

## Rules for normalizing algebraic functions

**2:**  $\int x^m (e (a + b x^n)^r)^p (f (c + d x^n)^s)^q dx$

Derivation: Piecewise constant extraction

Basis:  $\partial_x \frac{(e (a + b x^n)^r)^p (f (c + d x^n)^s)^q}{(a + b x^n)^{p r} (c + d x^n)^{q s}} = 0$

Rule 1.5.4.2:

$$\int x^m (e (a + b x^n)^r)^p (f (c + d x^n)^s)^q dx \rightarrow \frac{(e (a + b x^n)^r)^p (f (c + d x^n)^s)^q}{(a + b x^n)^{p r} (c + d x^n)^{q s}} \int x^m (a + b x^n)^{p r} (c + d x^n)^{q s} dx$$

Program code:

```
Int[x^m.*(e.*(a+b.*x^n.)^r.)^p.*(f.*(c+d.*x^n.)^s.)^q_,x_Symbol] :=
  (e*(a+b*x^n)^r)^p*(f*(c+d*x^n)^s)^q/((a+b*x^n)^(p*r)*(c+d*x^n)^(q*s))*
  Int[x^m*(a+b*x^n)^(p*r)*(c+d*x^n)^(q*s),x] /;
FreeQ[{a,b,c,d,e,f,m,n,p,q,r,s},x]
```

$$3. \int u \left( e^{\frac{a + b x^n}{c + d x^n}} \right)^p dx$$

$$\text{1: } \int u \left( e^{\frac{a + b x^n}{c + d x^n}} \right)^p dx \text{ when } b c - a d = 0$$

Derivation: Algebraic simplification

Basis: If  $b c - a d = 0$ , then  $\frac{a+bz}{c+dz} = \frac{b}{d}$

Rule 1.5.4.3.1: If  $b c - a d = 0$ , then

$$\int u \left( e^{\frac{a + b x^n}{c + d x^n}} \right)^p dx \rightarrow \left( \frac{b e}{d} \right)^p \int u dx$$

Program code:

```
Int[u_.*(e_.*(a_+b_.*x_^n_)/(c_+d_.*x_^n_))^p_,x_Symbol] :=
  (b*e/d)^p*Int[u,x] /;
FreeQ[{a,b,c,d,e,n,p},x] && EqQ[b*c-a*d,0]
```

$$2. \int u \left( e^{\frac{a+b x^n}{c+d x^n}} \right)^p dx \text{ when } b c - a d \neq 0$$

$$1: \int u \left( e^{\frac{a+b x^n}{c+d x^n}} \right)^p dx \text{ when } b d e > 0 \wedge c < \frac{a d}{b}$$

Derivation: Algebraic simplification

$$\text{Basis: If } b d e > 0 \wedge \frac{a d}{b} \leq c, \text{ then } \left( e^{\frac{a+b z}{c+d z}} \right)^p == \frac{(e^{(a+b z)})^p}{(c+d z)^p}$$

Rule 1.5.4.3.2.1: If  $b d e > 0 \wedge c < \frac{a d}{b}$ , then

$$\int u \left( e^{\frac{a+b x^n}{c+d x^n}} \right)^p dx \rightarrow \int \frac{u \left( e^{(a+b x^n)} \right)^p}{(c+d x^n)^p} dx$$

Program code:

```
Int[u_.*(e_.*(a_+b_.*x_^n_.)/(c_+d_.*x_^n_.))^p_,x_Symbol] :=
  Int[u*(e*(a+b*x^n))^p/(c+d*x^n)^p,x] /;
  FreeQ[{a,b,c,d,e,n,p},x] && GtQ[b*d*e,0] && GtQ[c-a*d/b,0]
```

$$2. \int u \left( e^{\frac{a+b x^n}{c+d x^n}} \right)^p dx \text{ when } \neg (b d e > 0 \wedge \frac{a d}{b} \leq c)$$

$$1: \int \left( e^{\frac{a+b x^n}{c+d x^n}} \right)^p dx \text{ when } \frac{1}{n} \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If  $\frac{1}{n} \in \mathbb{Z} \wedge q \in \mathbb{Z}^+$ , then

$$\left( e^{\frac{a+b x^n}{c+d x^n}} \right)^p == \frac{q e^{(b c - a d)}}{n} \text{Subst} \left[ \frac{x^q (p+1)^{-1} (-a e + c x^q)^{\frac{1}{n}-1}}{(b e - d x^q)^{\frac{1}{n}+1}}, x, \left( e^{\frac{a+b x^n}{c+d x^n}} \right)^{1/q} \right] \partial_x \left( e^{\frac{a+b x^n}{c+d x^n}} \right)^{1/q}$$

Rule 1.5.4.3.2.2.1: If  $\frac{1}{n} \in \mathbb{Z}$ , let  $q = \text{Denominator}[p]$ , then

$$\int \left( e \frac{a + b x^n}{c + d x^n} \right)^p dx \rightarrow \frac{q e (b c - a d)}{n} \text{Subst} \left[ \int \frac{x^{q(p+1)-1} (-a e + c x^q)^{\frac{1}{n}-1}}{(b e - d x^q)^{\frac{1}{n}+1}} dx, x, \left( e \frac{a + b x^n}{c + d x^n} \right)^{1/q} \right]$$

Program code:

```
Int[(e.*(a_.+b_.*x.^n_.)/(c_.+d_.*x.^n_.))^p_,x_Symbol] :=
  With[{q=Denominator[p]},
    q*e*(b*c-a*d)/n*Subst[
      Int[x^(q*(p+1)-1)*(-a*e+c*x^q)^(1/n-1)/(b*e-d*x^q)^(1/n+1),x],x,(e*(a+b*x^n)/(c+d*x^n))^(1/q)] /;
  FreeQ[{a,b,c,d,e},x] && FractionQ[p] && IntegerQ[1/n]
```

