

Rules for integrands of the form $u (a + b \operatorname{ArcSinh}[c x])^n$

$$1. \int (d + e x)^m (a + b \operatorname{ArcSinh}[c x])^n dx$$

$$1. \int (d + e x)^m (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } n \in \mathbb{Z}^+$$

$$\text{1: } \int \frac{(a + b \operatorname{ArcSinh}[c x])^n}{d + e x} dx$$

Derivation: Integration by substitution

$$\text{Basis: } \frac{1}{d+e x} = \text{Subst}\left[\frac{\cosh[x]}{c d + e \sinh[x]}, x, \operatorname{ArcSinh}[c x]\right] \partial_x \operatorname{ArcSinh}[c x]$$

Note: $\frac{(a+b x)^n \cosh[x]}{c d + e \sinh[x]}$ is not integrable unless $n \in \mathbb{Z}^+$.

Rule: If $n \in \mathbb{Z}^+$, then

$$\int \frac{(a + b \operatorname{ArcSinh}[c x])^n}{d + e x} dx \rightarrow \text{Subst}\left[\int \frac{(a + b x)^n \cosh[x]}{c d + e \sinh[x]} dx, x, \operatorname{ArcSinh}[c x]\right]$$

Program code:

```
Int[(a_.+b_.*ArcSinh[c_.*x_])^n_./(d_.+e_.*x_),x_Symbol] :=
  Subst[Int[(a+b*x)^n*Cosh[x]/(c*d+e*sinh[x]),x],x,ArcSinh[c*x] /;
FreeQ[{a,b,c,d,e},x] && IGtQ[n,0]
```

```
Int[(a_.+b_.*ArcCosh[c_.*x_])^n_./(d_.+e_.*x_),x_Symbol] :=
  Subst[Int[(a+b*x)^n*Sinh[x]/(c*d+e*cosh[x]),x],x,ArcCosh[c*x] /;
FreeQ[{a,b,c,d,e},x] && IGtQ[n,0]
```

2: $\int (d + e x)^m (a + b \operatorname{ArcSinh}[c x])^n dx$ when $n \in \mathbb{Z}^+ \wedge m \neq -1$

Reference: G&R 2.831, CRC 453, A&S 4.4.65

Reference: G&R 2.832, CRC 454, A&S 4.4.67

Derivation: Integration by parts

Basis: If $m \neq -1$, then $(d + e x)^m = \partial_x \frac{(d + e x)^{m+1}}{e(m+1)}$

Rule: If $n \in \mathbb{Z}^+ \wedge m \neq -1$, then

$$\int (d + e x)^m (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{(d + e x)^{m+1} (a + b \operatorname{ArcSinh}[c x])^n}{e(m+1)} - \frac{b c n}{e(m+1)} \int \frac{(d + e x)^{m+1} (a + b \operatorname{ArcSinh}[c x])^{n-1}}{\sqrt{1 + c^2 x^2}} dx$$

Program code:

```
Int[(d_+e_.**x_)^m_.*(a_+b_.**ArcSinh[c_.**x_])^n_,x_Symbol] :=
  (d+e*x)^(m+1)*(a+b*ArcSinh[c*x])^n/(e*(m+1)) -
  b*c*n/(e*(m+1))*Int[(d+e*x)^(m+1)*(a+b*ArcSinh[c*x])^(n-1)/Sqrt[1+c^2*x^2],x] /;
FreeQ[{a,b,c,d,e,m},x] && IGtQ[n,0] && NeQ[m,-1]
```

```
Int[(d_+e_.**x_)^m_.*(a_+b_.**ArcCosh[c_.**x_])^n_,x_Symbol] :=
  (d+e*x)^(m+1)*(a+b*ArcCosh[c*x])^n/(e*(m+1)) -
  b*c*n/(e*(m+1))*Int[(d+e*x)^(m+1)*(a+b*ArcCosh[c*x])^(n-1)/(Sqrt[-1+c*x]*Sqrt[1+c*x]),x] /;
FreeQ[{a,b,c,d,e,m},x] && IGtQ[n,0] && NeQ[m,-1]
```

2. $\int (d + e x)^m (a + b \operatorname{ArcSinh}[c x])^n dx$ when $m \in \mathbb{Z}^+$

1: $\int (d + e x)^m (a + b \operatorname{ArcSinh}[c x])^n dx$ when $m \in \mathbb{Z}^+ \wedge n < -1$

Derivation: Algebraic expansion

Rule: If $m \in \mathbb{Z}^+ \wedge n < -1$, then

$$\int (d + e x)^m (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \int \operatorname{ExpandIntegrand}[(d + e x)^m (a + b \operatorname{ArcSinh}[c x])^n, x] dx$$

Program code:

```
Int[(d_+e_.*x_)^m_.*(a_+b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x)^m*(a+b*ArcSinh[c*x])^n,x],x] /;
FreeQ[{a,b,c,d,e},x] && IGtQ[m,0] && LtQ[n,-1]
```

```
Int[(d_+e_.*x_)^m_.*(a_+b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x)^m*(a+b*ArcCosh[c*x])^n,x],x] /;
FreeQ[{a,b,c,d,e},x] && IGtQ[m,0] && LtQ[n,-1]
```

$$\mathbf{2:} \int (d + e x)^m (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } m \in \mathbb{Z}^+$$

Derivation: Integration by substitution

$$\text{Basis: } F[x] := \frac{1}{c} F\left[\frac{\sinh(\operatorname{ArcSinh}[c x])}{c}\right] \cosh[\operatorname{ArcSinh}[c x]] \partial_x \operatorname{ArcSinh}[c x]$$

Note: If $m \in \mathbb{Z}^+$, then $(a + b x)^n \cosh[x] (c d + e \sinh[x])^m$ is integrable in closed-form.

Rule: If $m \in \mathbb{Z}^+$, then

$$\int (d + e x)^m (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{1}{c^{m+1}} \operatorname{Subst}\left[\int (a + b x)^n \cosh[x] (c d + e \sinh[x])^m dx, x, \operatorname{ArcSinh}[c x]\right]$$

Program code:

```
Int[(d_+e_.*x_)^m_.*(a_+b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  1/c^(m+1)*Subst[Int[(a+b*x)^n*(c*d+e*Sinh[x])^m,x],x,ArcSinh[c*x]] /;
FreeQ[{a,b,c,d,e,n},x] && IGtQ[m,0]
```

```
Int[(d_+e_.*x_)^m_.*(a_+b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  1/c^(m+1)*Subst[Int[(a+b*x)^n*(c*d+e*Cosh[x])^m*Sinh[x],x],x,ArcCosh[c*x]] /;
FreeQ[{a,b,c,d,e,n},x] && IGtQ[m,0]
```

$$2. \int P_x (a + b \operatorname{ArcSinh}[c x])^n dx$$

$$\mathbf{1:} \int P_x (a + b \operatorname{ArcSinh}[c x]) dx$$

Derivation: Integration by parts

Rule: Let $u = \int P_x dx$, then

$$\int P_x (a + b \operatorname{ArcSinh}[c x]) \, dx \rightarrow u (a + b \operatorname{ArcSinh}[c x]) - b c \int \frac{u}{\sqrt{1 + c^2 x^2}} \, dx$$

$$\int P_x (a + b \operatorname{ArcCosh}[c x]) \, dx \rightarrow u (a + b \operatorname{ArcCosh}[c x]) - \frac{b c \sqrt{1 - c^2 x^2}}{\sqrt{-1 + c x} \sqrt{1 + c x}} \int \frac{u}{\sqrt{1 - c^2 x^2}} \, dx$$

