

Rules for integrands of the form $P_q[x] (a + b x^n + c x^{2n})^p$

1: $\int P_q[x] (a + b x^n + c x^{2n})^p dx$ when $p \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $p \in \mathbb{Z}^+$, then

$$\int P_q[x] (a + b x^n + c x^{2n})^p dx \rightarrow \int \text{ExpandIntegrand}[P_q[x] (a + b x^n + c x^{2n})^p, x] dx$$

Program code:

```
Int[Pq_*(a_+b_.*x_^n_+c_.*x_^n2_.)^p_,x_Symbol] :=
  Int[ExpandIntegrand[Pq*(a+b*x^n+c*x^(2*n))^p,x],x] /;
FreeQ[{a,b,c,n},x] && EqQ[n2,2*n] && PolyQ[Pq,x] && IGtQ[p,0]
```

2: $\int (d + e x^n + f x^{2n}) (a + b x^n + c x^{2n})^p dx$ when $a e - b d (n(p+1) + 1) = 0 \wedge a f - c d (2n(p+1) + 1) = 0$

Rule: If $a e - b d (n(p+1) + 1) = 0 \wedge a f - c d (2n(p+1) + 1) = 0$, then

$$\int (d + e x^n + f x^{2n}) (a + b x^n + c x^{2n})^p dx \rightarrow \frac{d x (a + b x^n + c x^{2n})^{p+1}}{a}$$

Program code:

```
Int[(d_+e_.*x_^n_+f_.*x_^n2_.)*(a_+b_.*x_^n_+c_.*x_^n2_.)^p_,x_Symbol] :=
  d*x*(a+b*x^n+c*x^(2*n))^(p+1)/a /;
FreeQ[{a,b,c,d,e,f,n,p},x] && EqQ[n2,2*n] && EqQ[a*e-b*d*(n*(p+1)+1),0] && EqQ[a*f-c*d*(2*n*(p+1)+1),0]
```

```
Int[(d_+f_.*x_^n2_.)*(a_+b_.*x_^n_+c_.*x_^n2_.)^p_,x_Symbol] :=
  d*x*(a+b*x^n+c*x^(2*n))^(p+1)/a /;
FreeQ[{a,b,c,d,f,n,p},x] && EqQ[n2,2*n] && EqQ[n*(p+1)+1,0] && EqQ[c*d+a*f,0]
```

3: $\int P_q[x] (a+bx^n+cx^{2n})^p dx$ when $b^2 - 4ac == 0 \wedge p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: If $b^2 - 4ac == 0$, then $\partial_x \frac{(a+bx^n+cx^{2n})^p}{(b+2cx^n)^{2p}} == 0$

Basis: If $b^2 - 4ac == 0$, then $\frac{(a+bx^n+cx^{2n})^p}{(b+2cx^n)^{2p}} == \frac{(a+bx^n+cx^{2n})^{\text{FracPart}[p]}}{(4c)^{\text{IntPart}[p]} (b+2cx^n)^{2\text{FracPart}[p]}}$

Rule: If $b^2 - 4ac == 0 \wedge p \notin \mathbb{Z}$, then

$$\int P_q[x] (a+bx^n+cx^{2n})^p dx \rightarrow \frac{(a+bx^n+cx^{2n})^{\text{FracPart}[p]}}{(4c)^{\text{IntPart}[p]} (b+2cx^n)^{2\text{FracPart}[p]}} \int P_q[x] (b+2cx^n)^{2p} dx$$

Program code:

```
Int[Pq*(a+b_.*x^_+c_.*x^2_.)^p_,x_Symbol] :=
  (a+b*x^n+c*x^(2*n))^FracPart[p]/((4*c)^IntPart[p]*(b+2*c*x^n)^(2*FracPart[p]))*Int[Pq*(b+2*c*x^n)^(2*p),x] /;
FreeQ[{a,b,c,n,p},x] && EqQ[n2,2*n] && PolyQ[Pq,x] && EqQ[b^2-4*a*c,0] && Not[IntegerQ[2*p]]
```

4: $\int P_q[x] (a + bx^n + cx^{2n})^p dx$ when $P_q[x, 0] = 0$

Derivation: Algebraic simplification

Rule: If $P_q[x, 0] = 0$, then

$$\int P_q[x] (a + bx^n + cx^{2n})^p dx \rightarrow \int x \text{PolynomialQuotient}[P_q[x], x, x] (a + bx^n + cx^{2n})^p dx$$

