

Rules for integrands of the form $(g \cos[e + f x])^p (a + b \sin[e + f x])^m (c + d \sin[e + f x])^n$

1. $\int \cos[e + f x]^p (a + b \sin[e + f x])^m (c + d \sin[e + f x])^n dx$ when $\frac{p-1}{2} \in \mathbb{Z}$

1: $\int \cos[e + f x] (a + b \sin[e + f x])^m (c + d \sin[e + f x])^n dx$

Derivation: Integration by substitution

Basis: $\cos[e + f x] F[\sin[e + f x]] = \frac{1}{b f} \text{Subst}[F[\frac{x}{b}], x, b \sin[e + f x]] \partial_x (b \sin[e + f x])$

Rule:

$$\int \cos[e + f x] (a + b \sin[e + f x])^m (c + d \sin[e + f x])^n dx \rightarrow \frac{1}{b f} \text{Subst}\left[\int (a + x)^m \left(c + \frac{d}{b} x\right)^n dx, x, b \sin[e + f x]\right]$$

Program code:

```
Int[cos[e_+f_.*x_]*(a_+b_.*sin[e_+f_.*x_])^m_.*(c_+d_.*sin[e_+f_.*x_])^n_,x_Symbol] :=
  1/(b*f)*Subst[Int[(a+x)^m*(c+d/b*x)^n,x],x,b*Sin[e+f*x]] /;
FreeQ[{a,b,c,d,e,f,m,n},x]
```

2: $\int \cos[e+fx]^p (d \sin[e+fx])^n (a+b \sin[e+fx]) dx$ when $\frac{p-1}{2} \in \mathbb{Z} \wedge n \in \mathbb{Z} \wedge (p < 0 \wedge a^2 - b^2 \neq 0 \vee 0 < n < p-1 \vee p+1 < -n < 2p+1)$

Derivation: Algebraic expansion

Rule: If $\frac{p-1}{2} \in \mathbb{Z} \wedge n \in \mathbb{Z} \wedge (p < 0 \wedge a^2 - b^2 \neq 0 \vee 0 < n < p-1 \vee p+1 < -n < 2p+1)$, then

$$\int \cos[e+fx]^p (d \sin[e+fx])^n (a+b \sin[e+fx]) dx \rightarrow a \int \cos[e+fx]^p (d \sin[e+fx])^n dx + \frac{b}{d} \int \cos[e+fx]^p (d \sin[e+fx])^{n+1} dx$$

Program code:

```
Int[cos[e_.+f_.*x_]^p_*(d_.*sin[e_.+f_.*x_]^n_*(a_+b_.*sin[e_.+f_.*x_]),x_Symbol] :=
  a*Int[Cos[e+f*x]^p*(d*sin[e+f*x])^n,x] + b/d*Int[Cos[e+f*x]^p*(d*sin[e+f*x])^(n+1),x] /;
FreeQ[{a,b,d,e,f,n,p},x] && IntegerQ[(p-1)/2] && IntegerQ[n] && (LtQ[p,0] && NeQ[a^2-b^2,0] || LtQ[0,n,p-1] || LtQ[p+1,-n,2*p+1])
```

3. $\int \cos[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $\frac{p-1}{2} \in \mathbb{Z} \wedge a^2 - b^2 = 0$

1: $\int \frac{\cos[e+fx]^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx$ when $\frac{p-1}{2} \in \mathbb{Z} \wedge a^2 - b^2 = 0 \wedge n \in \mathbb{Z} \wedge (0 < n < \frac{p+1}{2} \vee p \leq -n < 2p-3 \vee 0 < n \leq -p)$

Derivation: Algebraic expansion

Basis: If $a^2 - b^2 = 0$, then $\frac{\cos[z]^2}{a+b \sin[z]} = \frac{1}{a} - \frac{d \sin[z]}{b d}$

Rule: If $\frac{p-1}{2} \in \mathbb{Z} \wedge a^2 - b^2 = 0 \wedge n \in \mathbb{Z} \wedge (0 < n < \frac{p+1}{2} \vee p \leq -n < 2p-3 \vee 0 < n \leq -p)$, then

$$\int \frac{\cos[e+fx]^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \rightarrow \frac{1}{a} \int \cos[e+fx]^{p-2} (d \sin[e+fx])^n dx - \frac{1}{b d} \int \cos[e+fx]^{p-2} (d \sin[e+fx])^{n+1} dx$$

Program code:

```
Int[cos[e_.+f_.*x_]^p_*(d_.*sin[e_.+f_.*x_]^n_./(a_+b_.*sin[e_.+f_.*x_] ),x_Symbol] :=
  1/a*Int[Cos[e+f*x]^(p-2)*(d*sin[e+f*x])^n,x] -
  1/(b*d)*Int[Cos[e+f*x]^(p-2)*(d*sin[e+f*x])^(n+1),x] /;
FreeQ[{a,b,d,e,f,n,p},x] && IntegerQ[(p-1)/2] && EqQ[a^2-b^2,0] && IntegerQ[n] && (LtQ[0,n,(p+1)/2] || LeQ[p,-n] && LtQ[-n,2*p-3] ||
```

2: $\int \cos[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $\frac{p-1}{2} \in \mathbb{Z} \wedge a^2 - b^2 = 0$

Derivation: Integration by substitution

Basis: If $\frac{p-1}{2} \in \mathbb{Z} \wedge a^2 - b^2 = 0$, then

$$\cos[e+fx]^p (a+b \sin[e+fx])^m = \frac{1}{b^p f} \text{Subst}\left[(a+x)^{m+\frac{p-1}{2}} (a-x)^{\frac{p-1}{2}}, x, b \sin[e+fx]\right] \partial_x (b \sin[e+fx])$$

Rule: If $\frac{p-1}{2} \in \mathbb{Z} \wedge a^2 - b^2 = 0$, then

$$\int \cos[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow \frac{1}{b^p f} \text{Subst}\left[\int (a+x)^{m+\frac{p-1}{2}} (a-x)^{\frac{p-1}{2}} \left(c + \frac{d}{b} x\right)^n dx, x, b \sin[e+fx]\right]$$

Program code:

```
Int[cos[e_.+f_.*x_]^p_*(a_+b_.*sin[e_.+f_.*x_]^m_.*(c_.+d_.*sin[e_.+f_.*x_]^n_.,x_Symbol] :=
  1/(b^p*f)*Subst[Int[(a+x)^(m+(p-1)/2)*(a-x)^((p-1)/2)*(c+d/b*x)^n,x],x,b*Sin[e+f*x]] /;
FreeQ[{a,b,e,f,c,d,m,n},x] && IntegerQ[(p-1)/2] && EqQ[a^2-b^2,0]
```

4: $\int \cos[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $\frac{p-1}{2} \in \mathbb{Z} \wedge a^2 - b^2 \neq 0$

Derivation: Integration by substitution

Basis: If $\frac{p-1}{2} \in \mathbb{Z}$, then $\cos[e+fx]^p = \frac{1}{b^p f} \text{Subst}[(b^2 - x^2)^{\frac{p-1}{2}}, x, b \sin[e+fx]] \partial_x (b \sin[e+fx])$

Rule: If $\frac{p-1}{2} \in \mathbb{Z} \wedge a^2 - b^2 \neq 0$, then

$$\int \cos[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow \frac{1}{b^p f} \text{Subst}\left[\int (a+x)^m \left(c + \frac{d}{b} x\right)^n (b^2 - x^2)^{\frac{p-1}{2}} dx, x, b \sin[e+fx]\right]$$

Program code:

```
Int[cos[e_.+f_.*x_]^p_*(a_+b_.*sin[e_.+f_.*x_]^m_*(c_+d_.*sin[e_.+f_.*x_]^n_,x_Symbol] :=
  1/(b^p*f)*Subst[Int[(a+x)^m*(c+d/b*x)^n*(b^2-x^2)^((p-1)/2),x],x,b*Sin[e+f*x]] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && IntegerQ[(p-1)/2] && NeQ[a^2-b^2,0]
```

2: $\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx]) dx$

Derivation: Algebraic expansion

Rule:

$$\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx]) dx \rightarrow a \int (g \cos[e+fx])^p (d \sin[e+fx])^n dx + \frac{b}{d} \int (g \cos[e+fx])^p (d \sin[e+fx])^{n+1} dx$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_]^p_*(d_.*sin[e_.+f_.*x_]^n_*(a_+b_.*sin[e_.+f_.*x_]^1),x_Symbol] :=
  a*Int[(g*Cos[e+f*x])^p*(d*Sin[e+f*x])^n,x] + b/d*Int[(g*Cos[e+f*x])^p*(d*Sin[e+f*x])^(n+1),x] /;
FreeQ[{a,b,d,e,f,g,n,p},x]
```

3:
$$\int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \text{ when } a^2 - b^2 = 0$$

Derivation: Algebraic expansion

Basis: If $a^2 - b^2 = 0$, then $\frac{\cos[z]^2}{a+b \sin[z]} = \frac{1}{a} - \frac{d \sin[z]}{b d}$

Rule: If $a^2 - b^2 = 0$, then

$$\int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \rightarrow \frac{g^2}{a} \int (g \cos[e+fx])^{p-2} (d \sin[e+fx])^n dx - \frac{g^2}{b d} \int (g \cos[e+fx])^{p-2} (d \sin[e+fx])^{n+1} dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(d_.sin[e_.+f_.x_])^n_/(a_+b_.sin[e_.+f_.x_]),x_Symbol] :=
  g^2/a*Int[(g*cos[e+f*x])^(p-2)*(d*sin[e+f*x])^n,x] -
  g^2/(b*d)*Int[(g*cos[e+f*x])^(p-2)*(d*sin[e+f*x])^(n+1),x] /;
FreeQ[{a,b,d,e,f,g,n,p},x] && EqQ[a^2-b^2,0]
```

$$4. \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \text{ when } bc+ad=0 \wedge a^2-b^2=0$$

$$1: \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \text{ when } bc+ad=0 \wedge a^2-b^2=0 \wedge m \in \mathbb{Z}$$

Derivation: Algebraic simplification

Basis: If $bc+ad=0 \wedge a^2-b^2=0$, then $(a+b \sin[z])(c+d \sin[z])=ac \cos[z]^2$

Rule: If $bc+ad=0 \wedge a^2-b^2=0 \wedge m \in \mathbb{Z}$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow \frac{a^m c^m}{g^{2m}} \int (g \cos[e+fx])^{2m+p} (c+d \sin[e+fx])^{n-m} dx$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_])^p_*(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  a^m*c^m/g^(2*m)*Int[(g*Cos[e+f*x])^(2*m+p)*(c+d*Sin[e+f*x])^(n-m),x] /;
FreeQ[{a,b,c,d,e,f,g,n,p},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && IntegerQ[m] && Not[IntegerQ[n]] && LtQ[n^2,m^2]
```

2: $\int \cos[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $b c + a d = 0 \wedge a^2 - b^2 = 0 \wedge \frac{p}{2} \in \mathbb{Z}$

Derivation: Algebraic simplification

Basis: If $b c + a d = 0 \wedge a^2 - b^2 = 0$, then $\cos[z]^2 = \frac{1}{a c} (a + b \sin[z]) (c + d \sin[z])$

Rule: If $b c + a d = 0 \wedge a^2 - b^2 = 0 \wedge \frac{p}{2} \in \mathbb{Z}$, then

$$\int \cos[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow \frac{1}{a^{p/2} c^{p/2}} \int (a+b \sin[e+fx])^{m+\frac{p}{2}} (c+d \sin[e+fx])^{n+\frac{p}{2}} dx$$

Program code:

```
Int[cos[e_+f_.*x_]^p_*(a_+b_.*sin[e_+f_.*x_]^m_.*(c_+d_.*sin[e_+f_.*x_]^n_,x_Symbol] :=
  1/(a^(p/2)*c^(p/2))*Int[(a+b*sin[e+f*x])^(m+p/2)*(c+d*sin[e+f*x])^(n+p/2),x] /;
FreeQ[{a,b,c,d,e,f,n,p},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && IntegerQ[p/2]
```


3: $\int \frac{(g \cos[e+fx])^p}{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}} dx$ when $bc+ad == 0 \wedge a^2 - b^2 == 0$

Derivation: Piecewise constant extraction

Basis: If $bc+ad == 0 \wedge a^2 - b^2 == 0$, then $\partial_x \frac{\cos[e+fx]}{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}} == 0$

Rule: If $bc+ad == 0 \wedge a^2 - b^2 == 0$, then

$$\int \frac{(g \cos[e+fx])^p}{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}} dx \rightarrow \frac{g \cos[e+fx]}{\sqrt{a+b \sin[e+fx]} \sqrt{c+d \sin[e+fx]}} \int (g \cos[e+fx])^{p-1} dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_/(Sqrt[a_+b_.sin[e_.+f_.x_])*Sqrt[c_+d_.sin[e_.+f_.x_]]),x_Symbol] :=
  g*Cos[e+f*x]/(Sqrt[a+b*sin[e+f*x])*Sqrt[c+d*sin[e+f*x]])*Int[(g*Cos[e+f*x])^(p-1),x] /;
FreeQ[{a,b,c,d,e,f,g,p},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0]
```

4. $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $bc+ad == 0 \wedge a^2 - b^2 == 0 \wedge m + \frac{p}{2} + \frac{1}{2} \in \mathbb{Z}^+$

1. $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $bc+ad == 0 \wedge a^2 - b^2 == 0 \wedge m + \frac{p}{2} - \frac{1}{2} == 0$

1: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $bc+ad == 0 \wedge a^2 - b^2 == 0 \wedge 2m+p-1 == 0 \wedge m-n-1 == 0$

Derivation: Piecewise constant extraction

Basis: If $bc+ad == 0 \wedge a^2 - b^2 == 0$, then $\partial_x \frac{(a+b \sin[e+fx])^m (c+d \sin[e+fx])^m}{(g \cos[e+fx])^{2m}} == 0$

Rule: If $bc+ad == 0 \wedge a^2 - b^2 == 0 \wedge 2m+p-1 == 0 \wedge m-n-1 == 0$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow$$

$$\left((a^{\text{IntPart}[m]} c^{\text{IntPart}[m]} (a+b \sin[e+fx])^{\text{FracPart}[m]} (c+d \sin[e+fx])^{\text{FracPart}[m]} / (g^{2 \text{IntPart}[m]} (g \cos[e+fx])^{2 \text{FracPart}[m]}) \right) \int \frac{(g \cos[e+fx])^{2m+p}}{c+d \sin[e+fx]} dx$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_])^p_*(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  a^IntPart[m]*c^IntPart[m]*(a+b*sin[e+f*x])^FracPart[m]*(c+d*sin[e+f*x])^FracPart[m]/
  (g^(2*IntPart[m])*(g*cos[e+f*x])^(2*FracPart[m]))*Int[(g*cos[e+f*x])^(2*m+p)/(c+d*sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && EqQ[2*m+p-1,0] && EqQ[m-n-1,0]
```

2: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $b c + a d == 0 \wedge a^2 - b^2 == 0 \wedge 2 m + p - 1 == 0 \wedge m - n - 1 \neq 0$

Derivation: Doubly degenerate sine recurrence 1a

Rule: If $b c + a d == 0 \wedge a^2 - b^2 == 0 \wedge 2 m + p - 1 == 0 \wedge m - n - 1 \neq 0$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow \frac{b (g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^{m-1} (c+d \sin[e+fx])^n}{f g (m-n-1)}$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_])^p_*(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  b*(g*cos[e+f*x])^(p+1)*(a+b*sin[e+f*x])^(m-1)*(c+d*sin[e+f*x])^n/(f*g*(m-n-1)) /;
FreeQ[{a,b,c,d,e,f,g,m,n,p},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && EqQ[2*m+p-1,0] && NeQ[m-n-1,0]
```

$$2. \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \text{ when } bc+ad=0 \wedge a^2-b^2=0 \wedge m+\frac{p}{2}-\frac{1}{2} \in \mathbb{Z}^+$$

$$1: \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \text{ when } bc+ad=0 \wedge a^2-b^2=0 \wedge m+\frac{p}{2}-\frac{1}{2} \in \mathbb{Z}^+ \wedge n < -1$$

Derivation: Doubly degenerate sine recurrence 1a

Rule: If $bc+ad=0 \wedge a^2-b^2=0 \wedge m+\frac{p}{2}-\frac{1}{2} \in \mathbb{Z}^+ \wedge n < -1$, then

$$\begin{aligned} & \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow \\ & - \frac{2b (g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^{m-1} (c+d \sin[e+fx])^n}{fg(2n+p+1)} - \\ & \frac{b(2m+p-1)}{d(2n+p+1)} \int (g \cos[e+fx])^p (a+b \sin[e+fx])^{m-1} (c+d \sin[e+fx])^{n+1} dx \end{aligned}$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a+b_.sin[e_.+f_.x_])^m_*(c+d_.sin[e_.+f_.x_])^n_,x_Symbol] :=
-2*b*(g*cos[e+f*x])^(p+1)*(a+b*sin[e+f*x])^(m-1)*(c+d*sin[e+f*x])^n/(f*g*(2*n+p+1)) -
b*(2*m+p-1)/(d*(2*n+p+1))*Int[(g*cos[e+f*x])^p*(a+b*sin[e+f*x])^(m-1)*(c+d*sin[e+f*x])^(n+1),x] /;
FreeQ[{a,b,c,d,e,f,g,p},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && IGtQ[Simplify[m+p/2-1/2],0] && LtQ[n,-1] &&
NeQ[2*n+p+1,0] && Not[ILtQ[Simplify[m+n+p],0] && GtQ[Simplify[2*m+n+3*p/2+1],0]]
```

$$2: \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \text{ when } bc+ad=0 \wedge a^2-b^2=0 \wedge m+\frac{p}{2}-\frac{1}{2} \in \mathbb{Z}^+ \wedge n \not< -1$$

Derivation: Doubly degenerate sine recurrence 1b

Rule: If $bc+ad=0 \wedge a^2-b^2=0 \wedge m+\frac{p}{2}-\frac{1}{2} \in \mathbb{Z}^+ \wedge n \not< -1$, then

$$\begin{aligned} & \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow \\ & - \frac{b (g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^{m-1} (c+d \sin[e+fx])^n}{fg(m+n+p)} + \end{aligned}$$

$$\frac{a(2m+p-1)}{m+n+p} \int (g \cos[e+fx])^p (a+b \sin[e+fx])^{m-1} (c+d \sin[e+fx])^n dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_+d_.sin[e_.+f_.x_])^n_,x_Symbol] :=
  -b*(g*cos[e+f*x])^(p+1)*(a+b*sin[e+f*x])^(m-1)*(c+d*sin[e+f*x])^n/(f*g*(m+n+p)) +
  a*(2*m+p-1)/(m+n+p)*Int[(g*cos[e+f*x])^p*(a+b*sin[e+f*x])^(m-1)*(c+d*sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,g,n,p},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && IGtQ[Simplify[m+p/2-1/2],0] && Not[LtQ[n,-1]] &&
Not[IGtQ[Simplify[n+p/2-1/2],0] && GtQ[m-n,0]] && Not[ILtQ[Simplify[m+n+p],0] && GtQ[Simplify[2*m+n+3*p/2+1],0]]
```

5. $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $bc+ad=0 \wedge a^2-b^2=0 \wedge m+n+p \in \mathbb{Z}^-$

1. $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $bc+ad=0 \wedge a^2-b^2=0 \wedge m+n+p+1=0$

1: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $bc+ad=0 \wedge a^2-b^2=0 \wedge m+n+p+1=0 \wedge m-n=0$

Derivation: Piecewise constant extraction

Basis: If $bc+ad=0 \wedge a^2-b^2=0$, then $\partial_x \frac{(a+b \sin[e+fx])^m (c+d \sin[e+fx])^m}{(g \cos[e+fx])^{2m}} = 0$