Program code:

```
Int[Px_*(a_.+b_.*ArcSinh[c_.*x_]),x_Symbol] :=
  With[{u=IntHide[ExpandExpression[Px,x],x]},
    Dist[a+b*ArcSinh[c*x],u,x] - b*c*Int[SimplifyIntegrand[u/Sqrt[1+c^2*x^2],x],x] /;
  FreeQ[{a,b,c},x] && PolynomialQ[Px,x]
```

```
Int[Px_*(a_.+b_.*ArcCosh[c_.*x_]),x_Symbol] :=
  With[{u=IntHide[ExpandExpression[Px,x],x]},
    Dist[a+b*ArcCosh[c*x],u,x] - b*c*Sqrt[1-c^2*x^2]/(Sqrt[-1+c*x]*Sqrt[1+c*x])*Int[SimplifyIntegrand[u/Sqrt[1-c^2*x^2],x],x] /;
  FreeQ[{a,b,c},x] && PolynomialQ[Px,x]
```

x: $\int P_x (a + b \operatorname{ArcSinh}[c x])^n dx$ when $n \in \mathbb{Z}^+$

Derivation: Integration by parts

Rule: If $n \in \mathbb{Z}^+$, let $u = \int P_x dx$, then

$$\int P_x (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow u (a + b \operatorname{ArcSinh}[c x])^n - b c n \int \frac{u (a + b \operatorname{ArcSinh}[c x])^{n-1}}{\sqrt{1 + c^2 x^2}} dx$$

$$\int P_x (a + b \operatorname{ArcCosh}[c x])^n dx \rightarrow u (a + b \operatorname{ArcCosh}[c x])^n - \frac{b c n \sqrt{1 - c^2 x^2}}{\sqrt{-1 + c x} \sqrt{1 + c x}} \int \frac{u (a + b \operatorname{ArcCosh}[c x])^{n-1}}{\sqrt{1 - c^2 x^2}} dx$$

Program code:

```
(* Int[Px*(a_.+b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  With[{u=IntHide[Px,x]},
    Dist[(a+b*ArcSinh[c*x])^n,u,x] - b*c*n*Int[SimplifyIntegrand[u*(a+b*ArcSinh[c*x])^(n-1)/Sqrt[1+c^2*x^2],x],x] /;
  FreeQ[{a,b,c},x] && PolynomialQ[Px,x] && IGtQ[n,0] *)
```

```
(* Int[Px*(a_.+b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  With[{u=IntHide[Px,x]},
    Dist[(a+b*ArcCosh[c*x])^n,u,x] -
      b*c*n*Sqrt[1-c^2*x^2]/(Sqrt[-1+c*x]*Sqrt[1+c*x])*Int[SimplifyIntegrand[u*(a+b*ArcCosh[c*x])^(n-1)/Sqrt[1-c^2*x^2],x],x] /;
  FreeQ[{a,b,c},x] && PolynomialQ[Px,x] && IGtQ[n,0] *)
```

2: $\int P_x (a + b \operatorname{ArcSinh}[c x])^n dx$ when $n \neq 1$

Derivation: Algebraic expansion

Rule: If $n \neq 1$, then

$$\int P_x (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \int \operatorname{ExpandIntegrand}[P_x (a + b \operatorname{ArcSinh}[c x])^n, x] dx$$

Program code:

```
Int[Px*(a_.+b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  Int[ExpandIntegrand[Px*(a+b*ArcSinh[c*x])^n,x],x] /;
FreeQ[{a,b,c,n},x] && PolynomialQ[Px,x]
```

```
Int[Px*(a_.+b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  Int[ExpandIntegrand[Px*(a+b*ArcCosh[c*x])^n,x],x] /;
FreeQ[{a,b,c,n},x] && PolynomialQ[Px,x]
```

3. $\int P_x (d + e x)^m (a + b \operatorname{ArcSinh}[c x])^n dx$ when $n \in \mathbb{Z}^+$

1: $\int P_x (d + e x)^m (a + b \operatorname{ArcSinh}[c x]) dx$

Derivation: Integration by parts

Rule: Let $u = \int P_x (d + e x)^m dx$, then

$$\int P_x (d + e x)^m (a + b \operatorname{ArcSinh}[c x]) dx \rightarrow u (a + b \operatorname{ArcSinh}[c x]) - b c \int \frac{u}{\sqrt{1 + c^2 x^2}} dx$$

$$\int P_x (d + e x)^m (a + b \operatorname{ArcCosh}[c x]) dx \rightarrow u (a + b \operatorname{ArcCosh}[c x]) - \frac{b c \sqrt{1 - c^2 x^2}}{\sqrt{-1 + c x} \sqrt{1 + c x}} \int \frac{u}{\sqrt{1 - c^2 x^2}} dx$$

Program code:

```
Int[Px*(d_+e_*x_)^m_.*(a_+b_.*ArcSinh[c_*x_]),x_Symbol] :=
  With[{u=IntHide[Px*(d+e*x)^m,x]},
    Dist[a+b*ArcSinh[c*x],u,x] - b*c*Int[SimplifyIntegrand[u/Sqrt[1+c^2*x^2],x],x]] /;
  FreeQ[{a,b,c,d,e,m},x] && PolynomialQ[Px,x]
```

```
Int[Px*(d_+e_*x_)^m_.*(a_+b_.*ArcCosh[c_*x_]),x_Symbol] :=
  With[{u=IntHide[Px*(d+e*x)^m,x]},
    Dist[a+b*ArcCosh[c*x],u,x] - b*c*Sqrt[1-c^2*x^2]/(Sqrt[-1+c*x]*Sqrt[1+c*x])*Int[SimplifyIntegrand[u/Sqrt[1-c^2*x^2],x],x]] /;
  FreeQ[{a,b,c,d,e,m},x] && PolynomialQ[Px,x]
```


2: $\int (f + g x)^p (d + e x)^m (a + b \operatorname{ArcSinh}[c x])^n dx$ when $(n \mid p) \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^- \wedge m + p + 1 < 0$

Derivation: Integration by parts

Note: If $p \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^- \wedge m + p + 1 < 0$, then $\int (f + g x)^p (d + e x)^m dx$ is a rational function.

Rule: If $(n \mid p) \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}^- \wedge m + p + 1 < 0$, let $u = \int (f + g x)^p (d + e x)^m dx$, then

$$\int (f + g x)^p (d + e x)^m (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow u (a + b \operatorname{ArcSinh}[c x])^n - b c n \int \frac{u (a + b \operatorname{ArcSinh}[c x])^{n-1}}{\sqrt{1 + c^2 x^2}} dx$$

Program code:

```
Int[(f_.+g_.**x_)^p_.*(d_.+e_.**x_)^m_.*(a_.+b_.**ArcSinh[c_.**x_])^n_,x_Symbol] :=
  With[{u=IntHide[(f+g*x)^p*(d+e*x)^m,x]},
    Dist[(a+b*ArcSinh[c*x])^n,u,x] - b*c*n*Int[SimplifyIntegrand[u*(a+b*ArcSinh[c*x])^(n-1)/Sqrt[1+c^2*x^2],x],x] /;
  FreeQ[{a,b,c,d,e,f,g},x] && IGtQ[n,0] && IGtQ[p,0] && ILtQ[m,0] && LtQ[m+p+1,0]
```

```
Int[(f_.+g_.**x_)^p_.*(d_.+e_.**x_)^m_.*(a_.+b_.**ArcCosh[c_.**x_])^n_,x_Symbol] :=
  With[{u=IntHide[(f+g*x)^p*(d+e*x)^m,x]},
    Dist[(a+b*ArcCosh[c*x])^n,u,x] - b*c*n*Int[SimplifyIntegrand[u*(a+b*ArcCosh[c*x])^(n-1)/(Sqrt[1+c*x]*Sqrt[-1+c*x]),x],x] /;
  FreeQ[{a,b,c,d,e,f,g},x] && IGtQ[n,0] && IGtQ[p,0] && ILtQ[m,0] && LtQ[m+p+1,0]
```

$$3: \int \frac{(f + g x + h x^2)^p (a + b \operatorname{ArcSinh}[c x])^n}{(d + e x)^2} dx \text{ when } (n \mid p) \in \mathbb{Z}^+ \wedge e g - 2 d h = 0$$

Derivation: Integration by parts

Note: If $p \in \mathbb{Z}^+ \wedge e g - 2 d h = 0$, then $\int \frac{(f+g x+h x^2)^p}{(d+e x)^2} dx$ is a rational function.

