

Rules for integrands of the form $(c + d x)^m (a + b \sin[e + f x])^n$

$$1. \int (c + d x)^m (b \sin[e + f x])^n dx$$

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Reference: CRC 392, A&S 4.3.119

Reference: CRC 396, A&S 4.3.123

Derivation: Integration by parts

$$\text{Basis: } \sin[e + f x] = -\frac{1}{f} \partial_x \cos[e + f x]$$

Rule: If $m > 0$, then

$$\int (c + d x)^m \sin[e + f x] dx \rightarrow -\frac{(c + d x)^m \cos[e + f x]}{f} + \frac{d m}{f} \int (c + d x)^{m-1} \cos[e + f x] dx$$

Program code:

```
Int[(c_.+d_.**x_)^m_.**sin[e_.+f_.**x_],x_Symbol] :=
  -(c+d*x)^m**Cos[e+f*x]/f +
  d*m/f*Int[(c+d*x)^(m-1)*Cos[e+f*x],x] /;
FreeQ[{c,d,e,f},x] && GtQ[m,0]
```

2: $\int (c+dx)^m \sin[ex+f] dx$ when $m < -1$

Reference: CRC 405, A&S 4.3.120

Reference: CRC 406, A&S 4.3.124

Derivation: Integration by parts

Rule: If $m < -1$, then

$$\int (c+dx)^m \sin[ex+f] dx \rightarrow \frac{(c+dx)^{m+1} \sin[ex+f]}{d(m+1)} - \frac{f}{d(m+1)} \int (c+dx)^{m+1} \cos[ex+f] dx$$

Program code:

```
Int[(c_.+d_.*x_)^m_*sin[e_.+f_.*x_],x_Symbol] :=
  (c+d*x)^(m+1)*Sin[e+f*x]/(d*(m+1)) -
  f/(d*(m+1))*Int[(c+d*x)^(m+1)*Cos[e+f*x],x] /;
FreeQ[{c,d,e,f},x] && LtQ[m,-1]
```

3. $\int \frac{\sin[ex+f]}{c+dx} dx$

1: $\int \frac{\sin[ex+f]}{c+dx} dx$ when $de - cf = 0$

Derivation: Primitive rule

Basis: $\text{SinIntegral}[\pm z] == \pm \text{SinhIntegral}[z]$

Basis: $\partial_x \text{CosIntegral}[\pm F[x]] == \partial_x \text{CoshIntegral}[F[x]] == \partial_x \text{CoshIntegral}[-F[x]]$

Rule: If $de - cf = 0$, then

$$\int \frac{\sin[ex]}{c+dx} dx \rightarrow \frac{\text{SinIntegral}[ex]}{d}$$

$$\int \frac{\cos[ex]}{c+dx} dx \rightarrow \frac{\text{CosIntegral}[ex]}{d}$$

Program code:

```
Int[sin[e_+f_.*Complex[0,fz_]*x_]/(c_+d_.*x_),x_Symbol] :=
  I*SinhIntegral[c*f*fz/d+f*fz*x]/d /;
FreeQ[{c,d,e,f,fz},x] && EqQ[d*e-c*f*fz*I,0]
```

```
Int[sin[e_+f_.*x_]/(c_+d_.*x_),x_Symbol] :=
  SinIntegral[e+f*x]/d /;
FreeQ[{c,d,e,f},x] && EqQ[d*e-c*f,0]
```

```
Int[sin[e_+f_.*Complex[0,fz_]*x_]/(c_+d_.*x_),x_Symbol] :=
  CoshIntegral[-c*f*fz/d-f*fz*x]/d /;
FreeQ[{c,d,e,f,fz},x] && EqQ[d*(e-Pi/2)-c*f*fz*I,0] && NegQ[c*f*fz/d,0]
```

```
Int[sin[e_+f_.*Complex[0,fz_]*x_]/(c_+d_.*x_),x_Symbol] :=
  CoshIntegral[c*f*fz/d+f*fz*x]/d /;
FreeQ[{c,d,e,f,fz},x] && EqQ[d*(e-Pi/2)-c*f*fz*I,0]
```

```
Int[sin[e_+f_.*x_]/(c_+d_.*x_),x_Symbol] :=
  CosIntegral[e-Pi/2+f*x]/d /;
FreeQ[{c,d,e,f},x] && EqQ[d*(e-Pi/2)-c*f,0]
```

$$\text{2: } \int \frac{\sin[ex+f]}{c+dx} dx \text{ when } de - cf \neq 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \sin[ex+f] = \cos\left[\frac{de-cf}{d}\right] \sin\left[\frac{cf}{d} + ex\right] + \sin\left[\frac{de-cf}{d}\right] \cos\left[\frac{cf}{d} + ex\right]$$

