

Rules for integrands of the form $(c x)^m (a x^j + b x^n)^p$

1: $\int x^m (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{j}{n} \in \mathbb{Z} \wedge m - n + 1 \neq 0$

Derivation: Integration by substitution

Basis: $x^{n-1} F[x^n] = \frac{1}{n} \text{Subst}[F[x], x, x^n] \partial_x x^n$

Rule: If $p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{j}{n} \in \mathbb{Z} \wedge m - n + 1 \neq 0$, then

$$\int x^m (a x^j + b x^n)^p dx \rightarrow \int x^m (a (x^n)^{j/n} + b x^n)^p dx \rightarrow \frac{1}{n} \text{Subst}\left[\int (a x^{j/n} + b x)^p dx, x, x^n\right]$$

Program code:

```
Int[x_^m_.*(a_.*x_^j_.+b_.*x_^n_)^p_,x_Symbol] :=
  1/n*Subst[Int[(a*x^Simplify[j/n]+b*x)^p,x],x,x^n] /;
FreeQ[{a,b,j,m,n,p},x] && Not[IntegerQ[p]] && NeQ[n,j] && IntegerQ[Simplify[j/n]] && EqQ[Simplify[m-n+1],0]
```

2: $\int (c x)^m (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge j \neq n \wedge m + n p + n - j + 1 \neq 0 \wedge (j \in \mathbb{Z} \vee c > 0)$

Derivation: Generalized binomial recurrence 2a

Rule: If $p \notin \mathbb{Z} \wedge j \neq n \wedge m + n p + n - j + 1 \neq 0 \wedge (j \in \mathbb{Z} \vee c > 0)$, then

$$\int (c x)^m (a x^j + b x^n)^p dx \rightarrow - \frac{c^{j-1} (c x)^{m-j+1} (a x^j + b x^n)^{p+1}}{a (n-j) (p+1)}$$

Program code:

```
Int[(c_.*x_)^m_.*(a_.*x_^j_.+b_.*x_^n_)^p_,x_Symbol] :=
  -c^(j-1)*(c*x)^(m-j+1)*(a*x^j+b*x^n)^(p+1)/(a*(n-j)*(p+1)) /;
FreeQ[{a,b,c,j,m,n,p},x] && Not[IntegerQ[p]] && NeQ[n,j] && EqQ[m+n*p+n-j+1,0] && (IntegerQ[j] || GtQ[c,0])
```

$$3. \int (c x)^m (a x^j + b x^n)^p dx \text{ when } p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{m+n p+n-j+1}{n-j} \in \mathbb{Z}^-$$

$$1: \int (c x)^m (a x^j + b x^n)^p dx \text{ when } p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{m+n p+n-j+1}{n-j} \in \mathbb{Z}^- \wedge p < -1 \wedge (j \in \mathbb{Z} \vee c > 0)$$

Derivation: Generalized binomial recurrence 2b

Note: This rule increments $\frac{m+n p+n-j+1}{n-j}$ by 1 thus driving it to 0.

Rule: If $p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{m+n p+n-j+1}{n-j} \in \mathbb{Z}^- \wedge p < -1 \wedge (j \in \mathbb{Z} \vee c > 0)$, then

$$\int (c x)^m (a x^j + b x^n)^p dx \rightarrow -\frac{c^{j-1} (c x)^{m-j+1} (a x^j + b x^n)^{p+1}}{a (n-j) (p+1)} + \frac{c^j (m+n p+n-j+1)}{a (n-j) (p+1)} \int (c x)^{m-j} (a x^j + b x^n)^{p+1} dx$$

Program code:

```
Int[(c_.**x_)^m_.*(a_.**x_^j_.+b_.**x_^n_.)^p_,x_Symbol] :=
  -c^(j-1)*(c*x)^(m-j+1)*(a*x^j+b*x^n)^(p+1)/(a*(n-j)*(p+1)) +
  c^j*(m+n*p+n-j+1)/(a*(n-j)*(p+1))*Int[(c*x)^(m-j)*(a*x^j+b*x^n)^(p+1),x] /;
FreeQ[{a,b,c,j,m,n},x] && Not[IntegerQ[p]] && NeQ[n,j] && ILtQ[Simplify[(m+n*p+n-j+1)/(n-j)],0] && LtQ[p,-1] && (IntegerQ[j] || GtQ[
```

2: $\int (c x)^m (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{m+n p+n-j+1}{n-j} \in \mathbb{Z}^- \wedge m+j p+1 \neq 0 \wedge ((j | n) \in \mathbb{Z} \vee c > 0)$

Derivation: Generalized binomial recurrence 3b

Note: This rule increments $\frac{m+n p+n-j+1}{n-j}$ by 1 thus driving it to 0.

