

Rules for integrands of the form $(a x^j + b x^n)^p$

1: $\int (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge j \neq n \wedge j p - n + j + 1 = 0$

Derivation: Generalized binomial recurrence 2a with $m = 0$ and $j p - n + j + 1 = 0$

Rule: If $p \notin \mathbb{Z} \wedge j \neq n \wedge j p - n + j + 1 = 0$, then

$$\int (a x^j + b x^n)^p dx \rightarrow \frac{(a x^j + b x^n)^{p+1}}{b (n-j) (p+1) x^{n-1}}$$

Program code:

```
Int[(a_.*x_^j_.+b_.*x_^n_.)^p_,x_Symbol] :=
  (a*x^j+b*x^n)^(p+1)/(b*(n-j)*(p+1)*x^(n-1)) /;
FreeQ[{a,b,j,n,p},x] && Not[IntegerQ[p]] && NeQ[n,j] && EqQ[j*p-n+j+1,0]
```

$$2. \int (a x^j + b x^n)^p dx \text{ when } p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{n p + n - j + 1}{n - j} \in \mathbb{Z}^-$$

$$1: \int (a x^j + b x^n)^p dx \text{ when } p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{n p + n - j + 1}{n - j} \in \mathbb{Z}^- \wedge p < -1$$

Derivation: Generalized binomial recurrence 2b with $m = 0$

Note: This rule increments $\frac{n p + n - j + 1}{n - j}$ by 1 thus driving it to 0.

Rule: If $p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{n p + n - j + 1}{n - j} \in \mathbb{Z}^- \wedge p < -1 \wedge (j \in \mathbb{Z} \vee c > 0)$, then

$$\int (a x^j + b x^n)^p dx \rightarrow -\frac{(a x^j + b x^n)^{p+1}}{a (n - j) (p + 1) x^{j-1}} + \frac{n p + n - j + 1}{a (n - j) (p + 1)} \int \frac{(a x^j + b x^n)^{p+1}}{x^j} dx$$

Program code:

```
Int[(a_.**x_^j_.+b_.**x_^n_.)^p_,x_Symbol] :=
  -(a*x^j+b*x^n)^(p+1)/(a*(n-j)*(p+1)*x^(j-1)) +
  (n*p+n-j+1)/(a*(n-j)*(p+1))*Int[(a*x^j+b*x^n)^(p+1)/x^j,x] /;
FreeQ[{a,b,j,n},x] && Not[IntegerQ[p]] && NeQ[n,j] && ILtQ[Simplify[(n*p+n-j+1)/(n-j)],0] && LtQ[p,-1]
```

2: $\int (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{n p + n - j + 1}{n - j} \in \mathbb{Z}^- \wedge j p + 1 \neq 0$

Derivation: Generalized binomial recurrence 3b with $m = 0$

Note: This rule increments $\frac{n p + n - j + 1}{n - j}$ by 1 thus driving it to 0.

Rule: If $p \notin \mathbb{Z} \wedge j \neq n \wedge \frac{n p + n - j + 1}{n - j} \in \mathbb{Z}^- \wedge j p + 1 \neq 0$, then

$$\int (a x^j + b x^n)^p dx \rightarrow \frac{(a x^j + b x^n)^{p+1}}{a (j p + 1) x^{j-1}} - \frac{b (n p + n - j + 1)}{a (j p + 1)} \int x^{n-j} (a x^j + b x^n)^p dx$$

Program code:

```
Int[(a_.**x_^j_.+b_.**x_^n_.)^p_,x_Symbol] :=
  (a*x^j+b*x^n)^(p+1)/(a*(j*p+1)*x^(j-1)) -
  b*(n*p+n-j+1)/(a*(j*p+1))*Int[x^(n-j)*(a*x^j+b*x^n)^p,x] /;
FreeQ[{a,b,j,n,p},x] && Not[IntegerQ[p]] && NeQ[n,j] && ILtQ[Simplify[(n*p+n-j+1)/(n-j)],0] && NeQ[j*p+1,0]
```

4. $\int (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge 0 < j < n$

1. $\int (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge 0 < j < n \wedge p > 0$

1: $\int (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge 0 < j < n \wedge p > 0 \wedge j p + 1 < 0$

Derivation: Generalized binomial recurrence 1a with $m = 0$

Rule: If $p \notin \mathbb{Z} \wedge 0 < j < n \wedge p > 0 \wedge j p + 1 < 0$, then

$$\int (a x^j + b x^n)^p dx \rightarrow \frac{x (a x^j + b x^n)^p}{j p + 1} - \frac{b (n - j) p}{j p + 1} \int x^n (a x^j + b x^n)^{p-1} dx$$

