No.
Date.

2024/10/10 作业 3

3 (1) $|f(z)| = \dfrac{\sqrt{(x^3-y^3)^2+(x^3+y^3)^2}}{(x^2+y^2)}$

$= \dfrac{\sqrt{2x^6+2y^6}}{x^2+y^2}$

$\because x^6+y^6 \le (x^2+y^2)^3$

$\therefore 0 \le |f(z)| \le \dfrac{2\sqrt{(x^2+y^2)^3}}{x^2+y^2}$

$0 \le |f(z)| \le 2\sqrt{x^2+y^2}$

$\because \lim\limits_{\substack{x\to 0\\ y\to 0}} 2\sqrt{x^2+y^2} = 0$

$\therefore \lim\limits_{z\to 0} |f(z)| = 0$ 即 $\lim\limits_{z\to 0} f(z) = 0$

$\therefore f(z)$ 在 $z=0$ 处连续

(2) $u = \dfrac{x^3-y^3}{x^2+y^2}$, $v = \dfrac{x^3+y^3}{x^2+y^2}$

$\dfrac{\partial u}{\partial x} = \dfrac{3x^2(x^2+y^2)-2x(x^3-y^3)}{(x^2+y^2)^2}$, $\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0$.

$\dfrac{\partial v}{\partial y} = \dfrac{3y^2(x^2+y^2)-2y(x^3+y^3)}{(x^2+y^2)^2}$, $\left.\dfrac{\partial v}{\partial y}\right|_{y=0} = 0$.

$\dfrac{\partial u}{\partial y} = \dfrac{-3y^2(x^2+y^2)+2y(x^3-y^3)}{(x^2+y^2)^2}$, $\left.\dfrac{\partial u}{\partial y}\right|_{y=0} = 0$.

$\dfrac{\partial v}{\partial x} = \dfrac{3x^2(x^2+y^2)+2x(x^3+y^3)}{(x^2+y^2)^2}$, $\left.\dfrac{\partial v}{\partial x}\right|_{x=0} = 0$

$\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = \left.\dfrac{\partial v}{\partial y}\right|_{y=0}$; $\left.\dfrac{\partial u}{\partial y}\right|_{y=0} = -\left.\dfrac{\partial v}{\partial x}\right|_{x=0}$.

No.

Date. . .

$\therefore f(z)$ 在 $z=0$ 处满足 C-R 方程

(3) $\lim\limits_{\Delta z\to 0}\frac{f(\Delta z)-f(0)}{\Delta z}$

$=\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{(\Delta x)^3+(\Delta y)^3-i((\Delta x)^3+(\Delta y)^3)}{(\Delta x)^2+(\Delta y)^2}\cdot\frac{1}{\Delta x+i\Delta y}$

$=\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{(x^3+y^3-i(x^3+y^3))(x-iy)}{(x^2+y^2)^2}$

$=\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x^4-x^3y+xy^3-y^4-(x^4+x^3y+xy^3+y^4)i}{(x^2+y^2)^2}$

令 $y=kx$

上式 $=\frac{(1-k+k^3-k^4)-(1+k+k^3+k^4)i}{(1+k^2)^2}$ 与 k 有关.

故 $\lim\limits_{\Delta z\to 0}\frac{f(\Delta z)-f(0)}{\Delta z}$ 不存在.

即 $f(z)$ 在 $z=0$ 处导数不存在

4. (1). $f(z)=\frac{z+2}{(z+1)(z+i)(z-i)}$

∴ $f(z)$ 在 $z\neq-1$, $z\neq-i$, $z\neq i$ 时可导

$$f'(z)=\frac{-2z^3-5z^2+4z-3}{(z-1)^2(z^2+1)^2}$$

(2). $u=\frac{x+y}{x^2+y^2}$, $v=\frac{x-y}{x^2+y^2}$

$$u_x=\frac{(x^2+y^2)-2x(x+y)}{(x^2+y^2)^2}=\frac{-x^2-2xy+y^2}{(x^2+y^2)^2}$$

$$v_y=\frac{-(x^2+y^2)-2y(x-y)}{(x^2+y^2)^2}=\frac{-x^2-2xy+y^2}{(x^2+y^2)}$$

$$u_y=\frac{(x^2+y^2)-2y(x+y)}{(x^2+y^2)^2}=\frac{x^2-2xy-y^2}{(x^2+y^2)^2}$$

$$v_x=\frac{(x^2+y^2)-2x(x-y)}{(x^2+y^2)^2}=\frac{-x^2+2xy+y^2}{(x^2+y^2)^2}$$

∴ $u_x=v_y$, $-v_x=u_y$ 满足C-R条件

又∵u,v在除$(0,0)$外任意一阶偏导连续

即: $f(z)$在$z\neq0$时均可导

$$f'(z)=\frac{-x^2-2xy+y^2}{(x^2+y^2)^2}+i\frac{-x^2+2xy+y^2}{(x^2+y^2)^2}.$$

No.