Rule: If $(n \mid p) \in \mathbb{Z}^+ \wedge e g - 2 d h = 0$, let $u = \int \frac{(f+g x+h x^2)^p}{(d+e x)^2} dx$, then

$$\int \frac{(f + g x + h x^2)^p (a + b \operatorname{ArcSinh}[c x])^n}{(d + e x)^2} dx \rightarrow u (a + b \operatorname{ArcSinh}[c x])^n - b c n \int \frac{u (a + b \operatorname{ArcSinh}[c x])^{n-1}}{\sqrt{1 + c^2 x^2}} dx$$

Program code:

```
Int[(f_.+g_.**x_+h_.**x_^2)^p_.*(a_.+b_.*ArcSinh[c_.**x_])^n_/(d_+e_.**x_)^2,x_Symbol] :=
  With[{u=IntHide[(f+g*x+h*x^2)^p/(d+e*x)^2,x]},
    Dist[(a+b*ArcSinh[c*x])^n,u,x] - b*c*n*Int[SimplifyIntegrand[u*(a+b*ArcSinh[c*x])^(n-1)/Sqrt[1+c^2*x^2],x],x] /;
    FreeQ[{a,b,c,d,e,f,g,h},x] && IGtQ[n,0] && IGtQ[p,0] && EqQ[e*g-2*d*h,0]
```

```
Int[(f_.+g_.**x_+h_.**x_^2)^p_.*(a_.+b_.*ArcCosh[c_.**x_])^n_/(d_+e_.**x_)^2,x_Symbol] :=
  With[{u=IntHide[(f+g*x+h*x^2)^p/(d+e*x)^2,x]},
    Dist[(a+b*ArcCosh[c*x])^n,u,x] - b*c*n*Int[SimplifyIntegrand[u*(a+b*ArcCosh[c*x])^(n-1)/(Sqrt[1+c*x]*Sqrt[-1+c*x]),x],x] /;
    FreeQ[{a,b,c,d,e,f,g,h},x] && IGtQ[n,0] && IGtQ[p,0] && EqQ[e*g-2*d*h,0]
```

4: $\int P_x (d + e x)^m (a + b \operatorname{ArcSinh}[c x])^n dx$ when $n \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}$

Derivation: Algebraic expansion

Rule: If $n \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}$, then

$$\int P_x (d + e x)^m (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \int \operatorname{ExpandIntegrand}[P_x (d + e x)^m (a + b \operatorname{ArcSinh}[c x])^n, x] dx$$

Program code:

```
Int[Px_*(d_+e_.*x_)^m_.*(a_+b_.*ArcSinh[c_.*x_])^n_.,x_Symbol] :=
  Int[ExpandIntegrand[Px*(d+e*x)^m*(a+b*ArcSinh[c*x])^n,x],x] /;
FreeQ[{a,b,c,d,e},x] && PolynomialQ[Px,x] && IGtQ[n,0] && IntegerQ[m]
```

```
Int[Px_*(d_+e_.*x_)^m_.*(a_+b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=
  Int[ExpandIntegrand[Px*(d+e*x)^m*(a+b*ArcCosh[c*x])^n,x],x] /;
FreeQ[{a,b,c,d,e},x] && PolynomialQ[Px,x] && IGtQ[n,0] && IntegerQ[m]
```

4. $\int (f + g x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx$ when $e = c^2 d \wedge m \in \mathbb{Z} \wedge p - \frac{1}{2} \in \mathbb{Z}$

1. $\int (f + g x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx$ when $e = c^2 d \wedge m \in \mathbb{Z} \wedge p - \frac{1}{2} \in \mathbb{Z} \wedge d > 0$

1: $\int (f + g x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x]) dx$ when $e = c^2 d \wedge m \in \mathbb{Z} \wedge p + \frac{1}{2} \in \mathbb{Z}^- \wedge d > 0 \wedge m > 0$

Derivation: Integration by parts

Note: If $m \in \mathbb{Z} \wedge p + \frac{1}{2} \in \mathbb{Z} \wedge 0 < m < -2p - 1$, then $\int (f + g x)^m (d + e x^2)^p dx$ is an algebraic function.

Rule: If $e = c^2 d \wedge m \in \mathbb{Z} \wedge p + \frac{1}{2} \in \mathbb{Z}^- \wedge d > 0 \wedge m > 0$, let $u = \int (f + g x)^m (d + e x^2)^p dx$, then

$$\int (f + g x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow u (a + b \operatorname{ArcSinh}[c x]) - b c \int \frac{u}{\sqrt{1 + c^2 x^2}} dx$$

Program code:

```
Int[(f_+g_.*x_)^m_.*(d_+e_.*x_^2)^p_.*(a_+b_.*ArcSinh[c_.*x_]),x_Symbol] :=
  With[{u=IntHide[(f+g*x)^m*(d+e*x^2)^p,x]},
    Dist[a+b*ArcSinh[c*x],u,x] - b*c*Int[Dist[1/Sqrt[1+c^2*x^2],u,x],x] /;
  FreeQ[{a,b,c,d,e,f,g},x] && EqQ[e,c^2*d] && IGtQ[m,0] && ILtQ[p+1/2,0] && GtQ[d,0] && (LtQ[m,-2*p-1] || GtQ[m,3])
```

```
Int[(f_+g_.*x_)^m_.*(d1_+e1_.*x_)^p_.*(d2_+e2_.*x_)^p_.*(a_+b_.*ArcCosh[c_.*x_]),x_Symbol] :=
  With[{u=IntHide[(f+g*x)^m*(d1+e1*x)^p*(d2+e2*x)^p,x]},
    Dist[a+b*ArcCosh[c*x],u,x] - b*c*Int[Dist[1/(Sqrt[1+c*x]*Sqrt[-1+c*x]),u,x],x] /;
  FreeQ[{a,b,c,d1,e1,d2,e2,f,g},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && IGtQ[m,0] && ILtQ[p+1/2,0] && GtQ[d1,0] && LtQ[d2,0] &&
  (LtQ[m,-2*p-1] || GtQ[m,3])
```

2: $\int (f + g x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx$ when $e = c^2 d \wedge m \in \mathbb{Z} \wedge p + \frac{1}{2} \in \mathbb{Z} \wedge d > 0 \wedge n \in \mathbb{Z}^+ \wedge m > 0$

Derivation: Algebraic expansion

Rule: If $e = c^2 d \wedge m \in \mathbb{Z} \wedge p + \frac{1}{2} \in \mathbb{Z} \wedge d > 0 \wedge n \in \mathbb{Z}^+ \wedge m > 0$, then

$$\int (f + g x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \int (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n \operatorname{ExpandIntegrand}[(f + g x)^m, x] dx$$

