

Rules for integrands of the form $(g + h x)^m (a + b x + c x^2)^p (d + e x + f x^2)^q$

1. $\int (g + h x)^m (a + b x + c x^2)^p (d + e x + f x^2)^q dx$ when $cd - af == 0 \wedge bd - ae == 0$

1: $\int (g + h x)^m (a + b x + c x^2)^p (d + e x + f x^2)^q dx$ when $cd - af == 0 \wedge bd - ae == 0 \wedge (p \in \mathbb{Z} \vee \frac{c}{f} > 0)$

Derivation: Algebraic simplification

Basis: If $cd - af == 0 \wedge bd - ae == 0 \wedge (p \in \mathbb{Z} \vee \frac{c}{f} > 0)$, then $(a + b x + c x^2)^p == (\frac{c}{f})^p (d + e x + f x^2)^p$

Rule 1.2.1.6.1.1: If $cd - af == 0 \wedge bd - ae == 0 \wedge (p \in \mathbb{Z} \vee \frac{c}{f} > 0)$, then

$$\int (g + h x)^m (a + b x + c x^2)^p (d + e x + f x^2)^q dx \rightarrow \left(\frac{c}{f}\right)^p \int (g + h x)^m (d + e x + f x^2)^{p+q} dx$$

Program code:

```
Int[(g_.+h_.**x_)^m_.*(a_+b_.**x_+c_.**x_^2)^p_*(d_+e_.**x_+f_.**x_^2)^q_,x_Symbol] :=
  (c/f)^p*Int[(g+h*x)^m*(d+e*x+f*x^2)^(p+q),x] /;
FreeQ[{a,b,c,d,e,f,g,h,p,q},x] && EqQ[c*d-a*f,0] && EqQ[b*d-a*e,0] && (IntegerQ[p] || GtQ[c/f,0]) &&
  (Not[IntegerQ[q]] || LeafCount[d+e*x+f*x^2] ≤ LeafCount[a+b*x+c*x^2])
```

2: $\int (g+hx)^m (a+bx+cx^2)^p (d+ex+fx^2)^q dx$ when $cd - af == 0 \wedge bd - ae == 0 \wedge p \notin \mathbb{Z} \wedge q \notin \mathbb{Z} \wedge \neg \left(\frac{c}{f} > 0\right)$

Derivation: Piecewise constant extraction

Basis: If $cd - af == 0 \wedge bd - ae == 0$, then $a_x \frac{(a+bx+cx^2)^p}{(d+ex+fx^2)^p} == 0$

Rule 1.2.1.6.1.2: If $cd - af == 0 \wedge bd - ae == 0 \wedge p \notin \mathbb{Z} \wedge q \notin \mathbb{Z} \wedge \neg \left(\frac{c}{f} > 0\right)$, then

$$\int (g+hx)^m (a+bx+cx^2)^p (d+ex+fx^2)^q dx \rightarrow \frac{a^{\text{IntPart}[p]} (a+bx+cx^2)^{\text{FracPart}[p]}}{d^{\text{IntPart}[p]} (d+ex+fx^2)^{\text{FracPart}[p]}} \int (g+hx)^m (d+ex+fx^2)^{p+q} dx$$

Program code:

```
Int[(g_.+h_.**x_)^m_.*(a_+b_.**x_+c_.**x_^2)^p_*(d_+e_.**x_+f_.**x_^2)^q_,x_Symbol] :=
  a^IntPart[p]*(a+b*x+c*x^2)^FracPart[p]/(d^IntPart[p]*(d+e*x+f*x^2)^FracPart[p])*Int[(g+h*x)^m*(d+e*x+f*x^2)^(p+q),x] /;
FreeQ[{a,b,c,d,e,f,g,h,p,q},x] && EqQ[c*d-a*f,0] && EqQ[b*d-a*e,0] && Not[IntegerQ[p]] && Not[IntegerQ[q]] && Not[GtQ[c/f,0]]
```

2: $\int (g+hx)^m (a+bx+cx^2)^p (d+ex+fx^2)^q dx$ when $b^2 - 4ac = 0$

Derivation: Piecewise constant extraction

Basis: If $b^2 - 4ac = 0$, then $\partial_x \frac{(a+bx+cx^2)^p}{(b+2cx)^{2p}} = 0$

Rule 1.2.1.6.2: If $b^2 - 4ac = 0$, then

$$\int (g+hx)^m (a+bx+cx^2)^p (d+ex+fx^2)^q dx \rightarrow \frac{(a+bx+cx^2)^{\text{FracPart}[p]}}{(4c)^{\text{IntPart}[p]} (b+2cx)^{2\text{FracPart}[p]}} \int (g+hx)^m (b+2cx)^{2p} (d+ex+fx^2)^q dx$$

Program code:

```
Int[(g_.+h_.**x_)^m_.*(a_+b_.**x_+c_.**x_^2)^p_.*(d_+e_.**x_+f_.**x_^2)^q_,x_Symbol] :=
  (a+b*x+c*x^2)^FracPart[p]/((4*c)^IntPart[p]*(b+2*c*x)^(2*FracPart[p]))*Int[(g+h*x)^m*(b+2*c*x)^(2*p)*(d+e*x+f*x^2)^q,x] /;
FreeQ[{a,b,c,d,e,f,g,h,m,p,q},x] && EqQ[b^2-4*a*c,0]
```

```
Int[(g_.+h_.**x_)^m_.*(a_+b_.**x_+c_.**x_^2)^p_.*(d_+f_.**x_^2)^q_,x_Symbol] :=
  (a+b*x+c*x^2)^FracPart[p]/((4*c)^IntPart[p]*(b+2*c*x)^(2*FracPart[p]))*Int[(g+h*x)^m*(b+2*c*x)^(2*p)*(d+f*x^2)^q,x] /;
FreeQ[{a,b,c,d,f,g,h,m,p,q},x] && EqQ[b^2-4*a*c,0]
```

3: $\int (g+hx)^m (a+bx+cx^2)^p (d+ex+fx^2)^q dx$ when $cg^2 - bgh + ah^2 = 0 \wedge c^2 dg^2 - acegh + a^2 fh^2 = 0 \wedge q = m \wedge m \in \mathbb{Z}$

Derivation: Algebraic simplification

Basis: If $cg^2 - bgh + ah^2 = 0 \wedge c^2 dg^2 - acegh + a^2 fh^2 = 0$, then
 $(g+hx)(d+ex+fx^2) = \left(\frac{dg}{a} + \frac{fhx}{c}\right)(a+bx+cx^2)$

Rule 1.2.1.6.3: If $cg^2 - bgh + ah^2 = 0 \wedge c^2 dg^2 - acegh + a^2 fh^2 = 0 \wedge q = m \wedge m \in \mathbb{Z}$, then

$$\int (g+hx)^m (a+bx+cx^2)^p (d+ex+fx^2)^q dx \rightarrow \int \left(\frac{dg}{a} + \frac{fhx}{c}\right)^m (a+bx+cx^2)^{m+p} dx$$

Program code:

```
Int[(g+h_.**x_)^m_.*(a+b_.**x_+c_.**x_^2)^p_*(d_.+e_.**x_+f_.**x_^2)^m_.,x_Symbol] :=
  Int[(d*g/a+f*h*x/c)^m*(a+b*x+c*x^2)^(m+p),x] /;
FreeQ[{a,b,c,d,e,f,g,h,p},x] && EqQ[c*g^2-b*g*h+a*h^2,0] && EqQ[c^2*d*g^2-a*c*e*g*h+a^2*f*h^2,0] && IntegerQ[m]
```

```
Int[(g+h_.**x_)^m_.*(a+c_.**x_^2)^p_*(d_.+e_.**x_+f_.**x_^2)^m_.,x_Symbol] :=
  Int[(d*g/a+f*h*x/c)^m*(a+c*x^2)^(m+p),x] /;
FreeQ[{a,c,d,e,f,g,h,p},x] && EqQ[c*g^2+a*h^2,0] && EqQ[c^2*d*g^2-a*c*e*g*h+a^2*f*h^2,0] && IntegerQ[m]
```

```
Int[(g+h_.**x_)^m_.*(a+b_.**x_+c_.**x_^2)^p_*(d_.+f_.**x_^2)^m_.,x_Symbol] :=
  Int[(d*g/a+f*h*x/c)^m*(a+b*x+c*x^2)^(m+p),x] /;
FreeQ[{a,b,c,d,f,g,h,p},x] && EqQ[c*g^2-b*g*h+a*h^2,0] && EqQ[c^2*d*g^2+a^2*f*h^2,0] && IntegerQ[m]
```

```
Int[(g+h_.**x_)^m_.*(a+c_.**x_^2)^p_*(d_.+f_.**x_^2)^m_.,x_Symbol] :=
  Int[(d*g/a+f*h*x/c)^m*(a+c*x^2)^(m+p),x] /;
FreeQ[{a,c,d,f,g,h,p},x] && EqQ[c*g^2+a*h^2,0] && EqQ[c^2*d*g^2+a^2*f*h^2,0] && IntegerQ[m]
```

$$x. \int (g+hx)^m (a+bx+cx^2)^p (d+ex+fx^2)^q dx \text{ when } b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0 \wedge cg^2 - bgh + ah^2 = 0$$

$$1: \int (g+hx)^m (a+bx+cx^2)^p (d+ex+fx^2)^q dx \text{ when } cg^2 - bgh + ah^2 = 0 \wedge p \in \mathbb{Z}$$

