

Rules for integrands of the form $(a + b x^n)^p (c + d x^n)^q$

1: $\int (a + b x^n)^p (c + d x^n)^q dx$ when $b c - a d \neq 0 \wedge (p \mid q) \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule 1.1.3.3.1: If $b c - a d \neq 0 \wedge (p \mid q) \in \mathbb{Z}^+$, then

$$\int (a + b x^n)^p (c + d x^n)^q dx \rightarrow \int \text{ExpandIntegrand}[(a + b x^n)^p (c + d x^n)^q, x] dx$$

Program code:

```
Int[(a_+b_.**x_^n_)^p_.*(c_+d_.**x_^n_)^q_,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*x^n)^p*(c+d*x^n)^q,x],x] /;
FreeQ[{a,b,c,d,n},x] && NeQ[b*c-a*d,0] && IGtQ[p,0] && IGtQ[q,0]
```

2: $\int (a + b x^n)^p (c + d x^n)^q dx$ when $b c - a d \neq 0 \wedge (p \mid q) \in \mathbb{Z} \wedge n < 0$

Derivation: Algebraic expansion

Basis: If $p \in \mathbb{Z}$, then $(a + b x^n)^p = x^{np} (b + a x^{-n})^p$

Rule 1.1.3.3.2: If $b c - a d \neq 0 \wedge (p \mid q) \in \mathbb{Z} \wedge n < 0$, then

$$\int (a + b x^n)^p (c + d x^n)^q dx \rightarrow \int x^{n(p+q)} (b + a x^{-n})^p (d + c x^{-n})^q dx$$

Program code:

```
Int[(a_+b_.**x_^n_)^p_.*(c_+d_.**x_^n_)^q_,x_Symbol] :=
  Int[x^(n*(p+q))*(b+a*x^(-n))^p*(d+c*x^(-n))^q,x] /;
FreeQ[{a,b,c,d,n},x] && NeQ[b*c-a*d,0] && IntegersQ[p,q] && NegQ[n]
```

3: $\int (a+bx^n)^p (c+dx^n)^q dx$ when $b c - a d \neq 0 \wedge n \in \mathbb{Z}^-$

Derivation: Integration by substitution

Basis: $F[x] \Rightarrow -\text{Subst}\left[\frac{F\left[\frac{x^{-1}}{x^2}\right]}{x^2}, x, \frac{1}{x}\right] \partial_x \frac{1}{x}$

Rule 1.1.3.3: If $b c - a d \neq 0 \wedge n \in \mathbb{Z}^-$, then

$$\int (a+bx^n)^p (c+dx^n)^q dx \rightarrow -\text{Subst}\left[\int \frac{(a+bx^{-n})^p (c+dx^{-n})^q}{x^2} dx, x, \frac{1}{x}\right]$$

Program code:

```
Int[(a+b_.**x_^n_)^p_.*(c+d_.**x_^n_)^q_,x_Symbol] :=
  -Subst[Int[(a+b*x^(-n))^p*(c+d*x^(-n))^q/x^2,x],x,1/x] /;
FreeQ[{a,b,c,d,p,q},x] && NeQ[b*c-a*d,0] && ILtQ[n,0]
```

4: $\int (a + bx^n)^p (c + dx^n)^q dx$ when $b c - a d \neq 0 \wedge n \in \mathbb{F}$

Derivation: Integration by substitution

Basis: If $g \in \mathbb{Z}^+$, then $F[x^n] = g \text{Subst}[x^{g-1} F[x^g], x, x^{1/g}] \partial_x x^{1/g}$

Rule 1.1.3.3.4: If $b c - a d \neq 0 \wedge n \in \mathbb{F}$, let $g = \text{Denominator}[n]$, then

$$\int (a + bx^n)^p (c + dx^n)^q dx \rightarrow g \text{Subst}\left[\int x^{g-1} (a + bx^g)^p (c + dx^g)^q dx, x, x^{1/g}\right]$$

Program code:

```
Int[(a+b_.*x_^n_)^p_.*(c+d_.*x_^n_)^q_.,x_Symbol] :=
  With[{g=Denominator[n]},
    g*Subst[Int[x^(g-1)*(a+b*x^(g*n))^p*(c+d*x^(g*n))^q,x],x,x^(1/g)]] /;
  FreeQ[{a,b,c,d,p,q},x] && NeQ[b*c-a*d,0] && FractionQ[n]
```

5. $\int (a + b x^n)^p (c + d x^n)^q dx$ when $b c - a d \neq 0 \wedge n (p + q + 1) + 1 = 0$

1: $\int \frac{(a + b x^n)^p}{c + d x^n} dx$ when $b c - a d \neq 0 \wedge n p + 1 = 0 \wedge n \in \mathbb{Z}$

Derivation: Integration by substitution

Basis: If $n \in \mathbb{Z}$, then $\frac{1}{(a+bx^n)^{1/n} (c+dx^n)} = \text{Subst}\left[\frac{1}{c - (b c - a d) x^n}, x, \frac{x}{(a+bx^n)^{1/n}}\right] \partial_x \frac{x}{(a+bx^n)^{1/n}}$

Rule 1.1.3.3.5.1: If $b c - a d \neq 0 \wedge n p + 1 = 0 \wedge n \in \mathbb{Z}$, then

$$\int \frac{(a + b x^n)^p}{c + d x^n} dx \rightarrow \text{Subst}\left[\int \frac{1}{c - (b c - a d) x^n} dx, x, \frac{x}{(a + b x^n)^{1/n}}\right]$$

Program code:

```
Int[(a_+b_.*x_^n_)^p/(c_+d_.*x_^n_),x_Symbol] :=
  Subst[Int[1/(c-(b*c-a*d)*x^n),x],x,x/(a+b*x^n)^(1/n)] /;
FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && EqQ[n*p+1,0] && IntegerQ[n]
```

2: $\int (a+bx^n)^p (c+dx^n)^q dx$ when $bc - ad \neq 0 \wedge n(p+q+1) + 1 = 0 \wedge q > 0 \wedge p \neq -1$

Derivation: Binomial product recurrence 1 with $A = 1$, $B = 0$ and $n(p+q+1) + 1 = 0$

Note: If this kool rules applies, it will also apply to the resulting integrands until p and q are reduced to the interval $[-1,0)$.

Rule 1.1.3.3.5.2: If $bc - ad \neq 0 \wedge n(p+q+1) + 1 = 0 \wedge q > 0 \wedge p \neq -1$, then

$$\int (a+bx^n)^p (c+dx^n)^q dx \rightarrow -\frac{x(a+bx^n)^{p+1}(c+dx^n)^q}{an(p+1)} - \frac{cq}{a(p+1)} \int (a+bx^n)^{p+1}(c+dx^n)^{q-1} dx$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  -x*(a+b*x^n)^(p+1)*(c+d*x^n)^q/(a*n*(p+1)) -
  c*q/(a*(p+1))*Int[(a+b*x^n)^(p+1)*(c+d*x^n)^(q-1),x] /;
FreeQ[{a,b,c,d,n,p},x] && NeQ[b*c-a*d,0] && EqQ[n*(p+q+1)+1,0] && GtQ[q,0] && NeQ[p,-1]
```

3: $\int (a+bx^n)^p (c+dx^n)^q dx$ when $bc - ad \neq 0 \wedge n(p+q+1) + 1 = 0 \wedge p \in \mathbb{Z}^-$

Rule 1.1.3.3.5.3: If $bc - ad \neq 0 \wedge n(p+q+1) + 1 = 0 \wedge p \in \mathbb{Z}^-$, then

$$\int (a+bx^n)^p (c+dx^n)^q dx \rightarrow \frac{a^p x}{c^{p+1} (c+dx^n)^{1/n}} \text{Hypergeometric2F1}\left[\frac{1}{n}, -p, 1+\frac{1}{n}, -\frac{(bc-ad)x^n}{a(c+dx^n)}\right]$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  a^p*x/(c^(p+1)*(c+d*x^n)^(1/n))*Hypergeometric2F1[1/n,-p,1+1/n,-(b*c-a*d)*x^n/(a*(c+d*x^n))]/;
FreeQ[{a,b,c,d,n,q},x] && NeQ[b*c-a*d,0] && EqQ[n*(p+q+1)+1,0] && ILtQ[p,0]
```

4: $\int (a+bx^n)^p (c+dx^n)^q dx$ when $bc - ad \neq 0 \wedge n(p+q+1) + 1 = 0$

Rule 1.1.3.3.5.4: If $bc - ad \neq 0 \wedge n(p+q+1) + 1 = 0$, then

$$\int (a+bx^n)^p (c+dx^n)^q dx \rightarrow \frac{x(a+bx^n)^p}{c \left(\frac{c(a+bx^n)}{a(c+dx^n)} \right)^p (c+dx^n)^{\frac{1}{n}+p}} \text{Hypergeometric2F1} \left[\frac{1}{n}, -p, 1 + \frac{1}{n}, -\frac{(bc-ad)x^n}{a(c+dx^n)} \right]$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  x*(a+b*x^n)^p/(c*(c*(a+b*x^n)/(a*(c+d*x^n)))^p*(c+d*x^n)^(1/n+p))*
  Hypergeometric2F1[1/n,-p,1+1/n,-(b*c-a*d)*x^n/(a*(c+d*x^n))]/;
FreeQ[{a,b,c,d,n,p,q},x] && NeQ[b*c-a*d,0] && EqQ[n*(p+q+1)+1,0]
```

6. $\int (a+bx^n)^p (c+dx^n)^q dx$ when $bc - ad \neq 0 \wedge n(p+q+2) + 1 = 0$

1: $\int (a+bx^n)^p (c+dx^n)^q dx$ when $bc - ad \neq 0 \wedge n(p+q+2) + 1 = 0 \wedge ad(p+1) + bc(q+1) = 0$

Derivation: Binomial product recurrence 2a with $A = 1$, $B = 0$ and $n(p+q+2) + 1 = 0$

Rule 1.1.3.3.6.1: If $bc - ad \neq 0 \wedge n(p+q+2) + 1 = 0 \wedge ad(p+1) + bc(q+1) = 0$, then

$$\int (a+bx^n)^p (c+dx^n)^q dx \rightarrow \frac{x(a+bx^n)^{p+1} (c+dx^n)^{q+1}}{ac}$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  x*(a+b*x^n)^(p+1)*(c+d*x^n)^(q+1)/(a*c)/;
FreeQ[{a,b,c,d,n,p,q},x] && NeQ[b*c-a*d,0] && EqQ[n*(p+q+2)+1,0] && EqQ[a*d*(p+1)+b*c*(q+1),0]
```

```
(* Int[(a1_+b1_.*x_^n2_.)^p_*(a2_+b2_.*x_^n2_.)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  x*(a1+b1*x^(n/2))^ (p+1)*(a2+b2*x^(n/2))^ (p+1)*(c+d*x^n)^(q+1)/(a1*a2*c) /;
FreeQ[{a1,b1,a2,b2,c,d,n,p,q},x] && EqQ[n2,n/2] && EqQ[a2*b1+a1*b2,0] && EqQ[n*(p+q+2)+1,0] && EqQ[a1*a2*d*(p+1)+b1*b2*c*(q+1),0] *)
```

2: $\int (a+bx^n)^p (c+dx^n)^q dx$ when $bc - ad \neq 0 \wedge n(p+q+2) + 1 = 0 \wedge p < -1$

Derivation: Binomial product recurrence 2a with $A = 1$, $B = 0$ and $n(p+q+2) + 1 = 0$

Note: Note the resulting integrand is of the form $(a+bx^n)^p (c+dx^n)^q$ where $n(p+q+1) + 1 = 0$.

