

$$?. \int \frac{(dx)^m (e + f x^{n/4} + g x^{3n/4} + h x^n)}{(a + c x^n)^{3/2}} dx \text{ when } 4m - n + 4 == 0 \wedge ce + ah == 0$$

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Rule: If $4m - n + 4 == 0 \wedge ce + ah == 0$, then

$$\int \frac{x^m (e + f x^{n/4} + g x^{3n/4} + h x^n)}{(a + c x^n)^{3/2}} dx \rightarrow -\frac{2ag + 4ahx^{n/4} - 2cfx^{n/2}}{acn\sqrt{a + cx^n}}$$

Program code:

```
Int[x^m_.*(e_+f_.*x^q_+g_.*x^r_+h_.*x^n_)/(a+c_.*x^n_)^(3/2),x_Symbol] :=
-(2*a*g+4*a*h*x^(n/4)-2*c*f*x^(n/2))/(a*c*n*Sqrt[a+c*x^n]) /;
FreeQ[{a,c,e,f,g,h,m,n},x] && EqQ[q,n/4] && EqQ[r,3*n/4] && EqQ[4*m-n+4,0] && EqQ[c*e+a*h,0]
```

$$2: \int \frac{(dx)^m (e + f x^{n/4} + g x^{3n/4} + h x^n)}{(a + c x^n)^{3/2}} dx \text{ when } 4m - n + 4 == 0 \wedge ce + ah == 0$$

Rule: If $4m - n + 4 == 0 \wedge ce + ah == 0$, then

$$\int \frac{(dx)^m (e + f x^{n/4} + g x^{3n/4} + h x^n)}{(a + c x^n)^{3/2}} dx \rightarrow \frac{(dx)^m}{x^m} \int \frac{x^m (e + f x^{n/4} + g x^{3n/4} + h x^n)}{(a + c x^n)^{3/2}} dx$$

Program code:

```
Int[(d*x_)^m_.*(e_+f_.*x^q_+g_.*x^r_+h_.*x^n_)/(a+c_.*x^n_)^(3/2),x_Symbol] :=
(d*x)^m/x^m*Int[x^m_.*(e+f*x^(n/4)+g*x^((3*n)/4)+h*x^n)/(a+c*x^n)^(3/2),x] /;
FreeQ[{a,c,d,e,f,g,h,m,n},x] && EqQ[4*m-n+4,0] && EqQ[q,n/4] && EqQ[r,3*n/4] && EqQ[c*e+a*h,0]
```

Rules for integrands of the form $(c x)^m P_q[x] (a + b x^n)^p$

1: $\int (c x)^m P_q[x] (a + b x)^p dx$ when $p \in \mathbb{F} \wedge m + 1 \in \mathbb{Z}^-$

Derivation: Integration by substitution

Basis: If $n \in \mathbb{Z}^+$, then $F[x] (a + b x)^p = \frac{n}{b} \text{Subst}[x^{n p + n - 1} F[-\frac{a}{b} + \frac{x^n}{b}], x, (a + b x)^{1/n}] \partial_x (a + b x)^{1/n}$

Rule: If $p \in \mathbb{F} \wedge m + 1 \in \mathbb{Z}^-$, let $n = \text{Denominator}[p]$, then

$$\int (c x)^m P_q[x] (a + b x)^p dx \rightarrow \frac{n}{b} \text{Subst}\left[\int x^{n p + n - 1} \left(-\frac{a c}{b} + \frac{c x^n}{b}\right)^m P_q\left[-\frac{a}{b} + \frac{x^n}{b}\right] dx, x, (a + b x)^{1/n}\right]$$

Program code:

```
Int[(c_.**x_)^m_*Pq_*(a_+b_.**x_)^p_,x_Symbol] :=
  With[{n=Denominator[p]},
    n/b*Subst[Int[x^(n*p+n-1)*(-a*c/b+c*x^n/b)^m*ReplaceAll[Pq,x->-a/b+x^n/b],x],x,(a+b*x)^(1/n)] /;
  FreeQ[{a,b,c,m},x] && PolyQ[Pq,x] && FractionQ[p] && ILtQ[m,-1]
```

2: $\int x^m P_q[x^{m+1}] (a + b x^n)^p dx$ when $m \neq -1 \wedge \frac{n}{m+1} \in \mathbb{Z}^+$

Derivation: Integration by substitution

Basis: $x^m F[x^{m+1}] = \frac{1}{m+1} \text{Subst}[F[x], x, x^{m+1}] \partial_x x^{m+1}$

Rule: If $m \neq -1 \wedge \frac{n}{m+1} \in \mathbb{Z}^+$, then

$$\int x^m P_q[x^{m+1}] (a + b x^n)^p dx \rightarrow \frac{1}{m+1} \text{Subst}\left[\int P_q[x] (a + b x^{\frac{n}{m+1}})^p dx, x, x^{m+1}\right]$$

Program code:

```
Int[x_^m_.*Pq_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  1/(m+1)*Subst[Int[SubstFor[x^(m+1),Pq,x]*(a+b*x^Simplify[n/(m+1)])^p,x],x,x^(m+1)] /;
FreeQ[{a,b,m,n,p},x] && NeQ[m,-1] && IGtQ[Simplify[n/(m+1)],0] && PolyQ[Pq,x^(m+1)]
```

3: $\int (c x)^m P_q[x] (a + b x^n)^p dx$ when $p \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $p \in \mathbb{Z}^+$, then

$$\int (c x)^m P_q[x] (a + b x^n)^p dx \rightarrow \int \text{ExpandIntegrand}[(c x)^m P_q[x] (a + b x^n)^p, x] dx$$

