

Rules for integrands of the form $(d + e x)^m (a + b x + c x^2)^p$

x: $\int (d + e x)^m (a + b x + c x^2)^p dx$ when $b^2 - 4 a c == 0 \wedge p \in \mathbb{Z}$

Derivation: Algebraic simplification

Basis: If $b^2 - 4 a c == 0$, then $a + b x + c x^2 == \frac{1}{c} \left(\frac{b}{2} + c x \right)^2$

Rule 1.2.1.2.2.1: If $b^2 - 4 a c == 0 \wedge p \in \mathbb{Z}$, then

$$\int (d + e x)^m (a + b x + c x^2)^p dx \rightarrow \frac{1}{c^p} \int (d + e x)^m \left(\frac{b}{2} + c x \right)^{2p} dx$$

Program code:

```
(* Int[(d_.*e_.*x_)^m_.*(a_+b_.*x_+c_.*x_^2)^p_.,x_Symbol] :=
  1/c^p*Int[(d+e*x)^m*(b/2+c*x)^(2*p),x] /;
FreeQ[{a,b,c,d,e,m},x] && EqQ[b^2-4*a*c,0] && IntegerQ[p] *)
```

0: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0 \wedge (p \in \mathbb{Z} \vee b = 0 \wedge a > 0 \wedge d > 0 \wedge m+p \in \mathbb{Z})$

Derivation: Algebraic simplification

Basis: If $cd^2 - bde + ae^2 = 0$, then $a + bx + cx^2 = (d + ex) \left(\frac{a}{d} + \frac{cx}{e} \right)$

Basis: If $cd^2 + ae^2 = 0 \wedge a > 0 \wedge d > 0$, then $(a + cx^2)^p = \left(a - \frac{ae^2x^2}{d^2} \right)^p = (d + ex)^p \left(\frac{a}{d} + \frac{cx}{e} \right)^p$

Rule 1.2.1.2.3.1: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0 \wedge (p \in \mathbb{Z} \vee b = 0 \wedge a > 0 \wedge d > 0 \wedge m+p \in \mathbb{Z})$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \int (d+ex)^{m+p} \left(\frac{a}{d} + \frac{cx}{e} \right)^p dx$$

Program code:

```
Int[(d_+e_.*x_)^m_.*(a_.+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  Int[(d+e*x)^(m+p)*(a/d+c/e*x)^p,x] /;
FreeQ[{a,b,c,d,e,m},x] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && IntegerQ[p]
```

```
Int[(d_+e_.*x_)^m_.*(a_+c_.*x_^2)^p_,x_Symbol] :=
  Int[(d+e*x)^(m+p)*(a/d+c/e*x)^p,x] /;
FreeQ[{a,c,d,e,m,p},x] && EqQ[c*d^2+a*e^2,0] && (IntegerQ[p] || GtQ[a,0] && GtQ[d,0] && IntegerQ[m+p])
```

$$1. \int (d+e x) (a+b x+c x^2)^p dx$$

$$1. \int (d+e x) (a+b x+c x^2)^p dx \text{ when } 2 c d - b e == 0$$

$$\text{1: } \int \frac{d+e x}{a+b x+c x^2} dx \text{ when } 2 c d - b e == 0$$

Derivation: Integration by substitution

Basis: If $2 c d - b e == 0$, then $(d+e x) F[a+b x+c x^2] == \frac{d}{b} \text{Subst}[F[x], x, a+b x+c x^2] \partial_x (a+b x+c x^2)$

Rule 1.2.1.2.1.1.1: If $2 c d - b e == 0$, then

$$\int \frac{d+e x}{a+b x+c x^2} dx \rightarrow \frac{d}{b} \text{Subst}\left[\int \frac{1}{x} dx, x, a+b x+c x^2\right] \rightarrow \frac{d \text{Log}[a+b x+c x^2]}{b}$$

Program code:

```
Int[(d+_e_.*x_)/(a+_b_.*x+_c_.*x_^2),x_Symbol] :=
  d*Log[RemoveContent[a+b*x+c*x^2,x]]/b /;
FreeQ[{a,b,c,d,e},x] && EqQ[2*c*d-b*e,0]
```

2: $\int (d+ex) (a+bx+cx^2)^p dx$ when $2cd - be = 0 \wedge p \neq -1$

Derivation: Integration by substitution

Basis: If $2cd - be = 0$, then $(d+ex) F[a+bx+cx^2] = \frac{d}{b} \text{Subst}[F[x], x, a+bx+cx^2] \partial_x (a+bx+cx^2)$

Rule 1.2.1.2.1.1.2: If $2cd - be = 0 \wedge p \neq -1$, then

$$\int (d+ex) (a+bx+cx^2)^p dx \rightarrow \frac{d}{b} \text{Subst}\left[\int x^p dx, x, a+bx+cx^2\right] \rightarrow \frac{d (a+bx+cx^2)^{p+1}}{b (p+1)}$$

Program code:

```
Int[(d_+e_.**x_)*(a_+b_.**x_+c_.**x_^2)^p_,x_Symbol] :=
  d*(a+b*x+c*x^2)^(p+1)/(b*(p+1)) /;
FreeQ[{a,b,c,d,e,p},x] && EqQ[2*c*d-b*e,0] && NeQ[p,-1]
```

$$2. \int (d+ex) (a+bx+cx^2)^p dx \text{ when } 2cd - be \neq 0$$

$$1. \int (d+ex) (a+bx+cx^2)^p dx \text{ when } 2cd - be \neq 0 \wedge p \in \mathbb{Z} \wedge (p > 0 \vee a = 0)$$

$$1: \int (d+ex) (a+bx+cx^2)^p dx \text{ when } 2cd - be \neq 0 \wedge p \in \mathbb{Z}^+ \wedge cd^2 - bde + ae^2 = 0$$

Derivation: Algebraic simplification

Basis: If $cd^2 - bde + ae^2 = 0$, then $a + bx + cx^2 = (d + ex) \left(\frac{a}{d} + \frac{cx}{e} \right)$

Rule 1.2.1.2.1.2.1.1: If $2cd - be \neq 0 \wedge p \in \mathbb{Z}^+ \wedge cd^2 - bde + ae^2 = 0$, then

$$\int (d+ex) (a+bx+cx^2)^p dx \rightarrow \int (d+ex)^{p+1} \left(\frac{a}{d} + \frac{cx}{e} \right)^p dx$$

Program code:

```
Int[(d+_.*x_)*(a+_.*x_+c_.*x_^2)^p_.,x_Symbol] :=
  Int[(d+e*x)^(p+1)*(a/d+c/e*x)^p,x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[2*c*d-b*e,0] && IGtQ[p,0] && EqQ[c*d^2-b*d*e+a*e^2,0]
```

$$\mathbf{2:} \int (d+e x) (a+b x+c x^2)^p dx \text{ when } 2 c d-b e \neq 0 \wedge p \in \mathbb{Z} \wedge (p > 0 \vee a == 0)$$

Derivation: Algebraic expansion

Rule 1.2.1.2.1.2.1.2: If $2 c d-b e \neq 0 \wedge p \in \mathbb{Z} \wedge (p > 0 \vee a == 0)$, then

$$\int (d+e x) (a+b x+c x^2)^p dx \rightarrow \int \text{ExpandIntegrand}[(d+e x) (a+b x+c x^2)^p, x] dx$$

Program code:

```
Int[(d_+e_.x_)*(a_+b_.x_+c_.x_^2)^p_,x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x)*(a+b*x+c*x^2)^p,x],x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[2*c*d-b*e,0] && IntegerQ[p] && (GtQ[p,0] || EqQ[a,0])
```

2. $\int \frac{d+e x}{a+b x+c x^2} dx$ when $2 c d - b e \neq 0 \wedge b^2 - 4 a c \neq 0$

1: $\int \frac{d+e x}{a+b x+c x^2} dx$ when $2 c d - b e \neq 0 \wedge b^2 - 4 a c \neq 0 \wedge \text{NiceSqrtQ}[b^2 - 4 a c]$

Reference: G&R 2.161.1a & G&R 2.161.3

Derivation: Algebraic expansion

Basis: Let $q = \sqrt{b^2 - 4 a c}$, then $\frac{d+e x}{a+b x+c x^2} = \frac{c d - e \left(\frac{b}{2} - \frac{q}{2}\right)}{q \left(\frac{b}{2} - \frac{q}{2} + c x\right)} - \frac{c d - e \left(\frac{b}{2} + \frac{q}{2}\right)}{q \left(\frac{b}{2} + \frac{q}{2} + c x\right)}$

Rule 1.2.1.2.1.2.2.1: If $2 c d - b e \neq 0 \wedge b^2 - 4 a c \neq 0 \wedge \text{NiceSqrtQ}[b^2 - 4 a c]$, let $q \rightarrow \sqrt{b^2 - 4 a c}$, then

$$\int \frac{d+e x}{a+b x+c x^2} dx \rightarrow \frac{c d - e \left(\frac{b}{2} - \frac{q}{2}\right)}{q} \int \frac{1}{\frac{b}{2} - \frac{q}{2} + c x} dx - \frac{c d - e \left(\frac{b}{2} + \frac{q}{2}\right)}{q} \int \frac{1}{\frac{b}{2} + \frac{q}{2} + c x} dx$$

Program code:

```
Int[(d_+e_.x_)/(a_+b_.x_+c_.x_^2),x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    (c*d-e*(b/2-q/2))/q*Int[1/(b/2-q/2+c*x),x] - (c*d-e*(b/2+q/2))/q*Int[1/(b/2+q/2+c*x),x] /;
  FreeQ[{a,b,c,d,e},x] && NeQ[2*c*d-b*e,0] && NeQ[b^2-4*a*c,0] && NiceSqrtQ[b^2-4*a*c]
```

```
Int[(d_+e_.x_)/(a_+c_.x_^2),x_Symbol] :=
  With[{q=Rt[-a*c,2]},
    (e/2+c*d/(2*q))*Int[1/(-q+c*x),x] + (e/2-c*d/(2*q))*Int[1/(q+c*x),x] /;
  FreeQ[{a,c,d,e},x] && NiceSqrtQ[-a*c]
```

2: $\int \frac{d+e x}{a+b x+c x^2} dx$ when $2 c d - b e \neq 0 \wedge b^2 - 4 a c \neq 0 \wedge \neg \text{NiceSqrtQ}[b^2 - 4 a c]$

Reference: A&S 3.3.19

Derivation: Algebraic expansion

Basis: $\frac{d+e x}{a+b x+c x^2} = \left(d - \frac{b e}{2 c}\right) \frac{1}{a+b x+c x^2} + \frac{e (b+2 c x)}{2 c (a+b x+c x^2)}$

Note: $\frac{b+2 c x}{a+b x+c x^2}$ is easily integrated using the rules for when $2 c d - b e = 0$.

Rule 1.2.1.2.1.2.2.2: If $2 c d - b e \neq 0 \wedge b^2 - 4 a c \neq 0 \wedge \neg \text{NiceSqrtQ}[b^2 - 4 a c]$, then

$$\int \frac{d+e x}{a+b x+c x^2} dx \rightarrow \frac{2 c d - b e}{2 c} \int \frac{1}{a+b x+c x^2} dx + \frac{e}{2 c} \int \frac{b+2 c x}{a+b x+c x^2} dx$$

Program code:

```
Int[(d_+e_*x_)/(a_+b_*x_+c_*x_^2),x_Symbol] :=
(* (d-b*e/(2*c))*Int[1/(a+b*x+c*x^2),x] + *)
(2*c*d-b*e)/(2*c)*Int[1/(a+b*x+c*x^2),x] + e/(2*c)*Int[(b+2*c*x)/(a+b*x+c*x^2),x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[2*c*d-b*e,0] && NeQ[b^2-4*a*c,0] && Not[NiceSqrtQ[b^2-4*a*c]]
```

```
Int[(d_+e_*x_)/(a_+c_*x_^2),x_Symbol] :=
d*Int[1/(a+c*x^2),x] + e*Int[x/(a+c*x^2),x] /;
FreeQ[{a,c,d,e},x] && Not[NiceSqrtQ[-a*c]]
```


$$3. \int (d+e x) (a+b x+c x^2)^p dx \text{ when } 2 c d - b e \neq 0 \wedge b^2 - 4 a c \neq 0 \wedge p < -1$$

$$1: \int \frac{d+e x}{(a+b x+c x^2)^{3/2}} dx \text{ when } 2 c d - b e \neq 0 \wedge b^2 - 4 a c \neq 0$$

Derivation: Quadratic recurrence 2a

Rule 1.2.1.2.1.2.3.1: If $2 c d - b e \neq 0 \wedge b^2 - 4 a c \neq 0$, then

$$\int \frac{d+e x}{(a+b x+c x^2)^{3/2}} dx \rightarrow -\frac{2 (b d - 2 a e + (2 c d - b e) x)}{(b^2 - 4 a c) \sqrt{a+b x+c x^2}}$$

Program code:

```
Int[(d_+e_.*x_)/(a_+b_.*x_+c_.*x_^2)^(3/2),x_Symbol] :=
  -2*(b*d-2*a*e+(2*c*d-b*e)*x)/((b^2-4*a*c)*Sqrt[a+b*x+c*x^2]) /;
FreeQ[{a,b,c,d,e},x] && NeQ[2*c*d-b*e,0] && NeQ[b^2-4*a*c,0]
```

```
Int[(d_+e_.*x_)/(a_+c_.*x_^2)^(3/2),x_Symbol] :=
  (-a*e+c*d*x)/(a*c*Sqrt[a+c*x^2]) /;
FreeQ[{a,c,d,e},x]
```

$$\mathbf{2:} \int (d+ex) (a+bx+cx^2)^p dx \text{ when } 2cd-be \neq 0 \wedge b^2-4ac \neq 0 \wedge p < -1 \wedge p \neq -\frac{3}{2}$$

Derivation: Quadratic recurrence 2a

Rule 1.2.1.2.1.2.3.2: If $2cd-be \neq 0 \wedge b^2-4ac \neq 0 \wedge p < -1 \wedge p \neq -\frac{3}{2}$, then

$$\int (d+ex) (a+bx+cx^2)^p dx \rightarrow \frac{(bd-2ae+(2cd-be)x) (a+bx+cx^2)^{p+1}}{(p+1)(b^2-4ac)} - \frac{(2p+3)(2cd-be)}{(p+1)(b^2-4ac)} \int (a+bx+cx^2)^{p+1} dx$$

Program code:

```
Int[(d_+e_.x_)*(a_+b_.x_+c_.x_^2)^p_,x_Symbol] :=
  (b*d-2*a*e+(2*c*d-b*e)*x)/((p+1)*(b^2-4*a*c))*(a+b*x+c*x^2)^(p+1) -
  (2*p+3)*(2*c*d-b*e)/((p+1)*(b^2-4*a*c))*Int[(a+b*x+c*x^2)^(p+1),x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[2*c*d-b*e,0] && NeQ[b^2-4*a*c,0] && LtQ[p,-1] && NeQ[p,-3/2]
```

```
Int[(d_+e_.x_)*(a_+c_.x_^2)^p_,x_Symbol] :=
  (a*e-c*d*x)/(2*a*c*(p+1))*(a+c*x^2)^(p+1) +
  d*(2*p+3)/(2*a*(p+1))*Int[(a+c*x^2)^(p+1),x] /;
FreeQ[{a,c,d,e},x] && LtQ[p,-1] && NeQ[p,-3/2]
```

4: $\int (d+e x) (a+b x+c x^2)^p dx$ when $2 c d - b e \neq 0 \wedge p \neq -1$

Reference: G&R 2.181.1, CRC 119

Derivation: Special quadratic recurrence 3a

Rule 1.2.1.2.1.2.4: If $2 c d - b e \neq 0 \wedge p \neq -1$, then

$$\int (d+e x) (a+b x+c x^2)^p dx \rightarrow \frac{e (a+b x+c x^2)^{p+1}}{2 c (p+1)} + \frac{2 c d - b e}{2 c} \int (a+b x+c x^2)^p dx$$

Program code:

```
Int[(d_+e_.*x_)*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  e*(a+b*x+c*x^2)^(p+1)/(2*c*(p+1)) + (2*c*d-b*e)/(2*c)*Int[(a+b*x+c*x^2)^p,x] /;
FreeQ[{a,b,c,d,e,p},x] && NeQ[2*c*d-b*e,0] && NeQ[p,-1]
```

```
Int[(d_+e_.*x_)*(a_+c_.*x_^2)^p_,x_Symbol] :=
  e*(a+c*x^2)^(p+1)/(2*c*(p+1)) + d*Int[(a+c*x^2)^p,x] /;
FreeQ[{a,c,d,e,p},x] && NeQ[p,-1]
```

$$2. \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2-4 a c == 0 \wedge p \notin \mathbb{Z}$$

$$1. \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2-4 a c == 0 \wedge p \notin \mathbb{Z} \wedge 2 c d-b e == 0$$

$$1. \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2-4 a c == 0 \wedge p \notin \mathbb{Z} \wedge 2 c d-b e == 0 \wedge m \in \mathbb{Z}$$

$$\text{1: } \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2-4 a c == 0 \wedge p \notin \mathbb{Z} \wedge 2 c d-b e == 0 \wedge \frac{m}{2} \in \mathbb{Z}$$

Derivation: Algebraic simplification

Basis: If $b^2-4 a c == 0 \wedge 2 c d-b e == 0 \wedge \frac{m}{2} \in \mathbb{Z}$, then $(d+e x)^m (a+b x+c x^2)^p == \frac{e^m}{c^{m/2}} (a+b x+c x^2)^{p+\frac{m}{2}}$

Rule 1.2.1.2.2.1.1.1: If $b^2-4 a c == 0 \wedge p \notin \mathbb{Z} \wedge 2 c d-b e == 0 \wedge \frac{m}{2} \in \mathbb{Z}$, then

$$\int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow \frac{e^m}{c^{m/2}} \int (a+b x+c x^2)^{p+\frac{m}{2}} dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  e^m/c^(m/2)*Int[(a+b*x+c*x^2)^(p+m/2),x] /;
FreeQ[{a,b,c,d,e,p},x] && EqQ[b^2-4*a*c,0] && Not[IntegerQ[p]] && EqQ[2*c*d-b*e,0] && IntegerQ[m/2]
```

$$\mathbf{2:} \int (d+ex)^m (a+bx+cx^2)^p dx \text{ when } b^2 - 4ac = 0 \wedge p \notin \mathbb{Z} \wedge 2cd - be = 0 \wedge \frac{m-1}{2} \in \mathbb{Z} \wedge m \neq 1$$

Derivation: Algebraic simplification

Basis: If $b^2 - 4ac = 0 \wedge 2cd - be = 0 \wedge \frac{m-1}{2} \in \mathbb{Z}$, then $(d+ex)^m (a+bx+cx^2)^p = \frac{e^{m-1}}{c^{\frac{m-1}{2}}} (d+ex) (a+bx+cx^2)^{p+\frac{m-1}{2}}$

Rule 1.2.1.2.2.1.1.2: If $b^2 - 4ac = 0 \wedge p \notin \mathbb{Z} \wedge 2cd - be = 0 \wedge \frac{m-1}{2} \in \mathbb{Z} \wedge m \neq 1$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{e^{m-1}}{c^{\frac{m-1}{2}}} \int (d+ex) (a+bx+cx^2)^{p+\frac{m-1}{2}} dx$$

Program code:

```
Int[(d+_e*_x_)^m_*(a+_b*_x_+_c*_x_^2)^p_,x_Symbol] :=
  e^(m-1)/c^((m-1)/2)*Int[(d+e*x)*(a+b*x+c*x^2)^(p+(m-1)/2),x] /;
FreeQ[{a,b,c,d,e,p},x] && EqQ[b^2-4*a*c,0] && Not[IntegerQ[p]] && EqQ[2*c*d-b*e,0] && IntegerQ[(m-1)/2]
```

$$2: \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2 - 4 a c == 0 \wedge p \notin \mathbb{Z} \wedge 2 c d - b e == 0 \wedge m \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction

Basis: If $b^2 - 4 a c == 0 \wedge 2 c d - b e == 0$, then $\partial_x \frac{(a+b x+c x^2)^p}{(d+e x)^{2 p}} == 0$

Rule 1.2.1.2.2.1.2: If $b^2 - 4 a c == 0 \wedge p \notin \mathbb{Z} \wedge 2 c d - b e == 0 \wedge m \notin \mathbb{Z}$, then

$$\int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow \frac{(a+b x+c x^2)^p}{(d+e x)^{2 p}} \int (d+e x)^{m+2 p} dx$$

Program code:

```
Int[(d+_e_.*x_)^m_*(a+_b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  (a+b*x+c*x^2)^p/(d+e*x)^(2*p)*Int[(d+e*x)^(m+2*p),x] /;
FreeQ[{a,b,c,d,e,m,p},x] && EqQ[b^2-4*a*c,0] && Not[IntegerQ[p]] && EqQ[2*c*d-b*e,0] && Not[IntegerQ[m]]
```

$$2. \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2 - 4 a c == 0 \wedge p \notin \mathbb{Z} \wedge 2 c d - b e \neq 0$$

$$1: \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2 - 4 a c == 0 \wedge p \notin \mathbb{Z} \wedge 2 c d - b e \neq 0 \wedge m \in \mathbb{Z}^+ \wedge m - 2 p + 1 == 0$$

Derivation: Piecewise constant extraction and algebraic expansion

$$\text{Basis: If } b^2 - 4 a c == 0, \text{ then } \partial_x \frac{(a+b x+c x^2)^p}{\left(\frac{b}{2}+c x\right)^{2 p}} == 0$$

Rule 1.2.1.2.2.2.1: If $b^2 - 4 a c == 0 \wedge p \notin \mathbb{Z} \wedge 2 c d - b e \neq 0 \wedge m \in \mathbb{Z}^+ \wedge m - 2 p + 1 == 0$, then

$$\int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow \frac{(a+b x+c x^2)^{\text{FracPart}[p]}}{c^{\text{IntPart}[p]} \left(\frac{b}{2}+c x\right)^{2 \text{FracPart}[p]}} \int \text{ExpandLinearProduct}\left[\left(\frac{b}{2}+c x\right)^{2 p}, (d+e x)^m, \frac{b}{2}, c, x\right] dx$$

Program code:

```
Int[(d_+e_.x_)^m_*(a+b_.x_+c_.x_^2)^p_,x_Symbol] :=
  (a+b*x+c*x^2)^FracPart[p]/(c^IntPart[p]*(b/2+c*x)^(2*FracPart[p]))*
  Int[ExpandLinearProduct[(b/2+c*x)^(2*p),(d+e*x)^m,b/2,c,x],x] /;
FreeQ[{a,b,c,d,e,m,p},x] && EqQ[b^2-4*a*c,0] && Not[IntegerQ[p]] && NeQ[2*c*d-b*e,0] && IGtQ[m,0] && EqQ[m-2*p+1,0]
```

2: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac == 0 \wedge p \notin \mathbb{Z} \wedge 2cd - be \neq 0$

Derivation: Piecewise constant extraction

Basis: If $b^2 - 4ac == 0$, then $\partial_x \frac{(a+bx+cx^2)^p}{\left(\frac{b}{2}+cx\right)^{2p}} == 0$

Rule 1.2.1.2.2.2.2: If $b^2 - 4ac == 0 \wedge p \notin \mathbb{Z} \wedge 2cd - be \neq 0$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{(a+bx+cx^2)^{\text{FracPart}[p]}}{c^{\text{IntPart}[p]} \left(\frac{b}{2}+cx\right)^{2 \text{FracPart}[p]}} \int (d+ex)^m \left(\frac{b}{2}+cx\right)^{2p} dx$$

Program code:

```
Int[(d_+e_.**x_)^m_*(a_+b_.**x_+c_.**x_^2)^p_,x_Symbol] :=
  (a+b*x+c*x^2)^FracPart[p]/(c^IntPart[p]*(b/2+c*x)^(2*FracPart[p]))*Int[(d+e*x)^m*(b/2+c*x)^(2*p),x] /;
FreeQ[{a,b,c,d,e,m,p},x] && EqQ[b^2-4*a*c,0] && Not[IntegerQ[p]] && NeQ[2*c*d-b*e,0]
```


3. $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0$

0: $\int (ex)^m (bx+cx^2)^p dx$ when $p \in \mathbb{Z}$

Derivation: Algebraic simplification

Rule 1.2.1.2.3.0: If $p \in \mathbb{Z}$, then

$$\int (ex)^m (bx+cx^2)^p dx \rightarrow \frac{1}{e^p} \int (ex)^{m+p} (b+cx)^p dx$$

Program code:

```
Int[(e.*x_)^m_.*(b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  1/e^p*Int[(e*x)^(m+p)*(b+c*x)^p,x] /;
FreeQ[{b,c,e,m},x] && IntegerQ[p]
```

1: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0 \wedge p \notin \mathbb{Z} \wedge m+p = 0$

Reference: G&R 2.181.1, CRC 119 with $cd^2 - bde + ae^2 = 0 \wedge m+p = 0$

Derivation: Special quadratic recurrence 2a or 3a with $m+p = 0$

Rule 1.2.1.2.3.2.1: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0 \wedge p \notin \mathbb{Z} \wedge m+p = 0$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{e (d+ex)^{m-1} (a+bx+cx^2)^{p+1}}{c (p+1)}$$

Program code:

```
Int[(d_+e_.x_)^m_*(a_+b_.x_+c_.x_^2)^p_,x_Symbol] :=
  e*(d+e*x)^(m-1)*(a+b*x+c*x^2)^(p+1)/(c*(p+1)) /;
FreeQ[{a,b,c,d,e,m,p},x] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && Not[IntegerQ[p]] && EqQ[m+p,0]
```

```
Int[(d_+e_.x_)^m_*(a_+c_.x_^2)^p_,x_Symbol] :=
  e*(d+e*x)^(m-1)*(a+c*x^2)^(p+1)/(c*(p+1)) /;
FreeQ[{a,c,d,e,m,p},x] && EqQ[c*d^2+a*e^2,0] && Not[IntegerQ[p]] && EqQ[m+p,0]
```

2: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0 \wedge p \notin \mathbb{Z} \wedge m+2p+2 = 0$

Reference: G&R 2.181.4.4

Derivation: Special quadratic recurrence 2b or 3b with $m+2p+2 = 0$

Note: If $m+2p+2 = 0$ and $m \neq 0$, then $p+1 \neq 0$.

Rule 1.2.1.2.3.2.2: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0 \wedge p \notin \mathbb{Z} \wedge m+2p+2 = 0$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{e (d+ex)^m (a+bx+cx^2)^{p+1}}{(p+1) (2cd - be)}$$

Program code:

```
Int[(d_+e_.**x_)^m_*(a_+b_.**x_+c_.**x_^2)^p_,x_Symbol] :=
  e*(d+e*x)^m*(a+b*x+c*x^2)^(p+1)/((p+1)*(2*c*d-b*e)) /;
FreeQ[{a,b,c,d,e,m,p},x] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && Not[IntegerQ[p]] && EqQ[m+2*p+2,0]
```

```
Int[(d_+e_.**x_)^m_*(a_+c_.**x_^2)^p_,x_Symbol] :=
  e*(d+e*x)^m*(a+c*x^2)^(p+1)/(2*c*d*(p+1)) /;
FreeQ[{a,c,d,e,m,p},x] && EqQ[c*d^2+a*e^2,0] && Not[IntegerQ[p]] && EqQ[m+2*p+2,0]
```

3: $\int (d+ex)^2 (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0 \wedge p \notin \mathbb{Z} \wedge p < -1$

Derivation: Special quadratic recurrence 2a

Rule 1.2.1.2.3.2.3: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0 \wedge p \notin \mathbb{Z} \wedge p < -1$, then

$$\int (d+ex)^2 (a+bx+cx^2)^p dx \rightarrow \frac{e(d+ex)(a+bx+cx^2)^{p+1}}{c(p+1)} - \frac{e^2(p+2)}{c(p+1)} \int (a+bx+cx^2)^{p+1} dx$$

Program code:

```
Int[(d_+e_.**x_)^2*(a_+b_.**x_+c_.**x_^2)^p_,x_Symbol] :=
  e*(d+e*x)*(a+b*x+c*x^2)^(p+1)/(c*(p+1)) - e^2*(p+2)/(c*(p+1))*Int[(a+b*x+c*x^2)^(p+1),x] /;
FreeQ[{a,b,c,d,e,p},x] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && Not[IntegerQ[p]] && LtQ[p,-1]
```

```
Int[(d_+e_.**x_)^2*(a_+c_.**x_^2)^p_,x_Symbol] :=
  e*(d+e*x)*(a+c*x^2)^(p+1)/(c*(p+1)) - e^2*(p+2)/(c*(p+1))*Int[(a+c*x^2)^(p+1),x] /;
FreeQ[{a,c,d,e,p},x] && EqQ[c*d^2+a*e^2,0] && Not[IntegerQ[p]] && LtQ[p,-1]
```

4: $\int (d+e x)^m (a+b x+c x^2)^p dx$ when $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p \notin \mathbb{Z} \wedge m \in \mathbb{Z} \wedge (0 < -m < p \vee p < -m < 0)$

Derivation: Algebraic simplification

Basis: If $c d^2 - b d e + a e^2 = 0$, then $d + e x = \frac{a+b x+c x^2}{\frac{a}{d} + \frac{c x}{e}}$

Basis: If $c d^2 + a e^2 = 0$, then $d + e x = \frac{d^2 (a+c x^2)}{a (d-e x)}$

Rule 1.2.1.2.3.2.4: If $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p \notin \mathbb{Z} \wedge m \in \mathbb{Z} \wedge (0 < -m < p \vee p < -m < 0)$, then

$$\int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow \int \frac{(a+b x+c x^2)^{m+p}}{\left(\frac{a}{d} + \frac{c x}{e}\right)^m} dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  Int[(a+b*x+c*x^2)^(m+p)/(a/d+c*x/e)^m,x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && Not[IntegerQ[p]] && IntegerQ[m] &&
  RationalQ[p] && (LtQ[0,-m,p] || LtQ[p,-m,0]) && NeQ[m,2] && NeQ[m,-1]
```

```
Int[(d_+e_.*x_)^m_*(a_+c_.*x_^2)^p_,x_Symbol] :=
  d^(2*m)/a^m*Int[(a+c*x^2)^(m+p)/(d-e*x)^m,x] /;
FreeQ[{a,c,d,e,m,p},x] && EqQ[c*d^2+a*e^2,0] && Not[IntegerQ[p]] && IntegerQ[m] &&
  RationalQ[p] && (LtQ[0,-m,p] || LtQ[p,-m,0]) && NeQ[m,2] && NeQ[m,-1]
```

5: $\int (d+e x)^m (a+b x+c x^2)^p dx$ when $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p \notin \mathbb{Z} \wedge m+p \in \mathbb{Z}^+$

Reference: G&R 2.181.1, CRC 119 with $a e^2 - b d e + c d^2 = 0$

Derivation: Special quadratic recurrence 3a

Note: If $p \notin \mathbb{Z} \wedge m+p \in \mathbb{Z}^+$, then $m+2 p+1 \neq 0$.

Rule 1.2.1.2.3.2.5: If $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p \notin \mathbb{Z} \wedge m+p \in \mathbb{Z}^+$, then

$$\int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow \frac{e (d+e x)^{m-1} (a+b x+c x^2)^{p+1}}{c (m+2 p+1)} + \frac{(m+p) (2 c d-b e)}{c (m+2 p+1)} \int (d+e x)^{m-1} (a+b x+c x^2)^p dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  e*(d+e*x)^(m-1)*(a+b*x+c*x^2)^(p+1)/(c*(m+2*p+1)) +
  Simplify[m+p]*(2*c*d-b*e)/(c*(m+2*p+1))*Int[(d+e*x)^(m-1)*(a+b*x+c*x^2)^p,x] /;
FreeQ[{a,b,c,d,e,m,p},x] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && Not[IntegerQ[p]] && IGtQ[Simplify[m+p],0]
```

```
Int[(d_+e_.*x_)^m_*(a_+c_.*x_^2)^p_,x_Symbol] :=
  e*(d+e*x)^(m-1)*(a+c*x^2)^(p+1)/(c*(m+2*p+1)) +
  2*c*d*Simplify[m+p]/(c*(m+2*p+1))*Int[(d+e*x)^(m-1)*(a+c*x^2)^p,x] /;
FreeQ[{a,c,d,e,m,p},x] && EqQ[c*d^2+a*e^2,0] && Not[IntegerQ[p]] && IGtQ[Simplify[m+p],0]
```

6: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0 \wedge p \notin \mathbb{Z} \wedge m+2p+2 \in \mathbb{Z}^-$

Reference: G&R 2.181.4.4

Derivation: Special quadratic recurrence 3b

Note: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0$, then $2cd - be \neq 0$.

Note: If $p \notin \mathbb{Z} \wedge m+2p+2 \in \mathbb{Z}^-$, then $m+p+1 \neq 0$.

Rule 1.2.1.2.3.2.6: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0 \wedge p \notin \mathbb{Z} \wedge m+2p+2 \in \mathbb{Z}^-$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow -\frac{e (d+ex)^m (a+bx+cx^2)^{p+1}}{(m+p+1) (2cd-be)} + \frac{c (m+2p+2)}{(m+p+1) (2cd-be)} \int (d+ex)^{m+1} (a+bx+cx^2)^p dx$$

Program code:

```
Int[(d_+e_.**x_)^m_*(a_+b_.**x_+c_.**x_^2)^p_,x_Symbol] :=
  -e*(d+e*x)^m*(a+b*x+c*x^2)^(p+1)/((m+p+1)*(2*c*d-b*e)) +
  c*Simplify[m+2*p+2]/((m+p+1)*(2*c*d-b*e))*Int[(d+e*x)^(m+1)*(a+b*x+c*x^2)^p,x] /;
FreeQ[{a,b,c,d,e,m,p},x] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && Not[IntegerQ[p]] && ILtQ[Simplify[m+2*p+2],0]
```

```
Int[(d_+e_.**x_)^m_*(a_+c_.**x_^2)^p_,x_Symbol] :=
  -e*(d+e*x)^m*(a+c*x^2)^(p+1)/(2*c*d*(m+p+1)) +
  Simplify[m+2*p+2]/(2*d*(m+p+1))*Int[(d+e*x)^(m+1)*(a+c*x^2)^p,x] /;
FreeQ[{a,c,d,e,m,p},x] && EqQ[c*d^2+a*e^2,0] && Not[IntegerQ[p]] && ILtQ[Simplify[m+2*p+2],0]
```

7: $\int \frac{1}{\sqrt{d+e x} \sqrt{a+b x+c x^2}} dx$ when $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0$

Derivation: Integration by substitution

Basis: If $c d^2 - b d e + a e^2 = 0$, then $\frac{1}{\sqrt{d+e x} \sqrt{a+b x+c x^2}} = 2 e \operatorname{Subst}\left[\frac{1}{2 c d - b e + e^2 x^2}, x, \frac{\sqrt{a+b x+c x^2}}{\sqrt{d+e x}}\right] \partial_x \frac{\sqrt{a+b x+c x^2}}{\sqrt{d+e x}}$

Rule 1.2.1.2.3.2.7: If $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0$, then

$$\int \frac{1}{\sqrt{d+e x} \sqrt{a+b x+c x^2}} dx \rightarrow 2 e \operatorname{Subst}\left[\int \frac{1}{2 c d - b e + e^2 x^2} dx, x, \frac{\sqrt{a+b x+c x^2}}{\sqrt{d+e x}}\right]$$

Program code:

```
Int[1/(Sqrt[d_+e_*x_]*Sqrt[a_+b_*x_+c_*x_^2]),x_Symbol] :=
  2*e*Subst[Int[1/(2*c*d-b*e+e^2*x^2),x],x,Sqrt[a+b*x+c*x^2]/Sqrt[d+e*x]] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0]
```

```
Int[1/(Sqrt[d_+e_*x_]*Sqrt[a_+c_*x_^2]),x_Symbol] :=
  2*e*Subst[Int[1/(2*c*d+e^2*x^2),x],x,Sqrt[a+c*x^2]/Sqrt[d+e*x]] /;
FreeQ[{a,c,d,e},x] && EqQ[c*d^2+a*e^2,0]
```


8. $\int (d+e x)^m (a+b x+c x^2)^p dx$ when $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p > 0 \wedge m < 0$

1: $\int (d+e x)^m (a+b x+c x^2)^p dx$ when $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p > 0 \wedge (m < -2 \vee m + 2 p + 1 = 0) \wedge m + p + 1 \neq 0$

Reference: G&R 2.265b

Derivation: Special quadratic recurrence 1a

Rule 1.2.1.2.3.2.8.1: If

$b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p > 0 \wedge (m < -2 \vee m + 2 p + 1 = 0) \wedge m + p + 1 \neq 0$, then

$$\int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow \frac{(d+e x)^{m+1} (a+b x+c x^2)^p}{e (m+p+1)} - \frac{c p}{e^2 (m+p+1)} \int (d+e x)^{m+2} (a+b x+c x^2)^{p-1} dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  (d+e*x)^(m+1)*(a+b*x+c*x^2)^p/(e*(m+p+1)) -
  c*p/(e^2*(m+p+1))*Int[(d+e*x)^(m+2)*(a+b*x+c*x^2)^(p-1),x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && GtQ[p,0] && (LtQ[m,-2] || EqQ[m+2*p+1,0]) && NeQ[m+p+1,0] && I
```

```
Int[(d_+e_.*x_)^m_*(a_+c_.*x_^2)^p_,x_Symbol] :=
  (d+e*x)^(m+1)*(a+c*x^2)^p/(e*(m+p+1)) -
  c*p/(e^2*(m+p+1))*Int[(d+e*x)^(m+2)*(a+c*x^2)^(p-1),x] /;
FreeQ[{a,c,d,e},x] && EqQ[c*d^2+a*e^2,0] && GtQ[p,0] && (LtQ[m,-2] || EqQ[m+2*p+1,0]) && NeQ[m+p+1,0] && IntegerQ[2*p]
```

2: $\int (d+e x)^m (a+b x+c x^2)^p dx$ when $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p > 0 \wedge (-2 \leq m < 0 \vee m + p + 1 = 0) \wedge m + 2 p + 1 \neq 0$

Derivation: Special quadratic recurrence 1b

Rule 1.2.1.2.3.2.8.2: If

$b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p > 0 \wedge (-2 \leq m < 0 \vee m + p + 1 = 0) \wedge m + 2 p + 1 \neq 0$, then

$$\int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow \frac{(d+e x)^{m+1} (a+b x+c x^2)^p}{e (m+2 p+1)} - \frac{p (2 c d-b e)}{e^2 (m+2 p+1)} \int (d+e x)^{m+1} (a+b x+c x^2)^{p-1} dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  (d+e*x)^(m+1)*(a+b*x+c*x^2)^p/(e*(m+2*p+1)) -
  p*(2*c*d-b*e)/(e^2*(m+2*p+1))*Int[(d+e*x)^(m+1)*(a+b*x+c*x^2)^(p-1),x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && GtQ[p,0] && (LeQ[-2,m,0] || EqQ[m+p+1,0]) && NeQ[m+2*p+1,0] &&
```

```
Int[(d_+e_.*x_)^m_*(a_+c_.*x_^2)^p_,x_Symbol] :=
  (d+e*x)^(m+1)*(a+c*x^2)^p/(e*(m+2*p+1)) -
  2*c*d*p/(e^2*(m+2*p+1))*Int[(d+e*x)^(m+1)*(a+c*x^2)^(p-1),x] /;
FreeQ[{a,c,d,e},x] && EqQ[c*d^2+a*e^2,0] && GtQ[p,0] && (LeQ[-2,m,0] || EqQ[m+p+1,0]) && NeQ[m+2*p+1,0] && IntegerQ[2*p]
```

9. $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0 \wedge p < -1 \wedge m > 0$

1: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0 \wedge p < -1 \wedge 0 < m < 1$

Derivation: Special quadratic recurrence 2b

Rule 1.2.1.2.3.2.9.1: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0 \wedge p < -1 \wedge 0 < m < 1$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{(2cd-be)(d+ex)^m (a+bx+cx^2)^{p+1}}{e(p+1)(b^2-4ac)} - \frac{(2cd-be)(m+2p+2)}{(p+1)(b^2-4ac)} \int (d+ex)^{m-1} (a+bx+cx^2)^{p+1} dx$$

Program code:

```
Int[(d_+e_.x_)^m_*(a_+b_.x_+c_.x_^2)^p_,x_Symbol] :=
  (2*c*d-b*e)*(d+e*x)^m*(a+b*x+c*x^2)^(p+1)/(e*(p+1)*(b^2-4*a*c)) -
  (2*c*d-b*e)*(m+2*p+2)/((p+1)*(b^2-4*a*c))*Int[(d+e*x)^(m-1)*(a+b*x+c*x^2)^(p+1),x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && LtQ[p,-1] && LtQ[0,m,1] && IntegerQ[2*p]
```

```
Int[(d_+e_.x_)^m_*(a_+c_.x_^2)^p_,x_Symbol] :=
  -d*(d+e*x)^m*(a+c*x^2)^(p+1)/(2*a*e*(p+1)) +
  d*(m+2*p+2)/(2*a*(p+1))*Int[(d+e*x)^(m-1)*(a+c*x^2)^(p+1),x] /;
FreeQ[{a,c,d,e},x] && EqQ[c*d^2+a*e^2,0] && LtQ[p,-1] && LtQ[0,m,1] && IntegerQ[2*p]
```

2: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0 \wedge p < -1 \wedge m > 1$

Derivation: Special quadratic recurrence 2a

Rule 1.2.1.2.2.3.9.2: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0 \wedge p < -1 \wedge m > 1$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow$$

$$\frac{e (d+ex)^{m-1} (a+bx+cx^2)^{p+1}}{c (p+1)} - \frac{e^2 (m+p)}{c (p+1)} \int (d+ex)^{m-2} (a+bx+cx^2)^{p+1} dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  e*(d+e*x)^(m-1)*(a+b*x+c*x^2)^(p+1)/(c*(p+1)) -
  e^2*(m+p)/(c*(p+1))*Int[(d+e*x)^(m-2)*(a+b*x+c*x^2)^(p+1),x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && LtQ[p,-1] && GtQ[m,1] && IntegerQ[2*p]
```

```
Int[(d_+e_.*x_)^m_*(a_+c_.*x_^2)^p_,x_Symbol] :=
  e*(d+e*x)^(m-1)*(a+c*x^2)^(p+1)/(c*(p+1)) -
  e^2*(m+p)/(c*(p+1))*Int[(d+e*x)^(m-2)*(a+c*x^2)^(p+1),x] /;
FreeQ[{a,c,d,e},x] && EqQ[c*d^2+a*e^2,0] && LtQ[p,-1] && GtQ[m,1] && IntegerQ[2*p]
```