Rule 1.1.3.3.6.2: If $bc - ad \neq 0 \wedge n(p+q+2) + 1 = 0 \wedge p < -1$, then

$$\int (a+bx^n)^p (c+dx^n)^q dx \rightarrow -\frac{bx(a+bx^n)^{p+1}(c+dx^n)^{q+1}}{an(p+1)(bc-ad)} + \frac{bc+n(p+1)(bc-ad)}{an(p+1)(bc-ad)} \int (a+bx^n)^{p+1}(c+dx^n)^q dx$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  -b*x*(a+b*x^n)^(p+1)*(c+d*x^n)^(q+1)/(a*n*(p+1)*(b*c-a*d)) +
  (b*c+n*(p+1)*(b*c-a*d))/(a*n*(p+1)*(b*c-a*d))*Int[(a+b*x^n)^(p+1)*(c+d*x^n)^q,x] /;
FreeQ[{a,b,c,d,n,q},x] && NeQ[b*c-a*d,0] && EqQ[n*(p+q+2)+1,0] && (LtQ[p,-1] || Not[LtQ[q,-1]]) && NeQ[p,-1]
```

7. $\int (a + b x^n)^p (c + d x^n) dx$ when $bc - ad \neq 0$

1: $\int (a + b x^n)^p (c + d x^n) dx$ when $bc - ad \neq 0 \wedge ad - bc(n(p+1) + 1) = 0$

Derivation: Trinomial recurrence 2b with $c = 0, p = 0$ and $ad - bc(n(p+1) + 1) = 0$

Rule 1.1.3.3.7.1: If $bc - ad \neq 0 \wedge ad - bc(n(p+1) + 1) = 0$, then

$$\int (a + b x^n)^p (c + d x^n) dx \rightarrow \frac{c x (a + b x^n)^{p+1}}{a}$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_),x_Symbol] :=
  c*x*(a+b*x^n)^(p+1)/a /;
FreeQ[{a,b,c,d,n,p},x] && NeQ[b*c-a*d,0] && EqQ[a*d-b*c*(n*(p+1)+1),0]
```

```
Int[(a1_+b1_.*x_^non2_)^p_.*(a2_+b2_.*x_^non2_)^p_.*(c_+d_.*x_^n_),x_Symbol] :=
  c*x*(a1+b1*x^(n/2))^(p+1)*(a2+b2*x^(n/2))^(p+1)/(a1*a2) /;
FreeQ[{a1,b1,a2,b2,c,d,n,p},x] && EqQ[non2,n/2] && EqQ[a2*b1+a1*b2,0] && EqQ[a1*a2*d-b1*b2*c*(n*(p+1)+1),0]
```


2: $\int (a+bx^n)^p (c+dx^n) dx$ when $bc - ad \neq 0 \wedge p < -1$

Derivation: Trinomial recurrence 2b with $c = 0$ and $p = 0$

Rule 1.1.3.3.7.2: If $bc - ad \neq 0 \wedge p < -1$, then

$$\int (a+bx^n)^p (c+dx^n) dx \rightarrow -\frac{(bc-ad)x(a+bx^n)^{p+1}}{abn(p+1)} - \frac{ad-bc(n(p+1)+1)}{abn(p+1)} \int (a+bx^n)^{p+1} dx$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_),x_Symbol] :=
  -(b*c-a*d)*x*(a+b*x^n)^(p+1)/(a*b*n*(p+1)) -
  (a*d-b*c*(n*(p+1)+1))/(a*b*n*(p+1))*Int[(a+b*x^n)^(p+1),x] /;
FreeQ[{a,b,c,d,n,p},x] && NeQ[b*c-a*d,0] && (LtQ[p,-1] || ILtQ[1/n+p,0])
```

```
Int[(a1_+b1_.*x_^non2_)^p_.*(a2_+b2_.*x_^non2_)^p_*(c_+d_.*x_^n_),x_Symbol] :=
  -(b1*b2*c-a1*a2*d)*x*(a1+b1*x^(n/2))^(p+1)*(a2+b2*x^(n/2))^(p+1)/(a1*a2*b1*b2*n*(p+1)) -
  (a1*a2*d-b1*b2*c*(n*(p+1)+1))/(a1*a2*b1*b2*n*(p+1))*Int[(a1+b1*x^(n/2))^(p+1)*(a2+b2*x^(n/2))^(p+1),x] /;
FreeQ[{a1,b1,a2,b2,c,d,n},x] && EqQ[non2,n/2] && EqQ[a2*b1+a1*b2,0] && (LtQ[p,-1] || ILtQ[1/n+p,0])
```

3: $\int \frac{c+dx^n}{a+bx^n} dx$ when $bc-ad \neq 0 \wedge n < 0$

Derivation: Algebraic expansion

Basis: $\frac{c+dx^n}{a+bx^n} = \frac{c}{a} - \frac{bc-ad}{a(b+ax^{-n})}$

Rule 1.1.3.3.7.3: If $bc-ad \neq 0 \wedge n < 0$, then

$$\int \frac{c+dx^n}{a+bx^n} dx \rightarrow \frac{cx}{a} - \frac{bc-ad}{a} \int \frac{1}{b+ax^{-n}} dx$$

Program code:

```
Int[(c_+d_.*x_^n_)/(a_+b_.*x_^n_),x_Symbol] :=
  c*x/a - (b*c-a*d)/a*Int[1/(b+a*x^(-n)),x] /;
FreeQ[{a,b,c,d,n},x] && NeQ[b*c-a*d,0] && LtQ[n,0]
```

4: $\int (a+bx^n)^p (c+dx^n) dx$ when $bc - ad \neq 0 \wedge n(p+1) + 1 \neq 0$

Derivation: Trinomial recurrence 2b with $c = 0$ and $p = 0$ composed with binomial recurrence 1b with $p = 0$

Rule 1.1.3.3.7.4: If $bc - ad \neq 0 \wedge n(p+1) + 1 \neq 0$, then

$$\int (a+bx^n)^p (c+dx^n) dx \rightarrow \frac{dx (a+bx^n)^{p+1}}{b(n(p+1)+1)} - \frac{ad-bc(n(p+1)+1)}{b(n(p+1)+1)} \int (a+bx^n)^p dx$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_),x_Symbol] :=
  d*x*(a+b*x^n)^(p+1)/(b*(n*(p+1)+1)) -
  (a*d-b*c*(n*(p+1)+1))/(b*(n*(p+1)+1))*Int[(a+b*x^n)^p,x] /;
FreeQ[{a,b,c,d,n},x] && NeQ[b*c-a*d,0] && NeQ[n*(p+1)+1,0]
```

```
Int[(a1_+b1_.*x_^non2_)^p_.*(a2_+b2_.*x_^non2_)^p_*(c_+d_.*x_^n_),x_Symbol] :=
  d*x*(a1+b1*x^(n/2))^(p+1)*(a2+b2*x^(n/2))^(p+1)/(b1*b2*(n*(p+1)+1)) -
  (a1*a2*d-b1*b2*c*(n*(p+1)+1))/(b1*b2*(n*(p+1)+1))*Int[(a1+b1*x^(n/2))^p*(a2+b2*x^(n/2))^p,x] /;
FreeQ[{a1,b1,a2,b2,c,d,n,p},x] && EqQ[non2,n/2] && EqQ[a2*b1+a1*b2,0] && NeQ[n*(p+1)+1,0]
```

8: $\int (a+bx^n)^p (c+dx^n)^q dx$ when $bc-ad \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p \in \mathbb{Z}^+ \wedge q \in \mathbb{Z}^- \wedge p \geq -q$

Derivation: Algebraic expansion

Rule 1.1.3.3.8: If $bc-ad \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p \in \mathbb{Z}^+ \wedge q \in \mathbb{Z}^- \wedge p \geq -q$, then

$$\int (a+bx^n)^p (c+dx^n)^q dx \rightarrow \int \text{PolynomialDivide}[(a+bx^n)^p, (c+dx^n)^{-q}, x] dx$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  Int[PolynomialDivide[(a+b*x^n)^p,(c+d*x^n)^(-q),x],x] /;
FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && IGtQ[p,0] && ILtQ[q,0] && GeQ[p,-q]
```

9. $\int \frac{(a+bx^n)^p}{c+dx^n} dx$ when $bc - ad \neq 0$

1: $\int \frac{1}{(a+bx^n)(c+dx^n)} dx$ when $bc - ad \neq 0$

Derivation: Algebraic expansion

Basis: $\frac{1}{(a+bx)(c+dx)} = \frac{b}{(b c - a d)(a+bx)} - \frac{d}{(b c - a d)(c+dx)}$

Rule 1.1.3.3.9.1: If $bc - ad \neq 0$, then

$$\int \frac{1}{(a+bx^n)(c+dx^n)} dx \rightarrow \frac{b}{(bc-ad)} \int \frac{1}{a+bx^n} dx - \frac{d}{(bc-ad)} \int \frac{1}{c+dx^n} dx$$

Program code:

```
Int[1/((a+b_.**x_^n_)*(c+d_.**x_^n_)),x_Symbol] :=
  b/(b*c-a*d)*Int[1/(a+b*x^n),x] - d/(b*c-a*d)*Int[1/(c+d*x^n),x] /;
FreeQ[{a,b,c,d,n},x] && NeQ[b*c-a*d,0]
```

$$\begin{aligned}
2. \int \frac{(a+bx^2)^p}{c+dx^2} dx \text{ when } bc-ad \neq 0 \\
1. \int \frac{1}{(a+bx^2)^{1/3} (c+dx^2)} dx \text{ when } bc-ad \neq 0 \wedge (bc+3ad=0 \vee bc-9ad=0) \\
1. \int \frac{1}{(a+bx^2)^{1/3} (c+dx^2)} dx \text{ when } bc-ad \neq 0 \wedge bc+3ad=0 \\
1: \int \frac{1}{(a+bx^2)^{1/3} (c+dx^2)} dx \text{ when } bc-ad \neq 0 \wedge bc+3ad=0 \wedge \frac{b}{a} > 0
\end{aligned}$$

Derivation: Integration by substitution

$$\text{Basis: } F[(a+bx^2)^{1/3}, x^2] = \frac{3\sqrt{bx^2}}{2bx} \text{Subst}\left[\frac{x^2}{\sqrt{-a+x^3}} F\left[x, \frac{-a+x^3}{b}\right], x, (a+bx^2)^{1/3}\right] \partial_x (a+bx^2)^{1/3}$$

Rule 1.1.3.3.9.2.1.1.1: If $bc-ad \neq 0 \wedge bc+3ad=0 \wedge \frac{b}{a} > 0$, let $q \rightarrow \sqrt{\frac{b}{a}}$, then

$$\begin{aligned}
& \int \frac{1}{(a+bx^2)^{1/3} (c+dx^2)} dx \rightarrow \frac{3\sqrt{bx^2}}{2x} \text{Subst}\left[\int \frac{x}{\sqrt{-a+x^3} (bc-ad+dx^3)} dx, x, (a+bx^2)^{1/3}\right] \\
& \rightarrow \frac{q \text{ArcTanh}\left[\frac{\sqrt{3}}{qx}\right]}{2 \times 2^{2/3} \sqrt{3} a^{1/3} d} + \frac{q \text{ArcTanh}\left[\frac{\sqrt{3} (a^{1/3}-2^{1/3} (a+bx^2)^{1/3})}{a^{1/3} qx}\right]}{2 \times 2^{2/3} \sqrt{3} a^{1/3} d} + \frac{q \text{ArcTan}[qx]}{6 \times 2^{2/3} a^{1/3} d} - \frac{q \text{ArcTan}\left[\frac{a^{1/3} qx}{a^{1/3}+2^{1/3} (a+bx^2)^{1/3}}\right]}{2 \times 2^{2/3} a^{1/3} d}
\end{aligned}$$

Program code:

```

Int[1/((a+b_.**x^2)^(1/3)*(c+d_.**x^2)),x_Symbol] :=
  With[{q=Rt[b/a,2]},
    q*ArcTanh[Sqrt[3]/(q*x)]/(2*2^(2/3)*Sqrt[3]*a^(1/3)*d) +
    q*ArcTanh[Sqrt[3]*(a^(1/3)-2^(1/3)*(a+b*x^2)^(1/3))/(a^(1/3)*q*x)]/(2*2^(2/3)*Sqrt[3]*a^(1/3)*d) +
    q*ArcTan[q*x]/(6*2^(2/3)*a^(1/3)*d) -
    q*ArcTan[(a^(1/3)*q*x)/(a^(1/3)+2^(1/3)*(a+b*x^2)^(1/3))]/(2*2^(2/3)*a^(1/3)*d) /;
FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && EqQ[b*c+3*a*d,0] && PosQ[b/a]

```

2: $\int \frac{1}{(a+bx^2)^{1/3} (c+dx^2)} dx$ when $bc - ad \neq 0 \wedge bc + 3ad = 0 \wedge \frac{b}{a} \neq 0$

Rule 1.1.3.3.9.2.1.1.2: If $bc - ad \neq 0 \wedge bc + 3ad = 0 \wedge \frac{b}{a} \neq 0$, let $q \rightarrow \sqrt{-\frac{b}{a}}$, then

$$\int \frac{1}{(a+bx^2)^{1/3} (c+dx^2)} dx \rightarrow \frac{q \operatorname{ArcTan}\left[\frac{\sqrt{3}}{qx}\right]}{2 \times 2^{2/3} \sqrt{3} a^{1/3} d} + \frac{q \operatorname{ArcTan}\left[\frac{\sqrt{3} (a^{1/3} - 2^{1/3} (a+bx^2)^{1/3})}{a^{1/3} qx}\right]}{2 \times 2^{2/3} \sqrt{3} a^{1/3} d} - \frac{q \operatorname{ArcTanh}[qx]}{6 \times 2^{2/3} a^{1/3} d} + \frac{q \operatorname{ArcTanh}\left[\frac{a^{1/3} qx}{a^{1/3} + 2^{1/3} (a+bx^2)^{1/3}}\right]}{2 \times 2^{2/3} a^{1/3} d}$$

Program code:

```
Int[1/((a+b_.**x_^2)^(1/3)*(c+d_.**x_^2)),x_Symbol] :=
  With[{q=Rt[-b/a,2]},
    q*ArcTan[Sqrt[3]/(q*x)]/(2*2^(2/3)*Sqrt[3]*a^(1/3)*d) +
    q*ArcTan[Sqrt[3]*(a^(1/3)-2^(1/3)*(a+b*x^2)^(1/3))/(a^(1/3)*q*x)]/(2*2^(2/3)*Sqrt[3]*a^(1/3)*d) -
    q*ArcTanh[q*x]/(6*2^(2/3)*a^(1/3)*d) +
    q*ArcTanh[(a^(1/3)*q*x)/(a^(1/3)+2^(1/3)*(a+b*x^2)^(1/3))]/(2*2^(2/3)*a^(1/3)*d) /;
  FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && EqQ[b*c+3*a*d,0] && NegQ[b/a]
```

2. $\int \frac{1}{(a+bx^2)^{1/3} (c+dx^2)} dx$ when $bc - ad \neq 0 \wedge bc - 9ad = 0$

1: $\int \frac{1}{(a+bx^2)^{1/3} (c+dx^2)} dx$ when $bc - ad \neq 0 \wedge bc - 9ad = 0 \wedge \frac{b}{a} > 0$

Rule 1.1.3.3.9.2.1.2.1.1: If $bc - ad \neq 0 \wedge bc - 9ad = 0 \wedge \frac{b}{a} > 0$, let $q \rightarrow \sqrt{\frac{b}{a}}$, then

$$\int \frac{1}{(a+bx^2)^{1/3} (c+dx^2)} dx \rightarrow -\frac{q \operatorname{ArcTan}\left[\frac{qx}{3}\right]}{12 a^{1/3} d} + \frac{q \operatorname{ArcTan}\left[\frac{a^{1/3} - (a+bx^2)^{1/3}}{a^{1/3} qx}\right]}{12 a^{1/3} d} - \frac{q \operatorname{ArcTan}\left[\frac{a^{1/3} + 2 (a+bx^2)^{1/3}}{a^{1/3} qx}\right]}{12 a^{1/3} d} - \frac{q \operatorname{ArcTanh}\left[\frac{\sqrt{3} (a^{1/3} - (a+bx^2)^{1/3})}{a^{1/3} qx}\right]}{4 \sqrt{3} a^{1/3} d}$$

$$\int \frac{1}{(a+bx^2)^{1/3} (c+dx^2)} dx \rightarrow \frac{q \operatorname{ArcTan}\left[\frac{qx}{3}\right]}{12 a^{1/3} d} + \frac{q \operatorname{ArcTan}\left[\frac{(a^{1/3} - (a+bx^2)^{1/3})^2}{3 a^{2/3} q x}\right]}{12 a^{1/3} d} - \frac{q \operatorname{ArcTanh}\left[\frac{\sqrt{3} (a^{1/3} - (a+bx^2)^{1/3})}{a^{1/3} q x}\right]}{4 \sqrt{3} a^{1/3} d}$$

Program code:

```
Int[1/((a+_.*x_^2)^(1/3)*(c+_.*x_^2)),x_Symbol] :=
  With[{q=Rt[b/a,2]},
    q*ArcTan[q*x/3]/(12*Rt[a,3]*d) +
    q*ArcTan[(Rt[a,3]-(a+b*x^2)^(1/3))^2/(3*Rt[a,3]^2*q*x)]/(12*Rt[a,3]*d) -
    q*ArcTanh[(Sqrt[3]*(Rt[a,3]-(a+b*x^2)^(1/3)))/(Rt[a,3]*q*x)]/(4*Sqrt[3]*Rt[a,3]*d) /;
    FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && EqQ[b*c-9*a*d,0] && PosQ[b/a]
```

2: $\int \frac{1}{(a+bx^2)^{1/3} (c+dx^2)} dx$ when $bc - ad \neq 0 \wedge bc - 9ad = 0 \wedge \frac{b}{a} \neq 0$

Rule 1.1.3.3.9.2.1.2.1.1: If $bc - ad \neq 0 \wedge bc - 9ad = 0 \wedge \frac{b}{a} \neq 0$, let $q \rightarrow \sqrt{-\frac{b}{a}}$, then

$$\int \frac{1}{(a+bx^2)^{1/3} (c+dx^2)} dx \rightarrow -\frac{q \operatorname{ArcTanh}\left[\frac{qx}{3}\right]}{12 a^{1/3} d} + \frac{q \operatorname{ArcTanh}\left[\frac{a^{1/3} - (a+bx^2)^{1/3}}{a^{1/3} q x}\right]}{12 a^{1/3} d} - \frac{q \operatorname{ArcTanh}\left[\frac{a^{1/3} + 2(a+bx^2)^{1/3}}{a^{1/3} q x}\right]}{12 a^{1/3} d} - \frac{q \operatorname{ArcTan}\left[\frac{\sqrt{3} (a^{1/3} - (a+bx^2)^{1/3})}{a^{1/3} q x}\right]}{4 \sqrt{3} a^{1/3} d}$$

$$\int \frac{1}{(a+bx^2)^{1/3} (c+dx^2)} dx \rightarrow -\frac{q \operatorname{ArcTanh}\left[\frac{qx}{3}\right]}{12 a^{1/3} d} + \frac{q \operatorname{ArcTanh}\left[\frac{(a^{1/3} - (a+bx^2)^{1/3})^2}{3 a^{2/3} q x}\right]}{12 a^{1/3} d} - \frac{q \operatorname{ArcTan}\left[\frac{\sqrt{3} (a^{1/3} - (a+bx^2)^{1/3})}{a^{1/3} q x}\right]}{4 \sqrt{3} a^{1/3} d}$$

Program code:

```
Int[1/((a+_.*x_^2)^(1/3)*(c+_.*x_^2)),x_Symbol] :=
  With[{q=Rt[-b/a,2]},
    -q*ArcTanh[q*x/3]/(12*Rt[a,3]*d) +
    q*ArcTanh[(Rt[a,3]-(a+b*x^2)^(1/3))^2/(3*Rt[a,3]^2*q*x)]/(12*Rt[a,3]*d) -
    q*ArcTan[(Sqrt[3]*(Rt[a,3]-(a+b*x^2)^(1/3)))/(Rt[a,3]*q*x)]/(4*Sqrt[3]*Rt[a,3]*d) /;
    FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && EqQ[b*c-9*a*d,0] && NegQ[b/a]
```


$$\mathbf{2:} \int \frac{(a+bx^2)^{2/3}}{c+dx^2} dx \text{ when } bc - ad \neq 0 \wedge bc + 3ad = 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{(a+bx^2)^{2/3}}{c+dx^2} = \frac{b}{d(a+bx^2)^{1/3}} - \frac{bc-ad}{d(a+bx^2)^{1/3}(c+dx^2)}$$

Rule 1.1.3.3.9.2.2: If $bc - ad \neq 0 \wedge bc + 3ad = 0$, then

$$\int \frac{(a+bx^2)^{2/3}}{c+dx^2} dx \rightarrow \frac{b}{d} \int \frac{1}{(a+bx^2)^{1/3}} dx - \frac{bc-ad}{d} \int \frac{1}{(a+bx^2)^{1/3}(c+dx^2)} dx$$

Program code:

```
Int[(a+b_.**x_^2)^(2/3)/(c+d_.**x_^2),x_Symbol] :=
  b/d*Int[1/(a+b*x^2)^(1/3),x] - (b*c-a*d)/d*Int[1/((a+b*x^2)^(1/3)*(c+d*x^2)),x] /;
FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && EqQ[b*c+3*a*d,0]
```

$$3. \int \frac{1}{(a+bx^2)^{1/4} (c+dx^2)} dx \text{ when } bc - ad \neq 0$$

$$1. \int \frac{1}{(a+bx^2)^{1/4} (c+dx^2)} dx \text{ when } bc - 2ad = 0$$

$$1: \int \frac{1}{(a+bx^2)^{1/4} (c+dx^2)} dx \text{ when } bc - 2ad = 0 \wedge \frac{b^2}{a} > 0$$

Reference: Eneström index number E688 in The Euler Archive

Rule 1.1.3.3.9.2.3.1.1: If $bc - 2ad = 0 \wedge \frac{b^2}{a} > 0$, let $q \rightarrow \left(\frac{b^2}{a}\right)^{1/4}$, then

$$\int \frac{1}{(a+bx^2)^{1/4} (c+dx^2)} dx \rightarrow -\frac{b}{2adq} \operatorname{ArcTan}\left[\frac{b+q^2\sqrt{a+bx^2}}{q^3x(a+bx^2)^{1/4}}\right] - \frac{b}{2adq} \operatorname{ArcTanh}\left[\frac{b-q^2\sqrt{a+bx^2}}{q^3x(a+bx^2)^{1/4}}\right]$$

Program code:

```
Int[1/((a+b_.**x_^2)^(1/4)*(c+d_.**x_^2)),x_Symbol] :=
  With[{q=Rt[b^2/a,4]},
    -b/(2*a*d*q)*ArcTan[(b+q^2*Sqrt[a+b*x^2])/(q^3*x*(a+b*x^2)^(1/4))] -
    b/(2*a*d*q)*ArcTanh[(b-q^2*Sqrt[a+b*x^2])/(q^3*x*(a+b*x^2)^(1/4))]/;
  FreeQ[{a,b,c,d},x] && EqQ[b*c-2*a*d,0] && PosQ[b^2/a]
```

$$2: \int \frac{1}{(a+bx^2)^{1/4} (c+dx^2)} dx \text{ when } bc - 2ad == 0 \wedge \frac{b^2}{a} \neq 0$$

Reference: Eneström index number E688 in The Euler Archive

Derivation: Integration by substitution

Basis: If $bc - 2ad == 0$, then $\frac{1}{(a+bx^2)^{1/4} (c+dx^2)} == \frac{2b}{d} \text{Subst}\left[\frac{1}{4a+b^2x^4}, x, \frac{x}{(a+bx^2)^{1/4}}\right] \partial_x \frac{x}{(a+bx^2)^{1/4}}$

Rule 1.1.3.3.9.2.3.1.2: If $bc - 2ad == 0 \wedge \frac{b^2}{a} \neq 0$, let $q \rightarrow (-\frac{b^2}{a})^{1/4}$, then

$$\begin{aligned} \int \frac{1}{(a+bx^2)^{1/4} (c+dx^2)} dx &\rightarrow \frac{2b}{d} \text{Subst}\left[\int \frac{1}{4a+b^2x^4} dx, x, \frac{x}{(a+bx^2)^{1/4}}\right] \\ &\rightarrow \frac{b}{2\sqrt{2}adq} \text{ArcTan}\left[\frac{qx}{\sqrt{2}(a+bx^2)^{1/4}}\right] + \frac{b}{2\sqrt{2}adq} \text{ArcTanh}\left[\frac{qx}{\sqrt{2}(a+bx^2)^{1/4}}\right] \end{aligned}$$

Program code:

```
Int[1/((a+b_.**x_^2)^(1/4)*(c+d_.**x_^2)),x_Symbol] :=
  With[{q=Rt[-b^2/a,4]},
    b/(2*Sqrt[2]*a*d*q)*ArcTan[q*x/(Sqrt[2]*(a+b*x^2)^(1/4))] +
    b/(2*Sqrt[2]*a*d*q)*ArcTanh[q*x/(Sqrt[2]*(a+b*x^2)^(1/4))]] /;
  FreeQ[{a,b,c,d},x] && EqQ[b*c-2*a*d,0] && NegQ[b^2/a]
```

$$\text{x: } \int \frac{1}{(a+bx^2)^{1/4} (c+dx^2)} dx \text{ when } bc - 2ad == 0 \wedge \frac{b^2}{a} \neq 0$$

Reference: Eneström index number E688 in The Euler Archive

Derivation: Integration by substitution

Basis: If $bc - 2ad == 0$, then $\frac{1}{(a+bx^2)^{1/4} (c+dx^2)} == \frac{2b}{d} \text{Subst}\left[\frac{1}{4a+b^2x^4}, x, \frac{x}{(a+bx^2)^{1/4}}\right] \partial_x \frac{x}{(a+bx^2)^{1/4}}$

Note: Although this antiderivative is real and continuous when the integrand is real, it is unnecessarily discontinuous when the integrand is not real.

Rule 1.1.3.3.9.2.3.1.2: If $bc - 2ad == 0 \wedge \frac{b^2}{a} \neq 0$, let $q \rightarrow \left(-\frac{b^2}{a}\right)^{1/4}$, then

$$\begin{aligned} \int \frac{1}{(a+bx^2)^{1/4} (c+dx^2)} dx &\rightarrow \frac{2b}{d} \text{Subst}\left[\int \frac{1}{4a+b^2x^4} dx, x, \frac{x}{(a+bx^2)^{1/4}}\right] \\ &\rightarrow \frac{b}{2\sqrt{2}adq} \text{ArcTan}\left[\frac{qx}{\sqrt{2}(a+bx^2)^{1/4}}\right] + \frac{b}{4\sqrt{2}adq} \text{Log}\left[\frac{\sqrt{2}qx + 2(a+bx^2)^{1/4}}{\sqrt{2}qx - 2(a+bx^2)^{1/4}}\right] \end{aligned}$$

Program code:

```
(* Int[1/((a+b_.**x_^2)^(1/4)*(c+d_.**x_^2)),x_Symbol] :=
  With[{q=Rt[-b^2/a,4]},
    b/(2*Sqrt[2]*a*d*q)*ArcTan[q*x/(Sqrt[2]*(a+b*x^2)^(1/4))] +
    b/(4*Sqrt[2]*a*d*q)*Log[(Sqrt[2]*q*x+2*(a+b*x^2)^(1/4))/(Sqrt[2]*q*x-2*(a+b*x^2)^(1/4))]] /;
FreeQ[{a,b,c,d},x] && EqQ[b*c-2*a*d,0] && NegQ[b^2/a] *)
```

2: $\int \frac{1}{(a+bx^2)^{1/4} (c+dx^2)} dx$ when $bc - ad \neq 0$

Derivation: Piecewise constant extraction and integration by substitution

■ Basis: $\partial_x \sqrt{\frac{-bx^2}{a}} = 0$

Basis: $\frac{x}{\sqrt{-\frac{bx^2}{a}} (a+bx^2)^{1/4} (c+dx^2)} = 2 \text{Subst} \left[\frac{x^2}{\sqrt{1-\frac{x^4}{a}} (bc-ad+dx^4)}, x, (a+bx^2)^{1/4} \right] \partial_x (a+bx^2)^{1/4}$

Rule 1.1.3.3.9.2.3.2: If $bc - ad \neq 0$, then

$$\int \frac{1}{(a+bx^2)^{1/4} (c+dx^2)} dx \rightarrow \frac{\sqrt{-\frac{bx^2}{a}}}{x} \int \frac{x}{\sqrt{-\frac{bx^2}{a}} (a+bx^2)^{1/4} (c+dx^2)} dx \rightarrow$$

$$\frac{2\sqrt{-\frac{bx^2}{a}}}{x} \text{Subst} \left[\int \frac{x^2}{\sqrt{1-\frac{x^4}{a}} (bc-ad+dx^4)} dx, x, (a+bx^2)^{1/4} \right]$$

Program code:

```
Int[1/((a+b_.**x_^2)^(1/4)*(c+d_.**x_^2)),x_Symbol] :=
  2*Sqrt[-b*x^2/a]/x*Subst[Int[x^2/(Sqrt[1-x^4/a]*(b*c-a*d+d*x^4))],x],x,(a+b*x^2)^(1/4)] /;
FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0]
```

4. $\int \frac{1}{(a+bx^2)^{3/4} (c+dx^2)} dx$ when $bc - ad \neq 0$

$$1: \int \frac{1}{(a+bx^2)^{3/4} (c+dx^2)} dx \text{ when } bc - 2ad = 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{1}{(a+bx^2)^{3/4} (c+dx^2)} = \frac{1}{c (a+bx^2)^{3/4}} - \frac{dx^2}{c (a+bx^2)^{3/4} (c+dx^2)}$$

Note: There are terminal rules for $\int \frac{x^2}{(a+bx^2)^{3/4} (c+dx^2)} dx$ when $bc - 2ad = 0$.

Rule 1.1.3.3.9.2.4.1: If $bc - 2ad = 0$, then

$$\int \frac{1}{(a+bx^2)^{3/4} (c+dx^2)} dx \rightarrow \frac{1}{c} \int \frac{1}{(a+bx^2)^{3/4}} dx - \frac{d}{c} \int \frac{x^2}{(a+bx^2)^{3/4} (c+dx^2)} dx$$

Program code:

```
Int[1/((a+b*x^2)^(3/4)*(c+d*x^2)),x_Symbol] :=
  1/c*Int[1/(a+b*x^2)^(3/4),x] - d/c*Int[x^2/((a+b*x^2)^(3/4)*(c+d*x^2)),x] /;
FreeQ[{a,b,c,d},x] && EqQ[b*c-2*a*d,0]
```

$$2: \int \frac{1}{(a+bx^2)^{3/4} (c+dx^2)} dx \text{ when } bc - ad \neq 0$$

Derivation: Piecewise constant extranction and integration by substitution

■ Basis: $\partial_x \sqrt{\frac{bx^2}{a}} = 0$

$$\text{Basis: } x F[x^2] = \frac{1}{2} \text{Subst}[F[x], x, x^2] \partial_x x^2$$

Rule 1.1.3.3.9.2.4.2: If $bc - ad \neq 0$, then

$$\int \frac{1}{(a+bx^2)^{3/4} (c+dx^2)} dx \rightarrow \frac{\sqrt{-\frac{bx^2}{a}}}{x} \int \frac{x}{\sqrt{-\frac{bx^2}{a}} (a+bx^2)^{3/4} (c+dx^2)} dx \rightarrow$$

$$\frac{\sqrt{-\frac{bx^2}{a}}}{2x} \text{Subst} \left[\int \frac{1}{\sqrt{-\frac{bx}{a}} (a+bx)^{3/4} (c+dx)} dx, x, x^2 \right]$$

Program code:

```
Int[1/((a+b_.x_^2)^(3/4)*(c+d_.x_^2)),x_Symbol] :=
  Sqrt[-b*x^2/a]/(2*x)*Subst[Int[1/(Sqrt[-b*x/a]*(a+b*x)^(3/4)*(c+d*x)),x],x,x^2] /;
  FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0]
```

5: $\int \frac{(a+bx^2)^p}{c+dx^2} dx$ when $bc-ad \neq 0 \wedge p > 0$

Derivation: Algebraic expansion

Basis: $\frac{(a+bz)^p}{c+dz} = \frac{b(a+bz)^{p-1}}{d} - \frac{(bc-ad)(a+bz)^{p-1}}{d(c+dz)}$

Rule 1.1.3.3.9.2.5: If $bc-ad \neq 0 \wedge p > 0$, then

$$\int \frac{(a+bx^2)^p}{c+dx^2} dx \rightarrow \frac{b}{d} \int (a+bx^2)^{p-1} dx - \frac{bc-ad}{d} \int \frac{(a+bx^2)^{p-1}}{c+dx^2} dx$$

Program code:

```
Int[(a+b_.x_^2)^p_./(c+d_.x_^2),x_Symbol] :=
  b/d*Int[(a+b*x^2)^(p-1),x] - (b*c-a*d)/d*Int[(a+b*x^2)^(p-1)/(c+d*x^2),x] /;
  FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && GtQ[p,0] && (EqQ[p,1/2] || EqQ[Denominator[p],4])
```

$$\text{6: } \int \frac{(a+bx^2)^p}{c+dx^2} dx \text{ when } bc - ad \neq 0 \wedge p < -1$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{(a+bx)^p}{c+dz} = \frac{b(a+bx)^p}{b c - a d} - \frac{d(a+bx)^{p+1}}{(b c - a d)(c+dz)}$$

Rule 1.1.3.3.9.2.6: If $bc - ad \neq 0 \wedge p < -1$, then

$$\int \frac{(a+bx^2)^p}{c+dx^2} dx \rightarrow \frac{b}{(bc - ad)} \int (a+bx^2)^p dx - \frac{d}{(bc - ad)} \int \frac{(a+bx^2)^{p+1}}{c+dx^2} dx$$

Program code:

```
Int[(a+b_.**x_^2)^p/(c+d_.**x_^2),x_Symbol] :=
  b/(b*c-a*d)*Int[(a+b*x^2)^p,x] - d/(b*c-a*d)*Int[(a+b*x^2)^(p+1)/(c+d*x^2),x] /;
FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && LtQ[p,-1] && EqQ[Denominator[p],4] && (EqQ[p,-5/4] || EqQ[p,-7/4])
```

$$3. \int \frac{(a+bx^4)^p}{c+dx^4} dx \text{ when } bc - ad \neq 0$$

$$1. \int \frac{(a+bx^4)^p}{c+dx^4} dx \text{ when } bc - ad \neq 0 \wedge p > 0$$

$$1. \int \frac{\sqrt{a+bx^4}}{c+dx^4} dx \text{ when } bc - ad \neq 0$$

$$1. \int \frac{\sqrt{a+bx^4}}{c+dx^4} dx \text{ when } bc + ad = 0$$

$$\text{1: } \int \frac{\sqrt{a+bx^4}}{c+dx^4} dx \text{ when } bc + ad = 0 \wedge ab > 0$$

Derivation: Integration by substitution

Basis: If $bc + ad = 0$, then $\int \frac{\sqrt{a+bx^4}}{c+dx^4} dx = \frac{a}{c} \text{Subst} \left[\frac{1}{1-4abx^4}, x, \frac{x}{\sqrt{a+bx^4}} \right] \partial_x \frac{x}{\sqrt{a+bx^4}}$

Rule 1.1.3.3.9.3.1.1.1: If $bc + ad = 0 \wedge ab > 0$, then

$$\int \frac{\sqrt{a+bx^4}}{c+dx^4} dx \rightarrow \frac{a}{c} \text{Subst} \left[\int \frac{1}{1-4abx^4} dx, x, \frac{x}{\sqrt{a+bx^4}} \right]$$

Program code:

```
Int[Sqrt[a+_.*x_^4]/(c+_.*x_^4),x_Symbol] :=
  a/c*Subst[Int[1/(1-4*a*b*x^4),x],x,x/Sqrt[a+b*x^4]] /;
FreeQ[{a,b,c,d},x] && EqQ[b*c+a*d,0] && PosQ[a*b]
```

2: $\int \frac{\sqrt{a+bx^4}}{c+dx^4} dx$ when $bc + ad = 0 \wedge ab \neq 0$

Contributed by Martin Welz on 31 January 2017

Rule 1.1.3.3.9.3.1.1.2: If $bc + ad = 0 \wedge ab \neq 0$, let $q \rightarrow (-ab)^{1/4}$, then

$$\int \frac{\sqrt{a+bx^4}}{c+dx^4} dx \rightarrow \frac{a}{2cq} \text{ArcTan} \left[\frac{qx(a+q^2x^2)}{a\sqrt{a+bx^4}} \right] + \frac{a}{2cq} \text{ArcTanh} \left[\frac{qx(a-q^2x^2)}{a\sqrt{a+bx^4}} \right]$$

Program code:

```
Int[Sqrt[a+_.*x_^4]/(c+_.*x_^4),x_Symbol] :=
  With[{q=Rt[-a*b,4]},
    a/(2*c*q)*ArcTan[q*x*(a+q^2*x^2)/(a*Sqrt[a+b*x^4])] + a/(2*c*q)*ArcTanh[q*x*(a-q^2*x^2)/(a*Sqrt[a+b*x^4])]] /;
FreeQ[{a,b,c,d},x] && EqQ[b*c+a*d,0] && NegQ[a*b]
```

$$2: \int \frac{\sqrt{a+bx^4}}{c+dx^4} dx \text{ when } bc - ad \neq 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{\sqrt{a+bz}}{c+dz} = \frac{b}{d\sqrt{a+bz}} - \frac{bc-ad}{d\sqrt{a+bz}(c+dz)}$$

Rule 1.1.3.3.9.3.1.1.2: If $bc - ad \neq 0$, then

$$\int \frac{\sqrt{a+bx^4}}{c+dx^4} dx \rightarrow \frac{b}{d} \int \frac{1}{\sqrt{a+bx^4}} dx - \frac{bc-ad}{d} \int \frac{1}{\sqrt{a+bx^4}(c+dx^4)} dx$$

Program code:

```
Int[Sqrt[a+b_*x^4]/(c+d_*x^4),x_Symbol] :=
  b/d*Int[1/Sqrt[a+b*x^4],x] - (b*c-a*d)/d*Int[1/(Sqrt[a+b*x^4]*(c+d*x^4)),x] /;
FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0]
```

$$2: \int \frac{(a+bx^4)^{1/4}}{c+dx^4} dx \text{ when } bc - ad \neq 0$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: } \partial_x \sqrt{a+bx^4} \sqrt{\frac{a}{a+bx^4}} = 0$$

$$\text{Basis: } \frac{1}{\sqrt{\frac{a}{a+bx^4}} (a+bx^4)^{1/4} (c+dx^4)} = \text{Subst} \left[\frac{1}{\sqrt{1-bx^4} (c-(bc-ad)x^4)}, x, \frac{x}{(a+bx^4)^{1/4}} \right] \partial_x \frac{x}{(a+bx^4)^{1/4}}$$

Rule 1.1.3.3.9.3.1.1.2: If $bc - ad \neq 0$, then

$$\int \frac{(a+bx^4)^{1/4}}{c+dx^4} dx \rightarrow \sqrt{a+bx^4} \sqrt{\frac{a}{a+bx^4}} \int \frac{1}{\sqrt{\frac{a}{a+bx^4}} (a+bx^4)^{1/4} (c+dx^4)} dx$$

$$\rightarrow \sqrt{a+bx^4} \sqrt{\frac{a}{a+bx^4}} \text{Subst} \left[\int \frac{1}{\sqrt{1-bx^4} (c-(bc-ad)x^4)} dx, x, \frac{x}{(a+bx^4)^{1/4}} \right]$$

Program code:

```
Int[(a+b_.**x_^4)^(1/4)/(c+d_.**x_^4),x_Symbol] :=
  Sqrt[a+b*x^4]*Sqrt[a/(a+b*x^4)]*Subst[Int[1/(Sqrt[1-b*x^4]*(c-(b*c-a*d)*x^4)),x],x,x/(a+b*x^4)^(1/4)] /;
FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0]
```

$$\mathbf{3:} \int \frac{(a+bx^4)^p}{c+dx^4} dx \text{ when } bc-ad \neq 0 \wedge (p = \frac{3}{4} \vee p = \frac{5}{4})$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{(a+bz)^p}{c+dz} = \frac{b(a+bz)^{p-1}}{d} - \frac{(bc-ad)(a+bz)^{p-1}}{d(c+dz)}$$

Rule 1.1.3.3.9.3.1.3: If $bc-ad \neq 0 \wedge (p = \frac{3}{4} \vee p = \frac{5}{4})$, then

$$\int \frac{(a+bx^4)^p}{c+dx^4} dx \rightarrow \frac{b}{d} \int (a+bx^4)^{p-1} dx - \frac{bc-ad}{d} \int \frac{(a+bx^4)^{p-1}}{c+dx^4} dx$$

Program code:

```
Int[(a+b_.**x_^4)^p/(c+d_.**x_^4),x_Symbol] :=
  b/d*Int[(a+b*x^4)^(p-1),x] - (b*c-a*d)/d*Int[(a+b*x^4)^(p-1)/(c+d*x^4),x] /;
FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && (EqQ[p,3/4] || EqQ[p,5/4])
```

$$2. \int \frac{(a+bx^4)^p}{c+dx^4} dx \text{ when } bc-ad \neq 0 \wedge p < 0$$

$$1: \int \frac{1}{\sqrt{a+bx^4} (c+dx^4)} dx \text{ when } bc-ad \neq 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{1}{c+dx^4} = \frac{1}{2c \left(1 - \sqrt{-\frac{d}{c}} x^2\right)} + \frac{1}{2c \left(1 + \sqrt{-\frac{d}{c}} x^2\right)}$$

Rule 1.1.3.3.9.3.2.1: If $bc-ad \neq 0$, then

$$\int \frac{1}{\sqrt{a+bx^4} (c+dx^4)} dx \rightarrow \frac{1}{2c} \int \frac{1}{\sqrt{a+bx^4} \left(1 - \sqrt{-\frac{d}{c}} x^2\right)} dx + \frac{1}{2c} \int \frac{1}{\sqrt{a+bx^4} \left(1 + \sqrt{-\frac{d}{c}} x^2\right)} dx$$

Program code:

```
Int[1/(Sqrt[a+b_*x^4]*(c+d_*x^4)),x_Symbol] :=
  1/(2*c)*Int[1/(Sqrt[a+b*x^4]*(1-Rt[-d/c,2]*x^2)),x] + 1/(2*c)*Int[1/(Sqrt[a+b*x^4]*(1+Rt[-d/c,2]*x^2)),x] /;
FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0]
```

$$2: \int \frac{1}{(a+bx^4)^{3/4} (c+dx^4)} dx \text{ when } bc-ad \neq 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{(a+bz)^p}{c+dz} = \frac{b(a+bz)^p}{bc-ad} - \frac{d(a+bz)^{p+1}}{(bc-ad)(c+dz)}$$

Rule 1.1.3.3.9.3.2.2: If $bc-ad \neq 0$, then

$$\int \frac{1}{(a+bx^4)^{3/4} (c+dx^4)} dx \rightarrow \frac{b}{(bc-ad)} \int \frac{1}{(a+bx^4)^{3/4}} dx - \frac{d}{(bc-ad)} \int \frac{(a+bx^4)^{1/4}}{c+dx^4} dx$$

Program code:

```
Int[1/((a_+b_.*x_^4)^(3/4)*(c_+d_.*x_^4)),x_Symbol] :=
  b/(b*c-a*d)*Int[1/(a+b*x^4)^(3/4),x] - d/(b*c-a*d)*Int[(a+b*x^4)^(1/4)/(c+d*x^4),x] /;
FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0]
```

10. $\int (a+bx^n)^p (c+dx^n)^q dx$ when $bc - ad \neq 0 \wedge p < -1$

1. $\int (a+bx^n)^p (c+dx^n)^q dx$ when $bc - ad \neq 0 \wedge p < -1 \wedge q > 0$

1: $\int \frac{\sqrt{a+bx^2}}{(c+dx^2)^{3/2}} dx$ when $\frac{b}{a} > 0 \wedge \frac{d}{c} > 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2} \sqrt{\frac{c(a+bx^2)}{a(c+dx^2)}}} = 0$

Rule 1.1.3.3.10.1.1: If $\frac{b}{a} > 0 \wedge \frac{d}{c} > 0$, then

$$\int \frac{\sqrt{a+bx^2}}{(c+dx^2)^{3/2}} dx \rightarrow \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2} \sqrt{\frac{c(a+bx^2)}{a(c+dx^2)}}} \int \frac{\sqrt{\frac{c(a+bx^2)}{a(c+dx^2)}}}{c+dx^2} dx \rightarrow \frac{\sqrt{a+bx^2}}{c \sqrt{\frac{d}{c}} \sqrt{c+dx^2} \sqrt{\frac{c(a+bx^2)}{a(c+dx^2)}}} \text{EllipticE}\left[\text{ArcTan}\left[\sqrt{\frac{d}{c}} x\right], 1 - \frac{bc}{ad}\right]$$

$$\int \frac{\sqrt{a+bx^2}}{(c+dx^2)^{3/2}} dx \rightarrow \frac{a \sqrt{c+dx^2} \sqrt{\frac{c(a+bx^2)}{a(c+dx^2)}}}{c \sqrt{a+bx^2}} \int \frac{\sqrt{\frac{c(a+bx^2)}{a(c+dx^2)}}}{c+dx^2} dx \rightarrow \frac{a \sqrt{c+dx^2} \sqrt{\frac{c(a+bx^2)}{a(c+dx^2)}}}{c^2 \sqrt{\frac{d}{c}} \sqrt{a+bx^2}} \text{EllipticE}\left[\text{ArcTan}\left[\sqrt{\frac{d}{c}} x\right], 1 - \frac{bc}{ad}\right]$$

Program code:

```
Int[Sqrt[a_+b_.*x_^2]/(c_+d_.*x_^2)^(3/2),x_Symbol] :=
  Sqrt[a+b*x^2]/(c*Rt[d/c,2]*Sqrt[c+d*x^2]*Sqrt[c*(a+b*x^2)/(a*(c+d*x^2))])*EllipticE[ArcTan[Rt[d/c,2]*x],1-b*c/(a*d)] /;
FreeQ[{a,b,c,d},x] && PosQ[b/a] && PosQ[d/c]
```

```
(* Int[Sqrt[a_+b_.*x_^2]/(c_+d_.*x_^2)^(3/2),x_Symbol] :=
  a*Sqrt[c+d*x^2]*Sqrt[(c*(a+b*x^2))/(a*(c+d*x^2))]/(c^2*Rt[d/c,2]*Sqrt[a+b*x^2])*EllipticE[ArcTan[Rt[d/c,2]*x],1-b*c/(a*d)] /;
FreeQ[{a,b,c,d},x] && PosQ[b/a] && PosQ[d/c] *)
```

2: $\int (a+bx^n)^p (c+dx^n)^q dx$ when $bc-ad \neq 0 \wedge p < -1 \wedge 0 < q < 1$

Derivation: Binomial product recurrence 1 with $A = 1$ and $B = 0$

Rule 1.1.3.3.10.1.2: If $bc-ad \neq 0 \wedge p < -1 \wedge 0 < q < 1$, then

$$\int (a+bx^n)^p (c+dx^n)^q dx \rightarrow -\frac{x(a+bx^n)^{p+1}(c+dx^n)^q}{an(p+1)} + \frac{1}{an(p+1)} \int (a+bx^n)^{p+1} (c+dx^n)^{q-1} (c(n(p+1)+1) + d(n(p+q+1)+1)x^n) dx$$

Program code:

```
Int[(a+b*x^n)^p*(c+d*x^n)^q,x_Symbol] :=
  -x*(a+b*x^n)^(p+1)*(c+d*x^n)^q/(a*n*(p+1)) +
  1/(a*n*(p+1))*Int[(a+b*x^n)^(p+1)*(c+d*x^n)^(q-1)*Simp[c*(n*(p+1)+1)+d*(n*(p+q+1)+1)*x^n,x],x] /;
FreeQ[{a,b,c,d,n},x] && NeQ[b*c-a*d,0] && LtQ[p,-1] && LtQ[0,q,1] && IntBinomialQ[a,b,c,d,n,p,q,x]
```

3: $\int (a+bx^n)^p (c+dx^n)^q dx$ when $bc - ad \neq 0 \wedge p < -1 \wedge q > 1$

Derivation: Binomial product recurrence 1 with $A = c$, $B = d$ and $q = q - 1$

Rule 1.1.3.3.10.1.3: If $bc - ad \neq 0 \wedge p < -1 \wedge q > 1$, then

$$\int (a+bx^n)^p (c+dx^n)^q dx \rightarrow \frac{(ad-bc)x(a+bx^n)^{p+1}(c+dx^n)^{q-1}}{abn(p+1)} - \frac{1}{abn(p+1)} \int (a+bx^n)^{p+1} (c+dx^n)^{q-2} (c(ad-bc(n(p+1)+1)) + d(ad(n(q-1)+1) - bc(n(p+q)+1)) x^n) dx$$

Program code:

```
Int[(a+b_.**x_^n_)^p_*(c+d_.**x_^n_)^q_,x_Symbol] :=
  (a*d-c*b)**x*(a+b*x^n)^(p+1)*(c+d*x^n)^(q-1)/(a*b*n*(p+1)) -
  1/(a*b*n*(p+1))*
  Int[(a+b*x^n)^(p+1)*(c+d*x^n)^(q-2)*Simp[c*(a*d-c*b*(n*(p+1)+1))+d*(a*d*(n*(q-1)+1)-b*c*(n*(p+q)+1))*x^n,x],x] /;
FreeQ[{a,b,c,d,n},x] && NeQ[b*c-a*d,0] && LtQ[p,-1] && GtQ[q,1] && IntBinomialQ[a,b,c,d,n,p,q,x]
```


2: $\int (a+bx^n)^p (c+dx^n)^q dx$ when $bc - ad \neq 0 \wedge p < -1$

Derivation: Binomial product recurrence 2a with $A = 1$ and $B = 0$

Rule 1.1.3.3.10.1.2: If $bc - ad \neq 0 \wedge p < -1$, then

$$\int (a+bx^n)^p (c+dx^n)^q dx \rightarrow -\frac{bx(a+bx^n)^{p+1}(c+dx^n)^{q+1}}{an(p+1)(bc-ad)} + \frac{1}{an(p+1)(bc-ad)} \int (a+bx^n)^{p+1}(c+dx^n)^q (bc+n(p+1)(bc-ad) + db(n(p+q+2)+1)x^n) dx$$

Program code:

```
Int[(a+b_.**x_^n_)^p_*(c+d_.**x_^n_)^q_,x_Symbol] :=
  -b*x*(a+b*x^n)^(p+1)*(c+d*x^n)^(q+1)/(a*n*(p+1)*(b*c-a*d)) +
  1/(a*n*(p+1)*(b*c-a*d))*
  Int[(a+b*x^n)^(p+1)*(c+d*x^n)^q*Simp[b*c+n*(p+1)*(b*c-a*d)+d*b*(n*(p+q+2)+1)*x^n,x],x] /;
FreeQ[{a,b,c,d,n,q},x] && NeQ[b*c-a*d,0] && LtQ[p,-1] && Not[Not[IntegerQ[p]] && IntegerQ[q] && LtQ[q,-1]] &&
IntBinomialQ[a,b,c,d,n,p,q,x]
```

11: $\int (a + bx^n)^p (c + dx^n)^q dx$ when $bc - ad \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p \in \mathbb{Z} \wedge q \in \mathbb{Z} \wedge p + q > 0$

Derivation: Algebraic expansion

Rule 1.1.3.3.11: If $bc - ad \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p \in \mathbb{Z} \wedge q \in \mathbb{Z} \wedge p + q > 0$, then

$$\int (a + bx^n)^p (c + dx^n)^q dx \rightarrow \int \text{ExpandIntegrand}[(a + bx^n)^p (c + dx^n)^q, x] dx$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*x^n)^p*(c+d*x^n)^q,x],x] /;
FreeQ[{a,b,c,d},x] && NeQ[b*c-a*d,0] && IGtQ[n,0] && IntegersQ[p,q] && GtQ[p+q,0]
```

$$12. \int (a + b x^n)^p (c + d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge q > 0$$

$$1: \int (a + b x^n)^p (c + d x^n)^q dx \text{ when } b c - a d \neq 0 \wedge q > 1 \wedge n(p+q) + 1 \neq 0$$

Derivation: Binomial product recurrence 3a with $A = c$, $B = d$ and $q = q - 1$

Rule 1.1.3.3.12.1: If $b c - a d \neq 0 \wedge q > 1 \wedge n(p+q) + 1 \neq 0$, then

$$\int (a + b x^n)^p (c + d x^n)^q dx \rightarrow \frac{d x (a + b x^n)^{p+1} (c + d x^n)^{q-1}}{b (n(p+q) + 1)} + \frac{1}{b (n(p+q) + 1)} \int (a + b x^n)^p (c + d x^n)^{q-2} (c (b c (n(p+q) + 1) - a d) + d (b c (n(p+2q-1) + 1) - a d (n(q-1) + 1)) x^n) dx$$

Program code:

```
Int[(a_+b_.**x_^n_)^p_*(c_+d_.**x_^n_)^q_,x_Symbol] :=
  d**x*(a+b*x^n)^(p+1)*(c+d*x^n)^(q-1)/(b*(n*(p+q)+1)) +
  1/(b*(n*(p+q)+1))*
  Int[(a+b*x^n)^p*(c+d*x^n)^(q-2)*Simp[c*(b*c*(n*(p+q)+1)-a*d)+d*(b*c*(n*(p+2*q-1)+1)-a*d*(n*(q-1)+1))*x^n,x] /;
FreeQ[{a,b,c,d,n,p},x] && NeQ[b*c-a*d,0] && GtQ[q,1] && NeQ[n*(p+q)+1,0] && Not[IGtQ[p,1]] && IntBinomialQ[a,b,c,d,n,p,q,x]
```

2: $\int (a+bx^n)^p (c+dx^n)^q dx$ when $bc - ad \neq 0 \wedge q > 0 \wedge p > 0$

Derivation: Binomial product recurrence 2b with $m = 0$, $A = a$, $B = b$ and $p = p - 1$

Rule 1.1.3.3.12.2: If $bc - ad \neq 0 \wedge q > 0 \wedge p > 0$, then

$$\int (a+bx^n)^p (c+dx^n)^q dx \rightarrow \frac{x (a+bx^n)^p (c+dx^n)^q}{n(p+q)+1} + \frac{n}{n(p+q)+1} \int (a+bx^n)^{p-1} (c+dx^n)^{q-1} (ac(p+q) + (q(bc-ad) + ad(p+q))x^n) dx$$

Program code:

```
Int[(a+b*x^n)^p*(c+d*x^n)^q,x_Symbol] :=
  x*(a+b*x^n)^p*(c+d*x^n)^q/(n*(p+q)+1) +
  n/(n*(p+q)+1)*Int[(a+b*x^n)^(p-1)*(c+d*x^n)^(q-1)*Simp[a*c*(p+q)+(q*(b*c-a*d)+a*d*(p+q))*x^n,x] /;
FreeQ[{a,b,c,d,n},x] && NeQ[b*c-a*d,0] && GtQ[q,0] && GtQ[p,0] && IntBinomialQ[a,b,c,d,n,p,q,x]
```

$$13. \int \frac{(a+bx^2)^p}{\sqrt{c+dx^2}} dx \text{ when } bc - ad \neq 0 \wedge p^2 = \frac{1}{4}$$

$$1. \int \frac{1}{\sqrt{a+bx^2} \sqrt{c+dx^2}} dx \text{ when } bc - ad \neq 0$$

$$1: \int \frac{1}{\sqrt{a+bx^2} \sqrt{c+dx^2}} dx \text{ when } \frac{d}{c} > 0 \wedge \frac{b}{a} > 0$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{\sqrt{c+dx^2} \sqrt{\frac{c(a+bx^2)}{a(c+dx^2)}}}{\sqrt{a+bx^2}} = 0$$

