

Rules for integrands of the form $(g \sin[e + f x])^p (a + b \sin[e + f x])^m (c + d \sin[e + f x])^n$

$$1. \int \frac{(g \sin[e + f x])^p (a + b \sin[e + f x])^m}{c + d \sin[e + f x]} dx \text{ when } b c - a d \neq 0$$

$$1. \int \frac{(g \sin[e + f x])^p \sqrt{a + b \sin[e + f x]}}{c + d \sin[e + f x]} dx \text{ when } b c - a d \neq 0$$

$$1. \int \frac{\sqrt{g \sin[e + f x]} \sqrt{a + b \sin[e + f x]}}{c + d \sin[e + f x]} dx \text{ when } b c - a d \neq 0$$

$$1: \int \frac{\sqrt{g \sin[e + f x]} \sqrt{a + b \sin[e + f x]}}{c + d \sin[e + f x]} dx \text{ when } b c - a d \neq 0 \wedge (a^2 - b^2 = 0 \vee c^2 - d^2 = 0)$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{\sqrt{g z}}{c + d z} = \frac{g}{d \sqrt{g z}} - \frac{c g}{d \sqrt{g z} (c + d z)}$$

Rule: If $b c - a d \neq 0 \wedge (a^2 - b^2 = 0 \vee c^2 - d^2 = 0)$, then

$$\int \frac{\sqrt{g \sin[e + f x]} \sqrt{a + b \sin[e + f x]}}{c + d \sin[e + f x]} dx \rightarrow \frac{g}{d} \int \frac{\sqrt{a + b \sin[e + f x]}}{\sqrt{g \sin[e + f x]}} dx - \frac{c g}{d} \int \frac{\sqrt{a + b \sin[e + f x]}}{\sqrt{g \sin[e + f x]} (c + d \sin[e + f x])} dx$$

Program code:

```
Int[Sqrt[g_.*sin[e_.+f_.*x_]]*Sqrt[a+b_.*sin[e_.+f_.*x_]]/(c+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  g/d*Int[Sqrt[a+b*Sin[e+f*x]]/Sqrt[g*Sin[e+f*x]],x] -
  c*g/d*Int[Sqrt[a+b*Sin[e+f*x]]/(Sqrt[g*Sin[e+f*x]]*(c+d*Sin[e+f*x])),x] /;
FreeQ[{a,b,c,d,e,f,g},x] && NeQ[b*c-a*d,0] && (EqQ[a^2-b^2,0] || EqQ[c^2-d^2,0])
```

$$2: \int \frac{\sqrt{g \sin[e + f x]} \sqrt{a + b \sin[e + f x]}}{c + d \sin[e + f x]} dx \text{ when } b c - a d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0$$

Derivation: Algebraic expansion

Basis: $\frac{\sqrt{a+bz}}{c+dz} == \frac{b}{d\sqrt{a+bz}} - \frac{bc-ad}{d\sqrt{a+bz}(c+dz)}$

Rule: If $bc - ad \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0$, then

$$\int \frac{\sqrt{g \sin[e+fx]} \sqrt{a+b \sin[e+fx]}}{c+d \sin[e+fx]} dx \rightarrow \frac{b}{d} \int \frac{\sqrt{g \sin[e+fx]}}{\sqrt{a+b \sin[e+fx]}} dx - \frac{bc-ad}{d} \int \frac{\sqrt{g \sin[e+fx]}}{\sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx$$

Program code:

```
Int[Sqrt[g_.sin[e_.+f_.x_]]*Sqrt[a+b_.sin[e_.+f_.x_]]/(c+d_.sin[e_.+f_.x_]),x_Symbol] :=
  b/d*Int[Sqrt[g*sin[e+f*x]]/Sqrt[a+b*sin[e+f*x]],x] -
  (b*c-a*d)/d*Int[Sqrt[g*sin[e+f*x]]/(Sqrt[a+b*sin[e+f*x]]*(c+d*sin[e+f*x])),x] /;
FreeQ[{a,b,c,d,e,f,g},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

$$2. \int \frac{\sqrt{a+b \sin[e+fx]}}{\sqrt{g \sin[e+fx]} (c+d \sin[e+fx])} dx \text{ when } bc - ad \neq 0$$

$$1: \int \frac{\sqrt{a+b \sin[e+fx]}}{\sqrt{g \sin[e+fx]} (c+d \sin[e+fx])} dx \text{ when } bc - ad \neq 0 \wedge a^2 - b^2 == 0$$

Derivation: Integration by substitution

Basis: If $a^2 - b^2 == 0$, then

$$\frac{\sqrt{a+b \sin[e+fx]}}{\sqrt{g \sin[e+fx]} (c+d \sin[e+fx])} ==$$

$$- \frac{2b}{f} \text{Subst} \left[\frac{1}{b c + a d + c g x^2}, x, \frac{b \cos[e+fx]}{\sqrt{g \sin[e+fx]} \sqrt{a+b \sin[e+fx]}} \right] \partial_x \frac{b \cos[e+fx]}{\sqrt{g \sin[e+fx]} \sqrt{a+b \sin[e+fx]}}$$

Rule: If $bc - ad \neq 0 \wedge a^2 - b^2 == 0$, then

$$\int \frac{\sqrt{a+b \sin[e+fx]}}{\sqrt{g \sin[e+fx]} (c+d \sin[e+fx])} dx \rightarrow -\frac{2b}{f} \text{Subst}\left[\int \frac{1}{bc+ad+cgx^2} dx, x, \frac{b \cos[e+fx]}{\sqrt{g \sin[e+fx]} \sqrt{a+b \sin[e+fx]}}\right]$$

Program code:

```
Int[Sqrt[a+_.*sin[e_+f_.*x_]]/(Sqrt[g_.*sin[e_+f_.*x_]]*(c_+d_.*sin[e_+f_.*x_])),x_Symbol] :=
  -2*b/f*Subst[Int[1/(b*c+a*d+c*g*x^2),x],x,b*Cos[e+f*x]/(Sqrt[g*Sin[e+f*x]]*Sqrt[a+b*Sin[e+f*x]])] /;
FreeQ[{a,b,c,d,e,f,g},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0]
```

$$2. \int \frac{\sqrt{a+b \sin[e+fx]}}{\sqrt{g \sin[e+fx]} (c+d \sin[e+fx])} dx \text{ when } bc-ad \neq 0 \wedge a^2-b^2 \neq 0$$

$$1. \int \frac{\sqrt{a+b \sin[e+fx]}}{\sqrt{g \sin[e+fx]} (c+d \sin[e+fx])} dx \text{ when } bc-ad \neq 0 \wedge a^2-b^2 \neq 0 \wedge c^2-d^2 = 0$$

$$1: \int \frac{\sqrt{a+b \sin[e+fx]}}{\sqrt{\sin[e+fx]} (c+c \sin[e+fx])} dx \text{ when } a^2-b^2 > 0 \wedge b > 0$$

Basis: If $b-a > 0 \wedge b > 0$, then $\sqrt{a+bz} = \sqrt{1+z} \sqrt{\frac{a+bz}{1+z}}$

Rule: If $a^2-b^2 > 0 \wedge b > 0$, then

$$\int \frac{\sqrt{a+b \sin[e+fx]}}{\sqrt{\sin[e+fx]} (c+c \sin[e+fx])} dx \rightarrow -\frac{\sqrt{a+b}}{cf} \text{EllipticE}\left[\text{ArcSin}\left[\frac{\cos[e+fx]}{1+\sin[e+fx]}\right], -\frac{a-b}{a+b}\right]$$

Program code:

```
Int[Sqrt[a+_.*sin[e_+f_.*x_]]/(Sqrt[sin[e_+f_.*x_]]*(c_+d_.*sin[e_+f_.*x_])),x_Symbol] :=
  -Sqrt[a+b]/(c*f)*EllipticE[ArcSin[Cos[e+f*x]/(1+Sin[e+f*x])],-(a-b)/(a+b)] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[d,c] && GtQ[b^2-a^2,0] && GtQ[b,0]
```

$$\text{2: } \int \frac{\sqrt{a+b \sin[e+fx]}}{\sqrt{g \sin[e+fx]} (c+d \sin[e+fx])} dx \text{ when } bc-ad \neq 0 \wedge a^2-b^2 \neq 0 \wedge c^2-d^2 = 0$$

Rule: If $bc-ad \neq 0 \wedge a^2-b^2 \neq 0 \wedge c^2-d^2 = 0$, then

$$\int \frac{\sqrt{a+b \sin[e+fx]}}{\sqrt{g \sin[e+fx]} (c+d \sin[e+fx])} dx \rightarrow -\frac{\sqrt{a+b \sin[e+fx]} \sqrt{\frac{d \sin[e+fx]}{c+d \sin[e+fx]}}}{df \sqrt{g \sin[e+fx]} \sqrt{\frac{c^2 (a+b \sin[e+fx])}{(a+c \sin[e+fx]) (c+d \sin[e+fx])}}} \text{EllipticE}\left[\text{ArcSin}\left[\frac{c \cos[e+fx]}{c+d \sin[e+fx]}\right], \frac{bc-ad}{bc+ad}\right]$$

Program code:

```
Int[Sqrt[a+b_.sin[e_.+f_.x_]]/(Sqrt[g_.sin[e_.+f_.x_]]*(c+d_.sin[e_.+f_.x_])),x_Symbol] :=
  -Sqrt[a+b*Sin[e+f*x]]*Sqrt[d*Sin[e+f*x]/(c+d*Sin[e+f*x])]/
    (d*f*Sqrt[g*Sin[e+f*x]]*Sqrt[c^2*(a+b*Sin[e+f*x])/((a*c+b*d)*(c+d*Sin[e+f*x]))]) *
    EllipticE[ArcSin[c*Cos[e+f*x]/(c+d*Sin[e+f*x])],(b*c-a*d)/(b*c+a*d)] /;
FreeQ[{a,b,c,d,e,f,g},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && EqQ[c^2-d^2,0]
```

$$2: \int \frac{\sqrt{a+b \sin[e+fx]}}{\sqrt{g \sin[e+fx]} (c+d \sin[e+fx])} dx \text{ when } bc-ad \neq 0 \wedge a^2-b^2 \neq 0 \wedge c^2-d^2 \neq 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{\sqrt{a+bz}}{\sqrt{gz} (c+dz)} = \frac{a}{c \sqrt{gz} \sqrt{a+bz}} + \frac{(bc-ad) \sqrt{gz}}{cg \sqrt{a+bz} (c+dz)}$$

Rule: If $bc-ad \neq 0 \wedge a^2-b^2 \neq 0 \wedge c^2-d^2 \neq 0$, then

$$\int \frac{\sqrt{a+b \sin[e+fx]}}{\sqrt{g \sin[e+fx]} (c+d \sin[e+fx])} dx \rightarrow \frac{a}{c} \int \frac{1}{\sqrt{g \sin[e+fx]} \sqrt{a+b \sin[e+fx]}} dx + \frac{bc-ad}{cg} \int \frac{\sqrt{g \sin[e+fx]}}{\sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx$$

Program code:

```
Int[Sqrt[a+_.*sin[e_+f_.*x_]]/(Sqrt[g_.*sin[e_+f_.*x_]]*(c+_d_.*sin[e_+f_.*x_])),x_Symbol] :=
  a/c*Int[1/(Sqrt[g*Sin[e+f*x]]*Sqrt[a+b*Sin[e+f*x]]),x] +
  (b*c-a*d)/(c*g)*Int[Sqrt[g*Sin[e+f*x]]/(Sqrt[a+b*Sin[e+f*x]]*(c+d*Sin[e+f*x])),x] /;
FreeQ[{a,b,c,d,e,f,g},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

$$3. \int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] (c+d \sin[e+fx])} dx \text{ when } bc-ad \neq 0$$

$$1: \int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] (c+d \sin[e+fx])} dx \text{ when } bc-ad \neq 0 \wedge a^2-b^2 = 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{1}{z (c+dz)} = \frac{1}{cz} - \frac{d}{c (c+dz)}$$

Rule: If $bc-ad \neq 0 \wedge a^2-b^2 = 0$, then

$$\int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] (c+d \sin[e+fx])} dx \rightarrow \frac{1}{c} \int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx]} dx - \frac{d}{c} \int \frac{\sqrt{a+b \sin[e+fx]}}{c+d \sin[e+fx]} dx$$

Program code:

```
Int[Sqrt[a+_.*sin[e+_.*x_]]/(sin[e+_.*x_]*(c+_.*sin[e+_.*x_])),x_Symbol] :=
  1/c*Int[Sqrt[a+b*sin[e+fx]]/Sin[e+fx],x] -
  d/c*Int[Sqrt[a+b*sin[e+fx]]/(c+d*sin[e+fx]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0]
```

2: $\int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] (c+d \sin[e+fx])} dx$ when $b c - a d \neq 0 \wedge a^2 - b^2 \neq 0$

Derivation: Algebraic expansion

Basis: $\frac{\sqrt{a+bz}}{z(c+dz)} = \frac{a}{cz\sqrt{a+bz}} + \frac{bc-ad}{c\sqrt{a+bz}(c+dz)}$

Rule: If $b c - a d \neq 0 \wedge a^2 - b^2 \neq 0$, then

$$\int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] (c+d \sin[e+fx])} dx \rightarrow \frac{a}{c} \int \frac{1}{\sin[e+fx] \sqrt{a+b \sin[e+fx]}} dx + \frac{bc-ad}{c} \int \frac{1}{\sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx$$

Program code:

```
Int[Sqrt[a+_.*sin[e+_.*x_]]/(sin[e+_.*x_]*(c+_.*sin[e+_.*x_])),x_Symbol] :=
  a/c*Int[1/(Sin[e+fx]*Sqrt[a+b*sin[e+fx]]),x] +
  (b*c-a*d)/c*Int[1/(Sqrt[a+b*sin[e+fx]]*(c+d*sin[e+fx])),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0]
```

$$2. \int \frac{(g \sin[e+fx])^p}{\sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx \text{ when } bc - ad \neq 0$$

$$1. \int \frac{\sqrt{g \sin[e+fx]}}{\sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx \text{ when } bc - ad \neq 0$$

$$1: \int \frac{\sqrt{g \sin[e+fx]}}{\sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx \text{ when } bc - ad \neq 0 \wedge (a^2 - b^2 = 0 \vee c^2 - d^2 = 0)$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{\sqrt{g z}}{\sqrt{a+b z} (c+d z)} = -\frac{a g}{(b c - a d) \sqrt{g z} \sqrt{a+b z}} + \frac{c g \sqrt{a+b z}}{(b c - a d) \sqrt{g z} (c+d z)}$$

Rule: If $bc - ad \neq 0 \wedge (a^2 - b^2 = 0 \vee c^2 - d^2 = 0)$, then

$$\int \frac{\sqrt{g \sin[e+fx]}}{\sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx \rightarrow$$

$$-\frac{a g}{b c - a d} \int \frac{1}{\sqrt{g \sin[e+fx]} \sqrt{a+b \sin[e+fx]}} dx + \frac{c g}{b c - a d} \int \frac{\sqrt{a+b \sin[e+fx]}}{\sqrt{g \sin[e+fx]} (c+d \sin[e+fx])} dx$$

