

## Rules for integrands of the form $(d + e x^n)^q (a + b x^n + c x^{2n})^p$

0.  $\int (d + e x^n)^q (a + b x^n + c x^{2n})^p dx$  when  $b^2 - 4ac = 0$

**x:**  $\int (d + e x^n)^q (a + b x^n + c x^{2n})^p dx$  when  $b^2 - 4ac = 0 \wedge p \in \mathbb{Z}$

Derivation: Algebraic simplification

Basis: If  $b^2 - 4ac = 0$ , then  $a + bz + cz^2 = \frac{1}{c} \left( \frac{b}{2} + cz \right)^2$

Rule 1.2.3.2.4.1: If  $b^2 - 4ac = 0 \wedge p \in \mathbb{Z}$ , then

$$\int (d + e x^n)^q (a + b x^n + c x^{2n})^p dx \rightarrow \frac{1}{c^p} \int (d + e x^n)^q \left( \frac{b}{2} + c x^n \right)^{2p} dx$$

Program code:

```
(* Int[(d+e.*x_^n_.)^q_.*(a+b_.*x_^n_.+c_.*x_^n2_.)^p_,x_Symbol] :=
  1/c^p*Int[(d+e*x^n)^q*(b/2+c*x^n)^(2*p),x] /;
FreeQ[{a,b,c,d,e,n,p,q},x] && EqQ[n2,2*n] && EqQ[b^2-4*a*c,0] && IntegerQ[p] *)
```

$$2. \int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx \text{ when } b^2 - 4ac = 0 \wedge p \notin \mathbb{Z}$$

$$1: \int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx \text{ when } b^2 - 4ac = 0 \wedge p \notin \mathbb{Z} \wedge 2cd - be = 0 \quad \text{Necessary?}$$

Derivation: Piecewise constant extraction

Basis: If  $b^2 - 4ac = 0 \wedge 2cd - be = 0$ , then  $\partial_x \frac{(a+b x^n+c x^{2n})^p}{(d+e x^n)^{2p}} = 0$

Note: If  $b^2 - 4ac = 0 \wedge 2cd - be = 0$ , then  $a+bz+cz^2 = \frac{c}{e^2} (d+ez)^2$

Rule 1.2.3.3.0.1: If  $b^2 - 4ac = 0 \wedge p \notin \mathbb{Z} \wedge 2cd - be = 0$ , then

$$\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx \rightarrow \frac{(a+b x^n+c x^{2n})^p}{(d+e x^n)^{2p}} \int (d+e x^n)^{q+2p} dx$$

Program code:

```
Int[(d+_e_.**x_^n_.)^q_.*(a+_b_.**x_^n_.+c_.**x_^n2_.)^p_,x_Symbol] :=
  (a+b*x^n+c*x^(2*n))^p/(d+e*x^n)^(2*p)*Int[(d+e*x^n)^(q+2*p),x] /;
FreeQ[{a,b,c,d,e,n,p,q},x] && EqQ[n2,2*n] && EqQ[b^2-4*a*c,0] && Not[IntegerQ[p]] && EqQ[2*c*d-b*e,0]
```

$$2: \int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx \text{ when } b^2 - 4ac = 0 \wedge p \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction

Basis: If  $b^2 - 4ac = 0$ , then  $\partial_x \frac{(a+b x^n+c x^{2n})^p}{\left(\frac{b}{2}+c x^n\right)^{2p}} = 0$

Note: If  $b^2 - 4ac = 0$ , then  $a+bz+cz^2 = \frac{1}{c} \left(\frac{b}{2}+cz\right)^2$

Rule 1.2.3.3.0.2: If  $b^2 - 4ac = 0 \wedge p \notin \mathbb{Z}$ , then

$$\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx \rightarrow \frac{(a+b x^n+c x^{2n})^{\text{FracPart}[p]}}{c^{\text{IntPart}[p]} \left(\frac{b}{2}+c x^n\right)^{2 \text{FracPart}[p]}} \int (d+e x^n)^q \left(\frac{b}{2}+c x^n\right)^{2p} dx$$

Program code:

```
Int[(d+_e_.*x_^n_)^q_.*(a+_b_.*x_^n_+c_.*x_^n2_)^p_,x_Symbol] :=
  (a+b*x^n+c*x^(2*n))^FracPart[p]/(c^IntPart[p]*(b/2+c*x^n)^(2*FracPart[p]))*Int[(d+e*x^n)^q*(b/2+c*x^n)^(2*p),x] /;
FreeQ[{a,b,c,d,e,n,p,q},x] && EqQ[n2,2*n] && EqQ[b^2-4*a*c,0] && Not[IntegerQ[p]]
```

1:  $\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx$  when  $(p|q) \in \mathbb{Z} \wedge n < 0$

Derivation: Algebraic expansion

Basis: If  $(p|q) \in \mathbb{Z}$ , then  $(d+e x^n)^q (a+b x^n+c x^{2n})^p = x^{n(2p+q)} (e+d x^{-n})^q (c+b x^{-n}+a x^{-2n})^p$

Rule 1.2.3.3.1: If  $(p|q) \in \mathbb{Z} \wedge n < 0$ , then

$$\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx \rightarrow \int x^{n(2p+q)} (e+d x^{-n})^q (c+b x^{-n}+a x^{-2n})^p dx$$

Program code:

```
Int[(d+_e_.*x_^n_)^q_.*(a+_b_.*x_^n_+c_.*x_^n2_)^p_,x_Symbol] :=
  Int[x^(n*(2*p+q))*(e+d*x^(-n))^q*(c+b*x^(-n)+a*x^(-2*n))^p,x] /;
FreeQ[{a,b,c,d,e,n},x] && EqQ[n2,2*n] && IntegerQ[p,q] && NegQ[n]
```

```
Int[(d+_e_.*x_^n_)^q_.*(a+_c_.*x_^n2_)^p_,x_Symbol] :=
  Int[x^(n*(2*p+q))*(e+d*x^(-n))^q*(c+a*x^(-2*n))^p,x] /;
FreeQ[{a,c,d,e,n},x] && EqQ[n2,2*n] && IntegerQ[p,q] && NegQ[n]
```

**2:**  $\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx$  when  $n \in \mathbb{Z}^-$

Derivation: Integration by substitution

Basis: If  $n \in \mathbb{Z}$ , then  $F[x^n] = -\text{Subst}\left[\frac{F[\frac{x^{-n}}{x^2}]}{x^2}, x, \frac{1}{x}\right] \partial_x \frac{1}{x}$

Rule 1.2.3.3.2: If  $n \in \mathbb{Z}^-$ , then

$$\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx \rightarrow -\text{Subst}\left[\int \frac{(d+e x^{-n})^q (a+b x^{-n}+c x^{-2n})^p}{x^2} dx, x, \frac{1}{x}\right]$$

Program code:

```
Int[(d+_e_.*x_^n_)^q_.*(a+_b_.*x_^n+_c_.*x_^2n_)^p_,x_Symbol] :=
  -Subst[Int[(d+e*x^(-n))^q*(a+b*x^(-n)+c*x^(-2*n))^p/x^2,x],x,1/x] /;
FreeQ[{a,b,c,d,e,p,q},x] && EqQ[n2,2*n] && ILtQ[n,0]
```

```
Int[(d+_e_.*x_^n_)^q_.*(a+_c_.*x_^2n_)^p_,x_Symbol] :=
  -Subst[Int[(d+e*x^(-n))^q*(a+c*x^(-2*n))^p/x^2,x],x,1/x] /;
FreeQ[{a,c,d,e,p,q},x] && EqQ[n2,2*n] && ILtQ[n,0]
```

**3:**  $\int (d + e x^n)^q (a + b x^n + c x^{2n})^p dx$  when  $n \in \mathbb{F}$

Derivation: Integration by substitution

Basis: If  $g \in \mathbb{Z}^+$ , then  $F[x^n] = g \text{ Subst}[x^{g-1} F[x^g], x, x^{1/g}] \partial_x x^{1/g}$

Rule 1.2.3.3.3: If  $n \in \mathbb{F}$ , let  $g = \text{Denominator}[n]$ , then

$$\int (d + e x^n)^q (a + b x^n + c x^{2n})^p dx \rightarrow g \text{ Subst}\left[\int x^{g-1} (d + e x^g)^q (a + b x^g + c x^{2g})^p dx, x, x^{1/g}\right]$$

Program code:

```
Int[(d_+e_.*x_^n_)^q_.*(a_.+b_.*x_^n_+c_.*x_^n2_)^p_,x_Symbol] :=
  With[{g=Denominator[n]},
    g*Subst[Int[x^(g-1)*(d+e*x^(g*n))^q*(a+b*x^(g*n)+c*x^(2*g*n))^p,x],x,x^(1/g)]] /;
  FreeQ[{a,b,c,d,e,p,q},x] && EqQ[n2,2*n] && FractionQ[n]
```

```
Int[(d_+e_.*x_^n_)^q_.*(a_+c_.*x_^n2_)^p_,x_Symbol] :=
  With[{g=Denominator[n]},
    g*Subst[Int[x^(g-1)*(d+e*x^(g*n))^q*(a+c*x^(2*g*n))^p,x],x,x^(1/g)]] /;
  FreeQ[{a,c,d,e,p,q},x] && EqQ[n2,2*n] && FractionQ[n]
```

$$4. \int (d + e x^n)^q (b x^n + c x^{2n})^p dx \text{ when } p \notin \mathbb{Z}$$

$$1. \int (d + e x^n) (b x^n + c x^{2n})^p dx \text{ when } p \notin \mathbb{Z}$$

$$\mathbf{1:} \int (d + e x^n) (b x^n + c x^{2n})^p dx \text{ when } p \notin \mathbb{Z} \wedge n(2p+1)+1=0$$

Derivation: Trinomial recurrence 2a with  $a = 0$ ,  $m = 0$  and  $n(2p+1)+1=0$  composed with trinomial recurrence 5 with  $a = 0$

Rule 1.2.3.3.4.1.1: If  $p \notin \mathbb{Z} \wedge n(2p+1)+1=0$ , then

$$\int (d + e x^n) (b x^n + c x^{2n})^p dx \rightarrow -\frac{(cd - be) (b x^n + c x^{2n})^{p+1}}{bcn(p+1) x^{2n(p+1)}} + \frac{e}{c} \int x^{-n} (b x^n + c x^{2n})^{p+1} dx$$

Program code:

```
Int[(d_+e_.**x_^n_)*(b_.**x_^n_+c_.**x_^n2_)^p_,x_Symbol] :=
  (b*e-d*c)*(b**x^n+c**x^(2*n))^(p+1)/(b*c*n*(p+1)*x^(2*n*(p+1))) +
  e/c*Int[x^(-n)*(b**x^n+c**x^(2*n))^(p+1),x] /;
FreeQ[{b,c,d,e,n,p},x] && EqQ[n2,2*n] && Not[IntegerQ[p]] && EqQ[n*(2*p+1)+1,0]
```

$$\mathbf{2:} \int (d+e x^n) (b x^n + c x^{2n})^p dx \text{ when } p \notin \mathbb{Z} \wedge n(2p+1)+1 \neq 0 \wedge b e(np+1) - c d(n(2p+1)+1) = 0$$