Program code:

```
Int[Pq_*(a_+b_.*x_^n_.+c_.*x_^n2_.)^p_,x_Symbol] :=
  Int[x*PolynomialQuotient[Pq,x,x]*(a+b*x^n+c*x^(2*n))^p,x] /;
FreeQ[{a,b,c,n,p},x] && EqQ[n2,2*n] && PolyQ[Pq,x] && EqQ[Coeff[Pq,x,0],0] && Not[MatchQ[Pq,x^m_.*u_. /; IntegerQ[m]]]
```

5: $\int (d+ex^n+fx^{2n}+gx^{3n}) (a+bx^n+cx^{2n})^p dx$ when

$$b^2 - 4ac \neq 0 \wedge a^2 g(n+1) - c(n(2p+3)+1)(ae-bd(n(p+1)+1)) = 0 \wedge \\ a^2 f(n+1) - acd(n+1)(2n(p+1)+1) - b(n(p+2)+1)(ae-bd(n(p+1)+1)) = 0$$

Rule: If

$$b^2 - 4ac \neq 0 \wedge a^2 g(n+1) - c(n(2p+3)+1)(ae-bd(n(p+1)+1)) = 0 \wedge \quad , \text{ then} \\ a^2 f(n+1) - acd(n+1)(2n(p+1)+1) - b(n(p+2)+1)(ae-bd(n(p+1)+1)) = 0$$

$$\int (d+ex^n+fx^{2n}+gx^{3n}) (a+bx^n+cx^{2n})^p dx \rightarrow \frac{x(ad(n+1) + (ae-bd(n(p+1)+1))x^n)(a+bx^n+cx^{2n})^{p+1}}{a^2(n+1)}$$

Program code:

```
Int[(d+_e_.*x_^n+f_.*x_^n2_+g_.*x_^n3_)*(a+_b_.*x_^n+c_.*x_^n2_)^p_,x_Symbol] :=
  x*(a*d*(n+1)+(a*e-b*d*(n*(p+1)+1))*x^n)*(a+b*x^n+c*x^(2*n))^(p+1)/(a^2*(n+1)) /;
FreeQ[{a,b,c,d,e,f,g,n,p},x] && EqQ[n2,2*n] && EqQ[n3,3*n] && NeQ[b^2-4*a*c,0] &&
EqQ[a^2*g*(n+1)-c*(n*(2*p+3)+1)*(a*e-b*d*(n*(p+1)+1)),0] &&
EqQ[a^2*f*(n+1)-a*c*d*(n+1)*(2*n*(p+1)+1)-b*(n*(p+2)+1)*(a*e-b*d*(n*(p+1)+1)),0]
```

```
Int[(d+_f_.*x_^n2_+g_.*x_^n3_)*(a+_b_.*x_^n+c_.*x_^n2_)^p_,x_Symbol] :=
  d*x*(a*(n+1)-b*(n*(p+1)+1))*x^n*(a+b*x^n+c*x^(2*n))^(p+1)/(a^2*(n+1)) /;
FreeQ[{a,b,c,d,f,g,n,p},x] && EqQ[n2,2*n] && EqQ[n3,3*n] && NeQ[b^2-4*a*c,0] &&
EqQ[a^2*g*(n+1)+c*b*d*(n*(2*p+3)+1)*(n*(p+1)+1),0] &&
EqQ[a^2*f*(n+1)-a*c*d*(n+1)*(2*n*(p+1)+1)+b^2*d*(n*(p+2)+1)*(n*(p+1)+1),0]
```

```
Int[(d+_e_.*x_^n+g_.*x_^n3_)*(a+_b_.*x_^n+c_.*x_^n2_)^p_,x_Symbol] :=
  x*(a*d*(n+1)+(a*e-b*d*(n*(p+1)+1))*x^n)*(a+b*x^n+c*x^(2*n))^(p+1)/(a^2*(n+1)) /;
FreeQ[{a,b,c,d,e,g,n,p},x] && EqQ[n2,2*n] && EqQ[n3,3*n] && NeQ[b^2-4*a*c,0] &&
EqQ[a^2*g*(n+1)-c*(n*(2*p+3)+1)*(a*e-b*d*(n*(p+1)+1)),0] &&
EqQ[a*c*d*(n+1)*(2*n*(p+1)+1)+b*(n*(p+2)+1)*(a*e-b*d*(n*(p+1)+1)),0]
```

```

Int[(d_+g_.*x_^n3_.)*(a_+b_.*x_^n_+c_.*x_^n2_.)^p_.,x_Symbol] :=
  d*x*(a*(n+1)-b*(n*(p+1)+1)*x^n)*(a+b*x^n+c*x^(2*n))^(p+1)/(a^2*(n+1)) /;
FreeQ[{a,b,c,d,g,n,p},x] && EqQ[n2,2*n] && EqQ[n3,3*n] && NeQ[b^2-4*a*c,0] &&
  EqQ[a^2*g*(n+1)+c*b*d*(n*(2*p+3)+1)*(n*(p+1)+1),0] &&
  EqQ[a*c*d*(n+1)*(2*n*(p+1)+1)-b^2*d*(n*(p+2)+1)*(n*(p+1)+1),0]

