

1: $\int \frac{A + B \operatorname{Log}[c (d + e x)^n]}{\sqrt{a + b \operatorname{Log}[c (d + e x)^n]}} dx$

Rule:

$$\int \frac{A + B \operatorname{Log}[c (d + e x)^n]}{\sqrt{a + b \operatorname{Log}[c (d + e x)^n]}} dx \rightarrow$$

$$\frac{B (d + e x) \sqrt{a + b \operatorname{Log}[c (d + e x)^n]}}{b e} + \frac{2 A b - B (2 a + b n)}{2 b} \int \frac{1}{\sqrt{a + b \operatorname{Log}[c (d + e x)^n]}} dx$$

Program code:

```
Int[(A_.+B_.*Log[c_.*(d_.+e_.**x_)^n_.])/Sqrt[a_+b_.*Log[c_.*(d_.+e_.**x_)^n_.]],x_Symbol] :=
  B*(d+e*x)*Sqrt[a+b*Log[c*(d+e*x)^n]]/(b*e) +
  (2*A*b-B*(2*a+b*n))/(2*b)*Int[1/Sqrt[a+b*Log[c*(d+e*x)^n]],x] /;
FreeQ[{a,b,c,d,e,A,B,n},x]
```

Rules for integrands of the form $u (a + b \operatorname{Log}[c x^n])^p$

1. $\int (a + b \operatorname{Log}[c x^n])^p dx$

1: $\int (a + b \operatorname{Log}[c x^n])^p dx$ when $p > 0$

Reference: G&R 2.711.1, CRC 485, CRC 490

Derivation: Integration by parts

Rule: If $p > 0$, then

$$\int (a + b \operatorname{Log}[c x^n])^p dx \rightarrow x (a + b \operatorname{Log}[c x^n])^p - b n p \int (a + b \operatorname{Log}[c x^n])^{p-1} dx$$

Program code:

```
Int[Log[c_.**x_^n_.],x_Symbol] :=
  x*Log[c*x^n] - n*x /;
FreeQ[{c,n},x]
```

```
Int[(a_.+b_.*Log[c_.**x_^n_.])^p_.,x_Symbol] :=
  x*(a+b*Log[c*x^n])^p - b*n*p*Int[(a+b*Log[c*x^n])^(p-1),x] /;
FreeQ[{a,b,c,n},x] && GtQ[p,0] && IntegerQ[2*p]
```

2: $\int (a + b \operatorname{Log}[c x^n])^p dx$ when $p < -1$

Derivation: Inverted integration by parts

Rule: If $p < -1$, then

$$\int (a + b \operatorname{Log}[c x^n])^p dx \rightarrow \frac{x (a + b \operatorname{Log}[c x^n])^{p+1}}{b n (p + 1)} - \frac{1}{b n (p + 1)} \int (a + b \operatorname{Log}[c x^n])^{p+1} dx$$

Program code:

```
Int[(a_.+b_.*Log[c_.*x_^n_.])^p_,x_Symbol] :=
  x*(a+b*Log[c*x^n])^(p+1)/(b*n*(p+1)) - 1/(b*n*(p+1))*Int[(a+b*Log[c*x^n])^(p+1),x] /;
FreeQ[{a,b,c,n},x] && LtQ[p,-1] && IntegerQ[2*p]
```

$$3. \int (a + b \operatorname{Log}[c x^n])^p dx$$

$$1. \int (a + b \operatorname{Log}[c x^n])^p dx \text{ when } \frac{1}{n} \in \mathbb{Z}$$

$$1: \int \frac{1}{\operatorname{Log}[c x]} dx$$

Reference: CRC 492

Derivation: Integration by substitution and algebraic simplification

$$\text{Basis: } F[\operatorname{Log}[c x]] := \frac{1}{c} \operatorname{Subst}\left[\frac{e^x}{x} F[x], x, \operatorname{Log}[c x]\right] \partial_x \operatorname{Log}[c x]$$

$$\text{Basis: } \int \frac{e^x}{x} dx := \operatorname{ExpIntegralEi}[x]$$

$$\text{Basis: } \operatorname{ExpIntegralEi}[\operatorname{Log}[z]] := \operatorname{LogIntegral}[z]$$

Note: This rule is optional, but returns antiderivative expressed in terms of `LogIntegral` instead of `ExpIntegralEi`.

Rule:

$$\int \frac{1}{\operatorname{Log}[c x]} dx \rightarrow \frac{1}{c} \operatorname{Subst}\left[\int \frac{e^x}{x} dx, x, \operatorname{Log}[c x]\right] \rightarrow \frac{1}{c} \operatorname{ExpIntegralEi}[\operatorname{Log}[c x]] \rightarrow \frac{1}{c} \operatorname{LogIntegral}[c x]$$

Program code:

```
Int[1/Log[c_.*x_],x_Symbol] :=
  LogIntegral[c*x]/c /;
  FreeQ[c,x]
```

$$\mathbf{2:} \int (a + b \operatorname{Log}[c x^n])^p dx \text{ when } \frac{1}{n} \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If $\frac{1}{n} \in \mathbb{Z}$, then $F[\operatorname{Log}[c x^n]] = \frac{1}{n c^{1/n}} \operatorname{Subst}[e^{x/n} F[x], x, \operatorname{Log}[c x^n]] \partial_x \operatorname{Log}[c x^n]$

Rule: If $\frac{1}{n} \in \mathbb{Z}$, then

$$\begin{aligned} \int (a + b \operatorname{Log}[c x^n])^p dx &\rightarrow \frac{1}{n c^{1/n}} \operatorname{Subst}\left[\int e^{x/n} (a + b x)^p dx, x, \operatorname{Log}[c x^n]\right] \\ \int (a + b \operatorname{Log}[c x^n])^p dx &\rightarrow \frac{1}{b n c^{1/n} e^{\frac{a}{b n}}} \operatorname{Subst}\left[\int x^p e^{\frac{x}{b n}} dx, x, a + b \operatorname{Log}[c x^n]\right] \end{aligned}$$

Program code:

```
Int[(a_+b_*Log[c_*x^n_])^p_,x_Symbol] :=
  1/(n*c^(1/n))*Subst[Int[E^(x/n)*(a+b*x)^p_,x],x,Log[c*x^n]] /;
FreeQ[{a,b,c,p},x] && IntegerQ[1/n]
```

$$\mathbf{2:} \int (a + b \operatorname{Log}[c x^n])^p dx$$

Derivation: Piecewise constant extraction and integration by substitution

Basis: $\partial_x \frac{x}{(c x^n)^{1/n}} = 0$

Basis: $\frac{(c x^n)^k F[\operatorname{Log}[c x^n]]}{x} = \frac{1}{n} \operatorname{Subst}[e^{k x} F[x], x, \operatorname{Log}[c x^n]] \partial_x \operatorname{Log}[c x^n]$

Rule:

$$\int (a + b \operatorname{Log}[c x^n])^p dx \rightarrow \frac{x}{(c x^n)^{1/n}} \int \frac{(c x^n)^{1/n} (a + b \operatorname{Log}[c x^n])^p}{x} dx \rightarrow \frac{x}{n (c x^n)^{1/n}} \operatorname{Subst}\left[\int e^{x/n} (a + b x)^p dx, x, \operatorname{Log}[c x^n]\right]$$

$$\int (a + b \operatorname{Log}[c x^n])^p dx \rightarrow \frac{x}{(c x^n)^{1/n}} \int \frac{(c x^n)^{1/n} (a + b \operatorname{Log}[c x^n])^p}{x} dx \rightarrow \frac{x}{b n (c x^n)^{1/n} e^{\frac{a}{b n}}} \operatorname{Subst}\left[\int x^p e^{\frac{x}{b n}} dx, x, a + b \operatorname{Log}[c x^n]\right]$$

Program code:

```
Int[(a_.+b_.*Log[c_.*x_^n_.])^p_,x_Symbol] :=
  x/(n*(c*x^n)^(1/n))*Subst[Int[E^(x/n)*(a+b*x)^p,x],x,Log[c*x^n]] /;
FreeQ[{a,b,c,n,p},x]
```

$$2. \int (d x)^m (a + b \operatorname{Log}[c x^n])^p dx$$

$$1: \int \frac{(a + b \operatorname{Log}[c x^n])^p}{x} dx$$

Reference: CRC 491

Derivation: Integration by substitution

$$\text{Basis: } \frac{F[a+b \operatorname{Log}[c x^n]]}{x} == \frac{1}{b n} \operatorname{Subst}[F[x], x, a + b \operatorname{Log}[c x^n]] \partial_x (a + b \operatorname{Log}[c x^n])$$

Rule:

$$\int \frac{(a + b \operatorname{Log}[c x^n])^p}{x} dx \rightarrow \frac{1}{b n} \operatorname{Subst}\left[\int x^p dx, x, a + b \operatorname{Log}[c x^n]\right]$$

Program code:

```
Int[(a_.+b_.*Log[c_.*x_^n_.])/x_,x_Symbol] :=
  (a+b*Log[c*x^n])^2/(2*b*n) /;
FreeQ[{a,b,c,n},x]
```

```
Int[(a_.+b_.*Log[c_.*x_^n_.])^p_/x_,x_Symbol] :=
  1/(b*n)*Subst[Int[x^p,x],x,a+b*Log[c*x^n]] /;
FreeQ[{a,b,c,n,p},x]
```

$$2. \int (d x)^m (a + b \operatorname{Log}[c x^n])^p dx \text{ when } m \neq -1 \wedge p > 0$$

$$1: \int (d x)^m (a + b \operatorname{Log}[c x^n]) dx \text{ when } m \neq -1 \wedge a(m+1) - b n = 0$$

Note: Optional rule for special case returns a single term.

Rule: If $m \neq -1$, then

$$\int (d x)^m (a + b \operatorname{Log}[c x^n]) dx \rightarrow \frac{b (d x)^{m+1} \operatorname{Log}[c x^n]}{d (m+1)}$$

Program code:

```
Int[(d_.**x_)^m_.*(a_.+b_.*Log[c_.**x_^n_.]),x_Symbol] :=
  b*(d*x)^(m+1)*Log[c*x^n]/(d*(m+1)) /;
FreeQ[{a,b,c,d,m,n},x] && NeQ[m,-1] && EqQ[a*(m+1)-b*n,0]
```

$$\mathbf{2:} \int (dx)^m (a + b \operatorname{Log}[cx^n])^p dx \text{ when } m \neq -1 \wedge p > 0$$

Reference: G&R 2.721.1, CRC 496, A&S 4.1.51

Derivation: Integration by parts

$$\text{Basis: } \partial_x (a + b \operatorname{Log}[cx^n])^p = \frac{b n p (a + b \operatorname{Log}[cx^n])^{p-1}}{x}$$

Rule: If $m \neq -1 \wedge p > 0$, then

$$\int (dx)^m (a + b \operatorname{Log}[cx^n])^p dx \rightarrow \frac{(dx)^{m+1} (a + b \operatorname{Log}[cx^n])^p}{d(m+1)} - \frac{b n p}{m+1} \int (dx)^m (a + b \operatorname{Log}[cx^n])^{p-1} dx$$

Program code:

```
Int[(d.*x_)^m_.*(a_.+b_.*Log[c_.*x_^n_.]),x_Symbol] :=
  (d*x)^(m+1)*(a+b*Log[c*x^n])/(d*(m+1)) - b*n*(d*x)^(m+1)/(d*(m+1)^2) /;
FreeQ[{a,b,c,d,m,n},x] && NeQ[m,-1]
```

```
Int[(d.*x_)^m_.*(a_.+b_.*Log[c_.*x_^n_.])^p_,x_Symbol] :=
  (d*x)^(m+1)*(a+b*Log[c*x^n])^p/(d*(m+1)) - b*n*p/(m+1)*Int[(d*x)^m*(a+b*Log[c*x^n])^(p-1),x] /;
FreeQ[{a,b,c,d,m,n},x] && NeQ[m,-1] && GtQ[p,0]
```

$$\mathbf{3:} \int (dx)^m (a + b \operatorname{Log}[cx^n])^p dx \text{ when } m \neq -1 \wedge p < -1$$

Reference: G&R 2.724.1, CRC 495

Derivation: Inverted integration by parts

Rule: If $m \neq -1 \wedge p < -1$, then

$$\int (d x)^m (a + b \operatorname{Log}[c x^n])^p dx \rightarrow \frac{(d x)^{m+1} (a + b \operatorname{Log}[c x^n])^{p+1}}{b d n (p+1)} - \frac{m+1}{b n (p+1)} \int (d x)^m (a + b \operatorname{Log}[c x^n])^{p+1} dx$$

Program code:

```
Int[(d_.*x_)^m_.*(a_.+b_.*Log[c_.*x_^n_.])^p_,x_Symbol] :=
  (d*x)^(m+1)*(a+b*Log[c*x^n])^(p+1)/(b*d*n*(p+1)) - (m+1)/(b*n*(p+1))*Int[(d*x)^m*(a+b*Log[c*x^n])^(p+1),x] /;
FreeQ[{a,b,c,d,m,n},x] && NeQ[m,-1] && LtQ[p,-1]
```

4. $\int \frac{(d x)^m}{\operatorname{Log}[c x^n]} dx$ when $m == n - 1$

1: $\int \frac{x^m}{\operatorname{Log}[c x^n]} dx$ when $m == n - 1$

Derivation: Integration by substitution

Note: The resulting antiderivative of this unessential rule is expressed in terms of `LogIntegral` instead of `ExpIntegralEi`.

