

Rules for integrands of the form $(a + b x)^m (c + d x)^n (e + f x)^p (g + h x)^q$
when $b c - a d \neq 0 \wedge b e - a f \neq 0 \wedge b g - a h \neq 0 \wedge d e - c f \neq 0 \wedge d g - c h \neq 0 \wedge f g - e h \neq 0$

$$1. \int (a + b x)^m (c + d x)^n (e + f x)^p (g + h x)^q dx$$

$$1. \int (a + b x)^m (c + d x)^n (e + f x) (g + h x) dx$$

$$\text{1: } \int (a + b x)^m (c + d x)^n (e + f x) (g + h x) dx \text{ when } m \in \mathbb{Z}^+ \vee (m \mid n) \in \mathbb{Z}$$

Derivation: Algebraic expansion

Rule 1.1.1.4.1.1.1: If $m \in \mathbb{Z}^+ \vee (m \mid n) \in \mathbb{Z}$, then

$$\int (a + b x)^m (c + d x)^n (e + f x) (g + h x) dx \rightarrow \int \text{ExpandIntegrand}[(a + b x)^m (c + d x)^n (e + f x) (g + h x), x] dx$$

Program code:

```
Int[(a_.+b_.x_)^m_.*(c_.+d_.x_)^n_.*(e_.+f_.x_)*(g_.+h_.x_),x_Symbol] :=
  Int[ExpandIntegrand[(a+b*x)^m*(c+d*x)^n*(e+f*x)*(g+h*x),x],x] /;
FreeQ[{a,b,c,d,e,f,g,h},x] && (IGtQ[m,0] || IntegersQ[m,n])
```

2: $\int (a+bx)^m (c+dx)^n (e+fx) (g+hx) dx$ when $m+n+2 = 0 \wedge m \neq -1$

Derivation: ???

Rule 1.1.1.4.1.1.2: If $m+n+2 = 0 \wedge m \neq -1$, then

$$\int (a+bx)^m (c+dx)^n (e+fx) (g+hx) dx \rightarrow$$

$$\left((b^2deg - a^2dfhm - ab(d(fg+eh) - cfh(m+1)) + bfh(bc-ad)(m+1)x)(a+bx)^{m+1}(c+dx)^{n+1} / (b^2d(bc-ad)(m+1)) \right) +$$

$$\frac{adfhm + b(d(fg+eh) - cfh(m+2))}{b^2d} \int (a+bx)^{m+1} (c+dx)^n dx$$

Program code:

```
Int[(a_.+b_.x_)^m_*(c_.+d_.x_)^n_*(e_.+f_.x_)*(g_.+h_.x_),x_Symbol] :=
  (b^2*d*e*g-a^2*d*f*h*m-a*b*(d*(f*g+e*h)-c*f*h*(m+1))+b*f*h*(b*c-a*d)*(m+1)*x)*(a+b*x)^(m+1)*(c+d*x)^(n+1)/
  (b^2*d*(b*c-a*d)*(m+1)) +
  (a*d*f*h*m+b*(d*(f*g+e*h)-c*f*h*(m+2)))/(b^2*d)*Int[(a+b*x)^(m+1)*(c+d*x)^n,x] /;
FreeQ[{a,b,c,d,e,f,g,h,m,n},x] && EqQ[m+n+2,0] && NeQ[m,-1] && Not[SumSimplerQ[n,1] && Not[SumSimplerQ[m,1]]]
```

3. $\int (a+bx)^m (c+dx)^n (e+fx) (g+hx) dx$ when $m < -1$

1: $\int (a+bx)^m (c+dx)^n (e+fx) (g+hx) dx$ when $m < -1 \wedge n < -1$

Derivation: ???

Rule 1.1.1.4.1.1.3.1: If $m < -1 \wedge n < -1$, then

$$\int (a+bx)^m (c+dx)^n (e+fx) (g+hx) dx \rightarrow$$

$$((b^2cdeg(n+1) + a^2cdfh(n+1) + ab(d^2eg(m+1) + c^2fh(m+1) - cd(fg+eh)(m+n+2)) +$$

$$\frac{(a^2 d^2 f h (n+1) - a b d^2 (f g + e h) (n+1) + b^2 (c^2 f h (m+1) - c d (f g + e h) (m+1) + d^2 e g (m+n+2))) x}{(a+bx)^{m+1} (c+dx)^{n+1}} \cdot$$

$$\frac{(a^2 d^2 f h (2+3n+n^2) + a b d (n+1) (2 c f h (m+1) - d (f g + e h) (m+n+3)) + b^2 (c^2 f h (2+3m+m^2) - c d (f g + e h) (m+1) (m+n+3) + d^2 e g (6+m^2+5n+n^2+m(2n+5))))}{\int (a+bx)^{m+1} (c+dx)^{n+1} dx} \cdot$$

Program code:

```
Int[(a_.+b_.x_)^m_*(c_.+d_.x_)^n_*(e_.+f_.x_)*(g_.+h_.x_),x_Symbol] :=
  (b^2*c*d*e*g*(n+1)+a^2*c*d*f*h*(n+1)+a*b*(d^2*e*g*(m+1)+c^2*f*h*(m+1)-c*d*(f*g+e*h)*(m+n+2))+
    (a^2*d^2*f*h*(n+1)-a*b*d^2*(f*g+e*h)*(n+1)+b^2*(c^2*f*h*(m+1)-c*d*(f*g+e*h)*(m+1)+d^2*e*g*(m+n+2)))*x)/
  (b*d*(b*c-a*d)^2*(m+1)*(n+1))*(a+b*x)^(m+1)*(c+d*x)^(n+1) -
  (a^2*d^2*f*h*(2+3*n+n^2)+a*b*d*(n+1)*(2*c*f*h*(m+1)-d*(f*g+e*h)*(m+n+3))+
    b^2*(c^2*f*h*(2+3*m+m^2)-c*d*(f*g+e*h)*(m+1)*(m+n+3)+d^2*e*g*(6+m^2+5*n+n^2+m*(2*n+5))))/
  (b*d*(b*c-a*d)^2*(m+1)*(n+1))*Int[(a+b*x)^(m+1)*(c+d*x)^(n+1),x] /;
FreeQ[{a,b,c,d,e,f,g,h},x] && LtQ[m,-1] && LtQ[n,-1]
```

$$2. \int (a+bx)^m (c+dx)^n (e+fx) (g+hx) dx \text{ when } m < -1 \wedge n \neq -1$$

$$1: \int (a+bx)^m (c+dx)^n (e+fx) (g+hx) dx \text{ when } m < -2$$

Derivation: ???

Rule 1.1.1.4.1.1.3.2.1: If $m < -2$, then

$$\int (a+bx)^m (c+dx)^n (e+fx) (g+hx) dx \rightarrow$$

$$\frac{(b^3 c e g (m+2) - a^3 d f h (n+2) - a^2 b (c f h m - d (f g + e h) (m+n+3)) - a b^2 (c (f g + e h) + d e g (2 m + n + 4)) + b (a^2 d f h (m-n) - a b (2 c f h (m+1) - d (f g + e h) (n+1)) + b^2 (c (f g + e h) (m+1) - d e g (m+n+2))) x)}{(a+bx)^{m+1} (c+dx)^{n+1} +$$

$$\left(\frac{f h}{b^2} - (d (m+n+3) (a^2 d f h (m-n) - a b (2 c f h (m+1) - d (f g + e h) (n+1)) + b^2 (c (f g + e h) (m+1) - d e g (m+n+2))) \right) / (b^2 (b c - a d)^2 (m+1) (m+2))$$

$$\int (a + bx)^{m+2} (c + dx)^n dx$$

Program code:

```
Int[(a_.+b_.x_)^m_*(c_.+d_.x_)^n_*(e_.+f_.x_)*(g_.+h_.x_),x_Symbol] :=
  (b^3*c*e*g*(m+2)-a^3*d*f*h*(n+2)-a^2*b*(c*f*h*m-d*(f*g+e*h)*(m+n+3))-a*b^2*(c*(f*g+e*h)+d*e*g*(2*m+n+4))+
    b*(a^2*d*f*h*(m-n)-a*b*(2*c*f*h*(m+1)-d*(f*g+e*h)*(n+1))+b^2*(c*(f*g+e*h)*(m+1)-d*e*g*(m+n+2))*x)/
  (b^2*(b*c-a*d)^2*(m+1)*(m+2))*(a+b*x)^(m+1)*(c+d*x)^(n+1) +
  (f*h/b^2-(d*(m+n+3)*(a^2*d*f*h*(m-n)-a*b*(2*c*f*h*(m+1)-d*(f*g+e*h)*(n+1))+b^2*(c*(f*g+e*h)*(m+1)-d*e*g*(m+n+2))))/
  (b^2*(b*c-a*d)^2*(m+1)*(m+2))*
  Int[(a+b*x)^(m+2)*(c+d*x)^n,x] /;
FreeQ[{a,b,c,d,e,f,g,h,m,n},x] && (LtQ[m,-2] || EqQ[m+n+3,0] && Not[LtQ[n,-2]])
```

$$2: \int (a+bx)^m (c+dx)^n (e+fx) (g+hx) dx \text{ when } -2 \leq m < -1$$

Derivation: ???

Rule 1.1.1.4.1.1.3.2.2: If $-2 \leq m < -1$, then

$$\int (a+bx)^m (c+dx)^n (e+fx) (g+hx) dx \rightarrow$$

$$\left((a^2 d f h (n+2) + b^2 d e g (m+n+3) + a b (c f h (m+1) - d (f g + e h) (m+n+3)) + b f h (b c - a d) (m+1) x) / (b^2 d (b c - a d) (m+1) (m+n+3)) \right) (a+bx)^{m+1} (c+dx)^{n+1} -$$

$$\left((a^2 d^2 f h (n+1) (n+2) + a b d (n+1) (2 c f h (m+1) - d (f g + e h) (m+n+3)) + b^2 (c^2 f h (m+1) (m+2) - c d (f g + e h) (m+1) (m+n+3) + d^2 e g (m+n+2) (m+n+3))) / (b^2 d (b c - a d) (m+1) (m+n+3)) \right) \int (a+bx)^{m+1} (c+dx)^n dx$$

Program code:

```
Int[(a_.+b_.x_)^m_*(c_.+d_.x_)^n_*(e_.+f_.x_)*(g_.+h_.x_),x_Symbol] :=
  (a^2*d*f*h*(n+2)+b^2*d*e*g*(m+n+3)+a*b*(c*f*h*(m+1)-d*(f*g+e*h)*(m+n+3))+b*f*h*(b*c-a*d)*(m+1)*x)/
  (b^2*d*(b*c-a*d)*(m+1)*(m+n+3))*(a+b*x)^(m+1)*(c+d*x)^(n+1) -
  (a^2*d^2*f*h*(n+1)*(n+2)+a*b*d*(n+1)*(2*c*f*h*(m+1)-d*(f*g+e*h)*(m+n+3))+
  b^2*(c^2*f*h*(m+1)*(m+2)-c*d*(f*g+e*h)*(m+1)*(m+n+3)+d^2*e*g*(m+n+2)*(m+n+3)))/
  (b^2*d*(b*c-a*d)*(m+1)*(m+n+3))*Int[(a+b*x)^(m+1)*(c+d*x)^n,x] /;
FreeQ[{a,b,c,d,e,f,g,h,m,n},x] && (GeQ[m,-2] && LtQ[m,-1] || SumSimplerQ[m,1]) && NeQ[m,-1] && NeQ[m+n+3,0]
```

$$4: \int (a+bx)^m (c+dx)^n (e+fx) (g+hx) dx \text{ when } m \not< -1 \wedge m+n+2 \neq 0 \wedge m+n+3 \neq 0$$

Derivation: ???