Derivation: Trinomial recurrence 3a with  $a = 0$  with  $b e(np+1) - c d(n(2p+1)+1) = 0$

Rule 1.2.3.3.4.1.2: If  $p \notin \mathbb{Z} \wedge n(2p+1)+1 \neq 0 \wedge b e(np+1) - c d(n(2p+1)+1) = 0$ , then

$$\int (d+e x^n) (b x^n + c x^{2n})^p dx \rightarrow \frac{e x^{-n+1} (b x^n + c x^{2n})^{p+1}}{c(n(2p+1)+1)}$$

Program code:

```
Int[(d_+e_.*x_^n_)*(b_.*x_^n_+c_.*x_^n2_)^p_,x_Symbol] :=
  e*x^(-n+1)*(b*x^n+c*x^(2*n))^(p+1)/(c*(n*(2*p+1)+1)) /;
FreeQ[{b,c,d,e,n,p},x] && EqQ[n2,2*n] && Not[IntegerQ[p]] && NeQ[n*(2*p+1)+1,0] && EqQ[b*e*(n*p+1)-c*d*(n*(2*p+1)+1),0]
```

$$\mathbf{3:} \int (d+e x^n) (b x^n + c x^{2n})^p dx \text{ when } p \notin \mathbb{Z} \wedge n(2p+1)+1 \neq 0 \wedge b e(np+1) - c d(n(2p+1)+1) \neq 0$$

Derivation: Trinomial recurrence 3a with  $a = 0$

Rule 1.2.3.3.4.1.3: If  $p \notin \mathbb{Z} \wedge n(2p+1)+1 \neq 0 \wedge b e(np+1) - c d(n(2p+1)+1) \neq 0$ , then

$$\int (d+e x^n) (b x^n + c x^{2n})^p dx \rightarrow \frac{e x^{-n+1} (b x^n + c x^{2n})^{p+1}}{c(n(2p+1)+1)} - \frac{b e(np+1) - c d(n(2p+1)+1)}{c(n(2p+1)+1)} \int (b x^n + c x^{2n})^p dx$$

Program code:

```
Int[(d_+e_.*x_^n_)*(b_.*x_^n_+c_.*x_^n2_)^p_,x_Symbol] :=
  e*x^(-n+1)*(b*x^n+c*x^(2*n))^(p+1)/(c*(n*(2*p+1)+1)) -
  (b*e*(n*p+1)-c*d*(n*(2*p+1)+1))/(c*(n*(2*p+1)+1))*Int[(b*x^n+c*x^(2*n))^p,x] /;
FreeQ[{b,c,d,e,n,p},x] && EqQ[n2,2*n] && Not[IntegerQ[p]] && NeQ[n*(2*p+1)+1,0] && NeQ[b*e*(n*p+1)-c*d*(n*(2*p+1)+1),0]
```

**2:**  $\int (d+e x^n)^q (b x^n + c x^{2n})^p dx$  when  $p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis:  $\partial_x \frac{(b x^n + c x^{2n})^p}{x^{np} (b+c x^n)^p} == 0$

Basis:  $\frac{(b x^n + c x^{2n})^{\text{FracPart}[p]}}{x^{n \text{FracPart}[p]} (b+c x^n)^{\text{FracPart}[p]}} == \frac{(b x^n + c x^{2n})^{\text{FracPart}[p]}}{x^{n \text{FracPart}[p]} (b+c x^n)^{\text{FracPart}[p]}}$

Rule 1.2.3.3.4.2: If  $p \notin \mathbb{Z}$ , then

$$\int (d+e x^n)^q (b x^n + c x^{2n})^p dx \rightarrow \frac{(b x^n + c x^{2n})^{\text{FracPart}[p]}}{x^{n \text{FracPart}[p]} (b+c x^n)^{\text{FracPart}[p]}} \int x^{np} (d+e x^n)^q (b+c x^n)^p dx$$

Program code:

```
Int[(d_+e_.*x_^n_)^q_.*(b_.*x_^n_+c_.*x_^n2_)^p_,x_Symbol] :=
  (b*x^n+c*x^(2*n))^FracPart[p]/(x^(n*FracPart[p])*(b+c*x^n)^FracPart[p])*Int[x^(n*p)*(d+e*x^n)^q*(b+c*x^n)^p,x] /;
FreeQ[{b,c,d,e,n,p,q},x] && EqQ[n2,2*n] && Not[IntegerQ[p]]
```



6.  $\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx$  when  $b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 = 0$

1:  $\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx$  when  $b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p \in \mathbb{Z}$

Derivation: Algebraic simplification

Basis: If  $c d^2 - b d e + a e^2 = 0$ , then  $a + b z + c z^2 = (d + e z) \left( \frac{a}{d} + \frac{c z}{e} \right)$

Rule 1.2.3.3.6.1: If  $b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p \in \mathbb{Z}$ , then

$$\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx \rightarrow \int (d+e x^n)^{p+q} \left( \frac{a}{d} + \frac{c x^n}{e} \right)^p dx$$

Program code:

```
Int[(d+_e_.*x_^n_)^q_.*(a+_b_.*x_^n_+c_.*x_^n2_)^p_,x_Symbol] :=
  Int[(d+e*x^n)^(p+q)*(a/d+c/e*x^n)^p,x] /;
FreeQ[{a,b,c,d,e,n,q},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && IntegerQ[p]
```

```
Int[(d+_e_.*x_^n_)^q_.*(a+_c_.*x_^n2_)^p_,x_Symbol] :=
  Int[(d+e*x^n)^(p+q)*(a/d+c/e*x^n)^p,x] /;
FreeQ[{a,c,d,e,n,q},x] && EqQ[n2,2*n] && EqQ[c*d^2+a*e^2,0] && IntegerQ[p]
```

**2:**  $\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx$  when  $b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: If  $c d^2 - b d e + a e^2 = 0$ , then  $\partial_x \frac{(a+b x^n+c x^{2n})^p}{(d+e x^n)^p \left(\frac{a}{d} + \frac{c x^n}{e}\right)^p} = 0$

Basis: If  $c d^2 - b d e + a e^2 = 0$ , then  $\frac{(a+b x^n+c x^{2n})^p}{(d+e x^n)^p \left(\frac{a}{d} + \frac{c x^n}{e}\right)^p} = \frac{(a+b x^n+c x^{2n})^{\text{FracPart}[p]}}{(d+e x^n)^{\text{FracPart}[p]} \left(\frac{a}{d} + \frac{c x^n}{e}\right)^{\text{FracPart}[p]}}$

Rule 1.2.3.3.6.2: If  $b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p \notin \mathbb{Z}$ , then

$$\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx \rightarrow \frac{(a+b x^n+c x^{2n})^{\text{FracPart}[p]}}{(d+e x^n)^{\text{FracPart}[p]} \left(\frac{a}{d} + \frac{c x^n}{e}\right)^{\text{FracPart}[p]}} \int (d+e x^n)^{p+q} \left(\frac{a}{d} + \frac{c x^n}{e}\right)^p dx$$

Program code:

```
Int[(d+_e_.*x_^n_)^q_*(a+_b_.*x_^n+_c_.*x_^2n_)^p_,x_Symbol] :=
  (a+b*x^n+c*x^(2*n))^FracPart[p]/((d+e*x^n)^FracPart[p]*(a/d+c*x^n/e)^FracPart[p])*Int[(d+e*x^n)^(p+q)*(a/d+c/e*x^n)^p,x] /;
FreeQ[{a,b,c,d,e,n,p,q},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && Not[IntegerQ[p]]
```

```
Int[(d+_e_.*x_^n_)^q_*(a+_c_.*x_^2n_)^p_,x_Symbol] :=
  (a+c*x^(2*n))^FracPart[p]/((d+e*x^n)^FracPart[p]*(a/d+c*x^n/e)^FracPart[p])*Int[(d+e*x^n)^(p+q)*(a/d+c/e*x^n)^p,x] /;
FreeQ[{a,c,d,e,n,p,q},x] && EqQ[n2,2*n] && EqQ[c*d^2+a*e^2,0] && Not[IntegerQ[p]]
```

$$7. \int (d + e x^n)^q (a + b x^n + c x^{2n}) dx \text{ when } b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0$$

$$1: \int (d + e x^n)^q (a + b x^n + c x^{2n}) dx \text{ when } b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge q \in \mathbb{Z}^+$$

Derivation: Algebraic expansion

Rule 1.2.3.3.7.1: If  $b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge q \in \mathbb{Z}^+$ , then

$$\int (d + e x^n)^q (a + b x^n + c x^{2n}) dx \rightarrow \int \text{ExpandIntegrand}[(d + e x^n)^q (a + b x^n + c x^{2n}), x] dx$$

Program code:

```
Int[(d_+e_.*x_^n_)^q_.*(a_+b_.*x_^n_+c_.*x_^n2_),x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x^n)^q*(a+b*x^n+c*x^(2*n)),x],x] /;
FreeQ[{a,b,c,d,e,n},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && IGtQ[q,0]
```

```
Int[(d_+e_.*x_^n_)^q_.*(a_+c_.*x_^n2_),x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x^n)^q*(a+c*x^(2*n)),x],x] /;
FreeQ[{a,c,d,e,n},x] && EqQ[n2,2*n] && NeQ[c*d^2+a*e^2,0] && IGtQ[q,0]
```

2:  $\int (d+e x^n)^q (a+b x^n+c x^{2n}) dx$  when  $b^2-4ac \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge q < -1$

Derivation: ???