Program code:

```
Int[Sqrt[g_.*sin[e_.+f_.*x_]]/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*(c_+d_.*sin[e_.+f_.*x_])),x_Symbol] :=
  -a*g/(b*c-a*d)*Int[1/(Sqrt[g*Sin[e+f*x]]*Sqrt[a+b*Sin[e+f*x]]),x] +
  c*g/(b*c-a*d)*Int[Sqrt[a+b*Sin[e+f*x]]/(Sqrt[g*Sin[e+f*x]]*(c+d*Sin[e+f*x])),x] /;
FreeQ[{a,b,c,d,e,f,g},x] && NeQ[b*c-a*d,0] && (EqQ[a^2-b^2,0] || EqQ[c^2-d^2,0])
```

$$2: \int \frac{\sqrt{g \sin[e+fx]}}{\sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx \text{ when } bc - ad \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0$$

Rule: If $bc - ad \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0$, then

$$\int \frac{\sqrt{g \sin[e+fx]}}{\sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx \rightarrow$$

$$\frac{2 \sqrt{-\cot[e+fx]^2} \sqrt{g \sin[e+fx]}}{f (c+d) \cot[e+fx] \sqrt{a+b \sin[e+fx]}} \sqrt{\frac{b+a \csc[e+fx]}{a+b}} \text{EllipticPi}\left[\frac{2c}{c+d}, \text{ArcSin}\left[\frac{\sqrt{1-\csc[e+fx]}}{\sqrt{2}}\right], \frac{2a}{a+b}\right]$$

Program code:

```
Int[Sqrt[g_.*sin[e_+f_.*x_]]/(Sqrt[a_+b_.*sin[e_+f_.*x_]]*(c_+d_.*sin[e_+f_.*x_])),x_Symbol] :=
  2*Sqrt[-Cot[e+f*x]^2]*Sqrt[g*Sin[e+f*x]]/(f*(c+d)*Cot[e+f*x]*Sqrt[a+b*Sin[e+f*x]])*Sqrt[(b+a*Csc[e+f*x])/(a+b)]*
  EllipticPi[2*c/(c+d),ArcSin[Sqrt[1-Csc[e+f*x]]/Sqrt[2]],2*a/(a+b)] /;
FreeQ[{a,b,c,d,e,f,g},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```


$$2. \int \frac{1}{\sqrt{g \sin[e+fx]} \sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx \text{ when } bc - ad \neq 0$$

$$1: \int \frac{1}{\sqrt{g \sin[e+fx]} \sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx \text{ when } bc - ad \neq 0 \wedge (a^2 - b^2 = 0 \vee c^2 - d^2 = 0)$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{1}{\sqrt{a+bz} (c+dz)} = \frac{b}{(bc-ad) \sqrt{a+bz}} - \frac{d \sqrt{a+bz}}{(bc-ad) (c+dz)}$$

Rule: If $bc - ad \neq 0 \wedge (a^2 - b^2 = 0 \vee c^2 - d^2 = 0)$, then

$$\int \frac{1}{\sqrt{g \sin[e+fx]} \sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx \rightarrow$$

$$\frac{b}{bc-ad} \int \frac{1}{\sqrt{g \sin[e+fx]} \sqrt{a+b \sin[e+fx]}} dx - \frac{d}{bc-ad} \int \frac{\sqrt{a+b \sin[e+fx]}}{\sqrt{g \sin[e+fx]} (c+d \sin[e+fx])} dx$$

Program code:

```
Int[1/(Sqrt[g_.sin[e_.+f_.x_]]*Sqrt[a_+b_.sin[e_.+f_.x_]]*(c_+d_.sin[e_.+f_.x_])),x_Symbol] :=
  b/(b*c-a*d)*Int[1/(Sqrt[g*Sin[e+f*x]]*Sqrt[a+b*Sin[e+f*x]]),x] -
  d/(b*c-a*d)*Int[Sqrt[a+b*Sin[e+f*x]]/(Sqrt[g*Sin[e+f*x]]*(c+d*Sin[e+f*x])),x] /;
FreeQ[{a,b,c,d,e,f,g},x] && NeQ[b*c-a*d,0] && (EqQ[a^2-b^2,0] || EqQ[c^2-d^2,0])
```

$$2: \int \frac{1}{\sqrt{g \sin[e+fx]} \sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx \text{ when } bc - ad \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0$$

Derivation: Algebraic expansion

Basis: $\frac{1}{\sqrt{g z} \sqrt{a+b z} (c+d z)} = \frac{1}{c \sqrt{g z} \sqrt{a+b z}} - \frac{d \sqrt{g z}}{c g \sqrt{a+b z} (c+d z)}$

Rule: If $b c - a d \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0$, then

$$\int \frac{1}{\sqrt{g \sin[e+fx]} \sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx \rightarrow \frac{1}{c} \int \frac{1}{\sqrt{g \sin[e+fx]} \sqrt{a+b \sin[e+fx]}} dx - \frac{d}{c g} \int \frac{\sqrt{g \sin[e+fx]}}{\sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx$$

Program code:

```
Int[1/(Sqrt[g_*sin[e_+f_*x_])*Sqrt[a+b_*sin[e_+f_*x_]]*(c+d_*sin[e_+f_*x_])),x_Symbol] :=
  1/c*Int[1/(Sqrt[g*Sin[e+f*x]]*Sqrt[a+b*Sin[e+f*x]]),x] -
  d/(c*g)*Int[Sqrt[g*Sin[e+f*x]]/(Sqrt[a+b*Sin[e+f*x]]*(c+d*Sin[e+f*x])),x] /;
FreeQ[{a,b,c,d,e,f,g},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

$$3. \int \frac{1}{\sin[e+fx] \sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx \text{ when } bc - ad \neq 0$$

$$1: \int \frac{1}{\sin[e+fx] \sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx \text{ when } bc - ad \neq 0 \wedge a^2 - b^2 = 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{1}{z \sqrt{a+bz} (c+dz)} = \frac{bc-ad-bdz}{c(bc-ad)z \sqrt{a+bz}} + \frac{d^2 \sqrt{a+bz}}{c(bc-ad)(c+dz)}$$

Rule: If $bc - ad \neq 0 \wedge a^2 - b^2 = 0$, then

$$\int \frac{1}{\sin[e+fx] \sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx \rightarrow \frac{d^2}{c(bc-ad)} \int \frac{\sqrt{a+b \sin[e+fx]}}{c+d \sin[e+fx]} dx + \frac{1}{c(bc-ad)} \int \frac{bc-ad-bd \sin[e+fx]}{\sin[e+fx] \sqrt{a+b \sin[e+fx]}} dx$$

Program code:

```
Int[1/(sin[e_.+f_.*x_]*Sqrt[a+b_.sin[e_.+f_.*x_])*(c+d_.sin[e_.+f_.*x_])),x_Symbol] :=
  d^2/(c*(b*c-a*d))*Int[Sqrt[a+b*sin[e+f*x]]/(c+d*sin[e+f*x]),x] +
  1/(c*(b*c-a*d))*Int[(b*c-a*d-b*d*sin[e+f*x])/(sin[e+f*x]*Sqrt[a+b*sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0]
```

$$2: \int \frac{1}{\sin[e+fx] \sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx \text{ when } bc - ad \neq 0 \wedge a^2 - b^2 \neq 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{1}{z(c+dz)} = \frac{1}{cz} - \frac{d}{c(c+dz)}$$

