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20241107 作业7.

2. (1). $\lim\limits_{n\to+\infty}\left|\frac{C_{n+1}}{C_n}\right|$

$=\lim\limits_{n\to+\infty}\frac{(n+1)^2}{en^2}$

$=\frac{1}{e}$ $\therefore R=e$

(2). $\lim\limits_{n\to+\infty}\sqrt[n]{C_n}$

$=\lim\limits_{n\to+\infty}\frac{1}{n}(n!)^{\frac{1}{n}}$ 利用斯特林公式

$=\lim\limits_{n\to+\infty}\frac{1}{n}\left(\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n\right)^{\frac{1}{n}}$

$=\lim\limits_{n\to+\infty}\frac{1}{e}\cdot(2\pi n)^{\frac{1}{2n}}$

$=\frac{1}{e}$. $\therefore R=e$

(3). $\lim\limits_{n\to+\infty}\left|\frac{C_{n+1}}{C_n}\right|$

$=\lim\limits_{n\to+\infty}\frac{n+1}{2n}=\frac{1}{2}$ $\therefore R=2$

(4). $\lim\limits_{n\to+\infty}\sqrt[n]{C_n}$

$=\lim\limits_{n\to+\infty}e^{\frac{1}{n}(\ln n)^2}$

$=\lim\limits_{n\to+\infty}e^{\frac{(\ln n)^2}{n}}=e^0=1$ $\therefore R=1$

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3.(1). $f(z)=\frac{1}{z^2}$

$=-\frac{d}{dz}\left(\frac{1}{z}\right)$

$=-\frac{d}{dz}\left(\frac{1}{1-(1-z)}\right)$ 在 $z=1$ 处

上式 $=-\frac{d}{dz}\sum_{n=0}^{+\infty}(1-z)^n$

$=\sum_{n=1}^{+\infty}n(1-z)^{n-1}$，收敛域 $|z-1|<1$

(2). $f(z)=\frac{z-1}{z+1}$

$=1-\frac{2}{z+1}$

$=1-\frac{1}{1-\frac{1}{2}(1-z)}$

$=1-\sum_{n=0}^{+\infty}\frac{1}{2^n}(1-z)^n$，收敛域为 $|z|<2$

(3). $f(z) = -\frac{1}{z+1} + \frac{2}{z+2}$

$= -\frac{1}{3+(z-2)} + \frac{2}{4+(z-2)}$

$= -\frac{1}{3} \cdot \frac{1}{1-(-\frac{1}{3}(z-2))} + \frac{1}{2} \cdot \frac{1}{1-(-\frac{1}{4}(z-2))}$

$= -\frac{1}{3}\sum_{n=0}^{+\infty}(-\frac{1}{3}(z-2))^n + \frac{1}{2}\sum_{n=0}^{+\infty}(-\frac{1}{4}(z-2))^n$ 收敛域为 $|z-2|<3$.

4. 令 $z = re^{i\theta}$.

$r^n \cos n\theta = r^n \cdot \frac{(e^{i\theta})^n + (e^{-i\theta})^n}{2}$

$= \frac{1}{2}(z^n + \bar{z}^n)$

$\sum_{n=0}^{+\infty} r^n \cos n\theta$

$= \frac{1}{2}(\sum_{n=0}^{+\infty} z^n + \sum_{n=0}^{+\infty} \bar{z}^n)$

$= \frac{1}{2}(\frac{1}{1-z} + \frac{1}{1-\bar{z}})$

$= \frac{1}{2}(\frac{1}{1-re^{i\theta}} + \frac{1}{1-re^{-i\theta}})$

$= \frac{1}{2}(\frac{2-r(e^{i\theta}+e^{-i\theta})}{1+r^2-r(e^{i\theta}+e^{-i\theta})})$

$= \frac{1-r\cos\theta}{1+r^2-2r\cos\theta}$ ∴等式得证.

