20241128 作业.

6. (2). 令 $f(z)=\dfrac{z^2}{(z^2+1)(z^2+4)}$. 满足条件.

$f(z)$ 在上半平面奇点为 $z=i$ 和 $z=2i$

均为一阶极点

$$\operatorname{Res}[f(z),i]=\lim_{z\to i}\frac{z^2}{(z+i)(z^2+4)}=\frac{1}{6}i$$

$$\operatorname{Res}[f(z),2i]=\lim_{z\to 2i}\frac{z^2}{(z^2+1)(z+2i)}=-\frac{1}{3}i$$

$\because f(x)$ 为偶函数

$\therefore \displaystyle\int_0^{+\infty}\frac{x^2}{(x^2+1)(x^2+4)}dx$

$=\dfrac{1}{2}\displaystyle\int_{-\infty}^{+\infty}\frac{x^2}{(x^2+1)(x^2+4)}dx$

$=\dfrac{1}{2}\times 2\pi i\times\left(\dfrac{1}{6}i-\dfrac{1}{3}i\right)=\dfrac{1}{6}\pi$

(3). 令 $f(z)=\dfrac{z^2}{(z^2+a^2)^2}$, 满足条件

$f(z)$ 在上半平面奇点为 $z=ai$, 为二阶极点.

$$\operatorname{Res}[f(z),ai]=\lim_{z\to ai}\frac{d}{dz}\frac{z^2}{(z+ai)^2}=\frac{1}{4ai}$$

$\therefore \displaystyle\int_{-\infty}^{+\infty}\frac{x^2}{(x^2+a^2)^2}dx$

$=2\pi i\times\dfrac{1}{4ai}=\dfrac{\pi}{2a}$

7. (1) 令 $f(z)=\dfrac{z}{z^2+4z+20}$

$=\dfrac{z}{(z-(4i-2))(z-(-4i-2))}$

∴ 上半平面有一个一阶极点 $z=-2+4i$

$\mathrm{Res}[f(z)e^{iz}, -2+4i]$

$=\lim\limits_{z\to -2+4i}\dfrac{z}{(z-(-4i-2))}e^{iz}$

$=(\frac{1}{2}+\frac{1}{4}i)e^{(-4-2i)}$

$\therefore I=\mathrm{Im}\left(\int_{-\infty}^{+\infty}\dfrac{xe^{ix}}{x^2+4x+20}dx\right)$

$=\mathrm{Im}\left(2\pi i\times(\frac{1}{2}+\frac{1}{4}i)e^{(-4-2i)}\right)$

$=\mathrm{Im}\left(2\pi i\times(\frac{1}{2}+\frac{1}{4}i)e^{-4}(\cos(-2)+i\sin(-2))\right)$

$=\dfrac{\pi}{e^4}(\cos 2+\frac{1}{2}\sin 2)$

(2). 令 $f(z) = \dfrac{z}{(z^2+b^2)^2} = \dfrac{z}{(z+bi)^2(z-bi)^2}$

$f(z)$ 在上半平面有二阶极点 $z=bi$

$\mathrm{Res}[f(z)e^{iaz}, 2i]$

$= 2\pi i \times \lim\limits_{z \to bi} \dfrac{d}{dz}\left(\dfrac{ze^{iaz}}{(z+bi)^2}\right)$

$= 2\pi i \times \lim\limits_{z \to bi} \dfrac{(1+iaz)e^{iaz}\cdot(z+bi)^2 - ze^{iaz}\cdot 2(z+bi)}{(z+bi)^4}$

$= 2\pi i \times \left(\dfrac{(1-ab)e^{-ab}}{(2bi)^2} - \dfrac{2bi\,e^{-ab}}{(2bi)^3}\right)$

$\therefore$ 原式 $= \dfrac{1}{2}\mathrm{Im}[\mathrm{Res}[f(z)e^{iaz}, 2i]]$

$= \pi \times \left(\dfrac{(1-ab)}{(2bi)^2} - \dfrac{2bi}{(2bi)^3}\right)e^{-ab}$

$= \pi \times \dfrac{a}{4b} \times e^{-ab} = \dfrac{a}{4b}e^{-ab}\pi$

$\because \cos\omega x = \cos(-\omega x)$

8. $f(t) = \dfrac{1}{(t^2+1)(t^2+4)}$

在上半平面有一阶极点 $t=i$ 和 $t=2i$

$\mathrm{Res}[f(z)e^{i|\omega|z}, i]$

$= \lim\limits_{z \to i} \dfrac{e^{i|\omega|z}}{(z+i)(z^2+4)} = -\dfrac{1}{6} i e^{-|\omega|}$

$\mathrm{Res}[f(z)e^{i|\omega|z}, 2i]$

$= \lim\limits_{z \to 2i} \dfrac{e^{i|\omega|z}}{(z^2+1)(z+2i)} = \dfrac{1}{12} i e^{-2|\omega|}$

原式 $= \dfrac{1}{2} \mathrm{Re}\left(2\pi i \times \left(-\dfrac{1}{6} i e^{-|\omega|} + \dfrac{1}{3} i e^{-2|\omega|}\right)\right)$

$= \pi \times \left(\dfrac{1}{6} e^{-|\omega|} - \dfrac{1}{12} e^{-2|\omega|}\right)$

$= \left(\dfrac{1}{6e^{|\omega|}} - \dfrac{1}{12e^{2|\omega|}}\right)\pi$