Rules for integrands involving exponentials

1. $\int u \left(F^{c\ (a+b\ x)} \right)^n \, dx$

   1: $\int \left(F^{c\ (a+b\ x)} \right)^n \, dx$

   Reference: G&R 2.311, CRC 519, A&S 4.2.54

   Rule:

   $$\int \left(F^{c\ (a+b\ x)} \right)^n \, dx \;\longrightarrow\; \frac{\left(F^{c\ (a+b\ x)} \right)^n}{b\ c\ n\ \text{Log}[F]}$$

   Program code:

   ```
   Int[(F_^(c_.*(a_.+b_.*x_)))^n_.,x_Symbol] :=
     (F^(c*(a+b*x)))^n/(b*c*n*Log[F]) /;
   FreeQ[{F,a,b,c,n},x]
   ```

   2: $\int P_x\ F^{c\ v} \, dx$ when $v == a + b\ x$

   Derivation: Algebraic expansion

   Rule: If $v == a + b\ x$, then

   $$\int P_x\ F^{c\ v} \, dx \;\longrightarrow\; \int F^{c\ (a+b\ x)}\ \text{ExpandIntegrand}[P_x,\ x] \, dx$$

   Program code:

   ```
   Int[u_*F_^(c_.*v_),x_Symbol] :=
     Int[ExpandIntegrand[u*F^(c*ExpandToSum[v,x]),x],x] /;
   FreeQ[{F,c},x] && PolynomialQ[u,x] && LinearQ[v,x] && $UseGamma===True
   ```

```
Int[u_*F_^(c_.*v_),x_Symbol] :=
  Int[ExpandIntegrand[F^(c*ExpandToSum[v,x]),u,x],x] /;
FreeQ[{F,c},x] && PolynomialQ[u,x] && LinearQ[v,x] && Not[$UseGamma===True]
```

3: $\int (d + e\,x)^m\, F^{c\,(a+b\,x)}\, (f + g\,x)\, dx$ when $e\,g\,(m+1) - b\,c\,(e\,f - d\,g)\, Log[F] == 0$

Basis: $\partial_x \left(F^{f[x]}\, g[x]\right) == F^{f[x]}\, (Log[F]\, g[x]\, f'[x] + g'[x])$

Rule: If $v == a + b\,x \,\wedge\, u == d + e\,x \,\wedge\, w == f + g\,x \,\wedge\, e\,g\,(m+1) - b\,c\,(e\,f - d\,g)\, Log[F] == 0$, then

$$\int u^m\, F^{c\,v}\, w\, dx \;\longrightarrow\; \int (d + e\,x)^m\, F^{c\,(a+b\,x)}\, (f + g\,x)\, dx \;\longrightarrow\; \frac{g\,(d + e\,x)^{m+1}\, F^{c\,(a+b\,x)}}{b\,c\,e\, Log[F]}$$

Program code:

```
Int[u_^m_.*F_^(c_.*v_)*w_,x_Symbol] :=
  With[{b=Coefficient[v,x,1],d=Coefficient[u,x,0],e=Coefficient[u,x,1],f=Coefficient[w,x,0],g=Coefficient[w,x,1]},
   g*u^(m+1)*F^(c*v)/(b*c*e*Log[F]) /;
  EqQ[e*g*(m+1)-b*c*(e*f-d*g)*Log[F],0]] /;
FreeQ[{F,c,m},x] && LinearQ[{u,v,w},x]
```

4. $\int P_x \, u^m \, F^{c \, v} \, dx$ when $v == a + b \, x \, \wedge \, u == (d + e \, x)^n$

    1: $\int P_x \, u^m \, F^{c \, v} \, dx$ when $v == a + b \, x \, \wedge \, u == (d + e \, x)^n \, \wedge \, m \in \mathbb{Z}$

Derivation: Algebraic expansion

Rule: If $v == a + b \, x \, \wedge \, u == (d + e \, x)^n \, \wedge \, m \in \mathbb{Z}$, then

$$\int P_x \, u^m \, F^{c \, v} \, dx \; \longrightarrow \; \int F^{c \, (a + b \, x)} \, \text{ExpandIntegrand}\left[P_x \, (d + e \, x)^{m \, n}, \, x\right] \, dx$$

Program code:

```
Int[w_*u_^m_.*F_^(c_.*v_),x_Symbol] :=
  Int[ExpandIntegrand[w*NormalizePowerOfLinear[u,x]^m*F^(c*ExpandToSum[v,x]),x],x] /;
FreeQ[{F,c},x] && PolynomialQ[w,x] && LinearQ[v,x] && PowerOfLinearQ[u,x] && IntegerQ[m] && $UseGamma===True
```

```
Int[w_*u_^m_.*F_^(c_.*v_),x_Symbol] :=
  Int[ExpandIntegrand[F^(c*ExpandToSum[v,x]),w*NormalizePowerOfLinear[u,x]^m,x],x] /;
FreeQ[{F,c},x] && PolynomialQ[w,x] && LinearQ[v,x] && PowerOfLinearQ[u,x] && IntegerQ[m] && Not[$UseGamma===True]
```

2: $\int P_x\, u^m\, F^{c\,v}\, dx$ when $v == a + b\,x \wedge u == (d + e\,x)^n \wedge m \notin \mathbb{Z}$

Derivation: Algebraic expansion

Rule: If $v == a + b\,x \wedge u == (d + e\,x)^n \wedge m \notin \mathbb{Z}$, then

$$\int P_x\, u^m\, F^{c\,v}\, dx \;\longrightarrow\; \frac{\left(\left(d + e\,x\right)^n\right)^m}{\left(d + e\,x\right)^{m\,n}} \int F^{c\,(a+b\,x)}\; \text{ExpandIntegrand}\left[P_x\,(d + e\,x)^{m\,n},\, x\right]\, dx$$

Program code:

```
Int[w_*u_^m_.*F_^(c_.*v_),x_Symbol] :=
  Module[{uu=NormalizePowerOfLinear[u,x],z},
  z=If[PowerQ[uu] && FreeQ[uu[[2]],x], uu[[1]]^(m*uu[[2]]), uu^m];
  uu^m/z*Int[ExpandIntegrand[w*z*F^(c*ExpandToSum[v,x]),x],x]] /;
FreeQ[{F,c,m},x] && PolynomialQ[w,x] && LinearQ[v,x] && PowerOfLinearQ[u,x] && Not[IntegerQ[m]]
```

5. $\int u\, F^{c\,(a+b\,x)}\, \text{Log}\left[d\,x\right]^n\, dx$

1: $\int F^{c\,(a+b\,x)}\, \text{Log}\left[d\,x\right]^n \left(e + h\,(f + g\,x)\, \text{Log}\left[d\,x\right]\right)\, dx$ when $e == f\,h\,(n+1) \wedge g\,h\,(n+1) == b\,c\,e\,\text{Log}[F] \wedge n \neq -1$

Rule: If $e == f\,h\,(n+1) \wedge g\,h\,(n+1) == b\,c\,e\,\text{Log}[F] \wedge n \neq -1$, then

$$\int F^{c\,(a+b\,x)}\, \text{Log}\left[d\,x\right]^n \left(e + h\,(f + g\,x)\, \text{Log}\left[d\,x\right]\right)\, dx \;\longrightarrow\; \frac{e\,x\,F^{c\,(a+b\,x)}\, \text{Log}\left[d\,x\right]^{n+1}}{n+1}$$

Program code:

```
Int[F_^(c_.*(a_.+b_.*x_))*Log[d_.*x_]^n_.*(e_+h_.*(f_.+g_.*x_)*Log[d_.*x_]),x_Symbol] :=
  e*x*F^(c*(a+b*x))*Log[d*x]^(n+1)/(n+1) /;
FreeQ[{F,a,b,c,d,e,f,g,h,n},x] && EqQ[e-f*h*(n+1),0] && EqQ[g*h*(n+1)-b*c*e*Log[F],0] && NeQ[n,-1]
```

2: $\int x^m \, F^{c \, (a+b \, x)} \, \text{Log}[d \, x]^n \, (e + h \, (f + g \, x) \, \text{Log}[d \, x]) \, dx$ when $e \, (m + 1) == f \, h \, (n + 1) \ \wedge \ g \, h \, (n + 1) == b \, c \, e \, \text{Log}[F] \ \wedge \ n \neq -1$

Rule: If $e \, (m + 1) == f \, h \, (n + 1) \ \wedge \ g \, h \, (n + 1) == b \, c \, e \, \text{Log}[F] \ \wedge \ n \neq -1$, then

$$\int x^m \, F^{c \, (a+b \, x)} \, \text{Log}[d \, x]^n \, (e + h \, (f + g \, x) \, \text{Log}[d \, x]) \, dx \ \longrightarrow \ \frac{e \, x^{m+1} \, F^{c \, (a+b \, x)} \, \text{Log}[d \, x]^{n+1}}{n + 1}$$

Program code:

```
Int[x_^m_.*F_^(c_.*(a_.+b_.*x_))*Log[d_.*x_]^n_.*(e_+h_.*(f_.+g_.*x_)*Log[d_.*x_]),x_Symbol] :=
  e*x^(m+1)*F^(c*(a+b*x))*Log[d*x]^(n+1)/(n+1) /;
FreeQ[{F,a,b,c,d,e,f,g,h,m,n},x] && EqQ[e*(m+1)-f*h*(n+1),0] && EqQ[g*h*(n+1)-b*c*e*Log[F],0] && NeQ[n,-1]
```

2. $\displaystyle\int u\, F^{a+b\,(c+d\,x)^n}\,dx$

    1. $\displaystyle\int F^{a+b\,(c+d\,x)^n}\,dx$

        1. $\displaystyle\int F^{a+b\,(c+d\,x)^n}\,dx$ when $\frac{2}{n} \in \mathbb{Z}$

            1. $\displaystyle\int F^{a+b\,(c+d\,x)^n}\,dx$ when $\frac{2}{n} \in \mathbb{Z} \wedge n \in \mathbb{Z}$

                1. $\displaystyle\int F^{a+b\,(c+d\,x)^n}\,dx$ when $\frac{2}{n} \in \mathbb{Z} \wedge n \in \mathbb{Z}^+$

                    1: $\displaystyle\int F^{a+b\,(c+d\,x)}\,dx$

Reference: G&R 2.311, CRC 519, A&S 4.2.54

Rule:

$$\int F^{a+b\,(c+d\,x)}\,dx \;\longrightarrow\; \frac{F^{a+b\,(c+d\,x)}}{b\,d\,\text{Log}[F]}$$

Program code:

```
Int[F_^(a_.+b_.*(c_.+d_.*x_)),x_Symbol] :=
  F^(a+b*(c+d*x))/(b*d*Log[F]) /;
FreeQ[{F,a,b,c,d},x]
```

      2. $\displaystyle\int F^{a+b\,(c+d\,x)^2}\,dx$

        1: $\displaystyle\int F^{a+b\,(c+d\,x)^2}\,dx$ when $b > 0$

Basis: $\text{Erfi}'[z] == \dfrac{2\,e^{z^2}}{\sqrt{\pi}}$

Rule: If $b > 0$, then

$$\int F^{a+b\,(c+d\,x)^2}\,dx \;\longrightarrow\; \frac{F^a\,\sqrt{\pi}\,\text{Erfi}\left[(c+d\,x)\,\sqrt{b\,\text{Log}[F]}\right]}{2\,d\,\sqrt{b\,\text{Log}[F]}}$$

Program code:

```
Int[F_^(a_.+b_.*(c_.+d_.*x_)^2),x_Symbol] :=
  F^a*Sqrt[Pi]*Erfi[(c+d*x)*Rt[b*Log[F],2]]/(2*d*Rt[b*Log[F],2]) /;
FreeQ[{F,a,b,c,d},x] && PosQ[b]
```

2: $\displaystyle\int F^{a+b\,(c+d\,x)^2}\,dx$ when $\neg\,(b > 0)$

Basis: $\text{Erf}'[z] \;==\; \dfrac{2\,e^{-z^2}}{\sqrt{\pi}}$

Rule: If $\neg\,(b > 0)$, then

$$\int F^{a+b\,(c+d\,x)^2}\,dx \;\longrightarrow\; \frac{F^a\,\sqrt{\pi}\,\text{Erf}\left[(c+d\,x)\,\sqrt{-b\,\text{Log}[F]}\right]}{2\,d\,\sqrt{-b\,\text{Log}[F]}}$$

Program code:

```
Int[F_^(a_.+b_.*(c_.+d_.*x_)^2),x_Symbol] :=
  F^a*Sqrt[Pi]*Erf[(c+d*x)*Rt[-b*Log[F],2]]/(2*d*Rt[-b*Log[F],2]) /;
FreeQ[{F,a,b,c,d},x] && NegQ[b]
```

2: $\int F^{a+b\ (c+d\ x)^n}\ dx$ when $\frac{2}{n} \in \mathbb{Z} \wedge n \in \mathbb{Z}^-$

Derivation: Integration by parts

Basis: $1 \; {==} \; \partial_x \; \frac{c+d\ x}{d}$

Rule: If $\frac{2}{n} \in \mathbb{Z} \; \wedge \; n \in \mathbb{Z}^-$, then

$$\int F^{a+b\ (c+d\ x)^n}\ dx \; \longrightarrow \; \frac{(c+d\ x)\ F^{a+b\ (c+d\ x)^n}}{d} - b\ n\ \text{Log}[F] \int (c+d\ x)^n\ F^{a+b\ (c+d\ x)^n}\ dx$$

Program code:

```
Int[F_^(a_.+b_.*(c_.+d_.*x_)^n_),x_Symbol] :=
  (c+d*x)*F^(a+b*(c+d*x)^n)/d -
    b*n*Log[F]*Int[(c+d*x)^n*F^(a+b*(c+d*x)^n),x] /;
FreeQ[{F,a,b,c,d},x] && IntegerQ[2/n] && ILtQ[n,0]
```

2: $\int F^{a+b\,(c+d\,x)^n}\,\mathbb{d}x$ when $\frac{2}{n} \in \mathbb{Z} \wedge n \notin \mathbb{Z}$

Derivation: Integration by substitution

Basis: If $k \in \mathbb{Z}^+$, then $F[(c+d\,x)^n] \;==\; \frac{k}{d}\left((c+d\,x)^{1/k}\right)^{k-1} F\left[\left((c+d\,x)^{1/k}\right)^{k\,n}\right]\partial_x\,(c+d\,x)^{1/k}$

Rule: If $\frac{2}{n} \in \mathbb{Z} \wedge n \notin \mathbb{Z}^+$, let $k = \mathrm{Denominator}[n]$, then

$$\int F^{a+b\,x^n}\,\mathbb{d}x \;\longrightarrow\; \frac{k}{d}\,\mathrm{Subst}\left[\int x^{k-1}\,F^{a+b\,x^{k\,n}}\,\mathbb{d}x,\;x,\;(c+d\,x)^{1/k}\right]$$

Program code:

```
Int[F_^(a_.+b_.*(c_.+d_.*x_)^n_),x_Symbol] :=
  With[{k=Denominator[n]},
  k/d*Subst[Int[x^(k-1)*F^(a+b*x^(k*n)),x],x,(c+d*x)^(1/k)]] /;
FreeQ[{F,a,b,c,d},x] && IntegerQ[2/n] && Not[IntegerQ[n]]
```

2: $\int F^{a+b\ (c+d\ x)^n}\ dx$ when $\frac{2}{n} \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{(c+d\ x)}{(-b\ (c+d\ x)^n\ \text{Log}[F])^{1/n}} == 0$

Basis: $\partial_x \text{Gamma}\left[\frac{1}{n},\ -b\ (c+d\ x)^n\ \text{Log}[F]\right] == -\dfrac{d\ n\ F^{b\ (c+d\ x)^n}\ (-b\ (c+d\ x)^n\ \text{Log}[F])^{\frac{1}{n}}}{c+d\ x}$

Rule: If $\frac{2}{n} \notin \mathbb{Z}$, then

$$\int F^{a+b\ (c+d\ x)^n}\ dx\ \longrightarrow\ -\dfrac{F^a\ (c+d\ x)\ \text{Gamma}\left[\frac{1}{n},\ -b\ (c+d\ x)^n\ \text{Log}[F]\right]}{d\ n\ \left(-b\ (c+d\ x)^n\ \text{Log}[F]\right)^{1/n}}$$

Program code:

```
Int[F_^(a_.+b_.*(c_.+d_.*x_)^n_),x_Symbol] :=
  -F^a*(c+d*x)*Gamma[1/n,-b*(c+d*x)^n*Log[F]]/(d*n*(-b*(c+d*x)^n*Log[F])^(1/n)) /;
FreeQ[{F,a,b,c,d,n},x] && Not[IntegerQ[2/n]]
```

2. $\int (e + f\,x)^m\, F^{a+b\,(c+d\,x)^n}\, dx$

    1. $\int (e + f\,x)^m\, F^{a+b\,(c+d\,x)^n}\, dx$ when $d\,e - c\,f == 0$

        1. $\int (e + f\,x)^m\, F^{a+b\,(c+d\,x)^n}\, dx$ when $d\,e - c\,f == 0 \,\wedge\, \frac{2\,(m+1)}{n} \in \mathbb{Z}$

            1. $\int (c + d\,x)^m\, F^{a+b\,(c+d\,x)^n}\, dx$ when $\frac{2\,(m+1)}{n} \in \mathbb{Z}$

                1: $\int (e + f\,x)^{n-1}\, F^{a+b\,(c+d\,x)^n}\, dx$ when $d\,e - c\,f == 0$

Derivation: Piecewise constant extraction and integration by substitution

Rule: If $d\,e - c\,f == 0$, then $\partial_x \dfrac{(e+f\,x)^n}{(c+d\,x)^n} == 0$