Rule: If $p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{m+n p+n-j+1}{n-j} \in \mathbb{Z}^- \wedge m+j p+1 \neq 0 \wedge ((j | n) \in \mathbb{Z} \vee c > 0)$, then

$$\int (c x)^m (a x^j + b x^n)^p dx \rightarrow \frac{c^{j-1} (c x)^{m-j+1} (a x^j + b x^n)^{p+1}}{a (m+j p+1)} - \frac{b (m+n p+n-j+1)}{a c^{n-j} (m+j p+1)} \int (c x)^{m+n-j} (a x^j + b x^n)^p dx$$

Program code:

```
Int[(c_.x_)^m_.*(a_.x_^j_.+b_.x_^n_.)^p_,x_Symbol] :=
  c^(j-1)*(c*x)^(m-j+1)*(a*x^j+b*x^n)^(p+1)/(a*(m+j*p+1)) -
  b*(m+n*p+n-j+1)/(a*c^(n-j)*(m+j*p+1))*Int[(c*x)^(m+n-j)*(a*x^j+b*x^n)^p,x] /;
FreeQ[{a,b,c,j,m,n,p},x] && Not[IntegerQ[p]] && NeQ[n,j] && ILtQ[Simplify[(m+n*p+n-j+1)/(n-j)],0] && NeQ[m+j*p+1,0] && (IntegersQ[j,
```

$$\mathbf{3:} \int (c x)^m (a x^j + b x^n)^p dx \text{ when } p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{m+n p+n-j+1}{n-j} \in \mathbb{Z}^- \wedge c \neq 0$$

Derivation: Piecewise constant extraction

$$\text{Basis: } a_x \frac{(c x)^m}{x^m} == 0$$

$$\text{Basis: } \frac{(c x)^m}{x^m} == \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}}$$

Rule: If $p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{m+n p+n-j+1}{n-j} \in \mathbb{Z}^- \wedge c \neq 0$, then

$$\int (c x)^m (a x^j + b x^n)^p dx \rightarrow \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (a x^j + b x^n)^p dx$$

Program code:

```
Int[(c*x_)^m_.*(a_.*x_^j_.+b_.*x_^n_.)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a*x^j+b*x^n)^p,x] /;
FreeQ[{a,b,c,j,m,n,p},x] && Not[IntegerQ[p]] && NeQ[n,j] && ILtQ[Simplify[(m+n*p+n-j+1)/(n-j)],0]
```

$$4. \int (c x)^m (a x^j + b x^n)^p dx \text{ when } p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{j}{n} \in \mathbb{Z} \wedge \frac{m+1}{n} \in \mathbb{Z} \wedge n^2 \neq 1$$

$$1: \int x^m (a x^j + b x^n)^p dx \text{ when } p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{j}{n} \in \mathbb{Z} \wedge \frac{m+1}{n} \in \mathbb{Z} \wedge n^2 \neq 1$$

Derivation: Integration by substitution

Basis: If $\frac{m+1}{n} \in \mathbb{Z}$, then $x^m F[x^n] = \frac{1}{n} \text{Subst}\left[x^{\frac{m+1}{n}-1} F[x], x, x^n\right] \partial_x x^n$

Note: If $n \in \mathbb{Z} \wedge \frac{m+1}{n} \in \mathbb{Z}$, then $m \in \mathbb{Z}$, and $(c x)^m$ automatically evaluates to $c^m x^m$.

Rule: If $p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{j}{n} \in \mathbb{Z} \wedge \frac{m+1}{n} \in \mathbb{Z} \wedge n^2 \neq 1$, then

$$\int x^m (a x^j + b x^n)^p dx \rightarrow \frac{1}{n} \text{Subst}\left[\int x^{\frac{m+1}{n}-1} (a x^{j/n} + b x)^p dx, x, x^n\right]$$

Program code:

```
Int[x^m.*(a.*x^j_.+b.*x^n_)^p_,x_Symbol] :=
  1/n*Subst[Int[x^(Simplify[(m+1)/n]-1)*(a*x^Simplify[j/n]+b*x)^p_,x],x,x^n] /;
FreeQ[{a,b,j,m,n,p},x] && Not[IntegerQ[p]] && NeQ[n,j] && IntegerQ[Simplify[j/n]] && IntegerQ[Simplify[(m+1)/n]] && NeQ[n^2,1]
```