Rule: If $de - cf \neq 0$, then

$$\int \frac{\sin[ex+f]}{c+dx} dx \rightarrow \cos\left[\frac{de-cf}{d}\right] \int \frac{\sin\left[\frac{cf}{d} + ex\right]}{c+dx} dx + \sin\left[\frac{de-cf}{d}\right] \int \frac{\cos\left[\frac{cf}{d} + ex\right]}{c+dx} dx$$

Program code:

```
Int[sin[e_.+f_.*x_]/(c_.+d_.*x_),x_Symbol] :=
  Cos[(d*e-c*f)/d]*Int[Sin[c*f/d+f*x]/(c+d*x),x] +
  Sin[(d*e-c*f)/d]*Int[Cos[c*f/d+f*x]/(c+d*x),x] /;
FreeQ[{c,d,e,f},x] && NeQ[d*e-c*f,0]
```

$$4. \int \frac{\sin[ex+f]}{\sqrt{c+dx}} dx$$

$$\text{1: } \int \frac{\sin[ex+f]}{\sqrt{c+dx}} dx \text{ when } de - cf = 0$$

Derivation: Integration by substitution

$$\text{Basis: If } de - cf = 0, \text{ then } \frac{F[ex+f]}{\sqrt{c+dx}} = \frac{2}{d} \text{Subst}\left[F\left[\frac{fx^2}{d}\right], x, \sqrt{c+dx}\right] \partial_x \sqrt{c+dx}$$

Rule: If $de - cf = 0$, then

$$\int \frac{\sin[e + f x]}{\sqrt{c + d x}} \rightarrow \frac{2}{d} \text{Subst}\left[\int \sin\left[\frac{f x^2}{d}\right] dx, x, \sqrt{c + d x}\right]$$

Program code:

```
Int[sin[e_+Pi/2+f_.*x_]/Sqrt[c_+d_.*x_],x_Symbol] :=
  2/d*Subst[Int[Cos[f*x^2/d],x],x,Sqrt[c+d*x]] /;
FreeQ[{c,d,e,f},x] && ComplexFreeQ[f] && EqQ[d*e-c*f,0]
```

```
Int[sin[e_+f_.*x_]/Sqrt[c_+d_.*x_],x_Symbol] :=
  2/d*Subst[Int[Sin[f*x^2/d],x],x,Sqrt[c+d*x]] /;
FreeQ[{c,d,e,f},x] && ComplexFreeQ[f] && EqQ[d*e-c*f,0]
```

2: $\int \frac{\sin[e + f x]}{\sqrt{c + d x}} dx$ when $d e - c f \neq 0$

Derivation: Algebraic expansion

Basis: $\sin[e + f x] = \cos\left[\frac{d e - c f}{d}\right] \sin\left[\frac{c f}{d} + f x\right] + \sin\left[\frac{d e - c f}{d}\right] \cos\left[\frac{c f}{d} + f x\right]$

Rule: If $d e - c f \neq 0$, then

$$\int \frac{\sin[e + f x]}{\sqrt{c + d x}} dx \rightarrow \cos\left[\frac{d e - c f}{d}\right] \int \frac{\sin\left[\frac{c f}{d} + f x\right]}{\sqrt{c + d x}} dx + \sin\left[\frac{d e - c f}{d}\right] \int \frac{\cos\left[\frac{c f}{d} + f x\right]}{\sqrt{c + d x}} dx$$

Program code:

```
Int[sin[e_+f_.*x_]/Sqrt[c_+d_.*x_],x_Symbol] :=
  Cos[(d*e-c*f)/d]*Int[Sin[c*f/d+f*x]/Sqrt[c+d*x],x] +
  Sin[(d*e-c*f)/d]*Int[Cos[c*f/d+f*x]/Sqrt[c+d*x],x] /;
FreeQ[{c,d,e,f},x] && ComplexFreeQ[f] && NeQ[d*e-c*f,0]
```

5: $\int (c+dx)^m \sin[ex+f] dx$

Derivation: Algebraic expansion

Basis: $\sin[z] == \frac{1}{2} i e^{-i z} - \frac{1}{2} i e^{i z}$

Basis: $\cos[z] == \frac{1}{2} e^{-i z} + \frac{1}{2} e^{i z}$

Rule:

$$\int (c+dx)^m \sin[ex+f] dx \rightarrow \frac{i}{2} \int (c+dx)^m e^{-i(ex+f)} dx - \frac{i}{2} \int (c+dx)^m e^{i(ex+f)} dx$$