Derivation: Algebraic simplification

Basis: If $cg^2 - bgh + ah^2 = 0$, then $a + bx + cx^2 = (g + hx) \left(\frac{a}{g} + \frac{cx}{h} \right)$

Rule 1.2.1.6.x.1: If $cg^2 - bgh + ah^2 = 0 \wedge p \in \mathbb{Z}$, then

$$\int (g+hx)^m (a+bx+cx^2)^p (d+ex+fx^2)^q dx \rightarrow \int (g+hx)^{m+p} \left(\frac{a}{g} + \frac{cx}{h} \right)^p (d+ex+fx^2)^q dx$$

Program code:

```
(* Int[(g+h*x_)^m.*(a_.+b_.*x_+c_.*x_^2)^p.*(d_.+e_.*x_+f_.*x_^2)^q_,x_Symbol] :=
  Int[(g+h*x)^(m+p)*(a/g+c/h*x)^p*(d+e*x+f*x^2)^q,x] /;
FreeQ[{a,b,c,d,e,f,g,h,m,q},x] && EqQ[c*g^2-b*g*h+a*h^2,0] && IntegerQ[p] *)
```

```
(* Int[(g+h*x_)^m.*(a_+c_.*x_^2)^p.*(d_.+e_.*x_+f_.*x_^2)^q_,x_Symbol] :=
  Int[(g+h*x)^(m+p)*(a/g+c/h*x)^p*(d+e*x+f*x^2)^q,x] /;
FreeQ[{a,c,d,e,f,g,h,m,q},x] && NeQ[e^2-4*d*f,0] && EqQ[c*g^2+a*h^2,0] && IntegerQ[p] *)
```

```
(* Int[(g+h*x_)^m.*(a_.+b_.*x_+c_.*x_^2)^p.*(d_.+f_.*x_^2)^q_,x_Symbol] :=
  Int[(g+h*x)^(m+p)*(a/g+c/h*x)^p*(d+f*x^2)^q,x] /;
FreeQ[{a,b,c,d,f,g,h,m,q},x] && NeQ[b^2-4*a*c,0] && EqQ[c*g^2-b*g*h+a*h^2,0] && IntegerQ[p] *)
```

```
(* Int[(g+h*x_)^m.*(a_+c_.*x_^2)^p.*(d_.+f_.*x_^2)^q_,x_Symbol] :=
  Int[(g+h*x)^(m+p)*(a/g+c/h*x)^p*(d+f*x^2)^q,x] /;
FreeQ[{a,c,d,f,g,h,m,q},x] && EqQ[c*g^2+a*h^2,0] && IntegerQ[p] *)
```

2: $\int (g+hx)^m (a+bx+cx^2)^p (d+ex+fx^2)^q dx$ when $cg^2 - bgh + ah^2 = 0 \wedge p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: If $cg^2 - bgh + ah^2 = 0$, then $\partial_x \frac{(a+bx+cx^2)^p}{(g+hx)^p \left(\frac{a}{g} + \frac{cx}{h}\right)^p} = 0$

Rule 1.2.1.6.x.2: If $cg^2 - bgh + ah^2 = 0 \wedge p \notin \mathbb{Z}$, then

$$\int (g+hx)^m (a+bx+cx^2)^p (d+ex+fx^2)^q dx \rightarrow \frac{(a+bx+cx^2)^{\text{FracPart}[p]}}{(g+hx)^{\text{FracPart}[p]} \left(\frac{a}{g} + \frac{cx}{h}\right)^{\text{FracPart}[p]}} \int (g+hx)^{m+p} \left(\frac{a}{g} + \frac{cx}{h}\right)^p (d+ex+fx^2)^q dx$$

Program code:

```
(* Int[(g+h.*x_)^m.*(a_.+b_.*x_+c_.*x_^2)^p.*(d_.+e_.*x_+f_.*x_^2)^q_,x_Symbol] :=
  (a+b*x+c*x^2)^FracPart[p]/((g+h*x)^FracPart[p]*(a/g+(c*x)/h)^FracPart[p])*Int[(g+h*x)^(m+p)*(a/g+c/h*x)^p*(d+e*x+f*x^2)^q,x] /
  FreeQ[{a,b,c,d,e,f,g,h,m,q},x] && EqQ[c*g^2-b*g*h+a*h^2,0] && Not[IntegerQ[p]] *)
```

```
(* Int[(g+h.*x_)^m.*(a_.+c_.*x_^2)^p.*(d_.+e_.*x_+f_.*x_^2)^q_,x_Symbol] :=
  (a+c*x^2)^FracPart[p]/((g+h*x)^FracPart[p]*(a/g+(c*x)/h)^FracPart[p])*Int[(g+h*x)^(m+p)*(a/g+c/h*x)^p*(d+e*x+f*x^2)^q,x] /;
  FreeQ[{a,c,d,e,f,g,h,m,q},x] && NeQ[e^2-4*d*f,0] && EqQ[c*g^2+a*h^2,0] && Not[IntegerQ[p]] *)
```

```
(* Int[(g+h.*x_)^m.*(a_.+b_.*x_+c_.*x_^2)^p.*(d_.+f_.*x_^2)^q_,x_Symbol] :=
  (a+b*x+c*x^2)^FracPart[p]/((g+h*x)^FracPart[p]*(a/g+(c*x)/h)^FracPart[p])*Int[(g+h*x)^(m+p)*(a/g+c/h*x)^p*(d+f*x^2)^q,x] /;
  FreeQ[{a,b,c,d,f,g,h,m,q},x] && NeQ[b^2-4*a*c,0] && EqQ[c*g^2-b*g*h+a*h^2,0] && Not[IntegerQ[p]] *)
```

```
(* Int[(g+h.*x_)^m.*(a_.+c_.*x_^2)^p.*(d_.+f_.*x_^2)^q_,x_Symbol] :=
  (a+c*x^2)^FracPart[p]/((g+h*x)^FracPart[p]*(a/g+(c*x)/h)^FracPart[p])*Int[(g+h*x)^(m+p)*(a/g+c/h*x)^p*(d+f*x^2)^q,x] /;
  FreeQ[{a,c,d,f,g,h,m,q},x] && EqQ[c*g^2+a*h^2,0] && Not[IntegerQ[p]] *)
```

4: $\int x^p (a+bx+cx^2)^p (ex+fx^2)^q dx$ when $b^2 - 4ac \neq 0 \wedge ce^2 - bef + af^2 = 0 \wedge p \in \mathbb{Z}$

Derivation: Algebraic simplification

Basis: If $ce^2 - bef + af^2 = 0$, then $x(a+bx+cx^2) = \left(\frac{a}{e} + \frac{c}{f}x\right)(ex+fx^2)$

Rule 1.2.1.6.4: If $b^2 - 4ac \neq 0 \wedge ce^2 - bef + af^2 = 0 \wedge p \in \mathbb{Z}$, then

$$\int x^p (a+bx+cx^2)^p (ex+fx^2)^q dx \rightarrow \int \left(\frac{a}{e} + \frac{c}{f}x\right)^p (ex+fx^2)^{p+q} dx$$

Program code:

```
Int[x_^p_*(a_+b_.*x_+c_.*x_^2)^p_*(e_.*x_+f_.*x_^2)^q_,x_Symbol] :=
  Int[(a/e+c/f*x)^p*(e*x+f*x^2)^(p+q),x] /;
FreeQ[{a,b,c,e,f,q},x] && NeQ[b^2-4*a*c,0] && EqQ[c*e^2-b*e*f+a*f^2,0] && IntegerQ[p]
```

```
Int[x_^p_*(a+c_.*x_^2)^p_*(e_.*x_+f_.*x_^2)^q_,x_Symbol] :=
  Int[(a/e+c/f*x)^p*(e*x+f*x^2)^(p+q),x] /;
FreeQ[{a,c,e,f,q},x] && EqQ[c*e^2+a*f^2,0] && IntegerQ[p]
```

$$6. \int (g+hx) (a+bx+cx^2)^p (d+ex+fx^2)^q dx \text{ when } b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0$$

$$1. \int (g+hx) (a+cx^2)^p (d+fx^2)^q dx$$

$$1. \int \frac{g+hx}{(a+cx^2)^{1/3} (d+fx^2)} dx \text{ when } cd+3af=0 \wedge cg^2+9ah^2=0$$

$$1: \int \frac{g+hx}{(a+cx^2)^{1/3} (d+fx^2)} dx \text{ when } cd+3af=0 \wedge cg^2+9ah^2=0 \wedge a>0$$