Rule 1.1.1.4.1.1.4: If $m \not< -1 \wedge m+n+2 \neq 0 \wedge m+n+3 \neq 0$, then

$$\int (a+bx)^m (c+dx)^n (e+fx) (g+hx) dx \rightarrow$$

$$- \left((a d f h (n+2) + b c f h (m+2) - b d (f g + e h) (m+n+3) - b d f h (m+n+2) x) (a+bx)^{m+1} (c+dx)^{n+1} / (b^2 d^2 (m+n+2) (m+n+3)) \right) +$$

$$\frac{\left(a^2 d^2 f h (n+1) (n+2) + a b d (n+1) (2 c f h (m+1) - d (f g + e h) (m+n+3)) + b^2 (c^2 f h (m+1) (m+2) - c d (f g + e h) (m+1) (m+n+3) + d^2 e g (m+n+2) (m+n+3)) \right)}{(a+bx)^m (c+dx)^n} dx$$

Program code:

```
Int[(a_.+b_.**x_)^m_.*(c_.+d_.**x_)^n_.*(e_.+f_.**x_)*(g_.+h_.**x_),x_Symbol] :=
- (a*d*f*h*(n+2)+b*c*f*h*(m+2)-b*d*(f*g+e*h)*(m+n+3)-b*d*f*h*(m+n+2)**x)*(a+b*x)^(m+1)*(c+d*x)^(n+1)/
(b^2*d^2*(m+n+2)*(m+n+3)) +
(a^2*d^2*f*h*(n+1)*(n+2)+a*b*d*(n+1)*(2*c*f*h*(m+1)-d*(f*g+e*h)*(m+n+3))+
b^2*(c^2*f*h*(m+1)*(m+2)-c*d*(f*g+e*h)*(m+1)*(m+n+3)+d^2*e*g*(m+n+2)*(m+n+3)))/
(b^2*d^2*(m+n+2)*(m+n+3))*Int[(a+b*x)^m*(c+d*x)^n,x] /;
FreeQ[{a,b,c,d,e,f,g,h,m,n},x] && NeQ[m+n+2,0] && NeQ[m+n+3,0]
```

2: $\int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx) dx$ when $(m | n | p) \in \mathbb{Z} \vee (n | p) \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule 1.1.1.4.1.2: If $(m | n | p) \in \mathbb{Z} \vee (n | p) \in \mathbb{Z}^+$, then

$$\int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx) dx \rightarrow \int \text{ExpandIntegrand}[(a+bx)^m (c+dx)^n (e+fx)^p (g+hx), x] dx$$

Program code:

```
Int[(a_.+b_.**x_)^m_.*(c_.+d_.**x_)^n_.*(e_.+f_.**x_)^p_.*(g_.+h_.**x_),x_Symbol] :=
Int[ExpandIntegrand[(a+b*x)^m*(c+d*x)^n*(e+f*x)^p*(g+h*x),x],x] /;
FreeQ[{a,b,c,d,e,f,g,h,m},x] && (IntegersQ[m,n,p] || IGtQ[n,0] && IGtQ[p,0])
```

$$3. \int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx) dx \text{ when } m < -1$$

$$1: \int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx) dx \text{ when } m < -1 \wedge n > 0$$

Derivation: Nondegenerate trilinear recurrence 1

Rule 1.1.1.4.1.3.1: If $m < -1 \wedge n > 0$, then

$$\begin{aligned} & \int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx) dx \rightarrow \\ & \frac{(bg-ah)(a+bx)^{m+1}(c+dx)^n(e+fx)^{p+1}}{b(b e - a f)(m+1)} - \\ & \frac{1}{b(b e - a f)(m+1)} \int (a+bx)^{m+1}(c+dx)^{n-1}(e+fx)^p \cdot \\ & (bc(fg-eh)(m+1) + (bg-ah)(den+cf(p+1)) + d(b(fg-eh)(m+1) + f(bg-ah)(n+p+1))x) dx \end{aligned}$$

Program code:

```
Int[(a_.+b_.**x_)^m_*(c_.+d_.**x_)^n_*(e_.+f_.**x_)^p_*(g_.+h_.**x_),x_Symbol] :=
  (b*g-a*h)*(a+b*x)^(m+1)*(c+d*x)^n*(e+f*x)^(p+1)/(b*(b*e-a*f)*(m+1)) -
  1/(b*(b*e-a*f)*(m+1))*Int[(a+b*x)^(m+1)*(c+d*x)^(n-1)*(e+f*x)^p*
    Simp[b*c*(f*g-e*h)*(m+1)+(b*g-a*h)*(d*e*n+c*f*(p+1))+d*(b*(f*g-e*h)*(m+1)+f*(b*g-a*h)*(n+p+1))*x,x],x] /;
FreeQ[{a,b,c,d,e,f,g,h,p},x] && LtQ[m,-1] && GtQ[n,0] && IntegerQ[m]
```

```
Int[(a_.+b_.**x_)^m_*(c_.+d_.**x_)^n_*(e_.+f_.**x_)^p_*(g_.+h_.**x_),x_Symbol] :=
  (b*g-a*h)*(a+b*x)^(m+1)*(c+d*x)^n*(e+f*x)^(p+1)/(b*(b*e-a*f)*(m+1)) -
  1/(b*(b*e-a*f)*(m+1))*Int[(a+b*x)^(m+1)*(c+d*x)^(n-1)*(e+f*x)^p*
    Simp[b*c*(f*g-e*h)*(m+1)+(b*g-a*h)*(d*e*n+c*f*(p+1))+d*(b*(f*g-e*h)*(m+1)+f*(b*g-a*h)*(n+p+1))*x,x],x] /;
FreeQ[{a,b,c,d,e,f,g,h,p},x] && LtQ[m,-1] && GtQ[n,0] && IntegerQ[2*m,2*n,2*p]
```

2: $\int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx) dx$ when $m < -1 \wedge n \neq 0$

Derivation: Nondegenerate trilinear recurrence 3

Rule 1.1.1.4.1.3.2: If $m < -1 \wedge n \neq 0$, then

$$\begin{aligned} & \int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx) dx \rightarrow \\ & \frac{(bg-ah)(a+bx)^{m+1}(c+dx)^{n+1}(e+fx)^{p+1}}{(m+1)(bc-ad)(be-af)} + \\ & \frac{1}{(m+1)(bc-ad)(be-af)} \int (a+bx)^{m+1}(c+dx)^n (e+fx)^p \cdot \\ & ((adf g - b(de+cf)g + bceh)(m+1) - (bg-ah)(de(n+1) + cf(p+1)) - df(bg-ah)(m+n+p+3)x) dx \end{aligned}$$

Program code:

```
Int[(a_.+b_.*x_)^m.*(c_.+d_.*x_)^n.*(e_.+f_.*x_)^p.*(g_.+h_.*x_),x_Symbol] :=
  (b*g-a*h)*(a+b*x)^(m+1)*(c+d*x)^(n+1)*(e+f*x)^(p+1)/((m+1)*(b*c-a*d)*(b*e-a*f)) +
  1/((m+1)*(b*c-a*d)*(b*e-a*f))*Int[(a+b*x)^(m+1)*(c+d*x)^n*(e+f*x)^p*
    Simp[(a*d*f*g-b*(d*e+c*f)*g+b*c*e*h)*(m+1)-(b*g-a*h)*(d*e*(n+1)+c*f*(p+1))-d*f*(b*g-a*h)*(m+n+p+3)*x,x],x] /;
FreeQ[{a,b,c,d,e,f,g,h,n,p},x] && LtQ[m,-1] && IntegerQ[m]
```

```
Int[(a_.+b_.*x_)^m.*(c_.+d_.*x_)^n.*(e_.+f_.*x_)^p.*(g_.+h_.*x_),x_Symbol] :=
  (b*g-a*h)*(a+b*x)^(m+1)*(c+d*x)^(n+1)*(e+f*x)^(p+1)/((m+1)*(b*c-a*d)*(b*e-a*f)) +
  1/((m+1)*(b*c-a*d)*(b*e-a*f))*Int[(a+b*x)^(m+1)*(c+d*x)^n*(e+f*x)^p*
    Simp[(a*d*f*g-b*(d*e+c*f)*g+b*c*e*h)*(m+1)-(b*g-a*h)*(d*e*(n+1)+c*f*(p+1))-d*f*(b*g-a*h)*(m+n+p+3)*x,x],x] /;
FreeQ[{a,b,c,d,e,f,g,h,n,p},x] && LtQ[m,-1] && IntegersQ[2*m,2*n,2*p]
```


4: $\int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx) dx$ when $m > 0 \wedge m+n+p+2 \neq 0$

Derivation: Nondegenerate trilinear recurrence 2

Rule 1.1.1.4.1.4: If $m > 0 \wedge m+n+p+2 \neq 0$, then

$$\int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx) dx \rightarrow$$

$$\frac{h (a+bx)^m (c+dx)^{n+1} (e+fx)^{p+1}}{d f (m+n+p+2)} +$$

$$\frac{1}{d f (m+n+p+2)} \int (a+bx)^{m-1} (c+dx)^n (e+fx)^p dx.$$

$$(a d f g (m+n+p+2) - h (b c e m + a (d e (n+1) + c f (p+1))) + (b d f g (m+n+p+2) + h (a d f m - b (d e (m+n+1) + c f (m+p+1)))) x) dx$$

