

Rules for integrands of the form $(c x)^m (a + b x^n)^p$

D: $\int (c x)^m (a_1 + b_1 x^n)^p (a_2 + b_2 x^n)^p dx$ when $a_2 b_1 + a_1 b_2 = 0 \wedge (p \in \mathbb{Z} \vee (a_1 > 0 \wedge a_2 > 0))$

Derivation: Algebraic simplification

Basis: If $a_2 b_1 + a_1 b_2 = 0 \wedge (p \in \mathbb{Z} \vee (a_1 > 0 \wedge a_2 > 0))$, then $(a_1 + b_1 x^n)^p (a_2 + b_2 x^n)^p = (a_1 a_2 + b_1 b_2 x^{2n})^p$

Rule: If $a_2 b_1 + a_1 b_2 = 0 \wedge (p \in \mathbb{Z} \vee (a_1 > 0 \wedge a_2 > 0))$, then

$$\int (c x)^m (a_1 + b_1 x^n)^p (a_2 + b_2 x^n)^p dx \rightarrow \int (c x)^m (a_1 a_2 + b_1 b_2 x^{2n})^p dx$$

Program code:

```
Int[(c_.**x_)^m_.*(a1_+b1_.**x_^n_)^p_*(a2_+b2_.**x_^n_)^p_,x_Symbol] :=
  Int[(c*x)^m*(a1*a2+b1*b2*x^(2*n))^p,x] /;
FreeQ[{a1,b1,a2,b2,c,m,n,p},x] && EqQ[a2*b1+a1*b2,0] && (IntegerQ[p] || GtQ[a1,0] && GtQ[a2,0])
```

$$1. \int x^m (a + b x^n)^p dx \text{ when } m = n - 1$$

$$1: \int \frac{x^m}{a + b x^n} dx \text{ when } m = n - 1$$

Derivation: Integration by substitution and reciprocal rule for integration

Basis: If $m = n - 1$, then $x^m F[x^n] = \frac{1}{n} F[x^n] \partial_x x^n$

Rule 1.1.3.2.1.1: If $m = n - 1$, then

$$\int \frac{x^m}{a + b x^n} dx \rightarrow \frac{1}{n} \text{Subst} \left[\int \frac{1}{a + b x} dx, x, x^n \right] \rightarrow \frac{\text{Log}[a + b x^n]}{b n}$$

Program code:

```
Int[x_^m_./(a_+b_.*x_^n_),x_Symbol] :=
  Log[RemoveContent[a+b*x^n,x]]/(b*n) /;
FreeQ[{a,b,m,n},x] && EqQ[m,n-1]
```

$$2: \int x^m (a + b x^n)^p dx \text{ when } m = n - 1 \wedge p \neq -1$$

Reference: G&R 2.110.4, CRC 88a with $m = n - 1$

Derivation: Binomial recurrence 2a with $m = n - 1$

Derivation: Integration by substitution and power rule for integration

Basis: If $m = n - 1$, then $x^m F[x^n] = \frac{1}{n} F[x^n] \partial_x x^n$

Rule 1.1.3.2.1.2: If $m = n - 1 \wedge p \neq -1$, then

$$\int x^m (a + b x^n)^p dx \rightarrow \frac{1}{n} \text{Subst} \left[\int (a + b x)^p dx, x, x^n \right] \rightarrow \frac{(a + b x^n)^{p+1}}{b n (p + 1)}$$

Program code:

```
Int[x_^m_.*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  (a+b*x^n)^(p+1)/(b*n*(p+1)) /;
FreeQ[{a,b,m,n,p},x] && EqQ[m,n-1] && NeQ[p,-1]
```

```
Int[x_^m_.*(a1_+b1_.*x_^n_.)^p_.*(a2_+b2_.*x_^n_.)^p_,x_Symbol] :=
  (a1+b1*x^n)^(p+1)*(a2+b2*x^n)^(p+1)/(2*b1*b2*n*(p+1)) /;
FreeQ[{a1,b1,a2,b2,m,n,p},x] && EqQ[a2*b1+a1*b2,0] && EqQ[m,2*n-1] && NeQ[p,-1]
```

2: $\int x^m (a + b x^n)^p dx$ when $p \in \mathbb{Z} \wedge n < 0$

Derivation: Algebraic expansion

Basis: If $p \in \mathbb{Z}$, then $(a + b x^n)^p = x^{np} (b + a x^{-n})^p$

Rule 1.1.3.2.2: If $p \in \mathbb{Z} \wedge n < 0$, then

$$\int x^m (a + b x^n)^p dx \rightarrow \int x^{m+np} (b + a x^{-n})^p dx$$

Program code:

```
Int[x_^m_.*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  Int[x^(m+n*p)*(b+a*x^(-n))^p,x] /;
FreeQ[{a,b,m,n},x] && IntegerQ[p] && NegQ[n]
```

3: $\int (c x)^m (a + b x^n)^p dx$ when $\frac{m+1}{n} + p + 1 == 0 \wedge m \neq -1$

Reference: G&R 2.110.6, CRC 88c with $m + n p + n + 1 == 0$

Derivation: Binomial recurrence 3b with $m + n p + n + 1 == 0$

Rule 1.1.3.2.3: If $\frac{m+1}{n} + p + 1 == 0 \wedge m \neq -1$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow \frac{(c x)^{m+1} (a + b x^n)^{p+1}}{a c (m+1)}$$

Program code:

```
Int[(c_.**x_)^m_.*(a_+b_.**x_^n_)^p_,x_Symbol] :=
  (c*x)^(m+1)*(a+b*x^n)^(p+1)/(a*c*(m+1)) /;
FreeQ[{a,b,c,m,n,p},x] && EqQ[(m+1)/n+p+1,0] && NeQ[m,-1]
```

```
Int[(c_.**x_)^m_.*(a1_+b1_.**x_^n_)^p_*(a2_+b2_.**x_^n_)^p_,x_Symbol] :=
  (c*x)^(m+1)*(a1+b1*x^n)^(p+1)*(a2+b2*x^n)^(p+1)/(a1*a2*c*(m+1)) /;
FreeQ[{a1,b1,a2,b2,c,m,n,p},x] && EqQ[a2*b1+a1*b2,0] && EqQ[(m+1)/(2*n)+p+1,0] && NeQ[m,-1]
```

4. $\int (c x)^m (a + b x^n)^p dx$ when $\frac{m+1}{n} \in \mathbb{Z}$

1: $\int x^m (a + b x^n)^p dx$ when $\frac{m+1}{n} \in \mathbb{Z}$

Derivation: Integration by substitution

Basis: If $\frac{m+1}{n} \in \mathbb{Z}$, then $x^m F[x^n] = \frac{1}{n} \text{Subst}\left[x^{\frac{m+1}{n}-1} F[x], x, x^n\right] \partial_x x^n$

Note: If $n \in \mathbb{Z} \wedge \frac{m+1}{n} \in \mathbb{Z}$, then $m \in \mathbb{Z}$, and $(c x)^m$ automatically evaluates to $c^m x^m$.

Rule 1.1.3.2.4.1: If $\frac{m+1}{n} \in \mathbb{Z}$, then

$$\int x^m (a + b x^n)^p dx \rightarrow \frac{1}{n} \text{Subst}\left[\int x^{\frac{m+1}{n}-1} (a + b x)^p dx, x, x^n\right]$$

Program code:

```
Int[x^m_.*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  1/n*Subst[Int[x^(Simplify[(m+1)/n]-1)*(a+b*x)^p,x],x,x^n] /;
FreeQ[{a,b,m,n,p},x] && IntegerQ[Simplify[(m+1)/n]]
```

```
Int[x^m_.*(a1_+b1_.*x_^n_)^p_*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  1/n*Subst[Int[x^(Simplify[(m+1)/n]-1)*(a1+b1*x)^p*(a2+b2*x)^p,x],x,x^n] /;
FreeQ[{a1,b1,a2,b2,m,n,p},x] && EqQ[a2*b1+a1*b2,0] && IntegerQ[Simplify[(m+1)/(2*n)]]
```

2: $\int (c x)^m (a + b x^n)^p dx$ when $\frac{m+1}{n} \in \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(c x)^m}{x^m} = 0$

Rule 1.1.3.2.4.2: If $\frac{m+1}{n} \in \mathbb{Z}$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (a + b x^n)^p dx$$

Program code:

```
Int[(c*x_)^m_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a+b*x^n)^p,x] /;
FreeQ[{a,b,c,m,n,p},x] && IntegerQ[Simplify[(m+1)/n]]
```

```
Int[(c*x_)^m_*(a1_+b1_.*x_^n_)^p_*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a1+b1*x^n)^p*(a2+b2*x^n)^p,x] /;
FreeQ[{a1,b1,a2,b2,c,m,n,p},x] && EqQ[a2*b1+a1*b2,0] && IntegerQ[Simplify[(m+1)/(2*n)]]
```

5: $\int (c x)^m (a + b x^n)^p dx$ when $p \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule 1.1.3.2.5: If $p \in \mathbb{Z}^+$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow \int \text{ExpandIntegrand}[(c x)^m (a + b x^n)^p, x] dx$$

Program code:

```
Int[(c_.x_)^m_.*(a_+b_.x_^n_)^p_,x_Symbol] :=
  Int[ExpandIntegrand[(c*x)^m*(a+b*x^n)^p,x],x] /;
FreeQ[{a,b,c,m,n},x] && IGtQ[p,0]
```

6. $\int (c x)^m (a + b x^n)^p dx$ when $\frac{m+1}{n} + p + 1 \in \mathbb{Z}^-$

1: $\int x^m (a + b x^n)^p dx$ when $\frac{m+1}{n} + p + 1 \in \mathbb{Z}^- \wedge m \neq -1$

Reference: G&R 2.110.6, CRC 88c

Derivation: Binomial recurrence 3b

Note: This rule drives $\frac{m+1}{n} + p + 1$ to 0 by incrementing m by n .

Rule 1.1.3.2.6.1: If $\frac{m+1}{n} + p + 1 \in \mathbb{Z}^- \wedge m \neq -1$, then

$$\int x^m (a + b x^n)^p dx \rightarrow \frac{x^{m+1} (a + b x^n)^{p+1}}{a (m+1)} - \frac{b (m+n (p+1) + 1)}{a (m+1)} \int x^{m+n} (a + b x^n)^p dx$$

Program code:

```
Int[x^m*(a+b*x^n)^p,x_Symbol] :=
  x^(m+1)*(a+b*x^n)^(p+1)/(a*(m+1)) -
  b*(m+n*(p+1)+1)/(a*(m+1))*Int[x^(m+n)*(a+b*x^n)^p,x] /;
FreeQ[{a,b,m,n,p},x] && ILtQ[Simplify[(m+1)/n+p+1],0] && NeQ[m,-1]
```

```
Int[x^m*(a1+b1*x^n)^p*(a2+b2*x^n)^p,x_Symbol] :=
  x^(m+1)*(a1+b1*x^n)^(p+1)*(a2+b2*x^n)^(p+1)/(a1*a2*(m+1)) -
  b1*b2*(m+2*n*(p+1)+1)/(a1*a2*(m+1))*Int[x^(m+2*n)*(a1+b1*x^n)^p*(a2+b2*x^n)^p,x] /;
FreeQ[{a1,b1,a2,b2,m,n,p},x] && EqQ[a2*b1+a1*b2,0] && ILtQ[Simplify[(m+1)/(2*n)+p+1],0] && NeQ[m,-1]
```


2: $\int (c x)^m (a + b x^n)^p dx$ when $\frac{m+1}{n} + p + 1 \in \mathbb{Z}^- \wedge p \neq -1$

Reference: G&R 2.110.2, CRC 88d

Derivation: Binomial recurrence 2b

Derivation: Integration by parts

Basis: $x^m (a + b x^n)^p = x^{m+n p+n+1} \frac{(a+b x^n)^p}{x^{n(p+1)+1}}$

Basis: $\int \frac{(a+b x^n)^p}{x^{n(p+1)+1}} dx = - \frac{(a+b x^n)^{p+1}}{x^{n(p+1)} a n (p+1)}$

Note: This rule drives $\frac{m+1}{n} + p + 1$ to 0 by incrementing p by 1.

Rule 1.1.3.2.6.2: If $\frac{m+1}{n} + p + 1 \in \mathbb{Z}^- \wedge p \neq -1$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow - \frac{(c x)^{m+1} (a + b x^n)^{p+1}}{a c n (p+1)} + \frac{m + n (p+1) + 1}{a n (p+1)} \int (c x)^m (a + b x^n)^{p+1} dx$$

Program code:

```
Int[(c_.**x_)^m_.*(a_+b_.**x_^n_)^p_,x_Symbol] :=
  -(c*x)^(m+1)*(a+b*x^n)^(p+1)/(a*c*n*(p+1)) +
  (m+n*(p+1)+1)/(a*n*(p+1))*Int[(c*x)^m*(a+b*x^n)^(p+1),x] /;
FreeQ[{a,b,c,m,n,p},x] && ILtQ[Simplify[(m+1)/n+p+1],0] && NeQ[p,-1]
```

```
Int[(c_.**x_)^m_.*(a1_+b1_.**x_^n_)^p_*(a2_+b2_.**x_^n_)^p_,x_Symbol] :=
  -(c*x)^(m+1)*(a1+b1*x^n)^(p+1)*(a2+b2*x^n)^(p+1)/(2*a1*a2*c*n*(p+1)) +
  (m+2*n*(p+1)+1)/(2*a1*a2*n*(p+1))*Int[(c*x)^m*(a1+b1*x^n)^(p+1)*(a2+b2*x^n)^(p+1),x] /;
FreeQ[{a1,b1,a2,b2,c,m,n,p},x] && EqQ[a2*b1+a1*b2,0] && ILtQ[Simplify[(m+1)/(2*n)+p+1],0] && NeQ[p,-1]
```

$$7. \int (c x)^m (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}$$

$$1. \int (c x)^m (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+$$

$$\mathbf{1:} \int x^m (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+ \wedge m \in \mathbb{Z} \wedge \text{GCD}[m+1, n] \neq 1$$

Derivation: Integration by substitution

Basis: If $n \in \mathbb{Z} \wedge m \in \mathbb{Z}$, let $k = \text{GCD}[m+1, n]$, then $x^m F[x^n] = \frac{1}{k} \text{Subst}\left[x^{\frac{m+1}{k}-1} F[x^{n/k}], x, x^k\right] \partial_x x^k$

Rule 1.1.3.2.7.1.1: If $n \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}$, let $k = \text{GCD}[m+1, n]$, if $k \neq 1$, then

$$\int x^m (a + b x^n)^p dx \rightarrow \frac{1}{k} \text{Subst}\left[\int x^{\frac{m+1}{k}-1} (a + b x^{n/k})^p dx, x, x^k\right]$$

Program code:

```
Int[x^m_.*(a_+b_.*x^n_)^p_,x_Symbol] :=
  With[{k=GCD[m+1,n]},
    1/k*Subst[Int[x^((m+1)/k-1)*(a+b*x^(n/k))^p,x],x,x^k] /;
    k!=1] /;
FreeQ[{a,b,p},x] && IGtQ[n,0] && IntegerQ[m]
```

```
Int[x^m_.*(a1_+b1_.*x^n_)^p_*(a2_+b2_.*x^n_)^p_,x_Symbol] :=
  With[{k=GCD[m+1,2*n]},
    1/k*Subst[Int[x^((m+1)/k-1)*(a1+b1*x^(n/k))^p*(a2+b2*x^(n/k))^p,x],x,x^k] /;
    k!=1] /;
FreeQ[{a1,b1,a2,b2,p},x] && EqQ[a2+b1+a1*b2,0] && IGtQ[2*n,0] && IntegerQ[m]
```

$$2. \int (c x)^m (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+ \wedge p > 0$$

$$1: \int (c x)^m (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+ \wedge p > 0 \wedge m < -1$$

Reference: G&R 2.110.3

Derivation: Binomial recurrence 1a

Derivation: Integration by parts

Rule 1.1.3.2.7.1.2.1: If $n \in \mathbb{Z}^+ \wedge p > 0 \wedge m < -1$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow \frac{(c x)^{m+1} (a + b x^n)^p}{c (m+1)} - \frac{b n p}{c^n (m+1)} \int (c x)^{m+n} (a + b x^n)^{p-1} dx$$

Program code:

```
Int[(c_.x_)^m_.*(a_+b_.x_^n_)^p_,x_Symbol] :=
  (c*x)^(m+1)*(a+b*x^n)^p/(c*(m+1)) -
  b*n*p/(c^n*(m+1))*Int[(c*x)^(m+n)*(a+b*x^n)^(p-1),x] /;
FreeQ[{a,b,c},x] && IGtQ[n,0] && GtQ[p,0] && LtQ[m,-1] && Not[ILtQ[(m+n*p+n+1)/n,0]] &&
IntBinomialQ[a,b,c,n,m,p,x]
```

```
Int[(c_.x_)^m_.*(a1_+b1_.x_^n_)^p_*(a2_+b2_.x_^n_)^p_,x_Symbol] :=
  (c*x)^(m+1)*(a1+b1*x^n)^p*(a2+b2*x^n)^p/(c*(m+1)) -
  2*b1*b2*n*p/(c^(2*n)*(m+1))*Int[(c*x)^(m+2*n)*(a1+b1*x^n)^(p-1)*(a2+b2*x^n)^(p-1),x] /;
FreeQ[{a1,b1,a2,b2,c,m},x] && EqQ[a2*b1+a1*b2,0] && IGtQ[2*n,0] && GtQ[p,0] && LtQ[m,-1] && NeQ[m+2*n*p+1,0] &&
IntBinomialQ[a1*a2,b1*b2,c,2*n,m,p,x]
```

2: $\int (c x)^m (a + b x^n)^p dx$ when $n \in \mathbb{Z}^+ \wedge p > 0 \wedge m + n p + 1 \neq 0$