**2:**  $\int x^m \left( e \frac{a + b x^n}{c + d x^n} \right)^p dx$  when  $\frac{m+1}{n} \in \mathbb{Z}$

Derivation: Integration by substitution

Basis: If  $\frac{m+1}{n} \in \mathbb{Z} \wedge q \in \mathbb{Z}^+$ , then

$$x^m \left( e \frac{a + b x^n}{c + d x^n} \right)^p = \frac{q e (b c - a d)}{n} \text{Subst} \left[ \frac{x^{q(p+1)-1} (-a e + c x^q)^{\frac{m+1}{n}-1}}{(b e - d x^q)^{\frac{m+1}{n}+1}}, x, \left( e \frac{a + b x^n}{c + d x^n} \right)^{1/q} \right] \partial_x \left( e \frac{a + b x^n}{c + d x^n} \right)^{1/q}$$

Rule 1.5.4.3.2.2.2: If  $\frac{m+1}{n} \in \mathbb{Z}$ , let  $q = \text{Denominator}[p]$ , then

$$\int x^m \left( e \frac{a + b x^n}{c + d x^n} \right)^p dx \rightarrow \frac{q e (b c - a d)}{n} \text{Subst} \left[ \int \frac{x^{q(p+1)-1} (-a e + c x^q)^{\frac{m+1}{n}-1}}{(b e - d x^q)^{\frac{m+1}{n}+1}} dx, x, \left( e \frac{a + b x^n}{c + d x^n} \right)^{1/q} \right]$$

Program code:

```
Int[x.^m.*(e.*(a_.+b_.*x.^n_.)/(c_.+d_.*x.^n_.))^p_,x_Symbol] :=
  With[{q=Denominator[p]},
    q*e*(b*c-a*d)/n*Subst[
      Int[x^(q*(p+1)-1)*(-a*e+c*x^q)^(Simplify[(m+1)/n]-1)/(b*e-d*x^q)^(Simplify[(m+1)/n]+1),x],x,(e*(a+b*x^n)/(c+d*x^n))^(1/q)] /;
  FreeQ[{a,b,c,d,e,m,n},x] && FractionQ[p] && IntegerQ[Simplify[(m+1)/n]]
```

$$\mathbf{3:} \int \mathbf{P}_x^r \left( e^{\frac{a+b x^n}{c+d x^n}} \right)^p dx \text{ when } \frac{1}{n} \in \mathbb{Z} \wedge r \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If  $\frac{1}{n} \in \mathbb{Z} \wedge q \in \mathbb{Z}^+$ , then

$$\mathbf{F}[x] \left( e^{\frac{a+b x^n}{c+d x^n}} \right)^p == \frac{q e (b c - a d)}{n} \text{Subst} \left[ \frac{x^{q(p+1)-1} (-a e + c x^q)^{\frac{1}{n}-1}}{(b e - d x^q)^{\frac{1}{n}+1}} \mathbf{F} \left[ \frac{(-a e + c x^q)^{\frac{1}{n}}}{(b e - d x^q)^{\frac{1}{n}}} \right], x, \left( e^{\frac{a+b x^n}{c+d x^n}} \right)^{1/q} \right] \partial_x \left( e^{\frac{a+b x^n}{c+d x^n}} \right)^{1/q}$$

Rule 1.5.4.3.2.2.3: If  $\frac{1}{n} \in \mathbb{Z}$ , let  $q = \text{Denominator}[p]$ , then

$$\int \mathbf{P}_x^r \left( e^{\frac{a+b x^n}{c+d x^n}} \right)^p dx \rightarrow \frac{q e (b c - a d)}{n} \text{Subst} \left[ \int \frac{x^{q(p+1)-1} (-a e + c x^q)^{\frac{1}{n}-1}}{(b e - d x^q)^{\frac{1}{n}+1}} \text{Subst} \left[ \mathbf{P}_x, x, \frac{(-a e + c x^q)^{\frac{1}{n}}}{(b e - d x^q)^{\frac{1}{n}}} \right]^r dx, x, \left( e^{\frac{a+b x^n}{c+d x^n}} \right)^{1/q} \right]$$

Program code:

```
Int[u_^r.*(e.*(a_.+b_.*x_^n_.)/(c_.+d_.*x_^n_.))^p_,x_Symbol] :=
  With[{q=Denominator[p]},
    q*e*(b*c-a*d)/n*Subst[Int[SimplifyIntegrand[x^(q*(p+1)-1)*(-a*e+c*x^q)^(1/n-1)/(b*e-d*x^q)^(1/n+1)*
      ReplaceAll[u,x*(-a*e+c*x^q)^(1/n)/(b*e-d*x^q)^(1/n)]^r,x],x],x,(e*(a+b*x^n)/(c+d*x^n))^(1/q)]] /;
    FreeQ[{a,b,c,d,e},x] && PolynomialQ[u,x] && FractionQ[p] && IntegerQ[1/n] && IntegerQ[r]
```

$$\mathbf{4:} \int x^m \mathbf{P}_x^r \left( e^{\frac{a+b x^n}{c+d x^n}} \right)^p dx \text{ when } \frac{1}{n} \in \mathbb{Z} \wedge (m \mid r) \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If  $\frac{1}{n} \in \mathbb{Z} \wedge m \in \mathbb{Z} \wedge q \in \mathbb{Z}^+$ , then

$$x^m F[x] \left( e^{\frac{a+bx^n}{c+dx^n}} \right)^p ==$$

$$\frac{q e^{(bc-ad)}}{n} \text{Subst} \left[ \frac{x^{q(p+1)-1} (-ae+cx^q)^{\frac{m+1}{n}-1}}{(be-dx^q)^{\frac{m+1}{n}+1}} F \left[ \frac{(-ae+cx^q)^{\frac{1}{n}}}{(be-dx^q)^{\frac{1}{n}}} \right], x, \left( e^{\frac{a+bx^n}{c+dx^n}} \right)^{1/q} \right] \partial_x \left( e^{\frac{a+bx^n}{c+dx^n}} \right)^{1/q}$$

