Rules for integrands of the form $(a + b x)^m (c + d x)^n (e + f x)^p$
when $b c - a d \neq 0 \wedge b e - a f \neq 0 \wedge d e - c f \neq 0$

0: $\int \dfrac{(e + f x)^p}{(a + b x)\,(c + d x)} \, dx$ when $p \in \mathbb{Z}$

Derivation: Algebraic expansion

Rule 1.1.1.3.0: If $p \in \mathbb{Z}$, then

$$\int \frac{(e + f x)^p}{(a + b x)\,(c + d x)} \, dx \;\longrightarrow\; \int \text{ExpandIntegrand}\left[\frac{(e + f x)^p}{(a + b x)\,(c + d x)}, \; x\right] dx$$

Program code:

```
Int[(e_.+f_.*x_)^p_./((a_.+b_.*x_)*(c_.+d_.*x_)),x_Symbol] :=
  Int[ExpandIntegrand[(e+f*x)^p/((a+b*x)*(c+d*x)),x],x] /;
FreeQ[{a,b,c,d,e,f},x] && IntegerQ[p]
```

1: $\int (a + b x)^m (c + d x)^n (e + f x)^p \, dx$ when $b c + a d == 0 \wedge n == m \wedge m \in \mathbb{Z}$

Derivation: Algebraic simplification

Basis: If $b c + a d == 0 \wedge m \in \mathbb{Z}$, then $(a + b x)^m (c + d x)^m == (a c + b d x^2)^m$

Rule 1.1.1.3.1: If $b c + a d == 0 \wedge n == m \wedge m \in \mathbb{Z}$, then

$$\int (a + b x)^m (c + d x)^n (e + f x)^p \, dx \;\longrightarrow\; \int (a c + b d x^2)^m (e + f x)^p \, dx$$

Program code:

```
Int[(a_+b_.*x_)^m_.*(c_+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  Int[(a*c+b*d*x^2)^m*(e+f*x)^p,x] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && EqQ[b*c+a*d,0] && EqQ[n,m] && IntegerQ[m]
```

2. $\int (a + b\,x)\,(c + d\,x)^n\,(e + f\,x)^p\,dx$

1: $\int (a + b\,x)\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $n + p + 2 \neq 0 \wedge a\,d\,f\,(n + p + 2) - b\,(d\,e\,(n + 1) + c\,f\,(p + 1)) == 0$

Derivation: Quadratic recurrence 2b with $c = 0$: linear recurrence 2 with $a\,d\,f\,(n + p + 2) - b\,(d\,e\,(n + 1) + c\,f\,(p + 1)) == 0$

Rule 1.1.1.3.2.1: If $n + p + 2 \neq 0 \wedge a\,d\,f\,(n + p + 2) - b\,(d\,e\,(n + 1) + c\,f\,(p + 1)) == 0$, then

$$\int (a + b\,x)\,(c + d\,x)^n\,(e + f\,x)^p\,dx \longrightarrow \frac{b\,(c + d\,x)^{n+1}\,(e + f\,x)^{p+1}}{d\,f\,(n + p + 2)}$$

Program code:

```
Int[(a_.+b_.*x_)*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  b*(c+d*x)^(n+1)*(e+f*x)^(p+1)/(d*f*(n+p+2)) /;
FreeQ[{a,b,c,d,e,f,n,p},x] && NeQ[n+p+2,0] && EqQ[a*d*f*(n+p+2)-b*(d*e*(n+1)+c*f*(p+1)),0]
```

2: $\int (a + b\,x)\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $b\,c - a\,d \neq 0 \wedge ((n\,|\,p) \in \mathbb{Z}^- \vee p == 1 \vee p \in \mathbb{Z}^+ \wedge (n \notin \mathbb{Z} \vee 9\,p + 5\,(n + 2) \leq 0 \vee n + p + 1 \geq 0))$

Derivation: Algebraic expansion

Rule 1.1.1.3.2.2: If

$b\,c - a\,d \neq 0 \wedge ((n\,|\,p) \in \mathbb{Z}^- \vee p == 1 \vee p \in \mathbb{Z}^+ \wedge (n \notin \mathbb{Z} \vee 9\,p + 5\,(n + 2) \leq 0 \vee n + p + 1 \geq 0))$, then

$$\int (a + b\,x)\,(c + d\,x)^n\,(e + f\,x)^p\,dx \;\longrightarrow\; \int \mathtt{ExpandIntegrand}\big[(a + b\,x)\,(c + d\,x)^n\,(e + f\,x)^p,\,x\big]\,dx$$

Program code:

```
Int[(a_+b_.*x_)*(d_.*x_)^n_.*(e_+f_.*x_)^p_.,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*x)*(d*x)^n*(e+f*x)^p,x],x] /;
FreeQ[{a,b,d,e,f,n},x] && IGtQ[p,0] && EqQ[b*e+a*f,0] && Not[ILtQ[n+p+2,0] && GtQ[n+2*p,0]]
```

```
Int[(a_+b_.*x_)*(d_.*x_)^n_.*(e_+f_.*x_)^p_.,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*x)*(d*x)^n*(e+f*x)^p,x],x] /;
FreeQ[{a,b,d,e,f,n},x] && IGtQ[p,0] && (NeQ[n,-1] || EqQ[p,1]) && NeQ[b*e+a*f,0] &&
  (Not[IntegerQ[n]] || LtQ[9*p+5*n,0] || GeQ[n+p+1,0] || GeQ[n+p+2,0] && RationalQ[a,b,d,e,f]) && (NeQ[n+p+3,0] || EqQ[p,1])
```

```
Int[(a_.+b_.*x_)*(c_+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*x)*(c+d*x)^n*(e+f*x)^p,x],x] /;
FreeQ[{a,b,c,d,e,f,n},x] && NeQ[b*c-a*d,0] &&
  (ILtQ[n,0] && ILtQ[p,0] || EqQ[p,1] ||
    IGtQ[p,0] && (Not[IntegerQ[n]] || LeQ[9*p+5*(n+2),0] || GeQ[n+p+1,0] || GeQ[n+p+2,0] && RationalQ[a,b,c,d,e,f]))
```

3: $\int (a + b\,x)\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $p < -1 \wedge (n \not< -1 \vee p \in \mathbb{Z})$

Derivation: Quadratic recurrence 2b with c = 0

Derivation: Quadratic recurrence 3b with c = 0, n = p and p = n

Note: If n and p are both negative and one is an integer, best to drive that integer exponent toward -1 since the terms of the antiderivative of $\frac{(a+b\,x)^m}{c+d\,x}$ are of the form $g\,(a+b\,x)^k$.

Rule 1.1.1.3.2.3: If $p < -1 \;\wedge\; (n \not< -1 \vee p \in \mathbb{Z})$, then

$$\int (a+b\,x)\,(c+d\,x)^n\,(e+f\,x)^p\,\mathbb{d}x \;\longrightarrow$$

$$-\frac{(b\,e-a\,f)\,(c+d\,x)^{n+1}\,(e+f\,x)^{p+1}}{f\,(p+1)\,(c\,f-d\,e)} - \frac{a\,d\,f\,(n+p+2)-b\,(d\,e\,(n+1)+c\,f\,(p+1))}{f\,(p+1)\,(c\,f-d\,e)}\int (c+d\,x)^n\,(e+f\,x)^{p+1}\,\mathbb{d}x$$

Program code:

```
Int[(a_.+b_.*x_)*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  -(b*e-a*f)*(c+d*x)^(n+1)*(e+f*x)^(p+1)/(f*(p+1)*(c*f-d*e)) -
  (a*d*f*(n+p+2)-b*(d*e*(n+1)+c*f*(p+1)))/(f*(p+1)*(c*f-d*e))*Int[(c+d*x)^n*(e+f*x)^(p+1),x] /;
FreeQ[{a,b,c,d,e,f,n},x] && LtQ[p,-1] &&
  (Not[LtQ[n,-1]] || IntegerQ[p] || Not[IntegerQ[n]] || Not[EqQ[e,0]] || Not[EqQ[c,0]] || LtQ[p,n]]])
```

```
Int[(a_.+b_.*x_)*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  -(b*e-a*f)*(c+d*x)^(n+1)*(e+f*x)^(p+1)/(f*(p+1)*(c*f-d*e)) -
  (a*d*f*(n+p+2)-b*(d*e*(n+1)+c*f*(p+1)))/(f*(p+1)*(c*f-d*e))*Int[(c+d*x)^n*(e+f*x)^Simplify[p+1],x] /;
FreeQ[{a,b,c,d,e,f,n,p},x] && Not[RationalQ[p]] && SumSimplerQ[p,1]
```

4: $\int (a + b\,x)\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $n + p + 2 \neq 0$

Derivation: Quadratic recurrence 2b with $c = 0$: linear recurrence 2

Rule 1.1.1.3.2.4: If $n + p + 2 \neq 0$, then

$$\int (a + b\,x)\,(c + d\,x)^n\,(e + f\,x)^p\,dx \longrightarrow$$

$$\frac{b\,(c + d\,x)^{n+1}\,(e + f\,x)^{p+1}}{d\,f\,(n + p + 2)} + \frac{a\,d\,f\,(n + p + 2) - b\,(d\,e\,(n + 1) + c\,f\,(p + 1))}{d\,f\,(n + p + 2)}\int (c + d\,x)^n\,(e + f\,x)^p\,dx$$

Program code:

```
Int[(a_.+b_.*x_)*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  b*(c+d*x)^(n+1)*(e+f*x)^(p+1)/(d*f*(n+p+2)) +
  (a*d*f*(n+p+2)-b*(d*e*(n+1)+c*f*(p+1)))/(d*f*(n+p+2))*Int[(c+d*x)^n*(e+f*x)^p,x] /;
FreeQ[{a,b,c,d,e,f,n,p},x] && NeQ[n+p+2,0]
```

3: $\int (a + b\,x)^2\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when

$n + p + 2 \neq 0 \wedge n + p + 3 \neq 0 \wedge$
$d\,f\,(n + p + 2)\,(a^2\,d\,f\,(n + p + 3) - b\,(b\,c\,e + a\,(d\,e\,(n + 1) + c\,f\,(p + 1)))) - b\,(d\,e\,(n + 1) + c\,f\,(p + 1))\,(a\,d\,f\,(n + p + 4) - b\,(d\,e\,(n + 2) + c\,f\,(p + 2))) == 0$

Derivation: Nondegenerate trilinear recurrence 2 with A = a and B = b : quadratic recurrence 2b with $c = 0$: linear recurrence 2 with a d f $(n + p + 2) - b\ (d\ e\ (n + 1) + c\ f\ (p + 1)) == 0$

Rule 1.1.1.3.3: If $n + p + 2 \neq 0 \wedge n + p + 3 \neq 0 \wedge$, then

$$d\,f\,(n + p + 2)\,\left(a^2\,d\,f\,(n + p + 3) - b\,(b\,c\,e + a\,(d\,e\,(n + 1) + c\,f\,(p + 1)))\right) -$$
$$b\,(d\,e\,(n + 1) + c\,f\,(p + 1))\,(a\,d\,f\,(n + p + 4) - b\,(d\,e\,(n + 2) + c\,f\,(p + 2))) == 0$$
$$\int (a + b\,x)^2\,(c + d\,x)^n\,(e + f\,x)^p\,dx \longrightarrow$$

$$\left(\left(b\,(c+d\,x)^{n+1}\,(e+f\,x)^{p+1}\,\left(2\,a\,d\,f\,(n+p+3)-b\,\left(d\,e\,(n+2)+c\,f\,(p+2)\right)+b\,d\,f\,(n+p+2)\,x\right)\right)\Big/\left(d^2\,f^2\,(n+p+2)\,(n+p+3)\right)\right)$$

Program code:

```
Int[(a_.+b_.*x_)^2*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  b*(c+d*x)^(n+1)*(e+f*x)^(p+1)*(2*a*d*f*(n+p+3)-b*(d*e*(n+2)+c*f*(p+2))+b*d*f*(n+p+2)*x)/(d^2*f^2*(n+p+2)*(n+p+3)) /;
FreeQ[{a,b,c,d,e,f,n,p},x] && NeQ[n+p+2,0] && NeQ[n+p+3,0] &&
  EqQ[d*f*(n+p+2)*(a^2*d*f*(n+p+3)-b*(b*c*e+a*(d*e*(n+1)+c*f*(p+1))))-b*(d*e*(n+1)+c*f*(p+1))*(a*d*f*(n+p+4)-b*(d*e*(n+2)+c*f*(p+2))),
```

4: $\int (a+b\,x)^m\,(c+d\,x)^n\,(f\,x)^p\,dx$ when $b\,c+a\,d == 0 \wedge m-n == 1$

Derivation: Algebraic expansion

Note: Integrals of this form can be expressed as the sum of two hypergeometric functions.

Rule 1.1.1.3.4: If $b\,c+a\,d == 0 \wedge m-n == 1$, then

$$\int (a+b\,x)^m\,(c+d\,x)^n\,(f\,x)^p\,dx \;\longrightarrow\; a\int (a+b\,x)^n\,(c+d\,x)^n\,(f\,x)^p\,dx + \frac{b}{f}\int (a+b\,x)^n\,(c+d\,x)^n\,(f\,x)^{p+1}\,dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_.*(c_.+d_.*x_)^n_.*(f_.*x_)^p_.,x_Symbol] :=
  a*Int[(a+b*x)^n*(c+d*x)^n*(f*x)^p,x] + b/f*Int[(a+b*x)^n*(c+d*x)^n*(f*x)^(p+1),x] /;
FreeQ[{a,b,c,d,f,m,n,p},x] && EqQ[b*c+a*d,0] && EqQ[m-n-1,0] && Not[RationalQ[p]] && Not[IGtQ[m,0]] && NeQ[m+n+p+2,0]
```

5. $\int \dfrac{(e+f\,x)^p}{(a+b\,x)\,(c+d\,x)}\,dx$ when $p \notin \mathbb{Z}$

   1. $\int \dfrac{(e+f\,x)^p}{(a+b\,x)\,(c+d\,x)}\,dx$ when $p > 0$

1: $\displaystyle\int \frac{(e+fx)^p}{(a+bx)\ (c+dx)}\ dx$ when $0 < p < 1$

Derivation: Algebraic expansion

Basis: $\displaystyle\frac{e+f\,x}{(a+b\,x)\ (c+d\,x)} == \frac{b\,e-a\,f}{(b\,c-a\,d)\ (a+b\,x)} - \frac{d\,e-c\,f}{(b\,c-a\,d)\ (c+d\,x)}$

Rule 1.1.1.3.5.1.1: If $0 < p < 1$, then

$$\int \frac{(e+fx)^p}{(a+bx)\ (c+dx)}\ dx \;\longrightarrow\; \frac{b\,e-a\,f}{b\,c-a\,d}\int\frac{(e+fx)^{p-1}}{a+bx}\,dx - \frac{d\,e-c\,f}{b\,c-a\,d}\int\frac{(e+fx)^{p-1}}{c+dx}\,dx$$

Program code:

```
Int[(e_.+f_.*x_)^p_./((a_.+b_.*x_)*(c_.+d_.*x_)),x_Symbol] :=
  (b*e-a*f)/(b*c-a*d)*Int[(e+f*x)^(p-1)/(a+b*x),x] -
  (d*e-c*f)/(b*c-a*d)*Int[(e+f*x)^(p-1)/(c+d*x),x] /;
FreeQ[{a,b,c,d,e,f},x] && LtQ[0,p,1]
```

2: $\displaystyle\int \frac{(e + f\,x)^p}{(a + b\,x)\,(c + d\,x)}\,dx$ when $p > 1$

Derivation: Nondegenerate trilinear recurrence 2 with A $=$ a and B $=$ b

Rule 1.1.1.3.5.1.2: If $p > 1$, then

$$\int \frac{(e + f\,x)^p}{(a + b\,x)\,(c + d\,x)}\,dx \;\longrightarrow\; \frac{f\,(e + f\,x)^{p-1}}{b\,d\,(p - 1)} + \frac{1}{b\,d}\int \frac{\left(b\,d\,e^2 - a\,c\,f^2 + f\,(2\,b\,d\,e - b\,c\,f - a\,d\,f)\,x\right)\,(e + f\,x)^{p-2}}{(a + b\,x)\,(c + d\,x)}\,dx$$

Program code:

```
Int[(e_.+f_.*x_)^p_/((a_.+b_.*x_)*(c_.+d_.*x_)),x_Symbol] :=
  f*(e+f*x)^(p-1)/(b*d*(p-1)) +
  1/(b*d)*Int[(b*d*e^2-a*c*f^2+f*(2*b*d*e-b*c*f-a*d*f)*x)*(e+f*x)^(p-2)/((a+b*x)*(c+d*x)),x] /;
FreeQ[{a,b,c,d,e,f},x] && GtQ[p,1]
```

2: $\displaystyle \int \frac{(e+f\,x)^p}{(a+b\,x)\,(c+d\,x)}\,dx$ when $p < -1$

Derivation: Nondegenerate trilinear recurrence 3 with $A = 1$ and $B = 0$

Rule 1.1.1.3.5.2: If $p < -1$, then

$$\int \frac{(e+f\,x)^p}{(a+b\,x)\,(c+d\,x)}\,dx \longrightarrow$$

$$\frac{f\,(e+f\,x)^{p+1}}{(p+1)\,(b\,e-a\,f)\,(d\,e-c\,f)} + \frac{1}{(b\,e-a\,f)\,(d\,e-c\,f)}\int \frac{(b\,d\,e-b\,c\,f-a\,d\,f-b\,d\,f\,x)\,(e+f\,x)^{p+1}}{(a+b\,x)\,(c+d\,x)}\,dx$$

Program code:

```
Int[(e_.+f_.*x_)^p_/((a_.+b_.*x_)*(c_.+d_.*x_)),x_Symbol] :=
  f*(e+f*x)^(p+1)/((p+1)*(b*e-a*f)*(d*e-c*f)) +
  1/((b*e-a*f)*(d*e-c*f))*Int[(b*d*e-b*c*f-a*d*f-b*d*f*x)*(e+f*x)^(p+1)/((a+b*x)*(c+d*x)),x] /;
FreeQ[{a,b,c,d,e,f},x] && LtQ[p,-1]
```

3: $\int \dfrac{(e+f\,x)^p}{(a+b\,x)\,(c+d\,x)}\,dx$ when $p \notin \mathbb{Z}$

Derivation: Algebraic expansion

Basis: $\dfrac{1}{(a+b\,x)\,(c+d\,x)} == \dfrac{b}{(b\,c-a\,d)\,(a+b\,x)} - \dfrac{d}{(b\,c-a\,d)\,(c+d\,x)}$

Rule 1.1.1.3.5.3: If $p \notin \mathbb{Z}$, then

$$\int \dfrac{(e+f\,x)^p}{(a+b\,x)\,(c+d\,x)}\,dx \;\longrightarrow\; \dfrac{b}{b\,c-a\,d} \int \dfrac{(e+f\,x)^p}{a+b\,x}\,dx - \dfrac{d}{b\,c-a\,d} \int \dfrac{(e+f\,x)^p}{c+d\,x}\,dx$$

Program code:

```
Int[(e_.+f_.*x_)^p_/((a_.+b_.*x_)*(c_.+d_.*x_)),x_Symbol] :=
  b/(b*c-a*d)*Int[(e+f*x)^p/(a+b*x),x] -
  d/(b*c-a*d)*Int[(e+f*x)^p/(c+d*x),x] /;
FreeQ[{a,b,c,d,e,f,p},x] && Not[IntegerQ[p]]
```

6: $\displaystyle\int \frac{(c + d\,x)^n\,(e + f\,x)^p}{a + b\,x}\,dx$　when $n \in \mathbb{Z}^+ \wedge p < -1$

Derivation: Algebraic expansion

Rule 1.1.1.3.6: If $n \in \mathbb{Z}^+ \wedge p < -1$, then

$$\int \frac{(c + d\,x)^n\,(e + f\,x)^p}{a + b\,x}\,dx \;\longrightarrow$$

$$\int (e + f\,x)^{\text{FractionalPart}[p]}\,\text{ExpandIntegrand}\!\left[\frac{(c + d\,x)^n\,(e + f\,x)^{\text{IntegerPart}[p]}}{a + b\,x},\,x\right]dx$$

Program code:

```
Int[(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_/(a_.+b_.*x_),x_Symbol] :=
  Int[ExpandIntegrand[(e+f*x)^FractionalPart[p],(c+d*x)^n*(e+f*x)^IntegerPart[p]/(a+b*x),x],x] /;
FreeQ[{a,b,c,d,e,f},x] && IGtQ[n,0] && LtQ[p,-1] && FractionQ[p]
```

7: $\displaystyle\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$　when $(m \mid n) \in \mathbb{Z} \wedge (p \in \mathbb{Z} \vee (m > 0 \wedge n \geq -1))$

Derivation: Algebraic expansion

Rule 1.1.1.3.7: If $(m \mid n) \in \mathbb{Z} \wedge (p \in \mathbb{Z} \vee (m > 0 \wedge n \geq -1))$, then

$$\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx \;\longrightarrow\; \int \text{ExpandIntegrand}\!\left[(a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p,\,x\right]dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_.*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*x)^m*(c+d*x)^n*(e+f*x)^p,x],x] /;
FreeQ[{a,b,c,d,e,f,p},x] && IntegersQ[m,n] && (IntegerQ[p] || GtQ[m,0] && GeQ[n,-1])
```

8. $\int (a + b\,x)^2\,(c + d\,x)^n\,(e + f\,x)^p\,dx$

1: $\int (a + b\,x)^2\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $n < -1$

Derivation: ?

Rule 1.1.1.3.8.1: If $n < -1$, then

$$\int (a + b\,x)^2\,(c + d\,x)^n\,(e + f\,x)^p\,dx \;\longrightarrow$$

$$\frac{(b\,c - a\,d)^2\,(c + d\,x)^{n+1}\,(e + f\,x)^{p+1}}{d^2\,(d\,e - c\,f)\,(n + 1)} -$$

$$\frac{1}{d^2\,(d\,e - c\,f)\,(n + 1)}\int (c + d\,x)^{n+1}\,(e + f\,x)^p \cdot$$

$$\left(a^2\,d^2\,f\,(n + p + 2) + b^2\,c\,\left(d\,e\,(n + 1) + c\,f\,(p + 1)\right) - 2\,a\,b\,d\,\left(d\,e\,(n + 1) + c\,f\,(p + 1)\right) - b^2\,d\,\left(d\,e - c\,f\right)\,(n + 1)\,x\right)\,dx$$

Program code:

```
Int[(a_.+b_.*x_)^2*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  (b*c-a*d)^2*(c+d*x)^(n+1)*(e+f*x)^(p+1)/(d^2*(d*e-c*f)*(n+1)) -
  1/(d^2*(d*e-c*f)*(n+1))*Int[(c+d*x)^(n+1)*(e+f*x)^p*
    Simp[a^2*d^2*f*(n+p+2)+b^2*c*(d*e*(n+1)+c*f*(p+1))-2*a*b*d*(d*e*(n+1)+c*f*(p+1))-b^2*d*(d*e-c*f)*(n+1)*x,x],x] /;
FreeQ[{a,b,c,d,e,f,n,p},x] && (LtQ[n,-1] || EqQ[n+p+3,0] && NeQ[n,-1] && (SumSimplerQ[n,1] || Not[SumSimplerQ[p,1]]))
```

2: $\int (a + b\,x)^2\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $n + p + 3 \neq 0$

Derivation: Nondegenerate trilinear recurrence 2 with $A = a$ and $B = b$

Rule 1.1.1.3.8.2: If $n + p + 3 \neq 0$, then

$$\int (a + b\,x)^2\,(c + d\,x)^n\,(e + f\,x)^p\,dx \;\longrightarrow$$

$$\frac{b\,(a+b\,x)\,(c+d\,x)^{n+1}\,(e+f\,x)^{p+1}}{d\,f\,(n+p+3)}\,+$$

$$\frac{1}{d\,f\,(n+p+3)}\int (c+d\,x)^{n}\,(e+f\,x)^{p}\,\cdot$$

$$\left(a^{2}\,d\,f\,(n+p+3)-b\,\big(b\,c\,e+a\,(d\,e\,(n+1)+c\,f\,(p+1))\big)+b\,\big(a\,d\,f\,(n+p+4)-b\,(d\,e\,(n+2)+c\,f\,(p+2))\big)\,x\right)\,dx$$

Program code:

```
Int[(a_.+b_.*x_)^2*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  b*(a+b*x)*(c+d*x)^(n+1)*(e+f*x)^(p+1)/(d*f*(n+p+3)) +
  1/(d*f*(n+p+3))*Int[(c+d*x)^n*(e+f*x)^p*
    Simp[a^2*d*f*(n+p+3)-b*(b*c*e+a*(d*e*(n+1)+c*f*(p+1)))+b*(a*d*f*(n+p+4)-b*(d*e*(n+2)+c*f*(p+2)))*x,x],x] /;
FreeQ[{a,b,c,d,e,f,n,p},x] && NeQ[n+p+3,0]
```

9. $\int \dfrac{(a+bx)^m\,(c+dx)^n}{e+fx}\,dx$ when $m+n+1 == 0 \land -1 < m < 0$

1: $\int \dfrac{1}{(a+bx)^{1/3}\,(c+dx)^{2/3}\,(e+fx)}\,dx$

Rule 1.1.1.3.9.1: Let $q = \left(\dfrac{de-cf}{be-af}\right)^{1/3}$ then

$$\int \dfrac{1}{(a+bx)^{1/3}\,(c+dx)^{2/3}\,(e+fx)}\,dx \longrightarrow$$

$$-\dfrac{\sqrt{3}\,q\,\text{ArcTan}\!\left[\dfrac{1}{\sqrt{3}}+\dfrac{2q\,(a+bx)^{1/3}}{\sqrt{3}\,(c+dx)^{1/3}}\right]}{de-cf} + \dfrac{q\,\text{Log}[e+fx]}{2\,(de-cf)} - \dfrac{3q\,\text{Log}\!\left[q\,(a+bx)^{1/3}-(c+dx)^{1/3}\right]}{2\,(de-cf)}$$

Program code:

```
Int[1/((a_.+b_.*x_)^(1/3)*(c_.+d_.*x_)^(2/3)*(e_.+f_.*x_)),x_Symbol] :=
  With[{q=Rt[(d*e-c*f)/(b*e-a*f),3]},
  -Sqrt[3]*q*ArcTan[1/Sqrt[3]+2*q*(a+b*x)^(1/3)/(Sqrt[3]*(c+d*x)^(1/3))]/(d*e-c*f) +
  q*Log[e+f*x]/(2*(d*e-c*f)) -
  3*q*Log[q*(a+b*x)^(1/3)-(c+d*x)^(1/3)]/(2*(d*e-c*f))] /;
FreeQ[{a,b,c,d,e,f},x]
```

2: $\int \dfrac{1}{\sqrt{a+bx}\,\sqrt{c+dx}\,(e+fx)}\,dx$ when $2bde - f\,(bc+ad) == 0$

Derivation: Integration by substitution

Basis: If $2bde - f\,(bc+ad) == 0$, then

$$\dfrac{1}{\sqrt{a+bx}\,\sqrt{c+dx}\,(e+fx)} == b\,f\,\text{Subst}\!\left[\dfrac{1}{d\,(be-af)^2+b\,f^2\,x^2},\,x,\,\sqrt{a+bx}\,\sqrt{c+dx}\right]\partial_x\left(\sqrt{a+bx}\,\sqrt{c+dx}\right)$$

Rule 1.1.1.3.9.2: If $2bde - f\,(bc+ad) == 0$, then

$$\int \frac{1}{\sqrt{a+b\,x}\ \sqrt{c+d\,x}\ (e+f\,x)}\ dx \ \longrightarrow \ b\,f\,\text{Subst}\left[\int \frac{1}{d\,(b\,e-a\,f)^2 + b\,f^2\,x^2}\ dx,\ x,\ \sqrt{a+b\,x}\ \sqrt{c+d\,x}\ \right]$$

Program code:

```
Int[1/(Sqrt[a_.+b_.*x_]*Sqrt[c_.+d_.*x_]*(e_.+f_.*x_)),x_Symbol] :=
  b*f*Subst[Int[1/(d*(b*e-a*f)^2+b*f^2*x^2),x],x,Sqrt[a+b*x]*Sqrt[c+d*x]] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[2*b*d*e-f*(b*c+a*d),0]
```

3: $\displaystyle\int \frac{(a+b\,x)^m\ (c+d\,x)^n}{e+f\,x}\ dx$ when $m+n+1 == 0 \ \wedge\ -1 < m < 0$

Derivation: Integration by substitution

Basis: If $m+n+1 == 0 \ \wedge\ -1 < m < 0$, let $q = \text{Denominator}[m]$, then

$$\frac{(a+b\,x)^m\ (c+d\,x)^n}{e+f\,x} == q\,\text{Subst}\left[\frac{x^{q\,(m+1)-1}}{b\,e-a\,f-(d\,e-c\,f)\,x^q},\ x,\ \frac{(a+b\,x)^{1/q}}{(c+d\,x)^{1/q}}\right]\ \partial_x\,\frac{(a+b\,x)^{1/q}}{(c+d\,x)^{1/q}}$$

Rule 1.1.1.3.9.3: If $m+n+1 == 0 \ \wedge\ -1 < m < 0$, let $q = \text{Denominator}[m]$, then

$$\int \frac{(a+b\,x)^m\ (c+d\,x)^n}{e+f\,x}\ dx \ \longrightarrow \ q\,\text{Subst}\left[\int \frac{x^{q\,(m+1)-1}}{b\,e-a\,f-(d\,e-c\,f)\,x^q}\ dx,\ x,\ \frac{(a+b\,x)^{1/q}}{(c+d\,x)^{1/q}}\right]$$

Program code:

```
Int[(a_.+b_.*x_)^m_*(c_.+d_.*x_)^n_/(e_.+f_.*x_),x_Symbol] :=
  With[{q=Denominator[m]},
  q*Subst[Int[x^(q*(m+1)-1)/(b*e-a*f-(d*e-c*f)*x^q),x],x,(a+b*x)^(1/q)/(c+d*x)^(1/q)]] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[m+n+1,0] && RationalQ[n] && LtQ[-1,m,0] && SimplerQ[a+b*x,c+d*x]
```

10: $\int (a + b\,x)^m\ (c + d\,x)^n\ (e + f\,x)^p\, dx$ when $m + n + p + 2 == 0\ \wedge\ n > 0$

Derivation: Nondegenerate trilinear recurrence 1 with $A = 1, B = 0$ and $m + n + p + 2 == 0$

Rule 1.1.1.3.10: If $m + n + p + 2 == 0\ \wedge\ n > 0$, then

$$\int (a + b\,x)^m\ (c + d\,x)^n\ (e + f\,x)^p\, dx\ \longrightarrow$$

$$\frac{(a + b\,x)^{m+1}\ (c + d\,x)^n\ (e + f\,x)^{p+1}}{(m + 1)\ (b\,e - a\,f)} - \frac{n\ (d\,e - c\,f)}{(m + 1)\ (b\,e - a\,f)} \int (a + b\,x)^{m+1}\ (c + d\,x)^{n-1}\ (e + f\,x)^p\, dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  (a+b*x)^(m+1)*(c+d*x)^n*(e+f*x)^(p+1)/((m+1)*(b*e-a*f)) -
  n*(d*e-c*f)/((m+1)*(b*e-a*f))*Int[(a+b*x)^(m+1)*(c+d*x)^(n-1)*(e+f*x)^p,x] /;
FreeQ[{a,b,c,d,e,f,m,p},x] && EqQ[m+n+p+2,0] && GtQ[n,0] && Not[SumSimplerQ[p,1] && Not[SumSimplerQ[m,1]]]
```

11. $\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $m + n + p + 3 \,{=}{=}\, 0$

1: $\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $m + n + p + 3 \,{=}{=}\, 0 \,\wedge\, a\,d\,f\,(m + 1) + b\,c\,f\,(n + 1) + b\,d\,e\,(p + 1) \,{=}{=}\, 0 \,\wedge\, m \neq -1$

Derivation: Nondegenerate trilinear recurrence 3 with A = 1 and B = 0

Rule 1.1.1.3.11.1: If $m + n + p + 3 \,{=}{=}\, 0 \,\wedge\, a\,d\,f\,(m + 1) + b\,c\,f\,(n + 1) + b\,d\,e\,(p + 1) \,{=}{=}\, 0 \,\wedge\, m \neq -1$, then

$$\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx \;\longrightarrow\; \frac{b\,(a + b\,x)^{m+1}\,(c + d\,x)^{n+1}\,(e + f\,x)^{p+1}}{(m + 1)\,(b\,c - a\,d)\,(b\,e - a\,f)}$$

Program code:

```
Int[(a_.+b_.*x_)^m_*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  b*(a+b*x)^(m+1)*(c+d*x)^(n+1)*(e+f*x)^(p+1)/((m+1)*(b*c-a*d)*(b*e-a*f)) /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && EqQ[Simplify[m+n+p+3],0] && EqQ[a*d*f*(m+1)+b*c*f*(n+1)+b*d*e*(p+1),0] && NeQ[m,-1]
```

2: $\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $m + n + p + 3 == 0 \;\wedge\; m < -1$

Derivation: Nondegenerate trilinear recurrence 3 with A = 1 and B = 0

Rule 1.1.1.3.11.2: If $m + n + p + 3 == 0 \;\wedge\; m < -1$, then

$$\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx \;\longrightarrow$$

$$\frac{b\,(a + b\,x)^{m+1}\,(c + d\,x)^{n+1}\,(e + f\,x)^{p+1}}{(m+1)\,(b\,c - a\,d)\,(b\,e - a\,f)} + \frac{a\,d\,f\,(m+1) + b\,c\,f\,(n+1) + b\,d\,e\,(p+1)}{(m+1)\,(b\,c - a\,d)\,(b\,e - a\,f)}\int (a + b\,x)^{m+1}\,(c + d\,x)^n\,(e + f\,x)^p\,dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  b*(a+b*x)^(m+1)*(c+d*x)^(n+1)*(e+f*x)^(p+1)/((m+1)*(b*c-a*d)*(b*e-a*f)) +
  (a*d*f*(m+1)+b*c*f*(n+1)+b*d*e*(p+1))/((m+1)*(b*c-a*d)*(b*e-a*f))*Int[(a+b*x)^(m+1)*(c+d*x)^n*(e+f*x)^p,x] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && EqQ[Simplify[m+n+p+3],0] && (LtQ[m,-1] || SumSimplerQ[m,1])
```

12. $\int \left(a + b\, x\right)^m \left(c + d\, x\right)^n \left(e + f\, x\right)^p \, dx$ when $m < -1 \,\wedge\, n > 0$

    1: $\int \left(a + b\, x\right)^m \left(c + d\, x\right)^n \left(e + f\, x\right)^p \, dx$ when $m < -1 \,\wedge\, n > 0 \,\wedge\, p > 0$

Derivation: Nondegenerate trilinear recurrence 1 with A = e and B = f

Rule 1.1.1.3.12.1: If $m < -1 \,\wedge\, n > 0 \,\wedge\, p > 0$, then

$$\int \left(a + b\, x\right)^m \left(c + d\, x\right)^n \left(e + f\, x\right)^p \, dx \longrightarrow$$

$$\frac{\left(a + b\, x\right)^{m+1} \left(c + d\, x\right)^n \left(e + f\, x\right)^p}{b\,(m+1)} - \frac{1}{b\,(m+1)} \int \left(a + b\, x\right)^{m+1} \left(c + d\, x\right)^{n-1} \left(e + f\, x\right)^{p-1} \left(d\, e\, n + c\, f\, p + d\, f\,(n+p)\, x\right) \, dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  (a+b*x)^(m+1)*(c+d*x)^n*(e+f*x)^p/(b*(m+1)) -
  1/(b*(m+1))*Int[(a+b*x)^(m+1)*(c+d*x)^(n-1)*(e+f*x)^(p-1)*Simp[d*e*n+c*f*p+d*f*(n+p)*x,x],x] /;
FreeQ[{a,b,c,d,e,f},x] && LtQ[m,-1] && GtQ[n,0] && GtQ[p,0] && (IntegersQ[2*m,2*n,2*p] || IntegersQ[m,n+p] || IntegersQ[p,m+n])
```

2: $\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $m < -1 \wedge n > 1$

Derivation: ???

Rule 1.1.1.3.12.2: If $m < -1 \wedge n > 1$, then

$$\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx \;\longrightarrow$$

$$\frac{(b\,c - a\,d)\,(a + b\,x)^{m+1}\,(c + d\,x)^{n-1}\,(e + f\,x)^{p+1}}{b\,(b\,e - a\,f)\,(m + 1)} \;+$$

$$\frac{1}{b\,(b\,e - a\,f)\,(m + 1)}\int (a + b\,x)^{m+1}\,(c + d\,x)^{n-2}\,(e + f\,x)^p \;\cdot$$

$$\Big(a\,d\,\big(d\,e\,(n - 1) + c\,f\,(p + 1)\big) + b\,c\,\big(d\,e\,(m - n + 2) - c\,f\,(m + p + 2)\big) + d\,\big(a\,d\,f\,(n + p) + b\,\big(d\,e\,(m + 1) - c\,f\,(m + n + p + 1)\big)\big)\,x\Big)\,dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  (b*c-a*d)*(a+b*x)^(m+1)*(c+d*x)^(n-1)*(e+f*x)^(p+1)/(b*(b*e-a*f)*(m+1)) +
  1/(b*(b*e-a*f)*(m+1))*Int[(a+b*x)^(m+1)*(c+d*x)^(n-2)*(e+f*x)^p*
    Simp[a*d*(d*e*(n-1)+c*f*(p+1))+b*c*(d*e*(m-n+2)-c*f*(m+p+2))+d*(a*d*f*(n+p)+b*(d*e*(m+1)-c*f*(m+n+p+1)))*x,x],x] /;
FreeQ[{a,b,c,d,e,f,p},x] && LtQ[m,-1] && GtQ[n,1] && (IntegersQ[2*m,2*n,2*p] || IntegersQ[m,n+p] || IntegersQ[p,m+n])
```

3: $\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $m < -1 \wedge n > 0$

Derivation: Nondegenerate trilinear recurrence 1 with $A = 1$ and $B = 0$

Rule 1.1.1.3.12.3: If $m < -1 \wedge n > 0$, then

$$\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx \;\longrightarrow$$

$$\frac{(a + b\,x)^{m+1}\,(c + d\,x)^n\,(e + f\,x)^{p+1}}{(m + 1)\,(b\,e - a\,f)} \;-$$

$$\frac{1}{(m+1)\,(b\,e-a\,f)}\int (a+b\,x)^{m+1}\,(c+d\,x)^{n-1}\,(e+f\,x)^p\,\Big(d\,e\,n+c\,f\,(m+p+2)+d\,f\,(m+n+p+2)\,x\Big)\,dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  (a+b*x)^(m+1)*(c+d*x)^n*(e+f*x)^(p+1)/((m+1)*(b*e-a*f)) -
  1/((m+1)*(b*e-a*f))*Int[(a+b*x)^(m+1)*(c+d*x)^(n-1)*(e+f*x)^p*
    Simp[d*e*n+c*f*(m+p+2)+d*f*(m+n+p+2)*x,x],x] /;
FreeQ[{a,b,c,d,e,f,p},x] && LtQ[m,-1] && GtQ[n,0] && (IntegersQ[2*m,2*n,2*p] || IntegersQ[m,n+p] || IntegersQ[p,m+n])
```

13: $\int (a + b\, x)^m\, (c + d\, x)^n\, (e + f\, x)^p\, \mathbb{d}x$ when $m > 1 \,\wedge\, m + n + p + 1 \neq 0 \,\wedge\, m \in \mathbb{Z}$

Derivation: Nondegenerate trilinear recurrence 2 with A = a and B = b

Note: If the integrand has a positive integer exponent, decrementing it, rather than another positive fractional exponent, produces simpler antiderivatives.

Rule 1.1.1.3.13: If $m > 1 \,\wedge\, m + n + p + 1 \neq 0 \,\wedge\, m \in \mathbb{Z}$, then

$$\int (a + b\, x)^m\, (c + d\, x)^n\, (e + f\, x)^p\, \mathbb{d}x \;\longrightarrow$$

$$\frac{b\, (a + b\, x)^{m-1}\, (c + d\, x)^{n+1}\, (e + f\, x)^{p+1}}{d\, f\, (m + n + p + 1)} \; + $$

$$\frac{1}{d\, f\, (m + n + p + 1)}\, \int (a + b\, x)^{m-2}\, (c + d\, x)^n\, (e + f\, x)^p \;\cdot$$

$$\left(a^2\, d\, f\, (m + n + p + 1) - b\, \big(b\, c\, e\, (m - 1) + a\, \big(d\, e\, (n + 1) + c\, f\, (p + 1)\big)\big) + b\, \big(a\, d\, f\, (2\, m + n + p) - b\, \big(d\, e\, (m + n) + c\, f\, (m + p)\big)\big)\, x\right)\, \mathbb{d}x$$

Program code:

```
Int[(a_.+b_.*x_)^m_*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  b*(a+b*x)^(m-1)*(c+d*x)^(n+1)*(e+f*x)^(p+1)/(d*f*(m+n+p+1)) +
  1/(d*f*(m+n+p+1))*Int[(a+b*x)^(m-2)*(c+d*x)^n*(e+f*x)^p*
    Simp[a^2*d*f*(m+n+p+1)-b*(b*c*e*(m-1)+a*(d*e*(n+1)+c*f*(p+1)))+b*(a*d*f*(2*m+n+p)-b*(d*e*(m+n)+c*f*(m+p)))*x,x],x] /;
FreeQ[{a,b,c,d,e,f,n,p},x] && GtQ[m,1] && NeQ[m+n+p+1,0] && IntegerQ[m]
```

14: $\int (a + b\, x)^m\, (c + d\, x)^n\, (e + f\, x)^p\, \mathbb{d}x$ when $m > 0 \,\wedge\, n > 0 \,\wedge\, m + n + p + 1 \neq 0$

Derivation: Nondegenerate trilinear recurrence 2 with A = c and B = d

Rule 1.1.1.3.14: If $m > 0 \,\wedge\, n > 0 \,\wedge\, m + n + p + 1 \neq 0$, then

$$\int (a + b\, x)^m\, (c + d\, x)^n\, (e + f\, x)^p\, \mathbb{d}x \;\longrightarrow$$

$$\frac{(a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^{p+1}}{f\,(m + n + p + 1)} -$$

$$\frac{1}{f\,(m + n + p + 1)}\int (a + b\,x)^{m-1}\,(c + d\,x)^{n-1}\,(e + f\,x)^p\,\big(c\,m\,(b\,e - a\,f) + a\,n\,(d\,e - c\,f) + (d\,m\,(b\,e - a\,f) + b\,n\,(d\,e - c\,f))\,x\big)\,dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_.*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  (a+b*x)^m*(c+d*x)^n*(e+f*x)^(p+1)/(f*(m+n+p+1)) -
  1/(f*(m+n+p+1))*Int[(a+b*x)^(m-1)*(c+d*x)^(n-1)*(e+f*x)^p*
    Simp[c*m*(b*e-a*f)+a*n*(d*e-c*f)+(d*m*(b*e-a*f)+b*n*(d*e-c*f))*x,x],x] /;
FreeQ[{a,b,c,d,e,f,p},x] && GtQ[m,0] && GtQ[n,0] && NeQ[m+n+p+1,0] && (IntegersQ[2*m,2*n,2*p] || (IntegersQ[m,n+p] || IntegersQ[p,m+n]
```

15: $\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $m > 1 \wedge m + n + p + 1 \neq 0$

Derivation: Nondegenerate trilinear recurrence 2 with A = a and B = b

Rule 1.1.1.3.15: If $m > 1 \wedge m + n + p + 1 \neq 0$, then

$$\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx \longrightarrow$$

$$\frac{b\,(a + b\,x)^{m-1}\,(c + d\,x)^{n+1}\,(e + f\,x)^{p+1}}{d\,f\,(m + n + p + 1)} +$$

$$\frac{1}{d\,f\,(m + n + p + 1)}\int (a + b\,x)^{m-2}\,(c + d\,x)^n\,(e + f\,x)^p \cdot$$

$$\big(a^2\,d\,f\,(m + n + p + 1) - b\,(b\,c\,e\,(m - 1) + a\,(d\,e\,(n + 1) + c\,f\,(p + 1))) + b\,(a\,d\,f\,(2\,m + n + p) - b\,(d\,e\,(m + n) + c\,f\,(m + p)))\,x\big)\,dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  b*(a+b*x)^(m-1)*(c+d*x)^(n+1)*(e+f*x)^(p+1)/(d*f*(m+n+p+1)) +
  1/(d*f*(m+n+p+1))*Int[(a+b*x)^(m-2)*(c+d*x)^n*(e+f*x)^p*
    Simp[a^2*d*f*(m+n+p+1)-b*(b*c*e*(m-1)+a*(d*e*(n+1)+c*f*(p+1)))+b*(a*d*f*(2*m+n+p)-b*(d*e*(m+n)+c*f*(m+p)))*x,x],x] /;
FreeQ[{a,b,c,d,e,f,n,p},x] && GtQ[m,1] && NeQ[m+n+p+1,0] && IntegersQ[2*m,2*n,2*p]
```

16: $\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $m < -1$

Derivation: Nondegenerate trilinear recurrence 3 with A = 1 and B = 0

Note: If the integrand has a negative integer exponent, incrementing it, rather than another negative fractional exponent, produces simpler antiderivatives.

Rule 1.1.1.3.16: If $m < -1$, then

$$\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx \longrightarrow$$

$$\frac{b\,(a + b\,x)^{m+1}\,(c + d\,x)^{n+1}\,(e + f\,x)^{p+1}}{(m + 1)\,(b\,c - a\,d)\,(b\,e - a\,f)}\,+$$

$$\frac{1}{(m + 1)\,(b\,c - a\,d)\,(b\,e - a\,f)}\int (a + b\,x)^{m+1}\,(c + d\,x)^n\,(e + f\,x)^p\,\Big(a\,d\,f\,(m + 1) - b\,\big(d\,e\,(m + n + 2) + c\,f\,(m + p + 2)\big) - b\,d\,f\,(m + n + p + 3)\,x\Big)\,dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  b*(a+b*x)^(m+1)*(c+d*x)^(n+1)*(e+f*x)^(p+1)/((m+1)*(b*c-a*d)*(b*e-a*f)) +
  1/((m+1)*(b*c-a*d)*(b*e-a*f))*Int[(a+b*x)^(m+1)*(c+d*x)^n*(e+f*x)^p*
    Simp[a*d*f*(m+1)-b*(d*e*(m+n+2)+c*f*(m+p+2))-b*d*f*(m+n+p+3)*x,x],x] /;
FreeQ[{a,b,c,d,e,f,n,p},x] && LtQ[m,-1] && IntegerQ[m] && (IntegerQ[n] || IntegersQ[2*n,2*p])
```

```
Int[(a_.+b_.*x_)^m_*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  b*(a+b*x)^(m+1)*(c+d*x)^(n+1)*(e+f*x)^(p+1)/((m+1)*(b*c-a*d)*(b*e-a*f)) +
  1/((m+1)*(b*c-a*d)*(b*e-a*f))*Int[(a+b*x)^(m+1)*(c+d*x)^n*(e+f*x)^p*
    Simp[a*d*f*(m+1)-b*(d*e*(m+n+2)+c*f*(m+p+2))-b*d*f*(m+n+p+3)*x,x],x] /;
FreeQ[{a,b,c,d,e,f,n,p},x] && LtQ[m,-1] && IntegersQ[2*m,2*n,2*p]
```

17: $\int \dfrac{(a + b\,x)^m\ (c + d\,x)^n}{e + f\,x}\ dx$ when $m + n + 1 \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Basis: $\dfrac{(a+b\,x)^m}{e+f\,x} == \dfrac{b\ (a+b\,x)^{m-1}}{f} - \dfrac{(b\,e-a\,f)\ (a+b\,x)^{m-1}}{f\ (e+f\,x)}$

Note: Integrands of this form can be expressed in terms of the confluent hypergeometric function 2 F 1 instead of requiring the Appell hypergeometric function.