Rule: If $bc+ad=0 \wedge a^2-b^2=0 \wedge 2m+p+1=0$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^m dx \rightarrow$$

$$\left((a^{\text{IntPart}[m]} c^{\text{IntPart}[m]} (a+b \sin[e+fx])^{\text{FracPart}[m]} (c+d \sin[e+fx])^{\text{FracPart}[m]}) / (g^{2 \text{IntPart}[m]} (g \cos[e+fx])^{2 \text{FracPart}[m]}) \right) \int (g \cos[e+fx])^{2m+p} dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_+d_.sin[e_.+f_.x_])^m_,x_Symbol] :=
  a^IntPart[m]*c^IntPart[m]*(a+b*sin[e+f*x])^FracPart[m]*(c+d*sin[e+f*x])^FracPart[m]/
  (g^(2*IntPart[m])*(g*cos[e+f*x])^(2*FracPart[m]))*Int[(g*cos[e+f*x])^(2*m+p),x] /;
FreeQ[{a,b,c,d,e,f,g,m,p},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && EqQ[2*m+p+1,0]
```

2: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $bc+ad=0 \wedge a^2-b^2=0 \wedge m+n+p+1=0 \wedge m-n \neq 0$

Derivation: Doubly degenerate sine recurrence 1c with $n \rightarrow -m-p-1$

Rule: If $bc+ad=0 \wedge a^2-b^2=0 \wedge m+n+p+1=0 \wedge m-n \neq 0$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow \frac{b (g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n}{a f g (m-n)}$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_+d_.sin[e_.+f_.x_])^n_,x_Symbol] :=
  b*(g*cos[e+f*x])^(p+1)*(a+b*sin[e+f*x])^m*(c+d*sin[e+f*x])^n/(a*f*g*(m-n)) /;
FreeQ[{a,b,c,d,e,f,g,m,n,p},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && EqQ[m+n+p+1,0] && NeQ[m,n]
```

$$\mathbf{2:} \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \text{ when } bc+ad=0 \wedge a^2-b^2=0 \wedge m+n+p+1 \in \mathbb{Z}^- \wedge 2m+p+1 \neq 0$$

Derivation: Doubly degenerate sine recurrence 1c

Rule: If $bc+ad=0 \wedge a^2-b^2=0 \wedge m+n+p+1 \in \mathbb{Z}^- \wedge 2m+p+1 \neq 0$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow \frac{b (g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n}{afg(2m+p+1)} + \frac{m+n+p+1}{a(2m+p+1)} \int (g \cos[e+fx])^p (a+b \sin[e+fx])^{m+1} (c+d \sin[e+fx])^n dx$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_])^p_*(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  b*(g*cos[e+f*x])^(p+1)*(a+b*sin[e+f*x])^m*(c+d*sin[e+f*x])^n/(a*f*g*(2*m+p+1)) +
  (m+n+p+1)/(a*(2*m+p+1))*Int[(g*cos[e+f*x])^p*(a+b*sin[e+f*x])^(m+1)*(c+d*sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && ILtQ[Simplify[m+n+p+1],0] && NeQ[2*m+p+1,0] &&
(SumSimplerQ[m,1] || Not[SumSimplerQ[n,1]])
```

$$6. \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \text{ when } bc+ad=0 \wedge a^2-b^2=0 \wedge (2m|2n|2p) \in \mathbb{Z}$$

$$1. \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \text{ when } bc+ad=0 \wedge a^2-b^2=0 \wedge m > 0$$

$$\mathbf{1:} \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \text{ when } bc+ad=0 \wedge a^2-b^2=0 \wedge m > 0 \wedge n < -1 \wedge 2n+p+1 \neq 0$$

Derivation: Doubly degenerate sine recurrence 1a

Rule: If $bc+ad=0 \wedge a^2-b^2=0 \wedge m > 0 \wedge n < -1 \wedge 2n+p+1 \neq 0$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow - \frac{2b (g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^{m-1} (c+d \sin[e+fx])^n}{fg(2n+p+1)}$$

$$\frac{b(2m+p-1)}{d(2n+p+1)} \int (g \cos[e+fx])^p (a+b \sin[e+fx])^{m-1} (c+d \sin[e+fx])^{n+1} dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_+d_.sin[e_.+f_.x_])^n_,x_Symbol] :=
-2*b*(g*cos[e+f*x])^(p+1)*(a+b*sin[e+f*x])^(m-1)*(c+d*sin[e+f*x])^n/(f*g*(2*n+p+1)) -
b*(2*m+p-1)/(d*(2*n+p+1))*Int[(g*cos[e+f*x])^p*(a+b*sin[e+f*x])^(m-1)*(c+d*sin[e+f*x])^(n+1),x] /;
FreeQ[{a,b,c,d,e,f,g,p},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && GtQ[m,0] && LtQ[n,-1] && NeQ[2*n+p+1,0] && IntegersQ[2*m,2*n,2*p]
```

2: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $bc+ad=0 \wedge a^2-b^2=0 \wedge m>0 \wedge m+n+p \neq 0$

Derivation: Doubly degenerate sine recurrence 1b

Rule: If $bc+ad=0 \wedge a^2-b^2=0 \wedge m>0 \wedge m+n+p \neq 0$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow$$

$$-\frac{b(g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^{m-1} (c+d \sin[e+fx])^n}{f g (m+n+p)} + \frac{a(2m+p-1)}{m+n+p} \int (g \cos[e+fx])^p (a+b \sin[e+fx])^{m-1} (c+d \sin[e+fx])^n dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_+d_.sin[e_.+f_.x_])^n_,x_Symbol] :=
-b*(g*cos[e+f*x])^(p+1)*(a+b*sin[e+f*x])^(m-1)*(c+d*sin[e+f*x])^n/(f*g*(m+n+p)) +
a*(2*m+p-1)/(m+n+p)*Int[(g*cos[e+f*x])^p*(a+b*sin[e+f*x])^(m-1)*(c+d*sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,g,n,p},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && GtQ[m,0] && NeQ[m+n+p,0] && Not[LtQ[0,n,m]] && IntegersQ[2*m,2*n]
```

$$\mathbf{2:} \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \text{ when } bc+ad=0 \wedge a^2-b^2=0 \wedge m < -1 \wedge 2m+p+1 \neq 0$$

Derivation: Doubly degenerate sine recurrence 1c

Rule: If $bc+ad=0 \wedge a^2-b^2=0 \wedge m < -1 \wedge 2m+p+1 \neq 0$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow \frac{b (g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n}{a f g (2m+p+1)} + \frac{m+n+p+1}{a (2m+p+1)} \int (g \cos[e+fx])^p (a+b \sin[e+fx])^{m+1} (c+d \sin[e+fx])^n dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_+d_.sin[e_.+f_.x_])^n_,x_Symbol] :=
  b*(g*cos[e+f*x])^(p+1)*(a+b*sin[e+f*x])^m*(c+d*sin[e+f*x])^n/(a*f*g*(2*m+p+1)) +
  (m+n+p+1)/(a*(2*m+p+1))*Int[(g*cos[e+f*x])^p*(a+b*sin[e+f*x])^(m+1)*(c+d*sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,g,n,p},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && LtQ[m,-1] && NeQ[2*m+p+1,0] && Not[LtQ[m,n,-1]] &&
IntegersQ[2*m,2*n,2*p]
```


7: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $bc+ad=0 \wedge a^2-b^2=0 \wedge m \notin \mathbb{Z} \wedge n \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: If $bc+ad=0 \wedge a^2-b^2=0$, then $\partial_x \frac{(a+b \sin[e+fx])^m (c+d \sin[e+fx])^m}{(g \cos[e+fx])^{2m}} = 0$

Rule: If $bc+ad=0 \wedge a^2-b^2=0 \wedge m \notin \mathbb{Z} \wedge n \notin \mathbb{Z}$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow$$

$$\left((a^{\text{IntPart}[m]} c^{\text{IntPart}[m]} (a+b \sin[e+fx])^{\text{FracPart}[m]} (c+d \sin[e+fx])^{\text{FracPart}[m]} \right) / \left(g^{2 \text{IntPart}[m]} (g \cos[e+fx])^{2 \text{FracPart}[m]} \right)$$

$$\int (g \cos[e+fx])^{2m+p} (c+d \sin[e+fx])^{n-m} dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_+d_.sin[e_.+f_.x_])^n_,x_Symbol] :=
  a^IntPart[m]*c^IntPart[m]*(a+b*sin[e+f*x])^FracPart[m]*(c+d*sin[e+f*x])^FracPart[m]/
    (g^(2*IntPart[m])*(g*cos[e+f*x])^(2*FracPart[m]))*
    Int[(g*cos[e+f*x])^(2*m+p)*(c+d*sin[e+f*x])^(n-m),x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && (FractionQ[m] || Not[FractionQ[n]])
```

$$5. \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) \, dx$$

$$1. \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) \, dx \text{ when } a^2 - b^2 = 0$$

$$\text{1: } \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) \, dx \text{ when } a^2 - b^2 = 0 \wedge a d m + b c (m+p+1) = 0$$

Derivation: Singly degenerate sine recurrence 2c with $c \rightarrow 1$, $d \rightarrow 0$, $n \rightarrow 0$

Note: If $a^2 - b^2 = 0 \wedge a d m + b c (m+p+1) = 0$, then $m+p+1 \neq 0$.

Rule: If $a^2 - b^2 = 0 \wedge a d m + b c (m+p+1) = 0$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) \, dx \rightarrow - \frac{d (g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^m}{f g (m+p+1)}$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_])^p_*(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -d*(g*cos[e+f*x])^(p+1)*(a+b*sin[e+f*x])^m/(f*g*(m+p+1)) /;
FreeQ[{a,b,c,d,e,f,g,m,p},x] && EqQ[a^2-b^2,0] && EqQ[a*d*m+b*c*(m+p+1),0]
```

2: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) \, dx$ when $a^2 - b^2 = 0 \wedge m > -1 \wedge p < -1$

Derivation: Singly degenerate sine recurrence 4a with $c \rightarrow 1$, $d \rightarrow 0$

Rule: If $a^2 - b^2 = 0 \wedge m > -1 \wedge p < -1$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) \, dx \rightarrow$$

$$- \frac{(bc+ad) (g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^m}{afg(p+1)} + \frac{b(adm+bc(m+p+1))}{ag^2(p+1)} \int (g \cos[e+fx])^{p+2} (a+b \sin[e+fx])^{m-1} \, dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_.+d_.sin[e_.+f_.x_]),x_Symbol] :=
- (b*c+a*d)*(g*cos[e+f*x])^(p+1)*(a+b*sin[e+f*x])^m/(a*f*g*(p+1)) +
b*(a*d*m+b*c*(m+p+1))/(a*g^2*(p+1))*Int[(g*cos[e+f*x])^(p+2)*(a+b*sin[e+f*x])^(m-1),x] /;
FreeQ[{a,b,c,d,e,f,g},x] && EqQ[a^2-b^2,0] && GtQ[m,-1] && LtQ[p,-1]
```

$$\mathbf{3:} \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \text{ when } a^2 - b^2 = 0 \wedge \frac{2m+p+1}{2} \in \mathbb{Z}^+ \wedge m+p+1 \neq 0$$

Derivation: Singly degenerate sine recurrence 2c with $c \rightarrow 1$, $d \rightarrow 0$, $n \rightarrow 0$

Rule: If $a^2 - b^2 = 0 \wedge \frac{2m+p+1}{2} \in \mathbb{Z}^+ \wedge m+p+1 \neq 0$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow$$

$$- \frac{d (g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^m}{f g (m+p+1)} + \frac{a d m + b c (m+p+1)}{b (m+p+1)} \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_+d_.sin[e_.+f_.x_]),x_Symbol] :=
  -d*(g*cos[e+f*x])^(p+1)*(a+b*sin[e+f*x])^m/(f*g*(m+p+1)) +
  (a*d*m+b*c*(m+p+1))/(b*(m+p+1))*Int[(g*cos[e+f*x])^p*(a+b*sin[e+f*x])^m,x] /;
FreeQ[{a,b,c,d,e,f,g,m,p},x] && EqQ[a^2-b^2,0] && IGtQ[Simplify[(2*m+p+1)/2],0] && NeQ[m+p+1,0]
```

4. $\int \cos[e+fx]^2 (a+b \sin[e+fx])^m (c+d \sin[e+fx]) \, dx$ when $a^2 - b^2 = 0 \wedge m < 0$

1: $\int \cos[e+fx]^2 (a+b \sin[e+fx])^m (c+d \sin[e+fx]) \, dx$ when $a^2 - b^2 = 0 \wedge m < -\frac{3}{2}$

Rule: If $a^2 - b^2 = 0 \wedge m < -\frac{3}{2}$, then

$$\int \cos[e+fx]^2 (a+b \sin[e+fx])^m (c+d \sin[e+fx]) \, dx \rightarrow$$

$$\frac{2(b c - a d) \cos[e+fx] (a+b \sin[e+fx])^{m+1}}{b^2 f (2m+3)} + \frac{1}{b^3 (2m+3)} \int (a+b \sin[e+fx])^{m+2} (b c + 2 a d (m+1) - b d (2m+3) \sin[e+fx]) \, dx$$

Program code:

```
Int[cos[e_+f_*x_]^2*(a_+b_*sin[e_+f_*x_]^m*(c_+d_*sin[e_+f_*x_] ),x_Symbol] :=
  2*(b*c-a*d)*Cos[e+f*x]*(a+b*sin[e+f*x])^(m+1)/(b^2*f*(2*m+3)) +
  1/(b^3*(2*m+3))*Int[(a+b*sin[e+f*x])^(m+2)*(b*c+2*a*d*(m+1)-b*d*(2*m+3)*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[a^2-b^2,0] && LtQ[m,-3/2]
```

2: $\int \cos[e+fx]^2 (a+b \sin[e+fx])^m (c+d \sin[e+fx]) dx$ when $a^2 - b^2 = 0 \wedge -\frac{3}{2} \leq m < 0$

Rule: If $a^2 - b^2 = 0 \wedge -\frac{3}{2} \leq m < 0$, then

$$\int \cos[e+fx]^2 (a+b \sin[e+fx])^m (c+d \sin[e+fx]) dx \rightarrow$$

$$\frac{d \cos[e+fx] (a+b \sin[e+fx])^{m+2}}{b^2 f (m+3)} - \frac{1}{b^2 (m+3)} \int (a+b \sin[e+fx])^{m+1} (b d (m+2) - a c (m+3) + (b c (m+3) - a d (m+4)) \sin[e+fx]) dx$$

Program code:

```
Int[cos[e_.+f_.*x_]^2*(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  d*cos[e+f*x]*(a+b*sin[e+f*x])^(m+2)/(b^2*f*(m+3)) -
  1/(b^2*(m+3))*Int[(a+b*sin[e+f*x])^(m+1)*(b*d*(m+2)-a*c*(m+3)+(b*c*(m+3)-a*d*(m+4))*sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[a^2-b^2,0] && GeQ[m,-3/2] && LtQ[m,0]
```

5: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) \, dx$ when $a^2 - b^2 = 0 \wedge (m < -1 \vee m+p \in \mathbb{Z}^-) \wedge 2m+p+1 \neq 0$

Derivation: Singly degenerate sine recurrence 2a with $c \rightarrow 1$, $d \rightarrow 0$

Derivation: Singly degenerate sine recurrence 2b with $c \rightarrow 1$, $d \rightarrow 0$

Rule: If $a^2 - b^2 = 0 \wedge (m < -1 \vee m+p \in \mathbb{Z}^-) \wedge 2m+p+1 \neq 0$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) \, dx \rightarrow$$

$$\frac{(bc-ad)(g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^m}{afg(2m+p+1)} + \frac{adm+bc(m+p+1)}{ab(2m+p+1)} \int (g \cos[e+fx])^p (a+b \sin[e+fx])^{m+1} \, dx$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_])^p_*(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  (b*c-a*d)*(g*cos[e+f*x])^(p+1)*(a+b*sin[e+f*x])^m/(a*f*g*(2*m+p+1)) +
  (a*d*m+b*c*(m+p+1))/(a*b*(2*m+p+1))*Int[(g*cos[e+f*x])^p*(a+b*sin[e+f*x])^(m+1),x] /;
FreeQ[{a,b,c,d,e,f,g,m,p},x] && EqQ[a^2-b^2,0] && (LtQ[m,-1] || ILtQ[Simplify[m+p],0]) && NeQ[2*m+p+1,0]
```

6: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) dx$ when $a^2 - b^2 = 0 \wedge m+p+1 \neq 0$

Derivation: Singly degenerate sine recurrence 2c with $c \rightarrow 1$, $d \rightarrow 0$, $n \rightarrow 0$

Rule: If $a^2 - b^2 = 0 \wedge m+p+1 \neq 0$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) dx \rightarrow$$

$$-\frac{d (g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^m}{f g (m+p+1)} + \frac{a d m + b c (m+p+1)}{b (m+p+1)} \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_+d_.sin[e_.+f_.x_]),x_Symbol] :=
  -d*(g*cos[e+f*x])^(p+1)*(a+b*sin[e+f*x])^m/(f*g*(m+p+1)) +
  (a*d*m+b*c*(m+p+1))/(b*(m+p+1))*Int[(g*cos[e+f*x])^p*(a+b*sin[e+f*x])^m,x] /;
FreeQ[{a,b,c,d,e,f,g,m,p},x] && EqQ[a^2-b^2,0] && NeQ[m+p+1,0]
```

2. $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) dx$ when $a^2 - b^2 \neq 0$

1. $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) dx$ when $a^2 - b^2 \neq 0 \wedge m > 0$

1: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) dx$ when $a^2 - b^2 \neq 0 \wedge m > 0 \wedge p < -1$

Derivation: Nondegenerate sine recurrence 3a with $c \rightarrow 1$, $d \rightarrow 0$, $C \rightarrow 0$

Rule: If $a^2 - b^2 \neq 0 \wedge m > 0 \wedge p < -1$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) dx \rightarrow$$

$$-\frac{(g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^m (d+c \sin[e+fx])}{f g (p+1)} +$$

$$\frac{1}{g^2 (p+1)} \int (g \cos[e+fx])^{p+2} (a+b \sin[e+fx])^{m-1} (ac(p+2) + bdm + bc(m+p+2) \sin[e+fx]) dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a+b_.sin[e_.+f_.x_])^m_*(c_.+d_.sin[e_.+f_.x_]),x_Symbol] :=
  -(g*cos[e+fx])^(p+1)*(a+b*sin[e+fx])^m*(d+c*sin[e+fx])/(f*g*(p+1)) +
  1/(g^2*(p+1))*Int[(g*cos[e+fx])^(p+2)*(a+b*sin[e+fx])^(m-1)*Simp[a*c*(p+2)+b*d*m+b*c*(m+p+2)*Sin[e+fx],x],x] /;
FreeQ[{a,b,c,d,e,f,g},x] && NeQ[a^2-b^2,0] && GtQ[m,0] && LtQ[p,-1] && IntegerQ[2*m] &&
Not[EqQ[m,1] && NeQ[c^2-d^2,0] && SimplifierQ[c+d*x,a+b*x]]
```

2: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) dx$ when $a^2 - b^2 \neq 0 \wedge m > 0 \wedge p \neq -1$

Derivation: Nondegenerate sine recurrence 1b with $c \rightarrow 0$, $d \rightarrow 1$, $A \rightarrow 0$, $B \rightarrow A$, $C \rightarrow B$, $n \rightarrow -1$

Rule: If $a^2 - b^2 \neq 0 \wedge m > 0 \wedge p \neq -1$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) dx \rightarrow$$

$$- \frac{d (g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^m}{f g (m+p+1)} +$$

$$\frac{1}{m+p+1} \int (g \cos[e+fx])^p (a+b \sin[e+fx])^{m-1} (ac(m+p+1) + bdm + (adm + bc(m+p+1)) \sin[e+fx]) dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a+b_.sin[e_.+f_.x_])^m_*(c_.+d_.sin[e_.+f_.x_]),x_Symbol] :=
  -d*(g*cos[e+fx])^(p+1)*(a+b*sin[e+fx])^m/(f*g*(m+p+1)) +
  1/(m+p+1)*Int[(g*cos[e+fx])^p*(a+b*sin[e+fx])^(m-1)*Simp[a*c*(m+p+1)+b*d*m+(a*d*m+b*c*(m+p+1))*Sin[e+fx],x],x] /;
FreeQ[{a,b,c,d,e,f,g,p},x] && NeQ[a^2-b^2,0] && GtQ[m,0] && Not[LtQ[p,-1]] && IntegerQ[2*m] &&
Not[EqQ[m,1] && NeQ[c^2-d^2,0] && SimplifierQ[c+d*x,a+b*x]]
```

$$2. \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) \, dx \text{ when } a^2 - b^2 \neq 0 \wedge m < -1$$

$$1: \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) \, dx \text{ when } a^2 - b^2 \neq 0 \wedge m < -1 \wedge p > 1 \wedge m+p+1 \neq 0$$