10: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0 \wedge m > 1 \wedge m+2p+1 \neq 0$

Reference: G&R 2.181.1, CRC 119 with $ae^2 - bde + cd^2 = 0$

Derivation: Special quadratic recurrence 3a

Rule 1.2.1.2.3.2.10: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 = 0 \wedge m > 1 \wedge m+2p+1 \neq 0$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{e (d+ex)^{m-1} (a+bx+cx^2)^{p+1}}{c (m+2p+1)} + \frac{(m+p) (2cd-be)}{c (m+2p+1)} \int (d+ex)^{m-1} (a+bx+cx^2)^p dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  e*(d+e*x)^(m-1)*(a+b*x+c*x^2)^(p+1)/(c*(m+2*p+1)) +
  (m+p)*(2*c*d-b*e)/(c*(m+2*p+1))*Int[(d+e*x)^(m-1)*(a+b*x+c*x^2)^p,x] /;
FreeQ[{a,b,c,d,e,p},x] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && GtQ[m,1] && NeQ[m+2*p+1,0] && IntegerQ[2*p]
```

```
Int[(d_+e_.*x_)^m_*(a_+c_.*x_^2)^p_,x_Symbol] :=
  e*(d+e*x)^(m-1)*(a+c*x^2)^(p+1)/(c*(m+2*p+1)) +
  2*c*d*(m+p)/(c*(m+2*p+1))*Int[(d+e*x)^(m-1)*(a+c*x^2)^p,x] /;
FreeQ[{a,c,d,e,p},x] && EqQ[c*d^2+a*e^2,0] && GtQ[m,1] && NeQ[m+2*p+1,0] && IntegerQ[2*p]
```

11: $\int (d+e x)^m (a+b x+c x^2)^p dx$ when $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge m < 0 \wedge m+p+1 \neq 0$

Reference: G&R 2.181.4.4

Derivation: Special quadratic recurrence 3b

Note: If $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0$, then $2 c d - b e \neq 0$

Rule 1.2.1.2.3.2.11: If $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge m < 0 \wedge m+p+1 \neq 0$, then

$$\int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow$$

$$-\frac{e (d+e x)^m (a+b x+c x^2)^{p+1}}{(m+p+1) (2 c d - b e)} + \frac{c (m+2 p+2)}{(m+p+1) (2 c d - b e)} \int (d+e x)^{m+1} (a+b x+c x^2)^p dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  -e*(d+e*x)^m*(a+b*x+c*x^2)^(p+1)/((m+p+1)*(2*c*d-b*e)) +
  c*(m+2*p+2)/((m+p+1)*(2*c*d-b*e))*Int[(d+e*x)^(m+1)*(a+b*x+c*x^2)^p,x] /;
FreeQ[{a,b,c,d,e,p},x] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && LtQ[m,0] && NeQ[m+p+1,0] && IntegerQ[2*p]
```

```
Int[(d_+e_.*x_)^m_*(a_+c_.*x_^2)^p_,x_Symbol] :=
  -e*(d+e*x)^m*(a+c*x^2)^(p+1)/(2*c*d*(m+p+1)) +
  (m+2*p+2)/(2*d*(m+p+1))*Int[(d+e*x)^(m+1)*(a+c*x^2)^p,x] /;
FreeQ[{a,c,d,e,p},x] && EqQ[c*d^2+a*e^2,0] && LtQ[m,0] && NeQ[m+p+1,0] && IntegerQ[2*p]
```

12. $\int (d+e x)^m (a+b x+c x^2)^p dx$ when $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p \notin \mathbb{Z}$

1: $\int (e x)^m (b x+c x^2)^p dx$ when $p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(e x)^m (b x+c x^2)^p}{x^{m+p} (b+c x)^p} = 0$

Rule 1.2.1.2.3.2.12.1: If $p \notin \mathbb{Z}$, then

$$\int (e x)^m (b x+c x^2)^p dx \rightarrow \frac{(e x)^m (b x+c x^2)^p}{x^{m+p} (b+c x)^p} \int x^{m+p} (b+c x)^p dx$$

Program code:

```
Int[(e_.**x_)^m_*(b_.**x_+c_.**x_^2)^p_,x_Symbol] :=
  (e*x)^m*(b*x+c*x^2)^p/(x^(m+p)*(b+c*x)^p)*Int[x^(m+p)*(b+c*x)^p,x] /;
FreeQ[{b,c,e,m},x] && Not[IntegerQ[p]]
```

???: $\int (d+e x)^m (a+c x^2)^p dx$ when $c d^2 + a e^2 = 0 \wedge p \notin \mathbb{Z} \wedge a > 0 \wedge d > 0$

Derivation: Algebraic simplification

Basis: If $c d^2 + a e^2 = 0 \wedge a > 0 \wedge d > 0$, then $(a+c x^2)^p = \left(a - \frac{a e^2 x^2}{d^2}\right)^p = (d+e x)^p \left(\frac{a}{d} + \frac{c x}{e}\right)^p$

Rule 1.2.1.2.3.2.12.2: If $c d^2 + a e^2 = 0 \wedge p \notin \mathbb{Z} \wedge a > 0 \wedge d > 0$, then

$$\int (d+e x)^m (a+c x^2)^p dx \rightarrow \int (d+e x)^{m+p} \left(\frac{a}{d} + \frac{c x}{e}\right)^p dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+c_.*x_^2)^p_,x_Symbol] :=
  Int[(d+e*x)^(m+p)*(a/d+c/e*x)^p,x] /;
FreeQ[{a,c,d,e,m,p},x] && EqQ[c*d^2+a*e^2,0] && Not[IntegerQ[p]] && GtQ[a,0] && GtQ[d,0] && Not[IGtQ[m,0]]
```

$$?. \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p \notin \mathbb{Z} \wedge (m \in \mathbb{Z} \vee d > 0)$$

$$1: \int (d+e x)^m (a+c x^2)^p dx \text{ when } c d^2 + a e^2 = 0 \wedge p \notin \mathbb{Z} \wedge (m \in \mathbb{Z} \vee d > 0) \wedge a > 0$$

Derivation: Piecewise constant extraction

$$\text{Basis: If } c d^2 + a e^2 = 0, \text{ then } \partial_x \frac{(a+c x^2)^{p+1}}{\left(1+\frac{e x}{d}\right)^{p+1} \left(\frac{a}{d}+\frac{c x}{e}\right)^{p+1}} = 0$$

$$\text{Basis: If } c d^2 + a e^2 = 0 \wedge a > 0, \text{ then } \frac{(a+c x^2)^{p+1}}{\left(1+\frac{e x}{d}\right)^{p+1}} = a^{p+1} \left(\frac{d-e x}{d}\right)^{p+1}$$

Note: If $c d^2 - b d e + a e^2 = 0 \wedge m \in \mathbb{Z}^+ \wedge (3 p \in \mathbb{Z} \vee 4 p \in \mathbb{Z})$, then $(d+e x)^m (a+b x+c x^2)^p$ is integrable in terms of non-hypergeometric functions.

Rule 1.2.1.2.3.2.12.3: If $c d^2 + a e^2 = 0 \wedge p \notin \mathbb{Z} \wedge (m \in \mathbb{Z} \vee d > 0) \wedge a > 0$, then

$$\begin{aligned} \int (d+e x)^m (a+c x^2)^p dx &\rightarrow \frac{d^{m-1} (a+c x^2)^{p+1}}{\left(1+\frac{e x}{d}\right)^{p+1} \left(\frac{a}{d}+\frac{c x}{e}\right)^{p+1}} \int \left(1+\frac{e x}{d}\right)^{m+p} \left(\frac{a}{d}+\frac{c x}{e}\right)^p dx \\ &\rightarrow \frac{a^{p+1} d^{m-1} \left(\frac{d-e x}{d}\right)^{p+1}}{\left(\frac{a}{d}+\frac{c x}{e}\right)^{p+1}} \int \left(1+\frac{e x}{d}\right)^{m+p} \left(\frac{a}{d}+\frac{c x}{e}\right)^p dx \end{aligned}$$

Program code:

```
Int[(d+e.*x_)^m_*(a+c_.*x_^2)^p_,x_Symbol] :=
  a^(p+1)*d^(m-1)*((d-e*x)/d)^(p+1)/(a/d+c*x/e)^(p+1)*Int[(1+e*x/d)^(m+p)*(a/d+c/e*x)^p,x] /;
FreeQ[{a,c,d,e,m},x] && EqQ[c*d^2+a*e^2,0] && Not[IntegerQ[p]] && (IntegerQ[m] || GtQ[d,0]) && GtQ[a,0] &&
Not[IGtQ[m,0] && (IntegerQ[3*p] || IntegerQ[4*p])]
```


$$\mathbf{2:} \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p \notin \mathbb{Z} \wedge (m \in \mathbb{Z} \vee d > 0)$$

Derivation: Piecewise constant extraction

Basis: If $c d^2 - b d e + a e^2 = 0$, then $\partial_x \frac{(a+b x+c x^2)^p}{\left(1+\frac{e x}{d}\right)^p \left(\frac{a}{d}+\frac{c x}{e}\right)^p} = 0$

Note: If $c d^2 - b d e + a e^2 = 0 \wedge m \in \mathbb{Z}^+ \wedge (3 p \in \mathbb{Z} \vee 4 p \in \mathbb{Z})$, then $(d+e x)^m (a+b x+c x^2)^p$ is integrable in terms of non-hypergeometric functions.

Rule 1.2.1.2.3.2.12.3: If $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p \notin \mathbb{Z} \wedge (m \in \mathbb{Z} \vee d > 0)$, then

$$\int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow \frac{d^m (a+b x+c x^2)^{\text{FracPart}[p]}}{\left(1+\frac{e x}{d}\right)^{\text{FracPart}[p]} \left(\frac{a}{d}+\frac{c x}{e}\right)^{\text{FracPart}[p]}} \int \left(1+\frac{e x}{d}\right)^{m+p} \left(\frac{a}{d}+\frac{c x}{e}\right)^p dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  d^m*(a+b*x+c*x^2)^FracPart[p]/((1+e*x/d)^FracPart[p]*(a/d+(c*x)/e)^FracPart[p])*Int[(1+e*x/d)^(m+p)*(a/d+c/e*x)^p,x] /;
FreeQ[{a,b,c,d,e,m},x] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && Not[IntegerQ[p]] && (IntegerQ[m] || GtQ[d,0]) &&
Not[IGtQ[m,0] && (IntegerQ[3*p] || IntegerQ[4*p])]
```

```
Int[(d_+e_.*x_)^m_*(a_+c_.*x_^2)^p_,x_Symbol] :=
  d^(m-1)*(a+c*x^2)^(p+1)/((1+e*x/d)^(p+1)*(a/d+(c*x)/e)^(p+1))*Int[(1+e*x/d)^(m+p)*(a/d+c/e*x)^p,x] /;
FreeQ[{a,c,d,e,m},x] && EqQ[c*d^2+a*e^2,0] && Not[IntegerQ[p]] && (IntegerQ[m] || GtQ[d,0]) && Not[IGtQ[m,0] && (IntegerQ[3*p] || Inte
```

$$\mathbf{3:} \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2-4 a c \neq 0 \wedge c d^2-b d e+a e^2=0 \wedge p \notin \mathbb{Z} \wedge \neg (m \in \mathbb{Z} \vee d > 0)$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{(d+e x)^m}{\left(1+\frac{e x}{d}\right)^m} == 0$$

Rule 1.2.1.2.3.2.12.3: If $b^2-4 a c \neq 0 \wedge c d^2-b d e+a e^2=0 \wedge p \notin \mathbb{Z} \wedge \neg (m \in \mathbb{Z} \vee d > 0)$, then

$$\int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow \frac{d^{\text{IntPart}[m]} (d+e x)^{\text{FracPart}[m]}}{\left(1+\frac{e x}{d}\right)^{\text{FracPart}[m]}} \int \left(1+\frac{e x}{d}\right)^m (a+b x+c x^2)^p dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_.+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  d^IntPart[m]*(d+e*x)^FracPart[m]/(1+e*x/d)^FracPart[m]*Int[(1+e*x/d)^m*(a+b*x+c*x^2)^p,x] /;
FreeQ[{a,b,c,d,e,m},x] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && Not[IntegerQ[p]] && Not[IntegerQ[m] || GtQ[d,0]]
```

```
Int[(d_+e_.*x_)^m_*(a_+c_.*x_^2)^p_,x_Symbol] :=
  d^IntPart[m]*(d+e*x)^FracPart[m]/(1+e*x/d)^FracPart[m]*Int[(1+e*x/d)^m*(a+c*x^2)^p,x] /;
FreeQ[{a,c,d,e,m},x] && EqQ[c*d^2+a*e^2,0] && Not[IntegerQ[p]] && Not[IntegerQ[m] || GtQ[d,0]]
```

$$4. \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2 - 4 a c \neq 0 \wedge 2 c d - b e = 0$$

$$1. \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2 - 4 a c \neq 0 \wedge 2 c d - b e = 0 \wedge m+2 p+3 = 0$$

$$1: \int \frac{1}{(d+e x) (a+b x+c x^2)} dx \text{ when } b^2 - 4 a c \neq 0 \wedge 2 c d - b e = 0$$

Derivation: Algebraic expansion

$$\text{Basis: If } 2 c d - b e = 0, \text{ then } \frac{1}{(d+e x) (a+b x+c x^2)} = -\frac{4 b c}{d (b^2 - 4 a c) (b+2 c x)} + \frac{b^2 (d+e x)}{d^2 (b^2 - 4 a c) (a+b x+c x^2)}$$

Rule 1.2.1.2.3.1.1: If $b^2 - 4 a c \neq 0 \wedge 2 c d - b e = 0$, then

$$\int \frac{1}{(d+e x) (a+b x+c x^2)} dx \rightarrow -\frac{4 b c}{d (b^2 - 4 a c)} \int \frac{1}{b+2 c x} dx + \frac{b^2}{d^2 (b^2 - 4 a c)} \int \frac{d+e x}{a+b x+c x^2} dx$$

Program code:

```
Int[1/((d+_e_.*x_)*(a_+b_.*x_+c_.*x_^2)),x_Symbol] :=
  -4*b*c/(d*(b^2-4*a*c))*Int[1/(b+2*c*x),x] +
  b^2/(d^2*(b^2-4*a*c))*Int[(d+e*x)/(a+b*x+c*x^2),x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && EqQ[2*c*d-b*e,0]
```

$$2: \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2 - 4 a c \neq 0 \wedge 2 c d - b e = 0 \wedge m+2 p+3 = 0 \wedge p \neq -1$$

Derivation: Derivative divides quadratic recurrence 2b or 3b with $m+2 p+3 = 0$

Rule 1.2.1.2.3.1.2: If $b^2 - 4 a c \neq 0 \wedge 2 c d - b e = 0 \wedge m+2 p+3 = 0 \wedge p \neq -1$, then

$$\int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow \frac{2 c (d+e x)^{m+1} (a+b x+c x^2)^{p+1}}{e (p+1) (b^2 - 4 a c)}$$

Program code:

```
Int[(d_+e_.**x_)^m_*(a_.+b_.**x_+c_.**x_^2)^p_.,x_Symbol] :=
  2*c*(d+e*x)^(m+1)*(a+b*x+c*x^2)^(p+1)/(e*(p+1)*(b^2-4*a*c)) /;
FreeQ[{a,b,c,d,e,m,p},x] && NeQ[b^2-4*a*c,0] && EqQ[2*c*d-b*e,0] && EqQ[m+2*p+3,0] && NeQ[p,-1]
```

$$2: \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2 - 4 a c \neq 0 \wedge 2 c d - b e = 0 \wedge p \in \mathbb{Z}^+$$

Derivation: Algebraic expansion

Rule 1.2.1.2.3.2: If $b^2 - 4 a c \neq 0 \wedge 2 c d - b e = 0 \wedge p \in \mathbb{Z}^+$, then

$$\int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow \int \text{ExpandIntegrand}[(d+e x)^m (a+b x+c x^2)^p, x] dx$$

Program code:

```
Int[(d_+e_.**x_)^m_*(a_.+b_.**x_+c_.**x_^2)^p_.,x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x)^m*(a+b*x+c*x^2)^p,x],x] /;
FreeQ[{a,b,c,d,e,m},x] && NeQ[b^2-4*a*c,0] && EqQ[2*c*d-b*e,0] && IGtQ[p,0] && Not[EqQ[m,3] && NeQ[p,1]]
```

$$3. \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2 - 4 a c \neq 0 \wedge 2 c d - b e = 0 \wedge m+2 p+3 \neq 0 \wedge p > 0$$

$$1: \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2 - 4 a c \neq 0 \wedge 2 c d - b e = 0 \wedge m+2 p+3 \neq 0 \wedge p > 0 \wedge m < -1$$

Derivation: Derivative divides quadratic recurrence 1a

Derivation: Inverted integration by parts

Rule 1.2.1.2.3.3.1: If $b^2 - 4 a c \neq 0 \wedge 2 c d - b e = 0 \wedge m+2 p+3 \neq 0 \wedge p > 0 \wedge m < -1$, then

$$\int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow \frac{(d+e x)^{m+1} (a+b x+c x^2)^p}{e (m+1)} - \frac{b p}{d e (m+1)} \int (d+e x)^{m+2} (a+b x+c x^2)^{p-1} dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  (d+e*x)^(m+1)*(a+b*x+c*x^2)^p/(e*(m+1)) -
  b*p/(d*e*(m+1))*Int[(d+e*x)^(m+2)*(a+b*x+c*x^2)^(p-1),x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && EqQ[2*c*d-b*e,0] && NeQ[m+2*p+3,0] && GtQ[p,0] && LtQ[m,-1] &&
  Not[IntegerQ[m/2] && LtQ[m+2*p+3,0]] && IntegerQ[2*p]
```

2: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge 2cd - be = 0 \wedge m+2p+3 \neq 0 \wedge p > 0 \wedge m \neq -1$

Derivation: Derivative divides quadratic recurrence 1b

Rule 1.2.1.2.3.3.2: If $b^2 - 4ac \neq 0 \wedge 2cd - be = 0 \wedge m+2p+3 \neq 0 \wedge p > 0 \wedge m \neq -1$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{(d+ex)^{m+1} (a+bx+cx^2)^p}{e(m+2p+1)} - \frac{dp(b^2-4ac)}{be(m+2p+1)} \int (d+ex)^m (a+bx+cx^2)^{p-1} dx$$

Program code:

```
Int[(d+_e*_x_)^m_*(a+_b*_x_+c*_x_^2)^p_,x_Symbol] :=
  (d+e*x)^(m+1)*(a+b*x+c*x^2)^p/(e*(m+2*p+1)) -
  d*p*(b^2-4*a*c)/(b*e*(m+2*p+1))*Int[(d+e*x)^m*(a+b*x+c*x^2)^(p-1),x] /;
FreeQ[{a,b,c,d,e,m},x] && NeQ[b^2-4*a*c,0] && EqQ[2*c*d-b*e,0] && NeQ[m+2*p+3,0] && GtQ[p,0] &&
  Not[LtQ[m,-1]] && Not[IGtQ[(m-1)/2,0]] && (Not[IntegerQ[p]] || LtQ[m,2*p]) && RationalQ[m] && IntegerQ[2*p]
```

4. $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge 2cd - be = 0 \wedge m+2p+3 \neq 0 \wedge p < -1$

1: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge 2cd - be = 0 \wedge m+2p+3 \neq 0 \wedge p < -1 \wedge m > 1$

Derivation: Derivative divides quadratic recurrence 2a

Derivation: Integration by parts

Rule 1.2.1.2.3.4.1: If $b^2 - 4ac \neq 0 \wedge 2cd - be = 0 \wedge m+2p+3 \neq 0 \wedge p < -1 \wedge m > 1$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{d(d+ex)^{m-1} (a+bx+cx^2)^{p+1}}{b(p+1)} - \frac{de(m-1)}{b(p+1)} \int (d+ex)^{m-2} (a+bx+cx^2)^{p+1} dx$$

Program code:

```
Int[(d+_e*_x_)^m_*(a+_b*_x_+c*_x_^2)^p_,x_Symbol] :=
  d*(d+e*x)^(m-1)*(a+b*x+c*x^2)^(p+1)/(b*(p+1)) -
  d*e*(m-1)/(b*(p+1))*Int[(d+e*x)^(m-2)*(a+b*x+c*x^2)^(p+1),x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && EqQ[2*c*d-b*e,0] && NeQ[m+2*p+3,0] && LtQ[p,-1] && GtQ[m,1] && IntegerQ[2*p]
```