Program code:

```
Int[(c_.+d_.**x_)^m_.**sin[e_.+k_.*Pi+f_.**x_],x_Symbol] :=
  I/2*Int[(c+d*x)^m**E^(-I*k*Pi)*E^(-I*(e+f*x)),x] - I/2*Int[(c+d*x)^m**E^(I*k*Pi)*E^(I*(e+f*x)),x] /;
FreeQ[{c,d,e,f,m},x] && IntegerQ[2*k]
```

```
Int[(c_.+d_.**x_)^m_.**sin[e_.+f_.**x_],x_Symbol] :=
  I/2*Int[(c+d*x)^m**E^(-I*(e+f*x)),x] - I/2*Int[(c+d*x)^m**E^(I*(e+f*x)),x] /;
FreeQ[{c,d,e,f,m},x]
```

$$2. \int (c+dx)^m (b \sin[ex])^n dx \text{ when } n > 1$$

$$1: \int (c+dx)^m \sin[ex]^2 dx$$

Derivation: Algebraic expansion

$$\text{Basis: } \sin[z]^2 = \frac{1}{2} - \frac{\cos[2z]}{2}$$

Rule:

$$\int (c+dx)^m \sin[ex]^2 dx \rightarrow \frac{1}{2} \int (c+dx)^m dx - \frac{1}{2} \int (c+dx)^m \cos[2ex] dx$$

Program code:

```
Int[(c_.+d_.**x_)^m_.**sin[e_.+f_.**x_/2]^2,x_Symbol] :=
  1/2*Int[(c+d*x)^m,x] - 1/2*Int[(c+d*x)^m**Cos[2*e+f*x],x] /;
FreeQ[{c,d,e,f,m},x]
```

$$2. \int (c+dx)^m (b \sin[ex])^n dx \text{ when } n > 1 \wedge m \geq 1$$

$$1: \int (c+dx) (b \sin[ex])^n dx \text{ when } n > 1$$

Reference: G&R 2.631.2 with $m \rightarrow 1$

Reference: G&R 2.631.3 with $m \rightarrow 1$

Rule: If $n > 1$, then

$$\int (c+dx) (b \sin[ex])^n dx \rightarrow$$

$$\frac{d (b \sin[e + f x])^n}{f^2 n^2} - \frac{b (c + d x) \cos[e + f x] (b \sin[e + f x])^{n-1}}{f n} + \frac{b^2 (n-1)}{n} \int (c + d x) (b \sin[e + f x])^{n-2} dx$$

Program code:

```
Int[(c_.+d_.**x_)*(b_.**sin[e_.+f_.**x_])^n_,x_Symbol] :=
  d*(b**Sin[e+f**x])^n/(f^2*n^2) -
  b*(c+d*x)*Cos[e+f**x]*(b**Sin[e+f**x])^(n-1)/(f*n) +
  b^2*(n-1)/n*Int[(c+d*x)*(b**Sin[e+f**x])^(n-2),x] /;
FreeQ[{b,c,d,e,f},x] && GtQ[n,1]
```

2: $\int (c + d x)^m (b \sin[e + f x])^n dx$ when $n > 1 \wedge m > 1$

Reference: G&R 2.631.2

Reference: G&R 2.631.3

Rule: If $n > 1 \wedge m > 1$, then

$$\int (c + d x)^m (b \sin[e + f x])^n dx \rightarrow$$

$$\frac{d m (c + d x)^{m-1} (b \sin[e + f x])^n}{f^2 n^2} - \frac{b (c + d x)^m \cos[e + f x] (b \sin[e + f x])^{n-1}}{f n} +$$

$$\frac{b^2 (n-1)}{n} \int (c + d x)^m (b \sin[e + f x])^{n-2} dx - \frac{d^2 m (m-1)}{f^2 n^2} \int (c + d x)^{m-2} (b \sin[e + f x])^n dx$$

Program code:

```
Int[(c_.+d_.**x_)^m_*(b_.**sin[e_.+f_.**x_])^n_,x_Symbol] :=
  d**m*(c+d*x)^(m-1)*(b**Sin[e+f**x])^n/(f^2*n^2) -
  b*(c+d*x)^m**Cos[e+f**x]*(b**Sin[e+f**x])^(n-1)/(f*n) +
  b^2*(n-1)/n*Int[(c+d*x)^m*(b**Sin[e+f**x])^(n-2),x] -
  d^2*m*(m-1)/(f^2*n^2)*Int[(c+d*x)^(m-2)*(b**Sin[e+f**x])^n,x] /;
FreeQ[{b,c,d,e,f},x] && GtQ[n,1] && GtQ[m,1]
```


$$3. \int (c+dx)^m (b \sin[ex])^n dx \text{ when } n > 1 \wedge m < 1$$

$$1: \int (c+dx)^m \sin[ex]^n dx \text{ when } n \in \mathbb{Z} \wedge n > 1 \wedge -1 \leq m < 1$$

Derivation: Algebraic expansion

Rule: If $n \in \mathbb{Z} \wedge n > 1 \wedge -1 \leq m < 1$, then

$$\int (c+dx)^m \sin[ex]^n dx \rightarrow \int (c+dx)^m \text{TrigReduce}[\sin[ex]^n] dx$$