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$$r^n \sin n\theta = r^n \cdot \left(\frac{e^{ni\theta} - e^{-ni\theta}}{2i}\right)$$

$$= \frac{1}{2i}\left(r^n e^{ni\theta} - r^n e^{-ni\theta}\right)$$

$$\therefore \sum_{n=0}^{+\infty} r^n \sin n\theta$$

$$= \frac{1}{2i} \sum_{n=0}^{+\infty} \left(z^n - \bar{z}^n\right)$$

$$= \frac{1}{2i}\left(\frac{1}{1-z} - \frac{1}{1-\bar{z}}\right)$$

$$= \frac{1}{2i}\left(\frac{1}{1-re^{i\theta}} - \frac{1}{1-re^{-i\theta}}\right)$$

$$= \frac{1}{2i}\left(\frac{re^{i\theta} - re^{-i\theta}}{1+r^2 - r(e^{i\theta} + e^{-i\theta})}\right)$$

$$= \frac{r\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)}{1+r^2 - 2r\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)}$$

$$= \frac{r\sin\theta}{1+r^2 - 2r\cos\theta}$$ ∴第二式得记

5. (1). $f(z)=\frac{1}{z-3}-\frac{1}{z-2}$

$=\frac{1}{z}\left(\frac{1}{1-\frac{3}{z}}-\frac{1}{1-\frac{2}{z}}\right)$

$=\frac{1}{z}\sum_{n=0}^{+\infty}\left(\left(\frac{3}{z}\right)^n-\left(\frac{2}{z}\right)^n\right)$

$=\sum_{n=0}^{+\infty}\frac{3^n-2^n}{z^{n+1}}$

(2) $f(z)=\frac{1}{5(z-2)}-\frac{z+2}{5(z^2+1)}$

在 $1<|z|<2$ 时

$f(z)=-\frac{1}{10}\left(\frac{1}{1-\frac{1}{2}z}\right)-\frac{z+2}{5z^2}\cdot\frac{1}{1-(-\frac{1}{z^2})}$

$=-\frac{1}{10}\sum_{n=0}^{+\infty}(\frac{1}{2}z)^n-\frac{z+2}{5z^2}\cdot\sum_{n=0}^{+\infty}(-\frac{1}{z^2})^n$

$=-\frac{1}{10}\sum_{n=0}^{+\infty}\left(\frac{z^n}{2^n}\right)-\frac{1}{5}\left(\sum_{n=0}^{+\infty}(-1)^n z^{-(2n+1)}+\sum_{n=0}^{+\infty}(-1)^n\cdot 2z^{-(2n+2)}\right)$

$=-\frac{1}{10}\sum_{n=0}^{+\infty}\left(\frac{1}{2^n}z^n\right)-\frac{1}{5}\sum_{n=0}^{+\infty}\left((-1)^n\left(z^{-(2n+1)}+2z^{-(2n+2)}\right)\right)$

在 $|z|>2$ 时

$f(z)=\frac{1}{5z}\cdot\frac{1}{(1-\frac{2}{z})}-\frac{z+2}{5z^2}\cdot\frac{1}{1-(-\frac{1}{z^2})}$

$=\frac{1}{5z}\sum_{n=0}^{+\infty}(-\frac{2}{z})^n-\frac{z+2}{5z^2}\sum_{n=0}^{+\infty}(-1)^n z^{-2n}$

$=\frac{1}{5}\sum_{n=0}^{+\infty}(-1)^n\left(2^n z^{-(n+1)}-z^{-(2n+1)}-2z^{-(2n+2)}\right)$

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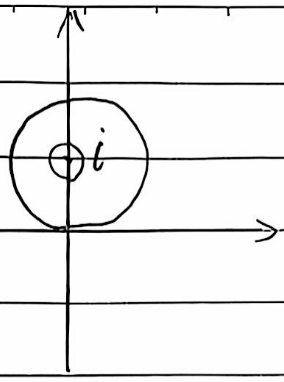
(3).



$$f(z) = \frac{1}{z^2} \cdot \frac{1}{z-i}$$

1°. 在 $0<|z-i|<1$

$$f(z) = \frac{z+i}{z^2} - \frac{1}{z-i}$$

$$= \frac{1}{z} + \frac{i}{z^2} - \frac{1}{z-i}$$

$$= \frac{1}{z} - i\frac{d}{dz}\left(\frac{1}{z}\right) - \frac{1}{z-i}$$

$$= \frac{i}{i(z-i)-1} - i\frac{d}{dz}\left(\frac{i}{i(z-i)-1}\right) - \frac{1}{z-i}$$

$$= -i\sum_{n=0}^{+\infty}(z-i)^n - \frac{d}{dz}\sum_{n=0}^{+\infty}(z-i)^n - (z-i)^{-1}$$