Basis: $(c + d\,x)^{n-1}\, F[(c + d\,x)^n] == \dfrac{1}{d\,n}\, F[(c + d\,x)^n]\, \partial_x\,(c + d\,x)^n$

Rule: If $d\,e - c\,f == 0$, then

$$\int (e + f\,x)^{n-1}\, F^{a+b\,(c+d\,x)^n}\, dx \;\longrightarrow\; \frac{(e + f\,x)^n\, F^{a+b\,(c+d\,x)^n}}{b\,f\,n\,(c + d\,x)^n\, \mathrm{Log}[F]}$$

Program code:

```
Int[(e_.+f_.*x_)^m_.*F_^(a_.+b_.*(c_.+d_.*x_)^n_),x_Symbol] :=
  (e+f*x)^n*F^(a+b*(c+d*x)^n)/(b*f*n*(c+d*x)^n*Log[F]) /;
FreeQ[{F,a,b,c,d,e,f,n},x] && EqQ[m,n-1] && EqQ[d*e-c*f,0]
```

2: $\displaystyle\int \frac{F^{a+b\,(c+d\,x)^n}}{e+f\,x}\,dx$ when $d\,e - c\,f == 0$

Basis: $\texttt{ExpIntegralEi'[z]} == \dfrac{e^z}{z}$

Rule: If $d\,e - c\,f == 0$, then

$$\int \frac{F^{a+b\,(c+d\,x)^n}}{e+f\,x}\,dx \;\longrightarrow\; \frac{F^a\,\texttt{ExpIntegralEi}\big[b\,(c+d\,x)^n\,\texttt{Log}[F]\big]}{f\,n}$$

Program code:

```
Int[F_^(a_.+b_.*(c_.+d_.*x_)^n_)/(e_.+f_.*x_),x_Symbol] :=
  F^a*ExpIntegralEi[b*(c+d*x)^n*Log[F]]/(f*n) /;
FreeQ[{F,a,b,c,d,e,f,n},x] && EqQ[d*e-c*f,0]
```

3: $\displaystyle\int (c+d\,x)^m\,F^{a+b\,(c+d\,x)^n}\,dx$ when $n == 2\,(m+1)$

Derivation: Integration by substitution

Basis: If $n == 2\,(m+1)$, then $(c+d\,x)^m\,F\big[(c+d\,x)^n\big] == \dfrac{1}{d\,(m+1)}\,F\Big[\big((c+d\,x)^{m+1}\big)^2\Big]\,\partial_x\,(c+d\,x)^{m+1}$

Rule: If $n == 2\,(m+1)$, then

$$\int (c+d\,x)^m\,F^{a+b\,(c+d\,x)^n}\,dx \;\longrightarrow\; \frac{1}{d\,(m+1)}\,\texttt{Subst}\Big[\int F^{a+b\,x^2}\,dx,\; x,\; (c+d\,x)^{m+1}\Big]$$

Program code:

```
Int[(c_.+d_.*x_)^m_.*F_^(a_.+b_.*(c_.+d_.*x_)^n_),x_Symbol] :=
  1/(d*(m+1))*Subst[Int[F^(a+b*x^2),x],x,(c+d*x)^(m+1)] /;
FreeQ[{F,a,b,c,d,m,n},x] && EqQ[n,2*(m+1)]
```

4. $\int (c + d\,x)^m\,F^{a+b\,(c+d\,x)^n}\,dx$ when $\frac{2\,(m+1)}{n} \in \mathbb{Z} \wedge n \in \mathbb{Z}$

   1: $\int (c + d\,x)^m\,F^{a+b\,(c+d\,x)^n}\,dx$ when $\frac{2\,(m+1)}{n} \in \mathbb{Z} \wedge n \in \mathbb{Z} \wedge (0 < n < m + 1 \vee m < n < 0)$

Reference: G&R 2.321.1, CRC 521, A&S 4.2.55

Derivation: Integration by parts

Basis: $(c + d\,x)^m\,F^{a+b\,(c+d\,x)^n} == (c + d\,x)^{m-n+1}\,\partial_x\,\dfrac{F^{a+b\,(c+d\,x)^n}}{b\,d\,n\,\mathrm{Log}[F]}$

Rule: If $\frac{2\,(m+1)}{n} \in \mathbb{Z} \wedge n \in \mathbb{Z} \wedge (0 < n < m + 1 \vee m < n < 0)$, then

$$\int (c + d\,x)^m\,F^{a+b\,(c+d\,x)^n}\,dx \;\longrightarrow\; \frac{(c + d\,x)^{m-n+1}\,F^{a+b\,(c+d\,x)^n}}{b\,d\,n\,\mathrm{Log}[F]} - \frac{m - n + 1}{b\,n\,\mathrm{Log}[F]}\int (c + d\,x)^{m-n}\,F^{a+b\,(c+d\,x)^n}\,dx$$

Program code:

```
Int[(c_.+d_.*x_)^m_.*F_^(a_.+b_.*(c_.+d_.*x_)^n_),x_Symbol] :=
  (c+d*x)^(m-n+1)*F^(a+b*(c+d*x)^n)/(b*d*n*Log[F]) -
  (m-n+1)/(b*n*Log[F])*Int[(c+d*x)^(m-n)*F^(a+b*(c+d*x)^n),x] /;
FreeQ[{F,a,b,c,d},x] && IntegerQ[2*(m+1)/n] && LtQ[0,(m+1)/n,5] && IntegerQ[n] && (LtQ[0,n,m+1] || LtQ[m,n,0])
```

```
Int[(c_.+d_.*x_)^m_.*F_^(a_.+b_.*(c_.+d_.*x_)^n_),x_Symbol] :=
  (c+d*x)^(m-n+1)*F^(a+b*(c+d*x)^n)/(b*d*n*Log[F]) -
  (m-n+1)/(b*n*Log[F])*Int[(c+d*x)^Simplify[m-n]*F^(a+b*(c+d*x)^n),x] /;
FreeQ[{F,a,b,c,d,m,n},x] && IntegerQ[2*Simplify[(m+1)/n]] && LtQ[0,Simplify[(m+1)/n],5] && Not[RationalQ[m]] && SumSimplerQ[m,-n]
```

2: $\int (c + d\,x)^m\,F^{a+b\,(c+d\,x)^n}\,dx$ when $\frac{2\,(m+1)}{n} \in \mathbb{Z} \wedge n \in \mathbb{Z} \wedge (n > 0 \wedge m < -1 \vee 0 < -n \leq m + 1)$

Reference: G&R 2.324.1, CRC 523, A&S 4.2.56

Derivation: Integration by parts

Rule: If $\frac{2\,(m+1)}{n} \in \mathbb{Z} \wedge n \in \mathbb{Z} \wedge (n > 0 \wedge m < -1 \vee 0 < -n \leq m + 1)$, then

$$\int (c + d\,x)^m\,F^{a+b\,(c+d\,x)^n}\,dx \;\longrightarrow\; \frac{(c + d\,x)^{m+1}\,F^{a+b\,(c+d\,x)^n}}{d\,(m+1)} - \frac{b\,n\,\text{Log}[F]}{m+1}\int (c + d\,x)^{m+n}\,F^{a+b\,(c+d\,x)^n}\,dx$$

Program code:

```
Int[(c_.+d_.*x_)^m_.*F_^(a_.+b_.*(c_.+d_.*x_)^n_),x_Symbol] :=
  (c+d*x)^(m+1)*F^(a+b*(c+d*x)^n)/(d*(m+1)) -
  b*n*Log[F]/(m+1)*Int[(c+d*x)^(m+n)*F^(a+b*(c+d*x)^n),x] /;
FreeQ[{F,a,b,c,d},x] && IntegerQ[2*(m+1)/n] && LtQ[-4,(m+1)/n,5] && IntegerQ[n] && (GtQ[n,0] && LtQ[m,-1] || GtQ[-n,0] && LeQ[-n,m+1])
```

```
Int[(c_.+d_.*x_)^m_.*F_^(a_.+b_.*(c_.+d_.*x_)^n_),x_Symbol] :=
  (c+d*x)^(m+1)*F^(a+b*(c+d*x)^n)/(d*(m+1)) -
  b*n*Log[F]/(m+1)*Int[(c+d*x)^Simplify[m+n]*F^(a+b*(c+d*x)^n),x] /;
FreeQ[{F,a,b,c,d,m,n},x] && IntegerQ[2*Simplify[(m+1)/n]] && LtQ[-4,Simplify[(m+1)/n],5] && Not[RationalQ[m]] && SumSimplerQ[m,n]
```

5: $\int (c + d\,x)^m\, F^{a+b\,(c+d\,x)^n}\, dx$ when $\frac{2\,(m+1)}{n} \in \mathbb{Z} \wedge n \notin \mathbb{Z}$

Derivation: Integration by substitution

Basis: If $k \in \mathbb{Z}^+$, then $(c + d\,x)^m\, F\,[\,(c + d\,x)^n\,] == \frac{k}{d}\,\left(\,(c + d\,x)^{1/k}\,\right)^{k\,(m+1)-1}\, F\,\left[\,\left(\,(c + d\,x)^{1/k}\,\right)^{k\,n}\,\right]\, \partial_x\,(c + d\,x)^{1/k}$

Rule: If $\frac{2\,(m+1)}{n} \in \mathbb{Z} \wedge n \notin \mathbb{Z}$, then

$$\int (c + d\,x)^m\, F^{a+b\,(c+d\,x)^n}\, dx \;\longrightarrow\; \frac{k}{d}\, \text{Subst}\left[\int x^{k\,(m+1)-1}\, F^{a+b\,x^{k\,n}}\, dx,\, x,\, (c + d\,x)^{1/k}\right]$$

Program code:

```
Int[(c_.+d_.*x_)^m_.*F_^(a_.+b_.*(c_.+d_.*x_)^n_),x_Symbol] :=
  With[{k=Denominator[n]},
  k/d*Subst[Int[x^(k*(m+1)-1)*F^(a+b*x^(k*n)),x],x,(c+d*x)^(1/k)]] /;
FreeQ[{F,a,b,c,d,m,n},x] && IntegerQ[2*(m+1)/n] && LtQ[0,(m+1)/n,5] && Not[IntegerQ[n]]
```

2: $\int (e + f\,x)^m\, F^{a+b\,(c+d\,x)^n}\, dx$ when $d\,e - c\,f == 0 \wedge \frac{2\,(m+1)}{n} \in \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: If $d\,e - c\,f == 0$, then $\partial_x \dfrac{(e + f\,x)^m}{(c + d\,x)^m} == 0$

Rule: If $d\,e - c\,f == 0 \wedge \dfrac{2\,(m+1)}{n} \in \mathbb{Z}$, then

$$\int (e + f\,x)^m\, F^{a+b\,(c+d\,x)^n}\, dx \;\longrightarrow\; \frac{(e + f\,x)^m}{(c + d\,x)^m} \int (c + d\,x)^m\, F^{a+b\,(c+d\,x)^n}\, dx$$

Program code:

```
Int[(e_.+f_.*x_)^m_.*F_^(a_.+b_.*(c_.+d_.*x_)^n_),x_Symbol] :=
  (e+f*x)^m/(c+d*x)^m*Int[(c+d*x)^m*F^(a+b*(c+d*x)^n),x] /;
FreeQ[{F,a,b,c,d,e,f,m,n},x] && EqQ[d*e-c*f,0] && IntegerQ[2*Simplify[(m+1)/n]] && NeQ[f,d] && Not[IntegerQ[m]] && NeQ[c*e,0]
```

2: $\int (e + f\,x)^m\, F^{a+b\,(c+d\,x)^n}\, \mathrm{d}x$ when $d\,e - c\,f == 0 \land \frac{2\,(m+1)}{n} \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{c + d\,x}{(-b\,(c+d\,x)^n\,\mathrm{Log}[F])^{1/n}} == 0$

Basis: $\partial_x\,\mathrm{Gamma}\left[\frac{m+1}{n},\ -b\,(c+d\,x)^n\,\mathrm{Log}[F]\right] == -\dfrac{d\,n\,F^{b\,(c+d\,x)^n}\,(-b\,(c+d\,x)^n\,\mathrm{Log}[F])^{\frac{m+1}{n}}}{c + d\,x}$

Note: This rule eliminates numerous steps and results in compact antiderivatives. When m or n is nonnumeric, *Mathematica* 8 and *Maple* 16 do not take advantage of it.

Note: To avoid introducing the incomplete gamma function when not absolutely necessary, apply the above substitution rule whenever $\frac{2\,(m+1)}{n} \in \mathbb{Z}$.

Note: The special case $d\,e - c\,f == 0$ is important because $\partial_x\,\mathrm{Gamma}[m,\ e + f\,x]$ equals $-f\,(e+f\,x)^{m-1}\,\mathrm{e}^{-(e+f\,x)}$.

Rule: If $d\,e - c\,f == 0 \land \frac{2\,(m+1)}{n} \notin \mathbb{Z}$, then

$$\int (e + f\,x)^m\, F^{a+b\,(c+d\,x)^n}\, \mathrm{d}x \longrightarrow -\frac{F^a\,(e+f\,x)^{m+1}}{f\,n}\,\mathrm{ExpIntegralE}\left[1 - \frac{m+1}{n},\ -b\,(c+d\,x)^n\,\mathrm{Log}[F]\right]$$

$$\int (e + f\,x)^m\, F^{a+b\,(c+d\,x)^n}\, \mathrm{d}x \longrightarrow -\frac{F^a\,(e+f\,x)^{m+1}}{f\,n\,\left(-b\,(c+d\,x)^n\,\mathrm{Log}[F]\right)^{\frac{m+1}{n}}}\,\mathrm{Gamma}\left[\frac{m+1}{n},\ -b\,(c+d\,x)^n\,\mathrm{Log}[F]\right]$$

Program code:

```
Int[(e_.+f_.*x_)^m_.*F_^(a_.+b_.*(c_.+d_.*x_)^n_),x_Symbol] :=
(*-F^a*(e+f*x)^(m+1)/(f*n)*ExpIntegralE[1-(m+1)/n,-b*(c+d*x)^n*Log[F]] *)
  -F^a*(e+f*x)^(m+1)/(f*n*(-b*(c+d*x)^n*Log[F])^((m+1)/n))*Gamma[(m+1)/n,-b*(c+d*x)^n*Log[F]] /;
FreeQ[{F,a,b,c,d,e,f,m,n},x] && EqQ[d*e-c*f,0]
```

2. $\int (e + f\,x)^m\, F^{a+b\,(c+d\,x)^n}\, dx$ when $d\,e - c\,f \neq 0$

   1. $\int (e + f\,x)^m\, F^{a+b\,(c+d\,x)^2}\, dx$ when $d\,e - c\,f \neq 0$

      1: $\int (e + f\,x)^m\, F^{a+b\,(c+d\,x)^2}\, dx$ when $d\,e - c\,f \neq 0 \;\wedge\; m > 1$

Derivation: Inverted integration by parts

Rule: If $d\,e - c\,f \neq 0 \;\wedge\; m > 1$, then

$$\int (e + f\,x)^m\, F^{a+b\,(c+d\,x)^2}\, dx \;\longrightarrow$$

$$\frac{f\,(e + f\,x)^{m-1}\, F^{a+b\,(c+d\,x)^2}}{2\,b\,d^2\,\mathrm{Log}[F]} + \frac{d\,e - c\,f}{d} \int (e + f\,x)^{m-1}\, F^{a+b\,(c+d\,x)^2}\, dx - \frac{(m - 1)\,f^2}{2\,b\,d^2\,\mathrm{Log}[F]} \int (e + f\,x)^{m-2}\, F^{a+b\,(c+d\,x)^2}\, dx$$

Program code:

```
Int[(e_.+f_.*x_)^m_*F_^(a_.+b_.*(c_.+d_.*x_)^2),x_Symbol] :=
  f*(e+f*x)^(m-1)*F^(a+b*(c+d*x)^2)/(2*b*d^2*Log[F]) +
  (d*e-c*f)/d*Int[(e+f*x)^(m-1)*F^(a+b*(c+d*x)^2),x] -
  (m-1)*f^2/(2*b*d^2*Log[F])*Int[(e+f*x)^(m-2)*F^(a+b*(c+d*x)^2),x] /;
FreeQ[{F,a,b,c,d,e,f},x] && NeQ[d*e-c*f,0] && FractionQ[m] && GtQ[m,1]
```

2: $\int (e + f x)^m \, F^{a+b \, (c+d \, x)^2} \, dx$ when $d\,e - c\,f \neq 0 \,\wedge\, m < -1$

Derivation: Integration by parts

Rule: If $d\,e - c\,f \neq 0 \,\wedge\, m < -1$, then

$$\int (e + f x)^m \, F^{a+b \, (c+d \, x)^2} \, dx \;\longrightarrow$$

$$\frac{f \, (e + f x)^{m+1} \, F^{a+b \, (c+d \, x)^2}}{(m + 1) \, f^2} + \frac{2 \, b \, d \, (d\,e - c\,f) \, \mathrm{Log}[F]}{f^2 \, (m + 1)} \int (e + f x)^{m+1} \, F^{a+b \, (c+d \, x)^2} \, dx - \frac{2 \, b \, d^2 \, \mathrm{Log}[F]}{f^2 \, (m + 1)} \int (e + f x)^{m+2} \, F^{a+b \, (c+d \, x)^2} \, dx$$

Program code:

```
Int[(e_.+f_.*x_)^m_*F_^(a_.+b_.*(c_.+d_.*x_)^2),x_Symbol] :=
  f*(e+f*x)^(m+1)*F^(a+b*(c+d*x)^2)/((m+1)*f^2) +
  2*b*d*(d*e-c*f)*Log[F]/(f^2*(m+1))*Int[(e+f*x)^(m+1)*F^(a+b*(c+d*x)^2),x] -
  2*b*d^2*Log[F]/(f^2*(m+1))*Int[(e+f*x)^(m+2)*F^(a+b*(c+d*x)^2),x] /;
FreeQ[{F,a,b,c,d,e,f},x] && NeQ[d*e-c*f,0] && LtQ[m,-1]
```