2: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge 2cd - be = 0 \wedge m+2p+3 \neq 0 \wedge p < -1 \wedge m \neq 1$

Derivation: Derivative divides quadratic recurrence 2b

Rule 1.2.1.2.3.4.2: If $b^2 - 4ac \neq 0 \wedge 2cd - be = 0 \wedge m+2p+3 \neq 0 \wedge p < -1 \wedge m \neq 1$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{2c(d+ex)^{m+1} (a+bx+cx^2)^{p+1}}{e(p+1)(b^2-4ac)} - \frac{2ce(m+2p+3)}{e(p+1)(b^2-4ac)} \int (d+ex)^m (a+bx+cx^2)^{p+1} dx$$

Program code:

```
Int[(d+e*x_)^m*(a_.+b_.x_+c_.x_^2)^p_,x_Symbol] :=
  2*c*(d+e*x)^(m+1)*(a+b*x+c*x^2)^(p+1)/(e*(p+1)*(b^2-4*a*c)) -
  2*c*e*(m+2*p+3)/(e*(p+1)*(b^2-4*a*c))*Int[(d+e*x)^m*(a+b*x+c*x^2)^(p+1),x] /;
FreeQ[{a,b,c,d,e,m},x] && NeQ[b^2-4*a*c,0] && EqQ[2*c*d-b*e,0] && NeQ[m+2*p+3,0] && LtQ[p,-1] && Not[GtQ[m,1]] && RationalQ[m] && Inte
```


5: $\int \frac{1}{(d+ex) \sqrt{a+bx+cx^2}} dx$ when $b^2 - 4ac \neq 0 \wedge 2cd - be = 0$

Derivation: Integration by substitution

Basis: If $2cd - be = 0$, then $\frac{1}{(d+ex) \sqrt{a+bx+cx^2}} = 4c \text{ Subst} \left[\frac{1}{b^2e-4ace+4cex^2}, x, \sqrt{a+bx+cx^2} \right] \partial_x \sqrt{a+bx+cx^2}$

Rule 1.2.1.2.3.5: If $b^2 - 4ac \neq 0 \wedge 2cd - be = 0$, then

$$\int \frac{1}{(d+ex) \sqrt{a+bx+cx^2}} dx \rightarrow 4c \text{ Subst} \left[\int \frac{1}{b^2e-4ace+4cex^2} dx, x, \sqrt{a+bx+cx^2} \right]$$

Program code:

```
Int[1/((d+_.*x_)*Sqrt[a+_.*x_+c_.*x_^2]),x_Symbol] :=
  4*c*Subst[Int[1/(b^2*e-4*a*c*e+4*c*e*x^2),x],x,Sqrt[a+b*x+c*x^2]] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && EqQ[2*c*d-b*e,0]
```

6. $\int \frac{(d+ex)^m}{\sqrt{a+bx+cx^2}} dx$ when $b^2 - 4ac \neq 0 \wedge 2cd - be = 0 \wedge m^2 = \frac{1}{4}$

1. $\int \frac{(d+ex)^m}{\sqrt{a+bx+cx^2}} dx$ when $b^2 - 4ac \neq 0 \wedge 2cd - be = 0 \wedge m^2 = \frac{1}{4} \wedge \frac{c}{b^2-4ac} < 0$

1: $\int \frac{1}{\sqrt{d+ex} \sqrt{a+bx+cx^2}} dx$ when $b^2 - 4ac \neq 0 \wedge 2cd - be = 0 \wedge \frac{c}{b^2-4ac} < 0$

Derivation: Integration by substitution

Basis: If $2cd - be = 0 \wedge \frac{c}{b^2-4ac} < 0$, then

$$\frac{1}{\sqrt{d+ex} \sqrt{a+bx+cx^2}} = \frac{4}{e} \sqrt{-\frac{c}{b^2-4ac}} \text{ Subst} \left[\frac{1}{\sqrt{1-\frac{b^2x^4}{d^2(b^2-4ac)}}}, x, \sqrt{d+ex} \right] \partial_x \sqrt{d+ex}$$

Rule 1.2.1.2.3.6.1.1: If $b^2 - 4 a c \neq 0 \wedge 2 c d - b e == 0 \wedge \frac{c}{b^2 - 4 a c} < 0$, then

$$\int \frac{1}{\sqrt{d+e x} \sqrt{a+b x+c x^2}} dx \rightarrow \frac{4}{e} \sqrt{-\frac{c}{b^2-4 a c}} \text{Subst}\left[\int \frac{1}{\sqrt{1-\frac{b^2 x^4}{d^2 (b^2-4 a c)}}} dx, x, \sqrt{d+e x}\right]$$

Program code:

```
Int[1/(Sqrt[d+_.*x_]*Sqrt[a+_.*x_+c_.*x_^2]),x_Symbol] :=
  4/e*Sqrt[-c/(b^2-4*a*c)]*Subst[Int[1/Sqrt[Simp[1-b^2*x^4/(d^2*(b^2-4*a*c))],x]],x],x,Sqrt[d+e*x]] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && EqQ[2*c*d-b*e,0] && LtQ[c/(b^2-4*a*c),0]
```

2: $\int \frac{\sqrt{d+e x}}{\sqrt{a+b x+c x^2}} dx$ when $b^2 - 4 a c \neq 0 \wedge 2 c d - b e == 0 \wedge \frac{c}{b^2 - 4 a c} < 0$

Derivation: Integration by substitution

Basis: If $2 c d - b e == 0 \wedge \frac{c}{b^2 - 4 a c} < 0$, then

$$\frac{\sqrt{d+e x}}{\sqrt{a+b x+c x^2}} == \frac{4}{e} \sqrt{-\frac{c}{b^2-4 a c}} \text{Subst}\left[\frac{x^2}{\sqrt{1-\frac{b^2 x^4}{d^2 (b^2-4 a c)}}}, x, \sqrt{d+e x}\right] \partial_x \sqrt{d+e x}$$

Rule 1.2.1.2.3.6.1.2: If $b^2 - 4 a c \neq 0 \wedge 2 c d - b e == 0 \wedge \frac{c}{b^2 - 4 a c} < 0$, then

$$\int \frac{\sqrt{d+e x}}{\sqrt{a+b x+c x^2}} dx \rightarrow \frac{4}{e} \sqrt{-\frac{c}{b^2-4 a c}} \text{Subst}\left[\int \frac{x^2}{\sqrt{1-\frac{b^2 x^4}{d^2 (b^2-4 a c)}}} dx, x, \sqrt{d+e x}\right]$$

Program code:

```
Int[Sqrt[d+_.*x_]/Sqrt[a+_.*x_+c_.*x_^2],x_Symbol] :=
  4/e*Sqrt[-c/(b^2-4*a*c)]*Subst[Int[x^2/Sqrt[Simp[1-b^2*x^4/(d^2*(b^2-4*a*c))],x]],x],x,Sqrt[d+e*x]] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && EqQ[2*c*d-b*e,0] && LtQ[c/(b^2-4*a*c),0]
```

2: $\int \frac{(d+e x)^m}{\sqrt{a+b x+c x^2}} dx$ when $b^2 - 4 a c \neq 0 \wedge 2 c d - b e == 0 \wedge m^2 == \frac{1}{4} \wedge \frac{c}{b^2-4 a c} \neq 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{\sqrt{-\frac{c(a+b x+c x^2)}{b^2-4 a c}}}{\sqrt{a+b x+c x^2}} == 0$

Rule 1.2.1.2.3.6.2: If $b^2 - 4 a c \neq 0 \wedge 2 c d - b e == 0 \wedge m^2 == \frac{1}{4} \wedge \frac{c}{b^2-4 a c} \neq 0$, then

$$\int \frac{(d+e x)^m}{\sqrt{a+b x+c x^2}} dx \rightarrow \frac{\sqrt{-\frac{c(a+b x+c x^2)}{b^2-4 a c}}}{\sqrt{a+b x+c x^2}} \int \frac{(d+e x)^m}{\sqrt{-\frac{a c}{b^2-4 a c} - \frac{b c x}{b^2-4 a c} - \frac{c^2 x^2}{b^2-4 a c}}} dx$$

Program code:

```
Int[(d+_e_.*x_)^m_/Sqrt[a_+_b_.*x_+_c_.*x_^2],x_Symbol] :=
  Sqrt[-c*(a+b*x+c*x^2)/(b^2-4*a*c)]/Sqrt[a+b*x+c*x^2]*
  Int[(d+e*x)^m/Sqrt[-a*c/(b^2-4*a*c)-b*c*x/(b^2-4*a*c)-c^2*x^2/(b^2-4*a*c)],x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && EqQ[2*c*d-b*e,0] && EqQ[m^2,1/4]
```

7: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge 2cd - be = 0 \wedge m+2p+3 \neq 0 \wedge m > 1 \wedge p \neq -1$

Derivation: Derivative divides quadratic recurrence 3a

Derivation: Integration by parts

Rule 1.2.1.2.3.7: If $b^2 - 4ac \neq 0 \wedge 2cd - be = 0 \wedge m+2p+3 \neq 0 \wedge m > 1 \wedge p \neq -1$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{2d(d+ex)^{m-1}(a+bx+cx^2)^{p+1}}{b(m+2p+1)} + \frac{d^2(m-1)(b^2-4ac)}{b^2(m+2p+1)} \int (d+ex)^{m-2}(a+bx+cx^2)^p dx$$

Program code:

```
Int[(d+e*x_)^m_*(a_+b_*x_+c_*x_^2)^p_,x_Symbol] :=
  2*d*(d+e*x)^(m-1)*(a+b*x+c*x^2)^(p+1)/(b*(m+2*p+1)) +
  d^2*(m-1)*(b^2-4*a*c)/(b^2*(m+2*p+1))*Int[(d+e*x)^(m-2)*(a+b*x+c*x^2)^p,x] /;
FreeQ[{a,b,c,d,e,p},x] && NeQ[b^2-4*a*c,0] && EqQ[2*c*d-b*e,0] && NeQ[m+2*p+3,0] && GtQ[m,1] &&
  NeQ[m+2*p+1,0] && (IntegerQ[2*p] || IntegerQ[m] && RationalQ[p] || OddQ[m])
```

8: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge 2cd - be = 0 \wedge m+2p+3 \neq 0 \wedge m < -1 \wedge p \neq 0$

Derivation: Derivative divides quadratic recurrence 3b

Rule 1.2.1.2.3.8: If $b^2 - 4ac \neq 0 \wedge 2cd - be = 0 \wedge m+2p+3 \neq 0 \wedge m < -1 \wedge p \neq 0$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow -\frac{2bd(d+ex)^{m+1}(a+bx+cx^2)^{p+1}}{d^2(m+1)(b^2-4ac)} + \frac{b^2(m+2p+3)}{d^2(m+1)(b^2-4ac)} \int (d+ex)^{m+2}(a+bx+cx^2)^p dx$$

Program code:

```
Int[(d+e*x_)^m*(a_.+b_.x_+c_.x_^2)^p_,x_Symbol] :=
  -2*b*d*(d+e*x)^(m+1)*(a+b*x+c*x^2)^(p+1)/(d^2*(m+1)*(b^2-4*a*c)) +
  b^2*(m+2*p+3)/(d^2*(m+1)*(b^2-4*a*c))*Int[(d+e*x)^(m+2)*(a+b*x+c*x^2)^p,x] /;
FreeQ[{a,b,c,d,e,p},x] && NeQ[b^2-4*a*c,0] && EqQ[2*c*d-b*e,0] && NeQ[m+2*p+3,0] && LtQ[m,-1] &&
(IntegerQ[2*p] || IntegerQ[m] && RationalQ[p] || IntegerQ[(m+2*p+3)/2])
```

9: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge 2cd - be = 0$

Derivation: Integration by substitution

Basis: If $2cd - be = 0$, then $F[a+bx+cx^2] = \frac{1}{e} \text{Subst}[F[a - \frac{b^2}{4c} + \frac{cx^2}{e^2}], x, d+ex] \partial_x (d+ex)$

Rule 1.2.1.2.3.9: If $b^2 - 4ac \neq 0 \wedge 2cd - be = 0$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{1}{e} \text{Subst}\left[\int x^m \left(a - \frac{b^2}{4c} + \frac{cx^2}{e^2}\right)^p dx, x, d+ex\right]$$

Program code:

```
Int[(d+_e*_x_)^m_*(a_+_b*_x_+_c*_x_^2)^p_,x_Symbol] :=
  1/e*Subst[Int[x^m*(a-b^2/(4*c)+(c*x^2)/e^2)^p,x],x,d+e*x] /;
FreeQ[{a,b,c,d,e,m,p},x] && NeQ[b^2-4*a*c,0] && EqQ[2*c*d-b*e,0]
```

?: $\int \frac{1}{(d+e x) (a+c x^2)^{1/4}} dx$ when $c d^2 + 2 a e^2 = 0 \wedge a < 0$

Reference: Eneström index number E688 in The Euler Archive

Rule 1.2.1.2.?: If $c d^2 + 2 a e^2 = 0 \wedge a < 0$, then

$$\int \frac{1}{(d+e x) (a+c x^2)^{1/4}} dx \rightarrow \frac{1}{2 (-a)^{1/4} e} \operatorname{ArcTan}\left[\frac{\left(-1 - \frac{c x^2}{a}\right)^{1/4}}{1 - \frac{c d x}{2 a e} - \sqrt{-1 - \frac{c x^2}{a}}}\right] + \frac{1}{4 (-a)^{1/4} e} \operatorname{Log}\left[\frac{1 - \frac{c d x}{2 a e} + \sqrt{-1 - \frac{c x^2}{a}} - \left(-1 - \frac{c x^2}{a}\right)^{1/4}}{1 - \frac{c d x}{2 a e} + \sqrt{-1 - \frac{c x^2}{a}} + \left(-1 - \frac{c x^2}{a}\right)^{1/4}}\right]$$

Program code:

```
Int[1/((d+_e_.*x_)*(a_+_c_.*x_^2)^(1/4)),x_Symbol] :=
  1/(2*(-a)^(1/4)*e)*ArcTan[(-1-c*x^2/a)^(1/4)/(1-c*d*x/(2*a*e)-Sqrt[-1-c*x^2/a])] +
  1/(4*(-a)^(1/4)*e)*Log[(1-c*d*x/(2*a*e)+Sqrt[-1-c*x^2/a]-(-1-c*x^2/a)^(1/4))/
    (1-c*d*x/(2*a*e)+Sqrt[-1-c*x^2/a]+(-1-c*x^2/a)^(1/4))]/;
FreeQ[{a,c,d,e},x] && EqQ[c*d^2+2*a*e^2,0] && LtQ[a,0]
```

5. $\int (d+e x)^m (a+b x+c x^2)^p dx$ when $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge 2 c d - b e \neq 0 \wedge p \in \mathbb{Z} \wedge (p > 0 \vee m \in \mathbb{Z})$

1. $\int (d+e x)^m (a+c x^2)^p dx$ when $c d^2 + a e^2 \neq 0 \wedge p \in \mathbb{Z}^+$

1: $\int (d+e x)^m (a+c x^2)^p dx$ when $c d^2 + a e^2 \neq 0 \wedge p - 1 \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m \leq p$

Derivation: Algebraic expansion and power rule for integration

Note: This rule removes the one degree term from the polynomial $(d+e x)^m$.

Rule: If $c d^2 + a e^2 \neq 0 \wedge p - 1 \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m \leq p$, then

$$\int (d+e x)^m (a+c x^2)^p dx \rightarrow e m d^{m-1} \int x (a+c x^2)^p dx + \int ((d+e x)^m - e m d^{m-1} x) (a+c x^2)^p dx$$

$$\rightarrow \frac{e m d^{m-1} (a + c x^2)^{p+1}}{2 c (p+1)} + \int ((d + e x)^m - e m d^{m-1} x) (a + c x^2)^p dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+c_.*x_^2)^p_,x_Symbol] :=
  e*m*d^(m-1)*(a+c*x^2)^(p+1)/(2*c*(p+1)) +
  Int[(d+e*x)^m-e*m*d^(m-1)*x*(a+c*x^2)^p,x] /;
FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0] && IGtQ[p,1] && IGtQ[m,0] && LeQ[m,p]
```

2: $\int (d + e x)^m (a + c x^2)^p dx$ when $c d^2 + a e^2 \neq 0 \wedge p \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule 1.2.1.2.5.2: If $c d^2 + a e^2 \neq 0 \wedge p \in \mathbb{Z}^+$, then

$$\int (d + e x)^m (a + b x + c x^2)^p dx \rightarrow \int \text{ExpandIntegrand}[(d + e x)^m (a + b x + c x^2)^p, x] dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+c_.*x_^2)^p_,x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x)^m*(a+c*x^2)^p,x],x] /;
FreeQ[{a,c,d,e,m},x] && NeQ[c*d^2+a*e^2,0] && IGtQ[p,0]
```


$$\mathbf{2:} \int (d+ex)^m (a+bx+cx^2)^p dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge p \in \mathbb{Z} \wedge (p > 0 \vee a = 0 \wedge m \in \mathbb{Z})$$

Derivation: Algebraic expansion

Rule 1.2.1.2.5.2: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge p \in \mathbb{Z} \wedge (p > 0 \vee a = 0 \wedge m \in \mathbb{Z})$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \int \text{ExpandIntegrand}[(d+ex)^m (a+bx+cx^2)^p, x] dx$$

Program code:

```
Int[(d_+e_.x_)^m_*(a_+b_.x_+c_.x_^2)^p_,x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x)^m*(a+b*x+c*x^2)^p,x],x] /;
  FreeQ[{a,b,c,d,e,m},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && IntegerQ[p] && (GtQ[p,0] || EqQ[a,0] &&
```

$$6. \int \frac{(d+ex)^m}{a+bx+cx^2} dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0$$

$$1. \int \frac{(d+ex)^m}{a+bx+cx^2} dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m > 0$$

$$x. \int \frac{\sqrt{d+ex}}{a+bx+cx^2} dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0$$

$$\mathbf{1:} \int \frac{\sqrt{d+ex}}{a+bx+cx^2} dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge b^2 - 4ac < 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \sqrt{d+ex} = \frac{d+q+ex}{2\sqrt{d+ex}} + \frac{d-q+ex}{2\sqrt{d+ex}}$$

Note: Resulting integrands are of the form $\frac{A+Bx}{\sqrt{d+ex} (a+bx+cx^2)}$ where $A^2 ce - 2ABcd + B^2 (bd - ae) = 0$.

Note: Although use of this rule when $b^2 - 4ac < 0$ results in antiderivatives superficially free of the imaginary unit but

significantly more complicated than those produced by the following rule.