Reference: G&R 2.110.1, CRC 88b

Derivation: Binomial recurrence 1b

Derivation: Inverted integration by parts

Rule 1.1.3.2.7.1.2.2: If $n \in \mathbb{Z}^+ \wedge p > 0 \wedge m + n p + 1 \neq 0$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow \frac{(c x)^{m+1} (a + b x^n)^p}{c (m + n p + 1)} + \frac{a n p}{m + n p + 1} \int (c x)^m (a + b x^n)^{p-1} dx$$

Program code:

```
Int[(c_.**x_)^m_.*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  (c*x)^(m+1)*(a+b*x^n)^p/(c*(m+n*p+1)) +
  a*n*p/(m+n*p+1)*Int[(c*x)^m*(a+b*x^n)^(p-1),x] /;
FreeQ[{a,b,c,m},x] && IGtQ[n,0] && GtQ[p,0] && NeQ[m+n*p+1,0] && IntBinomialQ[a,b,c,n,m,p,x]
```

```
Int[(c_.**x_)^m_.*(a1_+b1_.*x_^n_)^p_*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  (c*x)^(m+1)*(a1+b1*x^n)^p*(a2+b2*x^n)^p/(c*(m+2*n*p+1)) +
  2*a1*a2*n*p/(m+2*n*p+1)*Int[(c*x)^m*(a1+b1*x^n)^(p-1)*(a2+b2*x^n)^(p-1),x] /;
FreeQ[{a1,b1,a2,b2,c,m},x] && EqQ[a2*b1+a1*b2,0] && IGtQ[2*n,0] && GtQ[p,0] && NeQ[m+2*n*p+1,0] && IntBinomialQ[a1*a2,b1*b2,c,2*n,m,
```

$$3. \int (c x)^m (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+ \wedge p < -1$$

$$1. \int \frac{x^m}{(a + b x^4)^{5/4}} dx \text{ when } \frac{b}{a} > 0 \wedge \frac{m-2}{4} \in \mathbb{Z}$$

$$1: \int \frac{x^2}{(a + b x^4)^{5/4}} dx \text{ when } \frac{b}{a} > 0$$

Derivation: Piecewise constant extraction

$$\text{Basis: } a_x \frac{x \left(1 + \frac{a}{b x^4}\right)^{1/4}}{(a + b x^4)^{1/4}} = 0$$

Rule 1.1.3.2.7.1.3.1.1: If $\frac{b}{a} > 0$, then

$$\int \frac{x^2}{(a + b x^4)^{5/4}} dx \rightarrow \frac{x \left(1 + \frac{a}{b x^4}\right)^{1/4}}{b (a + b x^4)^{1/4}} \int \frac{1}{x^3 \left(1 + \frac{a}{b x^4}\right)^{5/4}} dx$$

Program code:

```
Int[x^2/(a+b_*x^4)^(5/4),x_Symbol] :=
  x*(1+a/(b*x^4))^(1/4)/(b*(a+b*x^4)^(1/4))*Int[1/(x^3*(1+a/(b*x^4))^(5/4)),x] /;
FreeQ[{a,b},x] && PosQ[b/a]
```

2: $\int \frac{x^m}{(a + b x^4)^{5/4}} dx$ when $\frac{b}{a} > 0 \wedge \frac{m-2}{4} \in \mathbb{Z}^+$

Reference: G&R 2.110.5, CRC 88a

Derivation: Binomial recurrence 3a

Derivation: Inverted integration by parts

Rule 1.1.3.2.7.1.3.1.2: If $\frac{b}{a} > 0 \wedge \frac{m-2}{4} \in \mathbb{Z}^+$, then

$$\int \frac{x^m}{(a + b x^4)^{5/4}} dx \rightarrow \frac{x^{m-3}}{b (m-4) (a + b x^4)^{1/4}} - \frac{a (m-3)}{b (m-4)} \int \frac{x^{m-4}}{(a + b x^4)^{5/4}} dx$$

Program code:

```
Int[x_^m/(a_+b_.*x_^4)^(5/4),x_Symbol] :=
  x^(m-3)/(b*(m-4)*(a+b*x^4)^(1/4)) - a*(m-3)/(b*(m-4))*Int[x^(m-4)/(a+b*x^4)^(5/4),x] /;
FreeQ[{a,b},x] && PosQ[b/a] && IGtQ[(m-2)/4,0]
```

$$\text{3: } \int \frac{x^m}{(a + b x^4)^{5/4}} dx \text{ when } \frac{b}{a} > 0 \wedge \frac{m-2}{4} \in \mathbb{Z}^-$$

Reference: G&R 2.110.6, CRC 88c

Derivation: Binomial recurrence 3b

Derivation: Integration by parts

Rule 1.1.3.2.7.1.3.1.3: If $\frac{b}{a} > 0 \wedge \frac{m-2}{4} \in \mathbb{Z}^-$, then

$$\int \frac{x^m}{(a + b x^4)^{5/4}} dx \rightarrow \frac{x^{m+1}}{a (m+1) (a + b x^4)^{1/4}} - \frac{b m}{a (m+1)} \int \frac{x^{m+4}}{(a + b x^4)^{5/4}} dx$$

Program code:

```
Int[x_^m/(a_+b_*x_^4)^(5/4),x_Symbol] :=
  x^(m+1)/(a*(m+1)*(a+b*x^4)^(1/4)) - b*m/(a*(m+1))*Int[x^(m+4)/(a+b*x^4)^(5/4),x] /;
FreeQ[{a,b},x] && PosQ[b/a] && ILtQ[(m-2)/4,0]
```

$$2. \int \frac{(c x)^m}{(a + b x^2)^{5/4}} dx \text{ when } \frac{b}{a} > 0 \wedge 2 m \in \mathbb{Z}$$

$$\text{1: } \int \frac{\sqrt{c x}}{(a + b x^2)^{5/4}} dx \text{ when } \frac{b}{a} > 0$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{\sqrt{c x} \left(1 + \frac{a}{b x^2}\right)^{1/4}}{(a + b x^2)^{1/4}} = 0$$

Rule 1.1.3.2.7.1.3.2.1: If $\frac{b}{a} > 0$, then

$$\int \frac{\sqrt{c x}}{(a + b x^2)^{5/4}} dx \rightarrow \frac{\sqrt{c x} \left(1 + \frac{a}{b x^2}\right)^{1/4}}{b (a + b x^2)^{1/4}} \int \frac{1}{x^2 \left(1 + \frac{a}{b x^2}\right)^{5/4}} dx$$

Program code:

```
Int[Sqrt[c_*x_]/(a_+b_*x_^2)^(5/4),x_Symbol] :=
  Sqrt[c*x]*(1+a/(b*x^2))^(1/4)/(b*(a+b*x^2)^(1/4))*Int[1/(x^2*(1+a/(b*x^2))^(5/4)),x] /;
FreeQ[{a,b,c},x] && PosQ[b/a]
```

2: $\int \frac{(c x)^m}{(a + b x^2)^{5/4}} dx$ when $\frac{b}{a} > 0 \wedge 2m \in \mathbb{Z} \wedge m > \frac{3}{2}$

Reference: G&R 2.110.5, CRC 88a

Derivation: Binomial recurrence 3a

Derivation: Inverted integration by parts

Rule 1.1.3.2.7.1.3.2.2: If $\frac{b}{a} > 0 \wedge 2m \in \mathbb{Z} \wedge m > \frac{3}{2}$, then

$$\int \frac{(c x)^m}{(a + b x^2)^{5/4}} dx \rightarrow \frac{2 c (c x)^{m-1}}{b (2m-3) (a + b x^2)^{1/4}} - \frac{2 a c^2 (m-1)}{b (2m-3)} \int \frac{(c x)^{m-2}}{(a + b x^2)^{5/4}} dx$$

Program code:

```
Int[(c_*x_)^m_/(a_+b_*x_^2)^(5/4),x_Symbol] :=
  2*c*(c*x)^(m-1)/(b*(2*m-3)*(a+b*x^2)^(1/4)) - 2*a*c^2*(m-1)/(b*(2*m-3))*Int[(c*x)^(m-2)/(a+b*x^2)^(5/4),x] /;
FreeQ[{a,b,c},x] && PosQ[b/a] && IntegerQ[2*m] && GtQ[m,3/2]
```


$$\text{3: } \int \frac{(c x)^m}{(a + b x^2)^{5/4}} dx \text{ when } \frac{b}{a} > 0 \wedge 2m \in \mathbb{Z} \wedge m < -1$$

Reference: G&R 2.110.6, CRC 88c

Derivation: Binomial recurrence 3b

Derivation: Integration by parts

Rule 1.1.3.2.7.1.3.2.3: If $\frac{b}{a} > 0 \wedge 2m \in \mathbb{Z} \wedge m < -1$, then

$$\int \frac{(c x)^m}{(a + b x^2)^{5/4}} dx \rightarrow \frac{(c x)^{m+1}}{a c (m+1) (a + b x^2)^{1/4}} - \frac{b (2m+1)}{2 a c^2 (m+1)} \int \frac{(c x)^{m+2}}{(a + b x^2)^{5/4}} dx$$

Program code:

```
Int[(c_.*x_)^m_/(a_+b_.*x_^2)^(5/4),x_Symbol] :=
  (c*x)^(m+1)/(a*c*(m+1)*(a+b*x^2)^(1/4)) - b*(2*m+1)/(2*a*c^2*(m+1))*Int[(c*x)^(m+2)/(a+b*x^2)^(5/4),x] /;
FreeQ[{a,b,c},x] && PosQ[b/a] && IntegerQ[2*m] && LtQ[m,-1]
```

3: $\int \frac{x^2}{(a + b x^4)^{5/4}} dx$ when $\frac{b}{a} \neq 0$

Reference: G&R 2.110.4

Derivation: Binomial recurrence 2a

Derivation: Integration by parts

Rule 1.1.3.2.7.1.3.3: If $\frac{b}{a} \neq 0$, then

$$\int \frac{x^2}{(a + b x^4)^{5/4}} dx \rightarrow -\frac{1}{b x (a + b x^4)^{1/4}} - \frac{1}{b} \int \frac{1}{x^2 (a + b x^4)^{1/4}} dx$$

Program code:

```
Int[x^2/(a+b_*x^4)^(5/4),x_Symbol] :=
  -1/(b*x*(a+b*x^4)^(1/4)) - 1/b*Int[1/(x^2*(a+b*x^4)^(1/4)),x] /;
FreeQ[{a,b},x] && NegQ[b/a]
```

$$\mathbf{4:} \int (c x)^m (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+ \wedge p < -1 \wedge m + 1 > n$$

Reference: G&R 2.110.4

Derivation: Binomial recurrence 2a

Derivation: Integration by parts

$$\text{Basis: } x^m (a + b x^n)^p = x^{m-n+1} (a + b x^n)^p x^{n-1}$$

$$\text{Basis: } \int (a + b x^n)^p x^{n-1} dx = \frac{(a + b x^n)^{p+1}}{b n (p+1)}$$

Rule 1.1.3.2.7.1.3.4: If $n \in \mathbb{Z}^+ \wedge p < -1 \wedge m + 1 > n$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow \frac{c^{n-1} (c x)^{m-n+1} (a + b x^n)^{p+1}}{b n (p+1)} - \frac{c^n (m-n+1)}{b n (p+1)} \int (c x)^{m-n} (a + b x^n)^{p+1} dx$$

Program code:

```
Int[(c_.*x_)^m_.*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  c^(n-1)*(c*x)^(m-n+1)*(a+b*x^n)^(p+1)/(b*n*(p+1)) -
  c^n*(m-n+1)/(b*n*(p+1))*Int[(c*x)^(m-n)*(a+b*x^n)^(p+1),x] /;
FreeQ[{a,b,c},x] && IGtQ[n,0] && LtQ[p,-1] && GtQ[m+1,n] && Not[ILtQ[(m+n*(p+1)+1)/n,0]] && IntBinomialQ[a,b,c,n,m,p,x]
```

```
(* Int[(c_.*x_)^m_.*u_^p_*v_^p_,x_Symbol] :=
  With[{a=BinomialParts[u,x][[1]],b=BinomialParts[u,x][[2]],n=BinomialParts[u,x][[3]]},
  c^(n-1)*(c*x)^(m-n+1)*u^(p+1)*v^(p+1)/(b*n*(p+1)) -
  c^n*(m-n+1)/(b*n*(p+1))*Int[(c*x)^(m-n)*u^(p+1)*v^(p+1),x] /;
  IGtQ[n,0] && m+1>n && Not[ILtQ[(m+n*(p+1)+1)/n,0]] &&
  IntBinomialQ[a,b,c,n,m,p,x] /;
FreeQ[c,x] && BinomialQ[u*v,x] && LtQ[p,-1] *)
```

```

Int[(c_.**x_)^m_.*(a1_+b1_.*x_^n_)^p_.*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  c^(2*n-1)*(c*x)^(m-2*n+1)*(a1+b1*x^n)^(p+1)*(a2+b2*x^n)^(p+1)/(2*b1*b2*n*(p+1)) -
  c^(2*n)*(m-2*n+1)/(2*b1*b2*n*(p+1))*Int[(c*x)^(m-2*n)*(a1+b1*x^n)^(p+1)*(a2+b2*x^n)^(p+1),x] /;
FreeQ[{a1,b1,a2,b2,c},x] && EqQ[a2*b1+a1*b2,0] && IGtQ[2*n,0] && LtQ[p,-1] && m+1>2*n &&
  Not[ILtQ[(m+2*n*(p+1)+1)/(2*n),0]] && IntBinomialQ[a1*a2,b1*b2,c,2*n,m,p,x]

```

5: $\int (c x)^m (a + b x^n)^p dx$ when $n \in \mathbb{Z}^+ \wedge p < -1$

Reference: G&R 2.110.2, CRC 88d

Derivation: Binomial recurrence 2b

Derivation: Integration by parts

Basis: $x^m (a + b x^n)^p == x^{m+n p+n+1} \frac{(a+b x^n)^p}{x^{n(p+1)+1}}$

Basis: $\int \frac{(a+b x^n)^p}{x^{n(p+1)+1}} dx == -\frac{(a+b x^n)^{p+1}}{x^{n(p+1)} a n(p+1)}$

Rule 1.1.3.2.7.1.3.5: If $n \in \mathbb{Z}^+ \wedge p < -1$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow -\frac{(c x)^{m+1} (a + b x^n)^{p+1}}{a c n (p+1)} + \frac{m + n (p+1) + 1}{a n (p+1)} \int (c x)^m (a + b x^n)^{p+1} dx$$

Program code:

```

Int[(c_.**x_)^m_.*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  -(c*x)^(m+1)*(a+b*x^n)^(p+1)/(a*c*n*(p+1)) +
  (m+n*(p+1)+1)/(a*n*(p+1))*Int[(c*x)^m*(a+b*x^n)^(p+1),x] /;
FreeQ[{a,b,c,m},x] && IGtQ[n,0] && LtQ[p,-1] && IntBinomialQ[a,b,c,n,m,p,x]

```

```

Int[(c_.*x_)^m_.*(a1_+b1_.*x_^n_)^p_.*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  -(c*x)^(m+1)*(a1+b1*x^n)^(p+1)*(a2+b2*x^n)^(p+1)/(2*a1*a2*c*n*(p+1)) +
  (m+2*n*(p+1)+1)/(2*a1*a2*n*(p+1))*Int[(c*x)^m*(a1+b1*x^n)^(p+1)*(a2+b2*x^n)^(p+1),x] /;
FreeQ[{a1,b1,a2,b2,c,m},x] && EqQ[a2*b1+a1*b2,0] && IGtQ[2*n,0] && LtQ[p,-1] && IntBinomialQ[a1*a2,b1*b2,c,2*n,m,p,x]

```

4. $\int \frac{x^m}{a + b x^n} dx$ when $n \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+$

1. $\int \frac{x^m}{a + b x^n} dx$ when $n \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n - 1$

1. $\int \frac{x^m}{a + b x^n} dx$ when $\frac{n-1}{2} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n - 1$

1: $\int \frac{x}{a + b x^3} dx$

Reference: G&R 2.126.2, CRC 75

Derivation: Algebraic expansion

Basis: $\frac{x}{a + b x^3} == -\frac{1}{3 a^{1/3} b^{1/3} (a^{1/3} + b^{1/3} x)} + \frac{a^{1/3} + b^{1/3} x}{3 a^{1/3} b^{1/3} (a^{2/3} - a^{1/3} b^{1/3} x + b^{2/3} x^2)}$

Rule 1.1.3.2.7.1.4.1.1.1:

$$\int \frac{x}{a + b x^3} dx \rightarrow -\frac{1}{3 a^{1/3} b^{1/3}} \int \frac{1}{a^{1/3} + b^{1/3} x} dx + \frac{1}{3 a^{1/3} b^{1/3}} \int \frac{a^{1/3} + b^{1/3} x}{a^{2/3} - a^{1/3} b^{1/3} x + b^{2/3} x^2} dx$$

Program code:

```

Int[x_/(a_+b_.*x_^3),x_Symbol] :=
  -1/(3*Rt[a,3]*Rt[b,3])*Int[1/(Rt[a,3]+Rt[b,3]*x),x] +
  1/(3*Rt[a,3]*Rt[b,3])*Int[(Rt[a,3]+Rt[b,3]*x)/(Rt[a,3]^2-Rt[a,3]*Rt[b,3]*x+Rt[b,3]^2*x^2),x] /;
FreeQ[{a,b},x]

```

$$2. \int \frac{x^m}{a+b x^5} dx \text{ when } m \in \mathbb{Z}^+ \wedge m < 4$$

$$\text{1: } \int \frac{x^m}{a+b x^5} dx \text{ when } m \in \mathbb{Z}^+ \wedge m < 4 \wedge \frac{a}{b} > 0$$

Derivation: Algebraic expansion

Basis: If $m \in \mathbb{Z} \wedge 0 \leq m < 5$, let $\frac{r}{s} = \left(\frac{a}{b}\right)^{1/5}$, then

$$\frac{x^m}{a+b x^5} = \frac{(-1)^m r^{m+1}}{5 a s^m (r+s x)} + \frac{2 r^{m+1}}{5 a s^m} \frac{r \cos\left[\frac{m\pi}{5}\right] - s \cos\left[\frac{(m+1)\pi}{5}\right] x}{r^2 - \frac{1}{2}(1+\sqrt{5}) r s x + s^2 x^2} + \frac{2 r^{m+1}}{5 a s^m} \frac{r \cos\left[\frac{3m\pi}{5}\right] - s \cos\left[\frac{3(m+1)\pi}{5}\right] x}{r^2 - \frac{1}{2}(1-\sqrt{5}) r s x + s^2 x^2}$$

Note: This rule not necessary for host systems that automatically simplify $\cos\left[\frac{k\pi}{5}\right]$ to radicals when k is an integer.