Rule: If $m == n - 1$, then

$$\int \frac{x^m}{\operatorname{Log}[c x^n]} dx \rightarrow \frac{1}{n} \operatorname{Subst}\left[\int \frac{1}{\operatorname{Log}[c x]} dx, x, x^n\right]$$

Program code:

```
Int[x_^m_/Log[c_.*x_^n_],x_Symbol] :=
  1/n*Subst[Int[1/Log[c*x],x],x,x^n] /;
FreeQ[{c,m,n},x] && EqQ[m,n-1]
```

2: $\int \frac{(d x)^m}{\text{Log}[c x^n]} dx$ when $m == n - 1$

Derivation: Piecewise constant extraction

Rule: If $m == n - 1$, then

$$\int \frac{(d x)^m}{\text{Log}[c x^n]} dx \rightarrow \frac{(d x)^m}{x^m} \int \frac{x^m}{\text{Log}[c x^n]} dx$$

Program code:

```
Int[(d*x_)^m_/Log[c_*x_^n_],x_Symbol] :=
  (d*x)^m/x^m*Int[x^m/Log[c*x^n],x] /;
FreeQ[{c,d,m,n},x] && EqQ[m,n-1]
```

5: $\int x^m (a + b \operatorname{Log}[c x])^p dx$ when $m \in \mathbb{Z}$

Derivation: Integration by substitution

Basis: If $m \in \mathbb{Z}$, then $x^m F[\operatorname{Log}[c x]] = \frac{1}{c^{m+1}} \operatorname{Subst}\left[e^{(m+1)x} F[x], x, \operatorname{Log}[c x]\right] \partial_x \operatorname{Log}[c x]$

Rule: If $m \in \mathbb{Z}$, then

$$\int x^m (a + b \operatorname{Log}[c x])^p dx \rightarrow \frac{1}{c^{m+1}} \operatorname{Subst}\left[\int e^{(m+1)x} (a + b x)^p dx, x, \operatorname{Log}[c x]\right]$$

Program code:

```
Int[x_^m_.*(a_.+b_.*Log[c_.*x_])^p_,x_Symbol] :=
  1/c^(m+1)*Subst[Int[E^((m+1)*x)*(a+b*x)^p,x],x,Log[c*x]] /;
FreeQ[{a,b,c,p},x] && IntegerQ[m]
```

6: $\int (dx)^m (a + b \log[cx^n])^p dx$

Derivation: Piecewise constant extraction and integration by substitution

Basis: $\partial_x \frac{(dx)^{m+1}}{(cx^n)^{\frac{m+1}{n}}} = 0$

Basis: $\frac{(cx^n)^k F[\log[cx^n]]}{x} = \frac{1}{n} \text{Subst}[e^{kx} F[x], x, \log[cx^n]] \partial_x \log[cx^n]$

Rule:

$$\int (dx)^m (a + b \log[cx^n])^p dx \rightarrow \frac{(dx)^{m+1}}{d (cx^n)^{\frac{m+1}{n}}} \int \frac{(cx^n)^{\frac{m+1}{n}} (a + b \log[cx^n])^p}{x} dx \rightarrow \frac{(dx)^{m+1}}{d n (cx^n)^{\frac{m+1}{n}}} \text{Subst}\left[\int e^{\frac{m+1}{n}x} (a + bx)^p dx, x, \log[cx^n]\right]$$

Program code:

```
Int[(d.*x_)^m.*(a_.+b_.*Log[c_.*x_^n_.])^p_,x_Symbol] :=
  (d*x)^(m+1)/(d*n*(c*x^n)^( (m+1)/n ))*Subst[Int[E^( (m+1)/n*x )*(a+b*x)^p,x],x,Log[c*x^n]] /;
FreeQ[{a,b,c,d,m,n,p},x]
```

P: $\int (d x^q)^m (a + b \operatorname{Log}[c x^n])^p dx$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(d x^q)^m}{x^{mq}} == 0$

Rule:

$$\int (d x^q)^m (a + b \operatorname{Log}[c x^n])^p dx \rightarrow \frac{(d x^q)^m}{x^{mq}} \int x^{mq} (a + b \operatorname{Log}[c x^n])^p dx$$

Program code:

```
Int[(d_.**x_^q_)^m_*(a_.+b_.**Log[c_.**x_^n_.])^p_,x_Symbol] :=
  (d*x^q)^m/x^(m*q)*Int[x^(m*q)*(a+b*Log[c*x^n])^p,x] /;
FreeQ[{a,b,c,d,m,n,p,q},x]
```

```
Int[(d1_.**x_^q1_)^m1_*(d2_.**x_^q2_)^m2_*(a_.+b_.**Log[c_.**x_^n_.])^p_,x_Symbol] :=
  (d1*x^q1)^m1*(d2*x^q2)^m2/x^(m1*q1+m2*q2)*Int[x^(m1*q1+m2*q2)*(a+b*Log[c*x^n])^p,x] /;
FreeQ[{a,b,c,d1,d2,m1,m2,n,p,q1,q2},x]
```

$$3. \int (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx$$

$$1: \int (d + e x^r)^q (a + b \operatorname{Log}[c x^n]) dx \text{ when } q \in \mathbb{Z}^+$$

Derivation: Integration by parts

$$\text{Basis: } \partial_x (a + b \operatorname{Log}[c x^n]) = \frac{b n}{x}$$

Rule: If $q \in \mathbb{Z}^+$, let $u \rightarrow \int (d + e x^r)^q dx$, then

$$\int (d + e x^r)^q (a + b \operatorname{Log}[c x^n]) dx \rightarrow u (a + b \operatorname{Log}[c x^n]) - b n \int \frac{u}{x} dx$$

Program code:

```
Int[(d_+e_.**x^r_.)^q_.*(a_.+b_.*Log[c_.**x^n_.]),x_Symbol] :=
  With[{u=IntHide[(d+e*x^r)^q,x]},
    u*(a+b*Log[c*x^n]) - b*n*Int[SimplifyIntegrand[u/x,x],x] /;
    FreeQ[{a,b,c,d,e,n,r},x] && IGtQ[q,0]
```

2: $\int (d + e x^r)^q (a + b \log[c x^n]) dx$ when $r (q + 1) + 1 = 0$

Derivation: Integration by parts

Basis: If $r (q + 1) + 1 = 0$, then $(d + e x^r)^q = \partial_x \frac{x (d + e x^r)^{q+1}}{d}$

Rule: If $r (q + 1) + 1 = 0$, then

$$\int (d + e x^r)^q (a + b \log[c x^n]) dx \rightarrow \frac{x (d + e x^r)^{q+1} (a + b \log[c x^n])}{d} - \frac{b n}{d} \int (d + e x^r)^{q+1} dx$$

Program code:

```
Int[(d+_.*x^r_.)^q_*(a_.+b_.*Log[c_.*x^n_.]),x_Symbol] :=
  x*(d+e*x^r)^(q+1)*(a+b*Log[c*x^n])/d - b*n/d*Int[(d+e*x^r)^(q+1),x] /;
FreeQ[{a,b,c,d,e,n,q,r},x] && EqQ[r*(q+1)+1,0]
```

$$3. \int (d + e x)^q (a + b \operatorname{Log}[c x^n])^p dx$$

$$1. \int (d + e x)^q (a + b \operatorname{Log}[c x^n])^p dx \text{ when } p > 0$$

$$1. \int \frac{(a + b \operatorname{Log}[c x^n])^p}{d + e x} dx \text{ when } p \in \mathbb{Z}^+$$

$$1. \int \frac{a + b \operatorname{Log}[c x]}{d + e x} dx \text{ when } -\frac{c d}{e} > 0$$

$$1: \int \frac{\operatorname{Log}[c x]}{d + e x} dx \text{ when } e + c d = 0$$

Rule: If $e + c d = 0$, then

$$\int \frac{\operatorname{Log}[c x]}{d + e x} dx \rightarrow -\frac{1}{e} \operatorname{PolyLog}[2, 1 - c x]$$

Program code:

```
Int[Log[c_.*x_]/(d_+e_.*x_),x_Symbol] :=
  -1/e*PolyLog[2,1-c*x] /;
FreeQ[{c,d,e},x] && EqQ[e+c*d,0]
```

$$2: \int \frac{a + b \operatorname{Log}[c x]}{d + e x} dx \text{ when } -\frac{c d}{e} > 0$$

Derivation: Algebraic expansion

Basis: If $-\frac{c d}{e} > 0$, then $\operatorname{Log}[c x] = \operatorname{Log}\left[-\frac{c d}{e}\right] + \operatorname{Log}\left[-\frac{e x}{d}\right]$

Note: Resulting integrand is of the form required by the above rule.

Rule: If $-\frac{c d}{e} > 0$, then

$$\int \frac{a + b \operatorname{Log}[c x]}{d + e x} dx \rightarrow \frac{\left(a + b \operatorname{Log}\left[-\frac{cd}{e}\right]\right) \operatorname{Log}[d + e x]}{e} + b \int \frac{\operatorname{Log}\left[-\frac{ex}{d}\right]}{d + e x} dx$$

Program code:

```
Int[(a_.+b_.*Log[c_.*x_])/(d_+e_.*x_),x_Symbol] :=
  (a+b*Log[-c*d/e])*Log[d+e*x]/e + b*Int[Log[-e*x/d]/(d+e*x),x] /;
FreeQ[{a,b,c,d,e},x] && GtQ[-c*d/e,0]
```

2: $\int \frac{(a + b \operatorname{Log}[c x^n])^p}{d + e x} dx$ when $p \in \mathbb{Z}^+$

Derivation: Integration by parts

Basis: $\frac{1}{d+ex} = \frac{1}{e} \partial_x \operatorname{Log}\left[1 + \frac{ex}{d}\right]$

Basis: $\partial_x (a + b \operatorname{Log}[c x^n])^p = \frac{b n p (a + b \operatorname{Log}[c x^n])^{p-1}}{x}$

Rule: If $p \in \mathbb{Z}^+$, then

$$\int \frac{(a + b \operatorname{Log}[c x^n])^p}{d + e x} dx \rightarrow \frac{\operatorname{Log}\left[1 + \frac{ex}{d}\right] (a + b \operatorname{Log}[c x^n])^p}{e} - \frac{b n p}{e} \int \frac{\operatorname{Log}\left[1 + \frac{ex}{d}\right] (a + b \operatorname{Log}[c x^n])^{p-1}}{x} dx$$

Program code:

```
Int[(a_.+b_.*Log[c_.*x_^n_.])^p_/(d_+e_.*x_),x_Symbol] :=
  Log[1+e*x/d]*(a+b*Log[c*x^n])^p/e - b*n*p/e*Int[Log[1+e*x/d]*(a+b*Log[c*x^n])^(p-1)/x,x] /;
FreeQ[{a,b,c,d,e,n},x] && IGtQ[p,0]
```

2: $\int \frac{(a + b \operatorname{Log}[c x^n])^p}{(d + e x)^2} dx$ when $p > 0$

Derivation: Integration by parts

Basis: $\frac{1}{(d+e x)^2} = \partial_x \frac{x}{d (d+e x)}$

Basis: $\partial_x (a + b \operatorname{Log}[c x^n])^p = \frac{b n p (a + b \operatorname{Log}[c x^n])^{p-1}}{x}$

Rule: If $p > 0$, then

$$\int \frac{(a + b \operatorname{Log}[c x^n])^p}{(d + e x)^2} dx \rightarrow \frac{x (a + b \operatorname{Log}[c x^n])^p}{d (d + e x)} - \frac{b n p}{d} \int \frac{(a + b \operatorname{Log}[c x^n])^{p-1}}{d + e x} dx$$

Program code:

```
Int[(a_.+b_.*Log[c_.*x_^n_.])^p_./(d_+e_.*x_)^2,x_Symbol] :=
  x*(a+b*Log[c*x^n])^p/(d*(d+e*x)) - b*n*p/d*Int[(a+b*Log[c*x^n])^(p-1)/(d+e*x),x] /;
FreeQ[{a,b,c,d,e,n,p},x] && GtQ[p,0]
```

3: $\int (d + e x)^q (a + b \operatorname{Log}[c x^n])^p dx$ when $p > 0 \wedge q \neq -1$

Reference: G&R 2.728.1, CRC 501, A&S 4.1.50'

Derivation: Integration by parts

Basis: $\partial_x (a + b \operatorname{Log}[c x^n])^p = \frac{b n p (a + b \operatorname{Log}[c x^n])^{p-1}}{x}$

Rule: If $p > 0 \wedge q \neq -1$, then

$$\int (d + e x)^q (a + b \operatorname{Log}[c x^n])^p dx \rightarrow \frac{(d + e x)^{q+1} (a + b \operatorname{Log}[c x^n])^p}{e (q + 1)} - \frac{b n p}{e (q + 1)} \int \frac{(d + e x)^{q+1} (a + b \operatorname{Log}[c x^n])^{p-1}}{x} dx$$

Program code:

```
Int[(d_+e_.*x_)^q_.*(a_+b_.*Log[c_.*x_^n_.])^p_,x_Symbol] :=
  (d+e*x)^(q+1)*(a+b*Log[c*x^n])^p/(e*(q+1)) - b*n*p/(e*(q+1))*Int[((d+e*x)^(q+1)*(a+b*Log[c*x^n])^(p-1))/x,x] /;
FreeQ[{a,b,c,d,e,n,p,q},x] && GtQ[p,0] && NeQ[q,-1] && (EqQ[p,1] || IntegersQ[2*p,2*q] && Not[IGtQ[q,0]] || EqQ[p,2] && NeQ[q,1])
```

2: $\int (d+e x)^q (a+b \operatorname{Log}[c x^n])^p dx$ when $p < -1 \wedge q > 0$

Rule: If $p < -1 \wedge q > 0$, then

$$\int (d+e x)^q (a+b \operatorname{Log}[c x^n])^p dx \rightarrow \frac{x (d+e x)^q (a+b \operatorname{Log}[c x^n])^{p+1}}{b n (p+1)} + \frac{d q}{b n (p+1)} \int (d+e x)^{q-1} (a+b \operatorname{Log}[c x^n])^{p+1} dx - \frac{q+1}{b n (p+1)} \int (d+e x)^q (a+b \operatorname{Log}[c x^n])^{p+1} dx$$