Program code:

```
Int[(c_.*x_)^m_.*Pq_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  Int[ExpandIntegrand[(c*x)^m*Pq*(a+b*x^n)^p,x],x] /;
FreeQ[{a,b,c,m,n},x] && PolyQ[Pq,x] && (IGtQ[p,0] || EqQ[n,1])
```

4. $\int (c x)^m P_q[x^n] (a + b x^n)^p dx$ when $\frac{m+1}{n} \in \mathbb{Z}$

1: $\int x^m P_q[x^n] (a + b x^n)^p dx$ when $\frac{m+1}{n} \in \mathbb{Z}$

Derivation: Integration by substitution

Basis: If $\frac{m+1}{n} \in \mathbb{Z}$, then $x^m F[x^n] = \frac{1}{n} \text{Subst}\left[x^{\frac{m+1}{n}-1} F[x], x, x^n\right] \partial_x x^n$

Note: If $n \in \mathbb{Z} \wedge \frac{m+1}{n} \in \mathbb{Z}$, then $m \in \mathbb{Z}$, and $(c x)^m$ automatically evaluates to $c^m x^m$.

Rule: If $\frac{m+1}{n} \in \mathbb{Z}$, then

$$\int x^m P_q[x^n] (a + b x^n)^p dx \rightarrow \frac{1}{n} \text{Subst}\left[\int x^{\frac{m+1}{n}-1} P_q[x] (a + b x)^p dx, x, x^n\right]$$

Program code:

```
Int[x^m_.*Pq_*(a_+b_.*x_^n_)^p_.,x_Symbol] :=
  1/n*Subst[Int[x^(Simplify[(m+1)/n]-1)*SubstFor[x^n,Pq,x]*(a+b*x)^p,x],x,x^n] /;
FreeQ[{a,b,m,n,p},x] && PolyQ[Pq,x^n] && IntegerQ[Simplify[(m+1)/n]]
```

2: $\int (c x)^m P_q[x^n] (a + b x^n)^p dx$ when $\frac{m+1}{n} \in \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(c x)^m}{x^m} = 0$

Basis: $\frac{(c x)^m}{x^m} = \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}}$

Rule: If $\frac{m+1}{n} \in \mathbb{Z}$, then

$$\int (c x)^m P_q[x^n] (a + b x^n)^p dx \rightarrow \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m P_q[x^n] (a + b x^n)^p dx$$

Program code:

```
Int[(c*x_)^m_.*Pq*(a+b_.*x_^n_)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m*Pq*(a+b*x^n)^p,x] /;
FreeQ[{a,b,c,m,n,p},x] && PolyQ[Pq,x^n] && IntegerQ[Simplify[(m+1)/n]]
```

5: $\int x^m P_q[x] (a + b x^n)^p dx$ when $m - n + 1 = 0 \wedge p < -1$

Derivation: Integration by parts

Basis: $x^{n-1} (a + b x^n)^p = \partial_x \frac{(a + b x^n)^{p+1}}{b n (p+1)}$

Rule: If $m - n + 1 = 0 \wedge p < -1$, then

$$\int x^m P_q[x] (a + b x^n)^p dx \rightarrow \frac{P_q[x] (a + b x^n)^{p+1}}{b n (p+1)} - \frac{1}{b n (p+1)} \int \partial_x P_q[x] (a + b x^n)^{p+1} dx$$

Program code:

```
Int[x^m_.*Pq_*(a_+b_.*x^n_)^p_,x_Symbol] :=
  Pq*(a+b*x^n)^(p+1)/(b*n*(p+1)) -
  1/(b*n*(p+1))*Int[D[Pq,x]*(a+b*x^n)^(p+1),x] /;
FreeQ[{a,b,m,n},x] && PolyQ[Pq,x] && EqQ[m-n+1,0] && LtQ[p,-1]
```

6: $\int (d x)^m P_q[x] (a+b x^n)^p dx$ when $P_q[x, 0] = 0$

Derivation: Algebraic simplification

Rule: If $P_q[x, 0] = 0$, then

$$\int (d x)^m P_q[x] (a+b x^n)^p dx \rightarrow \frac{1}{d} \int (d x)^{m+1} \text{PolynomialQuotient}[P_q[x], x, x] (a+b x^n)^p dx$$

Program code:

```
Int[(d_.**x_)^m_.*Pq_*(a_+b_.**x_^n_.)^p_,x_Symbol] :=
  1/d*Int[(d*x)^(m+1)*PolynomialQuotient[Pq,x,x]*(a+b*x^n)^p,x] /;
FreeQ[{a,b,d,m,n,p},x] && PolyQ[Pq,x] && EqQ[Coeff[Pq,x,0],0]
```

$$7. \int (c x)^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}$$

$$1. \int (c x)^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+$$

$$1. \int (c x)^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+ \wedge p > 0$$

$$\mathbf{1:} \int x^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+ \wedge p > 0 \wedge m + q + 1 < 0$$

Derivation: Integration by parts

Rule: If $n \in \mathbb{Z}^+ \wedge p > 0 \wedge m + q + 1 < 0$, let $u = \int x^m P_q[x] dx$ then

$$\int x^m P_q[x] (a + b x^n)^p dx \rightarrow u (a + b x^n)^p - b n p \int x^{m+n} (a + b x^n)^{p-1} \frac{u}{x^{m+1}} dx$$

Program code:

```
Int[x_^m_.*Pq_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
Module[{u=IntHide[x^m*Pq,x]},
u*(a+b*x^n)^p - b*n*p*Int[x^(m+n)*(a+b*x^n)^(p-1)*ExpandToSum[u/x^(m+1),x],x]] /;
FreeQ[{a,b},x] && PolyQ[Pq,x] && IGtQ[n,0] && GtQ[p,0] && LtQ[m+Expon[Pq,x]+1,0]
```


2: $\int (c x)^m P_q[x] (a + b x^n)^p dx$ when $\frac{n-1}{2} \in \mathbb{Z}^+ \wedge p > 0$

