Freie Universität Berlin

Leo-III

The World's Most Widely Applicable Modal Logic Theorem Prover

and its Associated Infrastructure

Alexander Steen

Freie Universität Berlin

RuleML Webinar September 29th

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[0] jww C. Benzmüller, T. Gleißner

Talk outline

1. Motivation
2. Flavours of modal logics
3. How it works (roughly)
4. Evaluation

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Reasoning in Non-Classical Logics

- Increasing interest in various fields
  - Artificial Intelligence (e.g. Agents, Knowledge)
  - Computer Linguistics (e.g. Semantics)
  - Mathematics (e.g. Geometry, Category theory)
  - Theoretical Philosophy (e.g. Metaphysics)
  - Legal Informatics (e.g. Computable/Smart contracts)
- Most powerful ATP/ITP: Classical logic only

Focus here: Modal logics

- Prover for (propositional) modal logics exist
  - ModLeanTAP, Molle, Bliksem, FaCT++,
  - MOLTAP, KtSeqC, STeP, TRP
  - ...
- Only few for quantified variants
  - MleanTAP, MleanCoP, MleanSeP (J. Otten)
  - f2p+MSPASS

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Motivation

1. **First-order quantification is (sometimes) not enough**
2. Semantic diversity/flexibility needed

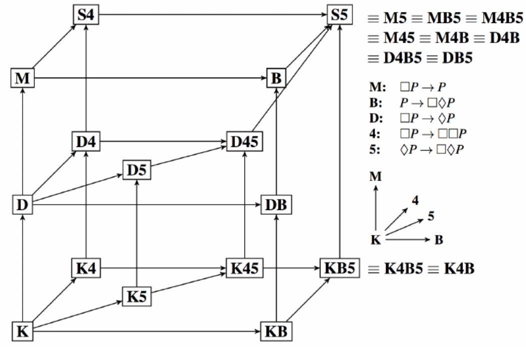
See studies in Metaphysics, e.g.

- Gödel's Ontological Argument [BenzmüllerW.-Paleo,2017] and several variants of it
- Anderson-Hájek Controversy [Benzm.WeberW.-Paleo,2017]

Motivation

1. First-order quantification is (sometimes) not enough
2. **Semantic diversity/flexibility needed:**

Properties of modal operators *necessary* ($\Box$) and *possibly* ($\Diamond$)



... but that's not all of it!

Motivation

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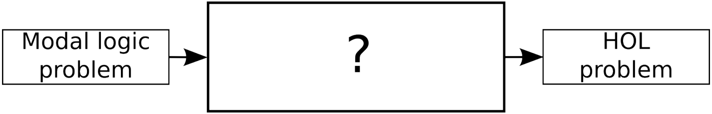
Automation approach

- Indirect: Via encoding into (classical) HOL
- Use existing general purpose HOL reasoners



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Advantages

- Sophisticated existing systems
  - ATPs: TPS, agsyHOL, Satallax, LEO-II, Leo-III
  - Further: Isabelle, Nitpick
- Not fixed to any one proving system
- Semantic variations with minor adjustments
  - Axiomatization
  - Quantification semantics
  - ...

Higher Order Modal Logic – Syntax

Based on Simple type theory [Church, J.Symb.L., 1940]
augmented with modalities

- Simple types $\mathcal{T}$ generated by base types and mappings ($\rightarrow$)
- Usually, base types are o and ι

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Type of truth-values

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Type of individuals

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- Terms defined by $(\alpha, \beta \in \mathcal{T},\ c_\alpha \in \Sigma,\ X_\alpha \in \mathcal{V},\ i \in I)$

$$s, t ::= c_\alpha \mid X_\alpha$$

- Allow infix notation for binary logical connectives
- Remaining logical connectives can be defined as usual
- Formulae of HOML are those terms with type o

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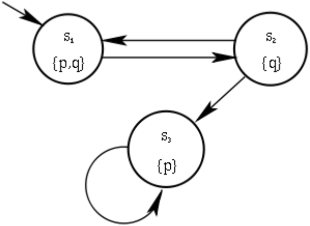
Higher Order Modal Logic – Semantics

Extend HOL models with Kripke structures

$$\mathcal{M} = \left(W, \{R^i\}_{i \in I}\ ,\ \{\mathcal{D}_w\}_{w \in W}\ ,\ \{\mathcal{I}_w\}_{w \in W}\ \right)$$