Rule 1.5.4.3.2.2.4: If  $\frac{1}{n} \in \mathbb{Z} \wedge (m \mid r) \in \mathbb{Z}$ , let  $q = \text{Denominator}[p]$ , then

$$\int x^m P_x^r \left( e^{\frac{a+bx^n}{c+dx^n}} \right)^p dx \rightarrow \frac{q e^{(bc-ad)}}{n} \text{Subst} \left[ \int \frac{x^{q(p+1)-1} (-ae+cx^q)^{\frac{m+1}{n}-1}}{(be-dx^q)^{\frac{m+1}{n}+1}} \text{Subst} \left[ P_x, x, \frac{(-ae+cx^q)^{\frac{1}{n}}}{(be-dx^q)^{\frac{1}{n}}} \right]^r dx, x, \left( e^{\frac{a+bx^n}{c+dx^n}} \right)^{1/q} \right]$$

Program code:

```
Int[x^m_.*u^r_.*(e_.*(a_.+b_.*x^n_.)/(c_.+d_.*x^n_.))^p_,x_Symbol] :=
  With[{q=Denominator[p]},
    q*e*(b*c-a*d)/n*Subst[Int[SimplifyIntegrand[x^(q*(p+1)-1)*(-a*e+c*x^q)^((m+1)/n-1)/(b*e-d*x^q)^((m+1)/n+1)*
      ReplaceAll[u,x*(-a*e+c*x^q)^(1/n)/(b*e-d*x^q)^(1/n)]^r,x],x],x,(e*(a+b*x^n)/(c+d*x^n))^(1/q)]] /;
  FreeQ[{a,b,c,d,e},x] && PolynomialQ[u,x] && FractionQ[p] && IntegerQ[1/n] && IntegersQ[m,r]
```

$$4. \int u \left( a + b \left( \frac{c}{x} \right)^n \right)^p dx$$

$$1: \int \left( a + b \left( \frac{c}{x} \right)^n \right)^p dx$$

Derivation: Integration by substitution

$$\text{Basis: } F \left[ \frac{c}{x} \right] == -c \text{ Subst} \left[ \frac{F[x]}{x^2}, x, \frac{c}{x} \right] \partial_x \frac{c}{x}$$

Rule 1.5.4.4.1:

$$\int \left( a + b \left( \frac{c}{x} \right)^n \right)^p dx \rightarrow -c \text{ Subst} \left[ \int \frac{(a + b x^n)^p}{x^2} dx, x, \frac{c}{x} \right]$$

Program code:

```
Int[(a_.+b_.*(c_/x_)^n_)^p_,x_Symbol] :=
  -c*Subst[Int[(a+b*x^n)^p/x^2,x],x,c/x] /;
FreeQ[{a,b,c,n,p},x]
```

$$2. \int (dx)^m \left( a + b \left( \frac{c}{x} \right)^n \right)^p dx$$

$$1: \int x^m \left( a + b \left( \frac{c}{x} \right)^n \right)^p dx \text{ when } m \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If  $m \in \mathbb{Z}$ , then  $x^m F\left[\frac{c}{x}\right] == -c^{m+1} \text{Subst}\left[\frac{F[x]}{x^{m+2}}, x, \frac{c}{x}\right] \partial_x \frac{c}{x}$

Rule 1.5.4.4.2.1: If  $m \in \mathbb{Z}$ , then

$$\int x^m \left( a + b \left( \frac{c}{x} \right)^n \right)^p dx \rightarrow -c^{m+1} \text{Subst}\left[\int \frac{(a + b x^n)^p}{x^{m+2}} dx, x, \frac{c}{x}\right]$$

Program code:

```
Int[x_^m_.*(a_.+b_.*(c_/x_)^n_)^p_,x_Symbol] :=
  -c^(m+1)*Subst[Int[(a+b*x^n)^p/x^(m+2),x],x,c/x] /;
FreeQ[{a,b,c,n,p},x] && IntegerQ[m]
```



**2:**  $\int (d x)^m \left(a + b \left(\frac{c}{x}\right)^n\right)^p dx$  when  $m \notin \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

Basis:  $\partial_x \left( (d x)^m \left(\frac{c}{x}\right)^m \right) = 0$

Basis:  $F\left[\frac{c}{x}\right] = -c \text{Subst}\left[\frac{F[x]}{x^2}, x, \frac{c}{x}\right] \partial_x \frac{c}{x}$

Rule 1.5.4.4.2.2: If  $m \notin \mathbb{Z}$ , then

$$\int (d x)^m \left(a + b \left(\frac{c}{x}\right)^n\right)^p dx \rightarrow (d x)^m \left(\frac{c}{x}\right)^m \int \frac{\left(a + b \left(\frac{c}{x}\right)^n\right)^p}{\left(\frac{c}{x}\right)^m} dx \rightarrow -c (d x)^m \left(\frac{c}{x}\right)^m \text{Subst}\left[\int \frac{(a + b x^n)^p}{x^{m+2}} dx, x, \frac{c}{x}\right]$$

Program code:

```
Int[(d_*x_)^m_*(a_+b_*(c_/x_)^n_)^p_,x_Symbol] :=
  -c*(d*x)^m*(c/x)^m*Subst[Int[(a+b*x^n)^p/x^(m+2),x],x,c/x] /;
FreeQ[{a,b,c,d,m,n,p},x] && Not[IntegerQ[m]]
```

$$5. \int u \left( a + b \left( \frac{d}{x} \right)^n + c \left( \frac{d}{x} \right)^{2n} \right)^p dx$$

$$1: \int \left( a + b \left( \frac{d}{x} \right)^n + c \left( \frac{d}{x} \right)^{2n} \right)^p dx$$

Derivation: Integration by substitution

$$\text{Basis: } F \left[ \frac{d}{x} \right] == -d \text{ Subst} \left[ \frac{F[x]}{x^2}, x, \frac{d}{x} \right] \partial_x \frac{d}{x}$$

Rule 1.5.4.5.1:

$$\int \left( a + b \left( \frac{d}{x} \right)^n + c \left( \frac{d}{x} \right)^{2n} \right)^p dx \rightarrow -d \text{ Subst} \left[ \int \frac{(a + b x^n + c x^{2n})^p}{x^2} dx, x, \frac{d}{x} \right]$$