Program code:

```
Int[(d+_e*_x_)^q_.*(a+_b_*Log[c*_x_^n_.])^p_,x_Symbol] :=
  x*(d+e*x)^q*(a+b*Log[c*x^n])^(p+1)/(b*n*(p+1)) +
  d*q/(b*n*(p+1))*Int[(d+e*x)^(q-1)*(a+b*Log[c*x^n])^(p+1),x] -
  (q+1)/(b*n*(p+1))*Int[(d+e*x)^q*(a+b*Log[c*x^n])^(p+1),x] /;
FreeQ[{a,b,c,d,e,n},x] && LtQ[p,-1] && GtQ[q,0]
```

$$4. \int (d + e x^2)^q (a + b \operatorname{Log}[c x^n]) dx$$

$$\mathbf{1:} \int (d + e x^2)^q (a + b \operatorname{Log}[c x^n]) dx \text{ when } q > 0$$

Rule: If $q > 0$, then

$$\int (d + e x^2)^q (a + b \operatorname{Log}[c x^n]) dx \rightarrow \frac{x (d + e x^2)^q (a + b \operatorname{Log}[c x^n])}{2q + 1} - \frac{b n}{2q + 1} \int (d + e x^2)^q dx + \frac{2 d q}{2q + 1} \int (d + e x^2)^{q-1} (a + b \operatorname{Log}[c x^n]) dx$$

Program code:

```
Int[(d_+e_.**x_^2)^q_.*(a_.+b_.*Log[c_.**x_^n_.]),x_Symbol] :=
  x*(d+e*x^2)^q*(a+b*Log[c*x^n])/(2*q+1) -
  b*n/(2*q+1)*Int[(d+e*x^2)^q,x] +
  2*d*q/(2*q+1)*Int[(d+e*x^2)^(q-1)*(a+b*Log[c*x^n]),x] /;
FreeQ[{a,b,c,d,e,n},x] && GtQ[q,0]
```

$$2. \int (d + e x^2)^q (a + b \operatorname{Log}[c x^n]) dx \text{ when } q < -1$$

$$1: \int \frac{a + b \operatorname{Log}[c x^n]}{(d + e x^2)^{3/2}} dx$$

Rule:

$$\int \frac{a + b \operatorname{Log}[c x^n]}{(d + e x^2)^{3/2}} dx \rightarrow \frac{x (a + b \operatorname{Log}[c x^n])}{d \sqrt{d + e x^2}} - \frac{b n}{d} \int \frac{1}{\sqrt{d + e x^2}} dx$$

Program code:

```
Int[(a_.+b_.*Log[c_.*x_^n_.])/(d_+e_.*x_^2)^(3/2),x_Symbol] :=
  x*(a+b*Log[c*x^n])/(d*Sqrt[d+e*x^2]) - b*n/d*Int[1/Sqrt[d+e*x^2],x] /;
FreeQ[{a,b,c,d,e,n},x]
```

$$2: \int (d + e x^2)^q (a + b \operatorname{Log}[c x^n]) dx \text{ when } q < -1$$

Rule: If $q < -1$, then

$$\int (d + e x^2)^q (a + b \operatorname{Log}[c x^n]) dx \rightarrow -\frac{x (d + e x^2)^{q+1} (a + b \operatorname{Log}[c x^n])}{2 d (q+1)} + \frac{b n}{2 d (q+1)} \int (d + e x^2)^{q+1} dx + \frac{2 q + 3}{2 d (q+1)} \int (d + e x^2)^{q+1} (a + b \operatorname{Log}[c x^n]) dx$$

Program code:

```
Int[(d_+e_.*x_^2)^q*(a_.+b_.*Log[c_.*x_^n_.]),x_Symbol] :=
  -x*(d+e*x^2)^(q+1)*(a+b*Log[c*x^n])/(2*d*(q+1)) +
  b*n/(2*d*(q+1))*Int[(d+e*x^2)^(q+1),x] +
  (2*q+3)/(2*d*(q+1))*Int[(d+e*x^2)^(q+1)*(a+b*Log[c*x^n]),x] /;
FreeQ[{a,b,c,d,e,n},x] && LtQ[q,-1]
```

3: $\int \frac{a + b \operatorname{Log}[c x^n]}{d + e x^2} dx$

Derivation: Integration by parts

Basis: $\partial_x (a + b \operatorname{Log}[c x^n]) = \frac{b n}{x}$

Rule: Let $u \rightarrow \int \frac{1}{d+e x^2} dx$, then

$$\int \frac{a + b \operatorname{Log}[c x^n]}{d + e x^2} dx \rightarrow u (a + b \operatorname{Log}[c x^n]) - b n \int \frac{u}{x} dx$$

Program code:

```
Int[(a_.+b_.*Log[c_.*x_^n_.])/(d_+e_.*x_^2),x_Symbol] :=
  With[{u=IntHide[1/(d+e*x^2),x]},
    u*(a+b*Log[c*x^n]) - b*n*Int[u/x,x] /;
  FreeQ[{a,b,c,d,e,n},x]
```

$$4. \int \frac{a + b \operatorname{Log}[c x^n]}{\sqrt{d + e x^2}} dx$$

$$1. \int \frac{a + b \operatorname{Log}[c x^n]}{\sqrt{d + e x^2}} dx \text{ when } d > 0$$

$$1: \int \frac{a + b \operatorname{Log}[c x^n]}{\sqrt{d + e x^2}} dx \text{ when } d > 0 \wedge e > 0$$

Derivation: Integration by parts

■ Basis: If $d > 0$, then $\frac{1}{\sqrt{d+e x^2}} = \partial_x \frac{\operatorname{ArcSinh}\left[\frac{\sqrt{e} x}{\sqrt{d}}\right]}{\sqrt{e}}$

Rule: If $d > 0 \wedge e > 0$, then

$$\int \frac{a + b \operatorname{Log}[c x^n]}{\sqrt{d + e x^2}} dx \rightarrow \frac{\operatorname{ArcSinh}\left[\frac{\sqrt{e} x}{\sqrt{d}}\right] (a + b \operatorname{Log}[c x^n])}{\sqrt{e}} - \frac{b n}{\sqrt{e}} \int \frac{\operatorname{ArcSinh}\left[\frac{\sqrt{e} x}{\sqrt{d}}\right]}{x} dx$$

Program code:

```
Int[(a_.+b_.*Log[c_.*x^n_.])/Sqrt[d_+e_.*x^2],x_Symbol] :=
  ArcSinh[Rt[e,2]*x/Sqrt[d]]*(a+b*Log[c*x^n])/Rt[e,2] - b*n/Rt[e,2]*Int[ArcSinh[Rt[e,2]*x/Sqrt[d]]/x,x] /;
FreeQ[{a,b,c,d,e,n},x] && GtQ[d,0] && PosQ[e]
```


2: $\int \frac{a + b \operatorname{Log}[c x^n]}{\sqrt{d + e x^2}} dx$ when $d > 0 \wedge e \neq 0$

Derivation: Integration by parts

■

Basis: If $d > 0$, then $\frac{1}{\sqrt{d + e x^2}} = \partial_x \frac{\operatorname{ArcSin}\left[\frac{\sqrt{-e} x}{\sqrt{d}}\right]}{\sqrt{-e}}$

Rule: If $d > 0 \wedge e \neq 0$, then

$$\int \frac{a + b \operatorname{Log}[c x^n]}{\sqrt{d + e x^2}} dx \rightarrow \frac{\operatorname{ArcSin}\left[\frac{\sqrt{-e} x}{\sqrt{d}}\right] (a + b \operatorname{Log}[c x^n])}{\sqrt{-e}} - \frac{b n}{\sqrt{-e}} \int \frac{\operatorname{ArcSin}\left[\frac{\sqrt{-e} x}{\sqrt{d}}\right]}{x} dx$$

Program code:

```
Int[(a_.+b_.*Log[c_.*x^n_.])/Sqrt[d_+e_.*x^2],x_Symbol] :=
  ArcSin[Rt[-e,2]*x/Sqrt[d]]*(a+b*Log[c*x^n])/Rt[-e,2] - b*n/Rt[-e,2]*Int[ArcSin[Rt[-e,2]*x/Sqrt[d]]/x,x] /;
FreeQ[{a,b,c,d,e,n},x] && GtQ[d,0] && NegQ[e]
```

2: $\int \frac{a + b \operatorname{Log}[c x^n]}{\sqrt{d + e x^2}} dx$ when $d \neq 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{\sqrt{1 + \frac{e}{d} x^2}}{\sqrt{d + e x^2}} = 0$

Rule: If $d \neq 0$, then

$$\int \frac{a + b \operatorname{Log}[c x^n]}{\sqrt{d + e x^2}} dx \rightarrow \frac{\sqrt{1 + \frac{e}{d} x^2}}{\sqrt{d + e x^2}} \int \frac{a + b \operatorname{Log}[c x^n]}{\sqrt{1 + \frac{e}{d} x^2}} dx$$

Program code:

```
Int[(a_.+b_.*Log[c_.**x_^n_.])/Sqrt[d_+e_.**x_^2],x_Symbol] :=
  Sqrt[1+e/d**x^2]/Sqrt[d+e**x^2]*Int[(a+b*Log[c*x^n])/Sqrt[1+e/d**x^2],x] /;
FreeQ[{a,b,c,d,e,n},x] && Not[GtQ[d,0]]
```

```
Int[(a_.+b_.*Log[c_.**x_^n_.])/(Sqrt[d1_+e1_.**x_] * Sqrt[d2_+e2_.**x_]),x_Symbol] :=
  Sqrt[1+e1*e2/(d1*d2)**x^2]/(Sqrt[d1+e1*x]*Sqrt[d2+e2*x])*Int[(a+b*Log[c*x^n])/Sqrt[1+e1*e2/(d1*d2)**x^2],x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,n},x] && EqQ[d2*e1+d1*e2,0]
```

5: $\int (d + e x^r)^q (a + b \log[c x^n]) dx$ when $2q \in \mathbb{Z} \wedge r \in \mathbb{Z}$

Derivation: Integration by parts

Basis: $\partial_x (a + b \log[c x^n]) = \frac{b n}{x}$

Note: If $q - \frac{1}{2} \in \mathbb{Z}$, then the terms of $\int (d + e x)^q dx$ will be algebraic functions or constants times an inverse function.

Rule: If $2q \in \mathbb{Z} \wedge r \in \mathbb{Z}$, let $u \rightarrow \int (d + e x^r)^q dx$, then

$$\int (d + e x^r)^q (a + b \log[c x^n]) dx \rightarrow u (a + b \log[c x^n]) - b n \int \frac{u}{x} dx$$

Program code:

```
Int[(d_+e_.**x_^r_.)^q_.*(a_.+b_.*Log[c_.**x_^n_.]),x_Symbol] :=
  With[{u=IntHide[(d+e*x^r)^q,x]},
    Dist[(a+b*Log[c*x^n]),u,x] - b*n*Int[SimplifyIntegrand[u/x,x],x] /;
    EqQ[r,1] && IntegerQ[q-1/2] || EqQ[r,2] && EqQ[q,-1] || InverseFunctionFreeQ[u,x] /;
    FreeQ[{a,b,c,d,e,n,q,r},x] && IntegerQ[2*q] && IntegerQ[r]
```

6: $\int (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx$ when $q \in \mathbb{Z} \wedge (q > 0 \vee p \in \mathbb{Z}^+ \wedge r \in \mathbb{Z})$

Derivation: Algebraic expansion

Rule: If $q \in \mathbb{Z} \wedge (q > 0 \vee p \in \mathbb{Z}^+ \wedge r \in \mathbb{Z})$, then

$$\int (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx \rightarrow \int (a + b \operatorname{Log}[c x^n])^p \operatorname{ExpandIntegrand}[(d + e x^r)^q, x] dx$$

Program code:

```
Int[(d_+e_.*x_^r_)^q_.*(a_+b_.*Log[c_.*x_^n_] )^p_,x_Symbol] :=
  With[{u=ExpandIntegrand[(a+b*Log[c*x^n])^p,(d+e*x^r)^q,x]},
    Int[u,x] /;
    SumQ[u] /;
    FreeQ[{a,b,c,d,e,n,p,q,r},x] && IntegerQ[q] && (GtQ[q,0] || IGtQ[p,0] && IntegerQ[r])
```

U: $\int (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx$

Rule:

$$\int (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx \rightarrow \int (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx$$

Program code:

```
Int[(d_+e_.*x_^r_)^q_.*(a_+b_.*Log[c_.*x_^n_] )^p_,x_Symbol] :=
  Unintegrable[(d+e*x^r)^q*(a+b*Log[c*x^n])^p,x] /;
  FreeQ[{a,b,c,d,e,n,p,q,r},x]
```

N: $\int u^q (a + b \operatorname{Log}[c x^n])^p dx$ when $u = d + e x^r$

Derivation: Algebraic normalization

Rule: If $u = d + e x^r$, then

$$\int u^q (a + b \operatorname{Log}[c x^n])^p dx \rightarrow \int (d + e x)^q (a + b \operatorname{Log}[c x^n])^p dx$$

Program code:

```
Int[u_^q_.*(a_.+b_.*Log[c_.*x_^n_.])^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^q*(a+b*Log[c*x^n])^p,x] /;
FreeQ[{a,b,c,n,p,q},x] && BinomialQ[u,x] && Not[BinomialMatchQ[u,x]]
```

4. $\int (f x)^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx$

O: $\int x^m \left(d + \frac{e}{x}\right)^q (a + b \operatorname{Log}[c x^n])^p dx$ when $m = q \wedge q \in \mathbb{Z}$

Derivation: Algebraic simplification

Rule: If $m = q \wedge q \in \mathbb{Z}$, then

$$\int x^m \left(d + \frac{e}{x}\right)^q (a + b \operatorname{Log}[c x^n])^p dx \rightarrow \int (e + d x)^q (a + b \operatorname{Log}[c x^n])^p dx$$