Rule 1.1.1.3.17: If $m + n + 1 \in \mathbb{Z}^+$, then

$$\int \frac{(a + b\,x)^m\ (c + d\,x)^n}{e + f\,x}\ dx \;\longrightarrow\; \frac{b}{f}\int (a + b\,x)^{m-1}\ (c + d\,x)^n\ dx - \frac{b\,e - a\,f}{f}\int \frac{(a + b\,x)^{m-1}\ (c + d\,x)^n}{e + f\,x}\ dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_*(c_.+d_.*x_)^n_/(e_.+f_.*x_),x_Symbol] :=
  b/f*Int[(a+b*x)^(m-1)*(c+d*x)^n,x] - (b*e-a*f)/f*Int[(a+b*x)^(m-1)*(c+d*x)^n/(e+f*x),x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && IGtQ[Simplify[m+n+1],0] &&
  (GtQ[m,0] || Not[RationalQ[m]] && (SumSimplerQ[m,-1] || Not[SumSimplerQ[n,-1]]))
```

x: $\int (a+b\,x)^m\,(c+d\,x)^n\,(e+f\,x)^p\,dx$ when $p \in \mathbb{Z}^- \wedge m+n+p+2 \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Basis: $(a+b\,x)^m\,(e+f\,x)^p == \frac{b}{f}\,(a+b\,x)^{m-1}\,(e+f\,x)^{p+1} - \frac{b\,e-a\,f}{f}\,(a+b\,x)^{m-1}\,(e+f\,x)^p$

Note: Integrands of this form can be expressed in terms of the confluent hypergeometric function 2 F 1 instead of requiring the Appell hypergeometric function.

Rule 1.1.1.3.x: If $p \in \mathbb{Z}^- \wedge m+n+p+2 \in \mathbb{Z}^+$, then

$$\int (a+b\,x)^m\,(c+d\,x)^n\,(e+f\,x)^p\,dx \longrightarrow \frac{b}{f}\int (a+b\,x)^{m-1}\,(c+d\,x)^n\,(e+f\,x)^{p+1}\,dx - \frac{b\,e-a\,f}{f}\int (a+b\,x)^{m-1}\,(c+d\,x)^n\,(e+f\,x)^p\,dx$$

Program code:

```
(* Int[(a_.+b_.*x_)^m_*(c_.+d_.*x_)^n_*(e_.+f_.*x_)^p_,x_Symbol] :=
  b/f*Int[(a+b*x)^(m-1)*(c+d*x)^n*(e+f*x)^(p+1),x] - (b*e-a*f)/f*Int[(a+b*x)^(m-1)*(c+d*x)^n*(e+f*x)^p,x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && ILtQ[p,0] && IGtQ[m+n+p+2,0] && Not[SimplerQ[c+d*x,a+b*x]] *)
```

18. $\int \frac{(e+f\,x)^p}{(a+b\,x)\,\sqrt{c+d\,x}}\,dx$

  1. $\int \frac{1}{(a+b\,x)\,\sqrt{c+d\,x}\,(e+f\,x)^{1/4}}\,dx$

    1: $\int \frac{1}{(a+b\,x)\,\sqrt{c+d\,x}\,(e+f\,x)^{1/4}}\,dx$ when $-\frac{f}{d\,e-c\,f} > 0$

Derivation: Integration by substitution

Basis: $\dfrac{1}{(a+b\,x)\,\sqrt{c+d\,x}\,\left(e+f\,x\right)^{1/4}} == -4\,\text{Subst}\left[\dfrac{x^2}{\left(b\,e-a\,f-b\,x^4\right)\,\sqrt{c-\frac{d\,e}{f}+\frac{d\,x^4}{f}}}\,,\ x,\ (e+f\,x)^{1/4}\right]\partial_x\,(e+f\,x)^{1/4}$

Rule 1.1.1.3.18.1.1: If $-\dfrac{f}{d\,e-c\,f}>0$, then

$$\int \dfrac{1}{(a+b\,x)\,\sqrt{c+d\,x}\,\left(e+f\,x\right)^{1/4}}\,dx \ \longrightarrow\ -4\,\text{Subst}\left[\int \dfrac{x^2}{\left(b\,e-a\,f-b\,x^4\right)\,\sqrt{c-\frac{d\,e}{f}+\frac{d\,x^4}{f}}}\,dx,\ x,\ \left(e+f\,x\right)^{1/4}\right]$$

Program code:

```
Int[1/((a_.+b_.*x_)*Sqrt[c_.+d_.*x_]*(e_.+f_.*x_)^(1/4)),x_Symbol] :=
  -4*Subst[Int[x^2/((b*e-a*f-b*x^4)*Sqrt[c-d*e/f+d*x^4/f]),x],x,(e+f*x)^(1/4)] /;
FreeQ[{a,b,c,d,e,f},x] && GtQ[-f/(d*e-c*f),0]
```

2: $\displaystyle\int \frac{1}{(a+bx)\ \sqrt{c+dx}\ (e+fx)^{1/4}}\ dx$ when $-\dfrac{f}{de-cf} \ngtr 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{\sqrt{-\dfrac{f\,(c+dx)}{de-cf}}}{\sqrt{c+dx}} == 0$

Rule 1.1.1.3.18.1.2: If $-\dfrac{f}{de-cf} \ngtr 0$, then

$$\int \frac{1}{(a+bx)\ \sqrt{c+dx}\ (e+fx)^{1/4}}\ dx \;\longrightarrow\; \frac{\sqrt{-\dfrac{f\,(c+dx)}{de-cf}}}{\sqrt{c+dx}} \int \frac{1}{(a+bx)\ \sqrt{-\dfrac{cf}{de-cf}-\dfrac{dfx}{de-cf}}\ (e+fx)^{1/4}}\ dx$$

Program code:

```
Int[1/((a_.+b_.*x_)*Sqrt[c_.+d_.*x_]*(e_.+f_.*x_)^(1/4)),x_Symbol] :=
  Sqrt[-f*(c+d*x)/(d*e-c*f)]/Sqrt[c+d*x]*Int[1/((a+b*x)*Sqrt[-c*f/(d*e-c*f)-d*f*x/(d*e-c*f)]*(e+f*x)^(1/4)),x] /;
FreeQ[{a,b,c,d,e,f},x] && Not[GtQ[-f/(d*e-c*f),0]]
```

2. $\displaystyle\int \frac{1}{(a+b\,x)\,\sqrt{c+d\,x}\,(e+f\,x)^{3/4}}\,dx$

1: $\displaystyle\int \frac{1}{(a+b\,x)\,\sqrt{c+d\,x}\,(e+f\,x)^{3/4}}\,dx$ when $-\dfrac{f}{d\,e-c\,f} > 0$

Derivation: Integration by substitution

Basis: $\displaystyle\frac{1}{(a+b\,x)\,\sqrt{c+d\,x}\,(e+f\,x)^{3/4}} == -4\,\mathrm{Subst}\left[\frac{1}{(b\,e-a\,f-b\,x^4)\,\sqrt{c-\frac{d\,e}{f}+\frac{d\,x^4}{f}}},\,x,\,(e+f\,x)^{1/4}\right]\partial_x\,(e+f\,x)^{1/4}$

Rule 1.1.1.3.18.2.1: If $-\dfrac{f}{d\,e-c\,f} > 0$, then

$$\int \frac{1}{(a+b\,x)\,\sqrt{c+d\,x}\,(e+f\,x)^{3/4}}\,dx \;\rightarrow\; -4\,\mathrm{Subst}\left[\int \frac{1}{(b\,e-a\,f-b\,x^4)\,\sqrt{c-\frac{d\,e}{f}+\frac{d\,x^4}{f}}}\,dx,\,x,\,(e+f\,x)^{1/4}\right]$$

Program code:

```
Int[1/((a_.+b_.*x_)*Sqrt[c_.+d_.*x_]*(e_.+f_.*x_)^(3/4)),x_Symbol] :=
  -4*Subst[Int[1/((b*e-a*f-b*x^4)*Sqrt[c-d*e/f+d*x^4/f]),x],x,(e+f*x)^(1/4)] /;
FreeQ[{a,b,c,d,e,f},x] && GtQ[-f/(d*e-c*f),0]
```

2: $\displaystyle\int \frac{1}{(a+bx)\ \sqrt{c+dx}\ (e+fx)^{3/4}}\ dx$ when $-\dfrac{f}{de-cf} \not> 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{\sqrt{-\dfrac{f\,(c+dx)}{de-cf}}}{\sqrt{c+dx}} == 0$

Rule 1.1.1.3.18.2.2: If $-\dfrac{f}{de-cf} \not> 0$, then

$$\int \frac{1}{(a+bx)\ \sqrt{c+dx}\ (e+fx)^{3/4}}\ dx \ \longrightarrow\ \frac{\sqrt{-\dfrac{f\,(c+dx)}{de-cf}}}{\sqrt{c+dx}} \int \frac{1}{(a+bx)\ \sqrt{-\dfrac{cf}{de-cf} - \dfrac{dfx}{de-cf}}\ (e+fx)^{3/4}}\ dx$$

Program code:

```
Int[1/((a_.+b_.*x_)*Sqrt[c_.+d_.*x_]*(e_.+f_.*x_)^(3/4)),x_Symbol] :=
  Sqrt[-f*(c+d*x)/(d*e-c*f)]/Sqrt[c+d*x]*Int[1/((a+b*x)*Sqrt[-c*f/(d*e-c*f)-d*f*x/(d*e-c*f)]*(e+f*x)^(3/4)),x] /;
FreeQ[{a,b,c,d,e,f},x] && Not[GtQ[-f/(d*e-c*f),0]]
```

19. $\int \dfrac{\sqrt{e + f\,x}}{\sqrt{a + b\,x}\ \sqrt{c + d\,x}}\ dx$

1. $\int \dfrac{\sqrt{e + f\,x}}{\sqrt{b\,x}\ \sqrt{c + d\,x}}\ dx$ when $d\,e - c\,f \neq 0$

   1. $\int \dfrac{\sqrt{e + f\,x}}{\sqrt{b\,x}\ \sqrt{c + d\,x}}\ dx$ when $d\,e - c\,f \neq 0 \wedge c > 0 \wedge e > 0$

      1: $\int \dfrac{\sqrt{e + f\,x}}{\sqrt{b\,x}\ \sqrt{c + d\,x}}\ dx$ when $d\,e - c\,f \neq 0 \wedge c > 0 \wedge e > 0 \wedge -\frac{b}{d} \not< 0$

Rule 1.1.1.3.19.1.1.1: If $d\,e - c\,f \neq 0 \wedge c > 0 \wedge e > 0 \wedge -\frac{b}{d} > 0$, then

$$\int \frac{\sqrt{e + f\,x}}{\sqrt{b\,x}\ \sqrt{c + d\,x}}\ dx \longrightarrow \frac{2\ \sqrt{e}}{b}\ \sqrt{-\frac{b}{d}}\ \text{EllipticE}\left[\text{ArcSin}\left[\frac{\sqrt{b\,x}}{\sqrt{c}\ \sqrt{-\frac{b}{d}}}\right],\ \frac{c\,f}{d\,e}\right]$$

Program code:

```
Int[Sqrt[e_+f_.*x_]/(Sqrt[b_.*x_]*Sqrt[c_+d_.*x_]),x_Symbol] :=
  2*Sqrt[e]/b*Rt[-b/d,2]*EllipticE[ArcSin[Sqrt[b*x]/(Sqrt[c]*Rt[-b/d,2])],c*f/(d*e)] /;
FreeQ[{b,c,d,e,f},x] && NeQ[d*e-c*f,0] && GtQ[c,0] && GtQ[e,0] && Not[LtQ[-b/d,0]]
```


      2: $\int \dfrac{\sqrt{e + f\,x}}{\sqrt{b\,x}\ \sqrt{c + d\,x}}\ dx$ when $d\,e - c\,f \neq 0 \wedge c > 0 \wedge e > 0 \wedge -\frac{b}{d} < 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{\sqrt{-F[x]}}{\sqrt{F[x]}} == 0$

Rule 1.1.1.3.19.1.1.2: If $d\,e - c\,f \neq 0 \wedge c > 0 \wedge e > 0 \wedge -\frac{b}{d} \not> 0$, then

$$\int \frac{\sqrt{e + f\,x}}{\sqrt{b\,x}\,\sqrt{c + d\,x}}\,dx \;\longrightarrow\; \frac{\sqrt{-b\,x}}{\sqrt{b\,x}}\int \frac{\sqrt{e + f\,x}}{\sqrt{-b\,x}\,\sqrt{c + d\,x}}\,dx$$

Program code:

```
Int[Sqrt[e_+f_.*x_]/(Sqrt[b_.*x_]*Sqrt[c_+d_.*x_]),x_Symbol] :=
  Sqrt[-b*x]/Sqrt[b*x]*Int[Sqrt[e+f*x]/(Sqrt[-b*x]*Sqrt[c+d*x]),x] /;
FreeQ[{b,c,d,e,f},x] && NeQ[d*e-c*f,0] && GtQ[c,0] && GtQ[e,0] && LtQ[-b/d,0]
```

2: $\displaystyle\int \frac{\sqrt{e + f\,x}}{\sqrt{b\,x}\,\sqrt{c + d\,x}}\,dx$ when $d\,e - c\,f \neq 0 \;\wedge\; \neg\,(c > 0 \wedge e > 0)$

Derivation: Piecewise constant extraction

■ Basis: $\displaystyle\partial_x \frac{\sqrt{e + f\,x}\,\sqrt{\frac{c + d\,x}{c}}}{\sqrt{c + d\,x}\,\sqrt{\frac{e + f\,x}{e}}} \;==\; 0$

Rule 1.1.1.3.19.1.2: If $d\,e - c\,f \neq 0 \;\wedge\; \neg\,(c > 0 \wedge e > 0)$, then

$$\int \frac{\sqrt{e + f\,x}}{\sqrt{b\,x}\,\sqrt{c + d\,x}}\,dx \;\longrightarrow\; \frac{\sqrt{e + f\,x}\,\sqrt{1 + \frac{d\,x}{c}}}{\sqrt{c + d\,x}\,\sqrt{1 + \frac{f\,x}{e}}}\int \frac{\sqrt{1 + \frac{f\,x}{e}}}{\sqrt{b\,x}\,\sqrt{1 + \frac{d\,x}{c}}}\,dx$$

Program code:

```
Int[Sqrt[e_+f_.*x_]/(Sqrt[b_.*x_]*Sqrt[c_+d_.*x_]),x_Symbol] :=
  Sqrt[e+f*x]*Sqrt[1+d*x/c]/(Sqrt[c+d*x]*Sqrt[1+f*x/e])*Int[Sqrt[1+f*x/e]/(Sqrt[b*x]*Sqrt[1+d*x/c]),x] /;
FreeQ[{b,c,d,e,f},x] && NeQ[d*e-c*f,0] && Not[GtQ[c,0] && GtQ[e,0]]
```

2. $\displaystyle\int \frac{\sqrt{e + f\,x}}{\sqrt{a + b\,x}\ \sqrt{c + d\,x}}\ dx$

   x: $\displaystyle\int \frac{\sqrt{e + f\,x}}{\sqrt{a + b\,x}\ \sqrt{c + d\,x}}\ dx$ when $b\,e == f\,(a - 1)$

Derivation: Algebraic expansion

■

Basis: If $b\,e == f\,(a - 1)$, then $\displaystyle\frac{\sqrt{e+f\,x}}{\sqrt{a+b\,x}} == \frac{f\,\sqrt{a+b\,x}}{b\,\sqrt{e+f\,x}} - \frac{f}{b\,\sqrt{a+b\,x}\ \sqrt{e+f\,x}}$

Note: Instead of a single elliptic integral term, this rule produces two simpler such terms.

Rule 1.1.1.3.19.2.x: If $b\,e == f\,(a - 1)$, then

$$\int \frac{\sqrt{e + f\,x}}{\sqrt{a + b\,x}\ \sqrt{c + d\,x}}\ dx \;\longrightarrow\; \frac{f}{b}\int \frac{\sqrt{a + b\,x}}{\sqrt{c + d\,x}\ \sqrt{e + f\,x}}\ dx - \frac{f}{b}\int \frac{1}{\sqrt{a + b\,x}\ \sqrt{c + d\,x}\ \sqrt{e + f\,x}}\ dx$$

Program code:

```
(* Int[Sqrt[e_.+f_.*x_]/(Sqrt[a_+b_.*x_]*Sqrt[c_+d_.*x_]),x_Symbol] :=
  f/b*Int[Sqrt[a+b*x]/(Sqrt[c+d*x]*Sqrt[e+f*x]),x] -
  f/b*Int[1/(Sqrt[a+b*x]*Sqrt[c+d*x]*Sqrt[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[b*e-f*(a-1),0] *)
```

x: $\displaystyle\int \frac{\sqrt{e+f\,x}}{\sqrt{a+b\,x}\;\sqrt{c+d\,x}}\;\mathbb{d}x$ when $\dfrac{b}{b\,c-a\,d}>0 \wedge \dfrac{b}{b\,e-a\,f}>0 \wedge -\dfrac{b\,c-a\,d}{d}\not<0$

Rule 1.1.1.3.19.2.x: If $\dfrac{b}{b\,c-a\,d}>0 \wedge \dfrac{b}{b\,e-a\,f}>0 \wedge -\dfrac{b\,c-a\,d}{d}\not<0$, then

$$\int \frac{\sqrt{e+f\,x}}{\sqrt{a+b\,x}\;\sqrt{c+d\,x}}\;\mathbb{d}x \;\rightarrow\; \frac{2}{b}\sqrt{-\frac{b\,c-a\,d}{d}}\;\sqrt{\frac{b\,e-a\,f}{b\,c-a\,d}}\;\text{EllipticE}\left[\text{ArcSin}\left[\frac{\sqrt{a+b\,x}}{\sqrt{-\frac{b\,c-a\,d}{d}}}\right],\frac{f\,(b\,c-a\,d)}{d\,(b\,e-a\,f)}\right]$$

Program code:

```
(* Int[Sqrt[e_.+f_.*x_]/(Sqrt[a_+b_.*x_]*Sqrt[c_+d_.*x_]),x_Symbol] :=
  2/b*Rt[-(b*c-a*d)/d,2]*Sqrt[(b*e-a*f)/(b*c-a*d)]*
    EllipticE[ArcSin[Sqrt[a+b*x]/Rt[-(b*c-a*d)/d,2]],f*(b*c-a*d)/(d*(b*e-a*f))] /;
FreeQ[{a,b,c,d,e,f},x] && GtQ[b/(b*c-a*d),0] && GtQ[b/(b*e-a*f),0] && Not[LtQ[-(b*c-a*d)/d,0]] &&
  Not[SimplerQ[c+d*x,a+b*x] && GtQ[-d/(b*c-a*d),0] && GtQ[d/(d*e-c*f),0] && Not[LtQ[(b*c-a*d)/b,0]]] *)
```

1: $\displaystyle\int \frac{\sqrt{e+f\,x}}{\sqrt{a+b\,x}\;\sqrt{c+d\,x}}\;\mathbb{d}x$ when $\dfrac{b}{b\,c-a\,d}>0 \wedge \dfrac{b}{b\,e-a\,f}>0 \wedge -\dfrac{b\,c-a\,d}{d}\not<0$

Derivation: Integration by substitution

- Basis: If $\dfrac{b}{b\,c-a\,d}>0 \wedge \dfrac{b}{b\,e-a\,f}>0$, then $\dfrac{\sqrt{e+f\,x}}{\sqrt{a+b\,x}\;\sqrt{c+d\,x}} == \dfrac{2\sqrt{\frac{-b\,e+a\,f}{d}}}{b\sqrt{-\frac{b\,c-a\,d}{d}}}\,\text{Subst}\left[\frac{\sqrt{1+\frac{f\,x^2}{b\,e-a\,f}}}{\sqrt{1+\frac{d\,x^2}{b\,c-a\,d}}},x,\sqrt{a+b\,x}\right]\partial_x\sqrt{a+b\,x}$

- Basis: $\displaystyle\int \frac{\sqrt{1+\frac{f\,x^2}{b\,e-a\,f}}}{\sqrt{1+\frac{d\,x^2}{b\,c-a\,d}}}\;\mathbb{d}x == \sqrt{-\frac{b\,c-a\,d}{d}}\;\text{EllipticE}\left[\text{ArcSin}\left[\frac{x}{\sqrt{-\frac{b\,c-a\,d}{d}}}\right],\frac{f\,(b\,c-a\,d)}{d\,(b\,e-a\,f)}\right]$

Rule 1.1.1.3.19.2.1: If $\dfrac{b}{b\,c-a\,d}>0 \wedge \dfrac{b}{b\,e-a\,f}>0 \wedge -\dfrac{b\,c-a\,d}{d}\not<0$, then

$$\int \frac{\sqrt{e + f\,x}}{\sqrt{a + b\,x}\ \sqrt{c + d\,x}}\ dx \ \rightarrow\ \frac{2\ \sqrt{-\dfrac{b\,e - a\,f}{d}}}{b\ \sqrt{-\dfrac{b\,c - a\,d}{d}}}\ \text{Subst}\left[\int \frac{\sqrt{1 + \dfrac{f\,x^2}{b\,e - a\,f}}}{\sqrt{1 + \dfrac{d\,x^2}{b\,c - a\,d}}}\ dx,\ x,\ \sqrt{a + b\,x}\ \right]$$

$$\rightarrow\ \frac{2}{b}\ \sqrt{-\frac{b\,e - a\,f}{d}}\ \text{EllipticE}\left[\text{ArcSin}\left[\frac{\sqrt{a + b\,x}}{\sqrt{-\dfrac{b\,c - a\,d}{d}}}\right],\ \frac{f\,(b\,c - a\,d)}{d\,(b\,e - a\,f)}\right]$$

Program code:

```
Int[Sqrt[e_.+f_.*x_]/(Sqrt[a_+b_.*x_]*Sqrt[c_+d_.*x_]),x_Symbol] :=
  2/b*Rt[-(b*e-a*f)/d,2]*EllipticE[ArcSin[Sqrt[a+b*x]/Rt[-(b*c-a*d)/d,2]],f*(b*c-a*d)/(d*(b*e-a*f))] /;
FreeQ[{a,b,c,d,e,f},x] && GtQ[b/(b*c-a*d),0] && GtQ[b/(b*e-a*f),0] && Not[LtQ[-(b*c-a*d)/d,0]] &&
  Not[SimplerQ[c+d*x,a+b*x] && GtQ[-d/(b*c-a*d),0] && GtQ[d/(d*e-c*f),0] && Not[LtQ[(b*c-a*d)/b,0]]]]
```

2: $\int \dfrac{\sqrt{e+f\,x}}{\sqrt{a+b\,x}\ \sqrt{c+d\,x}}\ dx$ when $\neg\left(\dfrac{b}{b\,c-a\,d}>0 \wedge \dfrac{b}{b\,e-a\,f}>0\right) \wedge -\dfrac{b\,c-a\,d}{d}\nless 0$

Derivation: Piecewise constant extraction

■

Basis: $\partial_x\ \dfrac{\sqrt{e+f\,x}\ \sqrt{r\ (c+d\,x)}}{\sqrt{c+d\,x}\ \sqrt{s\ (e+f\,x)}}\ ==\ 0$

Note: $-\dfrac{b\,c-a\,d}{d}\ ==\ \left(-\dfrac{b\,c-a\,d}{d}\ /.\ \left\{c\to\dfrac{b\,c}{b\,c-a\,d},\ d\to\dfrac{b\,d}{b\,c-a\,d},\ e\to\dfrac{b\,e}{b\,e-a\,f},\ f\to\dfrac{b\,f}{b\,e-a\,f}\right\}\right)$

Rule 1.1.1.3.19.2.2: If $\neg\left(\dfrac{b}{b\,c-a\,d}>0 \wedge \dfrac{b}{b\,e-a\,f}>0\right) \wedge -\dfrac{b\,c-a\,d}{d}\nless 0$, then

$$\int \dfrac{\sqrt{e+f\,x}}{\sqrt{a+b\,x}\ \sqrt{c+d\,x}}\ dx \ \rightarrow\ \dfrac{\sqrt{e+f\,x}\ \sqrt{\dfrac{b\ (c+d\,x)}{b\,c-a\,d}}}{\sqrt{c+d\,x}\ \sqrt{\dfrac{b\ (e+f\,x)}{b\,e-a\,f}}}\int \dfrac{\sqrt{\dfrac{b\,e}{b\,e-a\,f}+\dfrac{b\,f\,x}{b\,e-a\,f}}}{\sqrt{a+b\,x}\ \sqrt{\dfrac{b\,c}{b\,c-a\,d}+\dfrac{b\,d\,x}{b\,c-a\,d}}}\ dx$$

Program code:

```
Int[Sqrt[e_.+f_.*x_]/(Sqrt[a_+b_.*x_]*Sqrt[c_+d_.*x_]),x_Symbol] :=
  Sqrt[e+f*x]*Sqrt[b*(c+d*x)/(b*c-a*d)]/(Sqrt[c+d*x]*Sqrt[b*(e+f*x)/(b*e-a*f)])*
    Int[Sqrt[b*e/(b*e-a*f)+b*f*x/(b*e-a*f)]/(Sqrt[a+b*x]*Sqrt[b*c/(b*c-a*d)+b*d*x/(b*c-a*d)]),x] /;
FreeQ[{a,b,c,d,e,f},x] && Not[GtQ[b/(b*c-a*d),0] && GtQ[b/(b*e-a*f),0]] && Not[LtQ[-(b*c-a*d)/d,0]]
```

20. $\displaystyle\int \frac{1}{\sqrt{a+b\,x}\ \sqrt{c+d\,x}\ \sqrt{e+f\,x}}\, dx$

1. $\displaystyle\int \frac{1}{\sqrt{b\,x}\ \sqrt{c+d\,x}\ \sqrt{e+f\,x}}\, dx$

1: $\displaystyle\int \frac{1}{\sqrt{b\,x}\ \sqrt{c+d\,x}\ \sqrt{e+f\,x}}\, dx$　when $c > 0 \wedge e > 0$

Rule 1.1.1.3.20.1.1: If $c > 0 \wedge e > 0$, then

$$\int \frac{1}{\sqrt{b\,x}\ \sqrt{c+d\,x}\ \sqrt{e+f\,x}}\, dx \;\longrightarrow\; \frac{2}{b\,\sqrt{e}}\,\sqrt{-\frac{b}{d}}\ \text{EllipticF}\!\left[\text{ArcSin}\!\left[\frac{\sqrt{b\,x}}{\sqrt{c}\ \sqrt{-\frac{b}{d}}}\right],\frac{c\,f}{d\,e}\right]$$

Program code:

```
Int[1/(Sqrt[b_.*x_]*Sqrt[c_+d_.*x_]*Sqrt[e_+f_.*x_]),x_Symbol] :=
  2/(b*Sqrt[e])*Rt[-b/d,2]*EllipticF[ArcSin[Sqrt[b*x]/(Sqrt[c]*Rt[-b/d,2])],c*f/(d*e)] /;
FreeQ[{b,c,d,e,f},x] && GtQ[c,0] && GtQ[e,0] && (GtQ[-b/d,0] || LtQ[-b/f,0])
```

```
Int[1/(Sqrt[b_.*x_]*Sqrt[c_+d_.*x_]*Sqrt[e_+f_.*x_]),x_Symbol] :=
  2/(b*Sqrt[e])*Rt[-b/d,2]*EllipticF[ArcSin[Sqrt[b*x]/(Sqrt[c]*Rt[-b/d,2])],c*f/(d*e)] /;
FreeQ[{b,c,d,e,f},x] && GtQ[c,0] && GtQ[e,0] && (PosQ[-b/d] || NegQ[-b/f])
```

2: $\int \dfrac{1}{\sqrt{b\,x}\ \sqrt{c+d\,x}\ \sqrt{e+f\,x}}\,dx$ when $\neg\ (c > 0 \ \wedge\ e > 0)$

Derivation: Piecewise constant extraction

- Basis: $\partial_x\ \dfrac{\sqrt{1+\frac{d\,x}{c}}\ \sqrt{1+\frac{f\,x}{e}}}{\sqrt{c+d\,x}\ \sqrt{e+f\,x}}\ ==\ 0$

Rule 1.1.1.3.20.1.2: If $\neg\ \left(\dfrac{b}{b\,c-a\,d} > 0\ \wedge\ \dfrac{b}{b\,e-a\,f} > 0\right)$, then

$$\int \frac{1}{\sqrt{b\,x}\ \sqrt{c+d\,x}\ \sqrt{e+f\,x}}\,dx\ \longrightarrow\ \frac{\sqrt{1+\frac{d\,x}{c}}\ \sqrt{1+\frac{f\,x}{e}}}{\sqrt{c+d\,x}\ \sqrt{e+f\,x}}\int \frac{1}{\sqrt{b\,x}\ \sqrt{1+\frac{d\,x}{c}}\ \sqrt{1+\frac{f\,x}{e}}}\,dx$$

Program code:

```
Int[1/(Sqrt[b_.*x_]*Sqrt[c_+d_.*x_]*Sqrt[e_+f_.*x_]),x_Symbol] :=
  Sqrt[1+d*x/c]*Sqrt[1+f*x/e]/(Sqrt[c+d*x]*Sqrt[e+f*x])*Int[1/(Sqrt[b*x]*Sqrt[1+d*x/c]*Sqrt[1+f*x/e]),x] /;
FreeQ[{b,c,d,e,f},x] && Not[GtQ[c,0] && GtQ[e,0]]
```

2. $\int \dfrac{1}{\sqrt{a+bx}\ \sqrt{c+dx}\ \sqrt{e+fx}}\ dx$

1: $\int \dfrac{1}{\sqrt{a+bx}\ \sqrt{c+dx}\ \sqrt{e+fx}}\ dx$ when $\dfrac{d}{b} > 0 \ \wedge\ \dfrac{f}{b} > 0 \ \wedge\ c \le \dfrac{a\,d}{b} \ \wedge\ e \le \dfrac{a\,f}{b}$

Derivation: Algebraic expansion and integration by substitution

- Basis: If $\dfrac{d}{b} > 0 \ \wedge\ c \le \dfrac{a\,d}{b}$, then $\sqrt{c+dx} == \sqrt{\dfrac{d}{b}}\ \sqrt{a+bx}\ \sqrt{\dfrac{b\,(c+dx)}{d\,(a+bx)}}$

- Basis: If $\dfrac{f}{b} > 0 \ \wedge\ e \le \dfrac{a\,f}{b}$, then $\sqrt{e+fx} == \sqrt{\dfrac{f}{b}}\ \sqrt{a+bx}\ \sqrt{\dfrac{b\,(e+fx)}{f\,(a+bx)}}$

- Basis: $\dfrac{\sqrt{\dfrac{b\,(c+dx)}{d\,(a+bx)}}}{\sqrt{a+bx}\ (c+dx)\ \sqrt{\dfrac{b\,(e+fx)}{f\,(a+bx)}}} == \dfrac{2}{d}\ \mathrm{Subst}\left[\dfrac{1}{x^2\ \sqrt{1+\dfrac{b\,c-a\,d}{d\,x^2}}\ \sqrt{1+\dfrac{b\,e-a\,f}{f\,x^2}}},\ x,\ \sqrt{a+bx}\right] \partial_x \sqrt{a+bx}$

- Basis: $\int \dfrac{1}{x^2\ \sqrt{1+\dfrac{b\,c-a\,d}{d\,x^2}}\ \sqrt{1+\dfrac{b\,e-a\,f}{f\,x^2}}}\ dx == -\dfrac{1}{\sqrt{-\dfrac{b\,e-a\,f}{f}}}\ \mathrm{EllipticF}\left[\mathrm{ArcSin}\left[\dfrac{\sqrt{-\dfrac{b\,e-a\,f}{f}}}{x}\right],\ \dfrac{f\,(b\,c-a\,d)}{d\,(b\,e-a\,f)}\right]$

Rule 1.1.1.3.20.2.1: If $\dfrac{d}{b} > 0 \ \wedge\ \dfrac{f}{b} > 0 \ \wedge\ c \le \dfrac{a\,d}{b} \ \wedge\ e \le \dfrac{a\,f}{b}$, then

$$\int \dfrac{1}{\sqrt{a+bx}\ \sqrt{c+dx}\ \sqrt{e+fx}}\ dx \ \longrightarrow\ \sqrt{\dfrac{d}{f}}\ \int \dfrac{\sqrt{\dfrac{b\,(c+dx)}{d\,(a+bx)}}}{\sqrt{a+bx}\ (c+dx)\ \sqrt{\dfrac{b\,(e+fx)}{f\,(a+bx)}}}\ dx$$

$$\longrightarrow\ \dfrac{2\,\sqrt{\dfrac{d}{f}}}{d}\ \mathrm{Subst}\left[\int \dfrac{1}{x^2\ \sqrt{1+\dfrac{b\,c-a\,d}{d\,x^2}}\ \sqrt{1+\dfrac{b\,e-a\,f}{f\,x^2}}}\ dx,\ x,\ \sqrt{a+bx}\right]$$

$$\to \; - \frac{2 \sqrt{\frac{d}{f}}}{d \sqrt{-\frac{b\,e - a\,f}{f}}} \; \text{EllipticF}\left[\text{ArcSin}\left[\frac{\sqrt{-\frac{b\,e - a\,f}{f}}}{\sqrt{a + b\,x}}\right], \frac{f\,(b\,c - a\,d)}{d\,(b\,e - a\,f)}\right]$$

Program code:

```
Int[1/(Sqrt[a_+b_.*x_]*Sqrt[c_+d_.*x_]*Sqrt[e_+f_.*x_]),x_Symbol] :=
  -2*Sqrt[d/f]/(d*Rt[-(b*e-a*f)/f,2])*EllipticF[ArcSin[Rt[-(b*e-a*f)/f,2]/Sqrt[a+b*x]],f*(b*c-a*d)/(d*(b*e-a*f))] /;
FreeQ[{a,b,c,d,e,f},x] && GtQ[d/b,0] && GtQ[f/b,0] && LeQ[c,a*d/b] && LeQ[e,a*f/b]
```

x: $\displaystyle\int \frac{1}{\sqrt{a + b\,x}\ \sqrt{c + d\,x}\ \sqrt{e + f\,x}}\ dx$ when $-\frac{b\,e - a\,f}{f} > 0$

Derivation: Piecewise constant extraction and integration by substitution

■ Basis: $\displaystyle \partial_x \frac{\sqrt{c + d\,x}\ \sqrt{\frac{b\,(e + f\,x)}{f\,(a + b\,x)}}}{\sqrt{e + f\,x}\ \sqrt{\frac{b\,(c + d\,x)}{d\,(a + b\,x)}}} \;==\; 0$

■ Basis: $\displaystyle \frac{\sqrt{\frac{b\,(c + d\,x)}{d\,(a + b\,x)}}}{\sqrt{a + b\,x}\ (c + d\,x)\ \sqrt{\frac{b\,(e + f\,x)}{f\,(a + b\,x)}}} \;==\; \frac{2}{d} \, \text{Subst}\left[\frac{1}{x^2 \sqrt{1 + \frac{b\,c - a\,d}{d\,x^2}}\ \sqrt{1 + \frac{b\,e - a\,f}{f\,x^2}}}, x, \sqrt{a + b\,x}\right] \partial_x \sqrt{a + b\,x}$

■ Basis: $\displaystyle \int \frac{1}{x^2 \sqrt{1 + \frac{b\,c - a\,d}{d\,x^2}}\ \sqrt{1 + \frac{b\,e - a\,f}{f\,x^2}}}\ dx \;==\; -\frac{1}{\sqrt{-\frac{b\,e - a\,f}{f}}}\, \text{EllipticF}\left[\text{ArcSin}\left[\frac{\sqrt{-\frac{b\,e - a\,f}{f}}}{x}\right], \frac{f\,(b\,c - a\,d)}{d\,(b\,e - a\,f)}\right]$

Rule 1.1.1.3.20.2.1: If $-\dfrac{b\,e - a\,f}{f} > 0$, then

$$\int \frac{1}{\sqrt{a + b\,x}\ \sqrt{c + d\,x}\ \sqrt{e + f\,x}}\ dx \;\to\; \frac{\sqrt{c + d\,x}\ \sqrt{\frac{b\,(e + f\,x)}{f\,(a + b\,x)}}}{\sqrt{e + f\,x}\ \sqrt{\frac{b\,(c + d\,x)}{d\,(a + b\,x)}}} \int \frac{\sqrt{\frac{b\,(c + d\,x)}{d\,(a + b\,x)}}}{\sqrt{a + b\,x}\ (c + d\,x)\ \sqrt{\frac{b\,(e + f\,x)}{f\,(a + b\,x)}}}\ dx$$

$$\rightarrow \frac{2\sqrt{c+dx}\sqrt{\frac{b\,(e+fx)}{f\,(a+bx)}}}{d\sqrt{e+fx}\sqrt{\frac{b\,(c+dx)}{d\,(a+bx)}}}\ \text{Subst}\left[\int \frac{1}{x^2\sqrt{1+\frac{bc-ad}{d\,x^2}}\sqrt{1+\frac{be-af}{f\,x^2}}}\,dx,\ x,\ \sqrt{a+bx}\,\right]$$

$$\rightarrow -\frac{2\sqrt{c+dx}\sqrt{\frac{b\,(e+fx)}{f\,(a+bx)}}}{d\sqrt{-\frac{be-af}{f}}\sqrt{e+fx}\sqrt{\frac{b\,(c+dx)}{d\,(a+bx)}}}\ \text{EllipticF}\left[\text{ArcSin}\left[\frac{\sqrt{-\frac{be-af}{f}}}{\sqrt{a+bx}}\right],\ \frac{f\,(bc-ad)}{d\,(be-af)}\right]$$

Program code:

```
(* Int[1/(Sqrt[a_+b_.*x_]*Sqrt[c_+d_.*x_]*Sqrt[e_+f_.*x_]),x_Symbol] :=
  -2*Sqrt[c+d*x]*Sqrt[b*(e+f*x)/(f*(a+b*x))]/(d*Rt[-(b*e-a*f)/f,2]*Sqrt[e+f*x]*Sqrt[b*(c+d*x)/(d*(a+b*x))])*
    EllipticF[ArcSin[Rt[-(b*e-a*f)/f,2]/Sqrt[a+b*x]],f*(b*c-a*d)/(d*(b*e-a*f))] /;
FreeQ[{a,b,c,d,e,f},x] && PosQ[-(b*e-a*f)/f] && (* (LtQ[-a/b,-c/d,-e/f] || GtQ[-a/b,-c/d,-e/f]) *)
  Not[SimplerQ[c+d*x,a+b*x] && (PosQ[(-d*e+c*f)/f] || PosQ[(b*e-a*f)/b])] &&
  Not[SimplerQ[e+f*x,a+b*x] && (PosQ[(b*e-a*f)/b] || PosQ[(b*c-a*d)/b])] *)
```

2: $\int \dfrac{1}{\sqrt{a+bx}\,\sqrt{c+dx}\,\sqrt{e+fx}}\,dx$ when $\dfrac{bc-ad}{b}>0 \wedge \dfrac{be-af}{b}>0 \wedge -\dfrac{b}{d}>0$

Derivation: Integration by substitution

Basis: If $\dfrac{bc-ad}{b}>0 \wedge \dfrac{be-af}{b}>0$, then $\dfrac{1}{\sqrt{a+bx}\,\sqrt{c+dx}\,\sqrt{e+fx}} == \dfrac{2}{b\sqrt{\frac{bc-ad}{b}}\sqrt{\frac{be-af}{b}}}\ \text{Subst}\left[\dfrac{1}{\sqrt{1+\frac{dx^2}{bc-ad}}\sqrt{1+\frac{fx^2}{be-af}}},\ x,\ \sqrt{a+bx}\,\right]\partial_x\sqrt{a+bx}$

Basis: $\int \dfrac{1}{\sqrt{1+\frac{dx^2}{bc-ad}}\sqrt{1+\frac{fx^2}{be-af}}}\,dx == \sqrt{-\dfrac{b}{d}}\sqrt{\dfrac{bc-ad}{b}}\ \text{EllipticF}\left[\text{ArcSin}\left[\dfrac{x}{\sqrt{-\frac{b}{d}}\sqrt{\frac{bc-ad}{b}}}\right],\ \dfrac{f\,(bc-ad)}{d\,(be-af)}\right]$

Rule 1.1.1.3.20.2.2: If $\dfrac{bc-ad}{b}>0 \wedge \dfrac{be-af}{b}>0 \wedge -\dfrac{b}{d}>0$, then

$$\int \frac{1}{\sqrt{a+bx}\ \sqrt{c+dx}\ \sqrt{e+fx}}\ dx \quad \rightarrow \quad \frac{2}{b\ \sqrt{\frac{bc-ad}{b}}\ \sqrt{\frac{be-af}{b}}}\ \text{Subst}\left[\int \frac{1}{\sqrt{1+\frac{d\,x^2}{bc-ad}}\ \sqrt{1+\frac{f\,x^2}{be-af}}}\ dx,\ x,\ \sqrt{a+bx}\ \right]$$

$$\rightarrow \quad \frac{2\ \sqrt{-\frac{b}{d}}}{b\ \sqrt{\frac{be-af}{b}}}\ \text{EllipticF}\left[\text{ArcSin}\left[\frac{\sqrt{a+bx}}{\sqrt{-\frac{b}{d}}\ \sqrt{\frac{bc-ad}{b}}}\right],\ \frac{f\ (bc-ad)}{d\ (be-af)}\right]$$

Program code:

```
Int[1/(Sqrt[a_+b_.*x_]*Sqrt[c_+d_.*x_]*Sqrt[e_+f_.*x_]),x_Symbol] :=
  2*Rt[-b/d,2]/(b*Sqrt[(b*e-a*f)/b])*EllipticF[ArcSin[Sqrt[a+b*x]/(Rt[-b/d,2]*Sqrt[(b*c-a*d)/b])],f*(b*c-a*d)/(d*(b*e-a*f))] /;
FreeQ[{a,b,c,d,e,f},x] && GtQ[(b*c-a*d)/b,0] && GtQ[(b*e-a*f)/b,0] && PosQ[-b/d] &&
  Not[SimplerQ[c+d*x,a+b*x] && GtQ[(d*e-c*f)/d,0] && GtQ[-d/b,0]] &&
  Not[SimplerQ[c+d*x,a+b*x] && GtQ[(-b*e+a*f)/f,0] && GtQ[-f/b,0]] &&
  Not[SimplerQ[e+f*x,a+b*x] && GtQ[(-d*e+c*f)/f,0] && GtQ[(-b*e+a*f)/f,0] && (PosQ[-f/d] || PosQ[-f/b])]
```

```
Int[1/(Sqrt[a_+b_.*x_]*Sqrt[c_+d_.*x_]*Sqrt[e_+f_.*x_]),x_Symbol] :=
  2*Rt[-b/d,2]/(b*Sqrt[(b*e-a*f)/b])*EllipticF[ArcSin[Sqrt[a+b*x]/(Rt[-b/d,2]*Sqrt[(b*c-a*d)/b])],f*(b*c-a*d)/(d*(b*e-a*f))] /;
FreeQ[{a,b,c,d,e,f},x] && GtQ[b/(b*c-a*d),0] && GtQ[b/(b*e-a*f),0] && SimplerQ[a+b*x,c+d*x] && SimplerQ[a+b*x,e+f*x] &&
  (PosQ[-(b*c-a*d)/d] || NegQ[-(b*e-a*f)/f]) (* && PosQ[-b/d] *)
```

3: $\displaystyle\int \frac{1}{\sqrt{a+bx}\ \sqrt{c+dx}\ \sqrt{e+fx}}\ dx$ when $\frac{bc-ad}{b} \not> 0$

Derivation: Piecewise constant extraction

■ Basis: $\partial_x \dfrac{\sqrt{\frac{b\,(c+dx)}{bc-ad}}}{\sqrt{c+dx}} == 0$

Rule 1.1.1.3.20.2.3: If $\frac{bc-ad}{b} \not> 0$, then

$$\int \frac{1}{\sqrt{a+bx}\ \sqrt{c+dx}\ \sqrt{e+fx}}\,dx \ \rightarrow \ \frac{\sqrt{\frac{b(c+dx)}{bc-ad}}}{\sqrt{c+dx}} \int \frac{1}{\sqrt{a+bx}\ \sqrt{\frac{bc}{bc-ad}+\frac{bdx}{bc-ad}}\ \sqrt{e+fx}}\,dx$$

Program code:

```
Int[1/(Sqrt[a_+b_.*x_]*Sqrt[c_+d_.*x_]*Sqrt[e_+f_.*x_]),x_Symbol] :=
  Sqrt[b*(c+d*x)/(b*c-a*d)]/Sqrt[c+d*x]*Int[1/(Sqrt[a+b*x]*Sqrt[b*c/(b*c-a*d)+b*d*x/(b*c-a*d)]*Sqrt[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && Not[GtQ[(b*c-a*d)/b,0]] && SimplerQ[a+b*x,c+d*x] && SimplerQ[a+b*x,e+f*x]
```

4: $\int \dfrac{1}{\sqrt{a+bx}\ \sqrt{c+dx}\ \sqrt{e+fx}}\,dx$ when $\dfrac{be-af}{b} \ngtr 0$

Derivation: Piecewise constant extraction

■ Basis: $\partial_x \dfrac{\sqrt{\frac{b(e+fx)}{be-af}}}{\sqrt{e+fx}} == 0$

Rule 1.1.1.3.20.2.4: If $\dfrac{be-af}{b} \ngtr 0$, then

$$\int \frac{1}{\sqrt{a+bx}\ \sqrt{c+dx}\ \sqrt{e+fx}}\,dx \ \rightarrow \ \frac{\sqrt{\frac{b(e+fx)}{be-af}}}{\sqrt{e+fx}} \int \frac{1}{\sqrt{a+bx}\ \sqrt{c+dx}\ \sqrt{\frac{be}{be-af}+\frac{bfx}{be-af}}}\,dx$$

Program code:

```
Int[1/(Sqrt[a_+b_.*x_]*Sqrt[c_+d_.*x_]*Sqrt[e_+f_.*x_]),x_Symbol] :=
  Sqrt[b*(e+f*x)/(b*e-a*f)]/Sqrt[e+f*x]*Int[1/(Sqrt[a+b*x]*Sqrt[c+d*x]*Sqrt[b*e/(b*e-a*f)+b*f*x/(b*e-a*f)]),x] /;
FreeQ[{a,b,c,d,e,f},x] && Not[GtQ[(b*e-a*f)/b,0]]
```

21. $\displaystyle\int\frac{(a+bx)^m}{(c+dx)^{1/3}\,(e+fx)^{1/3}}\,dx$ when $2\,b\,d\,e-b\,c\,f-a\,d\,f==0 \,\wedge\, m \in \mathbb{Z}^-$

1: $\displaystyle\int\frac{1}{(a+bx)\,(c+dx)^{1/3}\,(e+fx)^{1/3}}\,dx$ when $2\,b\,d\,e-b\,c\,f-a\,d\,f==0$

Rule 1.1.1.3.21.1: If $2\,b\,d\,e-b\,c\,f-a\,d\,f==0$, let $q=\left(\dfrac{b\,(b\,e-a\,f)}{(b\,c-a\,d)^2}\right)^{1/3}$, then

$$\int\frac{1}{(a+bx)\,(c+dx)^{1/3}\,(e+fx)^{1/3}}\,dx \;\longrightarrow\; -\frac{\text{Log}[a+bx]}{2\,q\,(b\,c-a\,d)}-\frac{\sqrt{3}\ \text{ArcTan}\left[\frac{1}{\sqrt{3}}+\frac{2\,q\,(c+dx)^{2/3}}{\sqrt{3}\ (e+fx)^{1/3}}\right]}{2\,q\,(b\,c-a\,d)}+\frac{3\,\text{Log}\left[q\,(c+dx)^{2/3}-(e+fx)^{1/3}\right]}{4\,q\,(b\,c-a\,d)}$$

Program code:

```
Int[1/((a_.+b_.*x_)*(c_.+d_.*x_)^(1/3)*(e_.+f_.*x_)^(1/3)),x_Symbol] :=
  With[{q=Rt[b*(b*e-a*f)/(b*c-a*d)^2,3]},
  -Log[a+b*x]/(2*q*(b*c-a*d)) -
  Sqrt[3]*ArcTan[1/Sqrt[3]+2*q*(c+d*x)^(2/3)/(Sqrt[3]*(e+f*x)^(1/3))]/(2*q*(b*c-a*d)) +
  3*Log[q*(c+d*x)^(2/3)-(e+f*x)^(1/3)]/(4*q*(b*c-a*d))] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[2*b*d*e-b*c*f-a*d*f,0]
```

2: $\int \dfrac{(a+bx)^m}{(c+dx)^{1/3}\,(e+fx)^{1/3}}\,dx$ when $2\,b\,d\,e - b\,c\,f - a\,d\,f == 0 \;\wedge\; m+1 \in \mathbb{Z}^-$

Derivation: Nondegenerate trilinear recurrence 3 with $A = 1$ and $B = 0$

Rule 1.1.1.3.21.2: If $2\,b\,d\,e - b\,c\,f - a\,d\,f == 0 \;\wedge\; m+1 \in \mathbb{Z}^-$, then

$$\int \frac{(a+bx)^m}{(c+dx)^{1/3}\,(e+fx)^{1/3}}\,dx \;\longrightarrow$$

$$\frac{b\,(a+bx)^{m+1}\,(c+dx)^{2/3}\,(e+fx)^{2/3}}{(m+1)\,(bc-ad)\,(be-af)} + \frac{f}{6\,(m+1)\,(bc-ad)\,(be-af)} \int \frac{(a+bx)^{m+1}\,\big(a\,d\,(3m+1) - 3\,b\,c\,(3m+5) - 2\,b\,d\,(3m+7)\,x\big)}{(c+dx)^{1/3}\,(e+fx)^{1/3}}\,dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_/((c_.+d_.*x_)^(1/3)*(e_.+f_.*x_)^(1/3)),x_Symbol] :=
  b*(a+b*x)^(m+1)*(c+d*x)^(2/3)*(e+f*x)^(2/3)/((m+1)*(b*c-a*d)*(b*e-a*f)) +
  f/(6*(m+1)*(b*c-a*d)*(b*e-a*f))*
    Int[(a+b*x)^(m+1)*(a*d*(3*m+1)-3*b*c*(3*m+5)-2*b*d*(3*m+7)*x)/((c+d*x)^(1/3)*(e+f*x)^(1/3)),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[2*b*d*e-b*c*f-a*d*f,0] && ILtQ[m,-1]
```

22. $\int (a + b\,x)^m\,(c + d\,x)^n\,(f\,x)^p\,dx$ when $b\,c + a\,d == 0 \land m - n \in \mathbb{Z}$

  1. $\int (a + b\,x)^m\,(c + d\,x)^n\,(f\,x)^p\,dx$ when $b\,c + a\,d == 0 \land m - n == 0$

    1: $\int (a + b\,x)^m\,(c + d\,x)^n\,(f\,x)^p\,dx$ when $b\,c + a\,d == 0 \land m - n == 0 \land a > 0 \land c > 0$

Derivation: Algebraic simplification

Basis: If $b\,c + a\,d == 0 \land a > 0 \land c > 0$, then $(a + b\,x)^m\,(c + d\,x)^m == (a\,c + b\,d\,x^2)^m$

Rule 1.1.1.3.21.1.1: If $b\,c + a\,d == 0 \land m - n == 0 \land a > 0 \land c > 0$, then

$$\int (a + b\,x)^m\,(c + d\,x)^n\,(f\,x)^p\,dx \;\longrightarrow\; \int (a\,c + b\,d\,x^2)^m\,(f\,x)^p\,dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_.*(c_.+d_.*x_)^n_.*(f_.*x_)^p_.,x_Symbol] :=
  Int[(a*c+b*d*x^2)^m*(f*x)^p,x] /;
FreeQ[{a,b,c,d,f,m,n,p},x] && EqQ[b*c+a*d,0] && EqQ[m-n,0] && GtQ[a,0] && GtQ[c,0]
```

    2: $\int (a + b\,x)^m\,(c + d\,x)^n\,(f\,x)^p\,dx$ when $b\,c + a\,d == 0 \land m - n == 0$

Derivation: Piecewise constant extraction

Basis: If $b\,c + a\,d == 0$, then $\partial_x\,\dfrac{(a+b\,x)^m\,(c+d\,x)^m}{(a\,c+b\,d\,x^2)^m} == 0$

Basis: If $b\,c + a\,d == 0$, then $\dfrac{(a+b\,x)^m\,(c+d\,x)^m}{(a\,c+b\,d\,x^2)^m} == \dfrac{(a+b\,x)^{FracPart[m]}\,(c+d\,x)^{FracPart[m]}}{(a\,c+b\,d\,x^2)^{FracPart[m]}}$

Rule 1.1.1.3.21.1.2: If $b\,c + a\,d == 0 \land m - n == 0$, then

$$\int (a + b\,x)^m\,(c + d\,x)^n\,(f\,x)^p\,dx \;\longrightarrow\; \frac{(a + b\,x)^{\text{FracPart}[m]}\,(c + d\,x)^{\text{FracPart}[m]}}{(a\,c + b\,d\,x^2)^{\text{FracPart}[m]}}\int (a\,c + b\,d\,x^2)^m\,(f\,x)^p\,dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_.*(c_.+d_.*x_)^n_.*(f_.*x_)^p_.,x_Symbol] :=
  (a+b*x)^FracPart[m]*(c+d*x)^FracPart[m]/(a*c+b*d*x^2)^FracPart[m]*Int[(a*c+b*d*x^2)^m*(f*x)^p,x] /;
FreeQ[{a,b,c,d,f,m,n,p},x] && EqQ[b*c+a*d,0] && EqQ[m-n,0]
```

2: $\int (a + b\,x)^m\,(c + d\,x)^n\,(f\,x)^p\,dx$ when $b\,c + a\,d == 0 \;\wedge\; m - n \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Note: Integrals of this form can be expressed in terms of the confluent hypergeometric function 2 F 1 instead of requiring the Appell hypergeometric function.

Rule 1.1.1.3.21.2: If $b\,c + a\,d == 0 \;\wedge\; m - n \in \mathbb{Z}^+$, then

$$\int (a + b\,x)^m\,(c + d\,x)^n\,(f\,x)^p\,dx \;\longrightarrow\; \int (a + b\,x)^n\,(c + d\,x)^n\,(f\,x)^p\,\text{ExpandIntegrand}\big[(a + b\,x)^{m-n},\,x\big]\,dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_.*(c_.+d_.*x_)^n_.*(f_.*x_)^p_.,x_Symbol] :=
    Int[ExpandIntegrand[(a+b*x)^n*(c+d*x)^n*(f*x)^p,(a+b*x)^(m-n),x],x] /;
FreeQ[{a,b,c,d,f,m,n,p},x] && EqQ[b*c+a*d,0] && IGtQ[m-n,0] && NeQ[m+n+p+2,0]
```

23: $\int (a + b\, x)^m\, (c + d\, x)^n\, (e + f\, x)^p\, dx$　when $m \in \mathbb{Z}^+ \ \lor \ (m \mid n) \in \mathbb{Z}^-$

Derivation: Algebraic expansion

Rule 1.1.1.3.23: If $m \in \mathbb{Z}^+ \ \lor \ (m \mid n) \in \mathbb{Z}^-$, then

$$\int (a + b\, x)^m\, (c + d\, x)^n\, (e + f\, x)^p\, dx \ \longrightarrow \ \int \text{ExpandIntegrand}\left[(a + b\, x)^m\, (c + d\, x)^n\, (e + f\, x)^p, x\right] dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_.*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*x)^m*(c+d*x)^n*(e+f*x)^p,x],x] /;
FreeQ[{a,b,c,d,e,f,n,p},x] && (IGtQ[m,0] || ILtQ[m,0] && ILtQ[n,0])
```

24: $\int (a + b\,x)^m\ (c + d\,x)^n\ (e + f\,x)^p\ dx$ when $m + n + p + 2 \in \mathbb{Z}^-$

Derivation: Nondegenerate trilinear recurrence 3 with A = 1 and B = 0

Note: If $m + n + p + 2 \in \mathbb{Z}^-$, then $\int (a+b\,x)^m\ (c+d\,x)^n\ (e+f\,x)^p\ dx$ can be expressed in terms of the hypergeometric function 2F1.

Rule 1.1.1.3.24: If $m + n + p + 2 \in \mathbb{Z}^-$, then

$$\int (a + b\,x)^m\ (c + d\,x)^n\ (e + f\,x)^p\ dx \longrightarrow$$

$$\frac{b\ (a + b\,x)^{m+1}\ (c + d\,x)^{n+1}\ (e + f\,x)^{p+1}}{(m + 1)\ (b\,c - a\,d)\ (b\,e - a\,f)} +$$

$$\frac{1}{(m + 1)\ (b\,c - a\,d)\ (b\,e - a\,f)} \int (a + b\,x)^{m+1}\ (c + d\,x)^n\ (e + f\,x)^p\ \big(a\,d\,f\ (m + 1) - b\ (d\,e\ (m + n + 2) + c\,f\ (m + p + 2)) - b\,d\,f\ (m + n + p + 3)\ x\big)\ dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.,x_Symbol] :=
  b*(a+b*x)^(m+1)*(c+d*x)^(n+1)*(e+f*x)^(p+1)/((m+1)*(b*c-a*d)*(b*e-a*f)) +
  1/((m+1)*(b*c-a*d)*(b*e-a*f))*Int[(a+b*x)^(m+1)*(c+d*x)^n*(e+f*x)^p*
    Simp[a*d*f*(m+1)-b*(d*e*(m+n+2)+c*f*(m+p+2))-b*d*f*(m+n+p+3)*x,x],x] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && ILtQ[m+n+p+2,0] && NeQ[m,-1] &&
  (SumSimplerQ[m,1] || Not[NeQ[n,-1] && SumSimplerQ[n,1]] && Not[NeQ[p,-1] && SumSimplerQ[p,1]])
```

25: $\int (e\,x)^p\ (a + b\,x)^m\ (c + d\,x)^n\ dx$ when $b\,c - a\,d \neq 0 \land p \in \mathbb{F} \land m \in \mathbb{Z}$

Derivation: Integration by substitution

Basis: If $k \in \mathbb{Z}^+$, then $(e\,x)^p\ F[x] == \frac{k}{e}\ \mathrm{Subst}\big[x^{k\ (p+1)-1}\ F\big[\frac{x^k}{e}\big], x, (e\,x)^{1/k}\big]\ \partial_x\ (e\,x)^{1/k}$

Rule 1.1.13.25 If $b\,c - a\,d \neq 0 \land p \in \mathbb{F} \land m \in \mathbb{Z}$, let $k = \mathrm{Denominator}[p]$, then

$$\int (e\,x)^p\,(a+b\,x)^m\,(c+d\,x)^n\,dx \;\longrightarrow\; \frac{k}{e}\,\text{Subst}\left[\int x^{k\,(p+1)-1}\left(a+\frac{b\,x^k}{e}\right)^m\left(c+\frac{d\,x^k}{e}\right)^n dx,\; x,\; (e\,x)^{1/k}\right]$$

Program code:

```
Int[(e_.*x_)^p_*(a_+b_.*x_)^m_*(c_+d_.*x_)^n_,x_Symbol] :=
  With[{k=Denominator[p]},
  k/e*Subst[Int[x^(k*(p+1)-1)*(a+b*x^k/e)^m*(c+d*x^k/e)^n,x],x,(e*x)^(1/k)]] /;
FreeQ[{a,b,c,d,e,m,n},x] && NeQ[b*c-a*d,0] && FractionQ[p] && IntegerQ[m]
```

H. $\int (a+b\,x)^m\,(c+d\,x)^n\,(e+f\,x)^p\,dx$ when $m+n+p+2 == 0$

1: $\int (a+b\,x)^m\,(c+d\,x)^n\,(e+f\,x)^p\,dx$ when $m+n+p+2 == 0 \land n \in \mathbb{Z}^-$

Rule 1.1.1.3.H.1: If $m+n+p+2 == 0 \land n \in \mathbb{Z}^-$, then

$$\int (a+b\,x)^m\,(c+d\,x)^n\,(e+f\,x)^p\,dx \;\longrightarrow$$

$$\frac{(b\,c-a\,d)^n\,(a+b\,x)^{m+1}}{(m+1)\,(b\,e-a\,f)^{n+1}\,(e+f\,x)^{m+1}}\,\text{Hypergeometric2F1}\left[m+1,\; -n,\; m+2,\; -\frac{(d\,e-c\,f)\,(a+b\,x)}{(b\,c-a\,d)\,(e+f\,x)}\right]$$

Program code:

```
Int[(a_.+b_.*x_)^m_*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_,x_Symbol] :=
  (b*c-a*d)^n*(a+b*x)^(m+1)/((m+1)*(b*e-a*f)^(n+1)*(e+f*x)^(m+1))*
    Hypergeometric2F1[m+1,-n,m+2,-(d*e-c*f)*(a+b*x)/((b*c-a*d)*(e+f*x))] /;
FreeQ[{a,b,c,d,e,f,m,p},x] && EqQ[m+n+p+2,0] && ILtQ[n,0]
```

2: $\int (a+b\,x)^m\,(c+d\,x)^n\,(e+f\,x)^p\,dx$ when $m+n+p+2 == 0 \land n \notin \mathbb{Z}$

Rule 1.1.1.3.H.2: If $m+n+p+2 == 0 \land n \notin \mathbb{Z}$, then

$$\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx \;\longrightarrow$$

$$\frac{(a + b\,x)^{m+1}\,(c + d\,x)^n\,(e + f\,x)^{p+1}}{(b\,e - a\,f)\,(m + 1)}\left(\frac{(b\,e - a\,f)\,(c + d\,x)}{(b\,c - a\,d)\,(e + f\,x)}\right)^{-n}\text{Hypergeometric2F1}\left[m + 1,\;-n,\;m + 2,\;-\frac{(d\,e - c\,f)\,(a + b\,x)}{(b\,c - a\,d)\,(e + f\,x)}\right]$$

Program code:

```
Int[(a_.+b_.*x_)^m_*(c_.+d_.*x_)^n_*(e_.+f_.*x_)^p_,x_Symbol] :=
  (a+b*x)^(m+1)*(c+d*x)^n*(e+f*x)^(p+1)/((b*e-a*f)*(m+1))*((b*e-a*f)*(c+d*x)/((b*c-a*d)*(e+f*x)))^(-n)*
    Hypergeometric2F1[m+1,-n,m+2,-(d*e-c*f)*(a+b*x)/((b*c-a*d)*(e+f*x))] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && EqQ[m+n+p+2,0] && Not[IntegerQ[n]]
```

A. $\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $m \notin \mathbb{Z} \;\wedge\; n \notin \mathbb{Z}$

   1. $\int (b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $m \notin \mathbb{Z} \;\wedge\; n \notin \mathbb{Z}$

      1: $\int (b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $m \notin \mathbb{Z} \;\wedge\; n \notin \mathbb{Z} \;\wedge\; c > 0 \;\wedge\; (p \in \mathbb{Z} \;\vee\; e > 0)$

Rule 1.1.1.3.A.1.1: If $m \notin \mathbb{Z} \;\wedge\; n \notin \mathbb{Z} \;\wedge\; c > 0 \;\wedge\; (p \in \mathbb{Z} \;\vee\; e > 0)$, then

$$\int (b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx \;\longrightarrow\; \frac{c^n\,e^p\,(b\,x)^{m+1}}{b\,(m + 1)}\,\text{AppellF1}\left[m + 1,\;-n,\;-p,\;m + 2,\;-\frac{d\,x}{c},\;-\frac{f\,x}{e}\right]$$

Program code:

```
Int[(b_.*x_)^m_*(c_+d_.*x_)^n_*(e_+f_.*x_)^p_,x_Symbol] :=
  c^n*e^p*(b*x)^(m+1)/(b*(m+1))*AppellF1[m+1,-n,-p,m+2,-d*x/c,-f*x/e] /;
FreeQ[{b,c,d,e,f,m,n,p},x] && Not[IntegerQ[m]] && Not[IntegerQ[n]] && GtQ[c,0] && (IntegerQ[p] || GtQ[e,0])
```

2: $\int (b\,x)^m (c+d\,x)^n (e+f\,x)^p \, dx$ when $m \notin \mathbb{Z} \wedge n \notin \mathbb{Z} \wedge -\dfrac{d}{b\,c} > 0 \wedge \left(p \in \mathbb{Z} \vee \dfrac{d}{d\,e-c\,f} > 0 \right)$

Rule 1.1.1.3.A.1.2: If $m \notin \mathbb{Z} \wedge n \notin \mathbb{Z} \wedge -\dfrac{d}{b\,c} > 0 \wedge \left(p \in \mathbb{Z} \vee \dfrac{d}{d\,e-c\,f} > 0 \right)$, then

$$\int (b\,x)^m (c+d\,x)^n (e+f\,x)^p \, dx \longrightarrow$$

$$\frac{(c+d\,x)^{n+1}}{d\,(n+1) \left(-\frac{d}{b\,c}\right)^m \left(\frac{d}{d\,e-c\,f}\right)^p} \; \text{AppellF1}\left[n+1,\, -m,\, -p,\, n+2,\, 1+\frac{d\,x}{c},\, -\frac{f\,(c+d\,x)}{d\,e-c\,f}\right]$$

Program code:

```
Int[(b_.*x_)^m_*(c_+d_.*x_)^n_*(e_+f_.*x_)^p_,x_Symbol] :=
  (c+d*x)^(n+1)/(d*(n+1)*(-d/(b*c))^m*(d/(d*e-c*f))^p)*AppellF1[n+1,-m,-p,n+2,1+d*x/c,-f*(c+d*x)/(d*e-c*f)] /;
FreeQ[{b,c,d,e,f,m,n,p},x] && Not[IntegerQ[m]] && Not[IntegerQ[n]] && GtQ[-d/(b*c),0] && (IntegerQ[p] || GtQ[d/(d*e-c*f),0])
```

3: $\int (b\,x)^m\,(c+d\,x)^n\,(e+f\,x)^p\,dx$ when $m \notin \mathbb{Z} \,\wedge\, n \notin \mathbb{Z} \,\wedge\, c \ngtr 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{(c+d\,x)^n}{\left(\frac{c+d\,x}{c}\right)^n} == 0$

Rule 1.1.1.3.A.1.3: If $m \notin \mathbb{Z} \,\wedge\, n \notin \mathbb{Z} \,\wedge\, c \ngtr 0$, then

$$\int (b\,x)^m\,(c+d\,x)^n\,(e+f\,x)^p\,dx \;\longrightarrow\; \frac{c^{\text{IntPart}[n]}\,(c+d\,x)^{\text{FracPart}[n]}}{\left(1+\frac{d\,x}{c}\right)^{\text{FracPart}[n]}}\int (b\,x)^m\,\left(1+\frac{d\,x}{c}\right)^n\,(e+f\,x)^p\,dx$$

Program code:

```
Int[(b_.*x_)^m_*(c_+d_.*x_)^n_*(e_+f_.*x_)^p_,x_Symbol] :=
  c^IntPart[n]*(c+d*x)^FracPart[n]/(1+d*x/c)^FracPart[n]*Int[(b*x)^m*(1+d*x/c)^n*(e+f*x)^p,x] /;
FreeQ[{b,c,d,e,f,m,n,p},x] && Not[IntegerQ[m]] && Not[IntegerQ[n]] && Not[GtQ[c,0]]
```

2. $\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $m \notin \mathbb{Z} \wedge n \notin \mathbb{Z} \wedge p \in \mathbb{Z}$

    1: $\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $m \notin \mathbb{Z} \wedge n \notin \mathbb{Z} \wedge p \in \mathbb{Z} \wedge \dfrac{b}{b\,c - a\,d} > 0$

Rule 1.1.1.3.A.2.1: If $m \notin \mathbb{Z} \wedge n \notin \mathbb{Z} \wedge p \in \mathbb{Z} \wedge \dfrac{b}{b\,c-a\,d} > 0$, then

$$\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx \;\rightarrow$$

$$\frac{(b\,e - a\,f)^p\,(a + b\,x)^{m+1}}{b^{p+1}\,(m+1)\,\left(\frac{b}{b\,c-a\,d}\right)^n}\;\text{AppellF1}\!\left[m+1,\,-n,\,-p,\,m+2,\,-\frac{d\,(a + b\,x)}{b\,c - a\,d},\,-\frac{f\,(a + b\,x)}{b\,e - a\,f}\right]$$

Program code:

```
Int[(a_+b_.*x_)^m_*(c_.+d_.*x_)^n_*(e_.+f_.*x_)^p_,x_Symbol] :=
  (b*e-a*f)^p*(a+b*x)^(m+1)/(b^(p+1)*(m+1)*(b/(b*c-a*d))^n)*
    AppellF1[m+1,-n,-p,m+2,-d*(a+b*x)/(b*c-a*d),-f*(a+b*x)/(b*e-a*f)] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && Not[IntegerQ[m]] && Not[IntegerQ[n]] && IntegerQ[p] && GtQ[b/(b*c-a*d),0] &&
  Not[GtQ[d/(d*a-c*b),0]] && SimplerQ[c+d*x,a+b*x]]
```

2: $\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $m \notin \mathbb{Z} \wedge n \notin \mathbb{Z} \wedge p \in \mathbb{Z} \wedge \frac{b}{b\,c - a\,d} \not> 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(c + d\,x)^n}{\left(\frac{b\,(c + d\,x)}{b\,c - a\,d}\right)^n} == 0$

Rule 1.1.1.3.A.2.2: If $m \notin \mathbb{Z} \wedge n \notin \mathbb{Z} \wedge p \in \mathbb{Z} \wedge \frac{b}{b\,c - a\,d} \not> 0$, then

$$\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx \longrightarrow \frac{(c + d\,x)^{\text{FracPart}[n]}}{\left(\frac{b}{b\,c - a\,d}\right)^{\text{IntPart}[n]}\left(\frac{b\,(c + d\,x)}{b\,c - a\,d}\right)^{\text{FracPart}[n]}} \int (a + b\,x)^m\left(\frac{b\,c}{b\,c - a\,d} + \frac{b\,d\,x}{b\,c - a\,d}\right)^n (e + f\,x)^p\,dx$$

Program code:

```
Int[(a_+b_.*x_)^m_*(c_.+d_.*x_)^n_*(e_.+f_.*x_)^p_,x_Symbol] :=
  (c+d*x)^FracPart[n]/((b/(b*c-a*d))^IntPart[n]*(b*(c+d*x)/(b*c-a*d))^FracPart[n])*
    Int[(a+b*x)^m*(b*c/(b*c-a*d)+b*d*x/(b*c-a*d))^n*(e+f*x)^p,x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && Not[IntegerQ[m]] && Not[IntegerQ[n]] && IntegerQ[p] && Not[GtQ[b/(b*c-a*d),0]] &&
  Not[SimplerQ[c+d*x,a+b*x]]
```

3. $\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $m \notin \mathbb{Z} \wedge n \notin \mathbb{Z} \wedge p \notin \mathbb{Z}$

   1. $\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $m \notin \mathbb{Z} \wedge n \notin \mathbb{Z} \wedge p \notin \mathbb{Z} \wedge \frac{b}{b\,c - a\,d} > 0$

      1: $\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $m \notin \mathbb{Z} \wedge n \notin \mathbb{Z} \wedge p \notin \mathbb{Z} \wedge \frac{b}{b\,c - a\,d} > 0 \wedge \frac{b}{b\,e - a\,f} > 0$

Rule 1.1.1.3.A.3.1.1: If $m \notin \mathbb{Z} \wedge n \notin \mathbb{Z} \wedge p \notin \mathbb{Z} \wedge \frac{b}{b\,c - a\,d} > 0 \wedge \frac{b}{b\,e - a\,f} > 0$, then

$$\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx \longrightarrow$$

$$\frac{(a + b\,x)^{m+1}}{b\,(m+1)\,\left(\frac{b}{b\,c-a\,d}\right)^n\,\left(\frac{b}{b\,e-a\,f}\right)^p}\,\text{AppellF1}\left[m+1,\,-n,\,-p,\,m+2,\,-\frac{d\,(a+b\,x)}{b\,c-a\,d},\,-\frac{f\,(a+b\,x)}{b\,e-a\,f}\right]$$

Program code:

```
Int[(a_+b_.*x_)^m_*(c_.+d_.*x_)^n_*(e_.+f_.*x_)^p_,x_Symbol] :=
  (a+b*x)^(m+1)/(b*(m+1)*(b/(b*c-a*d))^n*(b/(b*e-a*f))^p)*AppellF1[m+1,-n,-p,m+2,-d*(a+b*x)/(b*c-a*d),-f*(a+b*x)/(b*e-a*f)] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && Not[IntegerQ[m]] && Not[IntegerQ[n]] && Not[IntegerQ[p]] &&
  GtQ[b/(b*c-a*d),0] && GtQ[b/(b*e-a*f),0] &&
  Not[GtQ[d/(d*a-c*b),0] && GtQ[d/(d*e-c*f),0] && SimplerQ[c+d*x,a+b*x]] &&
  Not[GtQ[f/(f*a-e*b),0] && GtQ[f/(f*c-e*d),0] && SimplerQ[e+f*x,a+b*x]]
```

2: $\displaystyle\int (a+b\,x)^m\,(c+d\,x)^n\,(e+f\,x)^p\,dx$ when $m \notin \mathbb{Z} \,\wedge\, n \notin \mathbb{Z} \,\wedge\, p \notin \mathbb{Z} \,\wedge\, \frac{b}{b\,c-a\,d} > 0 \,\wedge\, \frac{b}{b\,e-a\,f} \not> 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x\,\dfrac{(e+f\,x)^p}{\left(\frac{b\,(e+f\,x)}{b\,e-a\,f}\right)^p} == 0$

Rule 1.1.1.3.A.3.1.2: If $m \notin \mathbb{Z} \,\wedge\, n \notin \mathbb{Z} \,\wedge\, p \notin \mathbb{Z} \,\wedge\, \dfrac{b}{b\,c-a\,d} > 0 \,\wedge\, \dfrac{b}{b\,e-a\,f} \not> 0$, then

$$\int (a+b\,x)^m\,(c+d\,x)^n\,(e+f\,x)^p\,dx \;\longrightarrow\; \frac{(e+f\,x)^{\text{FracPart}[p]}}{\left(\frac{b}{b\,e-a\,f}\right)^{\text{IntPart}[p]}\,\left(\frac{b\,(e+f\,x)}{b\,e-a\,f}\right)^{\text{FracPart}[p]}}\,\int (a+b\,x)^m\,(c+d\,x)^n\,\left(\frac{b\,e}{b\,e-a\,f}+\frac{b\,f\,x}{b\,e-a\,f}\right)^p\,dx$$

Program code:

```
Int[(a_+b_.*x_)^m_*(c_.+d_.*x_)^n_*(e_.+f_.*x_)^p_,x_Symbol] :=
  (e+f*x)^FracPart[p]/((b/(b*e-a*f))^IntPart[p]*(b*(e+f*x)/(b*e-a*f))^FracPart[p])*
    Int[(a+b*x)^m*(c+d*x)^n*(b*e/(b*e-a*f)+b*f*x/(b*e-a*f))^p,x] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && Not[IntegerQ[m]] && Not[IntegerQ[n]] && Not[IntegerQ[p]] &&
  GtQ[b/(b*c-a*d),0] && Not[GtQ[b/(b*e-a*f),0]]
```

2: $\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx$ when $m \notin \mathbb{Z} \wedge n \notin \mathbb{Z} \wedge p \notin \mathbb{Z} \wedge \frac{b}{b\,c - a\,d} \not> 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{(c + d\,x)^n}{\left(\frac{b\,(c + d\,x)}{b\,c - a\,d}\right)^n} == 0$

Rule 1.1.1.3.A.3.2: If $m \notin \mathbb{Z} \wedge n \notin \mathbb{Z} \wedge p \notin \mathbb{Z} \wedge \frac{b}{b\,c - a\,d} \not> 0$, then

$$\int (a + b\,x)^m\,(c + d\,x)^n\,(e + f\,x)^p\,dx \;\longrightarrow\; \frac{(c + d\,x)^{\text{FracPart}[n]}}{\left(\frac{b}{b\,c - a\,d}\right)^{\text{IntPart}[n]} \left(\frac{b\,(c + d\,x)}{b\,c - a\,d}\right)^{\text{FracPart}[n]}} \int (a + b\,x)^m \left(\frac{b\,c}{b\,c - a\,d} + \frac{b\,d\,x}{b\,c - a\,d}\right)^n (e + f\,x)^p\,dx$$

Program code:

```
Int[(a_+b_.*x_)^m_*(c_.+d_.*x_)^n_*(e_.+f_.*x_)^p_,x_Symbol] :=
  (c+d*x)^FracPart[n]/((b/(b*c-a*d))^IntPart[n]*(b*(c+d*x)/(b*c-a*d))^FracPart[n])*
    Int[(a+b*x)^m*(b*c/(b*c-a*d)+b*d*x/(b*c-a*d))^n*(e+f*x)^p,x] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && Not[IntegerQ[m]] && Not[IntegerQ[n]] && Not[IntegerQ[p]] && Not[GtQ[b/(b*c-a*d),0]] &&
  Not[SimplerQ[c+d*x,a+b*x]] && Not[SimplerQ[e+f*x,a+b*x]]
```

S: $\int (a+bu)^m \, (c+du)^n \, (e+fu)^p \, \mathrm{d}x$ when $u == g + hx$

Derivation: Integration by substitution

Rule 1.1.1.3.S: If $u == g + hx$, then

$$\int (a+bu)^m \, (c+du)^n \, (e+fu)^p \, \mathrm{d}x \;\longrightarrow\; \frac{1}{h} \, \mathrm{Subst}\left[\int (a+bx)^m \, (c+dx)^n \, (e+fx)^p \, \mathrm{d}x, \, x, \, u\right]$$

Program code:

```
Int[(a_.+b_.*u_)^m_.*(c_.+d_.*u_)^n_.*(e_+f_.*u_)^p_.,x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(a+b*x)^m*(c+d*x)^n*(e+f*x)^p,x],x,u] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && LinearQ[u,x] && NeQ[u,x]
```