20241121 作业9

5.1 (1) 孤立奇点为 $z=1$ 和 $\infty$

$z=1$ 时 $\sum_{n=0}^{+\infty} \frac{1}{n!}\left(\frac{1}{1-z}\right)^n = f(z)$

则：$n=1$ 时，$C_{-n}=-1$

即 $Res[f(z), 1] = -1$

$\infty$ 点时：令 $\xi = \frac{1}{z}$

$f(z) = g(\xi) = e^{\frac{1}{1-\frac{1}{\xi}}}$

$= e^{\frac{\xi}{\xi-1}}$

$Res[f(z), \infty] = 1$

(4) 孤立奇点为 $z=1$ 和 $z=2$，和 $\infty$

1° $z=1$ 时

$Res[f(z), 1] = \lim_{z\to 1} \left(\frac{1}{(z-2)^{100}}\right)$

$= 1$

2° $z=2$ 时 $f(z) = \frac{1}{(1-(-(z-2))(z-2)^{100}}$

$= \frac{1}{(z-2)^{100}} \sum_{n=0}^{+\infty} (-1)^n (z-2)^n$

$\therefore Res[f(z), 2] = (-1)^{99} = -1$

3° $\infty$ $\lim_{z\to\infty} f(z) = 0$ $\therefore$ 为可去奇点

$\therefore Res[f(z), \infty] = 0$

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(5), 孤立奇点为 $z=0$ 和 $\infty$

1°. $z=0$

$$f(z)=z^n\sum_{n=0}^{+\infty}(-1)^n\cdot\frac{1}{(2n+1)!}\cdot\frac{1}{z^{2n+1}}$$

$$=\sum_{n=0}^{+\infty}(-1)^n\cdot\frac{1}{(2n+1)!}\cdot\frac{1}{z^{n+1}}$$

$n=0$ 时 $C_{-1}=1$

即: $Res[f(z),0]=1$

2°. $\infty$: 令 $g(z)=\frac{1}{z^2}f\left(\frac{1}{z}\right)$

$$=\frac{1}{z^2}\cdot\frac{1}{z^n}\sin z$$

$$=\frac{1}{z^{n+2}}\sin z$$

$$=\sum_{n=0}^{+\infty}(-1)^n\cdot\frac{1}{(2n+1)!}z^{n-1}$$

$n=0$ 时, $C_{-1}=1$

即: $Res[f(z),\infty]=-1$

(6). 奇点为 $z=(-1)^{\frac{1}{n}}$ 和 $\infty$

1°. $z=(-1)^{\frac{1}{n}}$ 记 $z_k=(-1)^{\frac{1}{k}}$

则: $z_k^{2n}=1$ $z_k$ 为一阶极点

$$f(z)=\lim_{z\to z_k}(z-z_k)\frac{z^{2n}}{1+z^n}$$

$$=\lim_{z\to z_k}(z-z_k)\frac{z^{2n}}{1+(z_k+z-z_k)^n}$$

$$=\lim_{z\to z_k}(z-z_k)\frac{z^{2n}}{1+z_k^n+nz_k^{n-1}(z-z_k)}\qquad \because 1+z_k^n=0$$

$$=\lim_{z\to z_k}\frac{z^{2n}}{nz_k^{n-1}}=\frac{1}{n}z_k^{n+1}=-\frac{z_k}{n}$$

即: $\operatorname{Res}[f(z),z_k]=-\frac{z_k}{n}$ 其中 $z_k=(-1)^{\frac{1}{k}}$

2°. $\infty$ 记 $g(z)=\frac{1}{z^2}f(\frac{1}{z})=\frac{1}{z^{n+2}(z^n+1)}$

$z=0$ 为 $n+2$ 阶极点

$$g(z)=\frac{1}{z^{n+2}}\cdot\frac{1}{1-(-z^n)}$$

$$=\frac{1}{z^{n+2}}\sum_{n=0}^{+\infty}(-1)^n(z^n)^n$$

$$=\sum_{n=0}^{+\infty}(-1)^n z^{n^2-n-2}\qquad C_{-1}=0$$

$\therefore \operatorname{Res}[f(z),\infty]=0$

$$f(z)=\frac{z}{\cos 2z}dz.$$

2.(1) C内奇点为 $z=\frac{\pi}{4}$ 和 $z=-\frac{\pi}{4}$ 均为一阶极点

$$\mathrm{Res}\left[f(z),\frac{\pi}{4}\right]=\lim_{z\to\frac{\pi}{4}}\frac{z}{\cos 2z}\left(z-\frac{\pi}{4}\right)=-\frac{\pi}{8}$$

$$\mathrm{Res}\left[f(z),-\frac{\pi}{4}\right]=\lim_{z\to-\frac{\pi}{4}}\frac{z}{\cos 2z}\left(z+\frac{\pi}{4}\right)=-\frac{\pi}{8}$$