Program code:

```
Int[(a_.**x_^j_.+b_.**x_^n_.)^p_,x_Symbol] :=
  x*(a*x^j+b*x^n)^p/(j*p+1) -
  b*(n-j)*p/(j*p+1)*Int[x^n*(a*x^j+b*x^n)^(p-1),x] /;
FreeQ[{a,b},x] && Not[IntegerQ[p]] && LtQ[0,j,n] && GtQ[p,0] && LtQ[j*p+1,0]
```

2: $\int (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge 0 < j < n \wedge p > 0 \wedge n p + 1 \neq 0$

Derivation: Generalized binomial recurrence 1b with $m = 0$

Rule: If $p \notin \mathbb{Z} \wedge 0 < j < n \wedge p > 0 \wedge n p + 1 \neq 0$, then

$$\int (a x^j + b x^n)^p dx \rightarrow \frac{x (a x^j + b x^n)^p}{n p + 1} + \frac{a (n - j) p}{n p + 1} \int x^j (a x^j + b x^n)^{p-1} dx$$

Program code:

```
Int[(a_.**x_^j_.+b_.**x_^n_.)^p_,x_Symbol] :=
  x*(a*x^j+b*x^n)^p/(n*p+1) +
  a*(n-j)*p/(n*p+1)*Int[x^j*(a*x^j+b*x^n)^(p-1),x] /;
FreeQ[{a,b},x] && Not[IntegerQ[p]] && LtQ[0,j,n] && GtQ[p,0] && NeQ[n*p+1,0]
```

$$2. \int (a x^j + b x^n)^p dx \text{ when } p \notin \mathbb{Z} \wedge 0 < j < n \wedge p < -1$$

$$1: \int (a x^j + b x^n)^p dx \text{ when } p \notin \mathbb{Z} \wedge 0 < j < n \wedge p < -1 \wedge j p + 1 > n - j$$

Derivation: Generalized binomial recurrence 2a with $m = 0$

Rule: If $p \notin \mathbb{Z} \wedge 0 < j < n \wedge p < -1 \wedge j p + 1 > n - j$, then

$$\int (a x^j + b x^n)^p dx \rightarrow \frac{(a x^j + b x^n)^{p+1}}{b (n-j) (p+1) x^{n-1}} - \frac{j p - n + j + 1}{b (n-j) (p+1)} \int \frac{(a x^j + b x^n)^{p+1}}{x^n} dx$$

Program code:

```
Int[(a_.**x_^j_.+b_.**x_^n_.)^p_,x_Symbol] :=
  (a**x^j+b*x^n)^(p+1)/(b*(n-j)*(p+1)*x^(n-1)) -
  (j*p-n+j+1)/(b*(n-j)*(p+1))*Int[(a**x^j+b*x^n)^(p+1)/x^n,x] /;
FreeQ[{a,b},x] && Not[IntegerQ[p]] && LtQ[0,j,n] && LtQ[p,-1] && GtQ[j*p+1,n-j]
```

2: $\int (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge 0 < j < n \wedge p < -1$

Derivation: Generalized binomial recurrence 2b with $m = 0$

Rule: If $p \notin \mathbb{Z} \wedge 0 < j < n \wedge p < -1$, then

$$\int (a x^j + b x^n)^p dx \rightarrow -\frac{(a x^j + b x^n)^{p+1}}{a (n-j) (p+1) x^{j-1}} + \frac{n p + n - j + 1}{a (n-j) (p+1)} \int \frac{(a x^j + b x^n)^{p+1}}{x^j} dx$$

Program code:

```
Int[(a_.**x_^j_.+b_.**x_^n_.)^p_,x_Symbol] :=
  -(a**x^j+b**x^n)^(p+1)/(a*(n-j)*(p+1)*x^(j-1)) +
  (n*p+n-j+1)/(a*(n-j)*(p+1))*Int[(a**x^j+b**x^n)^(p+1)/x^j,x] /;
FreeQ[{a,b},x] && Not[IntegerQ[p]] && LtQ[0,j,n] && LtQ[p,-1]
```

5. $\int (a x^j + b x^n)^p dx$ when $p + \frac{1}{2} \in \mathbb{Z} \wedge j \neq n \wedge j p + 1 = 0$