2: $\int (c x)^m (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{j}{n} \in \mathbb{Z} \wedge \frac{m+1}{n} \in \mathbb{Z} \wedge n^2 \neq 1$

Derivation: Piecewise constant extraction

Basis: $a_x \frac{(c x)^m}{x^m} == 0$

Basis: $\frac{(c x)^m}{x^m} == \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}}$

Rule: If $p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{j}{n} \in \mathbb{Z} \wedge \frac{m+1}{n} \in \mathbb{Z} \wedge n^2 \neq 1$, then

$$\int (c x)^m (a x^j + b x^n)^p dx \rightarrow \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (a x^j + b x^n)^p dx$$

Program code:

```
Int[(c*x_)^m_.*(a_.*x_^j_.+b_.*x_^n_)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a*x^j+b*x^n)^p,x] /;
FreeQ[{a,b,c,j,m,n,p},x] && Not[IntegerQ[p]] && NeQ[n,j] && IntegerQ[Simplify[j/n]] && IntegerQ[Simplify[(m+1)/n]] && NeQ[n^2,1]
```

$$5. \int (c x)^m (a x^j + b x^n)^p dx \text{ when } p \notin \mathbb{Z} \wedge 0 < j < n \wedge ((j | n) \in \mathbb{Z} \vee c > 0)$$

$$1. \int (c x)^m (a x^j + b x^n)^p dx \text{ when } p \notin \mathbb{Z} \wedge 0 < j < n \wedge ((j | n) \in \mathbb{Z} \vee c > 0) \wedge p > 0$$

$$\mathbf{1:} \int (c x)^m (a x^j + b x^n)^p dx \text{ when } p \notin \mathbb{Z} \wedge 0 < j < n \wedge ((j | n) \in \mathbb{Z} \vee c > 0) \wedge p > 0 \wedge m + j p + 1 < 0$$

Derivation: Generalized binomial recurrence 1a

Rule: If $p \notin \mathbb{Z} \wedge 0 < j < n \wedge ((j | n) \in \mathbb{Z} \vee c > 0) \wedge p > 0 \wedge m + j p + 1 < 0$, then

$$\int (c x)^m (a x^j + b x^n)^p dx \rightarrow \frac{(c x)^{m+1} (a x^j + b x^n)^p}{c (m + j p + 1)} - \frac{b p (n - j)}{c^n (m + j p + 1)} \int (c x)^{m+n} (a x^j + b x^n)^{p-1} dx$$

Program code:

```
Int[(c_.**x_)^m_*(a_.**x_^j_.+b_.**x_^n_.)^p_,x_Symbol] :=
  (c*x)^(m+1)*(a*x^j+b*x^n)^p/(c*(m+j*p+1)) -
  b*p*(n-j)/(c^n*(m+j*p+1))*Int[(c*x)^(m+n)*(a*x^j+b*x^n)^(p-1),x] /;
FreeQ[{a,b,c},x] && Not[IntegerQ[p]] && LtQ[0,j,n] && (IntegersQ[j,n] || GtQ[c,0]) && GtQ[p,0] && LtQ[m+j*p+1,0]
```

$$\mathbf{2:} \int (c x)^m (a x^j + b x^n)^p dx \text{ when } p \notin \mathbb{Z} \wedge 0 < j < n \wedge ((j | n) \in \mathbb{Z} \vee c > 0) \wedge p > 0 \wedge m + n p + 1 \neq 0$$

Derivation: Generalized binomial recurrence 1b

Rule: If $p \notin \mathbb{Z} \wedge 0 < j < n \wedge ((j | n) \in \mathbb{Z} \vee c > 0) \wedge p > 0 \wedge m + n p + 1 \neq 0$, then

$$\int (c x)^m (a x^j + b x^n)^p dx \rightarrow \frac{(c x)^{m+1} (a x^j + b x^n)^p}{c (m + n p + 1)} + \frac{a p (n - j)}{c^j (m + n p + 1)} \int (c x)^{m+j} (a x^j + b x^n)^{p-1} dx$$