$\therefore$ 原式 $=2\pi i\times\left(-\frac{\pi}{4}\right)=-\frac{1}{2}\pi^2 i$

(3). C外奇点为 $\infty$

$$\lim_{z\to\infty}\frac{1}{(z-1)^2(z^3-1)}=0$$ $\therefore$ 为可去奇点

$\therefore \mathrm{Res}[f(z),\infty]=0$

$$\therefore \oint_C\frac{1}{(z-1)^2(z^3-1)}dz=0.$$

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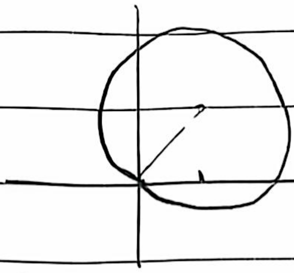
(4). C: $(x-1)^2+(y-1)^2=2$.

即：复平面上曲线为以 (1, 1) 为圆心的圆



曲线 C 内奇点为 $z=1$ 和 $z=i$

1°. $z=1$ 为二阶极点

$$\mathrm{Res}[f(z),1]=\lim_{z\to 1}\frac{d}{dz}\left(\frac{1}{z^2+1}\right)$$

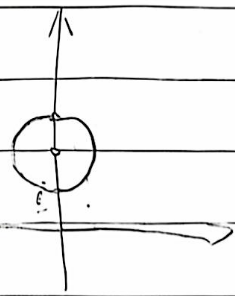
$$=\lim_{z\to 1}\left(-\frac{2z}{(z^2+1)^2}\right)=-\frac{1}{2}$$

2°. $z=i$ 为一阶极点

$$\mathrm{Res}[f(z),i]=\lim_{z\to i}\frac{1}{(z-1)^2(z+i)}=\frac{1}{2i(i-1)^2}=\frac{1}{4}$$

$$\therefore \oint_C \frac{1}{(z-1)^2(z^2+1)}dz=2\pi i\times\left(\frac{1}{4}-\frac{1}{2}\right)=-\frac{1}{2}\pi i$$

(5). $f(z)=\tanh z=\dfrac{e^{2z}-1}{e^{2z}+1}$



C 内奇点为 $\frac{\pi}{2}i$ 为一阶极点

$$\therefore \mathrm{Res}\left[f(z),\frac{\pi}{2}i\right]=\lim_{z\to\frac{\pi}{2}i}\left(z-\frac{\pi}{2}i\right)f(z)=1$$

$$\therefore \oint_C \tanh z\,dz=2\pi i$$

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4. C外奇点为∞

记 $g(z)=\frac{1}{z^2}f(\frac{1}{z})$

$$=\frac{e^z}{z^4(z+1)}$$

$$=\frac{1}{z^4}\cdot\sum_{n=0}^{+\infty}\frac{1}{n!}z^n\cdot\sum_{k=0}^{+\infty}(-1)^k z^k,\quad |z|<1$$

$\therefore n=0,\ k=3$ 时，$c=-1$

$n=1,\ k=2$ 时，$c=1$

$n=2,\ k=1$ 时 $c=-\frac{1}{2}$

$n=3,\ k=0$ 时，$c=\frac{1}{6}$

$\therefore C_{-1}=-\frac{1}{3}$ $\therefore Res[f(z),\infty]=\frac{1}{3}$

$\therefore \oint_C \frac{z^3e^{\frac{1}{z}}}{1+z}dz=\frac{2}{3}\pi i$

3. C内奇点：$z=1$ 和 $z=-i$

$z=1$ 时

记 $g(z)=ze^{\frac{1}{z-1}}$

$$=z\sum_{n=0}^{+\infty}\frac{1}{n!}(z-1)^{-n}$$

$$=(z-1+1)\sum_{n=0}^{+\infty}\frac{1}{n!}(z-1)^{-n}$$

$\therefore C_{-1}=1+\frac{1}{2}=\frac{3}{2}$ 即 $Res[f(z),1]=\frac{3}{2}$

$z=-i$ 时

记 $h(z)=\frac{z}{(z^2-9)(z+i)}$ 为一阶极点

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$\therefore Res[f(z), -i] = \lim\limits_{z \to -i} \frac{z}{(z^2-9)} = \frac{1}{10}i$

$\therefore I = 2\pi i\left(\frac{3}{2} + \frac{1}{10}i\right)$

4. (1). 令 $z = e^{ix}$ $\therefore \cos x = \frac{e^{ix}+e^{-ix}}{2} = \frac{z^2+1}{2z}$

$dx = \frac{1}{iz}dz$.