Program code:

```
Int[(f_+g_.*x_)^m_.*(d_+e_.*x_^2)^p_.*(a_+b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x^2)^p*(a+b*ArcSinh[c*x])^n,(f+g*x)^m,x],x] /;
  FreeQ[{a,b,c,d,e,f,g},x] && EqQ[e,c^2*d] && IGtQ[m,0] && IntegerQ[p+1/2] && GtQ[d,0] && IGtQ[n,0] &&
  (EqQ[n,1] && GtQ[p,-1] || GtQ[p,0] || EqQ[m,1] || EqQ[m,2] && LtQ[p,-2])
```

```

Int[(f_+g_.*x_)^m_.*(d1_+e1_.*x_)^p_.*(d2_+e2_.*x_)^p_.*(a_+b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=
  Int[ExpandIntegrand[(d1+e1*x)^p*(d2+e2*x)^p*(a+b*ArcCosh[c*x])^n,(f+g*x)^m,x],x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,g},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && IGtQ[m,0] && IntegerQ[p+1/2] && GtQ[d1,0] && LtQ[d2,0] && IGtQ[n,1] && GtQ[p,-1] || GtQ[p,0] || EqQ[m,1] || EqQ[m,2] && LtQ[p,-2])

```

$$3. \int (f+g x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge m \in \mathbb{Z} \wedge p + \frac{1}{2} \in \mathbb{Z}^+ \wedge d > 0$$

$$1: \int (f+g x)^m \sqrt{d+e x^2} (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge m \in \mathbb{Z} \wedge d > 0 \wedge n \in \mathbb{Z}^+ \wedge m < 0$$

Derivation: Integration by parts

Basis: If $e = c^2 d \wedge d > 0$, then $\frac{(a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{d+e x^2}} = \partial_x \frac{(a+b \operatorname{ArcSinh}[c x])^{n+1}}{b c \sqrt{d} (n+1)}$

Rule: If $e = c^2 d \wedge m \in \mathbb{Z} \wedge d > 0 \wedge n \in \mathbb{Z}^+ \wedge m < 0$, then

$$\begin{aligned} & \int (f+g x)^m \sqrt{d+e x^2} (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \\ & \frac{(f+g x)^m (d+e x^2) (a+b \operatorname{ArcSinh}[c x])^{n+1}}{b c \sqrt{d} (n+1)} - \\ & \frac{1}{b c \sqrt{d} (n+1)} \int (d g m + 2 e f x + e g (m+2) x^2) (f+g x)^{m-1} (a+b \operatorname{ArcSinh}[c x])^{n+1} dx \end{aligned}$$

Program code:

```

Int[(f_+g_.*x_)^m_*Sqrt[d_+e_.*x_^2]*(a_+b_.*ArcSinh[c_.*x_])^n_.,x_Symbol] :=
  (f+g*x)^m*(d+e*x^2)*(a+b*ArcSinh[c*x])^(n+1)/(b*c*Sqrt[d]*(n+1)) -
  1/(b*c*Sqrt[d]*(n+1))*Int[(d*g*m+2*e*f*x+e*g*(m+2)*x^2)*(f+g*x)^(m-1)*(a+b*ArcSinh[c*x])^(n+1),x] /;
FreeQ[{a,b,c,d,e,f,g},x] && EqQ[e,c^2*d] && ILtQ[m,0] && GtQ[d,0] && IGtQ[n,0]

```

```

Int[(f_+g_.*x_)^m*Sqrt[d1_+e1_.*x_]*Sqrt[d2_+e2_.*x_]*(a_+b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  (f+g*x)^m*(d1*d2+e1*e2*x^2)*(a+b*ArcCosh[c*x])^(n+1)/(b*c*Sqrt[-d1*d2]*(n+1)) -
  1/(b*c*Sqrt[-d1*d2]*(n+1))*Int[(d1*d2*g*m+2*e1*e2*f*x+e1*e2*g*(m+2)*x^2)*(f+g*x)^(m-1)*(a+b*ArcCosh[c*x])^(n+1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,g},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && ILtQ[m,0] && GtQ[d1,0] && LtQ[d2,0] && IGtQ[n,0]

```

2: $\int (f+g x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx$ when $e = c^2 d \wedge m \in \mathbb{Z} \wedge p + \frac{1}{2} \in \mathbb{Z}^+ \wedge d > 0 \wedge n \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $e = c^2 d \wedge m \in \mathbb{Z} \wedge p + \frac{1}{2} \in \mathbb{Z}^+ \wedge d > 0 \wedge n \in \mathbb{Z}^+$, then

$$\int (f+g x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \int \sqrt{d+e x^2} (a+b \operatorname{ArcSinh}[c x])^n \operatorname{ExpandIntegrand}[(f+g x)^m (d+e x^2)^{p-1/2}, x] dx$$

Program code:

```

Int[(f_+g_.*x_)^m_.*(d_+e_.*x_^2)^p_.*(a_+b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  Int[ExpandIntegrand[Sqrt[d+e*x^2]*(a+b*ArcSinh[c*x])^n,(f+g*x)^m*(d+e*x^2)^(p-1/2),x],x] /;
FreeQ[{a,b,c,d,e,f,g},x] && EqQ[e,c^2*d] && IntegerQ[m] && IGtQ[p+1/2,0] && GtQ[d,0] && IGtQ[n,0]

```

```

Int[(f_+g_.*x_)^m_.*(d1_+e1_.*x_)^p_.*(d2_+e2_.*x_)^p_.*(a_+b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  Int[ExpandIntegrand[Sqrt[d1+e1*x]*Sqrt[d2+e2*x]*(a+b*ArcCosh[c*x])^n,(f+g*x)^m*(d1+e1*x)^(p-1/2)*(d2+e2*x)^(p-1/2),x],x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,g},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && IntegerQ[m] && IGtQ[p+1/2,0] && GtQ[d1,0] && LtQ[d2,0] && IG

```

$$\text{3: } \int (f + g x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge m \in \mathbb{Z} \wedge p - \frac{1}{2} \in \mathbb{Z}^+ \wedge d > 0 \wedge n \in \mathbb{Z}^+ \wedge m < 0$$

Derivation: Integration by parts

$$\text{Basis: If } e = c^2 d \wedge d > 0, \text{ then } \frac{(a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{d+e x^2}} = \partial_x \frac{(a+b \operatorname{ArcSinh}[c x])^{n+1}}{b c \sqrt{d} (n+1)}$$

Rule: If $e = c^2 d \wedge m \in \mathbb{Z} \wedge p - \frac{1}{2} \in \mathbb{Z}^+ \wedge d > 0 \wedge n \in \mathbb{Z}^+ \wedge m < 0$, then

$$\int (f + g x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{(f + g x)^m (d + e x^2)^{p+\frac{1}{2}} (a + b \operatorname{ArcSinh}[c x])^{n+1}}{b c \sqrt{d} (n+1)} - \frac{1}{b c \sqrt{d} (n+1)} \int (f + g x)^{m-1} (a + b \operatorname{ArcSinh}[c x])^{n+1} \operatorname{ExpandIntegrand}[(d g m + e f (2 p + 1) x + e g (m + 2 p + 1) x^2) (d + e x^2)^{p-\frac{1}{2}}, x] dx$$

