

## Rules for integrands of the form $P[x]^p Q[x]^q$

0.  $\int \frac{\sqrt{a + b x^2 + c x^4}}{d + e x^4} dx$  when  $c d + a e == 0$

1:  $\int \frac{\sqrt{a + b x^2 + c x^4}}{d + e x^4} dx$  when  $c d + a e == 0 \wedge a c > 0$

Derivation: Integration by substitution

Basis: If  $c d + a e == 0$ , then  $\frac{\sqrt{a + b x^2 + c x^4}}{d + e x^4} == \frac{a}{d} \text{Subst}\left[\frac{1}{1 - 2 b x^2 + (b^2 - 4 a c) x^4}, x, \frac{x}{\sqrt{a + b x^2 + c x^4}}\right] \partial_x \frac{x}{\sqrt{a + b x^2 + c x^4}}$

Rule 1.3.3.4.4.1: If  $c d + a e == 0 \wedge a c > 0$ , then

$$\int \frac{\sqrt{a + b x^2 + c x^4}}{d + e x^4} dx \rightarrow \frac{a}{d} \text{Subst}\left[\int \frac{1}{1 - 2 b x^2 + (b^2 - 4 a c) x^4} dx, x, \frac{x}{\sqrt{a + b x^2 + c x^4}}\right]$$

Program code:

```
Int[Sqrt[v_]/(d_+e_.*x_^4),x_Symbol] :=
  With[{a=Coeff[v,x,0],b=Coeff[v,x,2],c=Coeff[v,x,4]},
    a/d*Subst[Int[1/(1-2*b*x^2+(b^2-4*a*c)*x^4),x],x,x/Sqrt[v]] /;
    EqQ[c*d+a*e,0] && PosQ[a*c] /;
    FreeQ[{d,e},x] && PolyQ[v,x^2,2]
```

2:  $\int \frac{\sqrt{a+bx^2+cx^4}}{d+ex^4} dx$  when  $cd+ae=0 \wedge ac \neq 0$

Rule 1.3.3.4.4.2: If  $cd+ae=0 \wedge ac \neq 0$ , let  $q \rightarrow \sqrt{b^2-4ac}$ , then

$$\int \frac{\sqrt{a+bx^2+cx^4}}{d+ex^4} dx \rightarrow$$

$$-\frac{a\sqrt{b+q}}{2\sqrt{2}\sqrt{-ac}d} \operatorname{ArcTan}\left[\frac{\sqrt{b+q}x(b-q+2cx^2)}{2\sqrt{2}\sqrt{-ac}\sqrt{a+bx^2+cx^4}}\right] + \frac{a\sqrt{-b+q}}{2\sqrt{2}\sqrt{-ac}d} \operatorname{ArcTanh}\left[\frac{\sqrt{-b+q}x(b+q+2cx^2)}{2\sqrt{2}\sqrt{-ac}\sqrt{a+bx^2+cx^4}}\right]$$

Program code:

```
Int[Sqrt[a_+b_.*x_^2+c_.*x_^4]/(d_+e_.*x_^4),x_Symbol] :=
  With[{q=Sqrt[b^2-4*a*c]},
    -a*Sqrt[b+q]/(2*Sqrt[2]*Rt[-a*c,2]*d)*ArcTan[Sqrt[b+q]**(b-q+2*c*x^2)/(2*Sqrt[2]*Rt[-a*c,2]*Sqrt[a+b*x^2+c*x^4])] +
    a*Sqrt[-b+q]/(2*Sqrt[2]*Rt[-a*c,2]*d)*ArcTanh[Sqrt[-b+q]**(b+q+2*c*x^2)/(2*Sqrt[2]*Rt[-a*c,2]*Sqrt[a+b*x^2+c*x^4])]] /;
  FreeQ[{a,b,c,d,e},x] && EqQ[c*d+a*e,0] && NegQ[a*c]
```

1.  $\int P[x]^p Q[x]^q dx$  when  $P[x] = P1[x] P2[x] \dots$

**1:**  $\int P[x^2]^p Q[x]^q dx$  when  $p \in \mathbb{Z}^- \wedge P[x] = P1[x] P2[x] \dots$

Derivation: Algebraic simplification

Note: This rule assumes host CAS distributes integer powers over products.

Rule: If  $p \in \mathbb{Z}^- \wedge P[x] = P1[x] P2[x] \dots$ , then

$$\int P[x^2]^p Q[x]^q dx \rightarrow \int P1[x^2]^p P2[x^2]^p \dots Q[x]^q dx$$

Program code:

```
Int[P_^p_*Q^q_,x_Symbol] :=
  With[{PP=Factor[ReplaceAll[P,x→Sqrt[x]]]},
    Int[ExpandIntegrand[ReplaceAll[PP,x→x^2]^p*Q^q,x],x] /;
    Not[SumQ[NonfreeFactors[PP,x]]] /;
    FreeQ[q,x] && PolyQ[P,x^2] && PolyQ[Q,x] && ILtQ[p,0]
```

**2:**  $\int P[x]^p Q[x]^q dx$  when  $p \in \mathbb{Z} \wedge P[x] = P_1[x] P_2[x] \dots$

Derivation: Algebraic expansion

Note: This rule assumes host CAS distributes integer powers over products.

Rule: If  $p \in \mathbb{Z} \wedge P[x] = P_1[x] P_2[x] \dots$ , then

$$\int P[x]^p Q[x]^q dx \rightarrow \int P_1[x]^p P_2[x]^p \dots Q[x]^q dx$$

Program code:

```
Int[P_^p_*Q_^q_,x_Symbol] :=
  With[{PP=Factor[P]},
    Int[ExpandIntegrand[PP^p*Q^q,x],x] /;
    Not[SumQ[NonfreeFactors[PP,x]]] /;
    FreeQ[q,x] && PolyQ[P,x] && PolyQ[Q,x] && IntegerQ[p] && NeQ[P,x]
```

**2:**  $\int P[x]^p Q[x] dx$  when  $p \in \mathbb{Z}^- \wedge P[x] = (a + bx + cx^2) (d + ex + fx^2) \dots$

Derivation: Algebraic expansion

Rule: If  $p \in \mathbb{Z}^- \wedge P[x] = (a + bx + cx^2) (d + ex + fx^2) \dots$ , then

$$\int P[x]^p Q[x] dx \rightarrow \int \text{ExpandIntegrand}[P[x]^p Q[x], x] dx$$

Program code:

```
Int[P_^p_*Qm_,x_Symbol] :=
  With[{PP=Factor[P]},
    Int[ExpandIntegrand[PP^p*Qm_,x],x] /;
    QuadraticProductQ[PP,x] /;
    PolyQ[Qm,x] && PolyQ[P,x] && ILtQ[p,0]
```

$$3. \int (e + f x)^m (a + b x + c x^2 + d x^3)^p dx$$

$$1. \int (e + f x)^m (a + b x + d x^3)^p dx$$

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Derivation: Algebraic expansion

Basis: If  $4b^3 + 27a^2d = 0$ , then  $a + bx + dx^3 = \frac{1}{3^3 a^2} (3a - bx)(3a + 2bx)^2$

Rule: If  $4b^3 + 27a^2d = 0 \wedge p \in \mathbb{Z}$ , then

$$\int (e + f x)^m (a + b x + d x^3)^p dx \rightarrow \frac{1}{3^{3p} a^{2p}} \int (e + f x)^m (3a - bx)^p (3a + 2bx)^{2p} dx$$

Program code:

```
Int[(e_.+f_.**x_)^m_.*(a_+b_.**x_+d_.**x_^3)^p_.,x_Symbol] :=
  1/(3^(3*p)*a^(2*p))*Int[(e+f**x)^m*(3*a-b*x)^p*(3*a+2*b*x)^(2*p),x] /;
FreeQ[{a,b,d,e,f,m},x] && EqQ[4*b^3+27*a^2*d,0] && IntegerQ[p]
```

$$2: \int (e + f x)^m (a + b x + d x^3)^p dx \text{ when } 4b^3 + 27a^2d = 0 \wedge p \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction

Basis: If  $4b^3 + 27a^2d = 0$ , then  $\partial_x \frac{(a+bx+dx^3)^p}{(3a-bx)^p (3a+2bx)^{2p}} = 0$

Rule: If  $4b^3 + 27a^2d = 0 \wedge p \notin \mathbb{Z}$ , then

$$\int (e+fx)^m (a+bx+dx^3)^p dx \rightarrow \frac{(a+bx+dx^3)^p}{(3a-bx)^p (3a+2bx)^{2p}} \int (e+fx)^m (3a-bx)^p (3a+2bx)^{2p} dx$$

Program code:

```
Int[(e_.+f_.**x_)^m_.*(a_.+b_.**x_+d_.**x_^3)^p_,x_Symbol] :=
  (a+b*x+d*x^3)^p/((3*a-b*x)^p*(3*a+2*b*x)^(2*p))*Int[(e+f*x)^m*(3*a-b*x)^p*(3*a+2*b*x)^(2*p),x] /;
FreeQ[{a,b,d,e,f,m,p},x] && EqQ[4*b^3+27*a^2*d,0] && Not[IntegerQ[p]]
```

2.  $\int (e+fx)^m (a+bx+dx^3)^p dx$  when  $4b^3 + 27a^2d \neq 0$

1.  $\int (e+fx)^m (a+bx+dx^3)^p dx$  when  $4b^3 + 27a^2d \neq 0 \wedge p \in \mathbb{Z}$

**1:**  $\int (e+fx)^m (a+bx+dx^3)^p dx$  when  $4b^3 + 27a^2d \neq 0 \wedge p \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If  $4b^3 + 27a^2d \neq 0 \wedge p \in \mathbb{Z}^+$ ,

$$\int (e+fx)^m (a+bx+dx^3)^p dx \rightarrow \int \text{ExpandIntegrand}[(e+fx)^m (a+bx+dx^3)^p, x] dx$$

Program code:

```
Int[(e_.+f_.**x_)^m_.*(a_.+b_.**x_+d_.**x_^3)^p_,x_Symbol] :=
  Int[ExpandIntegrand[(e+f*x)^m*(a+b*x+d*x^3)^p,x],x] /;
FreeQ[{a,b,d,e,f,m},x] && NeQ[4*b^3+27*a^2*d,0] && IGtQ[p,0]
```

$$\mathbf{2:} \int (e + f x)^m (a + b x + d x^3)^p dx \text{ when } 4 b^3 + 27 a^2 d \neq 0 \wedge p \in \mathbb{Z}^-$$

Derivation: Algebraic expansion

Basis: If  $r \rightarrow (-9 a d^2 + \sqrt{3} d \sqrt{4 b^3 d + 27 a^2 d^2})^{1/3}$ , then  $a + b x + d x^3 = \frac{2 b^3 d}{3 r^3} - \frac{r^3}{18 d^2} + b x + d x^3$

Basis:

$$\frac{2 b^3 d}{3 r^3} - \frac{r^3}{18 d^2} + b x + d x^3 = \frac{1}{d^2} \left( \frac{18^{1/3} b d}{3 r} - \frac{r}{18^{1/3}} + d x \right) \left( \frac{b d}{3} + \frac{12^{1/3} b^2 d^2}{3 r^2} + \frac{r^2}{3 \times 12^{1/3}} - d \left( \frac{2^{1/3} b d}{3^{1/3} r} - \frac{r}{18^{1/3}} \right) x + d^2 x^2 \right)$$

Rule: If  $4 b^3 + 27 a^2 d \neq 0 \wedge p \in \mathbb{Z}$ , let  $r \rightarrow (-9 a d^2 + \sqrt{3} d \sqrt{4 b^3 d + 27 a^2 d^2})^{1/3}$ , then

$$\int (e + f x)^m (a + b x + d x^3)^p dx \rightarrow \frac{1}{d^{2p}} \int (e + f x)^m \left( \frac{18^{1/3} b d}{3 r} - \frac{r}{18^{1/3}} + d x \right)^p \left( \frac{b d}{3} + \frac{12^{1/3} b^2 d^2}{3 r^2} + \frac{r^2}{3 \times 12^{1/3}} - d \left( \frac{2^{1/3} b d}{3^{1/3} r} - \frac{r}{18^{1/3}} \right) x + d^2 x^2 \right)^p dx$$

Program code:

```
Int[(e_.+f_.**x_)^m_.*(a_+b_.**x_+d_.**x_^3)^p_,x_Symbol] :=
  With[{r=Rt[-9*a*d^2+Sqrt[3]*d*Sqrt[4*b^3*d+27*a^2*d^2],3]},
    1/d^(2*p)*Int[(e+f*x)^m*Simp[18^(1/3)*b*d/(3*r)-r/18^(1/3)+d*x,x]^p*
      Simp[b*d/3+12^(1/3)*b^2*d^2/(3*r^2)+r^2/(3*12^(1/3))-d*(2^(1/3)*b*d/(3^(1/3)*r)-r/18^(1/3))*x+d^2*x^2,x]^p,x]] /;
    FreeQ[{a,b,d,e,f,m},x] && NeQ[4*b^3+27*a^2*d,0] && ILtQ[p,0]
```

$$\mathbf{2:} \int (e + f x)^m (a + b x + d x^3)^p dx \text{ when } 4 b^3 + 27 a^2 d \neq 0 \wedge p \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction

Basis: If  $r \rightarrow (-9 a d^2 + \sqrt{3} d \sqrt{4 b^3 d + 27 a^2 d^2})^{1/3}$ , then



$$\partial_x \left( (a + bx + dx^3)^p / \left( \left( \frac{18^{1/3} bd}{3r} - \frac{r}{18^{1/3}} + dx \right)^p \left( \frac{bd}{3} + \frac{12^{1/3} b^2 d^2}{3r^2} + \frac{r^2}{3 \times 12^{1/3}} - d \left( \frac{2^{1/3} bd}{3^{1/3} r} - \frac{r}{18^{1/3}} \right) x + d^2 x^2 \right)^p \right) \right) = 0$$

Rule: If  $4b^3 + 27a^2d \neq 0 \wedge p \notin \mathbb{Z}$ , let  $r \rightarrow (-9ad^2 + \sqrt{3}d\sqrt{4b^3d + 27a^2d^2})^{1/3}$ , then

$$\int (e + fx)^m (a + bx + dx^3)^p dx \rightarrow \left( (a + bx + dx^3)^p / \left( \left( \frac{18^{1/3} bd}{3r} - \frac{r}{18^{1/3}} + dx \right)^p \left( \frac{bd}{3} + \frac{12^{1/3} b^2 d^2}{3r^2} + \frac{r^2}{3 \times 12^{1/3}} - d \left( \frac{2^{1/3} bd}{3^{1/3} r} - \frac{r}{18^{1/3}} \right) x + d^2 x^2 \right)^p \right) \right) \cdot \int (e + fx)^m \left( \frac{18^{1/3} bd}{3r} - \frac{r}{18^{1/3}} + dx \right)^p \left( \frac{bd}{3} + \frac{12^{1/3} b^2 d^2}{3r^2} + \frac{r^2}{3 \times 12^{1/3}} - d \left( \frac{2^{1/3} bd}{3^{1/3} r} - \frac{r}{18^{1/3}} \right) x + d^2 x^2 \right)^p dx$$