Rule 1.2.3.3.7.2: If  $b^2-4ac \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge q < -1$ , then

$$\int (d+e x^n)^q (a+b x^n+c x^{2n}) dx \rightarrow -\frac{(c d^2-b d e+a e^2) x (d+e x^n)^{q+1}}{d e^2 n (q+1)} + \frac{1}{n (q+1) d e^2} \int (d+e x^n)^{q+1} (c d^2-b d e+a e^2 (n (q+1)+1) + c d e n (q+1) x^n) dx$$

Program code:

```
Int[(d_+e_.*x_^n_)^q_*(a_+b_.*x_^n_+c_.*x_^n2_),x_Symbol] :=
  -(c*d^2-b*d*e+a*e^2)*x*(d+e*x^n)^(q+1)/(d*e^2*n*(q+1)) +
  1/(n*(q+1)*d*e^2)*Int[(d+e*x^n)^(q+1)*Simp[c*d^2-b*d*e+a*e^2*(n*(q+1)+1)+c*d*e*n*(q+1)*x^n,x],x] /;
FreeQ[{a,b,c,d,e,n},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && LtQ[q,-1]
```

```
Int[(d_+e_.*x_^n_)^q_*(a_+c_.*x_^n2_),x_Symbol] :=
  -(c*d^2+a*e^2)*x*(d+e*x^n)^(q+1)/(d*e^2*n*(q+1)) +
  1/(n*(q+1)*d*e^2)*Int[(d+e*x^n)^(q+1)*Simp[c*d^2+a*e^2*(n*(q+1)+1)+c*d*e*n*(q+1)*x^n,x],x] /;
FreeQ[{a,c,d,e,n},x] && EqQ[n2,2*n] && NeQ[c*d^2+a*e^2,0] && LtQ[q,-1]
```

**3:**  $\int (d+e x^n)^q (a+b x^n+c x^{2n}) dx$  when  $b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0$

Derivation: Special case of rule for  $P_q[x] (d+e x^n)^q$

Rule 1.2.3.3.7.3: If  $b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0$ , then

$$\int (d+e x^n)^q (a+b x^n+c x^{2n}) dx \rightarrow \frac{c x^{n+1} (d+e x^n)^{q+1}}{e (n(q+2)+1)} + \frac{1}{e (n(q+2)+1)} \int (d+e x^n)^q (a e (n(q+2)+1) - (c d (n+1) - b e (n(q+2)+1)) x^n) dx$$

Program code:

```
Int[(d+_e_.*x_^n_)^q_*(a+_b_.*x_^n+_c_.*x_^n2_),x_Symbol] :=
  c*x^(n+1)*(d+e*x^n)^(q+1)/(e*(n*(q+2)+1)) +
  1/(e*(n*(q+2)+1))*Int[(d+e*x^n)^q*(a*e*(n*(q+2)+1)-(c*d*(n+1)-b*e*(n*(q+2)+1))*x^n),x] /;
FreeQ[{a,b,c,d,e,n,q},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0]
```

```
Int[(d+_e_.*x_^n_)^q_*(a+_c_.*x_^n2_),x_Symbol] :=
  c*x^(n+1)*(d+e*x^n)^(q+1)/(e*(n*(q+2)+1)) +
  1/(e*(n*(q+2)+1))*Int[(d+e*x^n)^q*(a*e*(n*(q+2)+1)-c*d*(n+1))*x^n),x] /;
FreeQ[{a,c,d,e,n,q},x] && EqQ[n2,2*n] && NeQ[c*d^2+a*e^2,0]
```

$$8. \int \frac{(d+e x^n)^q}{a+b x^n+c x^{2n}} dx \text{ when } b^2-4ac \neq 0 \wedge c d^2-b d e+a e^2 \neq 0$$

$$1. \int \frac{(d+e x^n)^q}{a+b x^n+c x^{2n}} dx \text{ when } b^2-4ac \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge q \in \mathbb{Z}$$

$$1. \int \frac{d+e x^n}{a+b x^n+c x^{2n}} dx \text{ when } b^2-4ac \neq 0 \wedge c d^2-b d e+a e^2 \neq 0$$

$$1. \int \frac{d+e x^n}{a+c x^{2n}} dx \text{ when } c d^2+a e^2 \neq 0$$

$$1. \int \frac{d+e x^n}{a+c x^{2n}} dx \text{ when } c d^2+a e^2 \neq 0 \wedge c d^2-a e^2 = 0 \wedge \frac{n}{2} \in \mathbb{Z}^+$$

$$1: \int \frac{d+e x^n}{a+c x^{2n}} dx \text{ when } c d^2-a e^2 = 0 \wedge \frac{n}{2} \in \mathbb{Z}^+ \wedge d e > 0$$

Derivation: Algebraic expansion

Basis: If  $c d^2 - a e^2 = 0$  and  $q \rightarrow \sqrt{2 d e}$ , then  $\frac{d+e z^2}{a+c z^4} = \frac{e^2}{2 c (d+q z+e z^2)} + \frac{e^2}{2 c (d-q z+e z^2)}$

Rule 1.2.3.3.8.1.1.1.1.1: If  $c d^2 - a e^2 = 0 \wedge \frac{n}{2} \in \mathbb{Z}^+ \wedge d e > 0$ , let  $q \rightarrow \sqrt{2 d e}$ , then

$$\int \frac{d+e x^n}{a+c x^{2n}} dx \rightarrow \frac{e^2}{2 c} \int \frac{1}{d+q x^{n/2}+e x^n} dx + \frac{e^2}{2 c} \int \frac{1}{d-q x^{n/2}+e x^n} dx$$

Program code:

```
Int[(d+_e_.**x_^n_)/(a+_c_.**x_^n2_),x_Symbol] :=
  With[{q=Rt[2*d*e,2]},
    e^2/(2*c)*Int[1/(d+q*x^(n/2)+e*x^n),x] + e^2/(2*c)*Int[1/(d-q*x^(n/2)+e*x^n),x] /;
    FreeQ[{a,c,d,e},x] && EqQ[n2,2*n] && EqQ[c*d^2-a*e^2,0] && IGtQ[n/2,0] && PosQ[d*e]
```

$$2: \int \frac{d+e x^n}{a+c x^{2n}} dx \text{ when } c d^2 - a e^2 = 0 \wedge \frac{n}{2} \in \mathbb{Z}^+ \wedge d e \neq 0$$

Derivation: Algebraic expansion

Basis: If  $c d^2 - a e^2 = 0$ , let  $q = \sqrt{-2 d e}$  then  $\frac{d+e z^2}{a+c z^4} = \frac{d (d-q z)}{2 a (d-q z-e z^2)} + \frac{d (d+q z)}{2 a (d+q z-e z^2)}$

Rule 1.2.3.3.8.1.1.1.2: If  $c d^2 - a e^2 = 0 \wedge \frac{n}{2} \in \mathbb{Z}^+ \wedge d e \neq 0$ , let  $q = \sqrt{-2 d e}$ , then

$$\int \frac{d+e x^n}{a+c x^{2n}} dx \rightarrow \frac{d}{2a} \int \frac{d-q x^{n/2}}{d-q x^{n/2}-e x^n} dx + \frac{d}{2a} \int \frac{d+q x^{n/2}}{d+q x^{n/2}-e x^n} dx$$

Program code:

```
Int[(d_+e_.*x_^n_)/(a_+c_.*x_^n2_),x_Symbol] :=
  With[{q=Rt[-2*d*e,2]},
    d/(2*a)*Int[(d-q*x^(n/2))/(d-q*x^(n/2)-e*x^n),x] +
    d/(2*a)*Int[(d+q*x^(n/2))/(d+q*x^(n/2)-e*x^n),x] /;
  FreeQ[{a,c,d,e},x] && EqQ[n2,2*n] && EqQ[c*d^2-a*e^2,0] && IGtQ[n/2,0] && NegQ[d*e]
```

**2:**  $\int \frac{d+e x^n}{a+c x^{2n}} dx$  when  $c d^2 + a e^2 \neq 0 \wedge c d^2 - a e^2 \neq 0 \wedge \frac{n}{2} \in \mathbb{Z}^+ \wedge a c > 0$

Derivation: Algebraic expansion

Basis: If  $q \rightarrow \left(\frac{a}{c}\right)^{1/4}$ , then  $\frac{d+e z^2}{a+c z^4} = \frac{\sqrt{2} d q - (d-e q^2) z}{2 \sqrt{2} c q^3 (q^2 - \sqrt{2} q z + z^2)} + \frac{\sqrt{2} d q + (d-e q^2) z}{2 \sqrt{2} c q^3 (q^2 + \sqrt{2} q z + z^2)}$

Rule 1.2.3.3.8.1.1.2.2: If  $c d^2 + a e^2 \neq 0 \wedge c d^2 - a e^2 \neq 0 \wedge \frac{n}{2} \in \mathbb{Z}^+ \wedge a c > 0$ , let  $q \rightarrow \left(\frac{a}{c}\right)^{1/4}$ , then

$$\int \frac{d+e x^n}{a+c x^{2n}} dx \rightarrow \frac{1}{2 \sqrt{2} c q^3} \int \frac{\sqrt{2} d q - (d-e q^2) x^{n/2}}{q^2 - \sqrt{2} q x^{n/2} + x^n} dx + \frac{1}{2 \sqrt{2} c q^3} \int \frac{\sqrt{2} d q + (d-e q^2) x^{n/2}}{q^2 + \sqrt{2} q x^{n/2} + x^n} dx$$

Program code:

```
Int[(d_+e_.*x_^n_)/(a_+c_.*x_^n2_),x_Symbol] :=
  With[{q=Rt[a/c,4]},
    1/(2*Sqrt[2]*c*q^3)*Int[(Sqrt[2]*d*q-(d-e*q^2)*x^(n/2))/(q^2-Sqrt[2]*q*x^(n/2)+x^n),x] +
    1/(2*Sqrt[2]*c*q^3)*Int[(Sqrt[2]*d*q+(d-e*q^2)*x^(n/2))/(q^2+Sqrt[2]*q*x^(n/2)+x^n),x] /;
    FreeQ[{a,c,d,e},x] && EqQ[n2,2*n] && NeQ[c*d^2+a*e^2,0] && NeQ[c*d^2-a*e^2,0] && IGtQ[n/2,0] && PosQ[a*c]
```



$$\mathbf{3:} \int \frac{d+e x^3}{a+c x^6} dx \text{ when } c d^2 + a e^2 \neq 0 \wedge \frac{c}{a} > 0$$

Derivation: Algebraic expansion

$$\text{Basis: Let } q \rightarrow \left(\frac{c}{a}\right)^{1/6}, \text{ then } \frac{d+e x^3}{a+c x^6} = \frac{q^2 d - e x}{3 a q^2 (1+q^2 x^2)} + \frac{2 q^2 d - (\sqrt{3} q^3 d - e) x}{6 a q^2 (1 - \sqrt{3} q x + q^2 x^2)} + \frac{2 q^2 d + (\sqrt{3} q^3 d + e) x}{6 a q^2 (1 + \sqrt{3} q x + q^2 x^2)}$$

Rule 1.2.3.3.8.1.1.1.3: If  $c d^2 + a e^2 \neq 0 \wedge \frac{c}{a} > 0$ , let  $q \rightarrow \left(\frac{c}{a}\right)^{1/6}$ , then

$$\int \frac{d+e x^3}{a+c x^6} dx \rightarrow \frac{1}{3 a q^2} \int \frac{q^2 d - e x}{1 + q^2 x^2} dx + \frac{1}{6 a q^2} \int \frac{2 q^2 d - (\sqrt{3} q^3 d - e) x}{1 - \sqrt{3} q x + q^2 x^2} dx + \frac{1}{6 a q^2} \int \frac{2 q^2 d + (\sqrt{3} q^3 d + e) x}{1 + \sqrt{3} q x + q^2 x^2} dx$$

Program code:

```
Int[(d_+e_.**x_^3)/(a_+c_.**x_^6),x_Symbol] :=
  With[{q=Rt[c/a,6]},
    1/(3*a*q^2)*Int[(q^2*d-e*x)/(1+q^2*x^2),x] +
    1/(6*a*q^2)*Int[(2*q^2*d-(Sqrt[3]*q^3*d-e)*x)/(1-Sqrt[3]*q*x+q^2*x^2),x] +
    1/(6*a*q^2)*Int[(2*q^2*d+(Sqrt[3]*q^3*d+e)*x)/(1+Sqrt[3]*q*x+q^2*x^2),x] /;
    FreeQ[{a,c,d,e},x] && NeQ[c*d^2+a*e^2,0] && PosQ[c/a]
```