Derivation: Nondegenerate sine recurrence 2a with $c \rightarrow 0$, $d \rightarrow 1$, $A \rightarrow 0$, $B \rightarrow A$, $C \rightarrow B$, $n \rightarrow -1$

Rule: If $a^2 - b^2 \neq 0 \wedge m < -1 \wedge p > 1 \wedge m+p+1 \neq 0$, then

$$\begin{aligned} & \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) \, dx \rightarrow \\ & \left((g (g \cos[e+fx])^{p-1} (a+b \sin[e+fx])^{m+1} (bc(m+p+1) - adp + bd(m+1) \sin[e+fx])) / (b^2 f(m+1)(m+p+1)) \right) + \\ & \frac{g^2 (p-1)}{b^2 (m+1)(m+p+1)} \int (g \cos[e+fx])^{p-2} (a+b \sin[e+fx])^{m+1} (bd(m+1) + (bc(m+p+1) - adp) \sin[e+fx]) \, dx \end{aligned}$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a+b_.sin[e_.+f_.x_])^m_*(c_.+d_.sin[e_.+f_.x_]),x_Symbol] :=
  g*(g*cos[e+f*x])^(p-1)*(a+b*sin[e+f*x])^(m+1)*(b*c*(m+p+1)-a*d*p+b*d*(m+1)*sin[e+f*x])/(b^2*f*(m+1)*(m+p+1)) +
  g^2*(p-1)/(b^2*(m+1)*(m+p+1))*
  Int[(g*cos[e+f*x])^(p-2)*(a+b*sin[e+f*x])^(m+1)*Simp[b*d*(m+1)+(b*c*(m+p+1)-a*d*p)*sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,g},x] && NeQ[a^2-b^2,0] && LtQ[m,-1] && GtQ[p,1] && NeQ[m+p+1,0] && IntegerQ[2*m]
```

$$2: \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) \, dx \text{ when } a^2 - b^2 \neq 0 \wedge m < -1$$

Derivation: Nondegenerate sine recurrence 1a with $c \rightarrow 1$, $d \rightarrow 0$, $C \rightarrow 0$

Derivation: Nondegenerate sine recurrence 1c with $c \rightarrow 1$, $d \rightarrow 0$, $C \rightarrow 0$

Rule: If $a^2 - b^2 \neq 0 \wedge m < -1$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) \, dx \rightarrow$$

$$- \frac{(bc - ad) (g \cos[e + fx])^{p+1} (a + b \sin[e + fx])^{m+1}}{f g (a^2 - b^2) (m + 1)} +$$

$$\frac{1}{(a^2 - b^2) (m + 1)} \int (g \cos[e + fx])^p (a + b \sin[e + fx])^{m+1} ((ac - bd) (m + 1) - (bc - ad) (m + p + 2) \sin[e + fx]) dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_.+d_.sin[e_.+f_.x_]),x_Symbol] :=
  -(b*c-a*d)*(g*cos[e+f*x])^(p+1)*(a+b*sin[e+f*x])^(m+1)/(f*g*(a^2-b^2)*(m+1)) +
  1/((a^2-b^2)*(m+1))*Int[(g*cos[e+f*x])^p*(a+b*sin[e+f*x])^(m+1)*Simp[(a*c-b*d)*(m+1)-(b*c-a*d)*(m+p+2)*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,g,p},x] && NeQ[a^2-b^2,0] && LtQ[m,-1] && IntegerQ[2*m]
```

3: $\int (g \cos[e + fx])^p (a + b \sin[e + fx])^m (c + d \sin[e + fx]) dx$ when $a^2 - b^2 \neq 0 \wedge p > 1 \wedge m + p \neq 0 \wedge m + p + 1 \neq 0$

Derivation: Nondegenerate sine recurrence 2b with $c \rightarrow 0$, $d \rightarrow 1$, $A \rightarrow 0$, $B \rightarrow A$, $C \rightarrow B$, $n \rightarrow -1$

Rule: If $a^2 - b^2 \neq 0 \wedge p > 1 \wedge m + p \neq 0 \wedge m + p + 1 \neq 0$, then

$$\int (g \cos[e + fx])^p (a + b \sin[e + fx])^m (c + d \sin[e + fx]) dx \rightarrow$$

$$\frac{((g \cos[e + fx])^{p-1} (a + b \sin[e + fx])^{m+1} (bc(m+p+1) - adp + bd(m+p) \sin[e + fx]))}{(b^2 f(m+p)(m+p+1))} +$$

$$\frac{g^2(p-1)}{b^2(m+p)(m+p+1)} \int (g \cos[e + fx])^{p-2} (a + b \sin[e + fx])^m (b(adm + bc(m+p+1)) + (abc(m+p+1) - d(a^2p - b^2(m+p))) \sin[e + fx]) dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_.+d_.sin[e_.+f_.x_]),x_Symbol] :=
  g*(g*cos[e+f*x])^(p-1)*(a+b*sin[e+f*x])^(m+1)*(b*c*(m+p+1)-a*d*p+b*d*(m+p)*Sin[e+f*x])/(b^2*f*(m+p)*(m+p+1)) +
  g^2*(p-1)/(b^2*(m+p)*(m+p+1))*
  Int[(g*cos[e+f*x])^(p-2)*(a+b*sin[e+f*x])^m*Simp[b*(a*d*m+b*c*(m+p+1))+(a*b*c*(m+p+1)-d*(a^2*p-b^2*(m+p)))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,g,m},x] && NeQ[a^2-b^2,0] && GtQ[p,1] && NeQ[m+p,0] && NeQ[m+p+1,0] && IntegerQ[2*m]
```

4: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) \, dx$ when $a^2 - b^2 \neq 0 \wedge p < -1$

Derivation: Nondegenerate sine recurrence 3b with $c \rightarrow 1$, $d \rightarrow 0$, $C \rightarrow 0$

Rule: If $a^2 - b^2 \neq 0 \wedge p < -1$, then

$$\frac{1}{g^2 (a^2 - b^2) (p+1)} \int (g \cos[e+fx])^{p+2} (a+b \sin[e+fx])^m (c (a^2 (p+2) - b^2 (m+p+2)) + a b d m + b (a c - b d) (m+p+3) \sin[e+fx]) \, dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_.+d_.sin[e_.+f_.x_]),x_Symbol] :=
  (g*cos[e+f*x])^(p+1)*(a+b*sin[e+f*x])^(m+1)*(b*c-a*d-(a*c-b*d)*Sin[e+f*x])/(f*g*(a^2-b^2)*(p+1)) +
  1/(g^2*(a^2-b^2)*(p+1))*
  Int[(g*cos[e+f*x])^(p+2)*(a+b*sin[e+f*x])^m*Simp[c*(a^2*(p+2)-b^2*(m+p+2))+a*b*d*m+b*(a*c-b*d)*(m+p+3)*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,g,m},x] && NeQ[a^2-b^2,0] && LtQ[p,-1] && IntegerQ[2*m]
```

$$\text{5: } \int \frac{(g \cos[e+fx])^p (c+d \sin[e+fx])}{a+b \sin[e+fx]} dx \text{ when } a^2 - b^2 \neq 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{c+d z}{a+b z} == \frac{d}{b} + \frac{b c - a d}{b (a+b z)}$$

Rule: If $a^2 - b^2 \neq 0$, then

$$\int \frac{(g \cos[e+fx])^p (c+d \sin[e+fx])}{a+b \sin[e+fx]} dx \rightarrow \frac{d}{b} \int (g \cos[e+fx])^p dx + \frac{b c - a d}{b} \int \frac{(g \cos[e+fx])^p}{a+b \sin[e+fx]} dx$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_])^p_*(c_.+d_.*sin[e_.+f_.*x_])/(a_.+b_.*sin[e_.+f_.*x_]),x_Symbol] :=
  d/b*Int[(g*cos[e+f*x])^p,x] + (b*c-a*d)/b*Int[(g*cos[e+f*x])^p/(a+b*sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f,g},x] && NeQ[a^2-b^2,0]
```

$$\text{6: } \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) dx \text{ when } a^2 - b^2 \neq 0 \wedge c^2 - d^2 == 0$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: } \partial_x \frac{(g \cos[e+fx])^{p-1}}{(1+\sin[e+fx])^{\frac{p-1}{2}} (1-\sin[e+fx])^{\frac{p-1}{2}}} == 0$$

$$\text{Basis: } \cos[e+fx] == \frac{1}{f} \partial_x \sin[e+fx]$$

Rule: If $a^2 - b^2 \neq 0 \wedge c^2 - d^2 == 0$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx]) dx \rightarrow$$

$$\frac{c g (g \cos[e + f x])^{p-1}}{(1 + \sin[e + f x])^{\frac{p-1}{2}} (1 - \sin[e + f x])^{\frac{p-1}{2}}} \int \cos[e + f x] \left(1 + \frac{d}{c} \sin[e + f x]\right)^{\frac{p+1}{2}} \left(1 - \frac{d}{c} \sin[e + f x]\right)^{\frac{p-1}{2}} (a + b \sin[e + f x])^m dx \rightarrow$$

$$\frac{c g (g \cos[e + f x])^{p-1}}{f (1 + \sin[e + f x])^{\frac{p-1}{2}} (1 - \sin[e + f x])^{\frac{p-1}{2}}} \text{Subst}\left[\int \left(1 + \frac{d}{c} x\right)^{\frac{p+1}{2}} \left(1 - \frac{d}{c} x\right)^{\frac{p-1}{2}} (a + b x)^m dx, x, \sin[e + f x]\right]$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_])^p_*(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  c*g*(g*Cos[e+f*x])^(p-1)/(f*(1+Sin[e+f*x])^((p-1)/2)*(1-Sin[e+f*x])^((p-1)/2))*
  Subst[Int[(1+d/c*x)^((p+1)/2)*(1-d/c*x)^((p-1)/2)*(a+b*x)^m,x,Sin[e+f*x]] /;
  FreeQ[{a,b,c,d,e,f,m,p},x] && NeQ[a^2-b^2,0] && EqQ[c^2-d^2,0]
```

6. $\int (g \cos[e + f x])^p (d \sin[e + f x])^n (a + b \sin[e + f x])^m dx$ when $a^2 - b^2 = 0$

1: $\int \cos[e + f x]^p (d \sin[e + f x])^n (a + b \sin[e + f x])^m dx$ when $a^2 - b^2 = 0 \wedge m \in \mathbb{Z} \wedge 2m + p = 0$

Derivation: Algebraic simplification

Basis: If $a^2 - b^2 = 0 \wedge m \in \mathbb{Z} \wedge 2m + p = 0$, then $\cos[z]^p (a + b \sin[z])^m = \frac{a^{2m}}{(a - b \sin[z])^m}$

Rule: If $a^2 - b^2 = 0 \wedge m \in \mathbb{Z} \wedge 2m + p = 0$, then

$$\int \cos[e + f x]^p (d \sin[e + f x])^n (a + b \sin[e + f x])^m dx \rightarrow a^{2m} \int \frac{(d \sin[e + f x])^n}{(a - b \sin[e + f x])^m} dx$$

Program code:

```
Int[cos[e_.+f_.*x_]^p_*(a_+b_.*sin[e_.+f_.*x_])^m_*(d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  a^(2*m)*Int[(d*Sin[e+f*x])^n/(a-b*Sin[e+f*x])^m,x] /;
  FreeQ[{a,b,d,e,f,n},x] && EqQ[a^2-b^2,0] && IntegersQ[m,p] && EqQ[2*m+p,0]
```

2: $\int (g \cos[e+fx])^p \sin[e+fx]^2 (a+b \sin[e+fx])^m dx$ when $a^2 - b^2 = 0 \wedge m = p$

Derivation: Nondegenerate sine recurrence 1b with $A \rightarrow a^2$, $B \rightarrow 2ab$, $C \rightarrow b^2$, $m \rightarrow 0$

Rule: If $a^2 - b^2 = 0 \wedge m = p$, then

$$\int (g \cos[e+fx])^p \sin[e+fx]^2 (a+b \sin[e+fx])^m dx \rightarrow$$

$$- \frac{(g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^{m+1}}{2bf g(m+1)} + \frac{a}{2g^2} \int (g \cos[e+fx])^{p+2} (a+b \sin[e+fx])^{m-1} dx$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_])^p_*sin[e_.+f_.*x_]^2*(a+b_.*sin[e_.+f_.*x_])^m_,x_Symbol] :=
  -(g*cos[e+f*x])^(p+1)*(a+b*sin[e+f*x])^(m+1)/(2*b*f*g*(m+1)) +
  a/(2*g^2)*Int[(g*cos[e+f*x])^(p+2)*(a+b*sin[e+f*x])^(m-1),x] /;
FreeQ[{a,b,e,f,g,m,p},x] && EqQ[a^2-b^2,0] && EqQ[m-p,0]
```

3: $\int (g \cos[e+fx])^p \sin[e+fx]^2 (a+b \sin[e+fx])^m dx$ when $a^2 - b^2 = 0 \wedge m+p+1 = 0$

Derivation: ???

Rule: If $a^2 - b^2 = 0 \wedge m+p+1 = 0$, then

$$\int (g \cos[e+fx])^p \sin[e+fx]^2 (a+b \sin[e+fx])^m dx \rightarrow \frac{b (g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^m}{a f g m} - \frac{1}{g^2} \int (g \cos[e+fx])^{p+2} (a+b \sin[e+fx])^m dx$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_])^p_*sin[e_.+f_.*x_]^2*(a_+b_.*sin[e_.+f_.*x_])^m_,x_Symbol] :=
  b*(g_*Cos[e+f*x])^(p+1)*(a+b*Sin[e+f*x])^m/(a*f*g*m) -
  1/g^2*Int[(g_*Cos[e+f*x])^(p+2)*(a+b*Sin[e+f*x])^m,x] /;
FreeQ[{a,b,e,f,g,m,p},x] && EqQ[a^2-b^2,0] && EqQ[m+p+1,0]
```


$$4. \int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 = 0 \wedge m \in \mathbb{Z}$$

$$1: \int \cos[e+fx]^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 = 0 \wedge m \in \mathbb{Z} \wedge \frac{p}{2} \in \mathbb{Z} \wedge m + \frac{p}{2} > 0$$

Derivation: Algebraic expansion

Basis: If $a^2 - b^2 = 0 \wedge \frac{p}{2} \in \mathbb{Z}$, then $\cos[z]^p = \frac{1}{a^p} (a - b \sin[z])^{p/2} (a + b \sin[z])^{p/2}$

Rule: If $a^2 - b^2 = 0 \wedge m \in \mathbb{Z} \wedge \frac{p}{2} \in \mathbb{Z} \wedge m + \frac{p}{2} > 0$, then

$$\int \cos[e+fx]^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow \frac{1}{a^p} \int \text{ExpandTrig}[(d \sin[e+fx])^n (a - b \sin[e+fx])^{p/2} (a + b \sin[e+fx])^{m+p/2}, x] dx$$

Program code:

```
Int[cos[e_.+f_.*x_]^p_*(d_.*sin[e_.+f_.*x_]^n_*(a_+b_.*sin[e_.+f_.*x_]^m_,x_Symbol] :=
  1/a^p*Int[ExpandTrig[(d*sin[e+f*x])^n*(a-b*sin[e+f*x])^(p/2)*(a+b*sin[e+f*x])^(m+p/2),x],x] /;
FreeQ[{a,b,d,e,f},x] && EqQ[a^2-b^2,0] && IntegersQ[m,n,p/2] && (GtQ[m,0] && GtQ[p,0] && LtQ[-m-p,n,-1] || GtQ[m,2] && LtQ[p,0] && G
```

2: $\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx$ when $a^2 - b^2 = 0 \wedge m \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $a^2 - b^2 = 0 \wedge m \in \mathbb{Z}^+$, then

$$\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow \int (g \cos[e+fx])^p \text{ExpandTrig}[(d \sin[e+fx])^n (a+b \sin[e+fx])^m, x] dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(d_.sin[e_.+f_.x_])^n_*(a+b_.sin[e_.+f_.x_])^m_,x_Symbol] :=
  Int[ExpandTrig[(g*cos[e+f*x])^p,(d*sin[e+f*x])^n*(a+b*sin[e+f*x])^m,x],x] /;
FreeQ[{a,b,d,e,f,g,n,p},x] && EqQ[a^2-b^2,0] && IGtQ[m,0]
```

$$3. \int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 = 0 \wedge m \in \mathbb{Z}^-$$

$$1: \int \cos[e+fx]^2 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 = 0 \wedge m \in \mathbb{Z}^-$$

Derivation: Algebraic simplification

Basis: If $a^2 - b^2 = 0$, then $\cos[z]^2 = \frac{1}{b^2} (a + b \sin[z]) (a - b \sin[z])$

Rule: If $a^2 - b^2 = 0 \wedge m \in \mathbb{Z}^-$, then

$$\int \cos[e+fx]^2 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow \frac{1}{b^2} \int (d \sin[e+fx])^n (a+b \sin[e+fx])^{m+1} (a-b \sin[e+fx]) dx$$

Program code:

```
Int[cos[e_.+f_.*x_]^2*(d_.*sin[e_.+f_.*x_]^n*(a+b_.*sin[e_.+f_.*x_]^m,x_Symbol] :=
  1/b^2*Int[(d*sin[e+f*x])^n*(a+b*sin[e+f*x])^(m+1)*(a-b*sin[e+f*x]),x] /;
FreeQ[{a,b,d,e,f,m,n},x] && EqQ[a^2-b^2,0] && (ILtQ[m,0] || Not[IGtQ[n,0]])
```

$$\mathbf{2:} \int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 = 0 \wedge m \in \mathbb{Z}^-$$

Derivation: Algebraic expansion

Basis: If $a^2 - b^2 = 0$, then $a + b \sin[z] = \frac{a^2 (g \cos[z])^2}{g^2 (a - b \sin[z])}$

Rule: If $a^2 - b^2 = 0 \wedge m \in \mathbb{Z}^-$, then

$$\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow \frac{a^{2m}}{g^{2m}} \int \frac{(g \cos[e+fx])^{2m+p} (d \sin[e+fx])^n}{(a - b \sin[e+fx])^m} dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(d_.sin[e_.+f_.x_])^n_*(a+b_.sin[e_.+f_.x_])^m_,x_Symbol] :=
  (a/g)^(2*m)*Int[(g*cos[e+f*x])^(2*m+p)*(d*sin[e+f*x])^n/(a-b*sin[e+f*x])^m,x] /;
FreeQ[{a,b,d,e,f,g,n,p},x] && EqQ[a^2-b^2,0] && ILtQ[m,0]
```

5: $\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx$ when $a^2 - b^2 = 0 \wedge m \in \mathbb{Z} \wedge (2m+p = 0 \vee 2m+p > 0 \wedge p < -1)$

Derivation: Algebraic expansion

Basis: If $a^2 - b^2 = 0$, then $a + b \sin[z] = \frac{a^2 (g \cos[z])^2}{g^2 (a - b \sin[z])}$

Note: By making the degree of the cosine factor in the integrand nonnegative, this rule removes the removable singularities from the integrand and hence from the resulting antiderivatives.

