

Rules for integrands of the form $(dx)^m P_q[x] (a + bx^2 + cx^4)^p$

1: $\int (dx)^m P_q[x] (a + bx^2 + cx^4)^p dx$ when $\neg P_q[x^2]$

Derivation: Algebraic expansion

Basis: $P_q[x] = \sum_{k=0}^{q/2+1} P_q[x, 2k] x^{2k} + x \sum_{k=0}^{(q-1)/2+1} P_q[x, 2k+1] x^{2k}$

Note: This rule transforms $P_q[x]$ into a sum of the form $Q_r[x^2] + x R_s[x^2]$.

Rule 1.2.2.6.3: If $\neg P_q[x^2]$, then

$$\int (dx)^m P_q[x] (a + bx^2 + cx^4)^p dx \rightarrow \int (dx)^m \left(\sum_{k=0}^{\frac{q}{2}+1} P_q[x, 2k] x^{2k} \right) (a + bx^2 + cx^4)^p dx + \frac{1}{d} \int (dx)^{m+1} \left(\sum_{k=0}^{\frac{q-1}{2}+1} P_q[x, 2k+1] x^{2k} \right) (a + bx^2 + cx^4)^p dx$$

Program code:

```
Int[(d.*x_)^m_.*Pq_*(a_+b_.*x_^2+c_.*x_^4)^p_,x_Symbol] :=
Module[{q=Expon[Pq,x],k},
Int[(d*x)^m*Sum[Coeff[Pq,x,2*k]*x^(2*k),{k,0,q/2+1}]* (a+b*x^2+c*x^4)^p,x] +
1/d*Int[(d*x)^(m+1)*Sum[Coeff[Pq,x,2*k+1]*x^(2*k),{k,0,(q-1)/2+1}]* (a+b*x^2+c*x^4)^p,x] /;
FreeQ[{a,b,c,d,m,p},x] && PolyQ[Pq,x] && Not[PolyQ[Pq,x^2]]
```

2: $\int x^m P_q[x^2] (a+bx^2+cx^4)^p dx$ when $\frac{m-1}{2} \in \mathbb{Z}$

Derivation: Integration by substitution

Basis: If $\frac{m-1}{2} \in \mathbb{Z}$, then $x^m F[x^2] = \frac{1}{2} \text{Subst}[x^{\frac{m-1}{2}} F[x], x, x^2] \partial_x x^2$

Rule 1.2.2.6.4: If $\frac{m-1}{2} \in \mathbb{Z}$, then

$$\int x^m P_q[x^2] (a+bx^2+cx^4)^p dx \rightarrow \frac{1}{2} \text{Subst}\left[\int x^{\frac{m-1}{2}} P_q[x] (a+bx+cx^2)^p dx, x, x^2\right]$$

Program code:

```
Int[x_^m_.*Pq_*(a_+b_.*x_^2+c_.*x_^4)^p_,x_Symbol] :=
  1/2*Subst[Int[x^((m-1)/2)*SubstFor[x^2,Pq,x]*(a+b*x+c*x^2)^p,x],x,x^2] /;
FreeQ[{a,b,c,p},x] && PolyQ[Pq,x^2] && IntegerQ[(m-1)/2]
```

3: $\int (dx)^m P_q[x^2] (a+bx^2+cx^4)^p dx$ when $p+2 \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule 1.2.2.6.1: If $p+2 \in \mathbb{Z}^+$, then

$$\int (dx)^m P_q[x^2] (a+bx^2+cx^4)^p dx \rightarrow \int \text{ExpandIntegrand}[(dx)^m P_q[x^2] (a+bx^2+cx^4)^p, x] dx$$

Program code:

```
Int[(d_.*x_)^m_.*Pq_*(a_+b_.*x_^2+c_.*x_^4)^p_,x_Symbol] :=
  Int[ExpandIntegrand[(dx)^m*Pq*(a+b*x^2+c*x^4)^p,x],x] /;
FreeQ[{a,b,c,d,m},x] && PolyQ[Pq,x^2] && IGtQ[p,-2]
```

4: $\int (dx)^m Pq[x^2] (a+bx^2+cx^4)^p dx$ when $Pq[x, 0] == 0$

Derivation: Algebraic expansion

Rule 1.2.2.6.2: If $Pq[x, 0] == 0$, then

$$\int (dx)^m Pq[x^2] (a+bx^2+cx^4)^p dx \rightarrow \frac{1}{d^2} \int (dx)^{m+2} \frac{Pq[x^2]}{x^2} (a+bx^2+cx^4)^p dx$$

