20241205 作业

6.1 (1). 记 $I=\int_{-1}^{1}(1-\tau^2)\cos\omega\tau d\tau$.

$\because f(t)$ 为偶函数.

$I=\int_{-1}^{1}\cos\omega\tau d\tau-\int_{-1}^{1}\tau^2\cos\omega\tau d\tau$

$=-\left(\tau^2\frac{\sin\omega\tau}{\omega}\Big|_{-1}^{1}-\int_{-1}^{1}\frac{1}{\omega}\sin\omega\tau\cdot 2\tau d\tau\right)+\int_{-1}^{1}\cos\omega\tau d\tau.$

$=-\left(\frac{2\sin\omega}{\omega}+\int_{-1}^{1}\frac{1}{\omega^2}\cdot 2\tau\cdot d(\cos\omega\tau)\right)+\int_{-1}^{1}\cos\omega\tau d\tau$

$=-\left(\frac{2\sin\omega}{\omega}+\left(\frac{2\tau}{\omega^2}\cos\omega\tau\Big|_{-1}^{1}-\int_{-1}^{1}\frac{2}{\omega^2}\cos\omega\tau d\tau\right)\right)+\int_{-1}^{1}\cos\omega\tau d\tau$

$=-\left(\frac{2\sin\omega}{\omega}+\frac{4}{\omega^2}\cos\omega\right)+\left(1+\frac{2}{\omega^2}\right)\frac{1}{\omega}\cdot 2\sin\omega$

$=\frac{4}{\omega^3}\sin\omega-\frac{4}{\omega^2}\cos\omega.$

即: $f(t)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\frac{4}{\omega^3}\sin\omega-\frac{4}{\omega^2}\cos\omega\right)\cos\omega t\, d\omega$

(2). $f(t)$ 为奇函数, 记 $I=\int_{0}^{+\infty}f(\tau)\sin\omega\tau d\tau$

$I=\int_{0}^{\pi}\sin\tau\sin\omega\tau d\tau$

$=-\int_{0}^{\pi}\frac{\cos(\omega+1)\tau-\cos(\omega-1)\tau}{2}d\tau$

$=-\frac{1}{2}\left(\frac{1}{\omega+1}\sin(\omega+1)\tau-\frac{1}{\omega-1}\sin(\omega-1)\tau\right)\Big|_{0}^{\pi}$

$=\frac{1}{2}\left(\frac{1}{\omega-1}\sin((\omega-1)\pi)-\frac{1}{\omega+1}\sin((\omega+1)\pi)\right)$

即 $f(t)=\frac{1}{\pi}\int_{0}^{+\infty}\left(\frac{1}{\omega-1}\sin((\omega-1)\pi)-\frac{1}{\omega+1}(\sin(\omega+1)\pi)\right)\cdot\sin\omega t\, d\omega$

(3). $f(t)$ 为奇函数. 在连续定义.

∴ 记 $I=\int_0^{+\infty} f(\tau)\sin\omega\tau\, d\tau$

$=\int_0^1 \sin\omega\tau\, d\tau$

$=-\frac{1}{\omega}\cos\omega\tau\Big|_0^1=\frac{1}{\omega}(1-\cos\omega)$

∴ 在 $t\neq 0$ 时, $f(t)=\frac{2}{\pi}\int_0^{+\infty}\frac{1}{\omega}(1-\cos\omega)\sin\omega t\, d\omega$

在 $t=0$ 时,

$\frac{f(t+0)+f(t-0)}{2}=\frac{2}{\pi}\int_0^{+\infty}\frac{1}{\omega}(1-\cos\omega)\sin\omega t\, d\omega$

(4). $f(t)$ 为偶函数.

记 $I=\int_0^{+\infty} f(\tau)\cos\omega\tau\, d\tau$

则 $I=\int_0^{\frac{\pi}{2}}(1-\cos\tau)\cos\omega\tau\, d\tau$

$=\int_0^{\frac{\pi}{2}}\cos\omega\tau\, d\tau-\int_0^{\frac{\pi}{2}}\cos\tau\cos\omega\tau\, d\tau$

$=\int_0^{\frac{\pi}{2}}\cos\omega\tau\, d\tau-\int_0^{\frac{\pi}{2}}\frac{1}{2}(\cos(\omega+1)\tau+\cos(\omega-1)\tau)\, d\tau$

$=\frac{1}{\omega}\sin\frac{\pi}{2}\omega-\frac{1}{2}\left(\frac{1}{\omega+1}\sin\frac{\pi}{2}(\omega+1)+\frac{1}{\omega-1}\sin\frac{\pi}{2}(\omega-1)\right)$

则: $f(t)=\frac{2}{\pi}\int_0^{+\infty}\left(\frac{1}{\omega}\sin\frac{\pi}{2}\omega-\frac{1}{2(\omega+1)}\sin\frac{\pi}{2}(\omega+1)-\frac{1}{2(\omega-1)}\sin\frac{\pi}{2}(\omega-1)\right)\cos\omega t\, d\omega$

2. (1). $f(t)$ 为偶函数

记 $I=\int_0^{+\infty} f(\tau)\cos\omega\tau\, d\tau$

$=\int_0^{\pi} \cos\tau\cos\omega\tau\, d\tau$

$=\int_0^{\pi} \frac{1}{2}(\cos(\omega+1)\tau+\cos(\omega-1)\tau)\,d\tau$

$=\frac{1}{2}\left(\frac{1}{\omega+1}\sin(\omega+1)\pi+\frac{1}{\omega-1}\sin(\omega-1)\pi\right)$