Derived from formula for this class of Goursat pseudo-elliptic integrands contributed by Martin Welz on 19 August 2016

Rule 1.2.1.6.6.1.1.1: If $cd+3af=0 \wedge cg^2+9ah^2=0 \wedge a>0$, then

$$\int \frac{g+hx}{(a+cx^2)^{1/3} (d+fx^2)} dx \rightarrow \frac{\sqrt{3} h \operatorname{ArcTan}\left[\frac{1}{\sqrt{3}} - \frac{2^{2/3} \left(1 - \frac{3hx}{g}\right)^{2/3}}{\sqrt{3} \left(1 + \frac{3hx}{g}\right)^{1/3}}\right]}{2^{2/3} a^{1/3} f} + \frac{h \operatorname{Log}[d+fx^2]}{2^{5/3} a^{1/3} f} - \frac{3 h \operatorname{Log}\left[\left(1 - \frac{3hx}{g}\right)^{2/3} + 2^{1/3} \left(1 + \frac{3hx}{g}\right)^{1/3}\right]}{2^{5/3} a^{1/3} f}$$

Program code:

```
Int[(g+h_.**x_)/((a+c_.**x_^2)^(1/3)*(d+f_.**x_^2)),x_Symbol] :=
  Sqrt[3]*h*ArcTan[1/Sqrt[3]-2^(2/3)*(1-3*h*x/g)^(2/3)/(Sqrt[3]*(1+3*h*x/g)^(1/3))]/(2^(2/3)*a^(1/3)*f) +
  h*Log[d+f*x^2]/(2^(5/3)*a^(1/3)*f) -
  3*h*Log[(1-3*h*x/g)^(2/3)+2^(1/3)*(1+3*h*x/g)^(1/3)]/(2^(5/3)*a^(1/3)*f) /;
FreeQ[{a,c,d,f,g,h},x] && EqQ[c*d+3*a*f,0] && EqQ[c*g^2+9*a*h^2,0] && GtQ[a,0]
```


$$\mathbf{2:} \int \frac{g+hx}{(a+cx^2)^{1/3} (d+fx^2)} dx \text{ when } cd+3af=0 \wedge cg^2+9ah^2=0 \wedge a \neq 0$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{\left(1 + \frac{cx^2}{a}\right)^{1/3}}{(a+cx^2)^{1/3}} = 0$$

Rule 1.2.1.6.6.1.1.2: If $cd+3af=0 \wedge cg^2+9ah^2=0 \wedge a \neq 0$, then

$$\int \frac{g+hx}{(a+cx^2)^{1/3} (d+fx^2)} dx \rightarrow \frac{\left(1 + \frac{cx^2}{a}\right)^{1/3}}{(a+cx^2)^{1/3}} \int \frac{g+hx}{\left(1 + \frac{cx^2}{a}\right)^{1/3} (d+fx^2)} dx$$

Program code:

```
Int[(g+h_.**x_)/(a+c_.**x_^2)^(1/3)*(d+f_.**x_^2)),x_Symbol] :=
  (1+c**x^2/a)^(1/3)/(a+c**x^2)^(1/3)*Int[(g+h**x)/((1+c**x^2/a)^(1/3)*(d+f**x^2)),x] /;
FreeQ[{a,c,d,f,g,h},x] && EqQ[c*d+3*a*f,0] && EqQ[c*g^2+9*a*h^2,0] && Not[GtQ[a,0]]
```

$$2: \int (g+hx) (a+cx^2)^p (d+fx^2)^q dx$$

Derivation: Algebraic expansion

Rule 1.2.1.6.6.1.2:

$$\int (g+hx) (a+cx^2)^p (d+fx^2)^q dx \rightarrow g \int (a+cx^2)^p (d+fx^2)^q dx + h \int x (a+cx^2)^p (d+fx^2)^q dx$$

Program code:

```
Int[(g+h_.**x_)*(a+c_.**x_^2)^p*(d+f_.**x_^2)^q_,x_Symbol] :=
  g*Int[(a+c*x^2)^p*(d+f*x^2)^q,x] + h*Int[x*(a+c*x^2)^p*(d+f*x^2)^q,x] /;
FreeQ[{a,c,d,f,g,h,p,q},x]
```

$$2: \int (a+bx+cx^2)^p (d+ex+fx^2)^q (g+hx) dx \text{ when } b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0 \wedge p \in \mathbb{Z}^+ \wedge q \in \mathbb{Z}$$

Derivation: Algebraic expansion

Rule 1.2.1.6.6.2: If $b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0 \wedge p \in \mathbb{Z}^+ \wedge q \in \mathbb{Z}$, then

$$\int (a+bx+cx^2)^p (d+ex+fx^2)^q (g+hx) dx \rightarrow \int \text{ExpandIntegrand}[(a+bx+cx^2)^p (d+ex+fx^2)^q (g+hx), x] dx$$

Program code:

```
Int[(a+b_.**x_+c_.**x_^2)^p*(d+e_.**x_+f_.**x_^2)^q*(g+h_.**x_),x_Symbol] :=
  Int[ExpandIntegrand[(a+b*x+c*x^2)^p*(d+e*x+f*x^2)^q*(g+h*x),x],x] /;
FreeQ[{a,b,c,d,e,f,g,h},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && IGtQ[p,0] && IntegerQ[q]
```

```

Int[(a+c_.**x_^2)^p_*(d+e_.**x_+f_.**x_^2)^q_*(g_.+h_.**x_),x_Symbol] :=
  Int[ExpandIntegrand[(a+c*x^2)^p*(d+e*x+f*x^2)^q*(g+h*x),x],x] /;
FreeQ[{a,c,d,e,f,g,h},x] && NeQ[e^2-4*d*f,0] && IntegersQ[p,q] && (GtQ[p,0] || GtQ[q,0])

```

3. $\int (a+bx+cx^2)^p (d+ex+fx^2)^q (g+hx) dx$ when $b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0 \wedge p < -1$

1: $\int (a+bx+cx^2)^p (d+ex+fx^2)^q (g+hx) dx$ when $b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0 \wedge p < -1 \wedge q > 0$

Derivation: Nondegenerate biquadratic recurrence 1

Rule 1.2.1.6.6.3.1: If $b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0 \wedge p < -1 \wedge q > 0$, then

$$\begin{aligned}
 & \int (a+bx+cx^2)^p (d+ex+fx^2)^q (g+hx) dx \rightarrow \\
 & \frac{(gb^2c - 2ahc - c(bh - 2gc)x)(a+bx+cx^2)^{p+1}(d+ex+fx^2)^q}{c(b^2 - 4ac)(p+1)} - \\
 & \frac{1}{(b^2 - 4ac)(p+1)} \int (a+bx+cx^2)^{p+1}(d+ex+fx^2)^{q-1} dx \\
 & (eq(gb - 2ah) - d(bh - 2gc)(2p+3) + (2fq(gb - 2ah) - e(bh - 2gc)(2p+q+3))x - f(bh - 2gc)(2p+2q+3)x^2) dx
 \end{aligned}$$

Program code:

```

Int[(a+b_.**x_+c_.**x_^2)^p_*(d+e_.**x_+f_.**x_^2)^q_*(g_.+h_.**x_),x_Symbol] :=
  (g*b-2*a*h-(b*h-2*g*c)*x)*(a+b*x+c*x^2)^(p+1)*(d+e*x+f*x^2)^q/((b^2-4*a*c)*(p+1)) -
  1/((b^2-4*a*c)*(p+1))*
  Int[(a+b*x+c*x^2)^(p+1)*(d+e*x+f*x^2)^(q-1)*
    Simp[e*q*(g*b-2*a*h)-d*(b*h-2*g*c)*(2*p+3)+
      (2*f*q*(g*b-2*a*h)-e*(b*h-2*g*c)*(2*p+q+3))*x-
      f*(b*h-2*g*c)*(2*p+2*q+3)*x^2,x],x] /;
FreeQ[{a,b,c,d,e,f,g,h},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && LtQ[p,-1] && GtQ[q,0]

```

```

Int[(a+c_.**x_^2)^p_*(d+e_.**x_+f_.**x_^2)^q_*(g_.+h_.**x_),x_Symbol] :=
  (a*h-g*c*x)*(a+c*x^2)^(p+1)*(d+e*x+f*x^2)^q/(2*a*c*(p+1)) +
  2/(4*a*c*(p+1))*
  Int[(a+c*x^2)^(p+1)*(d+e*x+f*x^2)^(q-1)*
    Simp[g*c*d*(2*p+3)-a*(h*e*q)+(g*c*e*(2*p+q+3)-a*(2*h*f*q))*x+g*c*f*(2*p+2*q+3)*x^2,x],x] /;
FreeQ[{a,c,d,e,f,g,h},x] && NeQ[e^2-4*d*f,0] && LtQ[p,-1] && GtQ[q,0]

