+d\,Sin[e+f\,x])^{n-1}}{a\,f\,(a+b\,Sin[e+f\,x])}-\frac{d}{a\,b}\int(c+d\,Sin[e+f\,x])^{n-2}\,(b\,d\,(n-1)-a\,c\,n+(b\,c\,(n-1)-a\,d\,n)\,Sin[e+f\,x]\,)\,dx$$

Program code:

```
Int[(c_.+d_.*sin[e_.+f_.*x_])^n_/(a_+b_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -(b*c-a*d)*Cos[e+f*x]*(c+d*Sin[e+f*x])^(n-1)/(a*f*(a+b*Sin[e+f*x])) -
  d/(a*b)*Int[(c+d*Sin[e+f*x])^(n-2)*Simp[b*d*(n-1)-a*c*n+(b*c*(n-1)-a*d*n)*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[n,1] && (IntegerQ[2*n] || EqQ[c,0])
```

2: $\displaystyle\int \frac{(c + d\,\mathrm{Sin}[e + f\,x])^n}{a + b\,\mathrm{Sin}[e + f\,x]}\,dx$ when $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge n < 0$

Derivation: Singly degenerate sine recurrence 2b with $A \to 1$, $B \to 0$, $m \to -1$, $p \to 0$

Rule: If $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge n < 0$, then

$$\int \frac{(c + d\,\mathrm{Sin}[e + f\,x])^n}{a + b\,\mathrm{Sin}[e + f\,x]}\,dx \longrightarrow$$

$$-\frac{b^2\,\mathrm{Cos}[e + f\,x]\,(c + d\,\mathrm{Sin}[e + f\,x])^{n+1}}{a\,f\,(b\,c - a\,d)\,(a + b\,\mathrm{Sin}[e + f\,x])} + \frac{d}{a\,(b\,c - a\,d)}\int (c + d\,\mathrm{Sin}[e + f\,x])^n\,(a\,n - b\,(n + 1)\,\mathrm{Sin}[e + f\,x])\,dx$$

Program code:

```
Int[(c_.+d_.*sin[e_.+f_.*x_])^n_/(a_+b_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -b^2*Cos[e+f*x]*(c+d*Sin[e+f*x])^(n+1)/(a*f*(b*c-a*d)*(a+b*Sin[e+f*x])) +
  d/(a*(b*c-a*d))*Int[(c+d*Sin[e+f*x])^n*(a*n-b*(n+1)*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[n,0] && (IntegerQ[2*n] || EqQ[c,0])
```

3: $\displaystyle\int \frac{(c + d\,\text{Sin}[e + f\,x])^n}{a + b\,\text{Sin}[e + f\,x]}\,dx$ when $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0$

Derivation: Singly degenerate sine recurrence 2a with $A \to 1$, $B \to 0$, $m \to -1$, $p \to 0$

Rule: If $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0$, then

$$\int \frac{(c + d\,\text{Sin}[e + f\,x])^n}{a + b\,\text{Sin}[e + f\,x]}\,dx \to$$

$$-\frac{b\,\text{Cos}[e + f\,x]\,(c + d\,\text{Sin}[e + f\,x])^n}{a\,f\,(a + b\,\text{Sin}[e + f\,x])} + \frac{d\,n}{a\,b}\int (c + d\,\text{Sin}[e + f\,x])^{n-1}\,(a - b\,\text{Sin}[e + f\,x])\,dx$$

Program code:

```
Int[(c_.+d_.*sin[e_.+f_.*x_])^n_/(a_+b_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -b*Cos[e+f*x]*(c+d*Sin[e+f*x])^n/(a*f*(a+b*Sin[e+f*x])) +
  d*n/(a*b)*Int[(c+d*Sin[e+f*x])^(n-1)*(a-b*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f,n},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && (IntegerQ[2*n] || EqQ[c,0])
```

6. $\displaystyle\int \sqrt{a + b\,\text{Sin}[e + f\,x]}\,(c + d\,\text{Sin}[e + f\,x])^n\,dx$ when $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0$

   1. $\displaystyle\int \sqrt{a + b\,\text{Sin}[e + f\,x]}\,(c + d\,\text{Sin}[e + f\,x])^n\,dx$ when $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge 2\,n \in \mathbb{Z}$

      1: $\displaystyle\int \sqrt{a + b\,\text{Sin}[e + f\,x]}\,(c + d\,\text{Sin}[e + f\,x])^n\,dx$ when $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge n > 0$

Derivation: Singly degenerate sine recurrence 1b with $A \to c$, $B \to d$, $m \to \frac{1}{2}$, $n \to n - 1$, $p \to 0$ and algebraic simplification

Rule: If $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge n > 0$, then

$$\int \sqrt{a + b \, Sin[e + f \, x]} \; (c + d \, Sin[e + f \, x])^n \, dx \longrightarrow$$

$$-\frac{2 \, b \, Cos[e + f \, x] \; (c + d \, Sin[e + f \, x])^n}{f \, (2 \, n + 1) \, \sqrt{a + b \, Sin[e + f \, x]}} + \frac{2 \, n \, (b \, c + a \, d)}{b \, (2 \, n + 1)} \int \sqrt{a + b \, Sin[e + f \, x]} \; (c + d \, Sin[e + f \, x])^{n-1} \, dx$$

Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -2*b*Cos[e+f*x]*(c+d*Sin[e+f*x])^n/(f*(2*n+1)*Sqrt[a+b*Sin[e+f*x]]) +
  2*n*(b*c+a*d)/(b*(2*n+1))*Int[Sqrt[a+b*Sin[e+f*x]]*(c+d*Sin[e+f*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[n,0] && IntegerQ[2*n]
```

2. $\int \sqrt{a + b \, Sin[e + f \, x]} \; (c + d \, Sin[e + f \, x])^n \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge n < -1$

1: $\int \frac{\sqrt{a + b \, Sin[e + f \, x]}}{(c + d \, Sin[e + f \, x])^{3/2}} \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0$

Derivation: Singly degenerate sine recurrence 1a with $A \to 1$, $B \to 0$, $m \to \frac{1}{2}$, $n \to -\frac{3}{2}$ $p \to 0$

Derivation: Singly degenerate sine recurrence 1c with $A \to a$, $B \to b$, $m \to -\frac{1}{2}$, $n \to -\frac{3}{2}$, $p \to 0$

Rule: If $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0$, then

$$\int \frac{\sqrt{a + b \, Sin[e + f \, x]}}{(c + d \, Sin[e + f \, x])^{3/2}} \, dx \longrightarrow -\frac{2 \, b^2 \, Cos[e + f \, x]}{f \, (b \, c + a \, d) \, \sqrt{a + b \, Sin[e + f \, x]} \, \sqrt{c + d \, Sin[e + f \, x]}}$$

Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/(c_.+d_.*sin[e_.+f_.*x_])^(3/2),x_Symbol] :=
  -2*b^2*Cos[e+f*x]/(f*(b*c+a*d)*Sqrt[a+b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]]) /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

2: $\int \sqrt{a + b \, Sin[e + f \, x]} \; (c + d \, Sin[e + f \, x])^n \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge n < -1$

Derivation: Singly degenerate sine recurrence 1c with $A \to a$, $B \to b$, $m \to -\frac{1}{2}$, $p \to 0$ and algebraic simplification

Rule: If $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge n < -1$, then

$$\int \sqrt{a + b \, Sin[e + f \, x]} \; (c + d \, Sin[e + f \, x])^n \, dx \; \longrightarrow$$

$$\frac{(b \, c - a \, d) \, Cos[e + f \, x] \, (c + d \, Sin[e + f \, x])^{n+1}}{f \, (n + 1) \, (c^2 - d^2) \, \sqrt{a + b \, Sin[e + f \, x]}} + \frac{(2 \, n + 3) \, (b \, c - a \, d)}{2 \, b \, (n + 1) \, (c^2 - d^2)} \int \sqrt{a + b \, Sin[e + f \, x]} \; (c + d \, Sin[e + f \, x])^{n+1} \, dx$$

Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  (b*c-a*d)*Cos[e+f*x]*(c+d*Sin[e+f*x])^(n+1)/(f*(n+1)*(c^2-d^2)*Sqrt[a+b*Sin[e+f*x]]) +
  (2*n+3)*(b*c-a*d)/(2*b*(n+1)*(c^2-d^2))*Int[Sqrt[a+b*Sin[e+f*x]]*(c+d*Sin[e+f*x])^(n+1),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[n,-1] && NeQ[2*n+3,0] && IntegerQ[2*n]
```

3: $\int \frac{\sqrt{a + b \, Sin[e + f \, x]}}{c + d \, Sin[e + f \, x]} \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0$

Author: Martin Welz on 24 June 2011; generalized by Albert Rich 14 April 2014

Derivation: Integration by substitution

Basis: If $a^2 - b^2 == 0$, then $\frac{\sqrt{a+b \, Sin[e+f \, x]}}{c+d \, Sin[e+f \, x]} == -\frac{2 \, b}{f} \, Subst[\frac{1}{b \, c+a \, d-d \, x^2}, x, \frac{b \, Cos[e+f \, x]}{\sqrt{a+b \, Sin[e+f \, x]}}] \, \partial_x \frac{b \, Cos[e+f \, x]}{\sqrt{a+b \, Sin[e+f \, x]}}$

Rule: If $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0$, then

$$\int \frac{\sqrt{a + b \, \text{Sin}[e + f \, x]}}{c + d \, \text{Sin}[e + f \, x]} \, dx \;\; \rightarrow \;\; -\frac{2 \, b}{f} \, \text{Subst}\left[\int \frac{1}{b \, c + a \, d - d \, x^2} \, dx, \, x, \, \frac{b \, \text{Cos}[e + f \, x]}{\sqrt{a + b \, \text{Sin}[e + f \, x]}}\right]$$

Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -2*b/f*Subst[Int[1/(b*c+a*d-d*x^2),x],x,b*Cos[e+f*x]/Sqrt[a+b*Sin[e+f*x]]] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

4. $\displaystyle\int \frac{\sqrt{a + b\,\text{Sin}[e + f\,x]}}{\sqrt{c + d\,\text{Sin}[e + f\,x]}}\,dx$ when $b\,c - a\,d \neq 0 \,\wedge\, a^2 - b^2 == 0 \,\wedge\, c^2 - d^2 \neq 0$

1: $\displaystyle\int \frac{\sqrt{a + b\,\text{Sin}[e + f\,x]}}{\sqrt{d\,\text{Sin}[e + f\,x]}}\,dx$ when $a^2 - b^2 == 0 \,\wedge\, d == \frac{a}{b}$

Author: Martin Welz on 24 June 2011

Derivation: Integration by substitution

Basis: If $a^2 - b^2 == 0 \,\wedge\, d == \frac{a}{b}$, then $\dfrac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\sqrt{d\,\text{Sin}[e+f\,x]}} == -\dfrac{2}{f}\,\text{Subst}\left[\dfrac{1}{\sqrt{1-\frac{x^2}{a}}},\,x,\,\dfrac{b\,\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\right]\partial_x \dfrac{b\,\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}$

Rule: If $a^2 - b^2 == 0 \,\wedge\, d == \frac{a}{b}$, then

$$\int \frac{\sqrt{a + b\,\text{Sin}[e + f\,x]}}{\sqrt{d\,\text{Sin}[e + f\,x]}}\,dx \;\longrightarrow\; -\frac{2}{f}\,\text{Subst}\left[\int \frac{1}{\sqrt{1 - \frac{x^2}{a}}}\,dx,\,x,\,\frac{b\,\text{Cos}[e + f\,x]}{\sqrt{a + b\,\text{Sin}[e + f\,x]}}\right]$$

Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/Sqrt[d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  -2/f*Subst[Int[1/Sqrt[1-x^2/a],x],x,b*Cos[e+f*x]/Sqrt[a+b*Sin[e+f*x]]] /;
FreeQ[{a,b,d,e,f},x] && EqQ[a^2-b^2,0] && EqQ[d,a/b]
```

2: $\displaystyle\int \frac{\sqrt{a + b\,\mathrm{Sin}[e + f\,x]}}{\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}}\,dx$ when $\mathbf{b\,c - a\,d \neq 0 \ \wedge\ a^2 - b^2 == 0 \ \wedge\ c^2 - d^2 \neq 0}$

Author: Martin Welz on 10 March 2011

Derivation: Integration by substitution

Basis: If $a^2 - b^2 == 0 \ \wedge\ c^2 - d^2 \neq 0$, then

$$\frac{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}} == -\frac{2\,b}{f}\,\mathrm{Subst}\Big[\frac{1}{b+d\,x^2}, x, \frac{b\,\mathrm{Cos}[e+f\,x]}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\Big]\,\partial_x\,\frac{b\,\mathrm{Cos}[e+f\,x]}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}$$

Note: The above identity is not valid if $b\,c - a\,d \neq 0 \ \wedge\ a^2 - b^2 == 0 \ \wedge\ c^2 - d^2 == 0$, since the derivative vanishes!

Rule: If $b\,c - a\,d \neq 0 \ \wedge\ a^2 - b^2 == 0 \ \wedge\ c^2 - d^2 \neq 0$, then

$$\int \frac{\sqrt{a + b\,\mathrm{Sin}[e + f\,x]}}{\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}}\,dx \ \longrightarrow\ -\frac{2\,b}{f}\,\mathrm{Subst}\Big[\int \frac{1}{b + d\,x^2}\,dx, x, \frac{b\,\mathrm{Cos}[e + f\,x]}{\sqrt{a + b\,\mathrm{Sin}[e + f\,x]}\,\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}}\Big]$$

Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/Sqrt[c_.+d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  -2*b/f*Subst[Int[1/(b+d*x^2),x],x,b*Cos[e+f*x]/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]])] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

2: $\int \sqrt{a + b\, \text{Sin}[e + f\, x]}\ (c + d\, \text{Sin}[e + f\, x])^n\, dx$ when $b\, c - a\, d \neq 0 \land a^2 - b^2 == 0 \land c^2 - d^2 \neq 0 \land 2\, n \notin \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

Basis: If $a^2 - b^2 == 0$, then $\partial_x \dfrac{\text{Cos}[e+f\,x]}{\sqrt{a-b\,\text{Sin}[e+f\,x]}\ \sqrt{a+b\,\text{Sin}[e+f\,x]}} == 0$

Basis: $\text{Cos}[e + f\, x] == \frac{1}{f}\, \partial_x\, \text{Sin}[e + f\, x]$

Rule: If $b\, c - a\, d \neq 0 \land a^2 - b^2 == 0 \land c^2 - d^2 \neq 0 \land 2\, n \notin \mathbb{Z}$, then

$$\int \sqrt{a + b\, \text{Sin}[e + f\, x]}\ (c + d\, \text{Sin}[e + f\, x])^n\, dx \longrightarrow$$

$$\frac{a^2\, \text{Cos}[e + f\, x]}{\sqrt{a + b\, \text{Sin}[e + f\, x]}\ \sqrt{a - b\, \text{Sin}[e + f\, x]}} \int \frac{\text{Cos}[e + f\, x]\ (c + d\, \text{Sin}[e + f\, x])^n}{\sqrt{a - b\, \text{Sin}[e + f\, x]}}\, dx \longrightarrow$$

$$\frac{a^2\, \text{Cos}[e + f\, x]}{f\, \sqrt{a + b\, \text{Sin}[e + f\, x]}\ \sqrt{a - b\, \text{Sin}[e + f\, x]}}\, \text{Subst}\left[\int \frac{(c + d\, x)^n}{\sqrt{a - b\, x}}\, dx,\ x,\ \text{Sin}[e + f\, x]\right]$$

Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  a^2*Cos[e+f*x]/(f*Sqrt[a+b*Sin[e+f*x]]*Sqrt[a-b*Sin[e+f*x]])*Subst[Int[(c+d*x)^n/Sqrt[a-b*x],x],x,Sin[e+f*x]] /;
FreeQ[{a,b,c,d,e,f,n},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && Not[IntegerQ[2*n]]
```