Program code:

```
Int[(e_.+f_.**x_)^m_.*(a_+b_.**x_+d_.**x_^3)^p_,x_Symbol] :=
  With[{r=Rt[-9*a*d^2+Sqrt[3]*d*Sqrt[4*b^3*d+27*a^2*d^2],3]},
    (a+b*x+d*x^3)^p/
    (Simp[18^(1/3)*b*d/(3*r)-r/18^(1/3)+d*x,x]^p*
     Simp[b*d/3+12^(1/3)*b^2*d^2/(3*r^2)+r^2/(3*12^(1/3))-d*(2^(1/3)*b*d/(3^(1/3)*r)-r/18^(1/3))*x+d^2*x^2,x]^p)*
    Int[(e+f*x)^m*Simp[18^(1/3)*b*d/(3*r)-r/18^(1/3)+d*x,x]^p*
     Simp[b*d/3+12^(1/3)*b^2*d^2/(3*r^2)+r^2/(3*12^(1/3))-d*(2^(1/3)*b*d/(3^(1/3)*r)-r/18^(1/3))*x+d^2*x^2,x]^p,x] /;
  FreeQ[{a,b,d,e,f,m,p},x] && NeQ[4*b^3+27*a^2*d,0] && Not[IntegerQ[p]]
```

**2:**  $\int (e + fx)^m (a + bx + cx^2 + dx^3)^p dx$

Derivation: Integration by substitution

Rule: If  $p \in \mathbb{Z}^- \wedge c^2 - 3bd \neq 0 \wedge b^2 - 3ac \neq 0$ , then

$$\int (e + fx)^m (a + bx + cx^2 + dx^3)^p dx \rightarrow \text{Subst} \left[ \int \left( \frac{3de - cf}{3d} + fx \right)^m \left( \frac{2c^3 - 9bcd + 27ad^2}{27d^2} - \frac{(c^2 - 3bd)x}{3d} + dx^3 \right)^p dx, x, x + \frac{c}{3d} \right]$$

Program code:

```
Int[(e_.+f_.**x_)^m_.*P3_^p_,x_Symbol] :=
  With[{a=Coeff[P3,x,0],b=Coeff[P3,x,1],c=Coeff[P3,x,2],d=Coeff[P3,x,3]},
    Subst[Int[(3*d*e-c*f)/(3*d)+f*x]^m*Simp[(2*c^3-9*b*c*d+27*a*d^2)/(27*d^2)-(c^2-3*b*d)*x/(3*d)+d*x^3,x]^p,x],x,x+c/(3*d)] /;
  NeQ[c,0] /;
  FreeQ[{e,f,m,p},x] && PolyQ[P3,x,3]
```

### Rules for integrands of the form $u (a + bx + cx^2 + dx^3 + ex^4)^p$

1.  $\int \frac{A + Bx}{\sqrt{a + bx + cx^2 + dx^3 + ex^4}} dx$  when

$$Bd - 4Ae = 0 \wedge d(141d^3 - 752cde - 400be^2) + 16e^2(71c^2 + 100ae) = 0 \wedge 144(3d^2 - 8ce)^3 + 125(d^3 - 4cde + 8be^2)^2 = 0$$

1:  $\int \frac{x}{\sqrt{a + bx + cx^2 + ex^4}} dx$  when  $71c^2 + 100ae = 0 \wedge 1152c^3 - 125b^2e = 0$

Reference: Bronstein

Rule: If  $71c^2 + 100ae = 0 \wedge 1152c^3 - 125b^2e = 0$ , let

$P[x] \rightarrow \frac{1}{320} (33b^2c + 6ac^2 + 40a^2e) - \frac{22}{5} ace x^2 +$ , then

$$\frac{22}{15} bce x^3 + \frac{1}{4} e (5c^2 + 4ae) x^4 + \frac{4}{3} be^2 x^5 + 2ce^2 x^6 + e^3 x^8$$

$$\int \frac{x}{\sqrt{a + bx + cx^2 + ex^4}} dx \rightarrow \frac{1}{8\sqrt{e}} \operatorname{Log} \left[ P[x] + \frac{\partial_x P[x]}{8\sqrt{e}x} \sqrt{a + bx + cx^2 + ex^4} \right]$$

Program code:

```
Int[x_/Sqrt[a_+b_.*x_+c_.*x_^2+e_.*x_^4],x_Symbol] :=
  With[{Px=1/320*(33*b^2*c+6*a*c^2+40*a^2*e)-22/5*a*c*e*x^2+22/15*b*c*e*x^3+1/4*e*(5*c^2+4*a*e)*x^4+
    4/3*b*e^2*x^5+2*c*e^2*x^6+e^3*x^8},
    1/(8*Rt[e,2])*Log[Px+Dist[1/(8*Rt[e,2]*x),D[Px,x],x]*Sqrt[a+b*x+c*x^2+e*x^4]]] /;
  FreeQ[{a,b,c,e},x] && EqQ[71*c^2+100*a*e,0] && EqQ[1152*c^3-125*b^2*e,0]
```

2:  $\int \frac{A+Bx}{\sqrt{a+bx+cx^2+dx^3+ex^4}} dx$  when  $Bd - 4Ae = 0 \wedge d(141d^3 - 752cde - 400be^2) + 16e^2(71c^2 + 100ae) = 0 \wedge 144(3d^2 - 8ce)^3 + 125(d^3 - 4cde + 8be^2)^2 = 0$

Derivation: Integration by substitution

Rule: If  $Bd - 4Ae = 0 \wedge d(141d^3 - 752cde - 400be^2) + 16e^2(71c^2 + 100ae) = 0 \wedge 144(3d^2 - 8ce)^3 + 125(d^3 - 4cde + 8be^2)^2 = 0$ , then

$$\int \frac{A+Bx}{\sqrt{a+bx+cx^2+dx^3+ex^4}} dx \rightarrow B \text{ Subst} \left[ \int \frac{x}{\sqrt{\frac{-3d^4+16cd^2e-64bd^2e^2+256ae^3}{256e^3} + \frac{(d^3-4cde+8be^2)x}{8e^2} - \frac{(3d^2-8ce)x^2}{8e} + ex^4}} dx, x, \frac{d}{4e} + x \right]$$

Program code:

```
Int[(A+B.*x_)/Sqrt[a+b.*x_+c.*x_^2+d.*x_^3+e.*x_^4],x_Symbol] :=
  B*Subst[Int[x/Sqrt[(-3*d^4+16*c*d^2*e-64*b*d*e^2+256*a*e^3)/(256*e^3)+(d^3-4*c*d*e+8*b*e^2)*x/(8*e^2)-
    (3*d^2-8*c*e)*x^2/(8*e)+e*x^4],x],x,d/(4*e)+x] /;
FreeQ[{a,b,c,d,e,A,B},x] && EqQ[B*d-4*A*e,0] &&
EqQ[d*(141*d^3-752*c*d*e-400*b*e^2)+16*e^2*(71*c^2+100*a*e),0] &&
EqQ[144*(3*d^2-8*c*e)^3+125*(d^3-4*c*d*e+8*b*e^2)^2,0]
```

$$2. \int \frac{f + gx^2}{(d + ex + dx^2) \sqrt{a + bx + cx^2 + bx^3 + ax^4}} dx \text{ when } bd - ae = 0 \wedge f + g = 0$$

$$1: \int \frac{f + gx^2}{(d + ex + dx^2) \sqrt{a + bx + cx^2 + bx^3 + ax^4}} dx \text{ when } bd - ae = 0 \wedge f + g = 0 \wedge a^2 (2a - c) > 0$$

Rule: If  $bd - ae = 0 \wedge f + g = 0 \wedge a^2 (2a - c) > 0$ , then

$$\int \frac{f + gx^2}{(d + ex + dx^2) \sqrt{a + bx + cx^2 + bx^3 + ax^4}} dx \rightarrow \frac{af}{d \sqrt{a^2 (2a - c)}} \operatorname{ArcTan} \left[ \frac{ab + (4a^2 + b^2 - 2ac)x + abx^2}{2 \sqrt{a^2 (2a - c)} \sqrt{a + bx + cx^2 + bx^3 + ax^4}} \right]$$