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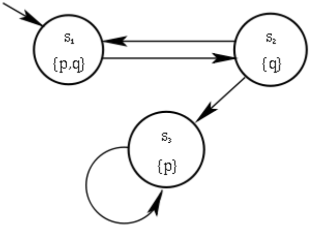
Set of possible worlds



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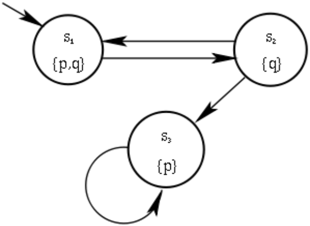
Family of accessibility relations $R^i \subseteq W \times W$



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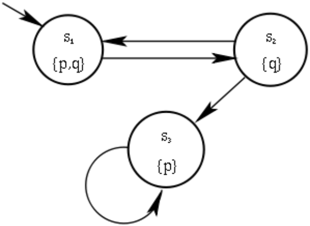

Family of frames, one for every world
Notion of frames $\mathcal{D} = (D_\tau)_{\tau \in \mathcal{T}}$ as in HOL:

$$D_\iota \neq \emptyset$$
$$D_o = \{T, F\}$$
$$D_{\tau \to \nu} = D_\nu^{D_\tau}$$

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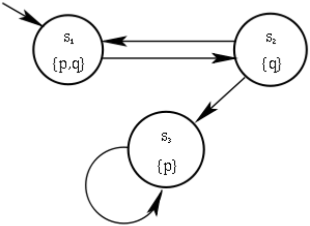

Family of interpretation functions $\mathcal{I}_w$

$c_\tau \overset{\mathcal{I}_w}{\mapsto} d \in D_\tau \in \mathcal{D}_w$

Assume $\mathcal{I}_w(\neg), \mathcal{I}_w(\vee)\ldots$ is standard.

Higher Order Modal Logic – Semantics

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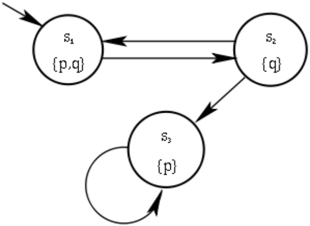

Value of a term (wrt. var. assignment g):

$$\|X_\tau\|^{\mathcal{M},g,w} = g_w(X)$$

$$\vdots \qquad \textit{(not all shown here ...)}$$

$$\|\Box^i_{o\to o}\, s_o\|^{\mathcal{M},g,w} = \begin{cases} T & \text{if } \|s_o\|^{\mathcal{M},g,v} = T \text{ for all } v \in W \text{ s.t. } (w,v) \in R^i \\ F & \text{otherwise} \end{cases}$$

Assume Henkin semantics

Semantic variants of HOML

1. **Axiomatization of $\Box^i$**
2. **Quantification**
3. **Rigidity**
4. **Consequence**

Semantic variants of HOML

1. **Axiomatization of $\Box^i$**
   - What properties does the box operators have?
   - Depending on the application domain

   Some popular axiom schemes:

Name	**Axiom scheme**	**Condition on r^i**	**Corr. formula**
K	$\Box^i(s \supset t) \supset (\Box^i s \supset \Box^i t)$	—	—
B	$s \supset \Box^i \Diamond^i s$	symmetric	$wR^i v \supset vR^i w$
D	$\Box^i s \supset \Diamond^i s$	serial	$\exists v. wR^i v$
T/M	$\Box^i s \supset s$	reflexive	$wR^i w$
4	$\Box^i s \supset \Box^i \Box^i s$	transitive	$(wR^i v \wedge vR^i u) \supset wR^i u$
5	$\Diamond^i s \supset \Box^i \Diamond^i s$	euclidean	$(wR^i v \wedge wR^i u) \supset vR^i u$
...	...	...	...

2. **Quantification**

3. **Rigidity**

4. **Consequence**

Semantic variants of HOML

1. **Axiomatization of $\Box^i$**
   - What properties does the box operators have?
2. **Quantification**
   - What is the meaning of $\forall$?
   - Several popular choices exist
     (1) Varying domains: As introduced (unrestricted frames)
     (2) Constant domains: $\mathcal{D}_w = \mathcal{D}_v$ for all worlds $w, v \in W$
     (3) Cumulative domains: $\mathcal{D}_w \subseteq \mathcal{D}_v$ whenever $(w, v) \in R^i$
     (4) Decreasing domains: $\mathcal{D}_w \supseteq \mathcal{D}_v$ whenever $(w, v) \in R^i$
3. **Rigidity**
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Semantic variants of HOML