Rule 1.2.1.2.6.1.x.1: If $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge 2 c d - b e \neq 0 \wedge b^2 - 4 a c < 0$, let

$q \rightarrow \sqrt{\frac{c d^2 - b d e + a e^2}{c}}$, then

$$\int \frac{\sqrt{d+e x}}{a+b x+c x^2} dx \rightarrow \frac{1}{2} \int \frac{d+q+e x}{\sqrt{d+e x} (a+b x+c x^2)} dx + \frac{1}{2} \int \frac{d-q+e x}{\sqrt{d+e x} (a+b x+c x^2)} dx$$

Program code:

```
(* Int[Sqrt[d_+e_*x_]/(a_+b_*x_+c_*x_^2),x_Symbol] :=
  With[{q=Rt[(c*d^2-b*d*e+a*e^2)/c,2]},
    1/2*Int[(d+q+e*x)/(Sqrt[d+e*x]*(a+b*x+c*x^2)),x] +
    1/2*Int[(d-q+e*x)/(Sqrt[d+e*x]*(a+b*x+c*x^2)),x] /;
  FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && LtQ[b^2-4*a*c,0] *)
```

```
(* Int[Sqrt[d_+e_*x_]/(a_+c_*x_^2),x_Symbol] :=
  With[{q=Rt[(c*d^2+a*e^2)/c,2]},
    1/2*Int[(d+q+e*x)/(Sqrt[d+e*x]*(a+c*x^2)),x] +
    1/2*Int[(d-q+e*x)/(Sqrt[d+e*x]*(a+c*x^2)),x] /;
  FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0] && LtQ[-a*c,0] *)
```

2: $\int \frac{\sqrt{d+e x}}{a+b x+c x^2} dx$ when $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge 2 c d - b e \neq 0 \wedge \neg (b^2 - 4 a c < 0)$

Derivation: Algebraic expansion

Basis: If $q = \sqrt{b^2 - 4 a c}$, then $\frac{\sqrt{d+e x}}{a+b x+c x^2} = \frac{2 c d - b e + e q}{q \sqrt{d+e x} (b-q+2 c x)} - \frac{2 c d - b e - e q}{q \sqrt{d+e x} (b+q+2 c x)}$

Rule 1.2.1.2.6.1.x.2: If $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge 2 c d - b e \neq 0 \wedge \neg (b^2 - 4 a c < 0)$, let

$q \rightarrow \sqrt{b^2 - 4 a c}$, then

$$\int \frac{\sqrt{d+ex}}{a+bx+cx^2} dx \rightarrow \frac{2cd-be+eq}{q} \int \frac{1}{\sqrt{d+ex} (b-q+2cx)} dx - \frac{2cd-be-eq}{q} \int \frac{1}{\sqrt{d+ex} (b+q+2cx)} dx$$

Program code:

```
(* Int[Sqrt[d_+e_*x_]/(a_+b_*x_+c_*x_^2),x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    (2*c*d-b*e+e*q)/q*Int[1/(Sqrt[d+e*x]*(b-q+2*c*x)),x] -
    (2*c*d-b*e-e*q)/q*Int[1/(Sqrt[d+e*x]*(b+q+2*c*x)),x] /;
  FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] (* && Not[LtQ[b^2-4*a*c,0]] *) *)
```

```
(* Int[Sqrt[d_+e_*x_]/(a_+c_*x_^2),x_Symbol] :=
  With[{q=Rt[-a*c,2]},
    (c*d+e*q)/(2*q)*Int[1/(Sqrt[d+e*x]*(-q+c*x)),x] -
    (c*d-e*q)/(2*q)*Int[1/(Sqrt[d+e*x]*(+q+c*x)),x] /;
  FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0] (* && Not[LtQ[-a*c,0]] *) *)
```

1: $\int \frac{\sqrt{d+ex}}{a+bx+cx^2} dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0$

Derivation: Integration by substitution

Basis: $(d+ex)^m F[x] = \frac{2}{e} \text{Subst}[x^{2m+1} F[\frac{-d+ex^2}{e}], x, \sqrt{d+ex}] \partial_x \sqrt{d+ex}$

Rule 1.2.1.2.6.1.1: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0$

$$\int \frac{\sqrt{d+ex}}{a+bx+cx^2} dx \rightarrow 2e \text{Subst}\left[\int \frac{x^2}{cd^2 - bde + ae^2 - (2cd - be)x^2 + cx^4} dx, x, \sqrt{d+ex}\right]$$

Program code:

```
Int[Sqrt[d_+e_*x_]/(a_+b_*x_+c_*x_^2),x_Symbol] :=
  2*e*Subst[Int[x^2/(c*d^2-b*d*e+a*e^2-(2*c*d-b*e)*x^2+c*x^4),x],x,Sqrt[d+e*x]] /;
  FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0]
```

```
Int[Sqrt[d+_.*x_]/(a+_.*x_^2),x_Symbol] :=
  2*e*Subst[Int[x^2/(c*d^2+a*e^2-2*c*d*x^2+c*x^4),x],x,Sqrt[d+e*x]] /;
FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0]
```

$$2. \int \frac{(d+ex)^m}{a+bx+cx^2} dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m > 1$$

$$1: \int \frac{(d+ex)^m}{a+bx+cx^2} dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m \in \mathbb{Z} \wedge m > 1 \wedge (d \neq 0 \vee m > 2)$$

Derivation: Algebraic expansion

Rule 1.2.1.2.6.1.2.1: If

$b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m \in \mathbb{Z} \wedge m > 1 \wedge (d \neq 0 \vee m > 2)$, then

$$\int \frac{(d+ex)^m}{a+bx+cx^2} dx \rightarrow \int \text{PolynomialDivide}[(d+ex)^m, a+bx+cx^2, x] dx$$

Program code:

```
Int[(d+_.*x_)^m/(a+_.*x_^2),x_Symbol] :=
  Int[PolynomialDivide[(d+e*x)^m,a+b*x+c*x^2,x],x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && IGtQ[m,1] && (NeQ[d,0] || GtQ[m,2])
```

```
Int[(d+_.*x_)^m/(a+_.*x_^2),x_Symbol] :=
  Int[PolynomialDivide[(d+e*x)^m,a+c*x^2,x],x] /;
FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0] && IGtQ[m,1] && (NeQ[d,0] || GtQ[m,2])
```

$$\mathbf{2:} \int \frac{(d+e x)^m}{a+b x+c x^2} dx \text{ when } b^2-4 a c \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge 2 c d-b e \neq 0 \wedge m > 1$$

Reference: G&R 2.160.3, G&R 2.174.1, CRC 119

Derivation: Quadratic recurrence 3a with $A = d$, $B = e$, $m = m - 1$ and $p = -1$

Note: G&R 2.174.1 is a special case of G&R 2.160.3.

Rule 1.2.1.2.6.1.2.2: If $b^2-4 a c \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge 2 c d-b e \neq 0 \wedge m > 1$, then

$$\int \frac{(d+e x)^m}{a+b x+c x^2} dx \rightarrow \frac{e (d+e x)^{m-1}}{c (m-1)} + \frac{1}{c} \int \frac{(d+e x)^{m-2} (c d^2-a e^2+e (2 c d-b e) x)}{a+b x+c x^2} dx$$

Program code:

```
Int[(d_+e_.*x_)^m_/(a_+b_.*x_+c_.*x_^2),x_Symbol] :=
  e*(d+e*x)^(m-1)/(c*(m-1)) +
  1/c*Int[(d+e*x)^(m-2)*Simp[c*d^2-a*e^2+e*(2*c*d-b*e)*x,x]/(a+b*x+c*x^2),x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && GtQ[m,1]
```

```
Int[(d_+e_.*x_)^m_/(a_+c_.*x_^2),x_Symbol] :=
  e*(d+e*x)^(m-1)/(c*(m-1)) +
  1/c*Int[(d+e*x)^(m-2)*Simp[c*d^2-a*e^2+2*c*d*e*x,x]/(a+c*x^2),x] /;
FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0] && GtQ[m,1]
```

$$2. \int \frac{(d+e x)^m}{a+b x+c x^2} dx \text{ when } b^2-4 a c \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge 2 c d-b e \neq 0 \wedge m < 0$$

$$1: \int \frac{1}{(d+e x)(a+b x+c x^2)} dx \text{ when } b^2-4 a c \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge 2 c d-b e \neq 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{1}{(d+e x)(a+b x+c x^2)} = \frac{e^2}{(c d^2-b d e+a e^2)(d+e x)} + \frac{c d-b e-c e x}{(c d^2-b d e+a e^2)(a+b x+c x^2)}$$

Rule 1.2.1.2.6.2.1: If $b^2-4 a c \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge 2 c d-b e \neq 0$, then

$$\int \frac{1}{(d+e x)(a+b x+c x^2)} dx \rightarrow \frac{e^2}{c d^2-b d e+a e^2} \int \frac{1}{d+e x} dx + \frac{1}{c d^2-b d e+a e^2} \int \frac{c d-b e-c e x}{a+b x+c x^2} dx$$

Program code:

```
Int[1/((d_+e_.*x_)*(a_+b_.*x_+c_.*x_^2)),x_Symbol] :=
  e^2/(c*d^2-b*d*e+a*e^2)*Int[1/(d+e*x),x] +
  1/(c*d^2-b*d*e+a*e^2)*Int[(c*d-b*e-c*e*x)/(a+b*x+c*x^2),x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0]
```

```
Int[1/((d_+e_.*x_)*(a_+c_.*x_^2)),x_Symbol] :=
  e^2/(c*d^2+a*e^2)*Int[1/(d+e*x),x] +
  1/(c*d^2+a*e^2)*Int[(c*d-c*e*x)/(a+c*x^2),x] /;
FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0]
```

$$x. \int \frac{1}{\sqrt{d+ex} (a+bx+cx^2)} dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0$$

$$1: \int \frac{1}{\sqrt{d+ex} (a+bx+cx^2)} dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge b^2 - 4ac < 0$$

Derivation: Algebraic expansion

$$\text{Basis: } 1 = \frac{d+q+ex}{2q} - \frac{d-q+ex}{2q}$$

Note: Resulting integrands are of the form $\frac{A+Bx}{\sqrt{d+ex} (a+bx+cx^2)}$ where $A^2 ce - 2ABcd + B^2 (bd - ae) = 0$.

Note: Although use of this rule when $b^2 - 4ac < 0$ results in antiderivatives superficially free of the imaginary unit but significantly more complicated than those produced by the following rule.

Rule 1.2.1.2.6.2.x.1: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge b^2 - 4ac < 0$, let

$$q \rightarrow \sqrt{\frac{cd^2 - bde + ae^2}{c}}, \text{ then}$$

$$\int \frac{1}{\sqrt{d+ex} (a+bx+cx^2)} dx \rightarrow \frac{1}{2q} \int \frac{d+q+ex}{\sqrt{d+ex} (a+bx+cx^2)} dx - \frac{1}{2q} \int \frac{d-q+ex}{\sqrt{d+ex} (a+bx+cx^2)} dx$$

Program code:

```
(* Int[1/(Sqrt[d_+e_.*x_]*(a_+b_.*x_+c_.*x_^2)),x_Symbol] :=
  With[{q=Rt[(c*d^2-b*d*e+a*e^2)/c,2]},
    1/(2*q)*Int[(d+q+e*x)/(Sqrt[d+e*x]*(a+b*x+c*x^2)),x] -
    1/(2*q)*Int[(d-q+e*x)/(Sqrt[d+e*x]*(a+b*x+c*x^2)),x] /;
  FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && LtQ[b^2-4*a*c,0] *)
```

```
(* Int[1/(Sqrt[d_+e_.*x_]*(a_+c_.*x_^2)),x_Symbol] :=
  With[{q=Rt[(c*d^2+a*e^2)/c,2]},
    1/(2*q)*Int[(d+q+e*x)/(Sqrt[d+e*x]*(a+c*x^2)),x] -
    1/(2*q)*Int[(d-q+e*x)/(Sqrt[d+e*x]*(a+c*x^2)),x] /;
  FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0] && LtQ[-a*c,0] *)
```

2: $\int \frac{1}{\sqrt{d+e x} (a+b x+c x^2)} dx$ when $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge 2 c d - b e \neq 0 \wedge \neg (b^2 - 4 a c < 0)$

Derivation: Algebraic expansion

Basis: If $q = \sqrt{b^2 - 4 a c}$, then $\frac{1}{a+b x+c x^2} = \frac{2 c}{q (b-q+2 c x)} - \frac{2 c}{q (b+q+2 c x)}$

Rule 1.2.1.2.6.2.x.2: If $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge 2 c d - b e \neq 0 \wedge \neg (b^2 - 4 a c < 0)$, let $q \rightarrow \sqrt{b^2 - 4 a c}$, then

$$\int \frac{1}{\sqrt{d+e x} (a+b x+c x^2)} dx \rightarrow \frac{2 c}{q} \int \frac{1}{\sqrt{d+e x} (b-q+2 c x)} dx - \frac{2 c}{q} \int \frac{1}{\sqrt{d+e x} (b+q+2 c x)} dx$$

Program code:

```
(* Int[1/(Sqrt[d_+e_.*x_]*(a_+b_.*x_+c_.*x_^2)),x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    2*c/q*Int[1/(Sqrt[d+e*x]*(b-q+2*c*x)),x] -
    2*c/q*Int[1/(Sqrt[d+e*x]*(b+q+2*c*x)),x] /;
  FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] (* && Not[LtQ[b^2-4*a*c,0]] *) *)
```

```
(* Int[1/(Sqrt[d_+e_.*x_]*(a_+c_.*x_^2)),x_Symbol] :=
  With[{q=Rt[-a*c,2]},
    c/(2*q)*Int[1/(Sqrt[d+e*x]*(-q+c*x)),x] -
    c/(2*q)*Int[1/(Sqrt[d+e*x]*(q+c*x)),x] /;
  FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0] (* && Not[LtQ[-a*c,0]] *) *)
```


$$2: \int \frac{1}{\sqrt{d+e x} (a+b x+c x^2)} dx \text{ when } b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge 2 c d - b e \neq 0$$

Derivation: Integration by substitution

$$\text{Basis: } (d+e x)^m F[x] \Rightarrow \frac{2}{e} \text{Subst}\left[x^{2m+1} F\left[\frac{-d+ex^2}{e}\right], x, \sqrt{d+e x}\right] \partial_x \sqrt{d+e x}$$

Rule 1.2.1.2.6.2.2: If $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge 2 c d - b e \neq 0$

$$\int \frac{1}{\sqrt{d+e x} (a+b x+c x^2)} dx \rightarrow 2 e \text{Subst}\left[\int \frac{1}{c d^2 - b d e + a e^2 - (2 c d - b e) x^2 + c x^4} dx, x, \sqrt{d+e x}\right]$$

Program code:

```
Int[1/(Sqrt[d_+e_.*x_]*(a_+b_.*x_+c_.*x_^2)),x_Symbol] :=
  2*e*Subst[Int[1/(c*d^2-b*d*e+a*e^2-(2*c*d-b*e)*x^2+c*x^4),x],x,Sqrt[d+e*x]] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0]
```

```
Int[1/(Sqrt[d_+e_.*x_]*(a_+c_.*x_^2)),x_Symbol] :=
  2*e*Subst[Int[1/(c*d^2+a*e^2-2*c*d*x^2+c*x^4),x],x,Sqrt[d+e*x]] /;
FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0]
```

$$3: \int \frac{(d+ex)^m}{a+bx+cx^2} dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m < -1$$

Reference: G&R 2.176, CRC 123

Derivation: Quadratic recurrence 3b with $A = 1$, $B = 0$ and $p = -1$

Rule 1.2.1.2.6.2.3: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m < -1$, then

$$\int \frac{(d+ex)^m}{a+bx+cx^2} dx \rightarrow \frac{e(d+ex)^{m+1}}{(m+1)(cd^2 - bde + ae^2)} + \frac{1}{cd^2 - bde + ae^2} \int \frac{(d+ex)^{m+1}(cd - be - cex)}{a+bx+cx^2} dx$$

Program code:

```
Int[(d_+e_.*x_)^m/(a_+b_.*x_+c_.*x_^2),x_Symbol] :=
  e*(d+e*x)^(m+1)/((m+1)*(c*d^2-b*d*e+a*e^2)) +
  1/(c*d^2-b*d*e+a*e^2)*Int[(d+e*x)^(m+1)*Simp[c*d-b*e-c*e*x,x]/(a+b*x+c*x^2),x] /;
FreeQ[{a,b,c,d,e,m},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && LtQ[m,-1]
```

```
Int[(d_+e_.*x_)^m/(a_+c_.*x_^2),x_Symbol] :=
  e*(d+e*x)^(m+1)/((m+1)*(c*d^2+a*e^2)) +
  c/(c*d^2+a*e^2)*Int[(d+e*x)^(m+1)*(d-e*x)/(a+c*x^2),x] /;
FreeQ[{a,c,d,e,m},x] && NeQ[c*d^2+a*e^2,0] && LtQ[m,-1]
```

$$3: \int \frac{(d+ex)^m}{a+bx+cx^2} dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m \notin \mathbb{Z}$$

Derivation: Algebraic expansion

■ Basis: If $q = \sqrt{b^2 - 4ac}$, then $\frac{1}{a+bz+cz^2} = \frac{2c}{q(b-q+2cz)} - \frac{2c}{q(b+q+2cz)}$

Rule 1.2.1.2.6.3: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m \notin \mathbb{Z}$, then

$$\int \frac{(d+ex)^m}{a+bx+cx^2} dx \rightarrow \int (d+ex)^m \text{ExpandIntegrand}\left[\frac{1}{a+bx+cx^2}, x\right] dx$$

Program code:

```
Int[(d_+e_.x_)^m_/(a_+b_.x_+c_.x_^2),x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x)^m,1/(a+b*x+c*x^2),x],x] /;
FreeQ[{a,b,c,d,e,m},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && Not[IntegerQ[m]]
```

```
Int[(d_+e_.x_)^m_/(a_+c_.x_^2),x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x)^m,1/(a+c*x^2),x],x] /;
FreeQ[{a,c,d,e,m},x] && NeQ[c*d^2+a*e^2,0] && Not[IntegerQ[m]]
```

7: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge bd + ae = 0 \wedge cd + be = 0 \wedge m - p \in \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: If $bd + ae = 0 \wedge cd + be = 0$, then $\partial_x \frac{(d+ex)^p (a+bx+cx^2)^p}{(ad+ce x^3)^p} = 0$

Rule 1.2.1.2.7: If $bd + ae = 0 \wedge cd + be = 0 \wedge m - p \in \mathbb{Z}$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{(d+ex)^{\text{FracPart}[p]} (a+bx+cx^2)^{\text{FracPart}[p]}}{(ad+ce x^3)^{\text{FracPart}[p]}} \int (d+ex)^{m-p} (ad+ce x^3)^p dx$$

Program code:

```
Int[(d_+e_.x_)^m_*(a_+b_.x_+c_.x_^2)^p_,x_Symbol] :=
  (d+e*x)^FracPart[p]*(a+b*x+c*x^2)^FracPart[p]/(a*d+c*e*x^3)^FracPart[p]*Int[(d+e*x)^(m-p)*(a*d+c*e*x^3)^p,x] /;
FreeQ[{a,b,c,d,e,m,p},x] && EqQ[b*d+a*e,0] && EqQ[c*d+b*e,0] && IGtQ[m-p+1,0] && Not[IntegerQ[p]]
```

$$8. \int \frac{(d+ex)^m}{\sqrt{a+bx+cx^2}} dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m^2 = \frac{1}{4}$$

$$1. \int \frac{(d+ex)^m}{\sqrt{bx+cx^2}} dx \text{ when } cd - be \neq 0 \wedge 2cd - be \neq 0 \wedge m^2 = \frac{1}{4}$$

$$1: \int \frac{(d+ex)^m}{\sqrt{bx+cx^2}} dx \text{ when } cd - be \neq 0 \wedge 2cd - be \neq 0 \wedge m^2 = \frac{1}{4} \wedge c < 0 \wedge b \in \mathbb{R}$$

Derivation: Algebraic expansion

Basis: If $c < 0 \wedge b > 0$, then $\sqrt{bx+cx^2} = \sqrt{-x} \sqrt{b+cx}$

Basis: If $c < 0 \wedge b < 0$, then $\sqrt{bx+cx^2} = \sqrt{-x} \sqrt{-b-cx}$

Basis: If $c < 0 \wedge b \in \mathbb{R}$, then $\sqrt{bx+cx^2} = \sqrt{bx} \sqrt{1 + \frac{cx}{b}}$

Rule 1.2.1.2.8.1.1: If $cd - be \neq 0 \wedge 2cd - be \neq 0 \wedge m^2 = \frac{1}{4} \wedge c < 0 \wedge b \in \mathbb{R}$, then

$$\int \frac{(d+ex)^m}{\sqrt{bx+cx^2}} dx \rightarrow \int \frac{(d+ex)^m}{\sqrt{bx} \sqrt{1 + \frac{cx}{b}}} dx$$

Program code:

```
Int[(d_+e_.x_)^m_/Sqrt[b_.x_+c_.x_^2],x_Symbol] :=
  Int[(d+e*x)^m/(Sqrt[b*x]*Sqrt[1+c/b*x]),x] /;
FreeQ[{b,c,d,e},x] && NeQ[c*d-b*e,0] && NeQ[2*c*d-b*e,0] && EqQ[m^2,1/4] && LtQ[c,0] && RationalQ[b]
```

$$2: \int \frac{(d+ex)^m}{\sqrt{bx+cx^2}} dx \text{ when } cd-be \neq 0 \wedge 2cd-be \neq 0 \wedge m^2 = \frac{1}{4}$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{\sqrt{x} \sqrt{b+cx}}{\sqrt{bx+cx^2}} = 0$$

Rule 1.2.1.2.8.1.2: If $cd-be \neq 0 \wedge 2cd-be \neq 0 \wedge m^2 = \frac{1}{4}$, then

$$\int \frac{(d+ex)^m}{\sqrt{bx+cx^2}} dx \rightarrow \frac{\sqrt{x} \sqrt{b+cx}}{\sqrt{bx+cx^2}} \int \frac{(d+ex)^m}{\sqrt{x} \sqrt{b+cx}} dx$$

Program code:

```
Int[(d_+e_.**x_)^m_/Sqrt[b_.**x_+c_.**x_^2],x_Symbol] :=
  Sqrt[x]*Sqrt[b+c*x]/Sqrt[b*x+c*x^2]*Int[(d+e*x)^m/(Sqrt[x]*Sqrt[b+c*x]),x] /;
FreeQ[{b,c,d,e},x] && NeQ[c*d-b*e,0] && NeQ[2*c*d-b*e,0] && EqQ[m^2,1/4]
```

$$2. \int \frac{(e x)^m}{\sqrt{a+b x+c x^2}} dx \text{ when } b^2 - 4 a c \neq 0 \wedge m^2 = \frac{1}{4}$$

$$1: \int \frac{x^m}{\sqrt{a+b x+c x^2}} dx \text{ when } b^2 - 4 a c \neq 0 \wedge m^2 = \frac{1}{4}$$

Derivation: Integration by substitution

Basis: $x^m F[x] = 2 \operatorname{Subst}[x^{2m+1} F[x^2], x, \sqrt{x}] \partial_x \sqrt{x}$

Rule 1.2.1.2.8.2.1: If $b^2 - 4 a c \neq 0 \wedge m^2 = \frac{1}{4}$, then

$$\int \frac{x^m}{\sqrt{a+b x+c x^2}} dx \rightarrow 2 \operatorname{Subst}\left[\int \frac{x^{2m+1}}{\sqrt{a+b x^2+c x^4}} dx, x, \sqrt{x}\right]$$