Program code:

```
Int[(f_+g_.*x^2)/((d+_e_.*x+d_.*x^2)*Sqrt[a+_b_.*x+c_.*x^2+b_.*x^3+a_.*x^4]),x_Symbol] :=
  a*f/(d*Rt[a^2*(2*a-c),2])*ArcTan[(a*b+(4*a^2+b^2-2*a*c)*x+a*b*x^2)/(2*Rt[a^2*(2*a-c),2]*Sqrt[a+b*x+c*x^2+b*x^3+a*x^4])]/;
FreeQ[{a,b,c,d,e,f,g},x] && EqQ[b*d-a*e,0] && EqQ[f+g,0] && PosQ[a^2*(2*a-c)]
```

$$2: \int \frac{f + gx^2}{(d + ex + dx^2) \sqrt{a + bx + cx^2 + bx^3 + ax^4}} dx \text{ when } bd - ae = 0 \wedge f + g = 0 \wedge a^2 (2a - c) \neq 0$$

Rule: If  $bd - ae = 0 \wedge f + g = 0 \wedge a^2 (2a - c) \neq 0$ , then

$$\int \frac{f + gx^2}{(d + ex + dx^2) \sqrt{a + bx + cx^2 + bx^3 + ax^4}} dx \rightarrow -\frac{af}{d \sqrt{-a^2 (2a - c)}} \operatorname{ArcTanh} \left[ \frac{ab + (4a^2 + b^2 - 2ac)x + abx^2}{2 \sqrt{-a^2 (2a - c)} \sqrt{a + bx + cx^2 + bx^3 + ax^4}} \right]$$

Program code:

```
Int[(f_+g_.*x^2)/((d+_e_.*x+d_.*x^2)*Sqrt[a+_b_.*x+c_.*x^2+b_.*x^3+a_.*x^4]),x_Symbol] :=
  -a*f/(d*Rt[-a^2*(2*a-c),2])*ArcTanh[(a*b+(4*a^2+b^2-2*a*c)*x+a*b*x^2)/(2*Rt[-a^2*(2*a-c),2]*Sqrt[a+b*x+c*x^2+b*x^3+a*x^4])]/;
FreeQ[{a,b,c,d,e,f,g},x] && EqQ[b*d-a*e,0] && EqQ[f+g,0] && NegQ[a^2*(2*a-c)]
```

$$3. \int \frac{u (A + Bx + Cx^2 + Dx^3)}{a + bx + cx^2 + bx^3 + ax^4} dx$$

$$1: \int \frac{A + Bx + Cx^2 + Dx^3}{a + bx + cx^2 + bx^3 + ax^4} dx$$

Derivation: Algebraic expansion

Basis: Let  $q \rightarrow \sqrt{8a^2 + b^2 - 4ac}$ , then

$$\frac{A+Bx+Cx^2+Dx^3}{a+bx+cx^2+bx^3+ax^4} = \frac{bA-2aB+2aD+Aq+(2aA-2aC+bD+Dq)x}{q(2a+(b+q)x+2ax^2)} - \frac{bA-2aB+2aD-Aq+(2aA-2aC+bD-Dq)x}{q(2a+(b-q)x+2ax^2)}$$

■

Rule: Let  $q \rightarrow \sqrt{8a^2 + b^2 - 4ac}$ , then

$$\int \frac{A + Bx + Cx^2 + Dx^3}{a + bx + cx^2 + bx^3 + ax^4} dx \rightarrow \frac{1}{q} \int \frac{bA - 2aB + 2aD + Aq + (2aA - 2aC + bD + Dq)x}{2a + (b+q)x + 2ax^2} dx - \frac{1}{q} \int \frac{bA - 2aB + 2aD - Aq + (2aA - 2aC + bD - Dq)x}{2a + (b-q)x + 2ax^2} dx$$

Program code:

```
Int[P3/(a+b.*x+c.*x^2+d.*x^3+e.*x^4),x_Symbol] :=
  With[{q=Sqrt[8*a^2+b^2-4*a*c],A=Coeff[P3,x,0],B=Coeff[P3,x,1],C=Coeff[P3,x,2],D=Coeff[P3,x,3]},
    1/q*Int[(bA-2*a*B+2*a*D+A*q+(2*aA-2*a*C+b*D+D*q)*x)/(2*a+(b+q)*x+2*a*x^2),x] -
    1/q*Int[(bA-2*a*B+2*a*D-A*q+(2*aA-2*a*C+b*D-D*q)*x)/(2*a+(b-q)*x+2*a*x^2),x] /;
    FreeQ[{a,b,c},x] && PolyQ[P3,x,3] && EqQ[a,e] && EqQ[b,d]
```

$$2: \int \frac{x^m (A + Bx + Cx^2 + Dx^3)}{a + bx + cx^2 + bx^3 + ax^4} dx$$

Derivation: Algebraic expansion

Basis: Let  $q \rightarrow \sqrt{8a^2 + b^2 - 4ac}$ , then

$$\frac{A+Bx+Cx^2+Dx^3}{a+bx+cx^2+bx^3+ax^4} = \frac{bA-2aB+2aD+Aq+(2aA-2aC+bD+Dq)x}{q(2a+(b+q)x+2ax^2)} - \frac{bA-2aB+2aD-Aq+(2aA-2aC+bD-Dq)x}{q(2a+(b-q)x+2ax^2)}$$

■

Rule: Let  $q \rightarrow \sqrt{8a^2 + b^2 - 4ac}$ , then

$$\int \frac{x^m (A+Bx+Cx^2+Dx^3)}{a+bx+cx^2+bx^3+ax^4} dx \rightarrow \frac{1}{q} \int \frac{x^m (bA-2aB+2aD+Aq+(2aA-2aC+bD+Dq)x)}{2a+(b+q)x+2ax^2} dx - \frac{1}{q} \int \frac{x^m (bA-2aB+2aD-Aq+(2aA-2aC+bD-Dq)x)}{2a+(b-q)x+2ax^2} dx$$

Program code:

```
Int[x^m_.*P3_/(a_+b_.x_+c_.x_^2+d_.x_^3+e_.x_^4),x_Symbol] :=
  With[{q=Sqrt[8*a^2+b^2-4*a*c],A=Coeff[P3,x,0],B=Coeff[P3,x,1],C=Coeff[P3,x,2],D=Coeff[P3,x,3]},
    1/q*Int[x^m*(b*A-2*a*B+2*a*D+A*q+(2*a*A-2*a*C+b*D+D*q)*x)/(2*a+(b+q)*x+2*a*x^2),x] -
    1/q*Int[x^m*(b*A-2*a*B+2*a*D-A*q+(2*a*A-2*a*C+b*D-D*q)*x)/(2*a+(b-q)*x+2*a*x^2),x] /;
    FreeQ[{a,b,c,m},x] && PolyQ[P3,x,3] && EqQ[a,e] && EqQ[b,d]
```

$$4. \int \frac{A+Bx+Cx^2}{a+bx+cx^2+dx^3+ex^4} dx \text{ when } B^2d+2C(bC+Ad)-2B(cC+2Ae) = 0 \wedge 2B^2cC-8aC^3-B^3d-4ABCD+4A(B^2+2AC)e = 0$$

$$1: \int \frac{A+Bx+Cx^2}{a+bx+cx^2+dx^3+ex^4} dx \text{ when}$$

$$B^2d+2C(bC+Ad)-2B(cC+2Ae) = 0 \wedge 2B^2cC-8aC^3-B^3d-4ABCD+4A(B^2+2AC)e = 0 \wedge C(2e(Bd-4Ae)+C(d^2-4ce)) > 0$$

