20241017 作业4

8.(1). $\frac{du}{dx}=2x+y$, $\frac{d^2u}{dx^2}=2$.

$\frac{du}{dy}=x-2y$, $\frac{d^2u}{dy^2}=-2$.

$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=0$ $\therefore u(x,y)$为调和函数.

由题: $\frac{dv}{dy}=\frac{du}{dx}=2x+y$ ① , $\frac{dv}{dx}=-\frac{du}{dy}=2y-x$ ②.

由①: $V=\frac{1}{2}y^2+2xy+\varphi(x)$

$\frac{dV}{dx}=2y+\varphi'(x)=2y-x$ $\therefore \varphi'(x)=-x$

$\therefore \varphi(x)=-\frac{1}{2}x^2+C$.

$V=\frac{1}{2}y^2+2xy-\frac{1}{2}x^2+C$ $\because f(i)=-1+i$

$\therefore C=\frac{1}{2}$

$\therefore f(z)=(x^2+xy-y^2)+i(\frac{1}{2}y^2+2xy-\frac{1}{2}x^2+\frac{1}{2})$

(3) $\frac{dV}{dx}=-\frac{y}{x^2+y^2}$, $\frac{dV}{dy}=\frac{x}{x^2+y^2}$

$\frac{d^2V}{dx^2}=\frac{2xy}{(x^2+y^2)^2}$, $\frac{d^2v}{dy^2}=-\frac{2xy}{(x^2+y^2)^2}$

$\therefore \frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}=0$ $V(x,y)$为调和函数.

由题, $\frac{du}{dx}=\frac{dv}{dy}=\frac{x}{x^2+y^2}$

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$\therefore u=\frac{1}{2}\ln(x^2+y^2)+\varphi(y)$

$\frac{\partial u}{\partial y}=\frac{y}{x^2+y^2}+\varphi'(y)$

又 $\because \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=\frac{y}{x^2+y^2}$

$\therefore \varphi'(y)=0,\quad \varphi(y)=C.$

$\therefore u=\frac{1}{2}\ln(x^2+y^2)+C$

又 $\because f(1)=0\quad \therefore C=0$

$\therefore f(z)=\frac{1}{2}\ln(x^2+y^2)+i\arctan\frac{y}{x}$

9. $\begin{cases} \frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}=-2x \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}=2y \\ \frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0 \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0 \end{cases}$

$\therefore \begin{cases} \frac{\partial u}{\partial x}=y-x \\ \frac{\partial v}{\partial x}=-y-x \\ \frac{\partial u}{\partial y}=x+y \\ \frac{\partial v}{\partial y}=y-x \end{cases}$

$u = yx - \frac{1}{2}x^2 + \varphi(y)$, $\frac{\partial u}{\partial y} = x + \varphi'(y) = x + y$

$\therefore \varphi'(y) = y$, $\varphi(y) = \frac{1}{2}y^2 + C$.

即 $u = -\frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + C$.

又 $\because u + v = y^2 - x^2$

$\therefore v = -\frac{1}{2}x^2 - xy + \frac{1}{2}y^2 - C$.

$f(z) = (-\frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + C) + i(-\frac{1}{2}x^2 - xy + \frac{1}{2}y^2 - C)$

10. (1). $e^{2z} = 1 - \sqrt{3}i$

$2z = \mathrm{Ln}(1 - \sqrt{3}i)$

$= \ln 2 - \frac{\pi}{3}i + 2k\pi$, $k \in Z$.

即 $z = \frac{1}{2}\ln 2 - \frac{\pi}{6}i + k\pi$,

(2). $\frac{e^{iz} - e^{-iz}}{2i} - \frac{e^{iz} + e^{-iz}}{2} = 2$.

$(1-i)(e^{iz})^2 - 4ie^{iz} - (1+i) = 0$.

$e^{iz} = -\frac{2+\sqrt{2}}{2} + \frac{2+\sqrt{2}}{2}i$ 或 $\frac{-2+\sqrt{2}}{2} + \frac{2-\sqrt{2}}{2}i$

若 $e^{iz} = -\frac{2+\sqrt{2}}{2} + \frac{2+\sqrt{2}}{2}i$, 则 $iz = \ln(1+\sqrt{2}) + i\frac{3\pi}{4} + 2k\pi i$, $k \in Z$

即 $z = -i\ln(1+\sqrt{2}) + \frac{3\pi}{4} + 2k\pi$, $k \in Z$

若 $e^{iz} = \frac{-2+\sqrt{2}}{2} + \frac{2-\sqrt{2}}{2}i$, 则 $iz = \ln(\sqrt{2}-1) + \frac{3\pi}{4} + 2k\pi$, $k \in Z$.

即 $z = -i\ln(\sqrt{2}-1) + \frac{3\pi}{4} + 2k\pi$, $k \in Z$

综上: $z = -i\ln(1+\sqrt{2}) + \frac{3\pi}{4} + 2k\pi$ 或 $z = -i\ln(\sqrt{2}-1) + \frac{3\pi}{4} + 2k\pi$, $k \in Z$