2: $\int (e + f\,x)^m\, F^{a+b\,(c+d\,x)^n}\, dx$ when $d\,e - c\,f \neq 0 \,\wedge\, n - 2 \in \mathbb{Z}^+ \wedge m < -1$

Derivation: Integration by parts

Basis: $(e + f\,x)^m == \partial_x \dfrac{(e + f\,x)^{m+1}}{f\,(m+1)}$

Rule: If $d\,e - c\,f \neq 0 \,\wedge\, n - 2 \in \mathbb{Z}^+ \wedge m < -1$, then

$$\int (e + f\,x)^m\, F^{a+b\,(c+d\,x)^n}\, dx \;\longrightarrow\; \frac{(e+f\,x)^{m+1}\, F^{a+b\,(c+d\,x)^n}}{f\,(m+1)} - \frac{b\,d\,n\,\text{Log}[F]}{f\,(m+1)} \int (e+f\,x)^{m+1}\,(c+d\,x)^{n-1}\, F^{a+b\,(c+d\,x)^n}\, dx$$

Program code:

```
Int[(e_.+f_.*x_)^m_*F_^(a_.+b_.*(c_.+d_.*x_)^n_),x_Symbol] :=
  (e+f*x)^(m+1)*F^(a+b*(c+d*x)^n)/(f*(m+1)) -
  b*d*n*Log[F]/(f*(m+1))*Int[(e+f*x)^(m+1)*(c+d*x)^(n-1)*F^(a+b*(c+d*x)^n),x] /;
FreeQ[{F,a,b,c,d,e,f},x] && NeQ[d*e-c*f,0] && IGtQ[n,2] && LtQ[m,-1]
```

3. $\displaystyle\int (e + f\,x)^m\, F^{a + \frac{b}{c + d\,x}}\, \mathrm{d}x$ when $d\,e - c\,f \neq 0 \land m \in \mathbb{Z}^-$

1: $\displaystyle\int \frac{F^{a + \frac{b}{c + d\,x}}}{e + f\,x}\, \mathrm{d}x$ when $d\,e - c\,f \neq 0$

Derivation: Algebraic expansion

Basis: $\displaystyle\frac{1}{e + f\,x} \; == \; \frac{d}{f\,(c + d\,x)} - \frac{d\,e - c\,f}{f\,(c + d\,x)\,(e + f\,x)}$

Rule: If $d\,e - c\,f \neq 0$, then

$$\int \frac{F^{a + \frac{b}{c + d\,x}}}{e + f\,x}\, \mathrm{d}x \;\longrightarrow\; \frac{d}{f} \int \frac{F^{a + \frac{b}{c + d\,x}}}{c + d\,x}\, \mathrm{d}x - \frac{d\,e - c\,f}{f} \int \frac{F^{a + \frac{b}{c + d\,x}}}{(c + d\,x)\,(e + f\,x)}\, \mathrm{d}x$$

Program code:

```
Int[F_^(a_.+b_./(c_.+d_.*x_))/(e_.+f_.*x_),x_Symbol] :=
  d/f*Int[F^(a+b/(c+d*x))/(c+d*x),x] -
  (d*e-c*f)/f*Int[F^(a+b/(c+d*x))/((c+d*x)*(e+f*x)),x] /;
FreeQ[{F,a,b,c,d,e,f},x] && NeQ[d*e-c*f,0]
```

2: $\int (e + f\,x)^m\, F^{a + \frac{b}{c + d\,x}}\, dx$ when $d\,e - c\,f \neq 0 \,\wedge\, m + 1 \in \mathbb{Z}^-$

Derivation: Integration by parts

Basis: $(e + f\,x)^m == \partial_x \dfrac{(e + f\,x)^{m+1}}{f\,(m+1)}$

Note: Although resulting integrand appears more complicated than the original one, it is amenable to partial fraction expansion.

Rule: If $d\,e - c\,f \neq 0 \,\wedge\, m + 1 \in \mathbb{Z}^-$, then

$$\int (e + f\,x)^m\, F^{a + \frac{b}{c + d\,x}}\, dx \;\longrightarrow\; \frac{(e + f\,x)^{m+1}\, F^{a + \frac{b}{c + d\,x}}}{f\,(m+1)} + \frac{b\,d\,\mathrm{Log}[F]}{f\,(m+1)} \int \frac{(e + f\,x)^{m+1}\, F^{a + \frac{b}{c + d\,x}}}{(c + d\,x)^2}\, dx$$

Program code:

```
Int[(e_.+f_.*x_)^m_*F_^(a_.+b_./(c_.+d_.*x_)),x_Symbol] :=
  (e+f*x)^(m+1)*F^(a+b/(c+d*x))/(f*(m+1)) +
  b*d*Log[F]/(f*(m+1))*Int[(e+f*x)^(m+1)*F^(a+b/(c+d*x))/(c+d*x)^2,x] /;
FreeQ[{F,a,b,c,d,e,f},x] && NeQ[d*e-c*f,0] && ILtQ[m,-1]
```

X: $\displaystyle\int \frac{F^{a+b\,(c+d\,x)^n}}{e+f\,x}\,dx$ when $d\,e-c\,f \neq 0$

Rule: If $d\,e-c\,f \neq 0$, then

$$\int \frac{F^{a+b\,(c+d\,x)^n}}{e+f\,x}\,dx \;\longrightarrow\; \int \frac{F^{a+b\,(c+d\,x)^n}}{e+f\,x}\,dx$$

Program code:

```
Int[F_^(a_.+b_.*(c_.+d_.*x_)^n_)/(e_.+f_.*x_),x_Symbol] :=
  Unintegrable[F^(a+b*(c+d*x)^n)/(e+f*x),x] /;
FreeQ[{F,a,b,c,d,e,f,n},x] && NeQ[d*e-c*f,0]
```

3: $\displaystyle\int u^m\,F^v\,dx$ when $u == e+f\,x \,\wedge\, v == a+b\,x^n$

Derivation: Algebraic normalization

Rule: If $u == e+f\,x \,\wedge\, v == a+b\,x^n$, then

$$\int u^m\,F^v\,dx \;\longrightarrow\; \int (e+f\,x)^m\,F^{a+b\,x^n}\,dx$$

Program code:

```
Int[u_^m_.*F_^v_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*F^ExpandToSum[v,x],x] /;
FreeQ[{F,m},x] && LinearQ[u,x] && BinomialQ[v,x] && Not[LinearMatchQ[u,x] && BinomialMatchQ[v,x]]
```

3. $\int P_x \, F^{a+b \, (c+d \, x)^n} \, dx$

    1: $\int P_x \, F^{a+b \, (c+d \, x)^n} \, dx$

Derivation: Algebraic expansion

Rule:

$$\int P_x \, F^{a+b \, (c+d \, x)^n} \, dx \;\longrightarrow\; \int F^{a+b \, (c+d \, x)^n} \, \text{ExpandLinearProduct}\big[P_x, \, c, \, d, \, x\big] \, dx$$

Program code:

```
Int[u_*F_^(a_.+b_.*(c_.+d_.*x_)^n_),x_Symbol] :=
  Int[ExpandLinearProduct[F^(a+b*(c+d*x)^n),u,c,d,x],x] /;
FreeQ[{F,a,b,c,d,n},x] && PolynomialQ[u,x]
```

    2: $\int P_x \, F^{a+b \, v} \, dx$ when $v == (c + d \, x)^n$

Derivation: Algebraic normalization

Rule: If $v == (c + d \, x)^n$, then

$$\int P_x \, F^{a+b \, v} \, dx \;\longrightarrow\; \int P_x \, F^{a+b \, (c+d \, x)^n} \, dx$$

Program code:

```
Int[u_.*F_^(a_.+b_.*v_),x_Symbol] :=
  Int[u*F^(a+b*NormalizePowerOfLinear[v,x]),x] /;
FreeQ[{F,a,b},x] && PolynomialQ[u,x] && PowerOfLinearQ[v,x] && Not[PowerOfLinearMatchQ[v,x]]
```

x: $\int P_x\, F^{a+b\, v^n}\, dx$ when $v == c + d\, x$

Derivation: Algebraic normalization

Rule: If $v == c + d\, x$, then

$$\int P_x\, F^{a+b\, v^n}\, dx \;\longrightarrow\; \int P_x\, F^{a+b\, (c+d\, x)^n}\, dx$$

Program code:

```
(* Int[u_.*F_^(a_.+b_.*v_^n_),x_Symbol] :=
  Int[u*F^(a+b*ExpandToSum[v,x]^n),x] /;
FreeQ[{F,a,b,n},x] && PolynomialQ[u,x] && LinearQ[v,x] && Not[LinearMatchQ[v,x]] *)
```

x: $\int P_x\, F^{v}\, dx$ when $v == a + b\, x^n$

Derivation: Algebraic normalization

Rule: If $v == a + b\, x^n$, then

$$\int P_x\, F^{v}\, dx \;\longrightarrow\; \int P_x\, F^{a+b\, x^n}\, dx$$

Program code:

```
(* Int[u_.*F_^u_,x_Symbol] :=
  Int[u*F^ExpandToSum[u,x],x] /;
FreeQ[F,x] && PolynomialQ[u,x] && BinomialQ[u,x] && Not[BinomialMatchQ[u,x]] *)
```

4: $\displaystyle\int \frac{F^{a+\frac{b}{c+d\,x}}}{(e+f\,x)\,(g+h\,x)}\,dx$ when $d\,e-c\,f == 0$

Derivation: Integration by substitution

Basis: If $d\,e-c\,f == 0$, then $\displaystyle\frac{F^{a+\frac{b}{c+d\,x}}}{(e+f\,x)\,(g+h\,x)} == -\frac{d}{f\,(d\,g-c\,h)}\frac{F^{a-\frac{b\,h}{d\,g-c\,h}+\frac{d\,b}{d\,g-c\,h}\frac{g+h\,x}{c+d\,x}}}{\frac{g+h\,x}{c+d\,x}}\,\partial_x\frac{g+h\,x}{c+d\,x}$

Rule: If $d\,e-c\,f == 0$, then

$$\int \frac{F^{a+\frac{b}{c+d\,x}}}{(e+f\,x)\,(g+h\,x)}\,dx \;\longrightarrow\; -\frac{d}{f\,(d\,g-c\,h)}\,\text{Subst}\left[\int \frac{F^{a-\frac{b\,h}{d\,g-c\,h}-\frac{-d\,b\,x}{d\,g-c\,h}}}{x}\,dx,\,x,\,\frac{g+h\,x}{c+d\,x}\right]$$

Program code:

```
Int[F_^(a_.+b_./(c_.+d_.*x_))/((e_.+f_.*x_)*(g_.+h_.*x_)),x_Symbol] :=
  -d/(f*(d*g-c*h))*Subst[Int[F^(a-b*h/(d*g-c*h)+d*b*x/(d*g-c*h))/x,x],x,(g+h*x)/(c+d*x)] /;
FreeQ[{F,a,b,c,d,e,f},x] && EqQ[d*e-c*f,0]
```

3. $\displaystyle\int u\; F^{e+f\,\frac{a+b\,x}{c+d\,x}}\; dx$

    1. $\displaystyle\int (g+h\,x)^m\; F^{e+f\,\frac{a+b\,x}{c+d\,x}}\; dx$

        1: $\displaystyle\int (g+h\,x)^m\; F^{e+f\,\frac{a+b\,x}{c+d\,x}}\; dx$　when　$b\,c - a\,d == 0$

Derivation: Algebraic simplification

Basis: If $b\,c - a\,d == 0$, then $\frac{a+b\,x}{c+d\,x} == \frac{b}{d}$

Rule: If $b\,c - a\,d == 0$, then

$$\int (g+h\,x)^m\; F^{e+f\,\frac{a+b\,x}{c+d\,x}}\; dx \;\longrightarrow\; F^{e+f\,\frac{b}{d}}\int (g+h\,x)^m\; dx$$

Program code:

```
Int[(g_.+h_.*x_)^m_.*F_^(e_.+f_.*(a_.+b_.*x_)/(c_.+d_.*x_)),x_Symbol] :=
  F^(e+f*b/d)*Int[(g+h*x)^m,x] /;
FreeQ[{F,a,b,c,d,e,f,g,h,m},x] && EqQ[b*c-a*d,0]
```

2. $\displaystyle\int (g + h\,x)^m\,F^{e + f\,\frac{a + b\,x}{c + d\,x}}\,dx$ when $b\,c - a\,d \neq 0$

    1: $\displaystyle\int (g + h\,x)^m\,F^{e + f\,\frac{a + b\,x}{c + d\,x}}\,dx$ when $b\,c - a\,d \neq 0\ \wedge\ d\,g - c\,h == 0$

Derivation: Algebraic normalization

Basis: $e + f\,\dfrac{a + b\,x}{c + d\,x} == \dfrac{d\,e + b\,f}{d} - f\,\dfrac{b\,c - a\,d}{d\,(c + d\,x)}$

Rule: If $b\,c - a\,d \neq 0\ \wedge\ d\,g - c\,h == 0$, then

$$\int (g + h\,x)^m\,F^{e + f\,\frac{a + b\,x}{c + d\,x}}\,dx\ \longrightarrow\ \int (g + h\,x)^m\,F^{\frac{d\,e + b\,f}{d} - f\,\frac{b\,c - a\,d}{d\,(c + d\,x)}}\,dx$$

Program code:

```
Int[(g_.+h_.*x_)^m_.*F_^(e_.+f_.*(a_.+b_.*x_)/(c_.+d_.*x_)),x_Symbol] :=
  Int[(g+h*x)^m*F^((d*e+b*f)/d-f*(b*c-a*d)/(d*(c+d*x))),x] /;
FreeQ[{F,a,b,c,d,e,f,g,h,m},x] && NeQ[b*c-a*d,0] && EqQ[d*g-c*h,0]
```

2. $\int (g + h\,x)^m\, F^{e + f\,\frac{a + b\,x}{c + d\,x}}\, dx$ when $b\,c - a\,d \neq 0 \wedge d\,g - c\,h \neq 0$

1: $\int \dfrac{F^{e + f\,\frac{a + b\,x}{c + d\,x}}}{g + h\,x}\, dx$ when $b\,c - a\,d \neq 0 \wedge d\,g - c\,h \neq 0$

Derivation: Algebraic expansion

Basis: $\dfrac{1}{g + h\,x} == \dfrac{d}{h\,(c + d\,x)} - \dfrac{d\,g - c\,h}{h\,(c + d\,x)\,(g + h\,x)}$

Rule: If $b\,c - a\,d \neq 0 \wedge d\,g - c\,h \neq 0$, then

$$\int \frac{F^{e + f\,\frac{a + b\,x}{c + d\,x}}}{g + h\,x}\, dx \;\longrightarrow\; \frac{d}{h} \int \frac{F^{e + f\,\frac{a + b\,x}{c + d\,x}}}{c + d\,x}\, dx - \frac{d\,g - c\,h}{h} \int \frac{F^{e + f\,\frac{a + b\,x}{c + d\,x}}}{(c + d\,x)\,(g + h\,x)}\, dx$$

Program code:

```
Int[F_^(e_.+f_.*(a_.+b_.*x_)/(c_.+d_.*x_))/(g_.+h_.*x_),x_Symbol] :=
  d/h*Int[F^(e+f*(a+b*x)/(c+d*x))/(c+d*x),x] -
  (d*g-c*h)/h*Int[F^(e+f*(a+b*x)/(c+d*x))/((c+d*x)*(g+h*x)),x] /;
FreeQ[{F,a,b,c,d,e,f,g,h},x] && NeQ[b*c-a*d,0] && NeQ[d*g-c*h,0]
```

2: $\int (g + h\,x)^m\, F^{e + f\,\frac{a + b\,x}{c + d\,x}}\, dx$ when $b\,c - a\,d \neq 0 \,\wedge\, d\,g - c\,h \neq 0 \,\wedge\, m + 1 \in \mathbb{Z}^-$

Derivation: Integration by parts

Basis: $(g + h\,x)^m == \partial_x \frac{(g + h\,x)^{m+1}}{h\,(m+1)}$

Note: Although resulting integrand appears more complicated than the original one, it is amenable to partial fraction expansion.