Program code:

```
Int[(a_.+b_.*(d_/x_)^n_+c_.*(d_/x_)^n2_.)^p_,x_Symbol] :=
  -d*Subst[Int[(a+b*x^n+c*x^(2*n))^p/x^2,x],x,d/x] /;
FreeQ[{a,b,c,d,n,p},x] && EqQ[n2,2*n]
```

$$2. \int (e x)^m \left( a + b \left( \frac{d}{x} \right)^n + c \left( \frac{d}{x} \right)^{2n} \right)^p dx$$

$$1: \int x^m \left( a + b \left( \frac{d}{x} \right)^n + c \left( \frac{d}{x} \right)^{2n} \right)^p dx \text{ when } m \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If  $m \in \mathbb{Z}$ , then  $x^m F \left[ \frac{d}{x} \right] == -d^{m+1} \text{Subst} \left[ \frac{F[x]}{x^{m+2}}, x, \frac{d}{x} \right] \partial_x \frac{d}{x}$

Rule 1.5.4.5.2.1: If  $m \in \mathbb{Z}$ , then

$$\int x^m \left( a + b \left( \frac{d}{x} \right)^n + c \left( \frac{d}{x} \right)^{2n} \right)^p dx \rightarrow -d^{m+1} \text{Subst} \left[ \int \frac{(a + b x^n + c x^{2n})^p}{x^{m+2}} dx, x, \frac{d}{x} \right]$$

Program code:

```
Int[x_^m_.*(a_+b_.*(d_/x_)^n+c_.*(d_/x_)^2n)^p_,x_Symbol] :=
  -d^(m+1)*Subst[Int[(a+b*x^n+c*x^(2*n))^p/x^(m+2),x],x,d/x] /;
FreeQ[{a,b,c,d,n,p},x] && EqQ[n2,2*n] && IntegerQ[m]
```

$$2: \int (e x)^m \left( a + b \left( \frac{d}{x} \right)^n + c \left( \frac{d}{x} \right)^{2n} \right)^p dx \text{ when } m \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction and integration by substitution

Basis:  $\partial_x \left( (e x)^m \left( \frac{d}{x} \right)^m \right) == 0$

Basis:  $F \left[ \frac{d}{x} \right] == -d \text{Subst} \left[ \frac{F[x]}{x^2}, x, \frac{d}{x} \right] \partial_x \frac{d}{x}$

Rule 1.5.4.5.2.2: If  $m \notin \mathbb{Z}$ , then

$$\int (e x)^m \left( a + b \left( \frac{d}{x} \right)^n + c \left( \frac{d}{x} \right)^{2n} \right)^p dx \rightarrow (e x)^m \left( \frac{d}{x} \right)^m \int \frac{\left( a + b \left( \frac{d}{x} \right)^n + c \left( \frac{d}{x} \right)^{2n} \right)^p}{\left( \frac{d}{x} \right)^m} dx \rightarrow -d (e x)^m \left( \frac{d}{x} \right)^m \text{Subst} \left[ \int \frac{(a + b x^n + c x^{2n})^p}{x^{m+2}} dx, x, \frac{d}{x} \right]$$

Program code:

```
Int[(e_.*x_)^m_*(a_+b_*(d_/x_)^n_+c_*(d_/x_)^n2_)^p_,x_Symbol] :=
  -d*(e*x)^m*(d/x)^m*Subst[Int[(a+b*x^n+c*x^(2*n))^p/x^(m+2),x],x,d/x] /;
FreeQ[{a,b,c,d,e,m,n,p},x] && EqQ[n2,2*n] && Not[IntegerQ[m]]
```

6.  $\int u \left( a + b \left( \frac{d}{x} \right)^n + c x^{-2n} \right)^p dx$  when  $2n \in \mathbb{Z}$

**1:**  $\int \left( a + b \left( \frac{d}{x} \right)^n + c x^{-2n} \right)^p dx$  when  $2n \in \mathbb{Z}$

Derivation: Integration by substitution

Basis:  $F \left[ \frac{d}{x} \right] == -d \text{Subst} \left[ \frac{F[x]}{x^2}, x, \frac{d}{x} \right] \partial_x \frac{d}{x}$

Rule 1.5.4.6.1: If  $2n \in \mathbb{Z}$ , then

$$\int \left( a + b \left( \frac{d}{x} \right)^n + c x^{-2n} \right)^p dx \rightarrow \int \left( a + b \left( \frac{d}{x} \right)^n + \frac{c}{d^{2n}} \left( \frac{d}{x} \right)^{2n} \right)^p dx \rightarrow -d \text{Subst} \left[ \int \frac{(a + b x^n + \frac{c}{d^{2n}} x^{2n})^p}{x^2} dx, x, \frac{d}{x} \right]$$

Program code:

```
Int[(a_+b_*(d_/x_)^n_+c_/x_^(2*n_)^p_,x_Symbol] :=
  -d*Subst[Int[(a+b*x^n+c/d^(2*n))*x^(2*n))^p/x^2,x],x,d/x] /;
FreeQ[{a,b,c,d,n,p},x] && EqQ[n2,-2*n] && IntegerQ[2*n]
```

$$2. \int (e x)^m \left( a + b \left( \frac{d}{x} \right)^n + c x^{-2n} \right)^p dx \text{ when } 2n \in \mathbb{Z}$$

$$1: \int x^m \left( a + b \left( \frac{d}{x} \right)^n + c x^{-2n} \right)^p dx \text{ when } 2n \in \mathbb{Z} \wedge m \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If  $m \in \mathbb{Z}$ , then  $x^m F\left[\frac{d}{x}\right] = -d^{m+1} \text{Subst}\left[\frac{F[x]}{x^{m+2}}, x, \frac{d}{x}\right] \partial_x \frac{d}{x}$

Rule 1.5.4.6.2.1: If  $2n \in \mathbb{Z} \wedge m \in \mathbb{Z}$ , then

$$\int x^m \left( a + b \left( \frac{d}{x} \right)^n + c x^{-2n} \right)^p dx \rightarrow \int x^m \left( a + b \left( \frac{d}{x} \right)^n + \frac{c}{d^{2n}} \left( \frac{d}{x} \right)^{2n} \right)^p dx \rightarrow -d^{m+1} \text{Subst}\left[\int \frac{(a + b x^n + \frac{c}{d^{2n}} x^{2n})^p}{x^{m+2}} dx, x, \frac{d}{x}\right]$$

