UNIT- 1

INFORMATION THEORY.

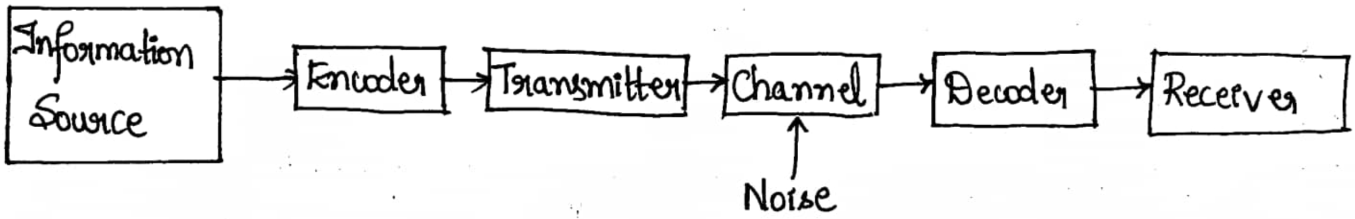
Mrs. Noor Afza
Asst professor
ECE dept
CIT gubbi

Information in simplest Terms is the intelligence or ideas conceived in one's mind. or

The meaning of word "Information" in information theory is message or intelligence. The message may be electrical such as voltage, current or power / speech message / picture message / Television / Music Message. A Source which produces these message is Called Information Source.

In Communication Systems, information is transmitted from Source to destination. Communication is a process of establishing Connection between 2 points, for information exchange.

For Communication to take place, 3 essential things must be present. They are transmitter, Receiver and Communication Channel. The Block diagram of an Information System is as below.



Information Sources can be classified into 2 categories
<1> Analog Information Source (Continuous Valued)
<2> Discrete Information Source.

Analog Information Sources such as microphone actuated by a voice signal emit a continuous amplitude, continuous time electrical waveform. The output of a discrete information source such as a Computer Consists of a Sequence of letters or symbols.

Analog information Sources can be transformed into discrete information sources through the process of Sampling and Quantising.

<u>Measure of Information</u> :- Information Content of a Message.

Some messages produced by the information Source contain more information than other messages. the amount of information received is different for different messages. the messages associated with an event least likely to occur contains most information.

• Consider an information Source emitting independent sequence of Symbols from $S=\{s_1, s_2 \ldots s_q\}$ with probability of Occurance $P=\{P_1, P_2, P_3 \ldots, P_q\}$ respectively with $P_1+P_2+\ldots+P_q=1$.

• Let S_K be the Symbol chosen for transmission at any instant of time with a probability equal to P_K.

Then the "Amount of Information" or "Self-Information" of message S_K is given by

$$I_K \propto \frac{1}{P_K} \quad \text{or} \quad I_K = \log_2 \frac{1}{P_K}$$

<u>To Calculate the Units of Information, we have.</u>

1> $I = \log_2 (1/P)$ ..... Bits

2> $I = \log_3 (1/P)$ .... kinits / ternary units

3> $I = \log_4 (1/P)$ .... Quaternary units / quadits.

4> $I = \log_e (1/P)$ .... Neper units / Nats

5> $I = \log_{10} (1/P)$ .... decits or Hartley.

Note :- $\log_2 x = \frac{\log_{10} x}{\log_{10} 2}$

The use of logarithmic function in a information

"Consider a information source emitting one of the q possible messages $m_1, m_2 \dots m_q$ with the probabilities of occurance $P_1, P_2 \dots P_q$ such that $P_1 + P_2 + \dots + P_q = 1$

The self-information or the amount of information in the K^{th} messaged denoted by $I(M_K)$

(i) must be inversly related to P_K.

$$I(m_K) > I(m_j) \text{ if } P_K < P_j$$

(ii) must approach zero as P_K approaches 1.

$$I(m_K) \longrightarrow 0(m_j) \text{ if } P_K \rightarrow 1$$

(iii) must be non-negative since each message contains some information & at the worst $I(m_K) = 0$

$$I(m_K) \geq 0 \quad \text{when} \quad 0 \leq P_K \leq 1$$

(iv) $I(M_K \text{ and } M_j) = I(M_K) + I(M_j) = I(M_K M_j)$

where M_K and M_j are 2 independent messages

A continuous function of P_K that satisfies all the above 4 conditions is a logarithmic function & we can define the measure of information as $I(M_K) = \log_2 (1/P_K)$

Problems.

1. The binary symbols '0' & '1' are transmitted with probabilities $1/4$ & $3/4$ respectively. find the corresponding Self-Information.

Soln Self Information in a '0' $= I_0 = \log_2 \frac{1}{P_0} = \log_2 \frac{1}{1/4} = 2$ bits.

Self Information in a '1' $= I_1 = \log_2 \frac{1}{P_1} = \log_2 \frac{1}{4/3} = 0.415$ bits.

2. Prove that the Independent Symbols are transmitted, the total Self information must be equal to the sum of individual self-inforn.

Soln To prove this statement, let us take 2 independent symbol I_1 and I_2 are transmitted with probabilities P_1 and P_2 respectively

$$I_1 = \log_2 (1/P_1) \quad \& \quad I_2 = \log_2 (1/P_2)$$

Total information is addition of 2 information,.
Then the total information obtained when both statements are made in additive i.e $I_1 + I_2$

$$I = \log \frac{1}{P(P_1 \text{ and } P_2)} = \log \frac{1}{P(P_1 \cap P_2)}$$

$$= \log \frac{1}{P(P_1)\, P(P_2)} = \log \frac{1}{P_1 . P_2}$$

$$I = \log \left(\frac{1}{P_1}\right) + \log \left(\frac{1}{P_2}\right)$$

$$\boxed{I = I_1 + I_2}$$

3. A Source puts out one of Five possible messages during each message interval. The probability of these messages are

$P_1 = 1/2$, $P_2 = 1/4$, $P_3 = 1/8$, $P_4 = 1/16$, $P_5 = 1/16$

Soln Self Information of each message.

$I(m_1) = \log_2 1/P_1 = \log_2 \frac{1}{1/2} = \log_2 2 = 1$ bit

$I(m_2) = \log_2 \frac{1}{P_2} = \log_2 \frac{1}{1/4} = \log_2 4 = 2$ bit

$I(m_3) = \log_2 \frac{1}{P_3} = \log_2 \frac{1}{1/8} = \log_2 8 = 3$ bit

$I(m_4) = \log_2 \frac{1}{P_4} = \log_2 \frac{1}{1/16} = \log_2 16 = 4$ bit

$I(m_5) = \log_2 \frac{1}{P_5} = \log_2 \frac{1}{1/16} = \log_2 16 = 4$ bit

4. Consider a source $S=\{S_1, S_2, S_3\}$ with $P=\{1/2, 1/4, 1/4\}$ find Self-information of each message.

→ To find Self-information $I_K = \log_2 1/P_K$

$S1 \Rightarrow I_1 = \log_2 \frac{1}{1/2} = \log_2 2 = 1$ bit.

$S2 \Rightarrow I_2 = \log_2 1/1/4 = \log_2 4 = 2$ bit.

$S3 \Rightarrow I_3 = \log_2 1/1/4 = \log_2 4 = 2$ bit.

5. A card is selected at random from a deck of playing cards. if you have been told that it is red in colour. how much information you have received. How much more information you needed to completely specify the card.

→ Total number of Cards = 52

Total number of cards in red = 26

Probability of getting a red card $= \frac{26}{52} = \frac{1}{2}$

∴ Amount of Information Contained $I = \log_2 \frac{1}{P} = \log_2 \frac{1}{1/2}$

$I = \log_2 2 = 1$ Bit.

$I = 1$ bit

for 1 unique card $= 1/52$ $\quad I = \log_2 52 = 5.7$ bits.

More needed ⇒ $5.7 - 1$ bit $= 4.7$ bits.

6. A single TV picture may be thought of as an array of black, white and grey dots with 500 rows & 600 columns. Suppose that each of these dots may take an any one of 10 distinguishable levels. What is the amount of information provided by 1 picture.

→ Total number of dots in the picture $= 500 \times 600 = 3 \times 10^5$

Total no of picture possible $= (10 \times 10 \times \ldots 10) = 10^{3 \times 10^5}$

Probability of each picture $= 1/10^{3 \times 10^5}$

Amount of Information in each picture $I = \log_2 10^{3 \times 10^5}$

$I = 996.578$ K bits

Average Information Content (Entropy) of Symbols in Long Independent Sequences.

A Source that emits one of 'Q' possible symbols $S_1, S_2 \ldots S_q$ in a statistically independent sequence. i.e the probability of Occurance of a particular symbol during a symbol time interval doesnot depend on the symbols emitted by the source during the preceeding symbol interval.

Average Information content gives an idea about the symbols emitted by the source. Average information is denoted by 'H'.

Let us consider a zero-memory source producing independent Sequence of symbols. The information source emitting a long sequence of 'q' symbols $(S_1 \ldots\ldots S_q)$ with probabilities $(P_1, P_2 \ldots P_q)$ then the self information of each of these symbols would be $(I_1, I_2 \ldots I_q)$

$$I_1 = \log_2 (1/P_1) \text{ bits}$$

$$I_2 = \log_2 (1/P_2) \text{ bits}$$

$$\vdots \qquad \vdots$$

$$I_q = \log_2 (1/P_q) \text{ bits}$$

Now in a long message containing N symbols, the symbol S_1 will occur on the average $P_1 N$ times,

Similarly S_2 will occur $P_2 N$ times

S_3 will occur $P_3 N$ times

$\vdots$

In general S_q will occur $P_q N$ times.

But we Known the Self information as

$$I_K = \log_2 \frac{1}{P_i} \text{ bits}.$$

$\therefore$ P_1N number of message of type $S1$ contain $P_1 N \log_2 1/P_1$ bits of Inforn.

P_2N number of message of type $S2$ contain $P_2N \log_2 \frac{1}{P_2}$ bits of Inforn.

$\vdots$

P_qN number of message of type S_q contain $P_q N \log_2 \frac{1}{P_q}$ bits of Inforn.

$\therefore$ The total Self-information content of all these message symbols is given by,

$$I_{total} = P_1 N \log_2 \frac{1}{P_1} + P_2 N \log_2 \frac{1}{P_2} + \dots \dots P_q N \log_2 \frac{1}{P_q}$$

$$I_{total} = N \sum_{i=1}^{q} P_i \log_2 \frac{1}{P_i} \text{ bits}$$

$$\text{Average Self-Information} = \frac{I_{total}}{N} \Rightarrow (\text{'H'})$$

Average Information per symbol is obtained by dividing total information content of message by the number of symbols.

$$H = \sum_{i=1}^{q} P_i \log_2 1/P_i \text{ bits/symbol}.$$

Average Self-Information is also called "Entropy" of source 'S' denoted by $H(s)$

$$\therefore H(s) = \sum_{i=1}^{q} P_i \log_2 1/P_i \text{ bits/symbol}.$$

Symbol Rate : The number of Symbols emitted from the source per second is Known as symbol rate denoted as 'r_s'
unit : symbols/sec.

Information Rate : The average source information rate "R_s" in bits/sec is defined as the product of the average information content per symbol & the message symbol rate r_s

$$\therefore R_s = r_s H(s) \text{ bits/sec or BPS}.$$

Note : Log Properties.

(1) $\log_a b = \dfrac{1}{\log_b a}$

(2) $\dfrac{\log_x b}{\log_x a} = \log_a b$

(3) $\log_e 10 = \ln$

Relationship Between Hartleys, Nats and Bits.

From definition we have

$$\text{Hartleys} \Rightarrow I = \log_{10}\frac{1}{P} \text{ Hartleys}$$
$$\text{Nats} \Rightarrow I = \log_e\frac{1}{P} \text{ Nats}$$
$$\text{Bits} \Rightarrow I = \log_2\frac{1}{P} \text{ bits.}$$

<1> Hartley $\longleftrightarrow$ Bits

$$1 \text{ Hartley} = \frac{I}{\log_{10}(1/P)} \rightarrow ①$$

But $I = \log_2(1/P)$ bits $\rightarrow$ ⓐ

Substitute a in ①

$$1 \text{ Hartley} = \frac{\log_2(1/P)}{\log_{10}(1/P)} \text{ bits}$$

$$1 \text{ Hartley} = \frac{-\log_2 P}{-\log_{10} P} \Rightarrow \frac{\log_2 P}{\log_{10} P}$$

$$1 \text{ Hartley} = \frac{\log_P 10}{\log_P 2} = \log_2 10 \text{ bits}$$

1 Hartley = 3.32 bits

⟨2⟩ Hartley ⟷ Nats

$$I = \log_{10}(1/P) \text{ Hartley}$$

$$1 \text{ Hartley} = \frac{I}{\log_{10} 1/P} \rightarrow ①$$

Nats ⇒ $I = \log_e 1/P$ Nats → ⓐ

Substitute ⓐ in ①

$$1 \text{ Hartley} = \frac{\log_e 1/P}{\log_{10} 1/P} \Rightarrow \frac{-\log_e P}{-\log_{10} P}$$

$$= \frac{1/\log_P e}{1/\log_P 10} = \frac{\log_P 10}{\log_P e}$$

$$1 \text{ Hartley} = \log_e 10 \Rightarrow \ln 10$$

$$\boxed{1 \text{ Hartley} = 2.302 \text{ Nats.}}$$

⟨3⟩ Nats ⟷ Bits

$$I = \log_e(1/P) \text{ Nats}$$

$$1 \text{ Nat} = \frac{I}{\log_e(1/P)} \longrightarrow ①$$

Bits ⇒ $I = \log_2 1/P$ bits → ⓐ

Substitute ⓐ in ①

$$1 \text{ Nat} = \frac{\log_2 1/P}{\log_e 1/P} = \frac{\cancel{-}\log_2 P}{\cancel{-}\log_e P} = \frac{1/\log_P 2}{1/\log_P e}$$

$$= \frac{\log_P e}{\log_P 2} = \log_2 e$$

$$\boxed{1 \text{ Nat} = 1.443 \text{ bits}}$$

Problems.

<1> The collector voltage of a certain circuit is to lie between -5 and -12 volts. The voltage can take on only these values $-5, -6, -7, -9, -11, -12$ volts with respective probabilities $1/6, 1/3, 1/12, 1/12, 1/6, 1/6$. This voltage is recorded with a pen recorder. Determine the average self-information associated with the record in terms of bits/level.

→ To calculate average self-information

$$H(S) = \sum_{i=1}^{6} P_i \log_2 \frac{1}{P_i} \text{ bits/level}$$

$$H(S) = \frac{1}{6}\log_2 6 + \frac{1}{3}\log_2 3 + \frac{1}{12}\log_2 12 + \frac{1}{12}\log_2 12 + \frac{1}{6}\log_2 6 + \frac{1}{6}\log_2 6.$$

$H(S) = 2.418$ bits/level.

<2> A discrete source emits one of six symbols once every m-sec. The symbol probabilities are $1/2, 1/4, 1/8, 1/16, 1/32$ & $1/32$ respectively. Find the source entropy and information rate.

→ To calculate source entropy

$$H(S) = \sum_{i=1}^{6} P_i \log_2 1/P_i$$

$$H(S) = 1/2\log_2 2 + \frac{1}{4}\log_2 4 + \frac{1}{8}\log_2 8 + \frac{1}{16}\log_2 16 + \frac{1}{32}\log_2 32 \times 2$$

$H(S) = 1.9375$ bits/message-symbol.

Information Rate R_s is given by $R_s = r_s H(S)$.

given $r_s = 1$ message symbol/m-sec

$= 10^3$ message symbol/sec.

$R_s = (10^3$ message symbols/sec$)(1.9375$ bits/msg symbol$)$

$R_s = 1937.5$ bits/sec.

<3> the output of an information source consists of 150 symbols, 32 of which occur with a probability of $1/64$ and the remaining 118 occur with a probability of $1/236$. the source emits 2000 symbols/sec. Assuming that the symbols are chosen independently, find the average information of this source.

→ The entropy is given by

$$H(S) = \sum_{i=1}^{150} P_i \log_2 1/P_i$$

$$= \sum_{i=1}^{32} P_i \log \frac{1}{P_i} + \sum_{i=33}^{150} P_i \log \frac{1}{P_i}$$

$$= \left[1/64 \log 64\right] \times 32 + \left[\frac{1}{236} \log 236\right] \times 118$$

$$H(S) = 6.9413 \text{ bits/message symbol.}$$

Given $r_s = 2000$ message symbols/sec

$\therefore$ Average information rate $= R_s = r_s H(S)$

$$= 2000 \times 6.9413$$

$$R_s = 13,882.6 \text{ bits/sec.}$$

<4> A code is composed of dots and dashes. Assuming that a dash is 3 times as long as a dot & has one-third the probability of occurence. calculate

(i) the information in a dot and a dash.

(ii) The entropy of dot-dash code.

(iii) the average rate of information if a dot lasts for 10m-sec and this time is allowed between symbols.

→ Since only dots and dashes are present, we must have

$$P_{dot} + P_{dash} = 1 \rightarrow (1)$$

Given $P_{dash} = 1/3 P_{dot}$

$\therefore$ substituting in (1), $P_{dot} + P_{dash} = 1$

$$P_{dot} + 1/3 P_{dot} = 1$$

$$\therefore P_{dash} = 1/4.$$

(i) Information in a dot = $I_{dot} = \log \frac{1}{P_{dot}} = \log \frac{4}{3} = 0.415$ bits.

Information in a dash = $I_{dash} = \log \frac{1}{P_{dash}} = \log 4 = 2$ bits.

(ii) the entropy of dot dash code is

$$H(S) = P_{dot} \log \frac{1}{P_{dot}} + P_{dash} \log \frac{1}{P_{dash}}$$

$$= 3/4 \log 4/3 + 1/4 \log 4$$

$$H(S) = 0.8113 \text{ bits/message symbol.}$$

(iii) Since $P_{dot} = 3/4$ and $P_{dash} = 1/4$, for every 4 symbols sent on an average, there will be 3 dots and 1 dash. With a dot lasting for 10 m-sec, a dash for 30 m-sec and with 10 m-sec gap in between two successive symbols, a total of 100 m-sec is required to transmit 4 symbols.

$$\therefore \text{Symbol rate} = r_s = 4 \text{ symbols/100 m-sec}$$

$$= 40 \text{ symbols/sec.}$$

$$\therefore \text{Information rate} = R_s = r_s H(S)$$

$$= (40 \text{ symbols/sec})(0.8113 \text{ bits/symbol})$$

$$R_s = 32.452 \text{ bits/sec.}$$

<5> A card is drawn from a deck.

(i) You are told it is a spade. How much information did you receive?

(ii) How much information did you receive if you are told that the card drawn is an ace?

(iii) If you are told that the card drawn is an ace of spades, how much information did you receive.

(iv) Is the information obtained in (iii) the sum of informations obtained in (i) and (ii)?

→ In a deck of cards we have

13 Spade
13 Heart
13 diamond
13 clubs.

Sol^n the Self information I_K is given by $I_K = \log_2 1/P_K$ bits

when $K=0$, $I_0 = \log_2 1/P_0 = \log_2 1/0.4 = 1.322$ bits

$K=1$, $I_1 = \log_2 \frac{1}{P_1} = \log_2 \frac{1}{0.3} = 1.737$ bits

$K=2$, $I_2 = \log_2 \frac{1}{P_2} = \log_2 \frac{1}{0.2} = 2.322$ bits

$K=3$, $I_3 = \log_2 \frac{1}{P_3} = \log_2 \frac{1}{0.1} = 3.322$ bits.

The entropy of the source is given by

$$H(X) = \sum_{K=0}^{3} P_K \log_2 1/P_K \text{ bits/message symbol.}$$

$$= \sum_{K=0}^{3} P_K I_K$$

$$= P_0 I_0 + P_1 I_1 + P_2 I_2 + P_3 I_3$$

$$= (0.4)(1.322) + 0.3 \times 1.737 + 0.2 \times 2.322 + 0.1 \times 3.322$$

$$H(X) = 1.8465 \text{ bits/message symbol.}$$

<7> Find the entropy of a source in nats/symbol of a source that emits one out of four symbols A, B, C and D in a statistically independent sequence with probabilities 1/2, 1/4, 1/8 & 1/8

Sol^n To find the entropy of a source in nats/symbol

$$H(S) = \sum_{i=1}^{D} P \log_e \frac{1}{P}$$

$$= 1/2 \log_e 2 + 1/4 \log_e 4 + 1/4 \log_e 8 \times 2$$

$$H(S) = 1.213 \text{ nats/symbol.}$$

<8> A pair of dice are tossed simultaneously. The outcome of the first dice is recorded as X_1 and that of second dice as X_2. Two events are defined as follows.

$$A = \{(X_1, X_2) \text{ such that } X_1 + X_2 \leq 7\}$$

$$B = \{(X_1, X_2) \text{ such that } X_1 > X_2\}$$

which event conveys more information ?

Solⁿ when a pair of dice are tossed simultaneously, then the sample space S consists of 36 combinations of (x_1, x_2) given by

$$S = \begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{bmatrix}$$

Each pair in the sample occur with a probability $= (1/6)(1/6) = 1/36$

∴ all the pairs are equiprobable.

The event A contains the pairs given by

$$A = \{(1,1)\ (1,2)(1,3)(1,4)(1,5)\ (1,6)\ (2,1)\ (2,2)\ (2,3)(2,4)(2,5)\ (3,1)\ (3,2)(3,3)(3,4)(4,1)\ (4,2)(4,3)\ (5,1)\ (5,2)\ (6,1)\}$$

And the event B contains the pairs

$$B = \{(2,1)(3,1)(3,2)(4,1)(4,2)(4,3)\ (5,1)\ (5,2)\ (5,3)(5,4)(6,1)\ (6,2)(6,3)\ (6,4)\ (6,5)\}$$

Since all the pairs are equiprobable.

$$P(A) = \frac{21}{36} = \frac{7}{12} \quad \text{and} \quad P(B) = \frac{15}{36} = \frac{5}{12}$$

∴ self information of event A $= I_A = \log \frac{1}{P(A)} = \log \frac{12}{7} = 0.776$ bits

∴ Self information of event B $= I_B = \log \frac{1}{P(B)} = \log \frac{12}{5} = 1.263$ bits

∴ $I_B > I_A$

The event B conveys more information than event A.

<9> An analog signal band limited to 6 KHz is sampled at thrice the Nyquist rate and then quantized into 11 levels $Q_1, Q_2, \ldots, Q_{11}$. of these, 3 levels occur with probability of 1/6 each, four levels with probability 1/12 each and the remaining four levels with probability 1/24 each. find the rate of information associated with the analog signal

Solⁿ Sampling rate r_s = 3 × Nyquist rate $= 3 \times 2B$
$= 3 \times 2 \times 6K$
$= 36$ K-sample/sec.

the entropy of the analog signal is given by

$$H(S) = \sum_{i=1}^{11} P_i \log_2 \frac{1}{P_i} \text{ bits/m-symbol or Sample.}$$

$$H(S) = 3 \times \frac{1}{6} \log 6 + 4 \times \frac{1}{12} \log 12 + 4 \times \frac{1}{24} \log 24$$

$$H(S) = 3.252 \text{ bits/sample}$$

$\therefore$ Rate of information $= r_s \cdot H(S)$

$$= (36 \text{ K-samples/sec}) \cdot (3.252 \text{ bits/sample})$$

$$R_s = 117.072 \text{ KBPS}$$

<10> A pair of dice are tossed simultaneously. The outcome of first dice is recorded as x_1 and that of second dice as x_2. Three events are defined as follows.

$A = \{(x_1, x_2)$ such that $(x_1 + x_2)$ is divisable exactly by $4\}$

$B = \{(x_1, x_2)$ such that $6 \leq (x_1 + x_2) \leq 8\}$

$C = \{(x_1, x_2)$ such that $x_1 x_2$ is divisable exactly by $3\}$

Which events conveys maximum information?

Soln The Sample Space S has thirty six combinations (refer Soln 8, page 8)

The event A contains the pairs given by

$A = \{(1,3), (2,2), (2,6), (3,1), (3,5), (4,4), (5,3), (6,2) (6,6)\}$

$$\therefore P(A) = \frac{9}{36} = 0.25$$

the event-B contains.

$B = \{(1,5), (1,6), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5) (4,2), (4,3)$
$(4,4) (5,1) (5,2) (5,3) (6,1) (6,2)\}$

$$P(B) = \frac{16}{36} = 0.444$$

the event-C contains.

$C = \{(1,3), (1,6), (2,3), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,3),$
$(4,6), (5,3), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$.

$$P(C) = \frac{20}{36} = 0.556$$

$\therefore$ Self Information of $A = I_A = \log \frac{1}{0.25} = 2$ bits.

Self Information of $B = I_B = \log \frac{1}{0.444} = 1.17$ bits.

Self Information of $C = I_C = \log \frac{1}{0.555} = 0.848$ bits.

<11> A voice signal in a PCM system is sampled at 2.5 times the Nyquist rate and is quantized into 16 levels with the following probabilities.

$$P_1 = P_2 = P_3 = P_4 = 0.08$$
$$P_5 = P_6 = P_7 = P_8 = 0.065$$
$$P_9 = P_{10} = P_{11} = P_{12} = 0.055$$
$$P_{13} = P_{14} = P_{15} = P_{16} = 0.05$$

calculate the entropy and information rate of the PCM signal if the bandwidth of signal is 3.5 KHz.

Solⁿ

Entropy $H(s) = \sum_{i=1}^{16} P_i \log \frac{1}{P_i}$

$$H(s) = 4 \times 0.08 \log \frac{1}{0.08} + 0.065 \log \frac{1}{0.065} * 4 + 4 \times 0.055 \log \frac{1}{0.055} + 4 \times 0.05 \log \frac{1}{0.05}$$

$H(s) = 3.9763$ bits/level or sample.

Sampling rate $= (2.5)$ (Nyquist rate)
$= (2.5)(2B)$
$= (2.5)(2)(3.5k)$
$= 17.5$ K-Sample/sec.

Information rate $R_s = r_s \cdot H(s)$
$= (17.5$ K-Samples$)$ $(3.9763$ bits/sample$)$

$R_s = 69.585$ KBPs.

<12> An analog signal is band limited to 500 Hz and is sampled at "Nyquist Rate". the Samples are quantized into 4 levels. the quantization levels are assumed to be independent and occur with probabilities $P_1 = P_4 = 1/8$, $P_2 = P_3 = 3/8$. find information rate of the source.

Sol^n $$H(S) = \sum_{i=1}^{4} P_i \log \frac{1}{P_i} = \left(\frac{1}{8}\log 8\right)(2) + \left(\frac{3}{8}\log\frac{8}{3}\right)(2)$$
$$= 1.8113 \text{ bits/level or symbol}$$

Since the signal is sampled at Nyquist rate, the symbol rate r_s is given by

$$r_s = 2B = 2 \times 500 = 1000 \text{ symbols/sec}$$

$\therefore$ Information rate $R_s = r_s H(s)$
$$= (1000)(1.81)$$
$$R_s = 1810 \text{ bits/sec}$$

<13> A binary source is emitting an independent sequence of 0's & 1's with probabilities p and 1-p respectively. Plot the entropy of the source versus p.

Sol^n The entropy of the binary source is given by

$$H(S) = \sum_{i=1}^{2} P_i \log_2 \frac{1}{P_i}$$
$$= P_1 \log \frac{1}{P_1} + P_2 \log \frac{1}{P_2}$$
$$H(S) = p \log \frac{1}{p} + (1-p) \log \frac{1}{(1-p)}$$

Let $p = 0.1$, $H(S) = 0.1 \log \frac{1}{0.1} + 0.9 \log \frac{1}{0.9} = 0.469$ bits/symbol.

Let $p = 0.2$, $H(S) = 0.2 \log \frac{1}{0.2} + 0.8 \log \frac{1}{0.8} = 0.722$ bits/symbol.

Let $p = 0.3$, $H(S) = 0.3 \log \frac{1}{0.3} + 0.7 \log \frac{1}{0.7} = 0.881$ bits/symbol.

Let $p = 0.4$, $H(S) = 0.4 \log \frac{1}{0.4} + 0.6 \log \frac{1}{0.6} = 0.971$ bits/symbol.

Let $p = 0.5$, $H(S) = 0.5 \log \frac{1}{0.5} + 0.5 \log \frac{1}{0.5} = 1$ bits/symbol.

Let $p = 0.6$, $H(S) = 0.6 \log \frac{1}{0.6} + 0.4 \log \frac{1}{0.4} = 0.971$ bits/symbol.

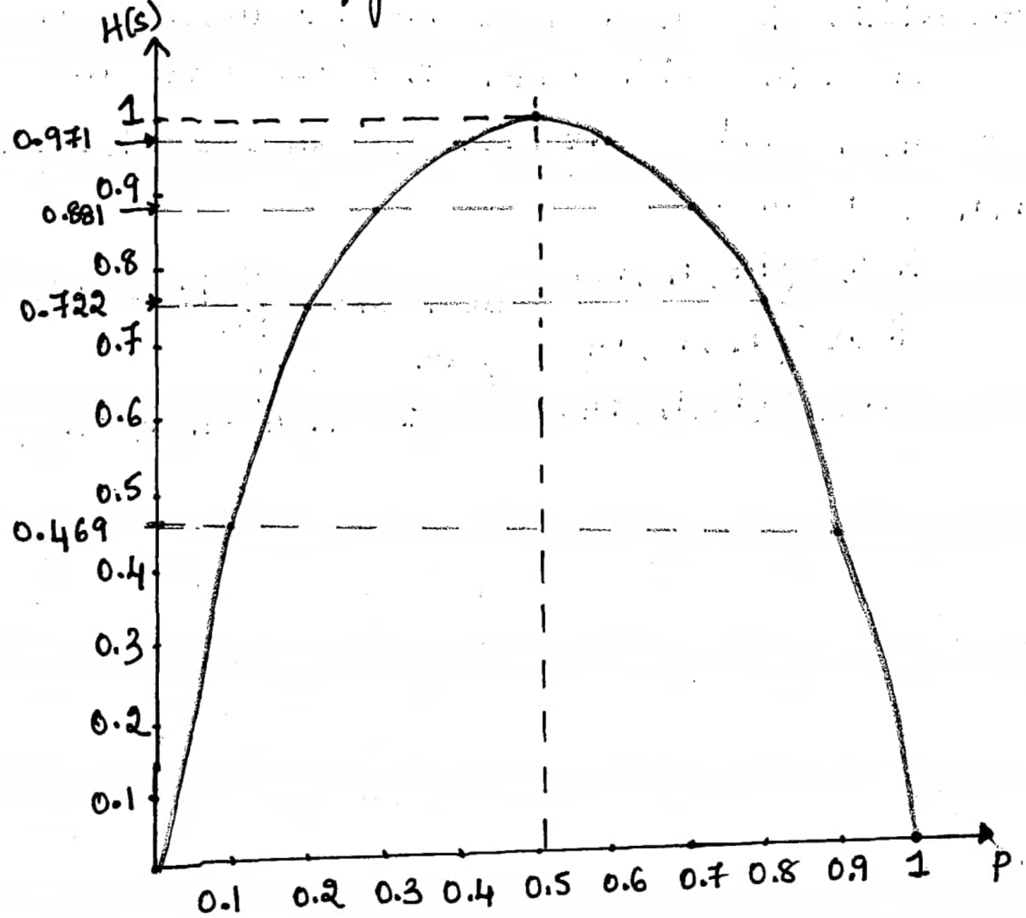
Let $p = 0.7$, $H(S) = 0.7 \log \frac{1}{0.7} + 0.3 \log \frac{1}{0.3} = 0.881$ bits/symbol.

Let $p=0.8$, $H(s) = 0.8 \log \frac{1}{0.8} + 0.2 \log \frac{1}{0.2} = 0.722$ bits/symbol

Let $p=0.9$, $H(s) = 0.9 \log \frac{1}{0.9} + 0.1 \log \frac{1}{0.1} = 0.469$ bits/symbol.

Let $p=1$, $H(s)=0$

The entropy $H(s)$ can now be plotted as a function of P as shown in the below figure.



Plot of 'H(s)' versus 'P'

Properties of Entropy

The entropy function is given by equation ① for a source alphabet $S = \{S_1, S_2 \ldots\ldots S_q\}$ with $P = \{P_1, P_2 \ldots\ldots P_q\}$ where q = number of source symbols, as

$$H(s) = \sum_{i=1}^{q} P_i \log_2 \frac{1}{P_i} \text{ bits/message symbol} \longrightarrow ①$$

The properties are listed:

① The entropy function is a continuous for every independent variable P_K in the interval $(0,1)$

i.e if P_K varies continuously from 0 to 1 (Entropy function Vanishes at both $P_K=0$ & $P_K=1$).

(2) Entropy is a symmetric function of its arguments.

i.e $H[P_K,(1-P_K)] = H[(1-P_K), P_K]$ for all $K=1,2,\ldots q$

the value of $H(s)$ remains the same irrespective of the locations of the probabilities, i.e as long as the probabilities are same, it doesnot matter in which order they are arranged.

ex:- If S_A, S_B and S_C represent 3 symbols with probabilities P_A, P_B & P_C then.

$$\left.\begin{array}{l} H(A)=\{P_A, P_B, P_C\} \\ H(B)=\{P_B, P_C, P_A\} \\ H(C)=\{P_C, P_B, P_A\} \end{array}\right]$$ such that $\sum_{i=1}^{3} P_i = 1$ will all have the same entropy.

i.e $H(A) = H(B) = H(C)$.

(3) Extremal Property : The extremal property defines the "upper bound" on the value of entropy. It gives the highest value of $H(s)$.

proof : Consider an information source S emitting 'q' symbols $S=\{S_1, S_2 \ldots, S_q\}$ with probabilities $P=\{P_1, P_2 \ldots P_q\}$ Such that $\sum_{i=1}^{q} P_i = 1$

• Let us now prove that the entropy $H(s)$ has an "upper bound" by considering a quantity.

$$\therefore \log(q) - H(s)$$

$$\therefore \log q - H(s) = \left[\sum_{i=1}^{q} P_i\right] \log q - \sum_{i=1}^{q} P_i \log \frac{1}{P_i}$$

$$= \sum_{i=1}^{q} P_i \left[\log q - \log 1/P_i\right] \longrightarrow (1)$$

the equality sign holds good when $p - \frac{1}{q} = 0$ for all value of i

i.e $P_i = 1/q$ for all $i=1,2,\ldots q$

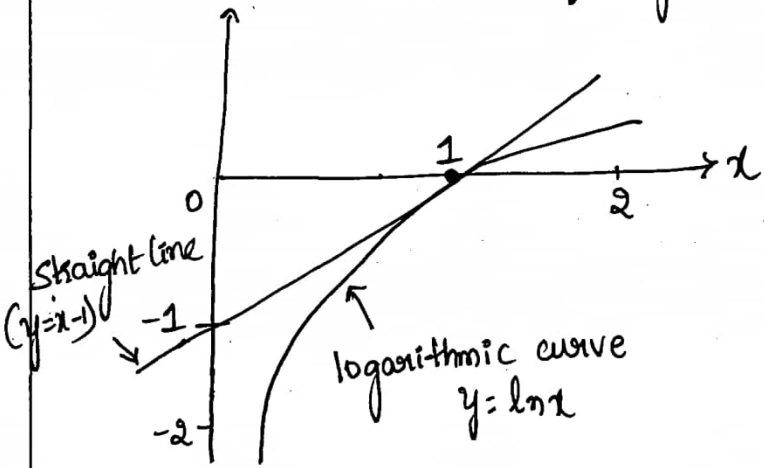
From ① $= \sum_{i=1}^{q} P_i \left[\log_2 q + \log_2 P_i\right)$

$= \sum_{i=1}^{q} P_i \log_2 qP_i$ $\left[\because \log_2 qp = \frac{\log qp}{\log 2}\right]$

$= \sum_{i=1}^{q} P_i \frac{\log_e qP_i}{\log_e 2} \rightarrow$ no item

$\therefore \log q - H(s) = \log_2 e \sum_{i=1}^{q} P_i \ln qP_i \rightarrow$ ②

Now consider a plot of $y = \ln x$ and $y = x-1$



From the graph it is evident that the straight line $y=x-1$ always lies above the logarithmic curve $y=\ln x$ except at $x=1$.

$\therefore \ln x \leq x-1$ or $-\ln x \geq 1-x$

$\ln \frac{1}{x} \geq 1-x \rightarrow$ ③

Substitute $x = 1/qP_i$ in equ^n ③ & x'ly equ^n ③ both sides by P_i and then taking summation for all $i = 1, 2 \ldots q$ we get

$$\sum_{i=1}^{q} P_i \ln qP_i \geq \sum_{i=1}^{q} P_i \left[1 - \frac{1}{qP_i}\right]$$

$$\sum_{i=1}^{q} P_i \ln [qP_i] \geq \sum_{i=1}^{q} P_i - \sum_{i=1}^{q} P_i - \frac{1}{qP_i}$$

$$\sum_{i=1}^{q} P_i \ln (qP_i) \geq 0 \rightarrow ④$$

x'ly the equ^n ④ throughout with $\log_e 2$

$$\log_e 2 \sum_{i=1}^{q} P_i \ln (qP_i) \geq 0 \rightarrow ⑤$$

combining ② & ⑤ we get $\log q - H(s) \geq 0$ or $\boxed{H(s) \leq \log_2 q}$

$\therefore$ The above equ^n is satisfied with an equality when all symbols emitted from the source are equiprobable.

$\therefore \boxed{H(s) = H(s)_{max} = \log_2 q \text{ bits/message-symbol}}$

⟨4⟩ Additive Property / Split Symbol Entropy.

When 'q' symbols are emitted from a source & one of them is split into 'n' sub symbols, then the split symbol entropy is always greater than or equal to the entropy of basic source.

- Consider a source emitting $S_1, S_2, S_3 \ldots, S_q$ with probabilities $P_1, P_2, P_3 \ldots, P_q$.
- Here the symbols S_q is split into sub symbols i.e $S_{q1}, S_{q2}, S_{q3} \ldots S_{qn}$ with probabilities $P_{q1}, P_{q2} \ldots P_{qn}$ such that $\sum_{j=1}^{n} P_{qj} = P_q$ and $\sum_{i=1}^{q} P_i = 1 \rightarrow (A)$

Consider the split symbol entropy as $H'(S)$

$$H'(S) = H(P_1, P_2, \ldots, P_{q-1}, P_{q1}, P_{q2}, \ldots P_{qn})$$

$$= \sum_{i=1}^{q-1} P_i \log \frac{1}{P_i} + \sum_{j=1}^{n} P_{qj} \log_2 1/P_{qj}$$

$$H'(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i} - P_q \log \frac{1}{P_q} + \sum_{j=1}^{n} P_{qj} \log \frac{1}{P_{qj}}$$

$$H'(S) = H(S) - \underbrace{\sum_{j=1}^{n} P_{qj}}_{\text{from (A)}} \log \frac{1}{P_q} + \sum_{j=1}^{n} P_{qj} \log \frac{1}{P_{qj}}$$

$$H'(S) = H(S) + \sum_{j=1}^{n} P_{qj} \left[\log\left(\frac{1}{P_{qj}}\right) - \log\left(\frac{1}{P_q}\right)\right]$$

$$= H(S) + \sum_{j=1}^{n} P_{qj} \left[\log \frac{P_q}{P_{qj}}\right]$$

x'lying & dividing by P_q

$$H'(S) = H(S) + P_q \sum_{j=1}^{n} \frac{P_{qj}}{P_q} \left[\log\left(\frac{P_q}{P_{qj}}\right)\right]$$

$H'(S) = H(S) +$ a positive quantity (value) $(\because P_q > P_{qj})$

hence $\boxed{H'(S) \geq H(S)}$

Split Symbol entropy of the source is always greather than the entropy of basic source.