Derivation: Binomial recurrence 1b applied q times

Rule: If $\frac{n-1}{2} \in \mathbb{Z}^+ \wedge p > 0$, then

$$\int (c x)^m P_q[x] (a + b x^n)^p dx \rightarrow (c x)^m (a + b x^n)^p \sum_{i=0}^q \frac{P_q[x, i] x^{i+1}}{m + n p + i + 1} + a n p \int (c x)^m (a + b x^n)^{p-1} \left(\sum_{i=0}^q \frac{P_q[x, i] x^i}{m + n p + i + 1} \right) dx$$

Program code:

```
Int[(c_.**x_)^m_.**Pq_*(a_+b_.**x_^n_)^p_,x_Symbol] :=
Module[{q=Expon[Pq,x],i},
(c*x)^m*(a+b*x^n)^p*Sum[Coeff[Pq,x,i]*x^(i+1)/(m+n*p+i+1),{i,0,q}] +
a*n*p*Int[(c*x)^m*(a+b*x^n)^(p-1)*Sum[Coeff[Pq,x,i]*x^i/(m+n*p+i+1),{i,0,q}],x] /;
FreeQ[{a,b,c,m},x] && PolyQ[Pq,x] && IGtQ[(n-1)/2,0] && GtQ[p,0]
```

$$2. \int x^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+ \wedge p < -1 \wedge m \in \mathbb{Z}$$

$$1. \int x^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+ \wedge p < -1 \wedge m \in \mathbb{Z}^+$$

$$1: \int \frac{x^2 (e + f x + h x^4)}{(a + b x^4)^{3/2}} dx \text{ when } b e - 3 a h = 0$$

Rule: If $b e - 3 a h = 0$, then

$$\int \frac{x^2 (e + f x + h x^4)}{(a + b x^4)^{3/2}} dx \rightarrow - \frac{f - 2 h x^3}{2 b \sqrt{a + b x^4}}$$

Program code:

```
Int[x_^2*P4_/(a_+b_.*x_^4)^(3/2),x_Symbol] :=
  With[{e=Coeff[P4,x,0],f=Coeff[P4,x,1],h=Coeff[P4,x,4]},
    -(f-2*h*x^3)/(2*b*Sqrt[a+b*x^4]) /;
    EqQ[b*e-3*a*h,0]] /;
FreeQ[{a,b},x] && PolyQ[P4,x,4] && EqQ[Coeff[P4,x,2],0] && EqQ[Coeff[P4,x,3],0]
```

$$2: \int x^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+ \wedge p < -1 \wedge m \in \mathbb{Z}^+ \wedge m + q \geq n$$

Derivation: Algebraic expansion and binomial recurrence 2b applied $n - 1$ times

Note: $\sum_{i=0}^q (i + 1) P_q[x, i] x^i = \partial_x (x P_q[x])$ contributed by Martin Welz on 5 June 2015

Rule: If $n \in \mathbb{Z}^+ \wedge p < -1 \wedge m \in \mathbb{Z}^+ \wedge m + q \geq n$, let $Q_{m+q-n}[x] \rightarrow \text{PolynomialQuotient}[x^m P_q[x], a + b x^n, x]$ and $R_{n-1}[x] \rightarrow \text{PolynomialRemainder}[x^m P_q[x], a + b x^n, x]$, then

$$\int x^m P_q[x] (a + b x^n)^p dx \rightarrow$$

$$\int R_{n-1}[x] (a + b x^n)^p dx + \int Q_{m+q-n}[x] (a + b x^n)^{p+1} dx \rightarrow$$

$$- \frac{x R_{n-1}[x] (a + b x^n)^{p+1}}{a n (p+1)} + \frac{1}{a n (p+1)} \int (a n (p+1) Q_{m+q-n}[x] + n (p+1) R_{n-1}[x] + \partial_x (x R_{n-1}[x])) (a + b x^n)^{p+1} dx$$

Program code:

```
Int[x_^m_.*Pq_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  With[{q=m+Expon[Pq,x]},
    Module[{Q=PolynomialQuotient[b^(Floor[(q-1)/n]+1)*x^m*Pq,a+b*x^n,x],
      R=PolynomialRemainder[b^(Floor[(q-1)/n]+1)*x^m*Pq,a+b*x^n,x]},
      -x*R*(a+b*x^n)^(p+1)/(a*n*(p+1)*b^(Floor[(q-1)/n]+1)) +
      1/(a*n*(p+1)*b^(Floor[(q-1)/n]+1))*Int[(a+b*x^n)^(p+1)*ExpandToSum[a*n*(p+1)*Q+n*(p+1)*R+D[x*R,x],x],x] /;
    GeQ[q,n] /;
    FreeQ[{a,b},x] && PolyQ[Pq,x] && IGtQ[n,0] && LtQ[p,-1] && IGtQ[m,0]
```

$$\mathbf{2:} \int x^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+ \wedge p < -1 \wedge m \in \mathbb{Z}^-$$

Derivation: Algebraic expansion and binomial recurrence 2b applied $n - 1$ times

Rule: If $n \in \mathbb{Z}^+ \wedge p < -1 \wedge m \in \mathbb{Z}^-$, let $Q_{q-n}[x] = \text{PolynomialQuotient}[x^m P_q[x], a + b x^n, x]$ and $R_{n-1}[x] = \text{PolynomialRemainder}[x^m P_q[x], a + b x^n, x]$, then

$$\begin{aligned} & \int x^m P_q[x] (a + b x^n)^p dx \rightarrow \\ & \int R_{n-1}[x] (a + b x^n)^p dx + \int Q_{q-n}[x] (a + b x^n)^{p+1} dx \rightarrow \\ & - \frac{x R_{n-1}[x] (a + b x^n)^{p+1}}{a n (p+1)} + \frac{1}{a n (p+1)} \int x^m \left(a n (p+1) x^{-m} Q_{q-n}[x] + \sum_{i=0}^{n-1} (n (p+1) + i + 1) R_{n-1}[x, i] x^{i-m} \right) (a + b x^n)^{p+1} dx \end{aligned}$$

