

Distribution Shifts

Medha Agarwal and Scott Geng



A friendly husky in the WILDS

WILDS: A Benchmark of In-the-Wild Distribution Shifts





Great Curassow



Spurfowl

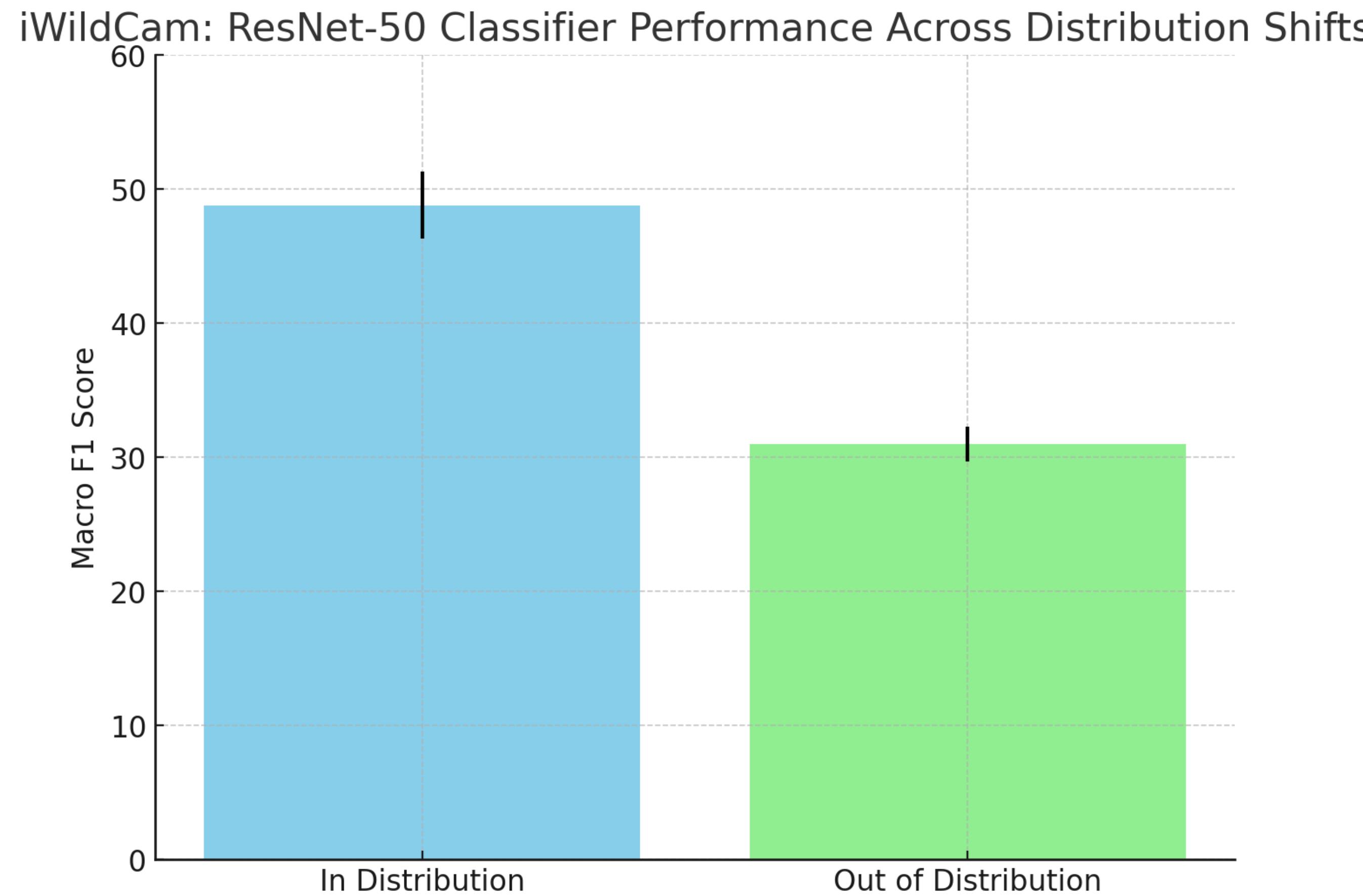




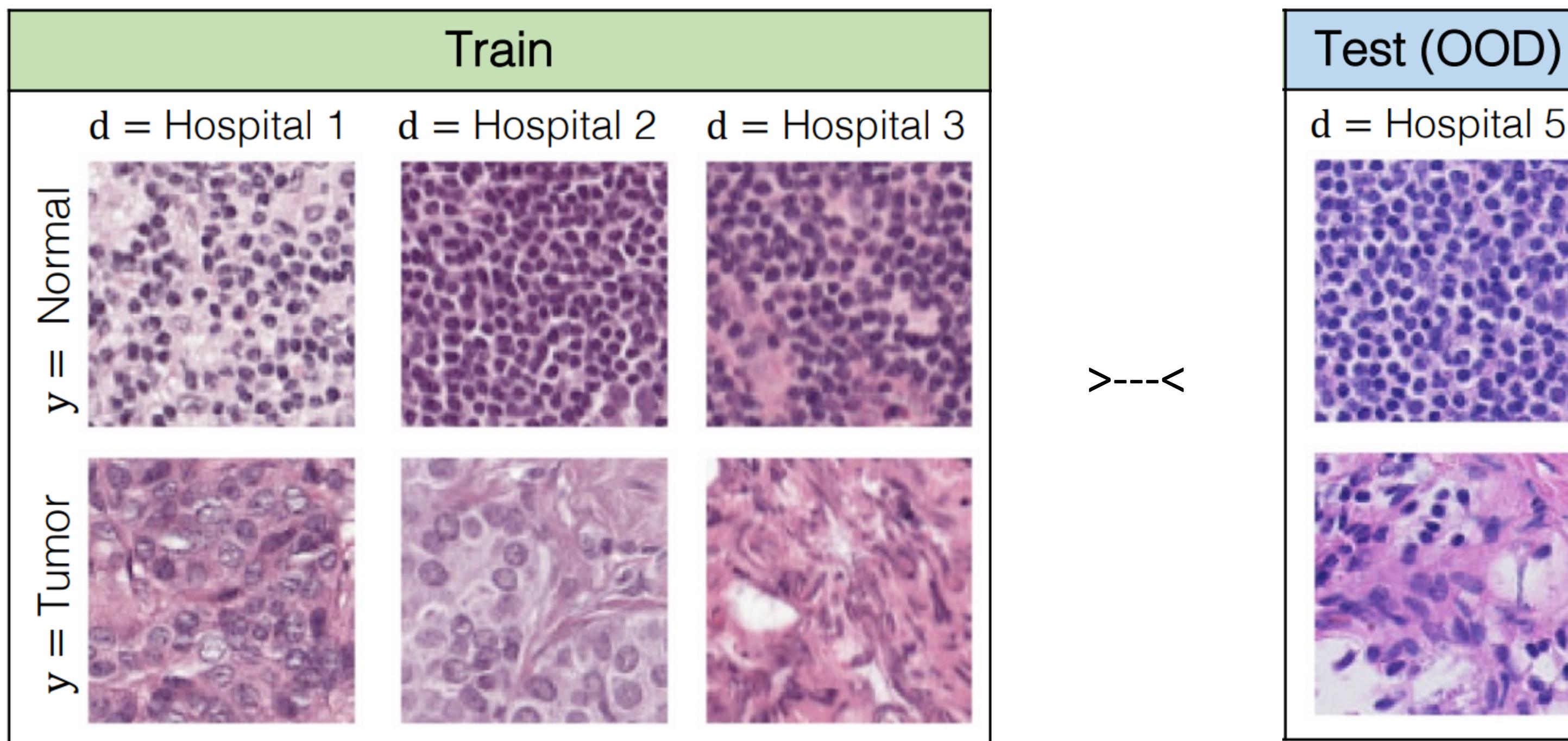




ML models often fail in the presence of distribution shifts...