Rule: If $b\,c - a\,d \neq 0 \,\wedge\, d\,g - c\,h \neq 0 \,\wedge\, m + 1 \in \mathbb{Z}^-$, then

$$\int (g + h\,x)^m\, F^{e + f\,\frac{a + b\,x}{c + d\,x}}\, dx \;\longrightarrow\; \frac{(g + h\,x)^{m+1}\, F^{e + f\,\frac{a + b\,x}{c + d\,x}}}{h\,(m + 1)} - \frac{f\,(b\,c - a\,d)\,\mathrm{Log}[F]}{h\,(m + 1)} \int \frac{(g + h\,x)^{m+1}\, F^{e + f\,\frac{a + b\,x}{c + d\,x}}}{(c + d\,x)^2}\, dx$$

Program code:

```
Int[(g_.+h_.*x_)^m_*F_^(e_.+f_.*(a_.+b_.*x_)/(c_.+d_.*x_)),x_Symbol] :=
  (g+h*x)^(m+1)*F^(e+f*(a+b*x)/(c+d*x))/(h*(m+1)) -
  f*(b*c-a*d)*Log[F]/(h*(m+1))*Int[(g+h*x)^(m+1)*F^(e+f*(a+b*x)/(c+d*x))/(c+d*x)^2,x] /;
FreeQ[{F,a,b,c,d,e,f,g,h},x] && NeQ[b*c-a*d,0] && NeQ[d*g-c*h,0] && ILtQ[m,-1]
```

2: $\int \frac{F^{e + f\,\frac{a + b\,x}{c + d\,x}}}{(g + h\,x)\,(i + j\,x)}\, dx$ when $d\,g - c\,h == 0$

Derivation: Integration by substitution

Basis: If $d\,g - c\,h == 0$, then $\dfrac{F^{e + f\,\frac{a + b\,x}{c + d\,x}}}{(g + h\,x)\,(i + j\,x)} == -\dfrac{d}{h\,(d\,i - c\,j)}\, F^{e + \frac{f\,(b\,i - a\,j)}{d\,i - c\,j} - \frac{(b\,c - a\,d)\,f}{d\,i - c\,j}\,\frac{i + j\,x}{c + d\,x}}\, \partial_x\,\dfrac{i + j\,x}{c + d\,x}$

Rule: If $d\,g - c\,h == 0$, then

$$\int \frac{F^{e+f\frac{a+b\,x}{c+d\,x}}}{(g+h\,x)\,(i+j\,x)}\,dx \;\longrightarrow\; -\frac{d}{h\,(d\,i-c\,j)}\,\mathrm{Subst}\!\left[\int \frac{F^{e+\frac{f\,(b\,i-a\,j)}{d\,i-c\,j}-\frac{(b\,c-a\,d)\,f\,x}{d\,i-c\,j}}}{x}\,dx,\,x,\,\frac{i+j\,x}{c+d\,x}\right]$$

Program code:

```
Int[F_^(e_.+f_.*(a_.+b_.*x_)/(c_.+d_.*x_))/((g_.+h_.*x_)*(i_.+j_.*x_)),x_Symbol] :=
  -d/(h*(d*i-c*j))*Subst[Int[F^(e+f*(b*i-a*j)/(d*i-c*j)-(b*c-a*d)*f*x/(d*i-c*j))/x,x],x,(i+j*x)/(c+d*x)] /;
FreeQ[{F,a,b,c,d,e,f,g,h},x] && EqQ[d*g-c*h,0]
```

4. $\displaystyle\int u\,F^{a+b\,x+c\,x^2}\,dx$

  1. $\displaystyle\int F^{a+b\,x+c\,x^2}\,dx$

    1: $\displaystyle\int F^{a+b\,x+c\,x^2}\,dx$

Derivation: Algebraic expansion

Basis: $a + b\,x + c\,x^2 == \dfrac{4\,a\,c-b^2}{4\,c} + \dfrac{(b+2\,c\,x)^2}{4\,c}$

Basis: $F^{z+w} == F^z\,F^w$

Rule:

$$\int F^{a+b\,x+c\,x^2}\,dx \;\longrightarrow\; F^{\frac{4\,a\,c-b^2}{4\,c}}\int F^{\frac{(b+2\,c\,x)^2}{4\,c}}\,dx$$

Program code:

```
Int[F_^(a_.+b_.*x_+c_.*x_^2),x_Symbol] :=
  F^(a-b^2/(4*c))*Int[F^((b+2*c*x)^2/(4*c)),x] /;
FreeQ[{F,a,b,c},x]
```

2: $\int F^v \, dx$ when $v == a + b\,x + c\,x^2$

Derivation: Algebraic normalization

Rule: If $v == a + b\,x + c\,x^2$, then

$$\int F^v \, dx \;\longrightarrow\; \int F^{a+b\,x+c\,x^2} \, dx$$

Program code:

```
Int[F_^v_,x_Symbol] :=
  Int[F^ExpandToSum[v,x],x] /;
FreeQ[F,x] && QuadraticQ[v,x] && Not[QuadraticMatchQ[v,x]]
```

2. $\int (d + e\, x)^m\, F^{a + b\, x + c\, x^2}\, dx$

    1. $\int (d + e\, x)^m\, F^{a + b\, x + c\, x^2}\, dx$ when $b\, e - 2\, c\, d == 0$

        1. $\int (d + e\, x)^m\, F^{a + b\, x + c\, x^2}\, dx$ when $b\, e - 2\, c\, d == 0 \wedge m > 0$

            1: $\int (d + e\, x)\, F^{a + b\, x + c\, x^2}\, dx$ when $b\, e - 2\, c\, d == 0$

Derivation: Integration by substitution

Rule: If $b\, e - 2\, c\, d == 0$, then

$$\int (d + e\, x)\, F^{a + b\, x + c\, x^2}\, dx \longrightarrow \frac{e\, F^{a + b\, x + c\, x^2}}{2\, c\, \text{Log}[F]}$$

Program code:

```
Int[(d_.+e_.*x_)*F_^(a_.+b_.*x_+c_.*x_^2),x_Symbol] :=
  e*F^(a+b*x+c*x^2)/(2*c*Log[F]) /;
FreeQ[{F,a,b,c,d,e},x] && EqQ[b*e-2*c*d,0]
```

2: $\int (d + e\, x)^m\, F^{a+b\, x+c\, x^2}\, dx$ when $b\, e - 2\, c\, d == 0 \wedge m > 1$

Derivation: Inverted integration by parts

Rule: If $b\, e - 2\, c\, d == 0 \wedge m > 1$, then

$$\int (d + e\, x)^m\, F^{a+b\, x+c\, x^2}\, dx \longrightarrow \frac{e\,(d + e\, x)^{m-1}\, F^{a+b\, x+c\, x^2}}{2\, c\, \text{Log}[F]} - \frac{(m - 1)\, e^2}{2\, c\, \text{Log}[F]} \int (d + e\, x)^{m-2}\, F^{a+b\, x+c\, x^2}\, dx$$

Program code:

```
Int[(d_.+e_.*x_)^m_*F_^(a_.+b_.*x_+c_.*x_^2),x_Symbol] :=
  e*(d+e*x)^(m-1)*F^(a+b*x+c*x^2)/(2*c*Log[F]) -
  (m-1)*e^2/(2*c*Log[F])*Int[(d+e*x)^(m-2)*F^(a+b*x+c*x^2),x] /;
FreeQ[{F,a,b,c,d,e},x] && EqQ[b*e-2*c*d,0] && GtQ[m,1]
```

2. $\int (d + e\, x)^m\, F^{a+b\, x+c\, x^2}\, dx$ when $b\, e - 2\, c\, d == 0 \wedge m < 0$

1: $\int \dfrac{F^{a+b\, x+c\, x^2}}{d + e\, x}\, dx$ when $b\, e - 2\, c\, d == 0$

Rule: If $b\, e - 2\, c\, d == 0$, then

$$\int \frac{F^{a+b\, x+c\, x^2}}{d + e\, x}\, dx \longrightarrow \frac{1}{2\, e}\, F^{a - \frac{b^2}{4\, c}}\, \text{ExpIntegralEi}\!\left[\frac{(b + 2\, c\, x)^2\, \text{Log}[F]}{4\, c}\right]$$

Program code:

```
Int[F_^(a_.+b_.*x_+c_.*x_^2)/(d_.+e_.*x_),x_Symbol] :=
  1/(2*e)*F^(a-b^2/(4*c))*ExpIntegralEi[(b+2*c*x)^2*Log[F]/(4*c)] /;
FreeQ[{F,a,b,c,d,e},x] && EqQ[b*e-2*c*d,0]
```

2: $\int (d + e\, x)^m\, F^{a+b\,x+c\,x^2}\, dx$ when $b\,e - 2\,c\,d == 0 \,\wedge\, m < -1$

Derivation: Integration by parts

Rule: If $b\,e - 2\,c\,d == 0 \,\wedge\, m < -1$, then

$$\int (d + e\,x)^m\, F^{a+b\,x+c\,x^2}\, dx \;\longrightarrow\; \frac{(d + e\,x)^{m+1}\, F^{a+b\,x+c\,x^2}}{e\,(m + 1)} - \frac{2\,c\, \text{Log}[F]}{e^2\,(m + 1)} \int (d + e\,x)^{m+2}\, F^{a+b\,x+c\,x^2}\, dx$$

Program code:

```
Int[(d_.+e_.*x_)^m_*F_^(a_.+b_.*x_+c_.*x_^2),x_Symbol] :=
  (d+e*x)^(m+1)*F^(a+b*x+c*x^2)/(e*(m+1)) -
  2*c*Log[F]/(e^2*(m+1))*Int[(d+e*x)^(m+2)*F^(a+b*x+c*x^2),x] /;
FreeQ[{F,a,b,c,d,e},x] && EqQ[b*e-2*c*d,0] && LtQ[m,-1]
```

2. $\int (d + e\,x)^m \, F^{a+b\,x+c\,x^2} \, dx$ when $b\,e - 2\,c\,d \neq 0$

   1. $\int (d + e\,x)^m \, F^{a+b\,x+c\,x^2} \, dx$ when $b\,e - 2\,c\,d \neq 0 \wedge m > 0$

      1: $\int (d + e\,x) \, F^{a+b\,x+c\,x^2} \, dx$ when $b\,e - 2\,c\,d \neq 0$

Derivation: Inverted integration by parts

Rule: If $b\,e - 2\,c\,d \neq 0$, then

$$\int (d + e\,x)\, F^{a+b\,x+c\,x^2}\, dx \;\longrightarrow\; \frac{e\, F^{a+b\,x+c\,x^2}}{2\,c\, \mathrm{Log}[F]} - \frac{b\,e - 2\,c\,d}{2\,c} \int F^{a+b\,x+c\,x^2}\, dx$$

Program code:

```
Int[(d_.+e_.*x_)*F_^(a_.+b_.*x_+c_.*x_^2),x_Symbol] :=
  e*F^(a+b*x+c*x^2)/(2*c*Log[F]) -
  (b*e-2*c*d)/(2*c)*Int[F^(a+b*x+c*x^2),x] /;
FreeQ[{F,a,b,c,d,e},x] && NeQ[b*e-2*c*d,0]
```

2: $\int (d + e\,x)^m\, F^{a + b\,x + c\,x^2}\, dx$ when $b\,e - 2\,c\,d \neq 0 \;\wedge\; m > 1$

Derivation: Inverted integration by parts

Rule: If $b\,e - 2\,c\,d \neq 0 \;\wedge\; m > 1$, then

$$\int (d + e\,x)^m\, F^{a + b\,x + c\,x^2}\, dx \;\longrightarrow$$

$$\frac{e\,(d + e\,x)^{m-1}\, F^{a + b\,x + c\,x^2}}{2\,c\,\mathrm{Log}[F]} - \frac{b\,e - 2\,c\,d}{2\,c} \int (d + e\,x)^{m-1}\, F^{a + b\,x + c\,x^2}\, dx - \frac{(m - 1)\,e^2}{2\,c\,\mathrm{Log}[F]} \int (d + e\,x)^{m-2}\, F^{a + b\,x + c\,x^2}\, dx$$

Program code:

```
Int[(d_.+e_.*x_)^m_*F_^(a_.+b_.*x_+c_.*x_^2),x_Symbol] :=
  e*(d+e*x)^(m-1)*F^(a+b*x+c*x^2)/(2*c*Log[F]) -
  (b*e-2*c*d)/(2*c)*Int[(d+e*x)^(m-1)*F^(a+b*x+c*x^2),x] -
  (m-1)*e^2/(2*c*Log[F])*Int[(d+e*x)^(m-2)*F^(a+b*x+c*x^2),x] /;
FreeQ[{F,a,b,c,d,e},x] && NeQ[b*e-2*c*d,0] && GtQ[m,1]
```

2: $\int (d + e\,x)^m\, F^{a+b\,x+c\,x^2}\, dx$ when $b\,e - 2\,c\,d \neq 0 \;\wedge\; m < -1$

Derivation: Integration by parts

Rule: If $b\,e - 2\,c\,d \neq 0 \;\wedge\; m < -1$, then

$$\int (d + e\,x)^m\, F^{a+b\,x+c\,x^2}\, dx \;\longrightarrow$$

$$\frac{(d + e\,x)^{m+1}\, F^{a+b\,x+c\,x^2}}{e\,(m+1)} - \frac{(b\,e - 2\,c\,d)\, \text{Log}[F]}{e^2\,(m+1)} \int (d + e\,x)^{m+1}\, F^{a+b\,x+c\,x^2}\, dx - \frac{2\,c\,\text{Log}[F]}{e^2\,(m+1)} \int (d + e\,x)^{m+2}\, F^{a+b\,x+c\,x^2}\, dx$$

Program code:

```
Int[(d_.+e_.*x_)^m_*F_^(a_.+b_.*x_+c_.*x_^2),x_Symbol] :=
  (d+e*x)^(m+1)*F^(a+b*x+c*x^2)/(e*(m+1)) -
  (b*e-2*c*d)*Log[F]/(e^2*(m+1))*Int[(d+e*x)^(m+1)*F^(a+b*x+c*x^2),x] -
  2*c*Log[F]/(e^2*(m+1))*Int[(d+e*x)^(m+2)*F^(a+b*x+c*x^2),x] /;
FreeQ[{F,a,b,c,d,e},x] && NeQ[b*e-2*c*d,0] && LtQ[m,-1]
```

X: $\int (d + e\,x)^m\, F^{a + b\,x + c\,x^2}\, dx$

Derivation: Algebraic normalization

Rule: If $u == d + e\,x \land v == a + b\,x + c\,x^2$, then

$$\int (d + e\,x)^m\, F^{a + b\,x + c\,x^2}\, dx \;\longrightarrow\; \int (d + e\,x)^m\, F^{a + b\,x + c\,x^2}\, dx$$

Program code:

```
Int[(d_.+e_.*x_)^m_.*F_^(a_.+b_.*x_+c_.*x_^2),x_Symbol] :=
  Unintegrable[(d+e*x)^m*F^(a+b*x+c*x^2),x] /;
FreeQ[{F,a,b,c,d,e,m},x]
```

4: $\int u^m\, F^v\, dx$ when $u == d + e\,x \land v == a + b\,x + c\,x^2$

Derivation: Algebraic normalization

Rule: If $u == d + e\,x \land v == a + b\,x + c\,x^2$, then

$$\int u^m\, F^v\, dx \;\longrightarrow\; \int (d + e\,x)^m\, F^{a + b\,x + c\,x^2}\, dx$$

Program code:

```
Int[u_^m_.*F_^v_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*F^ExpandToSum[v,x],x] /;
FreeQ[{F,m},x] && LinearQ[u,x] && QuadraticQ[v,x] && Not[LinearMatchQ[u,x] && QuadraticMatchQ[v,x]]
```

5. $\int u \left(a + b \left(F^{e\,(c+d\,x)}\right)^n\right)^p dx$

1: $\int x^m \, F^{e\,(c+d\,x)} \left(a + b \, F^{2\,e\,(c+d\,x)}\right)^p dx$ when $m > 0 \,\wedge\, p \in \mathbb{Z}^-$

Derivation: Integration by parts

Rule: If $m > 0 \,\wedge\, p \in \mathbb{Z}^-$, then

$$\int x^m \, F^{e\,(c+d\,x)} \left(a + b \, F^{2\,e\,(c+d\,x)}\right)^p dx \;\longrightarrow\; x^m \int F^{e\,(c+d\,x)} \left(a + b \, F^{2\,e\,(c+d\,x)}\right)^p dx - m \int x^{m-1} \left(\int F^{e\,(c+d\,x)} \left(a + b \, F^{2\,e\,(c+d\,x)}\right)^p dx\right) dx$$

Program code:

```
Int[x_^m_.*F_^(e_.*(c_.+d_.*x_))*(a_.+b_.*F_^v_)^p_,x_Symbol] :=
  With[{u=IntHide[F^(e*(c+d*x))*(a+b*F^v)^p,x]},
  Dist[x^m,u,x] - m*Int[x^(m-1)*u,x]] /;
FreeQ[{F,a,b,c,d,e},x] && EqQ[v,2*e*(c+d*x)] && GtQ[m,0] && ILtQ[p,0]
```

2. $\int \left(G^{h\ (f+g\ x)}\right)^{m} \left(a + b\ \left(F^{e\ (c+d\ x)}\right)^{n}\right)^{p} dx$ when $d\ e\ n\ Log[F] == g\ h\ m\ Log[G]$

    1: $\int \left(F^{e\ (c+d\ x)}\right)^{n} \left(a + b\ \left(F^{e\ (c+d\ x)}\right)^{n}\right)^{p} dx$

Derivation: Integration by substitution

Basis: $\left(F^{e\ (c+d\ x)}\right)^{n} \left(a + b\ \left(F^{e\ (c+d\ x)}\right)^{n}\right)^{p} == \dfrac{1}{d\ e\ n\ Log[F]} \ Subst\left[\left(a + b\ x\right)^{p},\ x,\ \left(F^{e\ (c+d\ x)}\right)^{n}\right] \partial_{x} \left(F^{e\ (c+d\ x)}\right)^{n}$

Rule:

$$\int \left(F^{e\ (c+d\ x)}\right)^{n} \left(a + b\ \left(F^{e\ (c+d\ x)}\right)^{n}\right)^{p} dx \ \longrightarrow \ \frac{1}{d\ e\ n\ Log[F]} \ Subst\left[\int \left(a + b\ x\right)^{p} dx,\ x,\ \left(F^{e\ (c+d\ x)}\right)^{n}\right]$$

Program code:

```
Int[(F_^(e_.*(c_.+d_.*x_)))^n_.*(a_+b_.*(F_^(e_.*(c_.+d_.*x_)))^n_.)^p_.,x_Symbol] :=
  1/(d*e*n*Log[F])*Subst[Int[(a+b*x)^p,x],x,(F^(e*(c+d*x)))^n] /;
FreeQ[{F,a,b,c,d,e,n,p},x]
```

2: $\int \left(G^{h\,(f+g\,x)}\right)^m \left(a + b\,\left(F^{e\,(c+d\,x)}\right)^n\right)^p dx$ when $d\,e\,n\,Log[F] == g\,h\,m\,Log[G]$