Program code:

```
Int[x_^m_.*(a_+b_.*(d_/x_)^n_+c_.*x_^n2_)^p_,x_Symbol] :=
  -d^(m+1)*Subst[Int[(a+b*x^n+c/d^(2*n))*x^(2*n)]^p/x^(m+2),x],x,d/x] /;
FreeQ[{a,b,c,d,n,p},x] && EqQ[n2,-2*n] && IntegerQ[2*n] && IntegerQ[m]
```

$$2: \int (e x)^m \left( a + b \left( \frac{d}{x} \right)^n + c x^{-2n} \right)^p dx \text{ when } 2n \in \mathbb{Z} \wedge m \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction and integration by substitution

Basis:  $\partial_x \left( (e x)^m \left( \frac{d}{x} \right)^m \right) = 0$

Basis:  $F\left[\frac{d}{x}\right] = -d \text{Subst}\left[\frac{F[x]}{x^2}, x, \frac{d}{x}\right] \partial_x \frac{d}{x}$

Rule 1.5.4.6.2.2: If  $2n \in \mathbb{Z} \wedge m \notin \mathbb{Z}$ , then

$$\int (e x)^m \left( a + b \left( \frac{d}{x} \right)^n + c x^{-2n} \right)^p dx \rightarrow (e x)^m \left( \frac{d}{x} \right)^m \int \frac{\left( a + b \left( \frac{d}{x} \right)^n + \frac{c}{d^{2n}} \left( \frac{d}{x} \right)^{2n} \right)^p}{\left( \frac{d}{x} \right)^m} dx \rightarrow -d (e x)^m \left( \frac{d}{x} \right)^m \text{Subst} \left[ \int \frac{\left( a + b x^n + \frac{c}{d^{2n}} x^{2n} \right)^p}{x^{m+2}} dx, x, \frac{d}{x} \right]$$

Program code:

```
Int[(e.*x_)^m.*(a_+b_.*(d_/x_)^n_+c_.*x_^n2_.)^p_,x_Symbol] :=
  -d*(e*x)^m*(d/x)^m*Subst[Int[(a+b*x^n+c/d^(2*n))*x^(2*n)]^p/x^(m+2),x],x,d/x] /;
FreeQ[{a,b,c,d,e,n,p},x] && EqQ[n2,-2*n] && Not[IntegerQ[m]] && IntegerQ[2*n]
```

## 7. Binomial products

### 1. Linear

**1:**  $\int u^m dx$  when  $u = a + b x$

Derivation: Algebraic normalization

Rule: If  $u = a + b x$ , then

$$\int u^m dx \rightarrow \int (a + b x)^m dx$$

Program code:

```
Int[u_^m_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m,x] /;
FreeQ[m,x] && LinearQ[u,x] && Not[LinearMatchQ[u,x]]
```

$$\mathbf{2:} \int u^m v^n dx \text{ when } u == a + b x \wedge v == c + d x$$

Derivation: Algebraic normalization

Rule: If  $u == a + b x \wedge v == c + d x$ , then

$$\int u^m v^n dx \rightarrow \int (a + b x)^m (c + d x)^n dx$$

Program code:

```
Int[u_^m_.*v_^n_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*ExpandToSum[v,x]^n,x] /;
  FreeQ[{m,n},x] && LinearQ[{u,v},x] && Not[LinearMatchQ[{u,v},x]]
```

$$\mathbf{3:} \int u^m v^n w^p dx \text{ when } u == a + b x \wedge v == c + d x \wedge w == e + f x$$

Derivation: Algebraic normalization

Rule: If  $u == a + b x \wedge v == c + d x \wedge w == e + f x$ , then

$$\int u^m v^n w^p dx \rightarrow \int (a + b x)^m (c + d x)^n (e + f x)^p dx$$

Program code:

```
Int[u_^m_.*v_^n_.*w_^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*ExpandToSum[v,x]^n*ExpandToSum[w,x]^p,x] /;
  FreeQ[{m,n,p},x] && LinearQ[{u,v,w},x] && Not[LinearMatchQ[{u,v,w},x]]
```

$$\mathbf{4:} \int u^m v^n w^p z^q dx \text{ when } u == a + b x \wedge v == c + d x \wedge w == e + f x \wedge z == g + h x$$

Derivation: Algebraic normalization

Rule: If  $u == a + b x \wedge v == c + d x \wedge w == e + f x \wedge z == g + h x$ , then

$$\int u^m v^n w^p z^q dx \rightarrow \int (a + b x)^m (c + d x)^n (e + f x)^p (g + h x)^q dx$$

Program code:

```
Int[u_^m_.*v_^n_.*w_^p_.*z_^q_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*ExpandToSum[v,x]^n*ExpandToSum[w,x]^p*ExpandToSum[z,x]^q,x] /;
  FreeQ[{m,n,p,q},x] && LinearQ[{u,v,w,z},x] && Not[LinearMatchQ[{u,v,w,z},x]]
```

### 3. General

$$\mathbf{1:} \int u^p dx \text{ when } u == a + b x^n$$

Derivation: Algebraic normalization

Rule: If  $u == a + b x^n$ , then

$$\int u^p dx \rightarrow \int (a + b x^n)^p dx$$

Program code:

```
Int[u_^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^p,x] /;
  FreeQ[p,x] && BinomialQ[u,x] && Not[BinomialMatchQ[u,x]]
```



**2:**  $\int (c x)^m u^p dx$  when  $u == a + b x^n$

Derivation: Algebraic normalization

Rule: If  $u == a + b x^n$ , then

$$\int (c x)^m u^p dx \rightarrow \int (c x)^m (a + b x^n)^p dx$$

Program code:

```
Int[(c_.*x_)^m_.*u_^p_,x_Symbol] :=
  Int[(c*x)^m*ExpandToSum[u,x]^p,x] /;
FreeQ[{c,m,p},x] && BinomialQ[u,x] && Not[BinomialMatchQ[u,x]]
```