Program code:

```
Int[(d.*x_)^m_.*Pq_*(a+_b_.*x_^2+c_.*x_^4)^p_,x_Symbol] :=
  1/d^2*Int[(d*x)^(m+2)*ExpandToSum[Pq/x^2,x]*(a+b*x^2+c*x^4)^p,x] /;
FreeQ[{a,b,c,d,m,p},x] && PolyQ[Pq,x^2] && EqQ[Coeff[Pq,x,0],0]
```

5: $\int (dx)^m (e+fx^2+gx^4) (a+bx^2+cx^4)^p dx$ when $a f(m+1) - b e(m+2p+3) == 0 \wedge a g(m+1) - c e(m+4p+5) == 0 \wedge m \neq -1$

Rule 1.2.2.6.5: If $a f(m+1) - b e(m+2p+3) == 0 \wedge a g(m+1) - c e(m+4p+5) == 0 \wedge m \neq -1$, then

$$\int (dx)^m (e+fx^2+gx^4) (a+bx^2+cx^4)^p dx \rightarrow \frac{e(dx)^{m+1} (a+bx^2+cx^4)^{p+1}}{a d(m+1)}$$

Program code:

```
Int[(d.*x_)^m_.*Pq_*(a+_b_.*x_^2+c_.*x_^4)^p_,x_Symbol] :=
  With[{e=Coeff[Pq,x,0],f=Coeff[Pq,x,2],g=Coeff[Pq,x,4]},
    e*(d*x)^(m+1)*(a+b*x^2+c*x^4)^(p+1)/(a*d*(m+1)) /;
    EqQ[a*f*(m+1)-b*e*(m+2*p+3),0] && EqQ[a*g*(m+1)-c*e*(m+4*p+5),0] && NeQ[m,-1] /;
    FreeQ[{a,b,c,d,m,p},x] && PolyQ[Pq,x^2] && EqQ[Expon[Pq,x],4]
```

6: $\int (dx)^m P_q[x^2] (a+bx^2+cx^4)^p dx$ when $q > 1 \wedge b^2 - 4ac = 0$

Derivation: Piecewise constant extraction

Basis: If $b^2 - 4ac = 0$, then $\partial_x \frac{(a+bx^2+cx^4)^p}{(b+2cx^2)^{2p}} = 0$

Rule 1.2.2.6.7: If $q > 1 \wedge b^2 - 4ac = 0$, then

$$\int (dx)^m P_q[x^2] (a+bx^2+cx^4)^p dx \rightarrow \frac{(a+bx^2+cx^4)^{\text{FracPart}[p]}}{(4c)^{\text{IntPart}[p]} (b+2cx^2)^{2\text{FracPart}[p]}} \int (dx)^m P_q[x^2] (b+2cx^2)^{2p} dx$$

Program code:

```
Int[(d_.x_)^m_.Pq_*(a+b_.x_^2+c_.x_^4)^p_,x_Symbol] :=
  (a+b*x^2+c*x^4)^FracPart[p]/((4*c)^IntPart[p]*(b+2*c*x^2)^(2*FracPart[p]))*Int[(d*x)^m*.Pq*(b+2*c*x^2)^(2*p),x] /;
  FreeQ[{a,b,c,d,m,p},x] && PolyQ[Pq,x^2] && GtQ[Expon[Pq,x^2],1] && EqQ[b^2-4*a*c,0]
```

7. $\int x^m P_q[x^2] (a+bx^2+cx^4)^p dx$ when $q > 1 \wedge b^2 - 4ac \neq 0 \wedge p < -1 \wedge \frac{m}{2} \in \mathbb{Z}$

1: $\int x^m P_q[x^2] (a+bx^2+cx^4)^p dx$ when $q > 1 \wedge b^2 - 4ac \neq 0 \wedge p < -1 \wedge \frac{m}{2} \in \mathbb{Z}^+$

Derivation: Algebraic expansion and trinomial recurrence 2b

Rule 1.2.2.6.8.1: If $q > 1 \wedge b^2 - 4ac \neq 0 \wedge p < -1 \wedge \frac{m}{2} \in \mathbb{Z}^+$, let