Program code:

```
Int[x^m*Pq_*(a_+b_.*x^n_)^p_,x_Symbol] :=
  With[{q=Expon[Pq,x]},
    Module[{Q=PolynomialQuotient[a*b^(Floor[(q-1)/n]+1)*x^m*Pq,a+b*x^n,x],
      R=PolynomialRemainder[a*b^(Floor[(q-1)/n]+1)*x^m*Pq,a+b*x^n,x],i},
      -x*R*(a+b*x^n)^(p+1)/(a^2*n*(p+1)*b^(Floor[(q-1)/n]+1)) +
      1/(a*n*(p+1)*b^(Floor[(q-1)/n]+1))*Int[x^m*(a+b*x^n)^(p+1)*
      ExpandToSum[n*(p+1)*x^(-m)*Q+Sum[(n*(p+1)+i+1)/a*Coeff[R,x,i]*x^(i-m),{i,0,n-1}],x,x]] /;
    FreeQ[{a,b},x] && PolyQ[Pq,x] && IGtQ[n,0] && LtQ[p,-1] && ILtQ[m,0]
```

3: $\int x^m P_q[x^n] (a + b x^n)^p dx$ when $n \in \mathbb{Z}^+ \wedge m \in \mathbb{Z} \wedge \text{GCD}[m+1, n] \neq 1$

Derivation: Integration by substitution

Basis: If $n \in \mathbb{Z} \wedge m \in \mathbb{Z}$, let $g = \text{GCD}[m+1, n]$, then $x^m F[x^n] = \frac{1}{g} \text{Subst}\left[x^{\frac{m+1}{g}-1} F\left[x^{\frac{n}{g}}\right], x, x^g\right] \partial_x x^g$

Rule: If $n \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}$, let $g = \text{GCD}[m+1, n]$, if $g \neq 1$, then

$$\int x^m P_q[x^n] (a + b x^n)^p dx \rightarrow \frac{1}{g} \text{Subst}\left[\int x^{\frac{m+1}{g}-1} P_q\left[x^{\frac{n}{g}}\right] (a + b x^{\frac{n}{g}})^p dx, x, x^g\right]$$

Program code:

```
Int[x^m_.*Pq_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  With[{g=GCD[m+1,n]},
    1/g*Subst[Int[x^((m+1)/g-1)*ReplaceAll[Pq,x->x^(1/g)]*(a+b*x^(n/g))^p,x],x,x^g] /;
    g!=1] /;
FreeQ[{a,b,p},x] && PolyQ[Pq,x^n] && IGtQ[n,0] && IntegerQ[m]
```

4: $\int \frac{(c x)^m P_q[x]}{a + b x^n} dx$ when $\frac{n}{2} \in \mathbb{Z}^+ \wedge q < n$

Derivation: Algebraic expansion

Basis: If $\frac{n}{2} \in \mathbb{Z} \wedge q < n$, then $P_q[x] = \sum_{i=0}^{q-1} x^i P_q[x, i] = \sum_{i=0}^{n/2-1} x^i (P_q[x, i] + P_q[x, \frac{n}{2} + i] x^{n/2})$

Note: The resulting integrands are of the form $\frac{(c x)^q (r + s x^{n/2})}{a + b x^n}$ for which there are rules.

Rule: If $\frac{n}{2} \in \mathbb{Z}^+ \wedge q < n$, then

$$\int \frac{(c x)^m P_q[x]}{a + b x^n} dx \rightarrow \int \sum_{i=0}^{n/2-1} \frac{(c x)^{m+i} (P_q[x, i] + P_q[x, \frac{n}{2} + i] x^{n/2})}{c^i (a + b x^n)} dx$$

Program code:

```
Int[(c_.**x_)^m_.**Pq_/(a_+b_.**x_^n_),x_Symbol] :=
  With[{v=Sum[(c**x)^(m+ii)*(Coeff[Pq,x,ii]+Coeff[Pq,x,n/2+ii]*x^(n/2))/(c^ii*(a+b*x^n)),{ii,0,n/2-1}]},
    Int[v,x] /;
    SumQ[v] /;
    FreeQ[{a,b,c,m},x] && PolyQ[Pq,x] && IGtQ[n/2,0] && Expon[Pq,x]<n
```

5: $\int \frac{P_q[x]}{x \sqrt{a+b x^n}} dx$ when $n \in \mathbb{Z}^+ \wedge P_q[x, 0] \neq 0$

Derivation: Algebraic expansion

Rule: If $n \in \mathbb{Z}^+ \wedge P_q[x, 0] \neq 0$, then

$$\int \frac{P_q[x]}{x \sqrt{a+b x^n}} dx \rightarrow P_q[x, 0] \int \frac{1}{x \sqrt{a+b x^n}} dx + \int \frac{P_q[x] - P_q[x, 0]}{x} \frac{1}{\sqrt{a+b x^n}} dx$$

Program code:

```
Int[Pq/(x*Sqrt[a+b_*x^n]),x_Symbol] :=
  Coeff[Pq,x,0]*Int[1/(x*Sqrt[a+b*x^n]),x] +
  Int[ExpandToSum[(Pq-Coeff[Pq,x,0])/x,x]/Sqrt[a+b*x^n],x] /;
FreeQ[{a,b},x] && PolyQ[Pq,x] && IGtQ[n,0] && NeQ[Coeff[Pq,x,0],0]
```

6: $\int (c x)^m P_q[x] (a + b x^n)^p dx$ when $\frac{n}{2} \in \mathbb{Z}^+ \wedge \neg \text{PolynomialQ}[P_q[x], x^{\frac{n}{2}}]$

Derivation: Algebraic expansion

Basis: If $n \in \mathbb{Z}^+$, then $P_q[x] = \sum_{j=0}^{q-1} x^j \sum_{k=0}^{(q-j)/n+1} P_q[x, j + k n] x^{k n}$

Note: This rule transform integrand into a sum of terms of the form $x^k Q_r[x^{\frac{n}{2}}] (a + b x^n)^p$.