```

```

Int[(a+b_.**x_+c_.**x_^2)^p_*(d+f_.**x_^2)^q_*(g_.+h_.**x_),x_Symbol] :=
  (g*b-2*a*h-(b*h-2*g*c)*x)*(a+b*x+c*x^2)^(p+1)*(d+f*x^2)^q/((b^2-4*a*c)*(p+1)) -
  1/((b^2-4*a*c)*(p+1))*
  Int[(a+b*x+c*x^2)^(p+1)*(d+f*x^2)^(q-1)*
    Simp[-d*(b*h-2*g*c)*(2*p+3)+(2*f*q*(g*b-2*a*h))*x-f*(b*h-2*g*c)*(2*p+2*q+3)*x^2,x],x] /;
FreeQ[{a,b,c,d,f,g,h},x] && NeQ[b^2-4*a*c,0] && LtQ[p,-1] && GtQ[q,0]

```

$$\text{2: } \int (a+bx+cx^2)^p (d+ex+fx^2)^q (g+hx) dx \text{ when } b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge p < -1 \wedge q \neq 0 \wedge (cd-af)^2 - (bd-ae)(ce-bf) \neq 0$$

Derivation: Nondegenerate biquadratic recurrence 3

Rule 1.2.1.6.6.3.2: If

$b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge p < -1 \wedge q \neq 0 \wedge (cd-af)^2 - (bd-ae)(ce-bf) \neq 0$, then

$$\begin{aligned}
 & \int (a+bx+cx^2)^p (d+ex+fx^2)^q (g+hx) dx \rightarrow \\
 & \frac{(a+bx+cx^2)^{p+1} (d+ex+fx^2)^{q+1}}{(b^2-4ac) ((cd-af)^2 - (bd-ae)(ce-bf)) (p+1)} \cdot \\
 & \frac{1}{(b^2-4ac) ((cd-af)^2 - (bd-ae)(ce-bf)) (p+1)} \int (a+bx+cx^2)^{p+1} (d+ex+fx^2)^q \cdot \\
 & ((bh-2gc) ((cd-af)^2 - (bd-ae)(ce-bf)) (p+1) + \\
 & (b^2gf - b(hcd+gce+ahf) + 2(gc(cd-af) + ahce)) (af(p+1) - cd(p+2)) - \\
 & e(gc(2ace-b(cd+af)) + (gb-ah)(2c^2d+b^2f-c(be+2af))) (p+q+2) - \\
 & (2f(gc(2ace-b(cd+af)) + (gb-ah)(2c^2d+b^2f-c(be+2af)))) (p+q+2) - \\
 & (b^2gf - b(hcd+gce+ahf) + 2(gc(cd-af) + ahce)) (bf(p+1) - ce(2p+q+4))) x -
 \end{aligned}$$

$$c f (b^2 g f - b (h c d + g c e + a h f) + 2 (g c (c d - a f) + a h c e)) (2 p + 2 q + 5) x^2 dx$$

Program code:

```
Int[(a+b*x+c*x^2)^p*(d+e*x+f*x^2)^q*(g_.+h_.x_),x_Symbol] :=
(a+b*x+c*x^2)^(p+1)*(d+e*x+f*x^2)^(q+1)/((b^2-4*a*c)*((c*d-a*f)^2-(b*d-a*e)*(c*e-b*f))*(p+1))*
(g*c*(2*a*c*e-b*(c*d+a*f))+(g*b-a*h)*(2*c^2*d+b^2*f-c*(b*e+2*a*f))+
c*(g*(2*c^2*d+b^2*f-c*(b*e+2*a*f))-h*(b*c*d-2*a*c*e+a*b*f))*x) +
1/((b^2-4*a*c)*((c*d-a*f)^2-(b*d-a*e)*(c*e-b*f))*(p+1))*
Int[(a+b*x+c*x^2)^(p+1)*(d+e*x+f*x^2)^q*
Simp[(b*h-2*g*c)*((c*d-a*f)^2-(b*d-a*e)*(c*e-b*f))*(p+1)+
(b^2*(g*f)-b*(h*c*d+g*c*e+a*h*f)+2*(g*c*(c*d-a*f)-a*(-h*c*e)))*(a*f*(p+1)-c*d*(p+2))-
e*(g*c)*(2*a*c*e-b*(c*d+a*f))+(g*b-a*h)*(2*c^2*d+b^2*f-c*(b*e+2*a*f)))*(p+q+2)-
(2*f*(g*c)*(2*a*c*e-b*(c*d+a*f))+(g*b-a*h)*(2*c^2*d+b^2*f-c*(b*e+2*a*f)))*(p+q+2)-
(b^2*g*f-b*(h*c*d+g*c*e+a*h*f)+2*(g*c*(c*d-a*f)-a*(-h*c*e)))*
(b*f*(p+1)-c*e*(2*p+q+4)))*x-
c*f*(b^2*(g*f)-b*(h*c*d+g*c*e+a*h*f)+2*(g*c*(c*d-a*f)+a*h*c*e))*(2*p+2*q+5)*x^2,x]/;
FreeQ[{a,b,c,d,e,f,g,h,q},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && LtQ[p,-1] &&
NeQ[(c*d-a*f)^2-(b*d-a*e)*(c*e-b*f),0] && Not[Not[IntegerQ[p]] && ILtQ[q,-1]]
```

```
Int[(a+c_.x^2)^p*(d+e_.x+f_.x^2)^q*(g_.+h_.x_),x_Symbol] :=
(a+c*x^2)^(p+1)*(d+e*x+f*x^2)^(q+1)/((-4*a*c)*(a*c*e^2+(c*d-a*f)^2)*(p+1))*
(g*c*(2*a*c*e)+(-a*h)*(2*c^2*d-c*(2*a*f))+
c*(g*(2*c^2*d-c*(2*a*f))-h*(-2*a*c*e))*x) +
1/((-4*a*c)*(a*c*e^2+(c*d-a*f)^2)*(p+1))*
Int[(a+c*x^2)^(p+1)*(d+e*x+f*x^2)^q*
Simp[(-2*g*c)*((c*d-a*f)^2-(-a*e)*(c*e))*(p+1)+
(2*(g*c*(c*d-a*f)-a*(-h*c*e)))*(a*f*(p+1)-c*d*(p+2))-
e*(g*c)*(2*a*c*e)+(-a*h)*(2*c^2*d-c*(2*a*f)))*(p+q+2)-
(2*f*(g*c)*(2*a*c*e)+(-a*h)*(2*c^2*d-c*(2*a*f)))*(p+q+2)-(2*(g*c*(c*d-a*f)-a*(-h*c*e)))*(-c*e*(2*p+q+4)))*x-
c*f*(2*(g*c*(c*d-a*f)-a*(-h*c*e)))*(2*p+2*q+5)*x^2,x]/;
FreeQ[{a,c,d,e,f,g,h,q},x] && NeQ[e^2-4*d*f,0] && LtQ[p,-1] && NeQ[a*c*e^2+(c*d-a*f)^2,0] && Not[Not[IntegerQ[p]] && ILtQ[q,-1]]
```

```

Int[(a+b_.**x_+c_.**x_^2)^p_*(d+f_.**x_^2)^q_*(g_.+h_.**x_),x_Symbol] :=
(a+b*x+c*x^2)^(p+1)*(d+f*x^2)^(q+1)/((b^2-4*a*c)*(b^2*d*f+(c*d-a*f)^2)*(p+1))*
((g*c)*(-b*(c*d+a*f))+(g*b-a*h)*(2*c^2*d+b^2*f-c*(2*a*f))+
c*(g*(2*c^2*d+b^2*f-c*(2*a*f))-h*(b*c*d+a*b*f))*x) +
1/((b^2-4*a*c)*(b^2*d*f+(c*d-a*f)^2)*(p+1))*
Int[(a+b*x+c*x^2)^(p+1)*(d+f*x^2)^q*
Simp[(b*h-2*g*c)*((c*d-a*f)^2-(b*d)*(-b*f))*(p+1)+
(b^2*(g*f)-b*(h*c*d+a*h*f)+2*(g*c*(c*d-a*f)))*(a*f*(p+1)-c*d*(p+2))-
(2*f*((g*c)*(-b*(c*d+a*f))+(g*b-a*h)*(2*c^2*d+b^2*f-c*(2*a*f)))*(p+q+2)-
(b^2*(g*f)-b*(h*c*d+a*h*f)+2*(g*c*(c*d-a*f)))*
(b*f*(p+1)))*x-
c*f*(b^2*(g*f)-b*(h*c*d+a*h*f)+2*(g*c*(c*d-a*f)))*(2*p+2*q+5)*x^2,x],x]/;
FreeQ[{a,b,c,d,f,g,h,q},x] && NeQ[b^2-4*a*c,0] && LtQ[p,-1] && NeQ[b^2*d*f+(c*d-a*f)^2,0] && Not[Not[IntegerQ[p]] && ILtQ[q,-1]]