Rule 1.1.3.2.7.1.4.1.1.2.1: If $m \in \mathbb{Z}^+ \wedge m < 4 \wedge \frac{a}{b} > 0$, let $\frac{r}{s} = \left(\frac{a}{b}\right)^{1/5}$, then

$$\int \frac{x^m}{a+b x^5} dx \rightarrow \frac{(-1)^m r^{m+1}}{5 a s^m} \int \frac{1}{r+s x} dx + \frac{2 r^{m+1}}{5 a s^m} \int \frac{r \cos\left[\frac{m\pi}{5}\right] - s \cos\left[\frac{(m+1)\pi}{5}\right] x}{r^2 - \frac{1}{2}(1+\sqrt{5}) r s x + s^2 x^2} dx + \frac{2 r^{m+1}}{5 a s^m} \int \frac{r \cos\left[\frac{3m\pi}{5}\right] - s \cos\left[\frac{3(m+1)\pi}{5}\right] x}{r^2 - \frac{1}{2}(1-\sqrt{5}) r s x + s^2 x^2} dx$$

Program code:

```
(* Int[x^m_./(a_+b_.*x^5),x_Symbol] :=
  With[{r=Numerator[Rt[a/b,5]], s=Denominator[Rt[a/b,5]]},
    (-1)^m*r^(m+1)/(5*a*s^m)*Int[1/(r+s*x),x] +
    2*r^(m+1)/(5*a*s^m)*Int[(r*cos[m*Pi/5]-s*cos[(m+1)*Pi/5]*x)/(r^2-1/2*(1+Sqrt[5])*r*s*x+s^2*x^2),x] +
    2*r^(m+1)/(5*a*s^m)*Int[(r*cos[3*m*Pi/5]-s*cos[3*(m+1)*Pi/5]*x)/(r^2-1/2*(1-Sqrt[5])*r*s*x+s^2*x^2),x]] /;
  FreeQ[{a,b},x] && IGtQ[m,0] && LtQ[m,4] && PosQ[a/b] *)
```

$$\text{2: } \int \frac{x^m}{a+b x^5} dx \text{ when } m \in \mathbb{Z}^+ \wedge m < 4 \wedge \frac{a}{b} \neq 0$$

Derivation: Algebraic expansion

Basis: If $m \in \mathbb{Z} \wedge 0 \leq m < 5$, let $\frac{r}{s} = \left(-\frac{a}{b}\right)^{1/5}$, then

$$\frac{x^m}{a+b x^5} = \frac{r^{m+1}}{5 a s^m (r-s x)} + \frac{2 (-1)^m r^{m+1}}{5 a s^m} \frac{r \cos\left[\frac{m\pi}{5}\right] + s \cos\left[\frac{(m+1)\pi}{5}\right] x}{r^2 + \frac{1}{2} (1+\sqrt{5}) r s x + s^2 x^2} + \frac{2 (-1)^m r^{m+1}}{5 a s^m} \frac{r \cos\left[\frac{3m\pi}{5}\right] + s \cos\left[\frac{3(m+1)\pi}{5}\right] x}{r^2 + \frac{1}{2} (1-\sqrt{5}) r s x + s^2 x^2}$$

Note: This rule not necessary for host systems that automatically simplify $\cos\left[\frac{k\pi}{5}\right]$ to radicals when k is an integer.

Rule 1.1.3.2.7.1.4.1.1.2.1: If $m \in \mathbb{Z}^+ \wedge m < 4 \wedge \frac{a}{b} \neq 0$, let $\frac{r}{s} = \left(-\frac{a}{b}\right)^{1/5}$, then

$$\int \frac{x^m}{a+b x^5} dx \rightarrow \frac{r^{m+1}}{5 a s^m} \int \frac{1}{r-s x} dx + \frac{2 (-1)^m r^{m+1}}{5 a s^m} \int \frac{r \cos\left[\frac{m\pi}{5}\right] + s \cos\left[\frac{(m+1)\pi}{5}\right] x}{r^2 + \frac{1}{2} (1+\sqrt{5}) r s x + s^2 x^2} dx + \frac{2 (-1)^m r^{m+1}}{5 a s^m} \int \frac{r \cos\left[\frac{3m\pi}{5}\right] + s \cos\left[\frac{3(m+1)\pi}{5}\right] x}{r^2 + \frac{1}{2} (1-\sqrt{5}) r s x + s^2 x^2} dx$$

Program code:

```
(* Int[x^m_./(a+_b_.x^5),x_Symbol] :=
  With[{r=Numerator[Rt[-a/b,5]], s=Denominator[Rt[-a/b,5]]},
    (r^(m+1)/(5*a*s^m))*Int[1/(r-s*x),x] +
    2*(-1)^m*r^(m+1)/(5*a*s^m)*Int[(r*cos[m*Pi/5]+s*cos[(m+1)*Pi/5]*x)/(r^2+1/2*(1+Sqrt[5])*r*s*x+s^2*x^2),x] +
    2*(-1)^m*r^(m+1)/(5*a*s^m)*Int[(r*cos[3*m*Pi/5]+s*cos[3*(m+1)*Pi/5]*x)/(r^2+1/2*(1-Sqrt[5])*r*s*x+s^2*x^2),x] /;
  FreeQ[{a,b},x] && IGtQ[m,0] && LtQ[m,4] && NegQ[a/b] *)
```

$$3. \int \frac{x^m}{a+b x^n} dx \text{ when } \frac{n-1}{2} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1 \wedge n > 3$$

$$1: \int \frac{x^m}{a+b x^n} dx \text{ when } \frac{n-1}{2} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1 \wedge \frac{a}{b} > 0$$

Derivation: Algebraic expansion

Basis: If $\frac{n-1}{2} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1$, let $\frac{r}{s} = \left(\frac{a}{b}\right)^{1/n}$, then

$$\frac{z^m}{a+b z^n} = -\frac{(-r)^{m+1}}{a n s^m (r+s z)} + \frac{2 r^{m+1}}{a n s^m} \sum_{k=1}^{\frac{n-1}{2}} \frac{r \cos\left[\frac{(2k-1)m\pi}{n}\right] - s \cos\left[\frac{(2k-1)(m+1)\pi}{n}\right] z}{r^2 - 2 r s \cos\left[\frac{(2k-1)\pi}{n}\right] z + s^2 z^2}$$

Rule 1.1.3.2.7.1.4.1.1.3.1: If $\frac{n-1}{2} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1 \wedge \frac{a}{b} > 0$, let $\frac{r}{s} = \left(\frac{a}{b}\right)^{1/n}$, then

$$\int \frac{x^m}{a + b x^n} dx \rightarrow -\frac{(-r)^{m+1}}{a n s^m} \int \frac{1}{r + s x} dx + \frac{2 r^{m+1}}{a n s^m} \sum_{k=1}^{\frac{n-1}{2}} \int \frac{r \cos\left[\frac{(2k-1)m\pi}{n}\right] - s \cos\left[\frac{(2k-1)(m+1)\pi}{n}\right] x}{r^2 - 2 r s \cos\left[\frac{(2k-1)\pi}{n}\right] x + s^2 x^2} dx$$

Program code:

```
Int[x^m_./(a+b_.*x^n_),x_Symbol] :=
Module[{r=Numerator[Rt[a/b,n]], s=Denominator[Rt[a/b,n]], k, u},
u=Int[(r*cos[(2*k-1)*m*Pi/n]-s*cos[(2*k-1)*(m+1)*Pi/n]*x)/(r^2-2*r*s*cos[(2*k-1)*Pi/n]*x+s^2*x^2),x];
-(-r)^(m+1)/(a*n*s^m)*Int[1/(r+s*x),x] + Dist[2*r^(m+1)/(a*n*s^m),Sum[u,{k,1,(n-1)/2}],x] /;
FreeQ[{a,b},x] && IGtQ[(n-1)/2,0] && IGtQ[m,0] && LtQ[m,n-1] && PosQ[a/b]
```

2: $\int \frac{x^m}{a + b x^n} dx$ when $\frac{n-1}{2} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1 \wedge \frac{a}{b} \neq 0$

Derivation: Algebraic expansion

Basis: If $\frac{n-1}{2} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1$, let $\frac{r}{s} = \left(-\frac{a}{b}\right)^{1/n}$, then

$$\frac{z^m}{a+b z^n} = \frac{r^{m+1}}{a n s^m (r-s z)} - \frac{2 (-r)^{m+1}}{a n s^m} \sum_{k=1}^{\frac{n-1}{2}} \frac{r \cos\left[\frac{(2k-1)m\pi}{n}\right] + s \cos\left[\frac{(2k-1)(m+1)\pi}{n}\right] z}{r^2 + 2 r s \cos\left[\frac{(2k-1)\pi}{n}\right] z + s^2 z^2}$$

Rule 1.1.3.2.7.1.4.1.1.3.2: If $\frac{n-1}{2} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1 \wedge \frac{a}{b} \neq 0$, let $\frac{r}{s} = \left(-\frac{a}{b}\right)^{1/n}$, then

$$\int \frac{x^m}{a + b x^n} dx \rightarrow \frac{r^{m+1}}{a n s^m} \int \frac{1}{r - s x} dx - \frac{2 (-r)^{m+1}}{a n s^m} \sum_{k=1}^{\frac{n-1}{2}} \int \frac{r \cos\left[\frac{(2k-1)m\pi}{n}\right] + s \cos\left[\frac{(2k-1)(m+1)\pi}{n}\right] x}{r^2 + 2 r s \cos\left[\frac{(2k-1)\pi}{n}\right] x + s^2 x^2} dx$$

Program code:

```
Int[x^m_./(a+b_.*x^n_),x_Symbol] :=
Module[{r=Numerator[Rt[-a/b,n]], s=Denominator[Rt[-a/b,n]], k, u},
u=Int[(r*cos[(2*k-1)*m*Pi/n]+s*cos[(2*k-1)*(m+1)*Pi/n]*x)/(r^2+2*r*s*cos[(2*k-1)*Pi/n]*x+s^2*x^2),x];
r^(m+1)/(a*n*s^m)*Int[1/(r-s*x),x] - Dist[2*(-r)^(m+1)/(a*n*s^m),Sum[u,{k,1,(n-1)/2}],x] /;
FreeQ[{a,b},x] && IGtQ[(n-1)/2,0] && IGtQ[m,0] && LtQ[m,n-1] && NegQ[a/b]
```


$$2. \int \frac{x^m}{a+b x^n} dx \text{ when } \frac{n}{2} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1$$

$$1. \int \frac{x^m}{a+b x^n} dx \text{ when } \frac{n-2}{4} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1$$

$$\text{1: } \int \frac{x^m}{a+b x^n} dx \text{ when } \frac{n-2}{4} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1 \wedge \frac{a}{b} > 0$$

Derivation: Algebraic expansion

Basis: If $\frac{n-2}{4} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1$, let $\frac{r}{s} = \left(\frac{a}{b}\right)^{1/n}$, then

$$\frac{z^m}{a+b z^n} = \frac{2(-1)^{\frac{m}{2}} r^{m+2}}{a n s^m (r^2+s^2 z^2)} + \frac{4 r^{m+2}}{a n s^m} \sum_{k=1}^{\frac{n-2}{4}} \frac{r^2 \cos\left[\frac{(2k-1)m\pi}{n}\right] - s^2 \cos\left[\frac{(2k-1)(m+2)\pi}{n}\right] z^2}{r^4 - 2 r^2 s^2 \cos\left[\frac{2(2k-1)\pi}{n}\right] z^2 + s^4 z^4}$$

$$\text{Basis: } \frac{r^2 \cos[\rho] - s^2 \cos[\rho+2\theta] z^2}{r^4 - 2 r^2 s^2 \cos[2\theta] z^2 + s^4 z^4} = \frac{1}{2r} \left(\frac{r \cos[\rho] - s \cos[\rho+\theta] z}{r^2 - 2 r s \cos[\theta] z + s^2 z^2} + \frac{r \cos[\rho] + s \cos[\rho+\theta] z}{r^2 + 2 r s \cos[\theta] z + s^2 z^2} \right)$$

Rule 1.1.3.2.7.1.4.1.2.1.1: If $\frac{n-2}{4} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1 \wedge \frac{a}{b} > 0$, let $\frac{r}{s} = \left(\frac{a}{b}\right)^{1/n}$, then

$$\begin{aligned} \int \frac{x^m}{a+b x^n} dx &\rightarrow \frac{2(-1)^{\frac{m}{2}} r^{m+2}}{a n s^m} \int \frac{1}{r^2 + s^2 x^2} dx + \frac{4 r^{m+2}}{a n s^m} \sum_{k=1}^{\frac{n-2}{4}} \int \frac{r^2 \cos\left[\frac{(2k-1)m\pi}{n}\right] - s^2 \cos\left[\frac{(2k-1)(m+2)\pi}{n}\right] x^2}{r^4 - 2 r^2 s^2 \cos\left[\frac{2(2k-1)\pi}{n}\right] x^2 + s^4 x^4} dx \\ &\rightarrow \frac{2(-1)^{\frac{m}{2}} r^{m+2}}{a n s^m} \int \frac{1}{r^2 + s^2 x^2} dx + \frac{2 r^{m+1}}{a n s^m} \sum_{k=1}^{\frac{n-2}{4}} \left(\int \frac{r \cos\left[\frac{(2k-1)m\pi}{n}\right] - s \cos\left[\frac{(2k-1)(m+1)\pi}{n}\right] x}{r^2 - 2 r s \cos\left[\frac{(2k-1)\pi}{n}\right] x + s^2 x^2} dx + \int \frac{r \cos\left[\frac{(2k-1)m\pi}{n}\right] + s \cos\left[\frac{(2k-1)(m+1)\pi}{n}\right] x}{r^2 + 2 r s \cos\left[\frac{(2k-1)\pi}{n}\right] x + s^2 x^2} dx \right) \end{aligned}$$

Program code:

```
Int[x_^m_/(a_+b_.*x_^n_),x_Symbol] :=
Module[{r=Numerator[Rt[a/b,n]], s=Denominator[Rt[a/b,n]], k, u},
u=Int[(r*cos[(2*k-1)*m*Pi/n]-s*cos[(2*k-1)*(m+1)*Pi/n]*x)/(r^2-2*r*s*cos[(2*k-1)*Pi/n]*x+s^2*x^2),x] +
Int[(r*cos[(2*k-1)*m*Pi/n]+s*cos[(2*k-1)*(m+1)*Pi/n]*x)/(r^2+2*r*s*cos[(2*k-1)*Pi/n]*x+s^2*x^2),x];
2*(-1)^(m/2)*r^(m+2)/(a*n*s^m)*Int[1/(r^2+s^2*x^2),x] + Dist[2*r^(m+1)/(a*n*s^m),Sum[u,{k,1,(n-2)/4}],x] /;
FreeQ[{a,b},x] && IGtQ[(n-2)/4,0] && IGtQ[m,0] && LtQ[m,n-1] && PosQ[a/b]
```

$$\mathbf{2:} \int \frac{x^m}{a+b x^n} dx \text{ when } \frac{n-2}{4} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1 \wedge \frac{a}{b} \neq 0$$

Derivation: Algebraic expansion

Basis: If $\frac{n-2}{4} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1$, let $\frac{r}{s} = \left(-\frac{a}{b}\right)^{1/n}$, then

$$\frac{z^m}{a+b z^n} = \frac{2 r^{m+2}}{a n s^m (r^2 - s^2 z^2)} + \frac{4 r^{m+2}}{a n s^m} \sum_{k=1}^{\frac{n-2}{4}} \frac{r^2 \cos\left[\frac{2 k m \pi}{n}\right] - s^2 \cos\left[\frac{2 k (m+2) \pi}{n}\right] z^2}{r^4 - 2 r^2 s^2 \cos\left[\frac{4 k \pi}{n}\right] z^2 + s^4 z^4}$$

$$\text{Basis: } \frac{r^2 \cos[\rho] - s^2 \cos[\rho+2\theta] z^2}{r^4 - 2 r^2 s^2 \cos[2\theta] z^2 + s^4 z^4} = \frac{1}{2r} \left(\frac{r \cos[\rho] - s \cos[\rho+\theta] z}{r^2 - 2 r s \cos[\theta] z + s^2 z^2} + \frac{r \cos[\rho] + s \cos[\rho+\theta] z}{r^2 + 2 r s \cos[\theta] z + s^2 z^2} \right)$$