**4:**  $\int \frac{d+e x^n}{a+c x^{2n}} dx$  when  $c d^2 + a e^2 \neq 0 \wedge a c \neq 0 \wedge n \in \mathbb{Z}$

Derivation: Algebraic expansion

Basis: If  $q \rightarrow \sqrt{-\frac{a}{c}}$ , then  $\frac{d+e z}{a+c z^2} == \frac{d+e q}{2 (a+c q z)} + \frac{d-e q}{2 (a-c q z)}$

Rule 1.2.3.3.8.1.1.1.4: If  $c d^2 + a e^2 \neq 0 \wedge a c \neq 0 \wedge n \in \mathbb{Z}$ , let  $q \rightarrow \sqrt{-\frac{a}{c}}$ , then

$$\int \frac{d+e x^n}{a+c x^{2n}} dx \rightarrow \frac{d+e q}{2} \int \frac{1}{a+c q x^n} dx + \frac{d-e q}{2} \int \frac{1}{a-c q x^n} dx$$

Program code:

```
Int[(d_+e_.*x_^n_)/(a_+c_.*x_^n2_),x_Symbol] :=
  With[{q=Rt[-a/c,2]},
    (d+e*q)/2*Int[1/(a+c*q*x^n),x] + (d-e*q)/2*Int[1/(a-c*q*x^n),x] /;
  FreeQ[{a,c,d,e,n},x] && EqQ[n2,2*n] && NeQ[c*d^2+a*e^2,0] && NegQ[a*c] && IntegerQ[n]
```

**5:**  $\int \frac{d+e x^n}{a+c x^{2n}} dx$  when  $c d^2 + a e^2 \neq 0 \wedge (a c > 0 \vee n \notin \mathbb{Z})$

Derivation: Algebraic expansion

Rule 1.2.3.3.8.1.1.1.5: If  $c d^2 + a e^2 \neq 0 \wedge (a c > 0 \vee n \notin \mathbb{Z})$ , then

$$\int \frac{d+e x^n}{a+c x^{2n}} dx \rightarrow d \int \frac{1}{a+c x^{2n}} dx + e \int \frac{x^n}{a+c x^{2n}} dx$$

Program code:

```
Int[(d_+e_.*x_^n_)/(a_+c_.*x_^n2_),x_Symbol] :=
  d*Int[1/(a+c*x^(2*n)),x] + e*Int[x^n/(a+c*x^(2*n)),x] /;
FreeQ[{a,c,d,e,n},x] && EqQ[n2,2*n] && NeQ[c*d^2+a*e^2,0] && (PosQ[a*c] || Not[IntegerQ[n]])
```

$$2. \int \frac{d+e x^n}{a+b x^n+c x^{2n}} dx \text{ when } b^2-4ac \neq 0 \wedge c d^2-b d e+a e^2 \neq 0$$

$$1. \int \frac{d+e x^n}{a+b x^n+c x^{2n}} dx \text{ when } b^2-4ac \neq 0 \wedge c d^2-a e^2 = 0 \wedge \frac{n}{2} \in \mathbb{Z}^+$$

$$1: \int \frac{d+e x^n}{a+b x^n+c x^{2n}} dx \text{ when } b^2-4ac \neq 0 \wedge c d^2-a e^2 = 0 \wedge \frac{n}{2} \in \mathbb{Z}^+ \wedge \frac{2d}{e} - \frac{b}{c} > 0$$

Derivation: Algebraic expansion

Basis: If  $c d^2 - a e^2 = 0$  and  $q \rightarrow \sqrt{\frac{2d}{e} - \frac{b}{c}}$ , then  $\frac{d+e z^2}{a+b z^2+c z^4} = \frac{e^2}{2c (d+e q z+e z^2)} + \frac{e^2}{2c (d-e q z+e z^2)}$

Rule 1.2.3.3.8.1.1.2.1.1: If  $b^2-4ac \neq 0 \wedge c d^2-a e^2 = 0 \wedge \frac{n}{2} \in \mathbb{Z}^+ \wedge \frac{2d}{e} - \frac{b}{c} > 0$ , let  $q \rightarrow \sqrt{\frac{2d}{e} - \frac{b}{c}}$ , then

$$\int \frac{d+e x^n}{a+b x^n+c x^{2n}} dx \rightarrow \frac{e}{2c} \int \frac{1}{\frac{d}{e}+q x^{n/2}+x^n} dx + \frac{e}{2c} \int \frac{1}{\frac{d}{e}-q x^{n/2}+x^n} dx$$

Program code:

```
Int[(d+_e_.**x_^n_)/(a+_b_.**x_^n_+_c_.**x_^n2_),x_Symbol] :=
  With[{q=Rt[2*d/e-b/c,2]},
    e/(2*c)*Int[1/Simp[d/e+q*x^(n/2)+x^n,x],x] +
    e/(2*c)*Int[1/Simp[d/e-q*x^(n/2)+x^n,x],x] /;
  FreeQ[{a,b,c,d,e},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-a*e^2,0] && IGtQ[n/2,0] && (GtQ[2*d/e-b/c,0] || Not[LtQ[2*d/e-b/c,0]])
```

**2:**  $\int \frac{d+e x^n}{a+b x^n+c x^{2n}} dx$  when  $b^2-4ac \neq 0 \wedge c d^2-a e^2 = 0 \wedge \frac{n}{2} \in \mathbb{Z}^+ \wedge b^2-4ac > 0$

Derivation: Algebraic expansion

■ Basis: Let  $q \rightarrow \sqrt{b^2-4ac}$ , then  $\frac{d+e z}{a+b z+c z^2} = \left(\frac{e}{2} + \frac{2cd-be}{2q}\right) \frac{1}{\frac{b}{2}-\frac{q}{2}+cz} + \left(\frac{e}{2} - \frac{2cd-be}{2q}\right) \frac{1}{\frac{b}{2}+\frac{q}{2}+cz}$

Rule 1.2.3.3.8.1.1.2.1.2: If  $b^2-4ac \neq 0 \wedge c d^2-a e^2 = 0 \wedge \frac{n}{2} \in \mathbb{Z}^+ \wedge b^2-4ac > 0$ , let  $q \rightarrow \sqrt{b^2-4ac}$ , then

$$\int \frac{d+e x^n}{a+b x^n+c x^{2n}} dx \rightarrow \left(\frac{e}{2} + \frac{2cd-be}{2q}\right) \int \frac{1}{\frac{b}{2}-\frac{q}{2}+c x^n} dx + \left(\frac{e}{2} - \frac{2cd-be}{2q}\right) \int \frac{1}{\frac{b}{2}+\frac{q}{2}+c x^n} dx$$

Program code:

```
Int[(d+_e_.**x_^n_)/(a+_b_.**x_^n+_c_.**x_^2n_),x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    (e/2+(2*c*d-b*e)/(2*q))*Int[1/(b/2-q/2+c*x^n),x] + (e/2-(2*c*d-b*e)/(2*q))*Int[1/(b/2+q/2+c*x^n),x] /;
  FreeQ[{a,b,c,d,e,n},x] && EqQ[n,2*n] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-a*e^2,0] && IGtQ[n/2,0] && GtQ[b^2-4*a*c,0]
```

**3:**  $\int \frac{d+e x^n}{a+b x^n+c x^{2n}} dx$  when  $b^2-4ac \neq 0 \wedge c d^2-a e^2 = 0 \wedge \frac{n}{2} \in \mathbb{Z}^+ \wedge b^2-4ac \neq 0$

Derivation: Algebraic expansion

Basis: If  $c d^2 - a e^2 = 0$  and  $q \rightarrow \sqrt{-\frac{2d}{e} - \frac{b}{c}}$ , then  $\frac{d+e z^2}{a+b z^2+c z^4} = \frac{e (q-2 z)}{2 c q (\frac{d}{e}+q z-z^2)} + \frac{e (q+2 z)}{2 c q (\frac{d}{e}-q z-z^2)}$

Rule 1.2.3.3.8.1.1.2.1.3: If  $b^2-4ac \neq 0 \wedge c d^2-a e^2 = 0 \wedge \frac{n}{2} \in \mathbb{Z}^+ \wedge b^2-4ac \neq 0$ , let  $q \rightarrow \sqrt{-\frac{2d}{e} - \frac{b}{c}}$ , then

$$\int \frac{d+e x^n}{a+b x^n+c x^{2n}} dx \rightarrow \frac{e}{2 c q} \int \frac{q-2 x^{n/2}}{\frac{d}{e}+q x^{n/2}-x^n} dx + \frac{e}{2 c q} \int \frac{q+2 x^{n/2}}{\frac{d}{e}-q x^{n/2}-x^n} dx$$

Program code:

```
Int[(d+_e_.*x_^n_)/(a+_b_.*x_^n_+_c_.*x_^2n_),x_Symbol] :=
  With[{q=Rt[-2*d/e-b/c,2]},
    e/(2*c*q)*Int[(q-2*x^(n/2))/Simp[d/e+q*x^(n/2)-x^n,x],x] +
    e/(2*c*q)*Int[(q+2*x^(n/2))/Simp[d/e-q*x^(n/2)-x^n,x],x] /;
  FreeQ[{a,b,c,d,e},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-a*e^2,0] && IGtQ[n/2,0] && Not[GtQ[b^2-4*a*c,0]]
```

**2:**  $\int \frac{d+e x^n}{a+b x^n+c x^{2n}} dx$  when  $b^2-4ac \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge (b^2-4ac > 0 \vee \frac{n}{2} \notin \mathbb{Z}^+)$

Derivation: Algebraic expansion

Basis: Let  $q \rightarrow \sqrt{b^2-4ac}$ , then  $\frac{d+e z}{a+b z+c z^2} = \left(\frac{e}{2} + \frac{2cd-be}{2q}\right) \frac{1}{\frac{b}{2}-\frac{q}{2}+cz} + \left(\frac{e}{2} - \frac{2cd-be}{2q}\right) \frac{1}{\frac{b}{2}+\frac{q}{2}+cz}$

Rule 1.2.3.3.8.1.1.2.2: If  $b^2-4ac \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge (b^2-4ac > 0 \vee \frac{n}{2} \notin \mathbb{Z}^+)$ , let  $q \rightarrow \sqrt{b^2-4ac}$ , then

$$\int \frac{d+e x^n}{a+b x^n+c x^{2n}} dx \rightarrow \left( \frac{e}{2} + \frac{2cd-be}{2q} \right) \int \frac{1}{\frac{b}{2} - \frac{a}{2} + c x^n} dx + \left( \frac{e}{2} - \frac{2cd-be}{2q} \right) \int \frac{1}{\frac{b}{2} + \frac{a}{2} + c x^n} dx$$

Program code:

```
Int[(d_+e_.*x_^n_)/(a_+b_.*x_^n_+c_.*x_^n2_),x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
    (e/2+(2*c*d-b*e)/(2*q))*Int[1/(b/2-q/2+c*x^n),x] + (e/2-(2*c*d-b*e)/(2*q))*Int[1/(b/2+q/2+c*x^n),x] /;
  FreeQ[{a,b,c,d,e,n},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && (PosQ[b^2-4*a*c] || Not[IGtQ[n/2,0]])
```

$$\mathbf{3:} \int \frac{d+e x^n}{a+b x^n+c x^{2n}} dx \text{ when } b^2-4ac \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge \frac{n}{2} \in \mathbb{Z}^+ \wedge b^2-4ac \neq 0$$

Derivation: Algebraic expansion

Basis: If  $q \rightarrow \sqrt{\frac{a}{c}}$  and  $r \rightarrow \sqrt{2q - \frac{b}{c}}$ , then  $\frac{d+e z^2}{a+b z^2+c z^4} = \frac{d r - (d-e q) z}{2 c q r (q-r z+z^2)} + \frac{d r + (d-e q) z}{2 c q r (q+r z+z^2)}$

Note: If  $(a | b | c) \in \mathbb{R} \wedge b^2-4ac < 0$ , then  $\frac{a}{c} > 0$  and  $2\sqrt{\frac{a}{c}} - \frac{b}{c} > 0$ .