Rule: If $a^2 - b^2 = 0 \wedge m \in \mathbb{Z} \wedge (2m+p = 0 \vee 2m+p > 0 \wedge p < -1)$, then

$$\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow \frac{a^{2m}}{g^{2m}} \int \frac{(g \cos[e+fx])^{2m+p} (d \sin[e+fx])^n}{(a - b \sin[e+fx])^m} dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(d_.sin[e_.+f_.x_])^n_*(a+b_.sin[e_.+f_.x_])^m_,x_Symbol] :=
  (a/g)^(2*m)*Int[(g*cos[e+f*x])^(2*m+p)*(d*sin[e+f*x])^n/(a-b*sin[e+f*x])^m,x] /;
FreeQ[{a,b,d,e,f,g,n},x] && EqQ[a^2-b^2,0] && IntegerQ[m] && RationalQ[p] && (EqQ[2*m+p,0] || GtQ[2*m+p,0] && LtQ[p,-1])
```

6. $\int (g \cos[e+fx])^p \sin[e+fx]^2 (a+b \sin[e+fx])^m dx$ when $a^2 - b^2 = 0$

1: $\int (g \cos[e+fx])^p \sin[e+fx]^2 (a+b \sin[e+fx])^m dx$ when $a^2 - b^2 = 0 \wedge m \leq -\frac{1}{2}$

Derivation: ???

Rule: If $a^2 - b^2 = 0 \wedge m \leq -\frac{1}{2}$, then

$$\int (g \cos[e+fx])^p \sin[e+fx]^2 (a+b \sin[e+fx])^m dx \rightarrow \frac{b (g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^m}{a f g (2m+p+1)} - \frac{1}{a^2 (2m+p+1)} \int (g \cos[e+fx])^p (a+b \sin[e+fx])^{m+1} (a m - b (2m+p+1) \sin[e+fx]) dx$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_])^p_*sin[e_.+f_.*x_]^2*(a_+b_.*sin[e_.+f_.*x_])^m_,x_Symbol] :=
  b*(g*Cos[e+f*x])^(p+1)*(a+b*Sin[e+f*x])^m/(a*f*g*(2*m+p+1)) -
  1/(a^2*(2*m+p+1))*Int[(g*Cos[e+f*x])^p*(a+b*Sin[e+f*x])^(m+1)*(a*m-b*(2*m+p+1)*Sin[e+f*x]),x] /;
FreeQ[{a,b,e,f,g,p},x] && EqQ[a^2-b^2,0] && LeQ[m,-1/2] && NeQ[2*m+p+1,0]
```

2: $\int (g \cos[e+fx])^p \sin[e+fx]^2 (a+b \sin[e+fx])^m dx$ when $a^2 - b^2 = 0 \wedge m \neq -\frac{1}{2}$

Derivation: Nondegenerate sine recurrence 1b with $A \rightarrow a^2$, $B \rightarrow 2ab$, $C \rightarrow b^2$, $m \rightarrow 0$

Rule: If $a^2 - b^2 = 0 \wedge m \neq -\frac{1}{2}$, then

$$\int (g \cos[e+fx])^p \sin[e+fx]^2 (a+b \sin[e+fx])^m dx \rightarrow$$

$$-\frac{(g \cos[e+fx])^{p+1} (a+b \sin[e+fx])^{m+1}}{b f g (m+p+2)} + \frac{1}{b (m+p+2)} \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (b(m+1) - a(p+1) \sin[e+fx]) dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_.sin[e_.+f_.x_]^2*(a+b_.sin[e_.+f_.x_])^m_,x_Symbol] :=
- (g*cos[e+f*x])^(p+1)*(a+b*sin[e+f*x])^(m+1)/(b*f*g*(m+p+2)) +
1/(b*(m+p+2))*Int[(g*cos[e+f*x])^p*(a+b*sin[e+f*x])^m*(b*(m+1)-a*(p+1)*sin[e+f*x]),x] /;
FreeQ[{a,b,e,f,g,m,p},x] && EqQ[a^2-b^2,0] && NeQ[m+p+2,0]
```

$$7. \int \cos[e+fx]^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 = 0 \wedge \frac{p}{2} \in \mathbb{Z}$$

$$1: \int \cos[e+fx]^2 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 = 0 \wedge (2m+2n) \in \mathbb{Z}$$

Derivation: Algebraic simplification

Basis: If $a^2 - b^2 = 0$, then $\cos[z]^2 = \frac{1}{b^2} (a + b \sin[z]) (a - b \sin[z])$

Rule: If $a^2 - b^2 = 0 \wedge (2m+2n) \in \mathbb{Z}$, then

$$\int \cos[e+fx]^2 (a+b \sin[e+fx])^m (d \sin[e+fx])^n dx \rightarrow \frac{1}{b^2} \int (d \sin[e+fx])^n (a+b \sin[e+fx])^{m+1} (a-b \sin[e+fx]) dx$$

Program code:

```
Int[cos[e_.+f_.*x_]^2*(d_.*sin[e_.+f_.*x_]^n*(a_.+b_.*sin[e_.+f_.*x_]^m_,x_Symbol] :=
  1/b^2*Int[(d*sin[e+f*x])^n*(a+b*sin[e+f*x])^(m+1)*(a-b*sin[e+f*x]),x] /;
FreeQ[{a,b,d,e,f,m,n},x] && EqQ[a^2-b^2,0] && IntegersQ[2*m,2*n]
```


$$2. \int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 = 0$$

$$1: \int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 = 0 \wedge m < -1$$

Derivation: Algebraic expansion

Basis: If $a^2 - b^2 = 0$, then $\cos[z]^4 = -\frac{2}{ab} \sin[z] (a+b \sin[z])^2 + \frac{1}{a^2} (1+\sin[z]^2) (a+b \sin[z])^2$

Rule: If $a^2 - b^2 = 0 \wedge 2m \in \mathbb{Z} \wedge m < -1$, then

$$\int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow$$

$$-\frac{2}{ab d} \int (d \sin[e+fx])^{n+1} (a+b \sin[e+fx])^{m+2} dx + \frac{1}{a^2} \int (d \sin[e+fx])^n (a+b \sin[e+fx])^{m+2} (1+\sin[e+fx]^2) dx$$

Program code:

```
Int[cos[e_.+f_.*x_]^4*(d_.*sin[e_.+f_.*x_]^n*(a_+b_.*sin[e_.+f_.*x_]^m_,x_Symbol] :=
  -2/(a*b*d)*Int[(d*sin[e+f*x])^(n+1)*(a+b*sin[e+f*x])^(m+2),x] +
  1/a^2*Int[(d*sin[e+f*x])^n*(a+b*sin[e+f*x])^(m+2)*(1+sin[e+f*x]^2),x] /;
FreeQ[{a,b,d,e,f,n},x] && EqQ[a^2-b^2,0] && LtQ[m,-1]
```

$$2: \int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 = 0 \wedge m \neq -1$$

Derivation: Algebraic expansion

Basis: $\cos[z]^4 = \sin[z]^4 + 1 - 2 \sin[z]^2$

Rule: If $a^2 - b^2 = 0 \wedge m \neq -1$, then

$$\int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow$$

$$\frac{1}{d^4} \int (d \sin[e+fx])^{n+4} (a+b \sin[e+fx])^m dx + \int (d \sin[e+fx])^n (a+b \sin[e+fx])^m (1-2 \sin[e+fx]^2) dx$$

Program code:

```
Int[cos[e_.+f_.*x_]^4*(d_.*sin[e_.+f_.*x_]^n*(a+b_.*sin[e_.+f_.*x_]^m_,x_Symbol] :=
  1/d^4*Int[(d*sin[e+f*x])^(n+4)*(a+b*sin[e+f*x])^m_,x] +
  Int[(d*sin[e+f*x])^n*(a+b*sin[e+f*x])^m*(1-2*sin[e+f*x]^2),x] /;
FreeQ[{a,b,d,e,f,m,n},x] && EqQ[a^2-b^2,0] && Not[IGtQ[m,0]]
```

$$\mathbf{3:} \int \cos[e+fx]^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 = 0 \wedge \frac{p}{2} \in \mathbb{Z} \wedge m \in \mathbb{Z}$$

Derivation: Algebraic expansion, piecewise constant extraction and integration by substitution

Basis: If $a^2 - b^2 = 0 \wedge \frac{p}{2} \in \mathbb{Z}$, then $\cos[z]^p = a^{-p} (a+b \sin[z])^{p/2} (a-b \sin[z])^{p/2}$

$$\text{Basis: } \partial_x \frac{\cos[e+fx]}{\sqrt{1+\sin[e+fx]} \sqrt{1-\sin[e+fx]}} = 0$$

$$\text{Basis: } \cos[e+fx] = \frac{1}{f} \partial_x \sin[e+fx]$$

Rule: If $a^2 - b^2 = 0 \wedge \frac{p}{2} \in \mathbb{Z} \wedge m \in \mathbb{Z}$, then

$$\int \cos[e+fx]^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow$$

$$a^{-p} \int (d \sin[e+fx])^n (a+b \sin[e+fx])^{m+p/2} (a-b \sin[e+fx])^{p/2} dx \rightarrow$$

$$\frac{a^m \cos[e+fx]}{\sqrt{1+\sin[e+fx]} \sqrt{1-\sin[e+fx]}} \int \cos[e+fx] (d \sin[e+fx])^n \left(1 + \frac{b}{a} \sin[e+fx]\right)^{m+\frac{p-1}{2}} \left(1 - \frac{b}{a} \sin[e+fx]\right)^{\frac{p-1}{2}} dx \rightarrow$$

$$\frac{a^m \cos[e+fx]}{f \sqrt{1+\sin[e+fx]} \sqrt{1-\sin[e+fx]}} \text{Subst} \left[\int (dx)^n \left(1 + \frac{b}{a} x\right)^{m+\frac{p-1}{2}} \left(1 - \frac{b}{a} x\right)^{\frac{p-1}{2}} dx, x, \sin[e+fx] \right]$$

Program code:

```
Int[cos[e_.+f_.*x_]^p_*(d_.*sin[e_.+f_.*x_]^n_*(a_+b_.*sin[e_.+f_.*x_]^m_,x_Symbol] :=
  a^m*Cos[e+f*x]/(f*Sqrt[1+Sin[e+f*x]]*Sqrt[1-Sin[e+f*x]])*
  Subst[Int[(d*x)^n*(1+b/a*x)^(m+(p-1)/2)*(1-b/a*x)^((p-1)/2),x],x,Sin[e+f*x]] /;
  FreeQ[{a,b,d,e,f,n},x] && EqQ[a^2-b^2,0] && IntegerQ[p/2] && IntegerQ[m]
```

4: $\int \cos[e+fx]^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx$ when $a^2 - b^2 = 0 \wedge \frac{p}{2} \in \mathbb{Z} \wedge m \notin \mathbb{Z}$

Derivation: Algebraic expansion, piecewise constant extraction and integration by substitution

Basis: If $a^2 - b^2 = 0 \wedge \frac{p}{2} \in \mathbb{Z}$, then $\cos[z]^p = a^{-p} (a+b \sin[z])^{p/2} (a-b \sin[z])^{p/2}$

Basis: If $a^2 - b^2 = 0$, then $\partial_x \frac{\cos[e+fx]}{\sqrt{a+b \sin[e+fx]} \sqrt{a-b \sin[e+fx]}} = 0$

Basis: $\cos[e+fx] = \frac{1}{f} \partial_x \sin[e+fx]$

Rule: If $a^2 - b^2 = 0 \wedge \frac{p}{2} \in \mathbb{Z} \wedge m \notin \mathbb{Z}$, then

$$\int \cos[e+fx]^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow$$

$$a^{-p} \int (d \sin[e+fx])^n (a+b \sin[e+fx])^{m+p/2} (a-b \sin[e+fx])^{p/2} dx \rightarrow$$

$$\frac{\cos[e+fx]}{a^{p-2} \sqrt{a+b \sin[e+fx]} \sqrt{a-b \sin[e+fx]}} \int \cos[e+fx] (d \sin[e+fx])^n (a+b \sin[e+fx])^{m+\frac{p}{2}-\frac{1}{2}} (a-b \sin[e+fx])^{\frac{p}{2}-\frac{1}{2}} dx \rightarrow$$

$$\frac{\text{Cos}[e+fx]}{a^{p-2} f \sqrt{a+b \sin[e+fx]} \sqrt{a-b \sin[e+fx]}} \text{Subst}\left[\int (dx)^n (a+bx)^{m+\frac{p-1}{2}} (a-bx)^{\frac{p-1}{2}} dx, x, \sin[e+fx]\right]$$

Program code:

```
Int[cos[e_.+f_.**x_]^p_*(d_.sin[e_.+f_.**x_]^n_*(a+b_.sin[e_.+f_.**x_]^m_,x_Symbol] :=
  Cos[e+f*x]/(a^(p-2)*f*Sqrt[a+b*Sin[e+f*x]]*Sqrt[a-b*Sin[e+f*x]])*
  Subst[Int[(d*x)^n*(a+b*x)^(m+p/2-1/2)*(a-b*x)^(p/2-1/2),x],x,Sin[e+f*x]] /;
FreeQ[{a,b,d,e,f,m,n},x] && EqQ[a^2-b^2,0] && IntegerQ[p/2] && Not[IntegerQ[m]]
```

8: $\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx$ when $a^2 - b^2 = 0 \wedge m \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $a^2 - b^2 = 0 \wedge m \in \mathbb{Z}^+$, then

$$\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow \int (g \cos[e+fx])^p \text{ExpandTrig}[(d \sin[e+fx])^n (a+b \sin[e+fx])^m, x] dx$$

Program code:

```
Int[(g_.cos[e_.+f_.**x_]^p_*(d_.sin[e_.+f_.**x_]^n_*(a+b_.sin[e_.+f_.**x_]^m_,x_Symbol] :=
  Int[ExpandTrig[(g*cos[e+f*x])^p,(d*sin[e+f*x])^n*(a+b*sin[e+f*x])^m,x],x] /;
FreeQ[{a,b,d,e,f,g,n,p},x] && EqQ[a^2-b^2,0] && IGtQ[m,0] && (IntegerQ[p] || IGtQ[n,0])
```

9. $\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx$ when $a^2 - b^2 = 0$

1: $\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx$ when $a^2 - b^2 = 0 \wedge m \in \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: } \partial_x \frac{(g \cos[e+fx])^{p-1}}{(1+\sin[e+fx])^{\frac{p-1}{2}} (1-\sin[e+fx])^{\frac{p-1}{2}}} == 0$$

$$\text{Basis: } \cos[e+fx] == \frac{1}{f} \partial_x \sin[e+fx]$$

Rule: If $a^2 - b^2 == 0 \wedge m \in \mathbb{Z}$, then

$$\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow$$

$$\frac{a^m g (g \cos[e+fx])^{p-1}}{(1+\sin[e+fx])^{\frac{p-1}{2}} (1-\sin[e+fx])^{\frac{p-1}{2}}} \int \cos[e+fx] (d \sin[e+fx])^n \left(1 + \frac{b}{a} \sin[e+fx]\right)^{m+\frac{p-1}{2}} \left(1 - \frac{b}{a} \sin[e+fx]\right)^{\frac{p-1}{2}} dx \rightarrow$$

$$\frac{a^m g (g \cos[e+fx])^{p-1}}{f (1+\sin[e+fx])^{\frac{p-1}{2}} (1-\sin[e+fx])^{\frac{p-1}{2}}} \text{Subst}\left[\int (dx)^n \left(1 + \frac{b}{a} x\right)^{m+\frac{p-1}{2}} \left(1 - \frac{b}{a} x\right)^{\frac{p-1}{2}} dx, x, \sin[e+fx]\right]$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_])^p_*(d_.*sin[e_.+f_.*x_])^n_*(a_+b_.*sin[e_.+f_.*x_])^m_,x_Symbol] :=
  a^m*g*(g*cos[e+f*x])^(p-1)/(f*(1+Sin[e+f*x])^((p-1)/2)*(1-Sin[e+f*x])^((p-1)/2))*
  Subst[Int[(d*x)^n*(1+b/a*x)^(m+(p-1)/2)*(1-b/a*x)^((p-1)/2),x],x,Sin[e+f*x]] /;
FreeQ[{a,b,d,e,f,n,p},x] && EqQ[a^2-b^2,0] && IntegerQ[m]
```

2: $\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx$ when $a^2 - b^2 = 0 \wedge m \notin \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

Basis: If $a^2 - b^2 = 0$, then $\partial_x \frac{\cos[e+fx]}{\sqrt{a+b \sin[e+fx]} \sqrt{a-b \sin[e+fx]}} = 0$

Basis: $\cos[e+fx] = \frac{1}{f} \partial_x \sin[e+fx]$

Rule: If $a^2 - b^2 = 0 \wedge m \notin \mathbb{Z}$, then

$$\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow$$

$$\frac{g (g \cos[e+fx])^{p-1}}{(a+b \sin[e+fx])^{\frac{p-1}{2}} (a-b \sin[e+fx])^{\frac{p-1}{2}}} \int \cos[e+fx] (d \sin[e+fx])^n (a+b \sin[e+fx])^{m+\frac{p-1}{2}} (a-b \sin[e+fx])^{\frac{p-1}{2}} dx \rightarrow$$

$$\frac{g (g \cos[e+fx])^{p-1}}{f (a+b \sin[e+fx])^{\frac{p-1}{2}} (a-b \sin[e+fx])^{\frac{p-1}{2}}} \text{Subst}\left[\int (dx)^n (a+bx)^{m+\frac{p-1}{2}} (a-bx)^{\frac{p-1}{2}} dx, x, \sin[e+fx]\right]$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(d_.sin[e_.+f_.x_])^n_*(a+b_.sin[e_.+f_.x_])^m_,x_Symbol] :=
  g*(g*cos[e+f*x])^(p-1)/(f*(a+b*sin[e+f*x])^((p-1)/2)*(a-b*sin[e+f*x])^((p-1)/2))*
  Subst[Int[(d*x)^n*(a+b*x)^(m+(p-1)/2)*(a-b*x)^((p-1)/2),x],x,Sin[e+f*x]] /;
FreeQ[{a,b,d,e,f,m,n,p},x] && EqQ[a^2-b^2,0] && Not[IntegerQ[m]]
```

7. $\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx$ when $a^2 - b^2 \neq 0$

$$1. \int \frac{(g \cos[e+fx])^p (a+b \sin[e+fx])^m}{\sqrt{d \sin[e+fx]}} dx \text{ when } a^2 - b^2 \neq 0$$

$$1: \int \frac{(g \cos[e+fx])^p (a+b \sin[e+fx])^m}{\sqrt{d \sin[e+fx]}} dx \text{ when } a^2 - b^2 \neq 0 \wedge m < -1 \wedge m+p+\frac{1}{2} = 0$$

Rule: If $a^2 - b^2 \neq 0 \wedge m < -1 \wedge m+p+\frac{1}{2} = 0$, then

$$\int \frac{(g \cos[e+fx])^p (a+b \sin[e+fx])^m}{\sqrt{d \sin[e+fx]}} dx \rightarrow$$

$$- \frac{g (g \cos[e+fx])^{p-1} \sqrt{d \sin[e+fx]} (a+b \sin[e+fx])^{m+1}}{a d f (m+1)} + \frac{g^2 (2m+3)}{2 a (m+1)} \int \frac{(g \cos[e+fx])^{p-2} (a+b \sin[e+fx])^{m+1}}{\sqrt{d \sin[e+fx]}} dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a+b_.sin[e_.+f_.x_])^m_/Sqrt[d_.sin[e_.+f_.x_]],x_Symbol] :=
-g*(g*cos[e+f*x])^(p-1)*Sqrt[d*sin[e+f*x]]*(a+b*sin[e+f*x])^(m+1)/(a*d*f*(m+1)) +
g^2*(2*m+3)/(2*a*(m+1))*Int[(g*cos[e+f*x])^(p-2)*(a+b*sin[e+f*x])^(m+1)/Sqrt[d*sin[e+f*x]],x] /;
FreeQ[{a,b,d,e,f,g},x] && NeQ[a^2-b^2,0] && LtQ[m,-1] && EqQ[m+p+1/2,0]
```

2: $\int \frac{(g \cos[e+fx])^p (a+b \sin[e+fx])^m}{\sqrt{d \sin[e+fx]}} dx$ when $a^2 - b^2 \neq 0 \wedge m > 0 \wedge m+p+\frac{3}{2} = 0$