Date. . .

5. (1). $u = xy^2$, $v = x^2y$

$u_x = y^2$, $v_y = x^2$

$u_y = 2xy$, $v_x = 2xy$

$\because \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases}$

$\therefore f(z)$ 仅在 $z=0$ 处可导，不解析

$f'(0)=0$.

(2). $u = x^3 - y^3$, $v = 2x^2y^2$.

$u_x = 3x^2$, $v_y = 4x^2y$

$v_x = 4y^2x$, $u_y = -3y^2$

$\because \begin{cases} u_x = v_y \\ v_x = -u_y \end{cases} \Rightarrow \begin{cases} 3x^2 = 4x^2y \\ 4y^2x = 3y^2 \end{cases}$

即：$\begin{cases} x=0 \\ y=0 \end{cases}$ 或 $\begin{cases} x = \frac{3}{4} \\ y = \frac{3}{4} \end{cases}$ 时，$f(z)$ 可导

$f(z)$ 在复平面上不解析

$x=0$, $y=0$ 时，$f'(z)=0$.

$x=\frac{3}{4}$, $y=\frac{3}{4}$ 时，$f'(z) = \frac{27}{16} + \frac{27}{16}i$

No.

Date. . .

(3). $z=x+iy$.

$f(z)=(x+iy)y-x$

$=xy-x+iy^2$.

$u=xy-x$, $V=y^2$

$u_x=y-1$, $V_y=2y$

$u_y=x$, $V_x=0$.

$\begin{cases} y-1=2y \\ x=0 \end{cases}$ 即 $\begin{cases} x=0 \\ y=-1 \end{cases}$

即 $z=-i$时, $f(z)$可导

$f(z)$在复平面上不解析.

$z=-i$时, $f'(z)=-2$

(4). $z=x+iy$.

$f(z)=x^2+y^2-i(x^2+y^2)$

$\therefore u=x^2+y^2$, $V=-(x^2+y^2)$

$u_x=2x$, $V_y=2y$

$u_y=2y$, $V_x=-2x$

$\therefore \begin{cases} 2x=2y \\ 2y=2x \end{cases}$ 即: $y=x$

$\therefore f(z)$在$y=x$处可导, 在复平面上不解析.

$y=x$时, $f'(z)=2x-2ix$

No.

Date.

6. (1) 即：$u^2+v^2=C$.

即：$2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=0$.

$$\begin{cases} u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}=0 \\ u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}=0. \end{cases}$$

又 ∵ $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$，$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$.

$$\therefore \begin{cases} u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}=0. & ① \\ -u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}=0. & ② \end{cases}$$

$①\times u+②\times v$：$(u^2+v^2)\frac{\partial u}{\partial x}=0$

又 ∵ $u^2+v^2\neq 0$ ∴ $\frac{\partial u}{\partial x}=0$.

同理：$\frac{\partial v}{\partial x}=0$.

∴ $\frac{\partial u}{\partial y}=0$，$\frac{\partial v}{\partial y}=0$.

∴ $u=C_1$，$v=C_2$. 即：$f(z)$是常数.

No.

Date. . .

(2). 先证明：区域D上的实值解析函数必定是常数.

设$\Phi(z)$为实值解析函数.

$\therefore v=0$, $\therefore \frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=0$.

根据C-R方程：$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$；$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

$\therefore \frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=0$

$\therefore u=C_1$, $v=C_2$, $\Phi(z)$为常数.

下面考虑$f(z)$

设$\arg f(z)=\theta_0$

$\therefore f(z)=|f(z)|e^{i\theta_0}$

$\because f(z)$在区域D上解析，$e^{i\theta_0}$为常数

$\therefore |f(z)|$为实值解析函数.

由上述过程可知$|f(z)|$为常数.

$\therefore f(z)$为常数.

No.

Date.

7. 由题：$\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}=0$，$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0$.

设 $m=u_y-v_x$，$n=u_x+v_y$.

$$m_x=\frac{\partial^2 u}{\partial y\partial x}-\frac{\partial^2 v}{\partial x^2}$$

$$n_y=\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}$$

$$m_y=\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 v}{\partial x\partial y}.$$

$$n_x=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial x}.$$

又∵ $\frac{\partial^2 v}{\partial x^2}=-\frac{\partial^2 v}{\partial y^2}$；$\frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2 u}{\partial y^2}$

又∵ u，v 为 D 内的调和函数.

∴ u，v 的二阶偏导数连续.

∴ $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$；$\frac{\partial^2 v}{\partial x\partial y}=\frac{\partial^2 v}{\partial y\partial x}$.

∴ $m_x=n_y$，$m_y=-n_x$.

又∵ $m(x,y)$，$n(x,y)$ 的一阶偏导数均连续

∴ $f(z)$ 在 D 内解析.