Rule: If

$$B^2d+2C(bC+Ad)-2B(cC+2Ae) = 0 \wedge ,$$

$$2B^2cC-8aC^3-B^3d-4ABCD+4A(B^2+2AC)e = 0 \wedge C(2e(Bd-4Ae)+C(d^2-4ce)) > 0$$

let  $q \rightarrow \sqrt{C(2e(Bd-4Ae)+C(d^2-4ce))}$ , then

$$\int \frac{A+Bx+Cx^2}{a+bx+cx^2+dx^3+ex^4} dx \rightarrow -\frac{2C^2}{q} \text{ArcTanh}\left[\frac{Cd-Be+2Cex}{q}\right] +$$

$$\frac{2C^2}{q} \operatorname{ArcTanh} \left[ \frac{1}{q(B^2 - 4AC)} C (4BcC - 3B^2d - 4ACd + 12ABe + 4C(2cC - Bd + 2Ae)x + 4C(2Cd - Be)x^2 + 8C^2ex^3) \right]$$

## Program code:

```
Int[(A_.+B_.**x_+C_.**x_^2)/(a_+b_.**x_+c_.**x_^2+d_.**x_^3+e_.**x_^4),x_Symbol] :=
  With[{q=Rt[C*(2*e*(B*d-4*A*e)+C*(d^2-4*C*e)),2]},
    -2*C^2/q*ArcTanh[(C*d-B*e+2*C*e*x)/q] +
    2*C^2/q*ArcTanh[C*(4*B*C*C-3*B^2*d-4*A*C*d+12*A*B*e+4*C*(2*c*C-B*d+2*A*e)*x+4*C*(2*C*d-B*e)*x^2+8*C^2*e*x^3)/(q*(B^2-4*A*C))] /;
FreeQ[{a,b,c,d,e,A,B,C},x] && EqQ[B^2*d+2*C*(b*C+A*d)-2*B*(c*C+2*A*e),0] &&
EqQ[2*B^2*c*C-8*a*C^3-B^3*d-4*A*B*C*d+4*A*(B^2+2*A*C)*e,0] && PosQ[C*(2*e*(B*d-4*A*e)+C*(d^2-4*C*e))]
```

```
Int[(A_.+C_.**x_^2)/(a_+b_.**x_+c_.**x_^2+d_.**x_^3+e_.**x_^4),x_Symbol] :=
  With[{q=Rt[C*(-8*A*e^2+C*(d^2-4*C*e)),2]},
    -2*C^2/q*ArcTanh[C*(d+2*e*x)/q] + 2*C^2/q*ArcTanh[C*(A*d-2*(c*C+A*e)*x-2*C*d*x^2-2*C*e*x^3)/(A*q)] /;
FreeQ[{a,b,c,d,e,A,C},x] && EqQ[b*C+A*d,0] && EqQ[a*C^2-A^2*e,0] && PosQ[C*(-8*A*e^2+C*(d^2-4*C*e))]
```



**2:**  $\int \frac{A+Bx+Cx^2}{a+bx+cx^2+dx^3+ex^4} dx$  when

$$B^2 d + 2 C (b C + A d) - 2 B (c C + 2 A e) = 0 \wedge 2 B^2 c C - 8 a C^3 - B^3 d - 4 A B C d + 4 A (B^2 + 2 A C) e = 0 \wedge C (2 e (B d - 4 A e) + C (d^2 - 4 c e)) \neq 0$$

Rule: If

$$B^2 d + 2 C (b C + A d) - 2 B (c C + 2 A e) = 0 \wedge$$

$$2 B^2 c C - 8 a C^3 - B^3 d - 4 A B C d + 4 A (B^2 + 2 A C) e = 0 \wedge C (2 e (B d - 4 A e) + C (d^2 - 4 c e)) \neq 0$$

let  $q = \sqrt{-C (2 e (B d - 4 A e) + C (d^2 - 4 c e))}$ , then

$$\int \frac{A+Bx+Cx^2}{a+bx+cx^2+dx^3+ex^4} dx \rightarrow$$

$$\frac{2 C^2}{q} \text{ArcTan}\left[\frac{C d - B e + 2 C e x}{q}\right] - \frac{2 C^2}{q} \text{ArcTan}\left[\frac{1}{q (B^2 - 4 A C)} C (4 B c C - 3 B^2 d - 4 A C d + 12 A B e + 4 C (2 c C - B d + 2 A e) x + 4 C (2 C d - B e) x^2 + 8 C^2 e x^3)\right]$$

Program code:

```
Int[(A_.+B_.**x_+C_.**x_^2)/(a_+b_.**x_+c_.**x_^2+d_.**x_^3+e_.**x_^4),x_Symbol] :=
  With[{q=Rt[-C*(2*e*(B*d-4*A*e)+C*(d^2-4*c*e)),2]},
    2*C^2/q*ArcTan[(C*d-B*e+2*C*e*x)/q] -
    2*C^2/q*ArcTan[C*(4*B*c*C-3*B^2*d-4*A*C*d+12*A*B*e+4*C*(2*c*C-B*d+2*A*e)*x+4*C*(2*C*d-B*e)*x^2+8*C^2*e*x^3)/(q*(B^2-4*A*C))]] /;
FreeQ[{a,b,c,d,e,A,B,C},x] && EqQ[B^2*d+2*C*(b*C+A*d)-2*B*(c*C+2*A*e),0] &&
EqQ[2*B^2*c*C-8*a*C^3-B^3*d-4*A*B*C*d+4*A*(B^2+2*A*C)*e,0] && NegQ[C*(2*e*(B*d-4*A*e)+C*(d^2-4*c*e))]
```

```
Int[(A_.+C_.**x_^2)/(a_+b_.**x_+c_.**x_^2+d_.**x_^3+e_.**x_^4),x_Symbol] :=
  With[{q=Rt[-C*(-8*A*e^2+C*(d^2-4*c*e)),2]},
    2*C^2/q*ArcTan[(C*d+2*C*e*x)/q] - 2*C^2/q*ArcTan[-C*(-A*d+2*(c*C+A*e)*x+2*C*d*x^2+2*C*e*x^3)/(A*q)]] /;
FreeQ[{a,b,c,d,e,A,C},x] && EqQ[b*C+A*d,0] && EqQ[a*C^2-A^2*e,0] && NegQ[C*(-8*A*e^2+C*(d^2-4*c*e))]
```

5:  $\int P[x] (a + bx + cx^2 + dx^3 + ex^4)^p dx$  when  $p \in \mathbb{Z}^- \wedge a \neq 0 \wedge c = \frac{b^2}{a} \wedge d = \frac{b^3}{a^2} \wedge e = \frac{b^4}{a^3}$

Derivation: Algebraic simplification

Basis: If  $a \neq 0 \wedge c = \frac{b^2}{a} \wedge d = \frac{b^3}{a^2} \wedge e = \frac{b^4}{a^3}$ , then  $a + bx + cx^2 + dx^3 + ex^4 = \frac{a^5 - b^5 x^5}{a^3 (a - bx)}$

Rule: If  $p \in \mathbb{Z}^- \wedge a \neq 0 \wedge c = \frac{b^2}{a} \wedge d = \frac{b^3}{a^2} \wedge e = \frac{b^4}{a^3}$ , then

$$\int P[x] (a + bx + cx^2 + dx^3 + ex^4)^p dx \rightarrow \frac{1}{a^{3p}} \int \text{ExpandIntegrand}\left[\frac{P[x] (a - bx)^{-p}}{(a^5 - b^5 x^5)^{-p}}, x\right] dx$$