Rule: If $\frac{n}{2} \in \mathbb{Z}^+ \wedge \neg \text{PolynomialQ}[P_q[x], x^{\frac{n}{2}}]$, then

$$\int (c x)^m P_q[x] (a + b x^n)^p dx \rightarrow \int \sum_{j=0}^{\frac{n}{2}-1} \frac{(c x)^{m+j}}{c^j} \left(\sum_{k=0}^{\frac{2(q-j)}{n}+1} P_q\left[x, j + \frac{k n}{2}\right] x^{\frac{k n}{2}} \right) (a + b x^n)^p dx$$

Program code:

```
Int[(c_.**x_)^m_.*Pq_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  Module[{q=Expon[Pq,x],j,k},
    Int[Sum[(c*x)^(m+j)/c^j*Sum[Coeff[Pq,x,j+k*n/2]*x^(k*n/2),{k,0,2*(q-j)/n+1}]]*(a+b*x^n)^p,{j,0,n/2-1}],x] /;
    FreeQ[{a,b,c,m,p},x] && PolyQ[Pq,x] && IGtQ[n/2,0] && Not[PolyQ[Pq,x^(n/2)]]
```


7: $\int \frac{(c x)^m P_q[x]}{a + b x^n} dx$ when $n \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $n \in \mathbb{Z}^+$, then

$$\int \frac{(c x)^m P_q[x]}{a + b x^n} dx \rightarrow \int \text{ExpandIntegrand}\left[\frac{(c x)^m P_q[x]}{a + b x^n}, x\right] dx$$

Program code:

```
Int[(c_.**x_)^m_.**Pq_/ (a_+b_.**x_^n_), x_Symbol] :=
  Int[ExpandIntegrand[(c**x)^m**Pq/(a+b**x^n), x], x] /;
FreeQ[{a,b,c,m}, x] && PolyQ[Pq, x] && IntegerQ[n] && Not[IGtQ[m, 0]]
```

$$8. \int (c x)^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+ \wedge q - n \geq -1$$

$$1: \int (c x)^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+ \wedge q - n \geq -1 \wedge m < -1 \wedge P_q[x, 0] \neq 0$$

Derivation: Algebraic expansion and binomial recurrence 3b

Note: This rule increments m *and* decrements the degree of the polynomial in the resulting integrand if $n - 1 < q$.

Rule: If $n \in \mathbb{Z}^+ \wedge m < -1 \wedge n - 1 \leq q \wedge P_q[x, 0] \neq 0$, then

$$\begin{aligned} & \int (c x)^m P_q[x] (a + b x^n)^p dx \rightarrow \\ & P_q[x, 0] \int (c x)^m (a + b x^n)^p dx + \frac{1}{c} \int (c x)^{m+1} \frac{P_q[x] - P_q[x, 0]}{x} (a + b x^n)^p dx \rightarrow \\ & \frac{P_q[x, 0] (c x)^{m+1} (a + b x^n)^{p+1}}{a c (m+1)} + \frac{1}{2 a c (m+1)} \int (c x)^{m+1} \left(2 a (m+1) \frac{P_q[x] - P_q[x, 0]}{x} - 2 b P_q[x, 0] (m+n(p+1)+1) x^{n-1} \right) (a + b x^n)^p dx \end{aligned}$$

Program code:

```
Int[(c_.*x_)^m_*Pq_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  With[{Pq0=Coeff[Pq,x,0]},
    Pq0*(c*x)^(m+1)*(a+b*x^n)^(p+1)/(a*c*(m+1)) +
    1/(2*a*c*(m+1))*Int[(c*x)^(m+1)*ExpandToSum[2*a*(m+1)*(Pq-Pq0)/x-2*b*Pq0*(m+n*(p+1)+1)*x^(n-1),x]*(a+b*x^n)^p,x] /;
  NeQ[Pq0,0] /;
  FreeQ[{a,b,c,p},x] && PolyQ[Pq,x] && IGtQ[n,0] && LtQ[m,-1] && LeQ[n-1,Expon[Pq,x]]
```

$$\mathbf{2:} \int (c x)^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^+ \wedge q - n \geq 0 \wedge m + q + n p + 1 \neq 0$$

Reference: G&R 2.110.5, CRC 88a

Derivation: Algebraic expansion and binomial recurrence 3a

Reference: G&R 2.104

Note: This rule reduces the degree of the polynomial in the resulting integrand.