7. $\int \dfrac{(c + d\, \text{Sin}[e + f\, x])^n}{\sqrt{a + b\, \text{Sin}[e + f\, x]}}\, dx$ when $b\, c - a\, d \neq 0 \land a^2 - b^2 == 0 \land c^2 - d^2 \neq 0$

1. $\displaystyle\int \frac{(c + d\,\text{Sin}[e + f\,x])^n}{\sqrt{a + b\,\text{Sin}[e + f\,x]}}\,dx$ when $b\,c - a\,d \ne 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \ne 0 \wedge n > 0$

1: $\displaystyle\int \frac{\sqrt{c + d\,\text{Sin}[e + f\,x]}}{\sqrt{a + b\,\text{Sin}[e + f\,x]}}\,dx$ when $b\,c - a\,d \ne 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \ne 0$

Derivation: Algebraic expansion

Basis: $\dfrac{\sqrt{c + d\,z}}{\sqrt{a + b\,z}} == \dfrac{b\,c - a\,d}{b\,\sqrt{a + b\,z}\,\sqrt{c + d\,z}} + \dfrac{d\,\sqrt{a + b\,z}}{b\,\sqrt{c + d\,z}}$

Rule: If $b\,c - a\,d \ne 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \ne 0$, then

$$\int \frac{\sqrt{c + d\,\text{Sin}[e + f\,x]}}{\sqrt{a + b\,\text{Sin}[e + f\,x]}}\,dx \;\longrightarrow\; \frac{d}{b}\int \frac{\sqrt{a + b\,\text{Sin}[e + f\,x]}}{\sqrt{c + d\,\text{Sin}[e + f\,x]}}\,dx + \frac{b\,c - a\,d}{b}\int \frac{1}{\sqrt{a + b\,\text{Sin}[e + f\,x]}\,\sqrt{c + d\,\text{Sin}[e + f\,x]}}\,dx$$

Program code:

```
Int[Sqrt[c_.+d_.*sin[e_.+f_.*x_]]/Sqrt[a_+b_.*sin[e_.+f_.*x_]],x_Symbol] :=
  d/b*Int[Sqrt[a+b*Sin[e+f*x]]/Sqrt[c+d*Sin[e+f*x]],x] +
  (b*c-a*d)/b*Int[1/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

2: $\displaystyle\int \frac{(c + d\,\text{Sin}[e + f\,x])^n}{\sqrt{a + b\,\text{Sin}[e + f\,x]}}\,dx$ when $b\,c - a\,d \ne 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \ne 0 \wedge n > 1$

Derivation: Singly degenerate sine recurrence 2c with $A \to c$, $B \to d$, $m \to \frac{1}{2}$, $n \to n - 1$, $p \to 0$

Rule: If $b\,c - a\,d \ne 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \ne 0 \wedge n > 1$, then

$$\int \frac{(c + d\,\text{Sin}[e + f\,x])^n}{\sqrt{a + b\,\text{Sin}[e + f\,x]}}\,dx \;\longrightarrow\;$$

$$-\frac{2 d \, \text{Cos}[e + f \, x] \, (c + d \, \text{Sin}[e + f \, x])^{n-1}}{f \, (2 \, n - 1) \, \sqrt{a + b \, \text{Sin}[e + f \, x]}} -$$

$$\frac{1}{b \, (2 \, n - 1)} \int \left(\left((c + d \, \text{Sin}[e + f \, x])^{n-2} \, \left(a \, c \, d - b \, (2 \, d^2 \, (n - 1) + c^2 \, (2 \, n - 1)) + d \, (a \, d - b \, c \, (4 \, n - 3)) \, \text{Sin}[e + f \, x] \right) \right) \middle/ \left(\sqrt{a + b \, \text{Sin}[e + f \, x]} \right) \right) dx$$

Program code:

```
Int[(c_.+d_.*sin[e_.+f_.*x_])^n_/Sqrt[a_+b_.*sin[e_.+f_.*x_]],x_Symbol] :=
  -2*d*Cos[e+f*x]*(c+d*Sin[e+f*x])^(n-1)/(f*(2*n-1)*Sqrt[a+b*Sin[e+f*x]]) -
  1/(b*(2*n-1))*Int[(c+d*Sin[e+f*x])^(n-2)/Sqrt[a+b*Sin[e+f*x]]*
    Simp[a*c*d-b*(2*d^2*(n-1)+c^2*(2*n-1))+d*(a*d-b*c*(4*n-3))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[n,1] && IntegerQ[2*n]
```

2: $\int \dfrac{(c + d \, \text{Sin}[e + f \, x])^n}{\sqrt{a + b \, \text{Sin}[e + f \, x]}} \, dx$ when $b \, c - a \, d \neq 0 \, \wedge \, a^2 - b^2 == 0 \, \wedge \, c^2 - d^2 \neq 0 \, \wedge \, n < -1$

Derivation: Singly degenerate sine recurrence 1c with A → 1, B → 0, p → 0

Rule: If $b \, c - a \, d \neq 0 \, \wedge \, a^2 - b^2 == 0 \, \wedge \, c^2 - d^2 \neq 0 \, \wedge \, n < -1$, then

$$\int \frac{(c + d \, \text{Sin}[e + f \, x])^n}{\sqrt{a + b \, \text{Sin}[e + f \, x]}} \, dx \longrightarrow$$

$$-\frac{d \, \text{Cos}[e + f \, x] \, (c + d \, \text{Sin}[e + f \, x])^{n+1}}{f \, (n + 1) \, (c^2 - d^2) \, \sqrt{a + b \, \text{Sin}[e + f \, x]}} - \frac{1}{2 \, b \, (n + 1) \, (c^2 - d^2)} \int \frac{(c + d \, \text{Sin}[e + f \, x])^{n+1} \, (a \, d - 2 \, b \, c \, (n + 1) + b \, d \, (2 \, n + 3) \, \text{Sin}[e + f \, x])}{\sqrt{a + b \, \text{Sin}[e + f \, x]}} \, dx$$

Program code:

```
Int[(c_.+d_.*sin[e_.+f_.*x_])^n_/Sqrt[a_+b_.*sin[e_.+f_.*x_]],x_Symbol] :=
  -d*Cos[e+f*x]*(c+d*Sin[e+f*x])^(n+1)/(f*(n+1)*(c^2-d^2)*Sqrt[a+b*Sin[e+f*x]]) -
  1/(2*b*(n+1)*(c^2-d^2))*Int[(c+d*Sin[e+f*x])^(n+1)*Simp[a*d-2*b*c*(n+1)+b*d*(2*n+3)*Sin[e+f*x],x]/Sqrt[a+b*Sin[e+f*x]],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[n,-1] && IntegerQ[2*n]
```

3: $\int \dfrac{1}{\sqrt{a + b\,\text{Sin}[e + f\,x]}\ (c + d\,\text{Sin}[e + f\,x])}\,dx$ when $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0$

Derivation: Algebraic expansion

Basis: $\dfrac{1}{\sqrt{a+b\,z}\ (c+d\,z)} == \dfrac{b}{(b\,c - a\,d)\ \sqrt{a+b\,z}} - \dfrac{d\,\sqrt{a+b\,z}}{(b\,c - a\,d)\ (c+d\,z)}$

Rule: If $b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0$, then

$$\int \dfrac{1}{\sqrt{a + b\,\text{Sin}[e + f\,x]}\ (c + d\,\text{Sin}[e + f\,x])}\,dx \longrightarrow \dfrac{b}{b\,c - a\,d}\int \dfrac{1}{\sqrt{a + b\,\text{Sin}[e + f\,x]}}\,dx - \dfrac{d}{b\,c - a\,d}\int \dfrac{\sqrt{a + b\,\text{Sin}[e + f\,x]}}{c + d\,\text{Sin}[e + f\,x]}\,dx$$

Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*(c_.+d_.*sin[e_.+f_.*x_])),x_Symbol] :=
  b/(b*c-a*d)*Int[1/Sqrt[a+b*Sin[e+f*x]],x] - d/(b*c-a*d)*Int[Sqrt[a+b*Sin[e+f*x]]/(c+d*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

4. $$\int \frac{1}{\sqrt{a + b\,Sin[e + f\,x]}\,\sqrt{c + d\,Sin[e + f\,x]}}\,dx \text{ when } b\,c - a\,d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0$$

1: $$\int \frac{1}{\sqrt{a + b\,Sin[e + f\,x]}\,\sqrt{d\,Sin[e + f\,x]}}\,dx \text{ when } a^2 - b^2 == 0 \wedge d == \frac{a}{b} \wedge a > 0$$

Author: Martin Welz on 24 June 2011

Derivation: Integration by substitution

Basis: If $a^2 - b^2 == 0 \wedge d == \frac{a}{b} \wedge a > 0$, then $\frac{1}{\sqrt{a+b\,Sin[e+f\,x]}\,\sqrt{d\,Sin[e+f\,x]}} == -\frac{\sqrt{2}}{\sqrt{a}\,f}\,Subst\left[\frac{1}{\sqrt{1-x^2}}, x, \frac{b\,Cos[e+f\,x]}{a+b\,Sin[e+f\,x]}\right] \partial_x \frac{b\,Cos[e+f\,x]}{a+b\,Sin[e+f\,x]}$

Basis: $F(z \mid 0) == z$

Note: This is a special case of the rule for $a^2 \neq b^2$.

Rule: If $a^2 - b^2 == 0 \wedge d == \frac{a}{b} \wedge a > 0$, then

$$\int \frac{1}{\sqrt{a + b\,Sin[e + f\,x]}\,\sqrt{d\,Sin[e + f\,x]}}\,dx \;\longrightarrow\; -\frac{\sqrt{2}}{\sqrt{a}\,f}\,Subst\left[\int \frac{1}{\sqrt{1 - x^2}}\,dx, x, \frac{b\,Cos[e + f\,x]}{a + b\,Sin[e + f\,x]}\right]$$

Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  -Sqrt[2]/(Sqrt[a]*f)*Subst[Int[1/Sqrt[1-x^2],x],x,b*Cos[e+f*x]/(a+b*Sin[e+f*x])] /;
FreeQ[{a,b,d,e,f},x] && EqQ[a^2-b^2,0] && EqQ[d,a/b] && GtQ[a,0]
```

2: $\int \dfrac{1}{\sqrt{a + b \, Sin[e + f \, x]} \, \sqrt{c + d \, Sin[e + f \, x]}} \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0$

Author: Martin Welz on 10 March 2011

Derivation: Integration by substitution

Basis: If $a^2 - b^2 == 0$, then

$$\dfrac{1}{\sqrt{a+b \, Sin[e+f \, x]} \, \sqrt{c+d \, Sin[e+f \, x]}} == -\dfrac{2 \, a}{f} \, Subst\left[\dfrac{1}{2 \, b^2 - (a \, c - b \, d) \, x^2}, \, x, \, \dfrac{b \, Cos[e+f \, x]}{\sqrt{a+b \, Sin[e+f \, x]} \, \sqrt{c+d \, Sin[e+f \, x]}}\right] \partial_x \dfrac{b \, Cos[e+f \, x]}{\sqrt{a+b \, Sin[e+f \, x]} \, \sqrt{c+d \, Sin[e+f \, x]}}$$

Note: The above identity is not valid if $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 == 0$, since the derivative vanishes!

Rule: If $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0$, then

$$\int \dfrac{1}{\sqrt{a + b \, Sin[e + f \, x]} \, \sqrt{c + d \, Sin[e + f \, x]}} \, dx \; \longrightarrow \; -\dfrac{2 \, a}{f} \, Subst\left[\int \dfrac{1}{2 \, b^2 - (a \, c - b \, d) \, x^2} \, dx, \, x, \, \dfrac{b \, Cos[e + f \, x]}{\sqrt{a + b \, Sin[e + f \, x]} \, \sqrt{c + d \, Sin[e + f \, x]}}\right]$$

Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[c_.+d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  -2*a/f*Subst[Int[1/(2*b^2-(a*c-b*d)*x^2),x],x,b*Cos[e+f*x]/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]])] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

8: $\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^n \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge n > 1$

Derivation: Singly degenerate sine recurrence 2c with $A \to c$, $B \to d$, $n \to n - 1$, $p \to 0$

Rule: If $b \, c - a \, d \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 \neq 0 \wedge n > 1$, then

$$\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^n \, dx \; \longrightarrow$$

$$-\frac{d \, \text{Cos}[e + f\,x] \, (a + b\,\text{Sin}[e + f\,x])^m \, (c + d\,\text{Sin}[e + f\,x])^{n-1}}{f\,(m + n)} +$$

$$\frac{1}{b\,(m + n)} \int (a + b\,\text{Sin}[e + f\,x])^m \, (c + d\,\text{Sin}[e + f\,x])^{n-2} \left(d\,\big(a\,c\,m + b\,d\,(n - 1)\big) + b\,c^2\,(m + n) + \big(d\,(a\,d\,m + b\,c\,(m + 2\,n - 1))\big)\,\text{Sin}[e + f\,x]\right) dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -d*Cos[e+f*x]*(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^(n-1)/(f*(m+n)) +
  1/(b*(m+n))*Int[(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^(n-2)*
    Simp[d*(a*c*m+b*d*(n-1))+b*c^2*(m+n)+d*(a*d*m+b*c*(m+2*n-1))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,m},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[n,1] && IntegerQ[n]
```

9. $\int (a + b\,\text{Sin}[e + f\,x])^m \, (c + d\,\text{Sin}[e + f\,x])^n \, dx$ when $b\,c - a\,d \neq 0 \,\wedge\, a^2 - b^2 == 0 \,\wedge\, c^2 - d^2 \neq 0$

   1: $\int (a + b\,\text{Sin}[e + f\,x])^m \, (c + d\,\text{Sin}[e + f\,x])^n \, dx$ when $b\,c - a\,d \neq 0 \,\wedge\, a^2 - b^2 == 0 \,\wedge\, c^2 - d^2 \neq 0 \,\wedge\, m \in \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

Basis: $\partial_x \dfrac{\text{Cos}[e + f\,x]}{\sqrt{1 + \text{Sin}[e + f\,x]} \, \sqrt{1 - \text{Sin}[e + f\,x]}} == 0$

Basis: $\text{Cos}[e + f\,x] == \frac{1}{f}\,\partial_x \, \text{Sin}[e + f\,x]$

Rule: If $b\,c - a\,d \neq 0 \,\wedge\, a^2 - b^2 == 0 \,\wedge\, c^2 - d^2 \neq 0 \,\wedge\, m \in \mathbb{Z}$, then

$$\int (a + b\,\text{Sin}[e + f\,x])^m \, (c + d\,\text{Sin}[e + f\,x])^n \, dx \;\longrightarrow$$

$$\frac{a^m\,\text{Cos}[e + f\,x]}{\sqrt{1 + \text{Sin}[e + f\,x]} \, \sqrt{1 - \text{Sin}[e + f\,x]}} \int \frac{\text{Cos}[e + f\,x]\,\left(1 + \frac{b}{a}\,\text{Sin}[e + f\,x]\right)^{m-\frac{1}{2}}\,(c + d\,\text{Sin}[e + f\,x])^n}{\sqrt{1 - \frac{b}{a}\,\text{Sin}[e + f\,x]}} \, dx \;\longrightarrow$$

$$\frac{a^m \, \text{Cos}[e+f\,x]}{f\,\sqrt{1+\text{Sin}[e+f\,x]}\,\sqrt{1-\text{Sin}[e+f\,x]}}\, \text{Subst}\left[\int \frac{\left(1+\frac{b}{a}\,x\right)^{m-\frac{1}{2}}\,(c+d\,x)^n}{\sqrt{1-\frac{b}{a}\,x}}\,\mathrm{d}x,\,x,\,\text{Sin}[e+f\,x]\right]$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_.,x_Symbol] :=
  a^m*Cos[e+f*x]/(f*Sqrt[1+Sin[e+f*x]]*Sqrt[1-Sin[e+f*x]])*Subst[Int[(1+b/a*x)^(m-1/2)*(c+d*x)^n/Sqrt[1-b/a*x],x],x,Sin[e+f*x]] /;
FreeQ[{a,b,c,d,e,f,n},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && IntegerQ[m]
```