Program code:

```
Int[(a_.+b_.**x_)^m_*(c_.+d_.**x_)^n_*(e_.+f_.**x_)^p_*(g_.+h_.**x_),x_Symbol] :=
  h*(a+b*x)^m*(c+d*x)^(n+1)*(e+f*x)^(p+1)/(d*f*(m+n+p+2)) +
  1/(d*f*(m+n+p+2))*Int[(a+b*x)^(m-1)*(c+d*x)^n*(e+f*x)^p*
    Simp[a*d*f*g*(m+n+p+2)-h*(b*c*e*m+a*(d*e*(n+1)+c*f*(p+1)))+(b*d*f*g*(m+n+p+2)+h*(a*d*f*m-b*(d*e*(m+n+1)+c*f*(m+p+1))))*x,x],x] /;
FreeQ[{a,b,c,d,e,f,g,h,n,p},x] && GtQ[m,0] && NeQ[m+n+p+2,0] && IntegerQ[m]
```

```
Int[(a_.+b_.**x_)^m_*(c_.+d_.**x_)^n_*(e_.+f_.**x_)^p_*(g_.+h_.**x_),x_Symbol] :=
  h*(a+b*x)^m*(c+d*x)^(n+1)*(e+f*x)^(p+1)/(d*f*(m+n+p+2)) +
  1/(d*f*(m+n+p+2))*Int[(a+b*x)^(m-1)*(c+d*x)^n*(e+f*x)^p*
    Simp[a*d*f*g*(m+n+p+2)-h*(b*c*e*m+a*(d*e*(n+1)+c*f*(p+1)))+(b*d*f*g*(m+n+p+2)+h*(a*d*f*m-b*(d*e*(m+n+1)+c*f*(m+p+1))))*x,x],x] /;
FreeQ[{a,b,c,d,e,f,g,h,n,p},x] && GtQ[m,0] && NeQ[m+n+p+2,0] && IntegersQ[2*m,2*n,2*p]
```

5: $\int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx) dx$ when $m+n+p+2 \in \mathbb{Z}^-$

Derivation: Nondegenerate trilinear recurrence 3

Note: If $m+n+p+2 \in \mathbb{Z}^-$, then $\int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx) dx$ can be expressed in terms of the hypergeometric function ${}_2F_1$.

Rule 1.1.1.4.1.5: If $m+n+p+2 \in \mathbb{Z}^-$, then

$$\begin{aligned} & \int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx) dx \rightarrow \\ & \frac{(bg-ah)(a+bx)^{m+1}(c+dx)^{n+1}(e+fx)^{p+1}}{(m+1)(bc-ad)(be-af)} + \\ & \frac{1}{(m+1)(bc-ad)(be-af)} \int (a+bx)^{m+1}(c+dx)^n (e+fx)^p dx \\ & ((adfg-b(de+cf)g+bceh)(m+1) - (bg-ah)(de(n+1)+cf(p+1)) - df(bg-ah)(m+n+p+3)x) dx \end{aligned}$$

Program code:

```
Int[(a_.+b_.x_)^m_*(c_.+d_.x_)^n_*(e_.+f_.x_)^p_*(g_.+h_.x_),x_Symbol] :=
  (b*g-a*h)*(a+b*x)^(m+1)*(c+d*x)^(n+1)*(e+f*x)^(p+1)/((m+1)*(b*c-a*d)*(b*e-a*f)) +
  1/((m+1)*(b*c-a*d)*(b*e-a*f))*Int[(a+b*x)^(m+1)*(c+d*x)^n*(e+f*x)^p*
    Simp[(a*d*f*g-b*(d*e+c*f)*g+b*c*e*h)*(m+1)-(b*g-a*h)*(d*e*(n+1)+c*f*(p+1))-d*f*(b*g-a*h)*(m+n+p+3)*x,x],x] /;
FreeQ[{a,b,c,d,e,f,g,h,n,p},x] && ILtQ[m+n+p+2,0] && NeQ[m,-1] &&
  (SumSimplerQ[m,1] || Not[NeQ[n,-1] && SumSimplerQ[n,1]] && Not[NeQ[p,-1] && SumSimplerQ[p,1]])
```

$$6. \int \frac{(c+dx)^n (e+fx)^p (g+hx)}{a+bx} dx$$

$$1: \int \frac{(e+fx)^p (g+hx)}{(a+bx)(c+dx)} dx$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{g+hx}{(a+bx)(c+dx)} = \frac{bg-ah}{(bc-ad)(a+bx)} - \frac{dg-ch}{(bc-ad)(c+dx)}$$

Rule 1.1.1.4.1.6.1:

$$\int \frac{(e+fx)^p (g+hx)}{(a+bx)(c+dx)} dx \rightarrow \frac{bg-ah}{bc-ad} \int \frac{(e+fx)^p}{a+bx} dx - \frac{dg-ch}{bc-ad} \int \frac{(e+fx)^p}{c+dx} dx$$

Program code:

```
Int[(e_.+f_.**x_)^p_*(g_.+h_.**x_)/((a_.+b_.**x_)*(c_.+d_.**x_)),x_Symbol] :=
  (b*g-a*h)/(b*c-a*d)*Int[(e+f*x)^p/(a+b*x),x] -
  (d*g-c*h)/(b*c-a*d)*Int[(e+f*x)^p/(c+d*x),x] /;
FreeQ[{a,b,c,d,e,f,g,h},x]
```

$$2: \int \frac{(c+dx)^n (e+fx)^p (g+hx)}{a+bx} dx$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{g+hx}{a+bx} = \frac{h}{b} + \frac{bg-ah}{b(a+bx)}$$

Rule 1.1.1.4.1.6.2:

$$\int \frac{(c+dx)^n (e+fx)^p (g+hx)}{a+bx} dx \rightarrow \frac{h}{b} \int (c+dx)^n (e+fx)^p dx + \frac{bg-ah}{b} \int \frac{(c+dx)^n (e+fx)^p}{a+bx} dx$$

Program code:

```
Int[(c_.+d_.**x_)^n_*(e_.+f_.**x_)^p_*(g_.+h_.**x_)/(a_.+b_.**x_),x_Symbol] :=
  h/b*Int[(c+d*x)^n*(e+f*x)^p,x] + (b*g-a*h)/b*Int[(c+d*x)^n*(e+f*x)^p/(a+b*x),x] /;
FreeQ[{a,b,c,d,e,f,g,h,n,p},x]
```

$$7: \int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx) dx$$

Derivation: Algebraic expansion

$$\text{Basis: } g + hx = \frac{h(a+bx)}{b} + \frac{bg-ah}{b}$$

Note: For $\frac{g+hx}{\sqrt{a+bx}\sqrt{c+dx}\sqrt{e+fx}}$, ensuring the simpler square-root factors remain in the denominator of the resulting integrands causes the two elliptic integrals in the antiderivative to have the same and simplest arguments.

Rule 1.1.1.4.1.7:

$$\int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx) dx \rightarrow \frac{h}{b} \int (a+bx)^{m+1} (c+dx)^n (e+fx)^p dx + \frac{bg-ah}{b} \int (a+bx)^m (c+dx)^n (e+fx)^p dx$$

Program code:

```
Int[(g_.+h_.*x_)/(Sqrt[a_.+b_.*x_]*Sqrt[c_.+d_.*x_]*Sqrt[e_.+f_.*x_]),x_Symbol] :=
  h/f*Int[Sqrt[e+f*x]/(Sqrt[a+b*x]*Sqrt[c+d*x]),x] + (f*g-e*h)/f*Int[1/(Sqrt[a+b*x]*Sqrt[c+d*x]*Sqrt[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f,g,h},x] && SimplerQ[a+b*x,e+f*x] && SimplerQ[c+d*x,e+f*x]
```

```
Int[(a_.+b_.*x_)^m*(c_.+d_.*x_)^n*(e_.+f_.*x_)^p*(g_.+h_.*x_),x_Symbol] :=
  h/b*Int[(a+b*x)^(m+1)*(c+d*x)^n*(e+f*x)^p,x] + (b*g-a*h)/b*Int[(a+b*x)^m*(c+d*x)^n*(e+f*x)^p,x] /;
FreeQ[{a,b,c,d,e,f,g,h,m,n,p},x] && (SumSimplerQ[m,1] || Not[SumSimplerQ[n,1]] && Not[SumSimplerQ[p,1]])
```

$$2. \int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx)^q dx \text{ when } 2m \in \mathbb{Z} \wedge n^2 = \frac{1}{4} \wedge p^2 = \frac{1}{4} \wedge q^2 = \frac{1}{4}$$

$$1. \int (a+bx)^m (c+dx)^n \sqrt{e+fx} \sqrt{g+hx} dx \text{ when } 2m \in \mathbb{Z} \wedge n^2 = \frac{1}{4}$$

$$1. \int (a+bx)^m \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx} dx \text{ when } 2m \in \mathbb{Z}$$

$$1: \int (a+bx)^m \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx} dx \text{ when } 2m \in \mathbb{Z} \wedge m < -1$$

Derivation: Integration by parts

$$\text{Basis: } \partial_x \left(\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx} \right) = \frac{deg+cf g+ce h+2(df g+de h+cf h)x+3df h x^2}{2\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}}$$

Rule 1.1.1.4.2.1.1.1: If $2m \in \mathbb{Z} \wedge m < -1$, then

$$\int (a+bx)^m \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx} dx \rightarrow \frac{(a+bx)^{m+1} \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}}{b(m+1)} - \frac{1}{2b(m+1)} \int \frac{(a+bx)^{m+1} (deg+cf g+ce h+2(df g+de h+cf h)x+3df h x^2)}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx$$

Program code:

```
Int[(a_.+b_.x_)^m_*Sqrt[c_.+d_.x_]_*Sqrt[e_.+f_.x_]_*Sqrt[g_.+h_.x_],x_Symbol] :=
  (a+b*x)^(m+1)*Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]/(b*(m+1)) -
  1/(2*b*(m+1))*Int[(a+b*x)^(m+1)/(Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x])*
    Simp[d*e*g+c*f*g+c*e*h+2*(d*f*g+d*e*h+c*f*h)*x+3*d*f*h*x^2,x],x] /;
FreeQ[{a,b,c,d,e,f,g,h,m},x] && IntegerQ[2*m] && LtQ[m,-1]
```

2: $\int (a+bx)^m \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx} \, dx$ when $2m \in \mathbb{Z} \wedge m \neq -1$