Rule 1.2.3.3.8.1.1.2.3: If  $b^2-4ac \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge \frac{n}{2} \in \mathbb{Z}^+ \wedge b^2-4ac \neq 0$ , let  $q \rightarrow \sqrt{\frac{a}{c}}$  and  $r \rightarrow \sqrt{2q - \frac{b}{c}}$ , then

$$\int \frac{d+e x^n}{a+b x^n+c x^{2n}} dx \rightarrow \frac{1}{2 c q r} \int \frac{d r - (d-e q) x^{n/2}}{q-r x^{n/2}+x^n} dx + \frac{1}{2 c q r} \int \frac{d r + (d-e q) x^{n/2}}{q+r x^{n/2}+x^n} dx$$

Program code:

```
Int[(d+e_.**x_^n_)/(a+b_.**x_^n_+c_.**x_^n2_),x_Symbol] :=
  With[{q=Rt[a/c,2]},
    With[{r=Rt[2*q-b/c,2]},
      1/(2*c*q*r)*Int[(d*r-(d-e*q)*x^(n/2))/(q-r*x^(n/2)+x^n),x] +
      1/(2*c*q*r)*Int[(d*r+(d-e*q)*x^(n/2))/(q+r*x^(n/2)+x^n),x]] /;
    FreeQ[{a,b,c,d,e},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && IGtQ[n/2,0] && NegQ[b^2-4*a*c]
```

$$\mathbf{2:} \int \frac{(d+e x^n)^q}{a+b x^n+c x^{2n}} dx \text{ when } b^2-4ac \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge q \in \mathbb{Z}$$

Derivation: Algebraic expansion

Rule 1.2.3.3.8.1.2: If  $b^2-4ac \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge q \in \mathbb{Z}$ , then



$$\int \frac{(d+e x^n)^q}{a+b x^n+c x^{2n}} dx \rightarrow \int \text{ExpandIntegrand}\left[\frac{(d+e x^n)^q}{a+b x^n+c x^{2n}}, x\right] dx$$

Program code:

```
Int[(d_+e_.*x_^n_)^q_/(a_+b_.*x_^n_+c_.*x_^n2_),x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x^n)^q/(a+b*x^n+c*x^(2*n)),x],x] /;
FreeQ[{a,b,c,d,e,n},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && IntegerQ[q]
```

```
Int[(d_+e_.*x_^n_)^q_/(a_+c_.*x_^n2_),x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x^n)^q/(a+c*x^(2*n)),x],x] /;
FreeQ[{a,c,d,e,n},x] && EqQ[n2,2*n] && NeQ[c*d^2+a*e^2,0] && IntegerQ[q]
```

2.  $\int \frac{(d+e x^n)^q}{a+b x^n+c x^{2n}} dx$  when  $b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge q \notin \mathbb{Z}$

1:  $\int \frac{(d+e x^n)^q}{a+b x^n+c x^{2n}} dx$  when  $b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge q \notin \mathbb{Z} \wedge q < -1$

Derivation: Algebraic expansion

Basis:  $\frac{1}{a+bz+cz^2} = \frac{e^2}{cd^2-bde+ae^2} + \frac{(d+ez)(cd-be-cex)}{(cd^2-bde+ae^2)(a+bz+cz^2)}$

Rule 1.2.3.3.8.2.1: If  $b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 \neq 0 \wedge q \notin \mathbb{Z} \wedge q < -1$ , then

$$\int \frac{(d+e x^n)^q}{a+b x^n+c x^{2n}} dx \rightarrow \frac{e^2}{c d^2 - b d e + a e^2} \int (d+e x^n)^q dx + \frac{1}{c d^2 - b d e + a e^2} \int \frac{(d+e x^n)^{q+1} (c d - b e - c e x^n)}{a+b x^n+c x^{2n}} dx$$

Program code:

```
Int[(d_+e_.*x_^n_)^q_/(a_+b_.*x_^n_+c_.*x_^n2_),x_Symbol] :=
  e^2/(c*d^2-b*d*e+a*e^2)*Int[(d+e*x^n)^q,x] +
  1/(c*d^2-b*d*e+a*e^2)*Int[(d+e*x^n)^(q+1)*(c*d-b*e-c*e*x^n)/(a+b*x^n+c*x^(2*n)),x] /;
FreeQ[{a,b,c,d,e,n},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && Not[IntegerQ[q]] && LtQ[q,-1]
```

```
Int[(d_+e_.*x_^n_)^q/(a_+c_.*x_^n2_),x_Symbol] :=
  e^2/(c*d^2+a*e^2)*Int[(d+e*x^n)^q,x] +
  c/(c*d^2+a*e^2)*Int[(d+e*x^n)^(q+1)*(d-e*x^n)/(a+c*x^(2*n)),x] /;
FreeQ[{a,c,d,e,n},x] && EqQ[n2,2*n] && NeQ[c*d^2+a*e^2,0] && Not[IntegerQ[q]] && LtQ[q,-1]
```

2:  $\int \frac{(d+e x^n)^q}{a+b x^n+c x^{2n}} dx$  when  $b^2-4ac \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge q \notin \mathbb{Z}$

Derivation: Algebraic expansion

■ Basis: If  $r = \sqrt{b^2-4ac}$ , then  $\frac{1}{a+b x^n+c x^{2n}} = \frac{2c}{r(b-r+2cx^n)} - \frac{2c}{r(b+r+2cx^n)}$

Rule 1.2.3.3.8.2.2: If  $b^2-4ac \neq 0 \wedge c d^2-b d e+a e^2 \neq 0 \wedge q \notin \mathbb{Z}$ , then

$$\int \frac{(d+e x^n)^q}{a+b x^n+c x^{2n}} dx \rightarrow \frac{2c}{r} \int \frac{(d+e x^n)^q}{b-r+2cx^n} dx - \frac{2c}{r} \int \frac{(d+e x^n)^q}{b+r+2cx^n} dx$$

Program code:

```
Int[(d_+e_.*x_^n_)^q/(a_+b_.*x_^n_+c_.*x_^n2_),x_Symbol] :=
  With[{r=Rt[b^2-4*a*c,2]},
    2*c/r*Int[(d+e*x^n)^q/(b-r+2*c*x^n),x] - 2*c/r*Int[(d+e*x^n)^q/(b+r+2*c*x^n),x] /;
  FreeQ[{a,b,c,d,e,n,q},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] && Not[IntegerQ[q]]
```

```
Int[(d_+e_.*x_^n_)^q/(a_+c_.*x_^n2_),x_Symbol] :=
  With[{r=Rt[-a*c,2]},
    -c/(2*r)*Int[(d+e*x^n)^q/(r-c*x^n),x] - c/(2*r)*Int[(d+e*x^n)^q/(r+c*x^n),x] /;
  FreeQ[{a,c,d,e,n,q},x] && EqQ[n2,2*n] && NeQ[c*d^2+a*e^2,0] && Not[IntegerQ[q]]
```

$$9. \int (d+e x^n) (a+b x^n+c x^{2n})^p dx \text{ when } b^2-4ac \neq 0$$

$$1: \int (d+e x^n) (a+b x^n+c x^{2n})^p dx \text{ when } b^2-4ac \neq 0 \wedge p < -1$$

Derivation: Trinomial recurrence 2b with  $m = 0$

Rule 1.2.3.3.9.1: If  $b^2-4ac \neq 0 \wedge p < -1$ , then

$$\int (d+e x^n) (a+b x^n+c x^{2n})^p dx \rightarrow -\frac{x (d b^2 - a b e - 2 a c d + (b d - 2 a e) c x^n) (a+b x^n+c x^{2n})^{p+1}}{a n (p+1) (b^2-4 a c)} + \frac{1}{a n (p+1) (b^2-4 a c)} \cdot \int ((n p+n+1) d b^2 - a b e - 2 a c d (2 n p+2 n+1) + (2 n p+3 n+1) (b d - 2 a e) c x^n) (a+b x^n+c x^{2n})^{p+1} dx$$

Program code:

```
Int[(d_+e_.*x_^n_)*(a_+b_.*x_^n_+c_.*x_^n2_)^p_,x_Symbol] :=
-x*(d*b^2-a*b*e-2*a*c*d+(b*d-2*a*e)*c*x^n)*(a+b*x^n+c*x^(2*n))^(p+1)/(a*n*(p+1)*(b^2-4*a*c)) +
1/(a*n*(p+1)*(b^2-4*a*c))*
Int[Simp[(n*p+n+1)*d*b^2-a*b*e-2*a*c*d*(2*n*p+2*n+1)+(2*n*p+3*n+1)*(d*b-2*a*e)*c*x^n,x]*
(a+b*x^n+c*x^(2*n))^(p+1),x] /;
FreeQ[{a,b,c,d,e,n},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && ILtQ[p,-1]
```

```
Int[(d_+e_.*x_^n_)*(a_+c_.*x_^n2_)^p_,x_Symbol] :=
-x*(d+e*x^n)*(a+c*x^(2*n))^(p+1)/(2*a*n*(p+1)) +
1/(2*a*n*(p+1))*Int[(d*(2*n*p+2*n+1)+e*(2*n*p+3*n+1)*x^n)*(a+c*x^(2*n))^(p+1),x] /;
FreeQ[{a,c,d,e,n},x] && EqQ[n2,2*n] && ILtQ[p,-1]
```

**2:**  $\int (d+e x^n) (a+b x^n+c x^{2n})^p dx$  when  $b^2 - 4ac \neq 0$

Derivation: Algebraic expansion

Rule 1.2.3.3.9.2: If  $b^2 - 4ac \neq 0$ , then

$$\int (d+e x^n) (a+b x^n+c x^{2n})^p dx \rightarrow \int \text{ExpandIntegrand}[(d+e x^n) (a+b x^n+c x^{2n})^p, x] dx$$

Program code:

```
Int[(d_+e_.*x_^n_)*(a_+b_.*x_^n_+c_.*x_^n2_)^p_,x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x^n)*(a+b*x^n+c*x^(2*n))^p,x],x] /;
FreeQ[{a,b,c,d,e,n},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0]
```

```
Int[(d_+e_.*x_^n_)*(a_+c_.*x_^n2_)^p_,x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x^n)*(a+c*x^(2*n))^p,x],x] /;
FreeQ[{a,c,d,e,n},x] && EqQ[n2,2*n]
```

**10:**  $\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx$  when  $b^2 - 4ac \neq 0 \wedge p \in \mathbb{Z}^+ \wedge 2np + nq + 1 \neq 0$

Reference: G&R 2.110.5, CRC 88a

Derivation: Binomial recurrence 3a

Note: This rule reduces the degree of the polynomial in the resulting integrand.