Rule 1.1.3.3.13.1.1: If $\frac{d}{c} > 0 \wedge \frac{b}{a} > 0$, then

$$\int \frac{1}{\sqrt{a+bx^2} \sqrt{c+dx^2}} dx \rightarrow \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2} \sqrt{\frac{c(a+bx^2)}{a(c+dx^2)}}} \int \frac{\sqrt{\frac{c(a+bx^2)}{a(c+dx^2)}}}{a+bx^2} dx \rightarrow \frac{\sqrt{a+bx^2}}{a \sqrt{\frac{d}{c}} \sqrt{c+dx^2} \sqrt{\frac{c(a+bx^2)}{a(c+dx^2)}}} \text{EllipticF}\left[\text{ArcTan}\left[\sqrt{\frac{d}{c}} x\right], 1 - \frac{bc}{ad}\right]$$

$$\int \frac{1}{\sqrt{a+bx^2} \sqrt{c+dx^2}} dx \rightarrow \frac{a \sqrt{c+dx^2} \sqrt{\frac{c(a+bx^2)}{a(c+dx^2)}}}{c \sqrt{a+bx^2}} \int \frac{\sqrt{\frac{c(a+bx^2)}{a(c+dx^2)}}}{a+bx^2} dx \rightarrow \frac{\sqrt{c+dx^2} \sqrt{\frac{c(a+bx^2)}{a(c+dx^2)}}}{c \sqrt{\frac{d}{c}} \sqrt{a+bx^2}} \text{EllipticF}\left[\text{ArcTan}\left[\sqrt{\frac{d}{c}} x\right], 1 - \frac{bc}{ad}\right]$$

Program code:

```
Int[1/(Sqrt[a+b_.x^2]*Sqrt[c+d_.x^2]),x_Symbol] :=
  Sqrt[a+b*x^2]/(a*Rt[d/c,2]*Sqrt[c+d*x^2]*Sqrt[c*(a+b*x^2)/(a*(c+d*x^2))])*EllipticF[ArcTan[Rt[d/c,2]*x],1-b*c/(a*d)] /;
FreeQ[{a,b,c,d},x] && PosQ[d/c] && PosQ[b/a] && Not[SimplerSqrtQ[b/a,d/c]]
```

```
(* Int[1/(Sqrt[a+b.*x_^2]*Sqrt[c+d.*x_^2]),x_Symbol] :=
  Sqrt[c+d*x^2]*Sqrt[c*(a+b*x^2)/(a*(c+d*x^2))]/(c*Rt[d/c,2]*Sqrt[a+b*x^2])*EllipticF[ArcTan[Rt[d/c,2]*x],1-b*c/(a*d)] /;
FreeQ[{a,b,c,d},x] && PosQ[d/c] && PosQ[b/a] && Not[SimplerSqrtQ[b/a,d/c]] *)
```

$$2. \int \frac{1}{\sqrt{a+bx^2} \sqrt{c+dx^2}} dx \text{ when } \frac{d}{c} \neq 0$$

$$\text{1: } \int \frac{1}{\sqrt{a+bx^2} \sqrt{c+dx^2}} dx \text{ when } \frac{d}{c} \neq 0 \wedge c > 0 \wedge a > 0$$

Rule 1.1.3.3.13.1.2.1: If $\frac{d}{c} \neq 0 \wedge c > 0 \wedge a > 0$, then

$$\int \frac{1}{\sqrt{a+bx^2} \sqrt{c+dx^2}} dx \rightarrow \frac{1}{\sqrt{a} \sqrt{c} \sqrt{-\frac{d}{c}}} \text{EllipticF}\left[\text{ArcSin}\left[\sqrt{-\frac{d}{c}} x\right], \frac{bc}{ad}\right]$$

Program code:

```
Int[1/(Sqrt[a+b.*x_^2]*Sqrt[c+d.*x_^2]),x_Symbol] :=
  1/(Sqrt[a]*Sqrt[c]*Rt[-d/c,2])*EllipticF[ArcSin[Rt[-d/c,2]*x],b*c/(a*d)] /;
FreeQ[{a,b,c,d},x] && NegQ[d/c] && GtQ[c,0] && GtQ[a,0] && Not[NegQ[b/a] && SimplerSqrtQ[-b/a,-d/c]]
```

2: $\int \frac{1}{\sqrt{a+bx^2} \sqrt{c+dx^2}} dx$ when $\frac{d}{c} \neq 0 \wedge c > 0 \wedge a - \frac{bc}{d} > 0$

Rule 1.1.3.3.13.1.2.2: If $\frac{d}{c} \neq 0 \wedge c > 0 \wedge a - \frac{bc}{d} > 0$, then

$$\int \frac{1}{\sqrt{a+bx^2} \sqrt{c+dx^2}} dx \rightarrow -\frac{1}{\sqrt{c} \sqrt{-\frac{d}{c}} \sqrt{a - \frac{bc}{d}}} \text{EllipticF}\left[\text{ArcCos}\left[\sqrt{-\frac{d}{c}} x\right], \frac{bc}{bc-ad}\right]$$

Program code:

```
Int[1/(Sqrt[a+b_.*x_^2]*Sqrt[c+d_.*x_^2]),x_Symbol] :=
  -1/(Sqrt[c]*Rt[-d/c,2]*Sqrt[a-b*c/d])*EllipticF[ArcCos[Rt[-d/c,2]*x],b*c/(b*c-a*d)] /;
FreeQ[{a,b,c,d},x] && NegQ[d/c] && GtQ[c,0] && GtQ[a-b*c/d,0]
```

$$\text{3: } \int \frac{1}{\sqrt{a+bx^2} \sqrt{c+dx^2}} dx \text{ when } \frac{d}{c} \neq 0 \wedge c \neq 0$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{\sqrt{1+\frac{d}{c}x^2}}{\sqrt{c+dx^2}} = 0$$

Rule 1.1.3.3.13.1.2.3: If $\frac{d}{c} \neq 0 \wedge c \neq 0$, then

$$\int \frac{1}{\sqrt{a+bx^2} \sqrt{c+dx^2}} dx \rightarrow \frac{\sqrt{1+\frac{d}{c}x^2}}{\sqrt{c+dx^2}} \int \frac{1}{\sqrt{a+bx^2} \sqrt{1+\frac{d}{c}x^2}} dx$$

Program code:

```
Int[1/(Sqrt[a+b_*x^2]*Sqrt[c+d_*x^2]),x_Symbol] :=
  Sqrt[1+d/c*x^2]/Sqrt[c+d*x^2]*Int[1/(Sqrt[a+b*x^2]*Sqrt[1+d/c*x^2]),x] /;
FreeQ[{a,b,c,d},x] && Not[GtQ[c,0]]
```

$$2. \int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2}} dx \text{ when } b \neq -a \neq 0$$

$$1. \int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2}} dx \text{ when } \frac{d}{c} > 0$$

$$\text{1: } \int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2}} dx \text{ when } \frac{d}{c} > 0 \wedge \frac{b}{a} > 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \sqrt{a+bx^2} == \frac{a}{\sqrt{a+bx^2}} + \frac{bx^2}{\sqrt{a+bx^2}}$$

Rule 1.1.3.3.13.2.1.1: If $\frac{d}{c} > 0 \wedge \frac{b}{a} > 0$, then

$$\int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2}} dx \rightarrow a \int \frac{1}{\sqrt{a+bx^2} \sqrt{c+dx^2}} dx + b \int \frac{x^2}{\sqrt{a+bx^2} \sqrt{c+dx^2}} dx$$

Program code:

```
Int[Sqrt[a+b_.**x_^2]/Sqrt[c+d_.**x_^2],x_Symbol] :=
  a*Int[1/(Sqrt[a+b*x^2]*Sqrt[c+d*x^2]),x] + b*Int[x^2/(Sqrt[a+b*x^2]*Sqrt[c+d*x^2]),x] /;
FreeQ[{a,b,c,d},x] && PosQ[d/c] && PosQ[b/a]
```

$$2: \int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2}} dx \text{ when } \frac{d}{c} > 0 \wedge \frac{b}{a} \neq 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2}} == \frac{b\sqrt{c+dx^2}}{d\sqrt{a+bx^2}} - \frac{bc-ad}{d\sqrt{a+bx^2}\sqrt{c+dx^2}}$$

Rule 1.1.3.3.13.2.1.2: If $\frac{d}{c} > 0 \wedge \frac{b}{a} \neq 0$, then

$$\int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2}} dx \rightarrow \frac{b}{d} \int \frac{\sqrt{c+dx^2}}{\sqrt{a+bx^2}} dx - \frac{bc-ad}{d} \int \frac{1}{\sqrt{a+bx^2} \sqrt{c+dx^2}} dx$$

Program code:

```
Int[Sqrt[a+b_.**x_^2]/Sqrt[c+d_.**x_^2],x_Symbol] :=
  b/d*Int[Sqrt[c+d*x^2]/Sqrt[a+b*x^2],x] - (b*c-a*d)/d*Int[1/(Sqrt[a+b*x^2]*Sqrt[c+d*x^2]),x] /;
FreeQ[{a,b,c,d},x] && PosQ[d/c] && NegQ[b/a]
```

$$2. \int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2}} dx \text{ when } \frac{d}{c} \neq 0$$

$$1. \int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2}} dx \text{ when } \frac{d}{c} \neq 0 \wedge c > 0$$

$$1: \int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2}} dx \text{ when } \frac{d}{c} \neq 0 \wedge c > 0 \wedge a > 0$$

Rule 1.1.3.3.13.2.2.1.1: If $\frac{d}{c} \neq 0 \wedge c > 0 \wedge a > 0$, then

$$\int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2}} dx \rightarrow \frac{\sqrt{a}}{\sqrt{c} \sqrt{-\frac{d}{c}}} \text{EllipticE}\left[\text{ArcSin}\left[\sqrt{-\frac{d}{c}} x\right], \frac{bc}{ad}\right]$$

Program code:

```
Int[Sqrt[a+b_*x_^2]/Sqrt[c+d_*x_^2],x_Symbol] :=
  Sqrt[a]/(Sqrt[c]*Rt[-d/c,2])*EllipticE[ArcSin[Rt[-d/c,2]*x],b*c/(a*d)] /;
FreeQ[{a,b,c,d},x] && NegQ[d/c] && GtQ[c,0] && GtQ[a,0]
```

2: $\int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2}} dx$ when $\frac{d}{c} \neq 0 \wedge c > 0 \wedge a - \frac{bc}{d} > 0$

Rule 1.1.3.3.13.2.2.1.2: If $\frac{d}{c} \neq 0 \wedge c > 0 \wedge a - \frac{bc}{d} > 0$, then

$$\int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2}} dx \rightarrow -\frac{\sqrt{a-\frac{bc}{d}}}{\sqrt{c}\sqrt{-\frac{d}{c}}} \text{EllipticE}\left[\text{ArcCos}\left[\sqrt{-\frac{d}{c}}x\right], \frac{bc}{bc-ad}\right]$$