**3:**  $\int u^p v^q dx$  when  $u == a + b x^n \wedge v == c + d x^n$

Derivation: Algebraic normalization

Rule: If  $u == a + b x^n \wedge v == c + d x^n$ , then

$$\int u^p v^q dx \rightarrow \int (a + b x^n)^p (c + d x^n)^q dx$$

Program code:

```
Int[u_^p_.*v_^q_,x_Symbol] :=
  Int[ExpandToSum[u,x]^p*ExpandToSum[v,x]^q,x] /;
FreeQ[{p,q},x] && BinomialQ[{u,v},x] && EqQ[BinomialDegree[u,x]-BinomialDegree[v,x],0] && Not[BinomialMatchQ[{u,v},x]]
```

$$4: \int (e x)^m u^p v^q dx \text{ when } u = a + b x^n \wedge v = c + d x^n$$

Derivation: Algebraic normalization

Rule: If  $u = a + b x^n \wedge v = c + d x^n$ , then

$$\int (e x)^m u^p v^q dx \rightarrow \int (e x)^m (a + b x^n)^p (c + d x^n)^q dx$$

Program code:

```
Int[(e_.*x_)^m_.*u_^p_.*v_^q_.,x_Symbol] :=
  Int[(e*x)^m*ExpandToSum[u,x]^p*ExpandToSum[v,x]^q,x] /;
  FreeQ[{e,m,p,q},x] && BinomialQ[{u,v},x] && EqQ[BinomialDegree[u,x]-BinomialDegree[v,x],0] && Not[BinomialMatchQ[{u,v},x]]
```

$$5: \int u^m v^p w^q dx \text{ when } u = a + b x^n \wedge v = c + d x^n \wedge w = e + f x^n$$

Derivation: Algebraic normalization

Rule: If  $u = a + b x^n \wedge v = c + d x^n \wedge w = e + f x^n$ , then

$$\int u^m v^p w^q dx \rightarrow \int (a + b x^n)^m (c + d x^n)^p (e + f x^n)^q dx$$

Program code:

```
Int[u_^m_.*v_^p_.*w_^q_.,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*ExpandToSum[v,x]^p*ExpandToSum[w,x]^q,x] /;
  FreeQ[{m,p,q},x] && BinomialQ[{u,v,w},x] && EqQ[BinomialDegree[u,x]-BinomialDegree[v,x],0] &&
  EqQ[BinomialDegree[u,x]-BinomialDegree[w,x],0] && Not[BinomialMatchQ[{u,v,w},x]]
```

$$\mathbf{6:} \int (g x)^m u^p v^q z^r dx \text{ when } u = a + b x^n \wedge v = c + d x^n \wedge z = e + f x^n$$

Derivation: Algebraic normalization

Rule: If  $u = a + b x^n \wedge v = c + d x^n \wedge z = e + f x^n$ , then

$$\int (g x)^m u^p v^q z^r dx \rightarrow \int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$$

Program code:

```
Int[(g_.*x_)^m_.*u_^p_.*v_^q_.*z_^r_. ,x_Symbol] :=
  Int[(g*x)^m*ExpandToSum[u,x]^p*ExpandToSum[v,x]^q*ExpandToSum[z,x]^r,x] /;
FreeQ[{g,m,p,q,r},x] && BinomialQ[{u,v,z},x] && EqQ[BinomialDegree[u,x]-BinomialDegree[v,x],0] &&
EqQ[BinomialDegree[u,x]-BinomialDegree[z,x],0] && Not[BinomialMatchQ[{u,v,z},x]]
```

$$\mathbf{7:} \int (c x)^m P_q[x] u^p dx \text{ when } u = a + b x^n$$

Derivation: Algebraic normalization

Rule: If  $u = a + b x^n$ , then

$$\int (c x)^m P_q[x] u^p dx \rightarrow \int (c x)^m P_q[x] (a + b x^n)^p dx$$

Program code:

```
Int[(c_.*x_)^m_.*Pq_*u_^p_. ,x_Symbol] :=
  Int[(c*x)^m*Pq*ExpandToSum[u,x]^p,x] /;
FreeQ[{c,m,p},x] && PolyQ[Pq,x] && BinomialQ[u,x] && Not[BinomialMatchQ[u,x]]
```

## 4. Improper

$$\mathbf{1:} \int u^p dx \text{ when } u = a x^j + b x^n$$

Derivation: Algebraic normalization

Rule: If  $u = a x^j + b x^n$ , then

$$\int u^p dx \rightarrow \int (a x^j + b x^n)^p dx$$

Program code:

```
Int[u_^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^p,x] /;
  FreeQ[p,x] && GeneralizedBinomialQ[u,x] && Not[GeneralizedBinomialMatchQ[u,x]]
```

$$\mathbf{2:} \int (c x)^m u^p dx \text{ when } u = a x^j + b x^n$$

Derivation: Algebraic normalization

Rule: If  $u = a x^j + b x^n$ , then

$$\int (c x)^m u^p dx \rightarrow \int (c x)^m (a x^j + b x^n)^p dx$$

Program code:

```
Int[(c_.*x_)^m_.*u_^p_,x_Symbol] :=
  Int[(c*x)^m*ExpandToSum[u,x]^p,x] /;
  FreeQ[{c,m,p},x] && GeneralizedBinomialQ[u,x] && Not[GeneralizedBinomialMatchQ[u,x]]
```

## 8 Trinomial products

### 1. Quadratic

$$\mathbf{1:} \int u^p dx \text{ when } u = a + b x + c x^2$$

Derivation: Algebraic normalization

Rule: If  $u = a + b x + c x^2$ , then

$$\int u^p dx \rightarrow \int (a + b x + c x^2)^p dx$$

Program code:

```
Int[u_^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^p,x] /;
  FreeQ[p,x] && QuadraticQ[u,x] && Not[QuadraticMatchQ[u,x]]
```

$$\mathbf{2:} \int u^m v^p dx \text{ when } u = d + e x \wedge v = a + b x + c x^2$$