Rule 1.2.3.3.10: If  $b^2 - 4ac \neq 0 \wedge p \in \mathbb{Z}^+ \wedge 2np + nq + 1 \neq 0$ , then

$$\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx \rightarrow \int (d+e x^n)^q ((a+b x^n+c x^{2n})^p - c^p x^{2np}) dx + c^p \int x^{2np} (d+e x^n)^q dx$$

$$\rightarrow \frac{c^p x^{2np-n+1} (d+e x^n)^{q+1}}{e (2np+nq+1)} + \int (d+e x^n)^q \left( (a+b x^n+c x^{2n})^p - c^p x^{2np} - \frac{d c^p (2np-n+1) x^{2np-n}}{e (2np+nq+1)} \right) dx$$

Program code:

```
Int[(d+_e_.*x_^n_)^q*(a+_b_.*x_^n+_c_.*x_^n2_)^p_,x_Symbol] :=
  c^p*x^(2*n*p-n+1)*(d+e*x^n)^(q+1)/(e*(2*n*p+n*q+1)) +
  Int[(d+e*x^n)^q*ExpandToSum[(a+b*x^n+c*x^(2*n))^p-c^p*x^(2*n*p)-d*c^p*(2*n*p-n+1)*x^(2*n*p-n)/(e*(2*n*p+n*q+1)),x],x] /;
FreeQ[{a,b,c,d,e,n,q},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && IGtQ[p,0] && NeQ[2*n*p+n*q+1,0] && IGtQ[n,0] && Not[IGtQ[q,0]]
```

```
Int[(d+_e_.*x_^n_)^q*(a+_c_.*x_^n2_)^p_,x_Symbol] :=
  c^p*x^(2*n*p-n+1)*(d+e*x^n)^(q+1)/(e*(2*n*p+n*q+1)) +
  Int[(d+e*x^n)^q*ExpandToSum[(a+c*x^(2*n))^p-c^p*x^(2*n*p)-d*c^p*(2*n*p-n+1)*x^(2*n*p-n)/(e*(2*n*p+n*q+1)),x],x] /;
FreeQ[{a,c,d,e,n,q},x] && EqQ[n2,2*n] && IGtQ[p,0] && NeQ[2*n*p+n*q+1,0] && IGtQ[n,0] && Not[IGtQ[q,0]]
```

**11:**  $\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx$  when  $b^2-4ac \neq 0 \wedge c d^2-bde+a e^2 \neq 0 \wedge ((p|q) \in \mathbb{Z} \vee p \in \mathbb{Z}^+ \vee q \in \mathbb{Z}^+)$

Derivation: Algebraic expansion

Rule 1.2.3.3.11: If  $b^2-4ac \neq 0 \wedge ((p|q) \in \mathbb{Z} \vee p \in \mathbb{Z}^+ \vee q \in \mathbb{Z}^+)$ , then

$$\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx \rightarrow \int \text{ExpandIntegrand}[(d+e x^n)^q (a+b x^n+c x^{2n})^p, x] dx$$

Program code:

```
Int[(d+_e_.*x_^n_)^q*(a+_b_.*x_^n+_c_.*x_^n2_)^p_,x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x^n)^q*(a+b*x^n+c*x^(2*n))^p,x],x] /;
FreeQ[{a,b,c,d,e,n,p,q},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && NeQ[c*d^2-b*d*e+a*e^2,0] &&
  (IntegersQ[p,q] && Not[IntegerQ[n]] || IGtQ[p,0] || IGtQ[q,0] && Not[IntegerQ[n]])
```

```

Int[(d_+e_.*x_^n_)^q*(a_+c_.*x_^n2_)^p_,x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x^n)^q*(a+c*x^(2*n))^p,x],x] /;
FreeQ[{a,c,d,e,n,p,q},x] && EqQ[n2,2*n] && NeQ[c*d^2+a*e^2,0] &&
  (IntegersQ[p,q] && Not[IntegerQ[n]] || IGtQ[p,0] || IGtQ[q,0] && Not[IntegerQ[n]])

```

**12:**  $\int (d + e x^n)^q (a + c x^{2n})^p dx$  when  $c d^2 + a e^2 \neq 0 \wedge p \notin \mathbb{Z} \wedge q \in \mathbb{Z}^-$

Derivation: Algebraic expansion

Basis: If  $q \in \mathbb{Z}$ , then  $(d + e x^n)^q = \left( \frac{d}{d^2 - e^2 x^{2n}} - \frac{e x^n}{d^2 - e^2 x^{2n}} \right)^{-q}$

Note: Resulting integrands are of the form  $x^m (a + b x^{2n})^p (c + d x^{2n})^q$  which are integrable in terms of the Appell hypergeometric function .

Rule 1.2.3.3.12: If  $c d^2 + a e^2 \neq 0 \wedge p \notin \mathbb{Z} \wedge q \in \mathbb{Z}^-$ , then

$$\int (d + e x^n)^q (a + c x^{2n})^p dx \rightarrow \int (a + c x^{2n})^p \text{ExpandIntegrand}\left[\left(\frac{d}{d^2 - e^2 x^{2n}} - \frac{e x^n}{d^2 - e^2 x^{2n}}\right)^{-q}, x\right] dx$$

Program code:

```

Int[(d_+e_.*x_^n_)^q*(a_+c_.*x_^n2_)^p_,x_Symbol] :=
  Int[ExpandIntegrand[(a+c*x^(2*n))^p,(d/(d^2-e^2*x^(2*n))-e*x^n/(d^2-e^2*x^(2*n)))^(-q),x],x] /;
FreeQ[{a,c,d,e,n,p},x] && EqQ[n2,2*n] && NeQ[c*d^2+a*e^2,0] && Not[IntegerQ[p]] && ILtQ[q,0]

```

**U:**  $\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx$

### Rule 1.2.3.3.X:

$$\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx \rightarrow \int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx$$

### Program code:

```
Int[(d_+e_.*x_^n_)^q_*(a_+b_.*x_^n_+c_.*x_^n2_)^p_,x_Symbol] :=
  Unintegrable[(d+e*x^n)^q*(a+b*x^n+c*x^(2*n))^p,x] /;
FreeQ[{a,b,c,d,e,n,p,q},x] && EqQ[n2,2*n]
```

```
Int[(d_+e_.*x_^n_)^q_*(a_+c_.*x_^n2_)^p_,x_Symbol] :=
  Unintegrable[(d+e*x^n)^q*(a+c*x^(2*n))^p,x] /;
FreeQ[{a,c,d,e,n,p,q},x] && EqQ[n2,2*n]
```

**S:**  $\int (d+e u^n)^q (a+b u^n+c u^{2n})^p dx$  when  $u == f + g x$

### Derivation: Integration by substitution

Rule 1.2.3.3.S: If  $u == f + g x$ , then

$$\int (d+e u^n)^q (a+b u^n+c u^{2n})^p dx \rightarrow \frac{1}{g} \text{Subst}\left[\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx, x, u\right]$$

### Program code:

```
Int[(d_+e_.*u_^n_)^q_*(a_+b_.*u_^n_+c_.*u_^n2_)^p_,x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(d+e*x^n)^q*(a+b*x^n+c*x^(2*n))^p,x],x,u] /;
FreeQ[{a,b,c,d,e,n,p,q},x] && EqQ[n2,2*n] && LinearQ[u,x] && NeQ[u,x]
```

```
Int[(d_+e_.*u_^n_)^q_.*(a_+c_.*u_^n2_)^p_.,x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(d+e*x^n)^q*(a+c*x^(2*n))^p,x],x,u] /;
FreeQ[{a,c,d,e,n,p,q},x] && EqQ[n2,2*n] && LinearQ[u,x] && NeQ[u,x]
```

### Rules for integrands of the form $(d + e x^{-n})^q (a + b x^n + c x^{2n})^p$

1.  $\int (d + e x^{-n})^q (a + b x^n + c x^{2n})^p dx$  when  $p \in \mathbb{Z} \vee q \in \mathbb{Z}$

**1:**  $\int (d + e x^{-n})^q (a + b x^n + c x^{2n})^p dx$  when  $q \in \mathbb{Z} \wedge (n > 0 \vee p \notin \mathbb{Z})$

Derivation: Algebraic simplification

Basis: If  $q \in \mathbb{Z}$ , then  $(d + e x^{-n})^q = x^{-nq} (e + d x^n)^q$

Rule: If  $q \in \mathbb{Z} \wedge (n > 0 \vee p \notin \mathbb{Z})$ , then

$$\int (d + e x^{-n})^q (a + b x^n + c x^{2n})^p dx \rightarrow \int x^{-nq} (e + d x^n)^q (a + b x^n + c x^{2n})^p dx$$

Program code:

```
Int[(d_+e_.*x_^mn_)^q_.*(a_+b_.*x_^n_+c_.*x_^n2_)^p_.,x_Symbol] :=
  Int[x^(-n*q)*(e+d*x^n)^q*(a+b*x^n+c*x^(2*n))^p,x] /;
FreeQ[{a,b,c,d,e,n,p},x] && EqQ[n2,2*n] && EqQ[mn,-n] && IntegerQ[q] && (PosQ[n] || Not[IntegerQ[p]])
```

```
Int[(d_+e_.*x_^mn_)^q_.*(a_+c_.*x_^n2_)^p_.,x_Symbol] :=
  Int[x^(mn*q)*(e+d*x^(-mn))^q*(a+c*x^n2)^p,x] /;
FreeQ[{a,c,d,e,mn,p},x] && EqQ[n2,-2*mn] && IntegerQ[q] && (PosQ[n2] || Not[IntegerQ[p]])
```



**2:**  $\int (d+e x^n)^q (a+b x^{-n}+c x^{-2n})^p dx$  when  $p \in \mathbb{Z}$

Derivation: Algebraic simplification

Basis: If  $p \in \mathbb{Z}$ , then  $(a+b x^{-n}+c x^{-2n})^p = x^{-2np} (c+b x^n+a x^{2n})^p$

Rule: If  $p \in \mathbb{Z}$ , then

$$\int (d+e x^n)^q (a+b x^{-n}+c x^{-2n})^p dx \rightarrow \int x^{-2np} (d+e x^n)^q (c+b x^n+a x^{2n})^p dx$$

Program code:

```
Int[(d+_e_.*x_^n_.)^q_.*(a+_b_.*x_^mn_.+c_.*x_^mn2_.)^p_,x_Symbol] :=
  Int[x^(-2*n*p)*(d+_e_.*x^n)^q*(c+_b_.*x^n+_a_.*x^(2*n))^p,x] /;
FreeQ[{a,b,c,d,e,n,q},x] && EqQ[mn,-n] && EqQ[mn2,2*mn] && IntegerQ[p]
```

```
Int[(d+_e_.*x_^n_.)^q_.*(a+_c_.*x_^mn2_.)^p_,x_Symbol] :=
  Int[x^(-2*n*p)*(d+_e_.*x^n)^q*(c+_a_.*x^(2*n))^p,x] /;
FreeQ[{a,c,d,e,n,q},x] && EqQ[mn2,-2*n] && IntegerQ[p]
```