Rule 1.1.3.2.7.1.4.1.2.1.2: If $\frac{n-2}{4} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1 \wedge \frac{a}{b} \neq 0$, let $\frac{r}{s} = \left(-\frac{a}{b}\right)^{1/n}$, then

$$\begin{aligned} \int \frac{x^m}{a+b x^n} dx &\rightarrow \frac{2 r^{m+2}}{a n s^m} \int \frac{1}{r^2 - s^2 x^2} dx + \frac{4 r^{m+2}}{a n s^m} \sum_{k=1}^{\frac{n-2}{4}} \int \frac{r^2 \cos\left[\frac{2 k m \pi}{n}\right] - s^2 \cos\left[\frac{2 k (m+2) \pi}{n}\right] x^2}{r^4 - 2 r^2 s^2 \cos\left[\frac{4 k \pi}{n}\right] x^2 + s^4 x^4} dx \\ &\rightarrow \frac{2 r^{m+2}}{a n s^m} \int \frac{1}{r^2 - s^2 x^2} dx + \frac{2 r^{m+1}}{a n s^m} \sum_{k=1}^{\frac{n-2}{4}} \left(\int \frac{r \cos\left[\frac{2 k m \pi}{n}\right] - s \cos\left[\frac{2 k (m+1) \pi}{n}\right] x}{r^2 - 2 r s \cos\left[\frac{2 k \pi}{n}\right] x + s^2 x^2} dx + \int \frac{r \cos\left[\frac{2 k m \pi}{n}\right] + s \cos\left[\frac{2 k (m+1) \pi}{n}\right] x}{r^2 + 2 r s \cos\left[\frac{2 k \pi}{n}\right] x + s^2 x^2} dx \right) \end{aligned}$$

Program code:

```
Int[x_^m_/(a_+b_.*x_^n_),x_Symbol] :=
Module[{r=Numerator[Rt[-a/b,n]], s=Denominator[Rt[-a/b,n]], k, u},
u=Int[(r*cos[2*k*m*Pi/n]-s*cos[2*k*(m+1)*Pi/n]*x)/(r^2-2*r*s*cos[2*k*Pi/n]*x+s^2*x^2),x] +
Int[(r*cos[2*k*m*Pi/n]+s*cos[2*k*(m+1)*Pi/n]*x)/(r^2+2*r*s*cos[2*k*Pi/n]*x+s^2*x^2),x];
2*r^(m+2)/(a*n*s^m)*Int[1/(r^2-s^2*x^2),x] + Dist[2*r^(m+1)/(a*n*s^m),Sum[u,{k,1,(n-2)/4}],x] /;
FreeQ[{a,b},x] && IGtQ[(n-2)/4,0] && IGtQ[m,0] && LtQ[m,n-1] && NegQ[a/b]
```

$$2. \int \frac{x^m}{a + b x^n} dx \text{ when } \frac{n}{4} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n - 1$$

$$1. \int \frac{x^2}{a + b x^4} dx$$

$$1: \int \frac{x^2}{a + b x^4} dx \text{ when } \frac{a}{b} > 0$$

Derivation: Algebraic expansion

Basis: If $\frac{r}{s} = \sqrt{\frac{a}{b}}$, then $\frac{x^2}{a + b x^4} = \frac{r + s x^2}{2 s (a + b x^4)} - \frac{r - s x^2}{2 s (a + b x^4)}$

Note: Resulting integrands are of the form $\frac{d + e x^2}{a + c x^4}$ where $c d^2 - a e^2 = 0$ as required by the algebraic trinomial rules.

Rule 1.1.3.2.7.1.4.1.2.2.1.1: If $\frac{a}{b} > 0$, let $\frac{r}{s} = \sqrt{\frac{a}{b}}$, then

$$\int \frac{x^2}{a + b x^4} dx \rightarrow \frac{1}{2 s} \int \frac{r + s x^2}{a + b x^4} dx - \frac{1}{2 s} \int \frac{r - s x^2}{a + b x^4} dx$$

Program code:

```
Int[x^2/(a+b_.*x^4),x_Symbol] :=
  With[{r=Numerator[Rt[a/b,2]], s=Denominator[Rt[a/b,2]]},
    1/(2*s)*Int[(r+s*x^2)/(a+b*x^4),x] -
    1/(2*s)*Int[(r-s*x^2)/(a+b*x^4),x] /;
    FreeQ[{a,b},x] && (GtQ[a/b,0] || PosQ[a/b] && AtomQ[SplitProduct[SumBaseQ,a]] && AtomQ[SplitProduct[SumBaseQ,b]])
```

$$2: \int \frac{x^2}{a + b x^4} dx \text{ when } \frac{a}{b} \neq 0$$

Reference: G&R 2.132.3.2', CRC 82'

Derivation: Algebraic expansion

Basis: If $\frac{r}{s} = \sqrt{-\frac{a}{b}}$, then $\frac{z}{a+b z^2} = \frac{s}{2 b (r+s z)} - \frac{s}{2 b (r-s z)}$

– Rule 1.1.3.2.7.1.4.1.2.2.1.2: If $\frac{a}{b} \neq 0$, let $\frac{r}{s} = \sqrt{-\frac{a}{b}}$, then

$$\int \frac{x^2}{a + b x^4} dx \rightarrow \frac{s}{2 b} \int \frac{1}{r + s x^2} dx - \frac{s}{2 b} \int \frac{1}{r - s x^2} dx$$

Program code:

```
Int[x^2/(a+b_*x^4),x_Symbol] :=
  With[{r=Numerator[Rt[-a/b,2]], s=Denominator[Rt[-a/b,2]]},
    s/(2*b)*Int[1/(r+s*x^2),x] -
    s/(2*b)*Int[1/(r-s*x^2),x] /;
  FreeQ[{a,b},x] && Not[GtQ[a/b,0]]
```

$$2. \int \frac{x^m}{a+b x^n} dx \text{ when } \frac{n}{4} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1 \wedge n > 4$$

$$\text{1: } \int \frac{x^m}{a+b x^n} dx \text{ when } \frac{n}{4} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1 \wedge \frac{a}{b} > 0$$

Reference: G&R 2.132.3.1', CRC 81'

Derivation: Algebraic expansion

Basis: If $\frac{r}{s} = \left(\frac{a}{b}\right)^{1/4}$, then $\frac{z}{a+b z^4} = \frac{s^3}{2\sqrt{2} b r (r^2 - \sqrt{2} r s z + s^2 z^2)} - \frac{s^3}{2\sqrt{2} b r (r^2 + \sqrt{2} r s z + s^2 z^2)}$

Rule 1.1.3.2.7.1.4.1.2.2.2.1: If $\frac{n}{4} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n-1 \wedge \frac{a}{b} > 0$, let $\frac{r}{s} = \left(\frac{a}{b}\right)^{1/4}$, then

$$\int \frac{x^m}{a+b x^n} dx \rightarrow \frac{s^3}{2\sqrt{2} b r} \int \frac{x^{m-n/4}}{r^2 - \sqrt{2} r s x^{n/4} + s^2 x^{n/2}} dx - \frac{s^3}{2\sqrt{2} b r} \int \frac{x^{m-n/4}}{r^2 + \sqrt{2} r s x^{n/4} + s^2 x^{n/2}} dx$$

Program code:

```
Int[x_^m_./(a_+b_.x_^n_),x_Symbol] :=
  With[{r=Numerator[Rt[a/b,4]], s=Denominator[Rt[a/b,4]]},
    s^3/(2*Sqrt[2]*b*r)*Int[x^(m-n/4)/(r^2-Sqrt[2]*r*s*x^(n/4)+s^2*x^(n/2)),x] -
    s^3/(2*Sqrt[2]*b*r)*Int[x^(m-n/4)/(r^2+Sqrt[2]*r*s*x^(n/4)+s^2*x^(n/2)),x] /;
  FreeQ[{a,b},x] && IGtQ[n/4,0] && IGtQ[m,0] && LtQ[m,n-1] && GtQ[a/b,0]
```

$$2. \int \frac{x^m}{a + b x^n} dx \text{ when } \frac{n}{4} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < n - 1 \wedge \frac{a}{b} \neq 0$$

$$1: \int \frac{x^m}{a + b x^n} dx \text{ when } \frac{n}{4} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < \frac{n}{2} \wedge \frac{a}{b} \neq 0$$

Reference: G&R 2.132.1.2', CRC 78'

Derivation: Algebraic expansion

Basis: If $\frac{r}{s} = \sqrt{-\frac{a}{b}}$, then $\frac{1}{a+b x^2} = \frac{r}{2 a (r+s x)} + \frac{r}{2 a (r-s x)}$

Rule 1.1.3.2.7.1.4.1.2.2.2.1: If $\frac{n}{4} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m < \frac{n}{2} \wedge \frac{a}{b} \neq 0$, let $\frac{r}{s} = \sqrt{-\frac{a}{b}}$, then

$$\int \frac{x^m}{a + b x^n} dx \rightarrow \frac{r}{2 a} \int \frac{x^m}{r + s x^{n/2}} dx + \frac{r}{2 a} \int \frac{x^m}{r - s x^{n/2}} dx$$

Program code:

```
Int[x^m/(a+b_*x^n),x_Symbol] :=
  With[{r=Numerator[Rt[-a/b,2]], s=Denominator[Rt[-a/b,2]]},
    r/(2*a)*Int[x^m/(r+s*x^(n/2)),x] +
    r/(2*a)*Int[x^m/(r-s*x^(n/2)),x] /;
  FreeQ[{a,b},x] && IGtQ[n/4,0] && IGtQ[m,0] && LtQ[m,n/2] && Not[GtQ[a/b,0]]
```

$$2: \int \frac{x^m}{a + b x^n} dx \text{ when } \frac{n}{4} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge \frac{n}{2} \leq m < n \wedge \frac{a}{b} \neq 0$$

Reference: G&R 2.132.3.2', CRC 82'

Derivation: Algebraic expansion

Basis: If $\frac{r}{s} = \sqrt{-\frac{a}{b}}$, then $\frac{z}{a + b z^2} = \frac{s}{2 b (r + s z)} - \frac{s}{2 b (r - s z)}$

-

Rule 1.1.3.2.7.1.4.1.2.2.2.2.2: If $\frac{n}{4} \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge \frac{n}{2} \leq m < n \wedge \frac{a}{b} \neq 0$, let $\frac{r}{s} = \sqrt{-\frac{a}{b}}$, then

$$\int \frac{x^m}{a + b x^n} dx \rightarrow \frac{s}{2 b} \int \frac{x^{m-n/2}}{r + s x^{n/2}} dx - \frac{s}{2 b} \int \frac{x^{m-n/2}}{r - s x^{n/2}} dx$$

Program code:

```
Int[x_^m/(a_+b_.*x_^n_),x_Symbol] :=
  With[{r=Numerator[Rt[-a/b,2]], s=Denominator[Rt[-a/b,2]]},
    s/(2*b)*Int[x^(m-n/2)/(r+s*x^(n/2)),x] -
    s/(2*b)*Int[x^(m-n/2)/(r-s*x^(n/2)),x] /;
  FreeQ[{a,b},x] && IGtQ[n/4,0] && IGtQ[m,0] && LeQ[n/2,m] && LtQ[m,n] && Not[GtQ[a/b,0]]
```

2: $\int \frac{x^m}{a + b x^n} dx$ when $n \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m > 2n - 1$

Derivation: Algebraic expansion

Rule 1.1.3.2.7.1.4.2: If $n \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+ \wedge m > 2n - 1$, then

$$\int \frac{x^m}{a + b x^n} dx \rightarrow \int \text{PolynomialDivide}[x^m, a + b x^n, x] dx$$

Program code:

```
Int[x_^m/(a+b_*x_^n_),x_Symbol] :=
  Int[PolynomialDivide[x^m,(a+b*x^n),x],x] /;
FreeQ[{a,b},x] && IGtQ[m,0] && IGtQ[n,0] && GtQ[m,2*n-1]
```


$$5. \int \frac{x^m}{\sqrt{a+b x^n}} dx \text{ when } n \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^+$$

$$1. \int \frac{x}{\sqrt{a+b x^3}} dx$$

$$1: \int \frac{x}{\sqrt{a+b x^3}} dx \text{ when } a > 0$$

Derivation: Algebraic expansion

Note: $\frac{\sqrt{2}}{\sqrt{2+\sqrt{3}}} = -1 + \sqrt{3}$

Rule: If $a > 0$, let $\frac{r}{s} \rightarrow \left(\frac{b}{a}\right)^{1/3}$, then

$$\int \frac{x}{\sqrt{a+b x^3}} dx \rightarrow \frac{\sqrt{2} s}{\sqrt{2+\sqrt{3}} r} \int \frac{1}{\sqrt{a+b x^3}} dx + \frac{1}{r} \int \frac{(1-\sqrt{3}) s + r x}{\sqrt{a+b x^3}} dx$$

Program code:

```
Int[x_/Sqrt[a+b_.*x_^3],x_Symbol] :=
  With[{r=Numer[Rt[b/a,3]], s=Denom[Rt[b/a,3]]},
    Sqrt[2]*s/(Sqrt[2+Sqrt[3]]*r)*Int[1/Sqrt[a+b*x^3],x] + 1/r*Int[((1-Sqrt[3])*s+r*x)/Sqrt[a+b*x^3],x] /;
    FreeQ[{a,b},x] && PosQ[a]
```

2: $\int \frac{x}{\sqrt{a+b x^3}} dx$ when $a \neq 0$

Derivation: Algebraic expansion

Note: $\frac{\sqrt{2}}{\sqrt{2-\sqrt{3}}} = 1 + \sqrt{3}$

Rule: If $a \neq 0$, let $\frac{r}{s} \rightarrow \left(\frac{b}{a}\right)^{1/3}$, then

$$\int \frac{x}{\sqrt{a+b x^3}} dx \rightarrow -\frac{\sqrt{2} s}{\sqrt{2-\sqrt{3}} r} \int \frac{1}{\sqrt{a+b x^3}} dx + \frac{1}{r} \int \frac{(1+\sqrt{3}) s + r x}{\sqrt{a+b x^3}} dx$$

Program code:

```
Int[x_/Sqrt[a+b_*x^3],x_Symbol] :=
  With[{r=Numer[Rt[b/a,3]], s=Denom[Rt[b/a,3]]},
    -Sqrt[2]*s/(Sqrt[2-Sqrt[3]]*r)*Int[1/Sqrt[a+b*x^3],x] + 1/r*Int[((1+Sqrt[3])*s+r*x)/Sqrt[a+b*x^3],x] /;
  FreeQ[{a,b},x] && NegQ[a]
```

$$2. \int \frac{x^2}{\sqrt{a + b x^4}} dx$$

$$\text{1: } \int \frac{x^2}{\sqrt{a + b x^4}} dx \text{ when } \frac{b}{a} > 0$$

Derivation: Algebraic expansion

Rule 1.1.3.2.7.1.5.2.1: If $\frac{b}{a} > 0$, let $q \rightarrow \sqrt{\frac{b}{a}}$, then

$$\int \frac{x^2}{\sqrt{a + b x^4}} dx \rightarrow \frac{1}{q} \int \frac{1}{\sqrt{a + b x^4}} dx - \frac{1}{q} \int \frac{1 - q x^2}{\sqrt{a + b x^4}} dx$$

Program code:

```
Int[x^2/Sqrt[a+b*x^4],x_Symbol] :=
  With[{q=Rt[b/a,2]},
    1/q*Int[1/Sqrt[a+b*x^4],x] - 1/q*Int[(1-q*x^2)/Sqrt[a+b*x^4],x] /;
    FreeQ[{a,b},x] && PosQ[b/a]
```

$$2. \int \frac{x^2}{\sqrt{a + b x^4}} dx \text{ when } \frac{b}{a} \neq 0$$

$$1: \int \frac{x^2}{\sqrt{a + b x^4}} dx \text{ when } a < 0 \wedge b > 0$$

Derivation: Algebraic expansion

Rule 1.1.3.2.7.1.5.2.2.1: If $\frac{b}{a} \neq 0 \wedge a < 0$, let $q \rightarrow \sqrt{-\frac{b}{a}}$, then

$$\int \frac{x^2}{\sqrt{a + b x^4}} dx \rightarrow \frac{1}{q} \int \frac{1}{\sqrt{a + b x^4}} dx - \frac{1}{q} \int \frac{1 - q x^2}{\sqrt{a + b x^4}} dx$$

Program code:

```
Int[x^2/Sqrt[a+b*x^4],x_Symbol] :=
  With[{q=Rt[-b/a,2]},
    1/q*Int[1/Sqrt[a+b*x^4],x] - 1/q*Int[(1-q*x^2)/Sqrt[a+b*x^4],x] /;
  FreeQ[{a,b},x] && LtQ[a,0] && GtQ[b,0]
```

2: $\int \frac{x^2}{\sqrt{a+b x^4}} dx$ when $\frac{b}{a} \neq 0 \wedge a \neq 0$

Derivation: Algebraic expansion

Rule 1.1.3.2.7.1.5.2.2.2: If $\frac{b}{a} \neq 0 \wedge a \neq 0$, let $q \rightarrow \sqrt{-\frac{b}{a}}$, then

$$\int \frac{x^2}{\sqrt{a+b x^4}} dx \rightarrow -\frac{1}{q} \int \frac{1}{\sqrt{a+b x^4}} dx + \frac{1}{q} \int \frac{1+q x^2}{\sqrt{a+b x^4}} dx$$

Program code:

```
Int[x_^2/Sqrt[a+b_.**x_^4],x_Symbol] :=
  With[{q=Rt[-b/a,2]},
    -1/q*Int[1/Sqrt[a+b*x^4],x] + 1/q*Int[(1+q*x^2)/Sqrt[a+b*x^4],x] /;
    FreeQ[{a,b},x] && NegQ[b/a]
```