Program code:

```
Int[x_^m_.*(d+e_/x_)^q_.*(a_.+b_.*Log[c_.*x_^n_.])^p_,x_Symbol] :=
  Int[(e+d*x)^q*(a+b*Log[c*x^n])^p,x] /;
FreeQ[{a,b,c,d,e,m,n,p},x] && EqQ[m,q] && IntegerQ[q]
```

1: $\int x^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n]) dx$ when $q \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}$

Derivation: Integration by parts

Basis: $\partial_x (a + b \operatorname{Log}[c x^n]) = \frac{b n}{x}$

Rule: If $q \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}$, let $u \rightarrow \int x^m (d + e x^r)^q dx$, then

$$\int x^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n]) dx \rightarrow u (a + b \operatorname{Log}[c x^n]) - b n \int \frac{u}{x} dx$$

Program code:

```
Int[x^m_.*(d_+e_.*x_^r_.)^q_.*(a_.+b_.*Log[c_.*x_^n_.]),x_Symbol] :=
  With[{u=IntHide[x^m_.*(d_+e_.*x_^r_.)^q_,x]},
    u*(a+b*Log[c*x^n]) - b*n*Int[SimplifyIntegrand[u/x,x],x] /;
  FreeQ[{a,b,c,d,e,n,r},x] && IGtQ[q,0] && IntegerQ[m] && Not[EqQ[q,1] && EqQ[m,-1]]
```

2: $\int (f x)^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n]) dx$ when $m + r (q + 1) + 1 = 0 \wedge m \neq -1$

Derivation: Integration by parts

■

Basis: If $m + r (q + 1) + 1 = 0 \wedge m \neq -1$, then $(f x)^m (d + e x^r)^q = \partial_x \frac{(f x)^{m+1} (d + e x^r)^{q+1}}{d f (m+1)}$

Rule: If $m + r (q + 1) + 1 = 0 \wedge m \neq -1$, then

$$\int (f x)^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n]) dx \rightarrow \frac{(f x)^{m+1} (d + e x^r)^{q+1} (a + b \operatorname{Log}[c x^n])}{d f (m+1)} - \frac{b n}{d (m+1)} \int (f x)^m (d + e x^r)^{q+1} dx$$

Program code:

```
Int[(f_.**x_)^m_.*(d_+e_.**x_^r_.)^q_*(a_.+b_.**Log[c_.**x_^n_.]),x_Symbol] :=
  (f*x)^(m+1)*(d+e*x^r)^(q+1)*(a+b*Log[c*x^n])/(d*f*(m+1)) -
  b*n/(d*(m+1))*Int[(f*x)^m*(d+e*x^r)^(q+1),x] /;
FreeQ[{a,b,c,d,e,f,m,n,q,r},x] && EqQ[m+r*(q+1)+1,0] && NeQ[m,-1]
```

$$3. \int (f x)^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx \text{ when } m = r - 1 \wedge p \in \mathbb{Z}^+$$

$$1. \int (f x)^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx \text{ when } m = r - 1 \wedge p \in \mathbb{Z}^+ \wedge (m \in \mathbb{Z} \vee f > 0)$$

$$1: \int (f x)^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx \text{ when } m = r - 1 \wedge p \in \mathbb{Z}^+ \wedge (m \in \mathbb{Z} \vee f > 0) \wedge r = n$$

Derivation: Integration by substitution

Rule: If $m = r - 1 \wedge p \in \mathbb{Z}^+ \wedge (m \in \mathbb{Z} \vee f > 0) \wedge r = n$, then

$$\int (f x)^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx \rightarrow \frac{f^m}{n} \operatorname{Subst}\left[\int (d + e x)^q (a + b \operatorname{Log}[c x])^p dx, x, x^n\right]$$

Program code:

```
Int[(f_.*x_)^m_.*(d+_e_.*x_^r_)^q_.*(a+_b_.*Log[c_.*x_^n_])^p_.,x_Symbol] :=
  f^m/n*Subst[Int[(d+e*x)^q*(a+b*Log[c*x])^p,x],x,x^n] /;
FreeQ[{a,b,c,d,e,f,m,n,q,r},x] && EqQ[m,r-1] && IGtQ[p,0] && (IntegerQ[m] || GtQ[f,0]) && EqQ[r,n]
```

$$2. \int (f x)^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx \text{ when } m = r - 1 \wedge p \in \mathbb{Z}^+ \wedge (m \in \mathbb{Z} \vee f > 0) \wedge r \neq n$$

$$1: \int \frac{(f x)^m (a + b \operatorname{Log}[c x^n])^p}{d + e x^r} dx \text{ when } m = r - 1 \wedge p \in \mathbb{Z}^+ \wedge (m \in \mathbb{Z} \vee f > 0) \wedge r \neq n$$

Derivation: Integration by parts

■ Basis: $\frac{(f x)^m}{d + e x^r} = \frac{f^m}{e r} \partial_x \operatorname{Log}\left[1 + \frac{e x^r}{d}\right]$

Rule: If $m = r - 1 \wedge p \in \mathbb{Z}^+ \wedge (m \in \mathbb{Z} \vee f > 0) \wedge r \neq n$, then

$$\int \frac{(f x)^m (a + b \operatorname{Log}[c x^n])^p}{d + e x^r} dx \rightarrow \frac{f^m \operatorname{Log}\left[1 + \frac{e x^r}{d}\right] (a + b \operatorname{Log}[c x^n])^p}{e r} - \frac{b f^m n p}{e r} \int \frac{\operatorname{Log}\left[1 + \frac{e x^r}{d}\right] (a + b \operatorname{Log}[c x^n])^{p-1}}{x} dx$$

Program code:

```
Int[(f_.x_)^m_.*(a_.+b_.*Log[c_.x_^n_.])^p_./(d_+e_.x_^r_),x_Symbol] :=
  f^m*Log[1+e*x^r/d]*(a+b*Log[c*x^n])^p/(e*r) -
  b*f^m*n*p/(e*r)*Int[Log[1+e*x^r/d]*(a+b*Log[c*x^n])^(p-1)/x,x] /;
FreeQ[{a,b,c,d,e,f,m,n,r},x] && EqQ[m,r-1] && IGtQ[p,0] && (IntegerQ[m] || GtQ[f,0]) && NeQ[r,n]
```

2: $\int (f x)^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx$ when $m = r - 1 \wedge p \in \mathbb{Z}^+ \wedge (m \in \mathbb{Z} \vee f > 0) \wedge r \neq n \wedge q \neq -1$

Derivation: Integration by parts

Rule: If $m = r - 1 \wedge p \in \mathbb{Z}^+ \wedge (m \in \mathbb{Z} \vee f > 0) \wedge r \neq n \wedge q \neq -1$, then

$$\int (f x)^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx \rightarrow \frac{f^m (d + e x^r)^{q+1} (a + b \operatorname{Log}[c x^n])^p}{e r (q + 1)} - \frac{b f^m n p}{e r (q + 1)} \int \frac{(d + e x^r)^{q+1} (a + b \operatorname{Log}[c x^n])^{p-1}}{x} dx$$

Program code:

```
Int[(f_.x_)^m_.*(d_+e_.x_^r_)^q_.*(a_.+b_.*Log[c_.x_^n_.])^p_. ,x_Symbol] :=
  f^m*(d+e*x^r)^(q+1)*(a+b*Log[c*x^n])^p/(e*r*(q+1)) -
  b*f^m*n*p/(e*r*(q+1))*Int[(d+e*x^r)^(q+1)*(a+b*Log[c*x^n])^(p-1)/x,x] /;
FreeQ[{a,b,c,d,e,f,m,n,q,r},x] && EqQ[m,r-1] && IGtQ[p,0] && (IntegerQ[m] || GtQ[f,0]) && NeQ[r,n] && NeQ[q,-1]
```

2: $\int (f x)^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx$ when $m = r - 1 \wedge p \in \mathbb{Z}^+ \wedge \neg (m \in \mathbb{Z} \vee f > 0)$

Derivation: Piecewise constant extraction

Rule: If $m = r - 1 \wedge p \in \mathbb{Z}^+ \wedge \neg (m \in \mathbb{Z} \vee f > 0)$, then

$$\int (f x)^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx \rightarrow \frac{(f x)^m}{x^m} \int x^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx$$

Program code:

```
Int[(f*x_)^m_.*(d_+e_*x_^r_)^q_.*(a_+b_*Log[c_*x_^n_.])^p_,x_Symbol] :=
  (f*x)^m/x^m*Int[x^m*(d+e*x^r)^q*(a+b*Log[c*x^n])^p,x] /;
FreeQ[{a,b,c,d,e,f,m,n,q,r},x] && EqQ[m,r-1] && IGtQ[p,0] && Not[(IntegerQ[m] || GtQ[f,0])]
```

4: $\int (f x)^m (d + e x^2)^q (a + b \operatorname{Log}[c x^n]) dx$ when $q < -1$

Rule: If $q < -1$, then

$$\int (f x)^m (d + e x^2)^q (a + b \operatorname{Log}[c x^n]) dx \rightarrow$$

$$- \frac{(f x)^{m+1} (d + e x^2)^{q+1} (a + b \operatorname{Log}[c x^n])}{2 d f (q+1)} + \frac{1}{2 d (q+1)} \int (f x)^m (d + e x^2)^{q+1} (a (m+2q+3) + b n + b (m+2q+3) \operatorname{Log}[c x^n]) dx$$

Program code:

```
Int[(f_*x_)^m_.*(d_+e_*x_^2)^q_.*(a_+b_*Log[c_*x_^n_.]),x_Symbol] :=
  - (f*x)^(m+1)*(d+e*x^2)^(q+1)*(a+b*Log[c*x^n])/(2*d*f*(q+1)) +
  1/(2*d*(q+1))*Int[(f*x)^m*(d+e*x^2)^(q+1)*(a*(m+2*q+3)+b*n+b*(m+2*q+3)*Log[c*x^n]),x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && ILtQ[q,-1] && ILtQ[m,0]
```

5: $\int x^m (d + e x^2)^q (a + b \operatorname{Log}[c x^n]) dx$ when $\frac{m}{2} \in \mathbb{Z} \wedge q - \frac{1}{2} \in \mathbb{Z} \wedge \neg (m + 2q < -2 \vee d > 0)$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(d+e x^2)^q}{(1+\frac{e}{d} x^2)^q} = 0$

Rule: If $\frac{m}{2} \in \mathbb{Z} \wedge q - \frac{1}{2} \in \mathbb{Z} \wedge \neg (m + 2q < -2 \vee d > 0)$, then

$$\int x^m (d + e x^2)^q (a + b \operatorname{Log}[c x^n]) dx \rightarrow \frac{d^{\operatorname{IntPart}[q]} (d + e x^2)^{\operatorname{FracPart}[q]}}{(1 + \frac{e}{d} x^2)^{\operatorname{FracPart}[q]}} \int x^m \left(1 + \frac{e}{d} x^2\right)^q (a + b \operatorname{Log}[c x^n]) dx$$

Program code:

```
Int[x^m_.*(d_+e_.*x_^2)^q_.*(a_.+b_.*Log[c_.*x_^n_.]),x_Symbol] :=
  d^IntPart[q]*(d+e*x^2)^FracPart[q]/(1+e/d*x^2)^FracPart[q]*Int[x^m*(1+e/d*x^2)^q*(a+b*Log[c*x^n]),x] /;
FreeQ[{a,b,c,d,e,n},x] && IntegerQ[m/2] && IntegerQ[q-1/2] && Not[LtQ[m+2*q,-2] || GtQ[d,0]]
```

```
Int[x^m_.*(d1_+e1_.*x_)^q_.*(d2_+e2_.*x_)^q_.*(a_.+b_.*Log[c_.*x_^n_.]),x_Symbol] :=
  (d1+e1*x)^q*(d2+e2*x)^q/(1+e1*e2/(d1*d2)*x^2)^q*Int[x^m*(1+e1*e2/(d1*d2)*x^2)^q*(a+b*Log[c*x^n]),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,n},x] && EqQ[d2*e1+d1*e2,0] && IntegerQ[m] && IntegerQ[q-1/2]
```

$$6. \int \frac{(d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p}{x} dx \text{ when } p \in \mathbb{Z}^+$$

$$1. \int \frac{(a + b \operatorname{Log}[c x^n])^p}{x (d + e x^r)} dx \text{ when } p \in \mathbb{Z}^+$$

$$1: \int \frac{a + b \operatorname{Log}[c x^n]}{x (d + e x^r)} dx \text{ when } \frac{r}{n} \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } \frac{F[x^n]}{x} == \frac{1}{n} \operatorname{Subst}\left[\frac{F[x]}{x}, x, x^n\right] \partial_x x^n$$

Rule: If $\frac{r}{n} \in \mathbb{Z}$, then

$$\int \frac{a + b \operatorname{Log}[c x^n]}{x (d + e x^r)} dx \rightarrow \frac{1}{n} \operatorname{Subst}\left[\int \frac{a + b \operatorname{Log}[c x]}{x (d + e x^{r/n})} dx, x, x^n\right]$$

Program code:

```
Int[(a_.+b_.*Log[c_.*x_^n_.])/(x_*(d_+e_.*x_^r_.)),x_Symbol] :=
  1/n*Subst[Int[(a+b*Log[c*x])/(x*(d+e*x^(r/n))),x],x,x^n] /;
FreeQ[{a,b,c,d,e,n,r},x] && IntegerQ[r/n]
```

$$2: \int \frac{(a + b \operatorname{Log}[c x^n])^p}{x (d + e x)} dx \text{ when } p \in \mathbb{Z}^+$$

Rule: Algebraic expansion

$$\text{Basis: } \frac{1}{x (d + e x)} == \frac{1}{d x} - \frac{e}{d (d + e x)}$$

Rule: If $p \in \mathbb{Z}^+$, then

$$\int \frac{(a + b \operatorname{Log}[c x^n])^p}{x (d + e x)} dx \rightarrow \frac{1}{d} \int \frac{(a + b \operatorname{Log}[c x^n])^p}{x} dx - \frac{e}{d} \int \frac{(a + b \operatorname{Log}[c x^n])^p}{d + e x} dx$$

Program code:

```
Int[(a_.+b_.*Log[c_.**x_^n_.])^p_./(x_*(d_+e_.**x_)),x_Symbol] :=
  1/d*Int[(a+b*Log[c*x^n])^p/x,x] - e/d*Int[(a+b*Log[c*x^n])^p/(d+e*x),x] /;
FreeQ[{a,b,c,d,e,n},x] && IGtQ[p,0]
```

$$\text{■ x: } \int \frac{(a + b \operatorname{Log}[c x^n])^p}{x (d + e x^r)} dx \text{ when } p \in \mathbb{Z}^+$$

Rule: Integration by parts

$$\text{■ Basis: } \frac{1}{x (d + e x^r)} = \partial_x \frac{r \operatorname{Log}[x] - \operatorname{Log}\left[1 + \frac{e x^r}{d}\right]}{d r}$$

$$\text{Basis: } \partial_x (a + b \operatorname{Log}[c x^n])^p = \frac{b n p (a + b \operatorname{Log}[c x^n])^{p-1}}{x}$$

Note: This rule returns antiderivatives in terms of x^r instead of x^{-r} , but requires more steps and larger antiderivatives.