Program code:

```
Int[(c_.+d_.**x_)^m_*sin[e_.+f_.**x_]^n_,x_Symbol] :=
  Int[ExpandTrigReduce[(c+d*x)^m,Sin[e+f*x]^n,x],x] /;
FreeQ[{c,d,e,f,m},x] && IGtQ[n,1] && (Not[RationalQ[m]] || GeQ[m,-1] && LtQ[m,1])
```

2: $\int (c+dx)^m \sin[ex]^n dx$ when $n \in \mathbb{Z} \wedge n > 1 \wedge -2 \leq m < -1$

Derivation: Integration by parts

Rule: If $n \in \mathbb{Z} \wedge n > 1 \wedge -2 \leq m < -1$, then

$$\int (c+dx)^m \sin[ex]^n dx \rightarrow \frac{(c+dx)^{m+1} \sin[ex]^n}{d(m+1)} - \frac{fn}{d(m+1)} \int (c+dx)^{m+1} \text{TrigReduce}[\cos[ex] \sin[ex]^{n-1}] dx$$

Program code:

```
Int[(c_.+d_.**x_)^m_*sin[e_.+f_.**x_]^n_,x_Symbol] :=
  (c+d*x)^(m+1)*Sin[e+f*x]^n/(d*(m+1)) -
  f*n/(d*(m+1))*Int[ExpandTrigReduce[(c+d*x)^(m+1),Cos[e+f*x]*Sin[e+f*x]^(n-1),x],x] /;
FreeQ[{c,d,e,f,m},x] && IGtQ[n,1] && GeQ[m,-2] && LtQ[m,-1]
```

3: $\int (c+dx)^m (b \sin[ex+f])^n dx$ when $n > 1 \wedge m < -2$

Reference: G&R 2.638.1

Reference: G&R 2.638.2

Rule: If $n > 1 \wedge m < -2$, then

$$\int (c+dx)^m (b \sin[ex+f])^n dx \rightarrow$$

$$\frac{(c+dx)^{m+1} (b \sin[ex+f])^n}{d(m+1)} - \frac{b f n (c+dx)^{m+2} \cos[ex+f] (b \sin[ex+f])^{n-1}}{d^2(m+1)(m+2)} -$$

$$\frac{f^2 n^2}{d^2(m+1)(m+2)} \int (c+dx)^{m+2} (b \sin[ex+f])^n dx + \frac{b^2 f^2 n(n-1)}{d^2(m+1)(m+2)} \int (c+dx)^{m+2} (b \sin[ex+f])^{n-2} dx$$

Program code:

```
Int[(c_.+d_.**x_)^m_*(b_.*sin[e_.+f_.**x_])^n_,x_Symbol] :=
  (c+d*x)^(m+1)*(b*sin[e+f*x])^n/(d*(m+1)) -
  b*f*n*(c+d*x)^(m+2)*Cos[e+f*x]*(b*sin[e+f*x])^(n-1)/(d^2*(m+1)*(m+2)) -
  f^2*n^2/(d^2*(m+1)*(m+2))*Int[(c+d*x)^(m+2)*(b*sin[e+f*x])^n,x] +
  b^2*f^2*n*(n-1)/(d^2*(m+1)*(m+2))*Int[(c+d*x)^(m+2)*(b*sin[e+f*x])^(n-2),x] /;
FreeQ[{b,c,d,e,f},x] && GtQ[n,1] && LtQ[m,-2]
```

$$2. \int (c+dx)^m (b \sin[ex+f])^n dx \text{ when } n < -1$$

$$1: \int (c+dx) (b \sin[ex+f])^n dx \text{ when } n < -1 \wedge n \neq -2$$

Reference: G&R 2.643.1 with $m \rightarrow 1$

Reference: G&R 2.643.2 with $m \rightarrow 1$

Rule: If $n < -1 \wedge n \neq -2$, then

$$\int (c+dx) (b \sin[ex+f])^n dx \rightarrow \frac{(c+dx) \cos[ex+f] (b \sin[ex+f])^{n+1}}{b f (n+1)} - \frac{d (b \sin[ex+f])^{n+2}}{b^2 f^2 (n+1) (n+2)} + \frac{n+2}{b^2 (n+1)} \int (c+dx) (b \sin[ex+f])^{n+2} dx$$

Program code:

```
Int[(c_.+d_.x_)*(b_.sin[e_.+f_.x_])^n_,x_Symbol] :=
  (c+d*x)*Cos[e+f*x]*(b*sin[e+f*x])^(n+1)/(b*f*(n+1)) -
  d*(b*sin[e+f*x])^(n+2)/(b^2*f^2*(n+1)*(n+2)) +
  (n+2)/(b^2*(n+1))*Int[(c+d*x)*(b*sin[e+f*x])^(n+2),x] /;
FreeQ[{b,c,d,e,f},x] && LtQ[n,-1] && NeQ[n,-2]
```

$$2: \int (c+dx)^m (b \sin[ex+f])^n dx \text{ when } n < -1 \wedge n \neq -2 \wedge m > 1$$