```

4: $\int (a+bx+cx^2)^p (d+ex+fx^2)^q (g+hx) dx$ when $b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0 \wedge p > 0 \wedge p+q+1 \neq 0 \wedge 2p+2q+3 \neq 0$

Derivation: Nondegenerate biquadratic recurrence 2

Rule 1.2.1.6.6.4: If $b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0 \wedge p > 0 \wedge p+q+1 \neq 0 \wedge 2p+2q+3 \neq 0$, then

$$\begin{aligned} & \int (a+bx+cx^2)^p (d+ex+fx^2)^q (g+hx) dx \rightarrow \\ & \frac{(hcf(2p+2q+3)) (a+bx+cx^2)^p (d+ex+fx^2)^{q+1}}{2cf^2(p+q+1)(2p+2q+3)} - \\ & \frac{1}{2f(p+q+1)} \int (a+bx+cx^2)^{p-1} (d+ex+fx^2)^q \cdot \\ & (h(bd-ae)p + a(he-2gf)(p+q+1) + (2h(cd-af)p + b(he-2gf)(p+q+1))x + (h(ce-bf)p + c(he-2gf)(p+q+1))x^2) dx \end{aligned}$$

Program code:

```
Int[(a_+b_.*x_+c_.*x_^2)^p_*(d_+e_.*x_+f_.*x_^2)^q_*(g_+h_.*x_),x_Symbol] :=
h*(a+b*x+c*x^2)^p*(d+e*x+f*x^2)^(q+1)/(2*f*(p+q+1)) -
(1/(2*f*(p+q+1)))*
Int[(a+b*x+c*x^2)^(p-1)*(d+e*x+f*x^2)^q*
Simp[h*p*(b*d-a*e)+a*(h*e-2*g*f)*(p+q+1)+
(2*h*p*(c*d-a*f)+b*(h*e-2*g*f)*(p+q+1))*x+
(h*p*(c*e-b*f)+c*(h*e-2*g*f)*(p+q+1))*x^2,x],x] /;
FreeQ[{a,b,c,d,e,f,g,h,q},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && GtQ[p,0] && NeQ[p+q+1,0]
```

```
Int[(a_+c_.*x_^2)^p_*(d_+e_.*x_+f_.*x_^2)^q_*(g_+h_.*x_),x_Symbol] :=
h*(a+c*x^2)^p*(d+e*x+f*x^2)^(q+1)/(2*f*(p+q+1)) +
(1/(2*f*(p+q+1)))*
Int[(a+c*x^2)^(p-1)*(d+e*x+f*x^2)^q*
Simp[a*h*e*p-a*(h*e-2*g*f)*(p+q+1)-2*h*p*(c*d-a*f)*x-(h*c*e*p+c*(h*e-2*g*f)*(p+q+1))*x^2,x],x] /;
FreeQ[{a,c,d,e,f,g,h,q},x] && NeQ[e^2-4*d*f,0] && GtQ[p,0] && NeQ[p+q+1,0]
```

```

Int[(a_+b_.*x_+c_.*x_^2)^p_*(d_+f_.*x_^2)^q_*(g_+.h_.*x_),x_Symbol] :=
  h*(a+b*x+c*x^2)^p*(d+f*x^2)^(q+1)/(2*f*(p+q+1)) -
  (1/(2*f*(p+q+1)))*
  Int[(a+b*x+c*x^2)^(p-1)*(d+f*x^2)^q*
    Simp[h*p*(b*d)+a*(-2*g*f)*(p+q+1)+
      (2*h*p*(c*d-a*f)+b*(-2*g*f)*(p+q+1))*x+
      (h*p*(-b*f)+c*(-2*g*f)*(p+q+1))*x^2,x],x] /;
FreeQ[{a,b,c,d,f,g,h,q},x] && NeQ[b^2-4*a*c,0] && GtQ[p,0] && NeQ[p+q+1,0]

```


5: $\int \frac{g+hx}{(a+bx+cx^2)(d+ex+fx^2)} dx$ when $b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0 \wedge c^2 d^2 - bcde + ace^2 + b^2 df - 2acdf - abef + a^2 f^2 \neq 0$

Derivation: Algebraic expansion

Basis: Let $q = c^2 d^2 - bcde + ace^2 + b^2 df - 2acdf - abef + a^2 f^2$, then $\frac{g+hx}{(a+bx+cx^2)(d+ex+fx^2)} =$
 $\frac{1}{q(a+bx+cx^2)} (gc^2 d - gbce + ahce + gb^2 f - abhf - agcf + c(hcd - gce + gbf - ahf)x) +$
 $\frac{1}{q(d+ex+fx^2)} (-hcde + gce^2 + bhd f - gcd f - gbef + agf^2 - f(hcd - gce + gbf - ahf)x)$

Rule 1.2.1.6.6.5: If $b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0$, let

$q = c^2 d^2 - bcde + ace^2 + b^2 df - 2acdf - abef + a^2 f^2$, if $q \neq 0$, then

$$\int \frac{g+hx}{(a+bx+cx^2)(d+ex+fx^2)} dx \rightarrow$$

$$\frac{1}{q} \int \frac{1}{a+bx+cx^2} (gc^2 d - gbce + ahce + gb^2 f - abhf - agcf + c(hcd - gce + gbf - ahf)x) dx +$$

$$\frac{1}{q} \int \frac{1}{d+ex+fx^2} (-hcde + gce^2 + bhd f - gcd f - gbef + agf^2 - f(hcd - gce + gbf - ahf)x) dx$$

Program code:

```
Int[(g_.+h_.**x_)/((a_+b_.**x_+c_.**x_^2)*(d_+e_.**x_+f_.**x_^2)),x_Symbol] :=
  With[{q=Simplify[c^2*d^2-b*c*d*e+a*c*e^2+b^2*d*f-2*a*c*d*f-a*b*e*f+a^2*f^2]},
    1/q*Int[Simp[g*c^2*d-g*b*c*e+a*h*c*e+g*b^2*f-a*b*h*f-a*g*c*f+c*(h*c*d-g*c*e+g*b*f-a*h*f)*x,x]/(a+b*x+c*x^2),x] +
    1/q*Int[Simp[-h*c*d*e+g*c*e^2+b*h*d*f-g*c*d*f-g*b*e*f+a*g*f^2-f*(h*c*d-g*c*e+g*b*f-a*h*f)*x,x]/(d+e*x+f*x^2),x] /;
    NeQ[q,0] /;
    FreeQ[{a,b,c,d,e,f,g,h},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0]
```

```

Int[(g_.+h_.**x_)/((a_+b_.**x_+c_.**x_^2)*(d_+f_.**x_^2)),x_Symbol] :=
  With[{q=Simplify[c^2*d^2+b^2*d*f-2*a*c*d*f+a^2*f^2]},
    1/q*Int[Simp[g*c^2*d+g*b^2*f-a*b*h*f-a*g*c*f+c*(h*c*d+g*b*f-a*h*f)**x,x]/(a+b*x+c*x^2),x] +
    1/q*Int[Simp[b*h*d*f-g*c*d*f+a*g*f^2-f*(h*c*d+g*b*f-a*h*f)**x,x]/(d+f*x^2),x] /;
  NeQ[q,0] /;
  FreeQ[{a,b,c,d,f,g,h},x] && NeQ[b^2-4*a*c,0]

```

$$\begin{aligned}
 6. \quad & \int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \text{ when } b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \\
 1. \quad & \int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \text{ when } b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge ce-bf = 0 \\
 1: \quad & \int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \text{ when } b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge ce-bf = 0 \wedge he-2gf = 0
 \end{aligned}$$

Derivation: Integration by substitution

Basis: If $ce-bf = 0 \wedge he-2gf = 0$, then

$$\frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} = -2g \operatorname{Subst}\left[\frac{1}{bd-ae-bx^2}, x, \sqrt{d+ex+fx^2}\right] \partial_x \sqrt{d+ex+fx^2}$$

Rule 1.2.1.6.6.1.1: If $b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge ce-bf = 0 \wedge he-2gf = 0$, then

$$\int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \rightarrow -2g \operatorname{Subst}\left[\int \frac{1}{bd-ae-bx^2} dx, x, \sqrt{d+ex+fx^2}\right]$$

Program code:

```

Int[(g_+h_.**x_)/((a_+b_.**x_+c_.**x_^2)*Sqrt[d_+e_.**x_+f_.**x_^2]),x_Symbol] :=
  -2*g*Subst[Int[1/(b*d-a*e-b*x^2),x],x,Sqrt[d+e*x+f*x^2]] /;
  FreeQ[{a,b,c,d,e,f,g,h},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && EqQ[c*e-b*f,0] && EqQ[h*e-2*g*f,0]

```

$$\mathbf{2:} \int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \text{ when } b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge ce-bf = 0 \wedge he-2gf \neq 0$$