$$= \sum_{n=0}^{+\infty}\left(-i(z-i)^n - n(z-i)^{n-1}\right) - (z-i)^{-1}$$

2°. 在 $|z-i|>1$ 上

$$f(z) = \frac{1}{z} - i\frac{d}{dz}\left(\frac{1}{z}\right) - \frac{1}{z-i}$$

$$= \frac{1}{z-i}\cdot\frac{1}{1-\frac{1}{i-z}} - i\frac{d}{dz}\left(\frac{1}{z}\right) - \frac{1}{z-i}$$

$$= \frac{1}{z-i}\sum_{n=0}^{+\infty}\left(\frac{1}{i-z}\right)^n - i\sum_{n=0}^{+\infty}(-1)^n\cdot(-(n+1))(z-i)^{-(n+2)} - (z-i)^{-1}$$

$$= \sum_{n=0}^{+\infty}\left((-1)^n(z-i)^{-(n+1)} - i(-1)^{n+1}\cdot(n+1)(z-i)^{-(n+2)}\right) - (z-i)^{-1}$$

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6. $f(z)=\frac{1}{a-b}\cdot\frac{1}{z-a}+\frac{1}{b-a}\cdot\frac{1}{z-b}$

$=\frac{1}{b-a}\left(\frac{1}{z-b}-\frac{1}{z-a}\right)$

(1). 在$|a|<|z|<|b|$时

$f(z)=\frac{1}{b-a}\left(-\frac{1}{b-z}-\frac{1}{1-\frac{a}{z}}\cdot\frac{1}{z}\right)$

$=\frac{1}{a-b}\left(\frac{1}{b}\sum_{n=0}^{+\infty}\left(\frac{z}{b}\right)^n+\frac{1}{z}\sum_{n=0}^{+\infty}\left(\frac{a}{z}\right)^n\right)$

$=\frac{1}{b(a-b)}\sum_{n=0}^{+\infty}\frac{z^n}{b^n}+\frac{1}{a-b}\sum_{n=0}^{+\infty}a^n z^{-(n+1)}$

(2). 在∞的邻域

令$w=\frac{1}{z}$ $\therefore z=\frac{1}{w}$

$f(z)=\frac{1}{(\frac{1}{w}-a)(\frac{1}{w}-b)}$

$=\frac{1}{b-a}\left(\frac{1}{\frac{1}{w}-b}-\frac{1}{\frac{1}{w}-a}\right)$

$=\frac{w}{b-a}\left(\frac{1}{1-wb}-\frac{1}{1-wa}\right)$

$=\frac{w}{b-a}\sum_{n=0}^{+\infty}w^n b^n-\frac{w}{b-a}\sum_{n=0}^{+\infty}w^n a^n$

$=\frac{1}{b-a}\sum_{n=0}^{+\infty}(b^n-a^n)w^{n+1}$

$=\frac{1}{b-a}\sum_{n=0}^{+\infty}(b^n-a^n)z^{-(n+1)}$

3. 在 a 的邻域：1°. $0<|z-a|<|b-a|$ 时

$$f(z)=\frac{1}{b-a}\cdot\frac{1}{z-b}-\frac{1}{b-a}\cdot\frac{1}{z-a}$$

$$=-\frac{1}{(b-a)^2}\cdot\frac{1}{1+\frac{z-a}{a-b}}-\frac{1}{b-a}\cdot\frac{1}{z-a}$$

$$=-\frac{1}{(b-a)^2}\cdot\sum_{n=0}^{+\infty}(-1)^n\left(\frac{z-a}{a-b}\right)^n-\frac{1}{b-a}\cdot\frac{1}{z-a}$$

$$=\sum_{n=0}^{+\infty}-\frac{(z-a)^n}{(b-a)^{n+2}}-\frac{1}{b-a}(z-a)^{-1}$$

2°. $|z-a|>|b-a|$

$$f(z)=\frac{1}{b-a}\cdot\frac{1}{z-a}\cdot\frac{1}{1-\frac{b-a}{z-a}}-\frac{1}{b-a}\cdot\frac{1}{z-a}$$

$$=\frac{1}{b-a}\cdot\frac{1}{z-a}\sum_{n=0}^{+\infty}\left(\frac{b-a}{z-a}\right)^n-\frac{1}{b-a}\cdot\frac{1}{z-a}$$