Program code:

```
Int[(c_.**x_)^m_.*(a_.**x_^j_.+b_.**x_^n_.)^p_,x_Symbol] :=
  (c*x)^(m+1)*(a*x^j+b*x^n)^p/(c*(m+n*p+1)) +
  a*(n-j)*p/(c^j*(m+n*p+1))*Int[(c*x)^(m+j)*(a*x^j+b*x^n)^(p-1),x] /;
FreeQ[{a,b,c,m},x] && Not[IntegerQ[p]] && LtQ[0,j,n] && (IntegersQ[j,n] || GtQ[c,0]) && GtQ[p,0] && NeQ[m+n*p+1,0]
```


$$2. \int (c x)^m (a x^j + b x^n)^p dx \text{ when } p \notin \mathbb{Z} \wedge 0 < j < n \wedge ((j | n) \in \mathbb{Z} \vee c > 0) \wedge p < -1$$

$$1: \int (c x)^m (a x^j + b x^n)^p dx \text{ when } p \notin \mathbb{Z} \wedge 0 < j < n \wedge ((j | n) \in \mathbb{Z} \vee c > 0) \wedge p < -1 \wedge m + j p + 1 > n - j$$

Derivation: Generalized binomial recurrence 2a

Note: If $\frac{m+n p+n-j+1}{n-j} \in \mathbb{Z}^-$ following rule is used to drive $m + n p + n - j + 1$ to zero instead.

Rule: If $p \notin \mathbb{Z} \wedge 0 < j < n \wedge ((j | n) \in \mathbb{Z} \vee c > 0) \wedge p < -1 \wedge m + j p + 1 > n - j$, then

$$\int (c x)^m (a x^j + b x^n)^p dx \rightarrow \frac{c^{n-1} (c x)^{m-n+1} (a x^j + b x^n)^{p+1}}{b (n-j) (p+1)} - \frac{c^n (m+j p - n + j + 1)}{b (n-j) (p+1)} \int (c x)^{m-n} (a x^j + b x^n)^{p+1} dx$$

Program code:

```
Int[(c_.*x_)^m_.*(a_.*x_^j_.+b_.*x_^n_.)^p_,x_Symbol] :=
  c^(n-1)*(c*x)^(m-n+1)*(a*x^j+b*x^n)^(p+1)/(b*(n-j)*(p+1)) -
  c^n*(m+j*p-n+j+1)/(b*(n-j)*(p+1))*Int[(c*x)^(m-n)*(a*x^j+b*x^n)^(p+1),x] /;
FreeQ[{a,b,c},x] && Not[IntegerQ[p]] && LtQ[0,j,n] && (IntegersQ[j,n] || GtQ[c,0]) && LtQ[p,-1] && GtQ[m+j*p+1,n-j]
```

2: $\int (c x)^m (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge 0 < j < n \wedge ((j | n) \in \mathbb{Z} \vee c > 0) \wedge p < -1$

Derivation: Generalized binomial recurrence 2b

Rule: If $p \notin \mathbb{Z} \wedge 0 < j < n \wedge ((j | n) \in \mathbb{Z} \vee c > 0) \wedge p < -1$, then

$$\int (c x)^m (a x^j + b x^n)^p dx \rightarrow -\frac{c^{j-1} (c x)^{m-j+1} (a x^j + b x^n)^{p+1}}{a (n-j) (p+1)} + \frac{c^j (m+n p+n-j+1)}{a (n-j) (p+1)} \int (c x)^{m-j} (a x^j + b x^n)^{p+1} dx$$

Program code:

```
Int[(c_.*x_)^m_.*(a_.*x_^j_.+b_.*x_^n_)^p_,x_Symbol] :=
  -c^(j-1)*(c*x)^(m-j+1)*(a*x^j+b*x^n)^(p+1)/(a*(n-j)*(p+1)) +
  c^j*(m+n*p+n-j+1)/(a*(n-j)*(p+1))*Int[(c*x)^(m-j)*(a*x^j+b*x^n)^(p+1),x] /;
FreeQ[{a,b,c,m},x] && Not[IntegerQ[p]] && LtQ[0,j,n] && (IntegersQ[j,n] || GtQ[c,0]) && LtQ[p,-1]
```

3: $\int (c x)^m (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge 0 < j < n \wedge ((j | n) \in \mathbb{Z} \vee c > 0) \wedge m+j p+1 > n-j \wedge m+n p+1 \neq 0$

Derivation: Generalized binomial recurrence 3a

Rule: If $p \notin \mathbb{Z} \wedge 0 < j < n \wedge ((j | n) \in \mathbb{Z} \vee c > 0) \wedge m+j p+1 > n-j \wedge m+n p+1 \neq 0$, then

$$\int (c x)^m (a x^j + b x^n)^p dx \rightarrow \frac{c^{n-1} (c x)^{m-n+1} (a x^j + b x^n)^{p+1}}{b (m+n p+1)} - \frac{a c^{n-j} (m+j p-n+j+1)}{b (m+n p+1)} \int (c x)^{m-(n-j)} (a x^j + b x^n)^p dx$$