<5> The "SOURCE EFFICIENCY" denoted by η_s is given by

$$\eta_s = \frac{H(s)}{H(s)_{max}}$$

& the "Source Redundancy" denoted by $R_{\eta s}$ is given by

$$R_{\eta s} = 1 - \eta_s.$$

Note :- η_s and $R_{\eta s}$ are expressed as percentage.

Problems.

<1> Find the maximum value for the given symbols.

$$S = \{s_1, s_2, s_3\} \text{ and } P = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right\}$$

→ we know to find $H(s) = \sum_{i=1}^{q} P_i \log_2 1/P_i$

$$H(s) = 1.5 \text{ bits/message-symbol}$$

$$\therefore H(s)_{max} = \log_2 q \quad \text{where } q = 3$$

$$H(s)_{max} = \log_2 3 \Rightarrow 1.585 \text{ bits/message symbol}$$

& the maximum value occurs when $P_i = \frac{1}{q}$ for all $i = 1, 2, 3$

i.e $P_1 = P_2 = P_3 = \frac{1}{3}$, when all messages become equiprobable.

<2> for $S = \{s_1, s_2, s_3, s_4\}$ and $P = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right\}$

→ The entropy is given by $H(s) = \sum_{i=1}^{q} P_i \log_2 \frac{1}{P_i}$

$$H(s) = 1/2 \log_2 2 + \frac{1}{4} \log_2 4 + \frac{1}{8} \log_2 8 \times 2$$

$$H(s) = 1.75 \text{ bits/message-symbol}$$

$$H(s)_{max} = \log_2 q = \log_2 4 = 2 \text{ bits/message-symbol}$$

which occurs when $P_1 = P_2 = P_3 = P_4 = \frac{1}{4}$ (equiprobable)

3. A black and white TV picture consists of 525 lines of picture information. Assume that each line consists of 525 picture elements (pixels) and that each element can have 256 brightness levels. Picture are repeated at the rate of 30 frames/sec. Calculate the average rate of information conveyed by a TV set to a viewer.

Soln A TV frame consisting of 525 horizontal lines with each line containing 525 pixels.

- Total number of pixels in one frame = $525 \times 525 = 2,75,625$ pixels.
- It is given that each pixel can have 256 different brightness levels from fully black to fully white. Then the total number of different frames possible.

$$= (256)^{2,75,625} \text{ frames.}$$

Let us assume that all these frames occur with equal probability, the net maximum information (average) content per frame is

$$I = H(S)_{max} = \log_2 q = \log_2 (256)^{2,75,625} = 275625 \log_2 256$$
$$= 22.05 \times 10^5 \text{ bits/frame.}$$

Given $r_s = 30$ frames/sec

$\therefore$ The average rate of information is given by

$$R_s = r_s I$$
$$= (30) \times (22.05 \times 10^5)$$
$$R_s = 66.15 \times 10^5 \text{ bits/sec.}$$

4. A discrete message source 'S' emits two independent symbols X and Y with probabilities 0.55 and 0.45 respectively. Calculate the efficiency of the source and its redundancy.

Soln Given:- $P(x) = 0.55$, $P(y) = 0.45$

Entropy $H(S) = \sum_{i=x}^{y} P_i \log_2 \frac{1}{P_i} \Rightarrow P_x \log_2 \frac{1}{P_x} + P_y \log_2 \frac{1}{P_y}$

$= 0.55 \log_2 \frac{1}{0.55} + 0.45 \log_2 \frac{1}{0.45} \Rightarrow 0.9928$ bits/msg symbol

To calculate the Source efficiency, the maximum entropy $H(s)_{max}$ has to be calculated

$$H(s)_{max} = \log_2 q = \log_2 2 = 1 \text{ bit/message symbol}$$

$$\therefore \text{Source efficiency } \eta_s = \frac{H(s)}{H(s)_{max}} = \frac{0.9928}{1}$$

$$\eta_s = 99.28\,\%$$

$$\text{Source Redundancy } R_{\eta_s} = 1 - \eta_s = 1 - 0.9928 = 0.0072$$

$$R_{\eta_s} = 0.72\,\%$$

5. In a facsimile transmission of picture, there are about 2.25×10^6 pixels/frame. for a good reproduction 12 brightness levels are necessary. Assume all these levels are equally likely to occur. find the rate of information if one picture is to be transmitted every 3 minutes. what is the source efficiency of this facsimile transmitter?

Sol^n Total no of pixels in one frame $= 2.25 \times 10^6$

Number of brightness levels $= 12$

$\therefore$ Total number of different frame possible $= (12)^{2.25 \times 10^6}$

Since all the levels are equally likely to occur, the net maximum information content/frame is $I = H(s)_{max} = \log_2 q$

$$I = \log_2 (12)^{2.25 \times 10^6} = 2.25 \times 10^6 \log_2 12$$

$$I = 8.066 \times 10^6 \text{ bits/picture}.$$

one picture is transmitted in 3 minutes.

$$\therefore \text{Rate of transmission } r_s = \frac{1}{3 \text{ minutes}} = \frac{1}{3 \times 60} \text{ picture/sec}$$

$\therefore$ The average rate of information is given by

$$R_s = r_s I = \frac{1}{3 \times 60} \times 8.066 \times 10^6 = 44812 \text{ bits/sec}$$

Since the information transmitted is maximum (all levels are equiprobable) the Source efficiency $\eta_s = \frac{H(s)}{H(s)_{max}} = \frac{H(s)_{max}}{H(s)_{max}} = 1$

$$\boxed{\eta_s = 100\,\%}$$

Extension of Zero-Memory Source.

The entropy of the n^{th} extension of a source is given by $H(S^n) = n\,H(S)$

To define:

Let us consider a binary source S emitting symbols $S1$ and $S2$ with probabilities P_1 and P_2 respectively such that

$$\boxed{P_1 + P_2 = 1}$$

Then the second extension of this binary source will have $2^2 = 4$ number of symbols given by

Symbol	Probability
S1S1	P_1^2
S1S2	$P_1 P_2$
S2S1	$P_2 P_1$
S2S2	P_2^2

The sum of all the probabilities of the 2^{nd} extended source is

$$P_1^2 + P_1P_2 + P_2P_1 + P_2^2 = (P_1+P_2)^2 = (1)^2 = 1 \longrightarrow (A)$$

i.e $\begin{pmatrix}\text{No of Symbols}\\ \text{in extended source}\end{pmatrix} = \begin{pmatrix}\text{No of}\\ \text{symbols in basic source}\end{pmatrix}$

$\Rightarrow$ No of Symbols in Second extension $= 2^2 = 4$ symbols

$\Rightarrow$ No of Symbols in third extension $= 2^3 = 8$ symbols

Symbol	Probability
S1S1S1	$P_1 P_1 P_1 = P_1^3$
S1S1S2	$P_1 P_1 P_2 = P_1^2 P_2$
S1S2S1	$P_1 P_2 P_1 = P_1^2 P_2$
S1S2S2	$P_1 P_2 P_2 = P_1 P_2^2$
S2S1S1	$P_2 P_1 P_1 = P_1^2 P_2$
S2S1S2	$P_2 P_1 P_2 = P_2^2 P_1$
S2S2S1	$P_2 P_2 P_1 = P_2^2 P_1$
S2S2S2	$P_2 P_2 P_2 = P_2^3$

Now, let us consider entropies of each of these sources

$H(s)$ = entropy of the basic binary source

$$H(s) = \sum_{i=1}^{2} P_i \log_2 \frac{1}{P_i}$$

$$H(s) = P_1 \log_2 \frac{1}{P_1} + P_2 \log_2 \frac{1}{P_2} \longrightarrow (1)$$

Note: $\log p^2 = 2 \log p$

$\log \left(\frac{1}{P_1}\right)^2 = 2 \log \frac{1}{P_1}$

The entropy of the 2nd extentend source is given by

$$H(s)^2 = \sum_{j=1}^{4} P_j \log \frac{1}{P_j}$$

$$H(s^2) = P_1^2 \log \frac{1}{P_1^2} + P_1 P_2 \log \frac{1}{P_1 P_2} + P_1 P_2 \log \frac{1}{P_1 P_2} + P_2^2 \log \frac{1}{P_2^2}$$

$$H(s^2) = 2P_1^2 \log \frac{1}{P_1} + 2 P_1 P_2 \log \frac{1}{P_1 P_2} + 2P_2^2 \log \frac{1}{P_2}$$

$$H(s^2) = 2P_1^2 \log (1/P_1) + 2P_1 P_2 \log\left(\frac{1}{P_1}\right) + 2P_1 P_2 \log \frac{1}{P_2} + 2P_2^2 \log \frac{1}{P_2}$$

$$H(s^2) = 2P_1 \left[P_1 \log\left(\frac{1}{P_1}\right) + P_2 \log\left(\frac{1}{P_2}\right)\right] + 2P_2 \left[P_2 \log\left(\frac{1}{P_2}\right) + P_1 \log\left(\frac{1}{P_1}\right)\right]$$

$$H(s^2) = 2P_1 \underbrace{(H(s)) + 2P_2 (H(s))}_{\text{from (1)}}$$

$$H(s^2) = H(s) \left[2P_1 + 2P_2\right]$$

$$H(s^2) = H(s) \left[2 \underbrace{(P_1 + P_2)}_{1 \text{ from A}}\right]$$

$$\boxed{H(s^2) = 2 H(s)}$$

$|||^{ly}$ $\quad H(s^3) = 3 H(s)$

$\vdots \qquad \vdots$

$$H(s^n) = n \cdot H(s)$$

Extension of the source is done to increase the efficiency of the source.

Problems.

1. A zero memory source has a source alphabet $S = \{s_1, s_2, s_3\}$ with $P = \{1/2, 1/4, 1/4\}$ find the entropy of this source. Also determine the entropy of its 2nd extension & verify that $H(S^2) = 2H(S)$.

→ From the Source, we have 3 symbols s_1, s_2, s_3 with $P = 1/2 \ 1/4 \ 1/4$.

To find the entropy $H(S) = \sum_{i=1}^{3} P_i \log_2 1/P_i$

$H(S) = 1/2 \log_2 2 + 1/4 \log_2 4 + 1/4 \log_2 4$

$\boxed{H(S) = 1.5 \text{ bits/symbol}}$

Second extension of the basic source with 3 symbols, we have $3^2 = 9$ symbols, which can be listed with probability

Symbol	Probability
s_1s_1	$1/2 \times 1/2 = 1/4$
s_1s_2	$1/2 \times 1/4 = 1/8$
s_1s_3	$1/2 \times 1/4 = 1/8$
s_2s_1	$1/4 \times 1/2 = 1/8$
s_2s_2	$1/4 \times 1/4 = 1/16$
s_2s_3	$1/4 \times 1/4 = 1/16$
s_3s_1	$1/4 \times 1/2 = 1/8$
s_3s_2	$1/4 \times 1/4 = 1/16$
s_3s_3	$1/4 \times 1/4 = 1/16$

where the sum of all probabilities of the 2nd extension source symbol must be $= 1$

$\therefore H(S^2) = \sum_{j=1}^{9} P_j \log_2 \frac{1}{P_j}$

$= \frac{1}{4}\log_2 4 + \frac{1}{8}\log_2 8 \times 4 + \frac{1}{16}\log_2 16 \times 4$

$H(S^2) = 3$ bits/message symbol

$\therefore H(S^2) = 2 \times H(S)$

$= 2 \times 1.5$ bits/symbol $= 3$ bits/msg symbol

$\boxed{H(S^2) = 2H(S)}$

2. Consider a discrete memoryless source with source alphabet $S = \{s_0, s_1, s_2\}$ with source statistics $\{0.7, 0.15, 0.15\}$

(a) calculate the entropy of source

(b) calculate the Entropy of second order extension of the source.

→ (a) To calculate the entropy $H(S) = \sum_{i=1}^{q} P_i \log_2 \frac{1}{P_i}$

$$\therefore H(s) = 0.7 \log_2 \frac{1}{0.7} + 0.15 \log_2 \frac{1}{0.15} + 0.15 \log_2 \frac{1}{0.15}$$

$$\boxed{H(s) = 1.8128 \text{ bits/symbol.}}$$

(b) the entropy of the second-order extension of the source is given by

$$H(s^2) = 2 \times H(s)$$
$$= 2 \times 1.8128$$
$$\boxed{H(s^2) = 2.3625 \text{ bits/symbol}}$$

3. A source emits one of four probable messages M1, M2, M3 & M4 with probability of 7/16, 5/16, 1/8 & 1/8 respectively. find the entropy of the source. List all the elements for the second extension of the source hence $H(s^2) = 2H(s)$.

Soln the entropy for the basic messages is found by

$$H(s) = \sum_{i=1}^{4} P(M_i) \log_2 \frac{1}{P(M_i)}$$

$$H(s) = \frac{7}{16} \log_2 \frac{16}{7} + \frac{5}{16} \log_2 \frac{16}{5} + \frac{1}{8} \log_2 8 \times 2$$

$$\boxed{H(s) = 1.7962 \text{ bits/message symbol.}}$$

the second extension s^2 of the basic source will have $4^2 = 16$ symbols.

Symbol	Probability
M1 M1	7/16 × 7/16 = 49/256
M1 M2	7/16 × 5/16 = 35/256
M1 M3	7/16 × 1/8 = 7/128
M1 M4	7/16 × 1/8 = 7/128
M2 M1	5/16 × 7/16 = 35/256
M2 M2	5/16 × 5/16 = 25/256
M2 M3	5/16 × 1/8 = 5/128
M2 M4	5/16 × 1/8 = 5/128
M3 M1	1/8 × 7/16 = 7/128
M3 M2	1/8 × 5/16 = 5/128
M3 M3	1/8 × 1/8 = 1/64
M3 M4	1/8 × 1/8 = 1/64
M4 M1	1/8 × 7/16 = 7/128
M4 M2	1/8 × 5/16 = 5/128
M4 M3	1/8 × 1/8 = 1/64
M4 M4	1/8 × 1/8 = 1/64

$$\therefore H(s^2) = \sum_{j=1}^{16} P_j \log_2 \frac{1}{P_j}$$

$$H(s^2) = \frac{49}{256} \log \frac{256}{49} + \left(\frac{35}{256} \log \frac{256}{35}\right)(2) + \frac{25}{256} \log \frac{256}{25} + \left(\frac{7}{128} \log \frac{128}{7}\right)(4) + \frac{5}{128} \log \frac{128}{5} + \left(\frac{1}{64} \log 64\right)(4)$$

$$H(s^2) = 3.5924 \text{ bits/message symbol}$$
$$= 2 \times 1.7962 \quad \text{—— " ——}$$

$$\therefore \boxed{H(s^2) = 2H(s) \quad \text{proved}}$$

Markoff Statistical Model for Information Sources.

In real-life sources, there is intersymbol influence present such that

"The occurance of x_i in the zeroth position s_0 of the message depends on the previous 'q' symbols $\{s_1, s_2, \ldots, s_q\}$ & these sources are generally specified by set of condition probabilities. $P\{x_i / s_1, s_2 \ldots s_q\}$

where $x_i \rightarrow$ symbol in the 0^{th} position & each s has an m-symbol alphabet $\{x_1, x_2 \ldots x_m\}$.

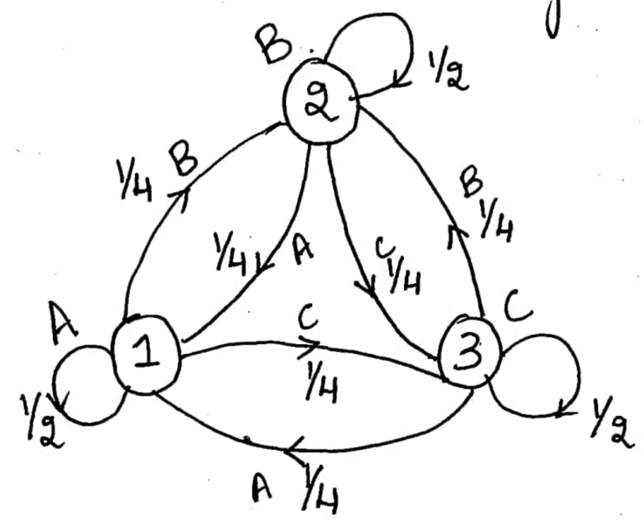
$\{s_1, s_2 \ldots s_q\}$ are the previous 'q' symbols emitted, since $P(x_i)$ depends on the earlier 'q' symbols, such a source is known as q^{th} Order of Mark-off source or "Markov Source".

Mark-off Source.

Mark-off source is represented in graphical form, where the states are represented by nodes of the graph & the transition between the states is represented by a directed line from initial state to final state.

The transition probabilities and symbol emitted corresponding to various transition are usually shown along the lines of the graph.

Let the source emits 3 symbols A, B & C



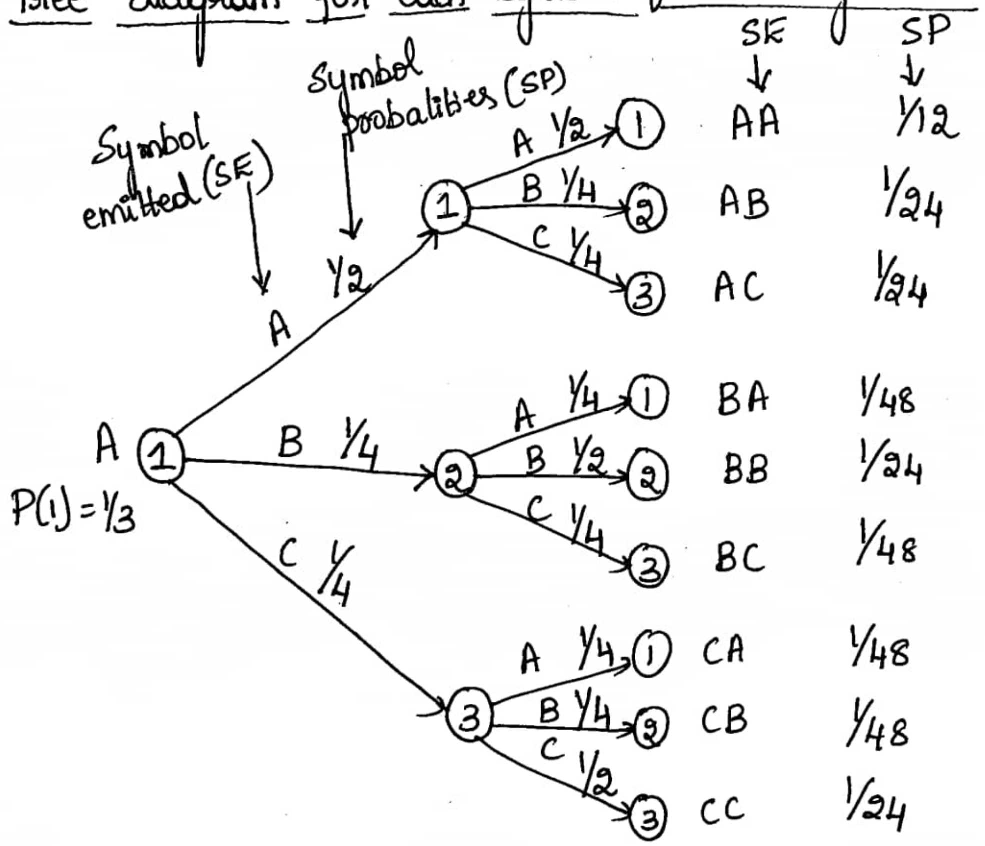
The Source emits 3 symbols A, B and C. The probability of occurance of a symbol depend on that particular symbol & the symbol immedidtely preceeding it i.e., the past influence lasts only for a duration of 1 symbol.

The state transition and symbol generation can also be illustrated using a TREE Diagram.

This is planar graph where nodes corresponds to state & branches correspond to transition.

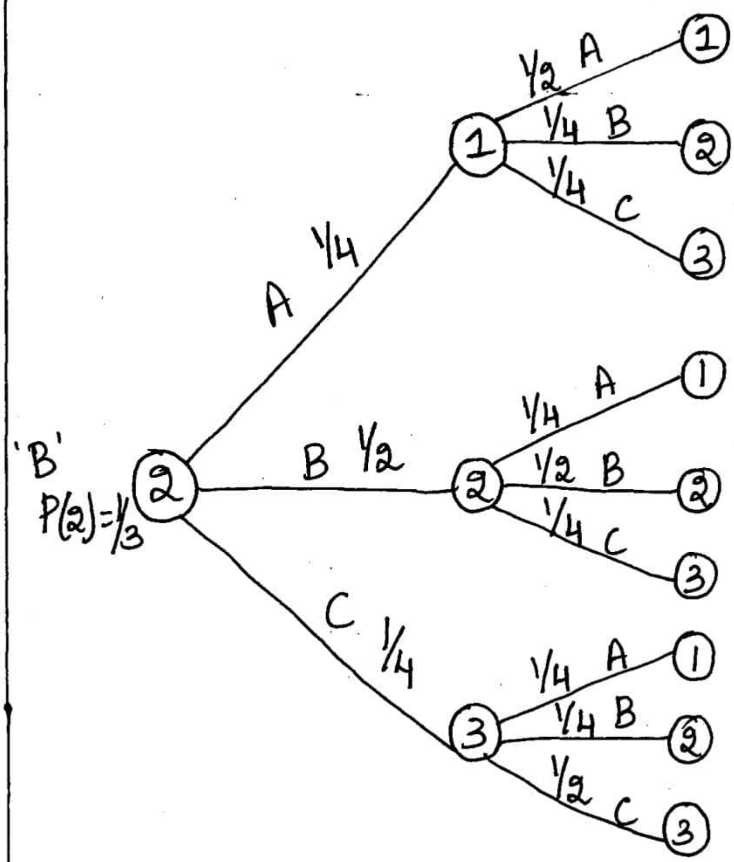
The tree diagram can be used to obtain the probabilities of generating various symbol sequeces.

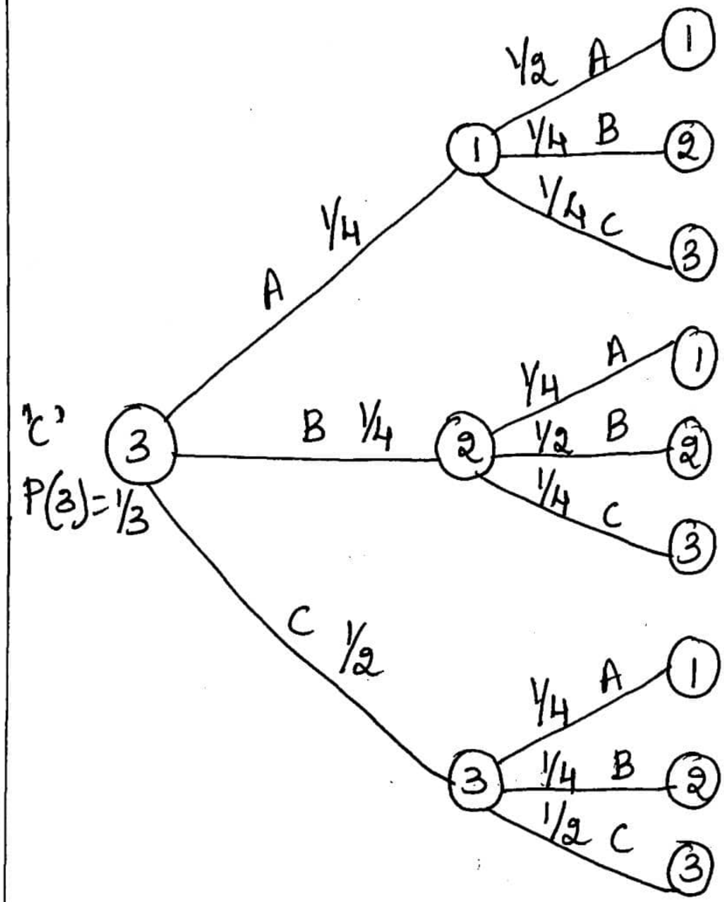
Tree diagram for each symbol from every state.



From the tree diagram, the symbol AA can be generated by either one of the following transitions $1 \to 1 \to 1$ or
$2 \to 1 \to 1$ or
$3 \to 1 \to 1$

$\therefore$ the Symbol probability for AA from state 1 is $\underbrace{1 \to 1}_{1/2} \underbrace{\to 1}_{1/2} = 1/4$, is x'ed with the probability of occurance of symbol A $\Rightarrow \therefore \; 1/3 \times 1/2 \times 1/2 = 1/12$.





SE	SP
AA	1/24
AB	1/48
AC	1/48
BA	1/24
BB	1/12
BC	1/24
CA	1/48
CB	1/48
CC	1/24
AA	1/24
AB	1/48
AC	1/48
BA	1/48
BB	1/24
BC	1/48
CA	1/24
CB	1/24
CC	1/12.

Entropy and Information Rate of Mark-off source.

The entropy of the Mark-off source is defined as the weighted average of the entropy of the symbols emitted from each state, where the entropy of state 'i', denoted by H_i, is defined as the average information content of the symbols emitted from the i^{th} state.

$$\therefore \boxed{H_i = \sum_{j=1}^{n} P_{ij} \log_2 \frac{1}{P_{ij}} \text{ bits/symbol}}$$

Where P_{ij} is the condition probability indicating the change in the state of the source from 'i' to 'j'.

The entropy of the source is then the average of the entropy of each state i.e.,

$$H = \sum_{i=1}^{n} P_i H_i$$

$$\boxed{H = \sum_{i=1}^{n} P_i \left[\sum_{j=1}^{n} P_{ij} \log_2 \frac{1}{P_{ij}} \right]} \text{ bits/msg-symbol.}$$

where P_i is the probability that the source is in state i

The average information rate 'R_s' for the source is defined as

$$R_s = r_s H \text{ bits/sec}$$

where r_s is the number of state transition per second or the symbol rate of the source.

Average Information Content per symbol.

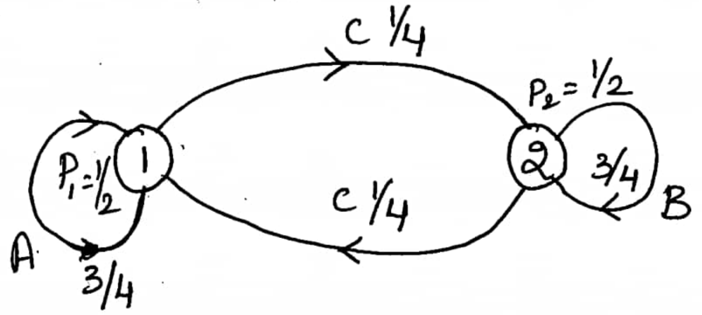
If $P(m_i)$ is the probability of a sequence m_i of N symbols from the source and if $\boxed{G_N = \frac{1}{N} \sum_i P(m_i) \log_2 \frac{1}{P(m_i)}} = \frac{1}{N} H(S^{-N})$

where the sum is over all sequences m_i containing N symbols, then G_N is monotonically decreasing function of N and

$$\lim_{N \to \infty} G_N = H \text{ bits/symbol.}$$

Problems based on Mark-off sources.

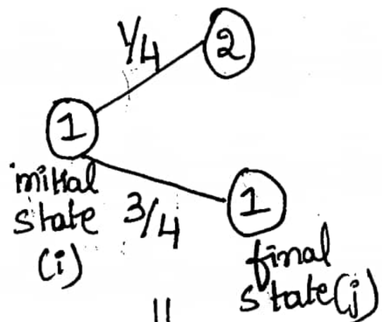
1. consider an Information Source whose graph is shown below. find the source entropy H & average Information Content per Symbol in messages containing 1, 2 & 3 symbols, i.e find G_1, G_2, G_3



→ To find the Source entropy $H = \sum_{=1}^{n} P_i H_i$

$H_i = \sum_{j=1}^{n} P_{ij} \log_2 1/P_{ij}$ bits/message symbol

$$H_1 = P_{11} \log \frac{1}{P_{11}} + P_{12} \log \frac{1}{P_{12}}$$



⇓

$P_{ij} \Rightarrow 3/4$ for P_{11}

$P_{ij} \Rightarrow 1/4$ for P_{12}

$$= \frac{3}{4} \log_2 \frac{4}{3} + \frac{1}{4} \log_2 4$$

$$H_1 = 0.3112 + 0.5$$

$$\boxed{H_1 = 0.8112 \text{ bits/symbols}}$$

$$H_2 = P_{21} \log P_{21} + P_{22} \log \frac{1}{P_{22}}$$

$$= \frac{1}{4} \log 4 + \frac{3}{4} \log \frac{4}{3}$$

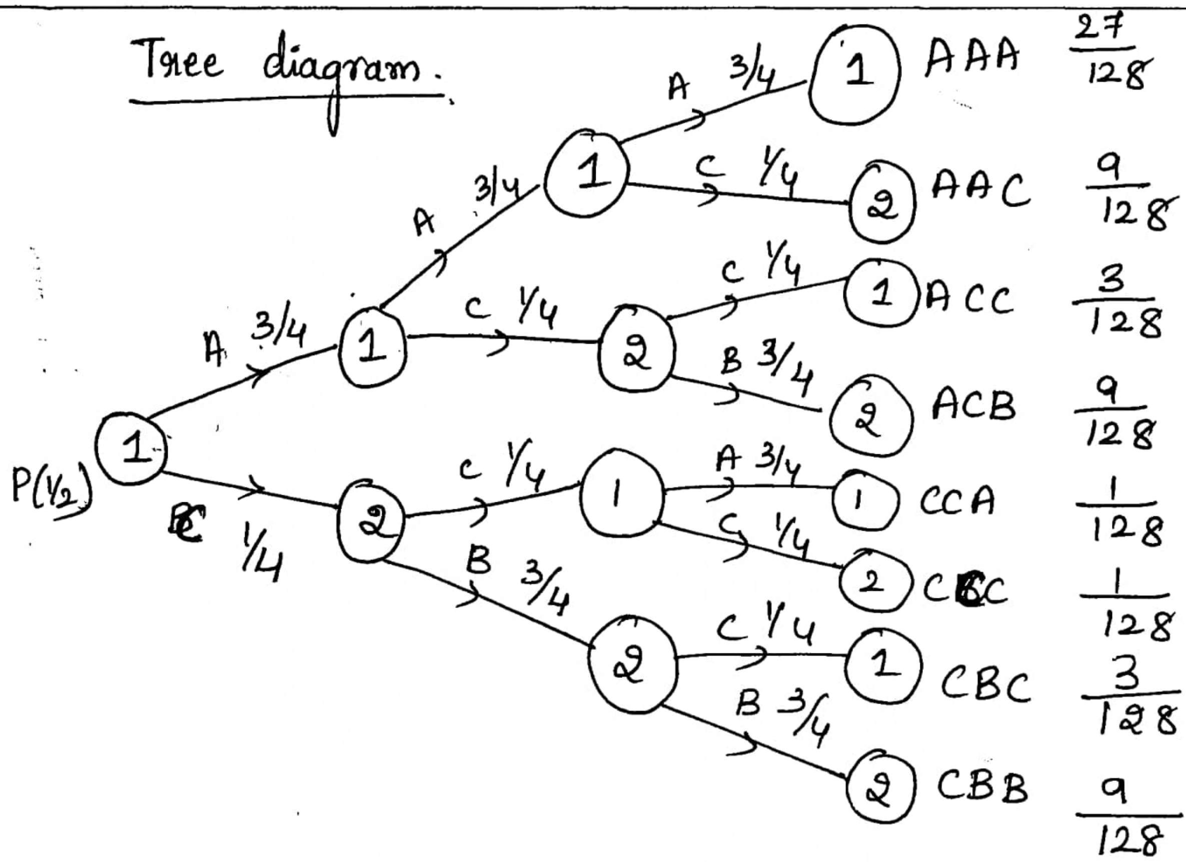
$$\boxed{H_2 = 0.8112 \text{ bits/symbol}}$$

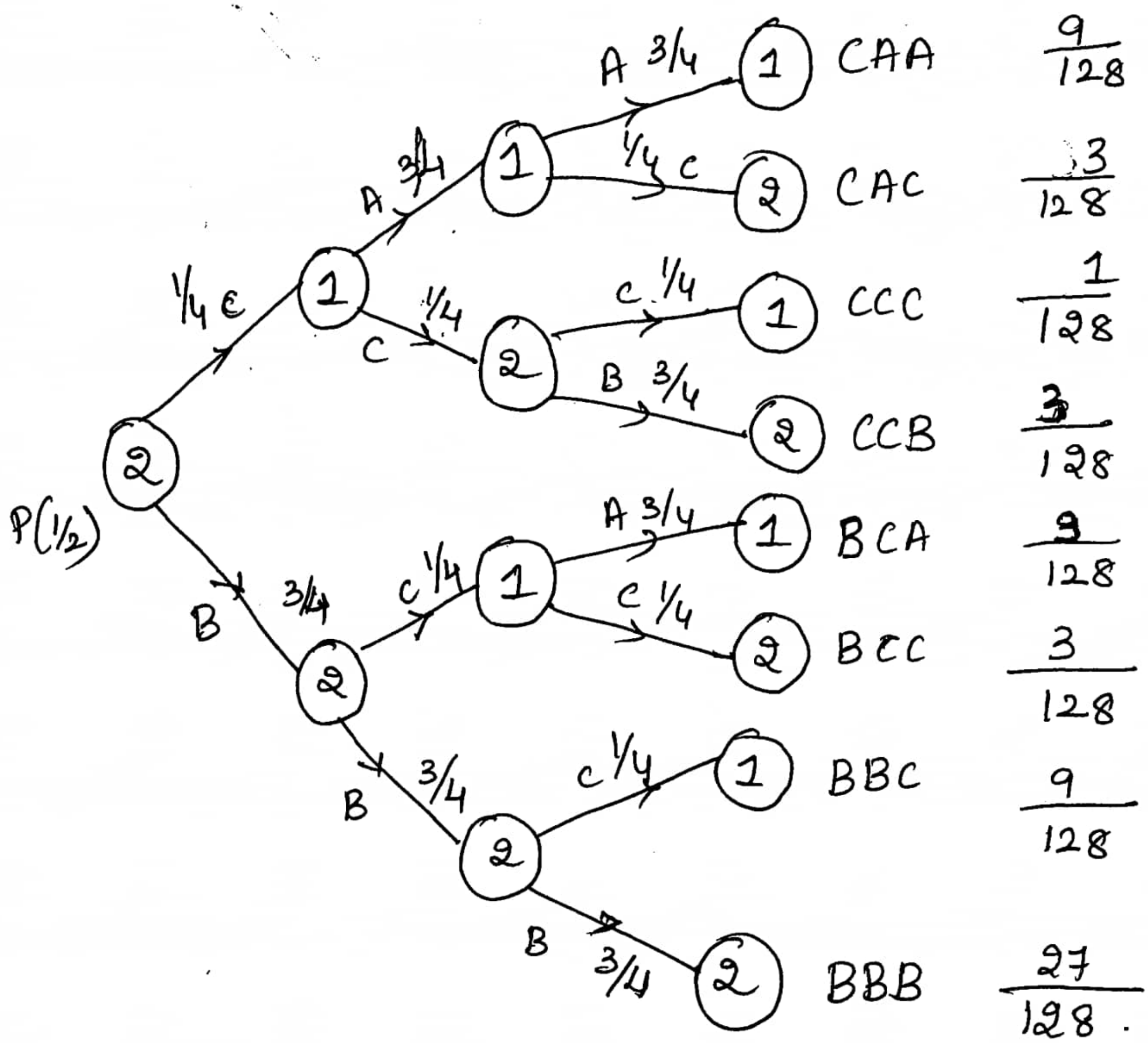
⟹ $H = P_1 H_1 + P_2 H_2$

$$= \frac{1}{2} \times 0.8112 + \frac{1}{2} \times 0.8112$$

$$\boxed{H = 0.8112 \text{ bits/symbol}}$$

Tree diagram.





Message of length 1 m(1)	P	Message of length 2 m(2)	P	Message of length 3 m(3)	P
A	$\frac{3}{8}$	AA	$\frac{9}{32}$	AAA	$\frac{27}{128}$
				AAC	9/128
B	$\frac{3}{8}$	AC	$\frac{3}{32}$	ACC	3/128
				ACB	9/128
C	$\frac{1}{4}$	CB	$\frac{3}{32}$	CCA	3/128
				CCC	2/128
				CBC	3/128
		CC	$\frac{2}{32}$	CBB	9/128
				CAA	9/128
		BB	$\frac{9}{32}$	CAC	3/128
				CCC	
		BC	$\frac{3}{32}$	CCB	3/128
		CA	$\frac{3}{32}$	BCA	9/128
				BCC	3/128
				BBC	9/128
				BBC	27/128.

So next we need to find the probability of a sequence of N symbols from the source, so G is defined as 3 symbols are a, b, c,

$$G_N = \frac{1}{N} \sum_{i=1}^{n} P(m_i) \log_2 \frac{1}{P(m_i)}.$$

$N = 1$ for length 1

$$\therefore G_1 = \frac{1}{1}\left[\frac{3}{8}\log\frac{8}{3} + \frac{3}{8}\log\frac{8}{3} + \frac{1}{4}\log_2 4\right]$$

$$G_1 = 0.5306 + 0.5306 + 0.5$$

$$\boxed{G_1 = 1.5612 \text{ bits/symbol}}$$

At length $N=2$

$$G_2 = \frac{1}{2}\left[2\times\frac{9}{32}\log_2\frac{32}{9} + \left(4\times\frac{3}{32}\log_2\frac{32}{3}\right) + \left(\frac{2}{32}\log\frac{32}{2}\right)\right]$$

$G_2 = 1.28$ bits/symbol

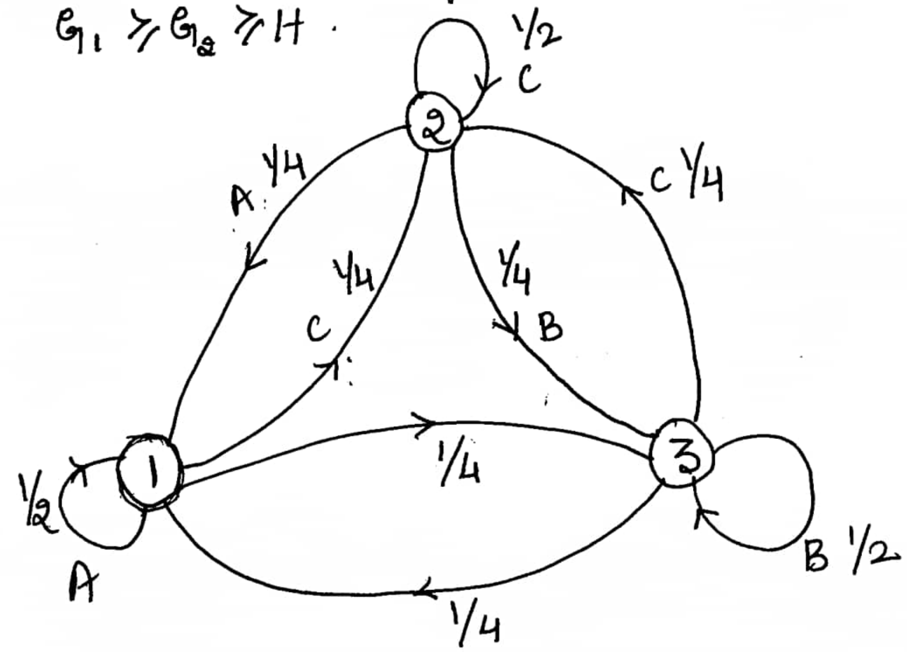
At length $N=3$

$$G_3 = \frac{1}{3}\left[\underbrace{\left(2\times\frac{27}{128}\log\frac{128}{27}\right)}_{0.9472} + \underbrace{\left(6\times\frac{9}{128}\log_2\frac{128}{9}\right)}_{1.6158} + \underbrace{\left(6\times\frac{3}{128}\log_2\frac{128}{3}\right)}_{0.7614} + \underbrace{\left(\frac{2}{128}\log\frac{128}{2}\right)}_{0.09375}\right]$$

$G_3 = 1.139$ bits/symbol.

$$\therefore \boxed{G_1 \geq G_2 \geq G_3 \geq H.}$$

(2) For the mark-off source given below, find the entropy of each state & entropy of the source and also S.T $G_1 \geq G_2 \geq H$.