$\therefore$ 原式 $= \oint_C \frac{1}{5+3\frac{z^2+1}{2z}} \cdot \frac{1}{iz}dz$

$= -2i\oint_C \frac{1}{3z^2+10z+3}\,dz$

$= -2i\oint_C \frac{1}{(3z+1)(z+3)}\,dz$ , $C: |z|<1$

$\therefore$ C内奇点 $z = -\frac{1}{3}$ 为一阶极点.

$Res\left(f(z), -\frac{1}{3}\right) = \lim\limits_{z \to -\frac{1}{3}} \frac{1}{z+3} = \frac{3}{8}$

$\therefore I = 2\pi i \cdot (-2i) \cdot \frac{3}{8} = \frac{3}{2}\pi$

(2). 原式 $= \int_0^{\frac{\pi}{2}} \frac{2}{3-\cos 2x}\,dx$

$= \frac{1}{2}\int_0^{\pi} \frac{2}{3-\cos\theta}\,d\theta$

$= \frac{1}{4}\int_{-\pi}^{\pi} \frac{2}{3-\cos\theta}\,d\theta = \frac{1}{2}\int_{-\pi}^{\pi}\frac{1}{3-\cos\theta}\,d\theta$

令 $z = e^{i\theta}$

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$$\text{原式} = -i\oint_C \frac{1}{z^2-6z+1}dz \quad , \quad C: |z|<1$$

$$= -i\oint_C \frac{1}{(z-(3-2\sqrt{2}))(z-(3+2\sqrt{2}))}dz$$

C内奇点为 $z=3-2\sqrt{2}$ 为一阶极点

$$\mathrm{Res}[f(z), 3-2\sqrt{2}] = \lim_{z\to 3-2\sqrt{2}} \frac{1}{z-(3+2\sqrt{2})}dz$$

$$= -\frac{1}{4\sqrt{2}}$$

$$\therefore \text{原式} = 2\pi i \times (-i\cdot(-\frac{1}{4\sqrt{2}})) = -\frac{\sqrt{2}}{4}\pi$$

5. 考虑 $g(\theta) = \dfrac{e^{3i\theta}}{1-2a\cos\theta+a^2}$，令 $z=e^{i\theta}$

$$I_1 = \oint_C g(\theta)d\theta = \oint_C \frac{z^3}{1-2a\cdot\frac{z^2+1}{2z}+a^2}\cdot\frac{1}{iz}dz \quad , \quad C: |z|<1$$

$$= i\oint_C \frac{z^3}{az^2-(a^2+1)z+a}dz$$

$$= i\oint_C \frac{z^3}{(az-1)(z-a)}dz \quad \because |a|>1$$

C内奇点为 $z=\frac{1}{a}$ 为一阶极点

$$\mathrm{Res}[f(z), \frac{1}{a}] = \lim_{z\to\frac{1}{a}} \frac{z^3}{z-a} = \frac{1}{a^2(1-a^2)}$$

$$\therefore I_1 = i\times 2\pi i \times \frac{1}{a^2(1-a^2)} = -2\pi\frac{1}{a^2(1-a^2)}$$

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又 $\because g(\theta) = \dfrac{\cos 3\theta + i\sin 3\theta}{1-2a\cos\theta+a^2}$.

$$\therefore \int_0^{2\pi} \frac{e^{3i\theta}}{1-2a\cos\theta+a^2}\,d\theta = \int_0^{2\pi} \frac{\cos 3\theta}{1-2a\cos\theta+a^2}\,d\theta + i\int_0^{2\pi} \frac{\sin 3\theta}{1-2a\cos\theta+a^2}\,d\theta$$

$$\therefore \int_0^{2\pi} \frac{\cos 3\theta}{1-2a\cos\theta+a^2}\,d\theta = -2\pi \frac{1}{a^2(1-a^2)}$$

$$= \frac{2\pi}{a^2(a^2-1)}$$