```

6. $\int P_q[x] (a+bx^n+cx^{2n})^p dx$ when $b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}$

1. $\int P_q[x] (a+bx^n+cx^{2n})^p dx$ when $b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+$

1. $\int P_q[x] (a+bx^n+cx^{2n})^p dx$ when $b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1$

1: $\int P_q[x] (a+bx^n+cx^{2n})^p dx$ when $b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge q < 2n$

Derivation: Trinomial recurrence 2b applied $n-1$ times

Rule: If $b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge q < 2n$, then

$$\begin{aligned} & \int P_q[x] (a+bx^n+cx^{2n})^p dx \rightarrow \\ & - \frac{1}{an(p+1)(b^2-4ac)} x (a+bx^n+cx^{2n})^{p+1} \sum_{i=0}^{n-1} (((b^2-2ac)P_q[x,i] - abP_q[x,n+i])x^i + c(bP_q[x,i] - 2aP_q[x,n+i])x^{n+i}) + \\ & \frac{1}{an(p+1)(b^2-4ac)} \int (a+bx^n+cx^{2n})^{p+1} dx \\ & \sum_{i=0}^{n-1} (((b^2(n(p+1)+i+1) - 2ac(2n(p+1)+i+1))P_q[x,i] - ab(i+1)P_q[x,n+i])x^i + c(n(2p+3)+i+1)(bP_q[x,i] - 2aP_q[x,n+i])x^{n+i}) dx \end{aligned}$$

Program code:

```
Int[Pq_*(a_+b_.*x_^n+c_.*x_^n2_)^p_,x_Symbol] :=
Module[{q=Expon[Pq,x],i},
-x*(a+b*x^n+c*x^(2*n))^ (p+1)/(a*n*(p+1)*(b^2-4*a*c))*
Sum[( (b^2-2*a*c)*Coeff[Pq,x,i]-a*b*Coeff[Pq,x,n+i])*x^i+
c*(b*Coeff[Pq,x,i]-2*a*Coeff[Pq,x,n+i])*x^(n+i),{i,0,n-1}] +
1/(a*n*(p+1)*(b^2-4*a*c))*Int[(a+b*x^n+c*x^(2*n))^ (p+1)*
Sum[( (b^2*(n*(p+1)+i+1)-2*a*c*(2*n*(p+1)+i+1))*Coeff[Pq,x,i]-a*b*(i+1)*Coeff[Pq,x,n+i])*x^i+
c*(n*(2*p+3)+i+1)*(b*Coeff[Pq,x,i]-2*a*Coeff[Pq,x,n+i])*x^(n+i),{i,0,n-1}],x] /;
LtQ[q,2*n] /;
FreeQ[{a,b,c},x] && EqQ[n2,2*n] && PolyQ[Pq,x] && NeQ[b^2-4*a*c,0] && IGtQ[n,0] && LtQ[p,-1]
```

$$\mathbf{2:} \int P_q[x] (a + b x^n + c x^{2n})^p dx \text{ when } b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge q \geq 2n$$

Derivation: Algebraic expansion and trinomial recurrence 2b applied $n-1$ times

Rule: If $b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge p < -1 \wedge q \geq 2n$, let $Q_{q-2n}[x] = \text{PolynomialQuotient}[P_q[x], a + b x^n + c x^{2n}, x]$ and $R_{2n-1}[x] = \text{PolynomialRemainder}[P_q[x], a + b x^n + c x^{2n}, x]$, then

$$\begin{aligned} & \int P_q[x] (a + b x^n + c x^{2n})^p dx \rightarrow \\ & \int R_{2n-1}[x] (a + b x^n + c x^{2n})^p dx + \int Q_{q-2n}[x] (a + b x^n + c x^{2n})^{p+1} dx \rightarrow \\ & - \left(\left(x (a + b x^n + c x^{2n})^{p+1} \sum_{i=0}^{n-1} \left((b^2 - 2ac) R_{2n-1}[x, i] - ab R_{2n-1}[x, n+i] \right) x^i + c (b R_{2n-1}[x, i] - 2a R_{2n-1}[x, n+i]) x^{n+i} \right) \right) / \\ & \quad \left(a n (p+1) (b^2 - 4ac) \right) + \\ & \quad \frac{1}{a n (p+1) (b^2 - 4ac)} \int (a + b x^n + c x^{2n})^{p+1} \left(a n (p+1) (b^2 - 4ac) Q_{q-2n}[x] + \right. \\ & \quad \sum_{i=0}^{n-1} \left((b^2 (n(p+1) + i + 1) - 2ac (2n(p+1) + i + 1)) R_{2n-1}[x, i] - ab (i+1) R_{2n-1}[x, n+i] \right) x^i + \\ & \quad \left. c (n(2p+3) + i + 1) (b R_{2n-1}[x, i] - 2a R_{2n-1}[x, n+i]) x^{n+i} \right) dx \end{aligned}$$