Derivation: Piecewise constant extraction

Basis: If $d\,e\,n\,Log[F] == g\,h\,m\,Log[G]$, then $\partial_x \dfrac{\left(G^{h\,(f+g\,x)}\right)^m}{\left(F^{e\,(c+d\,x)}\right)^n} == 0$

Rule: If $d\,e\,n\,Log[F] == g\,h\,m\,Log[G]$, then

$$\int \left(G^{h\,(f+g\,x)}\right)^m \left(a + b\,\left(F^{e\,(c+d\,x)}\right)^n\right)^p dx \longrightarrow \frac{\left(G^{h\,(f+g\,x)}\right)^m}{\left(F^{e\,(c+d\,x)}\right)^n} \int \left(F^{e\,(c+d\,x)}\right)^n \left(a + b\,\left(F^{e\,(c+d\,x)}\right)^n\right)^p dx$$

Program code:

```
Int[(G_^(h_.(f_.+g_.*x_)))^m_.*(a_+b_.*(F_^(e_.*(c_.+d_.*x_)))^n_.)^p_.,x_Symbol] :=
  (G^(h*(f+g*x)))^m/(F^(e*(c+d*x)))^n*Int[(F^(e*(c+d*x)))^n*(a+b*(F^(e*(c+d*x)))^n)^p,x] /;
FreeQ[{F,G,a,b,c,d,e,f,g,h,m,n,p},x] && EqQ[d*e*n*Log[F],g*h*m*Log[G]]
```

3. $\int G^{h\,(f+g\,x)} \left(a + b\,F^{e\,(c+d\,x)}\right)^p dx$

  1. $\int G^{h\,(f+g\,x)} \left(a + b\,F^{e\,(c+d\,x)}\right)^p dx$ when $\dfrac{g\,h\,Log[G]}{d\,e\,Log[F]} \in \mathbb{R}$

    1: $\int G^{h\,(f+g\,x)} \left(a + b\,F^{e\,(c+d\,x)}\right)^p dx$ when $Abs\left[\dfrac{g\,h\,Log[G]}{d\,e\,Log[F]}\right] \geq 1$

Derivation: Integration by substitution

Basis: If $k \in \mathbb{Z} \land k\,\dfrac{g\,h\,Log[G]}{d\,e\,Log[F]} \in \mathbb{Z}$, then

$G^{h\,(f+g\,x)} \left(a + b\,F^{e\,(c+d\,x)}\right)^p == \dfrac{k\,G^{f\,h - \frac{c\,g\,h}{d}}}{d\,e\,Log[F]}\,Subst\left[x^{k\,\frac{g\,h\,Log[G]}{d\,e\,Log[F]} - 1} \left(a + b\,x^k\right)^p,\ x,\ F^{\frac{e\,(c+d\,x)}{k}}\right] \partial_x F^{\frac{e\,(c+d\,x)}{k}}$

Rule: If $Abs\left[\dfrac{g\,h\,Log[G]}{d\,e\,Log[F]}\right] \geq 1$, then

$$\int G^{h\,(f+g\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^p\,\mathbb{d}x \;\longrightarrow\; \frac{k\,G^{f\,h-\frac{c\,g\,h}{d}}}{d\,e\,Log[F]}\,Subst\left[\int x^{k\,\frac{g\,h\,Log[G]}{d\,e\,Log[F]}-1}\,\left(a + b\,x^k\right)^p\,\mathbb{d}x,\,x,\,F^{\frac{e\,(c+d\,x)}{k}}\right]$$

Program code:

```
Int[G_^(h_.(f_.+g_.*x_))*(a_+b_.*F_^(e_.*(c_.+d_.*x_)))^p_.,x_Symbol] :=
  With[{m=FullSimplify[g*h*Log[G]/(d*e*Log[F])]},
  Denominator[m]*G^(f*h-c*g*h/d)/(d*e*Log[F])*Subst[Int[x^(Numerator[m]-1)*(a+b*x^Denominator[m])^p,x],x,F^(e*(c+d*x)/Denominator[m])]
  LeQ[m,-1] || GeQ[m,1]] /;
FreeQ[{F,G,a,b,c,d,e,f,g,h,p},x]
```

2: $\int G^{h\,(f+g\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^p\,\mathbb{d}x$ when $Abs\left[\frac{d\,e\,Log[F]}{g\,h\,Log[G]}\right] > 1$

Derivation: Integration by substitution

Basis: If $k \in \mathbb{Z} \,\wedge\, k\,\frac{d\,e\,Log[F]}{g\,h\,Log[G]} \in \mathbb{Z}$, then

$$G^{h\,(f+g\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^p == \frac{k}{g\,h\,Log[G]}\,Subst\left[x^{k-1}\,\left(a + b\,F^{c\,e-\frac{d\,e\,f}{g}}\,x^{k\,\frac{d\,e\,Log[F]}{g\,h\,Log[G]}}\right)^p,\,x,\,G^{\frac{h\,(f+g\,x)}{k}}\right]\,\partial_x\,G^{\frac{h\,(f+g\,x)}{k}}$$

Rule: If $Abs\left[\frac{d\,e\,Log[F]}{g\,h\,Log[G]}\right] > 1$, then

$$\int G^{h\,(f+g\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^p\,\mathbb{d}x \;\longrightarrow\; \frac{k}{g\,h\,Log[G]}\,Subst\left[\int x^{k-1}\,\left(a + b\,F^{c\,e-\frac{d\,e\,f}{g}}\,x^{k\,\frac{d\,e\,Log[F]}{g\,h\,Log[G]}}\right)^p\,\mathbb{d}x,\,x,\,G^{\frac{h\,(f+g\,x)}{k}}\right]$$

Program code:

```
Int[G_^(h_.(f_.+g_.*x_))*(a_+b_.*F_^(e_.*(c_.+d_.*x_)))^p_.,x_Symbol] :=
  With[{m=FullSimplify[d*e*Log[F]/(g*h*Log[G])]},
  Denominator[m]/(g*h*Log[G])*Subst[Int[x^(Denominator[m]-1)*(a+b*F^(c*e-d*e*f/g)*x^Numerator[m])^p,x],x,G^(h*(f+g*x)/Denominator[m])]
  LtQ[m,-1] || GtQ[m,1]] /;
FreeQ[{F,G,a,b,c,d,e,f,g,h,p},x]
```

2. $\int G^{h\,(f+g\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^{p}\,dx$ when $\frac{g\,h\,Log[G]}{d\,e\,Log[F]} \notin \mathbb{R}$

   1: $\int G^{h\,(f+g\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^{p}\,dx$ when $p \in \mathbb{Z}^{+}$

Rule: If $p \in \mathbb{Z}^{+}$, then

$$\int G^{h\,(f+g\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^{p}\,dx \;\longrightarrow\; \int \text{Expand}\left[G^{h\,(f+g\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^{p}\right]\,dx$$

Program code:

```
Int[G_^(h_.(f_.+g_.*x_))*(a_+b_.*F_^(e_.*(c_.+d_.*x_)))^p_.,x_Symbol] :=
  Int[Expand[G^(h*(f+g*x))*(a+b*F^(e*(c+d*x)))^p,x],x] /;
FreeQ[{F,G,a,b,c,d,e,f,g,h},x] && IGtQ[p,0]
```

   2: $\int G^{h\,(f+g\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^{p}\,dx$ when $p \in \mathbb{Z}^{-} \lor a > 0$

Rule: If $p \in \mathbb{Z}^{-} \lor a > 0$, then

$$\int G^{h\,(f+g\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^{p}\,dx \;\longrightarrow\; \frac{a^{p}\,G^{h\,(f+g\,x)}}{g\,h\,Log[G]}\,\text{Hypergeometric2F1}\left[-p,\,\frac{g\,h\,Log[G]}{d\,e\,Log[F]},\,\frac{g\,h\,Log[G]}{d\,e\,Log[F]}+1,\,-\frac{b}{a}\,F^{e\,(c+d\,x)}\right]$$

Program code:

```
Int[G_^(h_.(f_.+g_.*x_))*(a_+b_.*F_^(e_.*(c_.+d_.*x_)))^p_,x_Symbol] :=
  a^p*G^(h*(f+g*x))/(g*h*Log[G])*Hypergeometric2F1[-p,g*h*Log[G]/(d*e*Log[F]),g*h*Log[G]/(d*e*Log[F])+1,Simplify[-b/a*F^(e*(c+d*x))]]
FreeQ[{F,G,a,b,c,d,e,f,g,h,p},x] && (ILtQ[p,0] || GtQ[a,0])
```

3: $\displaystyle\int G^{h\ (f+g\ x)}\ \left(a + b\ F^{e\ (c+d\ x)}\right)^p\ \mathbb{d}x$ when $\neg\ (p \in \mathbb{Z}^- \lor a > 0)$

Derivation: Piecewise constant extraction

Basis: $\displaystyle\partial_x \frac{\left(a+b\ F^{e\ (c+d\ x)}\right)^p}{\left(1+\frac{b\ F^{e\ (c+d\ x)}}{a}\right)^p} == 0$

Rule: If $\neg\ (p \in \mathbb{Z}^- \lor a > 0)$, then

$$\int G^{h\ (f+g\ x)}\ \left(a + b\ F^{e\ (c+d\ x)}\right)^p\ \mathbb{d}x \longrightarrow \frac{\left(a + b\ F^{e\ (c+d\ x)}\right)^p}{\left(1 + \frac{b}{a}\ F^{e\ (c+d\ x)}\right)^p} \int G^{h\ (f+g\ x)}\ \left(1 + \frac{b}{a}\ F^{e\ (c+d\ x)}\right)^p\ \mathbb{d}x$$

Program code:

```
Int[G_^(h_.(f_.+g_.*x_))*(a_+b_.*F_^(e_.*(c_.+d_.*x_)))^p_,x_Symbol] :=
  (a+b*F^(e*(c+d*x)))^p/(1+(b/a)*F^(e*(c+d*x)))^p*Int[G^(h*(f+g*x))*(1+b/a*F^(e*(c+d*x)))^p,x] /;
FreeQ[{F,G,a,b,c,d,e,f,g,h,p},x] && Not[ILtQ[p,0] || GtQ[a,0]]
```

3: $\displaystyle\int G^{h\ u}\ \left(a + b\ F^{e\ v}\right)^p\ \mathbb{d}x$ when $u == f + g\ x \land v == c + d\ x$

Derivation: Algebraic normalization

Rule: If $u == f + g\ x \land v == c + d\ x$, then

$$\int G^{h\ u}\ \left(a + b\ F^{e\ v}\right)^p\ \mathbb{d}x \longrightarrow \int G^{h\ (f+g\ x)}\ \left(a + b\ F^{e\ (c+d\ x)}\right)^p\ \mathbb{d}x$$

Program code:

```
Int[G_^(h_.u_)*(a_+b_.*F_^(e_.*v_))^p_,x_Symbol] :=
  Int[G^(h*ExpandToSum[u,x])*(a+b*F^(e*ExpandToSum[v,x]))^p,x] /;
FreeQ[{F,G,a,b,e,h,p},x] && LinearQ[{u,v},x] && Not[LinearMatchQ[{u,v},x]]
```

4. $\int (e + f\,x)^m \left(a + b\,F^{g\,(i+j\,x)}\right)^p \left(c + d\,F^{h\,(i+j\,x)}\right)^q dx$ when $(p \mid q) \in \mathbb{Z} \wedge \frac{g}{h} \in \mathbb{R}$

x: $\int \frac{(c + d\,x)^m\,F^{g\,(e+f\,x)}}{a + b\,F^{h\,(e+f\,x)}} dx$ when $0 \leq \frac{g}{h} - 1 < \frac{g}{h}$

Derivation: Algebraic expansion

Basis: $\frac{F^{g\,z}}{a + b\,F^{h\,z}} == \frac{F^{(g-h)\,z}}{b} - \frac{a\,F^{(g-h)\,z}}{b\,\left(a + b\,F^{h\,z}\right)}$

Rule: If $0 \leq \frac{g}{h} - 1 < \frac{g}{h}$, then

$$\int \frac{(c + d\,x)^m\,F^{g\,(e+f\,x)}}{a + b\,F^{h\,(e+f\,x)}} dx \;\longrightarrow\; \frac{1}{b} \int (c + d\,x)^m\,F^{(g-h)\,(e+f\,x)} dx - \frac{a}{b} \int \frac{(c + d\,x)^m\,F^{(g-h)\,(e+f\,x)}}{a + b\,F^{h\,(e+f\,x)}} dx$$

Program code:

```
(* Int[(c_.+d_.*x_)^m_.*F_^(g_.*(e_.+f_.*x_))/(a_+b_.*F_^(h_.*(e_.+f_.*x_))),x_Symbol] :=
  1/b*Int[(c+d*x)^m*F^((g-h)*(e+f*x)),x] -
  a/b*Int[(c+d*x)^m*F^((g-h)*(e+f*x))/(a+b*F^(h*(e+f*x))),x] /;
FreeQ[{F,a,b,c,d,e,f,g,h,m},x] && LeQ[0,g/h-1,g/h] *)
```

x: $\displaystyle\int \frac{(c + d\, x)^m\, F^{g\,(e+f\,x)}}{a + b\, F^{h\,(e+f\,x)}}\, dx$ when $\frac{g}{h} < \frac{g}{h} + 1 \le 0$

Derivation: Algebraic expansion

Basis: $\displaystyle\frac{F^{g\,z}}{a+b\,F^{h\,z}} \,==\, \frac{F^{g\,z}}{a} - \frac{b\,F^{(g+h)\,z}}{a\,(a+b\,F^{h\,z})}$

Rule: If $\frac{g}{h} < \frac{g}{h} + 1 \le 0$, then

$$\int \frac{(c + d\, x)^m\, F^{g\,(e+f\,x)}}{a + b\, F^{h\,(e+f\,x)}}\, dx \;\longrightarrow\; \frac{1}{a} \int (c + d\, x)^m\, F^{g\,(e+f\,x)}\, dx - \frac{b}{a} \int \frac{(c + d\, x)^m\, F^{(g+h)\,(e+f\,x)}}{a + b\, F^{h\,(e+f\,x)}}\, dx$$

Program code:

```
(* Int[(c_.+d_.*x_)^m_.*F_^(g_.*(e_.+f_.*x_))/(a_+b_.*F_^(h_.*(e_.+f_.*x_))),x_Symbol] :=
  1/a*Int[(c+d*x)^m*F^(g*(e+f*x)),x] -
  b/a*Int[(c+d*x)^m*F^((g+h)*(e+f*x))/(a+b*F^(h*(e+f*x))),x] /;
FreeQ[{F,a,b,c,d,e,f,g,h,m},x] && LeQ[g/h,g/h+1,0] *)
```

1: $\int (e + f\,x)^m \, (a + b\,F^u)^p \, (c + d\,F^v)^q \, dx$ when $(p \mid q) \in \mathbb{Z} \wedge \frac{u}{v} \in \mathbb{R}$

Derivation: Algebraic expansion

Rule: If $(p \mid q) \in \mathbb{Z} \wedge \frac{u}{v} \in \mathbb{R}$, then

$$\int (e + f\,x)^m \, (a + b\,F^u)^p \, (c + d\,F^v)^q \, dx \;\longrightarrow\; \int (e + f\,x)^m \, \text{ExpandIntegrand}\big[(a + b\,F^u)^p \, (c + d\,F^v)^q, x\big] \, dx$$

Program code:

```
Int[(e_.+f_.*x_)^m_.*(a_.+b_.*F_^u_)^p_.*(c_.+d_.*F_^v_)^q_.,x_Symbol] :=
  With[{w=ExpandIntegrand[(e+f*x)^m,(a+b*F^u)^p*(c+d*F^v)^q,x]},
  Int[w,x] /;
 SumQ[w]] /;
FreeQ[{F,a,b,c,d,e,f,m},x] && IntegersQ[p,q] && LinearQ[{u,v},x] && RationalQ[Simplify[u/v]]
```

5. $\int G^{h\ (f+g\ x)}\ H^{t\ (r+s\ x)}\ \left(a + b\ F^{e\ (c+d\ x)}\right)^{p}\ dx$

    1: $\int G^{h\ (f+g\ x)}\ H^{t\ (r+s\ x)}\ \left(a + b\ F^{e\ (c+d\ x)}\right)^{p}\ dx$ when $\dfrac{g\ h\ \text{Log}[G] + s\ t\ \text{Log}[H]}{d\ e\ \text{Log}[F]} \in \mathbb{R}$

Derivation: Integration by substitution

Rule: If $k \in \mathbb{Z} \ \wedge\ k\ \dfrac{g\ h\ \text{Log}[G] + s\ t\ \text{Log}[H]}{d\ e\ \text{Log}[F]} \in \mathbb{Z}$, then

$G^{h\ (f+g\ x)}\ H^{t\ (r+s\ x)}\ \left(a + b\ F^{e\ (c+d\ x)}\right)^{p}\ ==$

$\dfrac{k\ G^{f\ h - \frac{c\ g\ h}{d}}\ H^{r\ t - \frac{c\ s\ t}{d}}}{d\ e\ \text{Log}[F]}\ \text{Subst}\left[x^{k\ \frac{g\ h\ \text{Log}[G] + s\ t\ \text{Log}[H]}{d\ e\ \text{Log}[F]} - 1}\ \left(a + b\ x^{k}\right)^{p},\ x,\ F^{\frac{e\ (c+d\ x)}{k}}\right]\ \partial_{x}\ F^{\frac{e\ (c+d\ x)}{k}}$

Rule: If $\dfrac{g\ h\ \text{Log}[G] + s\ t\ \text{Log}[H]}{d\ e\ \text{Log}[F]} \in \mathbb{R}$, then

