

Rules for integrands of the form $(a + b \sin[e + f x]^2)^p (A + B \sin[e + f x]^2)$

1. $\int (a + b \sin[e + f x]^2)^p (A + B \sin[e + f x]^2) dx$ when $p > 0$

1: $\int (a + b \sin[e + f x]^2) (A + B \sin[e + f x]^2) dx$

Derivation: Algebraic expansion

Basis: $(a + b z) (A + B z) = \frac{1}{8} (4A(2a + b) + B(4a + 3b)) - \frac{1}{8} (4Ab + B(4a + 3b)) (1 - 2z) - \frac{1}{4} bBz(3 - 4z)$

Rule:

$$\int (a + b \sin[e + f x]^2) (A + B \sin[e + f x]^2) dx \rightarrow \frac{(4A(2a + b) + B(4a + 3b))x}{8} - \frac{(4Ab + B(4a + 3b)) \cos[e + f x] \sin[e + f x]}{8f} - \frac{bB \cos[e + f x] \sin[e + f x]^3}{4f}$$

Program code:

```
Int[(a_+b_.sin[e_+f_.x_]^2)*(A_+B_.sin[e_+f_.x_]^2),x_Symbol] :=
  (4*A*(2*a+b)+B*(4*a+3*b))*x/8 -
  (4*A*b+B*(4*a+3*b))*Cos[e+f*x]*Sin[e+f*x]/(8*f) -
  b*B*Cos[e+f*x]*Sin[e+f*x]^3/(4*f) /;
FreeQ[{a,b,e,f,A,B},x]
```

2: $\int (a + b \sin[e + fx]^2)^p (A + B \sin[e + fx]^2) dx$ when $p > 0$

Rule: If $p > 0$, then

$$\int (a + b \sin[e + fx]^2)^p (A + B \sin[e + fx]^2) dx \rightarrow$$

$$- \frac{B \cos[e + fx] \sin[e + fx] (a + b \sin[e + fx]^2)^p}{2 f (p + 1)} +$$

$$\frac{1}{2 (p + 1)} \int (a + b \sin[e + fx]^2)^{p-1} (a B + 2 a A (p + 1) + (2 A b (p + 1) + B (b + 2 a p + 2 b p)) \sin[e + fx]^2) dx$$

Program code:

```
Int[(a+b_.sin[e_.+f_.x_]^2)^p_*(A_.+B_.sin[e_.+f_.x_]^2),x_Symbol] :=
  -B*cos[e+f*x]*sin[e+f*x]*(a+b*sin[e+f*x]^2)^p/(2*f*(p+1)) +
  1/(2*(p+1))*Int[(a+b*sin[e+f*x]^2)^(p-1)*
    Simp[a*B+2*a*A*(p+1)+(2*A*b*(p+1)+B*(b+2*a*p+2*b*p))*sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,e,f,A,B},x] && GtQ[p,0]
```

2. $\int (a + b \sin[ex + f])^p (A + B \sin[ex + f])^2 dx$ when $p < 0$

1: $\int \frac{A + B \sin[c + dx]^2}{a + b \sin[ex + f]^2} dx$

Derivation: Algebraic expansion

Basis: $\frac{A+Bz}{a+bz} == \frac{B}{b} + \frac{Ab-aB}{b(a+bz)}$

Rule:

$$\int \frac{A + B \sin[c + dx]^2}{a + b \sin[ex + f]^2} dx \rightarrow \frac{Bx}{b} + \frac{Ab - aB}{b} \int \frac{1}{a + b \sin[ex + f]^2} dx$$

Program code:

```
Int[(A_.+B_.sin[e_.+f_.x_]^2)/(a_.+b_.sin[e_.+f_.x_]^2),x_Symbol] :=
  B*x/b + (A*b-a*B)/b*Int[1/(a+b*sin[ex+f]^2),x] /;
FreeQ[{a,b,e,f,A,B},x]
```

2: $\int \frac{A + B \sin[c + dx]^2}{\sqrt{a + b \sin[ex + f]^2}} dx$

Derivation: Algebraic expansion

Basis: $\frac{A+B \sin[z]^2}{\sqrt{a+b \sin[z]^2}} == \frac{B \sqrt{a+b \sin[z]^2}}{b} + \frac{Ab-aB}{b \sqrt{a+b \sin[z]^2}}$

Rule:

$$\int \frac{A + B \sin[c + dx]^2}{\sqrt{a + b \sin[e + fx]^2}} dx \rightarrow \frac{B}{b} \int \sqrt{a + b \sin[e + fx]^2} dx + \frac{Ab - aB}{b} \int \frac{1}{\sqrt{a + b \sin[e + fx]^2}} dx$$

Program code:

```
Int[(A_+B_.*sin[e_+f_.*x_]^2)/Sqrt[a+b_.*sin[e_+f_.*x_]^2],x_Symbol] :=
  B/b*Int[Sqrt[a+b*Sin[e+f*x]^2],x] + (A*b-a*B)/b*Int[1/Sqrt[a+b*Sin[e+f*x]^2],x] /;
FreeQ[{a,b,e,f,A,B},x]
```

3: $\int (a + b \sin[e + fx]^2)^p (A + B \sin[e + fx]^2) dx$ when $p < -1 \wedge a + b \neq 0$

Rule: If $p < -1 \wedge a + b \neq 0$, then

$$\begin{aligned} & \int (a + b \sin[e + fx]^2)^p (A + B \sin[e + fx]^2) dx \rightarrow \\ & - \frac{(Ab - aB) \cos[e + fx] \sin[e + fx] (a + b \sin[e + fx]^2)^{p+1}}{2af(a+b)(p+1)} - \\ & \frac{1}{2a(a+b)(p+1)} \int (a + b \sin[e + fx]^2)^{p+1} (aB - A(2a(p+1) + b(2p+3)) + 2(Ab - aB)(p+2) \sin[e + fx]^2) dx \end{aligned}$$

Program code:

```
Int[(a_+b_.*sin[e_+f_.*x_]^2)^p_*(A_+B_.*sin[e_+f_.*x_]^2),x_Symbol] :=
  -(A*b-a*B)*Cos[e+f*x]*Sin[e+f*x]*(a+b*Sin[e+f*x]^2)^(p+1)/(2*a*f*(a+b)*(p+1)) -
  1/(2*a*(a+b)*(p+1))*Int[(a+b*Sin[e+f*x]^2)^(p+1)*
    Simp[a*B-A*(2*a*(p+1)+b*(2*p+3))+2*(A*b-a*B)*(p+2)*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,e,f,A,B},x] && LtQ[p,-1] && NeQ[a+b,0]
```

3: $\int (a + b \sin[e + fx]^2)^p (A + B \sin[e + fx]^2) dx$ when $p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

Basis: $A + B \sin[z]^2 == \frac{A + (A+B) \tan[z]^2}{1 + \tan[z]^2}$

Basis: $\partial_x \frac{(a + b \sin[e + fx]^2)^p (\sec[e + fx]^2)^p}{(a + (a+b) \tan[e + fx]^2)^p} == 0$

Basis: $F[\tan[e + fx]] == \frac{1}{f} \operatorname{Subst}\left[\frac{F[x]}{1+x^2}, x, \tan[e + fx]\right] \partial_x \tan[e + fx]$

Rule: If $p \notin \mathbb{Z}$, then

$$\begin{aligned} \int (a + b \sin[e + fx]^2)^p (A + B \sin[e + fx]^2) dx &\rightarrow \frac{(a + b \sin[e + fx]^2)^p (\sec[e + fx]^2)^p}{(a + (a+b) \tan[e + fx]^2)^p} \int \frac{(a + (a+b) \tan[e + fx]^2)^p (A + (A+B) \tan[e + fx]^2)}{(1 + \tan[e + fx]^2)^{p+1}} dx \\ &\rightarrow \frac{(a + b \sin[e + fx]^2)^p (\sec[e + fx]^2)^p}{f (a + (a+b) \tan[e + fx]^2)^p} \operatorname{Subst}\left[\int \frac{(a + (a+b) x^2)^p (A + (A+B) x^2)}{(1 + x^2)^{p+2}} dx, x, \tan[e + fx]\right] \end{aligned}$$

Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_]^2)^p_*(A_.+B_.*sin[e_.+f_.*x_]^2),x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    ff*(a+b*sin[e+f*x]^2)^p*(Sec[e+f*x]^2)^p/(f*(a+(a+b)*Tan[e+f*x]^2)^p)*
    Subst[Int[(a+(a+b)*ff^2*x^2)^p*(A+(A+B)*ff^2*x^2)/(1+ff^2*x^2)^(p+2),x],x,Tan[e+f*x]/ff]] /;
  FreeQ[{a,b,e,f,A,B},x] && Not[IntegerQ[p]]
```

Rules for integrands of the form $u (a + b \sin[e + f x]^2)^p$

1. $\int u (a + b \sin[e + f x]^2)^p dx$ when $a + b = 0$

1: $\int u (a + b \sin[e + f x]^2)^p dx$ when $a + b = 0 \wedge p \in \mathbb{Z}$

Derivation: Algebraic simplification

Basis: If $a + b = 0$, then $a + b \sin[z]^2 = a \cos[z]^2$

Rule: If $a + b = 0 \wedge p \in \mathbb{Z}$, then

$$\int u (a + b \sin[e + f x]^2)^p dx \rightarrow a^p \int u \cos[e + f x]^{2p} dx$$

Program code:

```
Int[u_.*(a_+b_.sin[e_+f_.x_]^2)^p_,x_Symbol] :=
  a^p*Int[ActivateTrig[u*cos[e+f*x]^(2*p)],x] /;
FreeQ[{a,b,e,f,p},x] && EqQ[a+b,0] && IntegerQ[p]
```

2: $\int u (a + b \sin[e + fx]^2)^p dx$ when $a + b = 0$

Derivation: Algebraic simplification

Basis: If $a + b = 0$, then $a + b \sin[z]^2 = a \cos[z]^2$

Rule: If $a + b = 0$, then

$$\int u (a + b \sin[e + fx]^2)^p dx \rightarrow \int u (a \cos[e + fx]^2)^p dx$$

Program code:

```
Int[u.*(a+b_.sin[e_.+f_.x_]^2)^p_,x_Symbol] :=
  Int[ActivateTrig[u*(a*cos[e+f*x]^2)^p],x] /;
FreeQ[{a,b,e,f,p},x] && EqQ[a+b,0]
```

$$2. \int (a + b \sin[e + f x]^2)^p dx$$

$$1. \int (a + b \sin[e + f x]^2)^p dx \text{ when } a + b \neq 0 \wedge p > 0$$

$$1. \int \sqrt{a + b \sin[e + f x]^2} dx$$

$$1: \int \sqrt{a + b \sin[e + f x]^2} dx \text{ when } a > 0$$

Rule: If $a > 0$, then

$$\int \sqrt{a + b \sin[e + f x]^2} dx \rightarrow \frac{\sqrt{a}}{f} \operatorname{EllipticE}\left[e + f x, -\frac{b}{a}\right]$$

Program code:

```
Int[Sqrt[a+b_.sin[e_.+f_.x_]^2],x_Symbol] :=
  Sqrt[a]/f*EllipticE[e+f*x,-b/a] /;
FreeQ[{a,b,e,f},x] && GtQ[a,0]
```

$$2: \int \sqrt{a + b \sin[e + f x]^2} dx \text{ when } a \neq 0$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{\sqrt{a + b \sin[e + f x]^2}}{\sqrt{1 + \frac{b \sin[e + f x]^2}{a}}} == 0$$

Rule: If $a \neq 0$, then

$$\int \sqrt{a + b \sin[e + f x]^2} \, dx \rightarrow \frac{\sqrt{a + b \sin[e + f x]^2}}{\sqrt{1 + \frac{b \sin[e + f x]^2}{a}}} \int \sqrt{1 + \frac{b \sin[e + f x]^2}{a}} \, dx$$

Program code:

```
Int[Sqrt[a+b_.sin[e_.+f_.x_]^2],x_Symbol] :=
  Sqrt[a+b*sin[e+f*x]^2]/Sqrt[1+b*sin[e+f*x]^2/a]*Int[Sqrt[1+(b*sin[e+f*x]^2)/a],x] /;
FreeQ[{a,b,e,f},x] && Not[GtQ[a,0]]
```

$$2: \int (a + b \sin[e + f x]^2)^2 \, dx$$

Derivation: Algebraic expansion

$$\text{Basis: } (a + b z)^2 = \frac{1}{8} (8 a^2 + 8 a b + 3 b^2) - \frac{b}{8} (8 a + 3 b) (1 - 2 z) - \frac{1}{4} b^2 (3 - 4 z) z$$

Rule:

$$\int (a + b \sin[e + f x]^2)^2 \, dx \rightarrow \frac{(8 a^2 + 8 a b + 3 b^2) x}{8} - \frac{b (8 a + 3 b) \cos[e + f x] \sin[e + f x]}{8 f} - \frac{b^2 \cos[e + f x] \sin[e + f x]^3}{4 f}$$

Program code:

```
Int[(a+b_.sin[e_.+f_.x_]^2)^2,x_Symbol] :=
  (8*a^2+8*a*b+3*b^2)*x/8 -
  b*(8*a+3*b)*Cos[e+f*x]*Sin[e+f*x]/(8*f) -
  b^2*Cos[e+f*x]*Sin[e+f*x]^3/(4*f) /;
FreeQ[{a,b,e,f},x]
```

3: $\int (a + b \sin[e + fx]^2)^p dx$ when $a + b \neq 0 \wedge p > 1$

Rule: If $a + b \neq 0 \wedge p > 1$, then

$$\int (a + b \sin[e + fx]^2)^p dx \rightarrow -\frac{b \cos[e + fx] \sin[e + fx] (a + b \sin[e + fx]^2)^{p-1}}{2 f p} + \frac{1}{2 p} \int (a + b \sin[e + fx]^2)^{p-2} (a (b + 2 a p) + b (2 a + b) (2 p - 1) \sin[e + fx]^2) dx$$

Program code:

```
Int[(a+b_.sin[e_.+f_.x_]^2)^p_,x_Symbol] :=
  -b*cos[e+f*x]*sin[e+f*x]*(a+b*sin[e+f*x]^2)^(p-1)/(2*f*p) +
  1/(2*p)*Int[(a+b*sin[e+f*x]^2)^(p-2)*Simp[a*(b+2*a*p)+b*(2*a+b)*(2*p-1)*sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,e,f},x] && NeQ[a+b,0] && GtQ[p,1]
```

2. $\int (a + b \sin[e + fx]^2)^p dx$ when $a + b \neq 0 \wedge p < 0$

1: $\int \frac{1}{a + b \sin[e + fx]^2} dx$

Derivation: Integration by substitution

Basis: $\sin[z]^2 = \frac{\tan[z]^2}{1 + \tan[z]^2}$

Basis: $F[\sin[e + fx]^2] = \frac{1}{f} \operatorname{Subst}\left[\frac{F\left[\frac{x^2}{1+x^2}\right]}{1+x^2}, x, \tan[e + fx]\right] \partial_x \tan[e + fx]$

Rule: If $p \in \mathbb{Z}$, then

$$\int \frac{1}{a+b \sin[e+fx]^2} dx \rightarrow \frac{1}{f} \operatorname{Subst}\left[\int \frac{1}{a+(a+b)x^2} dx, x, \tan[e+fx]\right]$$

Program code:

```
Int[1/(a+b_.*sin[e_.+f_.*x_]^2),x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    ff/f*Subst[Int[1/(a+(a+b)*ff^2*x^2),x],x,Tan[e+f*x]/ff]] /;
FreeQ[{a,b,e,f},x]
```

$$2. \int \frac{1}{\sqrt{a+b \sin[e+fx]^2}} dx$$

1: $\int \frac{1}{\sqrt{a+b \sin[e+fx]^2}} dx$ when $a > 0$

Rule: If $a > 0$, then

$$\int \frac{1}{\sqrt{a+b \sin[e+fx]^2}} dx \rightarrow \frac{1}{\sqrt{a} f} \operatorname{EllipticF}\left[e+fx, -\frac{b}{a}\right]$$

Program code:

```
Int[1/Sqrt[a+b_.*sin[e_.+f_.*x_]^2],x_Symbol] :=
  1/(Sqrt[a]*f)*EllipticF[e+f*x,-b/a] /;
FreeQ[{a,b,e,f},x] && GtQ[a,0]
```

$$2: \int \frac{1}{\sqrt{a+b \sin[e+fx]^2}} dx \text{ when } a \neq 0$$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{\sqrt{1 + \frac{b \sin[e+fx]^2}{a}}}{\sqrt{a+b \sin[e+fx]^2}} == 0$

Rule: If $a \neq 0$, then

$$\int \frac{1}{\sqrt{a+b \sin[e+fx]^2}} dx \rightarrow \frac{\sqrt{1 + \frac{b \sin[e+fx]^2}{a}}}{\sqrt{a+b \sin[e+fx]^2}} \int \frac{1}{\sqrt{1 + \frac{b \sin[e+fx]^2}{a}}} dx$$

Program code:

```
Int[1/Sqrt[a+b_.*sin[e_.+f_.*x_]^2],x_Symbol] :=
  Sqrt[1+b*Sin[e+f*x]^2/a]/Sqrt[a+b*Sin[e+f*x]^2]*Int[1/Sqrt[1+(b*Sin[e+f*x]^2)/a],x] /;
FreeQ[{a,b,e,f},x] && Not[GtQ[a,0]]
```

3: $\int (a + b \sin[e + fx]^2)^p dx$ when $a + b \neq 0 \wedge p < -1$

Rule: If $a + b \neq 0 \wedge p < -1$, then

$$\int (a + b \sin[e + fx]^2)^p dx \rightarrow -\frac{b \cos[e + fx] \sin[e + fx] (a + b \sin[e + fx]^2)^{p+1}}{2 a f (p+1) (a+b)} + \frac{1}{2 a (p+1) (a+b)} \int (a + b \sin[e + fx]^2)^{p+1} (2 a (p+1) + b (2 p+3) - 2 b (p+2) \sin[e + fx]^2) dx$$

Program code:

```
Int[(a_+b_.sin[e_+f_.x_]^2)^p_,x_Symbol] :=
  -b*cos[e+f*x]*Sin[e+f*x]*(a+b*sin[e+f*x]^2)^(p+1)/(2*a*f*(p+1)*(a+b)) +
  1/(2*a*(p+1)*(a+b))*Int[(a+b*sin[e+f*x]^2)^(p+1)*Simp[2*a*(p+1)+b*(2*p+3)-2*b*(p+2)*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,e,f},x] && NeQ[a+b,0] && LtQ[p,-1]
```

3: $\int (a + b \sin[e + fx]^2)^p dx$ when $p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

Basis: $\partial_x \frac{\sqrt{\cos[e + fx]^2}}{\cos[e + fx]} = 0$

Basis: $\cos[e + fx] F[\sin[e + fx]] = \frac{1}{f} \operatorname{Subst}[F[x], x, \sin[e + fx]] \partial_x \sin[e + fx]$

Rule: If $p \notin \mathbb{Z}$, then

$$\int (a + b \sin[e + fx]^2)^p dx \rightarrow \frac{\sqrt{\cos[e + fx]^2}}{\cos[e + fx]} \int \frac{\cos[e + fx] (a + b \sin[e + fx]^2)^p}{\sqrt{1 - \sin[e + fx]^2}} dx$$

$$\rightarrow \frac{\sqrt{\cos[e+fx]^2}}{f \cos[e+fx]} \operatorname{Subst}\left[\int \frac{(a+bx^2)^p}{\sqrt{1-x^2}} dx, x, \sin[e+fx]\right]$$

Program code:

```
Int[(a_+b_.*sin[e_+f_*x_]^2)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Sin[e+f*x],x]},
    ff*Sqrt[Cos[e+f*x]^2]/(f*Cos[e+f*x])*Subst[Int[(a+b*ff^2*x^2)^p/Sqrt[1-ff^2*x^2],x],x,Sin[e+f*x]/ff] /;
  FreeQ[{a,b,e,f,p},x] && Not[IntegerQ[p]]
```

$$3. \int (d \sin[e+fx])^m (a+b \sin[e+fx]^2)^p dx$$

$$1. \int \sin[e+fx]^m (a+b \sin[e+fx]^2)^p dx \text{ when } m \in \mathbb{Z}$$

$$1: \int \sin[e+fx]^m (a+b \sin[e+fx]^2)^p dx \text{ when } \frac{m-1}{2} \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } \sin[z]^2 == 1 - \cos[z]^2$$

Basis: If $\frac{m-1}{2} \in \mathbb{Z}$, then

$$\sin[e+fx]^m F[\sin[e+fx]^2] == -\frac{1}{f} \operatorname{Subst}\left[\left(1-x^2\right)^{\frac{m-1}{2}} F[1-x^2], x, \cos[e+fx]\right] \partial_x \cos[e+fx]$$

Rule: If $\frac{m-1}{2} \in \mathbb{Z}$, then

$$\int \sin[e+fx]^m (a+b \sin[e+fx]^2)^p dx \rightarrow -\frac{1}{f} \operatorname{Subst}\left[\int (1-x^2)^{\frac{m-1}{2}} (a+b-x^2)^p dx, x, \cos[e+fx]\right]$$

Program code:

```
Int[sin[e_+f_*x_]^m_.*(a_+b_*sin[e_+f_*x_]^2)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Cos[e+f*x],x]},
    -ff/f*Subst[Int[(1-ff^2*x^2)^(m-1)/2*(a+b-b*ff^2*x^2)^p,x],x,Cos[e+f*x]/ff]] /;
  FreeQ[{a,b,e,f,p},x] && IntegerQ[(m-1)/2]
```

$$2. \int \sin[e+fx]^m (a+b \sin[e+fx]^2)^p dx \text{ when } \frac{m}{2} \in \mathbb{Z}$$

$$1: \int \sin[e+fx]^m (a+b \sin[e+fx]^2)^p dx \text{ when } \frac{m}{2} \in \mathbb{Z} \wedge p \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } \sin[z]^2 = \frac{\tan[z]^2}{1+\tan[z]^2}$$

Basis: If $\frac{m}{2} \in \mathbb{Z}$, then

$$\sin[e+fx]^m F[\sin[e+fx]^2] = \frac{1}{f} \operatorname{Subst} \left[\frac{x^m F\left[\frac{x^2}{1+x^2}\right]}{(1+x^2)^{m/2+1}}, x, \tan[e+fx] \right] \partial_x \tan[e+fx]$$

Rule: If $\frac{m}{2} \in \mathbb{Z} \wedge p \in \mathbb{Z}$, then

$$\int \sin[e+fx]^m (a+b \sin[e+fx]^2)^p dx \rightarrow \frac{1}{f} \operatorname{Subst} \left[\int \frac{x^m (a+(a+b)x^2)^p}{(1+x^2)^{m/2+p+1}} dx, x, \tan[e+fx] \right]$$

Program code:

```
Int[sin[e_+f_*x_]^m_*(a_+b_*sin[e_+f_*x_]^2)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    ff^(m+1)/f*Subst[Int[x^m*(a+(a+b)*ff^2*x^2)^p/(1+ff^2*x^2)^(m/2+p+1),x],x,Tan[e+f*x]/ff]] /;
  FreeQ[{a,b,e,f},x] && IntegerQ[m/2] && IntegerQ[p]
```


$$\mathbf{2:} \int \sin[e+fx]^m (a+b \sin[e+fx]^2)^p dx \text{ when } \frac{m}{2} \in \mathbb{Z} \wedge p \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: } \partial_x \frac{\sqrt{\cos[e+fx]^2}}{\cos[e+fx]} == 0$$

$$\text{Basis: } \cos[e+fx] F[\sin[e+fx]] == \frac{1}{f} \operatorname{Subst}[F[x], x, \sin[e+fx]] \partial_x \sin[e+fx]$$

Rule: If $\frac{m}{2} \in \mathbb{Z} \wedge p \notin \mathbb{Z}$, then

$$\begin{aligned} \int \sin[e+fx]^m (a+b \sin[e+fx]^2)^p dx &\rightarrow \frac{\sqrt{\cos[e+fx]^2}}{\cos[e+fx]} \int \frac{\cos[e+fx] \sin[e+fx]^m (a+b \sin[e+fx]^2)^p}{\sqrt{1-\sin[e+fx]^2}} dx \\ &\rightarrow \frac{\sqrt{\cos[e+fx]^2}}{f \cos[e+fx]} \operatorname{Subst}\left[\int \frac{x^m (a+b x^2)^p}{\sqrt{1-x^2}} dx, x, \sin[e+fx]\right] \end{aligned}$$

Program code:

```
Int[sin[e_+f_*x_]^m_*(a_+b_*sin[e_+f_*x_]^2)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Sin[e+f*x],x]},
    ff^(m+1)*Sqrt[Cos[e+f*x]^2]/(f*Cos[e+f*x])*Subst[Int[x^m*(a+b*ff^2*x^2)^p/Sqrt[1-ff^2*x^2],x],x,Sin[e+f*x]/ff] /;
    FreeQ[{a,b,e,f,p},x] && IntegerQ[m/2] && Not[IntegerQ[p]]
```

$$\mathbf{2:} \int (d \sin[e+fx])^m (a+b \sin[e+fx]^2)^p dx \text{ when } m \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: } \partial_x \frac{(d \sin[e+fx])^{m-1}}{(\sin[e+fx]^2)^{\frac{m-1}{2}}} == 0$$

Basis: $\operatorname{Sin}[e+fx] F[\operatorname{Sin}[e+fx]^2] == -\frac{1}{f} \operatorname{Subst}[F[1-x^2], x, \operatorname{Cos}[e+fx]] \partial_x \operatorname{Cos}[e+fx]$

Rule: If $m \notin \mathbb{Z}$, then

$$\begin{aligned} \int (d \operatorname{Sin}[e+fx])^m (a+b \operatorname{Sin}[e+fx]^2)^p dx &\rightarrow d \int \operatorname{Sin}[e+fx] (d \operatorname{Sin}[e+fx])^{m-1} (a+b \operatorname{Sin}[e+fx]^2)^p dx \\ &\rightarrow \frac{d (d \operatorname{Sin}[e+fx])^{m-1}}{(\operatorname{Sin}[e+fx]^2)^{\frac{m-1}{2}}} \int \operatorname{Sin}[e+fx] (\operatorname{Sin}[e+fx]^2)^{\frac{m-1}{2}} (a+b \operatorname{Sin}[e+fx]^2)^p dx \\ &\rightarrow -\frac{d^{2 \operatorname{IntPart}[\frac{m-1}{2}]+1} (d \operatorname{Sin}[e+fx])^{2 \operatorname{FracPart}[\frac{m-1}{2}]}}{f (\operatorname{Sin}[e+fx]^2)^{\operatorname{FracPart}[\frac{m-1}{2}]}} \operatorname{Subst}\left[\int (1-x^2)^{\frac{m-1}{2}} (a+b-bx^2)^p dx, x, \operatorname{Cos}[e+fx]\right] \end{aligned}$$

Program code:

```
Int[(d_.*sin[e_.+f_.*x_])^m_*(a_+b_.*sin[e_.+f_.*x_]^2)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Cos[e+f*x],x]},
    -ff*d^(2*IntPart[(m-1)/2]+1)*(d*Sin[e+f*x])^(2*FracPart[(m-1)/2])/(f*(Sin[e+f*x]^2)^FracPart[(m-1)/2])*
      Subst[Int[(1-ff^2*x^2)^(m-1)/2*(a+b-b*ff^2*x^2)^p,x],x,Cos[e+f*x]/ff] /;
    FreeQ[{a,b,d,e,f,m,p},x] && Not[IntegerQ[m]]
```

$$4. \int (d \cos[e+fx])^m (a+b \sin[e+fx]^2)^p dx$$

$$1. \int \cos[e+fx]^m (a+b \sin[e+fx]^2)^p dx \text{ when } m \in \mathbb{Z}$$

$$\text{1: } \int \cos[e+fx]^m (a+b \sin[e+fx]^2)^p dx \text{ when } \frac{m-1}{2} \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If $\frac{m-1}{2} \in \mathbb{Z}$, then

$$\cos[e+fx]^m F[\sin[e+fx]] = \frac{1}{f} \operatorname{Subst} \left[(1-x^2)^{\frac{m-1}{2}} F[x], x, \sin[e+fx] \right] \partial_x \sin[e+fx]$$

Rule: If $\frac{m-1}{2} \in \mathbb{Z}$, then

$$\int \cos[e+fx]^m (a+b \sin[e+fx]^2)^p dx \rightarrow \frac{1}{f} \operatorname{Subst} \left[\int (1-x^2)^{\frac{m-1}{2}} (a+b x^2)^p dx, x, \sin[e+fx] \right]$$

Program code:

```
Int[cos[e_+f_.*x_]^m_.*(a_+b_.*sin[e_+f_.*x_]^2)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Sin[e+f*x],x]},
    ff/f*Subst[Int[(1-ff^2*x^2)^(m-1)/2*(a+b*ff^2*x^2)^p,x],x,Sin[e+f*x]/ff] /;
    FreeQ[{a,b,e,f,p},x] && IntegerQ[(m-1)/2]
```

$$2. \int \cos[e+fx]^m (a+b \sin[e+fx]^2)^p dx \text{ when } \frac{m}{2} \in \mathbb{Z}$$

$$1: \int \cos[e+fx]^m (a+b \sin[e+fx]^2)^p dx \text{ when } \frac{m}{2} \in \mathbb{Z} \wedge p \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } \cos[z]^2 == \frac{1}{1+\tan[z]^2}$$

$$\text{Basis: } \sin[z]^2 == \frac{\tan[z]^2}{1+\tan[z]^2}$$

Basis: If $\frac{m}{2} \in \mathbb{Z}$, then

$$\cos[e+fx]^m F[\sin[e+fx]^2] == \frac{1}{f} \operatorname{Subst} \left[\frac{F\left[\frac{x^2}{1+x^2}\right]}{(1+x^2)^{m/2+1}}, x, \tan[e+fx] \right] \partial_x \tan[e+fx]$$

Rule: If $\frac{m}{2} \in \mathbb{Z} \wedge p \in \mathbb{Z}$, then

$$\int \cos[e+fx]^m (a+b \sin[e+fx]^2)^p dx \rightarrow \frac{1}{f} \operatorname{Subst} \left[\int \frac{(a+(a+b)x^2)^p}{(1+x^2)^{m/2+p+1}} dx, x, \tan[e+fx] \right]$$

Program code:

```
Int[cos[e_+f_*x_]^m_*(a_+b_*sin[e_+f_*x_]^2)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    ff/f*Subst[Int[(a+(a+b)*ff^2*x^2)^p/(1+ff^2*x^2)^(m/2+p+1),x],x,Tan[e+f*x]/ff]] /;
  FreeQ[{a,b,e,f},x] && IntegerQ[m/2] && IntegerQ[p]
```

$$2: \int \cos[e+fx]^m (a+b \sin[e+fx]^2)^p dx \text{ when } \frac{m}{2} \in \mathbb{Z} \wedge p \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction and integration by substitution

Basis: $\partial_x \frac{\cos[e+fx]^{m-1}}{(\cos[e+fx]^2)^{\frac{m-1}{2}}} == 0$

Basis: If $\frac{m}{2} \in \mathbb{Z}$, then $\frac{\cos[e+fx]^{m-1}}{(\cos[e+fx]^2)^{\frac{m-1}{2}}} == \frac{\sqrt{\cos[e+fx]^2}}{\cos[e+fx]}$

Basis: $\cos[e+fx] F[\sin[e+fx]] == \frac{1}{f} \operatorname{Subst}[F[x], x, \sin[e+fx]] \partial_x \sin[e+fx]$

Rule: If $\frac{m}{2} \in \mathbb{Z} \wedge p \notin \mathbb{Z}$, then

$$\begin{aligned} \int \cos[e+fx]^m (a+b \sin[e+fx]^2)^p dx &\rightarrow \int \cos[e+fx] \cos[e+fx]^{m-1} (a+b \sin[e+fx]^2)^p dx \\ &\rightarrow \frac{\cos[e+fx]^{m-1}}{(\cos[e+fx]^2)^{\frac{m-1}{2}}} \int \cos[e+fx] (1 - \sin[e+fx]^2)^{\frac{m-1}{2}} (a+b \sin[e+fx]^2)^p dx \\ &\rightarrow \frac{\sqrt{\cos[e+fx]^2}}{f \cos[e+fx]} \operatorname{Subst}\left[\int (1-x^2)^{\frac{m-1}{2}} (a+b x^2)^p dx, x, \sin[e+fx]\right] \end{aligned}$$

Program code:

```
Int[cos[e_+f_*x_]^m*(a_+b_*sin[e_+f_*x_]^2)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Sin[e+fx],x]},
    ff*Sqrt[Cos[e+fx]^2]/(f*Cos[e+fx])*Subst[Int[(1-ff^2*x^2)^(m-1)/2*(a+b*ff^2*x^2)^p,x],x,Sin[e+fx]/ff]] /;
  FreeQ[{a,b,e,f,p},x] && IntegerQ[m/2] && Not[IntegerQ[p]]
```

2: $\int (d \cos[e+fx])^m (a+b \sin[e+fx]^2)^p dx$ when $m \notin \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

Basis: $\partial_x \frac{(d \cos[e+fx])^{m-1}}{(\cos[e+fx]^2)^{\frac{m-1}{2}}} = 0$

Basis: $\cos[e+fx] F[\sin[e+fx]] = \frac{1}{f} \operatorname{Subst}[F[x], x, \sin[e+fx]] \partial_x \sin[e+fx]$

Rule:

$$\begin{aligned} \int (d \cos[e+fx])^m (a+b \sin[e+fx]^2)^p dx &\rightarrow d \int \cos[e+fx] (d \cos[e+fx])^{m-1} (a+b \sin[e+fx]^2)^p dx \\ &\rightarrow \frac{d (d \cos[e+fx])^{m-1}}{(\cos[e+fx]^2)^{\frac{m-1}{2}}} \int \cos[e+fx] (1 - \sin[e+fx]^2)^{\frac{m-1}{2}} (a+b \sin[e+fx]^2)^p dx \\ &\rightarrow \frac{d^{2 \operatorname{IntPart}[\frac{m-1}{2}] + 1} (d \cos[e+fx])^{2 \operatorname{FracPart}[\frac{m-1}{2}]}}{f (\cos[e+fx]^2)^{\operatorname{FracPart}[\frac{m-1}{2}]}} \operatorname{Subst}\left[\int (1-x^2)^{\frac{m-1}{2}} (a+b x^2)^p dx, x, \sin[e+fx]\right] \end{aligned}$$

Program code:

```
Int[(d_.*cos[e_.+f_.*x_])^m_*(a_+b_.*sin[e_.+f_.*x_]^2)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Sin[e+f*x],x]},
    ff*d^(2*IntPart[(m-1)/2]+1)*(d*cos[e+f*x])^(2*FracPart[(m-1)/2])/(f*(Cos[e+f*x]^2)^FracPart[(m-1)/2])*
    Subst[Int[(1-ff^2*x^2)^(m-1)/2*(a+b*ff^2*x^2)^p,x],x,Sin[e+f*x]/ff] /;
    FreeQ[{a,b,d,e,f,m,p},x] && Not[IntegerQ[m]]
```

$$5. \int (d \operatorname{Tan}[e+fx])^m (a+b \operatorname{Sin}[e+fx]^2)^p dx$$

$$1: \int \operatorname{Tan}[e+fx]^m (a+b \operatorname{Sin}[e+fx]^2)^p dx \text{ when } \frac{m-1}{2} \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } \operatorname{Tan}[z]^2 == \frac{\operatorname{Sin}[z]^2}{1-\operatorname{Sin}[z]^2}$$

Basis: If $\frac{m-1}{2} \in \mathbb{Z}$, then

$$\operatorname{Tan}[e+fx]^m F[\operatorname{Sin}[e+fx]^2] == \frac{1}{2f} \operatorname{Subst}\left[\frac{x^{\frac{m-1}{2}} F[x]}{(1-x)^{\frac{m+1}{2}}}, x, \operatorname{Sin}[e+fx]^2\right] \partial_x \operatorname{Sin}[e+fx]^2$$

Rule: If $\frac{m-1}{2} \in \mathbb{Z}$, then

$$\int \operatorname{Tan}[e+fx]^m (a+b \operatorname{Sin}[e+fx]^2)^p dx \rightarrow \frac{1}{2f} \operatorname{Subst}\left[\int \frac{x^{\frac{m-1}{2}} (a+bx)^p}{(1-x)^{\frac{m+1}{2}}} dx, x, \operatorname{Sin}[e+fx]^2\right]$$

Program code:

```
Int[tan[e_+f_*x_]^m_*(a_+b_*sin[e_+f_*x_]^2)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Sin[e+f*x]^2,x]},
    ff^((m+1)/2)/(2*f)*Subst[Int[x^((m-1)/2)*(a+b*ff*x)^p/(1-ff*x)^((m+1)/2),x],x,Sin[e+f*x]^2/ff] /;
    FreeQ[{a,b,e,f,p},x] && IntegerQ[(m-1)/2]
```

$$2: \int (d \operatorname{Tan}[e+fx])^m (a+b \operatorname{Sin}[e+fx]^2)^p dx \text{ when } p \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } \operatorname{Sin}[z]^2 == \frac{\operatorname{Tan}[z]^2}{1+\operatorname{Tan}[z]^2}$$

$$\text{Basis: } (d \operatorname{Tan}[e+fx])^m F[\operatorname{Sin}[e+fx]^2] = \frac{1}{f} \operatorname{Subst}\left[\frac{(dx)^m F\left[\frac{x^2}{1+x^2}\right]}{1+x^2}, x, \operatorname{Tan}[e+fx]\right] \partial_x \operatorname{Tan}[e+fx]$$

Rule: If $p \in \mathbb{Z}$, then

$$\int (d \operatorname{Tan}[e+fx])^m (a+b \operatorname{Sin}[e+fx]^n)^p dx \rightarrow \frac{1}{f} \operatorname{Subst}\left[\int \frac{(dx)^m (a+(a+b)x^2)^p}{(1+x^2)^{p+1}} dx, x, \operatorname{Tan}[e+fx]\right]$$

Program code:

```
Int[(d_.*tan[e_.+f_.*x_])^m_*(a_+b_.*sin[e_.+f_.*x_]^2)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    ff/f*Subst[Int[(d*ff*x)^m*(a+(a+b)*ff^2*x^2)^p/(1+ff^2*x^2)^(p+1),x],x,Tan[e+f*x]/ff] /;
    FreeQ[{a,b,d,e,f,m},x] && IntegerQ[p]
```

$$\mathbf{3:} \int \operatorname{Tan}[e+fx]^m (a+b \operatorname{Sin}[e+fx]^2)^p dx \text{ when } \frac{m}{2} \in \mathbb{Z} \wedge p \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: If } \frac{m}{2} \in \mathbb{Z}, \text{ then } \operatorname{Tan}[e+fx]^m = \frac{\operatorname{Sin}[e+fx]^m}{(\operatorname{Cos}[e+fx]^2)^{m/2}}$$

$$\text{Basis: } \partial_x \frac{\sqrt{\operatorname{Cos}[e+fx]^2}}{\operatorname{Cos}[e+fx]} = 0$$

$$\text{Basis: } \operatorname{Cos}[e+fx] F[\operatorname{Sin}[e+fx]] = \frac{1}{f} \operatorname{Subst}[F[x], x, \operatorname{Sin}[e+fx]] \partial_x \operatorname{Sin}[e+fx]$$

Rule: If $\frac{m}{2} \in \mathbb{Z} \wedge p \notin \mathbb{Z}$, then

$$\begin{aligned} \int \operatorname{Tan}[e+fx]^m (a+b \operatorname{Sin}[e+fx]^2)^p dx &\rightarrow \int \frac{\operatorname{Sin}[e+fx]^m (a+b \operatorname{Sin}[e+fx]^2)^p}{(\operatorname{Cos}[e+fx]^2)^{m/2}} dx \\ &\rightarrow \frac{\sqrt{\operatorname{Cos}[e+fx]^2}}{\operatorname{Cos}[e+fx]} \int \frac{\operatorname{Cos}[e+fx] \operatorname{Sin}[e+fx]^m (a+b \operatorname{Sin}[e+fx]^2)^p}{(1-\operatorname{Sin}[e+fx]^2)^{\frac{m+1}{2}}} dx \end{aligned}$$

$$\rightarrow \frac{\sqrt{\cos[e+fx]^2}}{f \cos[e+fx]} \operatorname{Subst}\left[\int \frac{x^m (a+bx^2)^p}{(1-x^2)^{\frac{n+1}{2}}} dx, x, \sin[e+fx]\right]$$

Program code:

```
Int[tan[e_.+f_.*x_]^m_*(a_+b_.*sin[e_.+f_.*x_]^2)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Sin[e+f*x],x]},
    ff^(m+1)*Sqrt[Cos[e+f*x]^2]/(f*Cos[e+f*x])*
    Subst[Int[x^m*(a+b*ff^2*x^2)^p/(1-ff^2*x^2)^((m+1)/2),x],x,Sin[e+f*x]/ff] /;
  FreeQ[{a,b,e,f,p},x] && IntegerQ[m/2] && Not[IntegerQ[p]]
```

4: $\int (d \tan[e+fx])^m (a+b \sin[e+fx]^2)^p dx$ when $m \notin \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

Basis: If $\frac{m}{2} \in \mathbb{Z}$, then $\tan[e+fx]^m = \frac{\sin[e+fx]^m}{(\cos[e+fx]^2)^{m/2}}$

Basis: $\partial_x \frac{(d \tan[e+fx])^m (\cos[e+fx]^2)^{m/2}}{\sin[e+fx]^m} = 0$

Basis: $\cos[e+fx] F[\sin[e+fx]] = \frac{1}{f} \operatorname{Subst}[F[x], x, \sin[e+fx]] \partial_x \sin[e+fx]$

Rule: If $m \notin \mathbb{Z}$, then

$$\begin{aligned} \int (d \tan[e+fx])^m (a+b \sin[e+fx]^2)^p dx &\rightarrow \frac{(d \tan[e+fx])^m (\cos[e+fx]^2)^{m/2}}{\sin[e+fx]^m} \int \frac{\sin[e+fx]^m (a+b \sin[e+fx]^2)^p}{(\cos[e+fx]^2)^{m/2}} dx \\ &\rightarrow \frac{(d \tan[e+fx])^{m+1} (\cos[e+fx]^2)^{\frac{m+1}{2}}}{d \sin[e+fx]^{m+1}} \int \frac{\cos[e+fx] \sin[e+fx]^m (a+b \sin[e+fx]^2)^p}{(1-\sin[e+fx]^2)^{\frac{n+1}{2}}} dx \end{aligned}$$

$$\rightarrow \frac{(d \operatorname{Tan}[e+fx])^{m+1} (\operatorname{Cos}[e+fx]^2)^{\frac{m+1}{2}}}{d f \operatorname{Sin}[e+fx]^{m+1}} \operatorname{Subst}\left[\int \frac{x^m (a+bx^2)^p}{(1-x^2)^{\frac{m+1}{2}}} dx, x, \operatorname{Sin}[e+fx]\right]$$

Program code:

```
Int[(d_.*tan[e_.+f_.*x_])^m_*(a+_b_.*sin[e_.+f_.*x_]^2)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Sin[e+f*x],x]},
    ff*(d*Tan[e+f*x])^(m+1)*(Cos[e+f*x]^2)^((m+1)/2)/(d*f*Sin[e+f*x]^(m+1))*
    Subst[Int[(ff*x)^m*(a+b*ff^2*x^2)^p/(1-ff^2*x^2)^((m+1)/2),x],x,Sin[e+f*x]/ff] /;
    FreeQ[{a,b,d,e,f,m,p},x] && Not[IntegerQ[m]]
```

6. $\int (c \operatorname{Cos}[e+fx])^m (d \operatorname{Sin}[e+fx])^n (a+b \operatorname{Sin}[e+fx]^2)^p dx$

1: $\int \operatorname{Cos}[e+fx]^m (d \operatorname{Sin}[e+fx])^n (a+b \operatorname{Sin}[e+fx]^2)^p dx$ when $\frac{m-1}{2} \in \mathbb{Z}$

Derivation: Integration by substitution

Basis: If $\frac{m-1}{2} \in \mathbb{Z}$, then

$$\operatorname{Cos}[e+fx]^m F[\operatorname{Sin}[e+fx]] = \frac{1}{f} \operatorname{Subst}\left[(1-x^2)^{\frac{m-1}{2}} F[x], x, \operatorname{Sin}[e+fx]\right] \partial_x \operatorname{Sin}[e+fx]$$

Rule: If $\frac{m-1}{2} \in \mathbb{Z}$, then

$$\int \operatorname{Cos}[e+fx]^m (d \operatorname{Sin}[e+fx])^n (a+b \operatorname{Sin}[e+fx]^2)^p dx \rightarrow \frac{1}{f} \operatorname{Subst}\left[\int (dx)^n (1-x^2)^{\frac{m-1}{2}} (a+bx^2)^p dx, x, \operatorname{Sin}[e+fx]\right]$$

Program code:

```
Int[cos[e_.+f_.*x_]^m_*(d_.*sin[e_.+f_.*x_]^n_*(a+_b_.*sin[e_.+f_.*x_]^2)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Sin[e+f*x],x]},
    ff/f*Subst[Int[(d*ff*x)^n*(1-ff^2*x^2)^((m-1)/2)*(a+b*ff^2*x^2)^p,x],x,Sin[e+f*x]/ff] /;
    FreeQ[{a,b,d,e,f,n,p},x] && IntegerQ[(m-1)/2]
```

$$2: \int (c \cos[e+fx])^m \sin[e+fx]^n (a+b \sin[e+fx]^2)^p dx \text{ when } \frac{n-1}{2} \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } \sin[z]^2 == 1 - \cos[z]^2$$

Basis: If $\frac{n-1}{2} \in \mathbb{Z}$, then

$$\sin[e+fx]^n F[\sin[e+fx]^2] == -\frac{1}{f} \operatorname{Subst}\left[(1-x^2)^{\frac{n-1}{2}} F[1-x^2], x, \cos[e+fx]\right] \partial_x \cos[e+fx]$$

Rule: If $\frac{n-1}{2} \in \mathbb{Z}$, then

$$\int (c \cos[e+fx])^m \sin[e+fx]^n (a+b \sin[e+fx]^2)^p dx \rightarrow -\frac{1}{f} \operatorname{Subst}\left[\int (c x)^m (1-x^2)^{\frac{n-1}{2}} (a+b-bx^2)^p dx, x, \cos[e+fx]\right]$$

Program code:

```
Int[(c_.*sin[e_.+f_.*x_])^m_*sin[e_.+f_.*x_]^n_.*(a+b_.*sin[e_.+f_.*x_]^2)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Cos[e+f*x],x]},
    -ff/f*Subst[Int[(c*ff*x)^m*(1-ff^2*x^2)^((n-1)/2)*(a+b-b*ff^2*x^2)^p,x],x,Cos[e+f*x]/ff]] /;
  FreeQ[{a,b,c,e,f,m,p},x] && IntegerQ[(n-1)/2]
```

$$3. \int (c \cos[e+fx])^m (d \sin[e+fx])^n (a+b \sin[e+fx]^2)^p dx \text{ when } \frac{m}{2} \in \mathbb{Z}$$

$$1: \int \cos[e+fx]^m \sin[e+fx]^n (a+b \sin[e+fx]^2)^p dx \text{ when } \frac{m}{2} \in \mathbb{Z} \wedge \frac{n}{2} \in \mathbb{Z} \wedge p \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } \cos[z]^2 == \frac{1}{1+\tan[z]^2}$$

$$\text{Basis: } \sin[z]^2 == \frac{\tan[z]^2}{1+\tan[z]^2}$$

Basis: If $\frac{m}{2} \in \mathbb{Z}$, then

$$\operatorname{Cos}[e+fx]^m F[\operatorname{Sin}[e+fx]^2] == \frac{1}{f} \operatorname{Subst}\left[\frac{F\left[\frac{x^2}{1+x^2}\right]}{(1+x^2)^{m/2+1}}, x, \operatorname{Tan}[e+fx]\right] \partial_x \operatorname{Tan}[e+fx]$$

Rule: If $\frac{m}{2} \in \mathbb{Z} \wedge \frac{n}{2} \in \mathbb{Z} \wedge p \in \mathbb{Z}$, then

$$\int \operatorname{Cos}[e+fx]^m \operatorname{Sin}[e+fx]^n (a+b \operatorname{Sin}[e+fx]^2)^p dx \rightarrow \frac{1}{f} \operatorname{Subst}\left[\int \frac{x^n (a+(a+b)x^2)^p}{(1+x^2)^{(m+n)/2+p+1}} dx, x, \operatorname{Tan}[e+fx]\right]$$

Program code:

```
Int[cos[e_+f_*x_]^m*sin[e_+f_*x_]^n*(a+b_*sin[e_+f_*x_]^2)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    ff^(n+1)/f*Subst[Int[x^n*(a+(a+b)*ff^2*x^2)^p/(1+ff^2*x^2)^( (m+n)/2+p+1),x],x,Tan[e+f*x]/ff] /;
    FreeQ[{a,b,e,f},x] && IntegerQ[m/2] && IntegerQ[n/2] && IntegerQ[p]
```

$$\mathbf{2:} \int \operatorname{Cos}[e+fx]^m (d \operatorname{Sin}[e+fx])^n (a+b \operatorname{Sin}[e+fx]^2)^p dx \text{ when } \frac{m}{2} \in \mathbb{Z} \wedge \neg \left(\frac{n}{2} \in \mathbb{Z} \wedge p \in \mathbb{Z} \right)$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: } \partial_x \frac{\operatorname{Cos}[e+fx]^{m-1}}{(\operatorname{Cos}[e+fx]^2)^{\frac{m-1}{2}}} == 0$$

$$\text{Basis: If } \frac{m}{2} \in \mathbb{Z}, \text{ then } \frac{\operatorname{Cos}[e+fx]^{m-1}}{(\operatorname{Cos}[e+fx]^2)^{\frac{m-1}{2}}} == \frac{\sqrt{\operatorname{Cos}[e+fx]^2}}{\operatorname{Cos}[e+fx]}$$

$$\text{Basis: } \operatorname{Cos}[e+fx] F[\operatorname{Sin}[e+fx]] == \frac{1}{f} \operatorname{Subst}[F[x], x, \operatorname{Sin}[e+fx]] \partial_x \operatorname{Sin}[e+fx]$$

Rule: If $\frac{m}{2} \in \mathbb{Z}$, then

$$\int \operatorname{Cos}[e+fx]^m (d \operatorname{Sin}[e+fx])^n (a+b \operatorname{Sin}[e+fx]^2)^p dx \rightarrow \int \operatorname{Cos}[e+fx] \operatorname{Cos}[e+fx]^{m-1} (d \operatorname{Sin}[e+fx])^n (a+b \operatorname{Sin}[e+fx]^2)^p dx$$

$$\rightarrow \frac{\cos[e+fx]^{m-1}}{(\cos[e+fx]^2)^{\frac{m-1}{2}}} \int \cos[e+fx] (1 - \sin[e+fx]^2)^{\frac{m-1}{2}} (d \sin[e+fx])^n (a+b \sin[e+fx]^2)^p dx$$

$$\rightarrow \frac{\sqrt{\cos[e+fx]^2}}{f \cos[e+fx]} \operatorname{Subst}\left[\int (dx)^n (1-x^2)^{\frac{m-1}{2}} (a+bx^2)^p dx, x, \sin[e+fx]\right]$$

Program code:

```
Int[cos[e_.+f_.*x_]^m_*(d_.*sin[e_.+f_.*x_]^n_*(a_+b_.*sin[e_.+f_.*x_]^2)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Sin[e+f*x],x]},
    ff*Sqrt[Cos[e+f*x]^2]/(f*Cos[e+f*x])*Subst[Int[(d*ff*x)^n*(1-ff^2*x^2)^((m-1)/2)*(a+b*ff^2*x^2)^p,x],x,Sin[e+f*x]/ff] /;
    FreeQ[{a,b,d,e,f,n,p},x] && IntegerQ[m/2]
```

4: $\int (c \cos[e+fx])^m (d \sin[e+fx])^n (a+b \sin[e+fx]^2)^p dx$ when $m \notin \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

Basis: $\partial_x \frac{(c \cos[e+fx])^{m-1}}{(\cos[e+fx]^2)^{\frac{m-1}{2}}} = 0$

Basis: $\cos[e+fx] F[\sin[e+fx]] = \frac{1}{f} \operatorname{Subst}[F[x], x, \sin[e+fx]] \partial_x \sin[e+fx]$

Rule:

$$\begin{aligned} \int (c \cos[e+fx])^m (d \sin[e+fx])^n (a+b \sin[e+fx]^2)^p dx &\rightarrow c \int \cos[e+fx] (c \cos[e+fx])^{m-1} (d \sin[e+fx])^n (a+b \sin[e+fx]^2)^p dx \\ &\rightarrow \frac{c (c \cos[e+fx])^{m-1}}{(\cos[e+fx]^2)^{\frac{m-1}{2}}} \int \cos[e+fx] (1 - \sin[e+fx]^2)^{\frac{m-1}{2}} (d \sin[e+fx])^n (a+b \sin[e+fx]^2)^p dx \\ &\rightarrow \frac{c^{2 \operatorname{IntPart}[\frac{m-1}{2}] + 1} (c \cos[e+fx])^{2 \operatorname{FracPart}[\frac{m-1}{2}]}}{f (\cos[e+fx]^2)^{\operatorname{FracPart}[\frac{m-1}{2}]}} \operatorname{Subst}\left[\int (dx)^n (1-x^2)^{\frac{m-1}{2}} (a+b x^2)^p dx, x, \sin[e+fx]\right] \end{aligned}$$

Program code:

```
Int[(c_.cos[e_.+f_.x_])^m_*(d_.sin[e_.+f_.x_])^n_*(a_+b_.sin[e_.+f_.x_]^2)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Sin[e+f*x],x]},
    ff*c^(2*IntPart[(m-1)/2]+1)*(c*cos[e+f*x])^(2*FracPart[(m-1)/2])/(f*(Cos[e+f*x]^2)^FracPart[(m-1)/2])*
    Subst[Int[(d*ff*x)^n*(1-ff^2*x^2)^( (m-1)/2)*(a+b*ff^2*x^2)^p,x,Sin[e+f*x]/ff]] /;
    FreeQ[{a,b,c,d,e,f,m,n,p},x] && Not[IntegerQ[m]]
```

Rules for integrands of the form $(d \operatorname{Trig}[e + f x])^m (a + b (c \operatorname{Sin}[e + f x])^n)^p$

$$1. \int (d \operatorname{Trig}[e + f x])^m (b (c \operatorname{Sin}[e + f x])^n)^p dx \text{ when } p \notin \mathbb{Z}$$

$$1. \int (b \operatorname{Sin}[e + f x]^2)^p dx \text{ when } p \notin \mathbb{Z}$$

$$\mathbf{1:} \int (b \operatorname{Sin}[e + f x]^2)^p dx \text{ when } p \notin \mathbb{Z} \wedge p > 1$$

Rule: If $p \notin \mathbb{Z} \wedge p > 1$, then

$$\int (b \operatorname{Sin}[e + f x]^2)^p dx \rightarrow -\frac{\operatorname{Cot}[e + f x] (b \operatorname{Sin}[e + f x]^2)^p}{2 f p} + \frac{b (2 p - 1)}{2 p} \int (b \operatorname{Sin}[e + f x]^2)^{p-1} dx$$

Program code:

```
Int[(b_.*sin[e_.+f_.*x_]^2)^p_,x_Symbol] :=
  -Cot[e+f*x]*(b*Sin[e+f*x]^2)^p/(2*f*p) +
  b*(2*p-1)/(2*p)*Int[(b*Sin[e+f*x]^2)^(p-1),x] /;
FreeQ[{b,e,f},x] && Not[IntegerQ[p]] && GtQ[p,1]
```

2: $\int (b \sin[e+fx]^2)^p dx$ when $p \notin \mathbb{Z} \wedge p < -1$

Rule: If $p \notin \mathbb{Z} \wedge p < -1$, then

$$\int (b \sin[e+fx]^2)^p dx \rightarrow \frac{\cot[e+fx] (b \sin[e+fx]^2)^{p+1}}{b f (2p+1)} + \frac{2(p+1)}{b(2p+1)} \int (b \sin[e+fx]^2)^{p+1} dx$$

Program code:

```
Int[(b_.*sin[e_.+f_.**x_]^2)^p_,x_Symbol] :=
  Cot[e+f*x]*(b*sin[e+f*x]^2)^(p+1)/(b*f*(2*p+1)) +
  2*(p+1)/(b*(2*p+1))*Int[(b*sin[e+f*x]^2)^(p+1),x] /;
FreeQ[{b,e,f},x] && Not[IntegerQ[p]] && LtQ[p,-1]
```


$$2. \int \operatorname{Tan}[e+fx]^m (b(c \sin[e+fx])^n)^p dx \text{ when } \frac{m-1}{2} \in \mathbb{Z}$$

$$1: \int \operatorname{Tan}[e+fx]^m (b \sin[e+fx]^n)^p dx \text{ when } \frac{m-1}{2} \in \mathbb{Z} \wedge \frac{n}{2} \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } \operatorname{Tan}[z]^2 == \frac{\sin[z]^2}{1-\sin[z]^2}$$

Basis: If $\frac{m-1}{2} \in \mathbb{Z}$, then

$$\operatorname{Tan}[e+fx]^m F[\sin[e+fx]^2] == \frac{1}{2f} \operatorname{Subst}\left[\frac{x^{\frac{m-1}{2}} F[x]}{(1-x)^{\frac{m+1}{2}}}, x, \sin[e+fx]^2\right] \partial_x \sin[e+fx]^2$$

Rule: If $\frac{m-1}{2} \in \mathbb{Z} \wedge \frac{n}{2} \in \mathbb{Z}$, then

$$\int \operatorname{Tan}[e+fx]^m (b \sin[e+fx]^n)^p dx \rightarrow \frac{1}{2f} \operatorname{Subst}\left[\int \frac{x^{\frac{m-1}{2}} (b x^{n/2})^p}{(1-x)^{\frac{m+1}{2}}} dx, x, \sin[e+fx]^2\right]$$

Program code:

```
Int[tan[e_+f_*x_]^m_*(b_*sin[e_+f_*x_]^n_)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Sin[e+f*x]^2,x]},
    ff^( (m+1)/2)/(2*f)*Subst[Int[x^((m-1)/2)*(b*ff^(n/2)*x^(n/2))^p/(1-ff*x)^((m+1)/2),x],x,Sin[e+f*x]^2/ff] /;
    FreeQ[{b,e,f,p},x] && IntegerQ[(m-1)/2] && IntegerQ[n/2]
```

$$2: \int \operatorname{Tan}[e+fx]^m (b(c \sin[e+fx])^n)^p dx \text{ when } \frac{m-1}{2} \in \mathbb{Z}^-$$

Derivation: Integration by substitution

$$\text{Basis: } \operatorname{Tan}[z]^2 == \frac{\sin[z]^2}{1-\sin[z]^2}$$

Basis: If $\frac{m-1}{2} \in \mathbb{Z}$, then

$$\operatorname{Tan}[e+fx]^m F[\operatorname{Sin}[e+fx]] = \frac{1}{f} \operatorname{Subst}\left[\frac{x^m F[x]}{(1-x^2)^{\frac{m+1}{2}}}, x, \operatorname{Sin}[e+fx]\right] \partial_x \operatorname{Sin}[e+fx]$$

Rule: If $\frac{m+1}{2} \in \mathbb{Z}^-$, then

$$\int \operatorname{Tan}[e+fx]^m (b(c \operatorname{Sin}[e+fx])^n)^p dx \rightarrow \frac{1}{f} \operatorname{Subst}\left[\int \frac{x^m (b(c x)^n)^p}{(1-x^2)^{\frac{m+1}{2}}} dx, x, \operatorname{Sin}[e+fx]\right]$$

Program code:

```
Int[tan[e_+f_*x_]^m_.*(b_.*(c_.*sin[e_+f_*x_]^n_)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Sin[e+f*x],x]},
    ff^(m+1)/f*Subst[Int[x^m*(b*(c*ff*x)^n)^p/(1-ff^2*x^2)^((m+1)/2),x],x,Sin[e+f*x]/ff]] /;
  FreeQ[{b,c,e,f,n,p},x] && ILtQ[(m-1)/2,0]
```

3: $\int u (b \sin[e+fx]^n)^p dx$ when $p \notin \mathbb{Z} \wedge n \in \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(b \sin[e+fx]^n)^p}{\sin[e+fx]^{np}} = 0$

Rule: If $p \notin \mathbb{Z} \wedge n \in \mathbb{Z}$, then

$$\int u (b \sin[e+fx]^n)^p dx \rightarrow \frac{b^{\operatorname{IntPart}[p]} (b \sin[e+fx]^n)^{\operatorname{FracPart}[p]}}{\sin[e+fx]^{n \operatorname{FracPart}[p]}} \int u \sin[e+fx]^{np} dx$$

Program code:

```
Int[u.*(b.*sin[e.+f.*x_]^n_)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Sin[e+f*x],x]},
    (b*ff^n)^IntPart[p]*(b*Ssin[e+f*x]^n)^FracPart[p]/(Sin[e+f*x]/ff)^(n*FracPart[p])*
    Int[ActivateTrig[u]*(Sin[e+f*x]/ff)^(n*p),x] /;
  FreeQ[{b,e,f,n,p},x] && Not[IntegerQ[p]] && IntegerQ[n] &&
    (EqQ[u,1] || MatchQ[u,(d_.*trig_[e+f*x])^m_. /; FreeQ[{d,m},x] && MemberQ[{sin,cos,tan,cot,sec,csc},trig]))
```

4: $\int u (b (c \sin[e+fx])^n)^p dx$ when $p \notin \mathbb{Z} \wedge n \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(b (c \sin[e+fx])^n)^p}{(c \sin[e+fx])^{np}} == 0$

Rule: If $p \notin \mathbb{Z} \wedge n \notin \mathbb{Z}$, then

$$\int u (b (c \sin[e+fx])^n)^p dx \rightarrow \frac{b^{\operatorname{IntPart}[p]} (b (c \sin[e+fx])^n)^{\operatorname{FracPart}[p]}}{(c \sin[e+fx])^{n \operatorname{FracPart}[p]}} \int u (c \sin[e+fx])^{np} dx$$

Program code:

```
Int[u_.*(b_.*(c_.*sin[e_+f_*x_])^n_)^p_,x_Symbol] :=
  b^IntPart[p]*(b*(c*Sin[e+f*x])^n)^FracPart[p]/(c*Sin[e+f*x])^(n*FracPart[p])*
  Int[ActivateTrig[u]*(c*Sin[e+f*x])^(n*p),x] /;
FreeQ[{b,c,e,f,n,p},x] && Not[IntegerQ[p]] && Not[IntegerQ[n]] &&
(EqQ[u,1] || MatchQ[u,(d_.*trig_[e+f*x])^m_./; FreeQ[{d,m},x] && MemberQ[{sin,cos,tan,cot,sec,csc},trig]))
```

$$2. \int (a + b (c \sin[e + f x])^n)^p dx$$

$$1. \int (a + b \sin[e + f x]^4)^p dx$$

$$\mathbf{x:} \int (a + b \sin[e + f x]^4)^p dx \text{ when } p \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } \sin[z]^2 == \frac{1}{1 + \cot[z]^2}$$

$$\text{Basis: } F[\sin[e + f x]^2] == -\frac{1}{f} \operatorname{Subst}\left[\frac{F\left[\frac{1}{1+x^2}\right]}{1+x^2}, x, \cot[e + f x]\right] \partial_x \cot[e + f x]$$

Rule: If $p \in \mathbb{Z}$, then

$$\int (a + b \sin[e + f x]^4)^p dx \rightarrow -\frac{1}{f} \operatorname{Subst}\left[\int \frac{(a + b + 2 a x^2 + a x^4)^p}{(1 + x^2)^{2p+1}} dx, x, \cot[e + f x]\right]$$

Program code:

```
(* Int[(a_+b_.*sin[e_+f_.*x_]^4)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    -ff/f*Subst[Int[(a+b+2*a*ff^2*x^2+a*ff^4*x^4)^p/(1+ff^2*x^2)^(2*p+1),x],x,Cot[e+f*x]/ff]] /;
  FreeQ[{a,b,e,f},x] && IntegerQ[p] *)
```

$$\mathbf{1:} \int (a + b \sin[e + f x]^4)^p dx \text{ when } p \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } \sin[z]^2 == \frac{\tan[z]^2}{1 + \tan[z]^2}$$

$$\text{Basis: } F[\operatorname{Sin}[e+fx]^2] = \frac{1}{f} \operatorname{Subst}\left[\frac{F\left[\frac{x^2}{1+x^2}\right]}{1+x^2}, x, \operatorname{Tan}[e+fx]\right] \partial_x \operatorname{Tan}[e+fx]$$

Rule: If $p \in \mathbb{Z}$, then

$$\int (a+b \operatorname{Sin}[e+fx]^4)^p dx \rightarrow \frac{1}{f} \operatorname{Subst}\left[\int \frac{(a+2ax^2+(a+b)x^4)^p}{(1+x^2)^{2p+1}} dx, x, \operatorname{Tan}[e+fx]\right]$$

Program code:

```
Int[(a+b_.sin[e_.+f_.*x_]^4)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    ff/f*Subst[Int[(a+2*a*ff^2*x^2+(a+b)*ff^4*x^4)^p/(1+ff^2*x^2)^(2*p+1),x],x,Tan[e+f*x]/ff] /;
    FreeQ[{a,b,e,f},x] && IntegerQ[p]
```

$$\mathbf{2:} \int (a + b \sin[e + f x]^4)^p dx \text{ when } p - \frac{1}{2} \in \mathbb{Z}$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: } a + b \sin[z]^4 == \frac{a+2a \tan[z]^2 + (a+b) \tan[z]^4}{\sec[z]^4}$$

$$\text{Basis: } \partial_x \frac{(a+b \sin[e+fx]^4)^p (\sec[e+fx]^2)^{2p}}{(a+2a \tan[e+fx]^2 + (a+b) \tan[e+fx]^4)^p} == 0$$

$$\text{Basis: } F[\tan[e + f x]] == \frac{1}{f} \operatorname{Subst}\left[\frac{F\left[\frac{x}{1+x^2}\right]}{1+x^2}, x, \tan[e + f x]\right] \partial_x \tan[e + f x]$$

Rule: If $p - \frac{1}{2} \in \mathbb{Z}$, then

$$\begin{aligned} \int (a + b \sin[e + f x]^4)^p dx &\rightarrow \frac{(a + b \sin[e + f x]^4)^p (\sec[e + f x]^2)^{2p}}{(a + 2a \tan[e + f x]^2 + (a + b) \tan[e + f x]^4)^p} \int \frac{(a + 2a \tan[e + f x]^2 + (a + b) \tan[e + f x]^4)^p}{(1 + \tan[e + f x]^2)^{2p}} dx \\ &\rightarrow \frac{(a + b \sin[e + f x]^4)^p (\sec[e + f x]^2)^{2p}}{f (a + 2a \tan[e + f x]^2 + (a + b) \tan[e + f x]^4)^p} \operatorname{Subst}\left[\int \frac{(a + 2a x^2 + (a + b) x^4)^p}{(1 + x^2)^{2p+1}} dx, x, \tan[e + f x]\right] \end{aligned}$$

Program code:

```
Int[(a+b_.sin[e_.+f_.x_]^4)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    ff*(a+b*sin[e+f*x]^4)^p*(Sec[e+f*x]^2)^(2*p)/(f*(a+2*a*Tan[e+f*x]^2+(a+b)*Tan[e+f*x]^4)^p)*
    Subst[Int[(a+2*a*ff^2*x^2+(a+b)*ff^4*x^4)^p/(1+ff^2*x^2)^(2*p+1),x],x,Tan[e+f*x]/ff] /;
    FreeQ[{a,b,e,f,p},x] && IntegerQ[p-1/2]
```

2: $\int \frac{1}{a+b \sin[e+fx]^n} dx$ when $\frac{n}{2} \in \mathbb{Z}$

Derivation: Algebraic expansion

Basis: If $\frac{n}{2} \in \mathbb{Z}^+$, then $\frac{1}{a+b z^n} = \frac{2}{a n} \sum_{k=1}^{n/2} \frac{1}{1 - (-1)^{-4k/n} \left(-\frac{a}{b}\right)^{-2/n} z^2}$

Rule: If $\frac{n}{2} \in \mathbb{Z}$, then

$$\int \frac{1}{a+b \sin[e+fx]^n} dx \rightarrow \frac{2}{a n} \sum_{k=1}^{n/2} \int \frac{1}{1 - (-1)^{-4k/n} \left(-\frac{a}{b}\right)^{-2/n} \sin[e+fx]^2} dx$$

Program code:

```
Int[1/(a+b_.sin[e_.+f_.x_]^n_),x_Symbol] :=
  Module[{k},
    Dist[2/(a*n),Sum[Int[1/(1-Sin[e+f*x]^2/((-1)^(4*k/n)*Rt[-a/b,n/2]))],x],{k,1,n/2}],x] /;
  FreeQ[{a,b,e,f},x] && IntegerQ[n/2]
```


x: $\int (a + b \sin[e + f x]^n)^p dx$ when $\frac{n}{2} \in \mathbb{Z} \wedge p \in \mathbb{Z}^+$

Derivation: Integration by substitution

Basis: $F[\sin[e + f x]^2] = -\frac{1}{f} \operatorname{Subst}\left[\frac{F\left[\frac{1}{1+x^2}\right]}{1+x^2}, x, \cot[e + f x]\right] \partial_x \cot[e + f x]$

Rule: If $\frac{n}{2} \in \mathbb{Z} \wedge p \in \mathbb{Z}^+$, then

$$\int (a + b \sin[e + f x]^n)^p dx \rightarrow -\frac{1}{f} \operatorname{Subst}\left[\int \frac{(b + a(1+x^2)^{n/2})^p}{(1+x^2)^{np/2+1}} dx, x, \cot[e + f x]\right]$$

Program code:

```
(* Int[(a_+b_.*sin[e_+f_*x_]^n_)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    -ff/f*Subst[Int[(b+a*(1+ff^2*x^2)^(n/2))^p/(1+ff^2*x^2)^(n*p/2+1),x],x,Cot[e+f*x]/ff]] /;
  FreeQ[{a,b,e,f},x] && IntegerQ[n/2] && IGtQ[p,0] *)
```

3: $\int (a + b \sin[e + f x]^n)^p dx$ when $\frac{n}{2} \in \mathbb{Z} \wedge p \in \mathbb{Z}^+$

Derivation: Integration by substitution

Basis: $F[\sin[e + f x]^2] = \frac{1}{f} \operatorname{Subst}\left[\frac{F\left[\frac{x^2}{1+x^2}\right]}{1+x^2}, x, \tan[e + f x]\right] \partial_x \tan[e + f x]$

Rule: If $\frac{n}{2} \in \mathbb{Z} \wedge p \in \mathbb{Z}^+$, then

$$\int (a + b \sin[e + f x]^n)^p dx \rightarrow \frac{1}{f} \operatorname{Subst}\left[\int \frac{(b x^n + a (1 + x^2)^{n/2})^p}{(1 + x^2)^{n p/2 + 1}} dx, x, \tan[e + f x]\right]$$

Program code:

```
Int[(a+b_.sin[e_.+f_.x_]^n_)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    ff/f*Subst[Int[(b*ff^n*x^n+a*(1+ff^2*x^2)^(n/2))^p/(1+ff^2*x^2)^(n*p/2+1),x],x,Tan[e+f*x]/ff] /;
    FreeQ[{a,b,e,f},x] && IntegerQ[n/2] && IGtQ[p,0]
```

4: $\int (a + b (c \sin[e + f x])^n)^p dx$ when $p \in \mathbb{Z}^+ \vee (p = -1 \wedge n \in \mathbb{Z})$

Derivation: Algebraic expansion

Rule: If $p \in \mathbb{Z}^+ \vee (p = -1 \wedge n \in \mathbb{Z})$, then

$$\int (a + b (c \sin[e + f x])^n)^p dx \rightarrow \int \operatorname{ExpandTrig}[(a + b (c \sin[e + f x])^n)^p, x] dx$$

Program code:

```
Int[(a_+b_.*(c_.*sin[e_+f_.*x_])^n_)^p_,x_Symbol] :=
  Int[ExpandTrig[(a+b*(c*sin[e+f*x])^n)^p,x],x] /;
FreeQ[{a,b,c,e,f,n},x] && (IGtQ[p,0] || EqQ[p,-1] && IntegerQ[n])
```

U: $\int (a + b (c \sin[e + f x])^n)^p dx$

Rule:

$$\int (a + b (c \sin[e + f x])^n)^p dx \rightarrow \int (a + b (c \sin[e + f x])^n)^p dx$$

Program code:

```
Int[(a_+b_.*(c_.*sin[e_+f_.*x_])^n_)^p_,x_Symbol] :=
  Unintegrable[(a+b*(c*sin[e+f*x])^n)^p,x] /;
FreeQ[{a,b,c,e,f,n,p},x]
```

$$3. \int (d \sin[e+fx])^m (a+b(c \sin[e+fx])^n)^p dx$$

$$1: \int \sin[e+fx]^m (a+b \sin[e+fx]^n)^p dx \text{ when } \frac{m-1}{2} \in \mathbb{Z} \wedge \frac{n}{2} \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } \sin[z]^2 = 1 - \cos[z]^2$$

Basis: If $\frac{m-1}{2} \in \mathbb{Z}$, then

$$\sin[e+fx]^m F[\sin[e+fx]^2] = -\frac{1}{f} \operatorname{Subst}\left[(1-x^2)^{\frac{m-1}{2}} F[1-x^2], x, \cos[e+fx]\right] \partial_x \cos[e+fx]$$

Rule: If $\frac{m-1}{2} \in \mathbb{Z} \wedge \frac{n}{2} \in \mathbb{Z}$, then

$$\int \sin[e+fx]^m (a+b \sin[e+fx]^n)^p dx \rightarrow -\frac{1}{f} \operatorname{Subst}\left[\int (1-x^2)^{\frac{m-1}{2}} (a+b(1-x^2)^{n/2})^p dx, x, \cos[e+fx]\right]$$

Program code:

```
Int[sin[e_+f_.*x_]^m_.*(a_+b_.*sin[e_+f_.*x_]^4)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Cos[e+f*x],x]},
    -ff/f*Subst[Int[(1-ff^2*x^2)^(m-1)/2*(a+b-2*b*ff^2*x^2+b*ff^4*x^4)^p,x],x,Cos[e+f*x]/ff]] /;
  FreeQ[{a,b,e,f,p},x] && IntegerQ[(m-1)/2]
```

```
Int[sin[e_+f_.*x_]^m_.*(a_+b_.*sin[e_+f_.*x_]^n_)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Cos[e+f*x],x]},
    -ff/f*Subst[Int[(1-ff^2*x^2)^(m-1)/2*(a+b*(1-ff^2*x^2)^(n/2))^p,x],x,Cos[e+f*x]/ff]] /;
  FreeQ[{a,b,e,f,p},x] && IntegerQ[(m-1)/2] && IntegerQ[n/2]
```

2: $\int \sin[e+fx]^m (a+b \sin[e+fx]^n)^p dx$ when $\frac{m}{2} \in \mathbb{Z} \wedge \frac{n}{2} \in \mathbb{Z} \wedge p \in \mathbb{Z}$

Derivation: Integration by substitution

Basis: $\sin[z]^2 == \frac{\tan[z]^2}{1+\tan[z]^2}$

Basis: If $\frac{m}{2} \in \mathbb{Z}$, then

$$\sin[e+fx]^m F[\sin[e+fx]^2] == \frac{1}{f} \operatorname{Subst}\left[\frac{x^m F\left[\frac{x^2}{1+x^2}\right]}{(1+x^2)^{m/2+1}}, x, \tan[e+fx]\right] \partial_x \tan[e+fx]$$

Rule: If $\frac{m}{2} \in \mathbb{Z} \wedge \frac{n}{2} \in \mathbb{Z} \wedge p \in \mathbb{Z}$, then

$$\int \sin[e+fx]^m (a+b \sin[e+fx]^n)^p dx \rightarrow \frac{1}{f} \operatorname{Subst}\left[\int \frac{x^m (a(1+x^2)^{n/2} + b x^n)^p}{(1+x^2)^{m/2+n p/2+1}} dx, x, \tan[e+fx]\right]$$

Program code:

```
Int[sin[e_+f_*x_]^m_*(a_+b_*sin[e_+f_*x_]^4)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    ff^(m+1)/f*Subst[Int[x^m*(a+2*a*ff^2*x^2+(a+b)*ff^4*x^4)^p/(1+ff^2*x^2)^(m/2+2*p+1),x],x,Tan[e+f*x]/ff]] /;
  FreeQ[{a,b,e,f},x] && IntegerQ[m/2] && IntegerQ[p]
```

```
Int[sin[e_+f_*x_]^m_*(a_+b_*sin[e_+f_*x_]^n_)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    ff^(m+1)/f*Subst[Int[x^m*(a*(1+ff^2*x^2)^(n/2)+b*ff^n*x^n)^p/(1+ff^2*x^2)^(m/2+n*p/2+1),x],x,Tan[e+f*x]/ff]] /;
  FreeQ[{a,b,e,f},x] && IntegerQ[m/2] && IntegerQ[n/2] && IntegerQ[p]
```

3: $\int \sin[e+fx]^m (a+b \sin[e+fx]^4)^p dx$ when $\frac{m}{2} \in \mathbb{Z} \wedge p - \frac{1}{2} \in \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

Basis: If $\frac{m}{2} \in \mathbb{Z}$, then $\operatorname{Sin}[z]^m == \frac{\operatorname{Tan}[z]^m}{(1+\operatorname{Tan}[z]^2)^{m/2}}$

Basis: If $\frac{n}{2} \in \mathbb{Z}$, then $a + b \operatorname{Sin}[z]^n == \frac{a \operatorname{Sec}[z]^n + b \operatorname{Tan}[z]^n}{\operatorname{Sec}[z]^n} == \frac{a (1+\operatorname{Tan}[z]^2)^{n/2} + b \operatorname{Tan}[z]^n}{(1+\operatorname{Tan}[z]^2)^{n/2}}$

Basis: If $\frac{n}{2} \in \mathbb{Z}$, then $\partial_x \frac{(a+b \operatorname{Sin}[e+fx]^n)^p (\operatorname{Sec}[e+fx]^2)^{np/2}}{(a \operatorname{Sec}[e+fx]^n + b \operatorname{Tan}[e+fx]^n)^p} == 0$

Basis: $F[\operatorname{Tan}[e+fx]] == \frac{1}{f} \operatorname{Subst}\left[\frac{F[x]}{1+x^2}, x, \operatorname{Tan}[e+fx]\right] \partial_x \operatorname{Tan}[e+fx]$

Rule: If $\frac{m}{2} \in \mathbb{Z} \wedge p - \frac{1}{2} \in \mathbb{Z}$, then

$$\int \operatorname{Sin}[e+fx]^m (a+b \operatorname{Sin}[e+fx]^4)^p dx \rightarrow \frac{(a+b \operatorname{Sin}[e+fx]^4)^p (\operatorname{Sec}[e+fx]^2)^{2p}}{(a \operatorname{Sec}[e+fx]^4 + b \operatorname{Tan}[e+fx]^4)^p} \int \frac{\operatorname{Tan}[e+fx]^m (a (1+\operatorname{Tan}[e+fx]^2)^2 + b \operatorname{Tan}[e+fx]^4)^p}{(1+\operatorname{Tan}[e+fx]^2)^{m/2+2p}} dx$$

$$\rightarrow \frac{(a+b \operatorname{Sin}[e+fx]^4)^p (\operatorname{Sec}[e+fx]^2)^{2p}}{f (a \operatorname{Sec}[e+fx]^4 + b \operatorname{Tan}[e+fx]^4)^p} \operatorname{Subst}\left[\int \frac{x^m (a (1+x^2)^2 + b x^4)^p}{(1+x^2)^{m/2+2p+1}} dx, x, \operatorname{Tan}[e+fx]\right]$$

Program code:

```
Int[sin[e_+f_*x_]^m_*(a_+b_*sin[e_+f_*x_]^4)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    ff^(m+1)*(a+b*Sin[e+f*x]^4)^p*(Sec[e+f*x]^2)^(2*p)/(f*Apart[a*(1+Tan[e+f*x]^2)^2+b*Tan[e+f*x]^4]^p)*
    Subst[Int[x^m*ExpandToSum[a*(1+ff^2*x^2)^2+b*ff^4*x^4,x]^p/(1+ff^2*x^2)^(m/2+2*p+1),x],x,Tan[e+f*x]/ff] /;
    FreeQ[{a,b,e,f,p},x] && IntegerQ[m/2] && IntegerQ[p-1/2]
```

4: $\int \sin[e+fx]^m (a+b \sin[e+fx]^n)^p dx$ when $(m \mid p) \in \mathbb{Z} \wedge (n = 4 \vee p > 0 \vee p = -1 \wedge n \in \mathbb{Z})$

Derivation: Algebraic expansion

Rule: If $(m \mid p) \in \mathbb{Z} \wedge (n = 4 \vee p > 0 \vee p = -1 \wedge n \in \mathbb{Z})$, then

$$\int \sin[e+fx]^m (a+b \sin[e+fx]^n)^p dx \rightarrow \int \text{ExpandTrig}[\sin[e+fx]^m (a+b \sin[e+fx]^n)^p, x] dx$$

Program code:

```
Int[sin[e_.+f_.*x_]^m_.*(a_+b_.*sin[e_.+f_.*x_]^n_)^p_.,x_Symbol] :=
  Int[ExpandTrig[sin[e+f*x]^m*(a+b*sin[e+f*x]^n)^p,x],x] /;
FreeQ[{a,b,e,f},x] && IntegersQ[m,p] && (EqQ[n,4] || GtQ[p,0] || EqQ[p,-1] && IntegerQ[n])
```

5: $\int (d \sin[e+fx])^m (a+b(c \sin[e+fx])^n)^p dx$ when $p \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $p \in \mathbb{Z}^+$, then

$$\int (d \sin[e+fx])^m (a+b(c \sin[e+fx])^n)^p dx \rightarrow \int \text{ExpandTrig}[(d \sin[e+fx])^m (a+b(c \sin[e+fx])^n)^p, x] dx$$

Program code:

```
Int[(d_.*sin[e_.+f_.*x_])^m_.*(a_+b_.*(c_.*sin[e_.+f_.*x_]^n_)^p_.,x_Symbol] :=
  Int[ExpandTrig[(d*sin[e+f*x])^m*(a+b*(c*sin[e+f*x])^n)^p,x],x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && IGtQ[p,0]
```

$$\mathbf{U:} \int (d \sin[e+fx])^m (a+b(c \sin[e+fx])^n)^p dx$$

Rule:

$$\int (d \sin[e+fx])^m (a+b(c \sin[e+fx])^n)^p dx \rightarrow \int (d \sin[e+fx])^m (a+b(c \sin[e+fx])^n)^p dx$$

Program code:

```
Int[(d_.*sin[e_.+f_.*x_])^m_.*(a+_b_.*(c_.*sin[e_.+f_.*x_])^n_)^p_.,x_Symbol] :=
  Unintegrable[(d*Sin[e+f*x])^m*(a+b*(c*Sin[e+f*x])^n)^p,x] /;
  FreeQ[{a,b,c,d,e,f,m,n,p},x]
```

$$4. \int (d \cos[e+fx])^m (a+b(c \sin[e+fx])^n)^p dx$$

$$\mathbf{1:} \int \cos[e+fx]^m (a+b(c \sin[e+fx])^n)^p dx \text{ when } \frac{m-1}{2} \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If $\frac{m-1}{2} \in \mathbb{Z}$, then

$$\cos[e+fx]^m F[\sin[e+fx]] = \frac{1}{f} \operatorname{Subst} \left[(1-x^2)^{\frac{m-1}{2}} F[x], x, \sin[e+fx] \right] \partial_x \sin[e+fx]$$

Rule: If $\frac{m-1}{2} \in \mathbb{Z}$, then

$$\int \cos[e+fx]^m (a+b(c \sin[e+fx])^n)^p dx \rightarrow \frac{1}{f} \operatorname{Subst} \left[\int (1-x^2)^{\frac{m-1}{2}} (a+b(c x)^n)^p dx, x, \sin[e+fx] \right]$$

Program code:

```
Int[cos[e_.+f_.*x_]^m_.*(a+_b_.*(c_.*sin[e_.+f_.*x_])^n_)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Sin[e+f*x],x]},
    ff/f*Subst[Int[(1-ff^2*x^2)^(m-1)/2*(a+b*(c*ff*x)^n)^p,x],x,Sin[e+f*x]/ff] /;
    FreeQ[{a,b,c,e,f,n,p},x] && IntegerQ[(m-1)/2] && (EqQ[n,4] || GtQ[m,0] || IGtQ[p,0] || IntegersQ[m,p])
```


2: $\int \cos[e+fx]^m (a+b \sin[e+fx]^n)^p dx$ when $\frac{m}{2} \in \mathbb{Z} \wedge \frac{n}{2} \in \mathbb{Z} \wedge p \in \mathbb{Z}$

Derivation: Integration by substitution

Basis: $\cos[z]^2 == \frac{1}{1+\tan[z]^2}$

Basis: $\sin[z]^2 == \frac{\tan[z]^2}{1+\tan[z]^2}$

Basis: If $\frac{m}{2} \in \mathbb{Z}$, then

$\cos[e+fx]^m F[\sin[e+fx]^2] == \frac{1}{f} \operatorname{Subst}\left[\frac{F\left[\frac{x^2}{1+x^2}\right]}{(1+x^2)^{m/2+1}}, x, \tan[e+fx]\right] \partial_x \tan[e+fx]$

Rule: If $\frac{m}{2} \in \mathbb{Z} \wedge \frac{n}{2} \in \mathbb{Z} \wedge p \in \mathbb{Z}$, then

$$\int \cos[e+fx]^m (a+b \sin[e+fx]^n)^p dx \rightarrow \frac{1}{f} \operatorname{Subst}\left[\int \frac{(bx^n + a(1+x^2)^{n/2})^p}{(1+x^2)^{m/2+n p/2+1}} dx, x, \tan[e+fx]\right]$$

Program code:

```
Int[cos[e_.+f_.*x_]^m_*(a_+b_.*sin[e_.+f_.*x_]^4)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    ff/f*Subst[Int[(a+2*a*ff^2*x^2+(a+b)*ff^4*x^4)^p/(1+ff^2*x^2)^(m/2+2*p+1),x],x,Tan[e+f*x]/ff]] /;
  FreeQ[{a,b,e,f},x] && IntegerQ[m/2] && IntegerQ[p]
```

```
Int[cos[e_.+f_.*x_]^m_*(a_+b_.*sin[e_.+f_.*x_]^n_)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    ff/f*Subst[Int[(b*ff^n*x^n+a*(1+ff^2*x^2)^(n/2))^p/(1+ff^2*x^2)^(m/2+n*p/2+1),x],x,Tan[e+f*x]/ff]] /;
  FreeQ[{a,b,e,f},x] && IntegerQ[m/2] && IntegerQ[n/2] && IntegerQ[p]
```

3. $\int \frac{\cos[e+fx]^m}{a+b \sin[e+fx]^n} dx$ when $\frac{m}{2} \in \mathbb{Z} \wedge \frac{n-1}{2} \in \mathbb{Z}$

$$1: \int \frac{\cos[e+fx]^m}{a+b \sin[e+fx]^n} dx \text{ when } \frac{m}{2} \in \mathbb{Z}^+ \wedge \frac{n-1}{2} \in \mathbb{Z}$$

Derivation: Algebraic expansion

Basis: $\cos[z]^2 = 1 - \sin[z]^2$

Rule: If $\frac{m}{2} \in \mathbb{Z}^+ \wedge \frac{n-1}{2} \in \mathbb{Z}$, then

$$\int \frac{\cos[e+fx]^m}{a+b \sin[e+fx]^n} dx \rightarrow \int \operatorname{Expand}\left[\frac{(1 - \sin[e+fx]^2)^{m/2}}{a+b \sin[e+fx]^n}, x\right] dx$$

Program code:

```
Int[cos[e_.+f_.*x_]^m_/(a_.+b_.*sin[e_.+f_.*x_]^n_),x_Symbol] :=
  Int[Expand[(1-Sin[e+f*x]^2)^(m/2)/(a+b*Sin[e+f*x]^n),x],x] /;
FreeQ[{a,b,e,f},x] && IGtQ[m/2,0] && IntegerQ[(n-1)/2]
```

$$\mathbf{x:} \int \frac{\cos[e+fx]^m}{a+b \sin[e+fx]^n} dx \text{ when } \frac{m}{2} \in \mathbb{Z} \wedge \frac{n-1}{2} \in \mathbb{Z} \wedge p-1 \in \mathbb{Z}^- \wedge m < 0$$

Derivation: Algebraic expansion

Basis: $\cos[z]^2 = 1 - \sin[z]^2$

Rule: If $\frac{m}{2} \in \mathbb{Z} \wedge \frac{n-1}{2} \in \mathbb{Z} \wedge p-1 \in \mathbb{Z}^- \wedge m < 0$, then

$$\int \frac{\cos[e+fx]^m}{a+b \sin[e+fx]^n} dx \rightarrow \int \operatorname{ExpandTrig}\left[\frac{(1 - \sin[e+fx]^2)^{m/2}}{a+b \sin[e+fx]^n}, x\right] dx$$

Program code:

```
(* Int[cos[e_.+f_.**x_]^m_*(a_+b_.*sin[e_.+f_.**x_]^n_)^p_,x_Symbol] :=
  Int[ExpandTrig[(1-sin[e+f*x]^2)^(m/2)*(a+b*sin[e+f*x]^n)^p,x],x] /;
FreeQ[{a,b,e,f},x] && IntegerQ[m/2] && IntegerQ[(n-1)/2] && ILtQ[p,-1] && LtQ[m,0] *)
```

$$\mathbf{4:} \int (d \cos[e+fx])^m (a+b(c \sin[e+fx])^n)^p dx \text{ when } p \in \mathbb{Z}^+$$

Derivation: Algebraic expansion

Rule: If $p \in \mathbb{Z}^+$, then

$$\int (d \cos[e+fx])^m (a+b(c \cos[e+fx])^n)^p dx \rightarrow \int \operatorname{ExpandTrig}[(d \cos[e+fx])^m (a+b(c \sin[e+fx])^n)^p, x] dx$$

Program code:

```
Int[(d_.*cos[e_.+f_.**x_]^m_*(a_+b_.*(c_.*sin[e_.+f_.**x_]^n_)^p_,x_Symbol] :=
  Int[ExpandTrig[(d*cos[e+f*x])^m*(a+b*(c*sin[e+f*x]^n)^p,x],x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && IGtQ[p,0]
```

U: $\int (d \operatorname{Cos}[e + f x])^m (a + b (c \operatorname{Sin}[e + f x])^n)^p dx$

Rule:

$$\int (d \operatorname{Cos}[e + f x])^m (a + b (c \operatorname{Sin}[e + f x])^n)^p dx \rightarrow \int (d \operatorname{Cos}[e + f x])^m (a + b (c \operatorname{Sin}[e + f x])^n)^p dx$$

Program code:

```
Int[(d_.*cos[e_+f_.*x_])^m_.*(a_+b_.*(c_.*sin[e_+f_.*x_])^n_)^p_.,x_Symbol] :=
  Unintegrable[(d*Cos[e+f*x])^m*(a+b*(c*Sin[e+f*x])^n)^p,x] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x]
```

$$5. \int (d \operatorname{Tan}[e+fx])^m (a+b(c \operatorname{Sin}[e+fx])^n)^p dx$$

$$1. \int \operatorname{Tan}[e+fx]^m (a+b(c \operatorname{Sin}[e+fx])^n)^p dx \text{ when } \frac{m-1}{2} \in \mathbb{Z}$$

$$1: \int \operatorname{Tan}[e+fx]^m (a+b \operatorname{Sin}[e+fx]^n)^p dx \text{ when } \frac{m-1}{2} \in \mathbb{Z} \wedge \frac{n}{2} \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } \operatorname{Tan}[z]^2 == \frac{\operatorname{Sin}[z]^2}{1-\operatorname{Sin}[z]^2}$$

Basis: If $\frac{m-1}{2} \in \mathbb{Z}$, then

$$\operatorname{Tan}[e+fx]^m F[\operatorname{Sin}[e+fx]^2] == \frac{1}{2f} \operatorname{Subst}\left[\frac{x^{\frac{m-1}{2}} F[x]}{(1-x)^{\frac{m+1}{2}}}, x, \operatorname{Sin}[e+fx]^2\right] \partial_x \operatorname{Sin}[e+fx]^2$$

Rule: If $\frac{m-1}{2} \in \mathbb{Z} \wedge \frac{n}{2} \in \mathbb{Z}$, then

$$\int \operatorname{Tan}[e+fx]^m (a+b \operatorname{Sin}[e+fx]^n)^p dx \rightarrow \frac{1}{2f} \operatorname{Subst}\left[\int \frac{x^{\frac{m-1}{2}} (a+b x^{n/2})^p}{(1-x)^{\frac{m+1}{2}}} dx, x, \operatorname{Sin}[e+fx]^2\right]$$

Program code:

```
Int[tan[e_+f_*x_]^m_.*(a_+b_*sin[e_+f_*x_]^n_)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Sin[e+f*x]^2,x]},
    ff^((m+1)/2)/(2*f)*Subst[Int[x^((m-1)/2)*(a+b*ff^(n/2)*x^(n/2))^p/(1-ff*x)^((m+1)/2),x],x,Sin[e+f*x]^2/ff] /;
    FreeQ[{a,b,e,f,p},x] && IntegerQ[(m-1)/2] && IntegerQ[n/2]
```

$$2: \int \operatorname{Tan}[e+fx]^m (a+b(c \operatorname{Sin}[e+fx])^n)^p dx \text{ when } \frac{m-1}{2} \in \mathbb{Z}^-$$

Derivation: Integration by substitution

$$\text{Basis: } \operatorname{Tan}[z]^2 == \frac{\operatorname{Sin}[z]^2}{1-\operatorname{Sin}[z]^2}$$

Basis: If $\frac{m-1}{2} \in \mathbb{Z}$, then

$$\operatorname{Tan}[e+fx]^m F[\operatorname{Sin}[e+fx]] = \frac{1}{f} \operatorname{Subst}\left[\frac{x^m F[x]}{(1-x^2)^{\frac{m+1}{2}}}, x, \operatorname{Sin}[e+fx]\right] \partial_x \operatorname{Sin}[e+fx]$$

Rule: If $\frac{m-1}{2} \in \mathbb{Z}^-$, then

$$\int \operatorname{Tan}[e+fx]^m (a+b(c \operatorname{Sin}[e+fx])^n)^p dx \rightarrow \frac{1}{f} \operatorname{Subst}\left[\int \frac{x^m (a+b(c x)^n)^p}{(1-x^2)^{\frac{m+1}{2}}} dx, x, \operatorname{Sin}[e+fx]\right]$$

Program code:

```
Int[tan[e_+f_*x_]^m_.*(a_+b_.*(c_.*sin[e_+f_*x_])^n_)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Sin[e+f*x],x]},
    ff^(m+1)/f*Subst[Int[x^m*(a+b*(c*ff*x)^n)^p/(1-ff^2*x^2)^(m+1)/2,x],x,Sin[e+f*x]/ff] /;
  FreeQ[{a,b,c,e,f,n,p},x] && ILtQ[(m-1)/2,0]
```

$$2. \int (d \operatorname{Tan}[e+fx])^m (a+b \operatorname{Sin}[e+fx]^n)^p dx \text{ when } \frac{n}{2} \in \mathbb{Z}$$

$$1: \int (d \operatorname{Tan}[e+fx])^m (a+b \operatorname{Sin}[e+fx]^4)^p dx \text{ when } p \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } \operatorname{Sin}[z]^2 == \frac{\operatorname{Tan}[z]^2}{1+\operatorname{Tan}[z]^2}$$

$$\text{Basis: } (d \operatorname{Tan}[e+fx])^m F[\operatorname{Sin}[e+fx]^2] == \frac{1}{f} \operatorname{Subst}\left[\frac{(dx)^m F\left[\frac{x^2}{1+x^2}\right]}{1+x^2}, x, \operatorname{Tan}[e+fx]\right] \partial_x \operatorname{Tan}[e+fx]$$

Rule: If $p \in \mathbb{Z}$, then

$$\int (d \operatorname{Tan}[e+fx])^m (a+b \operatorname{Sin}[e+fx]^4)^p dx \rightarrow \frac{1}{f} \operatorname{Subst}\left[\int \frac{(dx)^m (a(1+x^2)^2 + b x^4)^p}{(1+x^2)^{2p+1}} dx, x, \operatorname{Tan}[e+fx]\right]$$

Program code:

```
Int[(d_.*tan[e_.+f_.*x_])^m_*(a_+b_.*sin[e_.+f_.*x_]^4)^p_.,x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    ff/f*Subst[Int[(d*ff*x)^m*ExpandToSum[a*(1+ff^2*x^2)^2+b*ff^4*x^4,x]^p/(1+ff^2*x^2)^(2*p+1),x],x,Tan[e+f*x]/ff]] /;
  FreeQ[{a,b,d,e,f,m},x] && IntegerQ[p]
```

$$2: \int (d \operatorname{Tan}[e+fx])^m (a+b \operatorname{Sin}[e+fx]^4)^p dx \text{ when } p - \frac{1}{2} \in \mathbb{Z}$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: If } \frac{n}{2} \in \mathbb{Z}, \text{ then } a + b \operatorname{Sin}[z]^n == \frac{a \operatorname{Sec}[z]^n + b \operatorname{Tan}[z]^n}{\operatorname{Sec}[z]^n} == \frac{a (1+\operatorname{Tan}[z]^2)^{n/2} + b \operatorname{Tan}[z]^n}{(1+\operatorname{Tan}[z]^2)^{n/2}}$$

$$\text{Basis: If } \frac{n}{2} \in \mathbb{Z}, \text{ then } \partial_x \frac{(a+b \operatorname{Sin}[e+fx]^n)^p (\operatorname{Sec}[e+fx]^2)^{np/2}}{(a \operatorname{Sec}[e+fx]^n + b \operatorname{Tan}[e+fx]^n)^p} == 0$$

Basis: $F[\operatorname{Tan}[e+fx]] = \frac{1}{f} \operatorname{Subst}\left[\frac{F[x]}{1+x^2}, x, \operatorname{Tan}[e+fx]\right] \partial_x \operatorname{Tan}[e+fx]$

Rule: If $p - \frac{1}{2} \in \mathbb{Z}$, then

$$\int (d \operatorname{Tan}[e+fx])^m (a+b \operatorname{Sin}[e+fx]^4)^p dx \rightarrow \frac{(a+b \operatorname{Sin}[e+fx]^4)^p (\operatorname{Sec}[e+fx]^2)^{2p}}{(a \operatorname{Sec}[e+fx]^4 + b \operatorname{Tan}[e+fx]^4)^p} \int \frac{(d \operatorname{Tan}[e+fx])^m (a(1+\operatorname{Tan}[e+fx]^2)^2 + b \operatorname{Tan}[e+fx]^5)^p}{(1+\operatorname{Tan}[e+fx]^2)^{2p+1}} dx$$

$$\rightarrow \frac{(a+b \operatorname{Sin}[e+fx]^4)^p (\operatorname{Sec}[e+fx]^2)^{2p}}{f (a \operatorname{Sec}[e+fx]^4 + b \operatorname{Tan}[e+fx]^4)^p} \operatorname{Subst}\left[\int \frac{(dx)^m (a(1+x^2)^2 + bx^4)^p}{(1+x^2)^{2p+1}} dx, x, \operatorname{Tan}[e+fx]\right]$$

Program code:

```
Int[(d_.*tan[e_.+f_.*x_])^m_*(a_+b_.*sin[e_.+f_.*x_]^4)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    ff*(a+b*Sin[e+f*x]^4)^p*(Sec[e+f*x]^2)^(2*p)/(f*Apart[a*(1+Tan[e+f*x]^2)^2+b*Tan[e+f*x]^4]^p)*
    Subst[Int[(d*ff*x)^m*ExpandToSum[a*(1+ff^2*x^2)^2+b*ff^4*x^4,x]^p/(1+ff^2*x^2)^(2*p+1),x],x,Tan[e+f*x]/ff] /;
    FreeQ[{a,b,d,e,f,m},x] && IntegerQ[p-1/2]
```


$$\mathbf{3:} \int (d \operatorname{Tan}[e+fx])^m (a+b \operatorname{Sin}[e+fx]^n)^p dx \text{ when } \frac{n}{2} \in \mathbb{Z} \wedge p \in \mathbb{Z}^+$$

Derivation: Integration by substitution

$$\text{Basis: } \operatorname{Sin}[z]^2 == \frac{\operatorname{Tan}[z]^2}{1+\operatorname{Tan}[z]^2}$$

$$\text{Basis: } (d \operatorname{Tan}[e+fx])^m F[\operatorname{Sin}[e+fx]^2] == \frac{1}{f} \operatorname{Subst}\left[\frac{(dx)^m F\left[\frac{x^2}{1+x^2}\right]}{1+x^2}, x, \operatorname{Tan}[e+fx]\right] \partial_x \operatorname{Tan}[e+fx]$$

Rule: If $\frac{n}{2} \in \mathbb{Z} \wedge p \in \mathbb{Z}^+$, then

$$\int (d \operatorname{Tan}[e+fx])^m (a+b \operatorname{Sin}[e+fx]^n)^p dx \rightarrow \frac{1}{f} \operatorname{Subst}\left[\int \frac{(dx)^m (bx^n + a(1+x^2)^{n/2})^p}{(1+x^2)^{np/2+1}} dx, x, \operatorname{Tan}[e+fx]\right]$$

Program code:

```
Int[(d_.*tan[e_.+f_.*x_])^m_*(a_+b_.*sin[e_.+f_.*x_]^n_)^p_,x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
    ff^(m+1)/f*Subst[Int[(d*x)^m*(b*ff^n*x^n+a*(1+ff^2*x^2)^(n/2))^p/(1+ff^2*x^2)^(n*p/2+1),x],x,Tan[e+f*x]/ff]] /;
  FreeQ[{a,b,d,e,f,m},x] && IntegerQ[n/2] && IGtQ[p,0]
```

3: $\int (d \operatorname{Tan}[e+fx])^m (a+b(c \operatorname{Sin}[e+fx])^n)^p dx$ when $p \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $p \in \mathbb{Z}^+$, then

$$\int (d \operatorname{Tan}[e+fx])^m (a+b(c \operatorname{Sin}[e+fx])^n)^p dx \rightarrow \int \operatorname{ExpandTrig}[(d \operatorname{Tan}[e+fx])^m (a+b(c \operatorname{Sin}[e+fx])^n)^p, x] dx$$

Program code:

```
Int[(d_.*tan[e_.+f_.*x_])^m_.*(a_+b_.*(c_.*sin[e_.+f_.*x_])^n_)^p_.,x_Symbol] :=
  Int[ExpandTrig[(d*tan[e+f*x])^m*(a+b*(c*sin[e+f*x])^n)^p,x],x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && IGtQ[p,0]
```

U: $\int (d \operatorname{Tan}[e+fx])^m (a+b(c \operatorname{Sin}[e+fx])^n)^p dx$

Rule:

$$\int (d \operatorname{Tan}[e+fx])^m (a+b(c \operatorname{Sin}[e+fx])^n)^p dx \rightarrow \int (d \operatorname{Tan}[e+fx])^m (a+b(c \operatorname{Sin}[e+fx])^n)^p dx$$

Program code:

```
Int[(d_.*tan[e_.+f_.*x_])^m_.*(a_+b_.*(c_.*sin[e_.+f_.*x_])^n_)^p_.,x_Symbol] :=
  Unintegrable[(d*Tan[e+f*x])^m*(a+b*(c*sin[e+f*x])^n)^p,x] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x]
```

6: $\int (d \operatorname{Cot}[e+fx])^m (a+b(c \operatorname{Sin}[e+fx])^n)^p dx$ when $m \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \left((d \operatorname{Cot}[e+fx])^m \left(\frac{\operatorname{Tan}[e+fx]}{d} \right)^m \right) == 0$$

Rule: If $m \notin \mathbb{Z}$, then

$$\int (d \operatorname{Cot}[e+fx])^m (a+b(c \operatorname{Sin}[e+fx])^n)^p dx \rightarrow (d \operatorname{Cot}[e+fx])^{\operatorname{FracPart}[m]} \left(\frac{\operatorname{Tan}[e+fx]}{d} \right)^{\operatorname{FracPart}[m]} \int \left(\frac{\operatorname{Tan}[e+fx]}{d} \right)^{-m} (a+b(c \operatorname{Sin}[e+fx])^n)^p dx$$

Program code:

```
Int[(d_.*cot[e_.+f_.*x_])^m_*(a_+b_*(c_.*sin[e_.+f_.*x_])^n_)^p_,x_Symbol] :=
  (d*Cot[e+f*x])^FracPart[m]*(Tan[e+f*x]/d)^FracPart[m]*Int[(Tan[e+f*x]/d)^(-m)*(a+b*(c*Sin[e+f*x])^n)^p,x] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && Not[IntegerQ[m]]
```

$$7: \int (d \operatorname{Sec}[e+fx])^m (a+b(c \operatorname{Sin}[e+fx])^n)^p dx \text{ when } m \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \left((d \operatorname{Sec}[e+fx])^m \left(\frac{\operatorname{Cos}[e+fx]}{d} \right)^m \right) == 0$$

Rule: If $m \notin \mathbb{Z}$, then

$$\int (d \operatorname{Sec}[e+fx])^m (a+b(c \operatorname{Sin}[e+fx])^n)^p dx \rightarrow (d \operatorname{Sec}[e+fx])^{\operatorname{FracPart}[m]} \left(\frac{\operatorname{Cos}[e+fx]}{d} \right)^{\operatorname{FracPart}[m]} \int \left(\frac{\operatorname{Cos}[e+fx]}{d} \right)^{-m} (a+b(c \operatorname{Sin}[e+fx])^n)^p dx$$

Program code:

```
Int[(d_.*sec[e_.+f_.*x_])^m_*(a_+b_*(c_.*sin[e_.+f_.*x_])^n_)^p_,x_Symbol] :=
  (d*Sec[e+f*x])^FracPart[m]*(Cos[e+f*x]/d)^FracPart[m]*Int[(Cos[e+f*x]/d)^(-m)*(a+b*(c*Sin[e+f*x])^n)^p,x] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && Not[IntegerQ[m]]
```

8. $\int (d \operatorname{Csc}[e+fx])^m (a+b(c \operatorname{Sin}[e+fx])^n)^p dx$ when $m \notin \mathbb{Z}$

1: $\int (d \operatorname{Csc}[e+fx])^m (a+b \operatorname{Sin}[e+fx]^n)^p dx$ when $m \notin \mathbb{Z} \wedge (n|p) \in \mathbb{Z}$

Derivation: Algebraic normalization

Basis: If $(n|p) \in \mathbb{Z}$, then $(a+b \operatorname{Sin}[e+fx]^n)^p = d^{np} (d \operatorname{Csc}[e+fx])^{-np} (b+a \operatorname{Csc}[e+fx]^n)^p$

Rule: If $m \notin \mathbb{Z} \wedge (n|p) \in \mathbb{Z}$, then

$$\int (d \operatorname{Csc}[e+fx])^m (a+b \operatorname{Sin}[e+fx]^n)^p dx \rightarrow d^{np} \int (d \operatorname{Csc}[e+fx])^{m-np} (b+a \operatorname{Csc}[e+fx]^n)^p dx$$

Program code:

```
Int[(d_.*csc[e_.+f_.*x_])^m_*(a_+b_.*sin[e_.+f_.*x_]^n_.)^p_,x_Symbol] :=
  d^(n*p)*Int[(d*Csc[e+f*x])^(m-n*p)*(b+a*Csc[e+f*x]^n)^p,x] /;
FreeQ[{a,b,d,e,f,m,n,p},x] && Not[IntegerQ[m]] && IntegersQ[n,p]
```

2: $\int (d \operatorname{Csc}[e+fx])^m (a+b(c \operatorname{Sin}[e+fx])^n)^p dx$ when $m \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \left((d \operatorname{Csc}[e+fx])^m \left(\frac{\operatorname{Sin}[e+fx]}{d} \right)^m \right) == 0$

Rule: If $m \notin \mathbb{Z}$, then

$$\int (d \operatorname{Csc}[e+fx])^m (a+b(c \operatorname{Sin}[e+fx])^n)^p dx \rightarrow (d \operatorname{Csc}[e+fx])^{\operatorname{FracPart}[m]} \left(\frac{\operatorname{Sin}[e+fx]}{d} \right)^{\operatorname{FracPart}[m]} \int \left(\frac{\operatorname{Sin}[e+fx]}{d} \right)^{-m} (a+b(c \operatorname{Sin}[e+fx])^n)^p dx$$

Program code:

```
Int[(d_.*csc[e_.+f_.**x_])^m_*(a_+b_.*(c_.*sin[e_.+f_.**x_])^n_)^p_,x_Symbol] :=
  (d*Csc[e+f*x])^FracPart[m]*(Sin[e+f*x]/d)^FracPart[m]*Int[(Sin[e+f*x]/d)^(-m)*(a+b*(c*Sin[e+f*x])^n)^p,x] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && Not[IntegerQ[m]]
```