1. **Axiomatization of $\Box^i$**
   - What properties does the box operators have?
2. **Quantification**
   - What is the meaning of $\forall$?
3. **Rigidity**
   - Do all constants $c \in \Sigma$ denote the same object at every world?
   - Several popular choices exist
     (1) Flexible constants: As introduced (unrestricted $\mathcal{I}_w$)
     (2) Rigid constants: $\mathcal{I}_w(c) = \mathcal{I}_v(c)$
     for all worlds $w, v \in W$ and all $c \in \Sigma$
4. **Consequence**

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   - Do all constants $c \in \Sigma$ denote the same object at every world?
4. **Consequence**
   - What is an appropriate notion of logical consequence $\models^{\mathrm{HOML}}$?
   - Many choices exist, two of them are

     (1) Local consequence: *... not displayed here ...*

     (2) Global consequence: *... not displayed here ...*

Semantic variants of HOML

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$\longrightarrow$ at least $10 \times 4 \times 2 \times 2 = 160$ distinct logics

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Example: Modal logic S5, constant domains, rigid symbols

Generated by **5**: $\Diamond^i s \supset \Box^i \Diamond^i s$

Theorems: *all classical tautologies*,

$s \supset \Box^i \Diamond^i s$, (symmetric r^i)

$\Box^i s \supset s$, (reflexive r^i)

$\forall X . \Box^i f\, X \supset \Box^i \forall X . f\, X$ (Barcan formula)

...

Embedding of HOML within HOL

Automation approach: Encode HOML semantics within (classical) HOL

HOL (meta-logic): $s, t ::=$

HOML (target logic): $s, t ::=$

Embedding of in

(1) Introduce new type μ for worlds

HOML formulas s_o are mapped to HOL predicates $s_{\mu \to o}$

(2) Introduce new constants $r^i_{\mu \to \mu \to o}$ for each $i \in I$

(3) Connectives:

$\neg_{o \to o} =$

$\vee_{o \to o \to o} =$

$\Pi^{\tau}_{(\tau \to o) \to o} =$

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(4) Meta-logical notions:

valid $=$

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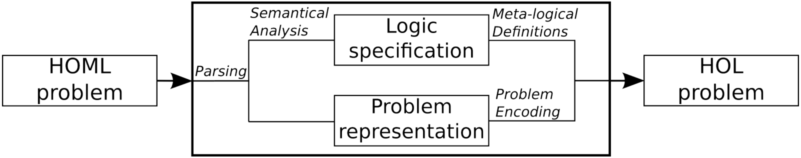
Stand-alone tool

Embedding procedure implemented as stand-alone tool



- Semantic specification is analyzed first
- (Meta-)logical notions are included as axioms/definitions
- Output format: ”Plain THF” (TH0)
- Integrated as pre-processor into Leo-III

Evaluation

Evaluation setting:

- Translated all 580 mono-modal QMLTP problems to modal THF
- Semantic setting:
  1. Modal operator axiom system $\in \{K, D, T, S4, S5\}$
  2. Quantification semantics $\in$ {constant, varying, cumul., decreasing}
  3. Rigid constants
  4. Consequence $\in$ {local, global}
- Native modal logic prover: MleanCoP (J. Otten)
- HOL reasoners: Satallax, LEO-II, Nitpick
- Timeout 60s (2x AMD Opteron 2376 Quad Core/16 GB RAM)

Comments on evaluation result:

- MleanCoP not applicable to modal logic K
- MleanCoP not applicable to decreasing domains semantics
- MleanCoP not applicable to problems with equality symbol
- MleanCoP not applicable for global consequence
- Only first-order modal logic problems
- Embedding approach **not restricted** to benchmark settings