Program code:

```

Int[Pq_*(a_+b_.*x_^n_+c_.*x_^n2_)^p_,x_Symbol] :=
  With[{q=Expon[Pq,x]},
    Module[{Q=PolynomialQuotient[(b*c)^(Floor[(q-1)/n]+1)*Pq,a+b*x^n+c*x^(2*n),x],
      R=PolynomialRemainder[(b*c)^(Floor[(q-1)/n]+1)*Pq,a+b*x^n+c*x^(2*n),x],i},
      -x*(a+b*x^n+c*x^(2*n))^(p+1)/(a*n*(p+1)*(b^2-4*a*c)*(b*c)^(Floor[(q-1)/n]+1))*
      Sum[(b^2-2*a*c)*Coeff[R,x,i]-a*b*Coeff[R,x,n+i])*x^i+
      c*(b*Coeff[R,x,i]-2*a*Coeff[R,x,n+i])*x^(n+i),{i,0,n-1}] +
      1/(a*n*(p+1)*(b^2-4*a*c)*(b*c)^(Floor[(q-1)/n]+1))*Int[(a+b*x^n+c*x^(2*n))^(p+1)*ExpandToSum[a*n*(p+1)*(b^2-4*a*c)*Q+
      Sum[(b^2*(n*(p+1)+i+1)-2*a*c*(2*n*(p+1)+i+1))*Coeff[R,x,i]-a*b*(i+1)*Coeff[R,x,n+i])*x^i+
      c*(n*(2*p+3)+i+1)*(b*Coeff[R,x,i]-2*a*Coeff[R,x,n+i])*x^(n+i),{i,0,n-1}],x,x]] /;
    GeQ[q,2*n]] /;
    FreeQ[{a,b,c},x] && EqQ[n2,2*n] && PolyQ[Pq,x] && NeQ[b^2-4*a*c,0] && IGtQ[n,0] && LtQ[p,-1]
  ]

```

$$2. \int P_q[x^n] (a + b x^n + c x^{2n})^p dx \text{ when } b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+$$

$$1: \int \frac{P_q[x^n]}{a + b x^n + c x^{2n}} dx \text{ when } b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge \text{NiceSqrtQ}[b^2 - 4ac]$$

Derivation: Algebraic expansion

Rule: If $b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge \text{NiceSqrtQ}[b^2 - 4ac]$, then

$$\int \frac{P_q[x^n]}{a + b x^n + c x^{2n}} dx \rightarrow \int \text{ExpandIntegrand}\left[\frac{P_q[x^n]}{a + b x^n + c x^{2n}}, x\right] dx$$

Program code:

```

Int[Pq_/(a_+b_.*x_^n_+c_.*x_^n2_),x_Symbol] :=
  Int[ExpandIntegrand[Pq/(a+b*x^n+c*x^(2*n)),x],x] /;
  FreeQ[{a,b,c},x] && EqQ[n2,2*n] && PolyQ[Pq,x^n] && NeQ[b^2-4*a*c,0] && IGtQ[n,0] && (NiceSqrtQ[b^2-4*a*c] || LtQ[Expon[Pq,x],n])

```


$$2. \int P_q[x] (a+bx+cx^2)^p dx \text{ when } b^2 - 4ac \neq 0 \wedge 2p \in \mathbb{Z}^- \wedge q+2p+1 = 0$$

$$1: \int P_q[x] (a+bx+cx^2)^p dx \text{ when } b^2 - 4ac \neq 0 \wedge p \in \mathbb{Z}^- \wedge q+2p+1 = 0$$

Note: This rule reduces the degree of the polynomial in the resulting integrand.