Program code:

```
Int[(f_+g_.*x_)^m_.*(d_+e_.*x_^2)^p_.*(a_+b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  (f+g*x)^m*(d+e*x^2)^(p+1/2)*(a+b*ArcSinh[c*x])^(n+1)/(b*c*Sqrt[d]*(n+1)) -
  1/(b*c*Sqrt[d]*(n+1))*
  Int[ExpandIntegrand[(f+g*x)^(m-1)*(a+b*ArcSinh[c*x])^(n+1), (d*g*m+e*f*(2*p+1)*x+e*g*(m+2*p+1)*x^2)*(d+e*x^2)^(p-1/2),x],x] /;
FreeQ[{a,b,c,d,e,f,g},x] && EqQ[e,c^2*d] && ILtQ[m,0] && IGtQ[p-1/2,0] && GtQ[d,0] && IGtQ[n,0]
```

```
Int[(f_+g_.*x_)^m_.*(d1_+e1_.*x_)^p_.*(d2_+e2_.*x_)^p_.*(a_+b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  (f+g*x)^m*(d1+e1*x)^(p+1/2)*(d2+e2*x)^(p+1/2)*(a+b*ArcCosh[c*x])^(n+1)/(b*c*Sqrt[-d1*d2]*(n+1)) -
  1/(b*c*Sqrt[-d1*d2]*(n+1))*
  Int[ExpandIntegrand[(f+g*x)^(m-1)*(a+b*ArcCosh[c*x])^(n+1),
  (d1*d2*g*m+e1*e2*f*(2*p+1)*x+e1*e2*g*(m+2*p+1)*x^2)*(d1+e1*x)^(p-1/2)*(d2+e2*x)^(p-1/2),x],x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,g},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && ILtQ[m,0] && IGtQ[p-1/2,0] && GtQ[d1,0] && LtQ[d2,0] && IGtQ
```

$$4. \int (f+g x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge m \in \mathbb{Z} \wedge p - \frac{1}{2} \in \mathbb{Z}^- \wedge d > 0$$

$$1. \int \frac{(f+g x)^m (a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{d+e x^2}} dx \text{ when } e = c^2 d \wedge m \in \mathbb{Z} \wedge d > 0$$

$$1: \int \frac{(f+g x)^m (a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{d+e x^2}} dx \text{ when } e = c^2 d \wedge m \in \mathbb{Z} \wedge d > 0 \wedge m > 0 \wedge n < -1$$

Derivation: Integration by parts

Basis: If $e = c^2 d \wedge d > 0$, then $\frac{(a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{d+e x^2}} = \partial_x \frac{(a+b \operatorname{ArcSinh}[c x])^{n+1}}{b c \sqrt{d} (n+1)}$

Rule: If $e = c^2 d \wedge m \in \mathbb{Z} \wedge d > 0 \wedge m > 0 \wedge n < -1$, then

$$\int \frac{(f+g x)^m (a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{d+e x^2}} dx \rightarrow \frac{(f+g x)^m (a+b \operatorname{ArcSinh}[c x])^{n+1}}{b c \sqrt{d} (n+1)} - \frac{g m}{b c \sqrt{d} (n+1)} \int (f+g x)^{m-1} (a+b \operatorname{ArcSinh}[c x])^{n+1} dx$$

Program code:

```
Int[(f+_g_.*x_)^m_.*(a+_b_.*ArcSinh[c_.*x_])^n_/Sqrt[d+_e_.*x_^2],x_Symbol] :=
  (f+g*x)^m*(a+b*ArcSinh[c*x])^(n+1)/(b*c*Sqrt[d]*(n+1)) -
  g*m/(b*c*Sqrt[d]*(n+1))*Int[(f+g*x)^(m-1)*(a+b*ArcSinh[c*x])^(n+1),x] /;
FreeQ[{a,b,c,d,e,f,g},x] && EqQ[e,c^2*d] && IGtQ[m,0] && GtQ[d,0] && LtQ[n,-1]
```

```
Int[(f+_g_.*x_)^m_.*(a+_b_.*ArcCosh[c_.*x_])^n_/Sqrt[d1+_e1_.*x_]*Sqrt[d2+_e2_.*x_]),x_Symbol] :=
  (f+g*x)^m*(a+b*ArcCosh[c*x])^(n+1)/(b*c*Sqrt[-d1*d2]*(n+1)) -
  g*m/(b*c*Sqrt[-d1*d2]*(n+1))*Int[(f+g*x)^(m-1)*(a+b*ArcCosh[c*x])^(n+1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,g},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && IGtQ[m,0] && GtQ[d1,0] && LtQ[d2,0] && LtQ[n,-1]
```


$$\text{2: } \int \frac{(f+g x)^m (a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{d+e x^2}} dx \text{ when } e == c^2 d \wedge m \in \mathbb{Z} \wedge d > 0 \wedge (m > 0 \vee n \in \mathbb{Z}^+)$$

Derivation: Integration by substitution

Basis: If $e == c^2 d \wedge d > 0$, then $\frac{F[x]}{\sqrt{d+e x^2}} == \frac{1}{c \sqrt{d}} \operatorname{Subst}\left[F\left[\frac{\operatorname{Sinh}[x]}{c}\right], x, \operatorname{ArcSinh}[c x]\right] \partial_x \operatorname{ArcSinh}[c x]$

Basis: If $d_1 > 0 \wedge d_2 < 0$, then

$$\frac{F[x]}{\sqrt{d_1+c d_1 x} \sqrt{d_2-c d_2 x}} == \frac{1}{c \sqrt{-d_1 d_2}} \operatorname{Subst}\left[F\left[\frac{\operatorname{Cosh}[x]}{c}\right], x, \operatorname{ArcCosh}[c x]\right] \partial_x \operatorname{ArcCosh}[c x]$$

Note: *Mathematica 8* is unable to validate antiderivatives of *ArcCosh* rule when c is symbolic.

Rule: If $e == c^2 d \wedge m \in \mathbb{Z} \wedge d > 0 \wedge (m > 0 \vee n \in \mathbb{Z}^+)$, then

$$\int \frac{(f+g x)^m (a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{d+e x^2}} dx \rightarrow \frac{1}{c^{m+1} \sqrt{d}} \operatorname{Subst}\left[\int (a+b x)^n (c f+g \operatorname{Sinh}[x])^m dx, x, \operatorname{ArcSinh}[c x]\right]$$

Program code:

```
Int[(f_+g_.*x_)^m_.*(a_+b_.*ArcSinh[c_.*x_])^n_/Sqrt[d_+e_.*x_^2],x_Symbol] :=
  1/(c^(m+1)*Sqrt[d])*Subst[Int[(a+b*x)^n*(c*f+g*Sinh[x])^m,x],x,ArcSinh[c*x]] /;
FreeQ[{a,b,c,d,e,f,g,n},x] && EqQ[e,c^2*d] && IntegerQ[m] && GtQ[d,0] && (GtQ[m,0] || IGtQ[n,0])
```

```
Int[(f_+g_.*x_)^m_.*(a_+b_.*ArcCosh[c_.*x_])^n_/(Sqrt[d1_+e1_.*x_]*Sqrt[d2_+e2_.*x_]),x_Symbol] :=
  1/(c^(m+1)*Sqrt[-d1*d2])*Subst[Int[(a+b*x)^n*(c*f+g*Cosh[x])^m,x],x,ArcCosh[c*x]] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,g,n},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && IntegerQ[m] && GtQ[d1,0] && LtQ[d2,0] && (GtQ[m,0] || IGtQ[n,0])
```

$$\text{2: } \int (f + g x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge m \in \mathbb{Z} \wedge p + \frac{1}{2} \in \mathbb{Z}^- \wedge d > 0 \wedge n \in \mathbb{Z}^+$$

Derivation: Algebraic expansion

Rule: If $e = c^2 d \wedge m \in \mathbb{Z} \wedge p + \frac{1}{2} \in \mathbb{Z}^- \wedge d > 0 \wedge n \in \mathbb{Z}^+$, then

$$\int (f + g x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \int \frac{(a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{d + e x^2}} \operatorname{ExpandIntegrand}[(f + g x)^m (d + e x^2)^{p+1/2}, x] dx$$