2. $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,\mathrm{d}x$ when $b\,c-a\,d \neq 0 \wedge a^2-b^2 == 0 \wedge c^2-d^2 \neq 0 \wedge m \notin \mathbb{Z}$

   1. $\int (a+b\,\text{Sin}[e+f\,x])^m\,(d\,\text{Sin}[e+f\,x])^n\,\mathrm{d}x$ when $a^2-b^2 == 0 \wedge m \notin \mathbb{Z}$

      1. $\int (a+b\,\text{Sin}[e+f\,x])^m\,(d\,\text{Sin}[e+f\,x])^n\,\mathrm{d}x$ when $a^2-b^2 == 0 \wedge m \notin \mathbb{Z} \wedge a > 0$

         1: $\int (a+b\,\text{Sin}[e+f\,x])^m\,(d\,\text{Sin}[e+f\,x])^n\,\mathrm{d}x$ when $a^2-b^2 == 0 \wedge m \notin \mathbb{Z} \wedge a > 0 \wedge \frac{d}{b} > 0$

Derivation: Piecewise constant extraction and integration by substitution

Basis: If $a^2-b^2 == 0$, then $\partial_x \dfrac{\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{a-b\,\text{Sin}[e+f\,x]}} == 0$

Basis: If $a^2-b^2 == 0$, then $\dfrac{b^2\,\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{a-b\,\text{Sin}[e+f\,x]}}\,\dfrac{\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{a-b\,\text{Sin}[e+f\,x]}} == 1$

Basis: $\dfrac{\text{Cos}[e+f\,x]\,(a+b\,\text{Sin}[e+f\,x])^{m-\frac{1}{2}}\,(b\,\text{Sin}[e+f\,x])^n}{\sqrt{a-b\,\text{Sin}[e+f\,x]}} ==$

$-\dfrac{1}{b\,f}\,\text{Subst}\left[\dfrac{(a-x)^n\,(2\,a-x)^{m-\frac{1}{2}}}{\sqrt{x}},\,x,\,a-b\,\text{Sin}[e+f\,x]\right]\,\partial_x\,(a-b\,\text{Sin}[e+f\,x])$

Note: If $a > 0$, then $\dfrac{(a-x)^n\,(2\,a-x)^{m-\frac{1}{2}}}{\sqrt{x}}$ is integrable in terms of the Appell function without the need for additional

piecewise constant extraction.

Rule: If $a^2 - b^2 == 0 \; \wedge \; m \notin \mathbb{Z} \; \wedge \; a > 0 \; \wedge \; \frac{d}{b} > 0$, then

$$\int \left(a + b \, Sin[e + f \, x]\right)^m \left(d \, Sin[e + f \, x]\right)^n dx \; \longrightarrow$$

$$\frac{b^2 \left(\frac{d}{b}\right)^n Cos[e + f \, x]}{\sqrt{a + b \, Sin[e + f \, x]} \; \sqrt{a - b \, Sin[e + f \, x]}} \int \frac{Cos[e + f \, x] \, \left(a + b \, Sin[e + f \, x]\right)^{m - \frac{1}{2}} \left(b \, Sin[e + f \, x]\right)^n}{\sqrt{a - b \, Sin[e + f \, x]}} dx \; \longrightarrow$$

$$- \frac{b \left(\frac{d}{b}\right)^n Cos[e + f \, x]}{f \, \sqrt{a + b \, Sin[e + f \, x]} \; \sqrt{a - b \, Sin[e + f \, x]}} \; Subst\left[\int \frac{(a - x)^n \, (2 \, a - x)^{m - \frac{1}{2}}}{\sqrt{x}} dx, \, x, \, a - b \, Sin[e + f \, x]\right]$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -b*(d/b)^n*Cos[e+f*x]/(f*Sqrt[a+b*Sin[e+f*x]]*Sqrt[a-b*Sin[e+f*x]])*
    Subst[Int[(a-x)^n*(2*a-x)^(m-1/2)/Sqrt[x],x],x,a-b*Sin[e+f*x]] /;
FreeQ[{a,b,d,e,f,m,n},x] && EqQ[a^2-b^2,0] && Not[IntegerQ[m]] && GtQ[a,0] && GtQ[d/b,0]
```

2: $\int \left(a + b \, \text{Sin}[e + f \, x]\right)^m \left(d \, \text{Sin}[e + f \, x]\right)^n dx$ when $a^2 - b^2 == 0 \wedge m \notin \mathbb{Z} \wedge a > 0 \wedge \frac{d}{b} \not> 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{\left(d \, \text{Sin}[e+f \, x]\right)^n}{\left(b \, \text{Sin}[e+f \, x]\right)^n} == 0$

Rule: If $a^2 - b^2 == 0 \wedge m \notin \mathbb{Z} \wedge a > 0 \wedge \frac{d}{b} \not> 0$, then

$$\int \left(a + b \, \text{Sin}[e + f \, x]\right)^m \left(d \, \text{Sin}[e + f \, x]\right)^n dx \longrightarrow \frac{\left(\frac{d}{b}\right)^{\text{IntPart}[n]} \left(d \, \text{Sin}[e + f \, x]\right)^{\text{FracPart}[n]}}{\left(b \, \text{Sin}[e + f \, x]\right)^{\text{FracPart}[n]}} \int \left(a + b \, \text{Sin}[e + f \, x]\right)^m \left(b \, \text{Sin}[e + f \, x]\right)^n dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(d_.*sin[e_.+f_.*x_])^n_.,x_Symbol] :=
  (d/b)^IntPart[n]*(d*Sin[e+f*x])^FracPart[n]/(b*Sin[e+f*x])^FracPart[n]*Int[(a+b*Sin[e+f*x])^m*(b*Sin[e+f*x])^n,x] /;
FreeQ[{a,b,d,e,f,m,n},x] && EqQ[a^2-b^2,0] && Not[IntegerQ[m]] && GtQ[a,0] && Not[GtQ[d/b,0]]
```

2: $\int \left(a + b \, Sin[e + f \, x]\right)^m \left(d \, Sin[e + f \, x]\right)^n \, dx$ when $a^2 - b^2 == 0 \,\wedge\, m \notin \mathbb{Z} \,\wedge\, a \ngtr 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{\left(a + b \, Sin[e + f \, x]\right)^m}{\left(1 + \frac{b}{a} \, Sin[e + f \, x]\right)^m} == 0$

Rule: If $a^2 - b^2 == 0 \,\wedge\, m \notin \mathbb{Z} \,\wedge\, a \ngtr 0,$ then

$$\int \left(a + b \, Sin[e + f \, x]\right)^m \left(d \, Sin[e + f \, x]\right)^n \, dx \;\longrightarrow\; \frac{a^{IntPart[m]} \, \left(a + b \, Sin[e + f \, x]\right)^{FracPart[m]}}{\left(1 + \frac{b}{a} \, Sin[e + f \, x]\right)^{FracPart[m]}} \int \left(1 + \frac{b}{a} \, Sin[e + f \, x]\right)^m \left(d \, Sin[e + f \, x]\right)^n \, dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(d_.*sin[e_.+f_.*x_])^n_.,x_Symbol] :=
  a^IntPart[m]*(a+b*Sin[e+f*x])^FracPart[m]/(1+b/a*Sin[e+f*x])^FracPart[m]*
    Int[(1+b/a*Sin[e+f*x])^m*(d*Sin[e+f*x])^n,x] /;
FreeQ[{a,b,d,e,f,m,n},x] && EqQ[a^2-b^2,0] && Not[IntegerQ[m]] && Not[GtQ[a,0]]
```

2: $\int (a + b\,\text{Sin}[e + f\,x])^m\,(c + d\,\text{Sin}[e + f\,x])^n\,dx$ when $b\,c - a\,d \neq 0 \;\wedge\; a^2 - b^2 == 0 \;\wedge\; c^2 - d^2 \neq 0 \;\wedge\; m \notin \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

Basis: If $a^2 - b^2 == 0$, then $\partial_x \dfrac{\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\;\sqrt{a-b\,\text{Sin}[e+f\,x]}} == 0$

Basis: If $a^2 - b^2 == 0$, then $\dfrac{a^2\,\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\;\sqrt{a-b\,\text{Sin}[e+f\,x]}}\;\dfrac{\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\;\sqrt{a-b\,\text{Sin}[e+f\,x]}} == 1$

Basis: $\text{Cos}[e + f\,x] == \dfrac{1}{f}\,\partial_x\,\text{Sin}[e + f\,x]$

Rule: If $b\,c - a\,d \neq 0 \;\wedge\; a^2 - b^2 == 0 \;\wedge\; c^2 - d^2 \neq 0 \;\wedge\; m \notin \mathbb{Z}$, then

$$\int (a + b\,\text{Sin}[e + f\,x])^m\,(c + d\,\text{Sin}[e + f\,x])^n\,dx \;\longrightarrow$$

$$\dfrac{a^2\,\text{Cos}[e + f\,x]}{\sqrt{a + b\,\text{Sin}[e + f\,x]}\;\sqrt{a - b\,\text{Sin}[e + f\,x]}}\int \dfrac{\text{Cos}[e + f\,x]\,(a + b\,\text{Sin}[e + f\,x])^{m-\frac{1}{2}}\,(c + d\,\text{Sin}[e + f\,x])^n}{\sqrt{a - b\,\text{Sin}[e + f\,x]}}\,dx \;\longrightarrow$$

$$\dfrac{a^2\,\text{Cos}[e + f\,x]}{f\,\sqrt{a + b\,\text{Sin}[e + f\,x]}\;\sqrt{a - b\,\text{Sin}[e + f\,x]}}\,\text{Subst}\!\left[\int \dfrac{(a + b\,x)^{m-\frac{1}{2}}\,(c + d\,x)^n}{\sqrt{a - b\,x}}\,dx, x, \text{Sin}[e + f\,x]\right]$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_.,x_Symbol] :=
  a^2*Cos[e+f*x]/(f*Sqrt[a+b*Sin[e+f*x]]*Sqrt[a-b*Sin[e+f*x]])*Subst[Int[(a+b*x)^(m-1/2)*(c+d*x)^n/Sqrt[a-b*x],x],x,Sin[e+f*x]] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && Not[IntegerQ[m]]
```

8. $\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^n \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0$

   1. $\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^2 \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0$

      1: $\int (b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^2 \, dx$

Derivation: Algebraic expansion

Basis: $(c + d \, z)^2 == \frac{2 \, c \, d}{b} (b \, z) + (c^2 + d^2 \, z^2)$

Rule:

$$\int (b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^2 \, dx \; \longrightarrow$$

$$\frac{2 \, c \, d}{b} \int (b \, Sin[e + f \, x])^{m+1} \, dx + \int (b \, Sin[e + f \, x])^m \, (c^2 + d^2 \, Sin[e + f \, x]^2) \, dx$$

Program code:

```
Int[(b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^2,x_Symbol] :=
  2*c*d/b*Int[(b*Sin[e+f*x])^(m+1),x]  +  Int[(b*Sin[e+f*x])^m*(c^2+d^2*Sin[e+f*x]^2),x] /;
FreeQ[{b,c,d,e,f,m},x]
```

      2: $\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^2 \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge m < -1$

Derivation: Nondegenerate sine recurrence 1a with $A \to c^2$, $B \to 2 \, c \, d$, $C \to d^2$, $n \to 0$, $p \to 0$

Rule: If $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge m < -1$, then

$$\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^2 \, dx \; \longrightarrow$$

$$- \frac{(b^2 \, c^2 - 2 \, a \, b \, c \, d + a^2 \, d^2) \, Cos[e + f \, x] \, (a + b \, Sin[e + f \, x])^{m+1}}{b \, f \, (m + 1) \, (a^2 - b^2)} -$$

$$\frac{1}{b \ (m+1) \ (a^2-b^2)} \int (a + b \ Sin[e + f \ x])^{m+1} \ (b \ (m+1) \ (2 \ b \ c \ d - a \ (c^2 + d^2)) + (a^2 \ d^2 - 2 \ a \ b \ c \ d \ (m+2) + b^2 \ (d^2 \ (m+1) + c^2 \ (m+2)))) \ Sin[e + f \ x]) \ dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^2,x_Symbol] :=
 -(b^2*c^2-2*a*b*c*d+a^2*d^2)*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+1)/(b*f*(m+1)*(a^2-b^2)) -
  1/(b*(m+1)*(a^2-b^2))*Int[(a+b*Sin[e+f*x])^(m+1)*
    Simp[b*(m+1)*(2*b*c*d-a*(c^2+d^2))+(a^2*d^2-2*a*b*c*d*(m+2)+b^2*(d^2*(m+1)+c^2*(m+2)))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && LtQ[m,-1]
```

3: $\int (a + b \ Sin[e + f \ x])^m \ (c + d \ Sin[e + f \ x])^2 \ dx$ when $b \ c - a \ d \neq 0 \ \wedge \ a^2 - b^2 \neq 0 \ \wedge \ m \not< -1$

Derivation: Nondegenerate sine recurrence 1b with $A \rightarrow a^2$, $B \rightarrow 2 \ a \ b$, $C \rightarrow b^2$, $m \rightarrow 0$, $p \rightarrow 0$

Rule: If $b \ c - a \ d \neq 0 \ \wedge \ a^2 - b^2 \neq 0 \ \wedge \ m \not< -1$, then

$$\int (a + b \ Sin[e + f \ x])^m \ (c + d \ Sin[e + f \ x])^2 \ dx \ \longrightarrow$$

$$-\frac{d^2 \ Cos[e + f \ x] \ (a + b \ Sin[e + f \ x])^{m+1}}{b \ f \ (m+2)} +$$

$$\frac{1}{b \ (m+2)} \int (a + b \ Sin[e + f \ x])^m \ (b \ (d^2 \ (m+1) + c^2 \ (m+2)) - d \ (a \ d - 2 \ b \ c \ (m+2)) \ Sin[e + f \ x]) \ dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^2,x_Symbol] :=
 -d^2*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+1)/(b*f*(m+2)) +
  1/(b*(m+2))*Int[(a+b*Sin[e+f*x])^m*Simp[b*(d^2*(m+1)+c^2*(m+2))-d*(a*d-2*b*c*(m+2))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,m},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && Not[LtQ[m,-1]]
```

x: $\int (a + b \, Sin[e + f \, x])^m \, (d \, Sin[e + f \, x])^n \, dx$ when $a^2 - b^2 \neq 0 \wedge m \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Note: If terms having the same powers of $sin[e + f \, x]$ are collected, this rule results in more compact antiderivatives; however, the number of steps required grows exponentially with m.