→ To find the entropy of the Source

$$H = H_1P_1 + H_2P_2 + H_3P_3$$

$$H_1 = P_{11}\log\frac{1}{P_{11}} + P_{12}\log\frac{1}{P_{12}} + P_{13}\log\frac{1}{P_{13}}$$

$$= \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{4}\log 4 \Rightarrow 1/2 + \frac{1}{2} + \frac{1}{2}$$

$H_1 = 1.5$ bits/symbol

$$H_2 = P_{21}\log\frac{1}{P_{21}} + P_{22}\log\frac{1}{P_{22}} + P_{23}\log\frac{1}{P_{23}}$$

$$= \frac{1}{4}\log 4 + \frac{1}{2}\log 2 + \frac{1}{4}\log 4$$

$H_2 = 1.5$ bits/symbol

$$H_3 = P_{31}\log\frac{1}{P_{31}} + P_{32}\log\frac{1}{P_{32}} + P_{33}\log\frac{1}{P_{33}}$$

$$= \frac{1}{4}\log 4 + \frac{1}{4}\log 4 + \frac{1}{2}\log 2$$

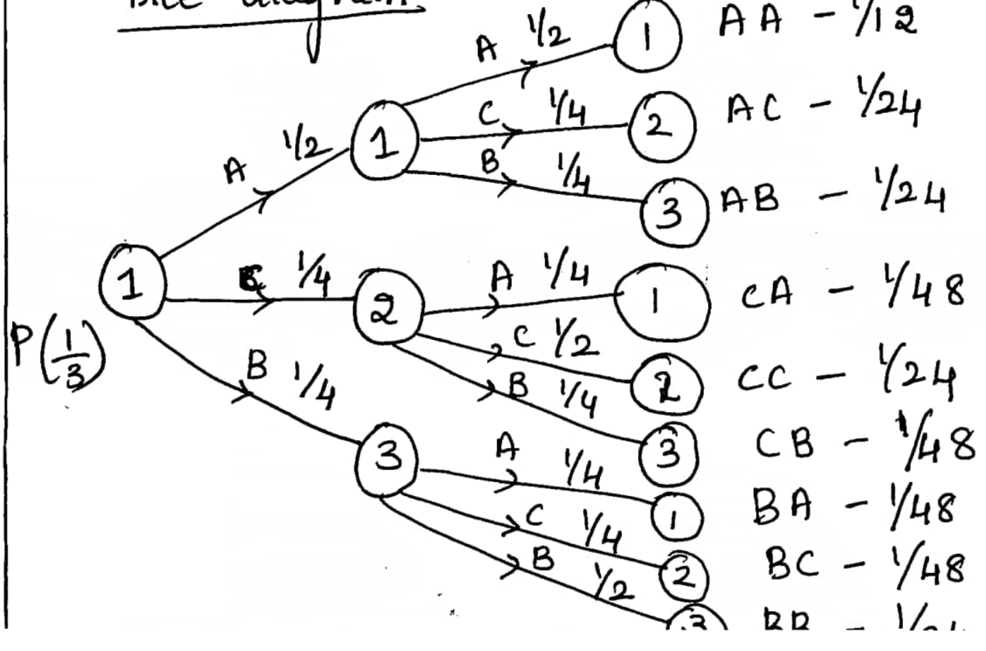
$H_3 = 1.5$ bits/symbol.

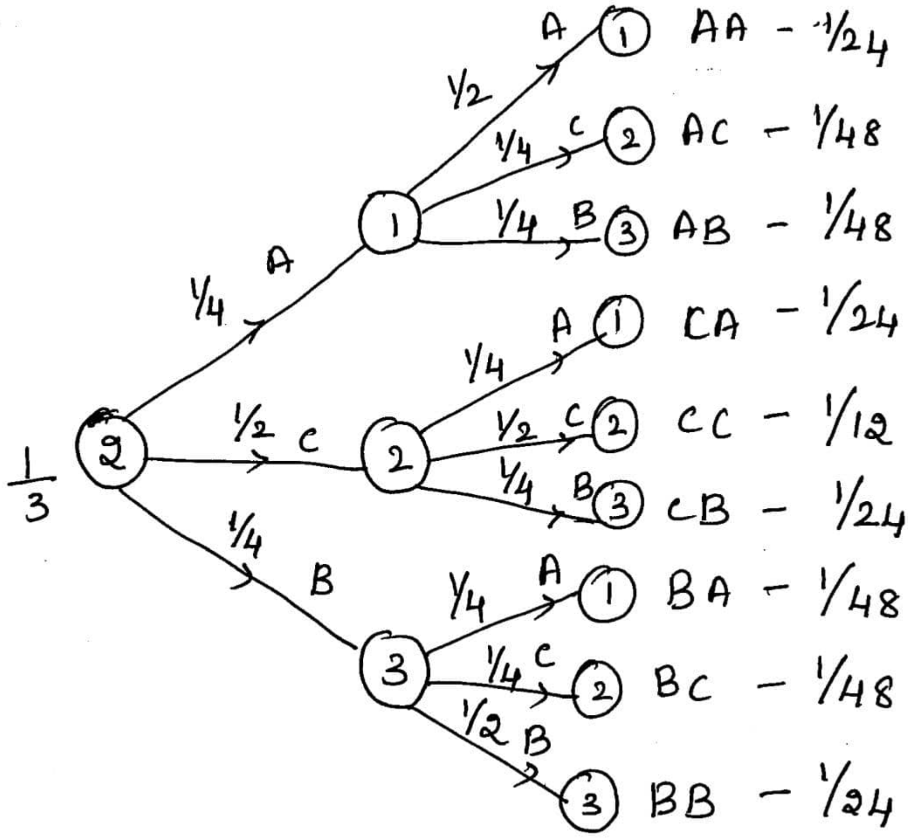
$$H = H_1P_1 + H_2P_2 + H_3P_3$$

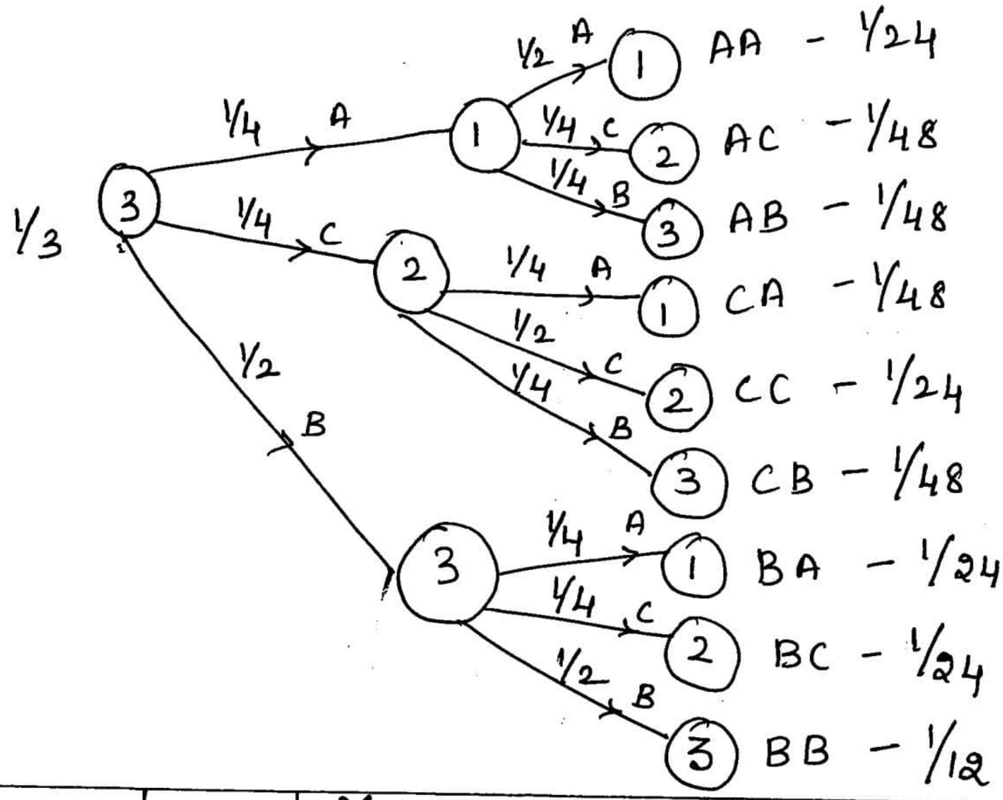
$$= 1.5\left(\frac{1}{3}\right) + 1.5\left(\frac{1}{3}\right) + 1.5\left(\frac{1}{3}\right)$$

$$\boxed{H = 1.5 \text{ bits/symbol}}$$

Tree diagram







Message of length 1 m(1)	P	Message of length 2 m(2)	P	
A	1/3	AA -	1/6	(1/12 + 1/24 + 1/24)
		AC -	1/12	
B	1/3	AB -	1/12	
		CA -	1/12	
C	1/3	CC -	1/6	
		CB -	1/12	
		BA -	1/12	
		BC -	1/12	

$$G_N = \frac{1}{N} \sum_{i=1}^{n} P_{mi} \log_2 \frac{1}{m_i}$$

$$G_1 = \frac{1}{1}\left[\frac{1}{3}\log_2 3 + \frac{1}{3}\log_2 3 + \frac{1}{3}\log_2 3\right]$$

$G_1 = 1.58$ bits/symbol.

$$G_2 = \frac{1}{2}\left[3 \times \frac{1}{6}\log_2 6 + 6 \times \frac{1}{12}\log_2 12\right]$$

$$= \frac{1}{2}\left[1.292 + 1.792\right]$$

$G_2 = 1.54$ bits/symbol

$\therefore \; G_1 \geq G_2 \geq H$

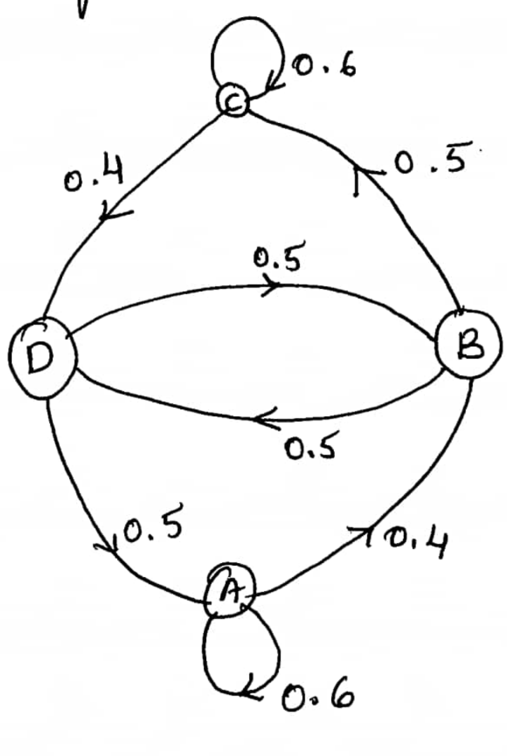
$1.58 \geq 1.54 \geq 1.5$.

(3.) Consider the state diagram of the mark-off Source

(a) compute the state probabilities

(b) find entropy of each state.

(c) find entropy of the Source.



⇒ State equations :-

Probability of Occurance of A

$$P(A) = 0.5\ P(D) + 0.6\ P(A) \rightarrow (1)$$

$$P(B) = 0.4\ P(A) + 0.5\ P(D) \rightarrow (2)$$

$$P(C) = 0.6\ P(C) + 0.5\ P(B) \rightarrow (3)$$

$$P(D) = 0.5\ P(B) + 0.4\ P(C) \rightarrow (4)$$

Path going to state

$$P(A) + P(B) + P(C) + P(D) = 1 \rightarrow (5)$$

from (1) we have

$$0.4\ P(A) = 0.5\ P(D)$$

$$P(A) = 1.25\ P(D) \rightarrow (6)$$

from (2), substitute P(A)

$$P(B) = 0.4 \times 1.25\ P(D) + 0.5\ P(D)$$

$$P(B) = P(D) \rightarrow (7)$$

Substituing $P(B) = P(D)$ in equⁿ (3)

$$0.4\ P(C) = 0.5\ P(D)$$

$$P(C) = 1.25\ P(D) \rightarrow (8)$$

Substitute for P(B) & P(C) in (4)

$$P(D) = 0.5\ P(B) + 0.4\ P(D) \rightarrow (9)$$

∴ apply in (5)

$$1.25\ P(D) + P(D) + 1.25\ P(D) + P(D) = 1$$

$$4.5\ P(D) = 1$$

$$\boxed{P(D) = 0.22}$$

∴ $P(A) = 0.275$

$P(B) = 0.22$

$P(C) = 0.275$

$P(D) = 0.22$

(b) Entropy of each state (symbols emitted by that state)

$$H_A = 0.4 \log_2 \frac{1}{0.4} + 0.6 \log_2 \frac{1}{0.6}$$

$$= 0.53 + 0.44 \quad = 0.97 \text{ bits/symbol}$$

$$H_B = 2 \times \frac{1}{2} \log_2 2 \quad = 1 \text{ bits/symbol}$$

$$H_C = 0.4 \log_2 \frac{1}{0.4} + 0.6 \log_2 \frac{1}{0.6} = 0.97 \text{ bits/symbol}$$

$$H_D = 2 \times 1/2 \log_2 2 \quad = 1 \text{ bits/symbol}$$

Source entropy

$$H = H(A) P(A) + H(B) P(B) + H(C) P(C) + H(D) P(D)$$

$$= 0.97 \times 0.275 + 0.22 + 0.97 \times 0.275 + 0.22$$

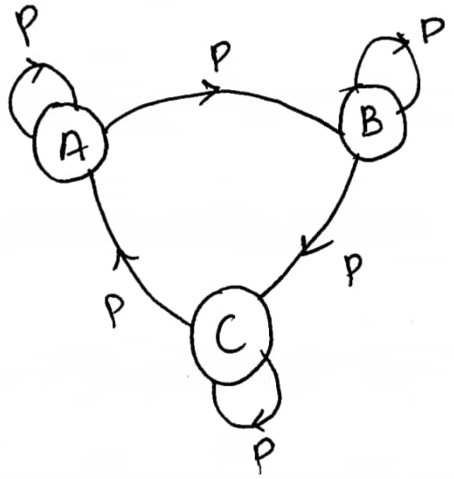
$$\boxed{H = 0.9735 \text{ bits/symbol}}$$

(4.) consider the state diagram of the mark-off source

(a) compute the state probabilities.

(b) find entropy of each state

(c) find entropy of the source.



→ State probabilities

$$P(A) = P\,P(A) + P\,P(C) \longrightarrow (1)$$

$$P(B) = P\,P(B) + P\,P(A) \longrightarrow (2)$$

$$P(C) = P\,P(B) + P\,P(C) \longrightarrow (3)$$

$$P(A) + P(B) + P(C) = 1 \longrightarrow (4).$$

Substituting (1), (2) & (3) in (4)

$$P\,P(A) + P\,P(C) + P\,P(B) + P\,P(A) + P\,P(B) + P\,P(C) = 1$$

$$2P\,P(A) + 2P\,P(B) + 2P\,P(C) = 1$$

$$2P\,(P_A + P_B + P_C] = 1 \quad \Rightarrow 2P = 1$$

$\therefore$ from ① $P_A = 0.5\, P(A) + 0.5\, P(C)$

$$0.5\, P(A) = 0.5\, P(C)$$

$$P(A) = P(C).$$

from ② $P_B = 0.5\, P_B + 0.5\, P_A$

$$0.5\, P_B = 0.5\, P_A$$

$$P_B = P_A$$

from ③ $P_C = 0.5\, P_B + 0.5\, P_C$

$$0.5\, P_C = 0.5\, P_B$$

$$P_C = P_B.$$

$$P(A) = P(B) = P(C)$$

$$P(A) + P(A) + P(A) = 1$$

$$3P(A) = 1$$

$$P(A) = 1/3.$$

therefore $P(B) = 1/3$

$P(C) = 1/3.$

(ii) Entropy of each state

$$H_A = 0.5 \log_2 \frac{1}{0.5} + 0.5 \log_2 \frac{1}{0.5} = 1 \text{ bits/symbol}$$

$H_B =$ ——— " ———

$H_C =$ ——— " ———

(iii) Entropy of the Source

$$H = P_A H_A + P_B H_B + P_C H_C$$

$$H = \frac{1}{3} \times 1 + \frac{1}{3} \times 1 + \frac{1}{3} \times 1$$

$$\boxed{H = 1 \text{ bit/symbol}}$$

(5) Design a system to report the heading of a collection of 400 cars. The heading levels are: heading straight (S), turning left (L) and turning right (R). This information is to be transmitted every second. Construct a model based on the test data given below.

(i) On the average during a given reporting interval, 200 cars were heading straight, 100 were turning left and remaining were turning right.

(ii) Out of 200 cars that reported heading straight, 100 of them reported going straight during the next reporting period. 50 of them turning left & remaining turning right during the next period

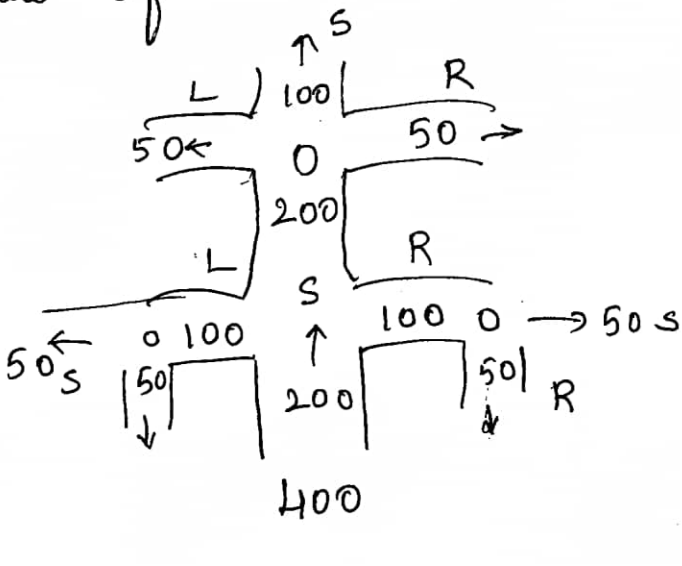
(iii) Out of 100 cars that reported as turning during a signalling period. 50 of them continued their turn and remaining headed straight during the next reporting period

(iv) The dynamics of the cars did not allow them to change their heading from left to right or right to left during subsequent reporting periods.

(1) find the entropy of each state (2) find the entropy of the system (3) find the rate of transmission

→ To find the entropy

$$H = H_S P_S + H_L P_L + H_R P_R$$



$\therefore P(S) = 0.5$

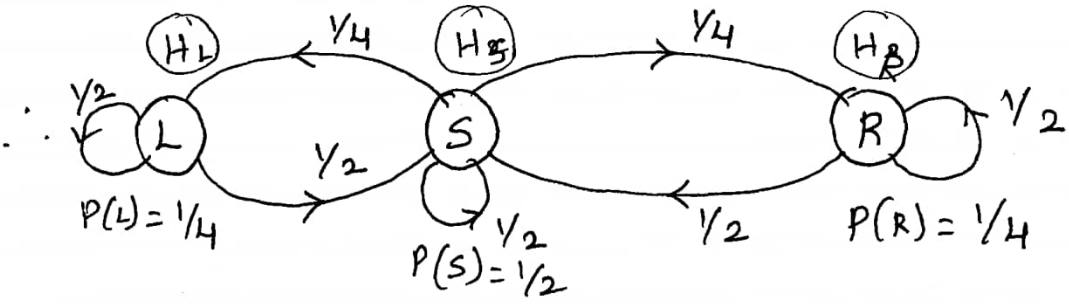
$P(L) = 0.25$

$P(R) = 0.25$

$P(L/L) = 0.5$ $\quad$ $P(S/S) = 0.5$ $\quad$ $P(R/R) = 0.5$

$P(S/L) = 0.5$ $\quad$ $P(L/S) = 0.25$ $\quad$ $P(S/R) = 0.5$

$P(R/S) = 0.25$



$\therefore \; H = H_L P_L + H_S P_S + H_R P_R$

$H_L = P_{LL} \log \frac{1}{P_{LL}} + P_{LS} \log \frac{1}{LS} = \frac{1}{2} \log 2 + \frac{1}{2} \log 2 = 1$ bit/symbol

$H_R = 1$ bits/symbol , $H_S = 3/2$ bits/symbol

$\therefore \; H = 1 \times 1/4 + \frac{3}{2} \times \frac{1}{2} + 1 \times 1/4 \Rightarrow H = 1.25$ bits/symbol

$R = r_s . H$ bits/sec. $\quad$ given : $r_s = 1$ message/sec

$R = 1 \times 1.25 \Rightarrow \boxed{R = 1.25 \text{ bits/sec.}}$

6. The international Morse code uses a sequence of dots and dashes to transmit letters of the English alphabet. The dash is represented by a current pulse that has a duration of 3 units and the dot has a duration of 1 unit. The probability of occurance of a dash is 1/3 of the probability of occurance of a dot

(a) calculate the information Content of a dot and a dash

(b) calculate the average information in the dot-dash code.

(c) Assume that the dot lasts 1 m sec, which is the Same time interval as the pause between symbols. find the average rate of information transmission.

→ Given : $P(\text{dash}) = \frac{1}{3} P \text{dot}$

Calculate : $I(\text{dash}) = ?$

$I(\text{dot}) = ?$

we Know that $P(dash) + P(dot) = 1$

$$\frac{1}{3} P(dot) + P(dot) = 1$$

$$\frac{4}{3} P(dot) = 1 \Rightarrow P(dot) = \frac{3}{4}$$

$$\therefore \quad P(dash) = \frac{1}{3} P(dot)$$

$$= \frac{1}{3} \times \frac{3}{4} \Rightarrow P(dash) = \frac{1}{4}$$

(i) Information in a dot $= I_{dot} = \log_2 \frac{1}{P_{dot}} \Rightarrow \log_2 \frac{4}{3} = 0.415$ bits

" —— " —— " dash $= I_{dash} = \log_2 \frac{1}{P_{dash}} \Rightarrow \log_2 4 = 2$ bits.

(ii) the entropy of dot dash code is

$$H(s) = P_{dot} \log_2 \frac{1}{P_{dot}} + P_{dash} \log_2 \frac{1}{P_{dash}}$$

$$= \frac{3}{4} \log_2 \frac{4}{3} + \frac{1}{4} \log_2 4$$

$H(s) = 0.8113$ bits/message - symbol.

(iii) $R = ?$ $\qquad R = r_s \times H$ bits/sec.

$t(dot) = 1\,ms \qquad t(pause) = 1\,ms \qquad t(dash) = 3\,ms$

T = avg time/symbol $= t(dot)\,P(dot) + t(Pause)\,P(Pause) + t(dash) + P(dash)$

$T = 1 \times 10^{-3} \times (3/4) + 1 \times 10^{-3} \times 1 + 3 \times 10^{-3} \times 1/4$

$T = 2.5\,ms$ $\qquad$ ($\because$ T = secs/symbol

$r_s = \frac{1}{T} = \frac{1}{2.5 \times 10^{-3}} \Rightarrow R = r_s \cdot H$ $\qquad$ r_s = symbol/sec)

$$\boxed{R = 324.76 \text{ bits/sec}}$$

(7) For a Source emitting Symbols in Independent Sequences, Show that the Source entropy is maximum when the symbols occur with equal probabilities.

→ Let us consider 2 symbols with probabilities P & $1-P$.

$$H = P \log_2 \frac{1}{P} + (1-P) \log_2 \frac{1}{1-p}$$

$$= -\left[P \log_2 P + (1-P) \log_2 (1-P)\right]$$

Diff^n H w.r.t p and equate to zero (to get max value)

$$H_{max} = \frac{dH}{dp} = 0 = -\left[\cancel{P} \times \frac{1}{\cancel{P}} + \log P + \frac{\cancel{(1-P)}}{\cancel{(1-P)}}(-1) + (-1)\log(1-P)\right]$$

$$0 = -\left[\log P - \log(1-P)\right]$$

$$\log_2 P - \log_2(1-P) = 0$$

$$\log_2\left(\frac{P}{1-P}\right) = 0$$

$$\frac{P}{1-P} = 1$$

$$P = 1-p$$

$$2p = 1$$

$$\boxed{p = 1/2}$$

Noor Afza CIT Gubbi

SOURCE CODING.

Source Encoding is a process by which the output of an information Source is converted into a binary sequence as its output. The functional block that performs this task in the Communication system is called "Source Encoder".

The input to the source encoder is a symbol sequence emitted by information Source.

The Source encoder assigns variable length binary code words to block of symbols and produce a binary Sequence as its output.

Properties of Codes

1. Block code : A block code is a code which maps each of the symbol of the source alphabet 'S' into some "finite Sequence" of code symbols from the code alphabet 'X' and each of these finite sequence is called a "code-word".

   ex:- Consider a source S emitting only four symbols which are to be enclosed with binary coding
   $S=\{S_1, S_2, S_3, S_4\}$ and $X=\{0,1\}$

   A block code can be constructed by considering a pair of code symbols for each source symbol

Source Symbols	code-A
S_1	00
S_2	01
S_3	10
S_4	11

(2) Non-Singular : A block code is said to be "non-singular" if and only if all the code-words are distinct & easily "distinguishable" from one another.

ex:- Consider a source emitting 4 symbols with $X=\{0,1\}$ i.e $S=\{S_1, S_2, S_3, S_4\}$

Let us assign code words for these source symbols

Source Symbols	Code-B
S_1	0
S_2	00
S_3	01
S_4	11

The code-word given, appears to be non-singular, but actually it is not so, this can be confirmed by considering the second extension of these code-words which is shown below

Source Symbol	code B	Source Symbol	code B	SS	code B	SS	code B
S_1S_1	00	S_2S_1	000	S_3S_1	010	S_4S_1	100
S_1S_2	000	S_2S_2	0000	S_3S_2	0100	S_4S_2	1100
S_1S_3	001	S_2S_3	0001	S_3S_3	0101	S_4S_3	1101
S_1S_4	011	S_2S_4	0011	S_4S_4	0111	S_4S_4	1111

∴ The code-word for S_1S_2 and S_2S_1 are the "same", thus we can conclude that the second extension of code-B are not "distinct" and hence they become "singular"

(3) Uniquely Decodable Codes : A Non-Singular code is said to be uniquely Decodable code at the receiver, if every symbol in a long received sequence can be uniquely identified.

To understand this property let us consider second extension of "code A" which is uniquely Decodable code.

Source Symbols	code A	Source Symbols	code A	Source Symbols	code A	Source Symbols	code A
S_1S_1	0000	S_2S_1	0100	S_3S_1	1000	S_4S_1	1100
S_1S_2	0001	S_2S_2	0101	S_3S_2	1001	S_4S_2	1101
S_1S_3	0010	S_2S_3	0110	S_3S_3	1010	S_4S_3	1110
S_1S_4	0011	S_2S_4	0111	S_3S_4	1011	S_4S_4	1111

From the above table we can conclude that all codes are non-singular hence they can be uniquely decoded.

(4.) Instantaneous codes:- A uniquely Decodable codes is said to be "instantaneous codes" if it is possible to recognize the end of any code-word in any received sequence, without reference to the succeeding symbols.

Ex:- Let us consider 3 uniquely decodable codes:-

Source Symbols	Code-C	code-D	code-E
S_1	00	0	0
S_2	01	10	01
S_3	10	110	011
S_4	11	1100	0111

Let the received sequence be 001100

If code-C is used, then it is decoded as $S_1S_4S_1$.

-"- code-D -"- as $S_1S_1S_3S_1$.

-"- Code-E -"- as $S_1S_3S_1S_1$.

When the code-C and D are used for Decoding the given sequence, the symbols arrive at the receiver are decoded without referring to the succeeding symbols.

When code-E is used for decoding & when symbol '0' arrives at the receiver, we are not in a position to decode it. Since we have to wait for the succeeding symbols to arrive at the receiver. coz all the code-words in code-E start with '0'.

When the second symbol to arrives is '0', then the first '0' can be decoded as 'S_1', if the second symbol is '1' then again we have to wait for the succeeding symbols to arrive.

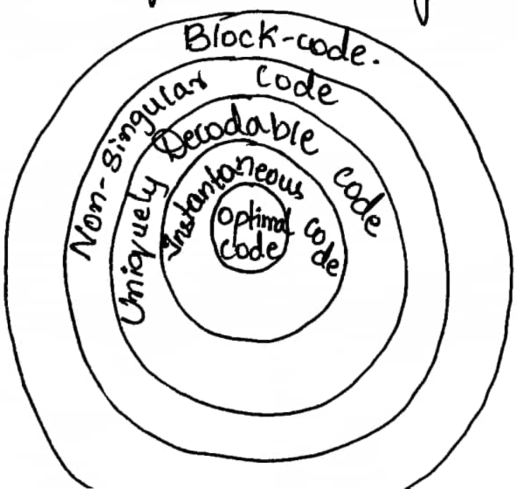
hence Code C, code D will be instantaneous codes
code E is not an instantaneous code.

(5) Optimal Codes: An instantaneous code is said to be "optimal code" if it has "minimum average length L" for a source with a probability for the source symbol.

Properties of code is represented as a tree-diagram.

- Code
  - Non-Block
  - Block
    - Singular
    - Non-Singular
      - Non Uniquely Decodable
      - Uniquely Decodable
        - Non-Instantaneous
        - Instantaneous
          - Non-optimal
          - Optimal

Code property Circle diagram.



There are many algorithms available for designing a source encoder. we have a simple most powerful source encoding algorithm given by Shannon.

Shannon's Encoding Algorithm

The design of the source encoder can be formulated as follows. The i/p to the source encoder consists of one of q possible messages, each message containing N symbols.

Let $P_1, P_2, P_3 \ldots P_q$ be the probabilities of these q messages. we would like to code the i^{th} message of m_i using a unique binary code-word C_i of length n_i bits, where n_i is an integer.

Our objective is to find n_i and C_i for $i=1,2,\ldots q$ Such that the average number of bits/symbol "$\hat{H}_N$" used in the coding scheme is as close to G_N as possible

i.e $$\hat{H}_N = \frac{1}{N}\sum_{i=1}^{q} P_i n_i \longrightarrow \frac{1}{N}\sum_{i=1}^{q} P_i \log_2 \frac{1}{P_i}$$

Several Solution have been proposed to the above problem & the algorithm given by Shannon (& Fano) is stated below.

"Suppose the q messages $m_1, m_2, \ldots m_q$ are arranged in order of decreasing probabilities Such that,

$$P_1 \geq P_2 \geq P_3 \ldots \geq P_q$$

Let, $$\boxed{F_i = \sum_{K=1}^{i-1} P_K \quad \text{with } F_1 = 0}$$

Let, n_i be an intger such that

$$\boxed{\log_2 \frac{1}{P_i} \leq n_i < 1 + \log_2 \frac{1}{P_i}}$$

Then the code word for the message m_i is the binary expansion of the fraction f_i up to n_i bits

$$\boxed{C_i = (f_i)_{binary}\ n_i} \text{ bits}$$

The algorithm yeild a source coding procedure that has the following properties.

1. Message of high probabilities are represented by short code words & those of low probabilities are represented by long code words.
2. The code word for m_i will differ from all succeding code words in one or more places & hence it is possible to decode messages uniquely from their code words.
3. The average number of bits/symbol used by the encoder is bounded by $\boxed{G_N \leq \hat{H}_N < G_N + \frac{1}{N}}$ where $N \rightarrow$ length of msg.
4. Rate efficiency e of encoder using block of 'N' symbo[ls] is defined as $\boxed{e = \frac{H}{\hat{H}_N}}$

Show that : $G_N \leq \hat{H}_N < G_N + \frac{1}{N}$

$\rightarrow$ if $\log_2(1/P_i) \leq n_i < 1 + \log_2\left(\frac{1}{P_i}\right) \longrightarrow (1)$

W.K.t $\hat{H}_N = \frac{1}{N}\sum_{i=1}^{q} P_i n_i$, $G_N = \frac{1}{N}\sum_{i=1}^{q} P_i \log_2 \frac{1}{P_i}$

Multiply eqn (1) by P_i/N

$$\frac{P_i}{N}\log_2\frac{1}{P_i} \leq \frac{n_i P_i}{N} < \frac{P_i}{N} + \frac{P_i}{N}\log_2\frac{1}{P_i}$$

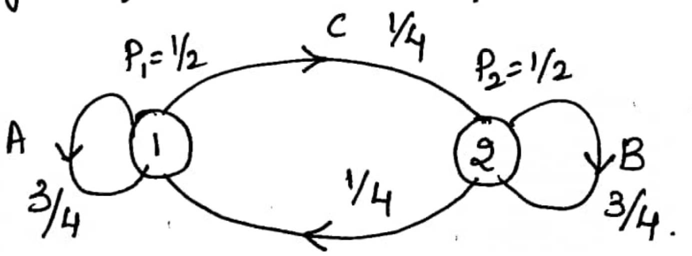
Taking $\sum_{i=1}^{q}$ on both sides

$$\frac{1}{N}\sum_{i=1}^{q} P_i\log_2\frac{1}{P_i} \leq \frac{1}{N}\sum_{i=1}^{q} n_i P_i < \frac{1}{N}\left[\sum_{i=1}^{q} P_i + \sum_{i=1}^{q} P_i \log_2\frac{1}{P_i}\right]$$

$\Rightarrow \boxed{G_N \leq \hat{H}_N < \frac{1}{N} + G_N.}$ $\quad \because \sum_{i=1}^{q} P_i = 1$

* Problems.

1. Design a source encoder for the information source given below. Compare the average output bit rate and efficiency of the encoder for $n = 1, 2$ and 3



$\Rightarrow$ $H = H_1 P_1 + H_2 P_2 + H_3 P_3$

$\boxed{H = 0.8112 \text{ bits/symbol}}$

→ for $\underline{N = 1}$

m_i	P_i	n_i	f_i	C_i
A	3/8	2	0	00
B	3/8	2	3/8	01
C	1/4	2	6/8	11

To find n_i

$\log_2 \frac{1}{P_i} \leq n_i < 1 + \log_2 \frac{1}{P_i}$

$\log_2 \frac{8}{3} \leq n_i < 1 + \log_2 \frac{8}{3}$

$1.4 \leq n_i < 2.4$

$n_i = 2$

∴ as per formula n_i should be greater than 1.4 and it should be less that 2.4, so for coding purpose we have taken '2' where n_i is an integer

To find f_i

we have $F_i = \sum_{K=1}^{i-1} P_K$ with $F_1 = 0$

∴ $F_1 = 0$.

$F_2 = \sum_{K=1}^{2-1} P_K = \boxed{F_2 = 3/8}$

$F_3 = \sum_{K=1}^{3-1} P_K \Rightarrow \frac{3}{8} + \frac{3}{8} \Rightarrow \boxed{F_2 = \frac{6}{8}}$

To find C_i

$C_1 = 00$ coz $f_i = 0$ and length $n_i = 2$ ∴ $C_i = 00$

$C_2 = \frac{3}{8} = 0.375 \times 2 = 0.75 \Rightarrow 0$
$\quad 0.75 \times 2 = 1.5 \Rightarrow 1$ } but we take $C_2 = 01$ coz $n_i = 2$

$\quad 0.5 \times 2 = 1.0 \Rightarrow 1$

$$\boxed{\hat{H}_1 = 1 \sum_{i=1}^{3} P_i n_i}$$

$$= \frac{3}{8} \times 2 + \frac{3}{8} \times 2 + \frac{1}{4} \times 2$$

$$\boxed{\hat{H}_1 = 2 \text{ bits / symbol}}$$

$$\boxed{G_1 = \frac{1}{1} \sum_{i=1}^{3} P_i \log_2 \frac{1}{P_i}}$$

$$G_1 = \frac{3}{8} \log_2 \frac{8}{3} + \frac{3}{8} \log_2 \frac{8}{3} + \frac{1}{4} \log_2 4 \Rightarrow \boxed{1.5612 \text{ bits/symbol}}$$

$$\boxed{G_N + \frac{1}{N}} \Rightarrow G_1 + \frac{1}{1} \Rightarrow 1.5612 + 1 \quad G_N + \frac{1}{N} \Rightarrow \boxed{2.5612}$$

Rate efficiency $\boxed{e : \frac{H}{\hat{H}_N}} \Rightarrow \boxed{40.56\,\%}$

for $N = 2$

m_i	P_i	n_i	F_i	C_i
AA	$\frac{9}{32}$	2	0	00
BB	$\frac{9}{32}$	2	$\frac{9}{32}$	01
AC	$\frac{3}{32}$	4	$\frac{18}{32}$	1001
CB	$\frac{3}{32}$	4	$\frac{21}{32}$	1010
BC	$\frac{3}{32}$	4	$\frac{24}{32}$	1100
CA	$\frac{3}{32}$	4	$\frac{27}{32}$	1101
CC	$\frac{2}{32}$	4	$\frac{30}{32}$	1111

$$\hat{H}_2 = \frac{1}{2} \sum_{i=1}^{7} P_i n_i$$

$$H_2 = \frac{1}{2} \left[\frac{9}{32} \times 2 \times 2 + \frac{3}{32} \times 4 \times 4 + \frac{2}{32} \times 4 \right]$$

$$H_2 = 1.4375$$

$$G_2 = \frac{1}{2} \sum_{i=1}^{7} P_i \log_2 \frac{1}{P_i}$$

$$G_2 = \frac{1}{2} \left[\frac{9}{32} \log_2 \frac{32}{9} \times 2 + \frac{3}{32} \log_2 \frac{32}{3} \times 4 + \frac{2}{32} \log_2 \frac{32}{2} \right]$$

$\boxed{G_2 = [illegible] \text{ bits/symbol}}$ $G_2 = \frac{1}{2}[0.294 + 1.286 + 0.25$

$$G_N + \frac{1}{N} \Rightarrow G_2 + \frac{1}{2} \Rightarrow 1.28 + \frac{1}{2} \Rightarrow 1.78$$

$$e = \frac{H}{\hat{H}_2} = \frac{0.8112}{1.4375} \quad e = 0.5643 \Rightarrow \boxed{e = 56.43\,\%}$$

for $N=3$

m_i	P_i	n_i	F_i	C_i
AAA	$\frac{27}{128}$	3	0	000
BBB	$\frac{27}{128}$	3	$\frac{27}{128}$	001
AAC	$\frac{9}{128}$	4	$\frac{54}{128}$	0110
ACB	$\frac{9}{128}$	4	$\frac{63}{128}$	0111
CBB	$\frac{9}{128}$	4	$\frac{72}{128}$	1001
CAA	$\frac{9}{128}$	4	$\frac{81}{128}$	1010
BCA	$\frac{9}{128}$	4	$\frac{90}{128}$	1011
BBC	$\frac{9}{128}$	4	$\frac{99}{128}$	1100
CBC	$\frac{3}{128}$	6	$\frac{108}{128}$	110110
CCA	$\frac{3}{128}$	6	$\frac{111}{128}$	110111
CAC	$\frac{3}{128}$	6	$\frac{114}{128}$	111001
CCB	$\frac{3}{128}$	6	$\frac{117}{128}$	111010
BCC	$\frac{3}{128}$	6	$\frac{120}{128}$	111100
ACC	$\frac{3}{128}$	6	$\frac{123}{128}$	111101
CCC	$\frac{2}{128}$	6	$\frac{126}{128}$	111111

$$\hat{H}_N = \frac{1}{N} \sum_{i=1}^{q} P_i n_i$$

$$\hat{H}_3 = \frac{1}{3} \sum_{i=1}^{15} P_i n_i$$

$$\hat{H}_3 = \frac{1}{3}\left[\frac{27}{128} \times 3 \times 2 + \frac{9}{128} \times 4 \times 6 + \frac{3}{128} \times 6 \times 6 + \frac{2}{128} \times 6\right]$$

$$\hat{H}_3 = \frac{1}{3}\left[1.2656 + 1.6875 + 0.84375 + 0.09375\right]$$

$$\hat{H}_3 = \frac{1}{3}\left[3.8906\right] \Rightarrow 1.30$$

$$\boxed{\hat{H}_3 = 1.30 \text{ bits/symbol}}$$

$$G_N = \frac{1}{N} \sum_{i=1}^{q} P_i \log_2 \frac{1}{P_i}$$

$$G_3 = \frac{1}{3} \sum_{i=1}^{15} P_i \log_2 \frac{1}{P_i}$$

$$\boxed{G_3 = 1.139}$$

$$G_N \leq \hat{H}_N < G_N + \frac{1}{N}$$

$$1.13 \leq 1.3 < 1.47$$

$$e = \frac{H}{\hat{H}_3} = \frac{0.8112}{1.30} = 0.624$$

$$\boxed{e = 62.40\,\%}$$

∴ Performance of the encoder

Average number of bits per symbol used	N=1	N=2	N=3
$\hat{H}_N$	2	1.44	1.30
G_N	1.561	1.28	1.139
$G_N + \frac{1}{N}$	2.561	1.78	1.47.
efficiency $= \frac{H}{\hat{H}_N}$	40.56%.	56.43%.	62.40%.
$H = 0.8113.$			

(2) A Source emits independent sequence of symbols from a Source alphabet containing 8 symbols $S_1, S_2, S_3, \ldots \ldots S_8$ with probabilities $1/4$, $1/4$, $1/8$, $1/8$, $1/16$, $1/16$, $1/16$, $1/16$. Design a Source encoder for a block size of 1.