1: $\int (a x^j + b x^n)^p dx$ when $p + \frac{1}{2} \in \mathbb{Z}^+ \wedge j \neq n \wedge j p + 1 = 0$

Derivation: Generalized binomial recurrence 1b

Rule: If $p + \frac{1}{2} \in \mathbb{Z}^+ \wedge j \neq n \wedge j p + 1 = 0$, then

$$\int (a x^j + b x^n)^p dx \rightarrow \frac{x (a x^j + b x^n)^p}{p (n - j)} + a \int x^j (a x^j + b x^n)^{p-1} dx$$

Program code:

```
Int[(a_.**x_^j_.+b_.**x_^n_.)^p_,x_Symbol] :=
  x*(a**x^j+b**x^n)^p/(p*(n-j)) + a*Int[x^j*(a**x^j+b**x^n)^(p-1),x] /;
FreeQ[{a,b,j,n},x] && IGtQ[p+1/2,0] && NeQ[n,j] && EqQ[Simplify[j*p+1],0]
```

2. $\int (a x^j + b x^n)^p dx$ when $p - \frac{1}{2} \in \mathbb{Z}^- \wedge j \neq n \wedge j p + 1 = 0$

1: $\int \frac{1}{\sqrt{a x^2 + b x^n}} dx$ when $n \neq 2$

Reference: G&R 2.261.1, CRC 237a, A&S 3.3.33

Reference: CRC 238

Derivation: Integration by substitution

Basis: If $n \neq 2$, then $\frac{1}{\sqrt{a x^2 + b x^n}} = \frac{2}{2-n} \text{Subst}\left[\frac{1}{1-a x^2}, x, \frac{x}{\sqrt{a x^2 + b x^n}}\right] \partial_x \frac{x}{\sqrt{a x^2 + b x^n}}$

Rule: If $n \neq 2$, then

$$\int \frac{1}{\sqrt{a x^2 + b x^n}} dx \rightarrow \frac{2}{2-n} \text{Subst} \left[\int \frac{1}{1 - a x^2} dx, x, \frac{x}{\sqrt{a x^2 + b x^n}} \right]$$

Program code:

```
Int[1/Sqrt[a_.*x_^2+b_.*x_^n_],x_Symbol] :=
  2/(2-n)*Subst[Int[1/(1-a*x^2),x],x,x/Sqrt[a*x^2+b*x^n]] /;
FreeQ[{a,b,n},x] && NeQ[n,2]
```

2: $\int (a x^j + b x^n)^p dx$ when $p + \frac{1}{2} \in \mathbb{Z}^- \wedge j \neq n \wedge j p + 1 = 0$

Derivation: Generalized binomial recurrence 2b

Rule: If $p + \frac{1}{2} \in \mathbb{Z}^- \wedge j \neq n \wedge j p + 1 = 0$, then

$$\int (a x^j + b x^n)^p dx \rightarrow -\frac{(a x^j + b x^n)^{p+1}}{a (n-j) (p+1) x^{j-1}} + \frac{n p + n - j + 1}{a (n-j) (p+1)} \int \frac{(a x^j + b x^n)^{p+1}}{x^j} dx$$

Program code:

```
Int[(a_.*x_^j_+b_.*x_^n_)^p_,x_Symbol] :=
  -(a*x^j+b*x^n)^(p+1)/(a*(n-j)*(p+1)*x^(j-1)) +
  (n*p+n-j+1)/(a*(n-j)*(p+1))*Int[(a*x^j+b*x^n)^(p+1)/x^j,x] /;
FreeQ[{a,b,j,n},x] && ILtQ[p+1/2,0] && NeQ[n,j] && EqQ[Simplify[j*p+1],0]
```

6: $\int \frac{1}{\sqrt{a x^j + b x^n}} dx$ when $2(n-1) < j < n$

Derivation: Generalized binomial recurrence 3a with $m = 0$ and $p = -\frac{1}{2}$

Rule: If $2(n-1) < j < n$, then

$$\int \frac{1}{\sqrt{a x^j + b x^n}} dx \rightarrow -\frac{2 \sqrt{a x^j + b x^n}}{b(n-2) x^{n-1}} - \frac{a(2n-j-2)}{b(n-2)} \int \frac{1}{x^{n-j} \sqrt{a x^j + b x^n}} dx$$