Rule: If $a^2 - b^2 \neq 0 \wedge m > 0 \wedge m+p+\frac{3}{2} = 0$, then

$$\int \frac{(g \cos[e+fx])^p (a+b \sin[e+fx])^m}{\sqrt{d \sin[e+fx]}} dx \rightarrow$$

$$\frac{2 (g \cos[e+fx])^{p+1} \sqrt{d \sin[e+fx]} (a+b \sin[e+fx])^m}{d f g (2m+1)} + \frac{2 a m}{g^2 (2m+1)} \int \frac{(g \cos[e+fx])^{p+2} (a+b \sin[e+fx])^{m-1}}{\sqrt{d \sin[e+fx]}} dx$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_])^p_*(a_+b_.*sin[e_.+f_.*x_])^m_/Sqrt[d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  2*(g_*Cos[e+f*x])^(p+1)*Sqrt[d*Sin[e+f*x]]*(a+b*Sin[e+f*x])^m/(d*f*g*(2*m+1)) +
  2*a*m/(g^2*(2*m+1))*Int[(g_*Cos[e+f*x])^(p+2)*(a+b*Sin[e+f*x])^(m-1)/Sqrt[d*Sin[e+f*x]],x] /;
FreeQ[{a,b,e,f,g},x] && NeQ[a^2-b^2,0] && GtQ[m,0] && EqQ[m+p+3/2,0]
```


$$2. \int \cos[e+fx]^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 \neq 0 \wedge (m \in \mathbb{Z}^+ \vee (2m | 2n) \in \mathbb{Z}) \wedge \frac{p}{2} \in \mathbb{Z}^+$$

$$1: \int \cos[e+fx]^2 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 \neq 0 \wedge (m \in \mathbb{Z}^+ \vee (2m | 2n) \in \mathbb{Z})$$

Derivation: Algebraic expansion

$$\text{Basis: } \cos[z]^2 = 1 - \sin[z]^2$$

Rule: If $a^2 - b^2 \neq 0 \wedge (m \in \mathbb{Z}^+ \vee (2m | 2n) \in \mathbb{Z})$, then

$$\int \cos[e+fx]^2 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow \int (d \sin[e+fx])^n (a+b \sin[e+fx])^m (1 - \sin[e+fx]^2) dx$$

Program code:

```
Int[cos[e_.+f_.x_]^2*(d_.sin[e_.+f_.x_]^n*(a+b_.sin[e_.+f_.x_]^m_,x_Symbol] :=
  Int[(d*sin[e+f*x])^n*(a+b*sin[e+f*x])^m*(1-Sin[e+f*x]^2),x] /;
  FreeQ[{a,b,d,e,f,m,n},x] && NeQ[a^2-b^2,0] && (IGtQ[m,0] || IntegersQ[2*m,2*n])
```

$$2. \int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 \neq 0 \wedge (m \in \mathbb{Z}^+ \vee (2m | 2n) \in \mathbb{Z})$$

$$1. \int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 \neq 0 \wedge (m \in \mathbb{Z}^+ \vee (2m | 2n) \in \mathbb{Z}) \wedge m < -1$$

$$x: \int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 \neq 0 \wedge (2m | 2n) \in \mathbb{Z} \wedge m < -1 \wedge n < -1$$

Derivation: Algebraic expansion

$$\text{Basis: } \cos[z]^4 = 1 - 2 \sin[z]^2 + \sin[z]^4$$

Note: This produces a slightly simpler antiderivative when $m = -2$.

Rule: If $a^2 - b^2 \neq 0 \wedge (2m | 2n) \in \mathbb{Z} \wedge m < -1 \wedge n < -1$, then

$$\int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow$$

$$\frac{(a^2 - b^2) \cos[e+fx] \sin[e+fx]^{n+1} (a+b \sin[e+fx])^{m+1}}{a b^2 d (m+1)} -$$

$$\left((a^2 (n+1) - b^2 (m+n+2)) \cos[e+fx] \sin[e+fx]^{n+1} (a+b \sin[e+fx])^{m+2} \right) / (a^2 b^2 d (n+1) (m+1)) +$$

$$\frac{1}{a^2 b (n+1) (m+1)} \int \sin[e+fx]^{n+1} (a+b \sin[e+fx])^{m+1} dx.$$

$$(a^2 (n+1) (n+2) - b^2 (m+n+2) (m+n+3) + a b (m+1) \sin[e+fx] - (a^2 (n+1) (n+3) - b^2 (m+n+2) (m+n+4)) \sin[e+fx]^2) dx$$

Program code:

```
(* Int[cos[e_.+f_.*x_] ^4*sin[e_.+f_.*x_] ^n*(a_+b_.*sin[e_.+f_.*x_] ) ^m_,x_Symbol] :=
(a^2-b^2)*Cos[e+f*x]*Sin[e+f*x]^(n+1)*(a+b*Sine+f*x])^(m+1)/(a*b^2*d*(m+1)) -
(a^2*(n+1)-b^2*(m+n+2))*Cos[e+f*x]*Sin[e+f*x]^(n+1)*(a+b*Sine+f*x])^(m+2)/(a^2*b^2*d*(n+1)*(m+1)) +
1/(a^2*b*(n+1)*(m+1))*Int[Sine+f*x]^(n+1)*(a+b*Sine+f*x])^(m+1)*
Simp[a^2*(n+1)*(n+2)-b^2*(m+n+2)*(m+n+3)+a*b*(m+1)*Sine+f*x]-(a^2*(n+1)*(n+3)-b^2*(m+n+2)*(m+n+4))*Sine+f*x]^2,x],x] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0] && IntegersQ[2*m,2*n] && LtQ[m,-1] && LtQ[n,-1] *)
```

$$1: \int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 \neq 0 \wedge (2m \mid 2n) \in \mathbb{Z} \wedge m < -1 \wedge n < -1$$

Derivation: Algebraic expansion and sine recurrence 3b with $A \rightarrow 1$, $B \rightarrow 0$, $C \rightarrow -2$, $m \rightarrow n$, $n \rightarrow p$, 2b with $A \rightarrow -b(m+n+2)$, $B \rightarrow -a n$, $C \rightarrow b(n+p+3)$, $m \rightarrow n+1$, $n \rightarrow p$ and
2a with $A \rightarrow 0$, $B \rightarrow 0$, $C \rightarrow 1$, $m \rightarrow n+4-2$, $n \rightarrow p$

Basis: $\cos[z]^4 = 1 - 2 \sin[z]^2 + \sin[z]^4$

Rule: If $a^2 - b^2 \neq 0 \wedge (2m \mid 2n) \in \mathbb{Z} \wedge m < -1 \wedge n < -1$, then

$$\int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow$$

$$\int (d \sin[e+fx])^n (a+b \sin[e+fx])^m (1 - 2 \sin[e+fx]^2) dx + \frac{1}{d^4} \int (d \sin[e+fx])^{n+4} (a+b \sin[e+fx])^m dx \rightarrow$$

$$\frac{\cos[e+fx] (d \sin[e+fx])^{n+1} (a+b \sin[e+fx])^{m+1}}{a d f (n+1)} -$$

$$\left((a^2 (n+1) - b^2 (m+n+2)) \cos[e+fx] (d \sin[e+fx])^{n+2} (a+b \sin[e+fx])^{m+1} / (a^2 b d^2 f (n+1) (m+1)) + \right.$$

$$\left. \frac{1}{a^2 b d (n+1) (m+1)} \int (d \sin[e+fx])^{n+1} (a+b \sin[e+fx])^{m+1} \cdot \right.$$

$$\left. \left((a^2 (n+1) (n+2) - b^2 (m+n+2) (m+n+3) + a b (m+1) \sin[e+fx] - (a^2 (n+1) (n+3) - b^2 (m+n+2) (m+n+4)) \sin[e+fx]^2 \right) dx \right.$$

Program code:

```
Int[cos[e_.+f_.*x_]^4*(d_.*sin[e_.+f_.*x_] )^n*(a_.+b_.*sin[e_.+f_.*x_] )^m,x_Symbol] :=
Cos[e+f*x]*(d*sin[e+f*x])^(n+1)*(a+b*sin[e+f*x])^(m+1)/(a*d*f*(n+1)) -
(a^2*(n+1)-b^2*(m+n+2))*Cos[e+f*x]*(d*sin[e+f*x])^(n+2)*(a+b*sin[e+f*x])^(m+1)/(a^2*b*d^2*f*(n+1)*(m+1)) +
1/(a^2*b*d*(n+1)*(m+1))*Int[(d*sin[e+f*x])^(n+1)*(a+b*sin[e+f*x])^(m+1)*
Simp[a^2*(n+1)*(n+2)-b^2*(m+n+2)*(m+n+3)+a*b*(m+1)*Sin[e+f*x]-(a^2*(n+1)*(n+3)-b^2*(m+n+2)*(m+n+4))*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0] && IntegersQ[2*m,2*n] && LtQ[m,-1] && LtQ[n,-1]
```

$$2. \int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 \neq 0 \wedge (2m \mid 2n) \in \mathbb{Z} \wedge m < -1 \wedge n \neq -1$$

1:

$$\int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 \neq 0 \wedge (2m \mid 2n) \in \mathbb{Z} \wedge m < -1 \wedge n \neq -1 \wedge (m < -2 \vee m+n+4 = 0)$$

Derivation: Algebraic expansion

$$\text{Basis: } \cos[z]^4 = 1 - 2 \sin[z]^2 + \sin[z]^4$$

Rule: If $a^2 - b^2 \neq 0 \wedge (2m \mid 2n) \in \mathbb{Z} \wedge m < -1 \wedge n \neq -1 \wedge (m < -2 \vee m+n+4 = 0)$, then

$$\begin{aligned} & \int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow \\ & \frac{(a^2 - b^2) \cos[e+fx] (d \sin[e+fx])^{n+1} (a+b \sin[e+fx])^{m+1}}{a b^2 d f (m+1)} + \\ & \frac{((a^2 (n-m+1) - b^2 (m+n+2)) \cos[e+fx] (d \sin[e+fx])^{n+1} (a+b \sin[e+fx])^{m+2})}{a^2 b^2 (m+1) (m+2)} - \\ & \frac{1}{a^2 b^2 (m+1) (m+2)} \int (d \sin[e+fx])^n (a+b \sin[e+fx])^{m+2} dx \\ & (a^2 (n+1) (n+3) - b^2 (m+n+2) (m+n+3) + a b (m+2) \sin[e+fx] - (a^2 (n+2) (n+3) - b^2 (m+n+2) (m+n+4)) \sin[e+fx]^2) dx \end{aligned}$$

Program code:

```
Int[cos[e_.+f_.*x_]^4*(d_.*sin[e_.+f_.*x_]^n*(a+b_.*sin[e_.+f_.*x_]^m_,x_Symbol] :=
  (a^2-b^2)*Cos[e+f*x]*(a+b*sin[e+f*x])^(m+1)*(d*sin[e+f*x])^(n+1)/(a*b^2*d*f*(m+1)) +
  (a^2*(n-m+1)-b^2*(m+n+2))*Cos[e+f*x]*(a+b*sin[e+f*x])^(m+2)*(d*sin[e+f*x])^(n+1)/(a^2*b^2*d*f*(m+1)*(m+2)) -
  1/(a^2*b^2*(m+1)*(m+2))*Int[(a+b*sin[e+f*x])^(m+2)*(d*sin[e+f*x])^n*
    Simp[a^2*(n+1)*(n+3)-b^2*(m+n+2)*(m+n+3)+a*b*(m+2)*Sin[e+f*x]-(a^2*(n+2)*(n+3)-b^2*(m+n+2)*(m+n+4))*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,d,e,f,n},x] && NeQ[a^2-b^2,0] && IntegersQ[2*m,2*n] && LtQ[m,-1] && Not[LtQ[n,-1]] && (LtQ[m,-2] || EqQ[m+n+4,0])
```

$$\mathbf{2:} \int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 \neq 0 \wedge (2m \mid 2n) \in \mathbb{Z} \wedge m < -1 \wedge n \neq -1 \wedge m+n+4 \neq 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \cos[z]^4 = 1 - 2 \sin[z]^2 + \sin[z]^4$$

Rule: If $a^2 - b^2 \neq 0 \wedge (2m \mid 2n) \in \mathbb{Z} \wedge m < -1 \wedge n \neq -1 \wedge m+n+4 \neq 0$, then

$$\begin{aligned} & \int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow \\ & \frac{(a^2 - b^2) \cos[e+fx] (d \sin[e+fx])^{n+1} (a+b \sin[e+fx])^{m+1}}{a b^2 d f (m+1)} - \frac{\cos[e+fx] (d \sin[e+fx])^{n+1} (a+b \sin[e+fx])^{m+2}}{b^2 d f (m+n+4)} - \\ & \frac{1}{a b^2 (m+1) (m+n+4)} \int (d \sin[e+fx])^n (a+b \sin[e+fx])^{m+1} \cdot \\ & (a^2 (n+1) (n+3) - b^2 (m+n+2) (m+n+4) + a b (m+1) \sin[e+fx] - (a^2 (n+2) (n+3) - b^2 (m+n+3) (m+n+4)) \sin[e+fx]^2) dx \end{aligned}$$

Program code:

```
Int[cos[e_.+f_.*x_]^4*(d_.*sin[e_.+f_.*x_]^n*(a+b_.*sin[e_.+f_.*x_]^m,x_Symbol) :=
(a^2-b^2)*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+1)*(d*Sin[e+f*x])^(n+1)/(a*b^2*d*f*(m+1)) -
Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+2)*(d*Sin[e+f*x])^(n+1)/(b^2*d*f*(m+n+4)) -
1/(a*b^2*(m+1)*(m+n+4))*Int[(a+b*Sin[e+f*x])^(m+1)*(d*Sin[e+f*x])^n*
Simp[a^2*(n+1)*(n+3)-b^2*(m+n+2)*(m+n+4)+a*b*(m+1)*Sin[e+f*x]-(a^2*(n+2)*(n+3)-b^2*(m+n+3)*(m+n+4))*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,d,e,f,n},x] && NeQ[a^2-b^2,0] && IntegersQ[2*m,2*n] && LtQ[m,-1] && Not[LtQ[n,-1]] && NeQ[m+n+4,0]
```

$$\mathbf{2.} \int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 \neq 0 \wedge (m \in \mathbb{Z}^+ \vee (2m \mid 2n) \in \mathbb{Z}) \wedge m \neq -1$$

$$\mathbf{1.} \int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 \neq 0 \wedge (m \in \mathbb{Z}^+ \vee (2m \mid 2n) \in \mathbb{Z}) \wedge m \neq -1 \wedge n < -1$$

1:

$$\int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 \neq 0 \wedge (m \in \mathbb{Z}^+ \vee (2m \mid 2n) \in \mathbb{Z}) \wedge m \neq -1 \wedge n < -1 \wedge (n < -2 \vee m+n+4 = 0)$$

Derivation: Algebraic expansion and sine recurrence 3b with $A \rightarrow 1$, $B \rightarrow 0$, $C \rightarrow 0$, $m \rightarrow n$, $n \rightarrow p$ and 3b with $A \rightarrow -b (n+p+2)$, $B \rightarrow a (n+2)$, $C \rightarrow b (n+p+3)$, $m \rightarrow n+1$, $n \rightarrow p$

Basis: $\cos[z]^4 = 1 - 2 \sin[z]^2 + \sin[z]^4$

Rule: If $a^2 - b^2 \neq 0 \wedge (m \in \mathbb{Z}^+ \vee (2m \mid 2n) \in \mathbb{Z}) \wedge m \neq -1 \wedge n < -1 \wedge (n < -2 \vee m+n+4 = 0)$, then

$$\int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow$$

$$\int (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx - \frac{1}{d^2} \int (d \sin[e+fx])^{n+2} (a+b \sin[e+fx])^m (2 - \sin[e+fx]^2) dx \rightarrow$$

$$\frac{\cos[e+fx] (d \sin[e+fx])^{n+1} (a+b \sin[e+fx])^{m+1}}{a d f (n+1)} - \frac{b (m+n+2) \cos[e+fx] (d \sin[e+fx])^{n+2} (a+b \sin[e+fx])^{m+1}}{a^2 d^2 f (n+1) (n+2)} -$$

$$\frac{1}{a^2 d^2 (n+1) (n+2)} \int (d \sin[e+fx])^{n+2} (a+b \sin[e+fx])^m dx \cdot$$

$$(a^2 n (n+2) - b^2 (m+n+2) (m+n+3) + a b m \sin[e+fx] - (a^2 (n+1) (n+2) - b^2 (m+n+2) (m+n+4)) \sin[e+fx]^2) dx$$

Program code:

```
Int[cos[e_+f_.*x_]^4*(d_.*sin[e_+f_.*x_] )^n_*(a_+b_.*sin[e_+f_.*x_] )^m_,x_Symbol] :=
  Cos[e+f*x]*(a+b*sin[e+f*x])^(m+1)*(d*sin[e+f*x])^(n+1)/(a*d*f*(n+1)) -
  b*(m+n+2)*Cos[e+f*x]*(a+b*sin[e+f*x])^(m+1)*(d*sin[e+f*x])^(n+2)/(a^2*d^2*f*(n+1)*(n+2)) -
  1/(a^2*d^2*(n+1)*(n+2))*Int[(a+b*sin[e+f*x])^m*(d*sin[e+f*x])^(n+2)*
    Simp[a^2*n*(n+2)-b^2*(m+n+2)*(m+n+3)+a*b*m*sin[e+f*x]-(a^2*(n+1)*(n+2)-b^2*(m+n+2)*(m+n+4))*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,d,e,f,m},x] && NeQ[a^2-b^2,0] && (IGtQ[m,0] || IntegersQ[2*m,2*n]) && Not[m<-1] && LtQ[n,-1] && (LtQ[n,-2] || EqQ[m+n+4,0])
```

2:

$$\int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 \neq 0 \wedge (m \in \mathbb{Z}^+ \vee (2m \mid 2n) \in \mathbb{Z}) \wedge m \neq -1 \wedge n < -1 \wedge m+n+4 \neq 0$$

Derivation: Algebraic expansion and sine recurrence 3b with $A \rightarrow 1$, $B \rightarrow 0$, $C \rightarrow -2$, $m \rightarrow n$, $n \rightarrow p$ and 3a with $A \rightarrow 0$, $B \rightarrow 0$, $C \rightarrow 1$, $m \rightarrow n+4-2$, $n \rightarrow p$

Basis: $\cos[z]^4 = 1 - 2 \sin[z]^2 + \sin[z]^4$

Rule: If $a^2 - b^2 \neq 0 \wedge (m \in \mathbb{Z}^+ \vee (2m \mid 2n) \in \mathbb{Z}) \wedge m \neq -1 \wedge n < -1 \wedge m+n+4 \neq 0$, then

$$\begin{aligned} & \int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow \\ & \int (d \sin[e+fx])^n (a+b \sin[e+fx])^m (1 - 2 \sin[e+fx]^2) dx + \int (d \sin[e+fx])^{n+4} (a+b \sin[e+fx])^m dx \rightarrow \\ & \frac{\cos[e+fx] (d \sin[e+fx])^{n+1} (a+b \sin[e+fx])^{m+1}}{a d f (n+1)} - \frac{\cos[e+fx] (d \sin[e+fx])^{n+2} (a+b \sin[e+fx])^{m+1}}{b d^2 f (m+n+4)} + \\ & \frac{1}{a b d (n+1) (m+n+4)} \int (d \sin[e+fx])^{n+1} (a+b \sin[e+fx])^m \cdot \\ & (a^2 (n+1) (n+2) - b^2 (m+n+2) (m+n+4) + a b (m+3) \sin[e+fx] - (a^2 (n+1) (n+3) - b^2 (m+n+3) (m+n+4)) \sin[e+fx]^2) dx \end{aligned}$$

Program code:

```
Int[cos[e_.+f_.x_]^4*(d_.sin[e_.+f_.x_]^n*(a+b_.sin[e_.+f_.x_]^m,x_Symbol) :=
  Cos[e+f*x]*(a+b*sin[e+f*x])^(m+1)*(d*sin[e+f*x])^(n+1)/(a*d*f*(n+1)) -
  Cos[e+f*x]*(a+b*sin[e+f*x])^(m+1)*(d*sin[e+f*x])^(n+2)/(b*d^2*f*(m+n+4)) +
  1/(a*b*d*(n+1)*(m+n+4))*Int[(a+b*sin[e+f*x])^m*(d*sin[e+f*x])^(n+1)*
  Simp[a^2*(n+1)*(n+2)-b^2*(m+n+2)*(m+n+4)+a*b*(m+3)*Sin[e+f*x]-(a^2*(n+1)*(n+3)-b^2*(m+n+3)*(m+n+4))*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,d,e,f,m},x] && NeQ[a^2-b^2,0] && (IGtQ[m,0] || IntegersQ[2*m,2*n]) && Not[m<-1] && LtQ[n,-1] && NeQ[m+n+4,0]
```

2:

$$\int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 \neq 0 \wedge (m \in \mathbb{Z}^+ \vee (2m | 2n) \in \mathbb{Z}) \wedge m \neq -1 \wedge n \neq -1 \wedge m+n+3 \neq 0 \wedge m+n+4 \neq 0$$

Derivation: Algebraic expansion and sine recurrence 3a with $A \rightarrow 0$, $B \rightarrow 0$, $C \rightarrow 1$, $m \rightarrow n+4-2$, $n \rightarrow p$ and 3a with $A \rightarrow a(n+2)$, $B \rightarrow b(n+p+3)$, $C \rightarrow -a(n+3)$, $m \rightarrow n+1$, $n \rightarrow p$

Basis: $\cos[z]^4 = 1 - 2 \sin[z]^2 + \sin[z]^4$

Rule: If $a^2 - b^2 \neq 0 \wedge (m \in \mathbb{Z}^+ \vee (2m | 2n) \in \mathbb{Z}) \wedge m \neq -1 \wedge n \neq -1 \wedge m+n+3 \neq 0 \wedge m+n+4 \neq 0$, then

$$\begin{aligned} & \int \cos[e+fx]^4 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow \\ & \int (d \sin[e+fx])^n (a+b \sin[e+fx])^m (1 - 2 \sin[e+fx]^2) dx + \frac{1}{d^4} \int (d \sin[e+fx])^{n+4} (a+b \sin[e+fx])^m dx \rightarrow \\ & \frac{a(n+3) \cos[e+fx] (d \sin[e+fx])^{n+1} (a+b \sin[e+fx])^{m+1}}{b^2 d f (m+n+3) (m+n+4)} - \frac{\cos[e+fx] (d \sin[e+fx])^{n+2} (a+b \sin[e+fx])^{m+1}}{b d^2 f (m+n+4)} - \\ & \frac{1}{b^2 (m+n+3) (m+n+4)} \int (d \sin[e+fx])^n (a+b \sin[e+fx])^m \cdot \\ & (a^2 (n+1) (n+3) - b^2 (m+n+3) (m+n+4) + a b m \sin[e+fx] - (a^2 (n+2) (n+3) - b^2 (m+n+3) (m+n+5)) \sin[e+fx]^2) dx \end{aligned}$$

Program code:

```
Int[cos[e_.+f_.x_]^4*(d_.sin[e_.+f_.x_]^n*(a+b_.sin[e_.+f_.x_]^m,x_Symbol] :=
  a*(n+3)*Cos[e+f*x]*(d*sin[e+f*x])^(n+1)*(a+b*sin[e+f*x])^(m+1)/(b^2*d*f*(m+n+3)*(m+n+4)) -
  Cos[e+f*x]*(d*sin[e+f*x])^(n+2)*(a+b*sin[e+f*x])^(m+1)/(b*d^2*f*(m+n+4)) -
  1/(b^2*(m+n+3)*(m+n+4))*Int[(d*sin[e+f*x])^n*(a+b*sin[e+f*x])^m*
    Simp[a^2*(n+1)*(n+3)-b^2*(m+n+3)*(m+n+4)+a*b*m*sin[e+f*x]-(a^2*(n+2)*(n+3)-b^2*(m+n+3)*(m+n+5))*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,d,e,f,m,n},x] && NeQ[a^2-b^2,0] && (IGtQ[m,0] || IntegersQ[2*m,2*n]) && Not[m<-1] && Not[LtQ[n,-1]] && NeQ[m+n+3,0] && NeQ[m+n+4,0]
```

$$\mathbf{3:} \int \cos[e+fx]^6 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 \neq 0 \wedge (2m | 2n) \in \mathbb{Z} \wedge n \neq -1 \wedge n \neq -2 \wedge m+n+5 \neq 0 \wedge m+n+6 \neq 0$$

Derivation: Algebraic expansion and sine recurrence 3b with $A \rightarrow 1$, $B \rightarrow 0$, $C \rightarrow -3$, $m \rightarrow n$, $n \rightarrow p$, 3b with $A \rightarrow -b(2+n+p)$, $B \rightarrow a(2+n-3(1+n))$, $C \rightarrow b(3+n+p)$, $m \rightarrow n+1$, $n \rightarrow p$,

3a with $A \rightarrow 3$, $B \rightarrow 0$, $C \rightarrow -1$, $m \rightarrow n+4$, $n \rightarrow p$ and 3a with
 $A \rightarrow -a(4+n)$, $B \rightarrow b(-5-n-p+3(6+n+p))$, $C \rightarrow a(5+n)$, $m \rightarrow n+3$, $n \rightarrow p$

Basis: $\cos[z]^6 = 1 - 3 \sin[z]^2 + \sin[z]^4 (3 - \sin[z]^2)$

Rule: If $a^2 - b^2 \neq 0 \wedge (2m \mid 2n) \in \mathbb{Z} \wedge n \neq -1 \wedge n \neq -2 \wedge m+n+5 \neq 0 \wedge m+n+6 \neq 0$, then

$$\int \cos[e+fx]^6 (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow$$

$$\int (d \sin[e+fx])^n (a+b \sin[e+fx])^m (1-3 \sin[e+fx]^2) dx + \frac{1}{d^4} \int (d \sin[e+fx])^{n+4} (a+b \sin[e+fx])^m (3-\sin[e+fx]^2) dx \rightarrow$$

$$\frac{\cos[e+fx] (d \sin[e+fx])^{n+1} (a+b \sin[e+fx])^{m+1}}{a d f (n+1)} - \frac{b (m+n+2) \cos[e+fx] (d \sin[e+fx])^{n+2} (a+b \sin[e+fx])^{m+1}}{a^2 d^2 f (n+1) (n+2)} -$$

$$\frac{a (n+5) \cos[e+fx] (d \sin[e+fx])^{n+3} (a+b \sin[e+fx])^{m+1}}{b^2 d^3 f (m+n+5) (m+n+6)} + \frac{\cos[e+fx] (d \sin[e+fx])^{n+4} (a+b \sin[e+fx])^{m+1}}{b d^4 f (m+n+6)} +$$

$$\frac{1}{a^2 b^2 d^2 (n+1) (n+2) (m+n+5) (m+n+6)} \int (d \sin[e+fx])^{n+2} (a+b \sin[e+fx])^m \cdot$$

$$(a^4 (n+1) (n+2) (n+3) (n+5) - a^2 b^2 (n+2) (2n+1) (m+n+5) (m+n+6) + b^4 (m+n+2) (m+n+3) (m+n+5) (m+n+6) +$$

$$a b m (a^2 (n+1) (n+2) - b^2 (m+n+5) (m+n+6)) \sin[e+fx] -$$

$$(a^4 (n+1) (n+2) (4+n) (n+5) + b^4 (m+n+2) (m+n+4) (m+n+5) (m+n+6) - a^2 b^2 (n+1) (n+2) (m+n+5) (2n+2m+13)) \sin[e+fx]^2 dx$$

Program code:

```
Int[cos[e_.+f_.*x_]^6*(d_.*sin[e_.+f_.*x_]^n*(a_.+b_.*sin[e_.+f_.*x_]^m,x_Symbol] :=
Cos[e+f*x]*(d*sin[e+f*x])^(n+1)*(a+b*sin[e+f*x])^(m+1)/(a*d*f*(n+1)) -
b*(m+n+2)*Cos[e+f*x]*(d*sin[e+f*x])^(n+2)*(a+b*sin[e+f*x])^(m+1)/(a^2*d^2*f*(n+1)*(n+2)) -
a*(n+5)*Cos[e+f*x]*(d*sin[e+f*x])^(n+3)*(a+b*sin[e+f*x])^(m+1)/(b^2*d^3*f*(m+n+5)*(m+n+6)) +
Cos[e+f*x]*(d*sin[e+f*x])^(n+4)*(a+b*sin[e+f*x])^(m+1)/(b*d^4*f*(m+n+6)) +
1/(a^2*b^2*d^2*(n+1)*(n+2)*(m+n+5)*(m+n+6))*
Int[(d*sin[e+f*x])^(n+2)*(a+b*sin[e+f*x])^m*
Simp[a^4*(n+1)*(n+2)*(n+3)*(n+5)-a^2*b^2*(n+2)*(2*n+1)*(m+n+5)*(m+n+6)+b^4*(m+n+2)*(m+n+3)*(m+n+5)*(m+n+6) +
a*b*m*(a^2*(n+1)*(n+2)-b^2*(m+n+5)*(m+n+6))*Sin[e+f*x] -
(a^4*(n+1)*(n+2)*(4+n)*(n+5)+b^4*(m+n+2)*(m+n+4)*(m+n+5)*(m+n+6)-a^2*b^2*(n+1)*(n+2)*(m+n+5)*(2*n+2*m+13))*Sin[e+f*x]^2,x],x
FreeQ[{a,b,d,e,f,m,n},x] && NeQ[a^2-b^2,0] && IntegersQ[2*m,2*n] && NeQ[n,-1] && NeQ[n,-2] && NeQ[m+n+5,0] && NeQ[m+n+6,0] && Not[IG
```

3: $\int \cos[e+fx]^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx$ when $a^2 - b^2 \neq 0 \wedge (m \mid 2n \mid \frac{p}{2}) \in \mathbb{Z} \wedge (m < -1 \vee m = 1 \wedge p > 0)$

Derivation: Algebraic expansion

Basis: $\cos[z]^2 = 1 - \sin[z]^2$

Rule: If $a^2 - b^2 \neq 0 \wedge (m \mid 2n \mid \frac{p}{2}) \in \mathbb{Z} \wedge (m < -1 \vee m = 1 \wedge p > 0)$, then

$$\int \cos[e+fx]^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow \int \text{ExpandTrig}[(d \sin[e+fx])^n (a+b \sin[e+fx])^m (1 - \sin[e+fx]^2)^{p/2}, x] dx$$

Program code:

```
Int[cos[e_.+f_.*x_]^p_*(d_.*sin[e_.+f_.*x_]^n_*(a_.+b_.*sin[e_.+f_.*x_]^m_,x_Symbol] :=
  Int[ExpandTrig[(d*sin[e+f*x])^n*(a+b*sin[e+f*x])^m*(1-sin[e+f*x]^2)^(p/2),x],x] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0] && IntegersQ[m,2*n,p/2] && (LtQ[m,-1] || EqQ[m,-1] && GtQ[p,0])
```

$$4. \int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \text{ when } a^2 - b^2 \neq 0$$

$$1: \int \frac{(g \cos[e+fx])^p \sin[e+fx]^n}{a+b \sin[e+fx]} dx \text{ when } a^2 - b^2 \neq 0 \wedge n \in \mathbb{Z} \wedge (n < 0 \vee p + \frac{1}{2} \in \mathbb{Z}^+)$$

Derivation: Algebraic expansion

Rule: If $a^2 - b^2 \neq 0 \wedge n \in \mathbb{Z} \wedge (n < 0 \vee p + \frac{1}{2} \in \mathbb{Z}^+)$, then

$$\int \frac{(g \cos[e+fx])^p \sin[e+fx]^n}{a+b \sin[e+fx]} dx \rightarrow \int (g \cos[e+fx])^p \text{ExpandTrig}\left[\frac{\sin[e+fx]^n}{a+b \sin[e+fx]}, x\right] dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_.sin[e_.+f_.x_]^n_/(a_.+b_.sin[e_.+f_.x_]),x_Symbol] :=
  Int[ExpandTrig[(g*cos[e+f*x])^p,sin[e+f*x]^n/(a+b*sin[e+f*x]),x],x] /;
FreeQ[{a,b,e,f,g,p},x] && NeQ[a^2-b^2,0] && IntegerQ[n] && (LtQ[n,0] || IGtQ[p+1/2,0])
```

$$2. \int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \text{ when } a^2 - b^2 \neq 0 \wedge (2n \mid 2p) \in \mathbb{Z} \wedge p > 1$$

$$1. \int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \text{ when } a^2 - b^2 \neq 0 \wedge (2n \mid 2p) \in \mathbb{Z} \wedge p > 1 \wedge n < -1$$

$$1: \int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \text{ when } a^2 - b^2 \neq 0 \wedge (2n \mid 2p) \in \mathbb{Z} \wedge p > 1 \wedge n \leq -2$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{\cos[z]^2}{a+b \sin[z]} = \frac{1}{a} - \frac{b \sin[z]}{a^2} - \frac{(a^2-b^2) \sin[z]^2}{a^2 (a+b \sin[z])}$$

Rule: If $a^2 - b^2 \neq 0 \wedge (2n \mid 2p) \in \mathbb{Z} \wedge p > 1 \wedge n \leq -2$, then

$$\int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \rightarrow$$

$$\frac{g^2}{a} \int (g \cos[e+fx])^{p-2} (d \sin[e+fx])^n dx - \frac{b g^2}{a^2 d} \int (g \cos[e+fx])^{p-2} (d \sin[e+fx])^{n+1} dx - \frac{g^2 (a^2 - b^2)}{a^2 d^2} \int \frac{(g \cos[e+fx])^{p-2} (d \sin[e+fx])^{n+2}}{a+b \sin[e+fx]} dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(d_.sin[e_.+f_.x_])^n_/(a_+b_.sin[e_.+f_.x_]),x_Symbol] :=
  g^2/a*Int[(g*cos[e+f*x])^(p-2)*(d*sin[e+f*x])^n,x] -
  b*g^2/(a^2*d)*Int[(g*cos[e+f*x])^(p-2)*(d*sin[e+f*x])^(n+1),x] -
  g^2*(a^2-b^2)/(a^2*d^2)*Int[(g*cos[e+f*x])^(p-2)*(d*sin[e+f*x])^(n+2)/(a+b*sin[e+f*x]),x] /;
FreeQ[{a,b,d,e,f,g},x] && NeQ[a^2-b^2,0] && IntegersQ[2*n,2*p] && GtQ[p,1] && (LeQ[n,-2] || EqQ[n,-3/2] && EqQ[p,3/2])
```

2: $\int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx$ when $a^2 - b^2 \neq 0 \wedge (2n \mid 2p) \in \mathbb{Z} \wedge p > 1 \wedge n < -1$

Derivation: Algebraic expansion

Basis: $\frac{(g \cos[z])^p (d \sin[z])^n}{a+b \sin[z]} == \frac{g^2 (g \cos[z])^{p-2} (d \sin[z])^n (b-a \sin[z])}{a b} + \frac{g^2 (a^2-b^2) (g \cos[z])^{p-2} (d \sin[z])^{n+1}}{a b d (a+b \sin[z])}$

Rule: If $a^2 - b^2 \neq 0 \wedge (2n \mid 2p) \in \mathbb{Z} \wedge p > 1 \wedge n < -1$, then

$$\int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \rightarrow$$

$$\frac{g^2}{a b} \int (g \cos[e+fx])^{p-2} (d \sin[e+fx])^n (b-a \sin[e+fx]) dx + \frac{g^2 (a^2 - b^2)}{a b d} \int \frac{(g \cos[e+fx])^{p-2} (d \sin[e+fx])^{n+1}}{a+b \sin[e+fx]} dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(d_.sin[e_.+f_.x_])^n_/(a_+b_.sin[e_.+f_.x_]),x_Symbol] :=
  g^2/(a*b)*Int[(g*cos[e+f*x])^(p-2)*(d*sin[e+f*x])^n*(b-a*sin[e+f*x]),x] +
  g^2*(a^2-b^2)/(a*b*d)*Int[(g*cos[e+f*x])^(p-2)*(d*sin[e+f*x])^(n+1)/(a+b*sin[e+f*x]),x] /;
FreeQ[{a,b,d,e,f,g},x] && NeQ[a^2-b^2,0] && IntegersQ[2*n,2*p] && GtQ[p,1] && (LtQ[n,-1] || EqQ[p,3/2] && EqQ[n,-1/2])
```

$$\mathbf{2:} \int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \text{ when } a^2 - b^2 \neq 0 \wedge (2n | 2p) \in \mathbb{Z} \wedge p > 1$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{(g \cos[z])^p}{a+b \sin[z]} = \frac{g^2 (g \cos[z])^{p-2} (a-b \sin[z])}{b^2} - \frac{g^2 (a^2-b^2) (g \cos[z])^{p-2}}{b^2 (a+b \sin[z])}$$

Rule: If $a^2 - b^2 \neq 0 \wedge (2n | 2p) \in \mathbb{Z} \wedge p > 1$, then

$$\int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \rightarrow \frac{g^2}{b^2} \int (g \cos[e+fx])^{p-2} (d \sin[e+fx])^n (a-b \sin[e+fx]) dx - \frac{g^2 (a^2-b^2)}{b^2} \int \frac{(g \cos[e+fx])^{p-2} (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p*(d_.sin[e_.+f_.x_])^n/(a+b_.sin[e_.+f_.x_]),x_Symbol] :=
  g^2/b^2*Int[(g*cos[e+f*x])^(p-2)*(d*sin[e+f*x])^n*(a-b*sin[e+f*x]),x] -
  g^2*(a^2-b^2)/b^2*Int[(g*cos[e+f*x])^(p-2)*(d*sin[e+f*x])^n/(a+b*sin[e+f*x]),x] /;
FreeQ[{a,b,d,e,f,g},x] && NeQ[a^2-b^2,0] && IntegersQ[2*n,2*p] && GtQ[p,1]
```

$$\mathbf{x:} \int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \text{ when } a^2 - b^2 \neq 0 \wedge (2n | 2p) \in \mathbb{Z} \wedge p > 1$$

Derivation: Algebraic expansion

$$\text{Basis: } (g \cos[z])^p (d \sin[z])^n = g^2 (g \cos[z])^{p-2} (d \sin[z])^n - \frac{g^2 (g \cos[z])^{p-2} (d \sin[z])^{n+2}}{d^2}$$

Rule: If $a^2 - b^2 \neq 0 \wedge (2n | 2p) \in \mathbb{Z} \wedge p > 1$, then

$$\int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \rightarrow g^2 \int \frac{(g \cos[e+fx])^{p-2} (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx - \frac{g^2}{d^2} \int \frac{(g \cos[e+fx])^{p-2} (d \sin[e+fx])^{n+2}}{a+b \sin[e+fx]} dx$$

Program code:

```
(* Int[(g_.cos[e_.+f_.x_])^p_*(d_.sin[e_.+f_.x_])^n_/(a_+b_.sin[e_.+f_.x_]),x_Symbol] :=
  g^2*Int[(g*cos[e+f*x])^(p-2)*(d*sin[e+f*x])^n/(a+b*sin[e+f*x]),x] -
  g^2/d^2*Int[(g*cos[e+f*x])^(p-2)*(d*sin[e+f*x])^(n+2)/(a+b*sin[e+f*x]),x] /;
FreeQ[{a,b,d,e,f,g},x] && NeQ[a^2-b^2,0] && IntegersQ[2*n,2*p] && GtQ[p,1] *)
```

$$3. \int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \text{ when } a^2 - b^2 \neq 0 \wedge (2n \mid 2p) \in \mathbb{Z} \wedge p < -1$$

$$1: \int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \text{ when } a^2 - b^2 \neq 0 \wedge (2n \mid 2p) \in \mathbb{Z} \wedge p < -1 \wedge n > 1$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{\sin[z]^2}{a+b \sin[z]} = \frac{a}{a^2-b^2} - \frac{b \sin[z]}{a^2-b^2} - \frac{a^2 \cos[z]^2}{(a^2-b^2)(a+b \sin[z])}$$

Rule: If $a^2 - b^2 \neq 0 \wedge (2n \mid 2p) \in \mathbb{Z} \wedge p < -1 \wedge n > 1$, then

$$\int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \rightarrow \frac{a d^2}{a^2 - b^2} \int (g \cos[e+fx])^p (d \sin[e+fx])^{n-2} dx - \frac{b d}{a^2 - b^2} \int (g \cos[e+fx])^p (d \sin[e+fx])^{n-1} dx - \frac{a^2 d^2}{g^2 (a^2 - b^2)} \int \frac{(g \cos[e+fx])^{p+2} (d \sin[e+fx])^{n-2}}{a+b \sin[e+fx]} dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p*(d_.sin[e_.+f_.x_])^n/(a_.+b_.sin[e_.+f_.x_]),x_Symbol] :=
  a*d^2/(a^2-b^2)*Int[(g*cos[e+f*x])^p*(d*sin[e+f*x])^(n-2),x] -
  b*d/(a^2-b^2)*Int[(g*cos[e+f*x])^p*(d*sin[e+f*x])^(n-1),x] -
  a^2*d^2/(g^2*(a^2-b^2))*Int[(g*cos[e+f*x])^(p+2)*(d*sin[e+f*x])^(n-2)/(a+b*sin[e+f*x]),x] /;
FreeQ[{a,b,d,e,f,g},x] && NeQ[a^2-b^2,0] && IntegersQ[2*n,2*p] && LtQ[p,-1] && GtQ[n,1]
```

$$\mathbf{2:} \int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \text{ when } a^2 - b^2 \neq 0 \wedge (2n \mid 2p) \in \mathbb{Z} \wedge p < -1 \wedge n > 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{(g \cos[z])^p (d \sin[z])^n}{a+b \sin[z]} == -\frac{d (g \cos[z])^p (d \sin[z])^{n-1} (b-a \sin[z])}{a^2-b^2} + \frac{a b d (g \cos[z])^{p+2} (d \sin[z])^{n-1}}{g^2 (a^2-b^2) (a+b \sin[z])}$$

Rule: If $a^2 - b^2 \neq 0 \wedge (2n \mid 2p) \in \mathbb{Z} \wedge p < -1 \wedge n > 0$, then

$$\int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \rightarrow$$

$$-\frac{d}{a^2-b^2} \int (g \cos[e+fx])^p (d \sin[e+fx])^{n-1} (b-a \sin[e+fx]) dx + \frac{a b d}{g^2 (a^2-b^2)} \int \frac{(g \cos[e+fx])^{p+2} (d \sin[e+fx])^{n-1}}{a+b \sin[e+fx]} dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(d_.sin[e_.+f_.x_])^n_/ (a+b_.sin[e_.+f_.x_]),x_Symbol] :=
  -d/(a^2-b^2)*Int[(g*cos[e+f*x])^p*(d*sin[e+f*x])^(n-1)*(b-a*sin[e+f*x]),x] +
  a*b*d/(g^2*(a^2-b^2))*Int[(g*cos[e+f*x])^(p+2)*(d*sin[e+f*x])^(n-1)/(a+b*sin[e+f*x]),x] /;
FreeQ[{a,b,d,e,f,g},x] && NeQ[a^2-b^2,0] && IntegersQ[2*n,2*p] && LtQ[p,-1] && GtQ[n,0]
```

$$\mathbf{3:} \int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \text{ when } a^2 - b^2 \neq 0 \wedge (2n \mid 2p) \in \mathbb{Z} \wedge p < -1$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{(g \cos[z])^p}{a+b \sin[z]} == \frac{g^2 (g \cos[z])^p (a-b \sin[z])}{g^2 (a^2-b^2)} - \frac{b^2 (g \cos[z])^{p+2}}{g^2 (a^2-b^2) (a+b \sin[z])}$$

Rule: If $a^2 - b^2 \neq 0 \wedge (2n \mid 2p) \in \mathbb{Z} \wedge p < -1$, then

$$\int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \rightarrow$$

$$\frac{1}{a^2 - b^2} \int (g \cos[e + f x])^p (d \sin[e + f x])^n (a - b \sin[e + f x]) \, dx - \frac{b^2}{g^2 (a^2 - b^2)} \int \frac{(g \cos[e + f x])^{p+2} (d \sin[e + f x])^n}{a + b \sin[e + f x]} \, dx$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_])^p_*(d_.*sin[e_.+f_.*x_])^n_/(a_+b_.*sin[e_.+f_.*x_]),x_Symbol] :=
  1/(a^2-b^2)*Int[(g*Cos[e+f*x])^p*(d*Sin[e+f*x])^n*(a-b*Sin[e+f*x]),x] -
  b^2/(g^2*(a^2-b^2))*Int[(g*Cos[e+f*x])^(p+2)*(d*Sin[e+f*x])^n/(a+b*Sin[e+f*x]),x] /;
FreeQ[{a,b,d,e,f,g},x] && NeQ[a^2-b^2,0] && IntegersQ[2*n,2*p] && LtQ[p,-1]
```

$$4. \int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \text{ when } a^2 - b^2 \neq 0 \wedge (2n | 2p) \in \mathbb{Z} \wedge -1 < p < 1$$

$$1. \int \frac{\sqrt{g \cos[e+fx]}}{\sqrt{d \sin[e+fx]} (a+b \sin[e+fx])} dx \text{ when } a^2 - b^2 \neq 0$$

$$1: \int \frac{\sqrt{g \cos[e+fx]}}{\sqrt{\sin[e+fx]} (a+b \sin[e+fx])} dx \text{ when } a^2 - b^2 \neq 0$$

Derivation: Integration by substitution

$$\text{Basis: } \frac{\sqrt{g \cos[e+fx]}}{\sqrt{\sin[e+fx]} (a+b \sin[e+fx])} = -\frac{4\sqrt{2}g}{f} \text{Subst}\left[\frac{x^2}{((a+b)g^2 + (a-b)x^4)\sqrt{1-\frac{x^4}{g^2}}}, x, \frac{\sqrt{g \cos[e+fx]}}{\sqrt{1+\sin[e+fx]}}\right] \partial_x \frac{\sqrt{g \cos[e+fx]}}{\sqrt{1+\sin[e+fx]}}$$

Rule: If $a^2 - b^2 \neq 0$, then

$$\int \frac{\sqrt{g \cos[e+fx]}}{\sqrt{\sin[e+fx]} (a+b \sin[e+fx])} dx \rightarrow -\frac{4\sqrt{2}g}{f} \text{Subst}\left[\int \frac{x^2}{((a+b)g^2 + (a-b)x^4)\sqrt{1-\frac{x^4}{g^2}}} dx, x, \frac{\sqrt{g \cos[e+fx]}}{\sqrt{1+\sin[e+fx]}}\right]$$

Program code:

```
Int[Sqrt[g_.cos[e_.+f_.x_]]/(Sqrt[sin[e_.+f_.x_]]*(a_+b_.sin[e_.+f_.x_])),x_Symbol] :=
  -4*Sqrt[2]*g/f*Subst[Int[x^2/(((a+b)*g^2+(a-b)*x^4)*Sqrt[1-x^4/g^2]),x],x,Sqrt[g*cos[e+f*x]]/Sqrt[1+Sin[e+f*x]]] /;
FreeQ[{a,b,e,f,g},x] && NeQ[a^2-b^2,0]
```

2:
$$\int \frac{\sqrt{g \cos[e+fx]}}{\sqrt{d \sin[e+fx]} (a+b \sin[e+fx])} dx \text{ when } a^2 - b^2 \neq 0$$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{\sqrt{\sin[e+fx]}}{\sqrt{d \sin[e+fx]}} = 0$

Rule: If $a^2 - b^2 \neq 0$, then

$$\int \frac{\sqrt{g \cos[e+fx]}}{\sqrt{d \sin[e+fx]} (a+b \sin[e+fx])} dx \rightarrow \frac{\sqrt{\sin[e+fx]}}{\sqrt{d \sin[e+fx]}} \int \frac{\sqrt{g \cos[e+fx]}}{\sqrt{\sin[e+fx]} (a+b \sin[e+fx])} dx$$

Program code:

```
Int[Sqrt[g_.cos[e_.+f_.x_]]/(Sqrt[d_.sin[e_.+f_.x_]]*(a_.+b_.sin[e_.+f_.x_])),x_Symbol] :=
  Sqrt[Sin[e+f*x]]/Sqrt[d*Sin[e+f*x]]*Int[Sqrt[g*cos[e+f*x]]/(Sqrt[Sin[e+f*x]]*(a+b*Sin[e+f*x])),x] /;
FreeQ[{a,b,d,e,f,g},x] && NeQ[a^2-b^2,0]
```

$$2. \int \frac{\sqrt{d \sin[e+fx]}}{\sqrt{g \cos[e+fx]} (a+b \sin[e+fx])} dx \text{ when } a^2 - b^2 \neq 0$$

$$1: \int \frac{\sqrt{d \sin[e+fx]}}{\sqrt{\cos[e+fx]} (a+b \sin[e+fx])} dx \text{ when } a^2 - b^2 \neq 0$$

Derivation: Integration by substitution and algebraic expansion

$$\text{Basis: } \frac{\sqrt{d \sin[e+fx]}}{\sqrt{\cos[e+fx]} (a+b \sin[e+fx])} = \frac{4\sqrt{2}d}{f} \text{Subst} \left[\frac{x^2}{(a d^2 + 2 b d x^2 + a x^4) \sqrt{1 - \frac{x^4}{d^2}}}, x, \frac{\sqrt{d \sin[e+fx]}}{\sqrt{1 + \cos[e+fx]}} \right] \partial_x \frac{\sqrt{d \sin[e+fx]}}{\sqrt{1 + \cos[e+fx]}}$$

$$\text{Basis: Let } q \rightarrow \sqrt{-a^2 + b^2}, \text{ then } \frac{x^2}{a d^2 + 2 b d x^2 + a x^4} = \frac{b+q}{2 q (d (b+q) + a x^2)} - \frac{b-q}{2 q (d (b-q) + a x^2)}$$

Rule: If $a^2 - b^2 \neq 0$, let $q \rightarrow \sqrt{-a^2 + b^2}$, then

$$\int \frac{\sqrt{d \sin[e+fx]}}{\sqrt{\cos[e+fx]} (a+b \sin[e+fx])} dx \rightarrow \frac{4\sqrt{2}d}{f} \text{Subst} \left[\int \frac{x^2}{(a d^2 + 2 b d x^2 + a x^4) \sqrt{1 - \frac{x^4}{d^2}}} dx, x, \frac{\sqrt{d \sin[e+fx]}}{\sqrt{1 + \cos[e+fx]}} \right]$$

$$\rightarrow \frac{2\sqrt{2}d(b+q)}{f q} \text{Subst} \left[\int \frac{1}{(d(b+q) + a x^2) \sqrt{1 - \frac{x^4}{d^2}}} dx, x, \frac{\sqrt{d \sin[e+fx]}}{\sqrt{1 + \cos[e+fx]}} \right] -$$

$$\frac{2\sqrt{2}d(b-q)}{f q} \text{Subst} \left[\int \frac{1}{(d(b-q) + a x^2) \sqrt{1 - \frac{x^4}{d^2}}} dx, x, \frac{\sqrt{d \sin[e+fx]}}{\sqrt{1 + \cos[e+fx]}} \right]$$

Program code:

```
Int[Sqrt[d_.sin[e_.+f_.x_]]/(Sqrt[cos[e_.+f_.x_]]*(a_+b_.sin[e_.+f_.x_])),x_Symbol] :=
  With[{q=Rt[-a^2+b^2,2]},
    2*Sqrt[2]*d*(b+q)/(f*q)*Subst[Int[1/((d*(b+q)+a*x^2)*Sqrt[1-x^4/d^2]),x],x,Sqrt[d*Sin[e+f*x]]/Sqrt[1+Cos[e+f*x]]] -
    2*Sqrt[2]*d*(b-q)/(f*q)*Subst[Int[1/((d*(b-q)+a*x^2)*Sqrt[1-x^4/d^2]),x],x,Sqrt[d*Sin[e+f*x]]/Sqrt[1+Cos[e+f*x]]]] /;
  FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0]
```

2:
$$\int \frac{\sqrt{d \sin[e+fx]}}{\sqrt{g \cos[e+fx]} (a+b \sin[e+fx])} dx \text{ when } a^2 - b^2 \neq 0$$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{\sqrt{\cos[e+fx]}}{\sqrt{g \cos[e+fx]}} = 0$

Rule: If $a^2 - b^2 \neq 0$, then

$$\int \frac{\sqrt{d \sin[e+fx]}}{\sqrt{g \cos[e+fx]} (a+b \sin[e+fx])} dx \rightarrow \frac{\sqrt{\cos[e+fx]}}{\sqrt{g \cos[e+fx]}} \int \frac{\sqrt{d \sin[e+fx]}}{\sqrt{\cos[e+fx]} (a+b \sin[e+fx])} dx$$

Program code:

```
Int[Sqrt[d_.sin[e_.+f_.x_]]/(Sqrt[g_.cos[e_.+f_.x_]]*(a_+b_.sin[e_.+f_.x_])),x_Symbol] :=
  Sqrt[Cos[e+f*x]]/Sqrt[g*cos[e+f*x]]*Int[Sqrt[d*Sin[e+f*x]]/(Sqrt[Cos[e+f*x]]*(a+b*Sin[e+f*x])),x] /;
  FreeQ[{a,b,d,e,f,g},x] && NeQ[a^2-b^2,0]
```

$$\text{3: } \int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \text{ when } a^2 - b^2 \neq 0 \wedge (2n \mid 2p) \in \mathbb{Z} \wedge -1 < p < 1 \wedge n > 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{(dz)^n}{a+bz} = \frac{d(dz)^{n-1}}{b} - \frac{ad(dz)^{n-1}}{b(a+bz)}$$

Rule: If $a^2 - b^2 \neq 0 \wedge (2n \mid 2p) \in \mathbb{Z} \wedge -1 < p < 1 \wedge n > 0$, then

$$\int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \rightarrow \frac{d}{b} \int (g \cos[e+fx])^p (d \sin[e+fx])^{n-1} dx - \frac{ad}{b} \int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^{n-1}}{a+b \sin[e+fx]} dx$$

Program code:

```
Int[(g_.*cos[e_.+f_.**x_])^p_*(d_.*sin[e_.+f_.**x_])^n_/(a_+b_.*sin[e_.+f_.**x_]),x_Symbol] :=
  d/b*Int[(g*Cos[e+f*x])^p*(d*Sin[e+f*x])^(n-1),x] -
  a*d/b*Int[(g*Cos[e+f*x])^p*(d*Sin[e+f*x])^(n-1)/(a+b*Sin[e+f*x]),x] /;
FreeQ[{a,b,d,e,f,g},x] && NeQ[a^2-b^2,0] && IntegersQ[2*n,2*p] && LtQ[-1,p,1] && GtQ[n,0]
```

$$4: \int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \text{ when } a^2 - b^2 \neq 0 \wedge (2n \mid 2p) \in \mathbb{Z} \wedge -1 < p < 1 \wedge n < 0$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{(dz)^n}{a+bz} = \frac{(dz)^n}{a} - \frac{b(dz)^{n+1}}{ad(a+bz)}$$

Rule: If $a^2 - b^2 \neq 0 \wedge (2n \mid 2p) \in \mathbb{Z} \wedge -1 < p < 1 \wedge n < 0$, then

$$\int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^n}{a+b \sin[e+fx]} dx \rightarrow \frac{1}{a} \int (g \cos[e+fx])^p (d \sin[e+fx])^n dx - \frac{b}{ad} \int \frac{(g \cos[e+fx])^p (d \sin[e+fx])^{n+1}}{a+b \sin[e+fx]} dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(d_.sin[e_.+f_.x_])^n_/(a_+b_.sin[e_.+f_.x_]),x_Symbol] :=
  1/a*Int[(g*cos[e+f*x])^p*(d*sin[e+f*x])^n,x] -
  b/(a*d)*Int[(g*cos[e+f*x])^p*(d*sin[e+f*x])^(n+1)/(a+b*sin[e+f*x]),x] /;
FreeQ[{a,b,d,e,f,g},x] && NeQ[a^2-b^2,0] && IntegersQ[2*n,2*p] && LtQ[-1,p,1] && LtQ[n,0]
```

$$5. \int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 \neq 0 \wedge m \in \mathbb{Z} \wedge (m > 0 \vee n \in \mathbb{Z})$$

$$1: \int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^2 dx \text{ when } a^2 - b^2 \neq 0$$

Derivation: Algebraic expansion

Rule: If $a^2 - b^2 \neq 0$, then

$$\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^2 dx \rightarrow \frac{2ab}{d} \int (g \cos[e+fx])^p (d \sin[e+fx])^{n+1} dx + \int (g \cos[e+fx])^p (d \sin[e+fx])^n (a^2 + b^2 \sin[e+fx]^2) dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(d_.sin[e_.+f_.x_])^n_*(a+b_.sin[e_.+f_.x_])^2,x_Symbol] :=
  2*a*b/d*Int[(g*cos[e+f*x])^p*(d*sin[e+f*x])^(n+1),x] +
  Int[(g*cos[e+f*x])^p*(d*sin[e+f*x])^n*(a^2+b^2*sin[e+f*x]^2),x] /;
FreeQ[{a,b,d,e,f,g,n,p},x] && NeQ[a^2-b^2,0]
```


$$\mathbf{2:} \int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \text{ when } a^2 - b^2 \neq 0 \wedge m \in \mathbb{Z} \wedge (m > 0 \vee n \in \mathbb{Z})$$

Derivation: Algebraic expansion

Rule: If $a^2 - b^2 \neq 0 \wedge m \in \mathbb{Z} \wedge (m > 0 \vee n \in \mathbb{Z})$, then

$$\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow \int (g \cos[e+fx])^p \text{ExpandTrig}[(d \sin[e+fx])^n (a+b \sin[e+fx])^m, x] dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(d_.sin[e_.+f_.x_])^n_*(a+b_.sin[e_.+f_.x_])^m_,x_Symbol] :=
  Int[ExpandTrig[(g*cos[e+f*x])^p,(d*sin[e+f*x])^n*(a+b*sin[e+f*x])^m,x],x] /;
FreeQ[{a,b,d,e,f,g,n,p},x] && NeQ[a^2-b^2,0] && IntegerQ[m] && (GtQ[m,0] || IntegerQ[n])
```

6: $\int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx$ when $a^2 - b^2 \neq 0 \wedge (m \mid 2n \mid 2p) \in \mathbb{Z} \wedge m < 0 \wedge p > 1 \wedge n \leq -2$

Derivation: Algebraic expansion

$$\text{Basis: } \cos[z]^2 == \frac{a+b \sin[z]}{a} - \frac{b \sin[z] (a+b \sin[z])}{a^2} - \frac{(a^2-b^2) \sin[z]^2}{a^2}$$

Rule: If $a^2 - b^2 \neq 0 \wedge (m \mid 2n \mid 2p) \in \mathbb{Z} \wedge m < 0 \wedge p > 1 \wedge n \leq -2$, then

$$\begin{aligned} & \int (g \cos[e+fx])^p (d \sin[e+fx])^n (a+b \sin[e+fx])^m dx \rightarrow \\ & \frac{g^2}{a} \int (g \cos[e+fx])^{p-2} (d \sin[e+fx])^n (a+b \sin[e+fx])^{m+1} dx - \\ & \frac{b g^2}{a^2 d} \int (g \cos[e+fx])^{p-2} (d \sin[e+fx])^{n+1} (a+b \sin[e+fx])^{m+1} dx - \\ & \frac{g^2 (a^2 - b^2)}{a^2 d^2} \int (g \cos[e+fx])^{p-2} (d \sin[e+fx])^{n+2} (a+b \sin[e+fx])^m dx \end{aligned}$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_])^p_*(d_.*sin[e_.+f_.*x_])^n_*(a+b_.*sin[e_.+f_.*x_])^m_,x_Symbol] :=
  g^2/a*Int[(g*cos[e+f*x])^(p-2)*(d*sin[e+f*x])^n*(a+b*sin[e+f*x])^(m+1),x] -
  b*g^2/(a^2*d)*Int[(g*cos[e+f*x])^(p-2)*(d*sin[e+f*x])^(n+1)*(a+b*sin[e+f*x])^(m+1),x] -
  g^2*(a^2-b^2)/(a^2*d^2)*Int[(g*cos[e+f*x])^(p-2)*(d*sin[e+f*x])^(n+2)*(a+b*sin[e+f*x])^m,x] /;
FreeQ[{a,b,d,e,f,g},x] && NeQ[a^2-b^2,0] && IntegersQ[m,2*n,2*p] && LtQ[m,0] && GtQ[p,1] && (LeQ[n,-2] || EqQ[m,-1] && EqQ[n,-3/2] &
```

8. $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $a^2 - b^2 = 0$

1: $\int \cos[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $a^2 - b^2 = 0 \wedge m \in \mathbb{Z} \wedge 2m+p = 0$

Derivation: Algebraic simplification

Basis: If $a^2 - b^2 = 0 \wedge m \in \mathbb{Z} \wedge 2m+p = 0$, then $\cos[z]^p (a+b \sin[z])^m = \frac{a^{2m}}{(a-b \sin[z])^m}$

Rule: If $a^2 - b^2 = 0 \wedge m \in \mathbb{Z} \wedge 2m+p = 0$, then

$$\int \cos[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow a^{2m} \int \frac{(c+d \sin[e+fx])^n}{(a-b \sin[e+fx])^m} dx$$

Program code:

```
Int[cos[e_.+f_.*x_]^p_*(a_+b_.*sin[e_.+f_.*x_]^m_*(c_+d_.*sin[e_.+f_.*x_]^n_,x_Symbol] :=
  a^(2*m)*Int[(c+d*sin[e+f*x])^n/(a-b*sin[e+f*x])^m,x] /;
FreeQ[{a,b,c,d,e,f,n},x] && EqQ[a^2-b^2,0] && IntegersQ[m,p] && EqQ[2*m+p,0]
```

2: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $a^2 - b^2 = 0 \wedge m \in \mathbb{Z} \wedge (2m+p = 0 \vee 2m+p > 0 \wedge p < -1)$

Derivation: Algebraic expansion

Basis: If $a^2 - b^2 = 0$, then $a+b \sin[z] = \frac{a^2 (g \cos[z])^2}{g^2 (a-b \sin[z])}$

Note: By making the degree of the cosine factor in the integrand nonnegative, this rule removes the removable singularities from the integrand and hence from the resulting antiderivatives.