3: $\int \frac{x^4}{\sqrt{a+b x^6}} dx$

Derivation: Algebraic expansion

Rule 1.1.3.2.7.1.5.3: Let $\frac{r}{s} \rightarrow \left(\frac{b}{a}\right)^{1/3}$, then

$$\int \frac{x^4}{\sqrt{a+b x^6}} dx \rightarrow \frac{(\sqrt{3}-1)s^2}{2r^2} \int \frac{1}{\sqrt{a+b x^6}} dx - \frac{1}{2r^2} \int \frac{(\sqrt{3}-1)s^2 - 2r^2 x^4}{\sqrt{a+b x^6}} dx$$

$$\rightarrow \frac{(1+\sqrt{3})rx\sqrt{a+b x^6}}{2b(s+(1+\sqrt{3})rx^2)} - \frac{3^{1/4}sx(s+rx^2)\sqrt{\frac{s^2-rsx^2+r^2x^4}{(s+(1+\sqrt{3})rx^2)^2}}}{2r^2\sqrt{a+b x^6}\sqrt{\frac{rx^2(s+rx^2)}{(s+(1+\sqrt{3})rx^2)^2}}} \text{EllipticE}\left[\text{ArcCos}\left[\frac{s+(1-\sqrt{3})rx^2}{s+(1+\sqrt{3})rx^2}\right], \frac{2+\sqrt{3}}{4}\right] -$$

$$\frac{(1 - \sqrt{3}) s x (s + r x^2) \sqrt{\frac{s^2 - r s x^2 + r^2 x^4}{(s + (1 + \sqrt{3}) r x^2)^2}}}{4 \times 3^{1/4} r^2 \sqrt{a + b x^6} \sqrt{\frac{r x^2 (s + r x^2)}{(s + (1 + \sqrt{3}) r x^2)^2}}} \text{EllipticF}\left[\text{ArcCos}\left[\frac{s + (1 - \sqrt{3}) r x^2}{s + (1 + \sqrt{3}) r x^2}\right], \frac{2 + \sqrt{3}}{4}\right]$$

Program code:

```
Int[x^4/Sqrt[a+b_.*x^6],x_Symbol] :=
  With[{r=Numer[Rt[b/a,3]], s=Denom[Rt[b/a,3]]},
    (Sqrt[3]-1)*s^2/(2*r^2)*Int[1/Sqrt[a+b*x^6],x] - 1/(2*r^2)*Int[((Sqrt[3]-1)*s^2-2*r^2*x^4)/Sqrt[a+b*x^6],x] /;
  FreeQ[{a,b},x]
```

```
(* Int[x^4/Sqrt[a+b_.*x^6],x_Symbol] :=
  With[{r=Numer[Rt[b/a,3]], s=Denom[Rt[b/a,3]]},
    (1+Sqrt[3])*r*x*Sqrt[a+b*x^6]/(2*b*(s+(1+Sqrt[3])*r*x^2)) -
    3^(1/4)*s*x*(s+r*x^2)*Sqrt[(s^2-r*s*x^2+r^2*x^4)/(s+(1+Sqrt[3])*r*x^2)^2]/
    (2*r^2*Sqrt[a+b*x^6]*Sqrt[r*x^2*(s+r*x^2)/(s+(1+Sqrt[3])*r*x^2)^2])*
    EllipticE[ArcCos[(s+(1-Sqrt[3])*r*x^2)/(s+(1+Sqrt[3])*r*x^2)],(2+Sqrt[3])/4] -
    (1-Sqrt[3])*s*x*(s+r*x^2)*Sqrt[(s^2-r*s*x^2+r^2*x^4)/(s+(1+Sqrt[3])*r*x^2)^2]/
    (4*3^(1/4)*r^2*Sqrt[a+b*x^6]*Sqrt[r*x^2*(s+r*x^2)/(s+(1+Sqrt[3])*r*x^2)^2])*
    EllipticF[ArcCos[(s+(1-Sqrt[3])*r*x^2)/(s+(1+Sqrt[3])*r*x^2)],(2+Sqrt[3])/4] /;
  FreeQ[{a,b},x] *)
```

4: $\int \frac{x^2}{\sqrt{a+b x^8}} dx$

Derivation: Algebraic expansion

Basis: $\frac{x^2}{\sqrt{a+b x^8}} = \frac{1 + \left(\frac{b}{a}\right)^{1/4} x^2}{2 \left(\frac{b}{a}\right)^{1/4} \sqrt{a+b x^8}} - \frac{1 - \left(\frac{b}{a}\right)^{1/4} x^2}{2 \left(\frac{b}{a}\right)^{1/4} \sqrt{a+b x^8}}$

Note: Integrands are of the form $\frac{c+d x^2}{\sqrt{a+b x^8}}$ where $b c^4 - a d^4 = 0$ for which there is a terminal rule.

Rule 1.1.3.2.7.1.5.4:

$$\int \frac{x^2}{\sqrt{a+b x^8}} dx \rightarrow \frac{1}{2 \left(\frac{b}{a}\right)^{1/4}} \int \frac{1 + \left(\frac{b}{a}\right)^{1/4} x^2}{\sqrt{a+b x^8}} dx - \frac{1}{2 \left(\frac{b}{a}\right)^{1/4}} \int \frac{1 - \left(\frac{b}{a}\right)^{1/4} x^2}{\sqrt{a+b x^8}} dx$$

Program code:

```
Int[x^2/Sqrt[a+b_*x^8],x_Symbol] :=
  1/(2*Rt[b/a,4])*Int[(1+Rt[b/a,4]*x^2)/Sqrt[a+b*x^8],x] -
  1/(2*Rt[b/a,4])*Int[(1-Rt[b/a,4]*x^2)/Sqrt[a+b*x^8],x] /;
FreeQ[{a,b},x]
```

$$6. \int \frac{x^m}{(a + b x^n)^{1/4}} dx \text{ when } n \in \mathbb{Z}^+ \wedge 2m \in \mathbb{Z}^+$$

$$1. \int \frac{x^2}{(a + b x^4)^{1/4}} dx$$

$$\text{1: } \int \frac{x^2}{(a + b x^4)^{1/4}} dx \text{ when } \frac{b}{a} > 0$$

Reference: G&R 2.110.1, CRC 88b

Derivation: Binomial recurrence 1b

Rule 1.1.3.2.7.1.6.1.1: If $\frac{b}{a} > 0$, then

$$\int \frac{x^2}{(a + b x^4)^{1/4}} dx \rightarrow \frac{x^3}{2 (a + b x^4)^{1/4}} - \frac{a}{2} \int \frac{x^2}{(a + b x^4)^{5/4}} dx$$

Program code:

```
Int[x^2/(a+b.*x^4)^(1/4),x_Symbol] :=
  x^3/(2*(a+b*x^4)^(1/4)) - a/2*Int[x^2/(a+b*x^4)^(5/4),x] /;
FreeQ[{a,b},x] && PosQ[b/a]
```

$$\text{2: } \int \frac{x^2}{(a + b x^4)^{1/4}} dx \text{ when } \frac{b}{a} \neq 0$$

Reference: G&R 2.110.5, CRC 88a

Derivation: Binomial recurrence 3a

Rule 1.1.3.2.7.1.6.1.2: If $\frac{b}{a} \neq 0$, then

$$\int \frac{x^2}{(a + b x^4)^{1/4}} dx \rightarrow \frac{(a + b x^4)^{3/4}}{2 b x} + \frac{a}{2 b} \int \frac{1}{x^2 (a + b x^4)^{1/4}} dx$$

Program code:

```
Int[x^2/(a+b_*x^4)^(1/4),x_Symbol] :=
  (a+b*x^4)^(3/4)/(2*b*x) + a/(2*b)*Int[1/(x^2*(a+b*x^4)^(1/4)),x] /;
FreeQ[{a,b},x] && NegQ[b/a]
```

$$2. \int \frac{1}{x^2 (a + b x^4)^{1/4}} dx$$

$$\textcolor{red}{1}: \int \frac{1}{x^2 (a + b x^4)^{1/4}} dx \text{ when } \frac{b}{a} > 0$$

Reference: G&R 2.110.3

Derivation: Binomial recurrence 1a

Derivation: Integration by parts

Rule 1.1.3.2.7.1.6.2.1: If $\frac{b}{a} > 0$, then

$$\int \frac{1}{x^2 (a + b x^4)^{1/4}} dx \rightarrow -\frac{1}{x (a + b x^4)^{1/4}} - b \int \frac{x^2}{(a + b x^4)^{5/4}} dx$$

Program code:

```
Int[1/(x^2*(a+b_*x^4)^(1/4)),x_Symbol] :=
  -1/(x*(a+b*x^4)^(1/4)) - b*Int[x^2/(a+b*x^4)^(5/4),x] /;
FreeQ[{a,b},x] && PosQ[b/a]
```

$$\text{2: } \int \frac{1}{x^2 (a + b x^4)^{1/4}} dx \text{ when } \frac{b}{a} \neq 0$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{x \left(1 + \frac{a}{b x^4}\right)^{1/4}}{(a + b x^4)^{1/4}} == 0$$

Rule 1.1.3.2.7.1.6.2.2: If $\frac{b}{a} \neq 0$, then

$$\int \frac{1}{x^2 (a + b x^4)^{1/4}} dx \rightarrow \frac{x \left(1 + \frac{a}{b x^4}\right)^{1/4}}{(a + b x^4)^{1/4}} \int \frac{1}{x^3 \left(1 + \frac{a}{b x^4}\right)^{1/4}} dx$$

Program code:

```
Int[1/(x^2*(a+b_*x^4)^(1/4)),x_Symbol] :=
  x*(1+a/(b*x^4))^(1/4)/(a+b*x^4)^(1/4)*Int[1/(x^3*(1+a/(b*x^4))^(1/4)),x] /;
FreeQ[{a,b},x] && NegQ[b/a]
```

$$3. \int \frac{\sqrt{c x}}{(a + b x^2)^{1/4}} dx$$

$$1: \int \frac{\sqrt{c x}}{(a + b x^2)^{1/4}} dx \text{ when } \frac{b}{a} > 0$$

Reference: G&R 2.110.1, CRC 88b

Derivation: Binomial recurrence 1b

Rule 1.1.3.2.7.1.6.3.2: If $\frac{b}{a} > 0$, then

$$\int \frac{\sqrt{c x}}{(a + b x^2)^{1/4}} dx \rightarrow \frac{x \sqrt{c x}}{(a + b x^2)^{1/4}} - \frac{a}{2} \int \frac{\sqrt{c x}}{(a + b x^2)^{5/4}} dx$$

Program code:

```
Int[Sqrt[c*x]/(a+b*x^2)^(1/4),x_Symbol] :=
  x*Sqrt[c*x]/(a+b*x^2)^(1/4) - a/2*Int[Sqrt[c*x]/(a+b*x^2)^(5/4),x] /;
FreeQ[{a,b,c},x] && PosQ[b/a]
```

2: $\int \frac{\sqrt{c x}}{(a + b x^2)^{1/4}} dx$ when $\frac{b}{a} \neq 0$

Reference: G&R 2.110.5, CRC 88a

Derivation: Binomial recurrence 3a

Rule 1.1.3.2.7.1.6.3.2: If $\frac{b}{a} \neq 0$, then

$$\int \frac{\sqrt{c x}}{(a + b x^2)^{1/4}} dx \rightarrow \frac{c (a + b x^2)^{3/4}}{b \sqrt{c x}} + \frac{a c^2}{2 b} \int \frac{1}{(c x)^{3/2} (a + b x^2)^{1/4}} dx$$

Program code:

```
Int[Sqrt[c*x_]/(a_+b_*x^2)^(1/4),x_Symbol] :=
  c*(a+b*x^2)^(3/4)/(b*Sqrt[c*x]) + a*c^2/(2*b)*Int[1/((c*x)^(3/2)*(a+b*x^2)^(1/4)),x] /;
FreeQ[{a,b,c},x] && NegQ[b/a]
```

4. $\int \frac{1}{(c x)^{3/2} (a + b x^2)^{1/4}} dx$

1: $\int \frac{1}{(c x)^{3/2} (a + b x^2)^{1/4}} dx$ when $\frac{b}{a} > 0$

Reference: G&R 2.110.3

Derivation: Binomial recurrence 1a

Derivation: Integration by parts

Rule 1.1.3.2.7.1.6.4.1: If $\frac{b}{a} > 0$, then

$$\int \frac{1}{(c x)^{3/2} (a + b x^2)^{1/4}} dx \rightarrow -\frac{2}{c \sqrt{c x} (a + b x^2)^{1/4}} - \frac{b}{c^2} \int \frac{\sqrt{c x}}{(a + b x^2)^{5/4}} dx$$

Program code:

```
Int[1/((c_.**x_)^(3/2)*(a_+b_.**x_^2)^(1/4)),x_Symbol] :=
  -2/(c*Sqrt[c*x]*(a+b*x^2)^(1/4)) - b/c^2*Int[Sqrt[c*x]/(a+b*x^2)^(5/4),x] /;
FreeQ[{a,b,c},x] && PosQ[b/a]
```

$$2: \int \frac{1}{(c x)^{3/2} (a + b x^2)^{1/4}} dx \text{ when } \frac{b}{a} \neq 0$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{\sqrt{c x} \left(1 + \frac{a}{b x^2}\right)^{1/4}}{(a + b x^2)^{1/4}} = 0$$

Rule 1.1.3.2.7.1.6.4.2: If $\frac{b}{a} \neq 0$, then

$$\int \frac{1}{(c x)^{3/2} (a + b x^2)^{1/4}} dx \rightarrow \frac{\sqrt{c x} \left(1 + \frac{a}{b x^2}\right)^{1/4}}{c^2 (a + b x^2)^{1/4}} \int \frac{1}{x^2 \left(1 + \frac{a}{b x^2}\right)^{1/4}} dx$$

Program code:

```
Int[1/((c_.**x_)^(3/2)*(a_+b_.**x_^2)^(1/4)),x_Symbol] :=
  Sqrt[c*x]*(1+a/(b*x^2))^(1/4)/(c^2*(a+b*x^2)^(1/4))*Int[1/(x^2*(1+a/(b*x^2))^(1/4)),x] /;
FreeQ[{a,b,c},x] && NegQ[b/a]
```

$$7. \int \frac{\sqrt{c x}}{\sqrt{a + b x^2}} dx \text{ when } -\frac{b}{a} > 0$$

$$1. \int \frac{\sqrt{x}}{\sqrt{a+b x^2}} dx \text{ when } -\frac{b}{a} > 0$$

$$1: \int \frac{\sqrt{x}}{\sqrt{a+b x^2}} dx \text{ when } -\frac{b}{a} > 0 \wedge a > 0$$

Derivation: Integration by substitution

$$\text{Basis: If } -\frac{b}{a} > 0 \wedge a > 0, \text{ then } \frac{\sqrt{x}}{\sqrt{a+b x^2}} = -\frac{2}{\sqrt{a} \left(-\frac{b}{a}\right)^{3/4}} \text{Subst}\left[\frac{\sqrt{1-2 x^2}}{\sqrt{1-x^2}}, x, \frac{\sqrt{1-\sqrt{-\frac{b}{a}} x}}{\sqrt{2}}\right] \partial_x \frac{\sqrt{1-\sqrt{-\frac{b}{a}} x}}{\sqrt{2}}$$

Rule 1.1.3.2.7.1.7.1.1: If $-\frac{b}{a} > 0 \wedge a > 0$, then

$$\int \frac{\sqrt{x}}{\sqrt{a+b x^2}} dx \rightarrow -\frac{2}{\sqrt{a} \left(-\frac{b}{a}\right)^{3/4}} \text{Subst}\left[\int \frac{\sqrt{1-2 x^2}}{\sqrt{1-x^2}} dx, x, \frac{\sqrt{1-\sqrt{-\frac{b}{a}} x}}{\sqrt{2}}\right]$$

Program code:

```
Int[Sqrt[x_]/Sqrt[a+b_*x_^2],x_Symbol] :=
  -2/(Sqrt[a]*(-b/a)^(3/4))*Subst[Int[Sqrt[1-2*x^2]/Sqrt[1-x^2],x],x,Sqrt[1-Sqrt[-b/a]*x]/Sqrt[2] /;
FreeQ[{a,b},x] && GtQ[-b/a,0] && GtQ[a,0]
```

$$2: \int \frac{\sqrt{x}}{\sqrt{a+b x^2}} dx \text{ when } -\frac{b}{a} > 0 \wedge a \neq 0$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{\sqrt{1+\frac{b x^2}{a}}}{\sqrt{a+b x^2}} = 0$$

Rule 1.1.3.2.7.1.7.1.2: If $-\frac{b}{a} > 0 \wedge a \neq 0$, then

$$\int \frac{\sqrt{x}}{\sqrt{a + b x^2}} dx \rightarrow \frac{\sqrt{1 + \frac{b x^2}{a}}}{\sqrt{a + b x^2}} \int \frac{\sqrt{x}}{\sqrt{1 + \frac{b x^2}{a}}} dx$$

Program code:

```
Int[Sqrt[x]/Sqrt[a_+b_.*x_^2],x_Symbol] :=
  Sqrt[1+b*x^2/a]/Sqrt[a+b*x^2]*Int[Sqrt[x]/Sqrt[1+b*x^2/a],x] /;
FreeQ[{a,b},x] && GtQ[-b/a,0] && Not[GtQ[a,0]]
```

2: $\int \frac{\sqrt{c x}}{\sqrt{a + b x^2}} dx$ when $-\frac{b}{a} > 0$

Derivation: Piecewise constant extraction

Rule 1.1.3.2.7.1.7.2: If $-\frac{b}{a} > 0$, then

$$\int \frac{\sqrt{c x}}{\sqrt{a + b x^2}} dx \rightarrow \frac{\sqrt{c x}}{\sqrt{x}} \int \frac{\sqrt{x}}{\sqrt{a + b x^2}} dx$$

Program code:

```
Int[Sqrt[c*x_]/Sqrt[a_+b_.*x_^2],x_Symbol] :=
  Sqrt[c*x]/Sqrt[x]*Int[Sqrt[x]/Sqrt[a+b*x^2],x] /;
FreeQ[{a,b,c},x] && GtQ[-b/a,0]
```