Rule: If $p \in \mathbb{Z}^+$, then

$$\int \frac{(a + b \operatorname{Log}[c x^n])^p}{x (d + e x^r)} dx \rightarrow \frac{\left(r \operatorname{Log}[x] - \operatorname{Log}\left[1 + \frac{e x^r}{d}\right]\right) (a + b \operatorname{Log}[c x^n])^p}{d r} - \frac{b n p}{d} \int \frac{\operatorname{Log}[x] (a + b \operatorname{Log}[c x^n])^{p-1}}{x} dx + \frac{b n p}{d r} \int \frac{\operatorname{Log}\left[1 + \frac{e x^r}{d}\right] (a + b \operatorname{Log}[c x^n])^{p-1}}{x} dx$$

Program code:

```
(* Int[(a_+b_*Log[c_*x_^n_])^p_/(x_*(d_+e_*x_^r_)),x_Symbol] :=
  (r*Log[x]-Log[1+(e*x^r)/d])*(a+b*Log[c*x^n])^p/(d*r) -
  b*n*p/d*Int[Log[x]*(a+b*Log[c*x^n])^(p-1)/x,x] +
  b*n*p/(d*r)*Int[Log[1+(e*x^r)/d]*(a+b*Log[c*x^n])^(p-1)/x,x] /;
FreeQ[{a,b,c,d,e,n,r},x] && IGtQ[p,0] *)
```

$$\mathbf{3:} \int \frac{(a + b \operatorname{Log}[c x^n])^p}{x (d + e x^r)} dx \text{ when } p \in \mathbb{Z}^+$$

Rule: Integration by parts

$$\text{Basis: } \frac{1}{x (d + e x^r)} == -\frac{1}{d r} \partial_x \operatorname{Log}\left[1 + \frac{d}{e x^r}\right]$$

Rule: If $p \in \mathbb{Z}^+$, then

$$\int \frac{(a + b \operatorname{Log}[c x^n])^p}{x (d + e x^r)} dx \rightarrow -\frac{\operatorname{Log}\left[1 + \frac{d}{e x^r}\right] (a + b \operatorname{Log}[c x^n])^p}{d r} + \frac{b n p}{d r} \int \frac{\operatorname{Log}\left[1 + \frac{d}{e x^r}\right] (a + b \operatorname{Log}[c x^n])^{p-1}}{x} dx$$

Program code:

```
Int[(a_.+b_.*Log[c_.*x_^n_.])^p_./(x_*(d_+e_.*x_^r_.)),x_Symbol] :=
  -Log[1+d/(e*x^r)]*(a+b*Log[c*x^n])^p/(d*r) +
  b*n*p/(d*r)*Int[Log[1+d/(e*x^r)]*(a+b*Log[c*x^n])^(p-1)/x,x] /;
FreeQ[{a,b,c,d,e,n,r},x] && IGtQ[p,0]
```

$$2. \int \frac{(d + e x)^q (a + b \operatorname{Log}[c x^n])^p}{x} dx \text{ when } p \in \mathbb{Z}^+$$

$$\mathbf{1:} \int \frac{(d + e x)^q (a + b \operatorname{Log}[c x^n])^p}{x} dx \text{ when } p \in \mathbb{Z}^+ \wedge q > 0$$

Rule: Algebraic expansion

$$\text{Basis: } \frac{(d + e x)^q}{x} == \frac{d (d + e x)^{q-1}}{x} + e (d + e x)^{q-1}$$

Rule: If $p \in \mathbb{Z}^+ \wedge q > 0$, then

$$\int \frac{(d+ex)^q (a+b \log[cx^n])^p}{x} dx \rightarrow d \int \frac{(d+ex)^{q-1} (a+b \log[cx^n])^p}{x} dx + e \int (d+ex)^{q-1} (a+b \log[cx^n])^p dx$$

Program code:

```
Int[(d+_.*x_)^q_.*(a+_.*Log[c_*x_^n_.])^p_/x_,x_Symbol] :=
  d*Int[(d+e*x)^(q-1)*(a+b*Log[c*x^n])^p/x,x] +
  e*Int[(d+e*x)^(q-1)*(a+b*Log[c*x^n])^p,x] /;
FreeQ[{a,b,c,d,e,n},x] && IGtQ[p,0] && GtQ[q,0] && IntegerQ[2*q]
```

2: $\int \frac{(d+ex)^q (a+b \log[cx^n])^p}{x} dx$ when $p \in \mathbb{Z}^+ \wedge q < -1$

Rule: Algebraic expansion

Basis: $\frac{(d+ex)^q}{x} == \frac{(d+ex)^{q+1}}{d x} - \frac{e (d+ex)^q}{d}$

Rule: If $p \in \mathbb{Z}^+ \wedge q < -1$, then

$$\int \frac{(d+ex)^q (a+b \log[cx^n])^p}{x} dx \rightarrow \frac{1}{d} \int \frac{(d+ex)^{q+1} (a+b \log[cx^n])^p}{x} dx - \frac{e}{d} \int (d+ex)^q (a+b \log[cx^n])^p dx$$

Program code:

```
Int[(d+_.*x_)^q_.*(a+_.*Log[c_*x_^n_.])^p_/x_,x_Symbol] :=
  1/d*Int[(d+e*x)^(q+1)*(a+b*Log[c*x^n])^p/x,x] -
  e/d*Int[(d+e*x)^q*(a+b*Log[c*x^n])^p,x] /;
FreeQ[{a,b,c,d,e,n},x] && IGtQ[p,0] && LtQ[q,-1] && IntegerQ[2*q]
```


$$\mathbf{3:} \int \frac{(d + e x^r)^q (a + b \operatorname{Log}[c x^n])}{x} dx \text{ when } q - \frac{1}{2} \in \mathbb{Z}$$

Derivation: Integration by parts

$$\text{Basis: } \partial_x (a + b \operatorname{Log}[c x^n]) = \frac{b n}{x}$$

Rule: If $q - \frac{1}{2} \in \mathbb{Z}$, let $u \rightarrow \int \frac{(d+e x^r)^q}{x} dx$, then

$$\int \frac{(d + e x^r)^q (a + b \operatorname{Log}[c x^n])}{x} dx \rightarrow u (a + b \operatorname{Log}[c x^n]) - b n \int \frac{u}{x} dx$$

Program code:

```
Int[(d_+e_.*x_^r_)^q_.*(a_.+b_.*Log[c_.*x_^n_.])/x_,x_Symbol] :=
  With[{u=IntHide[(d+e*x^r)^q/x,x]},
    u*(a+b*Log[c*x^n]) - b*n*Int[Dist[1/x,u,x],x] /;
  FreeQ[{a,b,c,d,e,n,r},x] && IntegerQ[q-1/2]
```

4: $\int \frac{(d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p}{x} dx$ when $p \in \mathbb{Z}^+ \wedge q + 1 \in \mathbb{Z}^-$

Rule: Algebraic expansion

Basis: $\frac{(d + e x^r)^q}{x} = \frac{(d + e x^r)^{q+1}}{d x} - \frac{e x^{r-1} (d + e x^r)^q}{d}$

Rule: If $p \in \mathbb{Z}^+ \wedge q + 1 \in \mathbb{Z}^-$, then

$$\int \frac{(d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p}{x} dx \rightarrow \frac{1}{d} \int \frac{(d + e x^r)^{q+1} (a + b \operatorname{Log}[c x^n])^p}{x} dx - \frac{e}{d} \int x^{r-1} (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx$$

Program code:

```
Int[(d_+e_.*x_^r_.)^q_*(a_+b_.*Log[c_.*x_^n_.])^p_/x_,x_Symbol] :=
  1/d*Int[(d+e*x^r)^(q+1)*(a+b*Log[c*x^n])^p/x,x] -
  e/d*Int[x^(r-1)*(d+e*x^r)^q*(a+b*Log[c*x^n])^p,x] /;
FreeQ[{a,b,c,d,e,n,r},x] && IGtQ[p,0] && ILtQ[q,-1]
```

7: $\int (f x)^m (d + e x^r)^q (a + b \log[c x^n]) dx$ when $m \in \mathbb{Z} \wedge 2q \in \mathbb{Z} \wedge r \in \mathbb{Z}$

Derivation: Integration by parts

Basis: $\partial_x (a + b \log[c x^n]) = \frac{b n}{x}$

Note: If $m \in \mathbb{Z} \wedge q - \frac{1}{2} \in \mathbb{Z}$, then the terms of $\int x^m (d + e x)^q dx$ will be algebraic functions or constants times an inverse function.

Rule: If $m \in \mathbb{Z} \wedge 2q \in \mathbb{Z} \wedge r \in \mathbb{Z}$, let $u \rightarrow \int (f x)^m (d + e x^r)^q dx$, then

$$\int (f x)^m (d + e x^r)^q (a + b \log[c x^n]) dx \rightarrow u (a + b \log[c x^n]) - b n \int \frac{u}{x} dx$$

Program code:

```
Int[(f_.x_)^m_.*(d+e_.x_^r_.)^q_.*(a_.+b_.Log[c_.x_^n_.]),x_Symbol] :=
  With[{u=IntHide[(f*x)^m*(d+e*x^r)^q,x]},
    Dist[(a+b*Log[c*x^n]),u,x] - b*n*Int[SimplifyIntegrand[u/x,x],x] /;
    (EqQ[r,1] || EqQ[r,2]) && IntegerQ[m] && IntegerQ[q-1/2] || InverseFunctionFreeQ[u,x] /;
    FreeQ[{a,b,c,d,e,f,m,n,q,r},x] && IntegerQ[2*q] && (IntegerQ[m] && IntegerQ[r] || IGtQ[q,0])
```

8: $\int (f x)^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n]) dx$ when $q \in \mathbb{Z} \wedge (q > 0 \vee m \in \mathbb{Z} \wedge r \in \mathbb{Z})$

Derivation: Algebraic expansion

Rule: If $q \in \mathbb{Z} \wedge (q > 0 \vee m \in \mathbb{Z} \wedge r \in \mathbb{Z})$, then

$$\int (f x)^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n]) dx \rightarrow \int (a + b \operatorname{Log}[c x^n]) \operatorname{ExpandIntegrand}[(f x)^m (d + e x^r)^q, x] dx$$

Program code:

```
Int[(f_.*x_)^m_.*(d+_e_.*x_^r_.)^q_.*(a_.+b_.*Log[c_.*x_^n_.]),x_Symbol] :=
  With[{u=ExpandIntegrand[(a+b*Log[c*x^n]), (f*x)^m*(d+e*x^r)^q,x]},
    Int[u,x] /;
    SumQ[u] /;
    FreeQ[{a,b,c,d,e,f,m,n,q,r},x] && IntegerQ[q] && (GtQ[q,0] || IntegerQ[m] && IntegerQ[r])
```

9: $\int x^m (d + e x^r)^q (a + b \log[c x^n])^p dx$ when $q \in \mathbb{Z} \wedge \frac{r}{n} \in \mathbb{Z} \wedge \frac{m+1}{n} \in \mathbb{Z} \wedge \left(\frac{m+1}{n} > 0 \vee p \in \mathbb{Z}^+\right)$

Derivation: Integration by substitution

– Basis: If $\frac{m+1}{n} \in \mathbb{Z}$, then $x^m F[x^n] = \frac{1}{n} \text{Subst}\left[x^{\frac{m+1}{n}-1} F[x], x, x^n\right] \partial_x x^n$

Rule: If $q \in \mathbb{Z} \wedge \frac{r}{n} \in \mathbb{Z} \wedge \frac{m+1}{n} \in \mathbb{Z} \wedge \left(\frac{m+1}{n} > 0 \vee p \in \mathbb{Z}^+\right)$, then

$$\int x^m (d + e x^r)^q (a + b \log[c x^n])^p dx \rightarrow \frac{1}{n} \text{Subst}\left[\int x^{\frac{m+1}{n}-1} (d + e x^{\frac{r}{n}})^q (a + b \log[c x])^p dx, x, x^n\right]$$

Program code:

```
Int[x_^m_.*(d_+e_.*x_^r_)^q_.*(a_+b_.*Log[c_.*x_^n_])^p_,x_Symbol] :=
  1/n*Subst[Int[x^(Simplify[(m+1)/n]-1)*(d+e*x^(r/n))^q*(a+b*Log[c*x])^p,x],x,x^n] /;
FreeQ[{a,b,c,d,e,m,n,p,q,r},x] && IntegerQ[q] && IntegerQ[r/n] && IntegerQ[Simplify[(m+1)/n]] && (GtQ[(m+1)/n,0] || IGtQ[p,0])
```