Program code:

```
Int[(f_+g_.*x_)^m_.*(d_+e_.*x_^2)^p_.*(a_+b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*ArcSinh[c*x])^n/Sqrt[d+e*x^2], (f+g*x)^m*(d+e*x^2)^(p+1/2), x], x] /;
FreeQ[{a,b,c,d,e,f,g},x] && EqQ[e,c^2*d] && IntegerQ[m] && ILtQ[p+1/2,0] && GtQ[d,0] && IGtQ[n,0]
```

```
Int[(f_+g_.*x_)^m_.*(d1_+e1_.*x_)^p_.*(d2_+e2_.*x_)^p_.*(a_+b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*ArcCosh[c*x])^n/(Sqrt[d1+e1*x]*Sqrt[d2+e2*x]), (f+g*x)^m*(d1+e1*x)^(p+1/2)*(d2+e2*x)^(p+1/2), x], x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,g},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && IntegerQ[m] && ILtQ[p+1/2,0] && GtQ[d1,0] && LtQ[d2,0] && IGtQ[n,0]
```

2: $\int (f + g x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx$ when $e = c^2 d \wedge m \in \mathbb{Z} \wedge p - \frac{1}{2} \in \mathbb{Z} \wedge d \neq 0$

Derivation: Piecewise constant extraction

Basis: If $e = c^2 d$, then $\partial_x \frac{(d+e x^2)^p}{(1+c^2 x^2)^p} = 0$

Rule: If $e = c^2 d \wedge m \in \mathbb{Z} \wedge p - \frac{1}{2} \in \mathbb{Z} \wedge d \neq 0$, then

$$\int (f + g x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{d^{\operatorname{IntPart}[p]} (d + e x^2)^{\operatorname{FracPart}[p]}}{(1 + c^2 x^2)^{\operatorname{FracPart}[p]}} \int (f + g x)^m (1 + c^2 x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx$$

Program code:

```
Int[(f+_.*x_)^m_.*(d+_.*x_^2)^p_.*(a+_.*b_.*ArcSinh[c_.*x_])^n_.,x_Symbol] :=
  d^IntPart[p]*(d+e*x^2)^FracPart[p]/(1+c^2*x^2)^FracPart[p]*Int[(f+g*x)^m*(1+c^2*x^2)^p*(a+b*ArcSinh[c*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,g,n},x] && EqQ[e,c^2*d] && IntegerQ[m] && IntegerQ[p-1/2] && Not[GtQ[d,0]]
```

```
Int[(f+_.*x_)^m_.*(d+_.*x_^2)^p_.*(a+_.*b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=
  (-d)^IntPart[p]*(d+e*x^2)^FracPart[p]/((1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
  Int[(f+g*x)^m*(1+c*x)^p*(-1+c*x)^p*(a+b*ArcCosh[c*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,g,n},x] && EqQ[c^2*d+e,0] && IntegerQ[m] && IntegerQ[p-1/2]
```

```
Int[(f+_.*x_)^m_.*(d1+_.*e1_.*x_)^p_.*(d2+_.*e2_.*x_)^p_.*(a+_.*b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=
  (-d1*d2)^IntPart[p]*(d1+e1*x)^FracPart[p]*(d2+e2*x)^FracPart[p]/(1-c^2*x^2)^FracPart[p]*
  Int[(f+g*x)^m*(1+c*x)^p*(-1+c*x)^p*(a+b*ArcCosh[c*x])^n,x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,g,n},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && IntegerQ[m] && IntegerQ[p-1/2] && Not[GtQ[d1,0] && LtQ[d2,0]]
```

5. $\int \operatorname{Log}[h (f + g x)^m] (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx$ when $e = c^2 d \wedge p - \frac{1}{2} \in \mathbb{Z}$

1. $\int \operatorname{Log}[h (f + g x)^m] (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx$ when $e = c^2 d \wedge p - \frac{1}{2} \in \mathbb{Z} \wedge d > 0$

$$1: \int \frac{\operatorname{Log}[h(f+g x)^m] (a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{d+e x^2}} dx \text{ when } e = c^2 d \wedge d > 0 \wedge n \in \mathbb{Z}^+$$

Derivation: Integration by parts

Basis: If $e = c^2 d \wedge d > 0$, then $\frac{(a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{d+e x^2}} = \partial_x \frac{(a+b \operatorname{ArcSinh}[c x])^{n+1}}{b c \sqrt{d} (n+1)}$

Note: If $n \in \mathbb{Z}^+$, then $\frac{(a+b \operatorname{ArcSinh}[c x])^{n+1}}{f+g x}$ is integrable in closed-form.

Rule: If $e = c^2 d \wedge d > 0 \wedge n \in \mathbb{Z}^+$, then

$$\int \frac{\operatorname{Log}[h(f+g x)^m] (a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{d+e x^2}} dx \rightarrow \frac{\operatorname{Log}[h(f+g x)^m] (a+b \operatorname{ArcSinh}[c x])^{n+1}}{b c \sqrt{d} (n+1)} - \frac{g m}{b c \sqrt{d} (n+1)} \int \frac{(a+b \operatorname{ArcSinh}[c x])^{n+1}}{f+g x} dx$$

Program code:

```
Int[Log[h_.*(f_+g_.*x_)^m_.*(a_+b_.*ArcSinh[c_.*x_])^n_./Sqrt[d_+e_.*x_^2],x_Symbol] :=
  Log[h*(f+g*x)^m]*(a+b*ArcSinh[c*x])^(n+1)/(b*c*Sqrt[d]*(n+1)) -
  g*m/(b*c*Sqrt[d]*(n+1))*Int[(a+b*ArcSinh[c*x])^(n+1)/(f+g*x),x] /;
FreeQ[{a,b,c,d,e,f,g,h,m},x] && EqQ[e,c^2*d] && GtQ[d,0] && IGtQ[n,0]
```

```
Int[Log[h_.*(f_+g_.*x_)^m_.*(a_+b_.*ArcCosh[c_.*x_])^n_./(Sqrt[d1_+e1_.*x_] * Sqrt[d2_+e2_.*x_] ),x_Symbol] :=
  Log[h*(f+g*x)^m]*(a+b*ArcCosh[c*x])^(n+1)/(b*c*Sqrt[-d1*d2]*(n+1)) -
  g*m/(b*c*Sqrt[-d1*d2]*(n+1))*Int[(a+b*ArcCosh[c*x])^(n+1)/(f+g*x),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,g,h,m},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[d1,0] && LtQ[d2,0] && IGtQ[n,0]
```

2: $\int \operatorname{Log}[h (f + g x)^m] (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx$ when $e = c^2 d \wedge p - \frac{1}{2} \in \mathbb{Z} \wedge d \neq 0$

Derivation: Piecewise constant extraction

Basis: If $e = c^2 d$, then $\partial_x \frac{(d+e x^2)^p}{(1+c^2 x^2)^p} = 0$

Rule: If $e = c^2 d \wedge p - \frac{1}{2} \in \mathbb{Z} \wedge d \neq 0$, then

$$\int \operatorname{Log}[h (f + g x)^m] (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{d^{\operatorname{IntPart}[p]} (d + e x^2)^{\operatorname{FracPart}[p]}}{(1 + c^2 x^2)^{\operatorname{FracPart}[p]}} \int \operatorname{Log}[h (f + g x)^m] (1 + c^2 x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx$$

