No.

Date.

2024/1024 作业5

3.1. (1). $z=t,\ -1\le t\le 1$

$\therefore \int_C |z|\,dz = \int_{-1}^{1} |t|\,dt = 1$

(2). $z=e^{it},\ \pi\le t\le 0,\ dz = ie^{it}dt,\ |e^{it}|=1$

$\therefore \int_C |z|\,dz = \int_{\pi}^{0} |e^{it}|\, ie^{it}\,dt$

$= i\int_{\pi}^{0} (\cos t + i\sin t)\,dt$

$= 2$

(3) $z=e^{it},\ \pi\le t\le 2\pi,\ dz = ie^{it}dt$

$\int_C |z|\,dz = \int_{\pi}^{2\pi} ie^{it}\,dt$

$= i\int_{\pi}^{2\pi} (\cos t + i\sin t)\,dt = 2$

2. $\left|\int_C (x^2+iy^2)\,dz\right| \le \int_C |x^2+iy^2|\,|dz| \quad \because z=it,\ t\in[-1,1]$

$\therefore |x^2+iy^2| \le 1$

$\therefore |\int| \le 1\times 2 = 2$

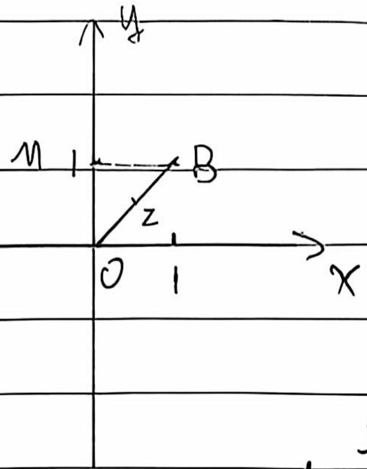
3. (1) $z=2e^{it}$, $|z|=2$, $\bar{z}=2e^{-it}$, $dz=2ie^{it}dt$

$\oint_C \frac{\bar{z}}{|z|}dz = 2i\int_0^{2\pi} dt = 4i\pi.$

(2) $z=4e^{it}$, $|z|=4$, $\bar{z}=4e^{-it}$, $dz=4ie^{it}dt$.

$\oint_C \frac{\bar{z}}{|z|}dz = 4i\int_0^{2\pi} dt = 8i\pi$

4.




$\because |MZ| \geq \frac{\sqrt{2}}{2}$ 其中 $M(0,1)$, $B(1,1)$

$\therefore \frac{1}{|MZ|} \leq \sqrt{2}$

即 $\left|\frac{1}{z-i}\right| \leq \sqrt{2}$

又 $\because |OB| = \sqrt{2}$

$\therefore \left|\int_C \frac{dz}{z-i}\right| \leq \int_C \left|\frac{1}{z-i}\right| |dz| \leq \sqrt{2} \times |OB| = 2.$

5. (1) 原式 $=(z^3+4z^2+z)\Big|_{-1}^{1} = 4$

(2) 原式

$= \int_0^i z e^{z^2} dz + \int_0^i \cos\frac{\pi}{2}iz\, dz$

$= \frac{1}{2}e^{z^2}\Big|_0^i + \frac{2}{\pi i}\sin\frac{\pi i}{2}z\Big|_0^i$

$= \frac{1}{2e} - \frac{1}{2} - \frac{2}{\pi i}$

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3. (3). 原式 $= \int_C R\,dz - \oint 2e^{z}\cos z\,dz$

$= 0$

6. (1). $\int_C \frac{1}{z+2}dz = 0$.

(2). $\because \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $d(e^{i\theta}) = ie^{i\theta}d\theta$.

取 $z = e^{i\theta}$ $\therefore \cos\theta = \frac{z^2+1}{2z}$

原式 $= \int_C \frac{1+2\frac{z^2+1}{2z}}{5+4\frac{z^2+1}{2z}} \cdot \frac{1}{iz}dz$

$= \int_C \frac{z^2+z+1}{(2z+1)(z+2)} \cdot \frac{1}{iz}dz$.

$= -i\int_C \left(-\frac{1}{2z+1} + \frac{1}{2}\cdot\frac{1}{z+2} + \frac{1}{2}\cdot\frac{1}{z}\right)dz$, 令为 I.

$\because \cos\theta$ 关于 $x=\theta$ 对称, 设 $C'$ 为半径为1的正向圆周, 中心为原点.

$\therefore 2I = -i\oint_{C'}\left(-\frac{1}{2z+1} + \frac{1}{2}\cdot\frac{1}{z+2} + \frac{1}{2z}\right)dz$.

$= -i\left(\oint_{C_1}\left(-\frac{1}{2z+1}\right)dz + \oint_{C_2}\frac{1}{2z}dz\right)$

$= -i(-\pi i + \pi i) = 0$.

7. (1) $f(z)=\frac{e^z}{z}dz$ 在 $z=0$ 处有奇点

$$\oint_{|z|=1}\frac{e^z}{z}dz = 2\pi i\times \mathrm{Res}_{z=0} f(z)$$

$$=2\pi i \times \lim_{z\to 0} z\frac{e^z}{z} = 2\pi i$$

(2). 设 $z=e^{i\theta}$, $|z|=1$

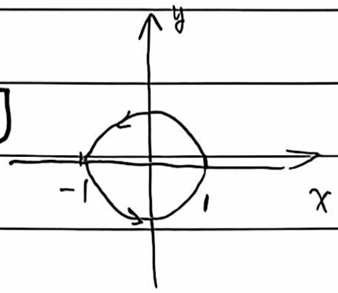
$dz = ie^{i\theta}d\theta$. 即 $d\theta=\frac{1}{iz}dz$

$$e^{\cos\theta}\cdot\cos(\sin\theta)d\theta$$

$$=e^{\cos\theta}\cdot\frac{e^{i\sin\theta}-e^{-i\sin\theta}}{2}d\theta$$

$$=\frac{1}{2}(e^{\cos\theta+i\sin\theta}-e^{\cos\theta-i\sin\theta})d\theta$$

$$=\frac{1}{2iz}(e^{z}-e^{\bar{z}})dz \quad \because \theta\in[0,\pi]$$

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$\therefore \int_0^{\pi} e^{\cos\theta}\cos(\sin\theta)d\theta$

$$=\oint_{|z|=1}\frac{1}{2i}\cdot\frac{e^z}{z}dz=\frac{1}{2i}\times 2\pi i=\pi.$$