Rule: If $b^2 - 4ac \neq 0 \wedge p \in \mathbb{Z}^- \wedge q+2p+1 = 0$, then

$$\int P_q[x] (a+bx+cx^2)^p dx \rightarrow \frac{c^p P_q[x, q] \text{Log}[a+bx+cx^2]}{2} + \frac{1}{2} \int \left(2 P_q[x] - \frac{c^p P_q[x, q] (b+2cx)}{(a+bx+cx^2)^{p+1}} \right) (a+bx+cx^2)^p dx$$

Program code:

```
Int[Pq_*(a+_.*x+_.*x^2)^p_,x_Symbol] :=
  With[{q=Expon[Pq,x]},
    With[{Pqq=Coeff[Pq,x,q]},
      c^p*Pqq*Log[a+b*x+c*x^2]/2 +
      1/2*Int[ExpandToSum[2*Pq-c^p*Pqq*(b+2*c*x)/(a+b*x+c*x^2)^(p+1),x]*(a+b*x+c*x^2)^p,x]] /;
    EqQ[q+2*p+1,0]] /;
  FreeQ[{a,b,c},x] && PolyQ[Pq,x] && NeQ[b^2-4*a*c,0] && ILtQ[p,0]
```

$$2. \int P_q[x] (a+bx+cx^2)^p dx \text{ when } b^2 - 4ac \neq 0 \wedge 2p \in \mathbb{Z}^- \wedge q+2p+1 = 0$$

$$1: \int P_q[x] (a+bx+cx^2)^p dx \text{ when } b^2 - 4ac \neq 0 \wedge p + \frac{1}{2} \in \mathbb{Z}^- \wedge q+2p+1 = 0 \wedge c > 0$$

Note: This rule reduces the degree of the polynomial in the resulting integrand.

Rule: If $b^2 - 4ac \neq 0 \wedge p + \frac{1}{2} \in \mathbb{Z}^- \wedge q+2p+1 = 0 \wedge c > 0$, then

$$\int Pq[x] (a+bx+cx^2)^p dx \rightarrow$$

$$c^p Pq[x, q] \operatorname{ArcTanh}\left[\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right] + \int \left(Pq[x] - \frac{c^{p+\frac{1}{2}} Pq[x, q]}{(a+bx+cx^2)^{p+\frac{1}{2}}} \right) (a+bx+cx^2)^p dx$$

Program code:

```
Int[Pq_*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  With[{q=Expon[Pq,x]},
    With[{Pqq=Coeff[Pq,x,q]},
      c^p*Pqq*ArcTanh[(b+2*c*x)/(2*Rt[c,2]*Sqrt[a+b*x+c*x^2])] +
      Int[ExpandToSum[Pq-c^(p+1/2)*Pqq/(a+b*x+c*x^2)^(p+1/2),x]*(a+b*x+c*x^2)^p,x] /;
      EqQ[q+2*p+1,0] ] /;
  FreeQ[{a,b,c},x] && PolyQ[Pq,x] && NeQ[b^2-4*a*c,0] && ILtQ[p+1/2,0] && PosQ[c]
```

$$\mathbf{2:} \int P_q[x] (a+bx+cx^2)^p dx \text{ when } b^2 - 4ac \neq 0 \wedge p + \frac{1}{2} \in \mathbb{Z}^- \wedge q+2p+1 = 0 \wedge c \neq 0$$

Note: This rule reduces the degree of the polynomial in the resulting integrand.

Rule: If $b^2 - 4ac \neq 0 \wedge p + \frac{1}{2} \in \mathbb{Z}^- \wedge q+2p+1 = 0 \wedge c \neq 0$, then

$$\int P_q[x] (a+bx+cx^2)^p dx \rightarrow -(-c)^p P_q[x, q] \operatorname{ArcTan}\left[\frac{b+2cx}{2\sqrt{-c}\sqrt{a+bx+cx^2}}\right] + \int \left(P_q[x] - \frac{(-c)^{p+\frac{1}{2}} P_q[x, q]}{(a+bx+cx^2)^{p+\frac{1}{2}}} \right) (a+bx+cx^2)^p dx$$

Program code:

```
Int[Pq_*(a_+b_.*x_+c_.*x_^2)^p_,x_Symbol] :=
  With[{q=Expon[Pq,x]},
    With[{Pqq=Coeff[Pq,x,q]},
      -(-c)^p*Pqq*ArcTan[(b+2*c*x)/(2*Rt[-c,2]*Sqrt[a+b*x+c*x^2])] +
      Int[ExpandToSum[Pq-(-c)^(p+1/2)*Pqq/(a+b*x+c*x^2)^(p+1/2),x]*(a+b*x+c*x^2)^p,x]] /;
    EqQ[q+2*p+1,0]] /;
  FreeQ[{a,b,c},x] && PolyQ[Pq,x] && NeQ[b^2-4*a*c,0] && ILtQ[p+1/2,0] && NegQ[c]
```

$$\mathbf{3:} \int P_q[x^n] (a+bx^n+cx^{2n})^p dx \text{ when } b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge q \geq 2n \wedge q+2np+1 \neq 0$$

Reference: G&R 2.160.3

Derivation: Trinomial recurrence 3a with $A = 0$, $B = 1$ and $m = m - n$

Reference: G&R 2.104

Note: This special case of the Ostrogradskiy-Hermite integration method reduces the degree of the polynomial in the resulting integrand.