Program code:

```
Int[Log[h_.*(f_+g_.*x_)^m_].*(d_+e_.*x_^2)^p_*(a_+b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  d^IntPart[p]*(d+e*x^2)^FracPart[p]/(1+c^2*x^2)^FracPart[p]*Int[Log[h*(f+g*x)^m]*(1+c^2*x^2)^p*(a+b*ArcSinh[c*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,g,h,m,n},x] && EqQ[e,c^2*d] && IntegerQ[p-1/2] && Not[GtQ[d,0]]
```

```
Int[Log[h_.*(f_+g_.*x_)^m_].*(d_+e_.*x_^2)^p_*(a_+b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  (-d)^IntPart[p]*(d+e*x^2)^FracPart[p]/((1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
  Int[Log[h*(f+g*x)^m]*(1+c*x)^p*(-1+c*x)^p*(a+b*ArcCosh[c*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,g,h,m,n},x] && EqQ[c^2*d+e,0] && IntegerQ[p-1/2]
```

```
Int[Log[h_.*(f_+g_.*x_)^m_].*(d1_+e1_.*x_)^p_*(d2_+e2_.*x_)^p_*(a_+b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  (-d1*d2)^IntPart[p]*(d1+e1*x)^FracPart[p]*(d2+e2*x)^FracPart[p]/((1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
  Int[Log[h*(f+g*x)^m]*(1+c*x)^p*(-1+c*x)^p*(a+b*ArcCosh[c*x])^n,x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,g,h,m,n},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && IntegerQ[p-1/2] && Not[GtQ[d1,0] && LtQ[d2,0]]
```

$$6. \int (d + e x)^m (f + g x)^m (a + b \operatorname{ArcSinh}[c x])^n dx$$

$$1: \int (d + e x)^m (f + g x)^m (a + b \operatorname{ArcSinh}[c x]) dx \text{ when } m + \frac{1}{2} \in \mathbb{Z}^-$$

Derivation: Integration by parts

Rule: If $m + \frac{1}{2} \in \mathbb{Z}^-$, let $u = \int (d + e x)^m (f + g x)^m dx$, then

$$\int (d + e x)^m (f + g x)^m (a + b \operatorname{ArcSinh}[c x]) dx \rightarrow u (a + b \operatorname{ArcSinh}[c x]) - b c \int \frac{u}{\sqrt{1 + c^2 x^2}} dx$$

Program code:

```
Int[(d_+e_.*x_)^m_*(f_+g_.*x_)^m_*(a_+b_.*ArcSinh[c_.*x_]),x_Symbol] :=
  With[{u=IntHide[(d+e*x)^m*(f+g*x)^m,x]},
    Dist[a+b*ArcSinh[c*x],u,x] - b*c*Int[Dist[1/Sqrt[1+c^2*x^2],u,x],x] /;
  FreeQ[{a,b,c,d,e,f,g},x] && ILtQ[m+1/2,0]
```

```
Int[(d_+e_.*x_)^m_*(f_+g_.*x_)^m_*(a_+b_.*ArcCosh[c_.*x_]),x_Symbol] :=
  With[{u=IntHide[(d+e*x)^m*(f+g*x)^m,x]},
    Dist[a+b*ArcCosh[c*x],u,x] - b*c*Int[Dist[1/(Sqrt[1+c*x]*Sqrt[-1+c*x]),u,x],x] /;
  FreeQ[{a,b,c,d,e,f,g},x] && ILtQ[m+1/2,0]
```

2: $\int (d + e x)^m (f + g x)^m (a + b \operatorname{ArcSinh}[c x])^n dx$ when $m \in \mathbb{Z}$

Derivation: Algebraic expansion

Rule: If $m \in \mathbb{Z}$, then

$$\int (d + e x)^m (f + g x)^m (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \int (a + b \operatorname{ArcSinh}[c x])^n \operatorname{ExpandIntegrand}[(d + e x)^m (f + g x)^m, x] dx$$

Program code:

```
Int[(d_+e_.*x_)^m_.*(f_+g_.*x_)^m_.*(a_+b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*ArcSinh[c*x])^n,(d+e*x)^m*(f+g*x)^m,x],x] /;
FreeQ[{a,b,c,d,e,f,g,n},x] && IntegerQ[m]
```

```
Int[(d_+e_.*x_)^m_.*(f_+g_.*x_)^m_.*(a_+b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*ArcCosh[c*x])^n,(d+e*x)^m*(f+g*x)^m,x],x] /;
FreeQ[{a,b,c,d,e,f,g,n},x] && IntegerQ[m]
```

7: $\int u (a + b \operatorname{ArcSinh}[c x]) \, dx$ when $\int u \, dx$ is free of inverse functions

Derivation: Integration by parts

Rule: Let $v = \int u \, dx$, if v is free of inverse functions, then

$$\int u (a + b \operatorname{ArcSinh}[c x]) \, dx \rightarrow v (a + b \operatorname{ArcSinh}[c x]) - b c \int \frac{v}{\sqrt{1 + c^2 x^2}} \, dx$$

$$\int u (a + b \operatorname{ArcCosh}[c x]) \, dx \rightarrow v (a + b \operatorname{ArcCosh}[c x]) - \frac{b c \sqrt{1 - c^2 x^2}}{\sqrt{-1 + c x} \sqrt{1 + c x}} \int \frac{v}{\sqrt{1 - c^2 x^2}} \, dx$$

Program code:

```
Int[u_*(a_.+b_.*ArcSinh[c_.*x_]),x_Symbol] :=
  With[{v=IntHide[u,x]},
    Dist[a+b*ArcSinh[c*x],v,x] - b*c*Int[SimplifyIntegrand[v/Sqrt[1+c^2*x^2],x],x] /;
    InverseFunctionFreeQ[v,x] /;
    FreeQ[{a,b,c},x]
```

```
Int[u_*(a_.+b_.*ArcCosh[c_.*x_]),x_Symbol] :=
  With[{v=IntHide[u,x]},
    Dist[a+b*ArcCosh[c*x],v,x] - b*c*Sqrt[1-c^2*x^2]/(Sqrt[-1+c*x]*Sqrt[1+c*x])*Int[SimplifyIntegrand[v/Sqrt[1-c^2*x^2],x],x] /;
    InverseFunctionFreeQ[v,x] /;
    FreeQ[{a,b,c},x]
```


$$8. \int P_x u (a + b \operatorname{ArcSinh}[c x])^n dx$$

$$1: \int P_x (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge p - \frac{1}{2} \in \mathbb{Z}$$

Derivation: Algebraic expansion

Rule: If $e = c^2 d \wedge p - \frac{1}{2} \in \mathbb{Z}$, then

$$\int P_x (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \int \operatorname{ExpandIntegrand}[P_x (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n, x] dx$$

Program code:

```
Int[Px*(d+e.*x_^2)^p*(a_.+b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  With[{u=ExpandIntegrand[Px*(d+e*x^2)^p*(a+b*ArcSinh[c*x])^n,x]},
    Int[u,x] /;
    SumQ[u] /;
    FreeQ[{a,b,c,d,e,n},x] && PolynomialQ[Px,x] && EqQ[e,c^2*d] && IntegerQ[p-1/2]
```

```
Int[Px*(d1+e1.*x_)^p*(d2+e2.*x_)^p*(a_.+b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  With[{u=ExpandIntegrand[Px*(d1+e1*x)^p*(d2+e2*x)^p*(a+b*ArcCosh[c*x])^n,x]},
    Int[u,x] /;
    SumQ[u] /;
    FreeQ[{a,b,c,d1,e1,d2,e2,n},x] && PolynomialQ[Px,x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && IntegerQ[p-1/2]
```

2: $\int P_x (f + g (d + e x^2)^p)^m (a + b \operatorname{ArcSinh}[c x])^n dx$ when $e = c^2 d \wedge p + \frac{1}{2} \in \mathbb{Z}^+ \wedge (m | n) \in \mathbb{Z}$

Derivation: Algebraic expansion

Rule: If $e = c^2 d \wedge p + \frac{1}{2} \in \mathbb{Z}^+ \wedge (m | n) \in \mathbb{Z}$, then

$$\int P_x (f + g (d + e x^2)^p)^m (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \int \operatorname{ExpandIntegrand}[P_x (f + g (d + e x^2)^p)^m (a + b \operatorname{ArcSinh}[c x])^n, x] dx$$