Soln

S_i	P_i	n_i	F_i	C_i
S_1	$1/4$	2	0	00
S_2	$1/4$	2	$1/4$	01
S_3	$1/8$	3	$2/4$	100
S_4	$1/8$	3	$5/8$	100
S_5	$1/16$	4	$6/8$	1100
S_6	$1/16$	4	$13/16$	1101
S_7	$1/16$	4	$14/16$	1110
S_8	$1/16$	4	$15/16$	1111

$\rightarrow \log_2 \frac{1}{P_i} \leq n_i < \log \frac{1}{P_i} + 1$

$2 \leq n_i < 3$

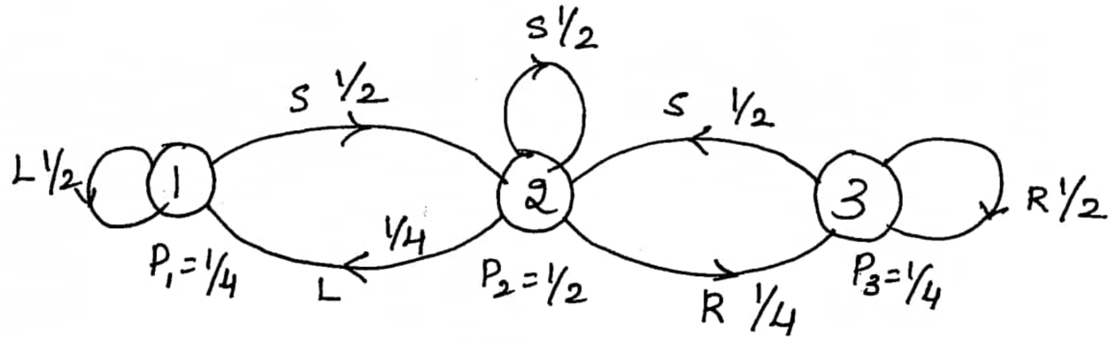
$\rightarrow F_i = \sum_{K=1}^{i-1} P_K.$

$\rightarrow \hat{H}_N = \frac{1}{N} \sum_{i=1}^{q} P_i n_i$

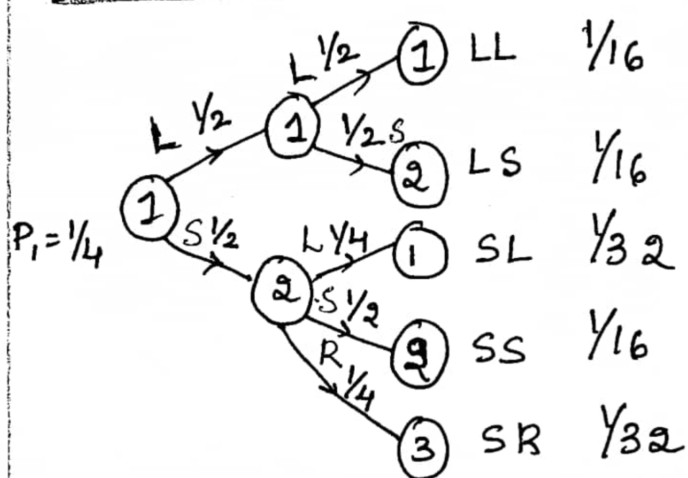
$$\hat{H}_N = \frac{1}{1} \sum_{i=1}^{8} P_i n_i$$

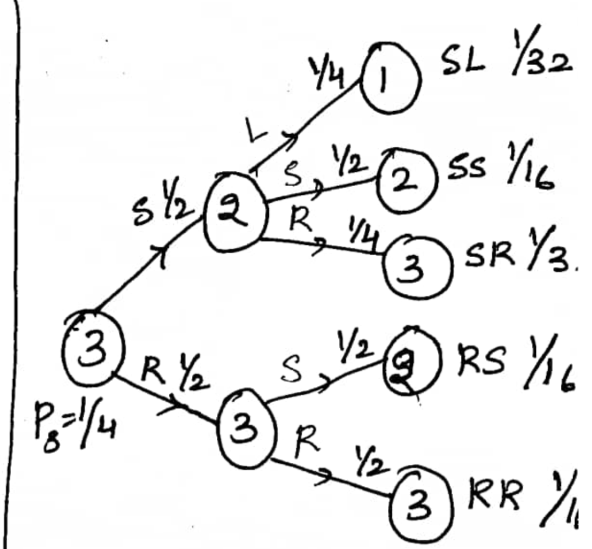
$$= \frac{1}{1}\left[\frac{1}{4} \times 2 \times 2 + \frac{1}{8} \times 3 \times 2 + \frac{1}{16} \times 4 \times 4\right]$$

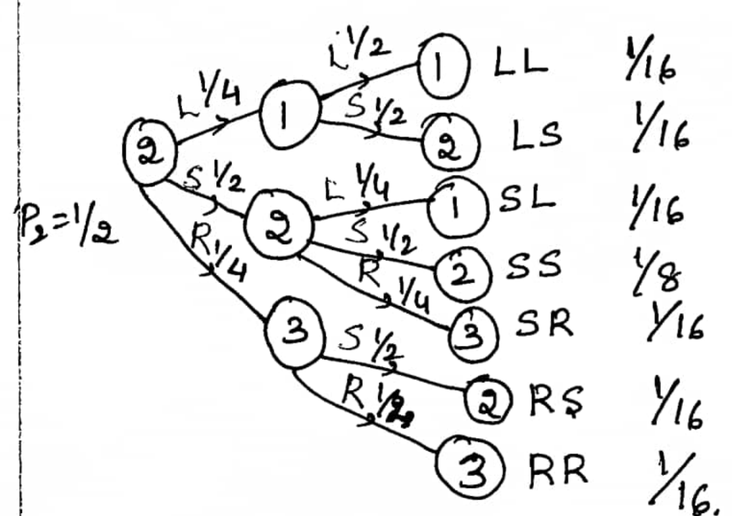
3. For the given source design the source encoding scheme using a block size of 2 symbols & variable length code words. Calculate the actual number of bits/symbol used by the encoder and verify that $\hat{H}_2 \le G_2 + 1/2$.
If the source is emitting symbols at a rate of 1000 symbols/sec. compute the output bit rate of the encoder



→ Given N=2.







∴

m_i	P_i
LL	1/8
LS	1/8
SL	1/8
SS	2/8
SR	1/8
RS	1/8
RR	1/8.

$\therefore$ So as per Source coding algorithm the probabilities should be arranged in decreasing order

m_i	P_i	n_i	f_i	C_i
SS	2/8	2	0	00
LS	1/8	3	2/8	010
SL	1/8	3	3/8	011
LL	1/8	3	4/8	100
SR	1/8	3	5/8	101
RS	1/8	3	6/8	110
RR	1/8	3	7/8	111

$\Rightarrow$ To calculate C_i

for SS $\Rightarrow$ As $f_1 = 0$ & $n_i = 2$

$C_i = 00$

LS $\Rightarrow F_2 = \frac{2}{8} \Rightarrow F_2 = 0.25$

$\therefore 0.25 \times 2 = 0.5 = 0$

$0.5 \times 2 = 1.0 = 1$

$0 \times 0 = 0 = 0$

SL $\Rightarrow F_3 = 3/8 \Rightarrow F_3 = 0.375$

$0.375 \times 2 = 0.75 = 0$

$0.75 \times 2 = 1.5 = 1$

$0.5 \times 2 = 1.0 = 1$

|||ly for F_4 to F_7

$$\therefore \hat{H}_N = \frac{1}{N} \sum_{i=1}^{q} P_i n_i$$

$$\hat{H}_2 = \frac{1}{2} \sum_{i=1}^{7} P_i n_i \Rightarrow \frac{1}{2}\left[2 \times \frac{2}{8} + \frac{1}{8} \times 3 \times 6\right]$$

$$\hat{H}_2 = \frac{1}{2}(2.75) \qquad \boxed{\hat{H}_2 = 1.375 \text{ bits/symbol}}$$

$$G_N = \frac{1}{N} \sum_{i=1}^{q} P_i \log_2 \frac{1}{P_i} \text{ bits/symbol}$$

$$G_2 = \frac{1}{2}\left[\frac{2}{8} \log_2 \frac{8}{2} + \frac{1}{8} \log_2 8 \times 6\right]$$

$$\boxed{G_2 = 1.375 \text{ bits/symbol}}$$

$$\hat{H}_2 \leq G_2 + \frac{1}{2} \Rightarrow \boxed{1.375 \leq 1.875}$$

$H = 1(1/4) + 1.5(1/2) + \frac{1}{4}(1) = 1.25$ bits/symbol

$R = r_s . H \Rightarrow R = (1000)(1.25) \quad \boxed{R = 1250 \text{ bits/sec}}$

Shannon-Fano encoding algorithm.

Shannon Fano encoding algorithm is an Another technique used in Constructing a source encoder.

- A Source encoder consists of arranging the messages in decreasing order of its probability and dividing the messages into two almost equally probable groups.
- The message in the 1st group is given bit '0', the message in the 2nd group are given bit 1.
- The procedure is now applied for each group seperately and continued until no further division is possible.
- Finally we get the code words for respective Symbol.

Problems.

① Find the code words Occuring in the probability 1/2, 1/4, 1/8 & 1/8 for symbols S_1, S_2, S_3 & S_4

→

Symbols	P_i	I	II	III	C_i	length n_i
s_1	1/2	0			0	1
s_2	1/4	1	$\frac{1}{4}$ 10		10	2
s_3	1/8	1	$\frac{1}{8}$ 11	$\frac{1}{8}$ 110	110	3
s_4	1/8	1	$\frac{1}{8}$ 11	$\frac{1}{8}$ 111	111	3

$$H = \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 \times 2 \Rightarrow \boxed{H = 1.75 \text{ bits/sy}}$$

$$\hat{H}_1 = \frac{1}{1}\sum_{i=1}^{4} P_i n_i \Longrightarrow \hat{H}_1 = 1.75 \text{ bits/symbol}$$

$$\therefore \quad e = \frac{H}{\hat{H}} \Rightarrow \frac{1.75}{1.75} = 100\%$$

2. Find the code word for the probability $1/4, 1/4, 1/8, 1/8, 1/16, 1/16, 1/16, 1/16$ for the symbols $S_1 \ldots S_8$.

→

Symbols	P_i					C_i	n_i
S_1	$1/4$	0	$1/4$] 00			00	2
S_2	$1/4$	0	$1/4$] 01			01	2
S_3	$1/8$	1	$1/8$] 10	$\frac{1}{8}$] 100		100	3
S_4	$1/8$	1	$1/8$] 10	$\frac{1}{8}$] 101		101	3
S_5	$1/16$	1	$1/16$] 11	$\frac{1}{16}$] 110	$\frac{1}{16}$] 1100	1100	4
S_6	$1/16$	1	$1/16$] 11	$\frac{1}{16}$] 110	$\frac{1}{16}$] 1101	1101	4
S_7	$1/16$	1	$1/16$] 11	$\frac{1}{16}$] 111	$\frac{1}{16}$] 1110	1110	4
S_8	$1/16$	1	$1/16$] 11	$\frac{1}{16}$] 111	$\frac{1}{16}$] 1111	1111	4

$$H = \left[\frac{1}{4}\log_2 4\right] \times 2 + \left[\frac{1}{8}\log_2 8\right] \times 2 + \left[\frac{1}{16}\log_2 16\right] \times 4$$

$H = 2.75$ bits/symbol

$$\hat{H}_1 = \frac{1}{1}\left[\frac{1}{4} \times 2 \times 2 + \frac{1}{8} \times 3 \times 2 + \frac{1}{16} \times 4 \times 4\right] \qquad \hat{H}_1 = 2.75 \text{ bits/symbol}$$

$$e = \frac{H}{\hat{H}} = \frac{2.75}{2.75} = 100\,\% \quad \Rightarrow \boxed{e \leq 100\,\%} \leftarrow \text{Note}$$

Redundancy : The Coding Redundancy denoted by R_e is defined as

$$R_e = 1 - e \Rightarrow 1 - 100\,\% = 0\,\%$$

where efficiency and Redundancy are expressed as percentages.

Using Shannon Fano encoding algorithm,
(3) Find the code word for the probability $33/64$, $1/4$, $1/8$, $1/16$, $1/32$, $1/64$ for the symbols $S_1 \ldots S_6$

→

Symbols	P_i									
S_1	$33/64$	0								
S_2	$1/4$	1	$\frac{1}{4}$	10						
S_3	$1/8$		$\frac{1}{8}$	11	$\frac{1}{8}$	110				
S_4	$1/16$		$\frac{1}{16}$		$\frac{1}{16}$	111	$\frac{1}{16}$	1110		
S_5	$1/32$		$\frac{1}{32}$		$\frac{1}{32}$		$\frac{1}{32}$	1111	$\frac{1}{32}$	11110
S_6	$1/64$		$\frac{1}{64}$		$\frac{1}{64}$		$\frac{1}{64}$		$\frac{1}{64}$	11111

C_i	n_i len
0	1
10	2
110	3
1110	4
11110	5
11111	5

$$H = \frac{33}{64}\log_2\frac{64}{33} + \frac{1}{4}\log_2 4 + \frac{1}{8}\log_2 8 + \frac{1}{16}\log_2 16 + \frac{1}{32}\log_2 32 + \frac{1}{64}\log_2 64$$

$H = 1.867$ bits/symbol

$$\hat{H}_N = \frac{1}{N}\sum_{i=1}^{n} P_i n_i \qquad \hat{H}_1 = \frac{1}{1}\sum_{i=1}^{6} n_i P_i$$

$$\hat{H}_1 = \frac{33}{64}\times 1 + \frac{1}{4}\times 2 + \frac{1}{8}\times 3 + \frac{1}{16}\times 4 + \frac{1}{32}\times 5 + \frac{1}{64}\times 5$$

$\hat{H}_1 = 1.875$ bits/symbol

(4)

Symbols	P_i			
S_1	$1/3$	0	$\frac{1}{3}$	00
S_2	$1/3$		$\frac{1}{3}$	01
S_3	$1/6$	1	$\frac{1}{6}$	10
S_4	$1/12$		$\frac{1}{12}$	11
S_5	$1/24$		$\frac{1}{24}$	
S_6	$1/24$		$\frac{1}{2\cdot}$	

2^{nd} Method.

Another way of generating code word for message consists of arranging the messages in decreasing order of probability and assigning the code words as follows.

The code word for 1^{st} message is 0,

the code word for i^{th} message ~~is~~ consists of $i-1$ bits of ones followed by a zero.

The code word for the last message is all ones. such that the no of bits in the code word for the last message is equal to the total no of messages that have to be encoded.

Problem.

a) find the code words & the avg number of bits/message used if the source emits one of 5 messages with probabilities $1/2$, $1/4$, $1/8$, $1/16$ and $1/16$.

→

m_i	P_i	C_i	n_i
m_1	$1/2$	0	1
m_2	$1/4$	10	2
m_3	$1/8$	110	3
m_4	$1/16$	1110	4
m_5	$1/16$	11111	5

$\therefore \hat{H}_N = 1.9375$ bits/symbol

$H = 1.875$ bits/symbol

$$e = \frac{H}{\hat{H}_N}$$

$e = 96.77\%$

(2) A Source producer 2 symbols with probabilities 3/4 and 1/4. Design a variable length code using Shanon-Fano algorithm, so that the coding efficiency is atleast 85%

→

S_i	P_i		n_i
S_1	3/4	0	1
S_2	1/4	1	1

$$H_1 = \sum_{i=1}^{2} P_i \log_2 \frac{1}{P_i} \qquad H = \frac{3}{4}\log_2 \frac{4}{3} + \frac{1}{4}\log_2 4$$

$$\boxed{H = 0.8112 \text{ bits/symbol}}$$

$$\hat{H}_N = \frac{1}{N}\sum_{i=1}^{n} P_i n_i \quad \text{where } N = 1$$

$$\hat{H}_N = \frac{1}{1} \times \left[\frac{3}{4}\times 1 + \frac{1}{4}\times 1\right] \qquad \boxed{\hat{H}_N = 1 \text{ bit/symbol}}$$

$$e = \frac{H}{\hat{H}_N} = \frac{0.8112}{1} = 81.12\%$$

As they have asked to find the coding efficiency for atleast 85%, we will go for length i.e $N=2$

S_i	P_i	I	II	III	n_i
S_1S_1	9/16	0			1
S_1S_2	3/16		3/16 10	3/16 110	2
S_2S_1	3/16	1	3/16 11	3/16 111	3
S_2S_2	1/16		3/16		3

$$\hat{H}_2 = \frac{1}{2}\sum_{i=1}^{4} P_i n_i = \frac{1}{2}\left[\qquad \right] = 0.8437$$

$$e = \frac{H}{\hat{H}_2} = \boxed{96.15\%}$$

HUFFMAN CODING.

The huffman encoding algorithm is

1. Arrange the symbols/messages in the decreasing order of probability.
2. Check the relation, $\boxed{n = r + \alpha(r-1)}$ is satisfied.

   where, n = no of symbols

   r = length of the code alphabet

   if $r = 2$ $(X = 0, 1)$ → Binary huffman coding

   $r = 3$ $(X = 0, 1, 2)$ → Ternary huffman coding.

   $r = 4$ $(X = 0, 1, 2, 3)$ → Quarternary huffman coding.

   α = no of reduction stages. & α should be integer.

   if the above relation doesnot give 'α' as integer, add a dummy symbol with probability equal to '0' so that the above relation is satisfied with α being an integer.
3. Combine last 'r' symbols and move the combined symbol as high as possible.
4. Repeat the above step till exactly 'r' symbols are left in last stage.
5. The code for each symbol is found by moving backward to forward direction.
6. Calculate efficiency by using formula $\boxed{e = \dfrac{H}{\bar{L} \log_2 r}}$

   where $\bar{L}$ is Average code word length $\boxed{\bar{L} = \sum_{i=1}^{n} P_i n_i}$
7. Calculate variance using $\boxed{\sigma^2 = \sum_{i=1}^{n} P_i (n_i - \bar{L})^2}$

Problems.

(1) A discrete memoryless source has a alphabet of 5 symbols with their probabilities given below. find huffman codes for the source and also find the efficiency and variance.

S_i	P_i	I	II	III	code	n_i
S_1	0.4	0.4	0.4	0.6 (0)	00	2
S_2	0.2	0.2	0.4 (0)	0.4 (1)	10	2
S_3	0.2	0.2 (0)	0.2 (1)		11	2
S_4	0.1 (0)	0.2 (1)			010	3
S_5	0.1 (1)				011	3

$$n = r + \alpha(r-1)$$
$$5 = 2 + \alpha(2-1)$$
$$\alpha = 5-2 \Rightarrow \boxed{\alpha = 3}$$

$$H = \sum_{i=1}^{5} P_i \log_2 \frac{1}{P_i} \Rightarrow H = 0.4 \log_2 \frac{1}{0.4} + 0.2 \log_2 \frac{1}{0.2} \times 2 + 0.1 \log_2 \frac{1}{0.1} \times$$

$$\boxed{H = 2.1216 \text{ bits/symbol}}$$

$$\bar{L} = \sum_{i=1}^{n} P_i n_i \Rightarrow 0.4 \times 2 + 0.2 \times 2 \times 2 + 0.1 \times 3 \times 2$$

$$\boxed{\bar{L} = 2.2 \text{ bits/symbol}}$$

$$e = \frac{H}{\bar{L} \log_2 r} \Rightarrow \frac{2.1216}{2.2 \log_2 2} \quad \boxed{e = 96.4\%}$$

$$\sigma^2 = \sum_{i=1}^{n} P_i (n_i - \bar{L})^2 = 0.4(2-2.2)^2 + 0.2(2-2.2)^2 \times 2 + 0.1(3-2.2)$$

$$\boxed{\sigma = 0.16}$$

The variance should be as low as possible.

- Huffman coding as high as possible is preferable.

Note: variance is a measure of code word length. we prefer as high as possible, since variance value is very less compared to as low as possible.

(2)

S_i	P_i	I	II	III	IV	V
S_1	0.4	0.4	0.4	0.4	0.4	0.6
S_2	0.2	0.2	0.2	0.2	0.4	0.4
S_3	0.2	0.2	0.2	0.2	0.2	
S_4	0.05	0.1	0.1	0.2		
S_5	0.05	0.05	0.1			
S_6	0.05	0.05				
S_7	0.05					

$H = 2.32175$ bits/symbol

codes	n_i
00	2
10	2
11	2
0100	4
0101	4
0110	4
0111	4

$\bar{L} = \sum_{i=1}^{7} P_i n_i = 2.4$ bits/symbol

$$\boxed{e = \frac{H}{\bar{L}} = 96.7\,\%}$$

(3)

S_i	P_i	I	II	code	n_i
S_1	1/2	1/2	1/2	1	1
S_2	1/4	1/4	1/2	01	2
S_3	1/8	1/4		000	3
S_4	1/8			001	3

$H = 0.5 + 0.5 + 0.75 = 1.75$ bits/symbol

$\bar{L} = \sum_{i=1}^{m} P_i n_i = 1.75$ bits/symbol

$$\boxed{e = \frac{H}{\bar{L}} = \frac{1.75}{1.75} = 100\,\%}$$

(4) A Source produces 9 symbols, that are equiprobable

(a) Construct binary huffman coding by moving the Combined symbol as high as possible. Also find efficiency and variance of coding.

(b) Construct binary huffman code by moving the Combined symbol as low as possible. Also find efficiency and variance of coding.

9 Symbols $\Rightarrow$ 0.11, 0.11, 0.11, 0.11, 0.11, 0.11, 0.11, 0.11, 0.11

→ (1) As high as possible $\Rightarrow$ $n = r + \alpha(r-1)$

$9 = 2 + \alpha(2-1) \Rightarrow \boxed{\alpha = 7}$

S_i	P_i	I	II	III	IV	V	V	VI
S_1	0.11	0.22	0.22	0.22	0.22	0.33	0.44	0.55
S_2	0.11	0.11	0.22	0.22	0.22	0.22	0.33 0	0.4
S_3	0.11	0.11	0.11	0.22	0.22	0.22 0	0.22 1	
S_4	0.11	0.11	0.11	0.11	0.22 0	0.22 1		
S_5	0.11	0.11	0.11	0.11 0	0.11 1			
S_6	0.11	0.11	0.11 0	0.11 1				
S_7	0.11	0.11 0	0.11 1					
S_8	0.11 0	0.11 1						
S_9	0.11 1							

C_i	n_i
001	3
010	3
011	3
100	3
101	3
110	3
111	3
0000	4
0001	4

$$H = \sum_{i=1}^{n} P_i \log_2 \frac{1}{P_i}$$

$$H = \sum_{i=1}^{9} 0.11 \log_2 \frac{1}{0.11}$$

$$H = 0.11 \log_2 \frac{1}{0.11} \times 9$$

$$\boxed{H = 3.152}$$

$$\bar{L} = \sum_{i=1}^{n} P_i n_i$$

$$\bar{L} = \sum_{i=1}^{9} P_i n_i \quad \boxed{\bar{L} = 3.19}$$

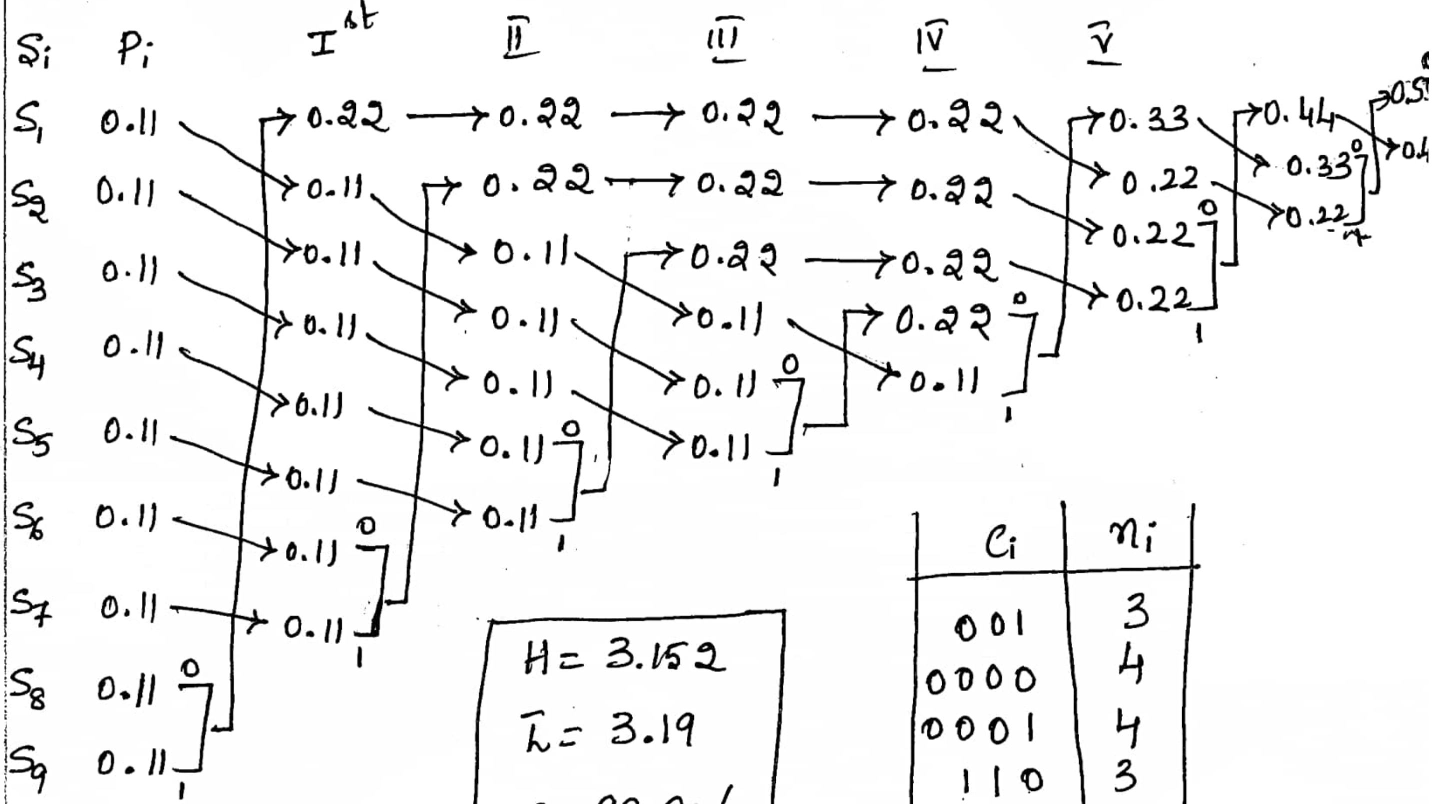
$$e = \frac{H}{\bar{L}\log_2 r} = \frac{3.152}{3.19 \log_2 2}$$

$$\boxed{e = 98.8\%}$$

$$\sigma^2 = \sum_{i=1}^{n} P_i (n_i - \bar{L})^2$$

$$\sigma^2 = 0.11(3-3.19)^2 \times 7 + 0.11(4-3.19)^2 \times 2 \Rightarrow \sigma^2 = 0.172 \Rightarrow \boxed{\sigma = 0.414}$$

② As low as possible.



$H = 3.152$

$\bar{L} = 3.19$

$e = 98.8\%$

$\sigma = 0.414$

C_i	n_i
001	3
0000	4
0001	4
110	3
011	3
100	3
011	3
010	3
110	3

③ A Source produces 9 symbols, Construct ternary, Quartenary huffman coding by moving symbols as high as possible. Also find efficiency & variance of coding.

$P_i = 0.11, 0.11, 0.11, 0.11, 0.11, 0.11, 0.11, 0.11, 0.11$

→ So Given : $n = 9$ symbols

$r = 3$ (∵ ternary huffman coding $r = 0, 1, 2$)

$\alpha = ?$

$\therefore n = r + \alpha(r-1)$

$9 = 3 + \alpha(3-1) \Rightarrow \boxed{\gamma = 3}$

S_i	P_i	1	2	3	C_i	n_i
S_1	0.11	0.33	0.33	0.33 0	00	2
S_2	0.11	0.11	0.33	0.33 1	01	2
S_3	0.11	0.11	0.11 0	0.33 2	02	2
S_4	0.11	0.11	0.11 1		10	2
S_5	0.11	0.11 0	0.11 2		11	2
S_6	0.11	0.11 1			12	2
S_7	0.11 0	0.11 2			20	2
S_8	0.11 1				21	2
S_9	0.11 2				22	2

$H = 3.169$ bits/symbol
$\bar{L} = 1.98$ bits/symbol
$e = 100\%$
$\sigma = 0.019$.

Quarternery Coding $n = r + \alpha(r-1)$
$9 = 4 + 3\alpha$ $\boxed{\alpha = 1.6}$

α should be integer. Add dummy symbol with prob = 0.

$10 = 4 + 3\alpha \Rightarrow \boxed{\alpha = 2}$

S_i	P_i	1st	2nd	C_i	n_i
S_1	0.11	0.33	0.44 0	2	1
S_2	0.11	0.11	0.33 1	3	1
S_3	0.11	0.11	0.11 2	00	2
S_4	0.11	0.11 0	0.11 3	01	2
S_5	0.11	0.11 1		02	2
S_6	0.11	0.11 2		03	2
S_7	0.11 0	0.11 3		10	2
S_8	0.11 1			11	2
S_9	0.11 2			12	2
S_{10}	0.0 3			13	2

$H = 3.169$
$\bar{L} = 1.76$
$e = 89.51\%$
$\sigma =$

(4) Construct Quarternery huffman Coding by moving symbols as high as possible. find efficiency and variance.

$S_1 = 1/3$, $S_2 = 1/4$, $S_3 = 1/8$, $S_4 = 1/8$, $S_5 = 1/12$, $S_6 = 1/12$

→ $n = 6$ symbols

$r = 4$ (Quarternery)

$\alpha = ?$

$n = r + \alpha(r-1)$

$6 = 4 + \alpha(4-1)$

$6 - 4 = 3\alpha$

$\boxed{\alpha = 0.667}$

α should be integer add dummy variable with probability equal to '0'

$n = 7 \Rightarrow \boxed{n = 6+1}$

$\Rightarrow 7 - 4 = 3\alpha$

$\boxed{\alpha = 1}$

S_i	P_i		1st stage		C_i	n_i
S_1	1/3		1/3 (0.33)	0	0	1
S_2	1/4		7/24 (0.29)	1	2	1
S_3	1/8		1/4 (0.25)	2	3	1
S_4	1/8	0	1/8 (0.125)	3	10	2
S_5	1/12	1			11	2
S_6	1/12	2			12	2
S_7	0	3			13	2

$\boxed{H = 2.3744 \text{ bits/symbol}}$

$\boxed{\bar{L} = 1.292 \text{ bits/symbol}}$

$$e = \frac{H}{\bar{L}\log_2 4}$$

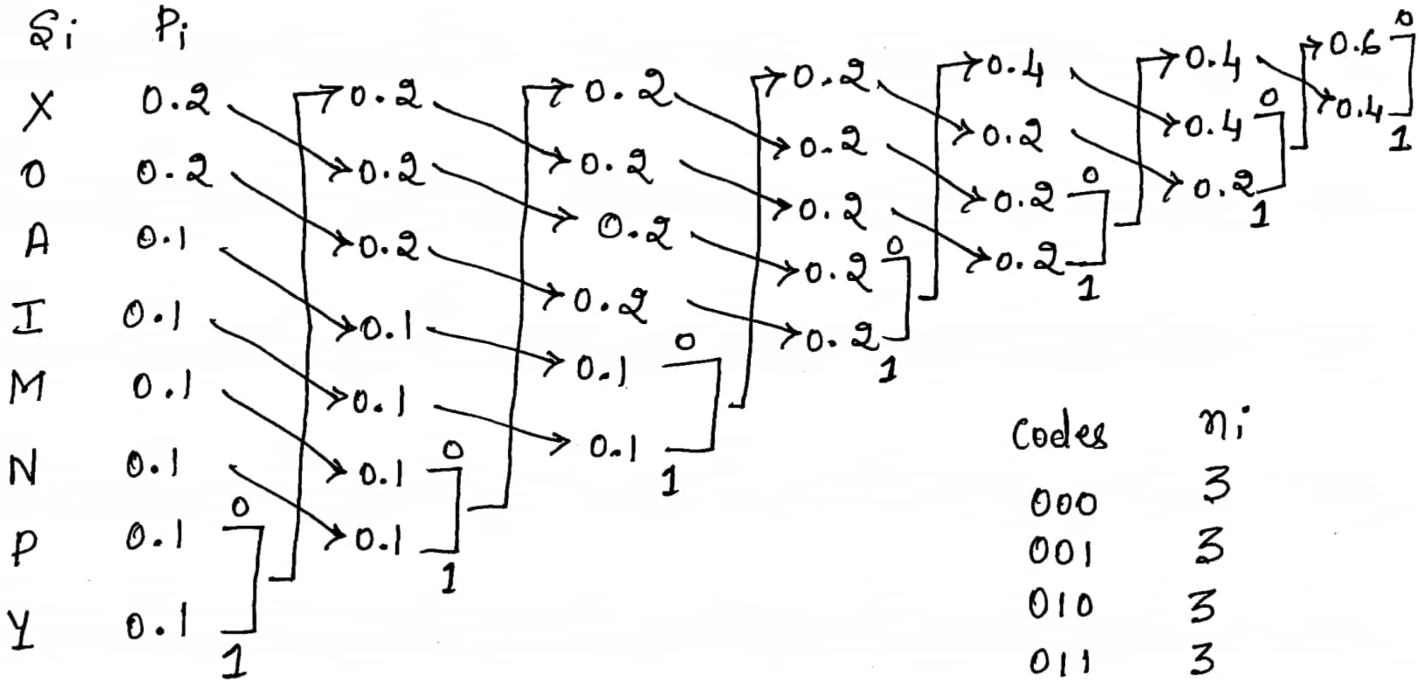
$\boxed{e = 91.96\%}$

$\boxed{\sigma = 0.2065}$

5. Consider a sequence of letters from english alphabet with their probability of occurance as

A	I	X	M	N	O	P	Y
0.1	0.1	0.2	0.1	0.1	0.2	0.1	0.1

Compute 2 different huffman codes for this alphabet. for the two codes find average code word length & variance of code word length.



Codes	n_i
000	3
001	3
010	3
011	3
100	3
101	3
110	3
111	3

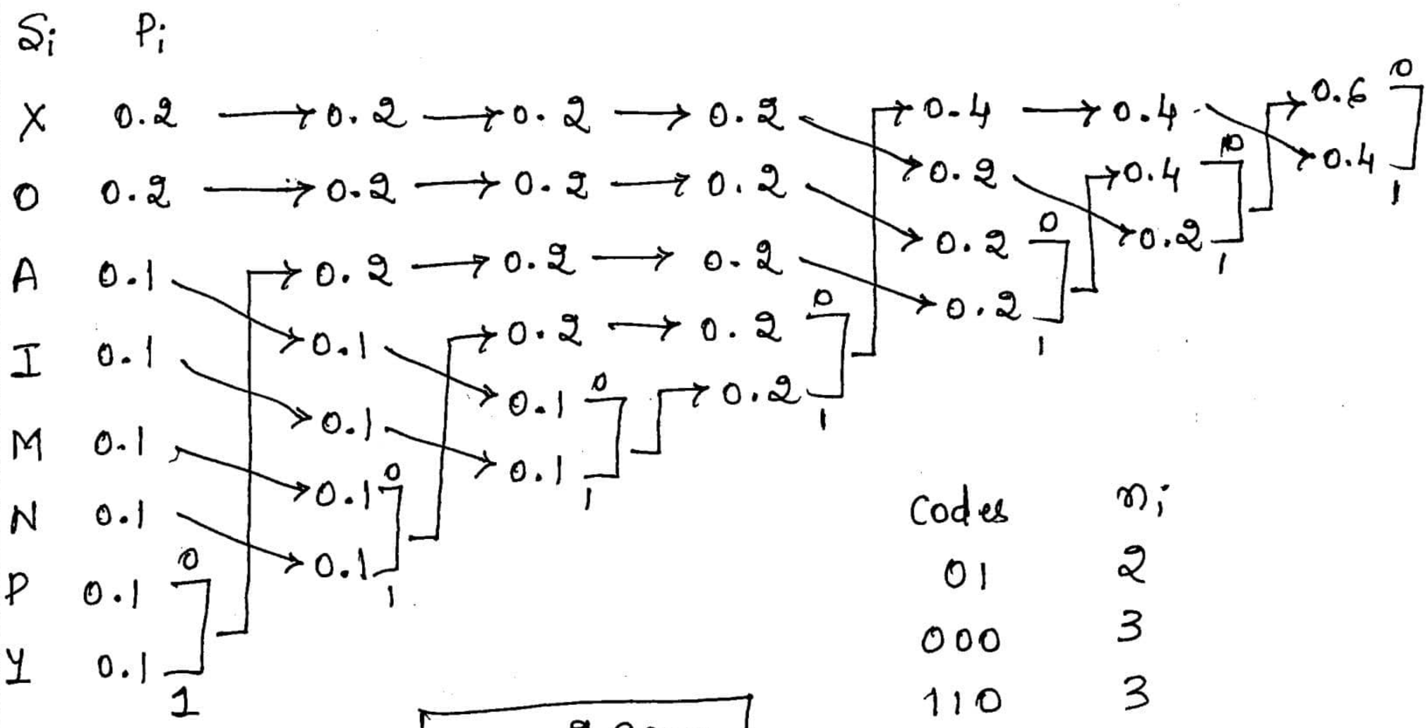
$\boxed{H = 2.9218}$

$\boxed{\bar{L} = 3 \text{ bits/symbol}}$

$$\sigma^2 = \sum_{i=1}^{n} P_i (n_i - \bar{L})^2$$

$$= 0.2(3-3)^2 \times 2 + 0.1(3-3)^2 \times 6$$

$\boxed{\sigma^2 = 0}$



Codes	n_i
01	2
000	3
110	3
111	3
100	3
101	3
~~0010~~	
0010	4
0011	4

$$\boxed{\begin{aligned} H &= 2.9218 \\ \bar{L} &= 3 \\ \sigma^2 &= 0.4 \end{aligned}}$$

So we prefer as high as possible.