Derivation: Algebraic normalization

Rule: If  $u = d + e x \wedge v = a + b x + c x^2$ , then

$$\int u^m v^p dx \rightarrow \int (d + e x)^m (a + b x + c x^2)^p dx$$

Program code:

```
Int[u_^m_.*v_^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*ExpandToSum[v,x]^p,x] /;
  FreeQ[{m,p},x] && LinearQ[u,x] && QuadraticQ[v,x] && Not[LinearMatchQ[u,x] && QuadraticMatchQ[v,x]]
```

$$\mathbf{3:} \int u^m v^n w^p dx \text{ when } u == d + e x \wedge v == f + g x \wedge w == a + b x + c x^2$$

Derivation: Algebraic normalization

Rule: If  $u == d + e x \wedge v == f + g x \wedge w == a + b x + c x^2$ , then

$$\int u^m v^n w^p dx \rightarrow \int (d + e x)^m (f + g x)^n (a + b x + c x^2)^p dx$$

Program code:

```
Int[u_^m_.*v_^n_.*w_^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*ExpandToSum[v,x]^n*ExpandToSum[w,x]^p,x] /;
  FreeQ[{m,n,p},x] && LinearQ[{u,v},x] && QuadraticQ[w,x] && Not[LinearMatchQ[{u,v},x] && QuadraticMatchQ[w,x]]
```

$$\mathbf{4:} \int u^p v^q dx \text{ when } u == a + b x + c x^2 \wedge v == d + e x + f x^2$$

Derivation: Algebraic normalization

Rule: If  $u == a + b x + c x^2 \wedge v == d + e x + f x^2$ , then

$$\int u^p v^q dx \rightarrow \int (a + b x + c x^2)^p (d + e x + f x^2)^q dx$$

Program code:

```
Int[u_^p_.*v_^q_,x_Symbol] :=
  Int[ExpandToSum[u,x]^p*ExpandToSum[v,x]^q,x] /;
  FreeQ[{p,q},x] && QuadraticQ[{u,v},x] && Not[QuadraticMatchQ[{u,v},x]]
```

$$\mathbf{5:} \int z^m u^p v^q dx \text{ when } z = g + h x \wedge u = a + b x + c x^2 \wedge v = d + e x + f x^2$$

Derivation: Algebraic normalization

Note: This normalization needs to be done before trying polynomial integration rules.

Rule 1.2.1.5.N: If  $z = g + h x \wedge u = a + b x + c x^2 \wedge v = d + e x + f x^2$ , then

$$\int z^m u^p v^q dx \rightarrow \int (g + h x)^m (a + b x + c x^2)^p (d + e x + f x^2)^q dx$$

Program code:

```
Int[z_^m_.*u_^p_.*v_^q_,x_Symbol] :=
  Int[ExpandToSum[z,x]^m*ExpandToSum[u,x]^p*ExpandToSum[v,x]^q,x] /;
  FreeQ[{m,p,q},x] && LinearQ[z,x] && QuadraticQ[{u,v},x] && Not[LinearMatchQ[z,x] && QuadraticMatchQ[{u,v},x]]
```

$$\mathbf{6:} \int P_q[x] u^p dx \text{ when } u = a + b x + c x^2$$

Derivation: Algebraic normalization

Rule: If  $u = a + b x + c x^2$ , then

$$\int P_q[x] u^p dx \rightarrow \int P_q[x] (a + b x + c x^2)^p dx$$

Program code:

```
Int[Pq_*u_^p_,x_Symbol] :=
  Int[Pq*ExpandToSum[u,x]^p,x] /;
  FreeQ[p,x] && PolyQ[Pq,x] && QuadraticQ[u,x] && Not[QuadraticMatchQ[u,x]]
```

$$7: \int u^m P_q[x] v^p dx \text{ when } u = d + e x \wedge v = a + b x + c x^2$$

Derivation: Algebraic normalization

Rule: If  $u = d + e x \wedge v = a + b x + c x^2$ , then

$$\int u^m P_q[x] v^p dx \rightarrow \int (d + e x)^m P_q[x] (a + b x + c x^2)^p dx$$

Program code:

```
Int[u_^m_.*Pq_*v_^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*Pq*ExpandToSum[v,x]^p,x] /;
  FreeQ[{m,p},x] && PolyQ[Pq,x] && LinearQ[u,x] && QuadraticQ[v,x] && Not[LinearMatchQ[u,x] && QuadraticMatchQ[v,x]]
```

### 3. General

$$1: \int u^p dx \text{ when } u = a + b x^n + c x^{2n}$$

Derivation: Algebraic normalization

Rule: If  $u = a + b x^n + c x^{2n}$ , then

$$\int u^p dx \rightarrow \int (a + b x^n + c x^{2n})^p dx$$

Program code:

```
Int[u_^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^p,x] /;
  FreeQ[p,x] && TrinomialQ[u,x] && Not[TrinomialMatchQ[u,x]]
```



$$\mathbf{2:} \int (d x)^m u^p dx \text{ when } u == a + b x^n + c x^{2 n}$$

Derivation: Algebraic normalization

Rule: If  $u == a + b x^n + c x^{2 n}$ , then

$$\int (d x)^m u^p dx \rightarrow \int (d x)^m (a + b x^n + c x^{2 n})^p dx$$

Program code:

```
Int[(d_.*x_)^m_.*u_^p_,x_Symbol] :=
  Int[(d*x)^m*ExpandToSum[u,x]^p,x] /;
FreeQ[{d,m,p},x] && TrinomialQ[u,x] && Not[TrinomialMatchQ[u,x]]
```

$$\mathbf{3:} \int u^q v^p dx \text{ when } u == d + e x^n \wedge v == a + b x^n + c x^{2 n}$$

Derivation: Algebraic normalization

Rule: If  $u == d + e x^n \wedge v == a + b x^n + c x^{2 n}$ , then

$$\int u^q v^p dx \rightarrow \int (d + e x^n)^q (a + b x^n + c x^{2 n})^p dx$$

Program code:

```
Int[u_^q_.*v_^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^q*ExpandToSum[v,x]^p,x] /;
FreeQ[{p,q},x] && BinomialQ[u,x] && TrinomialQ[v,x] && Not[BinomialMatchQ[u,x] && TrinomialMatchQ[v,x]]
```

```
Int[u_^q_.*v_^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^q*ExpandToSum[v,x]^p,x] /;
FreeQ[{p,q},x] && BinomialQ[u,x] && BinomialQ[v,x] && Not[BinomialMatchQ[u,x] && BinomialMatchQ[v,x]]
```