Reference: G&R 2.643.1

Reference: G&R 2.643.2

Rule: If $n < -1 \wedge n \neq -2 \wedge m > 1$, then

$$\int (c+dx)^m (b \sin[ex+f])^n dx \rightarrow$$

$$\frac{(c+dx)^m \cos[e+fx] (b \sin[e+fx])^{n+1}}{b f (n+1)} - \frac{d m (c+dx)^{m-1} (b \sin[e+fx])^{n+2}}{b^2 f^2 (n+1) (n+2)} +$$

$$\frac{n+2}{b^2 (n+1)} \int (c+dx)^m (b \sin[e+fx])^{n+2} dx + \frac{d^2 m (m-1)}{b^2 f^2 (n+1) (n+2)} \int (c+dx)^{m-2} (b \sin[e+fx])^{n+2} dx$$

Program code:

```
Int[(c_.+d_.**x_)^m_.*(b_.**sin[e_.+f_.**x_])^n_,x_Symbol] :=
  (c+d*x)^m*cos[e+f*x]*(b*sin[e+f*x])^(n+1)/(b*f*(n+1)) -
  d*m*(c+d*x)^(m-1)*(b*sin[e+f*x])^(n+2)/(b^2*f^2*(n+1)*(n+2)) +
  (n+2)/(b^2*(n+1))*Int[(c+d*x)^m*(b*sin[e+f*x])^(n+2),x] +
  d^2*m*(m-1)/(b^2*f^2*(n+1)*(n+2))*Int[(c+d*x)^(m-2)*(b*sin[e+f*x])^(n+2),x] /;
FreeQ[{b,c,d,e,f},x] && LtQ[n,-1] && NeQ[n,-2] && GtQ[m,1]
```

2: $\int (c+dx)^m (a+b \sin[e+fx])^n dx$ when $n \in \mathbb{Z}^+ \wedge (n = 1 \vee m \in \mathbb{Z}^+ \vee a^2 - b^2 \neq 0)$

Derivation: Algebraic expansion

Rule: If $n \in \mathbb{Z}^+ \wedge (n = 1 \vee m \in \mathbb{Z}^+ \vee a^2 - b^2 \neq 0)$, then

$$\int (c+dx)^m (a+b \sin[e+fx])^n dx \rightarrow \int (c+dx)^m \text{ExpandIntegrand}[(a+b \sin[e+fx])^n, x] dx$$

Program code:

```
Int[(c_.+d_.**x_)^m_.*(a_+b_.**sin[e_.+f_.**x_])^n_,x_Symbol] :=
  Int[ExpandIntegrand[(c+d*x)^m,(a+b*sin[e+f*x])^n,x],x] /;
FreeQ[{a,b,c,d,e,f,m},x] && IGtQ[n,0] && (EqQ[n,1] || IGtQ[m,0] || NeQ[a^2-b^2,0])
```

$$3. \int (c+dx)^m (a+b \sin[e+fx])^n dx \text{ when } a^2 - b^2 = 0 \wedge 2n \in \mathbb{Z} \wedge (n > 0 \vee m \in \mathbb{Z}^+)$$

$$1: \int (c+dx)^m (a+b \sin[e+fx])^n dx \text{ when } a^2 - b^2 = 0 \wedge n \in \mathbb{Z} \wedge (n > 0 \vee m \in \mathbb{Z}^+)$$

Derivation: Algebraic simplification

Basis: If $a^2 - b^2 = 0$, then $a + b \sin[e+fx] = 2a \sin\left[\frac{1}{2}\left(e + \frac{\pi a}{2b}\right) + \frac{fx}{2}\right]^2$

Rule: If $a^2 - b^2 = 0 \wedge n \in \mathbb{Z} \wedge (n > 0 \vee m \in \mathbb{Z}^+)$, then

$$\int (c+dx)^m (a+b \sin[e+fx])^n dx \rightarrow (2a)^n \int (c+dx)^m \sin\left[\frac{1}{2}\left(e + \frac{\pi a}{2b}\right) + \frac{fx}{2}\right]^{2n} dx$$

Program code:

```
Int[(c_.+d_.x_)^m_.*(a_+b_.sin[e_.+f_.x_])^n_,x_Symbol] :=
  (2*a)^n*Int[(c+d*x)^m*Sin[1/2*(e+Pi*a/(2*b))+f*x/2]^(2*n),x] /;
FreeQ[{a,b,c,d,e,f,m},x] && EqQ[a^2-b^2,0] && IntegerQ[n] && (GtQ[n,0] || IGtQ[m,0])
```