2.  $\int (d+e x^{-n})^q (a+b x^n+c x^{2n})^p dx$  when  $p \notin \mathbb{Z} \wedge q \notin \mathbb{Z}$

**1:**  $\int (d+e x^{-n})^q (a+b x^n+c x^{2n})^p dx$  when  $p \notin \mathbb{Z} \wedge q \notin \mathbb{Z} \wedge n > 0$

Derivation: Piecewise constant extraction

Basis:  $\partial_x \frac{x^n q (d+e x^{-n})^q}{(1+\frac{d x^n}{e})^q} = 0$

Rule: If  $p \notin \mathbb{Z} \wedge q \notin \mathbb{Z} \wedge n > 0$ , then

$$\int (d+e x^{-n})^q (a+b x^n+c x^{2n})^p dx \rightarrow \frac{e^{\text{IntPart}[q] x^n \text{FracPart}[q]} (d+e x^{-n})^{\text{FracPart}[q]}}{\left(1+\frac{d x^n}{e}\right)^{\text{FracPart}[q]}} \int x^{-n q} \left(1+\frac{d x^n}{e}\right)^q (a+b x^n+c x^{2n})^p dx$$

Program code:

```
Int[(d+_e_.**x_^mn_.)^q_*(a+_b_.**x_^n_.+c_.**x_^n2_.)^p_,x_Symbol] :=
  e^IntPart[q]*x^(n*FracPart[q])*(d+e*x^(-n))^FracPart[q]/(1+d*x^n/e)^FracPart[q]*Int[x^(-n*q)*(1+d*x^n/e)^q*(a+b*x^n+c*x^(2*n))^p,x],
FreeQ[{a,b,c,d,e,n,p,q},x] && EqQ[n2,2*n] && EqQ[mn,-n] && Not[IntegerQ[p]] && Not[IntegerQ[q]] && PosQ[n]
```

```
Int[(d+_e_.**x_^mn_.)^q_*(a+_c_.**x_^n2_.)^p_,x_Symbol] :=
  e^IntPart[q]*x^(-mn*FracPart[q])*(d+e*x^mn)^FracPart[q]/(1+d*x^(-mn)/e)^FracPart[q]*Int[x^(mn*q)*(1+d*x^(-mn)/e)^q*(a+c*x^n2)^p,x],
FreeQ[{a,c,d,e,mn,p,q},x] && EqQ[n2,-2*mn] && Not[IntegerQ[p]] && Not[IntegerQ[q]] && PosQ[n2]
```

**x:**  $\int (d+e x^{-n})^q (a+b x^n+c x^{2n})^p dx$  when  $p \notin \mathbb{Z} \wedge q \notin \mathbb{Z} \wedge n > 0$

Derivation: Piecewise constant extraction

Basis:  $\partial_x \frac{x^n q (d+e x^{-n})^q}{(e+d x^n)^q} == 0$

Rule: If  $p \notin \mathbb{Z} \wedge q \notin \mathbb{Z} \wedge n > 0$ , then

$$\int (d+e x^{-n})^q (a+b x^n+c x^{2n})^p dx \rightarrow \frac{x^n \text{FracPart}[q] (d+e x^{-n})^{\text{FracPart}[q]}}{(e+d x^n)^{\text{FracPart}[q]}} \int x^{-n q} (e+d x^n)^q (a+b x^n+c x^{2n})^p dx$$

Program code:

```
(* Int[(d+_e_.**x_^mn_.)^q_*(a+_b_.**x_^n_.+c_.**x_^n2_.)^p_,x_Symbol] :=
  x^(n*FracPart[q])*(d+e*x^(-n))^FracPart[q]/(e+d*x^n)^FracPart[q]*Int[x^(-n*q)*(e+d*x^n)^q*(a+b*x^n+c*x^(2*n))^p,x] /;
FreeQ[{a,b,c,d,e,n,p,q},x] && EqQ[n2,2*n] && EqQ[mn,-n] && Not[IntegerQ[p]] && Not[IntegerQ[q]] && PosQ[n] *)
```

```
(* Int[(d+_e_.**x_^mn_.)^q_*(a+_c_.**x_^n2_.)^p_,x_Symbol] :=
  x^(-mn*FracPart[q])*(d+e*x^mn)^FracPart[q]/(e+d*x^(-mn))^FracPart[q]*Int[x^(mn*q)*(e+d*x^(-mn))^q*(a+c*x^n2)^p,x] /;
FreeQ[{a,c,d,e,mn,p,q},x] && EqQ[n2,-2*mn] && Not[IntegerQ[p]] && Not[IntegerQ[q]] && PosQ[n2] *)
```

**2:**  $\int (d+e x^n)^q (a+b x^{-n}+c x^{-2n})^p dx$  when  $p \notin \mathbb{Z} \wedge q \notin \mathbb{Z} \wedge n > 0$

Derivation: Piecewise constant extraction

Basis:  $\partial_x \frac{x^{2n p} (a+b x^{-n}+c x^{-2n})^p}{(c+b x^n+a x^{2n})^p} = 0$

Rule: If  $p \notin \mathbb{Z} \wedge q \notin \mathbb{Z} \wedge n > 0$ , then

$$\int (d+e x^n)^q (a+b x^{-n}+c x^{-2n})^p dx \rightarrow \frac{x^{2n \text{FracPart}[p]} (a+b x^{-n}+c x^{-2n})^{\text{FracPart}[p]}}{(c+b x^n+a x^{2n})^{\text{FracPart}[p]}} \int x^{-2n p} (d+e x^n)^q (c+b x^n+a x^{2n})^p dx$$

Program code:

```
Int[(d_+e_.*x_^n_.)^q_.*(a_.+b_.*x_^mn_+.c_.*x_^mn2_.)^p_,x_Symbol] :=
  x^(2*n*FracPart[p])*(a+b*x^(-n)+c*x^(-2*n))^FracPart[p]/(c+b*x^n+a*x^(2*n))^FracPart[p]*
  Int[x^(-2*n*p)*(d+e*x^n)^q*(c+b*x^n+a*x^(2*n))^p,x] /;
FreeQ[{a,b,c,d,e,n,p,q},x] && EqQ[mn,-n] && EqQ[mn2,2*mn] && Not[IntegerQ[p]] && Not[IntegerQ[q]] && PosQ[n]
```

```
Int[(d_+e_.*x_^n_.)^q_.*(a_.+c_.*x_^mn2_.)^p_,x_Symbol] :=
  x^(2*n*FracPart[p])*(a+c*x^(-2*n))^FracPart[p]/(c+a*x^(2*n))^FracPart[p]*
  Int[x^(-2*n*p)*(d+e*x^n)^q*(c+a*x^(2*n))^p,x] /;
FreeQ[{a,c,d,e,n,p,q},x] && EqQ[mn2,-2*n] && Not[IntegerQ[p]] && Not[IntegerQ[q]] && PosQ[n]
```

### Rules for integrands of the form $(d + e x^n)^q (a + b x^{-n} + c x^n)^p$

**1:**  $\int (d + e x^n)^q (a + b x^{-n} + c x^n)^p dx$  when  $p \in \mathbb{Z}$

Derivation: Algebraic normalization

$$\text{Basis: } a + b x^{-n} + c x^n = x^{-n} (b + a x^n + c x^{2n})$$

Rule: If  $p \in \mathbb{Z}$ , then

$$\int (d + e x^n)^q (a + b x^{-n} + c x^n)^p dx \rightarrow \int x^{-np} (d + e x^n)^q (b + a x^n + c x^{2n})^p dx$$

Program code:

```
Int[(d_+e_.*x_^n_)^q_.*(a_+b_.*x_^mn_+c_.*x_^n_)^p_,x_Symbol] :=
  Int[x^(-n*p)*(d+e*x^n)^q*(b+a*x^n+c*x^(2*n))^p,x] /;
FreeQ[{a,b,c,d,e,n,q},x] && EqQ[mn,-n] && IntegerQ[p]
```

**2:**  $\int (d + e x^n)^q (a + b x^{-n} + c x^n)^p dx$  when  $p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{x^{np} (a + b x^{-n} + c x^n)^p}{(b + a x^n + c x^{2n})^p} = 0$$

$$\text{Basis: } \frac{x^{np} (a + b x^{-n} + c x^n)^p}{(b + a x^n + c x^{2n})^p} = \frac{x^{n \text{FracPart}[p]} (a + b x^{-n} + c x^n)^{\text{FracPart}[p]}}{(b + a x^n + c x^{2n})^{\text{FracPart}[p]}}$$

Rule: If  $p \notin \mathbb{Z}$ , then

$$\int (d+e x^n)^q (a+b x^n+c x^{2n})^p dx \rightarrow \frac{x^n \text{FracPart}[p] (a+b x^n+c x^{2n})^{\text{FracPart}[p]}}{(b+a x^n+c x^{2n})^{\text{FracPart}[p]}} \int x^{-n p} (d+e x^n)^q (b+a x^n+c x^{2n})^p dx$$

Program code:

```
Int[(d_+e_.*x_^n_)^q_.*(a_+b_.*x_^mn_+c_.*x_^n_)^p_,x_Symbol] :=
  x^(n*FracPart[p])*(a+b/x^n+c*x^n)^FracPart[p]/(b+a*x^n+c*x^(2*n))^FracPart[p]*
  Int[x^(-n*p)*(d+e*x^n)^q*(b+a*x^n+c*x^(2*n))^p,x] /;
FreeQ[{a,b,c,d,e,n,p,q},x] && EqQ[mn,-n] && Not[IntegerQ[p]]
```

Rules for integrands of the form  $(d+e x^n)^q (f+g x^n)^r (a+b x^n+c x^{2n})^p$

1:  $\int (d+e x^n)^q (f+g x^n)^r (a+b x^n+c x^{2n})^p dx$  when  $b^2 - 4ac == 0 \wedge p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: If  $b^2 - 4ac == 0$ , then  $\partial_x \frac{(a+b x^n+c x^{2n})^p}{(b+2 c x^n)^{2p}} == 0$

Basis: If  $b^2 - 4ac == 0$ , then  $\frac{(a+b x^n+c x^{2n})^p}{(b+2 c x^n)^{2p}} == \frac{(a+b x^n+c x^{2n})^{\text{FracPart}[p]}}{(4c)^{\text{IntPart}[p]} (b+2 c x^n)^{2 \text{FracPart}[p]}}$

Rule: If  $b^2 - 4ac == 0 \wedge 2p \notin \mathbb{Z}$ , then

$$\int (d+e x^n)^q (f+g x^n)^r (a+b x^n+c x^{2n})^p dx \rightarrow \frac{(a+b x^n+c x^{2n})^{\text{FracPart}[p]}}{(4c)^{\text{IntPart}[p]} (b+2 c x^n)^{2 \text{FracPart}[p]}} \int (d+e x^n)^q (f+g x^n)^r (b+2 c x^n)^{2p} dx$$