Program code:

```
Int[(c_.*x_)^m_.*(a_.*x_^j_.+b_.*x_^n_)^p_,x_Symbol] :=
  c^(n-1)*(c*x)^(m-n+1)*(a*x^j+b*x^n)^(p+1)/(b*(m+n*p+1)) -
  a*c^(n-j)*(m+j*p-n+j+1)/(b*(m+n*p+1))*Int[(c*x)^(m-(n-j))*(a*x^j+b*x^n)^p,x] /;
FreeQ[{a,b,c,m,p},x] && Not[IntegerQ[p]] && LtQ[0,j,n] && (IntegersQ[j,n] || GtQ[c,0]) && GtQ[m+j*p+1-n+j,0] && NeQ[m+n*p+1,0]
```

4: $\int (c x)^m (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge 0 < j < n \wedge ((j | n) \in \mathbb{Z} \vee c > 0) \wedge m + j p + 1 < 0$

Derivation: Generalized binomial recurrence 3b

Rule: If $p \notin \mathbb{Z} \wedge 0 < j < n \wedge ((j | n) \in \mathbb{Z} \vee c > 0) \wedge m + j p + 1 < 0$, then

$$\int (c x)^m (a x^j + b x^n)^p dx \rightarrow \frac{c^{j-1} (c x)^{m-j+1} (a x^j + b x^n)^{p+1}}{a (m + j p + 1)} - \frac{b (m + n p + n - j + 1)}{a c^{n-j} (m + j p + 1)} \int (c x)^{m+n-j} (a x^j + b x^n)^p dx$$

Program code:

```
Int[(c_.**x_)^m_.*(a_.**x_^j_.+b_.**x_^n_.)^p_,x_Symbol] :=
  c^(j-1)*(c*x)^(m-j+1)*(a*x^j+b*x^n)^(p+1)/(a*(m+j*p+1)) -
  b*(m+n*p+n-j+1)/(a*c^(n-j)*(m+j*p+1))*Int[(c*x)^(m+n-j)*(a*x^j+b*x^n)^p,x] /;
FreeQ[{a,b,c,m,p},x] && Not[IntegerQ[p]] && LtQ[0,j,n] && (IntegersQ[j,n] || GtQ[c,0]) && LtQ[m+j*p+1,0]
```

$$6. \int (c x)^m (a x^j + b x^n)^p dx \text{ when } p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{j}{n} \in \mathbb{Z} \wedge m+1 \neq 0 \wedge \frac{n}{m+1} \in \mathbb{Z}$$

$$1: \int x^m (a x^j + b x^n)^p dx \text{ when } p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{j}{n} \in \mathbb{Z} \wedge m+1 \neq 0 \wedge \frac{n}{m+1} \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If $\frac{n}{m+1} \in \mathbb{Z}$, then $x^m F[x^n] = \frac{1}{m+1} \text{Subst}[F[x^{\frac{n}{m+1}}], x, x^{m+1}] \partial_x x^{m+1}$

Rule: If $p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{j}{n} \in \mathbb{Z} \wedge m+1 \neq 0 \wedge \frac{n}{m+1} \in \mathbb{Z}$, then

$$\int x^m (a x^j + b x^n)^p dx \rightarrow \frac{1}{m+1} \text{Subst}\left[\int (a x^{\frac{j}{m+1}} + b x^{\frac{n}{m+1}})^p dx, x, x^{m+1}\right]$$

Program code:

```
Int[x_^m_.*(a_.*x_^j_.+b_.*x_^n_)^p_,x_Symbol] :=
  1/(m+1)*Subst[Int[(a*x^Simplify[j/(m+1)]+b*x^Simplify[n/(m+1)])^p,x],x,x^(m+1)] /;
FreeQ[{a,b,j,m,n,p},x] && Not[IntegerQ[p]] && NeQ[n,j] && IntegerQ[Simplify[j/n]] && NeQ[m,-1] && IntegerQ[Simplify[n/(m+1)]] && Not
```

2: $\int (c x)^m (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{j}{n} \in \mathbb{Z} \wedge m+1 \neq 0 \wedge \frac{n}{m+1} \in \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $a_x \frac{(c x)^m}{x^m} == 0$

Basis: $\frac{(c x)^m}{x^m} == \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}}$