Rule: If $bc - ad \neq 0 \wedge a^2 - b^2 \neq 0$, then

$$\int \frac{1}{\sin[e+fx] \sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx \rightarrow \frac{1}{c} \int \frac{1}{\sin[e+fx] \sqrt{a+b \sin[e+fx]}} dx - \frac{d}{c} \int \frac{1}{\sqrt{a+b \sin[e+fx]} (c+d \sin[e+fx])} dx$$

Program code:

```
Int[1/(sin[e_.+f_.**x_]*Sqrt[a_+b_.sin[e_.+f_.**x_])*(c_+d_.sin[e_.+f_.**x_])),x_Symbol] :=
  1/c*Int[1/(Sin[e+f*x]*Sqrt[a+b*Ssin[e+f*x]]),x] - d/c*Int[1/(Sqrt[a+b*Ssin[e+f*x]]*(c+d*Ssin[e+f*x])),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0]
```

$$2. \int \frac{(a+b \sin[e+fx])^m (c+d \sin[e+fx])^n}{\sin[e+fx]} dx \text{ when } bc - ad \neq 0 \wedge m^2 = n^2 = \frac{1}{4}$$

$$1. \int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] \sqrt{c+d \sin[e+fx]}} dx \text{ when } bc - ad \neq 0$$

$$1. \int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] \sqrt{c+d \sin[e+fx]}} dx \text{ when } bc - ad \neq 0 \wedge a^2 - b^2 = 0$$

$$1: \int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] \sqrt{c+d \sin[e+fx]}} dx \text{ when } bc - ad \neq 0 \wedge a^2 - b^2 = 0 \wedge bc + ad = 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{1}{z \sqrt{c+dz}} = -\frac{d}{c \sqrt{c+dz}} + \frac{\sqrt{c+dz}}{cz}$$

Rule: If $bc - ad \neq 0 \wedge a^2 - b^2 = 0 \wedge bc + ad = 0$, then

$$\int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] \sqrt{c+d \sin[e+fx]}} dx \rightarrow -\frac{d}{c} \int \frac{\sqrt{a+b \sin[e+fx]}}{\sqrt{c+d \sin[e+fx]}} dx + \frac{1}{c} \int \frac{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}}{\sin[e+fx]} dx$$

Program code:

```
Int[Sqrt[a+b_*sin[e_+f_*x_]]/(sin[e_+f_*x_]*Sqrt[c+d_*sin[e_+f_*x_]]),x_Symbol] :=
  -d/c*Int[Sqrt[a+b*sin[e+f*x]]/Sqrt[c+d*sin[e+f*x]],x] +
  1/c*Int[Sqrt[a+b*sin[e+f*x]]*Sqrt[c+d*sin[e+f*x]]/Sin[e+f*x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && EqQ[b*c+a*d,0]
```

2: $\int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] \sqrt{c+d \sin[e+fx]}} dx$ when $bc - ad \neq 0 \wedge a^2 - b^2 \neq 0 \wedge bc + ad \neq 0$

Derivation: Integration by substitution

Basis: If $a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0$, then

$$\frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] \sqrt{c+d \sin[e+fx]}} = -\frac{2a}{f} \text{Subst} \left[\frac{1}{1-acx^2}, x, \frac{\cos[e+fx]}{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}} \right] \partial_x \frac{\cos[e+fx]}{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}}$$

Rule: If $bc - ad \neq 0 \wedge a^2 - b^2 \neq 0 \wedge bc + ad \neq 0$, then

$$\int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] \sqrt{c+d \sin[e+fx]}} dx \rightarrow -\frac{2a}{f} \text{Subst} \left[\int \frac{1}{1-acx^2} dx, x, \frac{\cos[e+fx]}{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}} \right]$$

Program code:

```
Int[Sqrt[a+b_.sin[e_.+f_.*x_]]/(sin[e_.+f_.*x_]*Sqrt[c+d_.sin[e_.+f_.*x_]]),x_Symbol] :=
-2*a/f*Subst[Int[1/(1-a*c*x^2),x],x,Cos[e+f*x]/(Sqrt[a+b*sin[e+f*x]]*Sqrt[c+d*sin[e+f*x]])] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[b*c+a*d,0]
```

2. $\int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] \sqrt{c+d \sin[e+fx]}} dx$ when $bc - ad \neq 0 \wedge a^2 - b^2 \neq 0$

1: $\int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] \sqrt{c+d \sin[e+fx]}} dx$ when $bc - ad \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{\sqrt{a+bz}}{z\sqrt{c+dz}} == \frac{bc-ad}{c\sqrt{a+bz}\sqrt{c+dz}} + \frac{a\sqrt{c+dz}}{cz\sqrt{a+bz}}$$

Rule: If $bc - ad \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 = 0$, then

$$\int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] \sqrt{c+d \sin[e+fx]}} dx \rightarrow \frac{bc-ad}{c} \int \frac{1}{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}} dx + \frac{a}{c} \int \frac{\sqrt{c+d \sin[e+fx]}}{\sin[e+fx] \sqrt{a+b \sin[e+fx]}} dx$$

Program code:

```
Int[Sqrt[a+_.*sin[e+_.*x_]]/(sin[e+_.*x_]*Sqrt[c+_.*sin[e+_.*x_]]),x_Symbol] :=
  (b*c-a*d)/c*Int[1/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]]),x] +
  a/c*Int[Sqrt[c+d*Sin[e+f*x]]/(Sin[e+f*x]*Sqrt[a+b*Sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && EqQ[c^2-d^2,0]
```

2: $\int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] \sqrt{c+d \sin[e+fx]}} dx$ when $bc - ad \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0$

Rule: If $bc - ad \neq 0 \wedge a^2 - b^2 \neq 0 \wedge c^2 - d^2 \neq 0$, then

$$\int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] \sqrt{c+d \sin[e+fx]}} dx \rightarrow$$

$$-\frac{2(a+b \sin[e+fx])}{cf \sqrt{\frac{a+b}{c+d}} \cos[e+fx]} \sqrt{-\frac{(bc-ad)(1-\sin[e+fx])}{(c+d)(a+b \sin[e+fx])}}$$

$$\sqrt{\frac{(bc-ad)(1+\sin[e+fx])}{(c-d)(a+b \sin[e+fx])}} \text{EllipticPi}\left[\frac{a(c+d)}{c(a+b)}, \text{ArcSin}\left[\sqrt{\frac{a+b}{c+d}} \frac{\sqrt{c+d \sin[e+fx]}}{\sqrt{a+b \sin[e+fx]}}\right], \frac{(a-b)(c+d)}{(a+b)(c-d)}\right]$$