Rule: If $b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge q \geq 2n \wedge q+2np+1 \neq 0$, then

$$\begin{aligned} & \int P_q[x^n] (a+bx^n+cx^{2n})^p dx \rightarrow \\ & \int (P_q[x^n] - P_q[x, q] x^q) (a+bx^n+cx^{2n})^p dx + P_q[x, q] \int x^q (a+bx^n+cx^{2n})^p dx \rightarrow \\ & \frac{P_q[x, q] x^{q-2n+1} (a+bx^n+cx^{2n})^{p+1}}{c(q+2np+1)} + \\ & \int \left(P_q[x^n] - P_q[x, q] x^q - \frac{P_q[x, q] (a(q-2n+1)x^{q-2n} + b(q+n(p-1)+1)x^{q-n})}{c(q+2np+1)} \right) (a+bx^n+cx^{2n})^p dx \end{aligned}$$

Program code:

```
Int[Pq_*(a_+b_.*x_^n_+c_.*x_^n2_)^p_,x_Symbol] :=
  With[{q=Expon[Pq,x]},
    With[{Pqq=Coeff[Pq,x,q]},
      Pqq*x^(q-2*n+1)*(a+b*x^n+c*x^(2*n))^(p+1)/(c*(q+2*n*p+1)) +
      Int[ExpandToSum[Pq-Pqq*x^q-Pqq*(a*(q-2*n+1)*x^(q-2*n)+b*(q+n*(p-1)+1)*x^(q-n)]/(c*(q+2*n*p+1)),x]*(a+b*x^n+c*x^(2*n))^p,x]]
    GeQ[q,2*n] && NeQ[q+2*n*p+1,0] && (IntegerQ[2*p] || EqQ[n,1] && IntegerQ[4*p] || IntegerQ[p+(q+1)/(2*n)])] /;
    FreeQ[{a,b,c,p},x] && EqQ[n2,2*n] && PolyQ[Pq,x^n] && NeQ[b^2-4*a*c,0] && IGtQ[n,0]
```

$$\mathbf{3:} \int P_q[x] (a + b x^n + c x^{2n})^p dx \text{ when } b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge \neg \text{PolynomialQ}[P_q[x], x^n]$$

Derivation: Algebraic expansion

Basis: If $n \in \mathbb{Z}^+$, then $P_q[x] = \sum_{j=0}^{q-1} x^j \sum_{k=0}^{(q-j)/n+1} P_q[x, j+kn] x^{kn}$

Note: This rule transform integrand into a sum of terms of the form $(dx)^k Q_r[x^n] (a + b x^n + c x^{2n})^p$.

Rule: If $b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+ \wedge \neg \text{PolynomialQ}[P_q[x], x^n]$, then

$$\int P_q[x] (a + b x^n + c x^{2n})^p dx \rightarrow \int \sum_{j=0}^{q-1} x^j \left(\sum_{k=0}^{(q-j)/n+1} P_q[x, j+kn] x^{kn} \right) (a + b x^n + c x^{2n})^p dx$$

Program code:

```
Int[Pq_*(a_+b_.*x_^n_+c_.*x_^n2_)^p_,x_Symbol] :=
Module[{q=Expon[Pq,x],j,k},
Int[Sum[x^j*Sum[Coeff[Pq,x,j+k*n]*x^(k*n),{k,0,(q-j)/n+1}]*
(a+b*x^n+c*x^(2*n))^p,{j,0,n-1}],x]] /;
FreeQ[{a,b,c,p},x] && EqQ[n2,2*n] && PolyQ[Pq,x] && NeQ[b^2-4*a*c,0] && IGtQ[n,0] && Not[PolyQ[Pq,x^n]]
```

4: $\int \frac{P_q[x]}{a + b x^n + c x^{2n}} dx$ when $b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $b^2 - 4ac \neq 0 \wedge n \in \mathbb{Z}^+$, then

$$\int \frac{P_q[x]}{a + b x^n + c x^{2n}} dx \rightarrow \int \text{RationalFunctionExpand}\left[\frac{P_q[x]}{a + b x^n + c x^{2n}}, x\right] dx$$

Program code:

```
Int[Pq/(a+b.*x_^n_.+c_.*x_^n2_.),x_Symbol] :=
  Int[RationalFunctionExpand[Pq/(a+b*x^n+c*x^(2*n)),x],x] /;
FreeQ[{a,b,c},x] && EqQ[n2,2*n] && PolyQ[Pq,x] && NeQ[b^2-4*a*c,0] && IGtQ[n,0]
```