Rule: If $p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{j}{n} \in \mathbb{Z} \wedge m+1 \neq 0 \wedge \frac{n}{m+1} \in \mathbb{Z}$, then

$$\int (c x)^m (a x^j + b x^n)^p dx \rightarrow \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (a x^j + b x^n)^p dx$$

Program code:

```
Int[(c_*x_)^m_.*(a_*x_^j_.+b_*x_^n_)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a*x^j+b*x^n)^p,x] /;
FreeQ[{a,b,c,j,m,n,p},x] && Not[IntegerQ[p]] && NeQ[n,j] && IntegerQ[Simplify[j/n]] && NeQ[m,-1] && IntegerQ[Simplify[n/(m+1)]] && N
```

$$7. \int (c x)^m (a x^j + b x^n)^p dx \text{ when } p + \frac{1}{2} \in \mathbb{Z} \wedge j \neq n \wedge m + j p + 1 = 0$$

$$1. \int (c x)^m (a x^j + b x^n)^p dx \text{ when } p + \frac{1}{2} \in \mathbb{Z} \wedge j \neq n \wedge m + j p + 1 = 0 \wedge (j \in \mathbb{Z} \vee c > 0)$$

$$\mathbf{1:} \int (c x)^m (a x^j + b x^n)^p dx \text{ when } p + \frac{1}{2} \in \mathbb{Z}^+ \wedge j \neq n \wedge m + j p + 1 = 0 \wedge (j \in \mathbb{Z} \vee c > 0)$$

Derivation: Generalized binomial recurrence 1b

Rule: If $p + \frac{1}{2} \in \mathbb{Z}^+ \wedge j \neq n \wedge m + j p + 1 = 0 \wedge (j \in \mathbb{Z} \vee c > 0)$, then

$$\int (c x)^m (a x^j + b x^n)^p dx \rightarrow \frac{(c x)^{m+1} (a x^j + b x^n)^p}{c p (n - j)} + \frac{a}{c^j} \int (c x)^{m+j} (a x^j + b x^n)^{p-1} dx$$

Program code:

```
Int[(c_.**x_)^m_.*(a_.**x_^j_.+b_.**x_^n_.)^p_,x_Symbol] :=
  (c*x)^(m+1)*(a*x^j+b*x^n)^p/(c*p*(n-j)) + a/c^j*Int[(c*x)^(m+j)*(a*x^j+b*x^n)^(p-1),x] /;
FreeQ[{a,b,c,j,m,n},x] && IGtQ[p+1/2,0] && NeQ[n,j] && EqQ[Simplify[m+j*p+1],0] && (IntegerQ[j] || GtQ[c,0])
```

$$2. \int (c x)^m (a x^j + b x^n)^p dx \text{ when } p - \frac{1}{2} \in \mathbb{Z}^- \wedge j \neq n \wedge m + j p + 1 = 0 \wedge (j \in \mathbb{Z} \vee c > 0)$$

$$\mathbf{1:} \int \frac{x^m}{\sqrt{a x^j + b x^n}} dx \text{ when } m = \frac{j}{2} - 1 \wedge j \neq n$$

Derivation: Integration by substitution

$$\text{Basis: } \frac{x^{j/2-1}}{\sqrt{a x^j + b x^n}} = -\frac{2}{(n-j)} \text{Subst}\left[\frac{1}{1-a x^2}, x, \frac{x^{j/2}}{\sqrt{a x^j + b x^n}}\right] \partial_x \frac{x^{j/2}}{\sqrt{a x^j + b x^n}}$$

Rule: If $m = \frac{j}{2} - 1 \wedge j \neq n$, then

$$\int \frac{x^m}{\sqrt{a x^j + b x^n}} dx \rightarrow -\frac{2}{(n-j)} \text{Subst}\left[\int \frac{1}{1-a x^2} dx, x, \frac{x^{j/2}}{\sqrt{a x^j + b x^n}}\right]$$

Program code:

```
Int[x_^m_/Sqrt[a_*x_^j_.+b_*x_^n_.],x_Symbol] :=
  -2/(n-j)*Subst[Int[1/(1-a*x^2),x],x,x^(j/2)/Sqrt[a*x^j+b*x^n]] /;
FreeQ[{a,b,j,n},x] && EqQ[m,j/2-1] && NeQ[n,j]
```

2: $\int (c x)^m (a x^j + b x^n)^p dx$ when $p + \frac{1}{2} \in \mathbb{Z}^- \wedge j \neq n \wedge m + j p + 1 = 0 \wedge (j \in \mathbb{Z} \vee c > 0)$

