

## Rules for integrands of the form $(a + b x^n + c x^{2n})^p$

**1:**  $\int (a + b x^n + c x^{2n})^p dx$  when  $n < 0 \wedge p \in \mathbb{Z}$

Derivation: Algebraic expansion

Basis: If  $p \in \mathbb{Z}$ , then  $(a + b x^n + c x^{2n})^p = x^{2np} (c + b x^{-n} + a x^{-2n})^p$

Rule 1.2.3.1.1: If  $n < 0 \wedge p \in \mathbb{Z}$ , then

$$\int (a + b x^n + c x^{2n})^p dx \rightarrow \int x^{2np} (c + b x^{-n} + a x^{-2n})^p dx$$

Program code:

```
Int[(a_+b_.*x_^n_+c_.*x_^n2_.)^p_,x_Symbol] :=
  Int[x^(2*n*p)*(c+b*x^(-n)+a*x^(-2*n))^p,x] /;
FreeQ[{a,b,c},x] && EqQ[n2,2*n] && LtQ[n,0] && IntegerQ[p]
```

**2:**  $\int (a + bx^n + cx^{2n})^p dx$  when  $n \in \mathbb{F}$

Derivation: Integration by substitution

Basis: If  $k \in \mathbb{Z}^+$ , then  $F[x^n] = k \text{ Subst}[x^{k-1} F[x^{kn}], x, x^{1/k}] \partial_x x^{1/k}$

Rule 1.2.3.1.2: If  $b^2 - 4ac \neq 0 \wedge n \in \mathbb{F}$ , let  $k = \text{Denominator}[n]$ , then

$$\int (a + bx^n + cx^{2n})^p dx \rightarrow k \text{ Subst}\left[\int x^{k-1} (a + bx^{kn} + cx^{2kn})^p dx, x, x^{1/k}\right]$$

Program code:

```
Int[(a_+b_.*x_^n_+c_.*x_^n2_.)^p_,x_Symbol] :=
  With[{k=Denominator[n]},
    k*Subst[Int[x^(k-1)*(a+b*x^(k*n)+c*x^(2*k*n))^p,x],x,x^(1/k)]] /;
  FreeQ[{a,b,c,p},x] && EqQ[n2,2*n] && FractionQ[n]
```

**3:**  $\int (a + bx^n + cx^{2n})^p dx$  when  $n \in \mathbb{Z}^-$

Derivation: Integration by substitution

Basis: If  $n \in \mathbb{Z}$ , then  $F[x^n] = -\text{Subst}\left[\frac{F[\frac{x^{-n}}{x^2}]}{x}, x, \frac{1}{x}\right] \partial_x \frac{1}{x}$

Rule 1.2.3.1.3: If  $n \in \mathbb{Z}^-$ , then

$$\int (a + bx^n + cx^{2n})^p dx \rightarrow -\text{Subst}\left[\int \frac{(a + bx^{-n} + cx^{-2n})^p}{x^2} dx, x, \frac{1}{x}\right]$$

Program code:

```
Int[(a+b_.**x_^n_+c_.**x_^n2_.)^p_,x_Symbol] :=
  -Subst[Int[(a+b*x^(-n)+c*x^(-2*n))^p/x^2,x],x,1/x] /;
FreeQ[{a,b,c,p},x] && EqQ[n2,2*n] && ILtQ[n,0]
```

**4:**  $\int (a + b x^n + c x^{2n})^p dx$  when  $b^2 - 4ac = 0$

Derivation: Piecewise constant extraction

Basis: If  $b^2 - 4ac = 0$ , then  $\partial_x \frac{(a + b x^n + c x^{2n})^p}{(b + 2c x^n)^{2p}} = 0$

Note: If  $b^2 - 4ac = 0$ , then  $a + b z + c z^2 = \frac{1}{4c} (b + 2c z)^2$

Rule 1.2.3.1.4: If  $b^2 - 4ac = 0$ , then

$$\int (a + b x^n + c x^{2n})^p dx \rightarrow \frac{(a + b x^n + c x^{2n})^p}{(b + 2c x^n)^{2p}} \int (b + 2c x^n)^{2p} dx$$