Program code:

```
Int[1/Sqrt[a_.*x_^j_.+b_.*x_^n_.],x_Symbol] :=
  -2*Sqrt[a*x^j+b*x^n]/(b*(n-2)*x^(n-1)) -
  a*(2*n-j-2)/(b*(n-2))*Int[1/(x^(n-j)*Sqrt[a*x^j+b*x^n]),x] /;
FreeQ[{a,b},x] && LtQ[2*(n-1),j,n]
```

x. $\int (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge j \neq n$

1: $\int (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge j \neq n \wedge j p + 1 = 0$

Rule: If $p \notin \mathbb{Z} \wedge j \neq n \wedge j p + 1 = 0$, then

$$\int (a x^j + b x^n)^p dx \rightarrow \frac{x (a x^j + b x^n)^p}{p (n - j) \left(\frac{a x^j + b x^n}{b x^n} \right)^p} \text{Hypergeometric2F1} \left[-p, -p, 1 - p, -\frac{a}{b x^{n-j}} \right]$$

Program code:

```
(* Int[(a_.**x_^j_.+b_.**x_^n_.)^p_,x_Symbol] :=
  x*(a**x^j+b**x^n)^p/(p*(n-j)*((a**x^j+b**x^n)/(b**x^n))^p)*Hypergeometric2F1[-p,-p,1-p,-a/(b**x^(n-j))]/;
FreeQ[{a,b,j,n,p},x] && Not[IntegerQ[p]] && NeQ[n,j] && EqQ[j*p+1,0] *)
```

2: $\int (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge j \neq n \wedge j p + 1 \neq 0$

Rule: If $p \notin \mathbb{Z} \wedge j \neq n \wedge j p + 1 \neq 0$, then

$$\int (a x^j + b x^n)^p dx \rightarrow \frac{x (a x^j + b x^n)^p}{(j p + 1) \left(\frac{a x^j + b x^n}{a x^j} \right)^p} \text{Hypergeometric2F1} \left[-p, \frac{j p + 1}{n - j}, \frac{j p + 1}{n - j} + 1, -\frac{b x^{n-j}}{a} \right]$$

Program code:

```
(* Int[(a_.**x_^j_.+b_.**x_^n_.)^p_,x_Symbol] :=
  x*(a**x^j+b**x^n)^p/((j*p+1)*((a**x^j+b**x^n)/(a**x^j))^p)*
  Hypergeometric2F1[-p,(j*p+1)/(n-j),(j*p+1)/(n-j)+1,-b**x^(n-j)/a]/;
FreeQ[{a,b,j,n,p},x] && Not[IntegerQ[p]] && NeQ[n,j] && NeQ[j*p+1,0] *)
```

7: $\int (a x^j + b x^n)^p dx$ when $p \notin \mathbb{Z} \wedge j \neq n$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(a x^j + b x^n)^p}{x^j p (a + b x^{n-j})^p} == 0$

Basis: $\frac{(a x^j + b x^n)^p}{x^j p (a + b x^{n-j})^p} == \frac{(a x^j + b x^n)^{\text{FracPart}[p]}}{x^j \text{FracPart}[p] (a + b x^{n-j})^{\text{FracPart}[p]}}$

Rule: If $p \notin \mathbb{Z} \wedge j \neq n$, then

$$\int (a x^j + b x^n)^p dx \rightarrow \frac{(a x^j + b x^n)^{\text{FracPart}[p]}}{x^j \text{FracPart}[p] (a + b x^{n-j})^{\text{FracPart}[p]}} \int x^j p (a + b x^{n-j})^p dx$$

Program code:

```
Int[(a_.**x_^j_.+b_.**x_^n_.)^p_,x_Symbol] :=
  (a*x^j+b*x^n)^FracPart[p]/(x^(j*FracPart[p])*(a+b*x^(n-j))^FracPart[p])*Int[x^(j*p)*(a+b*x^(n-j))^p,x] /;
FreeQ[{a,b,j,n,p},x] && Not[IntegerQ[p]] && NeQ[n,j] && PosQ[n-j]
```

S: $\int (a u^j + b u^n)^p dx$ when $u = c + d x$

Derivation: Integration by substitution

Rule: If $u = c + d x$, then

$$\int (a u^j + b u^n)^p dx \rightarrow \frac{1}{d} \text{Subst} \left[\int (a x^j + b x^n)^p dx, x, u \right]$$

Program code:

```
Int[(a_.*u_^j_.+b_.*u_^n_.)^p_,x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(a*x^j+b*x^n)^p,x],x,u] /;
FreeQ[{a,b,j,n,p},x] && LinearQ[u,x] && NeQ[u,x]
```