Fundamental limits on Performance & Channels for Communication

Communication Channels - Introduction

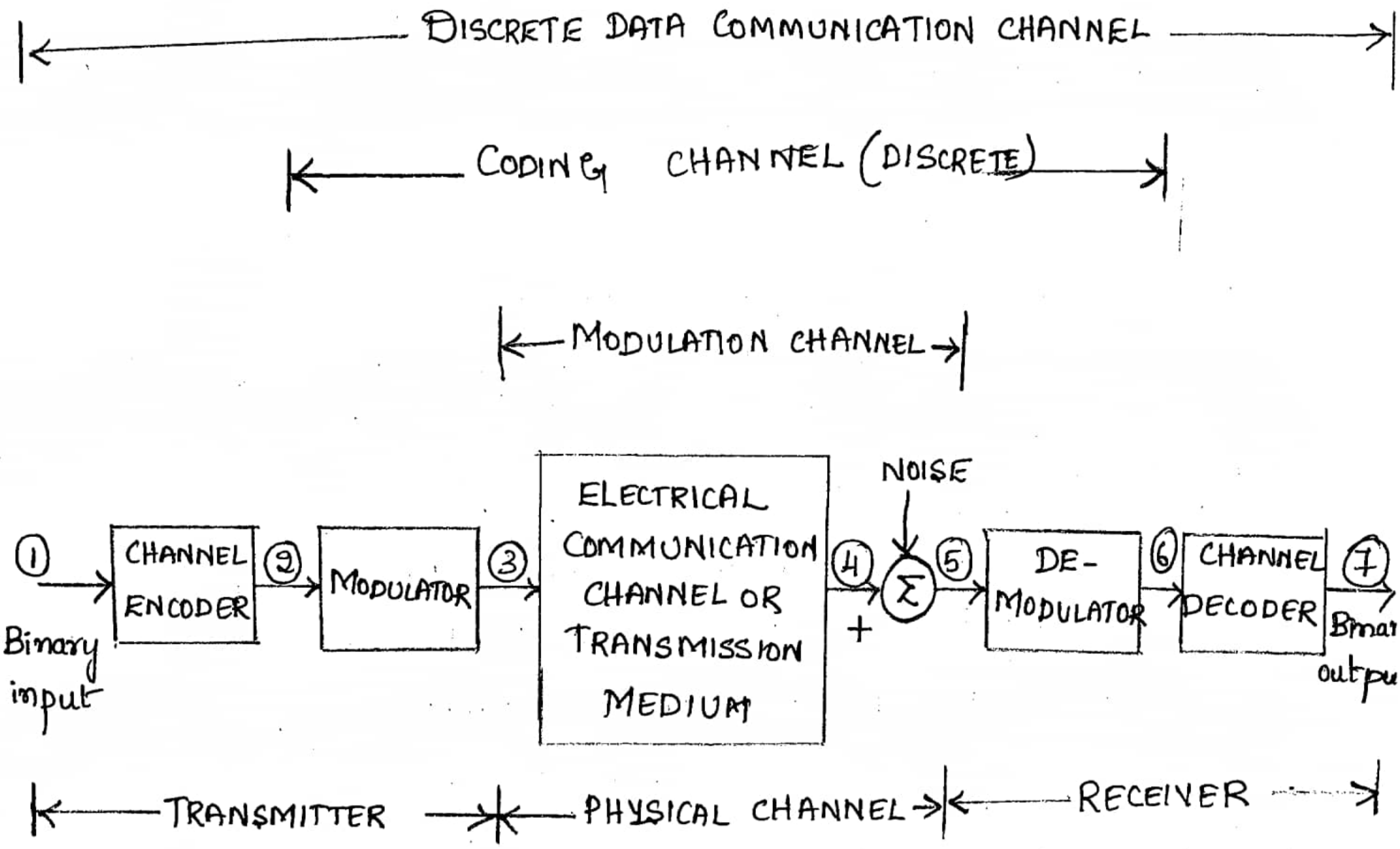


fig: Characterization of a binary Communication channel.

- A practical Communication system can be divided into a Transmitter, physical channel and a Receiver.
- Transmitter consists of an encoder and a modulator, while the Receiver consists of demodulator and a decoder.
- B/w points 2 & 6 in the system shown there is a discrete often refered to as a coding channel that accepts a sequence of symbols at its input & produces a sequence of symbols at its o/p
- The communication channel b/w 3 & 5 in the system provides electrical connection b/w the transmitter and Receiver.
- The i/p and o/p are analog electrical waveforms. This portion of the channel is called a Continuous or modulation channel.
- The channel corrupts the signal statistically due to various types of additive & multiplicative noise.
- The important characteristic of a data communication system is the "channel capacity" which represents maximum rate @ which the given data is transferred across the channel with small probability error.

Discrete Communication Channels.

The input to the channel is a symbol belonging to an alphabet "A" with "r" symbols. The output of the symbol belonging to some other alphabet "B" with "s" symbols. Due to error in the channel, the output symbol may be different from the input symbol. Errors are mainly due to noise in the analog portion of the channel.

Representation of a channel.

A Communication channel may be represented by a set of input alphabet $A = \{a_1, a_2, a_3, \ldots, a_r\}$ consisting of 'r' symbols, a set of output alphabet $B = \{b_1, b_2, b_3 \ldots \ldots, b_s\}$ Consisting of 'S' symbols and a set of conditional probabilities $P(b_j/a_i)$ with $i = 1, 2, 3, \ldots r$ and $j = 1, 2, \ldots s$ as shown below.

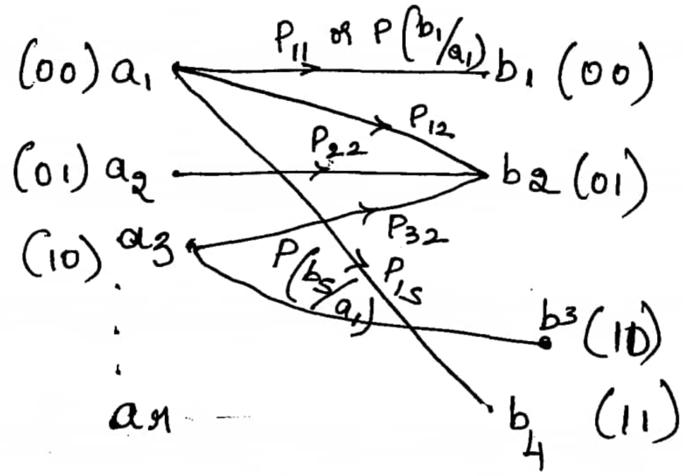
$$A \left\{ \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_r \end{array} \right. \longrightarrow \boxed{P(b_j/a_i)} \longrightarrow \left. \begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_s \end{array} \right\} B$$

The Conditional probabilities existance is due to the presence of noise in the channel. we have a different number of symbols 's' at the receiver from 'r' symbols at the transmitter.

$\therefore$ we have, $r \times s$ number of Conditional probabilities which are represented in a "<u>matrix</u>" form with input symbols represented row-wise and output symbols column wise. Such a matrix is called "<u>Channel Matrix</u> or <u>Noise Matrix</u>"

$$P(b_j/a_i) \text{ or } P(B/A) = \begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_r \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & \cdots & b_s \end{array} \\ \begin{bmatrix} P(b_1/a_1) & P(b_2/a_1) & & P(b_s/a_1) \\ P(b_1/a_2) & P(b_2/a_2) & & P(b_s/a_2) \\ \vdots & & & \\ P(b_1/a_r) & P(b_2/a_r) & & P(b_s/a_r) \end{bmatrix} \end{array}$$

A channel can be represented with conditional probabilities relating input and output symbols and the diagram. Showing these relation is known as channel/noise diagram.



In the above channel, the i/p symbol a_1 being transmitted having a code word "00". when these 2 binary symbols are transmitted over the channel, there is a chance that these 2 symbols are not affected and it is received as it is at the receiver.

this is the conditional probability that symbol b_1 is received given that a_1 is transmitted at the transmitter denoted by $P(b_1/a_1)$. this can be written as

$$P_{11} = P(b_1/a_1) \rightarrow ①$$

but at the second transmission to a_2 there is a chance of 00 getting converted to 01. this can be written as

$$P_{12} = P(b_2/a_1) \rightarrow ②$$

$$P_{13} = P(b_3/a_1) \rightarrow ③$$

$$\& \; P_{14} = P(b_4/a_1) \rightarrow ④$$

But we know that sum of all probabilities $= 1$

$$P_{11} + P_{12} + P_{13} + P_{14} = 1 \rightarrow ⑤$$

using ① to ④ in ⑤

$$P(b_1/a_1) + P(b_2/a_1) + P(b_3/a_1) + P(b_4/a_1) = 1 \rightarrow ⑥$$

Generalizing equation ⑥ for 'S' output symbols and 'r' input symbols.

$$P(b_1/a_1) + P(b_2/a_1) + P(b_3/a_1) + \dots P(b_s/a_1) = 1$$

writing with summation sign

$$\boxed{\sum_{j=1}^{s} P(b_j/a_1) = 1}$$

but it must hold good for all 'r' i/p symbols

$$\boxed{\sum_{j=1}^{s} P(b_j/a_i) = 1} \quad \text{for, } i = 1, 2 \dots r)$$

∴ the Sum of all the elements in any row of the channel Matrix is equal to unity.

Let us suppose take the r probabilities of the i/p symbols namely $P(a_1), P(a_2) \dots P(a_r)$ also $\boxed{\sum_{i=1}^{r} P(a_i) = 1}$. ∴ $P(a_1) + P(a_2) + \dots P(a_r) = 1$

Knowing the channel matrix elements $P(b_j/a_i)$ for all i and j & the input probabilities $P(a_i)$ for all i, the probabilities of the output symbol $P(b_j)$ for all j, can be found by "theorem of total probability"

$$P(b_1) = P(b_1/a_1)P(a_1) + P(b_1/a_2)P(a_2) + \dots + P(b_1/a_r)P(a_r)$$

$$P(b_2) = P(b_2/a_1)P(a_1) + P(b_2/a_2)P(a_2) + \dots + P(b_2/a_r)P(a_r)$$

$$\vdots$$

$$P(b_s) = P(b_s/a_1)P(a_1) + P(b_s/a_2)P(a_2) + \dots + P(b_s/a_r)P(a_r)$$

Knowing the input probabilities $P(a_i)$, Channel Matrix elements $P(b_j/a_i)$ and the output probabilities $P(b_j)$, the "Input Conditional Probabilities $P(a_i/b_j)$" can be found,

By using "Baye's Rule.

$$\boxed{P(a_i/b_j) = \frac{P(b_j/a_i)\,P(a_i)}{P(b_j)}}$$

JOINT PROBABILITY.

The joint probability between any input symbol "a_i" and any output symbol "b_j" is given by

$$P(a_i, b_j) = P(b_j/a_i)\, P(a_i) = P(a_i/b_j)\, P(b_j) \longrightarrow (1)$$

we know that

$$P(b_j/a_i) \text{ or } P(B/A) = \begin{array}{c|c} & \begin{array}{cccc} b_1 & b_2 & & b_s \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_r \end{array} & \begin{bmatrix} P(b_1/a_1) & P(b_2/a_1) & \cdots & P(b_s/a_1) \\ P(b_1/a_2) & P(b_2/a_2) & \cdots & P(b_s/a_2) \\ \vdots & \vdots & & \vdots \\ P(b_1/a_r) & P(b_2/a_r) & \cdots & P(b_s/a_r) \end{bmatrix} \end{array} \rightarrow (2)$$

So multiplying, all the elements of 1^{st} row of channel matrix by $P(a_1)$, 2^{nd} row $P(a_2)$ ...., r^{th} row by $P(a_r)$, we get

$$P(b_j/a_i)\, P(a_i) = \begin{array}{c|c} & \begin{array}{cccc} b_1 & b_2 & & b_s \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_r \end{array} & \begin{bmatrix} P(b_1/a_1)P(a_1) & P(b_2/a_1)P(a_1) & \cdots & P(b_s/a_1)P(a_1) \\ P(b_1/a_2)P(a_2) & P(b_2/a_2)P(a_2) & \cdots & P(b_s/a_2)P(a_2) \\ \vdots & \vdots & & \vdots \\ P(b_1/a_r)P(a_r) & P(b_2/a_r)P(a_r) & \cdots & P(b_s/a_r)P(a_r) \end{bmatrix} \end{array} \rightarrow (3)$$

using equation (1) in (3) we get

$$P(a_i, b_j) \text{ or } P(A,B) = \begin{array}{c|c} & \begin{array}{cccc} b_1 & b_2 & & b_s \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_r \end{array} & \begin{bmatrix} P(a_1,b_1) & P(a_1,b_2) & \cdots & P(a_1,b_s) \\ P(a_2,b_1) & P(a_2,b_2) & \cdots & P(a_2,b_s) \\ \vdots & \vdots & & \vdots \\ P(a_r,b_1) & P(a_r,b_2) & & P(a_r,b_s) \end{bmatrix} \end{array} \rightarrow (4)$$

The elements in the matrix are the various joint probabilities between input and output symbols is called "JOINT PROBABILITY MATRIX" popularly denoted as JPM denoted by $P(a_i, b_j)$ or $P(A,B)$.

Properties of JPM.

① From the equation ③, we have

$$P(b_1) = P(b_1/a_1)\,P(a_1) + P(b_1/a_2)\,P(a_2) + \cdots + P(b_1/a_r)\,P(a_r)$$

or from equⁿ ④ we have

$$P(b_1) = P(a_1, b_1) + P(a_2, b_1) + \cdots + P(a_r, b_1) \longrightarrow (5)$$

The R.H.S of equⁿ ⑤ are present in the 1^{st} column of JPM. hence we can conclude that by adding all the elements of the 1^{st} column, we get the probability of 1^{st} output symbol b_1.

$$|||^{ly} \quad P(b_2) = P(a_1, b_2) + P(a_2, b_2) + \cdots + P(a_r, b_2) \longrightarrow (6)$$

$$\vdots \qquad\qquad \vdots$$

$$P(b_s) = P(a_1, b_s) + P(a_2, b_s) + \cdots + P(a_r, b_s) \longrightarrow (7)$$

from equⁿ ⑤ to ⑦ we can derive the first property of JPM

i.e "BY adding the elements of JPM columnwise, we can obtain the probability of output symbols"

This property can be expressed as $\boxed{\sum_{i=1}^{r} P(a_i, b_j) = P(b_j)} \rightarrow (8)$

for j varying from 1, 2 ..... S symbo

② Using the theorem of total probability for the input symbols, we have for the 1^{st} symbol.

$$P(a_1) = P(a_1/b_1)P(b_1) + P(a_1/b_2)P(b_2) + \cdots + P(a_1/b_s)P(b_s)$$

∴ from ①

$$P(a_1) = P(a_1, b_1) + P(a_1, b_2) + \cdots + P(a_1, b_s) \longrightarrow (9)$$

$$|||^{ly} \quad P(a_2) = P(a_2, b_1) + P(a_2, b_2) + \cdots + P(a_2, b_s) \longrightarrow (10)$$

$$\vdots \qquad\qquad \vdots$$

$$P(a_r) = P(a_r, b_1) + P(a_r, b_2) + \cdots + P(a_r, b_s) \longrightarrow (11)$$

∴ "By adding the elements of JPM rowwise we can obtain the probability of input symbols"

Expressed as $\boxed{\sum_{j=1}^{s} P(a_i, b_j) = P(a_i)}$ ⟶ (12)

for i varying from 1, 2 ... r symbols

(3) By adding equn (9) to (11), we get

$$P(a_1)+P(a_2)+P(a_3)+\dots P(a_r) = [P(a_1,b_1)+P(a_1,b_2)+ \dots P(a_1,b_s)] + [P(a_2,b_1)+P(a_2,b_2)+\dots+P(a_2,b_s)] + \dots + [P(a_r,b_1)+P(a_r,b_2)+\dots+P(a_r,b_s)] \quad (13)$$

where L.H.S of equn (13) = 1

R.H.S is the sum of all the elements of JPM.

∴ "The sum of all the elements of JPM is equal to unity".

∴ Expressed as $\boxed{\sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) = 1}$

Problem.

1. In a Communication system, a transmitter has 3 input symbols $A = \{a_1, a_2, a_3\}$ and receiver also has 3 o/p symbols $B = \{b_1, b_2, b_3\}$. The matrix shows JPM with some marginal probabilities.

$a_i \backslash b_j$	b_1	b_2	b_3
a_1	$\frac{1}{12}$	*	$\frac{5}{36}$
a_2	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{5}{36}$
a_3	*	$\frac{1}{6}$	*
$P(b_j)$	$\frac{1}{3}$	$\frac{14}{36}$	*

(i) find the missing probabilities (*) in the table
(ii) find $P(b_3/a_1)$ and $P(a_1/b_3)$
(iii) Are the events a_1 and b_1 statistically independent ? why ?

Sol^n: we Know $\sum_{i=1}^{r} P(a_i) = 1$. for i/p symbols
similary $\sum_{j=1}^{s} P(b_j) = 1$ for o/p symbols.

(i) Given : $S = 3$

$$\therefore \sum_{j=1}^{3} P(b_j) = P(b_1) + P(b_2) + P(b_3) = 1$$

$$\therefore 1/3 + \frac{14}{36} + P(b_3) = 1$$

$$\boxed{P(b_3) = 5/18.}$$

from first property of JPM we Know, i.e by adding the elements of JPM column wise. $\sum_{i=1}^{r} P(a_i, b_j) = P(b_j)$

$$P(b_1) = P(a_1, b_1) + P(a_2, b_1) + P(a_3, b_1) \rightarrow 1^{st} \text{ column}$$

$$1/3 = 1/12 + 5/36 + P(a_3, b_1)$$

$$\boxed{P(a_3, b_1) = 1/9.}$$

III^{ly} $P(b_2) = P(a_1, b_2) + P(a_2, b_2) + P(a_3, b_2) \rightarrow 2^{nd}$ column

$$\frac{14}{36} = P(a_1, b_2) + 1/9 + 1/6$$

$$\therefore \boxed{P(a_1, b_2) = 1/9}$$

III^{ly} $P(b_3) = P(a_1, b_3) + P(a_2, b_3) + P(a_3, b_3) \rightarrow 3^{rd}$ column

$$\frac{5}{18} = \frac{5}{36} + \frac{5}{36} + P(a_3, b_3)$$

$$\therefore \boxed{P(a_3, b_3) = 0}$$

$\therefore$

$a_i \backslash b_j$	b_1	b_2	b_3
a_1	$1/12$	$1/9$	$5/36$
a_2	$5/36$	$1/9$	$5/36$
a_3	$1/9$	$1/6$	0
$P(b_j)$	$1/3$	$14/36$	$5/18$

(ii) To find $P(b_3/a_1)$ & $P(a_1/b_3)$

we have Conditional probability as,

$$\boxed{P(b_j/a_i) = \frac{P(a_i, b_j)}{P(a_i)}}$$

$$\therefore \quad P(b_3/a_1) = \frac{P(a_1, b_3)}{P(a_1)}$$

from property ② we have. By adding the elements of JPM Rowwise $\sum_{j=1}^{3} P(a_i, b_j) = P(a_i)$

$$P(a_1) = P(a_1, b_1) + P(a_1, b_2) + P(a, b_3) \rightarrow 1^{st} \text{ row}$$

$$P(a_1) = 1/12 + 1/9 + 5/36$$

$$\boxed{P(a_1) = 1/3}$$

$$\therefore \quad P(b_3/a_1) = \frac{5/36}{1/3} \Rightarrow \boxed{\frac{5}{12}}$$

$$\text{III}^{ly} \quad P(a_1/b_3) = \frac{P(a_1, b_3)}{P(b_3)} = \frac{(5/36)}{(5/18)} = \boxed{\frac{1}{2}}$$

(iii) To check weather a_1 & b_1 are stastically independent $P(a_1 \cap b_1) = P(a_1, b_1) = P(a_1)\,P(b_1)$

consider $P(a_1) \cdot P(b_1) = (1/3)(1/3) = 1/9$

& $P(a_1, b_1) = 1/12$

$\therefore P(a, b_1) \neq P(a_1)\,P(b_1)$

$\therefore$ a_1 & b_1 are not stastically independent.

coz, we will receive b_1 by transmitting a_1 with some probability.

hence b_1 is dependent on a_1.

Entropy Functions of Communication Channel.

① The input to the Communication Channel has 'r' symbols $(a_1, a_2 \ldots a_r)$ with probability of each symbol represented as $P(a_i)$ for i varying from 1 to r, then the average information or entropy of the input can be defined as

$$H(A) = \sum_{i=1}^{r} P(a_i) \log\left[\frac{1}{P(a_i)}\right] \text{ bits/symbol}$$

$$\& \quad 0 \leq H(A) \leq \log_2 r \quad \text{bits/symbol}$$

② The receiver receives 's' symbols $\{b_1, b_2, \ldots b_s\}$ and the probabilities are represented as $P(b_j)$ for j varying from 1 to s. The entropy of the receiver is defined as

$$H(B) = \sum_{j=1}^{s} P(b_j) \log\left[\frac{1}{P(b_j)}\right] \text{ bits/symbol}$$

$$\& \quad 0 \leq H(B) \leq \log_2 s \quad \text{bits/symbol}$$

Equivocation: The avg value of all the Conditional entropies.

③ Input and output Conditional entropies can be defined as

$$H(B/A) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log\left(1/P(b_j/a_i)\right)$$

$$H(A/B) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log\left(\frac{1}{P(a_i/b_j)}\right)$$

④ The joint entropy can be defined as

$$H(A,B) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log\left(\frac{1}{P(a_i, b_j)}\right)$$

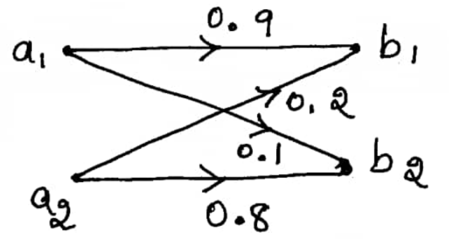
⑤ Relationships.

$$H(A,B) = H(B/A) + H(A)$$

$$H(A,B) = H(A/B) + H(B)$$

PROBLEMS

1. A binary channel is described by the following state diagram



and given $P(a_1) = 0.7$
$P(a_2) = 0.3$

find the entropies $H(A), H(B), H(A,B), H(A/B), H(B/A)$.

Sol^n

$$P(B/A) = \begin{array}{c|cc} & b_1 & b_2 \\ \hline a_1 & 0.9 & 0.1 \\ a_2 & 0.2 & 0.8 \end{array} \quad \begin{array}{c} \times 0.7 \\ \times 0.3 \end{array}$$

∵ "coz x^{lying} the first row of the channel matrix with $P(a_1)$, by using theorem of total probability".

$$\therefore \quad P(A,B) = \begin{bmatrix} 0.63 & 0.07 \\ 0.06 & 0.24 \end{bmatrix}$$

(i) $H(A) = \sum_{i=1}^{2} P(a_i) \log_2 \frac{1}{P(a_i)} = 0.7 \log_2 \frac{1}{0.7} + 0.3 \log_2 \frac{1}{0.3}$

$$\boxed{H(A) = 0.8813 \text{ bits/symbol}}$$

(ii) $H(B) = \sum_{j=1}^{2} P(b_j) \log_2 \frac{1}{P(b_j)}$

$P(b_1) = 0.63 + 0.06 = 0.69 \quad P(b_2) = 0.07 + 0.24 = 0.31$

$\therefore \; H(B) = 0.69 \log_2 \frac{1}{0.69} + 0.31 \log_2 \frac{1}{0.31}$

$$\boxed{H(B) = 0.8931 \text{ bits/symbol.}}$$

(iii) $H(A,B) = \sum_{i=1}^{2} \sum_{j=1}^{2} P(a_i, b_j) \log_2 \frac{1}{P(a_i, b_j)}$

$= 0.63 \log_2 \frac{1}{0.63} + 0.07 \log_2 \frac{1}{0.07} + 0.06 \log_2 \frac{1}{0.06} + 0.24 \log_2 \frac{1}{0.24}$

$$\boxed{H(A,B) = 1.426 \text{ bits/symbol}}$$

(iv) $H(B/A)$

we Know $H(A,B) = H(B/A) + H(A)$

$$H(B/A) = H(A,B) - H(A)$$

$$= 1.426 - 0.8813$$

$$\boxed{H(B/A) = 0.5447 \text{ bits/symbol}}$$

(v) $H(A/B)$

$|||^{ly}$ $H(A,B) = H(A/B) + H(B)$

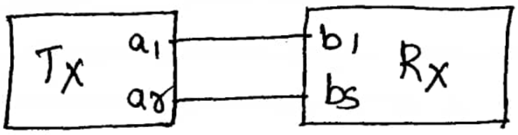
$$H(A/B) = H(A,B) - H(B)$$

$$= 1.426 - 0.8931$$

$$\boxed{H(A/B) = 0.533 \text{ bits/symbol}}$$

Mutual Information.

Consider a transmitter sending 'r' symbols of data over a channel and the received symbols are 's' in number & may be different from those transmitted as shown,



Now the average information coming from the transmitter is given by $H(A)$

$$H(A) = \sum_{i=1}^{r} P(a_i) \log \frac{1}{P(a_i)} \quad \text{bits/symbol}$$

if the channel was noiseless then the average information received should be equal to $H(A)$.

When an average information of $H(A)$ is transmitted over the channel, an average amount of information equal to equivocation $H(A/B)$ is lost in the channel due to noise.
This factor is represented as the mutual information b/w the transmitter & the receiver & is defined as "Mutual Information" or "Transinformation".

$$\boxed{I(A,B) = H(A) - H(A/B)} \quad \text{bits/symbol.}$$

Properties of Mutual Information.

1. The mutual information ~~of~~ a channel is symmetric.

i.e $\boxed{I(A,B) = I(B,A)}$

2. The mutual information is always non-negative

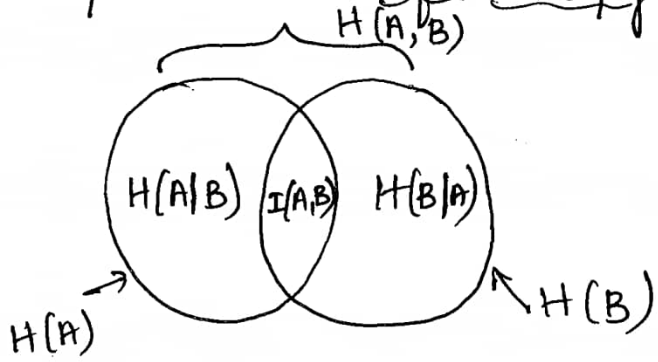
i.e $\boxed{I(A,B) \geq 0}$

3. The mutual information of a channel may be expressed, for the entropy of the channel output as $\boxed{I(A,B) = H(B) - H(B/A)}$

(4) The mutual information is related to the joint entropy of the channel,

$$\boxed{I(A,B) = H(A) + H(B) - H(A,B)}$$

Graphical representation of Entropy relations.



Problems.

1. A transmitter has an alphabet consisting of 5 letters $\{a_1, a_2, a_3, a_4, a_5\}$ and the receiver has an alphabet of four letters $\{b_1, b_2, b_3, b_4\}$. The joint probabilities of the systems are shown as.

$$P(A,B) = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 0.25 & 0 & 0 & 0 \\ 0.10 & 0.30 & 0 & 0 \\ 0 & 0.05 & 0.10 & 0 \\ 0 & 0 & 0.05 & 0.10 \\ 0 & 0 & 0.05 & 0 \end{bmatrix} \end{array}$$

Compute different entropies of this channel.

→ Different Entropies are $H(A), H(B), H(A,B)$
$H(B/A)$ & $H(A/B)$ &
Mutual Information $I(A,B) = H(A) - H(A/B)$.

To find $H(A)$ first we have to find the probabilities of a_1 to a_5, ∴ from property-2 of JPM, we have.

∴ $P(a_1) = 0.25$, $P(a_2) = 0.40$, $P(a_3) = 0.15$, $P(a_4) = 0.15$, $P(a_5) = 0.05$

$|||^{ly}$ $H(B)$, from property-1 of JPM

$P(b_1) = 0.35$, $P(b_2) = 0.35$, $P(b_3) = 0.20$, $P(b_4) = 0.10$.

To find the entropy of i/p symbol

$$H(A) = \sum_{i=1}^{s} P(a_i) \log_2 \frac{1}{P(a_i)} = 0.25 \log_2 \frac{1}{0.25} + 0.40 \log_2 \frac{1}{0.40} + 0.15 \log_2 \frac{1}{0.15}$$

$$+ 0.15 \log_2 \frac{1}{0.15} + 0.05 \log_2 \frac{1}{0.05} .$$

$$\boxed{H(A) = 2.066 \text{ bits/symbol}}.$$

To find the entropy of o/p symbol

$$H(B) = \sum_{j=1}^{r} P(b_j) \log_2 \frac{1}{P(b_j)}$$

$$= 0.35 \log_2 \frac{1}{0.35} + 0.35 \log_2 \frac{1}{0.35} + 0.20 \log_2 \frac{1}{0.20} + 0.10 \log_2 \frac{1}{0.10}$$

$$\boxed{H(B) = 1.857 \text{ bits/symbol}}$$

To find the joint entropy.

$$H(A,B) = \sum_{i=1}^{5} \sum_{j=1}^{4} P(a_i, b_j) \log \frac{1}{P(a_i, b_j)}$$

$$H(A,B) = 0.25 \log_2 \frac{1}{0.25} + 0.10 \log_2 \frac{1}{0.10} + 0.30 \log_2 \frac{1}{0.30}$$

$$+ 0.05 \log_2 \frac{1}{0.05} + 0.10 \log_2 \frac{1}{0.10} + 0.05 \log_2 \frac{1}{0.05}$$

$$+ 0.05 \log_2 \frac{1}{0.05} + 0.10 \log_2 \frac{1}{0.10} .$$

$$\boxed{H(A,B) = 2.666 \text{ bits/symbol}}$$

The equivocation $H(B/A)$ is

$$H(B/A) = H(A,B) - H(A)$$

$$= 2.666 - 2.066$$

$$\boxed{H(B/A) = 0.6 \text{ bits/symbol}}$$

The equivocation $H(A/B)$ is

$$H(A/B) = H(A,B) - H(B)$$

$$= 2.666 - 1.857$$

$$\boxed{H(A/B) = 0.809} \text{ bits/symbol} .$$

The mutual information $I(A,B)$ is found by

$$I(A,B) = H(A) - H(A/B)$$

$$= 2.066 - 0.809$$

$$= \boxed{I(A,B) = 1.257 \text{ bits/msg symbol.}}$$

(2) Show that $H(A,B) = H(A/B) + H(B)$

→ we Know,

$$H(A/B) = \sum_{i=1}^{r}\sum_{j=1}^{s} P(a_i, b_j) \log \frac{1}{P(a_i/b_j)} \text{ bits/symbol} \rightarrow (1)$$

$$H(B) = \sum_{j=1}^{s} P(b_j) \log \frac{1}{P(b_j)} \text{ bits/symbol} \rightarrow (2)$$

also $$H(A,B) = \sum_{i=1}^{r}\sum_{j=1}^{s} P(a_i, b_j) \log \frac{1}{P(a_i, b_j)} \rightarrow (3)$$

we Know $P(a_i, b_j) = P(a_i/b_j)\, P(b_j) \rightarrow (4)$

Replacing (4) in (3) we get

$$H(A,B) = \sum_{i=1}^{r}\sum_{j=1}^{s} P(a_i, b_j) \log \frac{1}{P(a_i/b_j)\, P(b_j)}$$

$$\therefore = \underbrace{\sum_{i=1}^{r}\sum_{j=1}^{s} P(a_i, b_j) \log \frac{1}{P(a_i/b_j)}}_{\text{from (1)}} + \sum_{i=1}^{r}\sum_{j=1}^{s} P(a_i, b_j) \log \frac{1}{P(b_j)}$$

$$\Rightarrow H(A/B) + \sum_{j=1}^{s} \underbrace{\left[\sum_{i=1}^{r} P(a_i, b_j)\right]}_{\text{from JPM } 1^{st} \text{ property}} \log \frac{1}{P(b_j)}$$

$$\Rightarrow H(A/B) + \sum_{j=1}^{s} P(b_j) \log \frac{1}{P(b_j)}$$

$$\Rightarrow H(A/B) + H(B)$$

$$\therefore \boxed{H(A,B) = H(A/B) + H(B)}$$

3. A transmitter transmits five symbols with probabilities 0.2, 0.3, 0.2, 0.1 and 0.2. Given the channel matrix $P(B/A)$, Calculate (i) $H(B)$ (ii) $H(A,B)$

$$P(B/A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

→ from JPM equ^n we have

$$\boxed{P(a_i, b_j) = P(a_i)\, P(b_j/a_i)} \longrightarrow (1)$$

∴ multiplying 1st row of $P(B/A)$ with 0.2,
2nd row ______ with 0.3
3rd ______ " 0.1
4th ______ 0.2

the JPM $P(a_i, b_j)$ can be constructed.

$$P(B/A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} \times 0.2 \\ \times 0.3 \\ \times 0.2 \\ \times 0.1 \\ \times 0.2 \end{matrix} \Rightarrow P(a_i, b_j) = \begin{bmatrix} 1/5 & 0 & 0 & 0 \\ 3/40 & 9/40 & 0 & 0 \\ 0 & 1/15 & 2/15 & 0 \\ 0 & 0 & 1/30 & 1/15 \\ 0 & 0 & 1/5 & 0 \end{bmatrix}$$

(i) To find $H(B)$, Adding the elements of each column, we get

$P(b_1) = 1/5 + 3/40 = 11/40$, $P(b_2) = 7/24$, $P(b_3) = 11/30$, $P(b_4) = 1/15$

$$\therefore H(B) = \sum_{j=1}^{4} P(b_j) \log_2 \frac{1}{P(b_j)} \Rightarrow \frac{11}{40}\log_2\frac{40}{11} + \frac{7}{24}\log_2\frac{24}{7} + \frac{11}{30}\log_2\frac{30}{11} + \frac{1}{15}\log_2 15$$

$$\therefore \boxed{H(B) = 1.822 \text{ bits/symbol}}$$

(ii) To find $H(A,B) \Rightarrow H(A,B) = \sum_{i=1}^{5}\sum_{j=1}^{4} P(a_i, b_j) \log \frac{1}{P(a_i, b_j)}$

$$= \frac{1}{5}\log_2 5 + \frac{3}{40}\log_2\frac{40}{3} + \frac{9}{40}\log_2\frac{40}{9} + \frac{1}{15}\log_2 15 + \frac{2}{15}\log_2\frac{15}{2} + \frac{1}{3}\log_2 30 + \frac{1}{5}\log_2 5 + \frac{1}{15}\log_2 15$$

$$\therefore \boxed{H(A,B) = 2.7653 \text{ bits/sym}}$$

(4) For the JPM given, compute individually $H(A)$, $H(B)$, $H(A,B)$, $H(A/B)$, $H(B/A)$ and $I(A,B)$. Verify the relationship among these entropies.

$$P(A,B) = \begin{bmatrix} 0.05 & 0 & 0.20 & 0.05 \\ 0 & 0.10 & 0.10 & 0 \\ 0 & 0 & 0.20 & 0.10 \\ 0.05 & 0.05 & 0 & 0.10 \end{bmatrix}$$

→ From property-1, adding columnwise

$P(b_1) = 0.10$, $P(b_2) = 0.15$, $P(b_3) = 0.50$, $P(b_4) = 0.25$

$$\therefore H(B) = \sum_{j=1}^{s} P(b_j) \log_2 \frac{1}{P(b_j)}$$ (The entropy of o/p symbol)

$$= 0.10 \log_2 \frac{1}{0.10} + 0.15 \log_2 \frac{1}{0.15} + 0.50 \log_2 \frac{1}{0.50} + 0.25 \log_2 \frac{1}{0.25}$$

$$\boxed{H(B) = 1.743 \text{ bits/symbol.}}$$

From property-2, adding rowwise.

$P(a_1) = 0.30$, $P(a_2) = 0.20$, $P(a_3) = 0.30$, $P(a_4) = 0.20$

$$\therefore H(A) = \sum_{i=1}^{r} P(a_i) \log_2 \frac{1}{P(a_i)}$$ (The entropy of i/p symbol)

$$= 0.30 \log_2 \frac{1}{0.30} + 0.20 \log_2 \frac{1}{0.20} + 0.30 \log_2 \frac{1}{0.30} + 0.20 \log_2 \frac{1}{0.20}$$

$$\boxed{H(A) = 1.971 \text{ bits/symbol}}$$

The Joint entropy is given by

$$H(A,B) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log_2 \frac{1}{P(a_i, b_j)}$$

$$= \left(0.05 \log_2 \frac{1}{0.05}\right) \times 4 + \left(0.10 \log_2 \frac{1}{0.10}\right) \times 4 + \left(0.2 \log_2 \frac{1}{0.2}\right) \times 2$$

$$\boxed{H(A,B) = 3.122 \text{ bits/symbol}}$$

$$H(A/B) = H(A,B) - H(B)$$
$$= 3.122 - 1.743$$
$$\boxed{H(A/B) = 1.379 \text{ bits/symbol.}}$$

$$H(B/A) = H(B,A) - H(A)$$
$$= 3.122 - 1.971$$
$$\boxed{H(B/A) = 1.151 \text{ bits/symbol}}$$

we can also calculate $H(A/B)$ & $H(B/A)$ by using equivocation

$\therefore$ we have $H(A/B) = \sum_{i=1}^{4} \sum_{j=1}^{4} P(a_i, b_j) \log_2 \frac{1}{P(a_i, b_j)}$

$\therefore$ using relationship $\boxed{P(a_i / b_j) = \frac{P(a_i, b_j)}{P(b_j)}}$

the $P(A/B)$ is constructed.

$$P(A/B) = \begin{bmatrix} \frac{0.05}{0.10} & 0 & \frac{0.2}{0.5} & \frac{0.05}{0.25} \\ 0 & \frac{0.10}{0.15} & \frac{0.1}{0.5} & 0 \\ 0 & 0 & \frac{0.2}{0.5} & \frac{0.10}{0.25} \\ \frac{0.05}{0.10} & \frac{0.05}{0.15} & 0 & \frac{0.10}{0.25} \end{bmatrix} \Rightarrow \begin{bmatrix} 1/2 & 0 & 2/5 & 1/5 \\ 0 & 2/3 & 1/5 & 0 \\ 0 & 0 & 2/5 & 2/5 \\ 1/2 & 1/3 & 0 & 2/5 \end{bmatrix}$$

$$\therefore H(A/B) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log_2 \frac{1}{P(a_i / b_j)}$$

$$\Rightarrow H(A/B) = 0.05 \log_2 2 + 0.05 \log_2 2 + 0.10 \log_2 \frac{3}{2} + 0.05 \log_2 3$$
$$+ 0.20 \log_2 \frac{5}{2} + 0.10 \log_2 5 + 0.20 \log_2 \frac{5}{2} + 0.05 \log_2 5$$
$$+ 0.10 \log_2 \frac{5}{2} + 0.10 \log_2 \frac{5}{2}$$

$$\boxed{H(A/B) = 1.379 \text{ bits/message symbol}}$$

Similarly $P(B/A) = \frac{P(a_i, b_j)}{P(\ \ )}$

$$\Rightarrow P(B/A) = \begin{bmatrix} \frac{0.05}{0.30} & 0 & \frac{0.20}{0.30} & \frac{0.05}{0.30} \\ 0 & \frac{0.10}{0.20} & \frac{0.10}{0.20} & 0 \\ 0 & 0 & \frac{0.20}{0.30} & \frac{0.10}{0.30} \\ \frac{0.05}{0.20} & \frac{0.05}{0.20} & 0 & \frac{0.10}{0.20} \end{bmatrix} = \begin{bmatrix} 1/6 & 0 & 2/3 & 1/6 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 2/3 & 1/3 \\ 1/4 & 1/4 & 0 & 1/2 \end{bmatrix}$$

$\therefore$ $\boxed{P(B/A) = 1.151 \text{ bits/message symbol.}}$

$\therefore$ The mutual Information $I(A,B)$ is given by

$$I(A,B) = H(A) - H(A/B)$$
$$= 1.971 - 1.379$$
$$\boxed{I(A,B) = 0.592 \text{ bits/message symbol}}$$

Rate of Information Transmission over a Discrete Channel

If a discrete memoryless channel accepts symbols at the rate of 'r_s' symbols/sec, then the average rate at which information is going into the channel is given by

$$\boxed{R_{in} = H(A)\, r_s \text{ bits/second (bps)}}$$

But at the receiver, since it is not possible to reconstruct the symbol sequence with certainity the error introduced by the channel also known as "Equivocation" represented by $H(A/B)$ should be taken into account. hence net information received in the mutual information, given as:

$$\boxed{I(A,B) = H(A) - H(A/B) \text{ bits/message-symbol}}$$

hence the average rate of information transmission "R_t" is given by

$$\boxed{R_t = I(A,B)\, r_s \text{ bits/second.}}$$

* Channel Capacity / Capacity of a Discrete Memoryless Channel

- The capacity of a discrete memoryless noisy channel is defined as the maximum possible rate of information transmission over the channel.