Derivation: Algebraic expansion

$$\text{Basis: } g+hx = -\frac{he-2gf}{2f} + \frac{h(e+2fx)}{2f}$$

Rule 1.2.1.6.6.1.2: If $b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge ce-bf = 0 \wedge he-2gf \neq 0$, then

$$\int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \rightarrow -\frac{he-2gf}{2f} \int \frac{1}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx + \frac{h}{2f} \int \frac{e+2fx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx$$

Program code:

```
Int[(g_.+h_.**x_)/((a_.+b_.**x_+c_.**x_^2)*Sqrt[d_.+e_.**x_+f_.**x_^2]),x_Symbol] :=
  -(h*e-2*g*f)/(2*f)*Int[1/((a+b*x+c*x^2)*Sqrt[d+e*x+f*x^2]),x] +
  h/(2*f)*Int[(e+2*f*x)/((a+b*x+c*x^2)*Sqrt[d+e*x+f*x^2]),x] /;
FreeQ[{a,b,c,d,e,f,g,h},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && EqQ[c*e-b*f,0] && NeQ[h*e-2*g*f,0]
```

$$2. \int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \text{ when } b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge bd-ae = 0$$

$$1: \int \frac{x}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \text{ when } b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge bd-ae = 0$$

Derivation: Integration by substitution

■ Basis: If $bd-ae = 0$, then $\frac{x}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} = -2e \text{ Subst} \left[\frac{1-dx^2}{ce-bf-e(2cd-be+2af)x^2+d^2(ce-bf)x^4}, x, \frac{1+\frac{(e+\sqrt{e^2-4df})x}{2d}}{\sqrt{d+ex+fx^2}} \right] \partial_x \frac{1+\frac{(e+\sqrt{e^2-4df})x}{2d}}{\sqrt{d+ex+fx^2}}$

Alternate basis: If $bd-ae = 0$, then

$$\frac{x}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} = -2e \text{ Subst} \left[\frac{d-x^2}{d^2(ce-bf)-e(2cd-be+2af)x^2+(ce-bf)x^4}, x, \frac{2d\sqrt{d+ex+fx^2}}{2d+(e+\sqrt{e^2-4df})x} \right] \partial_x \frac{2d\sqrt{d+ex+fx^2}}{2d+(e+\sqrt{e^2-4df})x}$$

Rule 1.2.1.6.6.2.1: If $b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge bd-ae = 0$, then

$$\int \frac{x}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \rightarrow -2e \text{ Subst} \left[\int \frac{1-dx^2}{ce-bf-e(2cd-be+2af)x^2+d^2(ce-bf)x^4} dx, x, \frac{1+\frac{(e+\sqrt{e^2-4df})x}{2d}}{\sqrt{d+ex+fx^2}} \right]$$

Program code:

```
Int[x_/((a+_.*x+_.*x^2)*Sqrt[d+_.*x+_.*x^2]),x_Symbol] :=
-2*e*Subst[Int[(1-d*x^2)/(c*e-b*f-e*(2*c*d-b*e+2*a*f)*x^2+d^2*(c*e-b*f)*x^4),x],x,
(1+(e+Sqrt[e^2-4*d*f])*x/(2*d))/Sqrt[d+e*x+f*x^2]] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && EqQ[b*d-a*e,0]
```

$$2: \int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \text{ when } b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge bd-ae=0 \wedge 2hd-ge=0$$

Derivation: Integration by substitution

Basis: If $bd-ae=0 \wedge 2hd-ge=0$, then $\frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} = g \operatorname{Subst}\left[\frac{1}{a+(cd-af)x^2}, x, \frac{x}{\sqrt{d+ex+fx^2}}\right] \partial_x \frac{x}{\sqrt{d+ex+fx^2}}$

Rule 1.2.1.6.6.2.2: If $b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge bd-ae=0 \wedge 2hd-ge=0$, then

$$\int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \rightarrow g \operatorname{Subst}\left[\int \frac{1}{a+(cd-af)x^2} dx, x, \frac{x}{\sqrt{d+ex+fx^2}}\right]$$

Program code:

```
Int[(g+h_*x)/((a+b_*x+c_*x^2)*Sqrt[d+e_*x+f_*x^2]),x_Symbol] :=
  g*Subst[Int[1/(a+(c*d-a*f)*x^2),x],x,x/Sqrt[d+e*x+f*x^2]] /;
FreeQ[{a,b,c,d,e,f,g,h},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && EqQ[b*d-a*e,0] && EqQ[2*h*d-g*e,0]
```

$$\mathbf{3:} \int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \text{ when } b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge bd-ae = 0 \wedge 2hd-ge \neq 0$$

Derivation: Algebraic expansion

$$\text{Basis: } g+hx = -\frac{2hd-ge}{e} + \frac{h(2d+ex)}{e}$$

Rule 1.2.1.6.6.2.3: If $b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge bd-ae = 0 \wedge 2hd-ge \neq 0$, then

$$\int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \rightarrow -\frac{2hd-ge}{e} \int \frac{1}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx + \frac{h}{e} \int \frac{2d+ex}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx$$

Program code:

```
Int[(g+h_.x)/((a+b_.x+c_.x^2)*Sqrt[d+e_.x+f_.x^2]),x_Symbol] :=
  -(2*h*d-g*e)/e*Int[1/((a+b*x+c*x^2)*Sqrt[d+e*x+f*x^2]),x] +
  h/e*Int[(2*d+e*x)/((a+b*x+c*x^2)*Sqrt[d+e*x+f*x^2]),x] /;
FreeQ[{a,b,c,d,e,f,g,h},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && EqQ[b*d-a*e,0] && NeQ[2*h*d-g*e,0]
```

$$\mathbf{3:} \int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \text{ when } b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge bd-ae \neq 0 \wedge h^2(bd-ae) - 2gh(cd-af) + g^2(ce-bf) = 0$$

Derivation: Integration by substitution

Basis: If $h^2(bd-ae) - 2gh(cd-af) + g^2(ce-bf) = 0$,

then

$$\frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} = -2g(gb-2ah) \operatorname{Subst}\left[\frac{1}{g(gb-2ah)(b^2-4ac)-(bd-ae)x^2}, x, \frac{gb-2ah-(bh-2gc)x}{\sqrt{d+ex+fx^2}}\right] \partial_x \frac{gb-2ah-(bh-2gc)x}{\sqrt{d+ex+fx^2}}$$

Rule 1.2.1.6.6.6.3: If

$b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge bd-ae \neq 0 \wedge h^2(bd-ae) - 2gh(cd-af) + g^2(ce-bf) = 0$,
then

$$\int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \rightarrow -2g(gb-2ah) \operatorname{Subst}\left[\int \frac{1}{g(gb-2ah)(b^2-4ac)-(bd-ae)x^2} dx, x, \frac{gb-2ah-(bh-2gc)x}{\sqrt{d+ex+fx^2}}\right]$$

Program code:

```
Int[(g_.+h_.**x_)/((a_.+b_.**x_+c_.**x_^2)*Sqrt[d_.+e_.**x_+f_.**x_^2]),x_Symbol] :=
-2*g*(g*b-2*a*h)*
Subst[Int[1/Simp[g*(g*b-2*a*h)*(b^2-4*a*c)-(b*d-a*e)**x^2,x],x],x,Simp[g*b-2*a*h-(b*h-2*g*c)**x,x]/Sqrt[d+e*x+f*x^2]] /;
FreeQ[{a,b,c,d,e,f,g,h},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && NeQ[b*d-a*e,0] &&
EqQ[h^2*(b*d-a*e)-2*g*h*(c*d-a*f)+g^2*(c*e-b*f),0]
```

```
Int[(g_.+h_.**x_)/((a_.+c_.**x_^2)*Sqrt[d_.+e_.**x_+f_.**x_^2]),x_Symbol] :=
-2*a*g*h*Subst[Int[1/Simp[2*a^2*g*h*c+a*e**x^2,x],x],x,Simp[a*h-g*c*x,x]/Sqrt[d+e*x+f*x^2]] /;
FreeQ[{a,c,d,e,f,g,h},x] && EqQ[a*h^2*e+2*g*h*(c*d-a*f)-g^2*c*e,0]
```

```
Int[(g+h_.**x_)/((a_.+b_.**x_+c_.**x_^2)*Sqrt[d_.+f_.**x_^2]),x_Symbol] :=
-2*g*(g*b-2*a*h)*Subst[Int[1/Simp[g*(g*b-2*a*h)*(b^2-4*a*c)-b*d*x^2,x],x],x,Simp[g*b-2*a*h-(b*h-2*g*c)*x,x]/Sqrt[d+f*x^2]] /;
FreeQ[{a,b,c,d,f,g,h},x] && NeQ[b^2-4*a*c,0] && EqQ[b*h^2*d-2*g*h*(c*d-a*f)-g^2*b*f,0]
```

$$4. \int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \text{ when } b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge h^2(bd-ae)-2gh(cd-af)+g^2(ce-bf) \neq 0$$

$$1: \int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \text{ when } b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge b^2-4ac > 0$$