Program code:

```
Int[x_^m/Sqrt[a_+b_.*x_+c_.*x_^2],x_Symbol] :=
  2*Subst[Int[x^(2*m+1)/Sqrt[a+b*x^2+c*x^4],x],x,Sqrt[x]] /;
FreeQ[{a,b,c},x] && NeQ[b^2-4*a*c,0] && EqQ[m^2,1/4]
```

$$2: \int \frac{(e x)^m}{\sqrt{a+b x+c x^2}} dx \text{ when } b^2 - 4 a c \neq 0 \wedge m^2 = \frac{1}{4}$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{(e x)^m}{x^m} = 0$$

Rule 1.2.1.2.8.2.2: If $b^2 - 4 a c \neq 0 \wedge m^2 = \frac{1}{4}$, then

$$\int \frac{(e x)^m}{\sqrt{a+b x+c x^2}} dx \rightarrow \frac{(e x)^m}{x^m} \int \frac{x^m}{\sqrt{a+b x+c x^2}} dx$$

```
Int[(e*x_)^m/Sqrt[a+b_*x_+c_*x_^2],x_Symbol] :=
  (e*x)^m/x^m*Int[x^m/Sqrt[a+b*x+c*x^2], x] /;
  FreeQ[{a,b,c,e},x] && NeQ[b^2-4*a*c,0] && EqQ[m^2,1/4]
```

$$3: \int \frac{(d+e x)^m}{\sqrt{a+b x+c x^2}} dx \text{ when } b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge 2 c d - b e \neq 0 \wedge m^2 = \frac{1}{4}$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: } \partial_x \frac{(d+e x)^m}{\sqrt{a+b x+c x^2}} \left(\frac{-\frac{c(a+b x+c x^2)}{b^2-4 a c}}{\frac{2 c(d+e x)}{2 c d-b e-e \sqrt{b^2-4 a c}}} \right)^m = 0$$

$$\text{Basis: } \frac{\left(\frac{2 c(d+e x)}{2 c d-b e-e \sqrt{b^2-4 a c}} \right)^m}{\sqrt{-\frac{c(a+b x+c x^2)}{b^2-4 a c}}} = \frac{2 \sqrt{b^2-4 a c}}{c} \text{Subst} \left[\frac{\left(1 + \frac{2 e \sqrt{b^2-4 a c} x^2}{2 c d-b e-e \sqrt{b^2-4 a c}} \right)^m}{\sqrt{1-x^2}}, x, \sqrt{\frac{b+\sqrt{b^2-4 a c}+2 c x}{2 \sqrt{b^2-4 a c}}} \right] \partial_x \sqrt{\frac{b+\sqrt{b^2-4 a c}+2 c x}{2 \sqrt{b^2-4 a c}}}$$

Rule 1.2.1.2.8.3: If $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge 2 c d - b e \neq 0 \wedge m^2 = \frac{1}{4}$, then

$$\int \frac{(d+e x)^m}{\sqrt{a+b x+c x^2}} dx \rightarrow \frac{(d+e x)^m}{\sqrt{a+b x+c x^2}} \left(\frac{-\frac{c(a+b x+c x^2)}{b^2-4 a c}}{\frac{2 c(d+e x)}{2 c d-b e-e \sqrt{b^2-4 a c}}} \right)^m \int \frac{\left(\frac{2 c(d+e x)}{2 c d-b e-e \sqrt{b^2-4 a c}} \right)^m}{\sqrt{-\frac{c(a+b x+c x^2)}{b^2-4 a c}}} dx$$

$$\rightarrow \frac{2\sqrt{b^2-4ac} (d+ex)^m \sqrt{-\frac{c(a+bx+cx^2)}{b^2-4ac}}}{c\sqrt{a+bx+cx^2} \left(\frac{2c(d+ex)}{2cd-be-e\sqrt{b^2-4ac}}\right)^m} \text{Subst} \left[\int \frac{\left(1 + \frac{2e\sqrt{b^2-4ac}x^2}{2cd-be-e\sqrt{b^2-4ac}}\right)^m}{\sqrt{1-x^2}} dx, x, \sqrt{\frac{b+\sqrt{b^2-4ac}+2cx}{2\sqrt{b^2-4ac}}} \right]$$

Rule 1.2.1.2.8.3: If $cd^2 + ae^2 \neq 0 \wedge m^2 = \frac{1}{4}$, then

$$\int \frac{(d+ex)^m}{\sqrt{a+cx^2}} dx \rightarrow \frac{(d+ex)^m \sqrt{1+\frac{cx^2}{a}}}{\sqrt{a+cx^2} \left(\frac{c(d+ex)}{cd-ae\sqrt{-c/a}}\right)^m} \int \frac{\left(\frac{c(d+ex)}{cd-ae\sqrt{-c/a}}\right)^m}{\sqrt{\frac{a+cx^2}{a}}} dx$$

$$\rightarrow \frac{2a\sqrt{-c/a} (d+ex)^m \sqrt{1+\frac{cx^2}{a}}}{c\sqrt{a+cx^2} \left(\frac{c(d+ex)}{cd-ae\sqrt{-c/a}}\right)^m} \text{Subst} \left[\int \frac{\left(1 + \frac{2ae\sqrt{-c/a}x^2}{cd-ae\sqrt{-c/a}}\right)^m}{\sqrt{1-x^2}} dx, x, \sqrt{\frac{1-\sqrt{-c/a}x}{2}} \right]$$

Program code:

```
Int[(d_+e_.x_)^m_/Sqrt[a_+b_.x_+c_.x_^2],x_Symbol] :=
  2*Rt[b^2-4*a*c,2]*(d+e*x)^m*Sqrt[-c*(a+b*x+c*x^2)/(b^2-4*a*c)]/
  (c*Sqrt[a+b*x+c*x^2]*(2*c*(d+e*x)/(2*c*d-b*e-e*Rt[b^2-4*a*c,2]))^m)*
  Subst[Int[(1+2*e*Rt[b^2-4*a*c,2]*x^2/(2*c*d-b*e-e*Rt[b^2-4*a*c,2]))^m/Sqrt[1-x^2],x],x,
  Sqrt[(b+Rt[b^2-4*a*c,2]+2*c*x)/(2*Rt[b^2-4*a*c,2])]] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && EqQ[m^2,1/4]
```

```
Int[(d_+e_.x_)^m_/Sqrt[a+c_.x_^2],x_Symbol] :=
  2*a*Rt[-c/a,2]*(d+e*x)^m*Sqrt[1+c*x^2/a]/(c*Sqrt[a+c*x^2]*(c*(d+e*x)/(c*d-a*e*Rt[-c/a,2]))^m)*
  Subst[Int[(1+2*a*e*Rt[-c/a,2]*x^2/(c*d-a*e*Rt[-c/a,2]))^m/Sqrt[1-x^2],x],x,Sqrt[(1-Rt[-c/a,2]*x)/2]] /;
FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0] && EqQ[m^2,1/4]
```


$$9. \int (d+ex)^m (a+bx+cx^2)^p dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m+2p+2 = 0$$

$$1: \int (d+ex)^m (a+bx+cx^2)^p dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m+2p+2 = 0 \wedge p > 0$$

Derivation: Quadratic recurrence 2a with $A = d$, $B = e$, $m = m - 1$ and $m + 2p + 2 = 0$ inverted

Rule 1.2.1.2.9.1: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m + 2p + 2 = 0 \wedge p > 0 \wedge p \notin \mathbb{Z}$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow -\frac{(d+ex)^{m+1} (db - 2ae + (2cd - be)x) (a+bx+cx^2)^p}{2(m+1)(cd^2 - bde + ae^2)} + \frac{p(b^2 - 4ac)}{2(m+1)(cd^2 - bde + ae^2)} \int (d+ex)^{m+2} (a+bx+cx^2)^{p-1} dx$$

Program code:

```
Int[(d_+e_.x_)^m_*(a_+b_.x_+c_.x_^2)^p_,x_Symbol] :=
  -(d+e*x)^(m+1)*(d*b-2*a*e+(2*c*d-b*e)*x)*(a+b*x+c*x^2)^p/(2*(m+1)*(c*d^2-b*d*e+a*e^2)) +
  p*(b^2-4*a*c)/(2*(m+1)*(c*d^2-b*d*e+a*e^2))*Int[(d+e*x)^(m+2)*(a+b*x+c*x^2)^(p-1),x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && EqQ[m+2*p+2,0] && GtQ[p,0]
```

```
Int[(d_+e_.x_)^m_*(a_+c_.x_^2)^p_,x_Symbol] :=
  -(d+e*x)^(m+1)*(-2*a*e+(2*c*d)*x)*(a+c*x^2)^p/(2*(m+1)*(c*d^2+a*e^2)) -
  4*a*c*p/(2*(m+1)*(c*d^2+a*e^2))*Int[(d+e*x)^(m+2)*(a+c*x^2)^(p-1),x] /;
FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0] && EqQ[m+2*p+2,0] && GtQ[p,0]
```

$$2: \int (d+ex)^m (a+bx+cx^2)^p dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m+2p+2 = 0 \wedge p < -1$$

Derivation: Quadratic recurrence 2a with $A = d$, $B = e$, $m = m - 1$ and $m + 2p + 2 = 0$

Rule 1.2.1.2.9.2: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m + 2p + 2 = 0 \wedge p < -1$, then

$$\int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow \frac{(d+e x)^{m-1} (d b-2 a e+(2 c d-b e) x) (a+b x+c x^2)^{p+1}}{(p+1) (b^2-4 a c)} - \frac{2 (2 p+3) (c d^2-b d e+a e^2)}{(p+1) (b^2-4 a c)} \int (d+e x)^{m-2} (a+b x+c x^2)^{p+1} dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  (d+e*x)^(m-1)*(d*b-2*a*e+(2*c*d-b*e)*x)*(a+b*x+c*x^2)^(p+1)/((p+1)*(b^2-4*a*c)) -
  2*(2*p+3)*(c*d^2-b*d*e+a*e^2)/((p+1)*(b^2-4*a*c))*Int[(d+e*x)^(m-2)*(a+b*x+c*x^2)^(p+1),x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && EqQ[m+2*p+2,0] && LtQ[p,-1]
```

```
Int[(d_+e_.*x_)^m_*(a_+c_.*x_^2)^p_,x_Symbol] :=
  (d+e*x)^(m-1)*(a*e-c*d*x)*(a+c*x^2)^(p+1)/(2*a*c*(p+1)) +
  (2*p+3)*(c*d^2+a*e^2)/(2*a*c*(p+1))*Int[(d+e*x)^(m-2)*(a+c*x^2)^(p+1),x] /;
FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0] && EqQ[m+2*p+2,0] && LtQ[p,-1]
```

$$3: \int \frac{1}{(d+e x) \sqrt{a+b x+c x^2}} dx \text{ when } b^2 - 4 a c \neq 0 \wedge 2 c d - b e \neq 0$$

Reference: G&R 2.266.1, CRC 258

Reference: G&R 2.266.3, CRC 259

Derivation: Integration by substitution

$$\text{Basis: } \frac{1}{(d+e x) \sqrt{a+b x+c x^2}} = -2 \operatorname{Subst}\left[\frac{1}{4 c d^2 - 4 b d e + 4 a e^2 - x^2}, x, \frac{2 a e - b d - (2 c d - b e) x}{\sqrt{a+b x+c x^2}}\right] \partial_x \frac{2 a e - b d - (2 c d - b e) x}{\sqrt{a+b x+c x^2}}$$

Rule 1.2.1.2.9.3: If $b^2 - 4 a c \neq 0 \wedge 2 c d - b e \neq 0$, then

$$\int \frac{1}{(d+e x) \sqrt{a+b x+c x^2}} dx \rightarrow -2 \operatorname{Subst}\left[\int \frac{1}{4 c d^2 - 4 b d e + 4 a e^2 - x^2} dx, x, \frac{2 a e - b d - (2 c d - b e) x}{\sqrt{a+b x+c x^2}}\right]$$

Program code:

```
Int[1/((d_+e_*x_)*Sqrt[a_+b_*x_+c_*x_^2]),x_Symbol] :=
  -2*Subst[Int[1/(4*c*d^2-4*b*d*e+4*a*e^2-x^2),x],x,(2*a*e-b*d-(2*c*d-b*e)*x)/Sqrt[a+b*x+c*x^2]] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && NeQ[2*c*d-b*e,0]
```

```
Int[1/((d_+e_*x_)*Sqrt[a_+c_*x_^2]),x_Symbol] :=
  -Subst[Int[1/(c*d^2+a*e^2-x^2),x],x,(a*e-c*d*x)/Sqrt[a+c*x^2]] /;
FreeQ[{a,c,d,e},x]
```

$$4: \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge 2 c d - b e \neq 0 \wedge m+2 p+2 = 0$$

Rule 1.2.1.2.9.4: If $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge 2 c d - b e \neq 0 \wedge p \notin \mathbb{Z} \wedge m+2 p+2 = 0$, then

$$\int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow$$

$$- \left(\left(\left(b - \sqrt{b^2 - 4ac} + 2cx \right) (d+ex)^{m+1} (a+bx+cx^2)^p \right) / \right. \\ \left. \left((m+1) \left(2cd - be + e\sqrt{b^2 - 4ac} \right) \left(\frac{(2cd - be + e\sqrt{b^2 - 4ac})(b + \sqrt{b^2 - 4ac} + 2cx)}{(2cd - be - e\sqrt{b^2 - 4ac})(b - \sqrt{b^2 - 4ac} + 2cx)} \right)^p \right) \right) \\ \text{Hypergeometric2F1} \left[m+1, -p, m+2, -\frac{4c\sqrt{b^2 - 4ac}(d+ex)}{(2cd - be - e\sqrt{b^2 - 4ac})(b - \sqrt{b^2 - 4ac} + 2cx)} \right]$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
- (b-Rt[b^2-4*a*c,2]+2*c*x)*(d+e*x)^(m+1)*(a+b*x+c*x^2)^p/
((m+1)*(2*c*d-b*e+e*Rt[b^2-4*a*c,2])*
((2*c*d-b*e+e*Rt[b^2-4*a*c,2])*(b+Rt[b^2-4*a*c,2]+2*c*x)/((2*c*d-b*e-e*Rt[b^2-4*a*c,2])*(b-Rt[b^2-4*a*c,2]+2*c*x)))^p)*
Hypergeometric2F1[m+1,-p,m+2,-4*c*Rt[b^2-4*a*c,2]*(d+e*x)/((2*c*d-b*e-e*Rt[b^2-4*a*c,2])*(b-Rt[b^2-4*a*c,2]+2*c*x))]/;
FreeQ[{a,b,c,d,e,m,p},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && Not[IntegerQ[p]] && EqQ[m+2*p+2,0]
```

```
Int[(d_+e_.*x_)^m_*(a_+c_.*x_^2)^p_,x_Symbol] :=
(Rt[-a*c,2]-c*x)*(d+e*x)^(m+1)*(a+c*x^2)^p/
((m+1)*(c*d+e*Rt[-a*c,2])*(c*d+e*Rt[-a*c,2])*(Rt[-a*c,2]+c*x)/((c*d-e*Rt[-a*c,2])*(-Rt[-a*c,2]+c*x)))^p)*
Hypergeometric2F1[m+1,-p,m+2,2*c*Rt[-a*c,2]*(d+e*x)/((c*d-e*Rt[-a*c,2])*(Rt[-a*c,2]-c*x))]/;
FreeQ[{a,c,d,e,m,p},x] && NeQ[c*d^2+a*e^2,0] && Not[IntegerQ[p]] && EqQ[m+2*p+2,0]
```

10. $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m+2p+3 \neq 0$

1: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m+2p+3 \neq 0 \wedge p < -1$

Derivation: Quadratic recurrence 2a with $A = 1$, $B = 0$ and $m+2p+3 \neq 0$

Rule 1.2.1.2.10.1: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m+2p+3 \neq 0 \wedge p < -1$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow$$

$$\frac{(d+ex)^m (b+2cx) (a+bx+cx^2)^{p+1}}{(p+1) (b^2-4ac)} + \frac{m(2cd-be)}{(p+1) (b^2-4ac)} \int (d+ex)^{m-1} (a+bx+cx^2)^{p+1} dx$$

Program code:

```
Int[(d_+e_.x_)^m_*(a_+b_.x_+c_.x_^2)^p_,x_Symbol] :=
  (d+e*x)^(m+1)*(b+2*c*x)*(a+b*x+c*x^2)^(p+1)/((p+1)*(b^2-4*a*c)) +
  m*(2*c*d-b*e)/((p+1)*(b^2-4*a*c))*Int[(d+e*x)^(m-1)*(a+b*x+c*x^2)^(p+1),x] /;
FreeQ[{a,b,c,d,e,m,p},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && EqQ[m+2*p+3,0] && LtQ[p,-1]
```

```
Int[(d_+e_.x_)^m_*(a_+c_.x_^2)^p_,x_Symbol] :=
  -(d+e*x)^(m+1)*(2*c*x)*(a+c*x^2)^(p+1)/(4*a*c*(p+1)) -
  m*(2*c*d)/(4*a*c*(p+1))*Int[(d+e*x)^(m-1)*(a+c*x^2)^(p+1),x] /;
FreeQ[{a,c,d,e,m,p},x] && NeQ[c*d^2+a*e^2,0] && EqQ[m+2*p+3,0] && LtQ[p,-1]
```

2: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2-4ac \neq 0 \wedge cd^2-bde+ae^2 \neq 0 \wedge 2cd-be \neq 0 \wedge m+2p+3 = 0 \wedge p \neq -1$

Reference: G&R 2.176, CRC 123

Derivation: Quadratic recurrence 3b with $A = 1$, $B = 0$ and $m+2p+3 = 0$

Rule 1.2.1.2.10.2: If $b^2-4ac \neq 0 \wedge cd^2-bde+ae^2 \neq 0 \wedge 2cd-be \neq 0 \wedge m+2p+3 = 0 \wedge p \neq -1$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{e(d+ex)^{m+1} (a+bx+cx^2)^{p+1}}{(m+1)(cd^2-bde+ae^2)} + \frac{(2cd-be)}{2(cd^2-bde+ae^2)} \int (d+ex)^{m+1} (a+bx+cx^2)^p dx$$

Program code:

```
Int[(d_+e_.x_)^m_*(a_+b_.x_+c_.x_^2)^p_,x_Symbol] :=
  e*(d+e*x)^(m+1)*(a+b*x+c*x^2)^(p+1)/((m+1)*(c*d^2-b*d*e+a*e^2)) +
  (2*c*d-b*e)/(2*(c*d^2-b*d*e+a*e^2))*Int[(d+e*x)^(m+1)*(a+b*x+c*x^2)^p,x] /;
FreeQ[{a,b,c,d,e,m,p},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && EqQ[m+2*p+3,0]
```

```

Int[(d+_e*_x_)^m_*(a+_c*_x_^2)^p_,x_Symbol] :=
  e*(d+e*x)^(m+1)*(a+c*x^2)^(p+1)/((m+1)*(c*d^2+a*e^2)) +
  c*d/(c*d^2+a*e^2)*Int[(d+e*x)^(m+1)*(a+c*x^2)^p,x] /;
FreeQ[{a,c,d,e,m,p},x] && NeQ[c*d^2+a*e^2,0] && EqQ[m+2*p+3,0]

```

11. $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge p > 0$

1: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge p > 0 \wedge m < -1 \wedge m+2p+1 \notin \mathbb{Z}^-$

Derivation: Quadratic recurrence 1a with $A = 1$ and $B = 0$

Rule 1.2.1.2.11.1: If

$b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge p > 0 \wedge m < -1 \wedge m+2p+1 \notin \mathbb{Z}^-$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{(d+ex)^{m+1} (a+bx+cx^2)^p}{e(m+1)} - \frac{p}{e(m+1)} \int (d+ex)^{m+1} (b+2cx) (a+bx+cx^2)^{p-1} dx$$

Program code:

```

Int[(d+_e*_x_)^m_*(a+_b*_x+_c*_x_^2)^p_,x_Symbol] :=
  (d+e*x)^(m+1)*(a+b*x+c*x^2)^p/(e*(m+1)) -
  p/(e*(m+1))*Int[(d+e*x)^(m+1)*(b+2*c*x)*(a+b*x+c*x^2)^(p-1),x] /;
FreeQ[{a,b,c,d,e,m},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && GtQ[p,0] &&
  (IntegerQ[p] || LtQ[m,-1]) && NeQ[m,-1] && Not[ILtQ[m+2*p+1,0]] && IntQuadraticQ[a,b,c,d,e,m,p,x]

```

```

Int[(d+_e*_x_)^m_*(a+_c*_x_^2)^p_,x_Symbol] :=
  (d+e*x)^(m+1)*(a+c*x^2)^p/(e*(m+1)) -
  2*c*p/(e*(m+1))*Int[x*(d+e*x)^(m+1)*(a+c*x^2)^(p-1),x] /;
FreeQ[{a,c,d,e,m},x] && NeQ[c*d^2+a*e^2,0] && GtQ[p,0] &&
  (IntegerQ[p] || LtQ[m,-1]) && NeQ[m,-1] && Not[ILtQ[m+2*p+1,0]] && IntQuadraticQ[a,0,c,d,e,m,p,x]