10: $\int (f x)^m (d + e x^r)^q (a + b \log[c x^n])^p dx$ when $q \in \mathbb{Z} \wedge (q > 0 \vee p \in \mathbb{Z}^+ \wedge m \in \mathbb{Z} \wedge r \in \mathbb{Z})$

Derivation: Algebraic expansion

Rule: If $q \in \mathbb{Z} \wedge (q > 0 \vee p \in \mathbb{Z}^+ \wedge m \in \mathbb{Z} \wedge r \in \mathbb{Z})$, then

$$\int (f x)^m (d + e x^r)^q (a + b \log[c x^n])^p dx \rightarrow \int (a + b \log[c x^n])^p \text{ExpandIntegrand}[(f x)^m (d + e x^r)^q, x] dx$$

Program code:

```
Int[(f_.*x_)^m_.*(d_+e_.*x_^r_)^q_.*(a_+b_.*Log[c_.*x_^n_.])^p_,x_Symbol] :=
  With[{u=ExpandIntegrand[(a+b*Log[c*x^n])^p,(f*x)^m*(d+e*x^r)^q,x]},
    Int[u,x] /;
    SumQ[u] /;
    FreeQ[{a,b,c,d,e,f,m,n,p,q,r},x] && IntegerQ[q] && (GtQ[q,0] || IGtQ[p,0] && IntegerQ[m] && IntegerQ[r])
```

U: $\int (f x)^m (d + e x^r)^q (a + b \log[c x^n])^p dx$

Rule:

$$\int (f x)^m (d + e x^r)^q (a + b \log[c x^n])^p dx \rightarrow \int (f x)^m (d + e x^r)^q (a + b \log[c x^n])^p dx$$

Program code:

```
Int[(f_.*x_)^m_.*(d_+e_.*x_^r_)^q_.*(a_+b_.*Log[c_.*x_^n_.])^p_,x_Symbol] :=
  Unintegrable[(f*x)^m*(d+e*x^r)^q*(a+b*Log[c*x^n])^p,x] /;
  FreeQ[{a,b,c,d,e,f,m,n,p,q,r},x]
```

N: $\int (f x)^m u^q (a + b \operatorname{Log}[c x^n])^p dx$ when $u = d + e x^r$

Derivation: Algebraic normalization

Rule: If $u = d + e x^r$, then

$$\int (f x)^m u^q (a + b \operatorname{Log}[c x^n])^p dx \rightarrow \int (f x)^m (d + e x^r)^q (a + b \operatorname{Log}[c x^n])^p dx$$

Program code:

```
Int[(f_.*x_)^m_.*u_^q_.*(a_.+b_.*Log[c_.*x_^n_.])^p_,x_Symbol] :=
  Int[(f*x)^m*ExpandToSum[u,x]^q*(a+b*Log[c*x^n])^p,x] /;
FreeQ[{a,b,c,f,m,n,p,q},x] && BinomialQ[u,x] && Not[BinomialMatchQ[u,x]]
```

5. $\int A F[x] (a + b \operatorname{Log}[c x^n])^p dx$

1: $\int \operatorname{Poly}[x] (a + b \operatorname{Log}[c x^n])^p dx$

Derivation: Algebraic expansion

Rule:

$$\int \operatorname{Poly}[x] (a + b \operatorname{Log}[c x^n])^p dx \rightarrow \int \operatorname{ExpandIntegrand}[\operatorname{Poly}[x] (a + b \operatorname{Log}[c x^n])^p, x] dx$$

Program code:

```
Int[Polym*(a_.+b_.*Log[c_.*x_^n_.])^p_,x_Symbol] :=
  Int[ExpandIntegrand[Polym*(a+b*Log[c*x^n])^p,x],x] /;
FreeQ[{a,b,c,n,p},x] && PolynomialQ[Polym,x]
```

2: $\int \text{RF}[x] (a + b \log[c x^n])^p dx$ when $p \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $p \in \mathbb{Z}^+$, then

$$\int \text{RF}[x] (a + b \log[c x^n])^p dx \rightarrow \int (a + b \log[c x^n])^p \text{ExpandIntegrand}[\text{RF}[x], x] dx$$

Program code:

```
Int[RFx_*(a_.+b_.*Log[c_.*x_^n_.])^p_.,x_Symbol] :=
  With[{u=ExpandIntegrand[(a+b*Log[c*x^n])^p,RFx,x]},
    Int[u,x] /;
    SumQ[u] /;
    FreeQ[{a,b,c,n},x] && RationalFunctionQ[RFx,x] && IGtQ[p,0]
```

```
Int[RFx_*(a_.+b_.*Log[c_.*x_^n_.])^p_.,x_Symbol] :=
  With[{u=ExpandIntegrand[RFx*(a+b*Log[c*x^n])^p,x]},
    Int[u,x] /;
    SumQ[u] /;
    FreeQ[{a,b,c,n},x] && RationalFunctionQ[RFx,x] && IGtQ[p,0]
```


U: $\int \text{AF}[x] (a + b \text{Log}[c x^n])^p dx$

Rule:

$$\int \text{AF}[x] (a + b \text{Log}[c x^n])^p dx \rightarrow \int \text{AF}[x] (a + b \text{Log}[c x^n])^p dx$$

Program code:

```
Int[AFx_*(a_+b_*Log[c_*x_^n_.])^p_.,x_Symbol] :=
  Unintegrable[AFx*(a+b*Log[c*x^n])^p,x] /;
  FreeQ[{a,b,c,n,p},x] && AlgebraicFunctionQ[AFx,x,True]
```

6. $\int (a + b \text{Log}[c x^n])^p (d + e \text{Log}[f x^r])^q dx$

1: $\int (a + b \text{Log}[c x^n])^p (d + e \text{Log}[c x^n])^q dx$ when $p \in \mathbb{Z} \wedge q \in \mathbb{Z}$

Derivation: Algebraic expansion

Rule: If $p \in \mathbb{Z} \wedge q \in \mathbb{Z}$, then

$$\int (a + b \text{Log}[c x^n])^p (d + e \text{Log}[c x^n])^q dx \rightarrow \int \text{ExpandIntegrand}[(a + b \text{Log}[c x^n])^p (d + e \text{Log}[c x^n])^q, x] dx$$

Program code:

```
Int[(a_+b_*Log[c_*x_^n_.])^p_.*(d_+e_*Log[c_*x_^n_.])^q_.,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*Log[c*x^n])^p*(d+e*Log[c*x^n])^q,x],x] /;
  FreeQ[{a,b,c,d,e,n},x] && IntegerQ[p] && IntegerQ[q]
```

2: $\int (a + b \operatorname{Log}[c x^n])^p (d + e \operatorname{Log}[f x^r]) \, dx$

Derivation: Integration by parts

Rule: Let $u \rightarrow \int (a + b \operatorname{Log}[c x^n])^p \, dx$, then

$$\int (a + b \operatorname{Log}[c x^n])^p (d + e \operatorname{Log}[f x^r]) \, dx \rightarrow u (d + e \operatorname{Log}[f x^r]) - e r \int \frac{u}{x} \, dx$$

Program code:

```
Int[(a_.+b_.*Log[c_.*x_^n_.])^p_.*(d_.+e_.*Log[f_.*x_^r_.]),x_Symbol] :=
  With[{u=IntHide[(a+b*Log[c*x^n])^p,x]},
    Dist[d+e*Log[f*x^r],u,x] - e*r*Int[SimplifyIntegrand[u/x,x],x] /;
  FreeQ[{a,b,c,d,e,f,n,p,r},x]
```

3: $\int (a + b \operatorname{Log}[c x^n])^p (d + e \operatorname{Log}[f x^r])^q dx$ when $p \in \mathbb{Z}^+ \wedge q \in \mathbb{Z}^+$

Derivation: Integration by parts

Rule: If $p \in \mathbb{Z}^+ \wedge q \in \mathbb{Z}^+$, then

$$\int (a + b \operatorname{Log}[c x^n])^p (d + e \operatorname{Log}[f x^r])^q dx \rightarrow$$

$$x (a + b \operatorname{Log}[c x^n])^p (d + e \operatorname{Log}[f x^r])^q - e q r \int (a + b \operatorname{Log}[c x^n])^p (d + e \operatorname{Log}[f x^r])^{q-1} dx - b n p \int (a + b \operatorname{Log}[c x^n])^{p-1} (d + e \operatorname{Log}[f x^r])^q dx$$

Program code:

```
Int[(a_.+b_.*Log[c_.*x_^n_.])^p_.*(d_.+e_.*Log[f_.*x_^r_.])^q_.,x_Symbol] :=
  x*(a+b*Log[c*x^n])^p*(d+e*Log[f*x^r])^q -
  e*q*r*Int[(a+b*Log[c*x^n])^p*(d+e*Log[f*x^r])^(q-1),x] -
  b*n*p*Int[(a+b*Log[c*x^n])^(p-1)*(d+e*Log[f*x^r])^q,x] /;
FreeQ[{a,b,c,d,e,f,n,r},x] && IGtQ[p,0] && IGtQ[q,0]
```

U: $\int (a + b \operatorname{Log}[c x^n])^p (d + e \operatorname{Log}[f x^r])^q dx$

Rule:

$$\int (a + b \operatorname{Log}[c x^n])^p (d + e \operatorname{Log}[f x^r])^q dx \rightarrow \int (g x)^m (a + b \operatorname{Log}[c x^n])^p (d + e \operatorname{Log}[f x^r])^q dx$$

Program code:

```
Int[(a_.+b_.*Log[c_.*x_^n_.])^p_.*(d_.+e_.*Log[f_.*x_^r_.])^q_.,x_Symbol] :=
  Unintegrable[(a+b*Log[c*x^n])^p*(d+e*Log[f*x^r])^q,x] /;
FreeQ[{a,b,c,d,e,f,n,p,q,r},x]
```

S: $\int (a + b \operatorname{Log}[v])^p (c + d \operatorname{Log}[v])^q dx$ when $v = g + h x \wedge g \neq 0$

Derivation: Integration by substitution

Rule: If $v = g + h x \wedge g \neq 0$, then

$$\int (a + b \operatorname{Log}[v])^p (c + d \operatorname{Log}[v])^q dx \rightarrow \frac{1}{h} \operatorname{Subst}\left[\int (a + b \operatorname{Log}[x])^p (c + d \operatorname{Log}[x])^q dx, x, g + h x\right]$$

Program code:

```
Int[(a_.+b_.*Log[v_])^p_.*(c_.+d_.*Log[v_])^q_,x_Symbol] :=
  1/Coeff[v,x,1]*Subst[Int[(a+b*Log[x])^p*(c+d*Log[x])^q,x],x,v] /;
FreeQ[{a,b,c,d,p,q},x] && LinearQ[v,x] && NeQ[Coeff[v,x,0],0]
```

$$7. \int (g x)^m (a + b \operatorname{Log}[c x^n])^p (d + e \operatorname{Log}[f x^r])^q dx$$

$$1: \int \frac{(a + b \operatorname{Log}[c x^n])^p (d + e \operatorname{Log}[c x^n])^q}{x} dx$$

Derivation: Integration by substitution

$$\text{Basis: } \frac{F[\operatorname{Log}[c x^n]]}{x} == \frac{1}{n} \operatorname{Subst}[F[x], x, \operatorname{Log}[c x^n]] \partial_x \operatorname{Log}[c x^n]$$

Rule:

$$\int \frac{(a + b \operatorname{Log}[c x^n])^p (d + e \operatorname{Log}[c x^n])^q}{x} dx \rightarrow \frac{1}{n} \operatorname{Subst}\left[\int (a + b x)^p (d + e x)^q dx, x, \operatorname{Log}[c x^n]\right]$$

Program code:

```
Int[(a_.+b_.*Log[c_.*x_^n_.])^p_.*(d_.+e_.*Log[c_.*x_^n_.])^q_. / x_, x_Symbol] :=
  1/n*Subst[Int[(a+b*x)^p*(d+e*x)^q,x],x,Log[c*x^n]] /;
FreeQ[{a,b,c,d,e,n,p,q},x]
```

2: $\int (g x)^m (a + b \log[c x^n])^p (d + e \log[f x^r]) dx$

Derivation: Integration by parts

Rule: Let $u \rightarrow \int (g x)^m (a + b \log[c x^n])^p dx$, then

$$\int (g x)^m (a + b \log[c x^n])^p (d + e \log[f x^r]) dx \rightarrow u (d + e \log[f x^r]) - e r \int \frac{u}{x} dx$$

Program code:

```
Int[(g_.**x_)^m_.*(a_.+b_.*Log[c_.**x_^n_.])^p_.*(d_.+e_.*Log[f_.**x_^r_.]),x_Symbol] :=
  With[{u=IntHide[(g**x)^m*(a+b*Log[c**x^n])^p,x]},
    Dist[(d+e*Log[f**x^r]),u,x] - e*r*Int[SimplifyIntegrand[u/x,x],x] /;
    FreeQ[{a,b,c,d,e,f,g,m,n,p,r},x] && Not[EqQ[p,1] && EqQ[a,0] && NeQ[d,0]]
```

3: $\int (g x)^m (a + b \operatorname{Log}[c x^n])^p (d + e \operatorname{Log}[f x^r])^q dx$ when $p \in \mathbb{Z}^+ \wedge q \in \mathbb{Z}^+ \wedge m \neq -1$