Derivation: Algebraic expansion

Basis: Let $q = \sqrt{b^2 - 4ac}$, then $\frac{g+hx}{a+bx+cx^2} = \frac{2cg-h(b-q)}{q} \frac{1}{(b-q+2cx)} - \frac{2cg-h(b+q)}{q} \frac{1}{(b+q+2cx)}$

■

Rule 1.2.1.6.6.4.1: If $b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0 \wedge b^2 - 4ac > 0$, let $q = \sqrt{b^2 - 4ac}$, then

$$\int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \rightarrow \frac{2cg-h(b-q)}{q} \int \frac{1}{(b-q+2cx)\sqrt{d+ex+fx^2}} dx - \frac{2cg-h(b+q)}{q} \int \frac{1}{(b+q+2cx)\sqrt{d+ex+fx^2}} dx$$

Program code:

```
Int[(g_.+h_.**x_)/((a_.+b_.**x_+c_.**x_^2)*Sqrt[d_.+e_.**x_+f_.**x_^2]),x_Symbol] :=
With[{q=Rt[b^2-4*a*c,2]},
(2*c*g-h*(b-q))/q*Int[1/((b-q+2*c*x)*Sqrt[d+e*x+f*x^2]),x] -
(2*c*g-h*(b+q))/q*Int[1/((b+q+2*c*x)*Sqrt[d+e*x+f*x^2]),x]] /;
FreeQ[{a,b,c,d,e,f,g,h},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && PosQ[b^2-4*a*c]
```



```
Int[(g_.+h_.**x_)/((a_+c_.**x_^2)*Sqrt[d_+e_.**x_+f_.**x_^2]),x_Symbol] :=
  With[{q=Rt[-a*c,2]},
    (h/2+c*g/(2*q))*Int[1/((-q+c*x)*Sqrt[d+e*x+f*x^2]),x] +
    (h/2-c*g/(2*q))*Int[1/((q+c*x)*Sqrt[d+e*x+f*x^2]),x] /;
  FreeQ[{a,c,d,e,f,g,h},x] && NeQ[e^2-4*d*f,0] && PosQ[-a*c]
```

```
Int[(g_.+h_.**x_)/((a_+b_.**x_+c_.**x_^2)*Sqrt[d_+f_.**x_^2]),x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    (2*c*g-h*(b-q))/q*Int[1/((b-q+2*c*x)*Sqrt[d+f*x^2]),x] -
    (2*c*g-h*(b+q))/q*Int[1/((b+q+2*c*x)*Sqrt[d+f*x^2]),x] /;
  FreeQ[{a,b,c,d,f,g,h},x] && NeQ[b^2-4*a*c,0] && PosQ[b^2-4*a*c]
```

$$2: \int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \text{ when } b^2-4ac \neq 0 \wedge e^2-4df \neq 0 \wedge b^2-4ac \not\asymp 0 \wedge bd-ae \neq 0$$

Derivation: Algebraic expansion

Note: If $b^2 - 4ac = \frac{1}{(ce-bf)^2}$, then

$$\left((b(ce-bf) - 2c(cd-af))^2 - 4c^2((cd-af)^2 - (bd-ae)(ce-bf)) \right) < 0$$

$$(cd-af)^2 - (bd-ae)(ce-bf) > 0 \text{ (noted by Martin Welz on sci.math.symbolic on 24 May 2015).}$$

Note: Resulting integrands are of the form $\frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}}$ where

$$h^2(bd-ae) - 2gh(cd-af) + g^2(ce-bf) = 0.$$

Rule 1.2.1.6.6.4.2: If $b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0 \wedge b^2 - 4ac \not\asymp 0 \wedge bd - ae \neq 0$, let

$q = \sqrt{(cd-af)^2 - (bd-ae)(ce-bf)}$, then

$$\int \frac{g+hx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}} dx \rightarrow$$

$$\frac{1}{2q} \int \frac{h(bd - ae) - g(cd - af - q) - (g(ce - bf) - h(cd - af + q))x}{(a + bx + cx^2) \sqrt{d + ex + fx^2}} dx -$$

$$\frac{1}{2q} \int \frac{h(bd - ae) - g(cd - af + q) - (g(ce - bf) - h(cd - af - q))x}{(a + bx + cx^2) \sqrt{d + ex + fx^2}} dx$$

Program code:

```
Int[(g_.+h_.x_)/((a_.+b_.x_+c_.x_^2)*Sqrt[d_.+e_.x_+f_.x_^2]),x_Symbol] :=
  With[{q=Rt[(c*d-a*f)^2-(b*d-a*e)*(c*e-b*f),2]},
    1/(2*q)*Int[Simp[h*(b*d-a*e)-g*(c*d-a*f-q)-(g*(c*e-b*f)-h*(c*d-a*f+q))*x,x]/((a+b*x+c*x^2)*Sqrt[d+e*x+f*x^2]),x] -
    1/(2*q)*Int[Simp[h*(b*d-a*e)-g*(c*d-a*f+q)-(g*(c*e-b*f)-h*(c*d-a*f-q))*x,x]/((a+b*x+c*x^2)*Sqrt[d+e*x+f*x^2]),x] /;
  FreeQ[{a,b,c,d,e,f,g,h},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0] && NeQ[b*d-a*e,0] && NegQ[b^2-4*a*c]
```

```
Int[(g_.+h_.x_)/((a_.+c_.x_^2)*Sqrt[d_.+e_.x_+f_.x_^2]),x_Symbol] :=
  With[{q=Rt[(c*d-a*f)^2+a*c*e^2,2]},
    1/(2*q)*Int[Simp[-a*h*e-g*(c*d-a*f-q)+(h*(c*d-a*f+q)-g*c*e)*x,x]/((a+c*x^2)*Sqrt[d+e*x+f*x^2]),x] -
    1/(2*q)*Int[Simp[-a*h*e-g*(c*d-a*f+q)+(h*(c*d-a*f-q)-g*c*e)*x,x]/((a+c*x^2)*Sqrt[d+e*x+f*x^2]),x] /;
  FreeQ[{a,c,d,e,f,g,h},x] && NeQ[e^2-4*d*f,0] && NegQ[-a*c]
```

```
Int[(g_.+h_.x_)/((a_.+b_.x_+c_.x_^2)*Sqrt[d_.+f_.x_^2]),x_Symbol] :=
  With[{q=Rt[(c*d-a*f)^2+b^2*d*f,2]},
    1/(2*q)*Int[Simp[h*b*d-g*(c*d-a*f-q)+(h*(c*d-a*f+q)+g*b*f)*x,x]/((a+b*x+c*x^2)*Sqrt[d+f*x^2]),x] -
    1/(2*q)*Int[Simp[h*b*d-g*(c*d-a*f+q)+(h*(c*d-a*f-q)+g*b*f)*x,x]/((a+b*x+c*x^2)*Sqrt[d+f*x^2]),x] /;
  FreeQ[{a,b,c,d,f,g,h},x] && NeQ[b^2-4*a*c,0] && NegQ[b^2-4*a*c]
```

7: $\int \frac{g+hx}{\sqrt{a+bx+cx^2} \sqrt{d+ex+fx^2}} dx$ when $b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0$

Derivation: Piecewise constant extraction

Basis: Let $s \rightarrow \sqrt{b^2 - 4ac}$, then $\partial_x \frac{\sqrt{b+s+2cx} \sqrt{2a+(b+s)x}}{\sqrt{a+bx+cx^2}} = 0$

■

Rule 1.2.1.6.6.7: If $b^2 - 4ac \neq 0 \wedge e^2 - 4df \neq 0$, let $s \rightarrow \sqrt{b^2 - 4ac}$ and $t \rightarrow \sqrt{e^2 - 4df}$, then

$$\int \frac{g+hx}{\sqrt{a+bx+cx^2} \sqrt{d+ex+fx^2}} dx \rightarrow \frac{\sqrt{b+s+2cx} \sqrt{2a+(b+s)x} \sqrt{e+t+2fx} \sqrt{2d+(e+t)x}}{\sqrt{a+bx+cx^2} \sqrt{d+ex+fx^2}} \int \frac{g+hx}{\sqrt{b+s+2cx} \sqrt{2a+(b+s)x} \sqrt{e+t+2fx} \sqrt{2d+(e+t)x}} dx$$