Program code:

```
Int[(a_+b_.*x_^n_+c_.*x_^n2_.)^p_,x_Symbol] :=
  (a+b*x^n+c*x^(2*n))^p/(b+2*c*x^n)^(2*p)*Int[(b+2*c*x^n)^(2*p),x] /;
FreeQ[{a,b,c,n,p},x] && EqQ[n2,2*n] && EqQ[b^2-4*a*c,0]
```

5.  $\int (a + bx^n + cx^{2n})^p dx$  when  $b^2 - 4ac \neq 0 \wedge p \in \mathbb{Z}$

**1:**  $\int (a + bx^n + cx^{2n})^p dx$  when  $b^2 - 4ac \neq 0 \wedge p \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule 1.2.3.1.5.1: If  $b^2 - 4ac \neq 0 \wedge p \in \mathbb{Z}^+$ , then

$$\int (a + bx^n + cx^{2n})^p dx \rightarrow \int \text{ExpandIntegrand}[(a + bx^n + cx^{2n})^p, x] dx$$

Program code:

```
Int[(a+b_.**x_^n_+c_.**x_^n2_.)^p_,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*x^n+c*x^(2*n))^p,x],x] /;
FreeQ[{a,b,c,n},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && IGtQ[p,0]
```

2:  $\int (a + bx^n + cx^{2n})^p dx$  when  $b^2 - 4ac \neq 0 \wedge p+1 \in \mathbb{Z}^-$

Reference: G&R 2.161.5

Derivation: Trinomial recurrence 2b with  $m = 0$ ,  $A = 1$  and  $B = 0$

Note: G&R 2.161.4 is a special case of G&R 2.161.5.

Rule 1.2.3.1.5.2: If  $b^2 - 4ac \neq 0 \wedge p+1 \in \mathbb{Z}^-$ , then

$$\int (a + bx^n + cx^{2n})^p dx \rightarrow -\frac{x(b^2 - 2ac + bcx^n)(a + bx^n + cx^{2n})^{p+1}}{an(p+1)(b^2 - 4ac)} + \frac{1}{an(p+1)(b^2 - 4ac)} \int (b^2 - 2ac + n(p+1)(b^2 - 4ac) + bc(n(2p+3)+1)x^n)(a + bx^n + cx^{2n})^{p+1} dx$$

Program code:

```
Int[(a+b_.**x_^n+c_.**x_^n2_.)^p_,x_Symbol] :=
  -x*(b^2-2*a*c+b*c*x^n)*(a+b*x^n+c*x^(2*n))^(p+1)/(a*n*(p+1)*(b^2-4*a*c)) +
  1/(a*n*(p+1)*(b^2-4*a*c))*
  Int[(b^2-2*a*c+n*(p+1)*(b^2-4*a*c)+b*c*(n*(2*p+3)+1)*x^n)*(a+b*x^n+c*x^(2*n))^(p+1),x] /;
FreeQ[{a,b,c,n},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && ILtQ[p,-1]
```

3.  $\int \frac{1}{a + bx^n + cx^{2n}} dx$  when  $b^2 - 4ac \neq 0$

1:  $\int \frac{1}{a + bx^n + cx^{2n}} dx$  when  $b^2 - 4ac \neq 0 \wedge \frac{n}{2} \in \mathbb{Z}^+ \wedge b^2 - 4ac \neq 0$

Derivation: Algebraic expansion

Basis: If  $q \rightarrow \sqrt{\frac{a}{c}}$  and  $r \rightarrow \sqrt{2q - \frac{b}{c}}$ , then  $\frac{1}{a+bz^2+cz^4} = \frac{r-z}{2cqr(q-rz+z^2)} + \frac{r+z}{2cqr(q+rz+z^2)}$

Note: If  $(a \mid b \mid c) \in \mathbb{R} \wedge b^2 - 4ac < 0$ , then  $\frac{a}{c} > 0$  and  $2\sqrt{\frac{a}{c} - \frac{b}{c}} > 0$ .