Program code:

```
Int[Px_*P4^p_,x_Symbol] :=
  With[{a=Coeff[P4,x,0],b=Coeff[P4,x,1],c=Coeff[P4,x,2],d=Coeff[P4,x,3],e=Coeff[P4,x,4]},
    1/a^(3*p)*Int[ExpandIntegrand[Px*(a-b*x)^(-p)/(a^5-b^5*x^5)^(-p),x],x] /;
    NeQ[a,0] && EqQ[c,b^2/a] && EqQ[d,b^3/a^2] && EqQ[e,b^4/a^3] /;
    FreeQ[p,x] && PolyQ[P4,x,4] && PolyQ[Px,x] && ILtQ[p,0]
```

Rules for integrands of the form  $P_m[x] Q_n[x]^p$ 

$$1. \int \frac{u (A + B x^n)}{a + b x^{2(m+1)} + c x^n + d x^{2n}} dx$$

$$1: \int \frac{A + B x^n}{a + b x^2 + c x^n + d x^{2n}} dx \text{ when } a B^2 - A^2 d (n-1)^2 = 0 \wedge B c + 2 A d (n-1) = 0$$

## Derivation: Integration by substitution

Basis: If  $a B^2 - A^2 d (n-1)^2 = 0 \wedge B c + 2 A d (n-1) = 0$ , then

$$\frac{A+B x^n}{a+b x^2+c x^n+d x^{2n}} = A^2 (n-1) \text{ Subst} \left[ \frac{1}{a+A^2 b (n-1)^2 x^2}, x, \frac{x}{A (n-1) - B x^n} \right] \partial_x \frac{x}{A (n-1) - B x^n}$$

Rule 1.3.3.16.1: If  $a B^2 - A^2 d (n-1)^2 = 0 \wedge B c + 2 A d (n-1) = 0$ , then

$$\int \frac{A + B x^n}{a + b x^2 + c x^n + d x^{2n}} dx \rightarrow A^2 (n-1) \text{ Subst} \left[ \int \frac{1}{a + A^2 b (n-1)^2 x^2} dx, x, \frac{x}{A (n-1) - B x^n} \right]$$

## Program code:

```
Int[(A_+B_.*x_^n_)/(a_+b_.*x_^2+c_.*x_^n_+d_.*x_^n2_), x_Symbol] :=
  A^2*(n-1)*Subst[Int[1/(a+A^2*b*(n-1)^2*x^2), x], x, x/(A*(n-1)-B*x^n)] /;
FreeQ[{a,b,c,d,A,B,n}, x] && EqQ[n2, 2*n] && NeQ[n, 2] && EqQ[a*B^2-A^2*d*(n-1)^2, 0] && EqQ[B*c+2*A*d*(n-1), 0]
```

$$2: \int \frac{x^m (A + B x^n)}{a + b x^{2(m+1)} + c x^n + d x^{2n}} dx \text{ when } a B^2 (m+1)^2 - A^2 d (m-n+1)^2 = 0 \wedge B c (m+1) - 2 A d (m-n+1) = 0$$

Derivation: Integration by substitution

Basis: If  $a B^2 (m+1)^2 - A^2 d (m-n+1)^2 = 0 \wedge B c (m+1) - 2 A d (m-n+1) = 0$ ,

$$\text{then } \frac{x^m (A + B x^n)}{a + b x^{2(m+1)} + c x^n + d x^{2n}} = \frac{A^2 (m-n+1)}{m+1} \text{Subst}\left[\frac{1}{a + A^2 b (m-n+1)^2 x^2}, x, \frac{x^{m+1}}{A (m-n+1) + B (m+1) x^n}\right] \partial_x \frac{x^{m+1}}{A (m-n+1) + B (m+1) x^n}$$

Rule 1.3.3.16.2: If  $a B^2 (m+1)^2 - A^2 d (m-n+1)^2 = 0 \wedge B c (m+1) - 2 A d (m-n+1) = 0$ , then

$$\int \frac{x^m (A + B x^n)}{a + b x^{2(m+1)} + c x^n + d x^{2n}} dx \rightarrow \frac{A^2 (m-n+1)}{m+1} \text{Subst}\left[\int \frac{1}{a + A^2 b (m-n+1)^2 x^2} dx, x, \frac{x^{m+1}}{A (m-n+1) + B (m+1) x^n}\right]$$

Program code:

```
Int[x^m_.*(A_+B_*x^n_)/(a_+b_*x^k_+c_*x^n_+d_*x^n2_), x_Symbol] :=
  A^2*(m-n+1)/(m+1)*Subst[Int[1/(a+A^2*b*(m-n+1)^2*x^2), x], x, x^(m+1)/(A*(m-n+1)+B*(m+1)*x^n)] /;
  FreeQ[{a,b,c,d,A,B,m,n}, x] && EqQ[n2, 2*n] && EqQ[k, 2*(m+1)] && EqQ[a*B^2*(m+1)^2-A^2*d*(m-n+1)^2, 0] && EqQ[B*c*(m+1)-2*A*d*(m-n+1), 0]
```

$$2. \int u Q_6[x]^p dx \text{ when } p \in \mathbb{Z}^-$$

$$1: \int \frac{a + b x^2 + c x^4}{d + e x^2 + f x^4 + g x^6} dx \text{ when } -9 c^3 d^2 + c d f (b^2 + 6 a c) - a^2 c f^2 - 2 a b g (3 c d + a f) + 12 a^3 g^2 = 0 \wedge \\ 3 c^4 d^2 e - 3 a^2 c^2 d f g + a^3 c f^2 g + 2 a^3 g^2 (b f - 6 a g) - c^3 d (2 b d f + a e f - 12 a d g) = 0 \wedge \frac{-a c f^2 + 12 a^2 g^2 + f (3 c^2 d - 2 a b g)}{c g (3 c d - a f)} > 0$$

Rule 1.3.3.17.1: If

$$\begin{aligned}
 & -9c^3d^2 + cdf(b^2 + 6ac) - a^2cf^2 - 2abg(3cd + af) + 12a^3g^2 = 0 \wedge \\
 & 3c^4d^2e - 3a^2c^2dfg + a^3cf^2g + 2a^3g^2(bf - 6ag) - c^3d(2bdf + aef - 12adg) = 0 \wedge \\
 & \frac{-acf^2 + 12a^2g^2 + f(3c^2d - 2abg)}{cg(3cd - af)} > 0
 \end{aligned}$$

$$\text{let } q \rightarrow \sqrt{\frac{-acf^2 + 12a^2g^2 + f(3c^2d - 2abg)}{cg(3cd - af)}} \text{ and } r \rightarrow \sqrt{\frac{acf^2 + 4g(bcd + a^2g) - f(3c^2d + 2abg)}{cg(3cd - af)}}, \text{ then}$$

$$\int \frac{a + bx^2 + cx^4}{d + ex^2 + fx^4 + gx^6} dx \rightarrow$$

$$\begin{aligned}
 & \frac{c}{gq} \text{ArcTan}\left[\frac{r+2x}{q}\right] - \frac{c}{gq} \text{ArcTan}\left[\frac{r-2x}{q}\right] - \\
 & \frac{c}{gq} \text{ArcTan}\left[\left((3cd - af)x(bcd - ab^2fg - 2a^2cfg + 6a^2bg^2 + c(3c^2df - acf^2 - bcdg + 2a^2g^2)x^2 + c^2g(3cd - af)x^4)\right) / \right. \\
 & \left. (gq(bcd - 2a^2g)(bcd - abf + 4a^2g))\right]
 \end{aligned}$$