$$=\sum_{n=0}^{+\infty}(b-a)^{n-1}\cdot(z-a)^{-(n+1)}-\frac{1}{b-a}(z-a)^{-1}$$

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7.(1). 1°. $0<|z|<1$

$$f(z) = -\frac{1}{2}\cdot\frac{1}{z-1} + \frac{3}{2}\cdot\frac{1}{z-3}$$

$$= \frac{1}{2}\cdot\frac{1}{1-z} - \frac{1}{2}\cdot\frac{1}{1-\frac{1}{3}z}$$

$$= \frac{1}{2}\left(\sum_{n=0}^{+\infty}\left(z^n - \left(\frac{1}{3}z\right)^n\right)\right)$$

$$= \frac{1}{2}\sum_{n=0}^{+\infty}\left(1-\frac{1}{3^n}\right)z^n$$

2°. $1<|z|<3$.

$$f(z) = -\frac{1}{2}\cdot\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} - \frac{1}{2}\cdot\frac{1}{1-\frac{1}{3}z}$$

$$= -\frac{1}{2z}\sum_{n=0}^{+\infty}\left(\frac{1}{z}\right)^n - \frac{1}{2}\sum_{n=0}^{+\infty}\left(\frac{1}{3}z\right)^n$$

$$= -\frac{1}{2}\sum_{n=0}^{+\infty}z^{-(n+1)} - \frac{1}{2}\sum_{n=0}^{+\infty}\frac{z^n}{3^n}$$

3°. $|z|>3$

$$f(z) = -\frac{1}{2}\cdot\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} + \frac{3}{2}\cdot\frac{1}{z}\cdot\frac{1}{1-\frac{3}{z}}$$

$$= -\frac{1}{2z}\sum_{n=0}^{+\infty}\left(\frac{1}{z}\right)^n + \frac{3}{2}\cdot\frac{1}{z}\cdot\sum_{n=0}^{+\infty}\left(\frac{3}{z}\right)^n$$

$$= -\frac{1}{2}\sum_{n=0}^{+\infty}z^{-(n+1)} + \frac{3}{2}\sum_{n=0}^{+\infty}3^n z^{-(n+1)}$$

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(2). 1°. $0<|z-1|<2$,

$$f(z) = -\frac{1}{2}\frac{1}{z-1} + \frac{3}{2}\cdot\frac{1}{z-3}$$

$$= -\frac{1}{2}\cdot\frac{1}{z-1} - \frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}(z-1)}$$

$$= -\frac{1}{2}(z-1)^{-1} - \frac{3}{4}\sum_{n=0}^{+\infty}\frac{1}{2^n}(z-1)^n$$

2°. $|z-1|>2$

$$f(z) = -\frac{1}{2}\cdot\frac{1}{z-1} + \frac{3}{2}\cdot\frac{1}{z-1}\cdot\frac{1}{1-\frac{2}{z-1}}$$

$$= -\frac{1}{2}(z-1)^{-1} + \frac{3}{2}\cdot\frac{1}{z-1}\cdot\sum_{n=0}^{+\infty}\left(\frac{2}{z-1}\right)^n$$

$$= -\frac{1}{2}(z-1)^{-1} + \frac{3}{2}\sum_{n=0}^{+\infty}2^n(z-1)^{-(n+1)}$$

(3). 1° $0<|z-3|<2$.

$$f(z) = -\frac{1}{2}\frac{1}{z-1} + \frac{3}{2}\cdot\frac{1}{z-3}$$

$$= -\frac{1}{4}\cdot\frac{1}{1-\frac{1}{2}(3-z)} + \frac{3}{2}(z-3)^{-1}$$

$$= -\frac{1}{4}\sum_{n=0}^{+\infty}\frac{1}{2^n}(3-z)^n + \frac{3}{2}(z-3)^{-1}$$

2°. $|z-3|>2$

$$f(z) = \frac{1}{2}\cdot\frac{1}{-2+(3-z)} + \frac{3}{2}\cdot\frac{1}{z-3}$$

$$= \frac{1}{2}\cdot\frac{1}{(3-z)}\cdot\frac{1}{1-\frac{2}{3-z}} + \frac{3}{2}\cdot\frac{1}{z-3}$$

$$= \frac{1}{2}\cdot\frac{1}{(3-z)}\cdot\sum_{n=0}^{+\infty}2^n(3-z)^{-n} + \frac{3}{2}\cdot(z-3)^{-1}$$

$$= \frac{1}{2}\sum_{n=0}^{+\infty}2^n(3-z)^{-(n+1)} + \frac{3}{2}(z-3)^{-1}$$