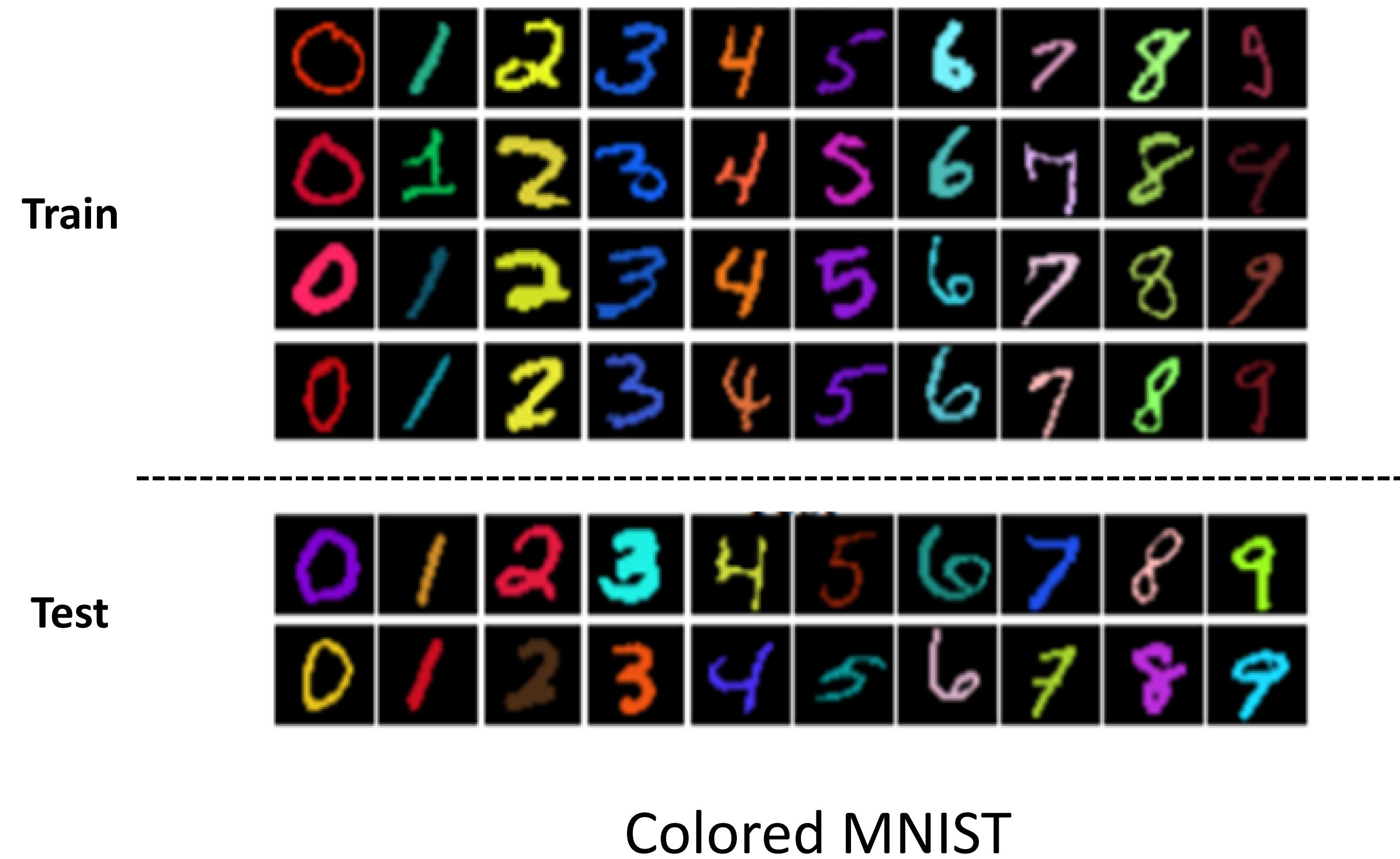


...and these failures can have severe real-world ramifications.

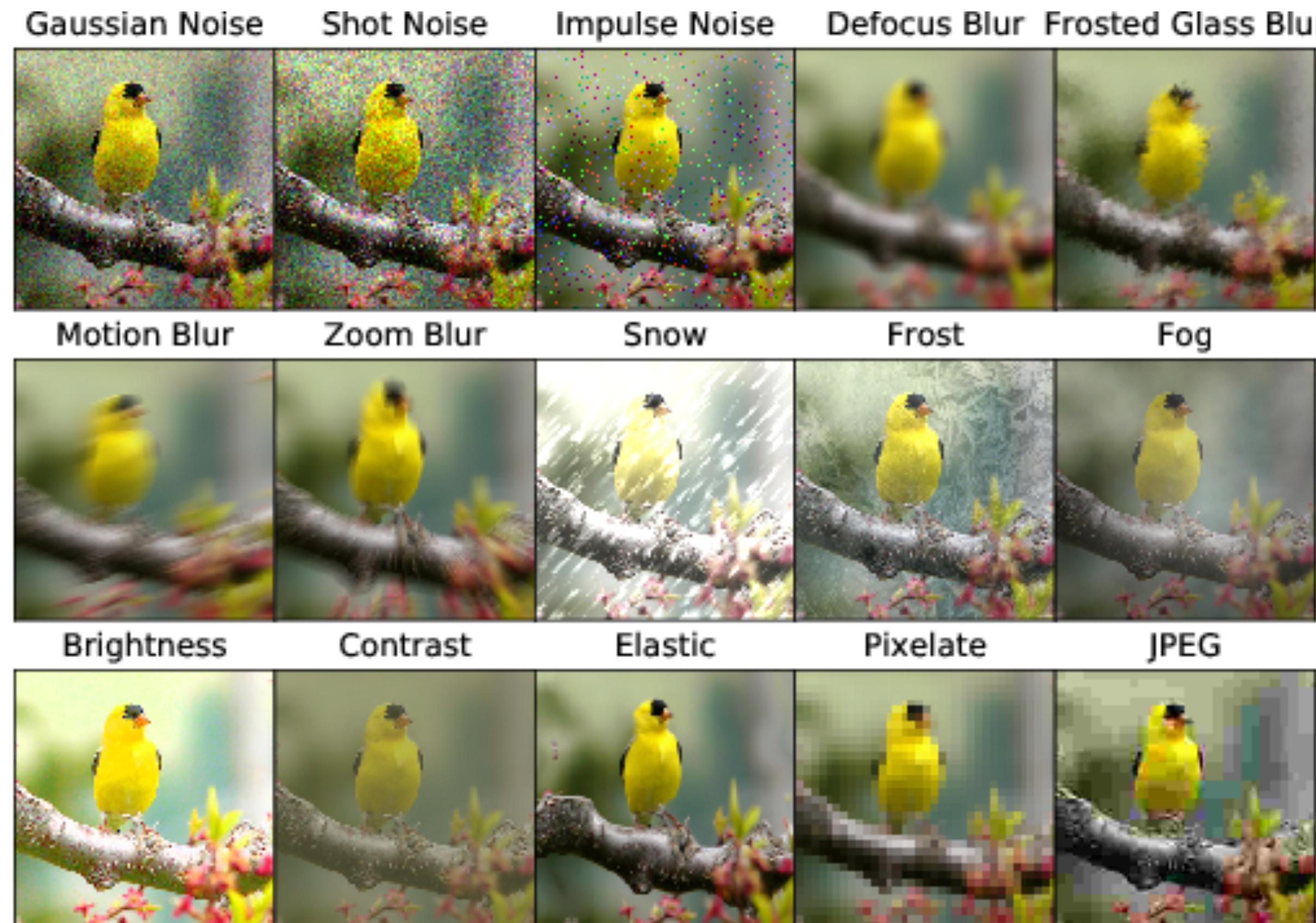


We hope to build models that can generalize well across distribution shifts.

Q: What sorts of ✨datasets✨ have ML researchers used to study this problem?