Program code:

```

Int[(a+b_.**x_^2+c_.**x_^4)/(d+e_.**x_^2+f_.**x_^4+g_.**x_^6),x_Symbol] :=
  With[{q=Rt[(-a*c*f^2+12*a^2*g^2+f*(3*c^2*d-2*a*b*g))/(c*g*(3*c*d-a*f)),2],
    r=Rt[(a*c*f^2+4*g*(b*c*d+a^2*g)-f*(3*c^2*d+2*a*b*g))/(c*g*(3*c*d-a*f)),2]},
  c/(g*q)*ArcTan[(r+2*x)/q] -
  c/(g*q)*ArcTan[(r-2*x)/q] -
  c/(g*q)*ArcTan[(3*c*d-a*f)*x/(g*q*(b*c*d-2*a^2*g)*(b*c*d-a*b*f+4*a^2*g))*
    (b*c^2*d*f-a*b^2*f*g-2*a^2*c*f*g+6*a^2*b*g^2+c*(3*c^2*d*f-a*c*f^2-b*c*d*g+2*a^2*g^2)*x^2+c^2*g*(3*c*d-a*f)*x^4)] /;
FreeQ[{a,b,c,d,e,f,g},x] && EqQ[9*c^3*d^2-c*(b^2+6*a*c)*d*f+a^2*c*f^2+2*a*b*(3*c*d+a*f)*g-12*a^3*g^2,0] &&
EqQ[3*c^4*d^2*e-3*a^2*c^2*d*f*g+a^3*c*f^2*g+2*a^3*g^2*(b*f-6*a*g)-c^3*d*(2*b*d*f+a*e*f-12*a*d*g),0] &&
NeQ[3*c*d-a*f,0] && NeQ[b*c*d-2*a^2*g,0] && NeQ[b*c*d-a*b*f+4*a^2*g,0] &&
PosQ[(-a*c*f^2+12*a^2*g^2+f*(3*c^2*d-2*a*b*g))/(c*g*(3*c*d-a*f))]

```

```

Int[(a_+c_.*x_^4)/(d_+e_.*x_^2+f_.*x_^4+g_.*x_^6),x_Symbol] :=
  With[{q=Rt[(-a*c*f^2+12*a^2*g^2+3*f*c^2*d)/(c*g*(3*c*d-a*f)),2],
    r=Rt[(a*c*f^2+4*a^2*g^2-3*c^2*d*f)/(c*g*(3*c*d-a*f)),2]},
    c/(g*q)*ArcTan[(r+2*x)/q] -
    c/(g*q)*ArcTan[(r-2*x)/q] -
    c/(g*q)*ArcTan[(c*(3*c*d-a*f)*x*(2*a^2*f*g-(3*c^2*d*f-a*c*f^2+2*a^2*g^2)*x^2-c*(3*c*d-a*f)*g*x^4))/(8*a^4*g^3*q)] /;
FreeQ[{a,c,d,e,f,g},x] && EqQ[9*c^3*d^2-6*a*c^2*d*f+a^2*c*f^2-12*a^3*g^2,0] &&
EqQ[3*c^4*d^2*e-3*a^2*c^2*d*f*g+a^3*c*f^2*g-12*a^4*g^3-a*c^3*d*(e*f-12*d*g),0] &&
NeQ[3*c*d-a*f,0] && PosQ[(-a*c*f^2+12*a^2*g^2+3*c^2*d*f)/(c*g*(3*c*d-a*f))]

```

**2:**  $\int u (a + bx^2 + cx^3 + dx^4 + ex^6)^p dx$  when  $p \in \mathbb{Z}^- \wedge b^2 - 3ad == 0 \wedge b^3 - 27a^2e == 0$

### Algebraic expansion

Basis: If  $b^2 - 3ad == 0 \wedge b^3 - 27a^2e == 0$ , then

$$a + bx^2 + cx^3 + dx^4 + ex^6 = \frac{1}{27a^2} (3a + 3a^{2/3}c^{1/3}x + bx^2) (3a - 3(-1)^{1/3}a^{2/3}c^{1/3}x + bx^2) (3a + 3(-1)^{2/3}a^{2/3}c^{1/3}x + bx^2)$$

Note: If  $\frac{m+1}{2} \in \mathbb{Z}^+$ , then  $cx^m + (a + bx^2)^m = \prod_{k=1}^m (a + (-1)^k (1 - \frac{1}{a}) c^{\frac{1}{3}} x + bx^2)$

Rule 1.3.3.17.2: If  $p \in \mathbb{Z}^- \wedge b^2 - 3ad == 0 \wedge b^3 - 27a^2e == 0$ , then

$$\int u (a + bx^2 + cx^3 + dx^4 + ex^6)^p dx \rightarrow \frac{1}{3^{3p} a^{2p}} \int \text{ExpandIntegrand}[u (3a + 3a^{2/3}c^{1/3}x + bx^2)^p (3a - 3(-1)^{1/3}a^{2/3}c^{1/3}x + bx^2)^p (3a + 3(-1)^{2/3}a^{2/3}c^{1/3}x + bx^2)^p, x] dx$$

### Program code:

```
Int[u_*Q6_^p_,x_Symbol] :=
  With[{a=Coeff[Q6,x,0],b=Coeff[Q6,x,2],c=Coeff[Q6,x,3],d=Coeff[Q6,x,4],e=Coeff[Q6,x,6]},
    1/(3^(3*p)*a^(2*p))*Int[ExpandIntegrand[u*
      (3*a+3*Rt[a,3]^2*Rt[c,3]*x+b*x^2)^p*
      (3*a-3*(-1)^(1/3)*Rt[a,3]^2*Rt[c,3]*x+b*x^2)^p*
      (3*a+3*(-1)^(2/3)*Rt[a,3]^2*Rt[c,3]*x+b*x^2)^p,x],x] /;
    EqQ[b^2-3*a*d,0] && EqQ[b^3-27*a^2*e,0] /;
    ILtQ[p,0] && PolyQ[Q6,x,6] && EqQ[Coeff[Q6,x,1],0] && EqQ[Coeff[Q6,x,5],0] && RationalFunctionQ[u,x]
```

3.  $\int P_m[x] Q_n[x]^p dx$  when  $m == n - 1$

1.  $\int P_m[x] Q_n[x]^p dx$  when  $m == n - 1 \wedge \partial_x \left( P_m[x] - \frac{P_m[x, m]}{n Q_n[x, n]} \partial_x Q_n[x] \right) == 0$

**1:**  $\int \frac{P_m[x]}{Q_n[x]} dx$  when  $m == n - 1 \wedge \partial_x \left( P_m[x] - \frac{P_m[x, m]}{n Q_n[x, n]} \partial_x Q_n[x] \right) == 0$

Derivation: Algebraic expansion and reciprocal integration rule

Rule 1.3.3.18.2.1: If  $m == n - 1 \wedge \partial_x \left( P_m[x] - \frac{P_m[x, m]}{n Q_n[x, n]} \partial_x Q_n[x] \right) == 0$ , then

$$\begin{aligned} \int \frac{P_m[x]}{Q_n[x]} dx &\rightarrow \frac{P_m[x, m]}{n Q_n[x, n]} \int \frac{\partial_x Q_n[x]}{Q_n[x]} dx + \left( P_m[x] - \frac{P_m[x, m]}{n Q_n[x, n]} \partial_x Q_n[x] \right) \int \frac{1}{Q_n[x]} dx \\ &\rightarrow \frac{P_m[x, m] \text{Log}[Q_n[x]]}{n Q_n[x, n]} + \left( P_m[x] - \frac{P_m[x, m]}{n Q_n[x, n]} \partial_x Q_n[x] \right) \int \frac{1}{Q_n[x]} dx \end{aligned}$$