Derivation: Generalized binomial recurrence 2b

Rule: If $p + \frac{1}{2} \in \mathbb{Z}^- \wedge j \neq n \wedge m + j p + 1 = 0 \wedge (j \in \mathbb{Z} \vee c > 0)$, then

$$\int (c x)^m (a x^j + b x^n)^p dx \rightarrow -\frac{c^{j-1} (c x)^{m-j+1} (a x^j + b x^n)^{p+1}}{a (n-j) (p+1)} + \frac{c^j (m+n p+n-j+1)}{a (n-j) (p+1)} \int (c x)^{m-j} (a x^j + b x^n)^{p+1} dx$$

Program code:

```
Int[(c_*x_)^m_.*(a_*x_^j_.+b_*x_^n_.)^p_,x_Symbol] :=
  -c^(j-1)*(c*x)^(m-j+1)*(a*x^j+b*x^n)^(p+1)/(a*(n-j)*(p+1)) +
  c^j*(m+n*p+n-j+1)/(a*(n-j)*(p+1))*Int[(c*x)^(m-j)*(a*x^j+b*x^n)^(p+1),x] /;
FreeQ[{a,b,c,j,m,n},x] && ILtQ[p+1/2,0] && NeQ[n,j] && EqQ[Simplify[m+j*p+1],0] && (IntegerQ[j] || GtQ[c,0])
```

2: $\int (c x)^m (a x^j + b x^n)^p dx$ when $p + \frac{1}{2} \in \mathbb{Z} \wedge j \neq n \wedge m + j p + 1 = 0 \wedge \neg (j \in \mathbb{Z} \vee c > 0)$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(c x)^m}{x^m} = 0$

Basis: $\frac{(c x)^m}{x^m} = \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}}$

Rule: If $p + \frac{1}{2} \in \mathbb{Z} \wedge j \neq n \wedge m + j p + 1 = 0$, then

$$\int (c x)^m (a x^j + b x^n)^p dx \rightarrow \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (a x^j + b x^n)^p dx$$

Program code:

```
Int[(c_*x_)^m_.*(a_*x_^j_.+b_*x_^n_)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a*x^j+b*x^n)^p,x] /;
FreeQ[{a,b,c,j,m,n,p},x] && IntegerQ[p+1/2] && NeQ[n,j] && EqQ[Simplify[m+j*p+1],0]
```


$$x. \int x^m (a x^j + b x^n)^p dx \text{ when } j \neq n$$

$$1: \int x^m (a x^j + b x^n)^p dx \text{ when } j \neq n \wedge m + j p + 1 \neq 0$$

Note: Although this antiderivative appears simpler than that produced using piecewise constant extraction, *Mathematica* 8 has a hard time differentiating it back to $x^m (a x^j + b x^n)^p$.

Rule: If $j \neq n \wedge m + j p + 1 \neq 0$, then

$$\int x^m (a x^j + b x^n)^p dx \rightarrow \frac{(a x^j + b x^n)^{p+1}}{b p (n - j) x^{n+j p}} \text{Hypergeometric2F1}\left[1, 1, 1 - p, -\frac{a}{b x^{n-j}}\right]$$

Program code:

```
(* Int[x_^m.*(a_.**x_^j_.+b_.**x_^n_.)^p_,x_Symbol] :=
  (a*x^j+b*x^n)^(p+1)/(b*p*(n-j)**x^(n+j*p))*Hypergeometric2F1[1,1,1-p,-a/(b*x^(n-j))]/;
FreeQ[{a,b,j,m,n,p},x] && NeQ[n,j] && EqQ[m+j*p+1,0] *)
```

2: $\int x^m (a x^j + b x^n)^p dx$ when $j \neq n \wedge m + n + (p - 1)j + 1 = 0$

Note: Although this antiderivative appears simpler than that produced using piecewise constant extraction, *Mathematica* 8 has a hard time differentiating it back to $x^m (a x^j + b x^n)^p$.