Rule: If $n \in \mathbb{Z}^+ \wedge m + q + n p + 1 \neq 0 \wedge q - n \geq 0$, then

$$\begin{aligned} & \int (c x)^m P_q[x] (a + b x^n)^p dx \rightarrow \\ & \frac{P_q[x, q]}{c^q} \int (c x)^{m+q} (a + b x^n)^p + \int (c x)^m (P_q[x] - P_q[x, q] x^q) (a + b x^n)^p dx dx \rightarrow \\ & \frac{P_q[x, q] (c x)^{m+q-n+1} (a + b x^n)^{p+1}}{b c^{q-n+1} (m + q + n p + 1)} + \\ & \frac{1}{b (m + q + n p + 1)} \int (c x)^m (b (m + q + n p + 1) (P_q[x] - P_q[x, q] x^q) - a P_q[x, q] (m + q - n + 1) x^{q-n}) (a + b x^n)^p dx \end{aligned}$$

Program code:

```
Int[(c_.*x_)^m_.*Pq_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  With[{q=Expon[Pq,x]},
    With[{Pqq=Coeff[Pq,x,q]},
      Pqq*(c*x)^(m+q-n+1)*(a+b*x^n)^(p+1)/(b*c^(q-n+1)*(m+q+n*p+1)) +
      1/(b*(m+q+n*p+1))*Int[(c*x)^m*ExpandToSum[b*(m+q+n*p+1)*(Pq-Pqq*x^q)-a*Pqq*(m+q-n+1)*x^(q-n),x]*(a+b*x^n)^p,x]] /;
    NeQ[m+q+n*p+1,0] && q-n>=0 && (IntegerQ[2*p] || IntegerQ[p+(q+1)/(2*n)])] /;
    FreeQ[{a,b,c,m,p},x] && PolyQ[Pq,x] && IGtQ[n,0]
```

$$2. \int (c x)^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^-$$

$$1. \int (c x)^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^- \wedge m \in \mathbb{Q}$$

$$\text{1: } \int x^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^- \wedge m \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } F[x] = -\text{Subst}\left[\frac{F[x^{-1}]}{x^2}, x, \frac{1}{x}\right] \partial_x \frac{1}{x}$$

Note: $x^q P_q[x^{-1}]$ is a polynomial in x .

Rule: If $n \in \mathbb{Z}^- \wedge m \in \mathbb{Z}$, then

$$\int x^m P_q[x] (a + b x^n)^p dx \rightarrow -\text{Subst}\left[\int \frac{x^q P_q[x^{-1}] (a + b x^{-n})^p}{x^{m+q+2}} dx, x, \frac{1}{x}\right]$$

Program code:

```
Int[x_^m_.*Pq_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  With[{q=Expon[Pq,x]},
    -Subst[Int[ExpandToSum[x^q*ReplaceAll[Pq,x->x^(-1)],x]*(a+b*x^(-n))^p/x^(m+q+2),x],x,1/x]] /;
  FreeQ[{a,b,p},x] && PolyQ[Pq,x] && ILtQ[n,0] && IntegerQ[m]
```

$$\mathbf{2:} \int (c x)^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^- \wedge m \in \mathbb{F}$$

Derivation: Integration by substitution

Basis: If $g > 1$, then $(c x)^m F[x] = -\frac{g}{c} \text{Subst}\left[\frac{F[c^{-1} x^{-g}]}{x^{g(m+1)+1}}, x, \frac{1}{(c x)^{1/g}}\right] \partial_x \frac{1}{(c x)^{1/g}}$

Note: $x^{gq} P_q[c^{-1} x^{-g}]$ is a polynomial in x .

Rule: If $n \in \mathbb{Z}^- \wedge m \in \mathbb{F}$, let $g = \text{Denominator}[m]$, then

$$\int (c x)^m P_q[x] (a + b x^n)^p dx \rightarrow -\frac{g}{c} \text{Subst}\left[\int \frac{x^{gq} P_q[c^{-1} x^{-g}] (a + b c^{-n} x^{-gn})^p}{x^{g(m+q+1)+1}} dx, x, \frac{1}{(c x)^{1/g}}\right]$$

Program code:

```
Int[(c_.*x_)^m_.*Pq_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  With[{g=Denominator[m],q=Expon[Pq,x]},
    -g/c*Subst[Int[ExpandToSum[x^(g*q)*ReplaceAll[Pq,x->c^(-1)*x^(-g)],x]*
      (a+b*c^(-n)*x^(-g*n))^p/x^(g*(m+q+1)+1),x],x,1/(c*x)^(1/g)]] /;
  FreeQ[{a,b,c,p},x] && PolyQ[Pq,x] && ILtQ[n,0] && FractionQ[m]
```

$$2: \int (c x)^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{Z}^- \wedge m \notin \mathbb{Q}$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: } \partial_x \left((c x)^m (x^{-1})^m \right) = 0$$

$$\text{Basis: } F[x] = -\text{Subst} \left[\frac{F[x^{-1}]}{x^2}, x, \frac{1}{x} \right] \partial_x \frac{1}{x}$$

Note: $x^q P_q[x^{-1}]$ is a polynomial in x .

Rule: If $n \in \mathbb{Z}^- \wedge m \notin \mathbb{Q}$, then

$$\begin{aligned} \int (c x)^m P_q[x] (a + b x^n)^p dx &\rightarrow (c x)^m (x^{-1})^m \int \frac{P_q[x] (a + b x^n)^p}{(x^{-1})^m} dx \\ &\rightarrow - (c x)^m (x^{-1})^m \text{Subst} \left[\int \frac{x^q P_q[x^{-1}] (a + b x^{-n})^p}{x^{m+q+2}} dx, x, \frac{1}{x} \right] \end{aligned}$$

Program code:

```
Int[(c_.*x_)^m_*Pq_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  With[{q=Expon[Pq,x]},
    -(c*x)^m*(x^(-1))^m*Subst[Int[ExpandToSum[x^q*ReplaceAll[Pq,x->x^(-1)],x]*(a+b*x^(-n))^p/x^(m+q+2),x],x,1/x]] /;
  FreeQ[{a,b,c,m,p},x] && PolyQ[Pq,x] && ILtQ[n,0] && Not[RationalQ[m]]
```

$$8. \int (c x)^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{F}$$

$$1: \int x^m P_q[x] (a + b x^n)^p dx \text{ when } n \in \mathbb{F}$$

Derivation: Integration by substitution

Basis: If $g \in \mathbb{Z}^+$, then $x^m P_q[x] F[x^n] = g \text{ Subst}[x^{g(m+1)-1} P_q[x^g] F[x^{g n}], x, x^{1/g}] \partial_x x^{1/g}$