Program code:

```
Int[Sqrt[a+b.*x_^2]/Sqrt[c+d.*x_^2],x_Symbol] :=
  -Sqrt[a-b*c/d]/(Sqrt[c]*Rt[-d/c,2])*EllipticE[ArcCos[Rt[-d/c,2]*x],b*c/(b*c-a*d)] /;
FreeQ[{a,b,c,d},x] && NegQ[d/c] && GtQ[c,0] && GtQ[a-b*c/d,0]
```

$$\text{3: } \int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2}} dx \text{ when } \frac{d}{c} \neq 0 \wedge c > 0 \wedge a \neq 0$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{\sqrt{a+bx^2}}{\sqrt{1+\frac{b}{a}x^2}} = 0$$

Rule 1.1.3.3.13.2.2.1.3: If $\frac{d}{c} \neq 0 \wedge c > 0 \wedge a \neq 0$, then

$$\int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2}} dx \rightarrow \frac{\sqrt{a+bx^2}}{\sqrt{1+\frac{b}{a}x^2}} \int \frac{\sqrt{1+\frac{b}{a}x^2}}{\sqrt{c+dx^2}} dx$$

Program code:

```
Int[Sqrt[a+b.*x^2]/Sqrt[c+d.*x^2],x_Symbol] :=
  Sqrt[a+b*x^2]/Sqrt[1+b/a*x^2]*Int[Sqrt[1+b/a*x^2]/Sqrt[c+d*x^2],x] /;
FreeQ[{a,b,c,d},x] && NegQ[d/c] && GtQ[c,0] && Not[GtQ[a,0]]
```

2: $\int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2}} dx$ when $\frac{d}{c} \neq 0 \wedge c \neq 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{\sqrt{1+\frac{d}{c}x^2}}{\sqrt{c+dx^2}} == 0$

Rule 1.1.3.3.13.2.2.2: If $\frac{d}{c} \neq 0 \wedge c \neq 0$, then

$$\int \frac{\sqrt{a+bx^2}}{\sqrt{c+dx^2}} dx \rightarrow \frac{\sqrt{1+\frac{d}{c}x^2}}{\sqrt{c+dx^2}} \int \frac{\sqrt{a+bx^2}}{\sqrt{1+\frac{d}{c}x^2}} dx$$

Program code:

```
Int[Sqrt[a+b.*x.^2]/Sqrt[c+d.*x.^2],x_Symbol] :=
  Sqrt[1+d/c*x^2]/Sqrt[c+d*x^2]*Int[Sqrt[a+b*x^2]/Sqrt[1+d/c*x^2],x] /;
FreeQ[{a,b,c,d},x] && NegQ[d/c] && Not[GtQ[c,0]]
```

14: $\int (a + bx^n)^p (c + dx^n)^q dx$ when $bc - ad \neq 0 \wedge p \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule 1.1.3.3.14: If $bc - ad \neq 0 \wedge p \in \mathbb{Z}^+$, then

$$\int (a + bx^n)^p (c + dx^n)^q dx \rightarrow \int \text{ExpandIntegrand}[(a + bx^n)^p (c + dx^n)^q, x] dx$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*x^n)^p*(c+d*x^n)^q,x],x] /;
FreeQ[{a,b,c,d,n,q},x] && NeQ[b*c-a*d,0] && IGtQ[p,0]
```

A. $\int (a + bx^n)^p (c + dx^n)^q dx$ when $bc - ad \neq 0 \wedge n \neq -1$

1: $\int (a + bx^n)^p (c + dx^n)^q dx$ when $bc - ad \neq 0 \wedge n \neq -1 \wedge (p \in \mathbb{Z} \vee a > 0) \wedge (q \in \mathbb{Z} \vee c > 0)$

Rule 1.1.3.3.A.1: If $bc - ad \neq 0 \wedge n \neq -1 \wedge (p \in \mathbb{Z} \vee a > 0) \wedge (q \in \mathbb{Z} \vee c > 0)$, then

$$\int (a + bx^n)^p (c + dx^n)^q dx \rightarrow a^p c^q x \text{AppellF1}\left[\frac{1}{n}, -p, -q, 1 + \frac{1}{n}, -\frac{bx^n}{a}, -\frac{dx^n}{c}\right]$$

Program code:

```
Int[(a_+b_.*x_^n_)^p_.*(c_+d_.*x_^n_)^q_,x_Symbol] :=
  a^p*c^q*x*AppellF1[1/n,-p,-q,1+1/n,-b*x^n/a,-d*x^n/c] /;
FreeQ[{a,b,c,d,n,p,q},x] && NeQ[b*c-a*d,0] && NeQ[n,-1] && (IntegerQ[p] || GtQ[a,0]) && (IntegerQ[q] || GtQ[c,0])
```

2: $\int (a+bx^n)^p (c+dx^n)^q dx$ when $bc-ad \neq 0 \wedge n \neq -1 \wedge \neg (p \in \mathbb{Z} \vee a > 0)$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(a+bx^n)^p}{\left(1+\frac{bx^n}{a}\right)^p} = 0$

Rule 1.1.3.3.A.2: If $bc-ad \neq 0 \wedge n \neq -1 \wedge \neg (p \in \mathbb{Z} \vee a > 0)$, then

$$\int (a+bx^n)^p (c+dx^n)^q dx \rightarrow \frac{a^{\text{IntPart}[p]} (a+bx^n)^{\text{FracPart}[p]}}{\left(1+\frac{bx^n}{a}\right)^{\text{FracPart}[p]}} \int \left(1+\frac{bx^n}{a}\right)^p (c+dx^n)^q dx$$

Program code:

```
Int[(a+b_.**x_^n_)^p_*(c+d_.**x_^n_)^q_,x_Symbol] :=
  a^IntPart[p]*(a+b*x^n)^FracPart[p]/(1+b*x^n/a)^FracPart[p]*Int[(1+b*x^n/a)^p*(c+d*x^n)^q,x] /;
FreeQ[{a,b,c,d,n,p,q},x] && NeQ[b*c-a*d,0] && NeQ[n,-1] && Not[IntegerQ[p] || GtQ[a,0]]
```

S: $\int (a + b u^n)^p (c + d u^n)^q dx$ when $u = e + f x$

Derivation: Integration by substitution

Rule 1.1.3.3.S: If $u = e + f x$, then

$$\int (a + b u^n)^p (c + d u^n)^q dx \rightarrow \frac{1}{f} \text{Subst} \left[\int (a + b x^n)^p (c + d x^n)^q dx, x, u \right]$$

Program code:

```
Int[(a_.+b_.*u_^n_)^p_.*(c_.+d_.*u_^n_)^q_.,x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(a+b*x^n)^p*(c+d*x^n)^q,x],x,u] /;
FreeQ[{a,b,c,d,n,p,q},x] && LinearQ[u,x] && NeQ[u,x]
```

N: $\int P_x^p Q_x^q dx$ when $P_x = a + b (e + f x)^n \wedge Q_x = c + d (e + f x)^n$

Derivation: Algebraic normalization

Rule 1.1.3.3.N: If $P_x = a + b (e + f x)^n \wedge Q_x = c + d (e + f x)^n$, then

$$\int P_x^p Q_x^q dx \rightarrow \int (a + b (e + f x)^n)^p (c + d (e + f x)^n)^q dx$$

Program code:

```
Int[u_^p_.*v_^q_.,x_Symbol] :=
  Int[NormalizePseudoBinomial[u,x]^p*NormalizePseudoBinomial[v,x]^q,x] /;
FreeQ[{p,q},x] && PseudoBinomialPairQ[u,v,x]
```



```

Int[x^m_.*u^p_.*v^q_.,x_Symbol] :=
  Int[NormalizePseudoBinomial[x^(m/p)*u,x]^p*NormalizePseudoBinomial[v,x]^q,x] /;
FreeQ[{p,q},x] && IntegersQ[p,m/p] && PseudoBinomialPairQ[x^(m/p)*u,v,x]

```

```

(* IntBinomialQ[a,b,c,d,n,p,q,x] returns True iff  $(a+bx^n)^p (c+dx^n)^q$  is integrable wrt x in terms of non-Appell functions. *)
IntBinomialQ[a_,b_,c_,d_,n_,p_,q_,x_Symbol] :=
  IntegersQ[p,q] || IGtQ[p,0] || IGtQ[q,0] ||
  (EqQ[n,2] || EqQ[n,4]) && (IntegersQ[p,4*q] || IntegersQ[4*p,q]) ||
  EqQ[n,2] && (IntegersQ[2*p,2*q] || IntegersQ[3*p,q] && EqQ[b*c+3*a*d,0] || IntegersQ[p,3*q] && EqQ[3*b*c+a*d,0])

```

Rules for integrands of the form $(a + b x^n)^p (c + d x^{-n})^q$

1: $\int (a + b x^n)^p (c + d x^{-n})^q dx$ when $q \in \mathbb{Z}$

Derivation: Algebraic normalization

Basis: If $q \in \mathbb{Z}$, then $(c + d x^{-n})^q = \frac{(d + c x^n)^q}{x^{nq}}$

Rule 1.1.3.3.15.1: If $q \in \mathbb{Z}$, then

$$\int (a + b x^n)^p (c + d x^{-n})^q dx \rightarrow \int \frac{(a + b x^n)^p (d + c x^n)^q}{x^{nq}} dx$$

Program code:

```

Int[(a_+b_.**x^n_.)^p_.*(c_+d_.**x^mn_.)^q_.,x_Symbol] :=
  Int[(a+bx^n)^p*(d+cx^n)^q/x^(n*q),x] /;
FreeQ[{a,b,c,d,n,p},x] && EqQ[mn,-n] && IntegerQ[q] && (PosQ[n] || Not[IntegerQ[p]])

```

2: $\int (a+bx^n)^p (c+dx^{-n})^q dx$ when $q \notin \mathbb{Z} \wedge p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{x^n q (c+dx^{-n})^q}{(d+cx^n)^q} == 0$

Basis: $\frac{x^n q (c+dx^{-n})^q}{(d+cx^n)^q} == \frac{x^n \text{FracPart}[q] (c+dx^{-n})^{\text{FracPart}[q]}}{(d+cx^n)^{\text{FracPart}[q]}}$

Rule 1.1.3.3.15.2: If $q \notin \mathbb{Z} \wedge p \notin \mathbb{Z}$, then

$$\int (a+bx^n)^p (c+dx^{-n})^q dx \rightarrow \frac{x^n \text{FracPart}[q] (c+dx^{-n})^{\text{FracPart}[q]}}{(d+cx^n)^{\text{FracPart}[q]}} \int \frac{(a+bx^n)^p (d+cx^n)^q}{x^{nq}} dx$$

Program code:

```
Int[(a+b_.**x_^n_.)^p_*(c+d_.**x_^mn_.)^q_,x_Symbol] :=
  x^(n*FracPart[q])*(c+d*x^(-n))^FracPart[q]/(d+c*x^n)^FracPart[q]*Int[(a+b*x^n)^p*(d+c*x^n)^q/x^(n*q),x] /;
FreeQ[{a,b,c,d,n,p,q},x] && EqQ[mn,-n] && Not[IntegerQ[q]] && Not[IntegerQ[p]]
```