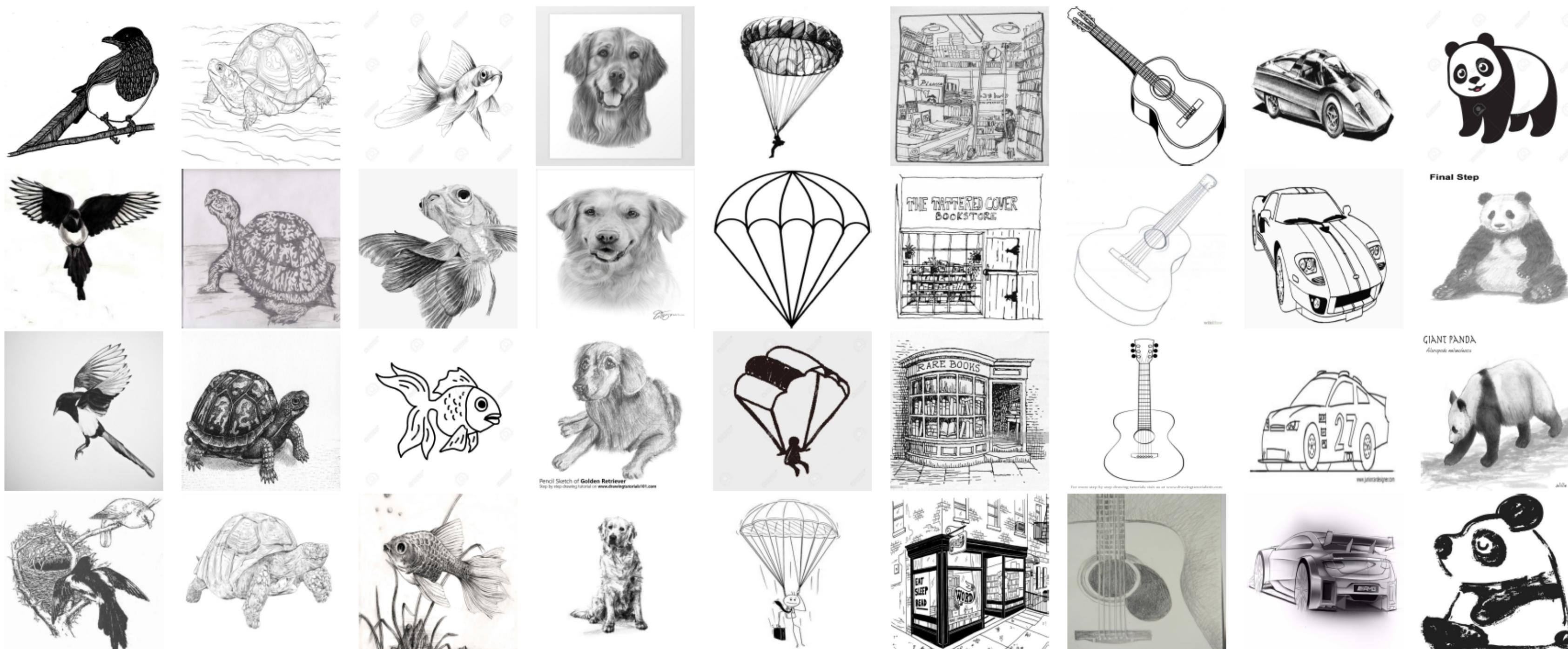


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ImageNet-C

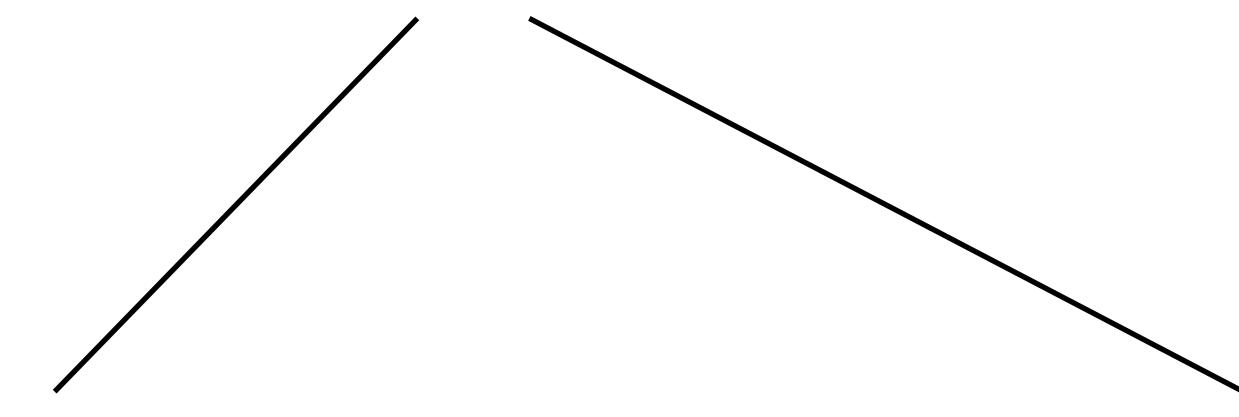
Q: What sorts of ✨datasets✨ have ML researchers used to study this problem?



ImageNet-Sketch

Q: What sorts of ✨datasets✨ have ML researchers used to study this problem?

A: ML researchers have predominantly studied datasets of **artificial** distribution shifts.



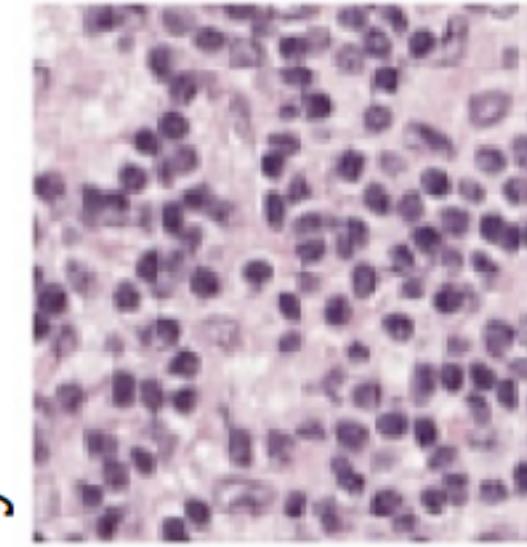
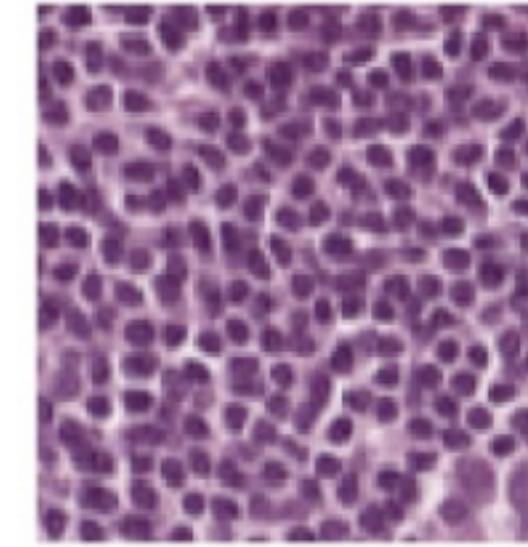
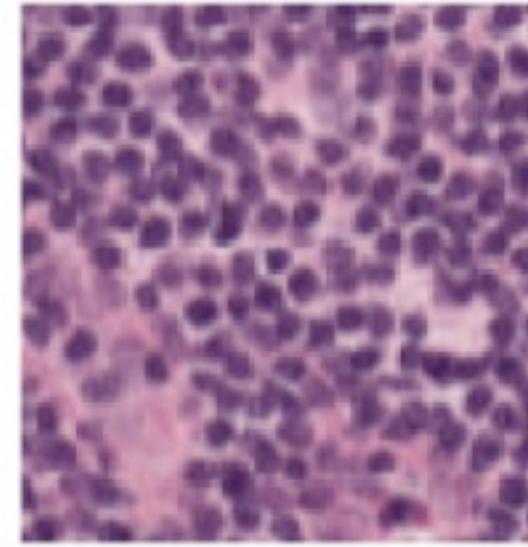
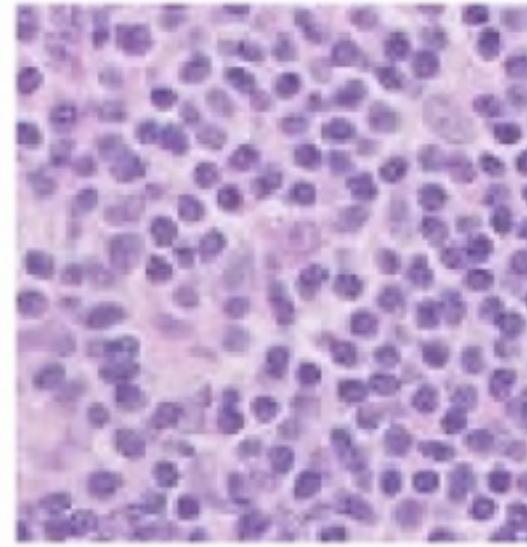
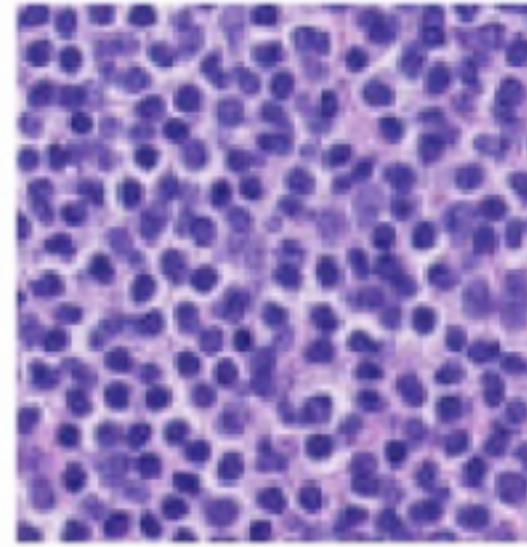
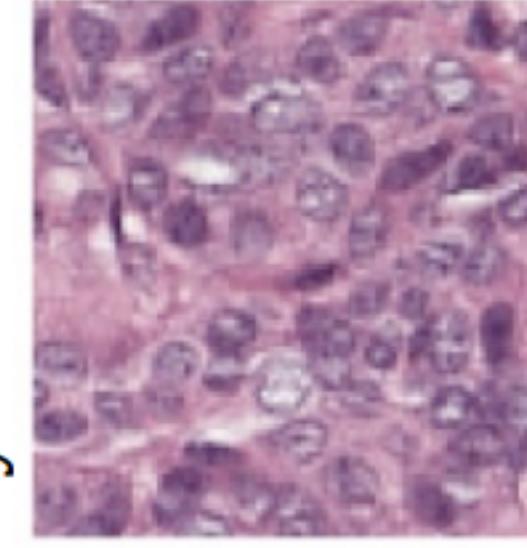
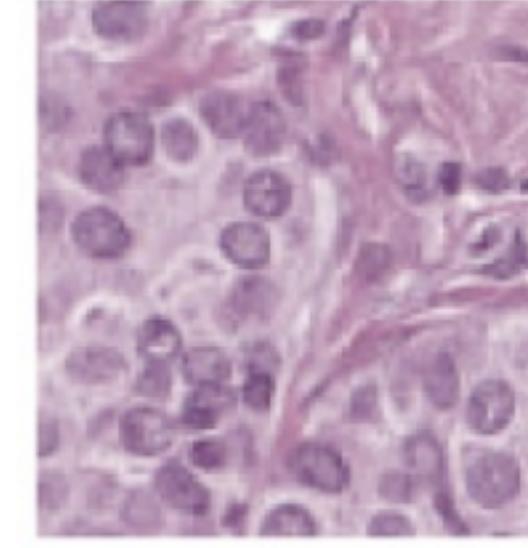
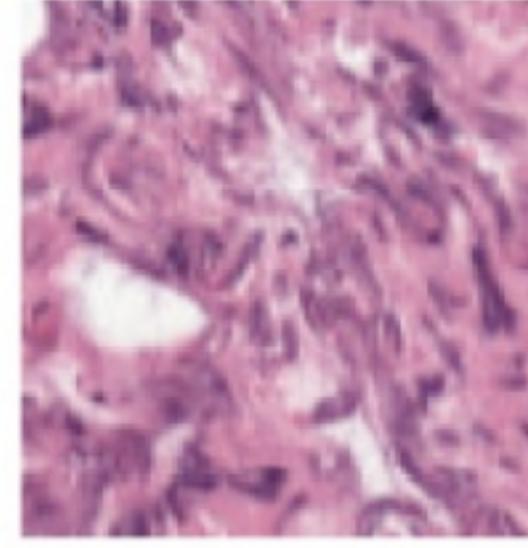
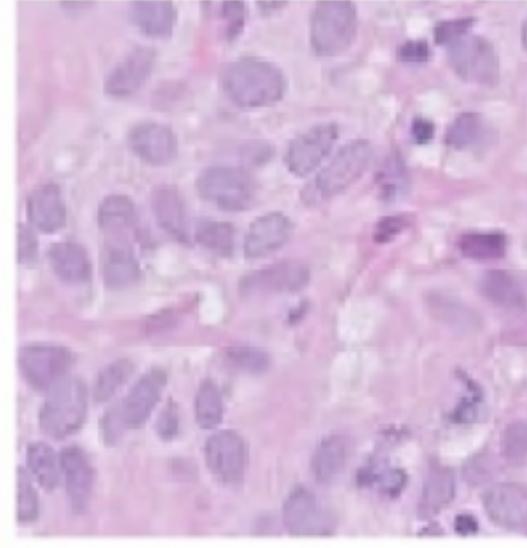
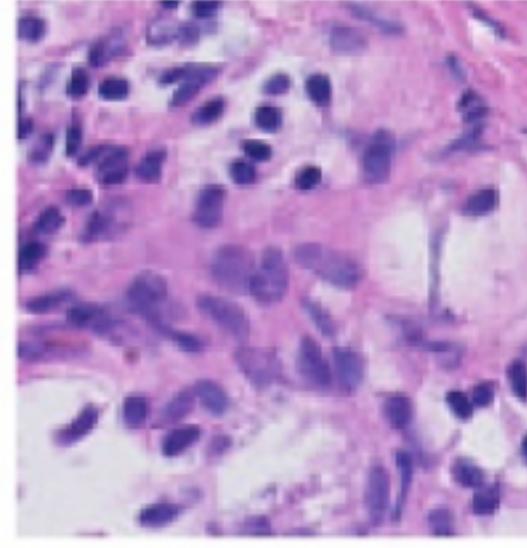
Synthetic transformations.