Rule 1.1.1.4.2.1.1.2: If $2m \in \mathbb{Z} \wedge m \neq -1$, then

$$\int (a+bx)^m \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx} \, dx \rightarrow$$

$$\frac{2(a+bx)^{m+1} \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}}{b(2m+5)} + \frac{1}{b(2m+5)} \int \frac{(a+bx)^m}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} \, dx$$

$$(3bceg - a(deg + cfg + ceh) + 2(b(deg + cfg + ceh) - a(dfg + deh + cfh))x - (3adf h - b(dfg + deh + cfh))x^2) \, dx$$

Program code:

```
Int[(a_.+b_.x_)^m*Sqrt[c_.+d_.x_]*Sqrt[e_.+f_.x_]*Sqrt[g_.+h_.x_],x_Symbol] :=
  2*(a+b*x)^(m+1)*Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]/(b*(2*m+5)) +
  1/(b*(2*m+5))*Int[((a+b*x)^m)/(Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]),x]
  Simp[3*b*c*e*g-a*(d*e*g+c*f*g+c*e*h)+2*(b*(d*e*g+c*f*g+c*e*h)-a*(d*f*g+d*e*h+c*f*h))*x-(3*a*d*f*h-b*(d*f*g+d*e*h+c*f*h))*x^2,x],x]
FreeQ[{a,b,c,d,e,f,g,h,m},x] && IntegerQ[2*m] && Not[LtQ[m,-1]]
```

2. $\int \frac{(a+bx)^m \sqrt{e+fx} \sqrt{g+hx}}{\sqrt{c+dx}} \, dx$ when $2m \in \mathbb{Z}$

1: $\int \frac{(a+bx)^m \sqrt{e+fx} \sqrt{g+hx}}{\sqrt{c+dx}} \, dx$ when $2m \in \mathbb{Z} \wedge m > 0$

Rule 1.1.1.4.2.1.2.1: If $2m \in \mathbb{Z} \wedge m > 0$, then

$$\int \frac{(a+bx)^m \sqrt{e+fx} \sqrt{g+hx}}{\sqrt{c+dx}} \, dx \rightarrow$$

$$\frac{2 (a+bx)^m \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}}{d (2m+3)} - \frac{1}{d (2m+3)} \int \frac{(a+bx)^{m-1}}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx$$

$$\frac{(2bceg m + a(c(fg+eh) - 2deg(m+1)) - (b(2deg - c(fg+eh)(2m+1)) - a(2cfh - d(2m+1)(fg+eh)))x - (2adfhm + b(d(fg+eh) - 2cfh(m+1)))x^2) dx}{d^2 (2m+3)^2}$$

Program code:

```
Int[(a_.+b_.x_)^m*Sqrt[e_.+f_.x_]*Sqrt[g_.+h_.x_]/Sqrt[c_.+d_.x_],x_Symbol] :=
  2*(a+b*x)^m*Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]/(d*(2*m+3)) -
  1/(d*(2*m+3))*Int[((a+b*x)^(m-1)/(Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]))*
  Simp[2*b*c*e*g*m+a*(c*(f*g+e*h)-2*d*e*g*(m+1)) -
  (b*(2*d*e*g-c*(f*g+e*h)*(2*m+1))-a*(2*c*f*h-d*(2*m+1)*(f*g+e*h)))*x -
  (2*a*d*f*h*m+b*(d*(f*g+e*h)-2*c*f*h*(m+1)))*x^2,x],x] /;
FreeQ[{a,b,c,d,e,f,g,h,m},x] && IntegerQ[2*m] && GtQ[m,0]
```


$$2. \int \frac{(a+bx)^m \sqrt{e+fx} \sqrt{g+hx}}{\sqrt{c+dx}} dx \text{ when } 2m \in \mathbb{Z} \wedge m < 0$$

$$1: \int \frac{\sqrt{e+fx} \sqrt{g+hx}}{(a+bx) \sqrt{c+dx}} dx$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{\sqrt{e+fx} \sqrt{g+hx}}{a+bx} = \frac{(be-af)(bg-ah)}{b^2(a+bx) \sqrt{e+fx} \sqrt{g+hx}} + \frac{bfg+beg-afh+bfx}{b^2 \sqrt{e+fx} \sqrt{g+hx}}$$

Rule 1.1.1.4.2.1.2.2.1:

$$\int \frac{\sqrt{e+fx} \sqrt{g+hx}}{(a+bx) \sqrt{c+dx}} dx \rightarrow \frac{(be-af)(bg-ah)}{b^2} \int \frac{1}{(a+bx) \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx + \frac{1}{b^2} \int \frac{bfg+beg-afh+bfx}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx$$

Program code:

```
Int[Sqrt[e_.+f_.**x_]*Sqrt[g_.+h_.**x_]/((a_.+b_.**x_)*Sqrt[c_.+d_.**x_]),x_Symbol] :=
  (b*e-a*f)*(b*g-a*h)/b^2*Int[1/((a+b*x)*Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]),x] +
  1/b^2*Int[Simp[b*f*g+b*e*h-a*f*h+b*f*h*x,x]/(Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]),x] /;
FreeQ[{a,b,c,d,e,f,g,h},x]
```

$$2: \int \frac{(a+bx)^m \sqrt{e+fx} \sqrt{g+hx}}{\sqrt{c+dx}} dx \text{ when } 2m \in \mathbb{Z} \wedge m < -1$$

Rule 1.1.1.4.2.1.2.2.2: If $2m \in \mathbb{Z} \wedge m < -1$, then

$$\int \frac{(a+bx)^m \sqrt{e+fx} \sqrt{g+hx}}{\sqrt{c+dx}} dx \rightarrow$$

$$\frac{(a+bx)^{m+1} \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}}{(m+1)(bc-ad)} - \frac{1}{2(m+1)(bc-ad)} \int \frac{(a+bx)^{m+1}}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx$$

$$(c(fg+eh) + deg(2m+3) + 2(cfhd + d(m+2)(fg+eh))x + dfh(2m+5)x^2) dx$$

Program code:

```
Int[(a_.+b_.**x_)^m_*Sqrt[e_.+f_.**x_]*Sqrt[g_.+h_.**x_]/Sqrt[c_.+d_.**x_],x_Symbol] :=
  (a+b*x)^(m+1)*Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]/((m+1)*(b*c-a*d)) -
  1/(2*(m+1)*(b*c-a*d))*Int[((a+b*x)^(m+1)/(Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]))*
    Simp[c*(f*g+e*h)+d*e*g*(2*m+3)+2*(c*f*h+d*(m+2)*(f*g+e*h))*x+d*f*h*(2*m+5)*x^2,x],x] /;
FreeQ[{a,b,c,d,e,f,g,h,m},x] && IntegerQ[2*m] && LtQ[m,-1]
```

$$2. \int \frac{(a+bx)^m (c+dx)^n}{\sqrt{e+fx} \sqrt{g+hx}} dx \text{ when } 2m \in \mathbb{Z} \wedge n^2 = \frac{1}{4}$$

$$1. \int \frac{(a+bx)^m}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx \text{ when } 2m \in \mathbb{Z}$$

$$1. \int \frac{(a+bx)^m}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx \text{ when } 2m \in \mathbb{Z} \wedge m > 0$$

$$1: \int \frac{\sqrt{a+bx}}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx$$

Derivation: Piecewise constant extraction and integration by substitution

$$\blacksquare \text{ Basis: } \partial_x \frac{(a+bx) \sqrt{\frac{(bg-ah)(c+dx)}{(dg-ch)(a+bx)}} \sqrt{\frac{(bg-ah)(e+fx)}{(fg-eh)(a+bx)}}}{\sqrt{c+dx} \sqrt{e+fx}} = 0$$

Basis:

$$\frac{1}{\sqrt{a+bx} \sqrt{\frac{(bg-ah)(c+dx)}{(dg-ch)(a+bx)}} \sqrt{\frac{(bg-ah)(e+fx)}{(fg-eh)(a+bx)}} \sqrt{g+hx}} = 2 \text{ Subst} \left[\frac{1}{(h-bx^2) \sqrt{1 + \frac{(bc-ad)x^2}{dg-ch}} \sqrt{1 + \frac{(be-af)x^2}{fg-eh}}}, x, \frac{\sqrt{g+hx}}{\sqrt{a+bx}} \right] \partial_x \frac{\sqrt{g+hx}}{\sqrt{a+bx}}$$

Rule 1.1.1.4.2.2.1.1.1:

$$\begin{aligned} \int \frac{\sqrt{a+bx}}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx &\rightarrow \frac{(a+bx) \sqrt{\frac{(bg-ah)(c+dx)}{(dg-ch)(a+bx)}} \sqrt{\frac{(bg-ah)(e+fx)}{(fg-eh)(a+bx)}}}{\sqrt{c+dx} \sqrt{e+fx}} \int \frac{1}{\sqrt{a+bx} \sqrt{\frac{(bg-ah)(c+dx)}{(dg-ch)(a+bx)}} \sqrt{\frac{(bg-ah)(e+fx)}{(fg-eh)(a+bx)}} \sqrt{g+hx}} dx \\ &\rightarrow \frac{2(a+bx) \sqrt{\frac{(bg-ah)(c+dx)}{(dg-ch)(a+bx)}} \sqrt{\frac{(bg-ah)(e+fx)}{(fg-eh)(a+bx)}}}{\sqrt{c+dx} \sqrt{e+fx}} \text{Subst} \left[\int \frac{1}{(h-bx^2) \sqrt{1 + \frac{(bc-ad)x^2}{dg-ch}} \sqrt{1 + \frac{(be-af)x^2}{fg-eh}}} dx, x, \frac{\sqrt{g+hx}}{\sqrt{a+bx}} \right] \end{aligned}$$

Program code:

```
Int[Sqrt[a_.+b_.x_]/(Sqrt[c_.+d_.x_]*Sqrt[e_.+f_.x_]*Sqrt[g_.+h_.x_]),x_Symbol] :=
  2*(a+b*x)*Sqrt[(b*g-a*h)*(c+d*x)/((d*g-c*h)*(a+b*x))]*Sqrt[(b*g-a*h)*(e+f*x)/((f*g-e*h)*(a+b*x))]/(Sqrt[c+d*x]*Sqrt[e+f*x])*
  Subst[Int[1/(h-b*x^2)*Sqrt[1+(b*c-a*d)*x^2/(d*g-c*h)]]*Sqrt[1+(b*e-a*f)*x^2/(f*g-e*h)],x],x,Sqrt[g+h*x]/Sqrt[a+b*x]] /;
FreeQ[{a,b,c,d,e,f,g,h},x]
```

$$2: \int \frac{(a+bx)^{3/2}}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{(a+bx)^{3/2}}{\sqrt{c+dx}} == \frac{b\sqrt{a+bx}\sqrt{c+dx}}{d} - \frac{(bc-ad)\sqrt{a+bx}}{d\sqrt{c+dx}}$$