2: $\int (c+dx)^m (a+b \sin[e+fx])^n dx$ when $a^2 - b^2 = 0 \wedge n + \frac{1}{2} \in \mathbb{Z} \wedge (n > 0 \vee m \in \mathbb{Z}^+)$

Derivation: Piecewise constant extraction

Basis: If $a^2 - b^2 = 0$, then $\partial_x \frac{(a+b \sin[e+fx])^n}{\sin[\frac{1}{2}(e+\frac{\pi a}{2b})+\frac{fx}{2}]^{2n}} = 0$

Rule: If $a^2 - b^2 = 0 \wedge n + \frac{1}{2} \in \mathbb{Z} \wedge (n > 0 \vee m \in \mathbb{Z}^+)$, then

$$\int (c+dx)^m (a+b \sin[e+fx])^n dx \rightarrow \frac{(2a)^{\text{IntPart}[n]} (a+b \sin[e+fx])^{\text{FracPart}[n]}}{\sin[\frac{e}{2} + \frac{a\pi}{4b} + \frac{fx}{2}]^{2\text{FracPart}[n]}} \int (c+dx)^m \sin[\frac{e}{2} + \frac{a\pi}{4b} + \frac{fx}{2}]^{2n} dx$$

Program code:

```
Int[(c_+d_*x_)^m_.*(a_+b_*sin[e_+f_*x_])^n_,x_Symbol] :=
  (2*a)^IntPart[n]*(a+b*sin[e+f*x])^FracPart[n]/Sin[e/2+a*Pi/(4*b)+f*x/2]^(2*FracPart[n])*
  Int[(c+d*x)^m*sin[e/2+a*Pi/(4*b)+f*x/2]^(2*n),x] /;
FreeQ[{a,b,c,d,e,f,m},x] && EqQ[a^2-b^2,0] && IntegerQ[n+1/2] && (GtQ[n,0] || IGtQ[m,0])
```

x: $\int (c+dx)^m (a+b \sin[e+fx])^n dx$ when $a^2 - b^2 = 0 \wedge n \in \mathbb{Z} \wedge (n > 0 \vee m \in \mathbb{Z}^+)$

Derivation: Algebraic simplification

Basis: If $a^2 - b^2 = 0$, then $a + b \sin[z] = 2a \cos\left[-\frac{\pi a}{4b} + \frac{z}{2}\right]^2$

Rule: If $a^2 - b^2 = 0 \wedge n \in \mathbb{Z} \wedge (n > 0 \vee m \in \mathbb{Z}^+)$, then

$$\int (c+dx)^m (a+b \sin[e+fx])^n dx \rightarrow (2a)^n \int (c+dx)^m \cos\left[\frac{1}{2}\left(e - \frac{\pi a}{2b}\right) + \frac{fx}{2}\right]^{2n} dx$$

Program code:

```
(* Int[(c_.+d_.*x_)^m_.*(a_+b_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  (2*a)^n*Int[(c+d*x)^m*Cos[1/2*(e-Pi*a/(2*b))+f*x/2]^(2*n),x] /;
FreeQ[{a,b,c,d,e,f,m},x] && EqQ[a^2-b^2,0] && IntegerQ[n] && (GtQ[n,0] || IGtQ[m,0]) *)
```


x: $\int (c+dx)^m (a+b \sin[e+fx])^n dx$ when $a^2 - b^2 = 0 \wedge n + \frac{1}{2} \in \mathbb{Z} \wedge (n > 0 \vee m \in \mathbb{Z}^+)$

Derivation: Piecewise constant extraction

Basis: If $a^2 - b^2 = 0$, then $\partial_x \frac{(a+b \sin[e+fx])^n}{\cos[\frac{1}{2}(e-\frac{\pi a}{2b})+\frac{fx}{2}]^{2n}} = 0$

Rule: If $a^2 - b^2 = 0 \wedge n + \frac{1}{2} \in \mathbb{Z} \wedge (n > 0 \vee m \in \mathbb{Z}^+)$, then

$$\int (c+dx)^m (a+b \sin[e+fx])^n dx \rightarrow \frac{(2a)^{\text{IntPart}[n]} (a+b \sin[e+fx])^{\text{FracPart}[n]}}{\cos[\frac{1}{2}(e-\frac{\pi a}{2b})+\frac{fx}{2}]^{2\text{FracPart}[n]}} \int (c+dx)^m \cos[\frac{1}{2}(e-\frac{\pi a}{2b})+\frac{fx}{2}]^{2n} dx$$

Program code:

```
(* Int[(c_.+d_.*x_)^m.*(a+b_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  (2*a)^IntPart[n]*(a+b*sin[e+fx])^FracPart[n]/Cos[1/2*(e-Pi*a/(2*b))+f*x/2]^(2*FracPart[n])*
  Int[(c+dx)^m*cos[1/2*(e-Pi*a/(2*b))+f*x/2]^(2*n),x] /;
FreeQ[{a,b,c,d,e,f,m},x] && EqQ[a^2-b^2,0] && IntegerQ[n+1/2] && (GtQ[n,0] || IGtQ[m,0]) *)
```