1. Colored MNIST
2. ImageNet-C
3. Waterbirds
4.

Artificially disparate data splits.

1. ImageNet-Sketch
2. ImageNet-Rendition
3. PACS
4.

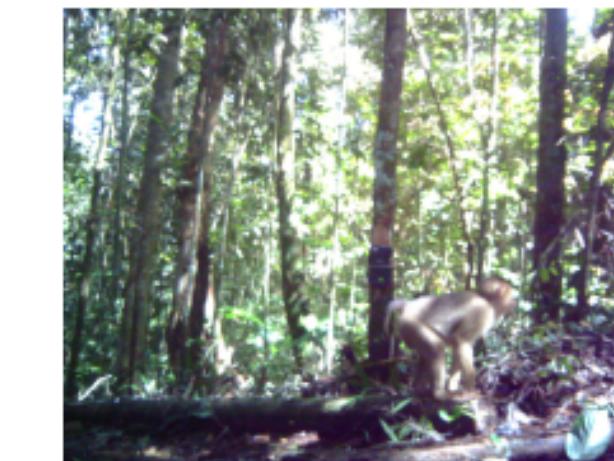
| Train | Test (OOD) |
|---|--|
| $d = \text{Location 1}$ | $d = \text{Location 246}$ |
|  Vulturine Guineafowl |  African Bush Elephant |
|  Cow |  Wild Horse |
|  Southern Pig-Tailed Macaque |  Great Curassow |
| ... | ... |

| Train | | | Val (OOD) | Test (OOD) |
|----------------|---|---|---|---|
| d = Hospital 1 | d = Hospital 2 | d = Hospital 3 | d = Hospital 4 | d = Hospital 5 |
| y = Normal |  |  |  |  |
| |  | | | |
| y = Tumor |  |  |  |  |
| |  | | | |

| Toxic | Comment Text | Male | Female | LGBTQ | White | Black | Asian | Christian |
|-------|---|------|--------|-------|-------|-------|-------|-----------|
| 0 | I applaud your father. He was a good man! We need more like him. | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | As a Christian, I will not be patronizing any of those businesses. | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | What do Black and LGBT people have to do with bicycle licensing? | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| 0 | Government agencies track down foreign baddies and protect law-abiding white citizens. How many shows does that describe? | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 1 | Maybe you should learn to write a coherent sentence so we can understand WTF your point is. | 0 | 0 | 0 | 0 | 0 | 0 | 0 |



Q: With our new dataset, what can we learn?

| Train | | | Test (OOD) |
|---|---|---|--|
| $d = \text{Location 1}$  Vulturine Guineafowl | $d = \text{Location 2}$  African Bush Elephant | $d = \text{Location 245}$  ... unknown | $d = \text{Location 246}$  ??? |
|  Cow |  Cow |  Southern Pig-Tailed Macaque |  ??? |