```

2: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge p > 0 \wedge m+2p \notin \mathbb{Z}^-$

Derivation: Quadratic recurrence 1b with $A = 1$ and $B = 0$

Derivation: Quadratic recurrence 1a with $A = d$, $B = e$ and $m = m - 1$

Rule 1.2.1.2.11.2: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge p > 0 \wedge m+2p \notin \mathbb{Z}^-$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{(d+ex)^{m+1} (a+bx+cx^2)^p}{e(m+2p+1)} - \frac{p}{e(m+2p+1)} \int (d+ex)^m (bd - 2ae + (2cd - be)x) (a+bx+cx^2)^{p-1} dx$$

Program code:

```
Int[(d_+e_.x_)^m_*(a_+b_.x_+c_.x_^2)^p_,x_Symbol] :=
  (d+e*x)^(m+1)*(a+b*x+c*x^2)^p/(e*(m+2*p+1)) -
  p/(e*(m+2*p+1))*Int[(d+e*x)^m*Simp[b*d-2*a*e+(2*c*d-b*e)*x,x]*(a+b*x+c*x^2)^(p-1),x] /;
FreeQ[{a,b,c,d,e,m},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && GtQ[p,0] &&
  NeQ[m+2*p+1,0] && (Not[RationalQ[m]] || LtQ[m,1]) && Not[ILtQ[m+2*p,0]] && IntQuadraticQ[a,b,c,d,e,m,p,x]
```

```
Int[(d_+e_.x_)^m_*(a_+c_.x_^2)^p_,x_Symbol] :=
  (d+e*x)^(m+1)*(a+c*x^2)^p/(e*(m+2*p+1)) +
  2*p/(e*(m+2*p+1))*Int[(d+e*x)^m*Simp[a*e-c*d*x,x]*(a+c*x^2)^(p-1),x] /;
FreeQ[{a,c,d,e,m},x] && NeQ[c*d^2+a*e^2,0] && GtQ[p,0] &&
  NeQ[m+2*p+1,0] && (Not[RationalQ[m]] || LtQ[m,1]) && Not[ILtQ[m+2*p,0]] && IntQuadraticQ[a,0,c,d,e,m,p,x]
```

$$12. \int (d+ex)^m (a+bx+cx^2)^p dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge p < -1$$

$$1. \int (d+ex)^m (a+bx+cx^2)^p dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge p < -1 \wedge m > 0$$

$$1: \int (d+ex)^m (a+bx+cx^2)^p dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge p < -1 \wedge 0 < m < 1$$

Derivation: Quadratic recurrence 2a with $A = 1$ and $B = 0$

Derivation: Quadratic recurrence 2b with $A = d$, $B = e$ and $m = m - 1$

Rule 1.2.1.2.12.1.1: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge p < -1 \wedge 0 < m < 1$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{(d+ex)^m (b+2cx) (a+bx+cx^2)^{p+1}}{(p+1) (b^2 - 4ac)} - \frac{1}{(p+1) (b^2 - 4ac)} \int (d+ex)^{m-1} (bem + 2cd(2p+3) + 2ce(m+2p+3)x) (a+bx+cx^2)^{p+1} dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  (d+e*x)^m*(b+2*c*x)*(a+b*x+c*x^2)^(p+1)/((p+1)*(b^2-4*a*c)) -
  1/((p+1)*(b^2-4*a*c))*Int[(d+e*x)^(m-1)*(b*e*m+2*c*d*(2*p+3)+2*c*e*(m+2*p+3)*x)*(a+b*x+c*x^2)^(p+1),x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] &&
  LtQ[p,-1] && GtQ[m,0] && (LtQ[m,1] || ILtQ[m+2*p+3,0] && NeQ[m,2]) && IntQuadraticQ[a,b,c,d,e,m,p,x]
```

```
Int[(d_+e_.*x_)^m_*(a_+c_.*x_^2)^p_,x_Symbol] :=
  -x*(d+e*x)^m*(a+c*x^2)^(p+1)/(2*a*(p+1)) +
  1/(2*a*(p+1))*Int[(d+e*x)^(m-1)*(d*(2*p+3)+e*(m+2*p+3)*x)*(a+c*x^2)^(p+1),x] /;
FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0] &&
  LtQ[p,-1] && GtQ[m,0] && (LtQ[m,1] || ILtQ[m+2*p+3,0] && NeQ[m,2]) && IntQuadraticQ[a,0,c,d,e,m,p,x]
```


2: $\int (d+e x)^m (a+b x+c x^2)^p dx$ when $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge 2 c d - b e \neq 0 \wedge p < -1 \wedge m > 1$

Derivation: Quadratic recurrence 2a with $A = d$, $B = e$ and $m = m - 1$

Rule 1.2.1.2.12.1.2: If $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge 2 c d - b e \neq 0 \wedge p < -1 \wedge m > 1$, then

$$\frac{\int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow \frac{(d+e x)^{m-1} (d b - 2 a e + (2 c d - b e) x) (a+b x+c x^2)^{p+1}}{(p+1) (b^2 - 4 a c)} + \frac{1}{(p+1) (b^2 - 4 a c)} \int (d+e x)^{m-2} (e (2 a e (m-1) + b d (2 p - m + 4)) - 2 c d^2 (2 p + 3) + e (b e - 2 d c) (m + 2 p + 2) x) (a+b x+c x^2)^{p+1} dx$$

Program code:

```
Int[(d_+e_.x_)^m_*(a_+b_.x_+c_.x_^2)^p_,x_Symbol] :=
  (d+e*x)^(m-1)*(d*b-2*a*e+(2*c*d-b*e)*x)*(a+b*x+c*x^2)^(p+1)/((p+1)*(b^2-4*a*c)) +
  1/((p+1)*(b^2-4*a*c))*
  Int[(d+e*x)^(m-2)*
    Simp[e*(2*a*e*(m-1)+b*d*(2*p-m+4))-2*c*d^2*(2*p+3)+e*(b*e-2*d*c)*(m+2*p+2)*x,x]*
    (a+b*x+c*x^2)^(p+1),x] /;
FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && LtQ[p,-1] && GtQ[m,1] && IntQuadraticQ[a,b
```

```
Int[(d_+e_.x_)^m_*(a_+c_.x_^2)^p_,x_Symbol] :=
  (d+e*x)^(m-1)*(a*e-c*d*x)*(a+c*x^2)^(p+1)/(2*a*c*(p+1)) +
  1/((p+1)*(-2*a*c))*
  Int[(d+e*x)^(m-2)*Simp[a*e^2*(m-1)-c*d^2*(2*p+3)-d*c*e*(m+2*p+2)*x,x]*(a+c*x^2)^(p+1),x] /;
FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0] && LtQ[p,-1] && GtQ[m,1] && IntQuadraticQ[a,0,c,d,e,m,p,x]
```

2: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge p < -1$

Derivation: Quadratic recurrence 2b with $A = 1$ and $B = 0$

Rule 1.2.1.2.12.2: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge p < -1$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{(d+ex)^{m+1} (bcd - b^2e + 2ace + c(2cd - be)x) (a+bx+cx^2)^{p+1}}{(p+1)(b^2 - 4ac)(cd^2 - bde + ae^2)} + \frac{1}{(p+1)(b^2 - 4ac)(cd^2 - bde + ae^2)} \cdot \int (d+ex)^m (bcde(2p-m+2) + b^2e^2(m+p+2) - 2c^2d^2(2p+3) - 2ace^2(m+2p+3) - ce(2cd-be)(m+2p+4)x) (a+bx+cx^2)^{p+1} dx$$

Program code:

```
Int[(d_+e_.**x_)^m_*(a_+b_.**x_+c_.**x_^2)^p_,x_Symbol] :=
  (d+e*x)^(m+1)*(b*c*d-b^2*e+2*a*c*e+c*(2*c*d-b*e)*x)*(a+b*x+c*x^2)^(p+1)/((p+1)*(b^2-4*a*c)*(c*d^2-b*d*e+a*e^2)) +
  1/((p+1)*(b^2-4*a*c)*(c*d^2-b*d*e+a*e^2))*
  Int[(d+e*x)^m*
    Simp[b*c*d*e*(2*p-m+2)+b^2*e^2*(m+p+2)-2*c^2*d^2*(2*p+3)-2*a*c*e^2*(m+2*p+3)-c*e*(2*c*d-b*e)*(m+2*p+4)*x,x]*
    (a+b*x+c*x^2)^(p+1),x] /;
FreeQ[{a,b,c,d,e,m},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && LtQ[p,-1] && IntQuadraticQ[a,b,c,d,e,m,p]
```

```
Int[(d_+e_.**x_)^m_*(a_+c_.**x_^2)^p_,x_Symbol] :=
  -(d+e*x)^(m+1)*(a+e*c*d*x)*(a+c*x^2)^(p+1)/(2*a*(p+1)*(c*d^2+a*e^2)) +
  1/(2*a*(p+1)*(c*d^2+a*e^2))*
  Int[(d+e*x)^m*Simp[c*d^2*(2*p+3)+a*e^2*(m+2*p+3)+c*e*d*(m+2*p+4)*x,x]*(a+c*x^2)^(p+1),x] /;
FreeQ[{a,c,d,e,m},x] && NeQ[c*d^2+a*e^2,0] && LtQ[p,-1] && IntQuadraticQ[a,0,c,d,e,m,p,x]
```

13: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m > 1 \wedge m+2p+1 \neq 0$

Reference: G&R 2.160.3, G&R 2.174.1, CRC 119

Derivation: Quadratic recurrence 3a with $A = d$, $B = e$ and $m = m - 1$

Note: G&R 2.174.1 is a special case of G&R 2.160.3.

Rule 1.2.1.2.13: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m > 1 \wedge m+2p+1 \neq 0$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{e (d+ex)^{m-1} (a+bx+cx^2)^{p+1}}{c (m+2p+1)} + \frac{1}{c (m+2p+1)} \int (d+ex)^{m-2} (cd^2 (m+2p+1) - e (ae (m-1) + bd (p+1)) + e (2cd - be) (m+p) x) (a+bx+cx^2)^p dx$$

Program code:

```
Int[(d_+e_.x_)^m_*(a_+b_.x_+c_.x_^2)^p_,x_Symbol] :=
  e*(d+e*x)^(m-1)*(a+b*x+c*x^2)^(p+1)/(c*(m+2*p+1)) +
  1/(c*(m+2*p+1))*
  Int[(d+e*x)^(m-2)*
    Simp[c*d^2*(m+2*p+1)-e*(a*e*(m-1)+b*d*(p+1))+e*(2*c*d-b*e)*(m+p)*x,x]*
    (a+b*x+c*x^2)^p,x] /;
FreeQ[{a,b,c,d,e,m,p},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] &&
If[RationalQ[m], GtQ[m,1], SumSimplerQ[m,-2]] && NeQ[m+2*p+1,0] && IntQuadraticQ[a,b,c,d,e,m,p,x]
```

```
Int[(d_+e_.x_)^m_*(a_+c_.x_^2)^p_,x_Symbol] :=
  e*(d+e*x)^(m-1)*(a+c*x^2)^(p+1)/(c*(m+2*p+1)) +
  1/(c*(m+2*p+1))*
  Int[(d+e*x)^(m-2)*Simp[c*d^2*(m+2*p+1)-a*e^2*(m-1)+2*c*d*e*(m+p)*x,x]*(a+c*x^2)^p,x] /;
FreeQ[{a,c,d,e,m,p},x] && NeQ[c*d^2+a*e^2,0] &&
If[RationalQ[m], GtQ[m,1], SumSimplerQ[m,-2]] && NeQ[m+2*p+1,0] && IntQuadraticQ[a,0,c,d,e,m,p,x]
```

14: $\int (d+ex)^m (a+bx+cx^2)^p dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m < -1$

Reference: G&R 2.176, CRC 123

Derivation: Quadratic recurrence 3b with $A = 1$ and $B = 0$

Rule 1.2.1.2.14: If $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge m < -1$, then

$$\int (d+ex)^m (a+bx+cx^2)^p dx \rightarrow \frac{e (d+ex)^{m+1} (a+bx+cx^2)^{p+1}}{(m+1) (cd^2 - bde + ae^2)} + \frac{1}{(m+1) (cd^2 - bde + ae^2)} \int (d+ex)^{m+1} (cd(m+1) - be(m+p+2) - ce(m+2p+3)x) (a+bx+cx^2)^p dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  e*(d+e*x)^(m+1)*(a+b*x+c*x^2)^(p+1)/((m+1)*(c*d^2-b*d*e+a*e^2)) +
  1/((m+1)*(c*d^2-b*d*e+a*e^2))*
  Int[(d+e*x)^(m+1)*Simp[c*d*(m+1)-b*e*(m+p+2)-c*e*(m+2*p+3)*x,x]*(a+b*x+c*x^2)^p,x] /;
FreeQ[{a,b,c,d,e,m,p},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && NeQ[m,-1] &&
(LtQ[m,-1] && IntQuadraticQ[a,b,c,d,e,m,p,x] || SumSimplerQ[m,1] && IntegerQ[p] || ILtQ[Simplify[m+2*p+3],0])
```

```
Int[(d_+e_.*x_)^m_*(a_+c_.*x_^2)^p_,x_Symbol] :=
  e*(d+e*x)^(m+1)*(a+c*x^2)^(p+1)/((m+1)*(c*d^2+a*e^2)) +
  c/((m+1)*(c*d^2+a*e^2))*
  Int[(d+e*x)^(m+1)*Simp[d*(m+1)-e*(m+2*p+3)*x,x]*(a+c*x^2)^p,x] /;
FreeQ[{a,c,d,e,m,p},x] && NeQ[c*d^2+a*e^2,0] && NeQ[m,-1] &&
(LtQ[m,-1] && IntQuadraticQ[a,0,c,d,e,m,p,x] || SumSimplerQ[m,1] && IntegerQ[p] || ILtQ[Simplify[m+2*p+3],0])
```

15. $\int \frac{(a+bx+cx^2)^p}{d+ex} dx$ when $b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge 4p \in \mathbb{Z}$

$$1. \int \frac{(a + c x^2)^p}{d + e x} dx \text{ when } c d^2 + a e^2 \neq 0 \wedge 4 p \in \mathbb{Z}$$

$$1: \int \frac{1}{(d + e x) (a + c x^2)^{1/4}} dx \text{ when } c d^2 + a e^2 \neq 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{1}{d+e x} = \frac{d}{d^2 - e^2 x^2} - \frac{e x}{d^2 - e^2 x^2}$$

Rule 1.2.1.2.15.1.1: If $c d^2 + a e^2 \neq 0$, then

$$\int \frac{1}{(d + e x) (a + c x^2)^{1/4}} dx \rightarrow d \int \frac{1}{(d^2 - e^2 x^2) (a + c x^2)^{1/4}} dx - e \int \frac{x}{(d^2 - e^2 x^2) (a + c x^2)^{1/4}} dx$$

Program code:

```
Int[1/((d_+e_.*x_)*(a_+c_.*x_^2)^(1/4)),x_Symbol] :=
  d*Int[1/((d^2-e^2*x^2)*(a+c*x^2)^(1/4)),x] - e*Int[x/((d^2-e^2*x^2)*(a+c*x^2)^(1/4)),x] /;
FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0]
```

2: $\int \frac{1}{(d+e x) (a+c x^2)^{3/4}} dx$ when $c d^2 + a e^2 \neq 0$

Derivation: Algebraic expansion

Basis: $\frac{1}{d+e x} = \frac{d}{d^2-e^2 x^2} - \frac{e x}{d^2-e^2 x^2}$

Rule 1.2.1.2.15.1.2: If $c d^2 + a e^2 \neq 0$, then

$$\int \frac{1}{(d+e x) (a+c x^2)^{3/4}} dx \rightarrow d \int \frac{1}{(d^2-e^2 x^2) (a+c x^2)^{3/4}} dx - e \int \frac{x}{(d^2-e^2 x^2) (a+c x^2)^{3/4}} dx$$

Program code:

```
Int[1/((d_+e_.*x_)*(a_+c_.*x_^2)^(3/4)),x_Symbol] :=
  d*Int[1/((d^2-e^2*x^2)*(a+c*x^2)^(3/4)),x] - e*Int[x/((d^2-e^2*x^2)*(a+c*x^2)^(3/4)),x] /;
FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0]
```

$$2. \int \frac{(a+bx+cx^2)^p}{d+ex} dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0 \wedge 4p \in \mathbb{Z}$$

$$1: \int \frac{(a+bx+cx^2)^p}{d+ex} dx \text{ when } 4a - \frac{b^2}{c} > 0 \wedge 4p \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If $4a - \frac{b^2}{c} > 0$, then $(a+bx+cx^2)^p F[x] = \frac{1}{2c \left(-\frac{4c}{b^2-4ac}\right)^p} \text{Subst}\left[\left(1 - \frac{x^2}{b^2-4ac}\right)^p F\left[-\frac{b}{2c} + \frac{x}{2c}\right], x, b+2cx\right] \partial_x (b+2cx)$

Rule 1.2.1.2.15.2.1: If $4a - \frac{b^2}{c} > 0 \wedge 4p \in \mathbb{Z}$, then

$$\int \frac{(a+bx+cx^2)^p}{d+ex} dx \rightarrow \frac{1}{\left(-\frac{4c}{b^2-4ac}\right)^p} \text{Subst}\left[\int \frac{\left(1 - \frac{x^2}{b^2-4ac}\right)^p}{2cd - be + ex} dx, x, b+2cx\right]$$

Program code:

```
Int[(a_.+b_.*x_+c_.*x_^2)^p/(d_.+e_.*x_),x_Symbol] :=
  1/(-4*c/(b^2-4*a*c))^p*Subst[Int[Simp[1-x^2/(b^2-4*a*c),x]^p/Simp[2*c*d-b*e+e*x,x],x],x,b+2*c*x] /;
FreeQ[{a,b,c,d,e,p},x] && GtQ[4*a-b^2/c,0] && IntegerQ[4*p]
```

$$2: \int \frac{(a+b x+c x^2)^p}{d+e x} dx \text{ when } 4a - \frac{b^2}{c} \neq 0 \wedge 4p \in \mathbb{Z}$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{(a+b x+c x^2)^p}{\left(-\frac{c(a+b x+c x^2)}{b^2-4ac}\right)^p} = 0$$

Rule 1.2.1.2.15.2.2: If $4a - \frac{b^2}{c} \neq 0 \wedge 4p \in \mathbb{Z}$, then

$$\int \frac{(a+b x+c x^2)^p}{d+e x} dx \rightarrow \frac{(a+b x+c x^2)^p}{\left(-\frac{c(a+b x+c x^2)}{b^2-4ac}\right)^p} \int \frac{\left(-\frac{ac}{b^2-4ac} - \frac{bcx}{b^2-4ac} - \frac{c^2x^2}{b^2-4ac}\right)^p}{d+e x} dx$$

Program code:

```
Int[(a_.+b_.**x_+c_.**x_^2)^p_/(d_.+e_.**x_),x_Symbol] :=
  (a+b*x+c*x^2)^p/(-c*(a+b*x+c*x^2)/(b^2-4*a*c))^p*
  Int[(-a*c/(b^2-4*a*c)-b*c*x/(b^2-4*a*c)-c^2*x^2/(b^2-4*a*c))^p/(d+e*x),x] /;
FreeQ[{a,b,c,d,e,p},x] && Not[GtQ[4*a-b^2/c,0]] && IntegerQ[4*p]
```


$$16. \int \frac{1}{(d+ex)(a+bx+cx^2)^{1/3}} dx \text{ when } b^2 - 4ac \neq 0 \wedge cd^2 - bde + ae^2 \neq 0 \wedge 2cd - be \neq 0$$

$$1. \int \frac{1}{(d+ex)(a+bx+cx^2)^{1/3}} dx \text{ when } 2cd - be \neq 0 \wedge c^2d^2 - bcde + b^2e^2 - 3ace^2 = 0$$

$$1: \int \frac{1}{(d+ex)(a+bx+cx^2)^{1/3}} dx \text{ when } 2cd - be \neq 0 \wedge c^2d^2 - bcde + b^2e^2 - 3ace^2 = 0 \wedge ce^2(2cd - be) > 0$$

Derived from formula for this class of Goursat pseudo-elliptic integrands contributed by Martin Welz on 19 September 2016

Rule 1.2.1.2.16.1.1: If $2cd - be \neq 0 \wedge c^2d^2 - bcde + b^2e^2 - 3ace^2 = 0 \wedge ce^2(2cd - be) > 0$, let $q \rightarrow (3ce^2(2cd - be))^{1/3}$, then

$$\int \frac{1}{(d+ex)(a+bx+cx^2)^{1/3}} dx \rightarrow$$

$$- \frac{\sqrt{3} ce \operatorname{ArcTan}\left[\frac{1}{\sqrt{3}} + \frac{2(cd-be-cex)}{\sqrt{3} q (a+bx+cx^2)^{1/3}}\right]}{q^2} - \frac{3ce \operatorname{Log}[d+ex]}{2q^2} + \frac{3ce \operatorname{Log}[cd-be-cex-q(a+bx+cx^2)^{1/3}]}{2q^2}$$