4. $\int (c+dx)^m (a+b \sin[e+fx])^n dx$ when $a^2 - b^2 \neq 0 \wedge n \in \mathbb{Z}^- \wedge m \in \mathbb{Z}^+$

1: $\int \frac{(c+dx)^m}{a+b \sin[e+fx]} dx$ when $a^2 - b^2 \neq 0 \wedge m \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Basis: $\frac{1}{a+b \sin[z]} = \frac{2e^{iz}}{ib+2ae^{iz}-ib e^{2iz}} = \frac{2e^{-iz}}{-ib+2ae^{-iz}+ib e^{-2iz}}$

Basis: $\frac{1}{a+b \cos[z]} = \frac{2e^{iz}}{b+2ae^{iz}+b e^{2iz}}$

Rule: If $a^2 - b^2 \neq 0 \wedge m \in \mathbb{Z}^+$, then

$$\int \frac{(c+dx)^m}{a+b \sin[e+fx]} dx \rightarrow -2i \int \frac{(c+dx)^m e^{i(e+fx)}}{b-2ia e^{i(e+fx)} - b e^{2i(e+fx)}} dx$$

$$\int \frac{(c+dx)^m}{a+b \cos[e+fx]} dx \rightarrow 2 \int \frac{(c+dx)^m e^{i(e+fx)}}{b+2ia e^{i(e+fx)} + b e^{2i(e+fx)}} dx$$

Program code:

```
Int[(c_.+d_.**x_)^m_./(a_+b_.*sin[e_.+k_.*Pi+f_.*Complex[0,fz_]*x_]),x_Symbol] :=
  2*Int[(c+d*x)^m*E^(-I*Pi*(k-1/2))*E^(-I*e+f*fz*x)/(b+2*a*E^(-I*Pi*(k-1/2))*E^(-I*e+f*fz*x)-b*E^(-2*I*k*Pi)*E^(2*(-I*e+f*fz*x))),x]
FreeQ[{a,b,c,d,e,f,fz},x] && IntegerQ[2*k] && NeQ[a^2-b^2,0] && IGtQ[m,0]
```

```
Int[(c_.+d_.**x_)^m_./(a_+b_.*sin[e_.+k_.*Pi+f_.**x_]),x_Symbol] :=
  2*Int[(c+d*x)^m*E^(I*Pi*(k-1/2))*E^(I*(e+f*x))/(b+2*a*E^(I*Pi*(k-1/2))*E^(I*(e+f*x))-b*E^(2*I*k*Pi)*E^(2*I*(e+f*x))),x] /;
FreeQ[{a,b,c,d,e,f},x] && IntegerQ[2*k] && NeQ[a^2-b^2,0] && IGtQ[m,0]
```

```
(* Int[(c_.+d_.**x_)^m_./(a_+b_.*sin[e_.+f_.*Complex[0,fz_]*x_]),x_Symbol] :=
  2*I*Int[(c+d*x)^m*E^(-I*e+f*fz*x)/(b+2*I*a*E^(-I*e+f*fz*x)-b*E^(2*(-I*e+f*fz*x))),x] /;
FreeQ[{a,b,c,d,e,f,fz},x] && NeQ[a^2-b^2,0] && IGtQ[m,0] *)
```

```
(* Int[(c_.+d_.**x_)^m_./(a_+b_.*sin[e_.+f_.**x_]),x_Symbol] :=
  -2*I*Int[(c+d*x)^m*E^(I*(e+f*x))/(b-2*I*a*E^(I*(e+f*x))-b*E^(2*I*(e+f*x))),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[a^2-b^2,0] && IGtQ[m,0] *)
```

```
Int[(c_.+d_.**x_)^m_./(a_+b_.*sin[e_.+f_.*Complex[0,fz_]*x_]),x_Symbol] :=
  2*Int[(c+d*x)^m*E^(-I*e+f*fz*x)/(-I*b+2*a*E^(-I*e+f*fz*x)+I*b*E^(2*(-I*e+f*fz*x))),x] /;
FreeQ[{a,b,c,d,e,f,fz},x] && NeQ[a^2-b^2,0] && IGtQ[m,0]
```

```
Int[(c_.+d_.**x_)^m_./(a_+b_.*sin[e_.+f_.**x_]),x_Symbol] :=
  2*Int[(c+d*x)^m*E^(I*(e+f*x))/(I*b+2*a*E^(I*(e+f*x))-I*b*E^(2*I*(e+f*x))),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[a^2-b^2,0] && IGtQ[m,0]
```

2: $\int \frac{(c+dx)^m}{(a+b \sin[ex])^2} dx$ when $a^2 - b^2 \neq 0 \wedge m \in \mathbb{Z}^+$