$\int G^{h\ (f+g\ x)}\ H^{t\ (r+s\ x)}\ \left(a + b\ F^{e\ (c+d\ x)}\right)^{p}\ dx\ \longrightarrow\ \dfrac{k\ G^{f\ h - \frac{c\ g\ h}{d}}\ H^{r\ t - \frac{c\ s\ t}{d}}}{d\ e\ \text{Log}[F]}\ \text{Subst}\left[\int x^{k\ \frac{g\ h\ \text{Log}[G] + s\ t\ \text{Log}[H]}{d\ e\ \text{Log}[F]} - 1}\ \left(a + b\ x^{k}\right)^{p}\ dx,\ x,\ F^{\frac{e\ (c+d\ x)}{k}}\right]$

Program code:

```
Int[G_^(h_.(f_.+g_.*x_))*H_^(t_.(r_.+s_.*x_))*(a_+b_.*F_^(e_.*(c_.+d_.*x_)))^p_.,x_Symbol] :=
  With[{m=FullSimplify[(g*h*Log[G]+s*t*Log[H])/(d*e*Log[F])]},
   Denominator[m]*G^(f*h-c*g*h/d)*H^(r*t-c*s*t/d)/(d*e*Log[F])*
     Subst[Int[x^(Numerator[m]-1)*(a+b*x^Denominator[m])^p,x],x,F^(e*(c+d*x)/Denominator[m])] /;
  RationalQ[m]] /;
FreeQ[{F,G,H,a,b,c,d,e,f,g,h,r,s,t,p},x]
```

2. $\int G^{h\,(f+g\,x)}\,H^{t\,(r+s\,x)}\,\left(a+b\,F^{e\,(c+d\,x)}\right)^p dx$ when $\frac{g\,h\,\text{Log}[G]+s\,t\,\text{Log}[H]}{d\,e\,\text{Log}[F]} \notin \mathbb{R}$

   1. $\int G^{h\,(f+g\,x)}\,H^{t\,(r+s\,x)}\,\left(a+b\,F^{e\,(c+d\,x)}\right)^p dx$ when $p \in \mathbb{Z}$

      1: $\int G^{h\,(f+g\,x)}\,H^{t\,(r+s\,x)}\,\left(a+b\,F^{e\,(c+d\,x)}\right)^p dx$ when $d\,e\,p\,\text{Log}[F]+g\,h\,\text{Log}[G] == 0 \,\wedge\, p \in \mathbb{Z}$

Derivation: Algebraic simplification

Basis: If $d\,e\,p\,\text{Log}[F]+g\,h\,\text{Log}[G] == 0 \,\wedge\, p \in \mathbb{Z}$, then $G^{h\,(f+g\,x)} == G^{\left(f-\frac{c\,g}{d}\right)\,h}\,\left(F^{e\,(c+d\,x)}\right)^{-p}$

Rule: If $d\,e\,p\,\text{Log}[F]+g\,h\,\text{Log}[G] == 0 \,\wedge\, p \in \mathbb{Z}$, then

$$\int G^{h\,(f+g\,x)}\,H^{t\,(r+s\,x)}\,\left(a+b\,F^{e\,(c+d\,x)}\right)^p dx \;\longrightarrow\; G^{\left(f-\frac{c\,g}{d}\right)\,h}\int\left(F^{e\,(c+d\,x)}\right)^{-p}\,H^{t\,(r+s\,x)}\,\left(a+b\,F^{e\,(c+d\,x)}\right)^p dx \;\longrightarrow\; G^{\left(f-\frac{c\,g}{d}\right)\,h}\int H^{t\,(r+s\,x)}\,\left(b+a\,F^{-e\,(c+d\,x)}\right)^p dx$$

Program code:

```
Int[G_^(h_.(f_.+g_.*x_))*H_^(t_.(r_.+s_.*x_))*(a_+b_.*F_^(e_.*(c_.+d_.*x_)))^p_.,x_Symbol] :=
  G^((f-c*g/d)*h)*Int[H^(t*(r+s*x))*(b+a*F^(-e*(c+d*x)))^p,x] /;
FreeQ[{F,G,H,a,b,c,d,e,f,g,h,r,s,t},x] && EqQ[d*e*p*Log[F]+g*h*Log[G],0] && IntegerQ[p]
```

2: $\int G^{h\,(f+g\,x)}\,H^{t\,(r+s\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^{p}\,\mathbb{d}x$ when $p \in \mathbb{Z}^{+}$

Rule: If $p \in \mathbb{Z}^{+}$, then

$$\int G^{h\,(f+g\,x)}\,H^{t\,(r+s\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^{p}\,\mathbb{d}x \;\longrightarrow\; \int \mathrm{Expand}\left[G^{h\,(f+g\,x)}\,H^{t\,(r+s\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^{p}\right]\,\mathbb{d}x$$

Program code:

```
Int[G_^(h_.(f_.+g_.*x_))*H_^(t_.(r_.+s_.*x_))*(a_+b_.*F_^(e_.*(c_.+d_.*x_)))^p_.,x_Symbol] :=
  Int[Expand[G^(h*(f+g*x))*H^(t*(r+s*x))*(a+b*F^(e*(c+d*x)))^p,x],x] /;
FreeQ[{F,G,H,a,b,c,d,e,f,g,h,r,s,t},x] && IGtQ[p,0]
```

3: $\int G^{h\,(f+g\,x)}\,H^{t\,(r+s\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^{p}\,\mathbb{d}x$ when $p \in \mathbb{Z}^{-}$

Rule: If $p \in \mathbb{Z}^{-}$, then

$$\int G^{h\,(f+g\,x)}\,H^{t\,(r+s\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^{p}\,\mathbb{d}x \;\longrightarrow\; \frac{a^{p}\,G^{h\,(f+g\,x)}\,H^{t\,(r+s\,x)}}{g\,h\,\mathrm{Log}[G] + s\,t\,\mathrm{Log}[H]}\,\mathrm{Hypergeometric2F1}\left[-p,\,\frac{g\,h\,\mathrm{Log}[G] + s\,t\,\mathrm{Log}[H]}{d\,e\,\mathrm{Log}[F]},\,\frac{g\,h\,\mathrm{Log}[G] + s\,t\,\mathrm{Log}[H]}{d\,e\,\mathrm{Log}[F]} + 1,\,-\frac{b}{a}\,F^{e\,(c+d\,x)}\right]$$

Program code:

```
Int[G_^(h_.(f_.+g_.*x_))*H_^(t_.(r_.+s_.*x_))*(a_+b_.*F_^(e_.*(c_.+d_.*x_)))^p_,x_Symbol] :=
  a^p*G^(h*(f+g*x))*H^(t*(r+s*x))/(g*h*Log[G]+s*t*Log[H])*
    Hypergeometric2F1[-p,(g*h*Log[G]+s*t*Log[H])/(d*e*Log[F]),(g*h*Log[G]+s*t*Log[H])/(d*e*Log[F])+1,Simplify[-b/a*F^(e*(c+d*x))]] /;
FreeQ[{F,G,H,a,b,c,d,e,f,g,h,r,s,t},x] && ILtQ[p,0]
```

2: $\int G^{h\,(f+g\,x)}\,H^{t\,(r+s\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^{p}\,dx$ when $p \notin \mathbb{Z}$

Rule: If $p \notin \mathbb{Z}$, then

$$\int G^{h\,(f+g\,x)}\,H^{t\,(r+s\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^{p}\,dx \;\longrightarrow\; \frac{G^{h\,(f+g\,x)}\,H^{t\,(r+s\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^{p}}{\left(g\,h\,\text{Log}[G] + s\,t\,\text{Log}[H]\right)\,\left(\frac{a+b\,F^{e\,(c+d\,x)}}{a}\right)^{p}}$$

$$\text{Hypergeometric2F1}\left[-p,\; \frac{g\,h\,\text{Log}[G] + s\,t\,\text{Log}[H]}{d\,e\,\text{Log}[F]},\; \frac{g\,h\,\text{Log}[G] + s\,t\,\text{Log}[H]}{d\,e\,\text{Log}[F]} + 1,\; -\frac{b}{a}\,F^{e\,(c+d\,x)}\right]$$

Program code:

```
Int[G_^(h_.(f_.+g_.*x_))*H_^(t_.(r_.+s_.*x_))*(a_+b_.*F_^(e_.*(c_.+d_.*x_)))^p_,x_Symbol] :=
  G^(h*(f+g*x))*H^(t*(r+s*x))*(a+b*F^(e*(c+d*x)))^p/((g*h*Log[G]+s*t*Log[H])*((a+b*F^(e*(c+d*x)))/a)^p)*
    Hypergeometric2F1[-p,(g*h*Log[G]+s*t*Log[H])/(d*e*Log[F]),(g*h*Log[G]+s*t*Log[H])/(d*e*Log[F])+1,Simplify[-b/a*F^(e*(c+d*x))]] /;
FreeQ[{F,G,H,a,b,c,d,e,f,g,h,r,s,t,p},x] && Not[IntegerQ[p]]
```

3: $\int G^{h\,u}\,H^{t\,w}\,\left(a + b\,F^{e\,v}\right)^{p}\,dx$ when $u == f + g\,x \,\wedge\, v == c + d\,x \,\wedge\, w == r + s\,x$

Derivation: Algebraic normalization

Rule: If $u == f + g\,x \,\wedge\, v == c + d\,x \,\wedge\, w == r + s\,x$, then

$$\int G^{h\,u}\,H^{t\,w}\,\left(a + b\,F^{e\,v}\right)^{p}\,dx \;\longrightarrow\; \int G^{h\,(f+g\,x)}\,H^{t\,(r+s\,x)}\,\left(a + b\,F^{e\,(c+d\,x)}\right)^{p}\,dx$$

Program code:

```
Int[G_^(h_.u_)*H_^(t_.w_)*(a_+b_.*F_^(e_.*v_))^p_,x_Symbol] :=
  Int[G^(h*ExpandToSum[u,x])*H^(t*ExpandToSum[w,x])*(a+b*F^(e*ExpandToSum[v,x]))^p,x] /;
FreeQ[{F,G,H,a,b,e,h,t,p},x] && LinearQ[{u,v,w},x] && Not[LinearMatchQ[{u,v,w},x]]
```

6. $\int u\, F^{e\ (c+d\ x)}\, \left(a\ x^n + b\ F^{e\ (c+d\ x)}\right)^p\, dx$

1: $\int F^{e\ (c+d\ x)}\, \left(a\ x^n + b\ F^{e\ (c+d\ x)}\right)^p\, dx$ when $p \neq -1$

Derivation: Integration by parts

Basis: $F^{e\ (c+d\ x)}\, \left(a\ x^n + b\ F^{e\ (c+d\ x)}\right)^p == \partial_x \dfrac{\left(a\ x^n + b\ F^{e\ (c+d\ x)}\right)^{p+1}}{b\ d\ e\ (p+1)\ \text{Log}[F]} - \dfrac{a\ n\ x^{n-1}\left(a\ x^n + b\ F^{e\ (c+d\ x)}\right)^p}{b\ d\ e\ \text{Log}[F]}$

Rule: If $p \neq -1$, then

$$\int F^{e\ (c+d\ x)}\, \left(a\ x^n + b\ F^{e\ (c+d\ x)}\right)^p\, dx \longrightarrow \dfrac{\left(a\ x^n + b\ F^{e\ (c+d\ x)}\right)^{p+1}}{b\ d\ e\ (p+1)\ \text{Log}[F]} - \dfrac{a\ n}{b\ d\ e\ \text{Log}[F]}\int x^{n-1}\left(a\ x^n + b\ F^{e\ (c+d\ x)}\right)^p\, dx$$

Program code:

```
Int[F_^(e_.*(c_.+d_.*x_))*(a_.*x_^n_.+b_.*F_^(e_.*(c_.+d_.*x_)))^p_.,x_Symbol] :=
  (a*x^n+b*F^(e*(c+d*x)))^(p+1)/(b*d*e*(p+1)*Log[F]) -
  a*n/(b*d*e*Log[F])*Int[x^(n-1)*(a*x^n+b*F^(e*(c+d*x)))^p,x] /;
FreeQ[{F,a,b,c,d,e,n,p},x] && NeQ[p,-1]
```

2: $\int x^m \, F^{e\,(c+d\,x)} \left(a\,x^n + b\,F^{e\,(c+d\,x)}\right)^p \, dx$ when $p \neq -1$

Derivation: Integration by parts

Basis: $x^m \, F^{e\,(c+d\,x)} \left(a\,x^n + b\,F^{e\,(c+d\,x)}\right)^p \;==\; x^m \, \partial_x \frac{\left(a\,x^n + b\,F^{e\,(c+d\,x)}\right)^{p+1}}{b\,d\,e\,(p+1)\,\text{Log}[F]} - \frac{a\,n\,x^{m+n-1} \left(a\,x^n + b\,F^{e\,(c+d\,x)}\right)^p}{b\,d\,e\,\text{Log}[F]}$

Rule: If $p \neq -1$, then

$$\int x^m \, F^{c+d\,x} \left(a\,x^n + b\,F^{c+d\,x}\right)^p \, dx \;\longrightarrow$$

$$\frac{x^m \left(a\,x^n + b\,F^{e\,(c+d\,x)}\right)^{p+1}}{b\,d\,e\,(p+1)\,\text{Log}[F]} - \frac{a\,n}{b\,d\,e\,\text{Log}[F]} \int x^{m+n-1} \left(a\,x^n + b\,F^{e\,(c+d\,x)}\right)^p \, dx - \frac{m}{b\,d\,e\,(p+1)\,\text{Log}[F]} \int x^{m-1} \left(a\,x^n + b\,F^{e\,(c+d\,x)}\right)^{p+1} \, dx$$

Program code:

```
Int[x_^m_.*F_^(e_.*(c_.+d_.*x_))*(a_.*x_^n_.+b_.*F_^(e_.*(c_.+d_.*x_)))^p_.,x_Symbol] :=
  x^m*(a*x^n+b*F^(e*(c+d*x)))^(p+1)/(b*d*e*(p+1)*Log[F]) -
  a*n/(b*d*e*Log[F])*Int[x^(m+n-1)*(a*x^n+b*F^(e*(c+d*x)))^p,x] -
  m/(b*d*e*(p+1)*Log[F])*Int[x^(m-1)*(a*x^n+b*F^(e*(c+d*x)))^(p+1),x] /;
FreeQ[{F,a,b,c,d,e,m,n,p},x] && NeQ[p,-1]
```

7. $\int \frac{u \left(f + g\,x\right)^m}{a + b\,F^{d+e\,x} + c\,F^{2\,(d+e\,x)}} \, dx$ when $\sqrt{b^2 - 4\,a\,c} \neq 0 \land m \in \mathbb{Z}^+$

1: $\int \frac{\left(f + g\,x\right)^m}{a + b\,F^{d+e\,x} + c\,F^{2\,(d+e\,x)}} \, dx$ when $\sqrt{b^2 - 4\,a\,c} \neq 0 \land m \in \mathbb{Z}^+$

Derivation: Algebraic expansion

■ Basis: If $q = \sqrt{b^2 - 4\,a\,c}$, then $\frac{1}{a+b\,z+c\,z^2} \;==\; \frac{2\,c}{q\,(b-q+2\,c\,z)} - \frac{2\,c}{q\,(b+q+2\,c\,z)}$

■ Rule: If $\sqrt{b^2 - 4\,a\,c} \neq 0 \land m \in \mathbb{Z}^+$, let $q = \sqrt{b^2 - 4\,a\,c}$, then

$$\int \frac{(f + g\,x)^m}{a + b\,F^{d+e\,x} + c\,F^{2\,(d+e\,x)}}\,dx \;\longrightarrow\; \frac{2\,c}{q}\int \frac{(f + g\,x)^m}{b - q + 2\,c\,F^{d+e\,x}}\,dx - \frac{2\,c}{q}\int \frac{(f + g\,x)^m}{b + q + 2\,c\,F^{d+e\,x}}\,dx$$

Program code:

```
Int[(f_.+g_.*x_)^m_./(a_.+b_.*F_^u_+c_.*F_^v_),x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
  2*c/q*Int[(f+g*x)^m/(b-q+2*c*F^u),x] - 2*c/q*Int[(f+g*x)^m/(b+q+2*c*F^u),x]] /;
FreeQ[{F,a,b,c,f,g},x] && EqQ[v,2*u] && LinearQ[u,x] && NeQ[b^2-4*a*c,0] && IGtQ[m,0]
```

2: $\displaystyle\int \frac{(f + g\,x)^m\,F^{d+e\,x}}{a + b\,F^{d+e\,x} + c\,F^{2\,(d+e\,x)}}\,dx$ when $\sqrt{b^2 - 4\,a\,c} \neq 0 \,\wedge\, m \in \mathbb{Z}^+$

Derivation: Algebraic expansion

- Basis: If $q = \sqrt{b^2 - 4\,a\,c}$, then $\dfrac{1}{a + b\,z + c\,z^2} == \dfrac{2\,c}{q\,(b - q + 2\,c\,z)} - \dfrac{2\,c}{q\,(b + q + 2\,c\,z)}$

- Rule: If $\sqrt{b^2 - 4\,a\,c} \neq 0 \,\wedge\, m \in \mathbb{Z}^+$, let $q = \sqrt{b^2 - 4\,a\,c}$, then

$$\int \frac{(f + g\,x)^m\,F^{d+e\,x}}{a + b\,F^{d+e\,x} + c\,F^{2\,(d+e\,x)}}\,dx \;\longrightarrow\; \frac{2\,c}{q}\int \frac{(f + g\,x)^m\,F^{d+e\,x}}{b - q + 2\,c\,F^{d+e\,x}}\,dx - \frac{2\,c}{q}\int \frac{(f + g\,x)^m\,F^{d+e\,x}}{b + q + 2\,c\,F^{d+e\,x}}\,dx$$