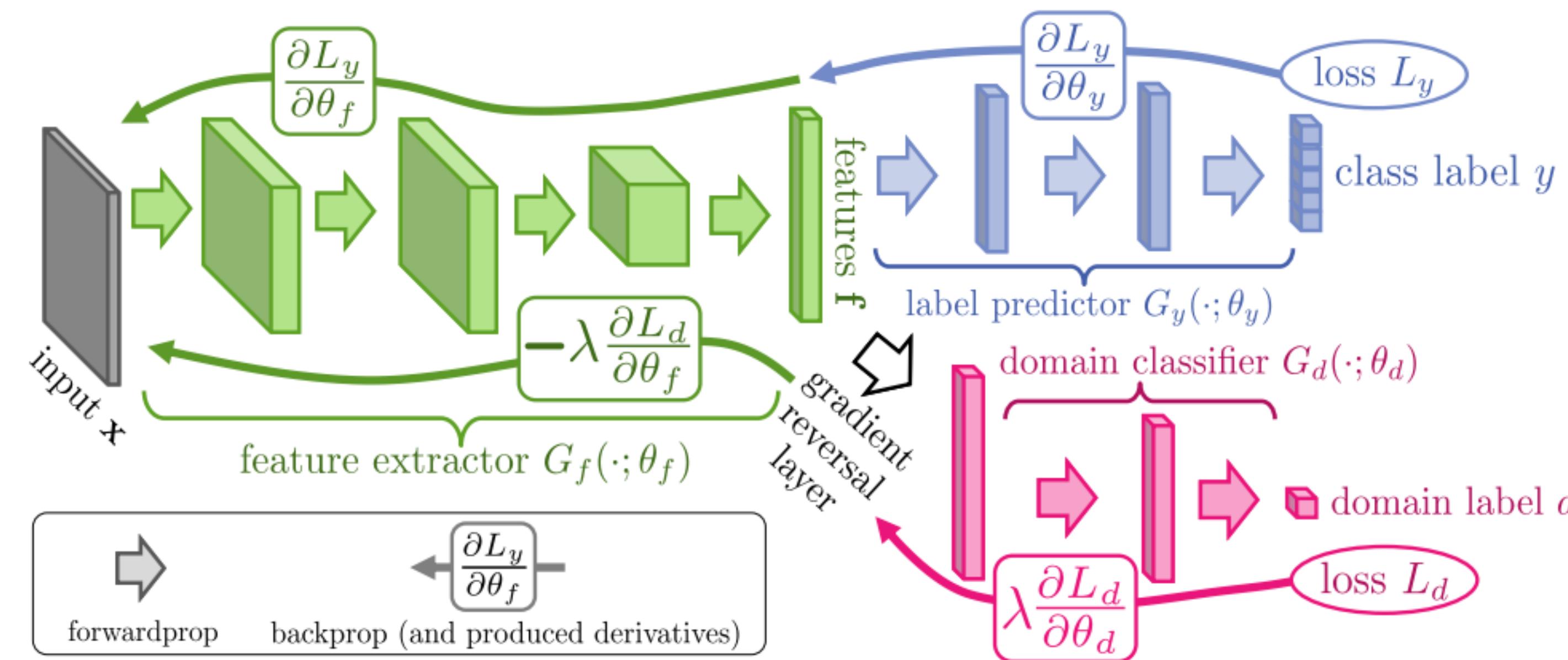


v2

Additional unlabeled examples
(possibly from test distribution)

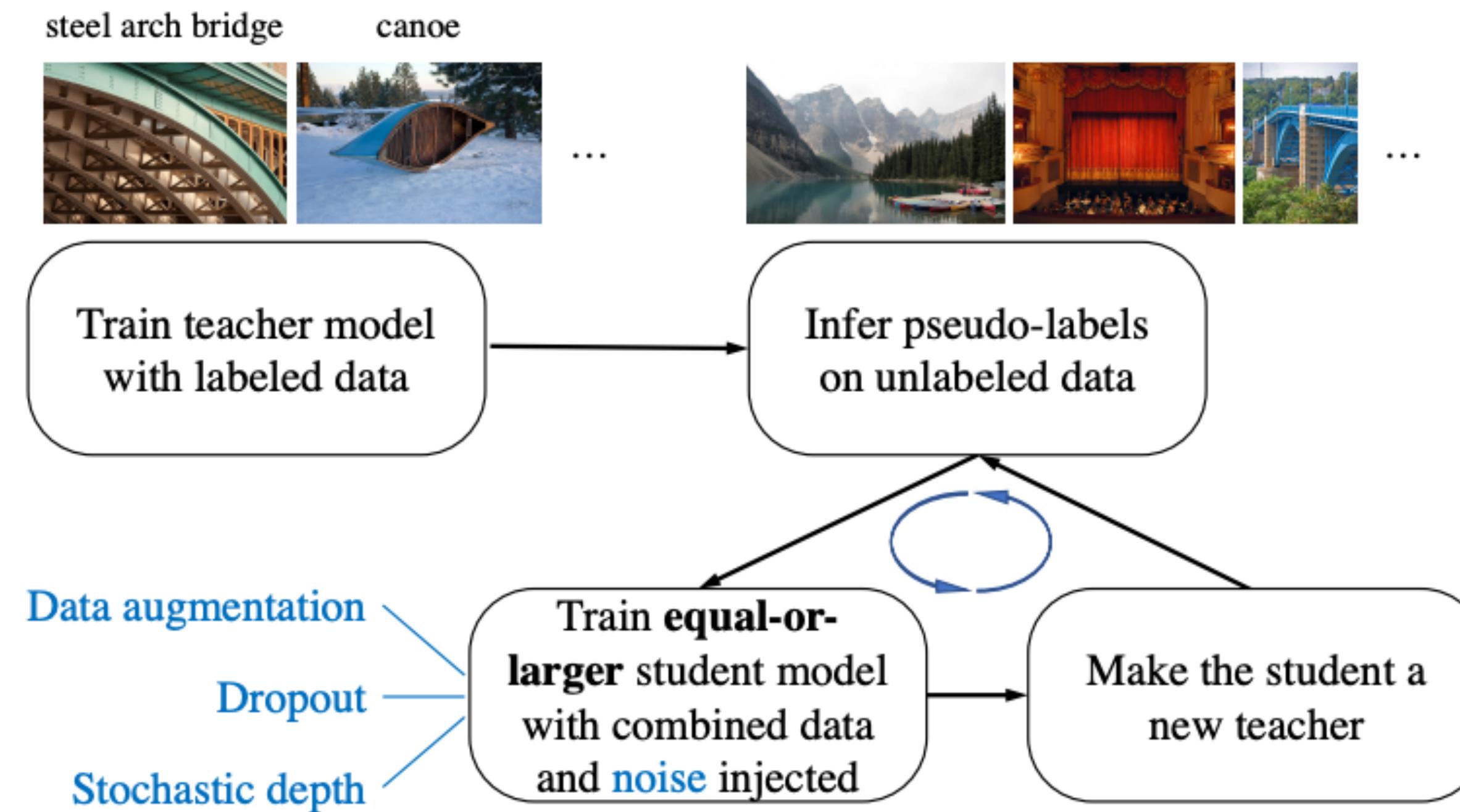
[Koh et al. 2021]

Domain adaptation approach: try to learn features that are invariant across domains.



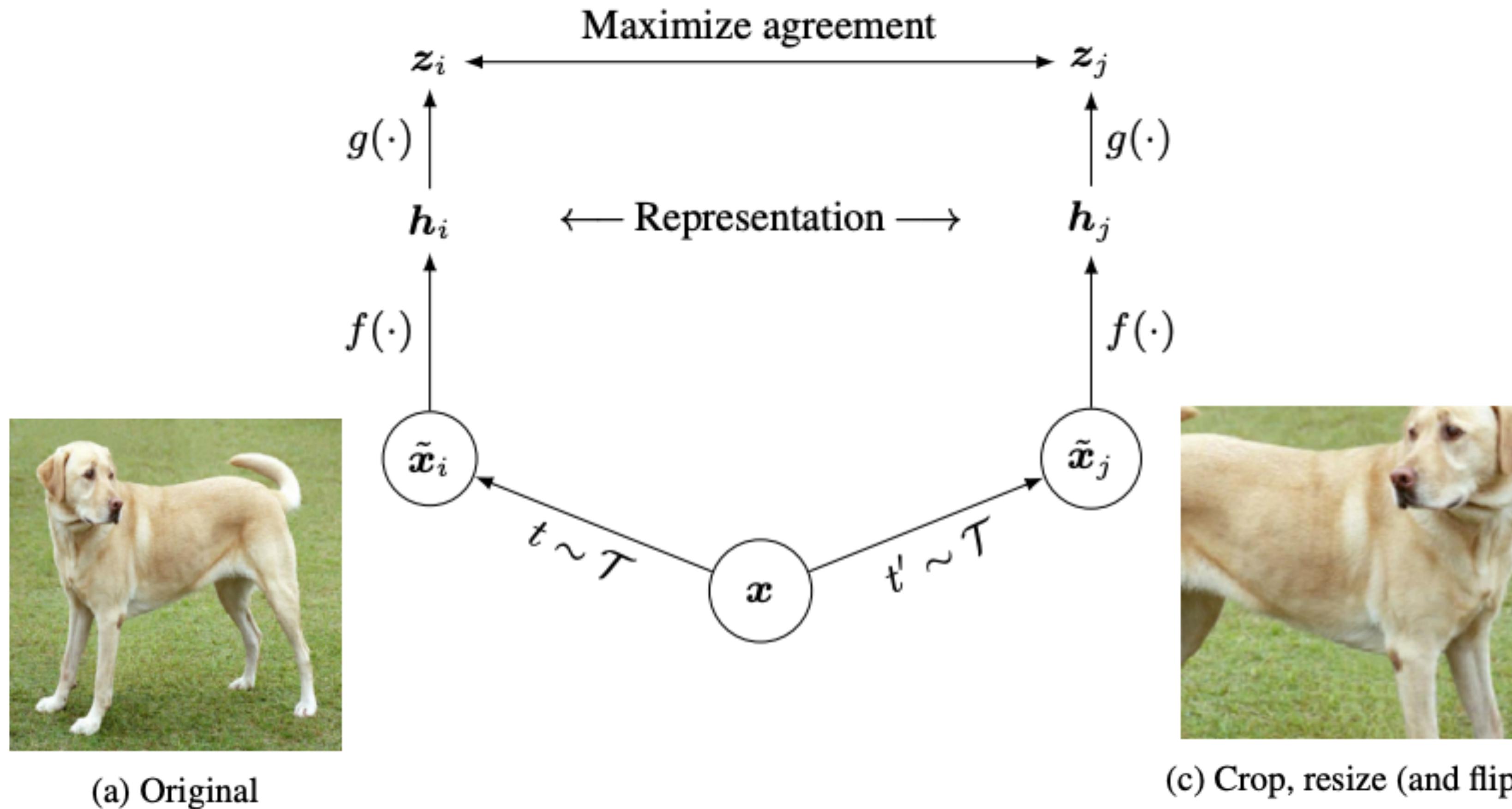
DANN: I want my **learned features** to achieve **low classification loss on my labeled data** and have **high domain identification loss across all data**.