Rule 1.2.3.1.5.3.1: If  $b^2 - 4ac \neq 0 \wedge \frac{n}{2} \in \mathbb{Z}^+ \wedge b^2 - 4ac \neq 0$ , let  $q \rightarrow \sqrt{\frac{a}{c}}$  and  $r \rightarrow \sqrt{2q - \frac{b}{c}}$ , then

$$\int \frac{1}{a + bx^n + cx^{2n}} dx \rightarrow \frac{1}{2cq} \int \frac{r - x^{n/2}}{q - rx^{n/2} + x^n} dx + \frac{1}{2cq} \int \frac{r + x^{n/2}}{q + rx^{n/2} + x^n} dx$$

Program code:

```
Int[1/(a+b_.*x_^n+c_.*x_^2n_),x_Symbol] :=
  With[{q=Rt[a/c,2]},
    With[{r=Rt[2*q-b/c,2]},
      1/(2*c*q*r)*Int[(r-x^(n/2))/(q-r*x^(n/2)+x^n),x] +
      1/(2*c*q*r)*Int[(r+x^(n/2))/(q+r*x^(n/2)+x^n),x]] /;
  FreeQ[{a,b,c},x] && EqQ[n,2*n] && NeQ[b^2-4*a*c,0] && IGtQ[n/2,0] && NegQ[b^2-4*a*c]
```

**2:**  $\int \frac{1}{a+bx^n+cx^{2n}} dx$  when  $b^2 - 4ac \neq 0 \wedge \left(\frac{n}{2} \notin \mathbb{Z}^+ \vee b^2 - 4ac > 0\right)$

Reference: G&R 2.161.1a

Derivation: Algebraic expansion

Basis: Let  $q \rightarrow \sqrt{b^2 - 4ac}$ , then  $\frac{1}{a+bx^n+cx^{2n}} = \frac{c}{q} \frac{1}{\frac{b}{2} - \frac{q}{2} + cx^n} - \frac{c}{q} \frac{1}{\frac{b}{2} + \frac{q}{2} + cx^n}$

■ Rule 1.2.3.1.5.3.2: If  $b^2 - 4ac \neq 0$ , let  $q \rightarrow \sqrt{b^2 - 4ac}$ , then

$$\int \frac{1}{a+bx^n+cx^{2n}} dx \rightarrow \frac{c}{q} \int \frac{1}{\frac{b}{2} - \frac{q}{2} + cx^n} dx - \frac{c}{q} \int \frac{1}{\frac{b}{2} + \frac{q}{2} + cx^n} dx$$

Program code:

```
Int[1/(a+_.*x_^n+c_.*x_^n2_),x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    c/q*Int[1/(b/2-q/2+c*x^n),x] - c/q*Int[1/(b/2+q/2+c*x^n),x]] /;
  FreeQ[{a,b,c},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0]
```

**6:**  $\int (a+bx^n+cx^{2n})^p dx$  when  $b^2 - 4ac \neq 0 \wedge p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis:  $\partial_x \frac{(a+bx^n+cx^{2n})^p}{\left(1 + \frac{2cx^n}{b+\sqrt{b^2-4ac}}\right)^p \left(1 + \frac{2cx^n}{b-\sqrt{b^2-4ac}}\right)^p} = 0$

Rule 1.2.3.1.6: If  $b^2 - 4ac \neq 0 \wedge p \notin \mathbb{Z}$ , then



$$\int (a + b x^n + c x^{2n})^p dx \rightarrow \frac{a^{\text{IntPart}[p]} (a + b x^n + c x^{2n})^{\text{FracPart}[p]}}{\left(1 + \frac{2 c x^n}{b + \sqrt{b^2 - 4 a c}}\right)^{\text{FracPart}[p]} \left(1 + \frac{2 c x^n}{b - \sqrt{b^2 - 4 a c}}\right)^{\text{FracPart}[p]}} \int \left(1 + \frac{2 c x^n}{b + \sqrt{b^2 - 4 a c}}\right)^p \left(1 + \frac{2 c x^n}{b - \sqrt{b^2 - 4 a c}}\right)^p dx$$