8: $\int (c x)^m (a + b x^n)^p dx$ when $n \in \mathbb{Z}^+ \wedge m > n - 1 \wedge m + n p + 1 \neq 0$

Reference: G&R 2.110.5, CRC 88a

Derivation: Binomial recurrence 3a

Derivation: Inverted integration by parts

Rule 1.1.3.2.7.1.8: If $n \in \mathbb{Z}^+ \wedge m > n - 1 \wedge m + n p + 1 \neq 0$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow \frac{c^{n-1} (c x)^{m-n+1} (a + b x^n)^{p+1}}{b (m + n p + 1)} - \frac{a c^n (m - n + 1)}{b (m + n p + 1)} \int (c x)^{m-n} (a + b x^n)^p dx$$

Program code:

```
Int[(c_.*x_)^m_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  c^(n-1)*(c*x)^(m-n+1)*(a+b*x^n)^(p+1)/(b*(m+n*p+1)) -
  a*c^n*(m-n+1)/(b*(m+n*p+1))*Int[(c*x)^(m-n)*(a+b*x^n)^p,x] /;
FreeQ[{a,b,c,p},x] && IGtQ[n,0] && GtQ[m,n-1] && NeQ[m+n*p+1,0] && IntBinomialQ[a,b,c,n,m,p,x]
```

```
Int[(c_.*x_)^m_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  c^(n-1)*(c*x)^(m-n+1)*(a+b*x^n)^(p+1)/(b*(m+n*p+1)) -
  a*c^n*(m-n+1)/(b*(m+n*p+1))*Int[(c*x)^(m-n)*(a+b*x^n)^p,x] /;
FreeQ[{a,b,c,m,p},x] && IGtQ[n,0] && SumSimplerQ[m,-n] && NeQ[m+n*p+1,0] && ILtQ[Simplify[(m+1)/n+p],0]
```

```
Int[(c_.*x_)^m_*(a1_+b1_.*x_^n_)^p_*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  c^(2*n-1)*(c*x)^(m-2*n+1)*(a1+b1*x^n)^(p+1)*(a2+b2*x^n)^(p+1)/(b1*b2*(m+2*n*p+1)) -
  a1*a2*c^(2*n)*(m-2*n+1)/(b1*b2*(m+2*n*p+1))*Int[(c*x)^(m-2*n)*(a1+b1*x^n)^p*(a2+b2*x^n)^p,x] /;
FreeQ[{a1,b1,a2,b2,c,p},x] && EqQ[a2*b1+a1*b2,0] && IGtQ[2*n,0] && GtQ[m,2*n-1] && NeQ[m+2*n*p+1,0] &&
  IntBinomialQ[a1*a2,b1*b2,c,2*n,m,p,x]
```



```

Int[(c_.*x_)^m_*(a1_+b1_.*x_^n_)^p_*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  c^(2*n-1)*(c*x)^(m-2*n+1)*(a1+b1*x^n)^(p+1)*(a2+b2*x^n)^(p+1)/(b1*b2*(m+2*n*p+1)) -
  a1*a2*c^(2*n)*(m-2*n+1)/(b1*b2*(m+2*n*p+1))*Int[(c*x)^(m-2*n)*(a1+b1*x^n)^p*(a2+b2*x^n)^p,x] /;
FreeQ[{a1,b1,a2,b2,c,m,p},x] && EqQ[a2*b1+a1*b2,0] && IGtQ[2*n,0] && SumSimplerQ[m,-2*n] && NeQ[m+2*n*p+1,0] &&
  ILtQ[Simplify[(m+1)/(2*n)+p],0]

```

9: $\int (c x)^m (a + b x^n)^p dx$ when $n \in \mathbb{Z}^+ \wedge m < -1$

Reference: G&R 2.110.6, CRC 88c

Derivation: Binomial recurrence 3b

Derivation: Integration by parts

$$\text{Basis: } x^m (a + b x^n)^p = \frac{x^m}{(a + b x^n)^{\frac{m+n+1}{n}}} (a + b x^n)^{\frac{m+1}{n} + p + 1}$$

$$\text{Basis: } \int \frac{x^m}{(a + b x^n)^{\frac{m+n+1}{n}}} dx = \frac{x^{m+1}}{(a + b x^n)^{\frac{m+1}{n}} (a(m+1))}$$

Note: Requirement that $m + 1 < n$ ensures new term is a proper fraction.

Rule 1.1.3.2.7.1.9: If $n \in \mathbb{Z}^+ \wedge m < -1$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow \frac{(c x)^{m+1} (a + b x^n)^{p+1}}{a c (m+1)} - \frac{b (m+n(p+1)+1)}{a c^n (m+1)} \int (c x)^{m+n} (a + b x^n)^p dx$$

Program code:

```

Int[(c_.*x_)^m_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  (c*x)^(m+1)*(a+b*x^n)^(p+1)/(a*c*(m+1)) -
  b*(m+n*(p+1)+1)/(a*c^n*(m+1))*Int[(c*x)^(m+n)*(a+b*x^n)^p,x] /;
FreeQ[{a,b,c,p},x] && IGtQ[n,0] && LtQ[m,-1] && IntBinomialQ[a,b,c,n,m,p,x]

```

```

Int[(c_.**x_)^m_*(a_+b_.**x_^n_)^p_,x_Symbol] :=
  (c*x)^(m+1)*(a+b*x^n)^(p+1)/(a*c*(m+1)) -
  b*(m+n*(p+1)+1)/(a*c^n*(m+1))*Int[(c*x)^(m+n)*(a+b*x^n)^p,x] /;
FreeQ[{a,b,c,m,p},x] && IGtQ[n,0] && SumSimplerQ[m,n] && ILtQ[Simplify[(m+1)/n+p],0]

```

```

Int[(c_.**x_)^m_*(a1_+b1_.**x_^n_)^p_*(a2_+b2_.**x_^n_)^p_,x_Symbol] :=
  (c*x)^(m+1)*(a1+b1*x^n)^(p+1)*(a2+b2*x^n)^(p+1)/(a1*a2*c*(m+1)) -
  b1*b2*(m+2*n*(p+1)+1)/(a1*a2*c^(2*n)*(m+1))*Int[(c*x)^(m+2*n)*(a1+b1*x^n)^p*(a2+b2*x^n)^p,x] /;
FreeQ[{a1,b1,a2,b2,c,p},x] && EqQ[a2*b1+a1*b2,0] && IGtQ[2*n,0] && LtQ[m,-1] && IntBinomialQ[a1*a2,b1*b2,c,2*n,m,p,x]

```

```

Int[(c_.**x_)^m_*(a1_+b1_.**x_^n_)^p_*(a2_+b2_.**x_^n_)^p_,x_Symbol] :=
  (c*x)^(m+1)*(a1+b1*x^n)^(p+1)*(a2+b2*x^n)^(p+1)/(a1*a2*c*(m+1)) -
  b1*b2*(m+2*n*(p+1)+1)/(a1*a2*c^(2*n)*(m+1))*Int[(c*x)^(m+2*n)*(a1+b1*x^n)^p*(a2+b2*x^n)^p,x] /;
FreeQ[{a1,b1,a2,b2,c,m,p},x] && EqQ[a2*b1+a1*b2,0] && IGtQ[2*n,0] && SumSimplerQ[m,2*n] && ILtQ[Simplify[(m+1)/(2*n)+p],0]

```

10: $\int (c x)^m (a + b x^n)^p dx$ when $n \in \mathbb{Z}^+ \wedge m \in \mathbb{F}$

Derivation: Integration by substitution

Basis: If $k \in \mathbb{Z}^+$, then $(c x)^m F[x] = \frac{k}{c} \text{Subst}\left[x^{k(m+1)-1} F\left[\frac{x^k}{c}\right], x, (c x)^{1/k}\right] \partial_x (c x)^{1/k}$

Rule 1.1.3.2.7.1.10: If $n \in \mathbb{Z}^+ \wedge m \in \mathbb{F}$, let $k = \text{Denominator}[m]$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow \frac{k}{c} \text{Subst}\left[\int x^{k(m+1)-1} \left(a + \frac{b x^{k n}}{c^n}\right)^p dx, x, (c x)^{1/k}\right]$$

Program code:

```

Int[(c_.**x_)^m_*(a_+b_.**x_^n_)^p_,x_Symbol] :=
  With[{k=Denominator[m]},
    k/c*Subst[Int[x^(k*(m+1)-1)*(a+b*x^(k*n))/c^n]^p,x,(c*x)^(1/k)]] /;
FreeQ[{a,b,c,p},x] && IGtQ[n,0] && FractionQ[m] && IntBinomialQ[a,b,c,n,m,p,x]

```

```

Int[(c_.*x_)^m_*(a1_+b1_.*x_^n_)^p_*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  With[{k=Denominator[m]},
    k/c*Subst[Int[x^(k*(m+1)-1)*(a1+b1*x^(k*n))/c^n]^p*(a2+b2*x^(k*n))/c^n^p,x],x,(c*x)^(1/k)] /;
FreeQ[{a1,b1,a2,b2,c,p},x] && EqQ[a2*b1+a1*b2,0] && IGtQ[2*n,0] && FractionQ[m] && IntBinomialQ[a1*a2,b1*b2,c,2*n,m,p,x]

```

$$11. \int x^m (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+ \wedge -1 < p < 0 \wedge p \neq -\frac{1}{2} \wedge m \in \mathbb{Z} \wedge \text{Denominator}\left[p + \frac{m+1}{n}\right] < \text{Denominator}[p]$$

$$1: \int x^m (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+ \wedge -1 < p < 0 \wedge p \neq -\frac{1}{2} \wedge m \in \mathbb{Z} \wedge p + \frac{m+1}{n} \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If $n \in \mathbb{Z} \wedge m \in \mathbb{Z} \wedge p + \frac{m+1}{n} \in \mathbb{Z}$, then

$$x^m (a + b x^n)^p = a^{p + \frac{m+1}{n}} \text{Subst}\left[\frac{x^m}{(1 - b x^n)^{p + \frac{m+1}{n} + 1}}, x, \frac{x}{(a + b x^n)^{1/n}}\right] \partial_x \frac{x}{(a + b x^n)^{1/n}}$$

Rule 1.1.3.2.7.1.11.1: If $n \in \mathbb{Z}^+ \wedge -1 < p < 0 \wedge p \neq -\frac{1}{2} \wedge m \in \mathbb{Z} \wedge p + \frac{m+1}{n} \in \mathbb{Z}$, then

$$\int x^m (a + b x^n)^p dx \rightarrow a^{p + \frac{m+1}{n}} \text{Subst}\left[\int \frac{x^m}{(1 - b x^n)^{p + \frac{m+1}{n} + 1}} dx, x, \frac{x}{(a + b x^n)^{1/n}}\right]$$

Program code:

```

Int[x_^m_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  a^(p+(m+1)/n)*Subst[Int[x^m/(1-b*x^n)^(p+(m+1)/n+1),x],x,x/(a+b*x^n)^(1/n)] /;
FreeQ[{a,b},x] && IGtQ[n,0] && LtQ[-1,p,0] && NeQ[p,-1/2] && IntegersQ[m,p+(m+1)/n]

```

```

Int[x_^m_*(a1_+b1_.*x_^n_)^p_*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  (a1*a2)^(p+(m+1)/(2*n))*
  Subst[Int[x^m/((1-b1*x^n)^(p+(m+1)/(2*n)+1)*(1-b2*x^n)^(p+(m+1)/(2*n)+1)),x],x,
    x/((a1+b1*x^n)^(1/(2*n))*(a2+b2*x^n)^(1/(2*n))))] /;
FreeQ[{a1,b1,a2,b2},x] && EqQ[a2*b1+a1*b2,0] && IGtQ[2*n,0] && LtQ[-1,p,0] && NeQ[p,-1/2] && IntegersQ[m,p+(m+1)/(2*n)]

```

$$\mathbf{2:} \int x^m (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+ \wedge -1 < p < 0 \wedge p \neq -\frac{1}{2} \wedge m \in \mathbb{Z} \wedge \text{Denominator}[p + \frac{m+1}{n}] < \text{Denominator}[p]$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: } \partial_x \left(\left(\frac{a}{a+b x^n} \right)^{p+\frac{m+1}{n}} (a+b x^n)^{p+\frac{m+1}{n}} \right) = 0$$

$$\text{Basis: If } n \in \mathbb{Z}, \text{ then } \frac{x^m}{\left(\frac{a}{a+b x^n} \right)^{p+\frac{m+1}{n}} (a+b x^n)^{\frac{m+1}{n}}} = \text{Subst} \left[\frac{x^m}{(1-b x^n)^{p+\frac{m+1}{n}+1}}, x, \frac{x}{(a+b x^n)^{1/n}} \right] \partial_x \frac{x}{(a+b x^n)^{1/n}}$$

Rule 1.1.3.2.7.1.11.2: If $n \in \mathbb{Z}^+ \wedge -1 < p < 0 \wedge p \neq -\frac{1}{2} \wedge m \in \mathbb{Z} \wedge \text{Denominator}[p + \frac{m+1}{n}] < \text{Denominator}[p]$, then

$$\begin{aligned} \int x^m (a + b x^n)^p dx &\rightarrow \left(\frac{a}{a+b x^n} \right)^{p+\frac{m+1}{n}} (a+b x^n)^{p+\frac{m+1}{n}} \int \frac{x^m}{\left(\frac{a}{a+b x^n} \right)^{p+\frac{m+1}{n}} (a+b x^n)^{\frac{m+1}{n}}} dx \\ &\rightarrow \left(\frac{a}{a+b x^n} \right)^{p+\frac{m+1}{n}} (a+b x^n)^{p+\frac{m+1}{n}} \text{Subst} \left[\int \frac{x^m}{(1-b x^n)^{p+\frac{m+1}{n}+1}} dx, x, \frac{x}{(a+b x^n)^{1/n}} \right] \end{aligned}$$

Program code:

```
Int[x^m_.*(a+b_.*x^n_)^p_,x_Symbol] :=
  (a/(a+b*x^n))^(p+(m+1)/n)*(a+b*x^n)^(p+(m+1)/n)*Subst[Int[x^m/(1-b*x^n)^(p+(m+1)/n+1),x],x,x/(a+b*x^n)^(1/n)] /;
FreeQ[{a,b},x] && IGtQ[n,0] && LtQ[-1,p,0] && NeQ[p,-1/2] && IntegerQ[m] && LtQ[Denominator[p+(m+1)/n],Denominator[p]]
```

```
Int[x^m_.*(a1+b1_.*x^n_)^p_*(a2+b2_.*x^n_)^p_,x_Symbol] :=
  (a1/(a1+b1*x^n))^(p+(m+1)/(2*n))*(a1+b1*x^n)^(p+(m+1)/(2*n))*(a2/(a2+b2*x^n))^(p+(m+1)/(2*n))*(a2+b2*x^n)^(p+(m+1)/(2*n))*
  Subst[Int[x^m/((1-b1*x^n)^(p+(m+1)/(2*n)+1)*(1-b2*x^n)^(p+(m+1)/(2*n)+1)),x],x,
  x/((a1+b1*x^n)^(1/(2*n))*(a2+b2*x^n)^(1/(2*n)))] /;
FreeQ[{a1,b1,a2,b2},x] && EqQ[a2*b1+a1*b2,0] && IGtQ[2*n,0] && LtQ[-1,p,0] && NeQ[p,-1/2] &&
IntegerQ[m] && LtQ[Denominator[p+(m+1)/(2*n)],Denominator[p]]
```

2. $\int (c x)^m (a + b x^n)^p dx$ when $n \in \mathbb{Z}^-$

1. $\int (c x)^m (a + b x^n)^p dx$ when $n \in \mathbb{Z}^- \wedge m \in \mathbb{Q}$

1: $\int x^m (a + b x^n)^p dx$ when $n \in \mathbb{Z}^- \wedge m \in \mathbb{Z}$

Derivation: Integration by substitution

Basis: If $n \in \mathbb{Z}^- \wedge m \in \mathbb{Z}$, then $x^m F[x^n] = -\text{Subst}\left[\frac{F\left[\frac{x^{-n}}{x^{m+2}}\right], x, \frac{1}{x}}{\partial_x \frac{1}{x}}\right]$

Rule 1.1.3.2.7.2.1.1: If $n \in \mathbb{Z}^- \wedge m \in \mathbb{Z}$, then

$$\int x^m (a + b x^n)^p dx \rightarrow -\text{Subst}\left[\int \frac{(a + b x^{-n})^p}{x^{m+2}} dx, x, \frac{1}{x}\right]$$

Program code:

```
Int[x^m_.*(a_+b_.*x^n_)^p_,x_Symbol] :=
  -Subst[Int[(a+b*x^(-n))^p/x^(m+2),x],x,1/x] /;
FreeQ[{a,b,p},x] && ILtQ[n,0] && IntegerQ[m]
```

```
Int[x^m_.*(a1_+b1_.*x^n_)^p_*(a2_+b2_.*x^n_)^p_,x_Symbol] :=
  -Subst[Int[(a1+b1*x^(-n))^p*(a2+b2*x^(-n))^p/x^(m+2),x],x,1/x] /;
FreeQ[{a1,b1,a2,b2,p},x] && EqQ[a2+b1+a1*b2,0] && ILtQ[2*n,0] && IntegerQ[m]
```

$$\mathbf{2:} \int (c x)^m (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^- \wedge m \in \mathbb{F}$$

Derivation: Integration by substitution

Basis: If $n \in \mathbb{Z}^- \wedge k > 1$, then $(c x)^m F[x^n] = -\frac{k}{c} \text{Subst}\left[\frac{F\left[\frac{c^{-n} x^{-k n}}{x^{k(m+1)+1}}\right], x, \frac{1}{(c x)^{1/k}}\right] \partial_x \frac{1}{(c x)^{1/k}}$