The maximum rate of transmission occur when the source is "matched" to the channel.

∴ Channel Capacity "C" is defined as

$$\boxed{C = Max\,[R_t]\ bps}$$

i.e $\boxed{C = Max\,[I(A,B)\, r_s]\ bps}$

$$\boxed{C = Max\left([H(A) - H(A/B)]\right) r_s \text{ bits per second.}}$$

* Shannon's theorem on Channel Capacity [Shannon's Second theorem]

It is stated in 2 ways

- Positive Statement : when the rate of information transmission is less than or equal to the capacity of the channel, then there exists a coding technique which enables transmission over the channel with as small a probability of error as possible even in the pressence of noise $\boxed{R \leq C}$

- Negative Statement : If $\boxed{R > C}$, then reliable transmission of information is not possible and no coding technique can control the errors.

* Channel Efficiency

the "channel efficiency" denoted by η_{ch} is given by

$$\eta_{ch} = \frac{R_t}{C} \times 100\%$$

Substituing for R_t and C

we have

$$\eta_{ch} = \frac{[H(A) - H(A/B)]\, r_s}{Max[H(A) - H(A/B)]\, r_s} \times 100\%$$

or

$$\boxed{\eta_{ch} = \frac{I(A,B)}{Max[I(A,B)]} \times 100\%}$$

* Channel Redundancy

The channel Redundancy denoted by $R_{\eta_{ch}}$ is given by

$\boxed{R_{\eta_{ch}} = 1 - \eta_{ch}}$ where η_{ch} is a fraction.

Special Channels

There are several special channels which are in the field of communication systems. Some of them are listed below here we need to find the channel capacity of these channels

1) Symmetric/uniform channels
2) Binary Symmetric Channels (BSC)
3) Binary Erasure Channels (BEC)
4) Noiseless channel
5) Deterministic channel
6) Cascaded Channel.

Capacity of different types of Channels.

⟨1⟩ **Noiseless Channel**: This channel is represented by a channel Matrix with one & only one non-zero element "every column" is defined as "noiseless channel".

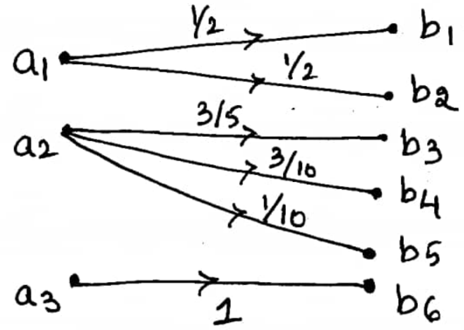
It has 2 properties.

1. Each column contains only one element with other elements in that column being zero
2. the sum of all the elements in any row is equal to unity.

ex:- $P(B/A)$ or $P(b_j/a_i) =$

$$\begin{array}{c|cccccc} & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ \hline a_1 & & & & & & \\ a_2 & & & & & & \\ a_3 & & & & & & \end{array} \quad \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3/5 & 3/10 & 1/10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Transmitter 'A'



Receiver 'B'

Since it is a noiseless Channel, the amount of information lost in the channel due to noise is zero, hence the channel becomes a noisless channel.

∴ the Mutual information $I(A,B) = H(A) - \cancel{H(A/B)}^{0}$

$$I(A,B) = H(A)$$

∴ Channel capacity $C = \text{Max}\,(I(A,B))\, r_s$

$$C = (\text{Max } H(A))\, r_s \qquad H(A)_{max} = \log_2 r$$

considering 'r' i/p symbols $\quad C = \log_2 r \cdot r_s \quad$ let $r_s = 1$ sym/sec

$$\boxed{C = \log_2 r \text{ bits/sec}}$$

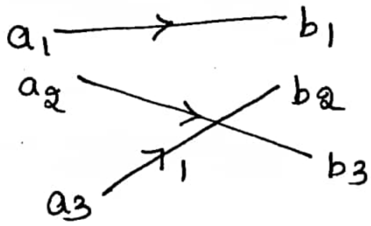
⟨2⟩ **Deterministic Channel**: A channel represented by a channel matrix with one & only one non zero element in "every row" is defined as "Deterministic Channel".

A deterministic channel must satisfy these 2 properties.

(1) The sum of the elements in every row must be equal to 1 &

(2) Each row must contain only one entry & the other elements in that row must be zero.

ex:- $P(B/A) = \begin{array}{c|ccc} & b_1 & b_2 & b_3 \\ \hline a_1 & 1 & 0 & 0 \\ a_2 & 0 & 0 & 1 \\ a_3 & 0 & 1 & 0 \end{array}$



In the deterministic channel, one can "determine" which symbol is to be received at the receiver & hence the channel is called deterministic channel. here the channel is characterized by the conditional entropy $H(B/A) = 0$.

$\therefore$ the mutual information of a deterministic channel is given by

$$I(A,B) = H(B) - H(B/A)$$

$$\therefore I(A,B) = H(B)$$

the channel capacity C is given by $C = Max[I(A,B)]r_s$

$C = Max\,[H(B)]\,r_s$

$$\therefore C = \log_2 S \cdot r_s$$

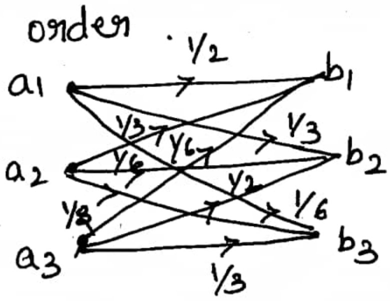
$\because H(B)max$ o/p symbol will be 'S'

if $r_s = 1$ sym/sec

$$\boxed{C = \log_2 S \text{ bits/symbol.}}$$

③ **Symmetric/Uniform Channel** : A channel is said to be symmetric or uniform channel, if the second & subsequent rows of the channel matrix contains the same elements as that of first row, but in a different order.

ex:- $P(B/A) = \begin{array}{c|ccc} & b_1 & b_2 & b_3 \\ \hline a_1 & 1/2 & 1/3 & 1/6 \\ a_2 & 1/3 & 1/6 & 1/2 \\ a_3 & 1/6 & 1/2 & 1/3 \end{array}$



Channel Capacity of symmetric/Uniform channel

In general the channel matrix of a symmetric/uniform channel can be written as.. , for r i/p symbols & S o/p symbols.

$$P(B/A) = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_r \end{array} \begin{array}{c} \begin{array}{ccccc} b_1 & b_2 & b_3 & \cdots & b_s \end{array} \\ \left[\begin{array}{ccccc} P_1 & P_2 & P_3 & \cdots & P_s \\ P_2 & P_3 & P_{s-1} & \cdots & P_1 \\ P_4 & P_{s-1} & P_{s-4} & \cdots & P_{s-3} \\ \vdots & \vdots & \vdots & & \\ P_3 & P_{s-1} & P_{s-2} & \cdots & P_2 \end{array}\right] \end{array} \rightarrow (1)$$

where $P_1, P_2, P_3 \cdots P_{s-4}, P_{s-3}, P_{s-2}, P_{s-1}$ are the conditional probabilities of $P(b_j/a_i)$,

we know the sum of all the elements of any row equal to unity

$$\sum_{j=1}^{s} P(b_j/a_i) = 1 \longrightarrow (2)$$

But all the 'r' number of rows have same elements as in the first row, it can be written as

$$\sum_{j=1}^{s} P_j = 1 \longrightarrow (3)$$

we also know $P(a_i, b_j) = P(b_j/a_i)\, P(a_i) \rightarrow (A)$ from Joint Probability

$\therefore$ The equivocation $H(B/A)$ is given by

$$H(B/A) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log_2 \frac{1}{P(b_j/a_i)} \rightarrow (4)$$

Applying (A) in (4)

$$H(B/A) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(b_j/a_i)\, P(a_i) \log_2 \frac{1}{P(b_j/a_i)}$$

$$= \underbrace{\sum_{i=1}^{r} P(a_i)}_{\Rightarrow 1} \; \underbrace{\sum_{j=1}^{s} P(b_j/a_i) \log_2 \frac{1}{P(b_j/a_i)}}_{\text{from } (3)}$$

$$H(B/A) = \sum_{j=1}^{s} P_j \log_2 \frac{1}{P_j} \Rightarrow \boxed{H(B/A) = h}$$

where $h = \sum_{j=1}^{s} P_j \log_2 \frac{1}{P_j}$

$\therefore$ The mutual information $I(A,B)$ is given by

$$I(A,B) = H(B) - H(B/A)$$

$$I(A,B) = H(B) - h$$

The channel capacity $C = \text{Max}\,[I(A,B)]\, r_s$

$$C = \text{Max}\,[H(B) - H(B/A)]\, r_s$$

$$C = \text{Max}\,(H(B) - h)\, r_s \quad \text{if } r_s = 1\ \text{sy/sec}$$

$$C = \text{Max}\, H(B) - h \quad \text{since } h \text{ is constant}$$

$\therefore$ we know from Deterministic channel $H_B \text{Max} = \log_2 s$

$$\therefore \quad H(B)_{max} = \log_2 s.$$

$$\therefore \quad \boxed{C = \log_2 s - h \quad \text{bits/sec}} \quad \text{for } h = \sum_{j=1}^{s} P_j \log_2 \frac{1}{P_j}$$

Problems

① For the channel matrix shown, find the channel capacity

$$P(b_j/a_i) = \begin{array}{c c} & \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 1/3 & 1/6 & 1/2 \\ 1/6 & 1/2 & 1/3 \end{bmatrix} \end{array}$$

→ The given matrix belongs to a symmetric or uniform channel.

$$H(B/A) = h = \sum_{j=1}^{s} P_j \log_2 1/P_j$$

$$= 1/2 \log_2 2 + 1/3 \log_2 3 + 1/6 \log_2 6$$

$$h = 1.4591 \text{ bits/symbol}$$

The channel capacity C is given by

$$C = (\log_2 s - h)\, r_s$$

$$C = (\log 3 - 1.4591)\, r_s$$

$$\boxed{C = 0.1258 \text{ bits/sec}} \quad \text{with } r_s = 1\ \text{sy/sec}$$

(2) For the channel matrix given below, compute the channel capacity.

$$P(Y/X) = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} \text{ with } r_s = 1000 \text{ symbols/sec.}$$

→ The given matrix is that of a symmetric channel

$$\therefore h = \sum_{j=1}^{3} P_j \log \frac{1}{P_j}$$

$$= 0.6 \log \frac{1}{0.6} + 2 \times 0.2 \log \frac{1}{0.2}$$

$$\boxed{h = 1.371 \text{ bits/message-symbol}}$$

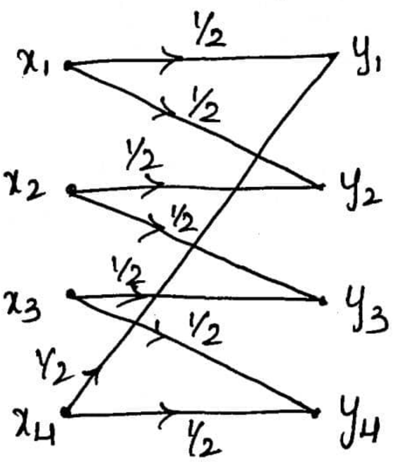
Given S = number of output symbols = 3

$\therefore$ Channel Capacity $C = (\log S - h) r_s$ bits/sec

$$= (\log 3 - 1.371)(1000) \text{ bits/sec}$$

$$\therefore \boxed{C = 214 \text{ bits/sec}}$$

(3) Determine the capacity of the channel shown



→ Solution : The channel matrix $P(Y/X)$ can be written as

$$P(Y/X) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix} \end{array}$$

The 2^{nd}, 3^{rd} and 4^{th} rows of the matrix consists of the Same elements as the 1^{st} row elements.
hence the given channel is a symmetric/uniform channel.

$$h = H(Y/X) = \sum_{j=1}^{4} P_j \log \frac{1}{P_j}$$

$$h = \frac{1}{2}\log 2 + \frac{1}{2}\log 2 + 0 + 0$$

$$\boxed{h = 1 \text{ bit/message-symbol}}$$

$\therefore$ The channel Capacity is given by

$$C = \log_2 S - h$$
$$= \log_2 4 - 1$$
$$= 2 - 1$$
$$\boxed{C = 1 \text{ bit/message-symbol}}$$

(4) A channel has the following characteristics.

$$P(Y/X) = \begin{array}{c} \\ X_1 \\ X_2 \end{array} \begin{array}{c} \begin{array}{cccc} Y_1 & Y_2 & Y_3 & Y_4 \end{array} \\ \begin{bmatrix} 1/3 & 1/3 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/3 & 1/3 \end{bmatrix} \end{array}$$

find $H(x)$, $H(y)$, $H(x,y)$ and channel capacity if $r = 1000$ symbols/sec.

$\rightarrow$ To get $P(X,Y)$ $=$ $\begin{array}{c} \\ X_1 \\ X_2 \end{array} \begin{array}{c} \begin{array}{cccc} Y_1 & Y_2 & Y_3 & Y_4 \end{array} \\ \begin{bmatrix} 1/6 & 1/6 & 1/12 & 1/12 \\ 1/12 & 1/12 & 1/6 & 1/6 \end{bmatrix} \end{array}$ $P(X_1) = 1/2$, $P(X_2) = 1/2$

The o/p probabilities are all equal given by

$$P(Y_1) = P(Y_2) = P(Y_3) = P(Y_4) = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$$

$$\therefore H(X) = \sum_{i=1}^{2} P(X_i) \log \frac{1}{P(X_i)} \Rightarrow \overset{P(X_1)}{\frac{1}{2}} \log 2 + \overset{P(X_2)}{\frac{1}{2}} \log 2$$

$$\boxed{H(X) = 1 \text{ bit/message-symbol}}$$

$$H(Y) = \sum_{j=1}^{4} P(Y_j) \log \frac{1}{P(Y_j)} \Rightarrow \frac{1}{4} \log 4 \times 4 \Rightarrow 2 \text{ bits/message symbol}$$

$$\boxed{H(Y) = 2 \text{ bits/message symbol}}$$

$\therefore$ The joint entropy $H(X, Y)$ is given by equ^n as

$$H(X, Y) = \sum_{i=1}^{2} \sum_{j=1}^{4} P(x_i, y_j) \log \frac{1}{P(x_i, y_j)}$$

$$H(X, Y) = \left(\frac{1}{6} \log 6\right)(4) + \left(\frac{1}{12} \log 12\right)(4)$$

$$H(X, Y) = 2.9183 \text{ bits/message-symbol}$$

The mutual information is given by

$$I(X, Y) = H(X) + H(Y) - H(X, Y)$$

The channel capacity given by

$$C = \text{Max}\left[I(X, Y)\right] = 1 + 2 - 2.9183$$

$$C = 0.0817 \text{ bits/message-symbol}$$

Given $r = 1000$ symbols/sec.

The channel capacity in bits/sec is given by

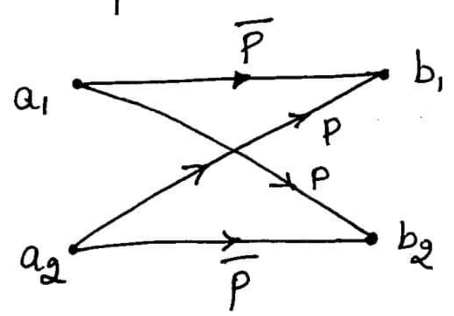
$$C = (r)(0.0817)$$

$$= (1000)(0.817)$$

$$\boxed{C = 81.7 \text{ bits/sec.}}$$

(4) Binary Symmetric Channel (BSC):

A Symmetric channel which has two inputs and two outputs, it is called a binary symmetric channel.



$$P(B/A) = \begin{array}{c c} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{bmatrix} \bar{P} & P \\ P & \bar{P} \end{bmatrix} \end{array}$$

$$\bar{P} = 1 - P$$

The procedure to find $H(B/A)$ and C is same as in the symmetric channel,

$\therefore$ we have $c = \log_2 s - h$ bps

we know $h = \sum_{j=1}^{s} P_j \log_2 \frac{1}{P_j}$

But in this case $s=2$, hence h is

$$h = \bar{P} \log_2 \frac{1}{\bar{P}} + P \log \frac{1}{P}$$

$$\therefore \quad c = \log_2 2 - \bar{P} \log_2 \frac{1}{P} + P \log \frac{1}{P}$$

$$c = 1 - \bar{P} \log\left(\frac{1}{P}\right) + P \log \frac{1}{P}$$

$$\therefore \boxed{c = 1 - h}$$

This is the capacity of a binary symmetric channel.

Problems.

(1) A binary symmetric channel has the following noise matrix with source probabilities of $P(X_1) = 2/3$ and $P(X_2) = 1/3$.

$$P(Y/X) = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} \end{array}$$

(i) Determine $H(X)$, $H(Y)$, $H(X,Y)$, $H(Y/X)$, $H(X/Y)$ & $I(X,Y)$

(ii) find the channel capacity C

(iii) find channel efficiency and redundancy.

→ (i) To find $H(X) = \sum_{i=1}^{2} P(x_i) \log_2 \frac{1}{P(x_i)}$

$$H(X) = \frac{2}{3} \log_2 \frac{3}{2} + \frac{1}{3} \log 3$$

$$H(X) = 0.9183 \text{ bits/symbol.}$$

To find H(Y)

$$P(X,Y) = \begin{array}{c} X_1 \\ X_2 \end{array}\begin{bmatrix} 1/2 & 1/6 \\ 1/12 & 1/4 \end{bmatrix}$$

$$P(b_1) = 7/12 \qquad P(b_2) = 5/12$$

$$H(Y) = \sum_{j=1}^{2} P(y_j) \log_2 \frac{1}{P(y_j)} = \frac{7}{12}\log_2\frac{12}{7} + \frac{5}{12}\log_2\frac{12}{5}$$

$$\boxed{H(Y) = 0.9799 \text{ bits/symbol}}$$

we know that $H(Y/X) = h = \bar{P}\log_2\frac{1}{\bar{P}} + P\log\frac{1}{P}$

$$= \frac{3}{4}\log_2\frac{4}{3} + \frac{1}{4}\log_2 4$$

$$\boxed{H(Y/X) = 0.8113 \text{ bits/symbol}}$$

$$H(X,Y) = H(X) + H(Y/X)$$
$$= 0.9183 + 0.8113$$
$$\boxed{H(X,Y) = 1.7296 \text{ bits/symbol}}$$

$$H(X/Y) = H(X,Y) - H(Y)$$
$$= 1.7296 - 0.9799$$
$$\boxed{H(X/Y) = 0.7497 \text{ bits/symbol}}$$

To find $I(X,Y) = H(X) - H(X/Y)$ or $H(Y) - H(Y/X)$

$$= 0.9183 - 0.7497$$

$$\boxed{I(X,Y) = 0.1686 \text{ bits/symbol}}$$

(ii) To find channel Capacity 'C'

$$C = 1 - h \Rightarrow 1 - H(Y/X) \Rightarrow 1 - 0.8113$$

$$\boxed{C = 0.1887 \text{ bits/symbol}}.$$

(iii) Channel efficiency $= \eta_{ch} = \frac{I(A,B)}{C} / \frac{I(X,Y)}{C}$.

$$\Rightarrow \eta_{ch} = \frac{0.1686}{0.1887} \Rightarrow \boxed{\eta_{ch} = 89.35\%}$$

Channel Redundancy $= R\eta_{ch} = 1 - \eta_{ch} =$

$$\boxed{R\eta_{ch} = 10.65\%}$$

⑤ **Binary Erasure Channel (BEC)**

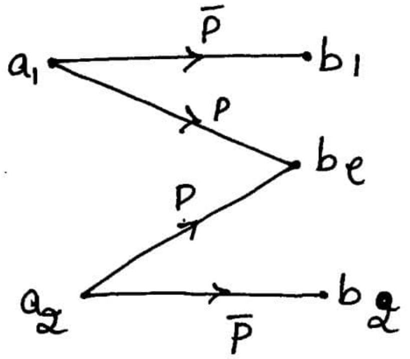
BEC is one of the most important channel used in Digital Communication.

If a symbol is received in error, it is just marked as error and discarded & a retransmission is requested from the transmitter using a reverse Channel, till a correct symbol is received at the output. This ensures 100% correct data recovery. Since the error is totally erased in this channel, it is called "Binary Erasure Channel".

Channel Diagram.

$\bar{P} = 1 - P$.

'A' Transmitter



'B' Receiver

ex:- So we have, $P(B/A) = \begin{array}{c|ccc} & b_1 & b_e & b_2 \\ \hline a_1 & \bar{P} & P & 0 \\ a_2 & 0 & P & \bar{P} \end{array}$

In order to characterize the channel, find $H(A/B)$

we Know $H(B/A) = h = \sum_{j=1}^{S} P_j \log_2 \frac{1}{P_j} \Rightarrow \boxed{\bar{P} \log \frac{1}{\bar{P}} + P \log \frac{1}{P}}$

if $P(a_1) = \omega$, $P(a_2) = \bar{\omega}$ then, $\omega + \bar{\omega} = 1$ & $P + \bar{P} = 1$

$\therefore$ we Know $H(A) = \sum_{i=1}^{2} P(a_i) \log_2 \frac{1}{P(a_i)}$

$$\boxed{H(A) = \omega \log_2 \frac{1}{\omega} + \bar{\omega} \log_2 \frac{1}{\bar{\omega}}}$$

$$P(A,B) = P(a_i, b_j) = \begin{array}{c|ccc} & b_1 & b_2 & b_3 \\ \hline a_1 & \bar{p}w & pw & 0 \\ a_2 & 0 & p\bar{w} & \bar{p}\bar{w} \end{array}$$

By x'lying the first row with w & second with $\bar{w}$

$\therefore P(b_1) = \bar{p}w$

$P(b_2) = pw + p\bar{w} = p$

$P(b_3) = \bar{p}\bar{w}$

$\therefore P(A/B) = P(a_i/b_j) = \dfrac{P(a_i, b_j)}{P(b_j)}$ from JPM

$$\therefore P(A/B) = \begin{bmatrix} \frac{\bar{p}w}{\bar{p}w} & \frac{pw}{p} & 0 \\ 0 & \frac{p\bar{w}}{p} & \frac{\bar{p}\bar{w}}{\bar{p}\bar{w}} \end{bmatrix}$$

$$\therefore P(A/B) = \begin{bmatrix} 1 & w & 0 \\ 0 & \bar{w} & 1 \end{bmatrix}$$

$$H(A/B) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log_2 \frac{1}{P(a_i/b_j)}$$

$$= \cancel{\bar{p}\bar{w} \log 1} + pw \log \frac{1}{w} + p\bar{w} \log \frac{1}{\bar{w}} + \cancel{\bar{p}\bar{w} \log 1}$$

$$= pw \log \frac{1}{w} + p\bar{w} \log \frac{1}{\bar{w}}$$

$$H(A/B) = p\left[w \log \frac{1}{w} + \bar{w} \log\left(\frac{1}{\bar{w}}\right)\right]$$

$$\boxed{H(A/B) = p \cdot H(A)}$$

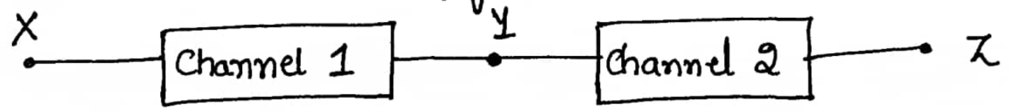
To find Capacity of the Channel.

$$C = \text{Max}\left[I(A,B)\right] r_s$$

$C = \text{Max}\left[H(A) - H(A/B)\right] r_s \implies C = \left[H(A) - p \cdot H(A)\right] r_s$

$C = \text{Max}\left[\bar{p}\, H(A)\right] r_s \implies C = \log_2 r \cdot \bar{p} \cdot r_s \implies$ for $r = 2, r_s = 1$ ... 84m/sec

(6) Cascaded channels : Two channels are connected in cascade as shown in below figure.



When the information is transmitted from X to Y through channel 1, there will be loss of information due to the noise in channel 1 & the mutual information at the o/p of channel 1 is

$$I(X,Y) = H(Y) - H(Y/X).$$

when the information is passed through Channel-II/2, there will be further loss of information & the mutual information at the output of the channel 2 would be

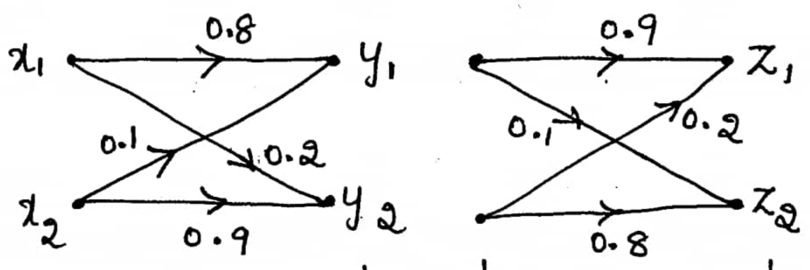
$$I(X,Z) = H(Z) - H(Z/X).$$

$$\therefore \boxed{I(X,Z) < I(X,Y)}$$

Problem.

(1) Two channels are cascaded as shown in the figure, find the following.

$H(X)$, $H(Y)$, $H(Z)$, $H(X,Z)$, $H(Z/X)$ & $H(X/Z)$ for given probability of $P(x_1) = P(x_2) = 0.5$. Show that $I(X,Z) < I(X,Y)$



Sol^n $P(Z/X)$ | $P(Y/X)$ | $P(Z/Y)$

We have $P(Y/X) = \begin{array}{c|cc} & y_1 & y_2 \\ x_1 & 0.8 & 0.2 \\ x_2 & 0.1 & 0.9 \end{array}$ & $P(Z/Y) = \begin{array}{c|cc} & z_1 & z_2 \\ y_1 & 0.9 & 0.1 \\ y_2 & 0.2 & 0.8 \end{array}$

$$P(Z/X) = \begin{array}{c|cc} & z_1 & z_2 \\ x_1 & 0.76 & 0.24 \\ x_2 & 0.27 & 0.73 \end{array}$$

$$\therefore P(X,Y) = \begin{array}{c|cc} & y_1 & y_2 \\ x_1 & 0.8 & 0.2 \\ x_2 & 0.1 & 0.9 \end{array} \begin{array}{l} \times P(x_1) = 0.5 \\ \times P(x_2) = 0.5 \end{array}$$

$$P(X,Y) = \begin{array}{c|cc} & y_1 & y_2 \\ x_1 & 0.4 & 0.1 \\ x_2 & 0.05 & 0.45 \end{array}$$

$P(y_1) = 0.45 \quad P(y_2) = 0.55$

$|||^{ly}$ $P(X,Z) = \begin{array}{c|cc} & z_1 & z_2 \\ x_1 & 0.38 & 0.12 \\ x_2 & 0.135 & 0.365 \end{array}$

$P(z_1) = 0.515 \quad P(z_2) = 0.48$

$$\therefore H(X) = 0.5 \log_2 \frac{1}{0.5} + 0.5 \log_2 \frac{1}{0.5} = \boxed{1 \text{ bits/symbol}}$$

$$H(Y) = 0.45 \log_2 \frac{1}{0.45} + 0.55 \log_2 \frac{1}{0.5} = \boxed{0.9927 \text{ bits/symbol}}$$

$$H(Z) = 0.515 \log_2 \frac{1}{0.515} + 0.485 \log_2 \frac{1}{0.485} = \boxed{0.9993 \text{ bits/symb}}$$

$$H(X,Z) = 0.38 \log_2 \frac{1}{0.38} + 0.12 \log_2 \frac{1}{0.12} + 0.135 \log_2 \frac{1}{0.135} + 0.365 \log_2 \frac{1}{0.365}$$

$$\boxed{H(X,Z) = 1.81825 \text{ bits/symbol}}$$

$$H(Z/X) = \sum_{i=1}^{2} \sum_{j=1}^{2} P(x_i, z_j) \log_2 \frac{1}{P(z_j/x_i)}$$

$$= 0.38 \log_2 \frac{1}{0.76} + 0.12 \log_2 \frac{1}{0.24} + 0.135 \log_2 \frac{1}{0.27} + 0.365 \log_2 \frac{1}{0.73}$$

$$\boxed{H(Z/X) = 0.8182 \text{ bits/symbol}}$$

$$H(X,Z) = H(X/Z) + H(Z)$$

$$H(X/Z) = H(X,Z) - H(Z) = 1.81825 - 0.9993$$

$$\boxed{H(X/Z) = 0.81895 \text{ bits/symbol}}$$

$$H(X,Y) = 0.4 \log_2 \frac{1}{0.4} + 0.1 \log_2 \frac{1}{0.1} + 0.05 \log_2 \frac{1}{0.05} + 0.45 \log_2 \frac{1}{0.45}$$

$$H(X,Y) = 1.595 \text{ bits/symbol}.$$

$$\therefore \; I(X,Y) = H(Y) - H(Y/X)$$

$$H(Y/X) = H(X,Y) - H(X)$$

$$= 1.595 - 1$$

$$H(Y/X) = 0.595.$$

$$\Rightarrow I(X,Y) = 0.9927 - 0.595$$

$$I(X,Y) = 0.3977.$$

$$\Rightarrow I(X,Z) = H(Z) - H(Z/X)$$

$$I(X,Z) = 0.9993 - 0.8182$$

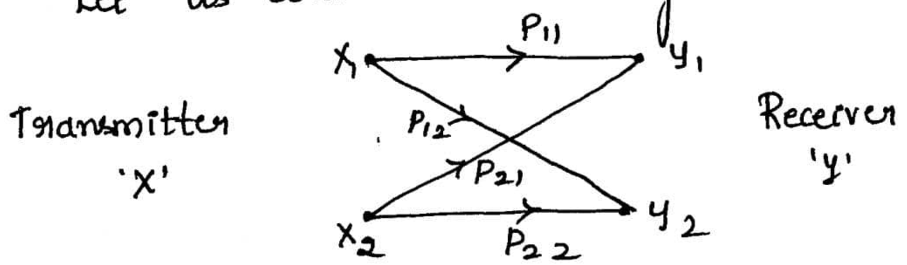
$$I(X,Z) = 0.1811.$$

$$\therefore I(X,Z) < I(X,Y)$$

$$\boxed{0.1811 < 0.3977}$$

Estimation of Channel Capacity By Muraga's Method.

Let us consider a binary channel as shown below



The channel matrix of the binary channel can be written as

$$P(y_j/x_i) = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \end{array}$$

To find the channel capacity of a binary channel, a method was suggested by Dr. S. Muroga, which is stated below:

Two Quantities θ_1 and θ_2 are found by using the matrix equ^n

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} P_{11} \log P_{11} + P_{12} \log P_{12} \\ P_{21} \log P_{21} + P_{22} \log P_{22} \end{bmatrix}$$

expanding we get

$$P_{11}\theta_1 + P_{12}\theta_2 = P_{11} \log P_{11} + P_{12} \log P_{12}$$
$$P_{21}\theta_1 + P_{22}\theta_2 = P_{21} \log P_{21} + P_{22} \log P_{22}$$

The above equ^n are simultaneous equ^n which are solved to get θ_1 and θ_2.

The channel capacity 'C' of the binary channel is then calculated using the relationship

$$\boxed{C = \log\left[2^{\theta_1} + 2^{\theta_2}\right] \text{ bits/symbol}}$$

lll^{ly} for three Quantities

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} P_{11} \log P_{11} + P_{12} \log P_{12} + P_{13} \log P_{13} \\ P_{21} \log P_{21} + P_{22} \log P_{22} + P_{23} \log P_{23} \\ P_{31} \log P_{31} + P_{32} \log P_{32} + P_{33} \log P_{33} \end{bmatrix}$$

here θ_1, θ_2 & θ_3 are found

$$\therefore C = \log\left[2^{\theta_1} + 2^{\theta_2} + 2^{\theta_3}\right] \text{ bits/symbol.}$$

$$\vdots$$

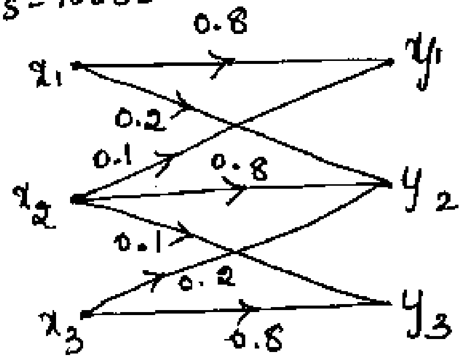
Generalizing the equation

$$\boxed{\text{Channel Capacity 'C'} = \log\left[2^{\theta_1} + 2^{\theta_2} + \dots + 2^{\theta_m}\right] \text{ bits/symbol}}$$

Problem.

① Find the Capacity of the discrete Channel shown in figure, for $r_s = 10000$

Transmitter 'X'



Receiver 'Y'

→ the channel matrix can be constructed as

$$P(y_j/x_i) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{bmatrix} \end{array}$$

Using Muroga's method to find the channel capacity

$$\begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 0.8 \log 0.8 + 0.2 \log 0.2 \\ 0.8 \log 0.8 + 2 \times 0.1 \log 0.1 \\ 0.8 \log 0.8 + 0.2 \log 0.2 \end{bmatrix}$$

$$0.8\, Q_1 + 0.2 Q_2 = -0.722 \rightarrow ①$$

$$0.1 Q_1 + 0.8 Q_2 + 0.1 Q_3 = -0.922 \rightarrow ②$$

$$0.2\, Q_2 + 0.8\, Q_3 = -0.722 \rightarrow ③$$

Substituting ①, ② & ③ equ^n in calci we get

$Q_1 = -0.6553 \quad Q_2 = -0.9887 \quad Q_3 = -0.6553$

∴ The channel capacity is given by

$$C = \log\left[2^{Q_1} + 2^{Q_2} + 2^{Q_3}\right] = \log\left[2^{-0.6553} + 2^{-0.9887} + 2^{-0.655}\right]$$

$$\boxed{C = 0.8269 \text{ bits/symbol}}$$

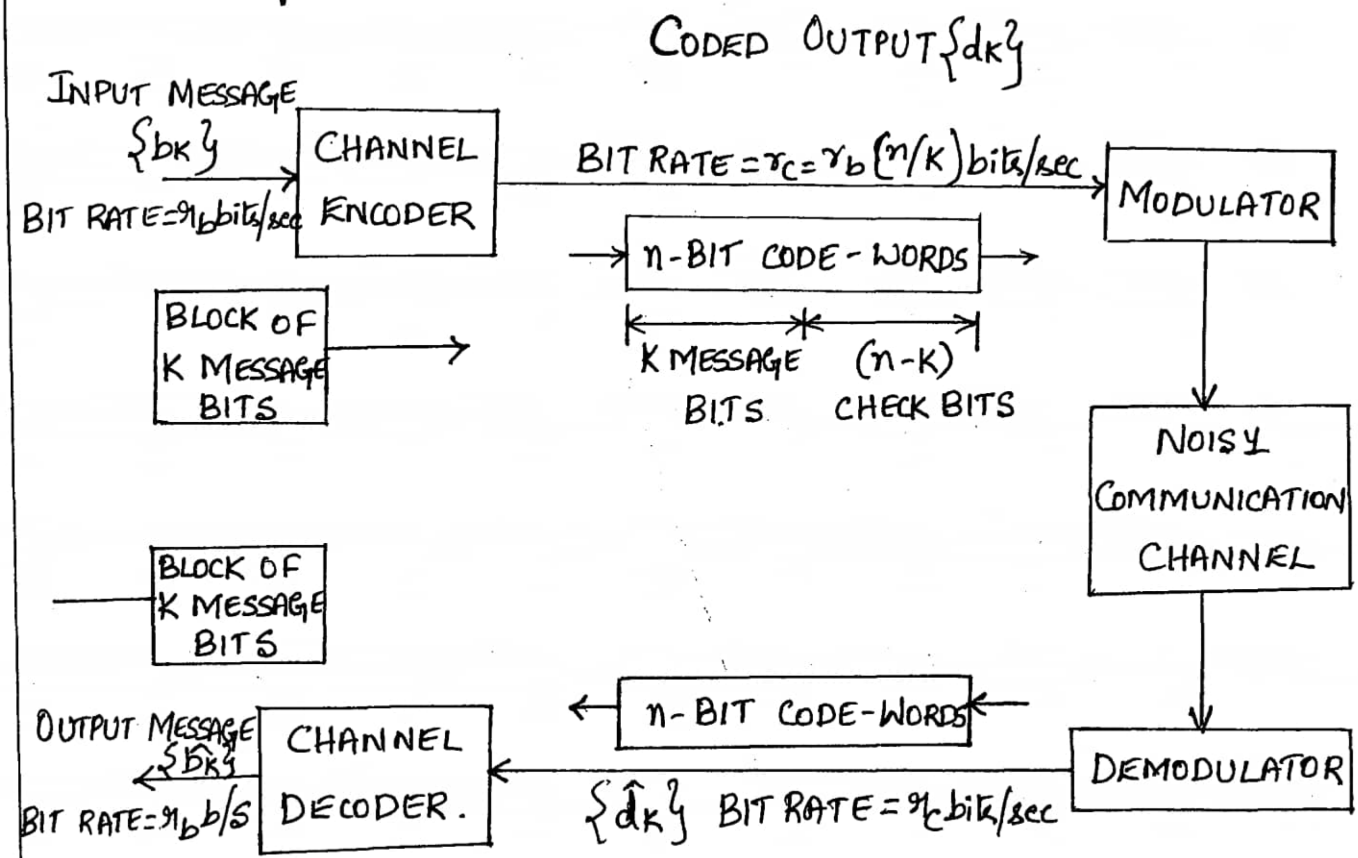
$$C = (0.8269)\, r_s \qquad \boxed{C = 8269 \text{ bits/sec}}$$

ITC.