Rule 1.1.1.4.2.2.1.1.2:

$$\int \frac{(a+bx)^{3/2}}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx \rightarrow \frac{b}{d} \int \frac{\sqrt{a+bx} \sqrt{c+dx}}{\sqrt{e+fx} \sqrt{g+hx}} dx - \frac{(bc-ad)}{d} \int \frac{\sqrt{a+bx}}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx$$

Program code:

```
Int[(a_.+b_.x_)^(3/2)/(Sqrt[c_.+d_.x_]*Sqrt[e_.+f_.x_]*Sqrt[g_.+h_.x_]),x_Symbol] :=
  b/d*Int[Sqrt[a+b*x]*Sqrt[c+d*x]/(Sqrt[e+f*x]*Sqrt[g+h*x]),x] -
  (b*c-a*d)/d*Int[Sqrt[a+b*x]/(Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]),x] /;
FreeQ[{a,b,c,d,e,f,g,h},x]
```

$$3: \int \frac{(a+bx)^m}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx \text{ when } 2m \in \mathbb{Z} \wedge m \geq 2$$

Rule 1.1.1.4.2.2.1.1.3: If $2m \in \mathbb{Z} \wedge m \geq 2$, then

$$\int \frac{(a+bx)^m}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx \rightarrow$$

$$\frac{2b^2(a+bx)^{m-2}\sqrt{c+dx}\sqrt{e+fx}\sqrt{g+hx}}{dfh(2m-1)} - \frac{1}{dfh(2m-1)} \int \frac{(a+bx)^{m-3}}{\sqrt{c+dx}\sqrt{e+fx}\sqrt{g+hx}} dx.$$

$$\begin{aligned} & \left(ab^2(deg+cfg+ceh) + 2b^3ceg(m-2) - a^3dfh(2m-1) + \right. \\ & \left. b(2ab(dfg+deh+cfh) + b^2(2m-3)(deg+cfg+ceh) - 3a^2dfh(2m-1))x - \right. \\ & \left. 2b^2(m-1)(3adfh - b(dfg+deh+cfh))x^2 \right) dx \end{aligned}$$

Program code:

```
Int[(a_.+b_.x_)^m/(Sqrt[c_.+d_.x_]*Sqrt[e_.+f_.x_]*Sqrt[g_.+h_.x_]),x_Symbol] :=
  2*b^2*(a+b*x)^(m-2)*Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]/(d*f*h*(2*m-1)) -
  1/(d*f*h*(2*m-1))*Int[((a+b*x)^(m-3)/(Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]))*
  Simp[a*b^2*(d*e*g+c*f*g+c*e*h)+2*b^3*c*e*g*(m-2)-a^3*d*f*h*(2*m-1) +
  b*(2*a*b*(d*f*g+d*e*h+c*f*h)+b^2*(2*m-3)*(d*e*g+c*f*g+c*e*h)-3*a^2*d*f*h*(2*m-1))*x -
  2*b^2*(m-1)*(3*a*d*f*h-b*(d*f*g+d*e*h+c*f*h))*x^2,x],x] /;
FreeQ[{a,b,c,d,e,f,g,h},x] && IntegerQ[2*m] && GeQ[m,2]
```

$$2. \int \frac{(a+bx)^m}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx \text{ when } 2m \in \mathbb{Z} \wedge m < 0$$

$$1: \int \frac{1}{(a+bx) \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx$$

Derivation: Integration by substitution

$$\text{Basis: } \frac{F[x]}{\sqrt{c+dx}} = \frac{2}{d} \text{Subst} \left[F \left[-\frac{c-x^2}{d} \right], x, \sqrt{c+dx} \right] \partial_x \sqrt{c+dx}$$

Rule 1.1.1.4.2.2.1.2.1:

$$\int \frac{1}{(a+bx) \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx \rightarrow -2 \text{Subst} \left[\int \frac{1}{(bc-ad-bx^2) \sqrt{\frac{de-cf}{d} + \frac{fx^2}{d}} \sqrt{\frac{dg-ch}{d} + \frac{hx^2}{d}}} dx, x, \sqrt{c+dx} \right]$$

Program code:

```
Int[1/((a_.+b_.*x_)*Sqrt[c_.+d_.*x_]*Sqrt[e_.+f_.*x_]*Sqrt[g_.+h_.*x_]),x_Symbol] :=
-2*Subst[Int[1/(Simp[b*c-a*d-b*x^2,x]*Sqrt[Simp[(d*e-c*f)/d+f*x^2/d,x]]*Sqrt[Simp[(d*g-c*h)/d+h*x^2/d,x]]),x],x,Sqrt[c+d*x]] /;
FreeQ[{a,b,c,d,e,f,g,h},x] && GtQ[(d*e-c*f)/d,0]
```

```
Int[1/((a_.+b_.*x_)*Sqrt[c_.+d_.*x_]*Sqrt[e_.+f_.*x_]*Sqrt[g_.+h_.*x_]),x_Symbol] :=
-2*Subst[Int[1/(Simp[b*c-a*d-b*x^2,x]*Sqrt[Simp[(d*e-c*f)/d+f*x^2/d,x]]*Sqrt[Simp[(d*g-c*h)/d+h*x^2/d,x]]),x],x,Sqrt[c+d*x]] /;
FreeQ[{a,b,c,d,e,f,g,h},x] && Not[SimplerQ[e+f*x,c+d*x]] && Not[SimplerQ[g+h*x,c+d*x]]
```

$$\mathbf{x:} \int \frac{1}{\sqrt{a+bx} \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx$$

Derivation: Piecewise constant extraction and integration by substitution

$$\blacksquare \text{ Basis: } \partial_x \frac{\sqrt{e+fx} \sqrt{\frac{(bg-ah)(c+dx)}{(dg-ch)(a+bx)}}}{\sqrt{c+dx} \sqrt{\frac{(bg-ah)(e+fx)}{(fg-eh)(a+bx)}}} = 0$$

Basis:

$$\frac{1}{(a+bx)^{3/2} \sqrt{g+hx} \sqrt{\frac{(bg-ah)(c+dx)}{(dg-ch)(a+bx)}} \sqrt{\frac{(bg-ah)(e+fx)}{(fg-eh)(a+bx)}}} = -\frac{2}{bg-ah} \text{Subst} \left[\frac{1}{\sqrt{1+\frac{(bc-ad)x^2}{dg-ch}} \sqrt{1+\frac{(be-af)x^2}{fg-eh}}}, x, \frac{\sqrt{g+hx}}{\sqrt{a+bx}} \right] \partial_x \frac{\sqrt{g+hx}}{\sqrt{a+bx}}$$

Rule 1.1.1.4.2.2.1.2.2:

$$\begin{aligned} \int \frac{1}{\sqrt{a+bx} \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx &\rightarrow \frac{(a+bx) \sqrt{\frac{(bg-ah)(c+dx)}{(dg-ch)(a+bx)}} \sqrt{\frac{(bg-ah)(e+fx)}{(fg-eh)(a+bx)}}}{\sqrt{c+dx} \sqrt{e+fx}} \int \frac{1}{(a+bx)^{3/2} \sqrt{g+hx} \sqrt{\frac{(bg-ah)(c+dx)}{(dg-ch)(a+bx)}} \sqrt{\frac{(bg-ah)(e+fx)}{(fg-eh)(a+bx)}}} dx \\ &\rightarrow -\frac{2(a+bx) \sqrt{\frac{(bg-ah)(c+dx)}{(dg-ch)(a+bx)}} \sqrt{\frac{(bg-ah)(e+fx)}{(fg-eh)(a+bx)}}}{(bg-ah) \sqrt{c+dx} \sqrt{e+fx}} \text{Subst} \left[\int \frac{1}{\sqrt{1+\frac{(bc-ad)x^2}{dg-ch}} \sqrt{1+\frac{(be-af)x^2}{fg-eh}}} dx, x, \frac{\sqrt{g+hx}}{\sqrt{a+bx}} \right] \end{aligned}$$

Program code:

```
(* Int[1/(Sqrt[a_.+b_.x_]*Sqrt[c_.+d_.x_]*Sqrt[e_.+f_.x_]*Sqrt[g_.+h_.x_]),x_Symbol] :=
-2*(a+b*x)*Sqrt[(b*g-a*h)*(c+d*x)/((d*g-c*h)*(a+b*x))]*Sqrt[(b*g-a*h)*(e+f*x)/((f*g-e*h)*(a+b*x))]/
((b*g-a*h)*Sqrt[c+d*x]*Sqrt[e+f*x])*
Subst[Int[1/(Sqrt[1+(b*c-a*d)*x^2/(d*g-c*h)]*Sqrt[1+(b*e-a*f)*x^2/(f*g-e*h)]),x],x,Sqrt[g+h*x]/Sqrt[a+b*x]] /;
FreeQ[{a,b,c,d,e,f,g,h},x] *)
```

$$2: \int \frac{1}{\sqrt{a+bx} \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx$$

Derivation: Piecewise constant extraction and integration by substitution

$$\blacksquare \text{ Basis: } \partial_x \frac{\sqrt{g+hx} \sqrt{\frac{(be-af)(c+dx)}{(de-cf)(a+bx)}}}{\sqrt{c+dx} \sqrt{-\frac{(be-af)(g+hx)}{(fg-eh)(a+bx)}}} == 0$$

Basis:

$$\frac{1}{(a+bx)^{3/2} \sqrt{e+fx} \sqrt{\frac{(be-af)(c+dx)}{(de-cf)(a+bx)}} \sqrt{\frac{(-be+af)(g+hx)}{(fg-eh)(a+bx)}}} == -\frac{2}{be-af} \text{Subst} \left[\frac{1}{\sqrt{1 + \frac{(bc-ad)x^2}{de-cf}} \sqrt{1 - \frac{(bg-ah)x^2}{fg-eh}}}, x, \frac{\sqrt{e+fx}}{\sqrt{a+bx}} \right] \partial_x \frac{\sqrt{e+fx}}{\sqrt{a+bx}}$$