Derivation: Integration by parts

Rule: If $p \in \mathbb{Z}^+ \wedge q \in \mathbb{Z}^+ \wedge m \neq -1$, then

$$\int (g x)^m (a + b \operatorname{Log}[c x^n])^p (d + e \operatorname{Log}[f x^r])^q dx \rightarrow$$

$$\frac{(g x)^{m+1} (a + b \operatorname{Log}[c x^n])^p (d + e \operatorname{Log}[f x^r])^q}{g (m+1)} -$$

$$\frac{e q r}{m+1} \int (g x)^m (a + b \operatorname{Log}[c x^n])^p (d + e \operatorname{Log}[f x^r])^{q-1} dx - \frac{b n p}{m+1} \int (g x)^m (a + b \operatorname{Log}[c x^n])^{p-1} (d + e \operatorname{Log}[f x^r])^q dx$$

Program code:

```
Int[(g_.x_)^m_.*(a_.+b_.*Log[c_.x_^n_.])^p_.*(d_.+e_.*Log[f_.x_^r_.])^q_.,x_Symbol] :=
  (g*x)^(m+1)*(a+b*Log[c*x^n])^p*(d+e*Log[f*x^r])^q/(g*(m+1)) -
  e*q*r/(m+1)*Int[(g*x)^m*(a+b*Log[c*x^n])^p*(d+e*Log[f*x^r])^(q-1),x] -
  b*n*p/(m+1)*Int[(g*x)^m*(a+b*Log[c*x^n])^(p-1)*(d+e*Log[f*x^r])^q,x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,r},x] && IGtQ[p,0] && IGtQ[q,0] && NeQ[m,-1]
```

U: $\int (g x)^m (a + b \log[c x^n])^p (d + e \log[f x^r])^q dx$

Rule:

$$\int (g x)^m (a + b \log[c x^n])^p (d + e \log[f x^r])^q dx \rightarrow \int (g x)^m (a + b \log[c x^n])^p (d + e \log[f x^r])^q dx$$

Program code:

```
Int[(g_.**x_)^m_.*(a_.+b_.*Log[c_.**x_^n_.])^p_.*(d_.+e_.*Log[f_.**x_^r_.])^q_.,x_Symbol] :=
  Unintegrable[(g*x)^m*(a+b*Log[c*x^n])^p*(d+e*Log[f*x^r])^q,x] /;
FreeQ[{a,b,c,d,e,f,g,m,n,p,q,r},x]
```

S: $\int u^m (a + b \log[v])^p (c + d \log[v])^q dx$ when $u == e + f x \wedge v == g + h x \wedge f g - e h == 0 \wedge g \neq 0$

Derivation: Integration by substitution

Rule: If $u == e + f x \wedge v == g + h x \wedge f g - e h == 0 \wedge g \neq 0$, then

$$\int u^m (a + b \log[v])^p (c + d \log[v])^q dx \rightarrow \frac{1}{h} \text{Subst}\left[\int \left(\frac{f x}{h}\right)^m (a + b \log[x])^p (c + d \log[x])^q dx, x, g + h x\right]$$

Program code:

```
Int[u_^m_.*(a_.+b_.*Log[v_])^p_.*(c_.+d_.*Log[v_])^q_.,x_Symbol] :=
  With[{e=Coeff[u,x,0],f=Coeff[u,x,1],g=Coeff[v,x,0],h=Coeff[v,x,1]},
    1/h*Subst[Int[(f*x/h)^m*(a+b*Log[x])^p*(c+d*Log[x])^q,x],x,v] /;
    EqQ[f*g-e*h,0] && NeQ[g,0] /;
    FreeQ[{a,b,c,d,m,p,q},x] && LinearQ[{u,v},x]
```


$$8. \int \text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^p dx$$

$$1: \int \text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^p dx \text{ when } p \in \mathbb{Z}^+ \wedge m \in \mathbb{R} \wedge (p == 1 \vee \frac{1}{m} \in \mathbb{Z} \vee r == 1 \wedge m == 1 \wedge d e == 1)$$

Derivation: Integration by parts

Note: If $m \in \mathbb{R}$, then $\frac{\int \text{Log}[d (e + f x^m)^r] dx}{x}$ is integrable.

Rule: If $p \in \mathbb{Z}^+ \wedge m \in \mathbb{R} \wedge (p == 1 \vee \frac{1}{m} \in \mathbb{Z} \vee r == 1 \wedge m == 1 \wedge d e == 1)$, let $u \rightarrow \int \text{Log}[d (e + f x^m)^r] dx$, then

$$\int \text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^p dx \rightarrow u (a + b \text{Log}[c x^n])^p - b n p \int \frac{u (a + b \text{Log}[c x^n])^{p-1}}{x} dx$$

Program code:

```
Int[Log[d_.*(e_+f_.*x_^m_)^r_].*(a_+b_.*Log[c_.*x_^n_])^p_,x_Symbol] :=
  With[{u=IntHide[Log[d*(e+f*x^m)^r],x]},
    Dist[(a+b*Log[c*x^n])^p,u,x] - b*n*p*Int[Dist[(a+b*Log[c*x^n])^(p-1)/x,u,x],x] /;
  FreeQ[{a,b,c,d,e,f,r,m,n},x] && IGtQ[p,0] && RationalQ[m] && (EqQ[p,1] || FractionQ[m] && IntegerQ[1/m] || EqQ[r,1] && EqQ[m,1] &&
```

2: $\int \text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^p dx$ when $p \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}$

Derivation: Integration by parts

Rule: If $p \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}$, let $u \rightarrow \int (a + b \text{Log}[c x^n])^p dx$, then

$$\int \text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^p dx \rightarrow u \text{Log}[d (e + f x^m)^r] - f m r \int \frac{u x^{m-1}}{e + f x^m} dx$$

Program code:

```
Int[Log[d_.*(e_+f_.*x_^m_)^r_.*(a_+b_.*Log[c_.*x_^n_])^p_,x_Symbol] :=
  With[{u=IntHide[(a+b*Log[c*x^n])^p,x]},
    Dist[Log[d*(e+f*x^m)^r],u,x] - f*m*r*Int[Dist[x^(m-1)/(e+f*x^m),u,x],x] /;
    FreeQ[{a,b,c,d,e,f,r,m,n},x] && IGtQ[p,0] && IntegerQ[m]
```

U: $\int \text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^p dx$

Rule:

$$\int \text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^p dx \rightarrow \int \text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^p dx$$

Program code:

```
Int[Log[d_.*(e_+f_.*x_^m_)^r_.*(a_+b_.*Log[c_.*x_^n_])^p_,x_Symbol] :=
  Unintegrable[Log[d*(e+f*x^m)^r]*(a+b*Log[c*x^n])^p,x] /;
  FreeQ[{a,b,c,d,e,f,r,m,n,p},x]
```

N: $\int \text{Log}[d u^r] (a + b \text{Log}[c x^n])^p dx$ when $u == e + f x^m$

Derivation: Algebraic normalization

Rule: If $u == e + f x^m$, then

$$\int (g x)^q \text{Log}[d u^r] (a + b \text{Log}[c x^n])^p dx \rightarrow \int (g x)^q \text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^p dx$$

Program code:

```
Int[Log[d_.*u_^r_.]*(a_.+b_.*Log[c_.*x_^n_.])^p_,x_Symbol] :=
  Int[Log[d*ExpandToSum[u,x]^r]*(a+b*Log[c*x^n])^p,x] /;
FreeQ[{a,b,c,d,r,n,p},x] && BinomialQ[u,x] && Not[BinomialMatchQ[u,x]]
```

$$9. \int (g x)^q \operatorname{Log}[d (e + f x^m)^r] (a + b \operatorname{Log}[c x^n])^p dx$$

$$1. \int \frac{\operatorname{Log}[d (e + f x^m)^r] (a + b \operatorname{Log}[c x^n])^p}{x} dx \text{ when } p \in \mathbb{Z}^+$$

$$\textcolor{red}{1:} \int \frac{\operatorname{Log}[d (e + f x^m)] (a + b \operatorname{Log}[c x^n])^p}{x} dx \text{ when } p \in \mathbb{Z}^+ \wedge d e = 1$$

Derivation: Integration by parts

■

Basis: If $d e = 1$, then $\frac{\operatorname{Log}[d (e + f x^m)]}{x} = -\partial_x \frac{\operatorname{PolyLog}[2, -d f x^m]}{m}$

Rule: If $p \in \mathbb{Z}^+ \wedge d e = 1$, then

$$\int \frac{\operatorname{Log}[d (e + f x^m)] (a + b \operatorname{Log}[c x^n])^p}{x} dx \rightarrow -\frac{\operatorname{PolyLog}[2, -d f x^m] (a + b \operatorname{Log}[c x^n])^p}{m} + \frac{b n p}{m} \int \frac{\operatorname{PolyLog}[2, -d f x^m] (a + b \operatorname{Log}[c x^n])^{p-1}}{x} dx$$

Program code:

```
Int[Log[d_.*(e_+f_.*x_^m_.)]*(a_.+b_.*Log[c_.*x_^n_.])^p_/x_,x_Symbol] :=
  -PolyLog[2,-d*f*x^m]*(a+b*Log[c*x^n])^p/m +
  b*n*p/m*Int[PolyLog[2,-d*f*x^m]*(a+b*Log[c*x^n])^(p-1)/x,x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && IGtQ[p,0] && EqQ[d*e,1]
```

$$2: \int \frac{\text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^p}{x} dx \text{ when } p \in \mathbb{Z}^+ \wedge d e \neq 1$$

Derivation: Integration by parts

$$\text{Basis: } \frac{(a+b \text{Log}[c x^n])^p}{x} == \partial_x \frac{(a+b \text{Log}[c x^n])^{p+1}}{b n (p+1)}$$

$$\text{Basis: } \partial_x \text{Log}[d (e + f x^m)^r] == \frac{f m r x^{m-1}}{e + f x^m}$$

Rule: If $p \in \mathbb{Z}^+ \wedge d e \neq 1$, then

$$\int \frac{\text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^p}{x} dx \rightarrow \frac{\text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^{p+1}}{b n (p+1)} - \frac{f m r}{b n (p+1)} \int \frac{x^{m-1} (a + b \text{Log}[c x^n])^{p+1}}{e + f x^m} dx$$

Program code:

```
Int[Log[d_.*(e_+f_.*x_^m_)^r_]*(a_+b_.*Log[c_.*x_^n_] )^p_/x_,x_Symbol] :=
  Log[d*(e+f*x^m)^r]*(a+b*Log[c*x^n])^(p+1)/(b*n*(p+1)) -
  f*m*r/(b*n*(p+1))*Int[x^(m-1)*(a+b*Log[c*x^n])^(p+1)/(e+f*x^m),x] /;
FreeQ[{a,b,c,d,e,f,r,m,n},x] && IGtQ[p,0] && NeQ[d*e,1]
```

2: $\int (g x)^q \operatorname{Log}[d (e + f x^m)^r] (a + b \operatorname{Log}[c x^n]) dx$ when $\left(\frac{q+1}{m} \in \mathbb{Z} \vee (m \mid q) \in \mathbb{R}\right) \wedge q \neq -1$

Derivation: Integration by parts

Note: If $\frac{q+1}{m} \in \mathbb{Z} \vee (m \mid q) \in \mathbb{R}$, then $\frac{\int (g x)^q \operatorname{Log}[d (e + f x^m)^r] dx}{x}$ is integrable.

Rule: If $\left(\frac{q+1}{m} \in \mathbb{Z} \vee (m \mid q) \in \mathbb{R}\right) \wedge q \neq -1$, let $u \rightarrow \int (g x)^q \operatorname{Log}[d (e + f x^m)^r] dx$, then

$$\int (g x)^q \operatorname{Log}[d (e + f x^m)^r] (a + b \operatorname{Log}[c x^n]) dx \rightarrow u (a + b \operatorname{Log}[c x^n]) - b n \int \frac{u}{x} dx$$

Program code:

```
Int[(g_.**x_)^q_.*Log[d_.*(e_+f_.*x_^m_)^r_.]*(a_.+b_.*Log[c_.*x_^n_.]),x_Symbol] :=
  With[{u=IntHide[(g*x)^q*Log[d*(e+f*x^m)^r],x]},
    Dist[(a+b*Log[c*x^n]),u,x] - b*n*Int[Dist[1/x,u,x],x] /;
    FreeQ[{a,b,c,d,e,f,g,r,m,n,q},x] && (IntegerQ[(q+1)/m] || RationalQ[m] && RationalQ[q]) && NeQ[q,-1]
```

$$\mathbf{3:} \int (g x)^q \operatorname{Log}[d (e + f x^m)] (a + b \operatorname{Log}[c x^n])^p dx \text{ when } p \in \mathbb{Z}^+ \wedge m \in \mathbb{R} \wedge q \in \mathbb{R} \wedge q \neq -1 \wedge (p = 1 \vee \frac{q+1}{m} \in \mathbb{Z} \vee (q \in \mathbb{Z}^+ \wedge \frac{q+1}{m} \in \mathbb{Z} \wedge d e = 1))$$

Derivation: Integration by parts

Rule: If $p \in \mathbb{Z}^+ \wedge m \in \mathbb{R} \wedge q \in \mathbb{R} \wedge q \neq -1 \wedge (p = 1 \vee \frac{q+1}{m} \in \mathbb{Z} \vee (q \in \mathbb{Z}^+ \wedge \frac{q+1}{m} \in \mathbb{Z} \wedge d e = 1))$, let $u \rightarrow \int (g x)^q \operatorname{Log}[d (e + f x^m)] dx$, then

$$\int (g x)^q \operatorname{Log}[d (e + f x^m)] (a + b \operatorname{Log}[c x^n])^p dx \rightarrow u (a + b \operatorname{Log}[c x^n])^p - b n p \int \frac{u (a + b \operatorname{Log}[c x^n])^{p-1}}{x} dx$$