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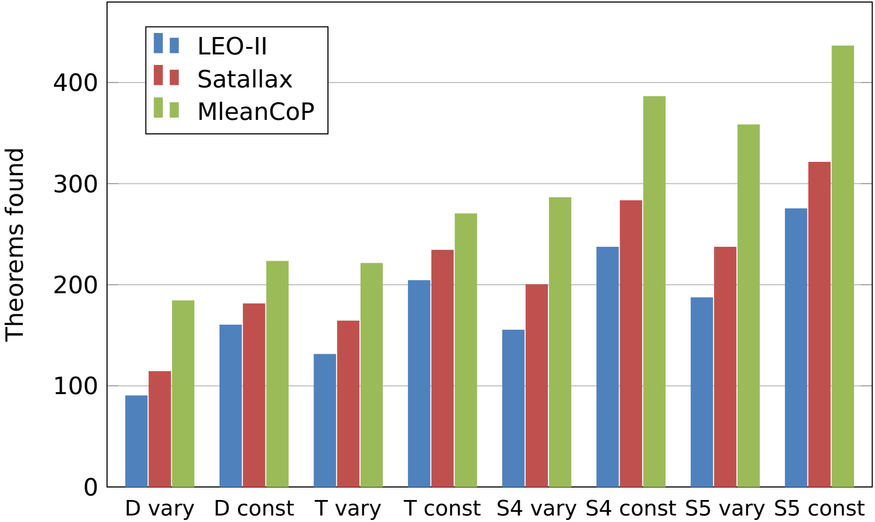
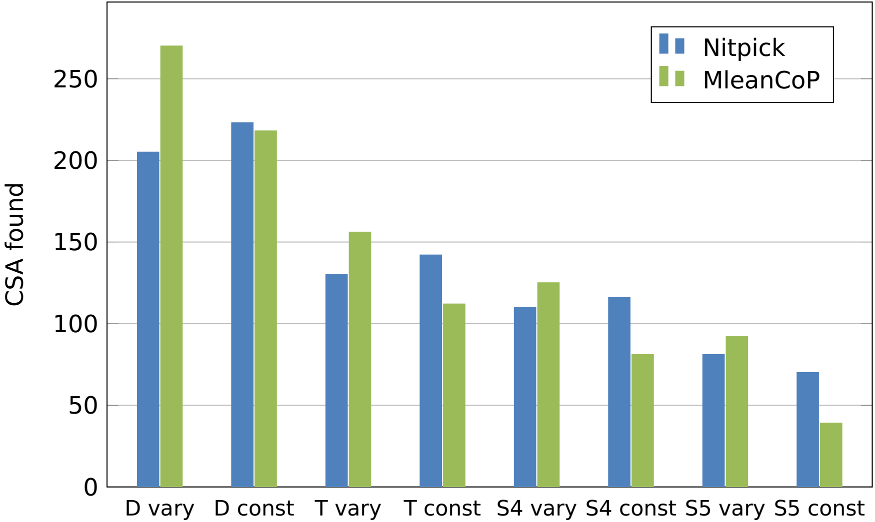
Evaluation #2

Result excerpt: Theorems



Result excerpt: Counter satisfiable (CSA)



Related work

- Generic theorem proving systems:
  The Logics Workbench,MetTeL2, LoTREC
- Embedding of further logics:
  Conditional logics, hybrid logics, many-valued logics, free logic, ...

Conclusion

- Provided a quite general semantics for HOML
- Presented a procedure that automatically converts HOML into HOL
- Implemented a stand-alone tool (e.g. as preprocessor)
  - standard HOL provers can be used to reason about problems encoded in the modal THF syntax
- Approach feasible (no evaluation for higher-order problems yet)
- Many new problems contributed in the modal THF format

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\vee_{o\to o\to o} &= \lambda S_{\mu\to o}.\lambda T_{\mu\to o}.\lambda W_\mu.\ (S\ W) \vee (T\ W) \\
\Pi^\tau_{(\tau\to o)\to o} &= \lambda P_{\tau\to\mu\to o}.\lambda W_\mu.\forall X_\tau.\ P\ X\ W \\
\Box_{o\to o} &= \lambda S_{\mu\to o}.\lambda W_\mu.\forall V_\mu.\ \neg(r^i\ W\ V) \vee S\ V
\end{aligned}$$

(4) Meta-logical notions:

$=$

Embedding of HOML within HOL

Automation approach: Encode HOML semantics within (classical) HOL

HOL (meta-logic): $s, t ::=$

HOML (target logic): $s, t ::=$

Embedding of in

(1) Introduce new type μ for worlds

HOML formulas s_o are mapped to HOL predicates $s_{\mu \to o}$

(2) Introduce new constants $r^i_{\mu \to \mu \to o}$ for each $i \in I$

(3) Connectives:

$$\neg_{o \to o} = \lambda S_{\mu \to o}.\lambda W_\mu.\ \neg(S\,W)$$
$$\vee_{o \to o \to o} = \lambda S_{\mu \to o}.\lambda T_{\mu \to o}.\lambda W_\mu.\ (S\,W) \vee (T\,W)$$
$$\Pi^\tau_{(\tau \to o) \to o} = \lambda P_{\tau \to \mu \to o}.\lambda W_\mu.\forall X_\tau.\ P\,X\,W$$
$$\Box_{o \to o} = \lambda S_{\mu \to o}.\lambda W_\mu.\forall V_\mu.\ \neg(r^i\,W\,V) \vee S\,V$$