Rule 1.2.1.2.16.1.1: If $c d^2 - 3 a e^2 = 0$, let $q \rightarrow \left(\frac{6 c^2 e^2}{d^2} \right)^{1/3}$, then

$$\int \frac{1}{(d+e x) (a+c x^2)^{1/3}} dx \rightarrow$$

$$-\frac{\sqrt{3} c e \operatorname{ArcTan}\left[\frac{1}{\sqrt{3}} + \frac{2 c (d-e x)}{\sqrt{3} d q (a+c x^2)^{1/3}}\right]}{d^2 q^2} - \frac{3 c e \operatorname{Log}[d+e x]}{2 d^2 q^2} + \frac{3 c e \operatorname{Log}[c d - c e x - d q (a+c x^2)^{1/3}]}{2 d^2 q^2}$$

Program code:

```
Int[1/((d_+e_*x_)*(a_+b_*x_+c_*x_^2)^(1/3)),x_Symbol] :=
  With[{q=Rt[3*c*e^2*(2*c*d-b*e),3]},
    -Sqrt[3]*c*e*ArcTan[1/Sqrt[3]+2*(c*d-b*e-c*e*x)/(Sqrt[3]*q*(a+b*x+c*x^2)^(1/3))]/q^2 -
    3*c*e*Log[d+e*x]/(2*q^2) +
    3*c*e*Log[c*d-b*e-c*e*x-q*(a+b*x+c*x^2)^(1/3)]/(2*q^2) ] /;
FreeQ[{a,b,c,d,e},x] && NeQ[2*c*d-b*e,0] && EqQ[c^2*d^2-b*c*d*e+b^2*e^2-3*a*c*e^2,0] && PosQ[c*e^2*(2*c*d-b*e)]
```

```
Int[1/((d_+e_*x_)*(a_+c_*x_^2)^(1/3)),x_Symbol] :=
  With[{q=Rt[6*c^2*e^2/d^2,3]},
    -Sqrt[3]*c*e*ArcTan[1/Sqrt[3]+2*c*(d-e*x)/(Sqrt[3]*d*q*(a+c*x^2)^(1/3))]/(d^2*q^2) -
    3*c*e*Log[d+e*x]/(2*d^2*q^2) +
    3*c*e*Log[c*d-c*e*x-d*q*(a+c*x^2)^(1/3)]/(2*d^2*q^2) ] /;
FreeQ[{a,c,d,e},x] && EqQ[c*d^2-3*a*e^2,0]
```

$$2: \int \frac{1}{(d+e x) (a+b x+c x^2)^{1/3}} dx \text{ when } 2 c d - b e \neq 0 \wedge c^2 d^2 - b c d e + b^2 e^2 - 3 a c e^2 = 0 \wedge c e^2 (2 c d - b e) \neq 0$$

Derived from formula for this class of Goursat pseudo-elliptic integrands contributed by Martin Welz on 19 September 2016

Rule 1.2.1.2.16.1.2: If $2 c d - b e \neq 0 \wedge c^2 d^2 - b c d e + b^2 e^2 - 3 a c e^2 = 0 \wedge c e^2 (2 c d - b e) \neq 0$, let $q \rightarrow \left(-3 c e^2 (2 c d - b e) \right)^{1/3}$, then

$$\int \frac{1}{(d+e x) (a+b x+c x^2)^{1/3}} dx \rightarrow$$

$$-\frac{\sqrt{3} c e \operatorname{ArcTan}\left[\frac{1}{\sqrt{3}} - \frac{2(c d-b e-c e x)}{\sqrt{3} q (a+b x+c x^2)^{1/3}}\right]}{q^2} - \frac{3 c e \operatorname{Log}[d+e x]}{2 q^2} + \frac{3 c e \operatorname{Log}[c d-b e-c e x+q (a+b x+c x^2)^{1/3}]}{2 q^2}$$

Program code:

```
Int[1/((d_+e_.*x_)*(a_+b_.*x_+c_.*x_^2)^(1/3)),x_Symbol] :=
  With[{q=Rt[-3*c*e^2*(2*c*d-b*e),3]},
    -Sqrt[3]*c*e*ArcTan[1/Sqrt[3]-2*(c*d-b*e-c*e*x)/(Sqrt[3]*q*(a+b*x+c*x^2)^(1/3)]/q^2 -
    3*c*e*Log[d+e*x]/(2*q^2) +
    3*c*e*Log[c*d-b*e-c*e*x+q*(a+b*x+c*x^2)^(1/3)]/(2*q^2)] /;
FreeQ[{a,b,c,d,e},x] && NeQ[2*c*d-b*e,0] && EqQ[c^2*d^2-b*c*d*e+b^2*e^2-3*a*c*e^2,0] && NegQ[c*e^2*(2*c*d-b*e)]
```

```
(* Int[1/((d_+e_.*x_)*(a_+c_.*x_^2)^(1/3)),x_Symbol] :=
  With[{q=Rt[-6*c^2*d*e^2,3]},
    -Sqrt[3]*c*e*ArcTan[1/Sqrt[3]-2*(c*d-c*e*x)/(Sqrt[3]*q*(a+c*x^2)^(1/3)]/q^2 -
    3*c*e*Log[d+e*x]/(2*q^2) +
    3*c*e*Log[c*d-c*e*x+q*(a+c*x^2)^(1/3)]/(2*q^2)] /;
FreeQ[{a,c,d,e},x] && EqQ[c*d^2-3*a*e^2,0] && NegQ[c^2*d*e^2] *)
```

2. $\int \frac{1}{(d+e x) (a+b x+c x^2)^{1/3}} dx$ when $b^2 - 4 a c \neq 0 \wedge c^2 d^2 - b c d e - 2 b^2 e^2 + 9 a c e^2 = 0$

1. $\int \frac{1}{(d+e x) (a+c x^2)^{1/3}} dx$ when $c d^2 + 9 a e^2 = 0$

1: $\int \frac{1}{(d+e x) (a+c x^2)^{1/3}} dx$ when $c d^2 + 9 a e^2 = 0 \wedge a > 0$

Derivation: Algebraic expansion

Basis: If $c d^2 + 9 a e^2 = 0 \wedge a > 0$, then $(a+c x^2)^{1/3} = a^{1/3} \left(1 - \frac{3 e x}{d}\right)^{1/3} \left(1 + \frac{3 e x}{d}\right)^{1/3}$

Rule 1.2.1.2.16.2.1.1: If $c d^2 + 9 a e^2 = 0 \wedge a > 0$, then

$$\int \frac{1}{(d+e x) (a+c x^2)^{1/3}} dx \rightarrow a^{1/3} \int \frac{1}{(d+e x) \left(1 - \frac{3ex}{d}\right)^{1/3} \left(1 + \frac{3ex}{d}\right)^{1/3}} dx$$

Program code:

```
Int[1/((d_+e_.*x_)*(a_+c_.*x_^2)^(1/3)),x_Symbol] :=
  a^(1/3)*Int[1/((d+e*x)*(1-3*e*x/d)^(1/3)*(1+3*e*x/d)^(1/3)),x] /;
FreeQ[{a,c,d,e},x] && EqQ[c*d^2+9*a*e^2,0] && GtQ[a,0]
```

2: $\int \frac{1}{(d+e x) (a+c x^2)^{1/3}} dx$ when $c d^2 + 9 a e^2 == 0 \wedge a \neq 0$

Derivation: Piecewise constant extraction

-

Basis: $\partial_x \frac{\left(1 + \frac{c x^2}{a}\right)^{1/3}}{(a+c x^2)^{1/3}} == 0$

Rule 1.2.1.2.16.2.1.2: If $c d^2 + 9 a e^2 == 0 \wedge a \neq 0$, then

$$\int \frac{1}{(d+e x) (a+c x^2)^{1/3}} dx \rightarrow \frac{\left(1 + \frac{c x^2}{a}\right)^{1/3}}{(a+c x^2)^{1/3}} \int \frac{1}{(d+e x) \left(1 + \frac{c x^2}{a}\right)^{1/3}} dx$$

Program code:

```
Int[1/((d_+e_.*x_)*(a_+c_.*x_^2)^(1/3)),x_Symbol] :=
  (1+c*x^2/a)^(1/3)/(a+c*x^2)^(1/3)*Int[1/((d+e*x)*(1+c*x^2/a)^(1/3)),x] /;
FreeQ[{a,c,d,e},x] && EqQ[c*d^2+9*a*e^2,0] && Not[GtQ[a,0]]
```

2: $\int \frac{1}{(d+e x) (a+b x+c x^2)^{1/3}} dx$ when $b^2 - 4 a c \neq 0 \wedge c^2 d^2 - b c d e - 2 b^2 e^2 + 9 a c e^2 = 0$

Derivation: Piecewise constant extraction

■ Basis: Let $q \rightarrow \sqrt{b^2 - 4 a c}$, then $\partial_x \frac{(b+q+2 c x)^{1/3} (b-q+2 c x)^{1/3}}{(a+b x+c x^2)^{1/3}} = 0$

■ Rule 1.2.1.2.16.2.2: If $b^2 - 4 a c \neq 0 \wedge c^2 d^2 - b c d e - 2 b^2 e^2 + 9 a c e^2 = 0$, let $q \rightarrow \sqrt{b^2 - 4 a c}$, then

$$\int \frac{1}{(d+e x) (a+b x+c x^2)^{1/3}} dx \rightarrow \frac{(b+q+2 c x)^{1/3} (b-q+2 c x)^{1/3}}{(a+b x+c x^2)^{1/3}} \int \frac{1}{(d+e x) (b+q+2 c x)^{1/3} (b-q+2 c x)^{1/3}} dx$$

Program code:

```
Int[1/((d_+e_.*x_)*(a_+b_.*x_+c_.*x_^2)^(1/3)),x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    (b+q+2*c*x)^(1/3)*(b-q+2*c*x)^(1/3)/(a+b*x+c*x^2)^(1/3)*Int[1/((d+e*x)*(b+q+2*c*x)^(1/3)*(b-q+2*c*x)^(1/3)),x] /;
    FreeQ[{a,b,c,d,e},x] && NeQ[b^2-4*a*c,0] && EqQ[c^2*d^2-b*c*d*e-2*b^2*e^2+9*a*c*e^2,0]
```

17: $\int (d+e x)^m (a+c x^2)^p dx$ when $c d^2 + a e^2 \neq 0 \wedge p \notin \mathbb{Z} \wedge a > 0 \wedge c < 0$

Derivation: Algebraic expansion

Basis: If $a > 0$, then $(a+c x^2)^p = (\sqrt{a} + \sqrt{-c} x)^p (\sqrt{a} - \sqrt{-c} x)^p$

Rule 1.2.1.2.17: If $c d^2 + a e^2 \neq 0 \wedge p \notin \mathbb{Z} \wedge a > 0 \wedge c < 0$, then

$$\int (d+e x)^m (a+c x^2)^p dx \rightarrow \int (d+e x)^m (\sqrt{a} + \sqrt{-c} x)^p (\sqrt{a} - \sqrt{-c} x)^p dx$$

Program code:

```
Int[(d+_e_.*x_)^m_*(a+_c_.*x_^2)^p_,x_Symbol] :=
  Int[(d+e*x)^m*(Rt[a,2]+Rt[-c,2]*x)^p*(Rt[a,2]-Rt[-c,2]*x)^p,x] /;
FreeQ[{a,c,d,e,m,p},x] && NeQ[c*d^2+a*e^2,0] && Not[IntegerQ[p]] && GtQ[a,0] && LtQ[c,0]
```

$$19. \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2-4 a c \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge 2 c d-b e \neq 0 \wedge p \notin \mathbb{Z}$$

$$1. \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2-4 a c \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge 2 c d-b e \neq 0 \wedge p \notin \mathbb{Z} \wedge m \in \mathbb{Z}^-$$

$$\mathbf{1:} \int (d+e x)^m (a+c x^2)^p dx \text{ when } c d^2+a e^2 \neq 0 \wedge p \notin \mathbb{Z} \wedge m \in \mathbb{Z}^-$$

Derivation: Algebraic expansion

Basis: If $m \in \mathbb{Z}$, then $(d+e x)^m = \left(\frac{d}{d^2-e^2 x^2} - \frac{e x}{d^2-e^2 x^2} \right)^{-m}$

Note: Resulting integrands are of the form $x^m (a+b x^2)^p (c+d x^2)^q$ which are integrable in terms of the Appell hypergeometric function .

Rule 1.2.1.2.18: If $c d^2+a e^2 \neq 0 \wedge p \notin \mathbb{Z} \wedge m \in \mathbb{Z}^-$, then

$$\int (d+e x)^m (a+c x^2)^p dx \rightarrow \int (a+c x^2)^p \text{ExpandIntegrand} \left[\left(\frac{d}{d^2-e^2 x^2} - \frac{e x}{d^2-e^2 x^2} \right)^{-m}, x \right] dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+c_.*x_^2)^p_,x_Symbol] :=
  Int[ExpandIntegrand[(a+c*x^2)^p,(d/(d^2-e^2*x^2)-e*x/(d^2-e^2*x^2))^(-m),x],x] /;
FreeQ[{a,c,d,e,p},x] && NeQ[c*d^2+a*e^2,0] && Not[IntegerQ[p]] && ILtQ[m,0]
```

$$\mathbf{2:} \int (d+e x)^m (a+b x+c x^2)^p dx \text{ when } b^2-4 a c \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge 2 c d-b e \neq 0 \wedge p \notin \mathbb{Z} \wedge m \in \mathbb{Z}^-$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: Let } q \rightarrow \sqrt{b^2-4 a c}, \text{ then } \partial_x \frac{\left(\frac{1}{d+e x}\right)^{2 p}(a+b x+c x^2)^p}{\left(\frac{e(b-q+2 c x)}{c(d+e x)}\right)^p\left(\frac{e(b+q+2 c x)}{c(d+e x)}\right)^p} = 0$$

$$\text{Basis: } F[x] = -\frac{1}{e} \text{Subst}\left[\frac{F\left[\frac{1-d x}{e x}\right]}{x^2}, x, \frac{1}{d+e x}\right] \partial_x \frac{1}{d+e x}$$

Rule 1.2.1.2.19.1: If $b^2-4 a c \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge 2 c d-b e \neq 0 \wedge p \notin \mathbb{Z} \wedge m \in \mathbb{Z}^-$, let $q \rightarrow \sqrt{b^2-4 a c}$, then

$$\begin{aligned} & \int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow \\ & \frac{\left(\frac{1}{d+e x}\right)^{2 p}(a+b x+c x^2)^p}{\left(\frac{e(b-q+2 c x)}{c(d+e x)}\right)^p\left(\frac{e(b+q+2 c x)}{c(d+e x)}\right)^p} \int \frac{\left(\frac{e(b-q+2 c x)}{c(d+e x)}\right)^p\left(\frac{e(b+q+2 c x)}{c(d+e x)}\right)^p}{\left(\frac{1}{d+e x}\right)^{m+2 p}} dx \rightarrow \\ & -\frac{\left(\frac{1}{d+e x}\right)^{2 p}(a+b x+c x^2)^p}{e\left(\frac{e(b-q+2 c x)}{2 c(d+e x)}\right)^p\left(\frac{e(b+q+2 c x)}{2 c(d+e x)}\right)^p} \text{Subst}\left[\int x^{-m-2(p+1)}\left(1-\left(d-\frac{e(b-q)}{2 c}\right) x\right)^p\left(1-\left(d-\frac{e(b+q)}{2 c}\right) x\right)^p dx, x, \frac{1}{d+e x}\right] \end{aligned}$$

Program code:

```
Int[(d_+e_.x_)^m_*(a_+b_.x_+c_.x_^2)^p_,x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    -(1/(d+e*x))^(2*p)*(a+b*x+c*x^2)^p/(e*(e*(b-q+2*c*x)/(2*c*(d+e*x)))^p*(e*(b+q+2*c*x)/(2*c*(d+e*x)))^p)*
    Subst[Int[x^(-m-2*(p+1))*Simp[1-(d-e*(b-q)/(2*c))*x,x]^p*Simp[1-(d-e*(b+q)/(2*c))*x,x]^p,x],x,1/(d+e*x)] /;
    FreeQ[{a,b,c,d,e,p},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && Not[IntegerQ[p]] && ILtQ[m,0]
```


2: $\int (d+e x)^m (a+b x+c x^2)^p dx$ when $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge 2 c d - b e \neq 0 \wedge p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

Basis: Let $q \rightarrow \sqrt{b^2 - 4 a c}$, then $\partial_x \frac{(a+b x+c x^2)^p}{\left(1 - \frac{d+e x}{d - \frac{e(b-q)}{2c}}\right)^p \left(1 - \frac{d+e x}{d - \frac{e(b+q)}{2c}}\right)^p} = 0$

Note: If $c d^2 - b d e + a e^2 \neq 0$ and $q = \sqrt{b^2 - 4 a c}$, then $d - \frac{e(b-q)}{2c} \neq 0$ and $d - \frac{e(b+q)}{2c} \neq 0$.

■ Rule 1.2.1.2.19.2: If $b^2 - 4 a c \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge 2 c d - b e \neq 0 \wedge p \notin \mathbb{Z}$, let $q \rightarrow \sqrt{b^2 - 4 a c}$, then

$$\int (d+e x)^m (a+b x+c x^2)^p dx \rightarrow$$

$$\frac{(a+b x+c x^2)^p}{\left(1 - \frac{d+e x}{d - \frac{e(b-q)}{2c}}\right)^p \left(1 - \frac{d+e x}{d - \frac{e(b+q)}{2c}}\right)^p} \int (d+e x)^m \left(1 - \frac{d+e x}{d - \frac{e(b-q)}{2c}}\right)^p \left(1 - \frac{d+e x}{d - \frac{e(b+q)}{2c}}\right)^p dx \rightarrow$$

$$\frac{(a+b x+c x^2)^p}{e \left(1 - \frac{d+e x}{d - \frac{e(b-q)}{2c}}\right)^p \left(1 - \frac{d+e x}{d - \frac{e(b+q)}{2c}}\right)^p} \text{Subst} \left[\int x^m \left(1 - \frac{x}{d - \frac{e(b-q)}{2c}}\right)^p \left(1 - \frac{x}{d - \frac{e(b+q)}{2c}}\right)^p dx, x, d+e x \right]$$

Program code:

```
Int[(d_+e_.*x_)^m_*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    (a+b*x+c*x^2)^p/(e*(1-(d+e*x)/(d-e*(b-q)/(2*c)) )^p*(1-(d+e*x)/(d-e*(b+q)/(2*c)) )^p)*
    Subst[Int[x^m*Simp[1-x/(d-e*(b-q)/(2*c)),x]^p*Simp[1-x/(d-e*(b+q)/(2*c)),x]^p,x],x,d+e*x] /;
  FreeQ[{a,b,c,d,e,m,p},x] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && NeQ[2*c*d-b*e,0] && Not[IntegerQ[p]]
```

```

Int[(d_+e_.*x_)^m_.*(a_+c_.*x_^2)^p_,x_Symbol] :=
  With[{q=Rt[-a*c,2]},
    (a+c*x^2)^p/(e*(1-(d+e*x)/(d+e*q/c))^p*(1-(d+e*x)/(d-e*q/c))^p)*
    Subst[Int[x^m*Simp[1-x/(d+e*q/c),x]^p*Simp[1-x/(d-e*q/c),x]^p,x],x,d+e*x]] /;
  FreeQ[{a,c,d,e,m,p},x] && NeQ[c*d^2+a*e^2,0] && Not[IntegerQ[p]]

```

S: $\int (d + e u)^m (a + b u + c u^2)^p dx$ when $u = f + g x$

Derivation: Integration by substitution

Rule 1.2.1.2.S: If $u = f + g x$, then

$$\int (d + e u)^m (a + b u + c u^2)^p dx \rightarrow \frac{1}{g} \text{Subst}\left[\int (d + e x)^m (a + b x + c x^2)^p dx, x, u\right]$$

Program code:

```

Int[(d_+e_.*u_)^m_.*(a_+b_.*u_+c_.*u_^2)^p_,x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(d+e*x)^m_.*(a+b*x+c*x^2)^p,x],x,u] /;
  FreeQ[{a,b,c,d,e,m,p},x] && LinearQ[u,x] && NeQ[u,x]

```

```

Int[(d_+e_.*u_)^m_.*(a_+c_.*u_^2)^p_,x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(d+e*x)^m_.*(a+c*x^2)^p,x],x,u] /;
  FreeQ[{a,c,d,e,m,p},x] && LinearQ[u,x] && NeQ[u,x]

```

```

(* IntQuadraticQ[a,b,c,d,e,m,p,x] returns True iff (d+e*x)^m_.*(a+b*x+c*x^2)^p is integrable wrt x in terms of non-Appell functions. *)
IntQuadraticQ[a_,b_,c_,d_,e_,m_,p_,x_] :=
  IntegerQ[p] || IGtQ[m,0] || IntegersQ[2*m,2*p] || IntegersQ[m,4*p] ||
  IntegersQ[m,p+1/3] && (EqQ[c^2*d^2-b*c*d*e+b^2*e^2-3*a*c*e^2,0] || EqQ[c^2*d^2-b*c*d*e-2*b^2*e^2+9*a*c*e^2,0])

```