Domain adaptation approach: try to self-train by producing pseudo-labels for our unlabeled data.

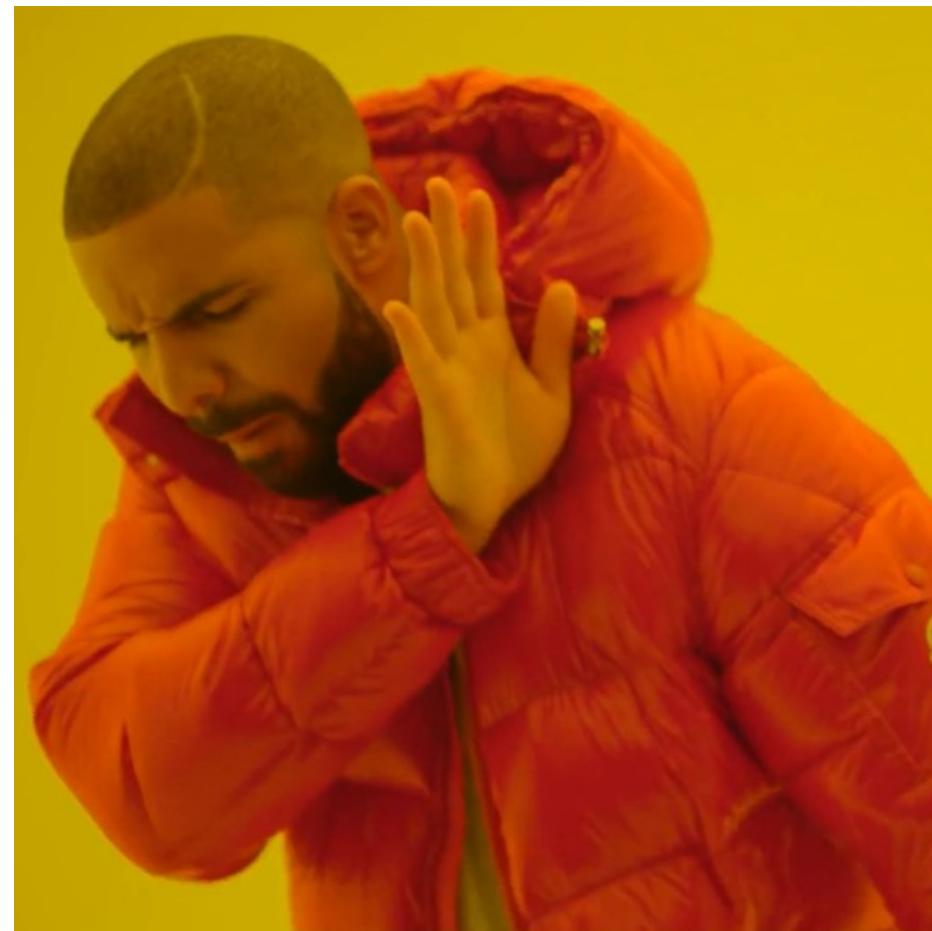


NoisyStudent intuition: **very strong regularization** allows us to avoid overfitting to wrong pseudo-labels.

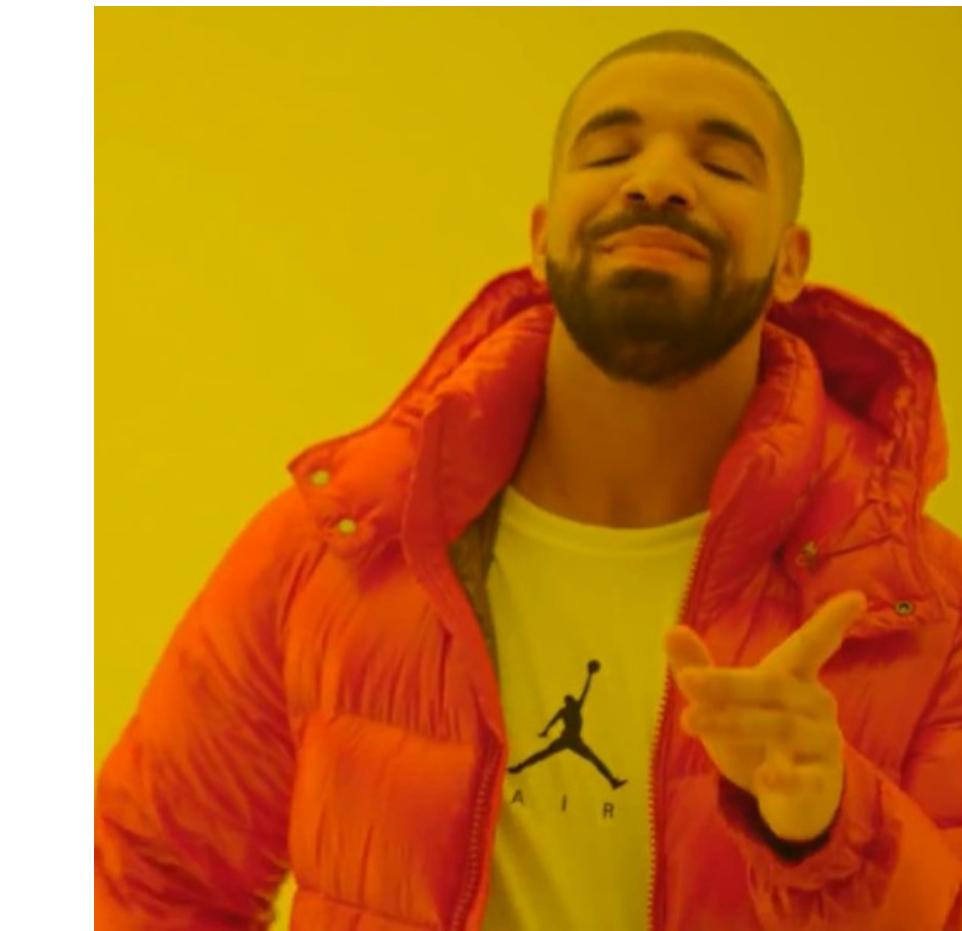
Domain adaptation approach: try to learn from unlabeled data via a self-supervised objective.



Baseline approach: ERM (+/- data augmentation)



| Test (OOD) |
|--|
| $d = \text{Location 246}$ |
| A small image of an antelope standing in a grassy field. |
| ??? |
| A small image of a bird standing in a field. |
| ??? |



| Train | |
|---|---|
| $d = \text{Location 1}$ | $d = \text{Location 2}$ |
| A small image of several cattle grazing in a field. | A small image of a herd of elephants walking across a grassy plain. |
| Vulturine Guineafowl | African Bush Elephant |
| ... | |
| A small image of a cow standing in a field. | A small image of a group of cattle standing in a field. |
| Cow | Cow |

Just pretend like our unlabeled data doesn't exist.

Q: With our new dataset, what can we learn?

SOTA on ImageNet-C 🍞

| | IWILDCAM2020-WILDS (Unlabeled extra, macro F1) | |
|---------------------|---|---------------------|
| | In-distribution | Out-of-distribution |
| ERM (-data aug) | 46.7 (0.6) | 30.6 (1.1) |
| ERM | 47.0 (1.4) | 32.2 (1.2) |
| CORAL | 40.5 (1.4) | 27.9 (0.4) |
| DANN | 48.5 (2.8) | 31.9 (1.4) |
| Pseudo-Label | 47.3 (0.4) | 30.3 (0.4) |
| FixMatch | 46.3 (0.5) | 31.0 (1.3) |
| Noisy Student | 47.5 (0.9) | 32.1 (0.7) |
| SwAV | 47.3 (1.4) | 29.0 (2.0) |
| ERM (fully-labeled) | 54.6 (1.5) | 44.0 (2.3) |

Q: With our new dataset, what can we learn?

A: Existing domain adaptation methods basically do not work*.

*they largely fail to significantly improve over an ERM baseline on the distribution shifts captured by WILDS.

Takeaway 1: As ML researchers, we should ground our work in (or at least by cognizant of) real-world use.

Takeaway 2: The field of domain adaptation is wide open!