(4) Meta-logical notions:

$$\text{valid} = \lambda s_{\mu \to o}.\forall W_\mu.\, s\,W$$

Embedding of HOML within HOL #2

Embedding semantic variants

1. **Axiomatization of $\Box^i$**
2. **Quantification**
3. **Rigidity**
4. **Consequence**

Embedding of HOML within HOL #2

Embedding semantic variants

1. **Axiomatization of $\Box^i$**

   Recall correspondences:

Name	Axiom scheme	Condition on r^i	Corr. formula
...	...	...	...
B	$s \supset \Box^i \Diamond^i s$	symmetric	$wR^i v \supset vR^i w$
...	...	...	...

   For each desired axiom scheme for $\Box^i$:
   Postulate frame condition on r^i as HOL axiom

2. **Quantification**
3. **Rigidity**
4. **Consequence**

Embedding semantic variants

1. **Axiomatization of $\Box^i$**
   Postulate frame condition on r^i as HOL axiom
2. **Quantification**
   Choose appropriate definition/axiomatization of quantifier:
   Constant domains quantifier:

$$\Pi^{\tau}_{(\tau\to o)\to o} = \lambda P_{\tau\to\mu\to o}.\lambda W_{\mu}.\forall X_{\tau}.\ P\,X\,W$$

   Varying domains quantifier:

$$\Pi^{\tau_{(\tau\to o)\to o},va} = \lambda P_{\tau\to\mu\to o}.\lambda W_{\mu}.\forall X_{\tau}.\ \neg(\mathrm{eiw}\,X\,W) \vee (P\,X\,W)$$

   Cumulative/decreasing domains quantifier:

   Add axioms on eiw
3. **Rigidity**
4. **Consequence**

Embedding of HOML within HOL #2

Embedding semantic variants

1. **Axiomatization of $\Box^i$**
   Postulate frame condition on r^i as HOL axiom
2. **Quantification**
   Choose appropriate definition/axiomatization of quantifier
3. **Rigidity**
   Rigid constants:
   Only translate Boolean types to predicates: $o = \mu \rightarrow o$

   Rigid constants:
   Also translate individuals types to predicates: $\iota = \mu \rightarrow \iota$
4. **Consequence**

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Embedding semantic variants

1. **Axiomatization of $\Box^i$**
   Postulate frame condition on r^i as HOL axiom
2. **Quantification**
   Choose appropriate definition/axiomatization of quantifier
3. **Rigidity**
   Appropriate type lifting
4. **Consequence**
   Global consequence: Apply $\text{valid}_{(\mu\to o)\to o}$ to all translated $s_{\mu\to o}$

   $$s_o = \text{valid}_{(\mu\to o)\to o}\, s_{\mu\to o}$$

   Local consequence: Apply *actuality* operator $\mathcal{A}$ to all translated $s_{\mu\to o}$

   $$s_o = \mathcal{A}_{(\mu\to o)\to o}\, s_{\mu\to o}$$

   where $\mathcal{A} = \lambda S_{\mu\to o}.\, s\, w_{\text{actual}}$ and w_{actual} is an uninterpreted symbol

Problem representation

Ongoing work: Extension of TPTP THF syntax for modal logic

(1) **Formula syntax**

```
thf( classical, axiom, ! [X:$i]: (p @ X)).
```

↓ Extend syntax with modalities

```
thf( modal, axiom, ! [X:$i]: ($box @ (p @ X))).
thf( multi_modal, axiom, ! [X:$i]: ($box_int @ 1 @ (p @ X))).
```

(2) **Semantics configuration**

Add "logic"-annotated statements to the problem:

```
thf(simple_s5, logic, ($modal := [
  $constants := $rigid,
  $quantification := $constant,
  $consequence := $global,
  $modalities := $modal_system_S5 ])).
```

- Intended semantics is attached to the problem

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(2) **Semantics configuration**

Add "logic"-annotated statements to the problem:

```
thf( mydomain_type , type , ( human : $tType ) ).
thf( myconstant_declaration , type , ( myconstant : $i ) ).
thf( complicated_s5 , logic , ( $modal := [
  $constants := [ $rigid , myconstant := $flexible ] ,
  $quantification := [ $constant , human := $varying ] ,
  $consequence := [ $global , myaxiom := $local ] ,
  $modalities := [ $modal_system_S5, $box_int @ 1 := $modal_system_T ] ] ) ).
```

- Intended semantics is attached to the problem