Program code:

```
Int[(a_+b_.x^n_+c_.x^2n_.)^p_,x_Symbol] :=
  a^IntPart[p]*(a+b*x^n+c*x^(2*n))^FracPart[p]/
  ((1+2*c*x^n/(b+Rt[b^2-4*a*c,2]))^FracPart[p]*(1+2*c*x^n/(b-Rt[b^2-4*a*c,2]))^FracPart[p])*
  Int[(1+2*c*x^n/(b+Sqrt[b^2-4*a*c]))^p*(1+2*c*x^n/(b-Sqrt[b^2-4*a*c]))^p,x] /;
FreeQ[{a,b,c,n,p},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && Not[IntegerQ[p]]
```

**S:**  $\int (a + b u^n + c u^{2n})^p dx$  when  $u == d + e x$

Derivation: Integration by substitution

Rule 1.2.3.1.S: If  $u == d + e x$ , then

$$\int (a + b u^n + c u^{2n})^p dx \rightarrow \frac{1}{e} \text{Subst}\left[\int (a + b x^n + c x^{2n})^p dx, x, u\right]$$

Program code:

```
Int[(a_+b_.u^n_+c_.u^2n_.)^p_,x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(a+b*x^n+c*x^(2*n))^p,x],x,u] /;
FreeQ[{a,b,c,n,p},x] && EqQ[n2,2*n] && LinearQ[u,x] && NeQ[u,x]
```

9.  $\int (a + bx^{-n} + cx^n)^p dx$

**1:**  $\int (a + bx^{-n} + cx^n)^p dx$  when  $p \in \mathbb{Z}$

Derivation: Algebraic normalization

Basis:  $a + bx^{-n} + cx^n = \frac{b+ax^n+cx^{2n}}{x^n}$

Rule 1.2.3.1.9.1: If  $p \in \mathbb{Z}$ , then

$$\int (a + bx^{-n} + cx^n)^p dx \rightarrow \int \frac{(b + ax^n + cx^{2n})^p}{x^{np}} dx$$

Program code:

```
Int[(a_+b_.**x^mn_+c_.**x^n_.)^p_,x_Symbol] :=
  Int[(b+a*x^n+c*x^(2*n))^p/x^(n*p),x] /;
FreeQ[{a,b,c,n},x] && EqQ[mn,-n] && IntegerQ[p] && PosQ[n]
```

**2:**  $\int (a + b x^{-n} + c x^n)^p dx$  when  $p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis:  $\partial_x \frac{x^n p (a + b x^{-n} + c x^n)^p}{(b + a x^n + c x^{2n})^p} == 0$

Basis:  $\frac{x^n p (a + b x^{-n} + c x^n)^p}{(b + a x^n + c x^{2n})^p} == \frac{x^n \text{FracPart}[p] (a + b x^{-n} + c x^n)^{\text{FracPart}[p]}}{(b + a x^n + c x^{2n})^{\text{FracPart}[p]}}$

Rule 1.2.3.1.9.2: If  $p \notin \mathbb{Z}$ , then

$$\int (a + b x^{-n} + c x^n)^p dx \rightarrow \frac{x^n \text{FracPart}[p] (a + b x^{-n} + c x^n)^{\text{FracPart}[p]}}{(b + a x^n + c x^{2n})^{\text{FracPart}[p]}} \int \frac{(b + a x^n + c x^{2n})^p}{x^{n p}} dx$$

Program code:

```
Int[(a_+b_.*x_^mn_+c_.*x_^n_)^p_,x_Symbol] :=
  x^(n*FracPart[p])*(a+b*x^(-n)+c*x^n)^FracPart[p]/(b+a*x^n+c*x^(2*n))^FracPart[p]*Int[(b+a*x^n+c*x^(2*n))^p/x^(n*p),x] /;
FreeQ[{a,b,c,n,p},x] && EqQ[mn,-n] && Not[IntegerQ[p]] && PosQ[n]
```