$Q \rightarrow \text{PolynomialQuotient}[x^m P_q[x^2], a+bx^2+cx^4, x]$ and $d+ex^2 \rightarrow \text{PolynomialRemainder}[x^m P_q[x^2], a+bx^2+cx^4, x]$, then

$$\int x^m P_q[x^2] (a+bx^2+cx^4)^p dx \rightarrow$$

$$\int (d+ex^2) (a+bx^2+cx^4)^p dx + \int Q (a+bx^2+cx^4)^{p+1} dx \rightarrow$$

$$\frac{x (a+b x^2+c x^4)^{p+1} (a b e-d (b^2-2 a c)-c (b d-2 a e) x^2)}{2 a (p+1) (b^2-4 a c)} +$$

$$\frac{1}{2 a (p+1) (b^2-4 a c)} \int (a+b x^2+c x^4)^{p+1} .$$

$$(2 a (p+1) (b^2-4 a c) Q + b^2 d (2 p+3) - 2 a c d (4 p+5) - a b e + c (4 p+7) (b d-2 a e) x^2) dx$$

Program code:

```
Int[x^m_*Pq_*(a_+b_.*x_^2+c_.*x_^4)^p_,x_Symbol] :=
  With[{d=Coeff[PolynomialRemainder[x^m*Pq,a+b*x^2+c*x^4,x],x,0],
    e=Coeff[PolynomialRemainder[x^m*Pq,a+b*x^2+c*x^4,x],x,2]},
  x*(a+b*x^2+c*x^4)^(p+1)*(a*b*e-d*(b^2-2*a*c)-c*(b*d-2*a*e)*x^2)/(2*a*(p+1)*(b^2-4*a*c)) +
  1/(2*a*(p+1)*(b^2-4*a*c))*Int[(a+b*x^2+c*x^4)^(p+1)*
  ExpandToSum[2*a*(p+1)*(b^2-4*a*c)*PolynomialQuotient[x^m*Pq,a+b*x^2+c*x^4,x]+
  b^2*d*(2*p+3)-2*a*c*d*(4*p+5)-a*b*e+c*(4*p+7)*(b*d-2*a*e)*x^2,x],x] /;
FreeQ[{a,b,c},x] && PolyQ[Pq,x^2] && GtQ[Expon[Pq,x^2],1] && NeQ[b^2-4*a*c,0] && LtQ[p,-1] && IGtQ[m/2,0]
```

2: $\int x^m P_q[x^2] (a+b x^2+c x^4)^p dx$ when $q > 1 \wedge b^2-4 a c \neq 0 \wedge p < -1 \wedge \frac{m}{2} \in \mathbb{Z}^-$

Derivation: Algebraic expansion and trinomial recurrence 2b

Rule 1.2.2.6.8.2: If $q > 1 \wedge b^2-4 a c \neq 0 \wedge p < -1 \wedge \frac{m}{2} \in \mathbb{Z}^-$, let

$Q \rightarrow \text{PolynomialQuotient}[x^m P_q[x^2], a+b x^2+c x^4, x]$ and $d+e x^2 \rightarrow \text{PolynomialRemainder}[x^m P_q[x^2], a+b x^2+c x^4, x]$, then

$$\int x^m P_q[x^2] (a+b x^2+c x^4)^p dx \rightarrow$$

$$\int (d+e x^2) (a+b x^2+c x^4)^p dx + \int Q (a+b x^2+c x^4)^{p+1} dx \rightarrow$$

$$\frac{x (a+b x^2+c x^4)^{p+1} (a b e-d (b^2-2 a c)-c (b d-2 a e) x^2)}{2 a (p+1) (b^2-4 a c)} +$$

$$\frac{1}{2 a (p+1) (b^2-4 a c)} \int x^m (a+b x^2+c x^4)^{p+1} .$$

$$(2a(p+1)(b^2-4ac)x^{-m}Q + (b^2d(2p+3) - 2acd(4p+5) - abe)x^{-m} + c(4p+7)(bd-2ae)x^{2-m}) dx$$

Program code:

```
Int[x_^m_*Pq_*(a_+b_.*x_^2+c_.*x_^4)^p_,x_Symbol] :=
  With[{d=Coeff[PolynomialRemainder[x^m*Pq,a+b*x^2+c*x^4,x],x,0],
    e=Coeff[PolynomialRemainder[x^m*Pq,a+b*x^2+c*x^4,x],x,2]},
    x*(a+b*x^2+c*x^4)^(p+1)*(a*b*e-d*(b^2-2*a*c)-c*(b*d-2*a*e)*x^2)/(2*a*(p+1)*(b^2-4*a*c)) +
    1/(2*a*(p+1)*(b^2-4*a*c))*Int[x^m*(a+b*x^2+c*x^4)^(p+1)*
      ExpandToSum[2*a*(p+1)*(b^2-4*a*c)*x^(-m)*PolynomialQuotient[x^m*Pq,a+b*x^2+c*x^4,x]+
        (b^2*d*(2*p+3)-2*a*c*d*(4*p+5)-a*b*e)*x^(-m)+c*(4*p+7)*(b*d-2*a*e)*x^(2-m),x],x] /;
  FreeQ[{a,b,c},x] && PolyQ[Pq,x^2] && GtQ[Expon[Pq,x^2],1] && NeQ[b^2-4*a*c,0] && LtQ[p,-1] && ILtQ[m/2,0]
```