Program code:

```
Int[(f_.+g_.*x_)^m_.*F_^u_/(a_.+b_.*F_^u_+c_.*F_^v_),x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
  2*c/q*Int[(f+g*x)^m*F^u/(b-q+2*c*F^u),x] - 2*c/q*Int[(f+g*x)^m*F^u/(b+q+2*c*F^u),x]] /;
FreeQ[{F,a,b,c,f,g},x] && EqQ[v,2*u] && LinearQ[u,x] && NeQ[b^2-4*a*c,0] && IGtQ[m,0]
```

3: $\int \dfrac{(f + g\,x)^m\,\left(h + i\,F^{d+e\,x}\right)}{a + b\,F^{d+e\,x} + c\,F^{2\,(d+e\,x)}}\,dx$ when $\sqrt{b^2 - 4\,a\,c} \neq 0 \wedge m \in \mathbb{Z}^+$

Derivation: Algebraic expansion

- **Basis:** If $q = \sqrt{b^2 - 4\,a\,c}$, then $\dfrac{h + i\,z}{a + b\,z + c\,z^2} == \left(\dfrac{2\,c\,h - b\,i}{q} + i\right)\,\dfrac{1}{b - q + 2\,c\,z} - \left(\dfrac{2\,c\,h - b\,i}{q} - i\right)\,\dfrac{1}{b + q + 2\,c\,z}$

- **Rule:** If $\sqrt{b^2 - 4\,a\,c} \neq 0 \wedge m \in \mathbb{Z}^+$, let $q = \sqrt{b^2 - 4\,a\,c}$, then

$$\int \dfrac{(f + g\,x)^m\,\left(h + i\,F^{d+e\,x}\right)}{a + b\,F^{d+e\,x} + c\,F^{2\,(d+e\,x)}}\,dx \longrightarrow \left(\dfrac{2\,c\,h - b\,i}{q} + i\right)\int \dfrac{(f + g\,x)^m}{b - q + 2\,c\,F^{d+e\,x}}\,dx - \left(\dfrac{2\,c\,h - b\,i}{q} - i\right)\int \dfrac{(f + g\,x)^m}{b + q + 2\,c\,F^{d+e\,x}}\,dx$$

Program code:

```
Int[(f_.+g_.*x_)^m_.*(h_+i_.*F_^u_)/(a_.+b_.*F_^u_+c_.*F_^v_),x_Symbol] :=
  With[{q=Rt[b^2-4*a*c,2]},
  (Simplify[(2*c*h-b*i)/q]+i)*Int[(f+g*x)^m/(b-q+2*c*F^u),x] -
  (Simplify[(2*c*h-b*i)/q]-i)*Int[(f+g*x)^m/(b+q+2*c*F^u),x]] /;
FreeQ[{F,a,b,c,f,g,h,i},x] && EqQ[v,2*u] && LinearQ[u,x] && NeQ[b^2-4*a*c,0] && IGtQ[m,0]
```

8. $\displaystyle\int \frac{u}{a + b\ F^{d+e\ x} + c\ F^{-(d+e\ x)}}\ dx$

   1: $\displaystyle\int \frac{x^m}{a\ F^{c+d\ x} + b\ F^{-(c+d*x)}}\ dx$ when $m > 0$

Derivation: Integration by parts

Rule: If $m > 0$, then

$$\int \frac{x^m}{a\ F^{c+d\ x} + b\ F^{-(c+d*x)}}\ dx \;\rightarrow\; x^m \int \frac{1}{a\ F^{c+d\ x} + b\ F^{-(c+d*x)}}\ dx - m \int x^{m-1} \int \frac{1}{a\ F^{c+d\ x} + b\ F^{-(c+d*x)}}\ dx\ dx$$

Program code:

```
Int[x_^m_./(a_.*F_^(c_.+d_.*x_)+b_.*F_^v_),x_Symbol] :=
  With[{u=IntHide[1/(a*F^(c+d*x)+b*F^v),x]},
   x^m*u - m*Int[x^(m-1)*u,x]] /;
FreeQ[{F,a,b,c,d},x] && EqQ[v,-(c+d*x)] && GtQ[m,0]
```

2: $\int \dfrac{u}{a + b\ F^{d+e\,x} + c\ F^{-(d+e\,x)}}\ dx$

Derivation: Algebraic simplification

Basis: $\dfrac{1}{a+b\ z+\frac{c}{z}} \ == \ \dfrac{z}{c+a\ z+b\ z^2}$

Rule:

$$\int \frac{u}{a + b\ F^{d+e\,x} + c\ F^{-(d+e\,x)}}\ dx \ \longrightarrow \ \int \frac{u\ F^{d+e\,x}}{c + a\ F^{d+e\,x} + b\ F^{2\ (d+e\,x)}}\ dx$$

Program code:

```
Int[u_/(a_+b_.*F_^v_+c_.*F_^w_),x_Symbol] :=
  Int[u*F^v/(c+a*F^v+b*F^(2*v)),x] /;
FreeQ[{F,a,b,c},x] && EqQ[w,-v] && LinearQ[v,x] &&
  If[RationalQ[Coefficient[v,x,1]], GtQ[Coefficient[v,x,1],0], LtQ[LeafCount[v],LeafCount[w]]]
```

9. $\int \dfrac{u\, F^{g\,(d+e\,x)^n}}{a + b\,x + c\,x^2}\, dx$

    1: $\int \dfrac{F^{g\,(d+e\,x)^n}}{a + b\,x + c\,x^2}\, dx$

Derivation: Algebraic expansion

Rule:

$$\int \frac{F^{g\,(d+e\,x)^n}}{a + b\,x + c\,x^2}\, dx \;\longrightarrow\; \int F^{g\,(d+e\,x)^n}\, \texttt{ExpandIntegrand}\!\left[\frac{1}{a + b\,x + c\,x^2},\, x\right] dx$$

Program code:

```
Int[F_^(g_.*(d_.+e_.*x_)^n_.)/(a_.+b_.*x_+c_.*x_^2),x_Symbol] :=
  Int[ExpandIntegrand[F^(g*(d+e*x)^n),1/(a+b*x+c*x^2),x],x] /;
FreeQ[{F,a,b,c,d,e,g,n},x]
```

```
Int[F_^(g_.*(d_.+e_.*x_)^n_.)/(a_+c_.*x_^2),x_Symbol] :=
  Int[ExpandIntegrand[F^(g*(d+e*x)^n),1/(a+c*x^2),x],x] /;
FreeQ[{F,a,c,d,e,g,n},x]
```

2: $\displaystyle\int \frac{P_x^m \, F^{g \, (d+e \, x)^n}}{a + b \, x + c \, x^2} \, dx$

Derivation: Algebraic expansion

Rule:

$$\int \frac{P_x^m \, F^{g \, (d+e \, x)^n}}{a + b \, x + c \, x^2} \, dx \;\longrightarrow\; \int F^{g \, (d+e \, x)^n} \, \texttt{ExpandIntegrand}\left[\frac{P_x^m}{a + b \, x + c \, x^2}, \, x\right] \, dx$$

Program code:

```
Int[u_^m_.*F_^(g_.*(d_.+e_.*x_)^n_.)/(a_.+b_.*x_+c_*x_^2),x_Symbol] :=
  Int[ExpandIntegrand[F^(g*(d+e*x)^n),u^m/(a+b*x+c*x^2),x],x] /;
FreeQ[{F,a,b,c,d,e,g,n},x] && PolynomialQ[u,x] && IntegerQ[m]
```

```
Int[u_^m_.*F_^(g_.*(d_.+e_.*x_)^n_.)/(a_+c_*x_^2),x_Symbol] :=
  Int[ExpandIntegrand[F^(g*(d+e*x)^n),u^m/(a+c*x^2),x],x] /;
FreeQ[{F,a,c,d,e,g,n},x] && PolynomialQ[u,x] && IntegerQ[m]
```

10: $\displaystyle\int F^{\frac{a+b\,x^4}{x^2}}\,dx$

Derivation: Integration by substitution

Rule:

$$\int F^{\frac{a+b\,x^4}{x^2}}\,dx \;\longrightarrow$$

$$\frac{\sqrt{\pi}\;\text{Exp}\left[2\,\sqrt{-a\,\text{Log}[F]}\;\sqrt{-b\,\text{Log}[F]}\right]\,\text{Erf}\left[\frac{\sqrt{-a\,\text{Log}[F]}\,+\sqrt{-b\,\text{Log}[F]}\;x^2}{x}\right]}{4\,\sqrt{-b\,\text{Log}[F]}} - \frac{\sqrt{\pi}\;\text{Exp}\left[-2\,\sqrt{-a\,\text{Log}[F]}\;\sqrt{-b\,\text{Log}[F]}\right]\,\text{Erf}\left[\frac{\sqrt{-a\,\text{Log}[F]}\,-\sqrt{-b\,\text{Log}[F]}\;x^2}{x}\right]}{4\,\sqrt{-b\,\text{Log}[F]}}$$

Program code:

```
Int[F_^((a_.+b_.*x_^4)/x_^2),x_Symbol] :=
  Sqrt[Pi]*Exp[2*Sqrt[-a*Log[F]]*Sqrt[-b*Log[F]]]*Erf[(Sqrt[-a*Log[F]]+Sqrt[-b*Log[F]]*x^2)/x]/
    (4*Sqrt[-b*Log[F]]) -
  Sqrt[Pi]*Exp[-2*Sqrt[-a*Log[F]]*Sqrt[-b*Log[F]]]*Erf[(Sqrt[-a*Log[F]]-Sqrt[-b*Log[F]]*x^2)/x]/
    (4*Sqrt[-b*Log[F]]) /;
FreeQ[{F,a,b},x]
```

11: $\int x^m \left(e^x + x^m\right)^n dx$ when $m > 0 \land n < 0 \land n \neq -1$

Derivation: Algebraic expansion

Basis: $x^m \left(e^x + x^m\right)^n == -\left(e^x + m\, x^{m-1}\right) \left(e^x + x^m\right)^n + \left(e^x + x^m\right)^{n+1} + m\, x^{m-1} \left(e^x + x^m\right)^n$

Rule: If $m > 0 \land n < 0 \land n \neq -1$, then

$$\int x^m \left(e^x + x^m\right)^n dx \longrightarrow -\frac{\left(e^x + x^m\right)^{n+1}}{n+1} + \int \left(e^x + x^m\right)^{n+1} dx + m \int x^{m-1} \left(e^x + x^m\right)^n dx$$

Program code:

```
Int[x_^m_.*(E^x_+x_^m_.)^n_,x_Symbol] :=
  -(E^x+x^m)^(n+1)/(n+1) +
  Int[(E^x+x^m)^(n+1),x] +
  m*Int[x^(m-1)*(E^x+x^m)^n,x] /;
RationalQ[m,n] && GtQ[m,0] && LtQ[n,0] && NeQ[n,-1]
```

12: $\int u \, F^{a \, (v+b \, Log[z])} \, dx$

Derivation: Algebraic simplification

Basis: $F^{a \, (v+b \, Log[z])} == F^{a \, v} \, z^{a \, b \, Log[F]}$

Rule:

$$\int u \, F^{a \, (v+b \, Log[z])} \, dx \longrightarrow \int u \, F^{a \, v} \, z^{a \, b \, Log[F]} \, dx$$

Program code:

```
Int[u_.*F_^(a_.*(v_.+b_.*Log[z_])),x_Symbol] :=
  Int[u*F^(a*v)*z^(a*b*Log[F]),x] /;
FreeQ[{F,a,b},x]
```

13. $\int u \, F^{d \, (a+b \, Log[c \, x^n]^2)} \, dx$

   1: $\int F^{d \, (a+b \, Log[c \, x^n]^2)} \, dx$

Derivation: Piecewise constant extraction and integration by substitution

Basis: $\partial_x \dfrac{x}{(c \, x^n)^{\frac{1}{n}}} == 0$

Basis: $\dfrac{G[Log[c \, x^n]]}{x} == \dfrac{1}{n} \, Subst[G[x], \, x, \, Log[c \, x^n]] \, \partial_x Log[c \, x^n]$

Rule:

$$\int F^{d \, (a+b \, Log[c \, x^n]^2)} \, dx \longrightarrow \dfrac{x}{(c \, x^n)^{\frac{1}{n}}} \int \dfrac{e^{Log[c \, x^n]/n} \, F^{d \, (a+b \, Log[c \, x^n]^2)}}{x} \, dx$$

$$\longrightarrow \quad \frac{x}{n\,(c\,x^n)^{\frac{1}{n}}}\ \text{Subst}\left[\int e^{a\,d\,\text{Log}[F]+x/n+b\,d\,\text{Log}[F]\,x^2}\,dx,\ x,\ \text{Log}\left[c\,x^n\right]\right]$$

Program code:

```
Int[F_^(d_.*(a_.+b_.*Log[c_.*x_^n_.]^2)),x_Symbol] :=
  x/(n*(c*x^n)^(1/n))*Subst[Int[E^(a*d*Log[F]+x/n+b*d*Log[F]*x^2),x],x,Log[c*x^n]] /;
FreeQ[{F,a,b,c,d,n},x]
```

2: $\displaystyle\int (e\,x)^m\ F^{d\,\left(a+b\,\text{Log}\left[c\,x^n\right]^2\right)}\,dx$

Derivation: Piecewise constant extraction and integration by substitution

Basis: $\partial_x\ \dfrac{(e\,x)^{m+1}}{(c\,x^n)^{\frac{m+1}{n}}}\ ==\ 0$

Basis: $\dfrac{G\left[\text{Log}\left[c\,x^n\right]\right]}{x}\ ==\ \dfrac{1}{n}\ \text{Subst}\left[G[x],\ x,\ \text{Log}\left[c\,x^n\right]\right]\ \partial_x\ \text{Log}\left[c\,x^n\right]$

Rule:

$$\int (e\,x)^m\ F^{d\,\left(a+b\,\text{Log}\left[c\,x^n\right]^2\right)}\,dx\ \longrightarrow\ \frac{(e\,x)^{m+1}}{e\,(c\,x^n)^{\frac{m+1}{n}}}\ \int \frac{e^{(m+1)\,\text{Log}\left[c\,x^n\right]/n}\ F^{d\,\left(a+b\,\text{Log}\left[c\,x^n\right]^2\right)}}{x}\,dx$$

$$\longrightarrow\ \frac{(e\,x)^{m+1}}{e\,n\,(c\,x^n)^{\frac{m+1}{n}}}\ \text{Subst}\left[\int e^{a\,d\,\text{Log}[F]+(m+1)\,x/n+b\,d\,\text{Log}[F]\,x^2}\,dx,\ x,\ \text{Log}\left[c\,x^n\right]\right]$$

Program code:

```
Int[(e_.*x_)^m_.*F_^(d_.*(a_.+b_.*Log[c_.*x_^n_.]^2)),x_Symbol] :=
  (e*x)^(m+1)/(e*n*(c*x^n)^((m+1)/n))*Subst[Int[E^(a*d*Log[F]+(m+1)*x/n+b*d*Log[F]*x^2),x],x,Log[c*x^n]] /;
FreeQ[{F,a,b,c,d,e,m,n},x]
```

14. $\int u\, F^{d\,(a+b\,\text{Log}[c\,x^n])^2}\, dx$

1: $\int F^{d\,(a+b\,\text{Log}[c\,x^n])^2}\, dx$

Derivation: Algebraic expansion

Rule:

$$\int F^{d\,(a+b\,\text{Log}[c\,x^n])^2}\, dx \;\longrightarrow\; \int F^{a^2\,d+2\,a\,b\,d\,\text{Log}[c\,x^n]+b^2\,d\,\text{Log}[c\,x^n]^2}\, dx$$

Program code:

```
Int[F_^(d_.*(a_.+b_.*Log[c_.*x_^n_.])^2),x_Symbol] :=
  Int[F^(a^2*d+2*a*b*d*Log[c*x^n]+b^2*d*Log[c*x^n]^2),x] /;
FreeQ[{F,a,b,c,d,n},x]
```

2: $\int (e\,x)^m\, F^{d\,(a+b\,\text{Log}[c\,x^n])^2}\, dx$

Derivation: Algebraic expansion

Rule:

$$\int (e\,x)^m\, F^{d\,(a+b\,\text{Log}[c\,x^n])^2}\, dx \;\longrightarrow\; \int (e\,x)^m\, F^{a^2\,d+2\,a\,b\,d\,\text{Log}[c\,x^n]+b^2\,d\,\text{Log}[c\,x^n]^2}\, dx$$

Program code:

```
Int[(e_.*x_)^m_.*F_^(d_.*(a_.+b_.*Log[c_.*x_^n_.])^2),x_Symbol] :=
  Int[(e*x)^m*F^(a^2*d+2*a*b*d*Log[c*x^n]+b^2*d*Log[c*x^n]^2),x] /;
FreeQ[{F,a,b,c,d,e,m,n},x]
```

15. $\int \text{Log}\left[a + b\left(F^{e\ (c+d\ x)}\right)^n\right] dx$

   1: $\int \text{Log}\left[a + b\left(F^{e\ (c+d\ x)}\right)^n\right] dx$ when $a > 0$

Derivation: Integration by substitution

Basis: $f\left[\left(F^{e\ (c+d\ x)}\right)^n\right] == \frac{1}{d\ e\ n\ \text{Log}[F]}\ \text{Subst}\left[\frac{f[x]}{x},\ x,\ \left(F^{e\ (c+d\ x)}\right)^n\right] \partial_x \left(F^{e\ (c+d\ x)}\right)^n$

Rule:

$$\int \text{Log}\left[a + b\left(F^{e\ (c+d\ x)}\right)^n\right] dx \;\longrightarrow\; \frac{1}{d\ e\ n\ \text{Log}[F]}\ \text{Subst}\left[\int \frac{\text{Log}[a + b\ x]}{x}\ dx,\ x,\ \left(F^{e\ (c+d\ x)}\right)^n\right]$$

Program code:

```
Int[Log[a_+b_.*(F_^(e_.*(c_.+d_.*x_)))^n_.],x_Symbol] :=
  1/(d*e*n*Log[F])*Subst[Int[Log[a+b*x]/x,x],x,(F^(e*(c+d*x)))^n] /;
FreeQ[{F,a,b,c,d,e,n},x] && GtQ[a,0]
```

2: $\int \text{Log}\left[a + b\left(F^{e\ (c+d\ x)}\right)^n\right] dx$ when $a \not> 0$

Derivation: Integration by parts

Rule: If $a \not> 0$, then

$$\int \text{Log}\left[a + b\left(F^{e\ (c+d\ x)}\right)^n\right] dx \;\longrightarrow\; x\,\text{Log}\left[a + b\left(F^{e\ (c+d\ x)}\right)^n\right] - b\,d\,e\,n\,\text{Log}[F] \int \frac{x\left(F^{e\ (c+d\ x)}\right)^n}{a + b\left(F^{e\ (c+d\ x)}\right)^n} dx$$

Program code:

```
Int[Log[a_+b_.*(F_^(e_.*(c_.+d_.*x_)))^n_.],x_Symbol] :=
  x*Log[a+b*(F^(e*(c+d*x)))^n] - b*d*e*n*Log[F]*Int[x*(F^(e*(c+d*x)))^n/(a+b*(F^(e*(c+d*x)))^n),x] /;
FreeQ[{F,a,b,c,d,e,n},x] && Not[GtQ[a,0]]
```

16. $\int u \, (a \, F^v)^n \, dx$

    x: $\int u \, (a \, F^v)^n \, dx$ when $n \in \mathbb{Z}$

Derivation: Algebraic simplification

Basis: If $n \in \mathbb{Z}$, then $(a \, F^v)^n == a^n \, F^{n \, v}$

Note: This rule not necessary since *Mathematica* automatically does this simplification.