Program code:

```
Int[Pm_/Qn_, x_Symbol] :=
  With[{m=Expon[Pm, x], n=Expon[Qn, x]},
    Coeff[Pm, x, m]*Log[Qn]/(n*Coeff[Qn, x, n]) + Simplify[Pm-Coeff[Pm, x, m]*D[Qn, x]/(n*Coeff[Qn, x, n])] * Int[1/Qn, x] /;
    EqQ[m, n-1] && EqQ[D[Simplify[Pm-Coeff[Pm, x, m]/(n*Coeff[Qn, x, n])*D[Qn, x]], x], 0] /;
    PolyQ[Pm, x] && PolyQ[Qn, x]
```



$$2: \int P_m[x] Q_n[x]^p dx \text{ when } m = n - 1 \wedge \partial_x \left( P_m[x] - \frac{P_m[x, m]}{n Q_n[x, n]} \partial_x Q_n[x] \right) = 0 \wedge p \neq -1$$

Derivation: Algebraic expansion and power integration rule

Rule 1.3.3.18.2.2: If  $m = n - 1 \wedge \partial_x \left( P_m[x] - \frac{P_m[x, m]}{n Q_n[x, n]} \partial_x Q_n[x] \right) = 0 \wedge p \neq -1$ , then

$$\begin{aligned} \int P_m[x] Q_n[x]^p dx &\rightarrow \frac{P_m[x, m]}{n Q_n[x, n]} \int Q_n[x]^p \partial_x Q_n[x] dx + \left( P_m[x] - \frac{P_m[x, m]}{n Q_n[x, n]} \partial_x Q_n[x] \right) \int Q_n[x]^p dx \\ &\rightarrow \frac{P_m[x, m] Q_n[x]^{p+1}}{n (p+1) Q_n[x, n]} + \left( P_m[x] - \frac{P_m[x, m]}{n Q_n[x, n]} \partial_x Q_n[x] \right) \int Q_n[x]^p dx \end{aligned}$$

Program code:

```
Int[Pm_*Qn^p_, x_Symbol] :=
  With[{m=Expon[Pm, x], n=Expon[Qn, x]},
    Coeff[Pm, x, m]*Qn^(p+1)/(n*(p+1)*Coeff[Qn, x, n]) + Simplify[Pm-Coeff[Pm, x, m]*D[Qn, x]/(n*Coeff[Qn, x, n])]*Int[Qn^p, x]/;
    EqQ[m, n-1] && EqQ[D[Simplify[Pm-Coeff[Pm, x, m]/(n*Coeff[Qn, x, n])*D[Qn, x]], x], 0]] /;
    FreeQ[p, x] && PolyQ[Pm, x] && PolyQ[Qn, x] && NeQ[p, -1]
```

$$2. \int P_m[x] Q_n[x]^p dx \text{ when } m == n - 1 \wedge \partial_x \left( P_m[x] - \frac{P_m[x, m]}{n Q_n[x, n]} \partial_x Q_n[x] \right) \neq 0$$

$$1: \int \frac{P_m[x]}{Q_n[x]} dx \text{ when } m == n - 1$$

Derivation: Algebraic expansion and reciprocal integration rule

Rule 1.3.3.18.2.1: If  $m == n - 1$ , then

$$\begin{aligned} \int \frac{P_m[x]}{Q_n[x]} dx &\rightarrow \frac{P_m[x, m]}{n Q_n[x, n]} \int \frac{\partial_x Q_n[x]}{Q_n[x]} dx + \frac{1}{n Q_n[x, n]} \int \frac{n Q_n[x, n] P_m[x] - P_m[x, m] \partial_x Q_n[x]}{Q_n[x]} dx \\ &\rightarrow \frac{P_m[x, m] \text{Log}[Q_n[x]]}{n Q_n[x, n]} + \frac{1}{n Q_n[x, n]} \int \frac{n Q_n[x, n] P_m[x] - P_m[x, m] \partial_x Q_n[x]}{Q_n[x]} dx \end{aligned}$$

Program code:

```
Int[Pm_/Qn_, x_Symbol] :=
  With[{m=Expon[Pm, x], n=Expon[Qn, x]},
    Coeff[Pm, x, m]*Log[Qn]/(n*Coeff[Qn, x, n]) +
    1/(n*Coeff[Qn, x, n]) Int[ExpandToSum[n*Coeff[Qn, x, n]*Pm - Coeff[Pm, x, m]*D[Qn, x], x]/Qn, x] /;
    EqQ[m, n-1] /;
    PolyQ[Pm, x] && PolyQ[Qn, x]
```

$$2: \int P_m[x] Q_n[x]^p dx \text{ when } m = n - 1 \wedge p \neq -1$$

Derivation: Algebraic expansion and power integration rule

Rule 1.3.3.18.2.2: If  $m = n - 1 \wedge p \neq -1$ , then

$$\begin{aligned} \int P_m[x] Q_n[x]^p dx &\rightarrow \frac{P_m[x, m]}{n Q_n[x, n]} \int Q_n[x]^p \partial_x Q_n[x] dx + \frac{1}{n Q_n[x, n]} \int (n Q_n[x, n] P_m[x] - P_m[x, m] \partial_x Q_n[x]) Q_n[x]^p dx \\ &\rightarrow \frac{P_m[x, m] Q_n[x]^{p+1}}{n (p+1) Q_n[x, n]} + \frac{1}{n Q_n[x, n]} \int (n Q_n[x, n] P_m[x] - P_m[x, m] \partial_x Q_n[x]) Q_n[x]^p dx \end{aligned}$$

Program code:

```
Int[Pm*Qn^p_, x_Symbol] :=
  With[{m=Expon[Pm,x], n=Expon[Qn,x]},
    Coeff[Pm,x,m]*Qn^(p+1)/(n*(p+1)*Coeff[Qn,x,n]) +
    1/(n*Coeff[Qn,x,n])*Int[ExpandToSum[n*Coeff[Qn,x,n]*Pm-Coeff[Pm,x,m]*D[Qn,x],x]*Qn^p,x]/;
    EqQ[m,n-1] /;
    FreeQ[p,x] && PolyQ[Pm,x] && PolyQ[Qn,x] && NeQ[p,-1]
```

$$4: \int P_m[x] Q_n[x]^p dx \text{ when } p < -1 \wedge 1 < n < m+1 \wedge m+n p+1 < 0$$

Reference: G&R 2.104

Note: Special case of the Ostrogradskiy-Hermite method without the need to solve a system of linear equations.

Note: Finds one term of the rational part of the antiderivative, thereby reducing the degree of the polynomial in the numerator of the integrand.

Note: Requirement that  $m < 2n - 1$  ensures new term is a proper fraction.

Rule 1.3.3.19: If  $p < -1 \wedge 1 < n < m+1 \wedge m+n p+1 < 0$ , then

$$\int P_m[x] Q_n[x]^p dx \rightarrow \frac{P_m[x, m] x^{m-n+1} Q_n[x]^{p+1}}{(m+n p+1) Q_n[x, n]} +$$

$$\frac{1}{(m+n p+1) Q_n[x, n]} \int \left( (m+n p+1) Q_n[x, n] P_m[x] - P_m[x, m] x^{m-n} \left( (m-n+1) Q_n[x] + (p+1) x \partial_x Q_n[x] \right) \right) Q_n[x]^p dx$$

Program code:

```
Int[Pm_*Qn^p_, x_Symbol] :=
  With[{m=Expon[Pm,x], n=Expon[Qn,x]},
    Coeff[Pm,x,m]*x^(m-n+1)*Qn^(p+1)/((m+n*p+1)*Coeff[Qn,x,n]) +
    1/((m+n*p+1)*Coeff[Qn,x,n])*
      Int[ExpandToSum[(m+n*p+1)*Coeff[Qn,x,n]*Pm-Coeff[Pm,x,m]*x^(m-n)*((m-n+1)*Qn+(p+1)*x*D[Qn,x]),x]*Qn^p,x] /;
    LtQ[1,n,m+1] && m+n*p+1<0] /;
  FreeQ[p,x] && PolyQ[Pm,x] && PolyQ[Qn,x] && LtQ[p,-1]
```