Rule: If $n \in \mathbb{F}$, let $g = \text{Denominator}[n]$, then

$$\int x^m P_q[x] (a + b x^n)^p dx \rightarrow g \text{ Subst}\left[\int x^{g(m+1)-1} P_q[x^g] (a + b x^{g n})^p dx, x, x^{1/g}\right]$$

Program code:

```
Int[x_^m_.*Pq_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  With[{g=Denominator[n]},
    g*Subst[Int[x^(g*(m+1)-1)*ReplaceAll[Pq,x->x^g]*(a+b*x^(g*n))^p,x],x,x^(1/g)]] /;
  FreeQ[{a,b,m,p},x] && PolyQ[Pq,x] && FractionQ[n]
```

2: $\int (c x)^m P_q[x] (a + b x^n)^p dx$ when $n \in \mathbb{F}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(c x)^m}{x^m} == 0$

Basis: $\frac{(c x)^m}{x^m} == \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}}$

Rule: If $n \in \mathbb{F}$, then

$$\int (c x)^m P_q[x] (a + b x^n)^p dx \rightarrow \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m P_q[x] (a + b x^n)^p dx$$

Program code:

```
Int[(c_*x_)^m_*Pq_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m*Pq*(a+b*x^n)^p,x] /;
FreeQ[{a,b,c,m,p},x] && PolyQ[Pq,x] && FractionQ[n]
```


9. $\int (c x)^m P_q[x^n] (a + b x^n)^p dx$ when $\frac{n}{m+1} \in \mathbb{Z}$

1: $\int x^m P_q[x^n] (a + b x^n)^p dx$ when $\frac{n}{m+1} \in \mathbb{Z}$

Derivation: Integration by substitution

Basis: If $\frac{n}{m+1} \in \mathbb{Z}$, then $x^m F[x^n] = \frac{1}{m+1} \text{Subst}[F[x^{\frac{n}{m+1}}], x, x^{m+1}] \partial_x x^{m+1}$

Rule: If $\frac{n}{m+1} \in \mathbb{Z}$

$$\int x^m P_q[x^n] (a + b x^n)^p dx \rightarrow \frac{1}{m+1} \text{Subst}\left[\int P_q\left[x^{\frac{n}{m+1}}\right] (a + b x^{\frac{n}{m+1}})^p dx, x, x^{m+1}\right]$$

Program code:

```
Int[x_^m_.*Pq_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  1/(m+1)*Subst[Int[ReplaceAll[SubstFor[x^n,Pq,x],x->x^Simplify[n/(m+1)]]*(a+b*x^Simplify[n/(m+1)])^p,x],x,x^(m+1)] /;
FreeQ[{a,b,m,n,p},x] && PolyQ[Pq,x^n] && IntegerQ[Simplify[n/(m+1)]] && Not[IntegerQ[n]]
```

2: $\int (c x)^m P_q[x^n] (a + b x^n)^p dx$ when $\frac{n}{m+1} \in \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(c x)^m}{x^m} = 0$

Basis: $\frac{(c x)^m}{x^m} = \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}}$

Rule: If $\frac{n}{m+1} \in \mathbb{Z}$, then

$$\int (c x)^m P_q[x^n] (a + b x^n)^p dx \rightarrow \frac{c^{\text{IntPart}[m]} (c x)^{\text{FracPart}[m]}}{x^{\text{FracPart}[m]}} \int x^m P_q[x^n] (a + b x^n)^p dx$$

Program code:

```
Int[(c*x_)^m_*Pq_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  c^IntPart[m]*(c*x)^FracPart[m]/x^FracPart[m]*Int[x^m*Pq*(a+b*x^n)^p,x] /;
FreeQ[{a,b,c,m,n,p},x] && PolyQ[Pq,x^n] && IntegerQ[Simplify[n/(m+1)]] && Not[IntegerQ[n]]
```

10: $\int (c x)^m P_q[x] (a + b x^n)^p dx$

Derivation: Algebraic expansion

Rule:

$$\int (c x)^m P_q[x] (a + b x^n)^p dx \rightarrow \int \text{ExpandIntegrand}[(c x)^m P_q[x] (a + b x^n)^p, x] dx$$

Program code:

```
Int[(c_.*x_)^m_.*Pq_*(a_+b_.*x_^n_)^p_,x_Symbol] :=
  Int[ExpandIntegrand[(c*x)^m*Pq*(a+b*x^n)^p,x],x] /;
FreeQ[{a,b,c,m,n,p},x] && (PolyQ[Pq,x] || PolyQ[Pq,x^n]) && Not[IGtQ[m,0]]
```

S: $\int u^m P_q[v^n] (a + b v^n)^p dx$ when $v == f + g x \wedge u == h v$

Derivation: Integration by substitution and piecewise constant extraction

Basis: If $u == h v$, then $\partial_x \frac{u^m}{v^m} == 0$

Rule: If $v == f + g x \wedge u == h v$, then

$$\int u^m P_q[v^n] (a + b v^n)^p dx \rightarrow \frac{u^m}{g v^m} \text{Subst}\left[\int x^m P_q[x^n] (a + b x^n)^p dx, x, v\right]$$

Program code:

```
Int[u^m_.*Pq_*(a_+b_.*v_^n_)^p_,x_Symbol] :=
  u^m/(Coeff[v,x,1]*v^m)*Subst[Int[x^m*SubstFor[v,Pq,x]*(a+b*x^n)^p,x],x,v] /;
FreeQ[{a,b,m,n,p},x] && LinearPairQ[u,v,x] && PolyQ[Pq,v^n]
```