Rule 1.1.3.2.7.2.1.2: If $n \in \mathbb{Z}^- \wedge m \in \mathbb{F}$, let $k = \text{Denominator}[m]$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow -\frac{k}{c} \text{Subst}\left[\int \frac{(a + b c^{-n} x^{-k n})^p}{x^{k(m+1)+1}} dx, x, \frac{1}{(c x)^{1/k}}\right]$$

Program code:

```
Int[(c_.**x_)^m_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  With[{k=Denominator[m]},
    -k/c*Subst[Int[(a+b*c^(-n))*x^(-k*n)]^p/x^(k*(m+1)+1),x],x,1/(c*x)^(1/k)] /;
  FreeQ[{a,b,c,p},x] && ILtQ[n,0] && FractionQ[m]
```

```
Int[(c_.**x_)^m_*(a1_+b1_.*x_^n_)^p_*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  With[{k=Denominator[m]},
    -k/c*Subst[Int[(a1+b1*c^(-n))*x^(-k*n)]^p*(a2+b2*c^(-n))*x^(-k*n)]^p/x^(k*(m+1)+1),x],x,1/(c*x)^(1/k)] /;
  FreeQ[{a1,b1,a2,b2,c,p},x] && EqQ[a2*b1+a1*b2,0] && ILtQ[2*n,0] && FractionQ[m]
```

$$2: \int (c x)^m (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^- \wedge m \notin \mathbb{Q}$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: } \partial_x \left((c x)^m (x^{-1})^m \right) = 0$$

$$\text{Basis: } F[x] = -\text{Subst} \left[\frac{F[x^{-1}]}{x^2}, x, \frac{1}{x} \right] \partial_x \frac{1}{x}$$

Rule 1.1.3.2.7.2.2: If $n \in \mathbb{Z}^- \wedge m \notin \mathbb{Q}$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow (c x)^m \left(\frac{1}{x} \right)^m \int \frac{(a + b x^n)^p}{\left(\frac{1}{x} \right)^m} d x \rightarrow -\frac{1}{c} (c x)^{m+1} \left(\frac{1}{x} \right)^{m+1} \text{Subst} \left[\int \frac{(a + b x^{-n})^p}{x^{m+2}} dx, x, \frac{1}{x} \right]$$

Program code:

```
Int[(c_.*x_)^m_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  -1/c*(c*x)^(m+1)*(1/x)^(m+1)*Subst[Int[(a+b*x^(-n))^p/x^(m+2),x],x,1/x] /;
FreeQ[{a,b,c,m,p},x] && ILtQ[n,0] && Not[RationalQ[m]]
```

```
Int[(c_.*x_)^m_*(a1_+b1_.*x_^n_)^p_*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  -1/c*(c*x)^(m+1)*(1/x)^(m+1)*Subst[Int[(a1+b1*x^(-n))^p*(a2+b2*x^(-n))^p/x^(m+2),x],x,1/x] /;
FreeQ[{a1,b1,a2,b2,c,m,p},x] && EqQ[a2*b1+a1*b2,0] && ILtQ[2*n,0] && Not[RationalQ[m]]
```

8. $\int (c x)^m (a + b x^n)^p dx$ when $n \in \mathbb{F}$

1: $\int x^m (a + b x^n)^p dx$ when $n \in \mathbb{F}$

Derivation: Integration by substitution

Basis: If $k \in \mathbb{Z}^+$, then $x^m F[x^n] = k \text{Subst}[x^{k(m+1)-1} F[x^{k n}], x, x^{1/k}] \partial_x x^{1/k}$

Rule 1.1.3.2.8.1: If $n \in \mathbb{F}$, let $k = \text{Denominator}[n]$, then

$$\int x^m (a + b x^n)^p dx \rightarrow k \text{Subst}\left[\int x^{k(m+1)-1} (a + b x^{k n})^p dx, x, x^{1/k}\right]$$

Program code:

```
Int[x^m_.*(a_+b_.*x^n_)^p_,x_Symbol] :=
  With[{k=Denominator[n]},
    k*Subst[Int[x^(k*(m+1)-1)*(a+b*x^(k*n))^p,x],x,x^(1/k)]] /;
  FreeQ[{a,b,m,p},x] && FractionQ[n]
```

```
Int[x^m_.*(a1_+b1_.*x^n_)^p_.*(a2_+b2_.*x^n_)^p_,x_Symbol] :=
  With[{k=Denominator[2*n]},
    k*Subst[Int[x^(k*(m+1)-1)*(a1+b1*x^(k*n))^p_*(a2+b2*x^(k*n))^p,x],x,x^(1/k)]] /;
  FreeQ[{a1,b1,a2,b2,m,p},x] && EqQ[a2*b1+a1*b2,0] && FractionQ[2*n]
```


2: $\int (c x)^m (a + b x^n)^p dx$ when $n \in \mathbb{F}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(c x)^m}{x^m} = 0$

Rule 1.1.3.2.8.2: If $n \in \mathbb{F}$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (a + b x^n)^p dx$$

Program code:

```
Int[(c*x_)^m_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a+b*x^n)^p,x] /;
FreeQ[{a,b,c,m,p},x] && FractionQ[n]
```

```
Int[(c*x_)^m_*(a1_+b1_.*x_^n_)^p_*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a1+b1*x^n)^p*(a2+b2*x^n)^p,x] /;
FreeQ[{a1,b1,a2,b2,c,m,p},x] && EqQ[a2*b1+a1*b2,0] && FractionQ[2*n]
```

9. $\int (c x)^m (a + b x^n)^p dx$ when $\frac{n}{m+1} \in \mathbb{Z}$

1: $\int x^m (a + b x^n)^p dx$ when $\frac{n}{m+1} \in \mathbb{Z}$

Derivation: Integration by substitution

Basis: If $\frac{n}{m+1} \in \mathbb{Z}$, then $x^m F[x^n] = \frac{1}{m+1} \text{Subst}[F[x^{\frac{n}{m+1}}], x, x^{m+1}] \partial_x x^{m+1}$

Rule 1.1.3.2.9.1: If $\frac{n}{m+1} \in \mathbb{Z}$, then

$$\int x^m (a + b x^n)^p dx \rightarrow \frac{1}{m+1} \text{Subst}\left[\int (a + b x^{\frac{n}{m+1}})^p dx, x, x^{m+1}\right]$$

Program code:

```
Int[x_^m_.*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  1/(m+1)*Subst[Int[(a+b*x^Simplify[n/(m+1)])^p,x],x,x^(m+1)] /;
FreeQ[{a,b,m,n,p},x] && IntegerQ[Simplify[n/(m+1)]] && Not[IntegerQ[n]]
```

```
Int[x_^m_.*(a1_+b1_.*x_^n_)^p_.*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  1/(m+1)*Subst[Int[(a1+b1*x^Simplify[n/(m+1)])^p_*(a2+b2*x^Simplify[n/(m+1)])^p,x],x,x^(m+1)] /;
FreeQ[{a1,b1,a2,b2,m,n,p},x] && EqQ[a2*b1+a1*b2,0] && IntegerQ[Simplify[2*n/(m+1)]] && Not[IntegerQ[2*n]]
```

2: $\int (c x)^m (a + b x^n)^p dx$ when $\frac{n}{m+1} \in \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(c x)^m}{x^m} = 0$

Rule 1.1.3.2.9.2: If $\frac{n}{m+1} \in \mathbb{Z}$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (a + b x^n)^p dx$$

Program code:

```
Int[(c*x_)^m_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a+b*x^n)^p,x] /;
FreeQ[{a,b,c,m,n,p},x] && IntegerQ[Simplify[n/(m+1)]] && Not[IntegerQ[n]]
```

```
Int[(c*x_)^m_*(a1_+b1_.*x_^n_)^p_*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a1+b1*x^n)^p*(a2+b2*x^n)^p,x] /;
FreeQ[{a1,b1,a2,b2,c,m,n,p},x] && EqQ[a2*b1+a1*b2,0] && IntegerQ[Simplify[2*n/(m+1)]] && Not[IntegerQ[2*n]]
```

$$10. \int (c x)^m (a + b x^n)^p dx \text{ when } \frac{m+1}{n} + p \in \mathbb{Z}$$

$$1. \int (c x)^m (a + b x^n)^p dx \text{ when } \frac{m+1}{n} + p \in \mathbb{Z} \wedge p > 0$$

$$1. \int (c x)^m (a + b x^n)^p dx \text{ when } \frac{m+1}{n} + p = 0 \wedge p > 0$$

$$\textcolor{red}{1}: \int x^m (a + b x^n)^p dx \text{ when } \frac{m+1}{n} + p = 0 \wedge p > 0$$

Reference: G&R 2.110.3

Derivation: Binomial recurrence 1a

Derivation: Integration by parts

Rule 1.1.3.2.10.1.1.1: If $\frac{m+1}{n} + p = 0 \wedge p > 0$, then

$$\int x^m (a + b x^n)^p dx \rightarrow \frac{x^{m+1} (a + b x^n)^p}{m+1} - \frac{b n p}{m+1} \int x^{m+n} (a + b x^n)^{p-1} dx$$

Program code:

```
Int[x_^m_.*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  x^(m+1)*(a+b*x^n)^p/(m+1) -
  b*n*p/(m+1)*Int[x^(m+n)*(a+b*x^n)^(p-1),x] /;
FreeQ[{a,b,m,n},x] && EqQ[(m+1)/n+p,0] && GtQ[p,0]
```

```
Int[x_^m_.*(a1_+b1_.*x_^n_)^p_.*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  x^(m+1)*(a1+b1*x^n)^p*(a2+b2*x^n)^p/(m+1) -
  2*b1*b2*n*p/(m+1)*Int[x^(m+2*n)*(a1+b1*x^n)^(p-1)*(a2+b2*x^n)^(p-1),x] /;
FreeQ[{a1,b1,a2,b2,m,n},x] && EqQ[a2*b1+a1*b2,0] && EqQ[(m+1)/(2*n)+p,0] && GtQ[p,0]
```

$$\mathbf{2:} \int (c x)^m (a + b x^n)^p dx \text{ when } \frac{m+1}{n} + p = 0 \wedge p > 0$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{(c x)^m}{x^m} = 0$$

Rule 1.1.3.2.10.1.1.2: If $\frac{m+1}{n} + p = 0 \wedge p > 0$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (a + b x^n)^p dx$$

Program code:

```
Int[(c*x_)^m*(a+b_.*x_^n_)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a+b*x^n)^p,x] /;
FreeQ[{a,b,c,m,n},x] && EqQ[(m+1)/n+p,0] && GtQ[p,0]
```

```
Int[(c*x_)^m*(a1+b1_.*x_^n_)^p_*(a2+b2_.*x_^n_)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a1+b1*x^n)^p*(a2+b2*x^n)^p,x] /;
FreeQ[{a1,b1,a2,b2,c,m,n},x] && EqQ[a2*b1+a1*b2,0] && EqQ[(m+1)/(2*n)+p,0] && GtQ[p,0]
```

$$\mathbf{2:} \int (c x)^m (a + b x^n)^p dx \text{ when } \frac{m+1}{n} + p \in \mathbb{Z} \wedge p > 0 \wedge m + n p + 1 \neq 0$$

Reference: G&R 2.110.1, CRC 88b

Derivation: Binomial recurrence 1b

Derivation: Inverted integration by parts

Rule 1.1.3.2.10.1.2: If $\frac{m+1}{n} + p \in \mathbb{Z} \wedge p > 0 \wedge m + n p + 1 \neq 0$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow \frac{(c x)^{m+1} (a + b x^n)^p}{c (m + n p + 1)} + \frac{a n p}{m + n p + 1} \int (c x)^m (a + b x^n)^{p-1} dx$$

Program code:

```
Int[(c_.**x_)^m_.*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  (c*x)^(m+1)*(a+b*x^n)^p/(c*(m+n*p+1)) +
  a*n*p/(m+n*p+1)*Int[(c*x)^m*(a+b*x^n)^(p-1),x] /;
FreeQ[{a,b,c,m,n},x] && IntegerQ[p+Simplify[(m+1)/n]] && GtQ[p,0] && NeQ[m+n*p+1,0]
```

```
Int[(c_.**x_)^m_.*(a1_+b1_.*x_^n_)^p_*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  (c*x)^(m+1)*(a1+b1*x^n)^p*(a2+b2*x^n)^p/(c*(m+2*n*p+1)) +
  2*a1*a2*n*p/(m+2*n*p+1)*Int[(c*x)^m*(a1+b1*x^n)^(p-1)*(a2+b2*x^n)^(p-1),x] /;
FreeQ[{a1,b1,a2,b2,c,m,n},x] && EqQ[a2*b1+a1*b2,0] && IntegerQ[p+Simplify[(m+1)/(2*n)]] && GtQ[p,0] && NeQ[m+2*n*p+1,0]
```

$$2. \int (c x)^m (a + b x^n)^p dx \text{ when } \frac{m+1}{n} + p \in \mathbb{Z} \wedge p < 0$$

$$1. \int (c x)^m (a + b x^n)^p dx \text{ when } \frac{m+1}{n} + p \in \mathbb{Z} \wedge -1 < p < 0$$

$$\text{1: } \int x^m (a + b x^n)^p dx \text{ when } \frac{m+1}{n} + p \in \mathbb{Z} \wedge -1 < p < 0$$

Derivation: Integration by substitution

Basis: If $\frac{m+1}{n} + p \in \mathbb{Z}$, let $k = \text{Denominator}[p]$, then

$$x^m (a + b x^n)^p = \frac{k a^{p + \frac{m+1}{n}}}{n} \text{Subst} \left[\frac{x^{\frac{k(m+1)}{n} - 1}}{(1 - b x^k)^{p + \frac{m+1}{n} + 1}}, x, \frac{x^{n/k}}{(a + b x^n)^{1/k}} \right] \partial_x \frac{x^{n/k}}{(a + b x^n)^{1/k}}$$

Basis: If $a_2 b_1 + a_1 b_2 = 0 \wedge \frac{m+1}{2n} + p \in \mathbb{Z}$, let $k = \text{Denominator}[p]$, then

$$x^m (a_1 + b_1 x^n)^p (a_2 + b_2 x^n)^p =$$

$$\frac{k (a_1 a_2)^{p + \frac{m+1}{2n}}}{2n} \text{Subst} \left[\frac{x^{\frac{k(m+1)}{2n} - 1}}{(1 - b_1 b_2 x^k)^{p + \frac{m+1}{2n} + 1}}, x, \frac{x^{2n/k}}{(a_1 + b_1 x^n)^{1/k} (a_2 + b_2 x^n)^{1/k}} \right] \partial_x \frac{x^{2n/k}}{(a_1 + b_1 x^n)^{1/k} (a_2 + b_2 x^n)^{1/k}}$$

Note: The exponents in the resulting integrand are integers.

Rule 1.1.3.2.10.2.1.1: If $\frac{m+1}{n} + p \in \mathbb{Z} \wedge -1 < p < 0$, let $k = \text{Denominator}[p]$, then

$$\int x^m (a + b x^n)^p dx \rightarrow \frac{k a^{p + \frac{m+1}{n}}}{n} \text{Subst} \left[\int \frac{x^{\frac{k(m+1)}{n} - 1}}{(1 - b x^k)^{p + \frac{m+1}{n} + 1}} dx, x, \frac{x^{n/k}}{(a + b x^n)^{1/k}} \right]$$

$$\int x^m (a_1 + b_1 x^n)^p (a_2 + b_2 x^n)^p dx \rightarrow \frac{k (a_1 a_2)^{p + \frac{m+1}{2n}}}{2n} \text{Subst} \left[\int \frac{x^{\frac{k(m+1)}{2n} - 1}}{(1 - b_1 b_2 x^k)^{p + \frac{m+1}{2n} + 1}} dx, x, \frac{x^{2n/k}}{(a_1 + b_1 x^n)^{1/k} (a_2 + b_2 x^n)^{1/k}} \right]$$

Program code:

```

Int[x_^m_.*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  With[{k=Denominator[p]},
    k*a^(p+Simplify[(m+1)/n])/n*
    Subst[Int[x^(k*Simplify[(m+1)/n]-1)/(1-b*x^k)^(p+Simplify[(m+1)/n]+1),x],x,x^(n/k)/(a+b*x^n)^(1/k)] /;
  FreeQ[{a,b,m,n},x] && IntegerQ[p+Simplify[(m+1)/n]] && LtQ[-1,p,0]

```

```

Int[x_^m_.*(a1_+b1_.*x_^n_)^p_.*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  With[{k=Denominator[p]},
    k*(a1*a2)^(p+Simplify[(m+1)/(2*n)])/(2*n)*
    Subst[Int[x^(k*Simplify[(m+1)/(2*n)]-1)/(1-b1*b2*x^k)^(p+Simplify[(m+1)/(2*n)]+1),x],x,x^(2*n/k)/((a1+b1*x^n)^(1/k)*(a2+b2*x^n)^(1/k)) /;
  FreeQ[{a1,b1,a2,b2,m,n},x] && EqQ[a2*b1+a1*b2,0] && IntegerQ[p+Simplify[(m+1)/(2*n)]] && LtQ[-1,p,0]

```

2: $\int (c x)^m (a + b x^n)^p dx$ when $\frac{m+1}{n} + p \in \mathbb{Z} \wedge -1 < p < 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(c x)^m}{x^m} = 0$

Rule 1.1.3.2.10.2.1.2: If $\frac{m+1}{n} + p \in \mathbb{Z} \wedge -1 < p < 0$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m (a + b x^n)^p dx$$

Program code:

```

Int[(c_*x_)^m_.*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a+b*x^n)^p,x] /;
  FreeQ[{a,b,c,m,n},x] && IntegerQ[p+Simplify[(m+1)/n]] && LtQ[-1,p,0]