Program code:

```
Int[(g_.x_)^q_.Log[d_.(e+f_.x_^m_.)]*(a_.+b_.Log[c_.x_^n_.])^p_,x_Symbol] :=
  With[{u=IntHide[(g*x)^q*Log[d*(e+f*x^m)],x]},
    Dist[(a+b*Log[c*x^n])^p,u,x] - b*n*p*Int[Dist[(a+b*Log[c*x^n])^(p-1)/x,u,x],x] /;
  FreeQ[{a,b,c,d,e,f,g,m,n,q},x] && IGtQ[p,0] && RationalQ[m] && RationalQ[q] && NeQ[q,-1] &&
    (EqQ[p,1] || FractionQ[m] && IntegerQ[(q+1)/m] || IGtQ[q,0] && IntegerQ[(q+1)/m] && EqQ[d*e,1])
```

4: $\int (g x)^q \text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^p dx$ when $p \in \mathbb{Z}^+ \wedge m \in \mathbb{R} \wedge q \in \mathbb{R}$

Derivation: Integration by parts

Rule: If $p \in \mathbb{Z}^+ \wedge m \in \mathbb{R} \wedge q \in \mathbb{R}$, let $u \rightarrow \int (g x)^q (a + b \text{Log}[c x^n])^p dx$, then

$$\int (g x)^q \text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^p dx \rightarrow u \text{Log}[d (e + f x^m)^r] - f m r \int \frac{u x^{m-1}}{e + f x^m} dx$$

Program code:

```
Int[(g_.**x_)^q_.*Log[d_.*(e_+f_.*x_^m_)^r_].*(a_+b_.*Log[c_.*x_^n_])^p_,x_Symbol] :=
  With[{u=IntHide[(g*x)^q*(a+b*Log[c*x^n])^p,x]},
    Dist[Log[d*(e+f*x^m)^r],u,x] - f*m*r*Int[Dist[x^(m-1)/(e+f*x^m),u,x],x] /;
    FreeQ[{a,b,c,d,e,f,g,r,m,n,q},x] && IGtQ[p,0] && RationalQ[m] && RationalQ[q]
```

U: $\int (g x)^q \text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^p dx$

Rule:

$$\int (g x)^q \text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^p dx \rightarrow \int (g x)^q \text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^p dx$$

Program code:

```
Int[(g_.**x_)^q_.*Log[d_.*(e_+f_.*x_^m_)^r_].*(a_+b_.*Log[c_.*x_^n_])^p_,x_Symbol] :=
  Unintegrable[(g*x)^q*Log[d*(e+f*x^m)^r]*(a+b*Log[c*x^n])^p,x] /;
  FreeQ[{a,b,c,d,e,f,g,r,m,n,p,q},x]
```


N: $\int (g x)^q \text{Log}[d u^r] (a + b \text{Log}[c x^n])^p dx$ when $u = e + f x^m$

Derivation: Algebraic normalization

Rule: If $u = e + f x^m$, then

$$\int (g x)^q \text{Log}[d u^r] (a + b \text{Log}[c x^n])^p dx \rightarrow \int (g x)^q \text{Log}[d (e + f x^m)^r] (a + b \text{Log}[c x^n])^p dx$$

Program code:

```
Int[(g_.x_)^q_.Log[d_.u_^r_.]*(a_.+b_.Log[c_.x_^n_.])^p_,x_Symbol] :=
  Int[(g*x)^q*Log[d*ExpandToSum[u,x]^r]*(a+b*Log[c*x^n])^p,x] /;
FreeQ[{a,b,c,d,g,r,n,p,q},x] && BinomialQ[u,x] && Not[BinomialMatchQ[u,x]]
```

$$10. \int \text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n])^p dx$$

$$1: \int \text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n]) dx \text{ when } k \in \mathbb{Z}^+$$

Derivation: Integration by parts

$$\text{Basis: } (a + b \text{Log}[c x^n]) = \partial_x (-b n x + x (a + b \text{Log}[c x^n]))$$

$$\text{Basis: } \partial_x \text{PolyLog}[k, e x^q] = \frac{q \text{PolyLog}[k-1, e x^q]}{x}$$

Rule: If $k \in \mathbb{Z}^+$, then

$$\begin{aligned} \int \text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n]) dx \rightarrow \\ -b n x \text{PolyLog}[k, e x^q] + x \text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n]) + \\ b n q \int \text{PolyLog}[k-1, e x^q] dx - q \int \text{PolyLog}[k-1, e x^q] (a + b \text{Log}[c x^n]) dx \end{aligned}$$

Program code:

```
Int[PolyLog[k, e_.**x^q_.]*(a_.+b_.**Log[c_.**x^n_.]), x_Symbol] :=
  -b*n*x*PolyLog[k, e**x^q] + x*PolyLog[k, e**x^q]*(a+b*Log[c*x^n]) +
  b*n*q*Int[PolyLog[k-1, e**x^q], x] - q*Int[PolyLog[k-1, e**x^q]*(a+b*Log[c*x^n]), x] /;
FreeQ[{a, b, c, e, n, q}, x] && IGtQ[k, 0]
```

U: $\int \text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n])^p dx$

Rule:

$$\int \text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n])^p dx \rightarrow \int \text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n])^p dx$$

Program code:

```
Int[PolyLog[k_, e_.**x_^q_.]*(a_.+b_.*Log[c_.**x_^n_.])^p_, x_Symbol] :=
  Unintegrable[PolyLog[k, e**x^q]*(a+b*Log[c*x^n])^p, x] /;
FreeQ[{a, b, c, e, n, p, q}, x]
```

$$11. \int (d x)^m \text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n])^p dx$$

$$1. \int \frac{\text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n])^p}{x} dx$$

$$1: \int \frac{\text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n])^p}{x} dx \text{ when } p > 0$$

Derivation: Integration by parts

$$\text{Basis: } \frac{\text{PolyLog}[k, e x^q]}{x} = \partial_x \frac{\text{PolyLog}[k+1, e x^q]}{q}$$

Rule: If $p > 0$, then

$$\int \frac{\text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n])^p}{x} dx \rightarrow \frac{\text{PolyLog}[k+1, e x^q] (a + b \text{Log}[c x^n])^p}{q} - \frac{b n p}{q} \int \frac{\text{PolyLog}[k+1, e x^q] (a + b \text{Log}[c x^n])^{p-1}}{x} dx$$

Program code:

```
Int[PolyLog[k_, e_.**x^q_.]*(a_.+b_.**Log[c_.**x^n_.])^p_/x_, x_Symbol] :=
  PolyLog[k+1, e**x^q]*(a+b*Log[c*x^n])^p/q - b*n*p/q*Int[PolyLog[k+1, e**x^q]*(a+b*Log[c*x^n])^(p-1)/x, x] /;
  FreeQ[{a,b,c,e,k,n,q}, x] && GtQ[p, 0]
```

$$2: \int \frac{\text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n])^p}{x} dx \text{ when } p < -1$$

Derivation: Integration by parts

$$\text{Basis: } \frac{(a+b \text{Log}[c x^n])^p}{x} == \partial_x \frac{(a+b \text{Log}[c x^n])^{p+1}}{b n (p+1)}$$

$$\text{Basis: } \partial_x \text{PolyLog}[k, e x^q] == \frac{q \text{PolyLog}[k-1, e x^q]}{x}$$

Rule: If $p < -1$, then

$$\int \frac{\text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n])^p}{x} dx \rightarrow \frac{\text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n])^{p+1}}{b n (p+1)} - \frac{q}{b n (p+1)} \int \frac{\text{PolyLog}[k-1, e x^q] (a + b \text{Log}[c x^n])^{p+1}}{x} dx$$

Program code:

```
Int[PolyLog[k_, e_. * x_^q_.] * (a_. + b_. * Log[c_. * x_^n_.])^p_./x_, x_Symbol] :=
  PolyLog[k, e * x^q] * (a + b * Log[c * x^n])^(p+1)/(b * n * (p+1)) - q/(b * n * (p+1)) * Int[PolyLog[k-1, e * x^q] * (a + b * Log[c * x^n])^(p+1)/x, x] /;
  FreeQ[{a, b, c, e, k, n, q}, x] && LtQ[p, -1]
```

$$2: \int (d x)^m \text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n]) dx \text{ when } k \in \mathbb{Z}^+$$

Derivation: Integration by parts

$$\text{Basis: } (d x)^m (a + b \text{Log}[c x^n]) == \partial_x \left(-\frac{b n (d x)^{m+1}}{d (m+1)^2} + \frac{(d x)^{m+1} (a + b \text{Log}[c x^n])}{d (m+1)} \right)$$

$$\text{Basis: } \partial_x \text{PolyLog}[k, e x^q] == \frac{q \text{PolyLog}[k-1, e x^q]}{x}$$

Rule: If $k \in \mathbb{Z}^+$, then

$$\int (d x)^m \text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n]) dx \rightarrow$$

$$- \frac{b n (d x)^{m+1} \text{PolyLog}[k, e x^q]}{d (m+1)^2} + \frac{(d x)^{m+1} \text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n])}{d (m+1)} +$$

$$\frac{b n q}{(m+1)^2} \int (d x)^m \text{PolyLog}[k-1, e x^q] dx - \frac{q}{(m+1)} \int (d x)^m \text{PolyLog}[k-1, e x^q] (a + b \text{Log}[c x^n]) dx$$

Program code:

```
Int[(d.*x_)^m_.*PolyLog[k_,e_.*x_^q_.]*(a_.+b_.*Log[c_.*x_^n_.]),x_Symbol] :=
  -b*n*(d*x)^(m+1)*PolyLog[k,e*x^q]/(d*(m+1)^2) +
  (d*x)^(m+1)*PolyLog[k,e*x^q]*(a+b*Log[c*x^n])/(d*(m+1)) +
  b*n*q/(m+1)^2*Int[(d*x)^m*PolyLog[k-1,e*x^q],x] -
  q/(m+1)*Int[(d*x)^m*PolyLog[k-1,e*x^q]*(a+b*Log[c*x^n]),x] /;
FreeQ[{a,b,c,d,e,m,n,q},x] && IGtQ[k,0]
```

U: $\int (d x)^m \text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n])^p dx$

Rule:

$$\int (d x)^m \text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n])^p dx \rightarrow \int (d x)^m \text{PolyLog}[k, e x^q] (a + b \text{Log}[c x^n])^p dx$$

Program code:

```
Int[(d.*x_)^m_.*PolyLog[k_,e_.*x_^q_.]*(a_.+b_.*Log[c_.*x_^n_.])^p_.,x_Symbol] :=
  Unintegrable[(d*x)^m*PolyLog[k,e*x^q]*(a+b*Log[c*x^n])^p,x] /;
FreeQ[{a,b,c,d,e,m,n,p,q},x]
```

$$12. \int P_x F[d(e+fx)]^m (a+b \log[cx^n]) dx$$

$$1: \int P_x F[d(e+fx)]^m (a+b \log[cx^n]) dx \text{ when } m \in \mathbb{Z}^+ \wedge F \in \{\text{ArcSin}, \text{ArcCos}, \text{ArcSinh}, \text{ArcCosh}\}$$

Derivation: Integration by parts

$$\text{Basis: } \partial_x (a + b \log[cx^n]) = \frac{bn}{x}$$

Note: If $m \in \mathbb{Z}^+ \wedge F \in \{\text{ArcSin}, \text{ArcCos}, \text{ArcSinh}, \text{ArcCosh}\}$, the terms of the antiderivative of $\frac{\int P_x F[d(e+fx)]^m dx}{x}$ will be integrable.

Rule: If $m \in \mathbb{Z}^+ \wedge F \in \{\text{ArcSin}, \text{ArcCos}, \text{ArcSinh}, \text{ArcCosh}\}$, let $u \rightarrow \int P_x F[d(e+fx)]^m dx$, then

$$\int P_x F[d(e+fx)]^m (a+b \log[cx^n]) dx \rightarrow u (a+b \log[cx^n]) - bn \int \frac{u}{x} dx$$

Program code:

```
Int[Px_.*F_[d_.*(e_.+f_.*x_)]^m_.*(a_.+b_.*Log[c_.*x_^n_.]),x_Symbol] :=
  With[{u=IntHide[Px*F[d*(e+f*x)]^m,x]},
    Dist[(a+b*Log[c*x^n]),u,x] - b*n*Int[Dist[1/x,u,x],x] /;
    FreeQ[{a,b,c,d,e,f,n},x] && PolynomialQ[Px,x] && IGtQ[m,0] && MemberQ[{ArcSin, ArcCos, ArcSinh, ArcCosh},F]
```

2: $\int P_x F[d(e + f x)] (a + b \log[c x^n]) dx$ when $F \in \{\text{ArcTan}, \text{ArcCot}, \text{ArcTanh}, \text{ArcCoth}\}$

Derivation: Integration by parts

Basis: $\partial_x (a + b \log[c x^n]) = \frac{b n}{x}$

■ Note: If $F \in \{\text{ArcTan}, \text{ArcCot}, \text{ArcTanh}, \text{ArcCoth}\}$, the terms of the antiderivative of $\frac{\int P_x F[d(e + f x)] dx}{x}$ will be integrable.

Rule: If $F \in \{\text{ArcTan}, \text{ArcCot}, \text{ArcTanh}, \text{ArcCoth}\}$, let $u \rightarrow \int P_x F[d(e + f x)] dx$, then

$$\int P_x F[d(e + f x)] (a + b \log[c x^n]) dx \rightarrow u (a + b \log[c x^n]) - b n \int \frac{u}{x} dx$$

Program code:

```
Int[Px_.*F[d_.*(e_.+f_.*x_)]*(a_.+b_.*Log[c_.*x_^n_.]),x_Symbol] :=
  With[{u=IntHide[Px*F[d*(e+f*x)],x]},
    Dist[(a+b*Log[c*x^n]),u,x] - b*n*Int[Dist[1/x,u,x],x] /;
    FreeQ[{a,b,c,d,e,f,n},x] && PolynomialQ[Px,x] && MemberQ[{ArcTan, ArcCot, ArcTanh, ArcCoth},F]
```