Noor Afza CIT Gubbi

ERROR CONTROL CODING.

Block diagram of Communication System Employing error control Coding.



The above block diagram gives us a digital communication system, employing error control coding. The main functional blocks are the channel encoder, the channel decoder, modulator and demodulator & the noisy communication channel with a Capacity C bits/sec.

The source generates a message block {bk} at a rate of r_b bits/sec and feeds it to the channel encoder.

The channel encoder, then adds (n-K) number of redundant bits to these K-bit messages to form n-bit code words.

These (n-K) number of additional bits also called "check bits" donot carry any information but helps channel decoder to detect & correct errors.

the bit rate of the coded output block $\{d_K\}$ will be $r_c = r_b(n/k)$ bits/sec. This is the rate at which the modem operates, to produce a message block $\{\hat{d}_K\}$ at the receiver. The channel decoder then decodes this message to get back the information block $\{\hat{b}_K\}$ at the receiver.

The information block $\{\hat{b}_K\}$ usually differs from the transmitted block $\{b_K\}$. Thus probability of error.

$$P_e = P\{\hat{b}_K \neq b_K\}$$

then the probability of error which depends on bit rate r_c is defined as

$$q_c = P\{\hat{d}_K \neq d_K\}$$

∴ It is required to design an error control coding scheme such that the overall probability of error is less than the desired value.

Methods of Controlling Errors.

1. Forward acting error correction : In this method the errors are detected and corrected by proper coding techniques. The check bits/redundant bits are used to detect & correct errors. The error detection & correction capability depends on the number of redundant bits in the transmitted message. The FAEC is faster but the probability of error is comparitively high.

2. Error Detection : In this method, the decoder checks the i/p sequence, when it detects any error it rejects that part of the sequence & request the Tx^r for retransmission. hence the decoder doesnot correct the errors. It just detects the errors & sends request to the Tx^r through a reverse channel for

Types of Errors :

(i) Random Error : The transmission errors that occur due to the presence of white gaussian noise are referred to as "random errors". Sources of Gaussian noise include thermal & shot noise in the transmitting and receiving equipment.

(ii) Burst error : These errors are generated by Impulsive noise in the channel. this noise is due to lightning & switching transients & man-made noise etc. this noise affect more than one symbol and the error caused is called "Burst Error".

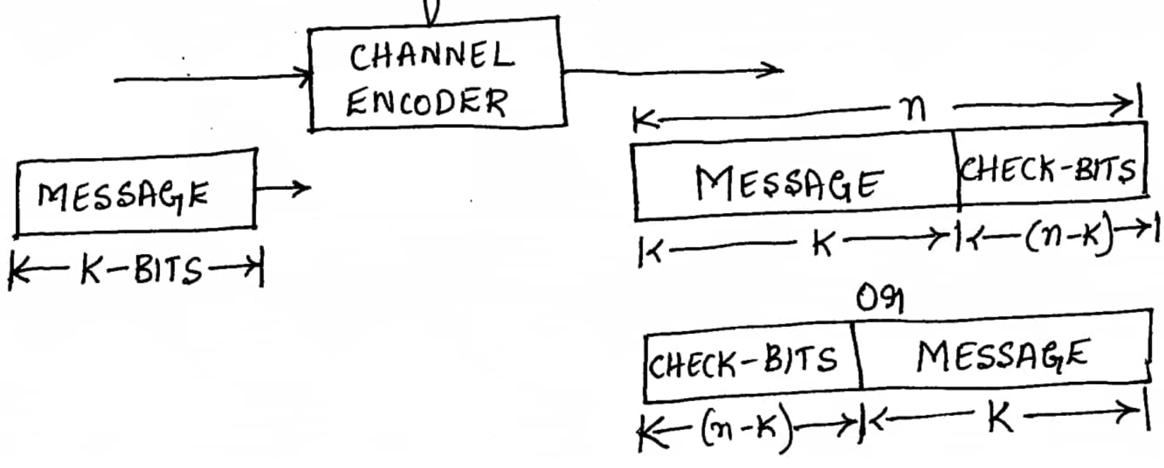
Types of Codes :

(i) Block Codes : these codes consists of 'n' number of bits in one block or codeword. This codeword consists of 'K' message bits and (n-K) redundant bits such block codes are called (n-K) block codes.

(ii) Convolution Codes : In this code, the check bits are continuously interleaved with information bits. the check bits verify the information bits not only in the block immediately preceeding them, but in other blocks as well.

LINEAR BLOCK CODES.

- Formation of linear Block Codes.



In channel encoder, a block of 'K' message bits is encoded into a block of 'n' bits by adding $(n-K)$ number of check bits as shown in figure.

where $n > K$, such a code formed is called (n,K) block code. A (n,K) block code is said to be a "(n,K) linear block code" if it satisfies the condition:

Let C_1 & C_2 be any 2 code-words belonging to a set of (n,k) block code. if $C_1 \oplus C_2$ is also a n-bit code-word belonging to the same set of (n,K) block code, then such a block code is called (n,K) linear block code.

A (n,K) linear block code is said to be "systematic" if the k-message bit appear either at the "beginning" of the code-word or at the end of the code-word as shown in figure.

Matrix Description of linear Block codes.

The encoding operation in a linear block encoding scheme consists of 2 basic steps.

1. The information sequence is segmented into message blocks each block consisting of 'K' successive information bits.
2. the encoder transforms each message block into a larger block of n bits according to some predetermined set of rules.

These n-k additional bits are generated from linear combinations of the message bits, & we can describe the encoding operations using matrices.

- Let the message block of K-bits be represented as a "row-vector" or "K-tuple" called "message-vector" given by

$$[D] = \{d_1, d_2, d_3 \ldots, d_K\} \longrightarrow (1)$$

where $d_1, d_2, \ldots d_K$ are either '0's or 1's.

$\therefore$ there are 2^K distinct message vectors.

the channel encoder systematically adds $(n-k)$ number of Check bits to form a (n,k) linear block code. Then the 2^K code vector can be represented by

$$C = \{c_1, c_2, \ldots, c_n\} \longrightarrow (2)$$

the ratio (K/n) is defined as the "rate efficiency" of the (n,K) linear block code.

In a systematic linear block code, the message bits appear at the beginning of the code-vector or at the end of the code vector

$$\therefore c_i = d_i \text{ for all } i = 1, 2, \ldots K \longrightarrow (3)$$

The remaining $(n-K)$ bits are check bits, hence equ^n (2) & (3) can be combined as

$$[C] = \{\underbrace{c_1, c_2 \ldots c_K}_{K\text{-message bits}}, \underbrace{c_{K+1}, c_{K+2}, \ldots, c_n}_{(n-K)\text{ Check bits}}\} \rightarrow (4)$$

These $(n-K)$ check bits $c_{K+1}, c_{K+2} \ldots, c_n$ are derived from K-message bits using a predetermined rule

$$\left.\begin{aligned}
\text{i.e}\quad c_{K+1} &= P_{11}d_1 + P_{21}d_2 + \ldots + P_{K1}d_K \\
c_{K+2} &= P_{12}d_1 + P_{22}d_2 + \ldots + P_{K2}d_K \\
&\vdots \\
c_n &= P_{1,n-K}d_1 + P_{2,n-K}d_2 + \ldots + P_{K,n-K}d_K
\end{aligned}\right\} \rightarrow (5)$$

where $P_{11}, P_{21}, P_{12}, P_{22} \ldots$ are either '0's & 1's and the addition above is performed using modulo-2 arithmetic.

$\therefore$ combining (3), (4) & (5) we can express the result in matrix form as

$$[c_1 \; c_2 \; \cdots \; c_k \; c_{k+1} \; c_{k+2} \; \cdots \; c_n] = [d_1 \; d_2 \cdots d_k] \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & p_{11} & p_{12} & \cdots & p_{1\,n-k} \\ 0 & 1 & \cdots & & 0 & p_{21} & p_{22} & \cdots & p_{2\,n-k} \\ \vdots & & & & & \vdots & & & \vdots \\ 0 & \cdots & & & 1 & p_{k1} & p_{k2} & \cdots & p_{k\,n-k} \end{bmatrix}$$

K-terms (n-k) terms

$$[C] = [D][G]$$

where $[G]$ is called as "Generator Matrix" of order $(K \times n)$ given by

$$[G] = [I_K \,\vdots\, P]_{K\times n}$$

where I_K = unit matrix of order 'K'

$[P]$ = arbitary matrix called "PARITY MATRIX" of order $K\times(n-k)$

& $\vdots$ = denotes the demarcation b/w the unit matrix I_K & parity matrix P.

The parity 'P' is selected to correct random & burst errors.

Modulo-2 arithmatic

Modulo-2 addition	Modulo-2 Multiplication	Mudulo-2 Subtraction
$0+0=0$	$0*0=0$	$0-0=0$
$0+1=1$	$0.1=0$	$0-1=1$
$1+0=1$	$1.0=0$	$1-0=1$
$1+1=0$	$1.1=1$	$1-1=0$

PROBLEMS.

(1) For a systematic (6,3) linear Block code, the parity matrix P is given by $[P] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ find all possible code-vectors.

→ Solution

we Know (n, K) linear block codes

Given : $(6,3)$ —— " ——

$\therefore n = 6, K = 3$

there are 2^K distinct message vectors $\Rightarrow 2^3 = 8$

$(000), (001), (010), (011), (100), (101), (110), (111)$

the code vectors are then found by using equn

$$[C] = [D][G]$$

where $[G] = [I_K \mid P]$

$$= [I_3 \mid P]$$

$$\therefore [G] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\therefore [C] = [D][G] = [d_1\ d_2\ d_3] \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$= [d_1, d_2, d_3, (d_1 + d_3), (d_2 + d_3), (d_1 + d_2)]$$

∴ The code vectors can be found which are listed

Code name	Message-vector	Code-vector for (6,3) linear block code
C_1	000	000000
C_2	001	001110
C_3	010	010011
C_4	011	011101
C_5	100	100101
C_6	101	101011
C_7	110	110110
C_8	111	111000

To verify it is linear block code

$$C_1 \oplus C_7 = \begin{array}{r} 000000 \\ 110110 \\ \hline 110110 \end{array} \Rightarrow C_7. \text{ which is present in the block code}$$

hence (6,3) is a linear block code.

2. For a systematic (7,4) linear Block code whose Generator Matrix is given (i) find all the Code words.

$$G = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

→ Given $(n,k) = (7,4)$ linear Block codes

∴ $2^k = 2^4$ message-vector. i.e 1to16 msg vectors

code vectors C_1 to C_{16}

message vectors d_1 to d_{16}

Code vectors are found by using $[C] = [D]\,[G]$

$$[C] = [d_1\ d_2\ d_3\ d_4] \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$[C] = \left[d_1, d_2, d_3, d_4, (d_1 + d_2 + d_3), (d_2, + d_3 + d_4), (d_1 + d_2 + d_4)\right]$$

The code-vectors are listed below

Code name	Message-vector	Code-vector
C_1	0000	0000000
C_2	0001	0001011
C_3	0010	0010110
C_4	0011	0011101
C_5	0100	0100111
C_6	0101	0101100
C_7	0110	0110001
C_8	0111	0111010
C_9	1000	1000101
C_{10}	1001	1001011
C_{11}	1010	1010110
C_{12}	1011	1011000
C_{13}	1100	1100010
C_{14}	1101	1101001
C_{15}	1110	1110100
C_{16}	1111	1111111

Parity check matrix

we have Generator matrix given by

$$[G] = [I_K \mid P]$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & & & 1 \end{bmatrix}}_{K \text{ columns}} \underbrace{\begin{matrix} P_{11} & P_{12} & \cdots & P_{1,n-K} \\ P_{21} & P_{22} & \cdots & P_{2,n-K} \\ \vdots & & & \\ P_{K1} & P_{K2} & \cdots & P_{K,n-K} \end{matrix}}_{(n-K) \text{ columns}} \rightarrow (1)$$

The generator matrix [G] having another matrix called "Parity check Matrix-H" given by

$$[H] = [P^T \mid I_{n-K}]$$

$$[H] = \underbrace{\begin{bmatrix} P_{11} & P_{21} & \cdots & P_{K1} \\ P_{12} & P_{22} & \cdots & P_{K2} \\ \vdots & & & \\ P_{1(n-K)} & P_{2,n-K} & \cdots & P_{K,(n-K)} \end{bmatrix}}_{K\text{-columns}} \underbrace{\begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & & 1 \end{matrix}}_{(n-K) \text{ columns}}$$

$\therefore$ [H] matrix is a $(n-K) \times (n)$ matrix.

Encoding circuit for (n,K) linear block codes:

we Know/have the matrix form as"

$$[c_1, c_2, c_3 \ldots c_K \, c_{K+1} \, c_{K+2} \ldots c_n] = [d_1 \, d_2 \ldots d_K] \underbrace{\begin{bmatrix} \overbrace{1 \; 0 \; 0 \; 0 \cdots 0}^{K\text{-terms}} & \overbrace{P_{11} \; P_{12} \cdots P_{1,n-K}}^{(n-K) \text{ terms}} \\ 0 \; 1 \; 0 \; 0 \cdots 0 & P_{21} \; P_{22} \cdots P_{2,n-K} \\ \vdots & \\ 0 \; 0 \; 0 \; 0 \cdots 1 & P_{K1} \; P_{K2} \cdots P_{K,n-K} \end{bmatrix}}_{K \times n}$$

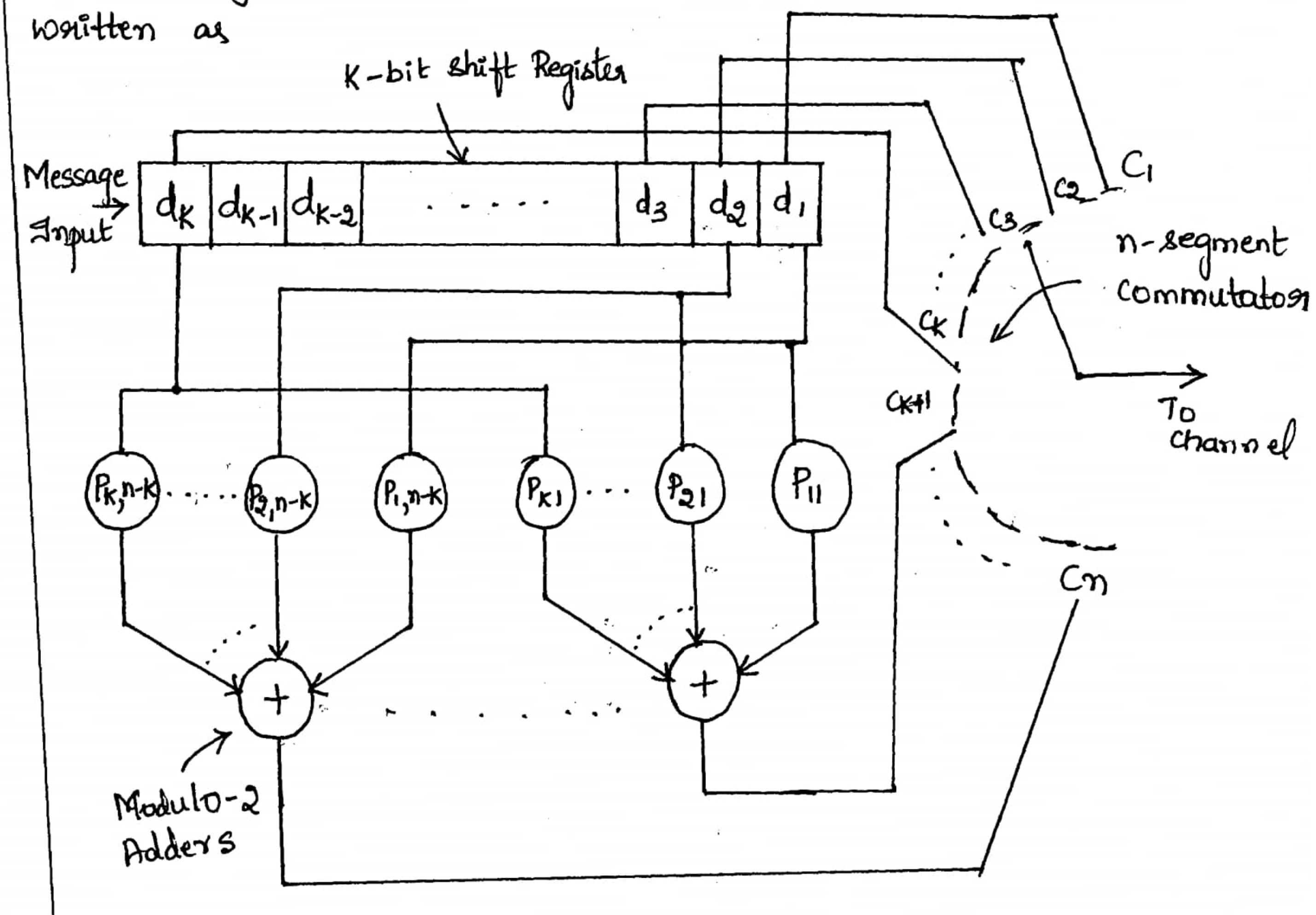
(1)

Then

$$\left.\begin{aligned} C_1 &= d_1 \\ C_2 &= d_2 \\ &\vdots \\ C_K &= d_K \\ C_{K+1} &= P_{11}d_1 + P_{21}d_2 + \cdots\cdots + P_{K1}d_K \\ C_{K+2} &= P_{12}d_1 + P_{22}d_2 + \cdots\cdots + P_{K2}d_K \\ &\vdots \\ C_n &= P_{1,n-K} + P_{2,n-K}d_2 + \cdots\cdots + P_{K,n-K}d_K \end{aligned}\right\} \rightarrow (2)$$

The design of encoder circuit consists of a K-bit shift register, a n-segment commutator & (n-K) number of modulo-2 adders.

∴ Encoding circuit for (n,K) linear block codes can be written as



The entire data $d_K, d_{K-1}, d_{K-2}, \ldots d_2, d_1$ is shifted into the K-bit shift register. The small circles $P_{11}, P_{21}, \ldots P_{K1}$ are either "open-circuit" or "short-circuit" depending on either '0' or '1'
ex:- if P_{11} is 0 then there is no connection from d_1 to the modulo-2 adder and if $P_{11} = 1$, then there is connection.

when the message is shifted into the shift register, the modulo-2 adders generate the "check bits" which are fed into the commutator segments along with message bit as shown in figure.

problem

1) Construct the encoding circuit for problem (1) in pg (4)

→ The code-vectors are given by

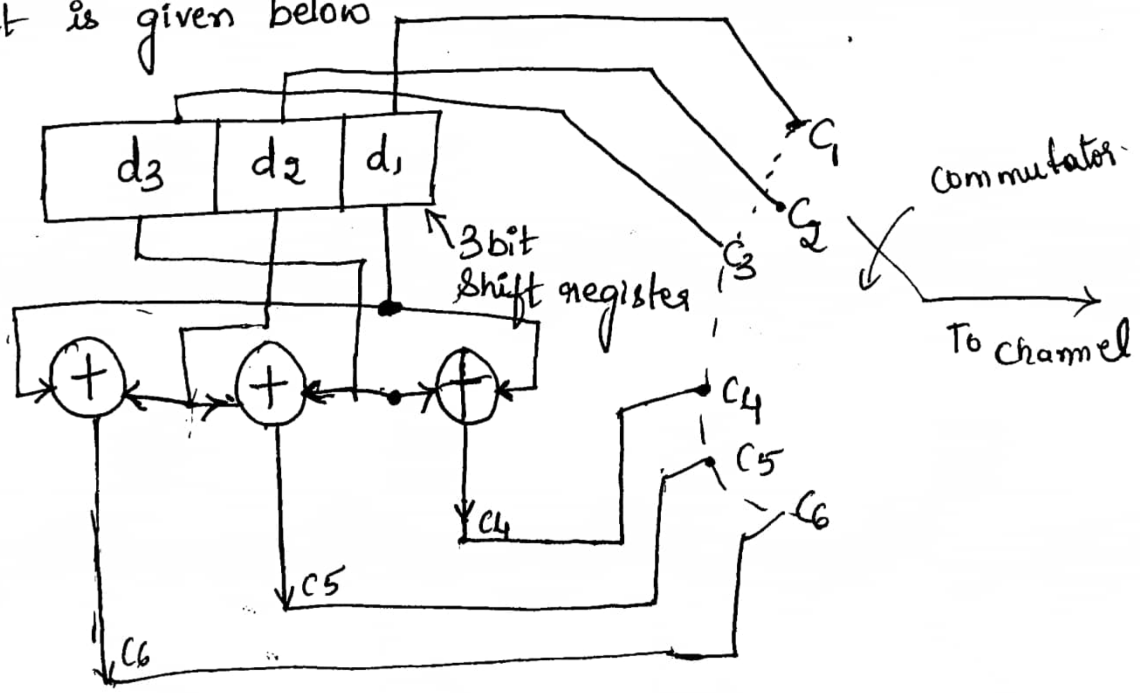
$c_1 = d_1$, $c_2 = d_2$, $c_3 = d_3$, $c_4 = d_1 + d_3$, $c_5 = d_2 + d_3$, $c_6 = d_1 + d_2$

we require 3 shift registers.

where we have (n, K) linear codes
(6, 3) — " —

$n - K = 6 - 3 = 3$, we require 3 modulo adders & 6 segment commutator.

the circuit is given below



2) Construct the encoding circuit for problem 2 in pg 4

→ The code vectors are given by

$C_1 = d_1$, $C_2 = d_2$, $C_3 = d_3$ $C_4 = d_4$, $C_5 = (d_1 + d_2 + d_3)$,

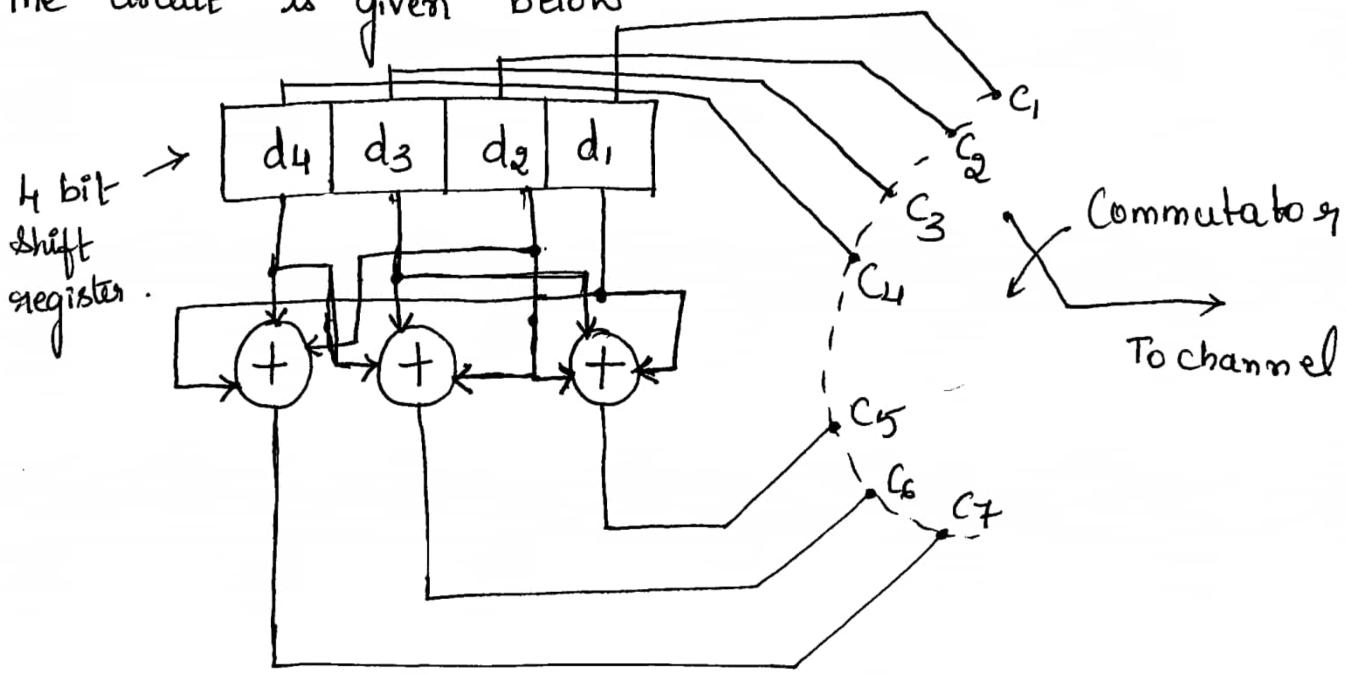
$C_6 = (d_2 + d_3 + d_4)$, $C_7 = (d_1 + d_2 + d_4)$

We have (7,4) linear block code

We require 4 shift registers, 7 segment Commutator,

$n - K = 6 - 3 = 3$, we require 3 modulo adders

The circuit is given below



3 If C is a Valid Code Vector, then prove that $CH^T = 0$ where H^T is the transpose of the parity check matrix H.

→ We have

$$[G] = [I_K \mid P] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & P_{11} & P_{12} \cdots P_{1j} & P_{1,n-k} \\ 0 & 1 & 0 & \cdots & 0 & P_{21} & P_{22} \cdots & P_{2,n-k} \\ \vdots & & & & & & & \\ 0 & 0 & \cdots & 1 \cdots & 0 & P_{i1} & P_{i2} \; P_{ij} \cdots & P_{i,n-k} \\ 0 & 0 & \cdots & & 1 & P_{K1} & P_{K2} \cdots & P_{K,n-k} \end{bmatrix}$$

∴ The i^{th} row of the [G] matrix is given by

$$g_i = [0 \; 0 \ldots \underset{\uparrow \; i^{th}\text{ element}}{1} \ldots 0 \quad P_{i1} \; P_{i2} \ldots \underset{\uparrow \; (K+j)^{th}\text{ elements}}{P_{ij}} \ldots P_{i,(n-k)}] \quad (1)$$

we have $[H] = [P^T \,\vdots\, I_{n-K}]$

$$\therefore [H] = \begin{bmatrix} P_{11} & P_{21} & \cdots & \cdots & P_{K1} & 1 & 0 & 0 & \cdots & \cdots & 0 \\ P_{12} & P_{22} & \cdots & \cdots & P_{K2} & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & & & & \vdots & & & & & \vdots \\ P_{1j} & P_{2j} & \cdots & P_{ij} & \cdots P_{Kj} & 0 & 0 & 0 & \cdots & 1 \cdots & 0 \\ \vdots & & & & & \vdots & & & & & \vdots \\ P_{1,n-K} & P_{2,(n-K)} & \cdots & \cdots & P_{K,n-K} & 0 & 0 & 0 & \cdots & \cdots & 1 \end{bmatrix}$$

$$h_j = [P_{1j} \; P_{2j} \cdots \underset{i^{th}\text{ element}}{P_{ij}} \cdots P_{Kj} \quad 0 \; 0 \cdots \underset{(K+j)^{th}\text{ element}}{1} \cdots 0] \longrightarrow (2)$$

Let us consider $g_i h_j^T = [0 \; 0 \cdots \underset{i^{th}\text{ element}}{1} \cdots 0 \; P_{i1} \; P_{i2} \cdots \underset{(K+j)^{th}\text{ element}}{P_{ij}} \cdots P_{i,(n-K)}] \begin{bmatrix} P_{1j} \\ P_{2j} \\ \vdots \\ P_{ij} \\ \vdots \\ P_{Kj} \\ \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \longrightarrow (3)$

Modulo-2 multiplication of the row and column yields

$$g_i h_j^T = (0)(P_{1j}) + (0)(P_{2j}) + \cdots + (1)(P_{ij}) + \cdots 0(P_{Kj}) + \cdots P_{i1}(0) + P_{i2}(0) + \cdots P_{ij}(1) + \cdots + P_{i(n-K)}(0)$$

$$g_i h_j^T = P_{ij} + P_{ij} = P_{ij}(1+1) = 0 \longrightarrow (4)$$ this equ^n (4) is true for every values of i & j

$\therefore$ we have $[G][H]^T = 0 \longrightarrow (5)$

pre-multiplying $[D]$ in both sides of equation (5), we get

$$[D][G][H]^T = [D][0]$$

But we have $[C] = [D][G]$

$$\therefore [C][H]^T = 0$$

$$\boxed{CH^T = 0}$$

Syndrome and Error Correction.

Consider the transmitted code-vector $[C] = [c_1, c_2, c_3 \ldots c_n]$ and received code vector $R = [r_1, r_2, \ldots r_n]$

If $R = C$, then there is no error

If $R \neq C$, then there is an error in received vector.

$\therefore$ The error vector 'e' can be represented as a vector by $E = (e_1, e_2, e_3 \ldots e_n)$ & it is defined as the difference b/w 'R' and 'C'

$$\therefore E = R - C = R + C \rightarrow (1)$$ ($\because$ add & sub is same in modulo-2 arith^c)

If $r_i = c_i$, $e_i = 0$, if $r_i \neq c_i$, $e_i = 1$

The Receiver Knows only 'R' & does not know 'C' and 'E'.

In order to detect the errors in the received vectors we define an $(n-k)$ bit vector known as the error syndrome of the circuit. It is defined as

$$[S] = [R][H]^T \longrightarrow (2)$$

for $S = (s_1, s_2 \ldots s_{n-k})$

But we know $\boxed{R = C + E}$ from (1) $\rightarrow$ (A)

$$\therefore \quad S = [C+E][H]^T$$

$$= CH^T + EH^T$$

$$\boxed{S = EH^T} \longrightarrow (3)$$

So from equ^n (3), the receiver finds 'E' where S & H^T are known.

— " — " (A), the code-vector 'C' can be found out

If $S = 0$, then the received vector is same as code vector i.e there is no error

$S \neq 0$, then an error has occured in R.

Problems.

1. Consider (7,4) LBC with the parity matrix as $P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

(i) find $[G]$, $[H]$

(ii) If there are 3 received vectors as

$R_1 = 0100000$, $R_2 = 1110111$, $R_3 = 1100100$

determine if there was an error during transmission, also detect & correct the errors.

Sol^n (i) $[G] = [I_K \,\vdots\, P]_{K \times n}$ we have $(n, K) = (7, 4)$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & \vdots & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & \vdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & \vdots & 1 & 1 & 1 \end{bmatrix}_{4 \times 7}$$

$$[H] = [P^T \,\vdots\, I_{n-k}]$$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}_{3 \times 7}$$

(i) $[S] = RH^T$

the given first $R_1 = 0100000$ $= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{7 \times 3}$

$S_1 = [1 \; 0 \; 1]$

$\therefore S_1 \neq 0$ There is an error

Match S_1 with rows of H^T, In H^T, 2 row matches with:

$\therefore$ The 2nd bit of R_1 is in error and corrected code vector is $\hat{C}_1 = 0000000$

(ii) given $S_2 = [1110111]\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 = [1110111]$

$$S_2 = [1 \quad 1 \quad 1]$$

$\therefore S_2 \neq 0 \Rightarrow$ There is an error

Match S_2 with rows of H^T, In H^T 4^{th} row matches with S_2

$\therefore$ the 4^{th} bit of R_2 is in error & corrected code vector is

$$\hat{C}_2 = [1111111]$$

(iii) given $S_3 = [1100100]\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_3 = [1100100]$

$$S_3 = [0 \quad 1 \quad 0]$$

$\therefore S_3 \neq 0$ there is an error $\Rightarrow$ 111^{ly} as above corrected vector $\hat{C}_3 = [1100110]$

(2.) For (6,3) linear block code the 'G' is given by

$$G = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

find (i) H

(ii) If $C = 111000$ find 'syndrome' if $R = 111001$ & determine the transmitted code word.

→ (i) $H = [P^T \vdots I_{n-k}]$

$$= \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\therefore H^T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(ii) $S = CH^T$

$$111000 \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $S=0$, the received bit is same as transmitted bit

Given $R = 111001$

$$S = RH^T \Rightarrow [111001] \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S = [0\ 0\ 1]$$

The error occurs in the 6^{th} row, this means that there is an error in 6^{th} bit of message transmitted

$\therefore$ Transmitted code word is $[111000]$

Syndrome Calculation Circuit.

We Know that $S = RH^T \longrightarrow (1)$

Expanding (1) in general format

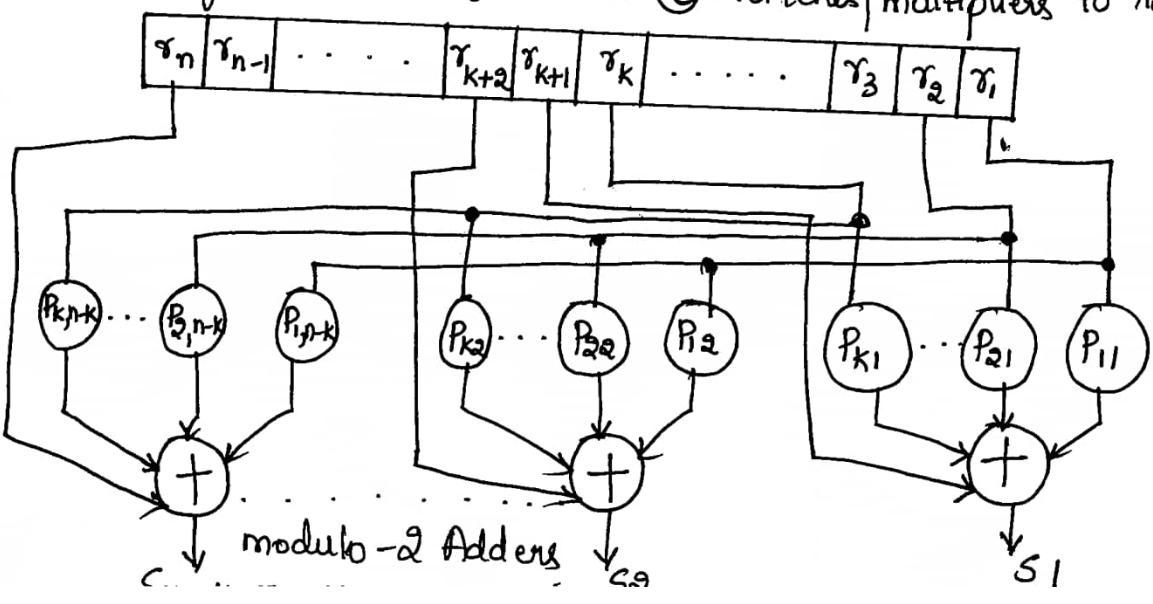
$$\therefore [s_1, s_2, \dots S_{n-k}] = [r_1, r_2 \dots r_n] \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1,n-k} \\ P_{21} & P_{22} & \dots & P_{2,n-k} \\ \vdots & \vdots & & \vdots \\ P_{k1} & P_{k2} & \dots & P_{k,n-k} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \dots & 1 \end{bmatrix} \longrightarrow (2)$$

Multiplying by using modulo-2 arithmatic, we get the syndrome bits as

$$\left.\begin{aligned} S_1 &= r_1 P_{11} + r_2 P_{21} + \dots\dots + r_k P_{k1} + r_{k+1} \\ S_2 &= r_1 P_{21} + r_2 P_{22} + \dots\dots + r_k P_{k2} + r_{k+2} \\ &\vdots \\ S_{n-k} &= r_1 P_{1,n-k} + r_2 P_{2,n-k} + \dots + r_k P_{k,n-k} + r_n \end{aligned}\right\} \longrightarrow (3)$$

So from (3), the hardware requirement for the syndrome calculation circuit for (n,k) linear block code can be summarised as follows:

(a) A n-bit shift register to hold the received vector R
(b) Max of $(n-k)$ mod-2 adders (c) switches/multipliers to realize P_{ij}.



Problems.

① For the given parity matrix, construct the syndrome circuit.

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

→ $H = [P^T \vdots I_{n-k}]$

$$= \begin{bmatrix} 1 & 0 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 1 & \vdots & 0 & 1 & 0 \\ 1 & 1 & 0 & \vdots & 0 & 0 & 1 \end{bmatrix}_{3\times 6}$$

$S = RH^T$

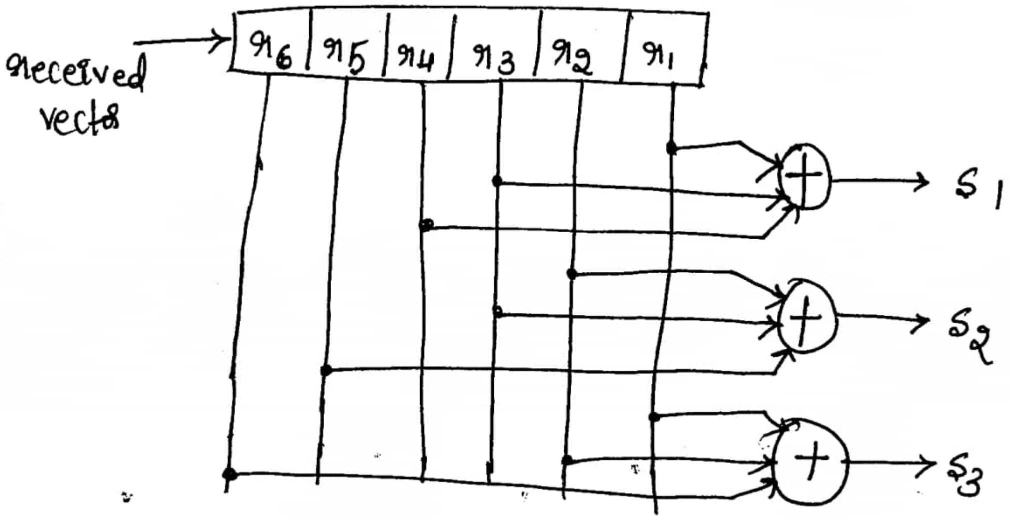
let $R = R_1, R_2, R_3\ R_4\ R_5\ R_6$

$$\therefore S \Rightarrow [R_1\ R_2\ R_3\ R_4\ R_5\ R_6] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$S_1 = r_1 + r_3 + r_4$, $S_2 = r_2 + r_3 + r_5$, $S_3 = r_1 + r_2 + r_6$

$\therefore n = 6$

$n - k$ = modulo adder $= 6 - 3 = 3$ syndrome.



(2) For a given (7,4) linear block code, the parity matrix P is given by

$$[P] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

① find all possible valid code-vectors
② Draw encoding circuit
③ A single error has occurred in each of these received vectors. Detect and correct those errors
ⓐ $R_A = [0111110]$ ⓑ $R_B = [1011100]$ ⓒ $R_C = [1010000]$.
④ Draw the syndrome Calculation circuit.