```

```

Int[(c_*x_)^m_.*(a1_+b1_.*x_^n_)^p_.*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m*(a1+b1*x^n)^p*(a2+b2*x^n)^p,x] /;
  FreeQ[{a1,b1,a2,b2,c,m,n},x] && EqQ[a2*b1+a1*b2,0] && IntegerQ[p+Simplify[(m+1)/(2*n)]] && LtQ[-1,p,0]

```


$$2: \int (c x)^m (a + b x^n)^p dx \text{ when } \frac{m+1}{n} + p \in \mathbb{Z} \wedge p < -1$$

Reference: G&R 2.110.2, CRC 88d

Derivation: Binomial recurrence 2b

Derivation: Integration by parts

$$\text{Basis: } x^m (a + b x^n)^p = x^{m+n(p+1)+1} \frac{(a + b x^n)^p}{x^{n(p+1)+1}}$$

$$\text{Basis: } \int \frac{(a + b x^n)^p}{x^{n(p+1)+1}} dx = - \frac{(a + b x^n)^{p+1}}{x^{n(p+1)} a n (p+1)}$$

Rule 1.1.3.2.10.2.2: If $\frac{m+1}{n} + p \in \mathbb{Z} \wedge p < -1$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow - \frac{(c x)^{m+1} (a + b x^n)^{p+1}}{a c n (p+1)} + \frac{m + n (p+1) + 1}{a n (p+1)} \int (c x)^m (a + b x^n)^{p+1} dx$$

Program code:

```
Int[(c_.**x_)^m_.*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  -(c*x)^(m+1)*(a+b*x^n)^(p+1)/(a*c*n*(p+1)) +
  (m+n*(p+1)+1)/(a*n*(p+1))*Int[(c*x)^m*(a+b*x^n)^(p+1),x] /;
FreeQ[{a,b,c,m,n},x] && IntegerQ[p+Simplify[(m+1)/n]] && LtQ[p,-1]
```

```
Int[(c_.**x_)^m_.*(a1_+b1_.*x_^n_)^p_.*(a2_+b2_.*x_^n_)^p_,x_Symbol] :=
  -(c*x)^(m+1)*(a1+b1*x^n)^(p+1)*(a2+b2*x^n)^(p+1)/(2*a1*a2*c*n*(p+1)) +
  (m+2*n*(p+1)+1)/(2*a1*a2*n*(p+1))*Int[(c*x)^m*(a1+b1*x^n)^(p+1)*(a2+b2*x^n)^(p+1),x] /;
FreeQ[{a1,b1,a2,b2,c,m,n},x] && EqQ[a2*b1+a1*b2,0] && IntegerQ[p+Simplify[(m+1)/(2*n)]] && LtQ[p,-1]
```

$$11. \int \frac{(c x)^m}{a + b x^n} dx \text{ when } \frac{m+1}{n} \in \mathbb{F}$$

$$1. \int \frac{x^m}{a + b x^n} dx \text{ when } \frac{m+1}{n} \in \mathbb{F}$$

$$1: \int \frac{x^m}{a + b x^n} dx \text{ when } \frac{m+1}{n} \in \mathbb{F} \wedge m - n \ll m$$

Reference: CRC 86

Derivation: Binomial recurrence 3a with $p = -1$

Rule 1.1.3.2.11.1.1: If $\frac{m+1}{n} \in \mathbb{F} \wedge m - n \ll m$, then

$$\int \frac{x^m}{a + b x^n} dx \rightarrow \frac{x^{m-n+1}}{b(m-n+1)} - \frac{a}{b} \int \frac{x^{m-n}}{a + b x^n} dx$$

Program code:

```
Int[x_^m_./(a_+b_.*x_^n_),x_Symbol] :=
  With[{mn=Simplify[m-n]},
    x^(mn+1)/(b*(mn+1)) -
    a/b*Int[x^mn/(a+b*x^n),x] /;
  FreeQ[{a,b,m,n},x] && FractionQ[Simplify[(m+1)/n]] && SumSimplerQ[m,-n]
```

$$2: \int \frac{x^m}{a + b x^n} dx \text{ when } \frac{m+1}{n} \in \mathbb{F} \wedge m + n \ll m$$

Reference: CRC 87

Derivation: Binomial recurrence 3b with $p = -1$

Rule 1.1.3.2.11.1.2: If $\frac{m+1}{n} \in \mathbb{F} \wedge m + n \ll m$, then

$$\int \frac{x^m}{a + b x^n} dx \rightarrow \frac{x^{m+1}}{a(m+1)} - \frac{b}{a} \int \frac{x^{m+n}}{a + b x^n} dx$$

Program code:

```
Int[x_^m/(a_+b_.*x_^n_),x_Symbol] :=
  x^(m+1)/(a*(m+1)) -
  b/a*Int[x^Simplify[m+n]/(a+b*x^n),x] /;
FreeQ[{a,b,m,n},x] && FractionQ[Simplify[(m+1)/n]] && SumSimplerQ[m,n]
```

2: $\int \frac{(c x)^m}{a + b x^n} dx$ when $\frac{m+1}{n} \in \mathbb{F}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(c x)^m}{x^m} = 0$

Rule 1.1.3.2.11.2: If $\frac{m+1}{n} \in \mathbb{F}$, then

$$\int \frac{(c x)^m}{a + b x^n} dx \rightarrow \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int \frac{x^m}{a + b x^n} dx$$

Program code:

```
Int[(c_*x_)^m/(a_+b_.*x_^n_),x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m/(a+b*x^n),x] /;
FreeQ[{a,b,c,m,n},x] && FractionQ[Simplify[(m+1)/n]] && (SumSimplerQ[m,n] || SumSimplerQ[m,-n])
```

12. $\int (c x)^m (a + b x^n)^p dx$ when $p \notin \mathbb{Z}^+$

1: $\int (c x)^m (a + b x^n)^p dx$ when $p \notin \mathbb{Z}^+ \wedge (p \in \mathbb{Z}^- \vee a > 0)$

Note: If $t = r + 1 \wedge r \in \mathbb{Z}$, then $\text{Hypergeometric2F1}[r, s, t, z] = \text{Hypergeometric2F1}[s, r, t, z]$ are elementary or undefined.

Rule 1.1.3.2.12.1: If $p \notin \mathbb{Z}^+ \wedge (p \in \mathbb{Z}^- \vee a > 0)$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow \frac{a^p (c x)^{m+1}}{c (m+1)} \text{Hypergeometric2F1}\left[-p, \frac{m+1}{n}, \frac{m+1}{n} + 1, -\frac{b x^n}{a}\right]$$

Program code:

```
Int[(c_.*x_)^m_.*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  a^p*(c*x)^(m+1)/(c*(m+1))*Hypergeometric2F1[-p,(m+1)/n,(m+1)/n+1,-b*x^n/a] /;
FreeQ[{a,b,c,m,n,p},x] && Not[IGtQ[p,0]] && (ILtQ[p,0] || GtQ[a,0])
```

x: $\int (c x)^m (a + b x^n)^p dx$ when $p \notin \mathbb{Z}^+ \wedge \neg (p \in \mathbb{Z}^- \vee a > 0)$

Note: If $r == 1 \wedge (s \in \mathbb{Z} \vee t \in \mathbb{Z})$, then $\text{Hypergeometric2F1}[r, s, t, z] == \text{Hypergeometric2F1}[s, r, t, z]$ are undefined or can be expressed in elementary form.

Note: *Mathematica* has a hard time simplifying the derivative of the following antiderivative to the integrand, so the following, more complicated, but easily differentiated, rule is used instead.

Rule 1.1.3.2.12.x: If $p \notin \mathbb{Z}^+ \wedge \neg (p \in \mathbb{Z}^- \vee a > 0)$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow \frac{(c x)^{m+1} (a + b x^n)^{p+1}}{a c (m+1)} \text{Hypergeometric2F1}\left[1, \frac{m+1}{n} + p + 1, \frac{m+1}{n} + 1, -\frac{b x^n}{a}\right]$$

Program code:

```
(* Int[(c_.*x_)^m_.*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  (c*x)^(m+1)*(a+b*x^n)^(p+1)/(a*c*(m+1))*Hypergeometric2F1[1,(m+1)/n+p+1,(m+1)/n+1,-b*x^n/a] /;
FreeQ[{a,b,c,m,n,p},x] && Not[IGtQ[p,0]] && Not[ILtQ[p,0] || GtQ[a,0]] *)
```

2: $\int (c x)^m (a + b x^n)^p dx$ when $p \notin \mathbb{Z}^+ \wedge \neg (p \in \mathbb{Z}^- \vee a > 0)$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(a + b x^n)^p}{\left(1 + \frac{b x^n}{a}\right)^p} == 0$

Rule 1.1.3.2.12.2: If $p \notin \mathbb{Z}^+ \wedge \neg (p \in \mathbb{Z}^- \vee a > 0)$, then

$$\int (c x)^m (a + b x^n)^p dx \rightarrow \frac{a^{\text{IntPart}[p]} (a + b x^n)^{\text{FracPart}[p]}}{\left(1 + \frac{b x^n}{a}\right)^{\text{FracPart}[p]}} \int (c x)^m \left(1 + \frac{b x^n}{a}\right)^p dx$$

Program code:

```
Int[(c_.*x_)^m_.*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  a^IntPart[p]*(a+b*x^n)^FracPart[p]/(1+b*x^n/a)^FracPart[p]*Int[(c*x)^m*(1+b*x^n/a)^p,x] /;
FreeQ[{a,b,c,m,n,p},x] && Not[IGtQ[p,0]] && Not[ILtQ[p,0] || GtQ[a,0]]
```

D: $\int (c x)^m (a_1 + b_1 x^n)^p (a_2 + b_2 x^n)^p dx$ when $a_2 b_1 + a_1 b_2 = 0 \wedge p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: If $a_2 b_1 + a_1 b_2 = 0$, then $\partial_x \frac{(a_1 + b_1 x^n)^p (a_2 + b_2 x^n)^p}{(a_1 a_2 + b_1 b_2 x^{2n})^p} = 0$

Rule: If $a_2 b_1 + a_1 b_2 = 0 \wedge p \notin \mathbb{Z}$, then

$$\int (c x)^m (a_1 + b_1 x^n)^p (a_2 + b_2 x^n)^p dx \rightarrow \frac{(a_1 + b_1 x^n)^{\text{FracPart}[p]} (a_2 + b_2 x)^{\text{FracPart}[p]}}{(a_1 a_2 + b_1 b_2 x^2)^{\text{FracPart}[p]}} \int (c x)^m (a_1 a_2 + b_1 b_2 x^{2n})^p dx$$

Program code:

```
Int[(c_.x_)^m_.*(a1_+b1_.x_^n_)^p_.*(a2_+b2_.x_^n_)^p_,x_Symbol] :=
  (a1+b1*x^n)^FracPart[p]*(a2+b2*x^n)^FracPart[p]/(a1*a2+b1*b2*x^(2*n))^FracPart[p]*Int[(c*x)^m*(a1*a2+b1*b2*x^(2*n))^p,x] /;
FreeQ[{a1,b1,a2,b2,c,m,n,p},x] && EqQ[a2*b1+a1*b2,0] && Not[IntegerQ[p]]
```

```
(* IntBinomialQ[a,b,c,n,m,p,x] returns True iff (c*x)^m*(a+b*x^n)^p is integrable wrt x in terms of non-hypergeometric functions. *)
IntBinomialQ[a_,b_,c_,n_,m_,p_,x_] :=
  IGtQ[p,0] || RationalQ[m] && IntegersQ[n,2*p] || IntegerQ[(m+1)/n+p] ||
  (EqQ[n,2] || EqQ[n,4]) && IntegersQ[2*m,4*p] ||
  EqQ[n,2] && IntegerQ[6*p] && (IntegerQ[m] || IntegerQ[m-p])
```

Rules for integrands of the form $(d x)^m (a + b (c x^q)^n)^p$

1: $\int (d x)^m (a + b (c x)^n)^p dx$

Derivation: Integration by substitution

Rule:

$$\int (d x)^m (a + b (c x)^n)^p dx \rightarrow \frac{1}{c} \text{Subst}\left[\int \left(\frac{d x}{c}\right)^m (a + b x^n)^p dx, x, c x\right]$$

Program code:

```
Int[(d_.*x_)^m_.*(a_+b_.*(c_.*x_)^n_)^p_,x_Symbol] :=
  1/c*Subst[Int[(d*x/c)^m*(a+b*x^n)^p,x],x,c*x] /;
FreeQ[{a,b,c,d,m,n,p},x]
```


2: $\int (d x)^m (a + b (c x^q)^n)^p dx$ when $n q \in \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

Basis: $\partial_x \frac{(d x)^{m+1}}{(c x^q)^{1/q})^{m+1}} = 0$

Basis: $\frac{F[(c x^q)^{1/q}]}{x} = \text{Subst}\left[\frac{F[x]}{x}, x, (c x^q)^{1/q}\right] \partial_x (c x^q)^{1/q}$

Rule: If $n q \in \mathbb{Z}$, then

$$\begin{aligned} \int (d x)^m (a + b (c x^q)^n)^p dx &\rightarrow \frac{(d x)^{m+1}}{d (c x^q)^{1/q})^{m+1}} \int \frac{((c x^q)^{1/q})^{m+1} (a + b ((c x^q)^{1/q})^{n q})^p}{x} dx \\ &\rightarrow \frac{(d x)^{m+1}}{d (c x^q)^{1/q})^{m+1}} \text{Subst}\left[\int x^m (a + b x^{n q})^p dx, x, (c x^q)^{1/q}\right] \end{aligned}$$

Program code:

```
Int[(d_.*x_)^m_.*(a_+b_.*(c_.*x_^q_)^n_)^p_,x_Symbol] :=
  (d*x)^(m+1)/(d*((c*x^q)^(1/q))^(m+1))*Subst[Int[x^m*(a+b*x^(n*q))^p,x],x,(c*x^q)^(1/q)] /;
FreeQ[{a,b,c,d,m,n,p,q},x] && IntegerQ[n*q] && NeQ[x,(c*x^q)^(1/q)]
```

3: $\int (d x)^m (a + b (c x^q)^n)^p dx$ when $n \in \mathbb{F}$

Derivation: Integration by substitution

Rule: If $n \in \mathbb{F}$, then

$$\int (d x)^m (a + b (c x^q)^n)^p dx \rightarrow \text{Subst} \left[\int (d x)^m (a + b c^n x^{n q})^p dx, x^{1/k}, \frac{(c x^q)^{1/k}}{c^{1/k} (x^{1/k})^{q-1}} \right]$$

Program code:

```
Int[(d_.**x_)^m_.*(a_+b_.*(c_.**x_^q_)^n_)^p_,x_Symbol] :=
  With[{k=Denominator[n]},
    Subst[Int[(d*x)^m*(a+b*c^n*x^(n*q))^p,x],x^(1/k),(c*x^q)^(1/k)/(c^(1/k)*(x^(1/k))^(q-1))]] /;
  FreeQ[{a,b,c,d,m,p,q},x] && FractionQ[n]
```

4: $\int (d x)^m (a + b (c x^q)^n)^p dx$ when $n \notin \mathbb{R}$

Derivation: Integration by substitution

Basis: $F[(c x^q)^n] = \text{Subst}\left[F[c^n x^{nq}], x^{nq}, \frac{(c x^q)^n}{c^n}\right]$

Rule: If $n \notin \mathbb{R}$, then

$$\int (d x)^m (a + b (c x^q)^n)^p dx \rightarrow \text{Subst}\left[\int (d x)^m (a + b c^n x^{nq})^p dx, x^{nq}, \frac{(c x^q)^n}{c^n}\right]$$

Program code:

```
Int[(d_.**x_)^m_.*(a_+b_.*(c_.**x_^q_)^n_)^p_,x_Symbol] :=
  Subst[Int[(d*x)^m*(a+b*c^n*x^(n*q))^p,x],x^(n*q),(c*x^q)^n/c^n] /;
FreeQ[{a,b,c,d,m,n,p,q},x] && Not[RationalQ[n]]
```

$$S. \int u^m (a + b v^n)^p dx$$

$$1: \int x^m (a + b v^n)^p dx \text{ when } v = c + d x \wedge m \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If $m \in \mathbb{Z}$, then $x^m F[c + d x] = \frac{1}{d^{m+1}} \text{Subst}[(x - c)^m F[x], x, c + d x] \partial_x (c + d x)$

Rule 1.1.3.2.S.2: If $v = c + d x \wedge m \in \mathbb{Z}$, then

$$\int x^m (a + b v^n)^p dx \rightarrow \frac{1}{d^{m+1}} \text{Subst}\left[\int (x - c)^m (a + b x^n)^p dx, x, v\right]$$

Program code:

```
Int[x^m_.*(a+b_.*v^n_)^p_.,x_Symbol] :=
  With[{c=Coefficient[v,x,0],d=Coefficient[v,x,1]},
    1/d^(m+1)*Subst[Int[SimplifyIntegrand[(x-c)^m*(a+b*x^n)^p,x],x],x,v] /;
    NeQ[c,0] /;
    FreeQ[{a,b,n,p},x] && LinearQ[v,x] && IntegerQ[m]
```

2: $\int u^m (a + b v^n)^p dx$ when $v = c + d x \wedge u = e v$

Derivation: Integration by substitution and piecewise constant extraction

Basis: If $u = e v$, then $\partial_x \frac{u^m}{v^m} = 0$

Rule 1.1.3.2.S.3: If $v = c + d x \wedge u = e v$, then

$$\int u^m (a + b v^n)^p dx \rightarrow \frac{u^m}{d v^m} \text{Subst} \left[\int x^m (a + b x^n)^p dx, x, v \right]$$

Program code:

```
Int[u^m_.*(a_+b_.*v_^n_)^p_,x_Symbol] :=
  u^m/(Coefficient[v,x,1]*v^m)*Subst[Int[x^m*(a+b*x^n)^p,x],x,v] /;
FreeQ[{a,b,m,n,p},x] && LinearPairQ[u,v,x]
```