Rule 1.1.1.4.2.2.1.2.2:

$$\begin{aligned} \int \frac{1}{\sqrt{a+bx} \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx &\rightarrow -\frac{(be-af) \sqrt{g+hx} \sqrt{\frac{(be-af)(c+dx)}{(de-cf)(a+bx)}}}{(fg-eh) \sqrt{c+dx} \sqrt{-\frac{(be-af)(g+hx)}{(fg-eh)(a+bx)}}} \int \frac{1}{(a+bx)^{3/2} \sqrt{e+fx} \sqrt{\frac{(be-af)(c+dx)}{(de-cf)(a+bx)}} \sqrt{\frac{(-be+af)(g+hx)}{(fg-eh)(a+bx)}}} dx \\ &\rightarrow \frac{2 \sqrt{g+hx} \sqrt{\frac{(be-af)(c+dx)}{(de-cf)(a+bx)}}}{(fg-eh) \sqrt{c+dx} \sqrt{-\frac{(be-af)(g+hx)}{(fg-eh)(a+bx)}}} \text{Subst} \left[\int \frac{1}{\sqrt{1 + \frac{(bc-ad)x^2}{de-cf}} \sqrt{1 - \frac{(bg-ah)x^2}{fg-eh}}} dx, x, \frac{\sqrt{e+fx}}{\sqrt{a+bx}} \right] \end{aligned}$$

Program code:

```
Int[1/(Sqrt[a_.+b_.x_]*Sqrt[c_.+d_.x_]*Sqrt[e_.+f_.x_]*Sqrt[g_.+h_.x_]),x_Symbol] :=
  2*Sqrt[g+h*x]*Sqrt[(b*e-a*f)*(c+d*x)/((d*e-c*f)*(a+b*x))]/
  ((f*g-e*h)*Sqrt[c+d*x]*Sqrt[-(b*e-a*f)*(g+h*x)/((f*g-e*h)*(a+b*x))]) *
  Subst[Int[1/(Sqrt[1+(b*c-a*d)*x^2/(d*e-c*f)]*Sqrt[1-(b*g-a*h)*x^2/(f*g-e*h)]),x],x,Sqrt[e+f*x]/Sqrt[a+b*x]] /;
FreeQ[{a,b,c,d,e,f,g,h},x]
```


$$\text{3: } \int \frac{1}{(a+bx)^{3/2} \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{1}{(a+bx)^{3/2} \sqrt{c+dx}} = -\frac{d}{(bc-ad) \sqrt{a+bx} \sqrt{c+dx}} + \frac{b\sqrt{c+dx}}{(bc-ad) (a+bx)^{3/2}}$$

Rule 1.1.1.4.2.2.1.2.3:

$$\int \frac{1}{(a+bx)^{3/2} \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx \rightarrow$$

$$-\frac{d}{bc-ad} \int \frac{1}{\sqrt{a+bx} \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx + \frac{b}{bc-ad} \int \frac{\sqrt{c+dx}}{(a+bx)^{3/2} \sqrt{e+fx} \sqrt{g+hx}} dx$$

Program code:

```
Int[1/((a_.+b_.*x_)^(3/2)*Sqrt[c_.+d_.*x_] *Sqrt[e_.+f_.*x_] *Sqrt[g_.+h_.*x_]),x_Symbol] :=
  -d/(b*c-a*d)*Int[1/(Sqrt[a+b*x]*Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]),x] +
  b/(b*c-a*d)*Int[Sqrt[c+d*x]/((a+b*x)^(3/2)*Sqrt[e+f*x]*Sqrt[g+h*x]),x] /;
FreeQ[{a,b,c,d,e,f,g,h},x]
```

$$\text{4: } \int \frac{(a+bx)^m}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx \text{ when } 2m \in \mathbb{Z} \wedge m \leq -2$$

Rule 1.1.1.4.2.2.1.2.4: If $2m \in \mathbb{Z} \wedge m \leq -2$, then

$$\int \frac{(a+bx)^m}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx \rightarrow$$

$$\frac{b^2 (a+bx)^{m+1} \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}}{(m+1)(bc-ad)(be-af)(bg-ah)} - \frac{1}{2(m+1)(bc-ad)(be-af)(bg-ah)} \int \frac{(a+bx)^{m+1}}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx$$

$$\frac{(2a^2dfh(m+1) - 2ab(m+1)(dfg+deh+cfh) + b^2(2m+3)(deg+cfg+ceh) - 2b(adfh(m+1) - b(m+2)(dfg+deh+cfh))x + dfhb^2(2m+5)x^2)}{(m+1)(bc-ad)(be-af)(bg-ah)} dx$$

Program code:

```
Int[(a_.+b_.x_)^m_/ (Sqrt[c_.+d_.x_] * Sqrt[e_.+f_.x_] * Sqrt[g_.+h_.x_]), x_Symbol] :=
  b^2*(a+b*x)^(m+1)*Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]/((m+1)*(b*c-a*d)*(b*e-a*f)*(b*g-a*h)) -
  1/(2*(m+1)*(b*c-a*d)*(b*e-a*f)*(b*g-a*h))*Int[((a+b*x)^(m+1)/(Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]))*
  Simp[2*a^2*d*f*h*(m+1)-2*a*b*(m+1)*(d*f*g+d*e*h+c*f*h)+b^2*(2*m+3)*(d*e*g+c*f*g+c*e*h) -
  2*b*(a*d*f*h*(m+1)-b*(m+2)*(d*f*g+d*e*h+c*f*h))*x + d*f*h*(2*m+5)*b^2*x^2,x],x] /;
FreeQ[{a,b,c,d,e,f,g,h},x] && IntegerQ[2*m] && LeQ[m,-2]
```

$$2. \int \frac{(a+bx)^m \sqrt{c+dx}}{\sqrt{e+fx} \sqrt{g+hx}} dx \text{ when } 2m \in \mathbb{Z}$$

$$1. \int \frac{(a+bx)^m \sqrt{c+dx}}{\sqrt{e+fx} \sqrt{g+hx}} dx \text{ when } 2m \in \mathbb{Z} \wedge m > 0$$

$$1: \int \frac{\sqrt{a+bx} \sqrt{c+dx}}{\sqrt{e+fx} \sqrt{g+hx}} dx$$

Derivation: Algebraic expansion

■

Basis: $\frac{\sqrt{a+bx} \sqrt{c+dx}}{\sqrt{e+fx} \sqrt{g+hx}} = \partial_x \frac{\sqrt{a+bx} \sqrt{c+dx} \sqrt{g+hx}}{h \sqrt{e+fx}} + \frac{(de-cf)(bfg+b eh-2afh)}{2f^2 h \sqrt{a+bx} \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} +$

$$\frac{(adfh-b(dfg+deh-cfh)) \sqrt{e+fx}}{2f^2 h \sqrt{a+bx} \sqrt{c+dx} \sqrt{g+hx}} - \frac{(de-cf)(fg-eh) \sqrt{a+bx}}{2fh \sqrt{c+dx} (e+fx)^{3/2} \sqrt{g+hx}}$$

Basis: $\frac{\sqrt{a+bx} \sqrt{c+dx}}{\sqrt{e+fx} \sqrt{g+hx}} ==$

$$\partial_x \frac{b \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}}{f h \sqrt{a+bx}} + \frac{(b c - a d) (b e - a f) (b g - a h)}{2 b f h (a+bx)^{3/2} \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} - \frac{(b d e h + f (b d g - b c h - a d h)) \sqrt{a+bx}}{2 b f h \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}}$$

Rule 1.1.1.4.2.2.1.1:

$$\int \frac{\sqrt{a+bx} \sqrt{c+dx}}{\sqrt{e+fx} \sqrt{g+hx}} dx \rightarrow$$

$$\frac{\sqrt{a+bx} \sqrt{c+dx} \sqrt{g+hx}}{h \sqrt{e+fx}} + \frac{(de - cf) (bfg + beh - 2afh)}{2 f^2 h} \int \frac{1}{\sqrt{a+bx} \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx +$$

$$\frac{(adfh - b(df g + de h - cf h))}{2 f^2 h} \int \frac{\sqrt{e+fx}}{\sqrt{a+bx} \sqrt{c+dx} \sqrt{g+hx}} dx - \frac{(de - cf) (fg - eh)}{2 f h} \int \frac{\sqrt{a+bx}}{\sqrt{c+dx} (e+fx)^{3/2} \sqrt{g+hx}} dx$$

Program code:

```
Int[Sqrt[a_.+b_.x_]*Sqrt[c_.+d_.x_]/(Sqrt[e_.+f_.x_]*Sqrt[g_.+h_.x_]),x_Symbol] :=
  Sqrt[a+b*x]*Sqrt[c+d*x]*Sqrt[g+h*x]/(h*Sqrt[e+f*x]) +
  (d*e-c*f)*(b*f*g+b*e*h-2*a*f*h)/(2*f^2*h)*Int[1/(Sqrt[a+b*x]*Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]),x] +
  (a*d*f*h-b*(d*f*g+d*e*h-c*f*h))/(2*f^2*h)*Int[Sqrt[e+f*x]/(Sqrt[a+b*x]*Sqrt[c+d*x]*Sqrt[g+h*x]),x] -
  (d*e-c*f)*(f*g-e*h)/(2*f*h)*Int[Sqrt[a+b*x]/(Sqrt[c+d*x]*(e+f*x)^(3/2)*Sqrt[g+h*x]),x] /;
FreeQ[{a,b,c,d,e,f,g,h},x]
```

2: $\int \frac{(a+bx)^m \sqrt{c+dx}}{\sqrt{e+fx} \sqrt{g+hx}} dx$ when $2m \in \mathbb{Z} \wedge m > 1$

Rule 1.1.1.4.2.2.1.2: If $2m \in \mathbb{Z} \wedge m > 1$, then

$$\int \frac{(a+bx)^m \sqrt{c+dx}}{\sqrt{e+fx} \sqrt{g+hx}} dx \rightarrow$$

$$\frac{2b(a+bx)^{m-1}\sqrt{c+dx}\sqrt{e+fx}\sqrt{g+hx}}{fh(2m+1)} - \frac{1}{fh(2m+1)} \int \frac{(a+bx)^{m-2}}{\sqrt{c+dx}\sqrt{e+fx}\sqrt{g+hx}}.$$

$$\begin{aligned} & (ab(deg+c(fg+eh)) + 2b^2ceg(m-1) - a^2cfh(2m+1) + \\ & (b^2(2m-1)(deg+c(fg+eh)) - a^2dfh(2m+1) + 2ab(df g+deh-2cfhm))x - \\ & b(adfh(4m-1) + b(cf h-2d(fg+eh)m))x^2) dx \end{aligned}$$