References

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A Theory of Learning from Different Domains

**Shai Ben-David · John Blitzer · Koby Crammer ·
Alex Kulesza · Fernando Pereira · Jennifer Wortman Vaughan**

CSE 599 Presentation

Medha Agarwal | February 02, 2024

Distribution Shifts

Source Domain \neq Target Domain

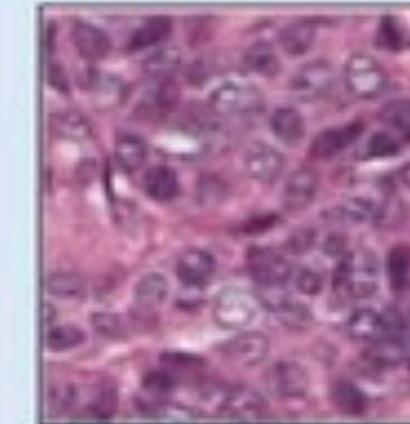
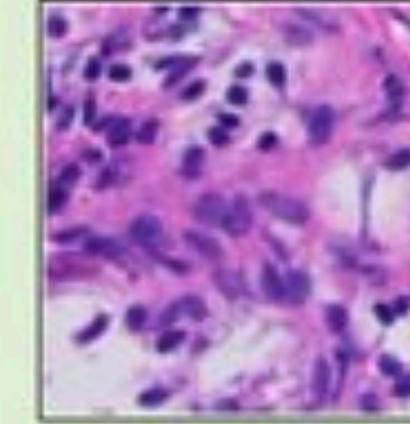
| | iWildCam | Camelyon17 | OGB-MolPCBA | CivilComments | Amazon | FMoW | PovertyMap | Py150 |
|--------------|--|---|---|--|--|---|---|---|
| Shift | camera | hospital | scaffold | demographic | user | time, region | country, rural-urban | git repository |
| Train |  |  | <chem>CCN1C=CC2=C1C(=O)NC(C)=C2C(=O)c3ccccc3</chem> | What do Black and LGBT people have to do with bicycle licensing? | Overall a solid package that has a good quality of construction for the price. |  |  | <pre>import numpy as np ... norm=np.____</pre> |
| Test |  |  | <chem>CC1=CSC2=C1C(O)C=C2Cc3ccccc3</chem> | As a Christian, I will not be patronizing any of those businesses. | I *loved* my French press, it's so perfect and came with all this fun stuff! |  |  | <pre>import subprocess as sp p=sp.Popen() stdout=p.____</pre> |
| Adapted from | Beery et al. 2020 | Bandi et al. 2018 | Hu et al. 2020 | Borkan et al. 2019 | Ni et al. 2019 | Christie et al. 2018 | Yeh et al. 2020 | Raychev et al. 2016 |

Image credits: Koh, Pang Wei, et al. (2021)

What is Domain?

Binary Classification Setting

| | |
|------------------------|--------------------------------------|
| Inputs | \mathcal{X} |
| Distribution on inputs | \mathcal{D} |
| Labeling function | $f: \mathcal{X} \rightarrow \{0,1\}$ |
| Domain | (\mathcal{D}, f) |

When source domain \neq target domain, let
 (\mathcal{D}_S, f_S) = source domain and (\mathcal{D}_T, f_T) = target domain.

TRAIN DATA

(\mathcal{D}_1, f_1)

$$\{(X_i, y_i)\}_{i=1}^{m_1}$$

(\mathcal{D}_2, f_2)

$$\{(X_i, y_i)\}_{i=1}^{m_2}$$

(\mathcal{D}_N, f_N)

$$\{(X_i, y_i)\}_{i=1}^{m_N}$$

...

TEST DATA

(\mathcal{D}_T, f_T)

$$\{(X_i, y_i)\}_{i=1}^{m_T}$$

TRAIN DATA

(\mathcal{D}_1, f_1)

$$\{(X_i, y_i)\}_{i=1}^{m_1}$$

(\mathcal{D}_2, f_2)

$$\{(X_i, y_i)\}_{i=1}^{m_2}$$

(\mathcal{D}_N, f_N)

$$\{(X_i, y_i)\}_{i=1}^{m_N}$$

...

TEST DATA

(\mathcal{D}_T, f_T)

$$\{X_i\}_{i=1}^{m_T}$$

Two Questions in Domain Adaptation

Question 1

Under what conditions can a classifier which performs well on a source data be expected to perform well on the target data?

Question 2

Given a small amount of labeled target data, how should we combine it during training with large amounts of labeled source data to achieve lowest target error at test time?

Quick Answers from the Paper

Answer 1

The authors bound a classifier's target domain error in terms of its **source domain error** and a measure of **divergence between the source & target domain.**

Answer 2

Minimize a **convex combination of the empirical source and target error**. The coefficients depend on the divergence between the domains and the size of source & target data.

Related Work - Theoretical

- Crammer et. al (2008) assume X_1, \dots, X_N follow same distribution but the deterministic labeling functions f_1, \dots, f_N are different. They minimize (uniformly weighted) source error.
- Blitzer et. al (2008) give error bounds for the hypothesis learned by minimizing weighted combination of source errors for the case of empirical risk minimization.
- Mansour et. al (2008) give theoretical analysis when the target is a mixture of source domains.
- Mansour et. al (2009) provide bounds on test error using a new discrepancy distance and provide generalized bounds for regularization based algorithms.

Related Work - Applications

- Deep Transfer Networks - Long et. al (2014), (2015), (2016)

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- Deep Transfer Networks - Long et. al (2014), (2015), (2016)
- [Multi-task Learning](#)

Related Work - Modeling Technology

- Deep Transfer Networks - Long et. al (2014), (2015), (2016)
- Multi-task Learning
- Multiple source adaptation model

Related Work - Modeling Technology

- Deep Transfer Networks - Long et. al (2014), (2015), (2016)
- Multi-task Learning
- Multiple source adaptation model
- Adversarial Learning - Cao et. al (2018)

Model for Domain Adaptation

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$$\epsilon_S(h, f) = \mathbb{E}_{X \sim \mathcal{D}_S} [|h(X) - f(X)|] = \mathbb{P}_{X \sim \mathcal{D}_S} (h(X) \neq f(X))$$

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- Risk of a hypothesis/source error: $\epsilon_S(h) = \epsilon_S(h, f_s)$
- Empirical source error $\hat{\epsilon}_S(h)$.
- Parallel notation for $\epsilon_S(h, f)$, $\epsilon_T(h)$, and $\hat{\epsilon}_T(h)$.

Answer 1

**Establishing bounds on target domain performance
of a classifier trained on source domain**

Some Definitions

The \mathcal{H} -divergence

Given a domain \mathcal{X} with two probability distributions \mathcal{D} and \mathcal{D}' .

Let \mathcal{H} be a hypothesis class on \mathcal{X} , and

$$I(h) = \{x \in \mathcal{X} : h(x) = 1\}.$$

$$d_{\mathcal{H}}(\mathcal{D}, \mathcal{D}') = 2 \sup_{h \in \mathcal{H}} |\Pr_{\mathcal{D}}[I(h)] - \Pr_{\mathcal{D}'}[I(h)]|$$

Some Definitions

Ideal Joint Hypothesis

$$h^* = \arg \min_{h \in \mathcal{H}} [\epsilon_S(h) + \epsilon_T(h)]$$

and

$$\lambda = \epsilon_S(h^*) + \epsilon_T(h^*)$$

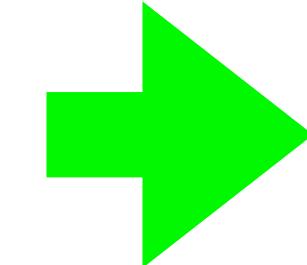
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and

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When this ideal joint hypothesis performs poorly, we cannot expect to learn a good target classifier by minimizing source error.

Some Definitions

Symmetric Difference Hypothesis

For a hypothesis space \mathcal{H} , the symmetric difference hypothesis

$$\mathcal{H} \Delta \mathcal{H} = \{g : g(x) = h(x) \oplus h'(x)\} \text{ for some } h, h' \in \mathcal{H},$$

Where \oplus is the XOR function

Every hypothesis $g \in \mathcal{H} \Delta \mathcal{H}$ is the set of disagreements between two hypotheses in \mathcal{H} .

$$d_{\mathcal{H} \Delta \mathcal{H}}(\mathcal{D}, \mathcal{D}') = 2 \sup_{h, h' \in \mathcal{H}} |\Pr_{X \sim \mathcal{D}_S}[h(X) \neq h'(X)] - \Pr_{X \sim \mathcal{D}_T}[h(X) \neq h'(X)]|$$

Main Result

Theorem 2 *Let \mathcal{H} be a hypothesis space of VC dimension d . If $\mathcal{U}_S, \mathcal{U}_T$ are unlabeled samples of size m' each, drawn from \mathcal{D}_S and \mathcal{D}_T respectively, then for any $\delta \in (0, 1)$, with probability at least $1 - \delta$ (over the choice of the samples), for every $h \in \mathcal{H}$:*

$$\epsilon_T(h) \leq \epsilon_S(h) + \frac{1}{2} \hat{d}_{\mathcal{H}\Delta\mathcal{H}}(\mathcal{U}_S, \mathcal{U}_T) + 4\sqrt{\frac{2d \log(2m') + \log(\frac{2}{\delta})}{m'}} + \lambda.$$

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When λ is small, domain adaptation is relevant => source error and unlabeled $\mathcal{H}\Delta\mathcal{H}$ -divergence are important for bounding target error.