Program code:

```
Int[(g_.+h_.x_)/(Sqrt[a_+b_.x_+c_.x_^2]*Sqrt[d_+e_.x_+f_.x_^2]),x_Symbol] :=
  With[{s=Rt[b^2-4*a*c,2],t=Rt[e^2-4*d*f,2]},
    Sqrt[b+s+2*c*x]*Sqrt[2*a+(b+s)*x]*Sqrt[e+t+2*f*x]*Sqrt[2*d+(e+t)*x]/(Sqrt[a+b*x+c*x^2]*Sqrt[d+e*x+f*x^2])*
    Int[(g+h*x)/(Sqrt[b+s+2*c*x]*Sqrt[2*a+(b+s)*x]*Sqrt[e+t+2*f*x]*Sqrt[2*d+(e+t)*x]),x] /;
  FreeQ[{a,b,c,d,e,f,g,h},x] && NeQ[b^2-4*a*c,0] && NeQ[e^2-4*d*f,0]
```

```
Int[(g_.+h_.x_)/(Sqrt[a_+b_.x_+c_.x_^2]*Sqrt[d_+f_.x_^2]),x_Symbol] :=
  With[{s=Rt[b^2-4*a*c,2],t=Rt[-4*d*f,2]},
    Sqrt[b+s+2*c*x]*Sqrt[2*a+(b+s)*x]*Sqrt[t+2*f*x]*Sqrt[2*d+t*x]/(Sqrt[a+b*x+c*x^2]*Sqrt[d+f*x^2])*
    Int[(g+h*x)/(Sqrt[b+s+2*c*x]*Sqrt[2*a+(b+s)*x]*Sqrt[t+2*f*x]*Sqrt[2*d+t*x]),x] /;
  FreeQ[{a,b,c,d,f,g,h},x] && NeQ[b^2-4*a*c,0]
```

$$8. \int \frac{g+hx}{(a+bx+cx^2)^{1/3} (d+ex+fx^2)} dx \text{ when } ce-bf = 0 \wedge c^2d-f(b^2-3ac) = 0 \wedge c^2g^2-bcg h-2b^2h^2+9ac h^2 = 0$$

$$1: \int \frac{g+hx}{(a+bx+cx^2)^{1/3} (d+ex+fx^2)} dx \text{ when } ce-bf = 0 \wedge c^2d-f(b^2-3ac) = 0 \wedge c^2g^2-bcg h-2b^2h^2+9ac h^2 = 0 \wedge -\frac{9ch^2}{(2cg-bh)^2} > 0$$

Derived from formula for this class of Goursat pseudo-elliptic integrands contributed by Martin Welz on 19 August 2016

Rule 1.2.1.6.6.8.1: If

$$ce-bf = 0 \wedge c^2d-f(b^2-3ac) = 0 \wedge c^2g^2-bcg h-2b^2h^2+9ac h^2 = 0 \wedge -\frac{9ch^2}{(2cg-bh)^2} > 0, \text{ let}$$

$$q \rightarrow \left(-\frac{9ch^2}{(2cg-bh)^2}\right)^{1/3}, \text{ then}$$

$$\int \frac{g+hx}{(a+bx+cx^2)^{1/3} (d+ex+fx^2)} dx \rightarrow$$

$$\frac{\sqrt{3} h q \operatorname{ArcTan}\left[\frac{1}{\sqrt{3}} - \frac{2^{2/3} \left(1 - \frac{3h(b+2cx)}{2cg-bh}\right)^{2/3}}{\sqrt{3} \left(1 + \frac{3h(b+2cx)}{2cg-bh}\right)^{1/3}}\right]}{f} + \frac{h q \operatorname{Log}[d+ex+fx^2]}{2f} - \frac{3 h q \operatorname{Log}\left[\left(1 - \frac{3h(b+2cx)}{2cg-bh}\right)^{2/3} + 2^{1/3} \left(1 + \frac{3h(b+2cx)}{2cg-bh}\right)^{1/3}\right]}{2f}$$

Program code:

```
Int[(g_.+h_.*x_)/((a_.+b_.*x_+c_.*x_^2)^(1/3)*(d_.+e_.*x_+f_.*x_^2)),x_Symbol] :=
  With[{q=(-9*c*h^2/(2*c*g-b*h)^2)^(1/3)},
    Sqrt[3]*h*q*ArcTan[1/Sqrt[3]-2^(2/3)*(1-(3*h*(b+2*c*x))/(2*c*g-b*h))^(2/3)/(Sqrt[3]*(1+(3*h*(b+2*c*x))/(2*c*g-b*h))^(1/3))]/f
    h*q*Log[d+e*x+f*x^2]/(2*f) -
    3*h*q*Log[(1-3*h*(b+2*c*x)/(2*c*g-b*h))^(2/3)+2^(1/3)*(1+3*h*(b+2*c*x)/(2*c*g-b*h))^(1/3)]/(2*f)] /;
  FreeQ[{a,b,c,d,e,f,g,h},x] && EqQ[c*e-b*f,0] && EqQ[c^2*d-f*(b^2-3*a*c),0] && EqQ[c^2*g^2-b*c*g*h-2*b^2*h^2+9*a*c*h^2,0] &&
  GtQ[-9*c*h^2/(2*c*g-b*h)^2,0]
```

$$2: \int \frac{g+hx}{(a+bx+cx^2)^{1/3} (d+ex+fx^2)} dx \text{ when } ce-bf = 0 \wedge c^2d-f(b^2-3ac) = 0 \wedge c^2g^2-bcg h-2b^2h^2+9ach^2 = 0 \wedge 4a-\frac{b^2}{c} \neq 0$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{(q(a+bx+cx^2))^{1/3}}{(a+bx+cx^2)^{1/3}} = 0$$

Rule 1.2.1.6.6.8.2: If

$ce-bf = 0 \wedge c^2d-f(b^2-3ac) = 0 \wedge c^2g^2-bcg h-2b^2h^2+9ach^2 = 0 \wedge 4a-\frac{b^2}{c} \neq 0$, let $q \rightarrow -\frac{c}{b^2-4ac}$, then

$$\int \frac{g+hx}{(a+bx+cx^2)^{1/3} (d+ex+fx^2)} dx \rightarrow \frac{(q(a+bx+cx^2))^{1/3}}{(a+bx+cx^2)^{1/3}} \int \frac{g+hx}{(qa+bx+cx^2)^{1/3} (d+ex+fx^2)} dx$$

Program code:

```
Int[(g_.+h_.x_)/((a_.+b_.x_+c_.x_^2)^(1/3)*(d_.+e_.x_+f_.x_^2)),x_Symbol] :=
  With[{q=-c/(b^2-4*a*c)},
    (q*(a+b*x+c*x^2))^(1/3)/(a+b*x+c*x^2)^(1/3)*Int[(g+h*x)/((q*a+b*q*x+c*q*x^2)^(1/3)*(d+e*x+f*x^2)),x] /;
  FreeQ[{a,b,c,d,e,f,g,h},x] && EqQ[c*e-b*f,0] && EqQ[c^2*d-f*(b^2-3*a*c),0] && EqQ[c^2*g^2-b*c*g*h-2*b^2*h^2+9*a*c*h^2,0] && Not[
```

U: $\int (g+hx) (a+bx+cx^2)^p (d+ex+fx^2)^q dx$

Rule 1.2.1.6.6.X:

$$\int (g+hx) (a+bx+cx^2)^p (d+ex+fx^2)^q dx \rightarrow \int (g+hx) (a+bx+cx^2)^p (d+ex+fx^2)^q dx$$

Program code:

```
Int[(a_.+b_.**x_+c_.**x_^2)^p_*(d_.+e_.**x_+f_.**x_^2)^q_*(g_.+h_.**x_),x_Symbol] :=
  Unintegrable[(a+b*x+c*x^2)^p*(d+e*x+f*x^2)^q*(g+h*x),x] /;
FreeQ[{a,b,c,d,e,f,g,h,p,q},x]
```

```
Int[(a_.+c_.**x_^2)^p_*(d_.+e_.**x_+f_.**x_^2)^q_*(g_.+h_.**x_),x_Symbol] :=
  Unintegrable[(a+c*x^2)^p*(d+e*x+f*x^2)^q*(g+h*x),x] /;
FreeQ[{a,c,d,e,f,g,h,p,q},x]
```

S: $\int (g+hu)^m (a+bu+cu^2)^p (d+eu+fu^2)^q dx$ when $u = g+hx$

Derivation: Integration by substitution

Rule 1.2.1.6.S: If $u = g+hx$, then

$$\int (g+hu)^m (a+bu+cu^2)^p (d+eu+fu^2)^q dx \rightarrow \frac{1}{h} \text{Subst}\left[\int (g+hx)^m (a+bx+cx^2)^p (d+ex+fx^2)^q dx, x, u\right]$$

Program code:

```
Int[(g_.+h_.**u_)^m_*(a_.+b_.**u_+c_.**u_^2)^p_*(d_.+e_.**u_+f_.**u_^2)^q_,x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(g+h*x)^m*(a+b*x+c*x^2)^p*(d+e*x+f*x^2)^q,x],x,u] /;
FreeQ[{a,b,c,d,e,f,g,h,m,p,q},x] && LinearQ[u,x] && NeQ[u,x]
```

```

Int[(g_.+h_.*u_)^m_.*(a_.+c_.*u_^2)^p_.*(d_.+e_.*u_+f_.*u_^2)^q_. ,x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(g+h*x)^m*(a+c*x^2)^p*(d+e*x+f*x^2)^q,x],x,u] /;
FreeQ[{a,c,d,e,f,g,h,m,p,q},x] && LinearQ[u,x] && NeQ[u,x]

```