Rules for integrands of the form $(h x)^m P_q[x] (a + b x^n)^p (c + d x^n)^q$

1. $\int (c x)^m P_q[x] (a_1 + b_1 x^n)^p (a_2 + b_2 x^n)^p dx$ when $a_2 b_1 + a_1 b_2 == 0$

1: $\int (c x)^m P_q[x] (a_1 + b_1 x^n)^p (a_2 + b_2 x^n)^p dx$ when $a_2 b_1 + a_1 b_2 == 0 \wedge (p \in \mathbb{Z} \vee a_1 > 0 \wedge a_2 > 0)$

Derivation: Algebraic simplification

Basis: If $a_2 b_1 + a_1 b_2 == 0 \wedge (p \in \mathbb{Z} \vee a_1 > 0 \wedge a_2 > 0)$, then $(a_1 + b_1 x^n)^p (a_2 + b_2 x^n)^p == (a_1 a_2 + b_1 b_2 x^{2n})^p$

Rule: If $a_2 b_1 + a_1 b_2 == 0 \wedge (p \in \mathbb{Z} \vee a_1 > 0 \wedge a_2 > 0)$, then

$$\int (c x)^m P_q[x] (a_1 + b_1 x^n)^p (a_2 + b_2 x^n)^p dx \rightarrow \int (c x)^m P_q[x] (a_1 a_2 + b_1 b_2 x^{2n})^p dx$$

Program code:

```
Int[(c_.*x_)^m_.*Pq_*(a1_+b1_.*x_^n_.)^p_.*(a2_+b2_.*x_^n_.)^p_.,x_Symbol] :=
  Int[(c*x)^m*Pq*(a1*a2+b1*b2*x^(2*n))^p,x] /;
FreeQ[{a1,b1,a2,b2,c,m,n,p},x] && PolyQ[Pq,x] && EqQ[a2*b1+a1*b2,0] && (IntegerQ[p] || GtQ[a1,0] && GtQ[a2,0])
```

2: $\int (c x)^m P_q[x] (a_1 + b_1 x^n)^p (a_2 + b_2 x^n)^p dx$ when $a_2 b_1 + a_1 b_2 == 0$

Derivation: Piecewise constant extraction

Basis: If $a_2 b_1 + a_1 b_2 == 0$, then $\partial_x \frac{(a_1+b_1 x^n)^p (a_2+b_2 x^n)^p}{(a_1 a_2+b_1 b_2 x^{2n})^p} == 0$

Rule: If $a_2 b_1 + a_1 b_2 == 0$, then

$$\int (c x)^m P_q[x] (a_1 + b_1 x^n)^p (a_2 + b_2 x^n)^p dx \rightarrow \frac{(a_1 + b_1 x^n)^{\text{FracPart}[p]} (a_2 + b_2 x^n)^{\text{FracPart}[p]}}{(a_1 a_2 + b_1 b_2 x^{2n})^{\text{FracPart}[p]}} \int (c x)^m P_q[x] (a_1 a_2 + b_1 b_2 x^{2n})^p dx$$

Program code:

```
Int[(c_.x_)^m_.*Pq_*(a1+b1_.x_^n_.)^p_.*(a2+b2_.x_^n_.)^p_,x_Symbol] :=
  (a1+b1*x^n)^FracPart[p]*(a2+b2*x^n)^FracPart[p]/(a1*a2+b1*b2*x^(2*n))^FracPart[p]*
  Int[(c*x)^m*Pq*(a1*a2+b1*b2*x^(2*n))^p,x] /;
FreeQ[{a1,b1,a2,b2,c,m,n,p},x] && PolyQ[Pq,x] && EqQ[a2*b1+a1*b2,0] && Not[EqQ[n,1] && LinearQ[Pq,x]]
```

2: $\int (h x)^m (e + f x^n + g x^{2n}) (a + b x^n)^p (c + d x^n)^p dx$ when $a c f (m+1) = e (b c + a d) (m + n (p+1) + 1) \wedge a c g (m+1) = b d e (m + 2 n (p+1) + 1) \wedge m \neq -1$

Rule: If

$a c f (m+1) = e (b c + a d) (m + n (p+1) + 1) \wedge a c g (m+1) = b d e (m + 2 n (p+1) + 1) \wedge m \neq -1$,
then

$$\int (h x)^m (e + f x^n + g x^{2n}) (a + b x^n)^p (c + d x^n)^p dx \rightarrow \frac{e (h x)^{m+1} (a + b x^n)^{p+1} (c + d x^n)^{p+1}}{a c h (m+1)}$$

Program code:

```
Int[(h_.x_)^m_.*(e+f_.x_^n_.+g_.x_^n2_.)*(a+b_.x_^n_.)^p_.*(c+d_.x_^n_.)^p_,x_Symbol] :=
  e*(h*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(p+1)/(a*c*h*(m+1)) /;
FreeQ[{a,b,c,d,e,f,g,h,m,n,p},x] && EqQ[n2,2*n] && EqQ[a*c*f*(m+1)-e*(b*c+a*d)*(m+n*(p+1)+1),0] &&
EqQ[a*c*g*(m+1)-b*d*e*(m+2*n*(p+1)+1),0] && NeQ[m,-1]
```

```
Int[(h_.x_)^m_.*(e+g_.x_^n2_.)*(a+b_.x_^n_.)^p_.*(c+d_.x_^n_.)^p_,x_Symbol] :=
  e*(h*x)^(m+1)*(a+b*x^n)^(p+1)*(c+d*x^n)^(p+1)/(a*c*h*(m+1)) /;
FreeQ[{a,b,c,d,e,g,h,m,n,p},x] && EqQ[n2,2*n] && EqQ[m+n*(p+1)+1,0] && EqQ[a*c*g*(m+1)-b*d*e*(m+2*n*(p+1)+1),0] &&
NeQ[m,-1]
```