Answer 2

A learning bound combining source and target data

Setup

- Sample $S = (S_S, S_T)$ of m instances.
- S_T consists of βm i.i.d. samples from \mathcal{D}_T .
- S_S consists of $(1 - \beta)m$ i.i.d. samples from \mathcal{D}_S .
- Goal: find a hypothesis h that minimizes $\epsilon_T(h)$.
- When β is small, minimizing empirical target error is not feasible.
- Consider minimizing: $\hat{\epsilon}_\alpha(h) := \alpha\hat{\epsilon}_T(h) + (1 - \alpha)\hat{\epsilon}_S(h)$

Main Result

Theorem 3 Let \mathcal{H} be a hypothesis space of VC dimension d . Let \mathcal{U}_S and \mathcal{U}_T be unlabeled samples of size m' each, drawn from \mathcal{D}_S and \mathcal{D}_T respectively. Let S be a labeled sample of size m generated by drawing βm points from \mathcal{D}_T and $(1 - \beta)m$ points from \mathcal{D}_S and labeling them according to f_S and f_T , respectively. If $\hat{h} \in \mathcal{H}$ is the empirical minimizer of $\hat{\epsilon}_\alpha(h)$ on S and $h_T^* = \min_{h \in \mathcal{H}} \epsilon_T(h)$ is the target error minimizer, then for any $\delta \in (0, 1)$, with probability at least $1 - \delta$ (over the choice of the samples),

$$\begin{aligned} \epsilon_T(\hat{h}) &\leq \epsilon_T(h_T^*) + 4 \sqrt{\frac{\alpha^2}{\beta} + \frac{(1 - \alpha)^2}{1 - \beta}} \sqrt{\frac{2d \log(2(m + 1)) + 2 \log(\frac{8}{\delta})}{m}} \\ &\quad + 2(1 - \alpha) \left(\frac{1}{2} \hat{d}_{\mathcal{H} \Delta \mathcal{H}}(\mathcal{U}_S, \mathcal{U}_T) + 4 \sqrt{\frac{2d \log(2m') + \log(\frac{8}{\delta})}{m'}} + \lambda \right). \end{aligned}$$

Observations

- When $\alpha = 0$ (ignore target data) and $\alpha = 1$ (ignore source data) the bound coincides with known bounds on target error.
- Choosing $\alpha \in (0,1)$ optimally allows us to tradeoff “small” amounts of “good” vs “large” amounts of “less relevant” source data.

Optimal Mixing

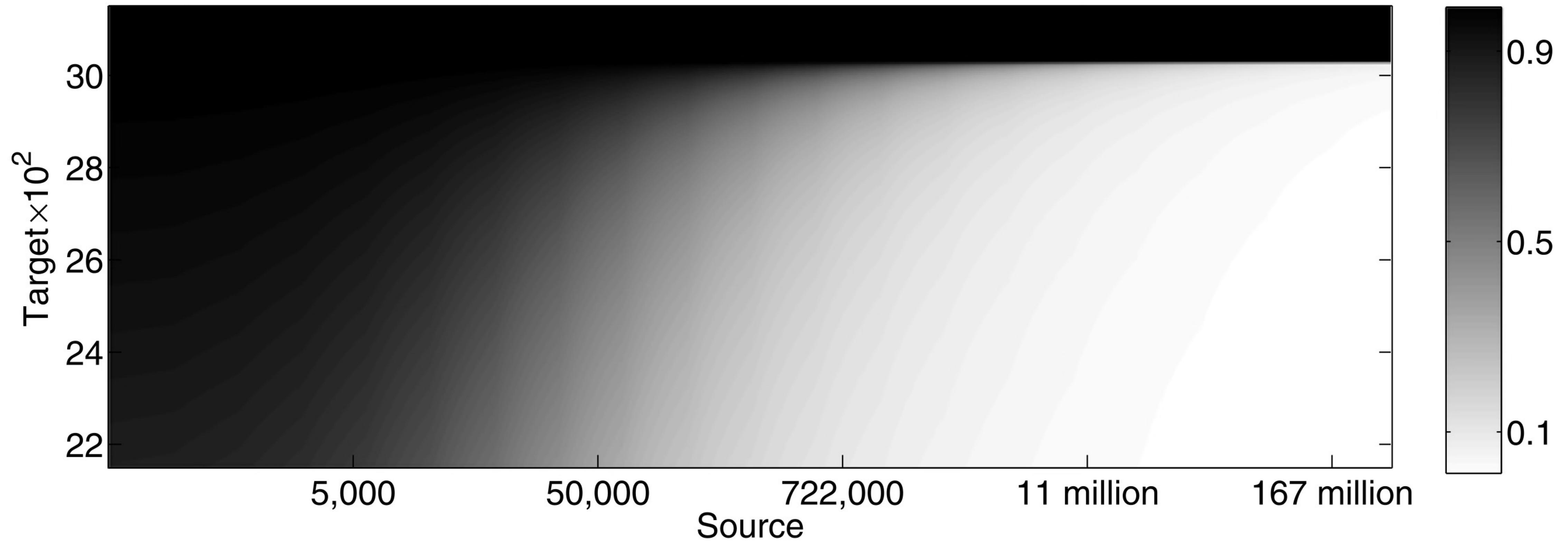
$$\alpha^*(m_T, m_S; D) = \begin{cases} 1 & m_T \geq D^2 \\ \min\{1, \nu\} & m_T \leq D^2, \end{cases}$$

$$\nu = \frac{m_T}{m_T + m_S} \left(1 + \frac{m_S}{\sqrt{D^2(m_S + m_T) - m_S m_T}} \right).$$

$D = \sqrt{d}/A$ where

$$A = \left(\frac{1}{2} \hat{d}_{\mathcal{H}\Delta\mathcal{H}}(\mathcal{U}_S, \mathcal{U}_T) + 4 \sqrt{\frac{2d \log(2m') + \log(\frac{4}{\delta})}{m'}} + \lambda \right)$$

Optimal Mixing Illustration



**Thank you!
Question?**

Bibliography

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Bonus

Combining Data from Multiple Sources

Combining Data from Multiple Sources

- Source data comes from N distinct sources.
- Each source S_j has distribution \mathcal{D}_j over inputs and labeling function f_j .
- Out of total m source samples, $\beta_j m$ are from source S_j .
- Minimizing convex combination of training error from different source using domain weights $\alpha = (\alpha_1, \dots, \alpha_N)$,

$$\hat{\epsilon}_{\alpha}(h) = \sum_{j=1}^N \alpha_j \hat{\epsilon}_j(h) = \sum_{j=1}^N \frac{\alpha_j}{\beta_j m} \sum_{x \in S_j} |h(x) - f_j(x)|$$

A bound using pairwise divergence

for any $\delta \in (0, 1)$, with probability at least $1 - \delta$,

$$\begin{aligned}\epsilon_T(\hat{h}) &\leq \epsilon_T(h_T^*) + 2 \sqrt{\left(\sum_{j=1}^N \frac{\alpha_j^2}{\beta_j} \right) \left(\frac{d \log(2m) - \log(\delta)}{2m} \right)} \\ &\quad + \sum_{j=1}^N \alpha_j (2\lambda_j + d_{\mathcal{H}\Delta\mathcal{H}}(D_j, D_T)),\end{aligned}$$

where $\lambda_j = \min_{h \in \mathcal{H}} \{\epsilon_T(h) + \epsilon_j(h)\}$.

A bound using combined divergence

for any $\delta \in (0, 1)$, with probability at least $1 - \delta$,

$$\epsilon_T(\hat{h}) \leq \epsilon_T(h_T^*) + 4 \sqrt{\left(\sum_{j=1}^N \frac{\alpha_j^2}{\beta_j} \right) \left(\frac{d \log(2m) - \log(\delta)}{2m} \right)} + 2\gamma_\alpha + d_{\mathcal{H}\Delta\mathcal{H}}(D_\alpha, D_T),$$

where $\gamma_\alpha = \min_h \{\epsilon_T(h) + \epsilon_\alpha(h)\} = \min_h \{\epsilon_T(h) + \sum_{j=1}^N \alpha_j \epsilon_j(h)\}$.