Rule: If $a^2 - b^2 = 0 \wedge m \in \mathbb{Z} \wedge (2m+p = 0 \vee 2m+p > 0 \wedge p < -1)$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow \frac{a^{2m}}{g^{2m}} \int \frac{(g \cos[e+fx])^{2m+p} (c+d \sin[e+fx])^n}{(a-b \sin[e+fx])^m} dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_+d_.sin[e_.+f_.x_])^n_,x_Symbol] :=
  (a/g)^(2*m)*Int[(g*cos[e+f*x])^(2*m+p)*(c+d*sin[e+f*x])^n/(a-b*sin[e+f*x])^m,x] /;
FreeQ[{a,b,c,d,e,f,g,n},x] && EqQ[a^2-b^2,0] && IntegerQ[m] && (EqQ[2*m+p,0] || GtQ[2*m+p,0] && LtQ[p,-1])
```

3. $\int \cos[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $a^2 - b^2 = 0 \wedge \frac{p}{2} \in \mathbb{Z}$

1: $\int \cos[e+fx]^2 (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $a^2 - b^2 = 0 \wedge (2m | 2n) \in \mathbb{Z}$

Derivation: Algebraic simplification

Basis: If $a^2 - b^2 = 0$, then $\cos[z]^2 = \frac{1}{b^2} (a + b \sin[z]) (a - b \sin[z])$

Rule: If $a^2 - b^2 = 0 \wedge (2m | 2n) \in \mathbb{Z}$, then

$$\int \cos[e+fx]^2 (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow \frac{1}{b^2} \int (a+b \sin[e+fx])^{m+1} (c+d \sin[e+fx])^n (a-b \sin[e+fx]) dx$$

Program code:

```
Int[cos[e_.+f_.x_]^2*(a_+b_.sin[e_.+f_.x_])^m_*(c_+d_.sin[e_.+f_.x_])^n_,x_Symbol] :=
  1/b^2*Int[(a+b*sin[e+f*x])^(m+1)*(c+d*sin[e+f*x])^n*(a-b*sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && EqQ[a^2-b^2,0] && IntegersQ[2*m,2*n]
```

$$2: \int \cos[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \text{ when } a^2 - b^2 = 0 \wedge \frac{p}{2} \in \mathbb{Z} \wedge m \in \mathbb{Z}$$

Derivation: Algebraic expansion, piecewise constant extraction and integration by substitution

$$\text{Basis: If } a^2 - b^2 = 0 \wedge \frac{p}{2} \in \mathbb{Z}, \text{ then } \cos[z]^p = a^{-p} (a+b \sin[z])^{p/2} (a-b \sin[z])^{p/2}$$

$$\text{Basis: } \partial_x \frac{\cos[e+fx]}{\sqrt{1+\sin[e+fx]} \sqrt{1-\sin[e+fx]}} = 0$$

$$\text{Basis: } \cos[e+fx] = \frac{1}{f} \partial_x \sin[e+fx]$$

$$\text{Rule: If } a^2 - b^2 = 0 \wedge \frac{p}{2} \in \mathbb{Z} \wedge m \in \mathbb{Z}, \text{ then}$$

$$\int \cos[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow$$

$$a^{-p} \int (a+b \sin[e+fx])^{m+p/2} (a-b \sin[e+fx])^{p/2} (c+d \sin[e+fx])^n dx \rightarrow$$

$$\frac{a^m \cos[e+fx]}{\sqrt{1+\sin[e+fx]} \sqrt{1-\sin[e+fx]}} \int \cos[e+fx] \left(1 + \frac{b}{a} \sin[e+fx]\right)^{m+\frac{p-1}{2}} \left(1 - \frac{b}{a} \sin[e+fx]\right)^{\frac{p-1}{2}} (c+d \sin[e+fx])^n dx \rightarrow$$

$$\frac{a^m \cos[e+fx]}{f \sqrt{1+\sin[e+fx]} \sqrt{1-\sin[e+fx]}} \text{Subst} \left[\int \left(1 + \frac{b}{a} x\right)^{m+\frac{p-1}{2}} \left(1 - \frac{b}{a} x\right)^{\frac{p-1}{2}} (c+dx)^n dx, x, \sin[e+fx] \right]$$

Program code:

```
Int[cos[e_.+f_.*x_]^p_*(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  a^m*Cos[e+f*x]/(f*Sqrt[1+Sin[e+f*x]]*Sqrt[1-Sin[e+f*x]])*
  Subst[Int[(1+b/a*x)^(m+(p-1)/2)*(1-b/a*x)^((p-1)/2)*(c+d*x)^n,x,Sin[e+f*x]] /;
  FreeQ[{a,b,c,d,e,f,n},x] && EqQ[a^2-b^2,0] && IntegerQ[p/2] && IntegerQ[m]
```

$$\mathbf{3:} \int \cos[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \text{ when } a^2 - b^2 = 0 \wedge \frac{p}{2} \in \mathbb{Z} \wedge m \notin \mathbb{Z}$$

Derivation: Algebraic expansion, piecewise constant extraction and integration by substitution

$$\text{Basis: If } a^2 - b^2 = 0 \wedge \frac{p}{2} \in \mathbb{Z}, \text{ then } \cos[z]^p = a^{-p} (a+b \sin[z])^{p/2} (a-b \sin[z])^{p/2}$$

$$\text{Basis: If } a^2 - b^2 = 0, \text{ then } \partial_x \frac{\cos[e+fx]}{\sqrt{a+b \sin[e+fx]} \sqrt{a-b \sin[e+fx]}} = 0$$

$$\text{Basis: } \cos[e+fx] = \frac{1}{f} \partial_x \sin[e+fx]$$

$$\text{Rule: If } a^2 - b^2 = 0 \wedge \frac{p}{2} \in \mathbb{Z} \wedge m \notin \mathbb{Z}, \text{ then}$$

$$\int \cos[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow$$

$$a^{-p} \int (a+b \sin[e+fx])^{m+p/2} (a-b \sin[e+fx])^{p/2} (c+d \sin[e+fx])^n dx \rightarrow$$

$$\frac{\cos[e+fx]}{a^{p-2} \sqrt{a+b \sin[e+fx]} \sqrt{a-b \sin[e+fx]}} \int \cos[e+fx] (a+b \sin[e+fx])^{m+\frac{p}{2}-\frac{1}{2}} (a-b \sin[e+fx])^{\frac{p}{2}-\frac{1}{2}} (c+d \sin[e+fx])^n dx \rightarrow$$

$$\frac{\cos[e+fx]}{a^{p-2} f \sqrt{a+b \sin[e+fx]} \sqrt{a-b \sin[e+fx]}} \text{Subst} \left[\int (a+bx)^{m+\frac{p}{2}-\frac{1}{2}} (a-bx)^{\frac{p}{2}-\frac{1}{2}} (c+dx)^n dx, x, \sin[e+fx] \right]$$

Program code:

```
Int[cos[e_.+f_.*x_]^p_*(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  Cos[e+f*x]/(a^(p-2)*f*Sqrt[a+b*sin[e+f*x]]*Sqrt[a-b*sin[e+f*x]])*
  Subst[Int[(a+b*x)^(m+p/2-1/2)*(a-b*x)^(p/2-1/2)*(c+d*x)^n,x],x,Sin[e+f*x]] /;
  FreeQ[{a,b,c,d,e,f,m,n},x] && EqQ[a^2-b^2,0] && IntegerQ[p/2] && Not[IntegerQ[m]]
```

4: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $a^2 - b^2 = 0 \wedge m \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $a^2 - b^2 = 0 \wedge m \in \mathbb{Z}^+$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow \int (g \cos[e+fx])^p \text{ExpandTrig}[(a+b \sin[e+fx])^m (c+d \sin[e+fx])^n, x] dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a+b_.sin[e_.+f_.x_])^m_*(c+d_.sin[e_.+f_.x_])^n_,x_Symbol] :=
  Int[ExpandTrig[(g*cos[e+f*x])^p,(a+b*sin[e+f*x])^m*(c+d*sin[e+f*x])^n,x],x] /;
FreeQ[{a,b,c,d,e,f,g,n,p},x] && EqQ[a^2-b^2,0] && IGtQ[m,0] && (IntegerQ[p] || IGtQ[n,0])
```

5. $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $a^2 - b^2 = 0$

1: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $a^2 - b^2 = 0 \wedge m \in \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

Basis: $\partial_x \frac{(g \cos[e+fx])^{p-1}}{(1+\sin[e+fx])^{\frac{p-1}{2}} (1-\sin[e+fx])^{\frac{p-1}{2}}} = 0$

Basis: $\cos[e+fx] = \frac{1}{f} \partial_x \sin[e+fx]$

Rule: If $a^2 - b^2 = 0 \wedge m \in \mathbb{Z}$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow$$

$$\frac{a^m g (g \cos[e+fx])^{p-1}}{(1+\sin[e+fx])^{\frac{p-1}{2}} (1-\sin[e+fx])^{\frac{p-1}{2}}} \int \cos[e+fx] \left(1+\frac{b}{a} \sin[e+fx]\right)^{m+\frac{p-1}{2}} \left(1-\frac{b}{a} \sin[e+fx]\right)^{\frac{p-1}{2}} (c+d \sin[e+fx])^n dx \rightarrow$$

$$\frac{a^m g (g \cos[e+fx])^{p-1}}{f (1+\sin[e+fx])^{\frac{p-1}{2}} (1-\sin[e+fx])^{\frac{p-1}{2}}} \text{Subst}\left[\int \left(1+\frac{b}{a} x\right)^{m+\frac{p-1}{2}} \left(1-\frac{b}{a} x\right)^{\frac{p-1}{2}} (c+dx)^n dx, x, \sin[e+fx]\right]$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_])^p_*(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  a^m*g*(g*cos[e+f*x])^(p-1)/(f*(1+Sin[e+f*x])^((p-1)/2)*(1-Sin[e+f*x])^((p-1)/2))*
  Subst[Int[(1+b/a*x)^(m+(p-1)/2)*(1-b/a*x)^((p-1)/2)*(c+d*x)^n,x,Sin[e+f*x]] /;
  FreeQ[{a,b,c,d,e,f,n,p},x] && EqQ[a^2-b^2,0] && IntegerQ[m]
```

2: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $a^2 - b^2 = 0 \wedge m \notin \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

Basis: If $a^2 - b^2 = 0$, then $\partial_x \frac{\cos[e+fx]}{\sqrt{a+b \sin[e+fx]} \sqrt{a-b \sin[e+fx]}} = 0$

Basis: $\cos[e+fx] = \frac{1}{f} \partial_x \sin[e+fx]$

Rule: If $a^2 - b^2 = 0 \wedge m \notin \mathbb{Z}$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow$$

$$\frac{g (g \cos[e+fx])^{p-1}}{(a+b \sin[e+fx])^{\frac{p-1}{2}} (a-b \sin[e+fx])^{\frac{p-1}{2}}} \int \cos[e+fx] (a+b \sin[e+fx])^{m+\frac{p-1}{2}} (a-b \sin[e+fx])^{\frac{p-1}{2}} (c+d \sin[e+fx])^n dx \rightarrow$$

$$\frac{g (g \cos[e + f x])^{p-1}}{f (a + b \sin[e + f x])^{\frac{p-1}{2}} (a - b \sin[e + f x])^{\frac{p-1}{2}}} \text{Subst} \left[\int (a + b x)^{m+\frac{p-1}{2}} (a - b x)^{\frac{p-1}{2}} (c + d x)^n dx, x, \sin[e + f x] \right]$$

Program code:

```
Int[(g_.*cos[e_.+f_.*x_])^p_*(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  g*(g*Cos[e+f*x])^(p-1)/(f*(a+b*Sin[e+f*x])^((p-1)/2)*(a-b*Sin[e+f*x])^((p-1)/2))*
  Subst[Int[(a+b*x)^(m+(p-1)/2)*(a-b*x)^((p-1)/2)*(c+d*x)^n,x],x,Sin[e+f*x]] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && EqQ[a^2-b^2,0] && Not[IntegerQ[m]]
```

9. $\int (g \cos[e + f x])^p (a + b \sin[e + f x])^m (c + d \sin[e + f x])^n dx$ when $a^2 - b^2 \neq 0$

1. $\int (g \cos[e + f x])^p (a + b \sin[e + f x])^m (c + d \sin[e + f x])^n dx$ when $a^2 - b^2 \neq 0 \wedge \frac{p}{2} \in \mathbb{Z}^+$

1: $\int \cos[e + f x]^2 (a + b \sin[e + f x])^m (c + d \sin[e + f x])^n dx$ when $a^2 - b^2 \neq 0$

Derivation: Algebraic expansion

Basis: $\cos[z]^2 = 1 - \sin[z]^2$

Rule: If $a^2 - b^2 \neq 0$, then

$$\int \cos[e + f x]^2 (a + b \sin[e + f x])^m (c + d \sin[e + f x])^n dx \rightarrow \int (a + b \sin[e + f x])^m (c + d \sin[e + f x])^n (1 - \sin[e + f x]^2) dx$$

Program code:

```
Int[cos[e_.+f_.*x_]^2*(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  Int[(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^n*(1-Sin[e+f*x]^2),x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && NeQ[a^2-b^2,0] && (IGtQ[m,0] || IntegerQ[2*m,2*n])
```

$$\mathbf{2:} \int \cos[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \text{ when } a^2 - b^2 \neq 0 \wedge \frac{p}{2} \in \mathbb{Z}^+$$

Derivation: Algebraic expansion

Basis: $\cos[z]^2 = 1 - \sin[z]^2$

Rule: If $a^2 - b^2 \neq 0 \wedge \frac{p}{2} \in \mathbb{Z}^+$, then

$$\int \cos[e+fx]^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow \int \text{ExpandTrig}[(a+b \sin[e+fx])^m (c+d \sin[e+fx])^n (1 - \sin[e+fx]^2)^{p/2}, x] dx$$

Program code:

```
Int[cos[e_.+f_.*x_]^p_*(a_+b_.*sin[e_.+f_.*x_]^m_*(c_+d_.*sin[e_.+f_.*x_]^n_,x_Symbol] :=
  Int[ExpandTrig[(a+b*sin[e+f*x])^m*(c+d*sin[e+f*x])^n*(1-sin[e+f*x]^2)^(p/2),x],x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && NeQ[a^2-b^2,0] && IGtQ[p/2,0] && (IGtQ[m,0] || IntegersQ[2*m,2*n])
```

2: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $a^2 - b^2 \neq 0 \wedge (m|n) \in \mathbb{Z}$

Derivation: Algebraic expansion

Rule: If $a^2 - b^2 \neq 0 \wedge (m|n) \in \mathbb{Z}$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow \int \text{ExpandTrig}[(g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n, x] dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_+d_.sin[e_.+f_.x_])^n_,x_Symbol] :=
  Int[ExpandTrig[(g*cos[e+f*x])^p*(a+b*sin[e+f*x])^m*(c+d*sin[e+f*x])^n,x],x] /;
FreeQ[{a,b,c,d,e,f,g,p},x] && NeQ[a^2-b^2,0] && IntegersQ[2*m,2*n]
```

X: $\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $a^2 - b^2 \neq 0$

Rule: If $a^2 - b^2 \neq 0$, then

$$\int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow \int (g \cos[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$$

Program code:

```
Int[(g_.cos[e_.+f_.x_])^p_*(a_+b_.sin[e_.+f_.x_])^m_*(c_+d_.sin[e_.+f_.x_])^n_,x_Symbol] :=
  Unintegrable[(g*cos[e+f*x])^p*(a+b*sin[e+f*x])^m*(c+d*sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p},x] && NeQ[a^2-b^2,0]
```

Rules for integrands of the form $(g \sec[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n$

1: $\int (g \sec[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx$ when $p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x ((g \cos[e+fx])^p (g \sec[e+fx])^p) = 0$

Rule: If $p \notin \mathbb{Z}$, then

$$\int (g \sec[e+fx])^p (a+b \sin[e+fx])^m (c+d \sin[e+fx])^n dx \rightarrow g^{2 \operatorname{IntPart}[p]} (g \cos[e+fx])^{\operatorname{FracPart}[p]} (g \sec[e+fx])^{\operatorname{FracPart}[p]} \int \frac{(a+b \sin[e+fx])^m (c+d \sin[e+fx])^n}{(g \cos[e+fx])^p} dx$$

Program code:

```
Int[(g_.*sec[e_.+f_.*x_])^p_*(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  g^(2*IntPart[p])*(g*cos[e+f*x])^FracPart[p]*(g*sec[e+f*x])^FracPart[p]*
  Int[(a+b*sin[e+f*x])^m*(c+d*sin[e+f*x])^n/(g*cos[e+f*x])^p,x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p},x] && Not[IntegerQ[p]]
```

```
Int[(g_.*csc[e_.+f_.*x_])^p_*(a_.+b_.*cos[e_.+f_.*x_])^m_*(c_.+d_.*cos[e_.+f_.*x_])^n_,x_Symbol] :=
  g^(2*IntPart[p])*(g*sin[e+f*x])^FracPart[p]*(g*csc[e+f*x])^FracPart[p]*
  Int[(a+b*cos[e+f*x])^m*(c+d*cos[e+f*x])^n/(g*sin[e+f*x])^p,x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p},x] && Not[IntegerQ[p]]
```