**4:**  $\int (f x)^m z^q u^p dx$  when  $z == d + e x^n \wedge u == a + b x^n + c x^{2n}$

## Derivation: Algebraic normalization

Rule: If  $z == d + e x^n \wedge u == a + b x^n + c x^{2n}$ , then

$$\int (f x)^m z^q u^p dx \rightarrow \int (f x)^m (d + e x^n)^q (a + b x^n + c x^{2n})^p dx$$

## Program code:

```
Int[(f_.*x_)^m_.*z_^q_.*u_^p_,x_Symbol] :=
  Int[(f*x)^m*ExpandToSum[z,x]^q*ExpandToSum[u,x]^p,x] /;
FreeQ[{f,m,p,q},x] && BinomialQ[z,x] && TrinomialQ[u,x] && Not[BinomialMatchQ[z,x] && TrinomialMatchQ[u,x]]
```

```
Int[(f_.*x_)^m_.*z_^q_.*u_^p_,x_Symbol] :=
  Int[(f*x)^m*ExpandToSum[z,x]^q*ExpandToSum[u,x]^p,x] /;
FreeQ[{f,m,p,q},x] && BinomialQ[z,x] && BinomialQ[u,x] && Not[BinomialMatchQ[z,x] && BinomialMatchQ[u,x]]
```

**5:**  $\int P_q[x] u^p dx$  when  $u = a + b x^n + c x^{2n}$

Derivation: Algebraic normalization

Rule: If  $u = a + b x^n + c x^{2n}$ , then

$$\int P_q[x] u^p dx \rightarrow \int P_q[x] (a + b x^n + c x^{2n})^p dx$$

Program code:

```
Int[Pq*u_^p_,x_Symbol] :=
  Int[Pq*ExpandToSum[u,x]^p,x] /;
FreeQ[p,x] && PolyQ[Pq,x] && TrinomialQ[u,x] && Not[TrinomialMatchQ[u,x]]
```

**6:**  $\int (dx)^m P_q[x] u^p dx$  when  $u = a + b x^n + c x^{2n}$

Derivation: Algebraic normalization

Rule: If  $u = a + b x^n + c x^{2n}$ , then

$$\int (dx)^m P_q[x] u^p dx \rightarrow \int (dx)^m P_q[x] (a + b x^n + c x^{2n})^p dx$$

Program code:

```
Int[(d.*x_)^m_.*Pq*u_^p_,x_Symbol] :=
  Int[(d*x)^m*Pq*ExpandToSum[u,x]^p,x] /;
FreeQ[{d,m,p},x] && PolyQ[Pq,x] && TrinomialQ[u,x] && Not[TrinomialMatchQ[u,x]]
```

## 4. Improper

$$\mathbf{1:} \int u^p dx \text{ when } u = a x^q + b x^n + c x^{2n-q}$$

Derivation: Algebraic normalization

Rule: If  $u = a x^q + b x^n + c x^{2n-q}$ , then

$$\int u^p dx \rightarrow \int (a x^q + b x^n + c x^{2n-q})^p dx$$

Program code:

```
Int[u_^p_, x_Symbol] :=
  Int[ExpandToSum[u, x]^p, x] /;
  FreeQ[p, x] && GeneralizedTrinomialQ[u, x] && Not[GeneralizedTrinomialMatchQ[u, x]]
```

$$\mathbf{2:} \int (dx)^m u^p dx \text{ when } u = a x^q + b x^n + c x^{2n-q}$$

Derivation: Algebraic normalization

Rule: If  $u = a x^q + b x^n + c x^{2n-q}$ , then

$$\int (dx)^m u^p dx \rightarrow \int (dx)^m (a x^q + b x^n + c x^{2n-q})^p dx$$

Program code:

```
Int[(d_. * x_)^m_. * u_^p_, x_Symbol] :=
  Int[(d * x)^m * ExpandToSum[u, x]^p, x] /;
  FreeQ[{d, m, p}, x] && GeneralizedTrinomialQ[u, x] && Not[GeneralizedTrinomialMatchQ[u, x]]
```

$$\mathbf{3:} \int z u^p dx \text{ when } z = A + B x^{n-q} \wedge u = a x^q + b x^n + c x^{2n-q}$$

Derivation: Algebraic normalization

Rule: If  $z = A + B x^{n-q} \wedge u = a x^q + b x^n + c x^{2n-q}$ , then

$$\int z u^p dx \rightarrow \int (A + B x^{n-q}) (a x^q + b x^n + c x^{2n-q})^p dx$$

Program code:

```
Int[z_*u^p_,x_Symbol] :=
  Int[ExpandToSum[z,x]*ExpandToSum[u,x]^p,x] /;
FreeQ[p,x] && BinomialQ[z,x] && GeneralizedTrinomialQ[u,x] &&
EqQ[BinomialDegree[z,x]-GeneralizedTrinomialDegree[u,x],0] && Not[BinomialMatchQ[z,x] && GeneralizedTrinomialMatchQ[u,x]]
```

$$\mathbf{4:} \int (f x)^m z u^p dx \text{ when } z = A + B x^{n-q} \wedge u = a x^q + b x^n + c x^{2n-q}$$

Derivation: Algebraic normalization

Rule: If  $z = A + B x^{n-q} \wedge u = a x^q + b x^n + c x^{2n-q}$ , then

$$\int (f x)^m z u^p dx \rightarrow \int (f x)^m (A + B x^{n-q}) (a x^q + b x^n + c x^{2n-q})^p dx$$

Program code:

```
Int[(f_*x_)^m_*z_*u^p_,x_Symbol] :=
  Int[(f*x)^m*ExpandToSum[z,x]*ExpandToSum[u,x]^p,x] /;
FreeQ[{f,m,p},x] && BinomialQ[z,x] && GeneralizedTrinomialQ[u,x] &&
EqQ[BinomialDegree[z,x]-GeneralizedTrinomialDegree[u,x],0] && Not[BinomialMatchQ[z,x] && GeneralizedTrinomialMatchQ[u,x]]
```