Rule: If $a^2 - b^2 \neq 0 \wedge m \in \mathbb{Z}^+$, then

$$\int \frac{(c+dx)^m}{(a+b \sin[ex])^2} dx \rightarrow \frac{b (c+dx)^m \cos[ex]}{f (a^2 - b^2) (a+b \sin[ex])} + \frac{a}{a^2 - b^2} \int \frac{(c+dx)^m}{a+b \sin[ex]} dx - \frac{b d m}{f (a^2 - b^2)} \int \frac{(c+dx)^{m-1} \cos[ex]}{a+b \sin[ex]} dx$$

Program code:

```
Int[(c_.+d_.**x_)^m_./(a_.+b_.**sin[e_.+f_.**x_])^2,x_Symbol] :=
  b*(c+d*x)^m**Cos[e+f*x]/(f*(a^2-b^2)*(a+b**Sin[e+f*x])) +
  a/(a^2-b^2)*Int[(c+d*x)^m/(a+b**Sin[e+f*x]),x] -
  b*d*m/(f*(a^2-b^2))*Int[(c+d*x)^(m-1)*Cos[e+f*x]/(a+b**Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[a^2-b^2,0] && IGtQ[m,0]
```

3: $\int (c+dx)^m (a+b \sin[e+fx])^n dx$ when $a^2 - b^2 \neq 0 \wedge n+2 \in \mathbb{Z}^- \wedge m \in \mathbb{Z}^+$

Rule: If $a^2 - b^2 \neq 0 \wedge n+2 \in \mathbb{Z}^- \wedge m \in \mathbb{Z}^+$, then

$$\begin{aligned} & \int (c+dx)^m (a+b \sin[e+fx])^n dx \rightarrow \\ & - \frac{b (c+dx)^m \cos[e+fx] (a+b \sin[e+fx])^{n+1}}{f (n+1) (a^2 - b^2)} + \frac{a}{a^2 - b^2} \int (c+dx)^m (a+b \sin[e+fx])^{n+1} dx + \\ & \frac{b d m}{f (n+1) (a^2 - b^2)} \int (c+dx)^{m-1} \cos[e+fx] (a+b \sin[e+fx])^{n+1} dx - \frac{b (n+2)}{(n+1) (a^2 - b^2)} \int (c+dx)^m \sin[e+fx] (a+b \sin[e+fx])^{n+1} dx \end{aligned}$$

Program code:

```
Int[(c_.+d_.x_)^m_.*(a_+b_.sin[e_.+f_.x_])^n_,x_Symbol] :=
  -b*(c+d*x)^m*cos[e+f*x]*(a+b*sin[e+f*x])^(n+1)/(f*(n+1)*(a^2-b^2)) +
  a/(a^2-b^2)*Int[(c+d*x)^m*(a+b*sin[e+f*x])^(n+1),x] +
  b*d*m/(f*(n+1)*(a^2-b^2))*Int[(c+d*x)^(m-1)*cos[e+f*x]*(a+b*sin[e+f*x])^(n+1),x] -
  b*(n+2)/((n+1)*(a^2-b^2))*Int[(c+d*x)^m*sin[e+f*x]*(a+b*sin[e+f*x])^(n+1),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[a^2-b^2,0] && ILtQ[n,-2] && IGtQ[m,0]
```

X: $\int (c+dx)^m (a+b \sin[ex+fx])^n dx$

Rule:

$$\int (c+dx)^m (a+b \sin[ex+fx])^n dx \rightarrow \int (c+dx)^m (a+b \sin[ex+fx])^n dx$$

Program code:

```
Int[(c_.+d_.**x_)^m_.*(a_.+b_.**sin[e_.+f_.**x_])^n_,x_Symbol] :=
  Unintegrable[(c+d*x)^m*(a+b*sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,m,n},x]
```

N: $\int u^m (a+b \sin[v])^n dx$ when $u = c+dx \wedge v = e+fx$

Derivation: Algebraic normalization

Rule: If $u = c+dx \wedge v = e+fx$, then

$$\int u^m (a+b \sin[v])^n dx \rightarrow \int (c+dx)^m (a+b \sin[ex+fx])^n dx$$

Program code:

```
Int[u_^m_.*(a_.+b_.**Sin[v_])^n_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*(a+b*sin[ExpandToSum[v,x]])^n,x] /;
FreeQ[{a,b,m,n},x] && LinearQ[{u,v},x] && Not[LinearMatchQ[{u,v},x]]
```

```
Int[u_^m_.*(a_.+b_.**Cos[v_])^n_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*(a+b*cos[ExpandToSum[v,x]])^n,x] /;
FreeQ[{a,b,m,n},x] && LinearQ[{u,v},x] && Not[LinearMatchQ[{u,v},x]]
```