7: $\int P_q[x] (a+bx^n+cx^{2n})^p dx$ when $b^2-4ac \neq 0 \wedge n \in \mathbb{F}$

Derivation: Integration by substitution

Basis: If $g \in \mathbb{Z}^+$, then $P_q[x] F[x^n] = g \text{Subst}[x^{g-1} P_q[x^g] F[x^{g n}], x, x^{1/g}] \partial_x x^{1/g}$

Rule: If $b^2-4ac \neq 0 \wedge n \in \mathbb{F}$, let $g = \text{Denominator}[n]$, then

$$\int P_q[x] (a+bx^n+cx^{2n})^p dx \rightarrow g \text{Subst}\left[\int x^{g-1} P_q[x^g] (a+bx^{g n}+cx^{2g n})^p dx, x, x^{1/g}\right]$$

Program code:

```
Int[Pq_*(a_+b_.*x_^n_+c_.*x_^n2_.)^p_,x_Symbol] :=
  With[{g=Denominator[n]},
    g*Subst[Int[x^(g-1)*ReplaceAll[Pq,x->x^g]*(a+b*x^(g*n)+c*x^(2*g*n))^p,x],x,x^(1/g)]] /;
  FreeQ[{a,b,c,p},x] && EqQ[n2,2*n] && PolyQ[Pq,x] && NeQ[b^2-4*a*c,0] && FractionQ[n]
```

8. $\int P_q[x] (a+bx^n+cx^{2n})^p dx$ when $b^2 - 4ac \neq 0 \wedge p \in \mathbb{Z}^-$

1: $\int \frac{P_q[x]}{a+bx^n+cx^{2n}} dx$ when $b^2 - 4ac \neq 0$

Reference: G&R 2.161.1a

Derivation: Algebraic expansion

Basis: Let $q = \sqrt{b^2 - 4ac}$, then $\frac{1}{a+bz+cz^2} = \frac{2c}{q} \frac{1}{b-q+2cz} - \frac{2c}{q} \frac{1}{b+q+2cz}$

■ Rule: If $b^2 - 4ac \neq 0$, let $q = \sqrt{b^2 - 4ac}$, then

$$\int \frac{P_q[x]}{a+bx^n+cx^{2n}} dx \rightarrow \frac{2c}{q} \int \frac{P_q[x]}{b-q+2cx^n} dx - \frac{2c}{q} \int \frac{P_q[x]}{b+q+2cx^n} dx$$

Program code:

```
Int[Pq/(a+b_.*x_^n_.+c_.*x_^2n_.),x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    2*c/q*Int[Pq/(b-q+2*c*x^n),x] -
    2*c/q*Int[Pq/(b+q+2*c*x^n),x] /;
    FreeQ[{a,b,c,n},x] && EqQ[n2,2*n] && PolyQ[Pq,x] && NeQ[b^2-4*a*c,0]
```

?: $\int (A+Bx^n+Cx^{2n}+Dx^{3n}) (a+bx^n+cx^{2n})^p dx$ when $b^2 - 4ac \neq 0 \wedge p+1 \in \mathbb{Z}^-$

Derivation: Two steps of OS and trinomial recurrence 2b

Note: This rule should be generalized for integrands of the form $P_q[x^n] (a+bx^n+cx^{2n})^p$ when n is symbolic.

Rule 1.3.3.17: If $b^2 - 4ac \neq 0 \wedge p+1 \in \mathbb{Z}^-$, then

$$\int (d + ex^n + fx^{2n} + gx^{3n}) (a + bx^n + cx^{2n})^p dx \rightarrow$$

$$- \left(x (b^2cd - 2ac(cd - af) - ab(ce + ag) + (bc(cd + af) - ab^2g - 2ac(ce - ag)) x^n) (a + bx^n + cx^{2n})^{p+1} \right) / (acn(p+1)(b^2 - 4ac)) -$$

$$\frac{1}{acn(p+1)(b^2 - 4ac)} \int (a + bx^n + cx^{2n})^{p+1} (ab(ce + ag) - b^2cd(n + np + 1) - 2ac(af - cd(2n(p+1) + 1)) +$$

$$(ab^2g(n(p+2) + 1) - bc(cd + af)(n(3 + 2p) + 1) - 2ac(ag(n+1) - ce(n(2p+3) + 1))) x^n) dx$$