Program code:

```
Int[(a_.+b_.x_)^m*Sqrt[c_.+d_.x_]/(Sqrt[e_.+f_.x_]*Sqrt[g_.+h_.x_]),x_Symbol] :=
  2*b*(a+b*x)^(m-1)*Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]/(f*h*(2*m+1)) -
  1/(f*h*(2*m+1))*Int[((a+b*x)^(m-2)/(Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]))*
  Simp[a*b*(d*e*g+c*(f*g+e*h))+2*b^2*c*e*g*(m-1)-a^2*c*f*h*(2*m+1) +
  (b^2*(2*m-1)*(d*e*g+c*(f*g+e*h))-a^2*d*f*h*(2*m+1)+2*a*b*(d*f*g+d*e*h-2*c*f*h*m))*x -
  b*(a*d*f*h*(4*m-1)+b*(c*f*h-2*d*(f*g+e*h)*m))*x^2,x],x] /;
FreeQ[{a,b,c,d,e,f,g,h,m},x] && IntegerQ[2*m] && GtQ[m,1]
```

$$2. \int \frac{(a+bx)^m \sqrt{c+dx}}{\sqrt{e+fx} \sqrt{g+hx}} dx \text{ when } 2m \in \mathbb{Z} \wedge m < 0$$

$$1: \int \frac{\sqrt{c+dx}}{(a+bx) \sqrt{e+fx} \sqrt{g+hx}} dx$$

Derivation: Algebraic expansion

$$\text{Basis: } \frac{\sqrt{c+dx}}{a+bx} == \frac{d}{b \sqrt{c+dx}} + \frac{bc-ad}{b(a+bx) \sqrt{c+dx}}$$

Rule 1.1.1.4.2.2.2.1:

$$\int \frac{\sqrt{c+dx}}{(a+bx) \sqrt{e+fx} \sqrt{g+hx}} dx \rightarrow \frac{d}{b} \int \frac{1}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx + \frac{bc-ad}{b} \int \frac{1}{(a+bx) \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx$$

Program code:

```
Int[Sqrt[c_.+d_.**x_]/((a_.+b_.**x_)*Sqrt[e_.+f_.**x_]*Sqrt[g_.+h_.**x_]),x_Symbol] :=
  d/b*Int[1/(Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]),x] +
  (b*c-a*d)/b*Int[1/((a+b*x)*Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]),x] /;
FreeQ[{a,b,c,d,e,f,g,h},x]
```

$$\text{■ } \int \frac{\sqrt{c+dx}}{(a+bx)^{3/2} \sqrt{e+fx} \sqrt{g+hx}} dx$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{■ Basis: } \partial_x \frac{\sqrt{c+dx} \sqrt{\frac{(bg-ah)(e+fx)}{(fg-eh)(a+bx)}}}{\sqrt{e+fx} \sqrt{\frac{(bg-ah)(c+dx)}{(dg-ch)(a+bx)}}} = 0$$

$$\text{■ Basis: } \frac{\sqrt{\frac{(bg-ah)(c+dx)}{(dg-ch)(a+bx)}}}{(a+bx)^{3/2} \sqrt{g+hx} \sqrt{\frac{(bg-ah)(e+fx)}{(fg-eh)(a+bx)}}} = -\frac{2}{bg-ah} \text{Subst} \left[\frac{\sqrt{1 + \frac{(bc-ad)x^2}{dg-ch}}}{\sqrt{1 + \frac{(be-af)x^2}{fg-eh}}}, x, \frac{\sqrt{g+hx}}{\sqrt{a+bx}} \right] \partial_x \frac{\sqrt{g+hx}}{\sqrt{a+bx}}$$

Rule 1.1.1.4.2.2.2.2.2:

$$\begin{aligned} \int \frac{\sqrt{c+dx}}{(a+bx)^{3/2} \sqrt{e+fx} \sqrt{g+hx}} dx &\rightarrow \frac{\sqrt{c+dx} \sqrt{\frac{(bg-ah)(e+fx)}{(fg-eh)(a+bx)}}}{\sqrt{e+fx} \sqrt{\frac{(bg-ah)(c+dx)}{(dg-ch)(a+bx)}}} \int \frac{\sqrt{\frac{(bg-ah)(c+dx)}{(dg-ch)(a+bx)}}}{(a+bx)^{3/2} \sqrt{g+hx} \sqrt{\frac{(bg-ah)(e+fx)}{(fg-eh)(a+bx)}}} dx \\ &\rightarrow -\frac{2\sqrt{c+dx} \sqrt{\frac{(bg-ah)(e+fx)}{(fg-eh)(a+bx)}}}{(bg-ah) \sqrt{e+fx} \sqrt{\frac{(bg-ah)(c+dx)}{(dg-ch)(a+bx)}}} \text{Subst} \left[\int \frac{\sqrt{1 + \frac{(bc-ad)x^2}{dg-ch}}}{\sqrt{1 + \frac{(be-af)x^2}{fg-eh}}} dx, x, \frac{\sqrt{g+hx}}{\sqrt{a+bx}} \right] \end{aligned}$$

Program code:

```
(* Int[Sqrt[c_+d_*x_]/((a_+b_*x_)^(3/2)*Sqrt[e_+f_*x_]*Sqrt[g_+h_*x_]),x_Symbol] :=
-2*Sqrt[c+d*x]*Sqrt[(b*g-a*h)*(e+f*x)/((f*g-e*h)*(a+b*x))]/
((b*g-a*h)*Sqrt[e+f*x]*Sqrt[(b*g-a*h)*(c+d*x)/((d*g-c*h)*(a+b*x))]) *
Subst[Int[Sqrt[1+(b*c-a*d)*x^2/(d*g-c*h)]]/Sqrt[1+(b*e-a*f)*x^2/(f*g-e*h)],x],x,Sqrt[g+h*x]/Sqrt[a+b*x]] /;
FreeQ[{a,b,c,d,e,f,g,h},x] *)
```

$$2: \int \frac{\sqrt{c+dx}}{(a+bx)^{3/2} \sqrt{e+fx} \sqrt{g+hx}} dx$$

Derivation: Piecewise constant extraction and integration by substitution

$$\blacksquare \text{ Basis: } \partial_x \frac{\sqrt{c+dx} \sqrt{-\frac{(be-af)(g+hx)}{(fg-eh)(a+bx)}}}{\sqrt{g+hx} \sqrt{\frac{(be-af)(c+dx)}{(de-cf)(a+bx)}}} = 0$$

$$\blacksquare \text{ Basis: } \frac{\sqrt{\frac{(be-af)(c+dx)}{(de-cf)(a+bx)}}}{(a+bx)^{3/2} \sqrt{e+fx} \sqrt{-\frac{(be-af)(g+hx)}{(fg-eh)(a+bx)}}} = -\frac{2}{be-af} \text{ Subst} \left[\sqrt{\frac{1 + \frac{(bc-ad)x^2}{de-cf}}{1 - \frac{(bg-ah)x^2}{fg-eh}}}, x, \frac{\sqrt{e+fx}}{\sqrt{a+bx}} \right] \partial_x \frac{\sqrt{e+fx}}{\sqrt{a+bx}}$$

Rule 1.1.1.4.2.2.2.2.2:

$$\begin{aligned} \int \frac{\sqrt{c+dx}}{(a+bx)^{3/2} \sqrt{e+fx} \sqrt{g+hx}} dx &\rightarrow \frac{\sqrt{c+dx} \sqrt{-\frac{(be-af)(g+hx)}{(fg-eh)(a+bx)}}}{\sqrt{g+hx} \sqrt{\frac{(be-af)(c+dx)}{(de-cf)(a+bx)}}} \int \frac{\sqrt{\frac{(be-af)(c+dx)}{(de-cf)(a+bx)}}}{(a+bx)^{3/2} \sqrt{e+fx} \sqrt{-\frac{(be-af)(g+hx)}{(fg-eh)(a+bx)}}} dx \\ &\rightarrow -\frac{2\sqrt{c+dx} \sqrt{-\frac{(be-af)(g+hx)}{(fg-eh)(a+bx)}}}{(be-af) \sqrt{g+hx} \sqrt{\frac{(be-af)(c+dx)}{(de-cf)(a+bx)}}} \text{Subst} \left[\int \frac{\sqrt{1 + \frac{(bc-ad)x^2}{de-cf}}}{\sqrt{1 - \frac{(bg-ah)x^2}{fg-eh}}} dx, x, \frac{\sqrt{e+fx}}{\sqrt{a+bx}} \right] \end{aligned}$$

Program code:

```
Int[Sqrt[c_.+d_.x_]/((a_.+b_.x_)^(3/2)*Sqrt[e_.+f_.x_]*Sqrt[g_.+h_.x_]),x_Symbol] :=
-2*Sqrt[c+d*x]*Sqrt[-(b*e-a*f)*(g+h*x)/((f*g-e*h)*(a+b*x))]/
((b*e-a*f)*Sqrt[g+h*x]*Sqrt[(b*e-a*f)*(c+d*x)/((d*e-c*f)*(a+b*x))])*
Subst[Int[Sqrt[1+(b*c-a*d)*x^2/(d*e-c*f)]/Sqrt[1-(b*g-a*h)*x^2/(f*g-e*h)],x],x,Sqrt[e+f*x]/Sqrt[a+b*x]] /;
FreeQ[{a,b,c,d,e,f,g,h},x]
```

$$\text{3: } \int \frac{(a+bx)^m \sqrt{c+dx}}{\sqrt{e+fx} \sqrt{g+hx}} dx \text{ when } 2m \in \mathbb{Z} \wedge m \leq -2$$

Rule 1.1.1.4.2.2.2.3: If $2m \in \mathbb{Z} \wedge m \leq -2$, then

$$\int \frac{(a+bx)^m \sqrt{c+dx}}{\sqrt{e+fx} \sqrt{g+hx}} dx \rightarrow$$

$$\frac{b(a+bx)^{m+1} \sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}}{(m+1)(be-af)(bg-ah)} + \frac{1}{2(m+1)(be-af)(bg-ah)} \int \frac{(a+bx)^{m+1}}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} dx$$

$$(2acfh(m+1) - b(deg+c(2m+3)(fg+eh)) + 2(adfh(m+1) - b(m+2)(dfg+deh+cfh))x - bdfh(2m+5)x^2) dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_*Sqrt[c_.+d_.*x_]/(Sqrt[e_.+f_.*x_]*Sqrt[g_.+h_.*x_]),x_Symbol] :=
  b*(a+b*x)^(m+1)*Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]/((m+1)*(b*e-a*f)*(b*g-a*h)) +
  1/(2*(m+1)*(b*e-a*f)*(b*g-a*h))*Int[((a+b*x)^(m+1))/(Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]))*
  Simp[2*a*c*f*h*(m+1)-b*(d*e*g+c*(2*m+3)*(f*g+e*h))+2*(a*d*f*h*(m+1)-b*(m+2)*(d*f*g+d*e*h+c*f*h))*x-b*d*f*h*(2*m+5)*x^2,x],x] /;
FreeQ[{a,b,c,d,e,f,g,h,m},x] && IntegerQ[2*m] && LeQ[m,-2]
```


3: $\int \frac{(e+fx)^p (g+hx)^q}{(a+bx)(c+dx)} dx$ when $0 < p < 1$

Derivation: Algebraic expansion

Basis: $\frac{e+fx}{(a+bx)(c+dx)} = \frac{be-af}{(bc-ad)(a+bx)} - \frac{de-cf}{(bc-ad)(c+dx)}$

Rule 1.1.1.4.3: If $0 < p < 1$, then

$$\int \frac{(e+fx)^p (g+hx)^q}{(a+bx)(c+dx)} dx \rightarrow \frac{be-af}{bc-ad} \int \frac{(e+fx)^{p-1} (g+hx)^q}{a+bx} dx - \frac{de-cf}{bc-ad} \int \frac{(e+fx)^{p-1} (g+hx)^q}{c+dx} dx$$

Program code:

```
Int[(e_.+f_.**x_)^p_*(g_.+h_.**x_)^q_/((a_.+b_.**x_)*(c_.+d_.**x_)),x_Symbol] :=
  (b*e-a*f)/(b*c-a*d)*Int[(e+f*x)^(p-1)*(g+h*x)^q/(a+b*x),x] -
  (d*e-c*f)/(b*c-a*d)*Int[(e+f*x)^(p-1)*(g+h*x)^q/(c+d*x),x] /;
FreeQ[{a,b,c,d,e,f,g,h,q},x] && LtQ[0,p,1]
```

4: $\int \frac{(a+bx)^m (c+dx)^n}{\sqrt{e+fx} \sqrt{g+hx}} dx$ when $m \in \mathbb{Z} \wedge n + \frac{1}{2} \in \mathbb{Z}$

Derivation: Algebraic expansion

Rule 1.1.1.4.4: If $m \in \mathbb{Z} \wedge n + \frac{1}{2} \in \mathbb{Z}$, then

$$\int \frac{(a+bx)^m (c+dx)^n}{\sqrt{e+fx} \sqrt{g+hx}} dx \rightarrow \int \frac{1}{\sqrt{c+dx} \sqrt{e+fx} \sqrt{g+hx}} \text{ExpandIntegrand}[(a+bx)^m (c+dx)^{n+\frac{1}{2}}, x] dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_.*(c_.+d_.*x_)^n_/ (Sqrt[e_.+f_.*x_] * Sqrt[g_.+h_.*x_]), x_Symbol] :=
  Int[ExpandIntegrand[1/(Sqrt[c+d*x]*Sqrt[e+f*x]*Sqrt[g+h*x]), (a+b*x)^m*(c+d*x)^(n+1/2), x], x] /;
  FreeQ[{a,b,c,d,e,f,g,h}, x] && IntegerQ[m] && IntegerQ[n+1/2]
```

5: $\int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx)^q dx$ when $(p \mid q) \in \mathbb{Z}$

Derivation: Algebraic expansion

Rule 1.1.1.4.5: If $(p \mid q) \in \mathbb{Z}$, then

$$\int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx)^q dx \rightarrow \int \text{ExpandIntegrand}[(a+bx)^m (c+dx)^n (e+fx)^p (g+hx)^q, x] dx$$

Program code:

```
Int[(a_.+b_.*x_)^m_.*(c_.+d_.*x_)^n_.*(e_.+f_.*x_)^p_.*(g_.+h_.*x_)^q_, x_Symbol] :=
  Int[ExpandIntegrand[(a+b*x)^m*(c+d*x)^n*(e+f*x)^p*(g+h*x)^q, x], x] /;
  FreeQ[{a,b,c,d,e,f,g,h,m,n}, x] && IntegerQ[p,q]
```

6: $\int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx)^q dx$ when $q \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Basis: $g + hx = \frac{h(a+bx)}{b} + \frac{bg-ah}{b}$

Rule 1.1.1.4.6: If $q \in \mathbb{Z}^+$, then

$$\int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx)^q dx \rightarrow \frac{h}{b} \int (a+bx)^{m+1} (c+dx)^n (e+fx)^p (g+hx)^{q-1} dx + \frac{bg-ah}{b} \int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx)^{q-1} dx$$

Program code:

```
Int[(a_.+b_.**x_)^m_*(c_.+d_.**x_)^n_*(e_.+f_.**x_)^p_*(g_.+h_.**x_)^q_,x_Symbol] :=
  h/b*Int[(a+b*x)^(m+1)*(c+d*x)^n*(e+f*x)^p*(g+h*x)^(q-1),x] +
  (b*g-a*h)/b*Int[(a+b*x)^m*(c+d*x)^n*(e+f*x)^p*(g+h*x)^(q-1),x] /;
FreeQ[{a,b,c,d,e,f,g,h,m,n,p},x] && IGtQ[q,0] && (SumSimplerQ[m,1] || Not[SumSimplerQ[n,1]] && Not[SumSimplerQ[p,1]])
```

C: $\int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx)^q dx$

Rule 1.1.1.4.C:

$$\int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx)^q dx \rightarrow \int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx)^q dx$$

Program code:

```
Int[(a_.+b_.**x_)^m_*(c_.+d_.**x_)^n_*(e_.+f_.**x_)^p_*(g_.+h_.**x_)^q_,x_Symbol] :=
  CannotIntegrate[(a+b*x)^m*(c+d*x)^n*(e+f*x)^p*(g+h*x)^q,x] /;
FreeQ[{a,b,c,d,e,f,g,h,m,n,p,q},x]
```

S: $\int (a+bu)^m (c+du)^n (e+fu)^p (g+hu)^q dx$ when $u = i + jx$

Derivation: Integration by substitution

Rule 1.1.1.4.S: If $u = i + jx$, then

$$\int (a+bu)^m (c+du)^n (e+fu)^p (g+hu)^q dx \rightarrow \frac{1}{j} \text{Subst} \left[\int (a+bx)^m (c+dx)^n (e+fx)^p (g+hx)^q dx, x, u \right]$$

Program code:

```
Int[(a_.+b_.*u_)^m_.*(c_.+d_.*u_)^n_.*(e_.+f_.*u_)^p_.*(g_.+h_.*u_)^q_. ,x_Symbol] :=
  1/Coefficient[u,x,1]*Subst[Int[(a+b*x)^m*(c+d*x)^n*(e+f*x)^p*(g+h*x)^q,x],x,u] /;
FreeQ[{a,b,c,d,e,f,g,h,m,n,p,q},x] && LinearQ[u,x] && NeQ[u,x]
```

Rules for integrands of the form $((a + bx)^m (c + dx)^n (e + fx)^p (g + hx)^q)^r$

1: $\int ((a + bx)^m (c + dx)^n (e + fx)^p (g + hx)^q)^r dx$

Derivation: Piecewise constant extraction

■ Basis: $\partial_x \frac{(i (a+bx)^m (c+dx)^n (e+fx)^p (g+hx)^q)^r}{(a+bx)^{mr} (c+dx)^{nr} (e+fx)^{pr} (g+hx)^{qr}} = 0$

Rule:

$$\int (i (a + bx)^m (c + dx)^n (e + fx)^p (g + hx)^q)^r dx \rightarrow \frac{(i (a + bx)^m (c + dx)^n (e + fx)^p (g + hx)^q)^r}{(a + bx)^{mr} (c + dx)^{nr} (e + fx)^{pr} (g + hx)^{qr}} \int (a + bx)^{mr} (c + dx)^{nr} (e + fx)^{pr} (g + hx)^{qr} dx$$

Program code:

```
Int[(i_*(a_+b_.*x_)^m_*(c_+d_.*x_)^n_*(e_+f_.*x_)^p_*(g_+h_.*x_)^q_)^r_,x_Symbol] :=
  (i*(a+b*x)^m*(c+d*x)^n*(e+f*x)^p*(g+h*x)^q)^r/((a+b*x)^(m*r)*(c+d*x)^(n*r)*(e+f*x)^(p*r)*(g+h*x)^(q*r))*
  Int[(a+b*x)^(m*r)*(c+d*x)^(n*r)*(e+f*x)^(p*r)*(g+h*x)^(q*r),x] /;
FreeQ[{a,b,c,d,e,f,g,h,i,m,n,p,q,r},x]
```

Normalize linear products

1: $\int u^m dx$ when $u == a + b x$

Derivation: Algebraic normalization

Rule: If $u == a + b x$, then

$$\int u^m dx \rightarrow \int (a + b x)^m dx$$

Program code:

```
Int[u_^m_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m,x] /;
  FreeQ[m,x] && LinearQ[u,x] && Not[LinearMatchQ[u,x]]
```

2: $\int u^m v^n dx$ when $u == a + b x \wedge v == c + d x$

Derivation: Algebraic normalization

Rule: If $u == a + b x \wedge v == c + d x$, then

$$\int u^m v^n dx \rightarrow \int (a + b x)^m (c + d x)^n dx$$

Program code:

```
Int[u_^m_.*v_^n_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*ExpandToSum[v,x]^n,x] /;
  FreeQ[{m,n},x] && LinearQ[{u,v},x] && Not[LinearMatchQ[{u,v},x]]
```

3: $\int u^m v^n w^p dx$ when $u = a + bx \wedge v = c + dx \wedge w = e + fx$

Derivation: Algebraic normalization

Rule: If $u = a + bx \wedge v = c + dx \wedge w = e + fx$, then

$$\int u^m v^n w^p dx \rightarrow \int (a + bx)^m (c + dx)^n (e + fx)^p dx$$

Program code:

```
Int[u_^m_.*v_^n_.*w_^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*ExpandToSum[v,x]^n*ExpandToSum[w,x]^p,x] /;
  FreeQ[{m,n,p},x] && LinearQ[{u,v,w},x] && Not[LinearMatchQ[{u,v,w},x]]
```

4: $\int u^m v^n w^p z^q dx$ when $u = a + bx \wedge v = c + dx \wedge w = e + fx \wedge z = g + hx$

Derivation: Algebraic normalization

Rule: If $u = a + bx \wedge v = c + dx \wedge w = e + fx \wedge z = g + hx$, then

$$\int u^m v^n w^p z^q dx \rightarrow \int (a + bx)^m (c + dx)^n (e + fx)^p (g + hx)^q dx$$

Program code:

```
Int[u_^m_.*v_^n_.*w_^p_.*z_^q_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*ExpandToSum[v,x]^n*ExpandToSum[w,x]^p*ExpandToSum[z,x]^q,x] /;
  FreeQ[{m,n,p,q},x] && LinearQ[{u,v,w,z},x] && Not[LinearMatchQ[{u,v,w,z},x]]
```