x: $\int x^m P_q[x^2] (a+bx^2+cx^4)^p dx$ when $q > 1 \wedge b^2 - 4ac \neq 0 \wedge p < -1 \wedge \frac{m-1}{2} \in \mathbb{Z}$

Derivation: Algebraic expansion and trinomial recurrence 2b

Note: Better to use the substitution $x \rightarrow x^2$.

Rule 1.2.2.6.8.2: If $q > 1 \wedge b^2 - 4ac \neq 0 \wedge p < -1 \wedge \frac{m-1}{2} \in \mathbb{Z}$, let

$Q \rightarrow \text{PolynomialQuotient}[x^m P_q[x^2], a+bx^2+cx^4, x]$ and $dx+ex^3 \rightarrow \text{PolynomialRemainder}[x^m P_q[x^2], a+bx^2+cx^4, x]$, then

$$\begin{aligned} & \int x^m P_q[x^2] (a+bx^2+cx^4)^p dx \rightarrow \\ & \int (dx+ex^3) (a+bx^2+cx^4)^p dx + \int Q (a+bx^2+cx^4)^{p+1} dx \rightarrow \\ & \frac{x^2 (a+bx^2+cx^4)^{p+1} (abe-d(b^2-2ac) - c(bd-2ae)x^2)}{2a(p+1)(b^2-4ac)} + \\ & \frac{1}{a(p+1)(b^2-4ac)} \int x^m (a+bx^2+cx^4)^{p+1} dx \\ & (a(p+1)(b^2-4ac) x^{-m} Q + (b^2 d(p+2) - 2acd(2p+3) - abe) x^{1-m} + 2c(p+2)(bd-2ae) x^{3-m}) dx \end{aligned}$$

Program code:

```
(* Int[x_^m_.*Pq_*(a_+b_.*x_^2+c_.*x_^4)^p_,x_Symbol] :=
  With[{d=Coeff[PolynomialRemainder[x^m*Pq,a+b*x^2+c*x^4,x],x,1],
    e=Coeff[PolynomialRemainder[x^m*Pq,a+b*x^2+c*x^4,x],x,3]},
  x^2*(a+b*x^2+c*x^4)^(p+1)*(a*b*e-d*(b^2-2*a*c)-c*(b*d-2*a*e)*x^2)/(2*a*(p+1)*(b^2-4*a*c)) +
  1/(a*(p+1)*(b^2-4*a*c))*Int[x^m*(a+b*x^2+c*x^4)^(p+1)*
  ExpandToSum[a*(p+1)*(b^2-4*a*c)*x^(-m)*PolynomialQuotient[x^m*Pq,a+b*x^2+c*x^4,x]+
  (b^2*d*(p+2)-2*a*c*d*(2*p+3)-a*b*e)*x^(1-m)+2*c*(p+2)*(b*d-2*a*e)*x^(3-m),x],x] /;
FreeQ[{a,b,c},x] && PolyQ[Pq,x^2] && GtQ[Expon[Pq,x^2],1] && NeQ[b^2-4*a*c,0] && LtQ[p,-1] && IntegerQ[(m-1)/2] *)
```

U: $\int (dx)^m P_q[x] (a+bx^2+cx^4)^p dx$

Rule 1.2.2.6.U:

$$\int (dx)^m P_q[x] (a+bx^2+cx^4)^p dx \rightarrow \int (dx)^m P_q[x] (a+bx^2+cx^4)^p dx$$

Program code:

```
Int[(d.*x_)^m_.*Pq*(a+b.*x_^2+c.*x_^4)^p_,x_Symbol] :=
  Unintegrable[(d*x)^m*Pq*(a+b*x^2+c*x^4)^p,x] /;
  FreeQ[{a,b,c,d,m,p},x] && PolyQ[Pq,x]
```