Program code:

```
Int[P3_*(a_+b_.*x_^n_+c_.*x_^n2_)^p_,x_Symbol] :=
  With[{d=Coeff[P3,x^n,0],e=Coeff[P3,x^n,1],f=Coeff[P3,x^n,2],g=Coeff[P3,x^n,3]},
    -x*(b^2*c*d-2*a*c*(c*d-a*f)-a*b*(c*e+a*g)+(b*c*(c*d+a*f)-a*b^2*g-2*a*c*(c*e-a*g))*x^n)*(a+b*x^n+c*x^(2*n))^(p+1)/
    (a*c*n*(p+1)*(b^2-4*a*c)) -
    1/(a*c*n*(p+1)*(b^2-4*a*c))*Int[(a+b*x^n+c*x^(2*n))^(p+1)*
      Simp[a*b*(c*e+a*g)-b^2*c*d*(n+n*p+1)-2*a*c*(a*f-c*d*(2*n*(p+1)+1))+
        (a*b^2*g*(n*(p+2)+1)-b*c*(c*d+a*f)*(n*(2*p+3)+1)-2*a*c*(a*g*(n+1)-c*e*(n*(2*p+3)+1)))*x^n,x],x] /;
  FreeQ[{a,b,c,n},x] && EqQ[n2,2*n] && PolyQ[P3,x^n,3] && NeQ[b^2-4*a*c,0] && ILtQ[p,-1]
```

```
Int[P2_*(a_+b_.*x_^n_+c_.*x_^n2_)^p_,x_Symbol] :=
  With[{d=Coeff[P2,x^n,0],e=Coeff[P2,x^n,1],f=Coeff[P2,x^n,2]},
    -x*(b^2*d-2*a*(c*d-a*f)-a*b*e+(b*(c*d+a*f)-2*a*c*e)*x^n)*(a+b*x^n+c*x^(2*n))^(p+1)/(a*n*(p+1)*(b^2-4*a*c)) -
    1/(a*n*(p+1)*(b^2-4*a*c))*Int[(a+b*x^n+c*x^(2*n))^(p+1)*
      Simp[a*b*e-b^2*d*(n+n*p+1)-2*a*(a*f-c*d*(2*n*(p+1)+1))-(b*(c*d+a*f)*(n*(2*p+3)+1)-2*a*c*e*(n*(2*p+3)+1))*x^n,x],x] /;
  FreeQ[{a,b,c,n},x] && EqQ[n2,2*n] && PolyQ[P2,x^n,2] && NeQ[b^2-4*a*c,0] && ILtQ[p,-1]
```

2: $\int P_q[x] (a+bx^n+cx^{2n})^p dx$ when $p+1 \in \mathbb{Z}^-$

Derivation: Algebraic expansion

Rule: If $p+1 \in \mathbb{Z}^-$, then

$$\int P_q[x] (a+bx^n+cx^{2n})^p dx \rightarrow \int \text{ExpandIntegrand}[P_q[x] (a+bx^n+cx^{2n})^p, x] dx$$

Program code:

```
Int[Pq*(a+b_.**x_^n_.+c_.**x_^n2_.)^p_,x_Symbol] :=
  Int[ExpandIntegrand[Pq*(a+b*x^n+c*x^(2*n))^p,x],x] /;
FreeQ[{a,b,c,n},x] && EqQ[n2,2*n] && PolyQ[Pq,x] && ILtQ[p,-1]
```

X: $\int P_q[x] (a+bx^n+cx^{2n})^p dx$

Rule:

$$\int P_q[x] (a+bx^n+cx^{2n})^p dx \rightarrow \int P_q[x] (a+bx^n+cx^{2n})^p dx$$

Program code:

```
Int[Pq*(a+b_.**x_^n_.+c_.**x_^n2_.)^p_,x_Symbol] :=
  Unintegrable[Pq*(a+b*x^n+c*x^(2*n))^p,x] /;
FreeQ[{a,b,c,n,p},x] && EqQ[n2,2*n] && (PolyQ[Pq,x] || PolyQ[Pq,x^n])
```

S: $\int P_q[v^n] (a + b v^n + c v^{2n})^p dx$ when $v = f + g x$

Derivation: Integration by substitution

Rule: If $v = f + g x$, then

$$\int P_q[v^n] (a + b v^n + c v^{2n})^p dx \rightarrow \frac{1}{g} \text{Subst} \left[\int P_q[x^n] (a + b x^n + c x^{2n})^p dx, x, v \right]$$

Program code:

```
Int[Pq_*(a_+b_.*v_^n_+c_.*v_^n2_.)^p_.,x_Symbol] :=
  1/Coefficient[v,x,1]*Subst[Int[SubstFor[v,Pq,x]*(a+b*x^n+c*x^(2*n))^p,x],x,v] /;
FreeQ[{a,b,c,n,p},x] && EqQ[n2,2*n] && LinearQ[v,x] && PolyQ[Pq,v^n]
```