Program code:

```
Int[Px_.*(f_+g_.*(d_+e_.*x_^2)^p_)^m_.*(a_+b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  With[{u=ExpandIntegrand[Px*(f+g*(d+e*x^2)^p)^m*(a+b*ArcSinh[c*x])^n,x]},
    Int[u,x] /;
    SumQ[u] /;
    FreeQ[{a,b,c,d,e,f,g},x] && PolynomialQ[Px,x] && EqQ[e,c^2*d] && IGtQ[p+1/2,0] && IntegersQ[m,n]
```

```
Int[Px_.*(f_+g_.*(d1_+e1_.*x_)^p_.*(d2_+e2_.*x_)^p_)^m_.*(a_+b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  With[{u=ExpandIntegrand[Px*(f+g*(d1+e1*x)^p*(d2+e2*x)^p)^m*(a+b*ArcCosh[c*x])^n,x]},
    Int[u,x] /;
    SumQ[u] /;
    FreeQ[{a,b,c,d1,e1,d2,e2,f,g},x] && PolynomialQ[Px,x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && IGtQ[p+1/2,0] && IntegersQ[m,n]
```

9. $\int \operatorname{RF}_x u (a + b \operatorname{ArcSinh}[c x])^n dx$ when $n \in \mathbb{Z}^+$

1. $\int \operatorname{RF}_x (a + b \operatorname{ArcSinh}[c x])^n dx$ when $n \in \mathbb{Z}^+$

1: $\int \operatorname{RF}_x \operatorname{ArcSinh}[c x]^n dx$ when $n \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $n \in \mathbb{Z}^+$, then

$$\int \operatorname{RF}_x \operatorname{ArcSinh}[c x]^n dx \rightarrow \int \operatorname{ArcSinh}[c x]^n \operatorname{ExpandIntegrand}[\operatorname{RF}_x, x] dx$$

Program code:

```
Int[RFx_*ArcSinh[c_.*x_]^n_,x_Symbol] :=
  With[{u=ExpandIntegrand[ArcSinh[c*x]^n,RFx,x]},
    Int[u,x] /;
    SumQ[u] /;
    FreeQ[c,x] && RationalFunctionQ[RFx,x] && IGtQ[n,0]
```

```
Int[RFx_*ArcCosh[c_.*x_]^n_,x_Symbol] :=
  With[{u=ExpandIntegrand[ArcCosh[c*x]^n,RFx,x]},
    Int[u,x] /;
    SumQ[u] /;
    FreeQ[c,x] && RationalFunctionQ[RFx,x] && IGtQ[n,0]
```

2: $\int \operatorname{RF}_x (a + b \operatorname{ArcSinh}[c x])^n dx$ when $n \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $n \in \mathbb{Z}^+$, then

$$\int \operatorname{RF}_x (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \int \operatorname{ExpandIntegrand}[\operatorname{RF}_x (a + b \operatorname{ArcSinh}[c x])^n, x] dx$$

Program code:

```
Int[RFx_*(a_+b_.*ArcSinh[c_.*x_])^n_.,x_Symbol] :=
  Int[ExpandIntegrand[RFx*(a+b*ArcSinh[c*x])^n,x],x] /;
FreeQ[{a,b,c},x] && RationalFunctionQ[RFx,x] && IGtQ[n,0]
```

```
Int[RFx_*(a_+b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=
  Int[ExpandIntegrand[RFx*(a+b*ArcCosh[c*x])^n,x],x] /;
FreeQ[{a,b,c},x] && RationalFunctionQ[RFx,x] && IGtQ[n,0]
```

$$2. \int \operatorname{RF}_x (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } n \in \mathbb{Z}^+ \wedge e = c^2 d \wedge p - \frac{1}{2} \in \mathbb{Z}$$

$$1: \int \operatorname{RF}_x (d + e x^2)^p \operatorname{ArcSinh}[c x]^n dx \text{ when } n \in \mathbb{Z}^+ \wedge e = c^2 d \wedge p - \frac{1}{2} \in \mathbb{Z}$$

Derivation: Algebraic expansion

Rule: If $n \in \mathbb{Z}^+ \wedge e = c^2 d \wedge p - \frac{1}{2} \in \mathbb{Z}$, then

$$\int \operatorname{RF}_x (d + e x^2)^p \operatorname{ArcSinh}[c x]^n dx \rightarrow \int (d + e x^2)^p \operatorname{ArcSinh}[c x]^n \operatorname{ExpandIntegrand}[\operatorname{RF}_x, x] dx$$

Program code:

```
Int[RFx_*(d_+e_.*x_^2)^p_*ArcSinh[c_.*x_]^n_,x_Symbol] :=
  With[{u=ExpandIntegrand[(d+e*x^2)^p*ArcSinh[c*x]^n,RFx,x]},
    Int[u,x] /;
    SumQ[u] /;
    FreeQ[{c,d,e},x] && RationalFunctionQ[RFx,x] && IGtQ[n,0] && EqQ[e,c^2*d] && IntegerQ[p-1/2]
```

```
Int[RFx_*(d1_+e1_.*x_)^p_*(d2_+e2_.*x_)^p_*ArcCosh[c_.*x_]^n_,x_Symbol] :=
  With[{u=ExpandIntegrand[(d1+e1*x)^p*(d2+e2*x)^p*ArcCosh[c*x]^n,RFx,x]},
    Int[u,x] /;
    SumQ[u] /;
    FreeQ[{c,d1,e1,d2,e2},x] && RationalFunctionQ[RFx,x] && IGtQ[n,0] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && IntegerQ[p-1/2]
```

2: $\int \operatorname{RF}_x (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx$ when $n \in \mathbb{Z}^+ \wedge e = c^2 d \wedge p - \frac{1}{2} \in \mathbb{Z}$

Derivation: Algebraic expansion

Rule: If $n \in \mathbb{Z}^+ \wedge e = c^2 d \wedge p - \frac{1}{2} \in \mathbb{Z}$, then

$$\int \operatorname{RF}_x (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \int (d + e x^2)^p \operatorname{ExpandIntegrand}[\operatorname{RF}_x (a + b \operatorname{ArcSinh}[c x])^n, x] dx$$

Program code:

```
Int[RFx_*(d_+e_.*x_^2)^p_*(a_+b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  Int[ExpandIntegrand[(d+e*x^2)^p,RFx*(a+b*ArcSinh[c*x])^n,x],x] /;
FreeQ[{a,b,c,d,e},x] && RationalFunctionQ[RFx,x] && IGtQ[n,0] && EqQ[e,c^2*d] && IntegerQ[p-1/2]
```

```
Int[RFx_*(d1_+e1_.*x_)^p_*(d2_+e2_.*x_)^p_*(a_+b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  Int[ExpandIntegrand[(d1+e1*x)^p*(d2+e2*x)^p,RFx*(a+b*ArcCosh[c*x])^n,x],x] /;
FreeQ[{a,b,c,d1,e1,d2,e2},x] && RationalFunctionQ[RFx,x] && IGtQ[n,0] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && IntegerQ[p-1/2]
```

X: $\int u (a + b \operatorname{ArcSinh}[c x])^n dx$

Rule:

$$\int u (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \int u (a + b \operatorname{ArcSinh}[c x])^n dx$$

Program code:

```
Int[u_.*(a_+b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  Unintegrable[u*(a+b*ArcSinh[c*x])^n,x] /;
FreeQ[{a,b,c,n},x]
```

```
Int[u_.*(a_.+b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=  
  Unintegrable[u*(a+b*ArcCosh[c*x])^n,x] /;  
FreeQ[{a,b,c,n},x]
```