Program code:

```
Int[(d_+e_.*x_^n_)^q_.*(f_+g_.*x_^n_)^r_.*(a_+b_.*x_^n_+c_.*x_^n2_)^p_,x_Symbol] :=
  (a+b*x^n+c*x^(2*n))^FracPart[p]/((4*c)^IntPart[p]*(b+2*c*x^n)^(2*FracPart[p]))*
  Int[(d+e*x^n)^q*(f+g*x^n)^r*(b+2*c*x^n)^(2*p),x] /;
FreeQ[{a,b,c,d,e,f,g,n,p,q,r},x] && EqQ[n2,2*n] && EqQ[b^2-4*a*c,0] && Not[IntegerQ[p]]
```

$$2. \int (d+e x^n)^q (f+g x^n)^r (a+b x^n+c x^{2n})^p dx \text{ when } b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 = 0$$

$$1: \int (d+e x^n)^q (f+g x^n)^r (a+b x^n+c x^{2n})^p dx \text{ when } b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p \in \mathbb{Z}$$

Derivation: Algebraic simplification

Basis: If  $c d^2 - b d e + a e^2 = 0$ , then  $a + b z + c z^2 = (d + e z) \left( \frac{a}{d} + \frac{c z}{e} \right)$

Rule: If  $b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p \in \mathbb{Z}$ , then

$$\int (d+e x^n)^q (f+g x^n)^r (a+b x^n+c x^{2n})^p dx \rightarrow \int (d+e x^n)^{p+q} (f+g x^n)^r \left( \frac{a}{d} + \frac{c x^n}{e} \right)^p dx$$

Program code:

```
Int[(d+_e_.*x_^n_)^q_.*(f+_g_.*x_^n_)^r_.*(a+_b_.*x_^n+_c_.*x_^n2_)^p_,x_Symbol] :=
  Int[(d+e*x^n)^(p+q)*(f+g*x^n)^r*(a/d+c/e*x^n)^p,x] /;
FreeQ[{a,b,c,d,e,f,g,n,q,r},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && IntegerQ[p]
```

```
Int[(d+_e_.*x_^n_)^q_.*(f+_g_.*x_^n_)^r_.*(a+_c_.*x_^n2_)^p_,x_Symbol] :=
  Int[(d+e*x^n)^(p+q)*(f+g*x^n)^r*(a/d+c/e*x^n)^p,x] /;
FreeQ[{a,c,d,e,f,g,n,q,r},x] && EqQ[n2,2*n] && EqQ[c*d^2+a*e^2,0] && IntegerQ[p]
```

**2:**  $\int (d+e x^n)^q (f+g x^n)^r (a+b x^n+c x^{2n})^p dx$  when  $b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: If  $c d^2 - b d e + a e^2 = 0$ , then  $\partial_x \frac{(a+b x^n+c x^{2n})^p}{(d+e x^n)^p \left(\frac{a}{d} + \frac{c x^n}{e}\right)^p} = 0$

Basis: If  $c d^2 - b d e + a e^2 = 0$ , then  $\frac{(a+b x^n+c x^{2n})^p}{(d+e x^n)^p \left(\frac{a}{d} + \frac{c x^n}{e}\right)^p} = \frac{(a+b x^n+c x^{2n})^{\text{FracPart}[p]}}{(d+e x^n)^{\text{FracPart}[p]} \left(\frac{a}{d} + \frac{c x^n}{e}\right)^{\text{FracPart}[p]}}$

Rule: If  $b^2 - 4ac \neq 0 \wedge c d^2 - b d e + a e^2 = 0 \wedge p \notin \mathbb{Z}$ , then

$$\int (d+e x^n)^q (f+g x^n)^r (a+b x^n+c x^{2n})^p dx \rightarrow \frac{(a+b x^n+c x^{2n})^{\text{FracPart}[p]}}{(d+e x^n)^{\text{FracPart}[p]} \left(\frac{a}{d} + \frac{c x^n}{e}\right)^{\text{FracPart}[p]}} \int (d+e x^n)^{p+q} (f+g x^n)^r \left(\frac{a}{d} + \frac{c x^n}{e}\right)^p dx$$

Program code:

```
Int[(d_+e_.*x_^n_)^q_.*(f_+g_.*x_^n_)^r_.*(a_+b_.*x_^n_+c_.*x_^n2_)^p_,x_Symbol] :=
  (a+b*x^n+c*x^(2*n))^FracPart[p]/((d+e*x^n)^FracPart[p]*(a/d+(c*x^n)/e)^FracPart[p])*
  Int[(d+e*x^n)^(p+q)*(f+g*x^n)^r*(a/d+c/e*x^n)^p,x] /;
FreeQ[{a,b,c,d,e,f,g,n,p,q,r},x] && EqQ[n2,2*n] && NeQ[b^2-4*a*c,0] && EqQ[c*d^2-b*d*e+a*e^2,0] && Not[IntegerQ[p]]
```

```
Int[(d_+e_.*x_^n_)^q_.*(f_+g_.*x_^n_)^r_.*(a_+c_.*x_^n2_)^p_,x_Symbol] :=
  (a+c*x^(2*n))^FracPart[p]/((d+e*x^n)^FracPart[p]*(a/d+(c*x^n)/e)^FracPart[p])*
  Int[(d+e*x^n)^(p+q)*(f+g*x^n)^r*(a/d+c/e*x^n)^p,x] /;
FreeQ[{a,c,d,e,f,g,n,p,q,r},x] && EqQ[n2,2*n] && EqQ[c*d^2+a*e^2,0] && Not[IntegerQ[p]]
```

$$3. \int (d_1 + e_1 x^{n/2})^q (d_2 + e_2 x^{n/2})^q (a + b x^n + c x^{2n})^p dx \text{ when } d_2 e_1 + d_1 e_2 = 0$$

$$1: \int (d_1 + e_1 x^{n/2})^q (d_2 + e_2 x^{n/2})^q (a + b x^n + c x^{2n})^p dx \text{ when } d_2 e_1 + d_1 e_2 = 0 \wedge (q \in \mathbb{Z} \vee d_1 > 0 \wedge d_2 > 0)$$

Derivation: Algebraic simplification

Basis: If  $d_2 e_1 + d_1 e_2 = 0 \wedge (q \in \mathbb{Z} \vee d_1 > 0 \wedge d_2 > 0)$ , then  $(d_1 + e_1 x^{n/2})^q (d_2 + e_2 x^{n/2})^q = (d_1 d_2 + e_1 e_2 x^n)^q$

Rule: If  $d_2 e_1 + d_1 e_2 = 0 \wedge (q \in \mathbb{Z} \vee d_1 > 0 \wedge d_2 > 0)$ , then

$$\int (d_1 + e_1 x^{n/2})^q (d_2 + e_2 x^{n/2})^q (a + b x^n + c x^{2n})^p dx \rightarrow \int (d_1 d_2 + e_1 e_2 x^n)^q (a + b x^n + c x^{2n})^p dx$$

Program code:

```
Int[(d1_+e1_.*x_^non2_)^q_.*(d2_+e2_.*x_^non2_)^q_.*(a_+b_.*x_^n+c_.*x_^n2_)^p_,x_Symbol] :=
  Int[(d1*d2+e1*e2*x^n)^q*(a+b*x^n+c*x^(2*n))^p,x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,n,p,q},x] && EqQ[n2,2*n] && EqQ[non2,n/2] && EqQ[d2*e1+d1*e2,0] && (IntegerQ[q] || GtQ[d1,0] && GtQ[d2,0])
```



**2:**  $\int (d_1 + e_1 x^{n/2})^q (d_2 + e_2 x^{n/2})^q (a + b x^n + c x^{2n})^p dx$  when  $d_2 e_1 + d_1 e_2 == 0$

Derivation: Piecewise constant extraction

— Basis: If  $d_2 e_1 + d_1 e_2 == 0$ , then  $\partial_x \frac{(d_1 + e_1 x^{n/2})^q (d_2 + e_2 x^{n/2})^q}{(d_1 d_2 + e_1 e_2 x^n)^q} == 0$

Rule: If  $d_2 e_1 + d_1 e_2 == 0$ , then

$$\int (d_1 + e_1 x^{n/2})^q (d_2 + e_2 x^{n/2})^q (a + b x^n + c x^{2n})^p dx \rightarrow \frac{(d_1 + e_1 x^{n/2})^{\text{FracPart}[q]} (d_2 + e_2 x^{n/2})^{\text{FracPart}[q]}}{(d_1 d_2 + e_1 e_2 x^n)^{\text{FracPart}[q]}} \int (d_1 d_2 + e_1 e_2 x^n)^q (a + b x^n + c x^{2n})^p dx$$

Program code:

```
Int[(d1_+e1_.*x^non2_)^q_.*(d2_+e2_.*x^non2_)^q_.*(a_+b_.*x^n_+c_.*x^2n_)^p_,x_Symbol] :=
  (d1+e1*x^(n/2))^FracPart[q]*(d2+e2*x^(n/2))^FracPart[q]/(d1*d2+e1*e2*x^n)^FracPart[q]*
  Int[(d1*d2+e1*e2*x^n)^q*(a+b*x^n+c*x^(2*n))^p,x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,n,p,q},x] && EqQ[n2,2*n] && EqQ[non2,n/2] && EqQ[d2*e1+d1*e2,0]
```

### Rules for integrands of the form $(A + B x^m) (d + e x^n)^q (a + b x^n + c x^{2n})^p$

**1:**  $\int (A + B x^m) (d + e x^n)^q (a + b x^n + c x^{2n})^p dx$  when  $m - n + 1 == 0$

Derivation: Algebraic expansion

Rule: If  $m - n + 1 == 0$ , then

$$\int (A + B x^m) (d + e x^n)^q (a + b x^n + c x^{2n})^p dx \rightarrow A \int (d + e x^n)^q (a + b x^n + c x^{2n})^p dx + B \int x^m (d + e x^n)^q (a + b x^n + c x^{2n})^p dx$$

Program code:

```
Int[(A_+B_.**x_^m_.)*(d_+e_.**x_^n_)^q_.*(a_+b_.**x_^n_+c_.**x_^2n_)^p_,x_Symbol] :=
  A*Int[(d+e*x^n)^q*(a+b*x^n+c*x^(2*n))^p,x] + B*Int[x^m*(d+e*x^n)^q*(a+b*x^n+c*x^(2*n))^p,x] /;
FreeQ[{a,b,c,d,e,A,B,m,n,p,q},x] && EqQ[n2,2*n] && EqQ[m-n+1,0]
```

```
Int[(A_+B_.**x_^m_.)*(d_+e_.**x_^n_)^q_.*(a_+c_.**x_^2n_)^p_,x_Symbol] :=
  A*Int[(d+e*x^n)^q*(a+c*x^(2*n))^p,x] + B*Int[x^m*(d+e*x^n)^q*(a+c*x^(2*n))^p,x] /;
FreeQ[{a,c,d,e,A,B,m,n,p,q},x] && EqQ[n2,2*n] && EqQ[m-n+1,0]
```