Rule: If $j \neq n \wedge m + n + (p - 1)j + 1 = 0$, then

$$\int x^m (a x^j + b x^n)^p dx \rightarrow \frac{(a x^j + b x^n)^{p+1}}{b (p - 1) (n - j) x^{2n+j (p-1)}} \text{Hypergeometric2F1}\left[1, 2, 2 - p, -\frac{a}{b x^{n-j}}\right]$$

Program code:

```
(* Int[x_^m_.*(a_.*x_^j_.+b_.*x_^n_.)^p_,x_Symbol] :=
  (a*x^j+b*x^n)^(p+1)/(b*(p-1)*(n-j)*x^(2*n+j*(p-1)))*Hypergeometric2F1[1,2,2-p,-a/(b*x^(n-j))]/;
FreeQ[{a,b,j,m,n,p},x] && NeQ[n,j] && EqQ[m+n+(p-1)*j+1,0] *)
```

3: $\int x^m (a x^j + b x^n)^p dx$ when $j \neq n \wedge m + j p + 1 \neq 0 \wedge m + n + (p - 1) j + 1 \neq 0$

Note: Although this antiderivative appears simpler than that produced using piecewise constant extraction, *Mathematica* 8 has a hard time differentiating it back to $x^m (a x^j + b x^n)^p$.

Rule: If $j \neq n \wedge m + j p + 1 \neq 0 \wedge m + n + (p - 1) j + 1 \neq 0$, then

$$\int x^m (a x^j + b x^n)^p dx \rightarrow \frac{x^{m-j+1} (a x^j + b x^n)^{p+1}}{a (m + j p + 1)} \text{Hypergeometric2F1}\left[1, \frac{m + n p + 1}{n - j} + 1, \frac{m + j p + 1}{n - j} + 1, -\frac{b x^{n-j}}{a}\right]$$

Program code:

```
(* Int[x^m.*(a.*x^j_.+b.*x^n_.)^p_,x_Symbol] :=
  (x^(m-j+1)*(a*x^j+b*x^n)^(p+1))/(a*(m+j*p+1))*Hypergeometric2F1[1,(m+n*p+1)/(n-j)+1,(m+j*p+1)/(n-j)+1,-b*x^(n-j)/a] /;
FreeQ[{a,b,j,m,n,p},x] && NeQ[n,j] && NeQ[m+j*p+1,0] && NeQ[m+n+(p-1)*j+1,0] *)
```

8: $\int (c x)^m (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge j \neq n$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(c x)^m (a x^j + b x^n)^p}{x^{m+j p} (a + b x^{n-j})^p} = 0$

Basis: $\frac{(c x)^m}{x^m} = \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}}$

Basis: $\frac{(a x^j + b x^n)^p}{x^j p (a + b x^{n-j})^p} = \frac{(a x^j + b x^n)^{\text{FracPart}[p]}}{x^j \text{FracPart}[p] (a + b x^{n-j})^{\text{FracPart}[p]}}$

Rule: If $p \notin \mathbb{Z} \wedge j \neq n$, then

$$\int (c x)^m (a x^j + b x^n)^p dx \rightarrow \frac{(c x)^m (a x^j + b x^n)^p}{x^{m+j p} (a + b x^{n-j})^p} \int x^{m+j p} (a + b x^{n-j})^p dx$$

$$\rightarrow \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]} (a x^j + b x^n)^{\text{FracPart}[p]}}{x^{\text{FracPart}[m] + j \text{FracPart}[p]} (a + b x^{n-j})^{\text{FracPart}[p]}} \int x^{m+j p} (a + b x^{n-j})^p dx$$

Program code:

```
Int[(c_.*x_)^m_.*(a_.*x_^j_.+b_.*x_^n_.)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]*(a*x^j+b*x^n)^FracPart[p]/
  (x^(FracPart[m]+j*FracPart[p])*(a+b*x^(n-j))^FracPart[p])*
  Int[x^(m+j*p)*(a+b*x^(n-j))^p,x] /;
FreeQ[{a,b,c,j,m,n,p},x] && Not[IntegerQ[p]] && NeQ[n,j] && PosQ[n-j]
```

S: $\int u^m (a v^j + b v^n)^p dx$ when $v == c + d x \wedge u == e v$

Derivation: Integration by substitution and piecewise constant extraction

Basis: If $u == e v$, then $\partial_x \frac{u^m}{v^m} == 0$

Rule: If $v == c + d x \wedge u == e v$, then

$$\int u^m (a v^j + b v^n)^p dx \rightarrow \frac{u^m}{d v^m} \text{Subst}\left[\int x^m (a x^j + b x^n)^p dx, x, v\right]$$

Program code:

```
Int[u_^m_.*(a_.*v_^j_.+b_.*v_^n_.)^p_,x_Symbol] :=
  u^m/(Coefficient[v,x,1]*v^m)*Subst[Int[x^m*(a*x^j+b*x^n)^p,x],x,v] /;
FreeQ[{a,b,j,m,n,p},x] && LinearPairQ[u,v,x]
```