Rule: If $n \in \mathbb{Z}$, then

$$\int u \, (a \, F^v)^n \, dx \; \longrightarrow \; a^n \int u \, F^{n \, v} \, dx$$

Program code:

```
(* Int[u_.*(a_.*F_^v_)^n_,x_Symbol] :=
  a^n*Int[u*F^(n*v),x] /;
FreeQ[{F,a},x] && IntegerQ[n] *)
```

2: $\int u \left(a \, F^v \right)^n \, dx$ when $n \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{\left(a \, F^{v[x]} \right)^n}{F^{n \, v[x]}} == 0$

Rule: If $n \notin \mathbb{Z}$, then

$$\int u \left(a \, F^v \right)^n \, dx \;\longrightarrow\; \frac{\left(a \, F^v \right)^n}{F^{n \, v}} \int u \, F^{n \, v} \, dx$$

Program code:

```
Int[u_.*(a_.*F_^v_)^n_,x_Symbol] :=
  (a*F^v)^n/F^(n*v)*Int[u*F^(n*v),x] /;
FreeQ[{F,a,n},x] && Not[IntegerQ[n]]
```

17: $\int f\left[F^{a+b\,x}\right]\,dx$

Derivation: Integration by substitution

Basis: $f\left[F^{a+b\,x}\right] \;==\; \frac{1}{b\,\text{Log}[F]}\;\text{Subst}\left[\frac{f[x]}{x},\,x,\,F^{a+b\,x}\right]\partial_x F^{a+b\,x}$

Basis: $\frac{1}{b\,\text{Log}[F]} \;==\; \frac{F^{a+b\,x}}{\partial_x F^{a+b\,x}}$

Rule:

$$\int f\left[F^{a+b\,x}\right]\,dx \;\longrightarrow\; \frac{F^{a+b\,x}}{\partial_x F^{a+b\,x}}\;\text{Subst}\left[\int \frac{f[x]}{x}\,dx,\,x,\,F^{a+b\,x}\right]$$

Program code:

```
Int[u_,x_Symbol] :=
  With[{v=FunctionOfExponential[u,x]},
  v/D[v,x]*Subst[Int[FunctionOfExponentialFunction[u,x]/x,x],x,v]] /;
FunctionOfExponentialQ[u,x] &&
  Not[MatchQ[u,w_*(a_.*v_^n_)^m_ /; FreeQ[{a,m,n},x] && IntegerQ[m*n]]] &&
  Not[MatchQ[u,E^(c_.*(a_.+b_.*x))*F_[v_] /; FreeQ[{a,b,c},x] && InverseFunctionQ[F[x]]]]
```

18. $\int u \left(a\ F^v + b\ G^w \right)^n dx$

    1. $\int u \left(a\ F^v + b\ G^w \right)^n dx$ when $n \in \mathbb{Z}^-$

        1: $\int u \left(a\ F^v + b\ F^w \right)^n dx$ when $n \in \mathbb{Z}^-$

Derivation: Algebraic simplification

Rule: If $n \in \mathbb{Z}^-$, then

$$\int u \left(a\ F^v + b\ F^w \right)^n dx \;\longrightarrow\; \int u\ F^{n\,v} \left(a + b\ F^{w-v} \right)^n dx$$

Program code:

```
Int[u_.*(a_.*F_^v_+b_.*F_^w_)^n_,x_Symbol] :=
  Int[u*F^(n*v)*(a+b*F^ExpandToSum[w-v,x])^n,x] /;
FreeQ[{F,a,b,n},x] && ILtQ[n,0] && LinearQ[{v,w},x]
```

        2: $\int u \left(a\ F^v + b\ G^w \right)^n dx$ when $n \in \mathbb{Z}^-$

Derivation: Algebraic simplification

Rule: If $n \in \mathbb{Z}^-$, then

$$\int u \left(a\ F^v + b\ G^w \right)^n dx \;\longrightarrow\; \int u\ F^{n\,v} \left(a + b\ E^{\text{Log}[G]\,w - \text{Log}[F]\,v} \right)^n dx$$

Program code:

```
Int[u_.*(a_.*F_^v_+b_.*G_^w_)^n_,x_Symbol] :=
  Int[u*F^(n*v)*(a+b*E^ExpandToSum[Log[G]*w-Log[F]*v,x])^n,x] /;
FreeQ[{F,G,a,b,n},x] && ILtQ[n,0] && LinearQ[{v,w},x]
```

2. $\int u \left(a\, F^v + b\, G^w\right)^n dx$ when $n \notin \mathbb{Z}$

   1: $\int u \left(a\, F^v + b\, F^w\right)^n dx$ when $n \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{\left(a\, F^{f[x]} + b\, F^{g[x]}\right)^n}{F^{n\, f[x]} \left(a + b\, F^{g[x]-f[x]}\right)^n} == 0$

Rule: If $n \notin \mathbb{Z}$, then

$$\int u \left(a\, F^v + b\, F^w\right)^n dx \;\longrightarrow\; \frac{\left(a\, F^v + b\, F^w\right)^n}{F^{n\, v} \left(a + b\, F^{w-v}\right)^n} \int u\, F^{n\, v} \left(a + b\, F^{w-v}\right)^n dx$$

Program code:

```
Int[u_.*(a_.*F_^v_+b_.*F_^w_)^n_,x_Symbol] :=
  (a*F^v+b*F^w)^n/(F^(n*v)*(a+b*F^ExpandToSum[w-v,x])^n)*Int[u*F^(n*v)*(a+b*F^ExpandToSum[w-v,x])^n,x] /;
FreeQ[{F,a,b,n},x] && Not[IntegerQ[n]] && LinearQ[{v,w},x]
```

2: $\int u \left(a\, F^v + b\, G^w\right)^n dx$ when $n \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: $\partial_x \dfrac{\left(a\, F^{f[x]} + b\, G^{g[x]}\right)^n}{F^{n\, f[x]} \left(a + b\, E^{Log[G]\, g[x] - Log[F]\, f[x]}\right)^n} \;==\; 0$

Rule: If $n \notin \mathbb{Z}$, then

$$\int u \left(a\, F^v + b\, G^w\right)^n dx \;\longrightarrow\; \frac{\left(a\, F^v + b\, G^w\right)^n}{F^{n\, v} \left(a + b\, E^{Log[G]\, w - Log[F]\, v}\right)^n} \int u\, F^{n\, v} \left(a + b\, E^{Log[G]\, w - Log[F]\, v}\right)^n dx$$

Program code:

```
Int[u_.*(a_.*F_^v_+b_.*G_^w_)^n_,x_Symbol] :=
  (a*F^v+b*G^w)^n/(F^(n*v)*(a+b*E^ExpandToSum[Log[G]*w-Log[F]*v,x])^n)*Int[u*F^(n*v)*(a+b*E^ExpandToSum[Log[G]*w-Log[F]*v,x])^n,x] /;
FreeQ[{F,G,a,b,n},x] && Not[IntegerQ[n]] && LinearQ[{v,w},x]
```

19: $\int u\, F^v\, G^w\, dx$

Derivation: Algebraic simplification

Basis: $F^v\, G^w == E^{v\,\text{Log}[F]+w\,\text{Log}[G]}$

Rule:

$$\int u\, F^v\, G^w\, dx \;\longrightarrow\; \int u\, E^{v\,\text{Log}[F]+w\,\text{Log}[G]}\, dx$$

Program code:

```
Int[u_.*F_^v_*G_^w_,x_Symbol] :=
  With[{z=v*Log[F]+w*Log[G]},
  Int[u*NormalizeIntegrand[E^z,x],x] /;
 BinomialQ[z,x] || PolynomialQ[z,x] && LeQ[Exponent[z,x],2]] /;
FreeQ[{F,G},x]
```

20: $\displaystyle\int F^u \ (v + w) \ y \ \mathbb{d}x$ when $\partial_x \ \dfrac{v \ y}{Log[F] \ \partial_x u} \ == \ w \ y$

Basis: $\partial_x \left(F^{f[x]} \ g[x] \right) \ == \ F^{f[x]} \ (Log[F] \ g[x] \ f'[x] + g'[x])$

Rule: Let $z = \dfrac{v \ y}{Log[F] \ \partial_x u}$, if $\partial_x z \ == \ w \ y$, then

$$\int F^u \ (v + w) \ y \ \mathbb{d}x \ \longrightarrow \ F^{f[x]} \ z$$

Program code:

```
Int[F_^u_*(v_+w_)*y_.,x_Symbol] :=
  With[{z=v*y/(Log[F]*D[u,x])},
  F^u*z /;
 EqQ[D[z,x],w*y]] /;
FreeQ[F,x]
```

21: $\int F^u \, v^n \, w \, dx$ when $\text{Log}[F] \, v \, \partial_x u + (n + 1) \, \partial_x v$ divides w

Basis: $\partial_x \left(F^{f[x]} \, g[x]^{n+1} \right) == F^{f[x]} \, g[x]^n \, (\text{Log}[F] \, g[x] \, f'[x] + (n + 1) \, g'[x])$

Rule: Let $z = \text{Log}[F] \, v \, \partial_x u + (n + 1) \, \partial_x v$, if z divides w, then

$$\int F^u \, v^n \, w \, dx \longrightarrow \frac{w}{z} \, F^u \, v^{n+1}$$

Program code:

```
Int[F_^u_*v_^n_.*w_,x_Symbol] :=
  With[{z=Log[F]*v*D[u,x]+(n+1)*D[v,x]},
  Coefficient[w,x,Exponent[w,x]]/Coefficient[z,x,Exponent[z,x]]*F^u*v^(n+1) /;
 EqQ[Exponent[w,x],Exponent[z,x]] && EqQ[w*Coefficient[z,x,Exponent[z,x]],z*Coefficient[w,x,Exponent[w,x]]]] /;
FreeQ[{F,n},x] && PolynomialQ[u,x] && PolynomialQ[v,x] && PolynomialQ[w,x]
```

22. $\int u \, \dfrac{\left(a + b \, F^{c \frac{\sqrt{d+e \, x}}{\sqrt{f+g \, x}}} \right)^n}{A + B \, x + C \, x^2} \, dx$ when $C \, d \, f - A \, e \, g == 0 \wedge B \, e \, g - C \, (e \, f + d \, g) == 0$

1: $\int \dfrac{\left(a + b \, F^{c \frac{\sqrt{d+e \, x}}{\sqrt{f+g \, x}}} \right)^n}{A + B \, x + C \, x^2} \, dx$ when $C \, d \, f - A \, e \, g == 0 \wedge B \, e \, g - C \, (e \, f + d \, g) == 0 \wedge n \in \mathbb{Z}^+$

Derivation: Integration by substitution

Basis: $F[x] == 2 \, (e \, f - d \, g) \, \text{Subst}\left[\dfrac{x}{(e-g \, x^2)^2} \, F\left[-\dfrac{d-f \, x^2}{e-g \, x^2} \right], x, \dfrac{\sqrt{d+e \, x}}{\sqrt{f+g \, x}} \right] \partial_x \dfrac{\sqrt{d+e \, x}}{\sqrt{f+g \, x}}$

Basis: If $C \, d \, f - A \, e \, g == 0 \wedge B \, e \, g - C \, (e \, f + d \, g) == 0$, then

$$\frac{1}{A+B\,x+C\,x^2} \;==\; \frac{2\,e\,g}{C\,(e\,f-d\,g)}\;\text{Subst}\!\left[\frac{1}{x},\;x,\;\frac{\sqrt{d+e\,x}}{\sqrt{f+g\,x}}\right]\,\partial_x\,\frac{\sqrt{d+e\,x}}{\sqrt{f+g\,x}}$$

Rule: If $C\,d\,f-A\,e\,g == 0 \;\wedge\; B\,e\,g-C\,(e\,f+d\,g) == 0 \;\wedge\; n\in\mathbb{Z}^{+}$, then

$$\int \frac{\left(a+b\,F^{\;c\,\frac{\sqrt{d+e\,x}}{\sqrt{f+g\,x}}}\right)^n}{A+B\,x+C\,x^2}\;\mathbb{d}x \;\longrightarrow\; \frac{2\,e\,g}{C\,(e\,f-d\,g)}\;\text{Subst}\!\left[\int \frac{\left(a+b\,F^{c\,x}\right)^n}{x}\,\mathbb{d}x,\;x,\;\frac{\sqrt{d+e\,x}}{\sqrt{f+g\,x}}\right]$$

Program code:

```
Int[(a_.+b_.*F_^(c_.*Sqrt[d_.+e_.*x_]/Sqrt[f_.+g_.*x_]))^n_./(A_.+B_.*x_+C_.*x_^2),x_Symbol] :=
  2*e*g/(C*(e*f-d*g))*Subst[Int[(a+b*F^(c*x))^n/x,x],x,Sqrt[d+e*x]/Sqrt[f+g*x]] /;
FreeQ[{a,b,c,d,e,f,g,A,B,C,F},x] && EqQ[C*d*f-A*e*g,0] && EqQ[B*e*g-C*(e*f+d*g),0] && IGtQ[n,0]
```

```
Int[(a_.+b_.*F_^(c_.*Sqrt[d_.+e_.*x_]/Sqrt[f_.+g_.*x_]))^n_./(A_+C_.*x_^2),x_Symbol] :=
  2*e*g/(C*(e*f-d*g))*Subst[Int[(a+b*F^(c*x))^n/x,x],x,Sqrt[d+e*x]/Sqrt[f+g*x]] /;
FreeQ[{a,b,c,d,e,f,g,A,C,F},x] && EqQ[C*d*f-A*e*g,0] && EqQ[e*f+d*g,0] && IGtQ[n,0]
```

2: $\displaystyle\int \frac{\left(a + b\, F^{\,c\,\frac{\sqrt{d+e\,x}}{\sqrt{f+g\,x}}}\right)^{n}}{A + B\,x + C\,x^{2}}\; dx$ when $C\,d\,f - A\,e\,g == 0 \;\wedge\; B\,e\,g - C\,(e\,f + d\,g) == 0 \;\wedge\; n \notin \mathbb{Z}^{+}$

Rule: If $C\,d\,f - A\,e\,g == 0 \;\wedge\; B\,e\,g - C\,(e\,f + d\,g) == 0 \;\wedge\; n \notin \mathbb{Z}^{+}$, then

$$\int \frac{\left(a + b\, F^{\,c\,\frac{\sqrt{d+e\,x}}{\sqrt{f+g\,x}}}\right)^{n}}{A + B\,x + C\,x^{2}}\; dx \;\longrightarrow\; \int \frac{\left(a + b\, F^{\,c\,\frac{\sqrt{d+e\,x}}{\sqrt{f+g\,x}}}\right)^{n}}{A + B\,x + C\,x^{2}}\; dx$$

Program code:

```
Int[(a_.+b_.*F_^(c_.*Sqrt[d_.+e_.*x_]/Sqrt[f_.+g_.*x_]))^n_/(A_.+B_.*x_+C_.*x_^2),x_Symbol] :=
  Unintegrable[(a+b*F^(c*Sqrt[d+e*x]/Sqrt[f+g*x]))^n/(A+B*x+C*x^2),x] /;
FreeQ[{a,b,c,d,e,f,g,A,B,C,F,n},x] && EqQ[C*d*f-A*e*g,0] && EqQ[B*e*g-C*(e*f+d*g),0] && Not[IGtQ[n,0]]
```

```
Int[(a_.+b_.*F_^(c_.*Sqrt[d_.+e_.*x_]/Sqrt[f_.+g_.*x_]))^n_/(A_+C_.*x_^2),x_Symbol] :=
  Unintegrable[(a+b*F^(c*Sqrt[d+e*x]/Sqrt[f+g*x]))^n/(A+C*x^2),x] /;
FreeQ[{a,b,c,d,e,f,g,A,C,F,n},x] && EqQ[C*d*f-A*e*g,0] && EqQ[e*f+d*g,0] && Not[IGtQ[n,0]]
```