Program code:

```
Int[Sqrt[a+_.*sin[e_+f_.*x_]]/(sin[e_+f_.*x_]*Sqrt[c+_.*sin[e_+f_.*x_]]),x_Symbol] :=
-2*(a+b*sin[e+f*x])/(c*f*Rt[(a+b)/(c+d),2]*Cos[e+f*x])*
Sqrt[-(b*c-a*d)*(1-Sin[e+f*x])/((c+d)*(a+b*sin[e+f*x]))]*Sqrt[(b*c-a*d)*(1+Sin[e+f*x])/((c-d)*(a+b*sin[e+f*x]))]*
EllipticPi[a*(c+d)/(c*(a+b)),ArcSin[Rt[(a+b)/(c+d),2]*Sqrt[c+d*sin[e+f*x]]/Sqrt[a+b*sin[e+f*x]]],(a-b)*(c+d)/((a+b)*(c-d))]/;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

2. $\int \frac{1}{\sin[e+fx] \sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}} dx$ when $bc - ad \neq 0$

1: $\int \frac{1}{\sin[e+fx] \sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}} dx$ when $bc - ad \neq 0 \wedge a^2 - b^2 = 0 \wedge c^2 - d^2 = 0$

Derivation: Piecewise constant extraction

Basis: If $a^2 - b^2 == 0 \wedge c^2 - d^2 == 0$, then $\partial_x \frac{\cos[e+fx]}{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}} == 0$

Rule: If $bc - ad \neq 0 \wedge a^2 - b^2 == 0 \wedge c^2 - d^2 == 0$, then

$$\int \frac{1}{\sin[e+fx] \sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}} dx \rightarrow \frac{\cos[e+fx]}{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}} \int \frac{1}{\cos[e+fx] \sin[e+fx]} dx$$

Program code:

```
Int[1/(sin[e_.+f_.*x_]*Sqrt[a+b_.sin[e_.+f_.*x_])*Sqrt[c+d_.sin[e_.+f_.*x_]]],x_Symbol] :=
  Cos[e+f*x]/(Sqrt[a+b*sin[e+f*x])*Sqrt[c+d*sin[e+f*x]])*Int[1/(Cos[e+f*x]*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && EqQ[c^2-d^2,0]
```

2: $\int \frac{1}{\sin[e+fx] \sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}} dx$ when $bc - ad \neq 0 \wedge (a^2 - b^2 \neq 0 \vee c^2 - d^2 \neq 0)$

Derivation: Algebraic expansion

Basis: $\frac{1}{z \sqrt{a+bz}} == -\frac{b}{a \sqrt{a+bz}} + \frac{\sqrt{a+bz}}{az}$

Rule: If $bc - ad \neq 0 \wedge (a^2 - b^2 \neq 0 \vee c^2 - d^2 \neq 0)$, then

$$\int \frac{1}{\sin[e+fx] \sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}} dx \rightarrow -\frac{b}{a} \int \frac{1}{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}} dx + \frac{1}{a} \int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] \sqrt{c+d \sin[e+fx]}} dx$$

Program code:

```
Int[1/(sin[e_.+f_.*x_]*Sqrt[a+b_.sin[e_.+f_.*x_])*Sqrt[c+d_.sin[e_.+f_.*x_]]],x_Symbol] :=
  -b/a*Int[1/(Sqrt[a+b*sin[e+f*x])*Sqrt[c+d*sin[e+f*x]]),x] +
  1/a*Int[Sqrt[a+b*sin[e+f*x]]/(Sin[e+f*x]*Sqrt[c+d*sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && (NeQ[a^2-b^2,0] || NeQ[c^2-d^2,0])
```

$$3. \int \frac{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}}{\sin[e+fx]} dx \text{ when } bc - ad \neq 0$$

$$1: \int \frac{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}}{\sin[e+fx]} dx \text{ when } bc - ad \neq 0 \wedge a^2 - b^2 = 0 \wedge c^2 - d^2 = 0$$

Derivation: Piecewise constant extraction

Basis: If $a^2 - b^2 = 0 \wedge c^2 - d^2 = 0$, then $\partial_x \frac{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}}{\cos[e+fx]} = 0$

Rule: If $bc - ad \neq 0 \wedge a^2 - b^2 = 0 \wedge c^2 - d^2 = 0$, then

$$\int \frac{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}}{\sin[e+fx]} dx \rightarrow \frac{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}}{\cos[e+fx]} \int \cot[e+fx] dx$$

Program code:

```
Int[Sqrt[a+_.*sin[e+_.*x_]]*Sqrt[c+_.*sin[e+_.*x_]]/sin[e+_.*x_],x_Symbol] :=
  Sqrt[a+b*sin[e+fx]]*Sqrt[c+d*sin[e+fx]]/Cos[e+fx]*Int[Cot[e+fx],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && EqQ[c^2-d^2,0]
```

$$2: \int \frac{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}}{\sin[e+fx]} dx \text{ when } bc - ad \neq 0 \wedge (a^2 - b^2 \neq 0 \vee c^2 - d^2 \neq 0)$$

Derivation: Algebraic expansion

Basis: $\frac{\sqrt{c+dz}}{z} = \frac{d}{\sqrt{c+dz}} + \frac{c}{z \sqrt{c+dz}}$

Rule: If $bc - ad \neq 0 \wedge (a^2 - b^2 \neq 0 \vee c^2 - d^2 \neq 0)$, then

$$\int \frac{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}}{\sin[e+fx]} dx \rightarrow d \int \frac{\sqrt{a+b \sin[e+fx]}}{\sqrt{c+d \sin[e+fx]}} dx + c \int \frac{\sqrt{a+b \sin[e+fx]}}{\sin[e+fx] \sqrt{c+d \sin[e+fx]}} dx$$

Program code:

```
Int[Sqrt[a+_.*sin[e+_.*x_]]*Sqrt[c+_.*sin[e+_.*x_]]/sin[e+_.*x_],x_Symbol] :=
  d*Int[Sqrt[a+b*sin[e+f*x]]/Sqrt[c+d*sin[e+f*x]],x] +
  c*Int[Sqrt[a+b*sin[e+f*x]]/(Sin[e+f*x]*Sqrt[c+d*sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && (NeQ[a^2-b^2,0] || NeQ[c^2-d^2,0])
```

3: $\int \sin[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $bc+ad=0 \wedge a^2-b^2=0 \wedge p+2n=0 \wedge n \in \mathbb{Z}$

Derivation: Algebraic simplification

Basis: If $bc+ad=0 \wedge a^2-b^2=0 \wedge p+2n=0 \wedge n \in \mathbb{Z}$, then

$$\sin[e+fx]^p (c+d \sin[e+fx])^n = a^n c^n \tan[e+fx]^p (a+b \sin[e+fx])^{-n}$$

Rule: If $bc+ad=0 \wedge a^2-b^2=0 \wedge p+2n=0 \wedge n \in \mathbb{Z}$, then

$$\int \sin[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow a^n c^n \int \tan[e+fx]^p (a+b \sin[e+fx])^{m-n} dx$$

Program code:

```
Int[sin[e+_.*x_]^p_*(a+b_*sin[e+_.*x_] )^m_*(c+d_*sin[e+_.*x_] )^n_,x_Symbol] :=
  a^n*c^n*Int[Tan[e+f*x]^p_*(a+b*sin[e+f*x])^(m-n),x] /;
FreeQ[{a,b,c,d,e,f,m},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && EqQ[p+2*n,0] && IntegerQ[n]
```

4: $\int (g \sin[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $bc-ad \neq 0 \wedge a^2-b^2=0 \wedge c^2-d^2 \neq 0 \wedge m-\frac{1}{2} \in \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

Basis: If $a^2 - b^2 = 0$, then $\partial_x \frac{\sqrt{a-b \sin[e+fx]} \sqrt{a+b \sin[e+fx]}}{\cos[e+fx]} = 0$

Basis: $\cos[e+fx] = \frac{1}{f} \partial_x \sin[e+fx]$

Rule: If $bc - ad \neq 0 \wedge a^2 - b^2 = 0 \wedge c^2 - d^2 \neq 0 \wedge m - \frac{1}{2} \in \mathbb{Z}$, then

$$\int (g \sin[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow$$

$$\frac{\sqrt{a-b \sin[e+fx]} \sqrt{a+b \sin[e+fx]}}{\cos[e+fx]} \int \left((\cos[e+fx] (g \sin[e+fx])^p (a+b \sin[e+fx])^{m-\frac{1}{2}} (c+d \sin[e+fx])^n) / (\sqrt{a-b \sin[e+fx]}) \right) dx \rightarrow$$

$$\frac{\sqrt{a-b \sin[e+fx]} \sqrt{a+b \sin[e+fx]}}{f \cos[e+fx]} \text{Subst} \left[\int \frac{(gx)^p (a+bx)^{m-\frac{1}{2}} (c+dx)^n}{\sqrt{a-bx}} dx, x, \sin[e+fx] \right]$$

Program code:

```
Int[(g_.sin[e_.+f_.x_])^p_*(a+b_.sin[e_.+f_.x_])^m_*(c+d_.sin[e_.+f_.x_])^n_,x_Symbol] :=
  Sqrt[a-b*sin[e*f*x]]*Sqrt[a+b*sin[e*f*x]]/(f*cos[e*f*x])*
  Subst[Int[(g*x)^p*(a+b*x)^(m-1/2)*(c+d*x)^n/Sqrt[a-b*x],x],x,Sin[e*f*x]] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && IntegerQ[m-1/2]
```

5: $\int (g \sin[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $bc - ad \neq 0 \wedge ((m|n) \in \mathbb{Z} \vee (m|p) \in \mathbb{Z} \vee (n|p) \in \mathbb{Z})$

Derivation: Algebraic expansion

Note: If p equal 1 or 2, better to use rules for integrands of the form $(a+b \sin[e+fx])^m (c+d \sin[e+fx])^n (A+B \sin[e+fx])$ or $(a+b \sin[e+fx])^m (c+d \sin[e+fx])^n (A+B \sin[e+fx] + C \sin[e+fx]^2)$ respectively.

Rule: If $bc - ad \neq 0 \wedge ((m|n) \in \mathbb{Z} \vee (m|p) \in \mathbb{Z} \vee (n|p) \in \mathbb{Z})$, then

$$\int (g \sin[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow$$

$$\int \text{ExpandTrig}[(g \sin[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n, x] dx$$

Program code:

```
Int[(g_.sin[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_+d_.sin[e_.+f_.x_])^n_,x_Symbol] :=
  Int[ExpandTrig[(g*sin[e+f*x])^p*(a+b*sin[e+f*x])^m*(c+d*sin[e+f*x])^n,x],x] /;
FreeQ[{a,b,c,d,e,f,g,n,p},x] && NeQ[b*c-a*d,0] && (IntegersQ[m,n] || IntegersQ[m,p] || IntegersQ[n,p]) && NeQ[p,2]
```

X: $\int (g \sin[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$

Rule:

$$\int (g \sin[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow \int (g \sin[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$$

Program code:

```
Int[(g_.sin[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_+d_.sin[e_.+f_.x_])^n_,x_Symbol] :=
  Unintegrable[(g*sin[e+f*x])^p*(a+b*sin[e+f*x])^m*(c+d*sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p},x] && NeQ[p,2]
```

Rules for integrands of the form $(g \sin[e + f x])^p (a + b \csc[e + f x])^m (c + d \csc[e + f x])^n$

1. $\int (g \sin[e + f x])^p (a + b \csc[e + f x])^m (c + d \csc[e + f x])^n dx$ when $b c - a d \neq 0 \wedge p \notin \mathbb{Z}$

1: $\int (g \sin[e + f x])^p (a + b \csc[e + f x])^m (c + d \csc[e + f x])^n dx$ when $b c - a d \neq 0 \wedge p \notin \mathbb{Z} \wedge m \in \mathbb{Z} \wedge n \in \mathbb{Z}$

Derivation: Algebraic normalization

Basis: $a + b \csc[z] = \frac{b+a \sin[z]}{\sin[z]}$

Rule: If $b c - a d \neq 0 \wedge p \notin \mathbb{Z} \wedge m \in \mathbb{Z} \wedge n \in \mathbb{Z}$, then

$$\int (g \sin[e + f x])^p (a + b \csc[e + f x])^m (c + d \csc[e + f x])^n dx \rightarrow g^{m+n} \int (g \sin[e + f x])^{p-m-n} (b + a \sin[e + f x])^m (d + c \sin[e + f x])^n dx$$

Program code:

```
Int[(g_.*sin[e_.+f_.*x_])^p_.*(a_.+b_.*csc[e_.+f_.*x_])^m_.*(c_.+d_.*csc[e_.+f_.*x_])^n_,x_Symbol] :=
  g^(m+n)*Int[(g*Sin[e+f*x])^(p-m-n)*(b+a*Sin[e+f*x])^m*(d+c*Sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,g,p},x] && NeQ[b*c-a*d,0] && Not[IntegerQ[p]] && IntegerQ[m] && IntegerQ[n]
```

2: $\int (g \sin[e+fx])^p (a+b \csc[e+fx])^m (c+d \csc[e+fx])^n dx$ when $b c - a d \neq 0 \wedge p \notin \mathbb{Z} \wedge \neg (m \in \mathbb{Z} \wedge n \in \mathbb{Z})$

Derivation: Piecewise constant extraction

Basis: $\partial_x ((g \cos[e+fx])^p (g \sec[e+fx])^p) = 0$

Rule: If $b c - a d \neq 0 \wedge p \notin \mathbb{Z} \wedge \neg (m \in \mathbb{Z} \wedge n \in \mathbb{Z})$, then

$$\int (g \sin[e+fx])^p (a+b \csc[e+fx])^m (c+d \csc[e+fx])^n dx \rightarrow (g \csc[e+fx])^p (g \sin[e+fx])^p \int \frac{(a+b \csc[e+fx])^m (c+d \csc[e+fx])^n}{(g \csc[e+fx])^p} dx$$

Program code:

```
Int[(g_.*sin[e_.+f_.*x_])^p_.*(a_.+b_.*csc[e_.+f_.*x_])^m_.*(c_.+d_.*csc[e_.+f_.*x_])^n_,x_Symbol] :=
  (g*csc[e+f*x])^p*(g*sin[e+f*x])^p*Int[(a+b*csc[e+f*x])^m*(c+d*csc[e+f*x])^n/(g*csc[e+f*x])^p,x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p},x] && NeQ[b*c-a*d,0] && Not[IntegerQ[p]] && Not[IntegerQ[m]] && IntegerQ[n]
```

Rules for integrands of the form $(g \sin[e + f x])^p (a + b \sin[e + f x])^m (c + d \csc[e + f x])^n$

1: $\int (g \sin[e + f x])^p (a + b \sin[e + f x])^m (c + d \csc[e + f x])^n dx$ when $n \in \mathbb{Z}$

Derivation: Algebraic normalization

Basis: $c + d \csc[z] = \frac{d+c \sin[z]}{\sin[z]}$

Rule: If $n \in \mathbb{Z}$, then

$$\int (g \sin[e + f x])^p (a + b \sin[e + f x])^m (c + d \csc[e + f x])^n dx \rightarrow g^n \int (g \sin[e + f x])^{p-n} (a + b \sin[e + f x])^m (d + c \sin[e + f x])^n dx$$

Program code:

```
Int[(g_.*sin[e_.+f_.*x_])^p_.*(a_+b_.*sin[e_.+f_.*x_])^m_.*(c_+d_.*csc[e_.+f_.*x_])^n_,x_Symbol] :=
  g^n*Int[(g*Sin[e+f*x])^(p-n)*(a+b*Sin[e+f*x])^m*(d+c*Sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,g,m,p},x] && IntegerQ[n]
```


$$2. \int (g \sin[e+fx])^p (a+b \sin[e+fx])^m (c+d \csc[e+fx])^n dx \text{ when } n \notin \mathbb{Z}$$

$$1. \int (g \sin[e+fx])^p (a+b \sin[e+fx])^m (c+d \csc[e+fx])^n dx \text{ when } n \notin \mathbb{Z} \wedge m \in \mathbb{Z}$$

$$1: \int \sin[e+fx]^p (a+b \sin[e+fx])^m (c+d \csc[e+fx])^n dx \text{ when } n \notin \mathbb{Z} \wedge m \in \mathbb{Z} \wedge p \in \mathbb{Z}$$

Derivation: Algebraic normalization

$$\text{Basis: } a + b \sin[z] == \frac{b+a \csc[z]}{\csc[z]}$$

Rule: If $n \notin \mathbb{Z} \wedge m \in \mathbb{Z} \wedge p \in \mathbb{Z}$, then

$$\int \sin[e+fx]^p (a+b \sin[e+fx])^m (c+d \csc[e+fx])^n dx \rightarrow \int \frac{(b+a \csc[e+fx])^m (c+d \csc[e+fx])^n}{\csc[e+fx]^{m+p}} dx$$

Program code:

```
Int[sin[e_.+f_.*x_]^p_.*(a_+b_.*sin[e_.+f_.*x_])^m_.*(c_+d_.*csc[e_.+f_.*x_])^n_,x_Symbol] :=
  Int[(b+a*Csc[e+f*x])^m*(c+d*Csc[e+f*x])^n/Csc[e+f*x]^(m+p),x] /;
  FreeQ[{a,b,c,d,e,f,n},x] && Not[IntegerQ[n]] && IntegerQ[m] && IntegerQ[p]
```

$$2: \int (g \sin[e+fx])^p (a+b \sin[e+fx])^m (c+d \csc[e+fx])^n dx \text{ when } n \notin \mathbb{Z} \wedge m \in \mathbb{Z} \wedge p \notin \mathbb{Z}$$

Derivation: Algebraic normalization and piecewise constant extraction

$$\text{Basis: } a + b \sin[z] == \frac{b+a \csc[z]}{\csc[z]}$$

$$\text{Basis: } \partial_x (\csc[e+fx]^p (g \sin[e+fx])^p) == 0$$

Rule: If $n \notin \mathbb{Z} \wedge m \in \mathbb{Z} \wedge p \notin \mathbb{Z}$, then

$$\int (g \sin[e+fx])^p (a+b \sin[e+fx])^m (c+d \csc[e+fx])^n dx \rightarrow \csc[e+fx]^p (g \sin[e+fx])^p \int \frac{(b+a \csc[e+fx])^m (c+d \csc[e+fx])^n}{\csc[e+fx]^{m+p}} dx$$

Program code:

```
Int[(g_.sin[e_.+f_.x_])^p_.*(a_+b_.sin[e_.+f_.x_])^m_.*(c_+d_.csc[e_.+f_.x_])^n_,x_Symbol] :=
  Csc[e+f*x]^p*(g*sin[e+f*x])^p*Int[(b+a*Csc[e+f*x])^m*(c+d*Csc[e+f*x])^n/Csc[e+f*x]^(m+p),x] /;
FreeQ[{a,b,c,d,e,f,g,n,p},x] && Not[IntegerQ[n]] && IntegerQ[m] && Not[IntegerQ[p]]
```

2: $\int (g \sin[e+fx])^p (a+b \sin[e+fx])^m (c+d \csc[e+fx])^n dx$ when $n \notin \mathbb{Z} \wedge m \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(g \sin[e+fx])^n (c+d \csc[e+fx])^n}{(d+c \sin[e+fx])^n} == 0$

Rule: If $n \notin \mathbb{Z} \wedge m \notin \mathbb{Z}$, then

$$\int (g \sin[e+fx])^p (a+b \sin[e+fx])^m (c+d \csc[e+fx])^n dx \rightarrow \frac{(g \sin[e+fx])^n (c+d \csc[e+fx])^n}{(d+c \sin[e+fx])^n} \int (g \sin[e+fx])^{p-n} (a+b \sin[e+fx])^m (d+c \sin[e+fx])^n dx$$

Program code:

```
Int[(g_.sin[e_.+f_.x_])^p_.*(a_+b_.sin[e_.+f_.x_])^m_.*(c_+d_.csc[e_.+f_.x_])^n_,x_Symbol] :=
  (g*sin[e+f*x])^n*(c+d*Csc[e+f*x])^n/(d+c*sin[e+f*x])^n*Int[(g*sin[e+f*x])^(p-n)*(a+b*sin[e+f*x])^m*(d+c*sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p},x] && Not[IntegerQ[n]] && Not[IntegerQ[m]]
```

Rules for integrands of the form $(g \operatorname{Csc}[e + f x])^p (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x])^n$

$$1. \int (g \operatorname{Csc}[e + f x])^p (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x])^n dx \text{ when } b c - a d \neq 0 \wedge p \notin \mathbb{Z}$$

$$\mathbf{1:} \int (g \operatorname{Csc}[e + f x])^p (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x])^n dx \text{ when } b c - a d \neq 0 \wedge p \notin \mathbb{Z} \wedge m \in \mathbb{Z} \wedge n \in \mathbb{Z}$$

Derivation: Algebraic normalization

$$\text{Basis: } a + b \operatorname{Sin}[z] == \frac{b+a \operatorname{Csc}[z]}{\operatorname{Csc}[z]}$$

Rule: If $b c - a d \neq 0 \wedge p \notin \mathbb{Z} \wedge m \in \mathbb{Z} \wedge n \in \mathbb{Z}$, then

$$\int (g \operatorname{Csc}[e + f x])^p (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x])^n dx \rightarrow g^{m+n} \int (g \operatorname{Csc}[e + f x])^{p-m-n} (b + a \operatorname{Csc}[e + f x])^m (d + c \operatorname{Csc}[e + f x])^n dx$$

Program code:

```
Int[(g_.*csc[e_.+f_.*x_])^p_.*(a_.+b_.*sin[e_.+f_.*x_])^m_.*(c_.+d_.*sin[e_.+f_.*x_])^n_.,x_Symbol] :=
  g^(m+n)*Int[(g*Csc[e+f*x])^(p-m-n)*(b+a*Csc[e+f*x])^m*(d+c*Csc[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,g,p},x] && NeQ[b*c-a*d,0] && Not[IntegerQ[p]] && IntegerQ[m] && IntegerQ[n]
```

2: $\int (g \operatorname{Csc}[e+fx])^p (a+b \operatorname{Sin}[e+fx])^m (c+d \operatorname{Sin}[e+fx])^n dx$ when $b c - a d \neq 0 \wedge p \notin \mathbb{Z} \wedge \neg (m \in \mathbb{Z} \wedge n \in \mathbb{Z})$

Derivation: Piecewise constant extraction

Basis: $\partial_x ((g \operatorname{Csc}[e+fx])^p (g \operatorname{Sin}[e+fx])^p) = 0$

Rule: If $b c - a d \neq 0 \wedge p \notin \mathbb{Z} \wedge \neg (m \in \mathbb{Z} \wedge n \in \mathbb{Z})$, then

$$\int (g \operatorname{Csc}[e+fx])^p (a+b \operatorname{Sin}[e+fx])^m (c+d \operatorname{Sin}[e+fx])^n dx \rightarrow (g \operatorname{Csc}[e+fx])^p (g \operatorname{Sin}[e+fx])^p \int \frac{(a+b \operatorname{Sin}[e+fx])^m (c+d \operatorname{Sin}[e+fx])^n}{(g \operatorname{Sin}[e+fx])^p} dx$$

Program code:

```
Int[(g_.*csc[e_+f_.*x_])^p_.*(a_+b_.*sin[e_+f_.*x_])^m_.*(c_+d_.*sin[e_+f_.*x_])^n_.,x_Symbol] :=
  (g*Csc[e+f*x])^p*(g*Sin[e+f*x])^p*Int[(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^n/(g*Sin[e+f*x])^p,x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p},x] && NeQ[b*c-a*d,0] && Not[IntegerQ[p]] && Not[IntegerQ[m]] && IntegerQ[n]
```

Rules for integrands of the form $(g \operatorname{Csc}[e + f x])^p (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Csc}[e + f x])^n$

1: $\int (g \operatorname{Csc}[e + f x])^p (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Csc}[e + f x])^n dx$ when $m \in \mathbb{Z}$

Derivation: Algebraic normalization

Basis: $a + b \operatorname{Sin}[z] = \frac{b+a \operatorname{Csc}[z]}{\operatorname{Csc}[z]}$

Rule: If $m \in \mathbb{Z}$, then

$$\int (g \operatorname{Csc}[e + f x])^p (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Csc}[e + f x])^n dx \rightarrow g^m \int (g \operatorname{Csc}[e + f x])^{p-m} (b + a \operatorname{Csc}[e + f x])^m (c + d \operatorname{Csc}[e + f x])^n dx$$

Program code:

```
Int[(g_.*csc[e_+f_.*x_])^p_.*(a_+b_.*sin[e_+f_.*x_])^m_.*(c_+d_.*csc[e_+f_.*x_])^n_.,x_Symbol] :=
  g^m*Int[(g*Csc[e+f*x])^(p-m)*(b+a*Csc[e+f*x])^m*(c+d*Csc[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,g,n,p},x] && IntegerQ[m]
```

$$2. \int (g \operatorname{Csc}[e+fx])^p (a+b \operatorname{Sin}[e+fx])^m (c+d \operatorname{Csc}[e+fx])^n dx \text{ when } m \notin \mathbb{Z}$$

$$1. \int (g \operatorname{Csc}[e+fx])^p (a+b \operatorname{Sin}[e+fx])^m (c+d \operatorname{Csc}[e+fx])^n dx \text{ when } m \notin \mathbb{Z} \wedge n \in \mathbb{Z}$$

$$\mathbf{1:} \int \operatorname{Csc}[e+fx]^p (a+b \operatorname{Sin}[e+fx])^m (c+d \operatorname{Csc}[e+fx])^n dx \text{ when } m \notin \mathbb{Z} \wedge n \in \mathbb{Z} \wedge p \in \mathbb{Z}$$

Derivation: Algebraic normalization

$$\text{Basis: } c + d \operatorname{Csc}[z] == \frac{d+c \operatorname{Sin}[z]}{\operatorname{Sin}[z]}$$

Rule: If $m \notin \mathbb{Z} \wedge n \in \mathbb{Z} \wedge p \in \mathbb{Z}$, then

$$\int \operatorname{Csc}[e+fx]^p (a+b \operatorname{Sin}[e+fx])^m (c+d \operatorname{Csc}[e+fx])^n dx \rightarrow \int \frac{(a+b \operatorname{Sin}[e+fx])^m (d+c \operatorname{Sin}[e+fx])^n}{\operatorname{Sin}[e+fx]^{n+p}} dx$$

Program code:

```
Int[Csc[e_.+f_.*x_]^p_.*(a+_b_.*sin[e_.+f_.*x_]^m_.*(c+_d_.*csc[e_.+f_.*x_]^n_.,x_Symbol] :=
  Int[(a+b*Sin[e+f*x])^m*(d+c*Sin[e+f*x])^n/Sin[e+f*x]^(n+p),x] /;
FreeQ[{a,b,c,d,e,f,m},x] && Not[IntegerQ[m]] && IntegerQ[n] && IntegerQ[p]
```

$$\mathbf{2:} \int (g \operatorname{Csc}[e+fx])^p (a+b \operatorname{Sin}[e+fx])^m (c+d \operatorname{Csc}[e+fx])^n dx \text{ when } m \notin \mathbb{Z} \wedge n \in \mathbb{Z} \wedge p \notin \mathbb{Z}$$

Derivation: Algebraic normalization and piecewise constant extraction

$$\text{Basis: } c + d \operatorname{Csc}[z] == \frac{d+c \operatorname{Sin}[z]}{\operatorname{Sin}[z]}$$

$$\text{Basis: } \partial_x (\operatorname{Sin}[e+fx])^p (g \operatorname{Csc}[e+fx])^p == 0$$

Rule: If $m \notin \mathbb{Z} \wedge n \in \mathbb{Z} \wedge p \notin \mathbb{Z}$, then

$$\int (g \operatorname{Csc}[e+fx])^p (a+b \operatorname{Sin}[e+fx])^m (c+d \operatorname{Csc}[e+fx])^n dx \rightarrow \operatorname{Sin}[e+fx]^p (g \operatorname{Csc}[e+fx])^p \int \frac{(a+b \operatorname{Sin}[e+fx])^m (d+c \operatorname{Sin}[e+fx])^n}{\operatorname{Sin}[e+fx]^{n+p}} dx$$

Program code:

```
Int[(g_*csc[e_+f_*x_])^p_*(a_+b_*sin[e_+f_*x_])^m_*(c_+d_*csc[e_+f_*x_])^n_,x_Symbol] :=
  Sin[e+f*x]^p*(g*Csc[e+f*x])^p*Int[(a+b*Sin[e+f*x])^m*(d+c*Sin[e+f*x])^n/Sin[e+f*x]^(n+p),x] /;
FreeQ[{a,b,c,d,e,f,g,m,p},x] && Not[IntegerQ[m]] && IntegerQ[n] && Not[IntegerQ[p]]
```

2: $\int (a+b \operatorname{Sin}[e+fx])^m (c+d \operatorname{Csc}[e+fx])^n (g \operatorname{Csc}[e+fx])^p dx$ when $m \notin \mathbb{Z} \wedge n \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(g \operatorname{Csc}[e+fx])^m (a+b \operatorname{Sin}[e+fx])^m}{(b+a \operatorname{Csc}[e+fx])^m} == 0$

Rule: If $m \notin \mathbb{Z} \wedge n \notin \mathbb{Z}$, then

$$\int (a+b \operatorname{Sin}[e+fx])^m (c+d \operatorname{Csc}[e+fx])^n (g \operatorname{Csc}[e+fx])^p dx \rightarrow \frac{(a+b \operatorname{Sin}[e+fx])^m (g \operatorname{Csc}[e+fx])^m}{(b+a \operatorname{Csc}[e+fx])^m} \int (g \operatorname{Csc}[e+fx])^{p-m} (b+a \operatorname{Csc}[e+fx])^m (c+d \operatorname{Csc}[e+fx])^n dx$$

Program code:

```
Int[(g_*csc[e_+f_*x_])^p_*(a_+b_*sin[e_+f_*x_])^m_*(c_+d_*csc[e_+f_*x_])^n_,x_Symbol] :=
  (a+b*Sin[e+f*x])^m*(g*Csc[e+f*x])^m/(b+a*Csc[e+f*x])^m*
  Int[(g*Csc[e+f*x])^(p-m)*(b+a*Csc[e+f*x])^m*(c+d*Csc[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p},x] && Not[IntegerQ[m]] && Not[IntegerQ[n]]
```