Rule: If $a^2 - b^2 \neq 0 \wedge m \in \mathbb{Z}^+$, then

$$\int (a + b \, Sin[e + f \, x])^m \, (d \, Sin[e + f \, x])^n \, dx \;\longrightarrow\; \int ExpandTrig[(a + b \, Sin[e + f \, x])^m \, (d \, Sin[e + f \, x])^n, x] \, dx$$

Program code:

```
(* Int[(a_+b_.*sin[e_.+f_.*x_])^m_.*(d_.*sin[e_.+f_.*x_])^n_.,x_Symbol] :=
  Int[ExpandTrig[(a+b*sin[e+f*x])^m*(d*sin[e+f*x])^n,x],x] /;
FreeQ[{a,b,d,e,f,n},x] && NeQ[a^2-b^2,0] && IGtQ[m,0] *)
```

2. $\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^n \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0 \wedge m > 2$

1: $\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^n \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0 \wedge m > 2 \wedge n < -1$

Derivation: Nondegenerate sine recurrence 1a with $A \to c^2$, $B \to 2 \, c \, d$, $C \to d^2$, $n \to n - 2$, $p \to 0$

Rule: If $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0 \wedge m > 2 \wedge n < -1$, then

$$\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^n \, dx \;\longrightarrow$$

$$- (((b^2 \, c^2 - 2 \, a \, b \, c \, d + a^2 \, d^2) \, Cos[e + f \, x] \, (a + b \, Sin[e + f \, x])^{m-2} \, (c + d \, Sin[e + f \, x])^{n+1}) / (d \, f \, (n + 1) \, (c^2 - d^2))) +$$

$$\frac{1}{d \, (n + 1) \, (c^2 - d^2)} \int (a + b \, Sin[e + f \, x])^{m-3} \, (c + d \, Sin[e + f \, x])^{n+1} \cdot$$

$$(b \, (m - 2) \, (b \, c - a \, d)^2 + a \, d \, (n + 1) \, (c \, (a^2 + b^2) - 2 \, a \, b \, d) +$$

$$(b \, (n + 1) \, (a \, b \, c^2 + c \, d \, (a^2 + b^2) - 3 \, a \, b \, d^2) - a \, (n + 2) \, (b \, c - a \, d)^2) \, Sin[e + f \, x] +$$

$$b \left(b^2 \left(c^2 - d^2\right) - m \left(b\,c - a\,d\right)^2 + d\,n \left(2\,a\,b\,c - d \left(a^2 + b^2\right)\right)\right) \mathtt{Sin}\left[e + f\,x\right]^2\right)\, \mathtt{d}x$$

Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -(b^2*c^2-2*a*b*c*d+a^2*d^2)*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m-2)*(c+d*Sin[e+f*x])^(n+1)/(d*f*(n+1)*(c^2-d^2)) +
   1/(d*(n+1)*(c^2-d^2))*Int[(a+b*Sin[e+f*x])^(m-3)*(c+d*Sin[e+f*x])^(n+1)*
     Simp[b*(m-2)*(b*c-a*d)^2+a*d*(n+1)*(c*(a^2+b^2)-2*a*b*d)+
      (b*(n+1)*(a*b*c^2+c*d*(a^2+b^2)-3*a*b*d^2)-a*(n+2)*(b*c-a*d)^2)*Sin[e+f*x]+
      b*(b^2*(c^2-d^2)-m*(b*c-a*d)^2+d*n*(2*a*b*c-d*(a^2+b^2)))*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[m,2] && LtQ[n,-1] && (IntegerQ[m] || IntegersQ[2
```

2: $\int \left(a + b \, Sin[e + f \, x]\right)^m \left(c + d \, Sin[e + f \, x]\right)^n \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0 \wedge m > 2 \wedge n \not< -1$

Derivation: Nondegenerate sine recurrence 1b with $A \to a^2$, $B \to 2 \, a \, b$, $C \to b^2$, $m \to m - 2$, $p \to 0$

Rule: If $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0 \wedge m > 2 \wedge n \not< -1$, then

$$\int \left(a + b \, Sin[e + f \, x]\right)^m \left(c + d \, Sin[e + f \, x]\right)^n \, dx \longrightarrow$$

$$-\frac{b^2 \, Cos[e + f \, x] \, \left(a + b \, Sin[e + f \, x]\right)^{m-2} \left(c + d \, Sin[e + f \, x]\right)^{n+1}}{d \, f \, (m + n)} +$$

$$\frac{1}{d \, (m + n)} \int \left(a + b \, Sin[e + f \, x]\right)^{m-3} \left(c + d \, Sin[e + f \, x]\right)^n \cdot$$

$$\left(a^3 \, d \, (m + n) + b^2 \, \left(b \, c \, (m - 2) + a \, d \, (n + 1)\right) - b \, \left(a \, b \, c - b^2 \, d \, (m + n - 1) - 3 \, a^2 \, d \, (m + n)\right) Sin[e + f \, x] - b^2 \, \left(b \, c \, (m - 1) - a \, d \, (3 \, m + 2 \, n - 2)\right) Sin[e + f \, x]^2\right) dx$$

Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
 -b^2*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m-2)*(c+d*Sin[e+f*x])^(n+1)/(d*f*(m+n)) +
  1/(d*(m+n))*Int[(a+b*Sin[e+f*x])^(m-3)*(c+d*Sin[e+f*x])^n*
   Simp[a^3*d*(m+n)+b^2*(b*c*(m-2)+a*d*(n+1))-
    b*(a*b*c-b^2*d*(m+n-1)-3*a^2*d*(m+n))*Sin[e+f*x]-
    b^2*(b*c*(m-1)-a*d*(3*m+2*n-2))*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f,n},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[m,2] &&
  (IntegerQ[m] || IntegersQ[2*m,2*n]) && Not[IGtQ[n,2] && (Not[IntegerQ[m]] || EqQ[a,0] && NeQ[c,0])]
```

3. $\int \left(a + b \, Sin[e + f \, x]\right)^m \left(c + d \, Sin[e + f \, x]\right)^n \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0 \wedge m < -1$

  1. $\int \left(a + b \, Sin[e + f \, x]\right)^m \left(c + d \, Sin[e + f \, x]\right)^n \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0 \wedge m < -1 \wedge 0 < n < 2$

    1. $\int \left(a + b \, Sin[e + f \, x]\right)^m \left(c + d \, Sin[e + f \, x]\right)^n \, dx$ when $b \, c - a \, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0 \wedge m < -1 \wedge 0 < n < 1$

1. $\displaystyle\int \frac{\sqrt{c + d\,Sin[e + f\,x]}}{(a + b\,Sin[e + f\,x])^{3/2}}\,dx$ when $b\,c - a\,d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0$

1: $\displaystyle\int \frac{\sqrt{d\,Sin[e + f\,x]}}{(a + b\,Sin[e + f\,x])^{3/2}}\,dx$ when $a^2 - b^2 \neq 0$

Derivation: Nondegenerate sine recurrence 1a with $A \to 0$, $B \to d$, $C \to 0$, $m \to -\frac{3}{2}$, $n \to -\frac{1}{2}$, $p \to 0$

Rule: If $a^2 - b^2 \neq 0$, then

$$\int \frac{\sqrt{d\,Sin[e + f\,x]}}{(a + b\,Sin[e + f\,x])^{3/2}}\,dx \longrightarrow -\frac{2\,a\,d\,Cos[e + f\,x]}{f\,(a^2 - b^2)\,\sqrt{a + b\,Sin[e + f\,x]}\,\sqrt{d\,Sin[e + f\,x]}} - \frac{d^2}{a^2 - b^2}\int \frac{\sqrt{a + b\,Sin[e + f\,x]}}{(d\,Sin[e + f\,x])^{3/2}}\,dx$$

Program code:

```
Int[Sqrt[d_.*sin[e_.+f_.*x_]]/(a_+b_.*sin[e_.+f_.*x_])^(3/2),x_Symbol] :=
  -2*a*d*Cos[e+f*x]/(f*(a^2-b^2)*Sqrt[a+b*Sin[e+f*x]]*Sqrt[d*Sin[e+f*x]]) -
  d^2/(a^2-b^2)*Int[Sqrt[a+b*Sin[e+f*x]]/(d*Sin[e+f*x])^(3/2),x] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0]
```

2: $\displaystyle\int \frac{\sqrt{c + d\,Sin[e + f\,x]}}{(a + b\,Sin[e + f\,x])^{3/2}}\,dx$ when $b\,c - a\,d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0$

Derivation: Algebraic expansion

Basis: $\displaystyle\frac{\sqrt{c + d\,z}}{(a + b\,z)^{3/2}} == \frac{c - d}{a - b}\,\frac{1}{\sqrt{a + b\,z}\,\sqrt{c + d\,z}} - \frac{b\,c - a\,d}{a - b}\,\frac{1 + z}{(a + b\,z)^{3/2}\,\sqrt{c + d\,z}}$

Rule: If $b\,c - a\,d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0$, then

$$\int \frac{\sqrt{c + d\,Sin[e + f\,x]}}{(a + b\,Sin[e + f\,x])^{3/2}}\,dx \longrightarrow$$

$$\frac{c-d}{a-b}\int\frac{1}{\sqrt{a+b\,Sin[e+f\,x]}\,\sqrt{c+d\,Sin[e+f\,x]}}\,dx - \frac{b\,c-a\,d}{a-b}\int\frac{1+Sin[e+f\,x]}{(a+b\,Sin[e+f\,x])^{3/2}\,\sqrt{c+d\,Sin[e+f\,x]}}\,dx$$

Program code:

```
Int[Sqrt[c_+d_.*sin[e_.+f_.*x_]]/(a_.+b_.*sin[e_.+f_.*x_])^(3/2),x_Symbol] :=
  (c-d)/(a-b)*Int[1/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]]),x] -
  (b*c-a*d)/(a-b)*Int[(1+Sin[e+f*x])/((a+b*Sin[e+f*x])^(3/2)*Sqrt[c+d*Sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

2: $\int (a+b\,Sin[e+f\,x])^m\,(c+d\,Sin[e+f\,x])^n\,dx$ when $b\,c-a\,d \neq 0 \wedge a^2-b^2 \neq 0 \wedge c^2-d^2 \neq 0 \wedge m < -1 \wedge 0 < n < 1$

Derivation: Nondegenerate sine recurrence 1a with $A \to 1$, $B \to 0$, $C \to 0$, $p \to 0$

Derivation: Nondegenerate sine recurrence 1c with $A \to c$, $B \to d$, $C \to 0$, $n \to n-1$, $p \to 0$

Rule: If $b\,c-a\,d \neq 0 \wedge a^2-b^2 \neq 0 \wedge c^2-d^2 \neq 0 \wedge m < -1 \wedge 0 < n < 1$, then

$$\int (a+b\,Sin[e+f\,x])^m\,(c+d\,Sin[e+f\,x])^n\,dx \longrightarrow$$

$$-\frac{b\,Cos[e+f\,x]\,(a+b\,Sin[e+f\,x])^{m+1}\,(c+d\,Sin[e+f\,x])^n}{f\,(m+1)\,(a^2-b^2)} +$$

$$\frac{1}{(m+1)\,(a^2-b^2)}\int (a+b\,Sin[e+f\,x])^{m+1}\,(c+d\,Sin[e+f\,x])^{n-1}\cdot$$

$$\left(a\,c\,(m+1)+b\,d\,n+\left(a\,d\,(m+1)-b\,c\,(m+2)\right)\,Sin[e+f\,x]-b\,d\,(m+n+2)\,Sin[e+f\,x]^2\right)\,dx$$

Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -b*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^n/(f*(m+1)*(a^2-b^2)) +
  1/((m+1)*(a^2-b^2))*Int[(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^(n-1)*
    Simp[a*c*(m+1)+b*d*n+(a*d*(m+1)-b*c*(m+2))*Sin[e+f*x]-b*d*(m+n+2)*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[m,-1] && LtQ[0,n,1] && IntegersQ[2*m,2*n]
```

2. $\int (a + b\,\text{Sin}[e + f\,x])^m\,(c + d\,\text{Sin}[e + f\,x])^n\,dx$ when $b\,c - a\,d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0 \,\wedge\, m < -1 \,\wedge\, 1 < n < 2$

   1. $\int \dfrac{(c + d\,\text{Sin}[e + f\,x])^{3/2}}{(a + b\,\text{Sin}[e + f\,x])^{3/2}}\,dx$ when $b\,c - a\,d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0$

      1: $\int \dfrac{(d\,\text{Sin}[e + f\,x])^{3/2}}{(a + b\,\text{Sin}[e + f\,x])^{3/2}}\,dx$ when $a^2 - b^2 \neq 0$

Derivation: Algebraic expansion

Basis: $\dfrac{(d\,z)^{3/2}}{(a+b\,z)^{3/2}} == \dfrac{d\,\sqrt{d\,z}}{b\,\sqrt{a+b\,z}} - \dfrac{a\,d\,\sqrt{d\,z}}{b\,(a+b\,z)^{3/2}}$

Rule: If $a^2 - b^2 \neq 0$, then

$$\int \frac{(d\,\text{Sin}[e + f\,x])^{3/2}}{(a + b\,\text{Sin}[e + f\,x])^{3/2}}\,dx \;\to\; \frac{d}{b}\int \frac{\sqrt{d\,\text{Sin}[e + f\,x]}}{\sqrt{a + b\,\text{Sin}[e + f\,x]}}\,dx - \frac{a\,d}{b}\int \frac{\sqrt{d\,\text{Sin}[e + f\,x]}}{(a + b\,\text{Sin}[e + f\,x])^{3/2}}\,dx$$

Program code:

```
Int[(d_.*sin[e_.+f_.*x_])^(3/2)/(a_+b_.*sin[e_.+f_.*x_])^(3/2),x_Symbol] :=
  d/b*Int[Sqrt[d*Sin[e+f*x]]/Sqrt[a+b*Sin[e+f*x]],x] -
  a*d/b*Int[Sqrt[d*Sin[e+f*x]]/(a+b*Sin[e+f*x])^(3/2),x] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0]
```

2: $\displaystyle\int \frac{\left(c + d \, \mathrm{Sin}[e + f \, x]\right)^{3/2}}{\left(a + b \, \mathrm{Sin}[e + f \, x]\right)^{3/2}} \, dx$ when $b \, c - a \, d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0$

Derivation: Algebraic expansion

Basis: $\displaystyle \frac{(c + d \, z)^{3/2}}{(a + b \, z)^{3/2}} == \frac{d^2 \, \sqrt{a + b \, z}}{b^2 \, \sqrt{c + d \, z}} + \frac{(b \, c - a \, d) \; (b \, c + a \, d + 2 \, b \, d \, z)}{b^2 \; (a + b \, z)^{3/2} \, \sqrt{c + d \, z}}$