$\therefore f(t)=\frac{1}{\pi}\int_0^{+\infty}\left(\frac{1}{\omega+1}\sin(\omega+1)\pi+\frac{1}{\omega-1}\sin(\omega-1)\pi\right)\cos\omega t\, d\omega$

$=\frac{1}{\pi}\int_0^{+\infty}\left(\frac{1}{\omega^2-1}\left((\omega-1)(-\sin\omega\pi)+(\omega+1)(-\sin\omega\pi)\right)\right)\cos\omega t\, d\omega$

$=\frac{1}{\pi}\int_0^{+\infty}\frac{2\omega}{1-\omega^2}\sin\omega\pi\cos\omega t\, d\omega.$

(2): $|t|<\pi$ 时, $f(t)=\cos t.$

即: $\frac{1}{\pi}\int_0^{+\infty}\frac{2\omega}{1-\omega^2}\sin\omega\pi\cos\omega t\, d\omega=\cos t$

即: $\int_0^{+\infty}\frac{\omega\sin\omega\pi\cos\omega t}{1-\omega^2}\,d\omega=\frac{\pi}{2}\cos t.$

$|t|=\pi$ 时, $f(t)=-1$, $t=\pi$ 为 $f(t)$ 的间断点, $f(t+0)=0$, $f(t-0)=-1$

即: $\frac{1}{\pi}\int_0^{+\infty}\frac{2\omega}{1-\omega^2}\sin\omega\pi\cos\omega t\, d\omega=-\frac{1}{2}$

即: $\int_0^{+\infty}\frac{\omega}{1-\omega^2}\sin\omega\pi\cos\omega t\, d\omega=-\frac{\pi}{4}$

$t>\pi$ 时, $f(t)=0$, 即 $\int_0^{+\infty}\frac{\omega\sin\omega\pi\cos\omega t}{1-\omega^2}\,d\omega=0.$

∴得证!

$$\int_0^{+\infty} \frac{\omega \sin \omega\pi \cos \omega t}{1-\omega^2} d\omega = \begin{cases} \frac{\pi}{2}\cos t, & |t| < \pi \\ -\frac{\pi}{4}, & |t| = \pi \\ 0, & |t| > \pi \end{cases}$$

(2). $f(t)$ 为偶函数

记 $I = \int_0^{+\infty} f(\tau)\cos\omega\tau \, d\tau$

则 $I = \int_0^{+\infty} e^{-\tau}\cos\tau\cos\omega\tau \, d\tau$

$$= \frac{1}{2}\int_0^{+\infty} e^{-\tau}[\cos(1-\omega)\tau + \cos(1+\omega)\tau]d\tau$$

$$= \frac{1}{2}\left(\int_0^{+\infty} e^{-\tau}\cos(1-\omega)\tau + \int_0^{+\infty} e^{-\tau}\cos(1+\omega)\tau \, d\tau\right)$$

$$= \frac{1}{2}\left(\frac{1}{1+(1-\omega)^2} + \frac{1}{1+(1+\omega)^2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{\omega^2-2\omega+2} + \frac{1}{\omega^2+2\omega+2}\right)$$

$$= \frac{1}{2}\cdot\frac{2\omega^2+4}{\omega^4+4} = \frac{\omega^2+2}{\omega^4+4}.$$

$$\therefore f(t) = \frac{2}{\pi}\int_0^{+\infty} \frac{\omega^2+2}{\omega^4+4}\cos\omega t \, d\omega$$

也即: $\int_0^{+\infty} \frac{\omega^2+2}{\omega^4+4}\cos\omega t \, d\omega = \frac{\pi}{2}e^{-|t|}\cos t$

3.(1). $F(\omega)=\int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt$

∵ $f(t)$ 为偶函数

∴ $F(\omega)=\int_{0}^{+\infty} f(t)\cos\omega t\,dt$

$$=\int_0^1 \cos 2\pi t\cos\omega t\,dt$$

$$=\frac{1}{2}\int_0^1\left[\cos(2\pi-\omega)t+\cos(2\pi+\omega)t\right]dt$$

$$=\frac{1}{2}\left(\frac{1}{2\pi-\omega}\sin(2\pi-\omega)t+\frac{1}{2\pi+\omega}\sin(2\pi+\omega)t\right)\Big|_0^1=\frac{1}{2}\left(\frac{\sin(2\pi-\omega)}{2\pi-\omega}+\frac{\sin(2\pi+\omega)}{2\pi+\omega}\right)$$

(2). $F(\omega)=\int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt$

$$=\int_0^{+\infty} e^{-t}\sin 2\pi t\, e^{-i\omega t}dt$$

$$=\int_0^{+\infty} e^{-t}\cdot\frac{1}{2i}\left(e^{i(2\pi t)}-e^{-i(2\pi t)}\right)e^{-i\omega t}dt$$

$$=\frac{1}{2i}\int_0^{+\infty}\left(e^{(2\pi i-\omega i-1)t}-e^{(-2\pi i-\omega i-1)t}\right)dt$$

$$=\frac{1}{2i}\left[\frac{e^{(2\pi i-\omega i-1)t}}{2\pi i-\omega i-1}-\frac{e^{(-2\pi i-\omega i-1)t}}{-2\pi i-\omega i-1}\right]\Bigg|_0^{+\infty}$$

$$=\frac{1}{2i}\left[\frac{1}{1+\omega i-2\pi i}-\frac{1}{1+\omega i+2\pi i}\right]$$

$$=\frac{2\pi}{(1+i\omega)^2+4\pi^2}$$