Rule: If $b \, c - a \, d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0$, then

$$\int \frac{\left(c + d \, \mathrm{Sin}[e + f \, x]\right)^{3/2}}{\left(a + b \, \mathrm{Sin}[e + f \, x]\right)^{3/2}} \, dx \;\longrightarrow\; \frac{d^2}{b^2} \int \frac{\sqrt{a + b \, \mathrm{Sin}[e + f \, x]}}{\sqrt{c + d \, \mathrm{Sin}[e + f \, x]}} \, dx + \frac{(b \, c - a \, d)}{b^2} \int \frac{b \, c + a \, d + 2 \, b \, d \, \mathrm{Sin}[e + f \, x]}{\left(a + b \, \mathrm{Sin}[e + f \, x]\right)^{3/2} \sqrt{c + d \, \mathrm{Sin}[e + f \, x]}} \, dx$$

Program code:

```
Int[(c_+d_.*sin[e_.+f_.*x_])^(3/2)/(a_.+b_.*sin[e_.+f_.*x_])^(3/2),x_Symbol] :=
  d^2/b^2*Int[Sqrt[a+b*Sin[e+f*x]]/Sqrt[c+d*Sin[e+f*x]],x] +
  (b*c-a*d)/b^2*Int[Simp[b*c+a*d+2*b*d*Sin[e+f*x],x]/((a+b*Sin[e+f*x])^(3/2)*Sqrt[c+d*Sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

2: $\int \left(a + b \, \text{Sin}\left[e + f \, x\right]\right)^m \left(c + d \, \text{Sin}\left[e + f \, x\right]\right)^n \, dx$ when $bc - ad \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0 \wedge m < -1 \wedge 1 < n < 2$

Derivation: Nondegenerate sine recurrence 1a with A → c, B → d, C → 0, n → n − 1, p → 0

Rule: If $bc - ad \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0 \wedge m < -1 \wedge 1 < n < 2$, then

$$\int \left(a + b \, \text{Sin}\left[e + f \, x\right]\right)^m \left(c + d \, \text{Sin}\left[e + f \, x\right]\right)^n \, dx \longrightarrow$$

$$- \left(\left(\left(b \, c - a \, d\right) \, \text{Cos}\left[e + f \, x\right] \left(a + b \, \text{Sin}\left[e + f \, x\right]\right)^{m+1} \left(c + d \, \text{Sin}\left[e + f \, x\right]\right)^{n-1}\right) \Big/ \left(f \, (m + 1) \, \left(a^2 - b^2\right)\right)\right) +$$

$$\frac{1}{(m + 1) \, \left(a^2 - b^2\right)} \int \left(a + b \, \text{Sin}\left[e + f \, x\right]\right)^{m+1} \left(c + d \, \text{Sin}\left[e + f \, x\right]\right)^{n-2} \, \cdot$$

$$\left(c \, \left(a \, c - b \, d\right) \, (m + 1) + d \, \left(b \, c - a \, d\right) \, (n - 1) + \left(d \, \left(a \, c - b \, d\right) \, (m + 1) - c \, \left(b \, c - a \, d\right) \, (m + 2)\right) \, \text{Sin}\left[e + f \, x\right] - d \, \left(b \, c - a \, d\right) \, (m + n + 1) \, \text{Sin}\left[e + f \, x\right]^2\right) \, dx$$

Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -(b*c-a*d)*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^(n-1)/(f*(m+1)*(a^2-b^2)) +
  1/((m+1)*(a^2-b^2))*Int[(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^(n-2)*
    Simp[c*(a*c-b*d)*(m+1)+d*(b*c-a*d)*(n-1)+(d*(a*c-b*d)*(m+1)-c*(b*c-a*d)*(m+2))*Sin[e+f*x]-d*(b*c-a*d)*(m+n+1)*Sin[e+f*x]^2,x],x]
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[m,-1] && LtQ[1,n,2] && IntegersQ[2*m,2*n]
```

2. $\int \left(a + b \, \text{Sin}[e + f \, x]\right)^m \left(c + d \, \text{Sin}[e + f \, x]\right)^n \, dx$ when $b \, c - a \, d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0 \,\wedge\, m < -1 \,\wedge\, n \neq 0$

   1. $\int \dfrac{1}{\left(a + b \, \text{Sin}[e + f \, x]\right)^{3/2} \sqrt{c + d \, \text{Sin}[e + f \, x]}} \, dx$ when $b \, c - a \, d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0$

      1: $\int \dfrac{1}{\left(a + b \, \text{Sin}[e + f \, x]\right)^{3/2} \sqrt{d \, \text{Sin}[e + f \, x]}} \, dx$ when $a^2 - b^2 \neq 0$

Derivation: Nondegenerate sine recurrence 1a with $c \to 0$, $A \to 1$, $B \to 0$, $C \to 0$, $p \to 0$, $m \to -\frac{3}{2}$, $n \to -\frac{1}{2}$

Rule: If $a^2 - b^2 \neq 0$, then

$$\int \frac{1}{\left(a + b \, \text{Sin}[e + f \, x]\right)^{3/2} \sqrt{d \, \text{Sin}[e + f \, x]}} \, dx \;\to\; \frac{2 \, b \, \text{Cos}[e + f \, x]}{f \, (a^2 - b^2) \sqrt{a + b \, \text{Sin}[e + f \, x]} \, \sqrt{d \, \text{Sin}[e + f \, x]}} + \frac{d}{a^2 - b^2} \int \frac{b + a \, \text{Sin}[e + f \, x]}{\sqrt{a + b \, \text{Sin}[e + f \, x]} \, \left(d \, \text{Sin}[e + f \, x]\right)^{3/2}} \, dx$$

Program code:

```
Int[1/((a_+b_.*sin[e_.+f_.*x_])^(3/2)*Sqrt[d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  2*b*Cos[e+f*x]/(f*(a^2-b^2)*Sqrt[a+b*Sin[e+f*x]]*Sqrt[d*Sin[e+f*x]]) +
  d/(a^2-b^2)*Int[(b+a*Sin[e+f*x])/(Sqrt[a+b*Sin[e+f*x]]*(d*Sin[e+f*x])^(3/2)),x] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0]
```

      2: $\int \dfrac{1}{\left(a + b \, \text{Sin}[e + f \, x]\right)^{3/2} \sqrt{c + d \, \text{Sin}[e + f \, x]}} \, dx$ when $b \, c - a \, d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0$

Derivation: Algebraic expansion

Basis: $\dfrac{1}{(a+b \, z)^{3/2}} \;==\; \dfrac{1}{(a-b) \sqrt{a+b \, z}} - \dfrac{b \, (1+z)}{(a-b) \, (a+b \, z)^{3/2}}$

Rule: If $b \, c - a \, d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0$, then

$$\int \frac{1}{(a + b \, \text{Sin}[e + f \, x])^{3/2} \sqrt{c + d \, \text{Sin}[e + f \, x]}} \, dx \; \longrightarrow$$

$$\frac{1}{a - b} \int \frac{1}{\sqrt{a + b \, \text{Sin}[e + f \, x]} \, \sqrt{c + d \, \text{Sin}[e + f \, x]}} \, dx - \frac{b}{a - b} \int \frac{1 + \text{Sin}[e + f \, x]}{(a + b \, \text{Sin}[e + f \, x])^{3/2} \sqrt{c + d \, \text{Sin}[e + f \, x]}} \, dx$$

Program code:

```
Int[1/((a_.+b_.*sin[e_.+f_.*x_])^(3/2)*Sqrt[c_.+d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  1/(a-b)*Int[1/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]]),x] -
  b/(a-b)*Int[(1+Sin[e+f*x])/((a+b*Sin[e+f*x])^(3/2)*Sqrt[c+d*Sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

2: $\int (a + b \sin[e + f\,x])^m\ (c + d \sin[e + f\,x])^n\,dx$ when $b\,c - a\,d \neq 0\ \wedge\ a^2 - b^2 \neq 0\ \wedge\ c^2 - d^2 \neq 0\ \wedge\ m < -1\ \wedge\ n \not> 0$

Derivation: Nondegenerate sine recurrence 1c with $A \to 1$, $B \to 0$, $C \to 0$, $p \to 0$

Rule: If $b\,c - a\,d \neq 0\ \wedge\ a^2 - b^2 \neq 0\ \wedge\ c^2 - d^2 \neq 0\ \wedge\ m < -1\ \wedge\ n \not> 0$, then

$$\int (a + b \sin[e + f\,x])^m\ (c + d \sin[e + f\,x])^n\,dx\ \longrightarrow$$

$$-\,\frac{b^2 \cos[e + f\,x]\ (a + b \sin[e + f\,x])^{m+1}\ (c + d \sin[e + f\,x])^{n+1}}{f\ (m + 1)\ (b\,c - a\,d)\ (a^2 - b^2)}\ +$$

$$\frac{1}{(m + 1)\ (b\,c - a\,d)\ (a^2 - b^2)} \int (a + b \sin[e + f\,x])^{m+1}\ (c + d \sin[e + f\,x])^n\ \cdot$$

$$\Big(a\ (b\,c - a\,d)\ (m + 1) + b^2\,d\ (m + n + 2) - \big(b^2\,c + b\ (b\,c - a\,d)\ (m + 1)\big) \sin[e + f\,x] - b^2\,d\ (m + n + 3) \sin[e + f\,x]^2\Big)\,dx$$

Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -b^2*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^(n+1)/(f*(m+1)*(b*c-a*d)*(a^2-b^2)) +
  1/((m+1)*(b*c-a*d)*(a^2-b^2))*Int[(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^n*
    Simp[a*(b*c-a*d)*(m+1)+b^2*d*(m+n+2)-(b^2*c+b*(b*c-a*d)*(m+1))*Sin[e+f*x]-b^2*d*(m+n+3)*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f,n},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[m,-1] && IntegersQ[2*m,2*n] &&
  (EqQ[a,0] && IntegerQ[m] && Not[IntegerQ[n]] || Not[IntegerQ[2*n] && LtQ[n,-1] && (IntegerQ[n] && Not[IntegerQ[m]] || EqQ[a,0])])
```

4: $\displaystyle\int \frac{\sqrt{c + d\,\text{Sin}[e + f\,x]}}{a + b\,\text{Sin}[e + f\,x]}\,dx$ when $\mathbf{b\,c - a\,d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0}$

Derivation: Algebraic expansion

Basis: $\displaystyle\frac{\sqrt{c + d\,z}}{a + b\,z} == \frac{d}{b\,\sqrt{c + d\,z}} + \frac{b\,c - a\,d}{b\,(a + b\,z)\,\sqrt{c + d\,z}}$

Rule: If $b\,c - a\,d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0$, then

$$\int \frac{\sqrt{c + d\,\text{Sin}[e + f\,x]}}{a + b\,\text{Sin}[e + f\,x]}\,dx \;\longrightarrow\; \frac{d}{b}\int \frac{1}{\sqrt{c + d\,\text{Sin}[e + f\,x]}}\,dx + \frac{b\,c - a\,d}{b}\int \frac{1}{(a + b\,\text{Sin}[e + f\,x])\,\sqrt{c + d\,\text{Sin}[e + f\,x]}}\,dx$$

Program code:

```
Int[Sqrt[c_.+d_.*sin[e_.+f_.*x_]]/(a_.+b_.*sin[e_.+f_.*x_]),x_Symbol] :=
  d/b*Int[1/Sqrt[c+d*Sin[e+f*x]],x] +
  (b*c-a*d)/b*Int[1/((a+b*Sin[e+f*x])*Sqrt[c+d*Sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

5: $\int \dfrac{(a + b \, \mathrm{Sin}[e + f \, x])^{3/2}}{c + d \, \mathrm{Sin}[e + f \, x]} \, dx$ when $b \, c - a \, d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0$

Derivation: Algebraic expansion

Basis: $\dfrac{a + b \, z}{c + d \, z} == \dfrac{b}{d} - \dfrac{b \, c - a \, d}{d \, (c + d \, z)}$

Rule: If $b \, c - a \, d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0$, then

$$\int \dfrac{(a + b \, \mathrm{Sin}[e + f \, x])^{3/2}}{c + d \, \mathrm{Sin}[e + f \, x]} \, dx \;\longrightarrow\; \dfrac{b}{d} \int \sqrt{a + b \, \mathrm{Sin}[e + f \, x]} \; dx - \dfrac{b \, c - a \, d}{d} \int \dfrac{\sqrt{a + b \, \mathrm{Sin}[e + f \, x]}}{c + d \, \mathrm{Sin}[e + f \, x]} \, dx$$

Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^(3/2)/(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  b/d*Int[Sqrt[a+b*Sin[e+f*x]],x] - (b*c-a*d)/d*Int[Sqrt[a+b*Sin[e+f*x]]/(c+d*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

6. $\int \dfrac{1}{(a + b \, \mathrm{Sin}[e + f \, x]) \, \sqrt{c + d \, \mathrm{Sin}[e + f \, x]}} \, dx$ when $b \, c - a \, d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0$

1: $\int \dfrac{1}{(a + b \, \mathrm{Sin}[e + f \, x]) \, \sqrt{c + d \, \mathrm{Sin}[e + f \, x]}} \, dx$ when $b \, c - a \, d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0 \,\wedge\, c + d > 0$

Derivation: Primitive rule

Basis: If $c + d > 0$, then $\partial_x \, \mathtt{EllipticPi}\left[\dfrac{2 \, b}{a + b}, \; \dfrac{1}{2}\left(x - \dfrac{\pi}{2}\right), \; \dfrac{2 \, d}{c + d}\right] == \dfrac{(a+b) \, \sqrt{c+d}}{2 \, (a + b \, \mathrm{Sin}[x]) \, \sqrt{c + d \, \mathrm{Sin}[x]}}$

Rule: If $b \, c - a \, d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0 \,\wedge\, c + d > 0$, then

$$\int \frac{1}{(a + b \, \text{Sin}[e + f \, x]) \, \sqrt{c + d \, \text{Sin}[e + f \, x]}} \, dx \;\rightarrow\; \frac{2}{f \, (a + b) \, \sqrt{c + d}} \, \text{EllipticPi}\left[\frac{2 \, b}{a + b}, \frac{1}{2}\left(e - \frac{\pi}{2} + f \, x\right), \frac{2 \, d}{c + d}\right]$$

Program code:

```
Int[1/((a_.+b_.*sin[e_.+f_.*x_])*Sqrt[c_.+d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  2/(f*(a+b)*Sqrt[c+d])*EllipticPi[2*b/(a+b),1/2*(e-Pi/2+f*x),2*d/(c+d)] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[c+d,0]
```

2: $\int \dfrac{1}{(a + b \, \text{Sin}[e + f \, x]) \, \sqrt{c + d \, \text{Sin}[e + f \, x]}} \, dx$ when $b \, c - a \, d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0 \,\wedge\, c - d > 0$

Derivation: Primitive rule

Basis: If $c - d > 0$, then $\partial_x \, \text{EllipticPi}\left[-\frac{2 \, b}{a - b}, \frac{1}{2}\left(x + \frac{\pi}{2}\right), -\frac{2 \, d}{c - d}\right] \;==\; \dfrac{(a - b) \, \sqrt{c - d}}{2 \, (a + b \, \text{Sin}[x]) \, \sqrt{c + d \, \text{Sin}[x]}}$

Rule: If $b \, c - a \, d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0 \,\wedge\, c - d > 0$, then

$$\int \frac{1}{(a + b \, \text{Sin}[e + f \, x]) \, \sqrt{c + d \, \text{Sin}[e + f \, x]}} \, dx \;\rightarrow\; \frac{2}{f \, (a - b) \, \sqrt{c - d}} \, \text{EllipticPi}\left[-\frac{2 \, b}{a - b}, \frac{1}{2}\left(e + \frac{\pi}{2} + f \, x\right), -\frac{2 \, d}{c - d}\right]$$

Program code:

```
Int[1/((a_.+b_.*sin[e_.+f_.*x_])*Sqrt[c_.+d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  2/(f*(a-b)*Sqrt[c-d])*EllipticPi[-2*b/(a-b),1/2*(e+Pi/2+f*x),-2*d/(c-d)] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[c-d,0]
```

3: $\displaystyle\int \frac{1}{(a + b\,\mathrm{Sin}[e + f\,x])\,\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}}\,\mathbb{d}x$ when $\mathbf{b\,c - a\,d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0 \wedge c + d \not> 0}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{\sqrt{\frac{c + d\,F[x]}{c + d}}}{\sqrt{c + d\,F[x]}} == 0$

Rule: If $\mathrm{b\,c - a\,d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0 \wedge c + d \not> 0}$, then

$$\int \frac{1}{(a + b\,\mathrm{Sin}[e + f\,x])\,\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}}\,\mathbb{d}x \;\longrightarrow\; \frac{\sqrt{\frac{c + d\,\mathrm{Sin}[e + f\,x]}{c + d}}}{\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}} \int \frac{1}{(a + b\,\mathrm{Sin}[e + f\,x])\,\sqrt{\frac{c}{c+d} + \frac{d}{c+d}\,\mathrm{Sin}[e + f\,x]}}\,\mathbb{d}x$$

Program code:

```
Int[1/((a_.+b_.*sin[e_.+f_.*x_])*Sqrt[c_.+d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  Sqrt[(c+d*Sin[e+f*x])/(c+d)]/Sqrt[c+d*Sin[e+f*x]]*Int[1/((a+b*Sin[e+f*x])*Sqrt[c/(c+d)+d/(c+d)*Sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && Not[GtQ[c+d,0]]
```

7. $\displaystyle\int \frac{\sqrt{a + b\,\mathrm{Sin}[e + f\,x]}}{\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}}\,\mathbb{d}x$ when $\mathbf{b\,c - a\,d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0}$

   1. $\displaystyle\int \frac{\sqrt{b\,\mathrm{Sin}[e + f\,x]}}{\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}}\,\mathbb{d}x$ when $\mathbf{c^2 - d^2 \neq 0}$

     1. $\displaystyle\int \frac{\sqrt{b\,\mathrm{Sin}[e + f\,x]}}{\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}}\,\mathbb{d}x$ when $\mathbf{c^2 - d^2 \neq 0 \wedge \frac{c+d}{b} > 0}$

1: $\displaystyle\int \frac{\sqrt{b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $c^2 - d^2 > 0 \wedge \frac{c+d}{b} > 0 \wedge c^2 > 0$

Rule: If $c^2 - d^2 > 0 \wedge \frac{c+d}{b} > 0 \wedge c^2 > 0$, then

$$\int \frac{\sqrt{b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx \longrightarrow$$

$$\frac{2\,c\,\sqrt{b\,(c+d)}\,\,\text{Tan}[e+f\,x]\,\sqrt{1+\text{Csc}[e+f\,x]}\,\sqrt{1-\text{Csc}[e+f\,x]}}{d\,f\,\sqrt{c^2-d^2}}\,\text{EllipticPi}\left[\frac{c+d}{d},\,\text{ArcSin}\left[\frac{\sqrt{c+d\,\text{Sin}[e+f\,x]}}{\sqrt{b\,\text{Sin}[e+f\,x]}}\Big/\sqrt{\frac{c+d}{b}}\right],\,-\frac{c+d}{c-d}\right]$$

Program code:

```
Int[Sqrt[b_.*sin[e_.+f_.*x_]]/Sqrt[c_+d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  2*c*Rt[b*(c+d),2]*Tan[e+f*x]*Sqrt[1+Csc[e+f*x]]*Sqrt[1-Csc[e+f*x]]/(d*f*Sqrt[c^2-d^2])*
    EllipticPi[(c+d)/d,ArcSin[Sqrt[c+d*Sin[e+f*x]]/Sqrt[b*Sin[e+f*x]]/Rt[(c+d)/b,2]],-(c+d)/(c-d)] /;
FreeQ[{b,c,d,e,f},x] && GtQ[c^2-d^2,0] && PosQ[(c+d)/b] && GtQ[c^2,0]
```

2: $\displaystyle\int \frac{\sqrt{b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,\mathrm{d}x$　when $c^2 - d^2 \neq 0 \wedge \frac{c+d}{b} > 0$

Rule: If $c^2 - d^2 \neq 0 \wedge \frac{c+d}{b} > 0$, then

$$\int \frac{\sqrt{b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,\mathrm{d}x \longrightarrow$$

$$\frac{2\,b\,\mathrm{Tan}[e+f\,x]}{d\,f}\sqrt{\frac{c+d}{b}}\sqrt{\frac{c\,(1+\mathrm{Csc}[e+f\,x])}{c-d}}\sqrt{\frac{c\,(1-\mathrm{Csc}[e+f\,x])}{c+d}}\;\mathrm{EllipticPi}\left[\frac{c+d}{d},\ \mathrm{ArcSin}\left[\frac{\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}{\sqrt{b\,\mathrm{Sin}[e+f\,x]}}\Big/\sqrt{\frac{c+d}{b}}\right],\ -\frac{c+d}{c-d}\right]$$

Program code:

```
Int[Sqrt[b_.*sin[e_.+f_.*x_]]/Sqrt[c_+d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  2*b*Tan[e+f*x]/(d*f)*Rt[(c+d)/b,2]*Sqrt[c*(1+Csc[e+f*x])/(c-d)]*Sqrt[c*(1-Csc[e+f*x])/(c+d)]*
    EllipticPi[(c+d)/d,ArcSin[Sqrt[c+d*Sin[e+f*x]]/Sqrt[b*Sin[e+f*x]]/Rt[(c+d)/b,2]],-(c+d)/(c-d)] /;
FreeQ[{b,c,d,e,f},x] && NeQ[c^2-d^2,0] && PosQ[(c+d)/b]
```

2: $\displaystyle\int \frac{\sqrt{b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $c^2 - d^2 \neq 0 \,\wedge\, \frac{c+d}{b} \ngtr 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{\sqrt{F[x]}}{\sqrt{-F[x]}} == 0$

Rule: If $c^2 - d^2 \neq 0 \,\wedge\, \frac{c+d}{b} \ngtr 0$, then

$$\int \frac{\sqrt{b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx \;\longrightarrow\; \frac{\sqrt{b\,\text{Sin}[e+f\,x]}}{\sqrt{-b\,\text{Sin}[e+f\,x]}}\int \frac{\sqrt{-b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$$

Program code:

```
Int[Sqrt[b_.*sin[e_.+f_.*x_]]/Sqrt[c_+d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  Sqrt[b*Sin[e+f*x]]/Sqrt[-b*Sin[e+f*x]]*Int[Sqrt[-b*Sin[e+f*x]]/Sqrt[c+d*Sin[e+f*x]],x] /;
FreeQ[{b,c,d,e,f},x] && NeQ[c^2-d^2,0] && NegQ[(c+d)/b]
```

2. $\displaystyle\int \frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $b\,c - a\,d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0$

x: $\displaystyle\int \frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\sqrt{d\,\text{Sin}[e+f\,x]}}\,dx$ when $a^2 - b^2 \neq 0$

Derivation: Algebraic expansion

Basis: $\dfrac{\sqrt{a+b\,z}}{\sqrt{d\,z}} == \dfrac{a}{\sqrt{a+b\,z}\,\sqrt{d\,z}} + \dfrac{b\,\sqrt{d\,z}}{d\,\sqrt{a+b\,z}}$

Rule: If $a^2 - b^2 \neq 0$, then

$$\int \frac{\sqrt{a + b \, \text{Sin}[e + f \, x]}}{\sqrt{d \, \text{Sin}[e + f \, x]}} \, dx \;\longrightarrow\; a \int \frac{1}{\sqrt{a + b \, \text{Sin}[e + f \, x]} \; \sqrt{d \, \text{Sin}[e + f \, x]}} \, dx + \frac{b}{d} \int \frac{\sqrt{d \, \text{Sin}[e + f \, x]}}{\sqrt{a + b \, \text{Sin}[e + f \, x]}} \, dx$$

Program code:

```
(* Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/Sqrt[d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  a*Int[1/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[d*Sin[e+f*x]]),x] +
  b/d*Int[Sqrt[d*Sin[e+f*x]]/Sqrt[a+b*Sin[e+f*x]],x] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0] *)
```

x: $\displaystyle \int \frac{\sqrt{a + b \, \text{Sin}[e + f \, x]}}{\sqrt{d \, \text{Sin}[e + f \, x]}} \, dx$ when $a^2 - b^2 \neq 0 \;\wedge\; \frac{a+b}{d} > 0$

Rule: If $a^2 - b^2 \neq 0 \;\wedge\; \frac{a+b}{d} > 0$, then

$$\int \frac{\sqrt{a + b \, \text{Sin}[e + f \, x]}}{\sqrt{d \, \text{Sin}[e + f \, x]}} \, dx \;\longrightarrow$$

$$\frac{2 \, (a + b \, \text{Sin}[e + f \, x])}{d \, f \sqrt{\frac{a+b}{d}} \; \text{Cos}[e + f \, x]} \sqrt{\frac{a \, (1 - \text{Sin}[e + f \, x])}{a + b \, \text{Sin}[e + f \, x]}} \sqrt{\frac{a \, (1 + \text{Sin}[e + f \, x])}{a + b \, \text{Sin}[e + f \, x]}} \; \text{EllipticPi}\left[\frac{b}{a + b}, \text{ArcSin}\left[\sqrt{\frac{a + b}{d}} \; \frac{\sqrt{d \, \text{Sin}[e + f \, x]}}{\sqrt{a + b \, \text{Sin}[e + f \, x]}}\right], -\frac{a - b}{a + b}\right]$$

Program code:

```
(* Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/Sqrt[d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  2*(a+b*Sin[e+f*x])/(d*f*Rt[(a+b)/d,2]*Cos[e+f*x])*Sqrt[a*(1-Sin[e+f*x])/(a+b*Sin[e+f*x])]*Sqrt[a*(1+Sin[e+f*x])/(a+b*Sin[e+f*x])]*
    EllipticPi[b/(a+b),ArcSin[Rt[(a+b)/d,2]*(Sqrt[d*Sin[e+f*x]]/Sqrt[a+b*Sin[e+f*x]])],-(a-b)/(a+b)] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0] && PosQ[(a+b)/d] *)
```

1: $\displaystyle\int \frac{\sqrt{a + b\,\mathrm{Sin}[e + f\,x]}}{\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}}\,dx$ when $b\,c - a\,d \neq 0 \;\wedge\; a^2 - b^2 \neq 0 \;\wedge\; c^2 - d^2 \neq 0 \;\wedge\; \frac{a+b}{c+d} > 0$

Rule: If $b\,c - a\,d \neq 0 \;\wedge\; a^2 - b^2 \neq 0 \;\wedge\; c^2 - d^2 \neq 0 \;\wedge\; \frac{a+b}{c+d} > 0$, then

$$\int \frac{\sqrt{a + b\,\mathrm{Sin}[e + f\,x]}}{\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}}\,dx \;\longrightarrow$$

$$\frac{2\,(a + b\,\mathrm{Sin}[e + f\,x])}{d\,f\,\sqrt{\frac{a+b}{c+d}}\,\mathrm{Cos}[e + f\,x]} \sqrt{\frac{(b\,c - a\,d)\,(1 + \mathrm{Sin}[e + f\,x])}{(c - d)\,(a + b\,\mathrm{Sin}[e + f\,x])}}$$

$$\sqrt{-\frac{(b\,c - a\,d)\,(1 - \mathrm{Sin}[e + f\,x])}{(c + d)\,(a + b\,\mathrm{Sin}[e + f\,x])}}\;\mathrm{EllipticPi}\left[\frac{b\,(c + d)}{d\,(a + b)},\;\mathrm{ArcSin}\left[\sqrt{\frac{a+b}{c+d}}\;\frac{\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}}{\sqrt{a + b\,\mathrm{Sin}[e + f\,x]}}\right],\;\frac{(a - b)\,(c + d)}{(a + b)\,(c - d)}\right]$$

Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/Sqrt[c_.+d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  2*(a+b*Sin[e+f*x])/(d*f*Rt[(a+b)/(c+d),2]*Cos[e+f*x])*
    Sqrt[(b*c-a*d)*(1+Sin[e+f*x])/((c-d)*(a+b*Sin[e+f*x]))]*
    Sqrt[-(b*c-a*d)*(1-Sin[e+f*x])/((c+d)*(a+b*Sin[e+f*x]))]*
    EllipticPi[b*(c+d)/(d*(a+b)),ArcSin[Rt[(a+b)/(c+d),2]*Sqrt[c+d*Sin[e+f*x]]/Sqrt[a+b*Sin[e+f*x]]],(a-b)*(c+d)/((a+b)*(c-d))] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && PosQ[(a+b)/(c+d)]
```

2: $\int \dfrac{\sqrt{a + b\,\mathrm{Sin}[e + f\,x]}}{\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}}\,\mathrm{d}x$ when $\mathbf{b\,c - a\,d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0 \wedge \frac{a+b}{c+d} \not> 0}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{\sqrt{F[x]}}{\sqrt{-F[x]}} == 0$

Rule: If $b\,c - a\,d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0 \wedge \dfrac{a+b}{c+d} \not> 0$, then

$$\int \frac{\sqrt{a + b\,\mathrm{Sin}[e + f\,x]}}{\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}}\,\mathrm{d}x \;\longrightarrow\; \frac{\sqrt{-c - d\,\mathrm{Sin}[e + f\,x]}}{\sqrt{c + d\,\mathrm{Sin}[e + f\,x]}} \int \frac{\sqrt{a + b\,\mathrm{Sin}[e + f\,x]}}{\sqrt{-c - d\,\mathrm{Sin}[e + f\,x]}}\,\mathrm{d}x$$

Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/Sqrt[c_.+d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  Sqrt[-c-d*Sin[e+f*x]]/Sqrt[c+d*Sin[e+f*x]]*Int[Sqrt[a+b*Sin[e+f*x]]/Sqrt[-c-d*Sin[e+f*x]],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && NegQ[(a+b)/(c+d)]
```

8. $\displaystyle\int \frac{1}{\sqrt{a + b \operatorname{Sin}[e + f\, x]}\,\sqrt{c + d \operatorname{Sin}[e + f\, x]}}\; dx$ when $b\, c - a\, d \neq 0 \;\wedge\; a^2 - b^2 \neq 0 \;\wedge\; c^2 - d^2 \neq 0$

   1. $\displaystyle\int \frac{1}{\sqrt{a + b \operatorname{Sin}[e + f\, x]}\,\sqrt{d \operatorname{Sin}[e + f\, x]}}\; dx$ when $a^2 - b^2 \neq 0$

     1. $\displaystyle\int \frac{1}{\sqrt{a + b \operatorname{Sin}[e + f\, x]}\,\sqrt{d \operatorname{Sin}[e + f\, x]}}\; dx$ when $a^2 - b^2 < 0 \;\wedge\; b^2 > 0$

       1: $\displaystyle\int \frac{1}{\sqrt{a + b \operatorname{Sin}[e + f\, x]}\,\sqrt{d \operatorname{Sin}[e + f\, x]}}\; dx$ when $a^2 - b^2 < 0 \;\wedge\; d^2 == 1 \;\wedge\; b\, d > 0$

Derivation: Integration by substitution

Basis: If $a^2 - b^2 < 0 \;\wedge\; d^2 == 1 \;\wedge\; b\, d > 0$, then

$$\frac{1}{\sqrt{a+b \operatorname{Sin}[e+f\,x]}\,\sqrt{d \operatorname{Sin}[e+f\,x]}} == -\frac{2\,d}{f\,\sqrt{a+b\,d}}\; \operatorname{Subst}\!\left[\frac{1}{\sqrt{1-x^2}\,\sqrt{1+\frac{(a-b\,d)\,x^2}{a+b\,d}}}, \; x, \; \frac{\operatorname{Cos}[e+f\,x]}{1+d \operatorname{Sin}[e+f\,x]}\right] \partial_x \frac{\operatorname{Cos}[e+f\,x]}{1+d \operatorname{Sin}[e+f\,x]}$$

Rule: If $a^2 - b^2 < 0 \;\wedge\; d^2 == 1 \;\wedge\; b\, d > 0$, then

$$\int \frac{1}{\sqrt{a + b \operatorname{Sin}[e + f\, x]}\,\sqrt{d \operatorname{Sin}[e + f\, x]}}\; dx \;\rightarrow\; -\frac{2\,d}{f\,\sqrt{a + b\,d}}\; \operatorname{Subst}\!\left[\int \frac{1}{\sqrt{1 - x^2}\,\sqrt{1 + \frac{(a-b\,d)\,x^2}{a+b\,d}}}\; dx, \; x, \; \frac{\operatorname{Cos}[e + f\, x]}{1 + d \operatorname{Sin}[e + f\, x]}\right]$$

$$\rightarrow\; -\frac{2\,d}{f\,\sqrt{a + b\,d}}\; \operatorname{EllipticF}\!\left[\operatorname{ArcSin}\!\left[\frac{\operatorname{Cos}[e + f\, x]}{1 + d \operatorname{Sin}[e + f\, x]}\right], \; -\frac{a - b\,d}{a + b\,d}\right]$$

Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  -2*d/(f*Sqrt[a+b*d])*EllipticF[ArcSin[Cos[e+f*x]/(1+d*Sin[e+f*x])],-(a-b*d)/(a+b*d)] /;
FreeQ[{a,b,d,e,f},x] && LtQ[a^2-b^2,0] && EqQ[d^2,1] && GtQ[b*d,0]
```

2: $\int \dfrac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\ \sqrt{d\,\text{Sin}[e+f\,x]}}\,dx$ when $a^2 - b^2 < 0 \ \wedge\ b^2 > 0 \ \wedge\ \neg\ (d^2 == 1 \ \wedge\ b\,d > 0)$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{\sqrt{b\,F[x]}}{\sqrt{d\,F[x]}} == 0$

Rule: If $a^2 - b^2 < 0 \ \wedge\ b^2 > 0 \ \wedge\ \neg\ \left(d^2 == 1 \ \wedge\ b\,d > 0\right)$, then

$$\int \dfrac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\ \sqrt{d\,\text{Sin}[e+f\,x]}}\,dx \ \longrightarrow\ \dfrac{\sqrt{\text{Sign}[b]\,\text{Sin}[e+f\,x]}}{\sqrt{d\,\text{Sin}[e+f\,x]}} \int \dfrac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\ \sqrt{\text{Sign}[b]\,\text{Sin}[e+f\,x]}}\,dx$$

Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  Sqrt[Sign[b]*Sin[e+f*x]]/Sqrt[d*Sin[e+f*x]]*Int[1/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[Sign[b]*Sin[e+f*x]]),x] /;
FreeQ[{a,b,d,e,f},x] && LtQ[a^2-b^2,0] && GtQ[b^2] && Not[EqQ[d^2,1] && GtQ[b*d,0]]
```

2. $\displaystyle\int \frac{1}{\sqrt{a + b\,\text{Sin}[e + f\,x]}\,\sqrt{d\,\text{Sin}[e + f\,x]}}\,dx$ when $a^2 - b^2 \neq 0 \;\wedge\; \frac{a+b}{d} > 0$

1: $\displaystyle\int \frac{1}{\sqrt{a + b\,\text{Sin}[e + f\,x]}\,\sqrt{d\,\text{Sin}[e + f\,x]}}\,dx$ when $a^2 - b^2 > 0 \;\wedge\; \frac{a+b}{d} > 0 \;\wedge\; a^2 > 0$

Rule: If $a^2 - b^2 > 0 \;\wedge\; \frac{a+b}{d} > 0 \;\wedge\; a^2 > 0$, then

$$\int \frac{1}{\sqrt{a + b\,\text{Sin}[e + f\,x]}\,\sqrt{d\,\text{Sin}[e + f\,x]}}\,dx \;\longrightarrow\; -\frac{2\sqrt{a^2}\,\sqrt{-\text{Cot}[e + f\,x]^2}}{a\,f\,\sqrt{a^2 - b^2}\,\text{Cot}[e + f\,x]}\,\sqrt{\frac{a + b}{d}}\;\text{EllipticF}\left[\text{ArcSin}\left[\frac{\sqrt{a + b\,\text{Sin}[e + f\,x]}}{\sqrt{d\,\text{Sin}[e + f\,x]}}\right/\sqrt{\frac{a + b}{d}}\right], -\frac{a + b}{a - b}\right]$$

Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  -2*Sqrt[a^2]*Sqrt[-Cot[e+f*x]^2]/(a*f*Sqrt[a^2-b^2]*Cot[e+f*x])*Rt[(a+b)/d,2]*
    EllipticF[ArcSin[Sqrt[a+b*Sin[e+f*x]]/Sqrt[d*Sin[e+f*x]]/Rt[(a+b)/d,2]],-(a+b)/(a-b)] /;
FreeQ[{a,b,d,e,f},x] && GtQ[a^2-b^2,0] && PosQ[(a+b)/d] && GtQ[a^2,0]
```

2: $\displaystyle\int \frac{1}{\sqrt{a + b\,\text{Sin}[e + f\,x]}\,\sqrt{d\,\text{Sin}[e + f\,x]}}\,dx$ when $a^2 - b^2 \neq 0 \wedge \frac{a+b}{d} > 0$

Rule: If $a^2 - b^2 \neq 0 \wedge \frac{a+b}{d} > 0$, then

$$\int \frac{1}{\sqrt{a + b\,\text{Sin}[e + f\,x]}\,\sqrt{d\,\text{Sin}[e + f\,x]}}\,dx \longrightarrow$$

$$-\frac{2\,\text{Tan}[e + f\,x]}{a\,f}\sqrt{\frac{a + b}{d}}\sqrt{\frac{a\,(1 - \text{Csc}[e + f\,x])}{a + b}}\sqrt{\frac{a\,(1 + \text{Csc}[e + f\,x])}{a - b}}\;\text{EllipticF}\left[\text{ArcSin}\left[\frac{\sqrt{a + b\,\text{Sin}[e + f\,x]}}{\sqrt{d\,\text{Sin}[e + f\,x]}}\Big/\sqrt{\frac{a + b}{d}}\right], -\frac{a + b}{a - b}\right]$$

Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  -2*Tan[e+f*x]/(a*f)*Rt[(a+b)/d,2]*Sqrt[a*(1-Csc[e+f*x])/(a+b)]*Sqrt[a*(1+Csc[e+f*x])/(a-b)]*
    EllipticF[ArcSin[Sqrt[a+b*Sin[e+f*x]]/Sqrt[d*Sin[e+f*x]]/Rt[(a+b)/d,2]],-(a+b)/(a-b)] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0] && PosQ[(a+b)/d]
```

3: $\displaystyle\int \frac{1}{\sqrt{a + b\, Sin[e + f\, x]}\ \sqrt{d\, Sin[e + f\, x]}}\, dx$ when $a^2 - b^2 \neq 0 \wedge \frac{a+b}{d} \not> 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{\sqrt{-F[x]}}{\sqrt{F[x]}} == 0$

Rule: If $a^2 - b^2 \neq 0 \wedge \frac{a+b}{d} \not> 0$, then

$$\int \frac{1}{\sqrt{a + b\, Sin[e + f\, x]}\ \sqrt{d\, Sin[e + f\, x]}}\, dx \;\rightarrow\; \frac{\sqrt{-d\, Sin[e + f\, x]}}{\sqrt{d\, Sin[e + f\, x]}} \int \frac{1}{\sqrt{a + b\, Sin[e + f\, x]}\ \sqrt{-d\, Sin[e + f\, x]}}\, dx$$

Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  Sqrt[-d*Sin[e+f*x]]/Sqrt[d*Sin[e+f*x]]*Int[1/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[-d*Sin[e+f*x]]),x] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0] && NegQ[(a+b)/d]
```

2. $\displaystyle\int \frac{1}{\sqrt{a + b\, Sin[e + f\, x]}\ \sqrt{c + d\, Sin[e + f\, x]}}\, dx$ when $b\, c - a\, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0$

1: $\displaystyle\int \frac{1}{\sqrt{a + b\, Sin[e + f\, x]}\ \sqrt{c + d\, Sin[e + f\, x]}}\, dx$ when $b\, c - a\, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0 \wedge \frac{c+d}{a+b} > 0$

Note: Alternative antiderivative contributed via email by Martin Welz on 12 April 2014.

Rule: If $b\, c - a\, d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0 \wedge \frac{c+d}{a+b} > 0$, then

$$\int \frac{1}{\sqrt{a + b\, Sin[e + f\, x]}\ \sqrt{c + d\, Sin[e + f\, x]}}\, dx \;\rightarrow$$

$$\frac{2\ (c+d\ Sin[e+f\ x])}{f\ (b\ c-a\ d)\ \sqrt{\frac{c+d}{a+b}}\ Cos[e+f\ x]}\ \sqrt{\frac{(b\ c-a\ d)\ (1-Sin[e+f\ x])}{(a+b)\ (c+d\ Sin[e+f\ x])}}$$

$$\sqrt{-\frac{(b\ c-a\ d)\ (1+Sin[e+f\ x])}{(a-b)\ (c+d\ Sin[e+f\ x])}}\ EllipticF\left[ArcSin\left[\sqrt{\frac{c+d}{a+b}}\ \frac{\sqrt{a+b\ Sin[e+f\ x]}}{\sqrt{c+d\ Sin[e+f\ x]}}\right],\ \frac{(a+b)\ (c-d)}{(a-b)\ (c+d)}\right]$$

$$\int \frac{1}{\sqrt{a+b\ Sin[e+f\ x]}\ \sqrt{c+d\ Sin[e+f\ x]}}\ dx\ \longrightarrow$$

$$\frac{2\ (1-Sin[e+f\ x])}{f\ \sqrt{-\frac{a+b}{a-b}}\ \sqrt{a+b\ Sin[e+f\ x]}\ \sqrt{c+d\ Sin[e+f\ x]}}\ \sqrt{\frac{a+b\ Sin[e+f\ x]}{(a-b)\ (1-Sin[e+f\ x])}}$$

$$\sqrt{\frac{c+d\ Sin[e+f\ x]}{(c-d)\ (1-Sin[e+f\ x])}}\ EllipticF\left[ArcSin\left[\sqrt{-\frac{a+b}{a-b}}\ \frac{1+Sin[e+f\ x]}{Cos[e+f\ x]}\right],\ \frac{(a-b)\ (c+d)}{(a+b)\ (c-d)}\right]$$

Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[c_+d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  2*(c+d*Sin[e+f*x])/(f*(b*c-a*d)*Rt[(c+d)/(a+b),2]*Cos[e+f*x])*
    Sqrt[(b*c-a*d)*(1-Sin[e+f*x])/((a+b)*(c+d*Sin[e+f*x]))]*
    Sqrt[-(b*c-a*d)*(1+Sin[e+f*x])/((a-b)*(c+d*Sin[e+f*x]))]*
    EllipticF[ArcSin[Rt[(c+d)/(a+b),2]*(Sqrt[a+b*Sin[e+f*x]]/Sqrt[c+d*Sin[e+f*x]])],(a+b)*(c-d)/((a-b)*(c+d))] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && PosQ[(c+d)/(a+b)]
```

2: $\displaystyle \int \frac{1}{\sqrt{a+b\ Sin[e+f\ x]}\ \sqrt{c+d\ Sin[e+f\ x]}}\ dx$ when $b\ c-a\ d \neq 0\ \wedge\ a^2-b^2 \neq 0\ \wedge\ c^2-d^2 \neq 0\ \wedge\ \frac{c+d}{a+b} \not> 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{\sqrt{-F[x]}}{\sqrt{F[x]}} == 0$

Rule: If $b\,c - a\,d \neq 0 \,\wedge\, a^2 - b^2 \neq 0 \,\wedge\, c^2 - d^2 \neq 0 \,\wedge\, \dfrac{c+d}{a+b} \not> 0$, then

$$\int \frac{1}{\sqrt{a + b\,\operatorname{Sin}[e + f\,x]}\,\sqrt{c + d\,\operatorname{Sin}[e + f\,x]}}\,dx \;\rightarrow\; \frac{\sqrt{-a - b\,\operatorname{Sin}[e + f\,x]}}{\sqrt{a + b\,\operatorname{Sin}[e + f\,x]}}\int \frac{1}{\sqrt{-a - b\,\operatorname{Sin}[e + f\,x]}\,\sqrt{c + d\,\operatorname{Sin}[e + f\,x]}}\,dx$$

Program code:

```
Int[1/(Sqrt[a_.+b_.*sin[e_.+f_.*x_]]*Sqrt[c_+d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  Sqrt[-a-b*Sin[e+f*x]]/Sqrt[a+b*Sin[e+f*x]]*Int[1/(Sqrt[-a-b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && NegQ[(c+d)/(a+b)]
```

9: $\displaystyle\int \frac{(d\,\operatorname{Sin}[e + f\,x])^{3/2}}{\sqrt{a + b\,\operatorname{Sin}[e + f\,x]}}\,dx$ when $a^2 - b^2 \neq 0$

Derivation: Algebraic expansion

Basis: $(d\,z)^{3/2} == -\dfrac{a\,d\,\sqrt{d\,z}}{2\,b} + \dfrac{d\,\sqrt{d\,z}\,(a + 2\,b\,z)}{2\,b}$

Rule: If $a^2 - b^2 \neq 0$, then

$$\int \frac{(d\,\operatorname{Sin}[e + f\,x])^{3/2}}{\sqrt{a + b\,\operatorname{Sin}[e + f\,x]}}\,dx \;\rightarrow\; -\frac{a\,d}{2\,b}\int \frac{\sqrt{d\,\operatorname{Sin}[e + f\,x]}}{\sqrt{a + b\,\operatorname{Sin}[e + f\,x]}}\,dx + \frac{d}{2\,b}\int \frac{\sqrt{d\,\operatorname{Sin}[e + f\,x]}\,(a + 2\,b\,\operatorname{Sin}[e + f\,x])}{\sqrt{a + b\,\operatorname{Sin}[e + f\,x]}}\,dx$$

Program code:

```
Int[(d_.*sin[e_.+f_.*x_])^(3/2)/Sqrt[a_.+b_.*sin[e_.+f_.*x_]],x_Symbol] :=
  -a*d/(2*b)*Int[Sqrt[d*Sin[e+f*x]]/Sqrt[a+b*Sin[e+f*x]],x] +
  d/(2*b)*Int[Sqrt[d*Sin[e+f*x]]*(a+2*b*Sin[e+f*x])/Sqrt[a+b*Sin[e+f*x]],x] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0]
```

10: $\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^n \, dx$ when $b \, c - a \, d \neq 0 \, \wedge \, a^2 - b^2 \neq 0 \, \wedge \, c^2 - d^2 \neq 0 \, \wedge \, 0 < m < 2 \, \wedge \, -1 < n < 2$

Derivation: Nondegenerate sine recurrence 1b with $A \to a \, c$, $B \to b \, c + a \, d$, $C \to b \, d$, $m \to m - 1$, $n \to n - 1$, $p \to 0$

Rule: If $b \, c - a \, d \neq 0 \, \wedge \, a^2 - b^2 \neq 0 \, \wedge \, c^2 - d^2 \neq 0 \, \wedge \, 0 < m < 2 \, \wedge \, -1 < n < 2$, then

$$\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^n \, dx \longrightarrow$$

$$-\frac{b \, Cos[e + f \, x] \, (a + b \, Sin[e + f \, x])^{m-1} \, (c + d \, Sin[e + f \, x])^n}{f \, (m + n)} \, +$$

$$\frac{1}{d \, (m + n)} \int (a + b \, Sin[e + f \, x])^{m-2} \, (c + d \, Sin[e + f \, x])^{n-1} \, \cdot$$

$$\left(a^2 \, c \, d \, (m + n) + b \, d \, (b \, c \, (m - 1) + a \, d \, n) + \left(a \, d \, (2 \, b \, c + a \, d) \, (m + n) - b \, d \, (a \, c - b \, d \, (m + n - 1)) \right) \, Sin[e + f \, x] + b \, d \, (b \, c \, n + a \, d \, (2 \, m + n - 1)) \, Sin[e + f \, x]^2 \right) \, dx$$

Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -b*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m-1)*(c+d*Sin[e+f*x])^n/(f*(m+n)) +
  1/(d*(m+n))*Int[(a+b*Sin[e+f*x])^(m-2)*(c+d*Sin[e+f*x])^(n-1)*
    Simp[a^2*c*d*(m+n)+b*d*(b*c*(m-1)+a*d*n)+
      (a*d*(2*b*c+a*d)*(m+n)-b*d*(a*c-b*d*(m+n-1)))*Sin[e+f*x]+
      b*d*(b*c*n+a*d*(2*m+n-1))*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[0,m,2] && LtQ[-1,n,2] && NeQ[m+n,0] &&
  (IntegerQ[m] || IntegersQ[2*m,2*n])
```

11: $\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^n \, dx$ when $b \, c - a \, d \neq 0 \wedge m \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Basis: $a + b \, z == \dfrac{b \, (c + d \, z)}{d} - \dfrac{b \, c - a \, d}{d}$

Rule: If $b \, c - a \, d \neq 0 \wedge m \in \mathbb{Z}^+$, then

$$\int (a + b \, Sin[e + f \, x])^m \, (c + d \, Sin[e + f \, x])^n \, dx \rightarrow$$

$$\frac{b}{d} \int (a + b \, Sin[e + f \, x])^{m-1} \, (c + d \, Sin[e + f \, x])^{n+1} \, dx - \frac{b \, c - a \, d}{d} \int (a + b \, Sin[e + f \, x])^{m-1} \, (c + d \, Sin[e + f \, x])^n \, dx$$

Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  b/d*Int[(a+b*Sin[e+f*x])^(m-1)*(c+d*Sin[e+f*x])^(n+1),x] -
  (b*c-a*d)/d*Int[(a+b*Sin[e+f*x])^(m-1)*(c+d*Sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,n},x] && NeQ[b*c-a*d,0] && IGtQ[m,0]
```

12. $\int (d \, Sin[e + f \, x])^n \, (a + b \, Sin[e + f \, x])^m \, dx$ when $a^2 - b^2 == 0 \wedge m \in \mathbb{Z}^-$

1: $\int \dfrac{(d \, Sin[e + f \, x])^n}{a + b \, Sin[e + f \, x]} \, dx$ when $a^2 - b^2 \neq 0$

Derivation: Algebraic expansion

Basis: $\dfrac{1}{a + b \, z} == \dfrac{a}{a^2 - b^2 \, z^2} - \dfrac{b \, z}{a^2 - b^2 \, z^2}$

Rule: If $a^2 - b^2 \neq 0$, then

$$\int \frac{(d\,Sin[e+f\,x])^n}{a+b\,Sin[e+f\,x]}\,dx \;\rightarrow\; a\int \frac{(d\,Sin[e+f\,x])^n}{a^2-b^2\,Sin[e+f\,x]^2}\,dx - \frac{b}{d}\int \frac{(d\,Sin[e+f\,x])^{n+1}}{a^2-b^2\,Sin[e+f\,x]^2}\,dx$$

Program code:

```
Int[(d_.*sin[e_.+f_.*x_])^n_./(a_+b_.*sin[e_.+f_.*x_]),x_Symbol] :=
  a*Int[(d*Sin[e+f*x])^n/(a^2-b^2*Sin[e+f*x]^2),x] -
  b/d*Int[(d*Sin[e+f*x])^(n+1)/(a^2-b^2*Sin[e+f*x]^2),x] /;
FreeQ[{a,b,d,e,f,n},x] && NeQ[a^2-b^2,0]
```

2: $\int (d\,Sin[e+f\,x])^n\,(a+b\,Sin[e+f\,x])^m\,dx$ when $a^2-b^2 \neq 0 \wedge m+1 \in \mathbb{Z}^-$

Derivation: Algebraic expansion

Basis: $\dfrac{1}{a+b\,z} \;==\; \dfrac{a-b\,z}{a^2-b^2\,z^2}$

Rule: If $a^2-b^2 \neq 0 \wedge m+1 \in \mathbb{Z}^-$, then

$$\int (d\,Sin[e+f\,x])^n\,(a+b\,Sin[e+f\,x])^m\,dx \;\rightarrow\; \int ExpandTrig\left[\frac{(d\,Sin[e+f\,x])^n\,(a-b\,Sin[e+f\,x])^{-m}}{(a^2-b^2\,Sin[e+f\,x]^2)^{-m}},\,x\right]dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_.*(d_.*sin[e_.+f_.*x_])^n_.,x_Symbol] :=
  Int[ExpandTrig[(d*sin[e+f*x])^n*(a-b*sin[e+f*x])^(-m)/(a^2-b^2*sin[e+f*x]^2)^(-m),x],x] /;
FreeQ[{a,b,d,e,f,n},x] && NeQ[a^2-b^2,0] && ILtQ[m,-1]
```

X: $\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x])^n \, dx$ when $b \, c - a \, d \neq 0 \, \wedge \, a^2 - b^2 \neq 0 \, \wedge \, c^2 - d^2 \neq 0$

Rule: If $b \, c - a \, d \neq 0 \, \wedge \, a^2 - b^2 \neq 0 \, \wedge \, c^2 - d^2 \neq 0$, then

$$\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x])^n \, dx \; \longrightarrow \; \int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Sin}[e + f \, x])^n \, dx$$

Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  Unintegrable[(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

Rules for integrands of the form $(a + b \, Sin[e + f \, x])^m \, (c \, (d \, Sin[e + f \, x])^p)^n$

x: $\int (a + b \, Sin[e + f \, x])^m \, (d \, Csc[e + f \, x])^n \, dx$ when $n \notin \mathbb{Z} \wedge m \in \mathbb{Z}$

Derivation: Algebraic normalization

Basis: If $m \in \mathbb{Z}$, then $(a + b \, Sin[z])^m == \frac{d^m \, (b + a \, Csc[z])^m}{(d \, Csc[z])^m}$

Note: Although this rule does not introduce a piecewise constant factor, it is better to stay in the sine/cosine world than the secant/cosecant world.

Rule: If $n \notin \mathbb{Z} \wedge m \in \mathbb{Z}$, then

$$\int (a + b \, Sin[e + f \, x])^m \, (d \, Csc[e + f \, x])^n \, dx \;\rightarrow\; d^m \int (d \, Csc[e + f \, x])^{n-m} \, (b + a \, Csc[e + f \, x])^m \, dx$$

Program code:

```
(* Int[(a_.+b_.*sin[e_.+f_.*x_])^m_.*(d_./sin[e_.+f_.*x_])^n_,x_Symbol] :=
  d^m*Int[(d*Csc[e+f*x])^(n-m)*(b+a*Csc[e+f*x])^m,x] /;
FreeQ[{a,b,d,e,f,n},x] && Not[IntegerQ[n]] && IntegerQ[m] *)
```

```
(* Int[(a_.+b_.*cos[e_.+f_.*x_])^m_.*(d_./cos[e_.+f_.*x_])^n_,x_Symbol] :=
  d^m*Int[(d*Sec[e+f*x])^(n-m)*(b+a*Sec[e+f*x])^m,x] /;
FreeQ[{a,b,d,e,f,n},x] && Not[IntegerQ[n]] && IntegerQ[m] *)
```

1: $\int (a + b \, Sin[e + f \, x])^m \, (c \, (d \, Sin[e + f \, x])^p)^n \, dx$ when $n \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{\left(c\,\left(d\,Sin[e+f\,x]\right)^p\right)^n}{\left(d\,Sin[e+f\,x]\right)^{np}} \;==\; 0$

Rule: If $n \notin \mathbb{Z}$, then

$$\int \left(a + b\,Sin[e + f\,x]\right)^m \left(c\,\left(d\,Sin[e + f\,x]\right)^p\right)^n \, dx \;\longrightarrow\; \frac{c^{IntPart[n]}\,\left(c\,\left(d\,Sin[e + f\,x]\right)^p\right)^{FracPart[n]}}{\left(d\,Sin[e + f\,x]\right)^{p\,FracPart[n]}} \int \left(a + b\,Sin[e + f\,x]\right)^m \left(d\,Sin[e + f\,x]\right)^{np}\, dx$$

Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_.*(c_.*(d_.*sin[e_.+f_.*x_])^p_)^n_,x_Symbol] :=
  c^IntPart[n]*(c*(d*Sin[e + f*x])^p)^FracPart[n]/(d*Sin[e + f*x])^(p*FracPart[n])*
    Int[(a+b*Sin[e+f*x])^m*(d*Sin[e+f*x])^(n*p),x] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && Not[IntegerQ[n]]
```

```
Int[(a_.+b_.*cos[e_.+f_.*x_])^m_.*(c_.*(d_.*cos[e_.+f_.*x_])^p_)^n_,x_Symbol] :=
  c^IntPart[n]*(c*(d*Cos[e + f*x])^p)^FracPart[n]/(d*Cos[e + f*x])^(p*FracPart[n])*
    Int[(a+b*Cos[e+f*x])^m*(d*Cos[e+f*x])^(n*p),x] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && Not[IntegerQ[n]]
```

Rules for integrands of the form $(a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Csc}[e + f \, x])^n$

1: $\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Csc}[e + f \, x])^n \, dx$ when $n \in \mathbb{Z}$

Derivation: Algebraic normalization

Basis: $c + d \, \text{Csc}[z] \; == \; \dfrac{d + c \, \text{Sin}[z]}{\text{Sin}[z]}$

Rule: If $n \in \mathbb{Z}$, then

$$\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Csc}[e + f \, x])^n \, dx \;\longrightarrow\; \int \frac{(a + b \, \text{Sin}[e + f \, x])^m \, (d + c \, \text{Sin}[e + f \, x])^n}{\text{Sin}[e + f \, x]^n} \, dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_.*(c_+d_.*csc[e_.+f_.*x_])^n_.,x_Symbol] :=
  Int[(a+b*Sin[e+f*x])^m*(d+c*Sin[e+f*x])^n/Sin[e+f*x]^n,x] /;
FreeQ[{a,b,c,d,e,f,m},x] && IntegerQ[n]
```

2. $\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Csc}[e + f \, x])^n \, dx$ when $n \notin \mathbb{Z}$

1: $\int (a + b \, \text{Sin}[e + f \, x])^m \, (c + d \, \text{Csc}[e + f \, x])^n \, dx$ when $n \notin \mathbb{Z} \, \wedge \, m \in \mathbb{Z}$

Derivation: Algebraic normalization

Basis: $a + b \, \text{Sin}[z] \; == \; \dfrac{b + a \, \text{Csc}[z]}{\text{Csc}[z]}$

Rule: If $n \notin \mathbb{Z} \, \wedge \, m \in \mathbb{Z}$, then

$$\int (a + b \, Sin[e + f\,x])^m \, (c + d \, Csc[e + f\,x])^n \, dx \;\rightarrow\; \int \frac{(b + a\,Csc[e + f\,x])^m \, (c + d\,Csc[e + f\,x])^n}{Csc[e + f\,x]^m} \, dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_.*(c_+d_.*csc[e_.+f_.*x_])^n_,x_Symbol] :=
  Int[(b+a*Csc[e+f*x])^m*(c+d*Csc[e+f*x])^n/Csc[e+f*x]^m,x] /;
FreeQ[{a,b,c,d,e,f,n},x] && Not[IntegerQ[n]] && IntegerQ[m]
```

```
Int[(a_+b_.*cos[e_.+f_.*x_])^m_.*(c_+d_.*sec[e_.+f_.*x_])^n_,x_Symbol] :=
  Int[(b+a*Sec[e+f*x])^m*(c+d*Sec[e+f*x])^n/Sec[e+f*x]^m,x] /;
FreeQ[{a,b,c,d,e,f,n},x] && Not[IntegerQ[n]] && IntegerQ[m]
```

2: $\int (a + b \, Sin[e + f\,x])^m \, (c + d \, Csc[e + f\,x])^n \, dx$ when $n \notin \mathbb{Z} \;\wedge\; m \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{(c + d \, Csc[e + f\,x])^n \, Sin[e + f\,x]^n}{(d + c \, Sin[e + f\,x])^n} \;==\; 0$

Rule: If $n \notin \mathbb{Z} \;\wedge\; m \notin \mathbb{Z}$, then

$$\int (a + b \, Sin[e + f\,x])^m \, (c + d \, Csc[e + f\,x])^n \, dx \;\rightarrow\; \frac{(c + d \, Csc[e + f\,x])^n \, Sin[e + f\,x]^n}{(d + c \, Sin[e + f\,x])^n} \int \frac{(a + b \, Sin[e + f\,x])^m \, (d + c \, Sin[e + f\,x])^n}{Sin[e + f\,x]^n} \, dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*csc[e_.+f_.*x_])^n_,x_Symbol] :=
  Sin[e+f*x]^n*(c+d*Csc[e+f*x])^n/(d+c*Sin[e+f*x])^n*Int[(a+b*Sin[e+f*x])^m*(d+c*Sin[e+f*x])^n/Sin[e+f*x]^n,x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && Not[IntegerQ[n]] && Not[IntegerQ[m]]
```

```
Int[(a_+b_.*cos[e_.+f_.*x_])^m_*(c_+d_.*sec[e_.+f_.*x_])^n_,x_Symbol] :=
  Cos[e+f*x]^n*(c+d*Sec[e+f*x])^n/(d+c*Cos[e+f*x])^n*Int[(a+b*Cos[e+f*x])^m*(d+c*Cos[e+f*x])^n/Cos[e+f*x]^n,x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && Not[IntegerQ[n]] && Not[IntegerQ[m]]
```