→ To find code-vectors, we have $[C] = [D][G]$

$$2^K = 2^4 = 16$$

where $[G] = [I_K \mid P]$

$$= \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

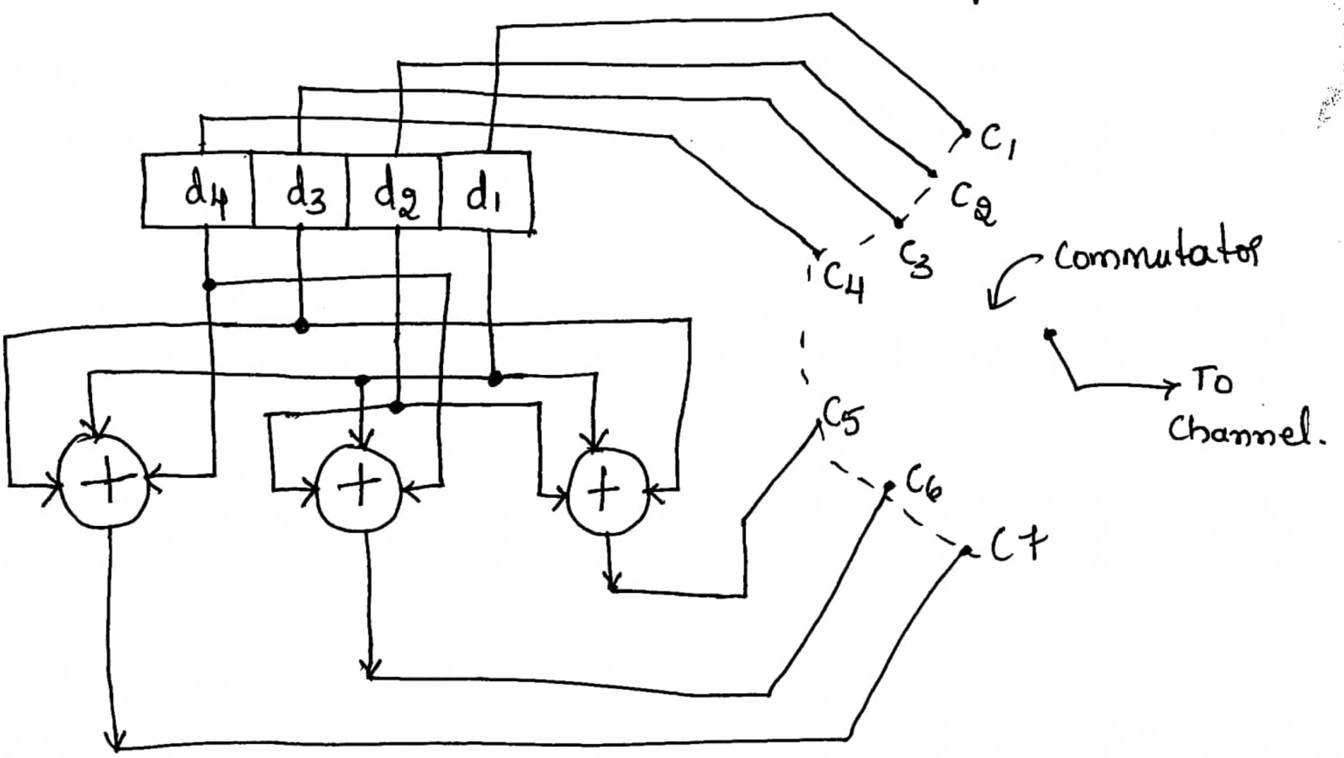
$$[C] = [D]\ [G]$$

$$= [d_1\ d_2\ d_3\ d_4] \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$[C] = [d_1, d_2, d_3, d_4, d_1+d_2+d_3, d_1+d_2+d_4, d_1+d_3+d_4]$$

Message vector (D)	code-Vectors (C)	Message Vector (D)	code-Vector (C)
0000	0000000	1000	1000111
0001	0001011	1001	1001100
0010	0010101	1010	1010010
0011	0011110	1011	1011001
0100	0100110	1100	1100001
0101	0101101	1101	1101010
0110	0110011	1110	1110100
0111	0111000	1111	[illegible]

(ii) The encoding circuit which requires a 4-bit shift register, $(n-k) = (7-4) \Rightarrow$ 3 modulo-2 adders & 7 segment commutator



(iii) To detect and correct errors

$$[S] = RH^T$$

$$[H] = \left[P^T \,\vdots\, I_{n-k}\right] = \left[P^T \,\vdots\, I_3\right] = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Given $R_A = [0111110] \Rightarrow [S_1] = R_A H^T$

$$= [0111110] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore [S] = [1 \ 1 \ 0]$

$\therefore [S_1] \neq 0$, In H^T the 2nd row matches with S_1, $\therefore$ the 2nd bit of R_A is in error and corrected code vector is $C_A = 0011110$ which is valid code vector.

Given $R_B = [1011100]$

So $[S_2] = R_B H^T$

$$= [1\,0\,1\,1\,1\,0\,0] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$[S_2] = [1\,0\,1]$

$\therefore [S_2] \neq 0$, 3^{rd} row matches with S_2, $\therefore$ the corrected code vector $C_B = 1001100$

Given $R_C = [1010000]$

So $[S_3] = R_C H^T$

$$= [1010000] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$[S_3] = [0\,1\,0]$

$\therefore [S_3] \neq 0$, 6^{th} row matches with S_3, $\therefore$ The corrected code vector $C_c = 1010010$.

(iv) Syndrome Calculation Circuit.

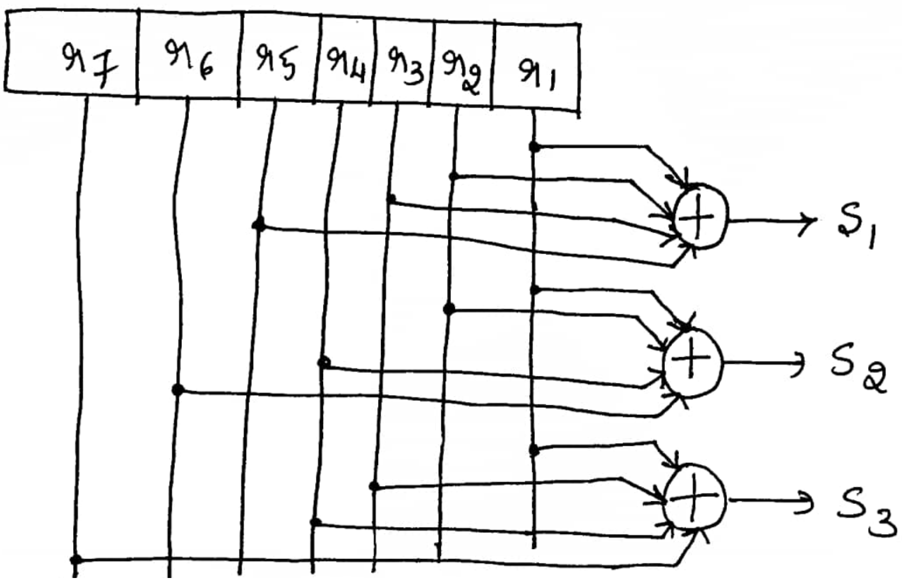
Let the received vector be represented in general form

$$R = [r_1\, r_2\, r_3\, r_4\, r_5\, r_6\, r_7]$$

The received vector R is

$$S = RH^T = r_1\, r_2\, r_3\, r_4\, r_5\, r_6\, r_7 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore S = [S_1, S_2, S_3] = [(r_1 + r_2 + r_3 + r_5), (r_1 + r_2 + r_4 + r_6), (r_1 + r_3 + r_4 + r_7)]$$



Hamming Weight, hamming distance & minimum distance of Linear block codes.

Hamming Weight : It is a measure of the total number of non-zero elements (1's) in a given code vector.

eg:- If $C_2 = 0110110$ → then $H_w = 4$. (i.e no of 1's is 4)

Hamming Distance : Hamming distance between any two code-vectors C_1 and C_2 is defined as the number of components in which they differ.

$$
\begin{array}{lllllllll}
 & 1^{st} & & 3^{rd} & 4^{th} & & 6^{th} & \\
C_1 = & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
C_2 = & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
 & \uparrow & & \uparrow & \uparrow & & \uparrow & \\
 & 1^{st} & & 2^{nd} & 3^{rd} & & 4^{th} &
\end{array}
$$

eg:-

The hamming distance between C_1 and C_2 is said to be 4.

Minimum Distance [dmin] : Minimum distance is defined as the Smallest hamming distance between any two code-vectors in a code.

(OR)

dmin can also be measured by Knowing all the hamming weights of a given code and $d_{min} = H_{w\,min}$.

eg:- If $C_3 = 0110100$ → $h_w = 3$

$C_4 = 1100110$ → $h_w = 4$

$C_5 = 0001100$ → $h_w = 2$

$\therefore \boxed{d_{min} = 2}$

RCET

2-Assessment
Question Bank
Information Theory & Coding (A, B & C sec)

<1> Explain Representation of a Channel ?

<2> Explain Properties of JPM.

<3> Define Mutual information ? Rate of Discrete channel, Channel capacity, Channel efficiency.

<4> Prove that $H(A/B) = P.H(A)$ for a binary erasure channel.

<5> Show that $H(A,B) = H(A/B) + H(B)$.

<6> Define the properties of Mutual Information.

<7> A Binary symmetric channel is illustrated below which has the following source symbol Probabilities

$P(x_1) = 2/3$ $\quad P(x_2) = 1/3$ $\quad P(y/x) = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$

(i) Determine $H(X), H(Y), H(X,Y), H(X/Y), H(Y/X)$ & $I(X,Y)$

(ii) Also determine channel capacity.

(iii) find Channel efficiency & Redundancy.

<8> The input of the channel consists of 5 letters $X_1 = \{x_1, x_2, x_3, x_4, x_5\}$ & o/p consists of four letters $Y = \{y_1, y_2, y_3, y_4\}$. The JPM of the channel is given below

$$\begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} & \begin{bmatrix} 0.25 & 0 & 0 & 0 \\ 0.1 & 0.3 & 0 & 0 \\ 0 & 0.05 & 0.1 & 0 \\ 0 & 0 & 0.05 & 0.1 \\ 0 & 0 & 0.05 & 0 \end{bmatrix} \end{array}$$

<i> compute $H(X), H(Y), H(X,Y), H(Y/X)$ & $H(X/Y)$

<ii> Rate of data transmission & Mutual Information

<iii> Channel Capacity, Efficiency & Redundancy.

<9> Two noisy channels are cascaded whose channel matrices are given by

$$P(y/x) = \begin{bmatrix} 1/6 & 1/6 & 2/3 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}, \quad P[z/y] = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 2/3 & 0 \\ 0 & 1/3 & 2/3 \end{bmatrix}$$

with $P(x_1) = P(x_2) = 1/2$ find $I(x,y)$ & $I(x,z)$

<10> For the JPM given, compute $H(x)$, $H(y)$, $H(x,y)$, $H(x/y)$, $H(y/x)$ & $I(x,y)$. verify the relationship among these entropies

$$P(x,y) = \begin{bmatrix} 0.05 & 0.00 & 0.20 & 0.05 \\ 0.00 & 0.10 & 0.10 & 0.00 \\ 0.00 & 0.00 & 0.20 & 0.10 \\ 0.05 & 0.05 & 0.00 & 0.10 \end{bmatrix}$$

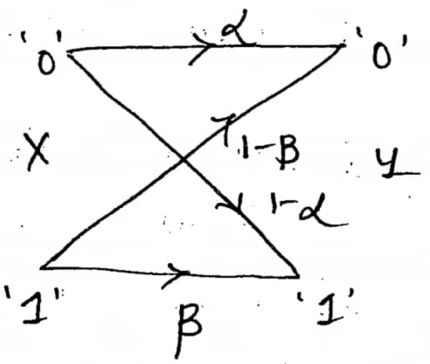
<11> A non symmetric binary channel shown in figure has a symbol rate of 1000 symbols/sec.

<i> find $H(x)$, $H(y)$, $H(x,y)$, $H(x/y)$, $H(y/x)$, $I(x,y)$

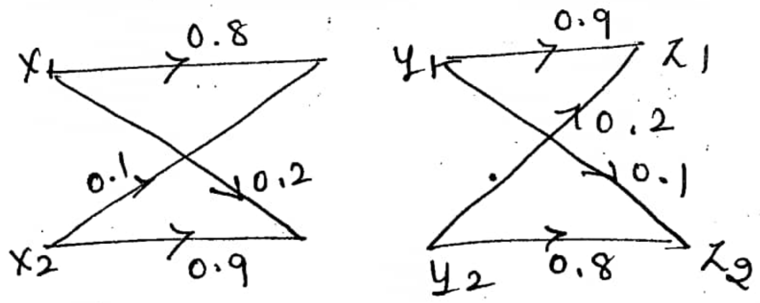
Take $P(x=0) = 1/4$, $P(x=1) = 3/4$, $\alpha = 0.75$, $\beta = 0.9$

(ii) find the capacity of channel for case(i)

(iii) find the capacity of the binary symmetric channel when $\alpha = \beta = 0.75$



<12> Two channels are cascaded as shown in the figure, find the following, $H(X)$, $H(Y)$, $H(Z)$, $H(X,Z)$, $H(Z/X)$ & $H(X/Z)$ for given probability of $P(X_1) = P(X_2) = 0.5$ Show that $I(X,Z) < I(X,Y)$



<13> With a neat block diagram, explain the digital communication system indicating the various types of communication channels.

<14> Derive an expⁿ for channel capacity of a Binary symmetric channel.

<15> Explain <1> Methods of controlling errors
<2> Types of Errors
<3> Types of codes

<16> Explain the Matrix Description of Linear Block code.

<17> For a systematic (7, 4) linear Block code, the parity matrix P is given by
$$P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

<i> find all possible code vectors
<ii> verify (7,4) is linear block code.
<iii> Draw Encoding circuit.

<18> If C is a Valid code vector, then prove that $CH^T = 0$, where H^T is the transpose of the parity Check Matrix H

<19> for a given (7,4) Linear block code, the parity matrix P is given by $[P] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

(i) find all possible valid code-vectors

<ii> Draw encoding circuit

<iii> Verify given (n,k) is Linear block code

<iv> A single error has occured in each of these received vectors. Detect & correct those errors

$R_1 = [0111110]$ $R_2 = [1011100]$ $R_3 = [1010000]$

<v> Draw the syndrome calculation circuit.

~~<20> Define~~

Single Error Correcting Hamming Codes (7)

we have the parity check matrix as $H = [P^T \mid I_{n-k}]$

$$\therefore H^T = \begin{bmatrix} P \\ \hline I_{n-k} \end{bmatrix} \longrightarrow (1)$$

- From (1) we have $(n-k)$ number of columns.
- In each row of H^T, it has $(n-k)$ number of entries each of which could be '0' or a '1'. Thus we can have 2^{n-k} number of distinct rows. But a row of 0's cannot be used as this represents the syndrome of no error. Thus we are left with $[2^{n-k}-1]$ number of distint rows.

$\therefore$ The condition for all the rows of H^T to be distinct & it is written as

$$2^{n-k} - 1 \geq n \longrightarrow (2)$$

$$2^{n-k} \geq n+1$$

$$n-k \geq \log_2(n+1)$$

or

$$k \leq n - \log_2(n+1) \longrightarrow (3)$$

Thus, the single error correcting (n,k) hamming code has the following parameters.

Code length : $n \leq 2^{n-k} - 1$
No of message bits : $k \leq n - \log_2(n+1)$
No of parity check bits : $(n-k)$
Error correcting capability : $t = \frac{d_{min}-1}{2}$

Hamming Bound: "If an (n,k) Linear block code is to be capable of correcting upto 't' errors, then the total no of syndromes shall not be less than the total no of all possible error patterns.

$$\text{i.e } 2^{n-k} \geq \sum_{i=0}^{t} {}^nC_i \longrightarrow (1)$$

equn (1) is referred to as the "Hamming Bound".
A Binary code for which the hamming bound turns out to be an equality is called a "perfect code"

Problems.

① Consider the code vectors $C_1 = 10010$, $C_2 = 01101$, $C_3 = 11001$.
find (i) $d(C_1, C_2)$ (ii) $d(C_1, C_3)$ (iii) $d(C_2, C_3)$ and prove that
$$d(C_1, C_2) + d(C_2, C_3) \geq d(C_1, C_3)$$

→ (i) $d(C_1, C_2) \Rightarrow C_1 = 10010$
$C_2 = 01101$

$$d(C_1, C_2) = 5$$

(ii) $d(C_1, C_3) = 3$ (iii) $d(C_2, C_3) = 2$

$$\therefore d(C_1, C_2) + d(C_2, C_3) \geq d(C_1, C_3)$$
$$5 + 2 \geq 3 = \boxed{7 \geq 3.}$$

② For a (6,3) linear Block code, the 3 parity check bits are formed from the following equations $C_4 = d_1 \oplus d_3$, $C_5 = d_1 \oplus d_2 \oplus d_3$, $C_6 = d_2 \oplus d_3$.
write 'G' suppose that the received word is 111010 decode the transmitted code word by finding the location of the error and transmitted data bits.

→ $$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow G = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

given $R = 111010$ $$H^T = \left[\begin{array}{c} P \\ \hline I_{n-k} \end{array}\right]$$

$\therefore S = RH^T$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$S = 000$

No error since the syndrome is zero.
The transmitted and received bits are same.

(3) Consider a (6,3) Linear Block code whose generator matrix is given below (i) find all the code words for this code. (ii) Minimum weight of the code. (iii) Hamming distance.

Given $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

→ $G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$

2^K code words can be determined

$\therefore 2^3 = 8$ Code words (D_1 to D_8)

$\therefore C = [d_1 \; d_2 \; d_3 \; d_1+d_3 \; d_2+d_3 \; d_1+d_2]$

Message	codewords	minimum weight	Hamming distance
000	000000	0	3
001	001110	3	4
010	010011	3	3
011	011101	4	3
100	100101	3	3
101	101011	4	4
110	110110	4	3
111	111000	3	3

(4) The parity check bits for (8,4) linear Block codes are generated by

$c_5 = d_2 \oplus d_3 \oplus d_4$

$c_6 = d_1 \oplus d_2 \oplus d_4$

$c_7 = d_1 \oplus d_2 \oplus d_3$

$c_8 = d_1 \oplus d_3 \oplus d_4$

where d_1, d_2, d_3, d_4 are message b find (i) G (ii) H (iii) Minimum weight of the co

→ $P = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

(i) $G = [I_K \mid P]_{K \times n} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$

(ii) $H = [P^T \vdots I_{n-k}]$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$c = [d_1, d_2, d_3, d_4, d_4+d_2+d_3, d_1+d_2+d_4, d_1+d_2+d_3, d_1+d_3+d_4]$

(iii) $C = DG \longrightarrow 2^4 = 16$ codewords

Message	Codewords	minimum weight
0000	00000000	0
0001	00011101	4
0010	00101011	4
0011	00110110	4
0100	01001110	4
0101	01010010	4
0110	01100101	4
0111	01111000	4
1000	10000111	4
1001	10011010	4
1010	10101100	4
1011	10110001	4
1100	11001001	4
1101	11010100	4
1110	11100010	4
1111	11111111	8

$d_{min} = 4$

⑤ Design an (n, K) hamming code with a $d_{min} = 3$ & message length $= 4$ bits, find all code vector Also show that it can correct single errors.

→ Given: $K = 4$, $d_{min} = 3$

By trial & error, the minimum value of codelength n which satisfies the inequality is $n \leq 2^{n-k} - 1$

$\therefore$ if $n = 5$, $5 \nleq 2^{5-4} - 1$ ×

$n = 6$, $6 \nleq 3$ ×

$n = 7$ $7 \leq 7$ ✓

$\therefore n = 7$ (minimum value)

$\therefore (7,4)$ LBC

To choose H^T,

we know $H^T = \left[\frac{P}{I_{n-k}}\right] = \left[\frac{P}{I_3}\right]$

$$H^T = \begin{bmatrix} & [P] & \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

'P' is choosen such as way that ① H^T should not contain a row of 0's. ② No two rows of H^T must be same i.e all 7 rows of H^T must be distinct.

$\therefore 2^{n-k} = 2^{7-4} = 2^3 = 8$ combinations are there

they are
000 → it cannot be used.
001 ✓
010 ✓
011
100 ✓
101
110
111

(001, 010, 100) cannot be used as they are allready present as the row of unit matrix.

$\therefore$ only four combinations are left. 011, 101, 110, 111 is choosen & they can be arranged in any order of [P].

$$\therefore \; [P] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\therefore \; H^T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies H = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \; [G] = [I_k \,\vdots\, P]$$

$$= \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\therefore \; [C] = [D]\,[G]$$

$$[C] = \{d_1, d_2, d_3, d_4, d_2+d_3+d_4, d_1+d_3+d_4, d_1+d_2+d_4]$$

Message vector [D]	code-vector [C]	Message vector [D]	code-vector [C]
0000	0000000	1000	1000011
0001	0001111	1001	1001100
0010	0010110	1010	1010101
0011	0011001	1011	1011010
0100	0100101	1100	1100110
0101	0101010	1101	1101001
0110	0110011	1110	1110000
0111	0111100	1111	1111111

To show that it correct single error $d_{min} = 3$

$$\therefore t = \frac{d_{min} - 1}{2} = \frac{3-1}{2} = 1$$

So let the received vector with single error be $R = [1111001]$

$$\therefore S = RH^T = [1111001]\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$S = [1\ 1\ 0]$ which is present in 3^{rd} row

So we need to correct 3^{rd} bit of R $\therefore R = [1101001]$

(2) Consider (7,4) hamming code, with $d_{min} = 3$, show that it is a perfect code.

→ To find it is a perfect code

we have $2^{n-k} \geq \sum_{i=0}^{t} {}_nC_i$

where $t = \frac{d_{min} - 1}{2} = \frac{3-1}{2} = 1$

$$\therefore 2^{(7-4)} \geq \sum_{i=0}^{1} {}_7C_0$$

$$2^3 \geq {}_7C_0 + {}_7C_1$$

$$2^3 \geq 1 + 7$$

$$8 \geq 8$$

$$\therefore \boxed{8 = 8}$$

$\therefore$ (7,4) is a perfect code.

Table look up Decoding (Syndrome Decoding) Using the standard Array.

Let $C_i = C_1, C_2, C_3 \ldots C_{2^K}$ be the 2^K distinct code vectors of a (n,K) Linear Block code.

let 'E' be any error pattern.

$$\therefore E_i = E + C_i \longrightarrow (1)$$

Where the set of vectors E_i is called the "CO-SET" of the code.

X'lying equation (1) by H^T on both the sides

$$E_i H^T = EH^T + C_i H^T \qquad \text{where } CH^T = 0$$

$$\therefore \boxed{E_i H^T = EH^T}$$

Each element in the standard array is distinct.

Steps to Construct the standard Array.

(i) 2^K valid code-vectors of the codes are placed in a row with all zero code-vector as first element

(ii) From the remaining an Error E_2 is chosen and is placed below all zero-code vector. The second row can now be formed by placing $E_2 + C_i$ where $i = 2, 3 \ldots 2^K$ under C_i

(iii) Now an unused E_3 is taken & the 3rd row is completed as given in step (ii)

(iv) The process is continued till all the combination of codes are used. The resulting array for a (n,K) linear block code is shown below.

$$\begin{array}{ccccc}
C_1 = \text{all } 0\text{'s} & C_2 & C_3 & \cdots & C_{2^K} \\
E_2 & C_2 + E_2 & C_3 + E_2 & \cdots & C_{2^K} + E_2 \\
E_3 & C_2 + E_3 & C_3 + E_3 & \cdots & C_{2^K} + E_3 \\
\vdots & \vdots & \vdots & & \vdots \\
E_{2^{n-K}} & C_2 + E_{2^{n-K}} & C_3 + E_{2^{n-K}} & & C_{2^K} + E_{2^{n-K}}
\end{array}$$

properties of Standard Array.

1. the standard array consists of 2^n codevectors which are partioned into 2^K disjoint cosets as shown
2. the first entry of each row or coset is known as the coset leader & it represents a correctable error pattern for a particular code.
3. Each element in the array is distinct. Therefore different columns in array are disjoint.
4. Generally error patterns of smallest weight are chosen as coset leaders and decoding which results from, such an array is known as min distance decoding or maximum likelihood decoding. Such an array can be used to correct all such errors induced by the channel only if these errors correspond /match with the coset leaders.
5. All 2^K error vectors in a single row will have the same syndrome.

Problems

1. Construct the standard array for a (6,3) LBC given $P=\begin{bmatrix}1&0&1\\0&1&1\\1&1&1\end{bmatrix}$ Detect and correct errors for the received vectors $R_1 = 100100$ and $R_2 = 000011$. Draw the error correction for the same.

→ $C = [D][G]$, we have 2^K code vectors $\Rightarrow 2^3 = 8$

$$[G] = [I_K \,\vdots\, P] = \begin{bmatrix}1&0&0&1&0&1\\0&1&0&0&1&1\\0&0&1&1&1&1\end{bmatrix} \qquad H^T = \begin{bmatrix}1&0&1\\0&1&1\\1&1&1\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\therefore \; c = [d_1 \; d_2 \; d_3] \begin{bmatrix}1&0&0&1&0&1\\0&1&0&0&1&1\\0&0&1&1&1&1\end{bmatrix}$$

$C_1 = d_1 \quad C_2 = d_2 \quad C_3 = d_3 \quad C_4 = d_1 + d_3 \quad C_5 = d_2 + d_3 \quad C_6 = d_1 + d_2 + d_3$

D	[C]
000	000000
001	001111
010	010011
011	011100
100	100101
101	101010
110	110110
111	111001

Standard Array

Syndrome	Coset leader	C_1	C_2	C_3	C_4	C_5	C_6	C_7
000	000000	001111	010011	011100	100101	101010	110110	111001
101	100000	101111	110011	111100	000101	001010	010110	011001
011	010000	011111	000011	001100	110101	111010	100110	101001
111	001000	000111	011011	010100	101101	100010	111110	110001
100	000100	001011	010111	011000	100001	101110	110010	111101
010	000010	001101	010001	011110	100111	101000	110100	111011
001	000001	001110	010010	011101	100100	101011	110111	111000
110	110000	111111	100011	101100	010101	011010	000110	001001

(b) $S = RH^T$

$$= [100100] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

match the S with H^T, it is present in 6^{th} row.

$\therefore$ 6^{th} Bit is in error

$\therefore$ corrected code vector $= [100101]$

$S = [0\ 0\ 1]$

Also match the R_1 with the elements of table. The Received Vector R_1 is present in the 4^{th} column of 7^{th} row. hence the corrected code vector corresponds to the entry in top most row of the column 4 i.e 100101

(c) Error correction circuit.

$S_1 = r_1 + r_3 + r_4$

$S_2 = r_2 + r_3 + r_5$

$S_3 = r_1 + r_2 + r_3 + r_6$

$\therefore$ $e_1 = S_1 \bar{S_1} S_2$

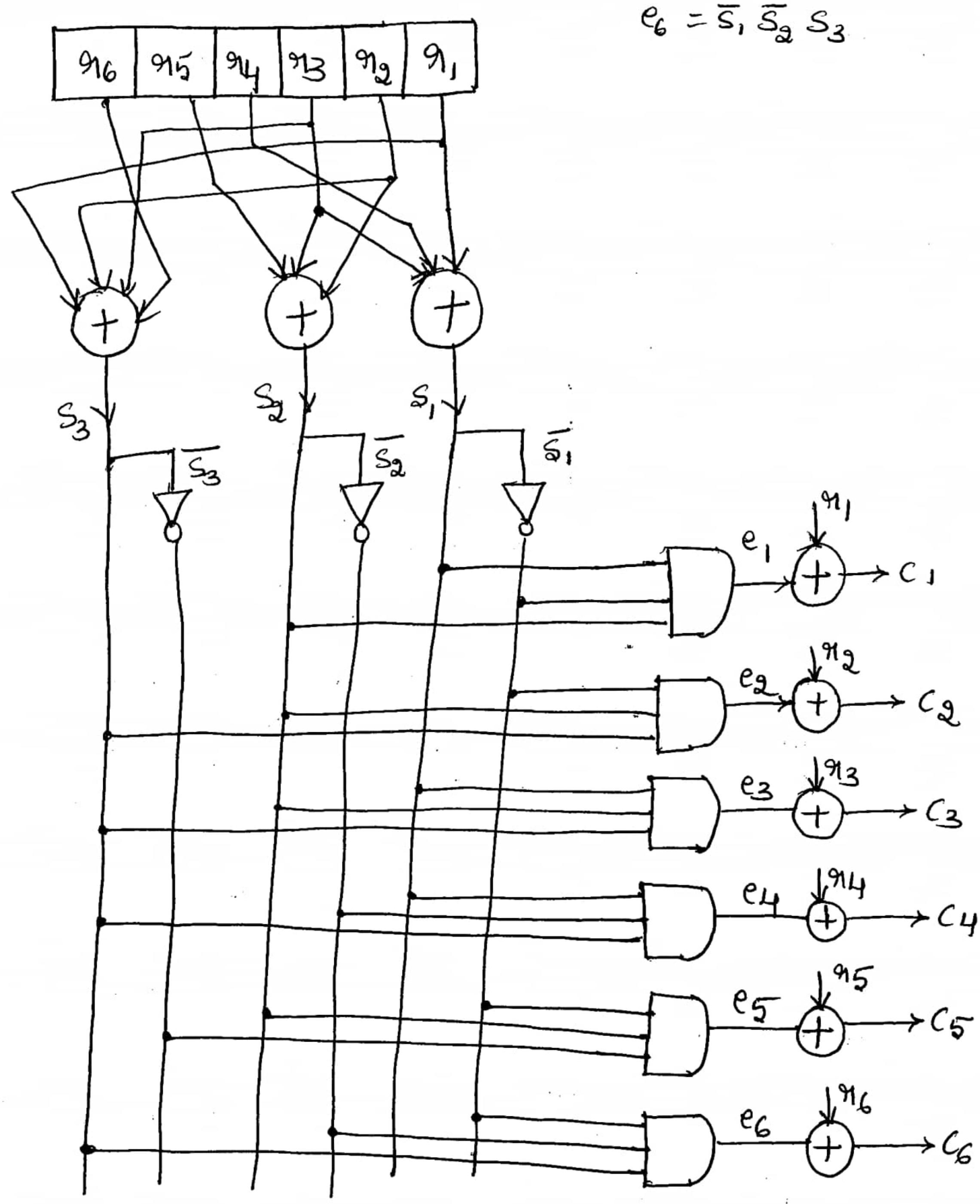
$e_2 = \bar{S_1} S_2 S_3$

$e_3 = S_1 S_2 S_3$

$e_4 = S_1 \bar{S_2} S_3$

$e_5 = \bar{S_1} S_2 \bar{S_3}$

$e_6 = \bar{S_1} \bar{S_2} S_3$



(2) For a (5,2) LBC, the generator matrix is of the form $[I_k \mid P]$ where $[P]$ is given by $[P] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

find (a) generator Matrix (b) Parity Check matrix (c) all possible valid code-vectors (d) d_{min} and error correcting capability of the code (e) Construct standard array by listing all the 32 five-tuple patterns.

(a) Generator matrix is given by $[G] = [I_k \mid P] = [I_2 \mid P]$

$$[G] = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

(b) The parity check matrix will be of the form $[P^T \mid I_{n-k}]$ given by

$$[H] = [P^T \mid I_{5-2}] = [P^T \mid I_3]$$

$$[H] = \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

(c) Code-Vectors are generated Using

$$[C] = [D][G] = [d_1 \ d_2] \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$= [d_1, d_2, (d_1 + d_2), (d_1 + d_2), d_1]$$

Message Vector	Code-Vector	Hw
00	00000	0
01	01110	3
10	10111	4
11	11001	3

(d) the lowest hamming weight of a non-zero code vector is 3

$$t = \frac{d_{min} - 1}{2} = \frac{3-1}{2} = 1$$

$\therefore$ (5,2) code is a SEC code

(e) Standard array

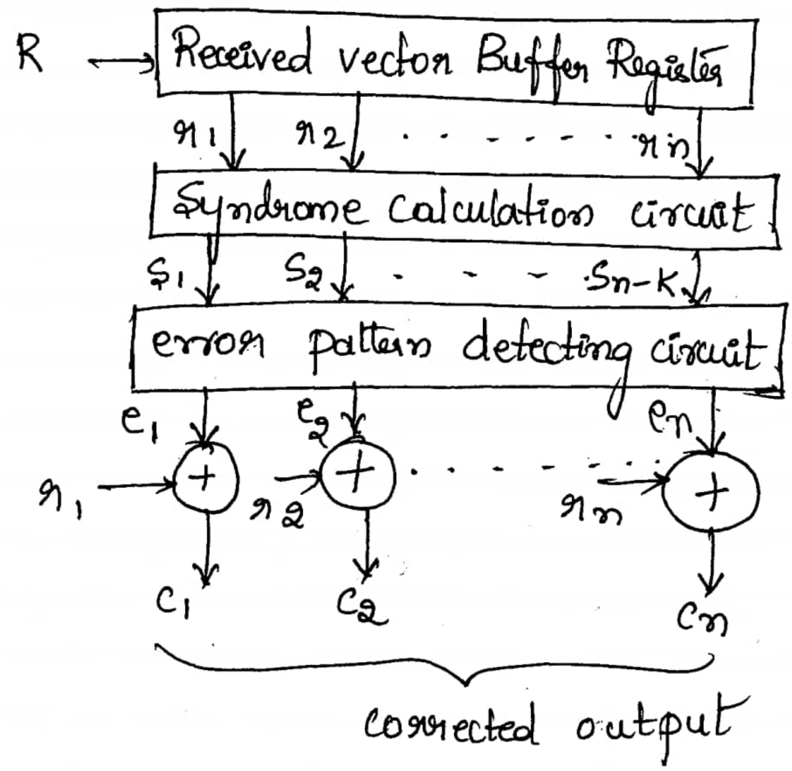
Syndrome	Coset leader			
000	00000	01110	10111	11001
111	10000	11110	00111	01001
110	01000	00110	11111	10001
100	00100	01010	10011	11101
010	00010	01100	10101	11011
001	00001	01111	10110	11000
011	00011	01101	10100	11010
101	00101	01011	10010	11100

General Decoding circuit for (n, K) LBC.

The implementation of the decoding circuit (error correcting circuit), the standard array can be regarded as the truth table of n-switching functions.

The Truth Table consists of 2 columns

1. the Syndrome column
2. Coset leader column.



$e_1 = f_1(S_1, S_2 \ldots, S_{n-1}$
$e_2 = f_2(S_1, S_2 \ldots S_{n-}$
$\vdots$
$e_n = f_n(S_1, S_2 \ldots, S_{n-}$

where $S_1, S_2 \ldots, S_{n-k}$ are syndrome digits
$e_1, e_2 \ldots, e_n$ are error digits of co-set leader.

for Problem (1) in page (10).

→ Draw the complete error correcting circuit

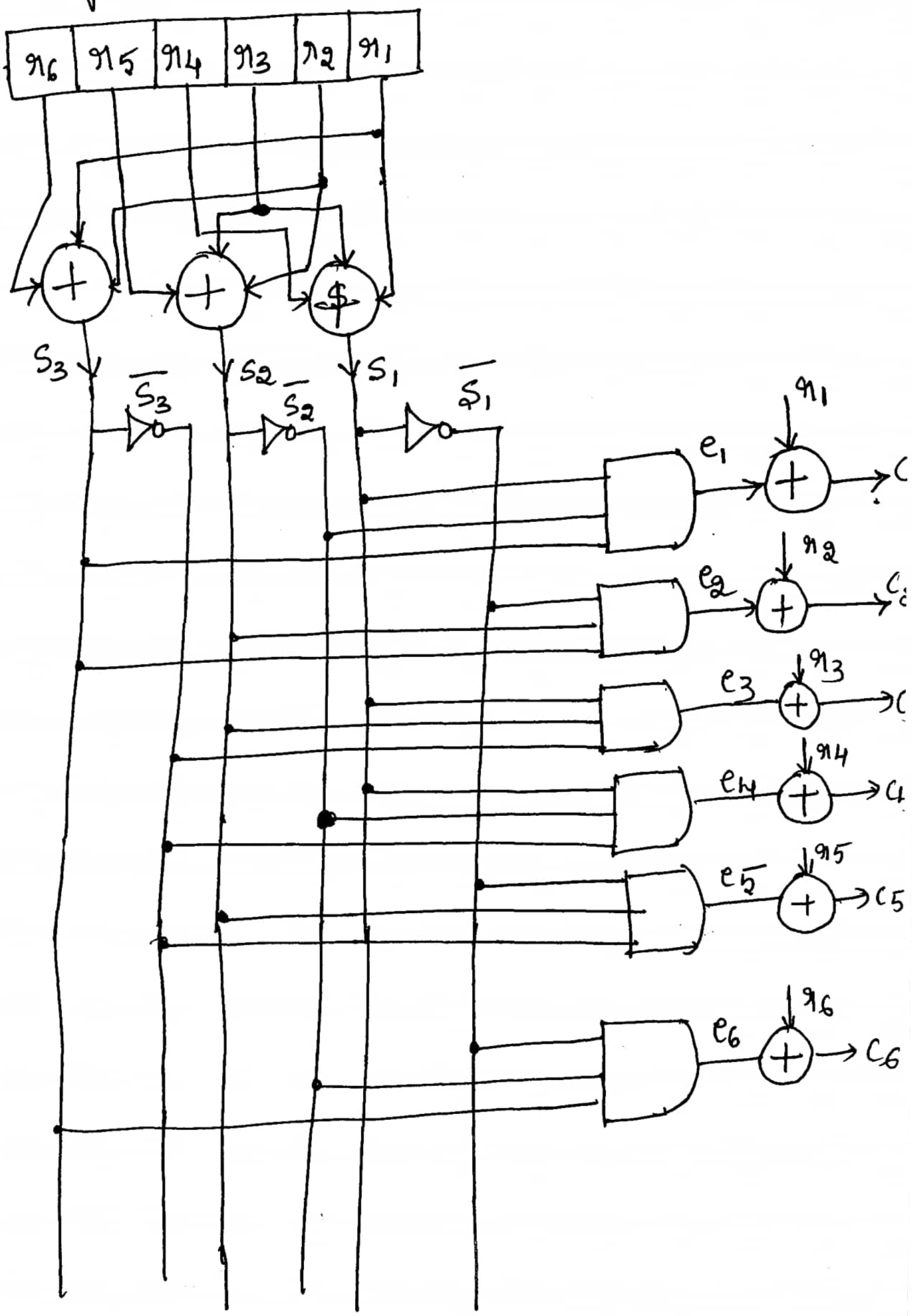
$$S = [S_1 S_2 S_3] = [(r_1 + r_3 + r_4), (r_2 + r_3 + r_5), (r_1 + r_2 + r_6)]$$

The error bits of the coset leader can be expressed in terms of syndrome bits.

$\therefore e_1 = S_1 \bar{S_2} S_3$ $\quad e_2 = \bar{S_1} S_2 S_3$ $\quad e_3 = S_1 S_2 \bar{S_3}$ $\quad e_4 = S_1 \bar{S_2} \bar{S_3}$

$e_5 = \bar{S_1} S_2 \bar{S_3}$ $\quad e_6 = \bar{S_1} \bar{S_2} S_3$

Error Correcting circuit can be written as:-



BINARY CYCLIC CODES.

Noor Afza CIT Gubbi

Binary Cyclic codes are subclass of linear Block codes. (16)

Its advantages are :-

1. the design of encoder and syndrome calculator hardware are more simple involving a shift register with feedback connections.
2. Cyclic codes have a very good mathematical structure, that makes it possible to design codes with useful error-correcting properties.

Algebraic structure of cyclic codes.

Definition of cyclic code.

A (n,K) linear block code C is said to be a cyclic code if every cyclic shifts of the code is also a code-vector of C.

ex:- let $C_1 = 0111001$ be a code vector of C.

if the last 1 of C_1 has moved into the first positio we get $C_2 = 1011100$, if C_2 is also a code vector of C then it is called as "cyclic code".

General Representation of cyclic codes and their Analysis.

Let $V = (V_0 \; V_1 \; V_2 \; \dots \; V_{n-1}) \longrightarrow (1)$

If V belongs to a cyclic code, then

$$\left.\begin{aligned} V^{(1)} &= (V_{n-1} \; V_0 \; V_1 \; V_2 \dots V_{n-2}) \\ V^{(2)} &= (V_{n-2} \; V_{n-1} \; V_0 \; V_1 \; V_2 \dots V_{n-3}) \\ &\vdots \\ V^{(i)} &= (V_{n-i} \; V_{n-i+1} \dots V_0 \; V_1 \; V_2 \dots V_{n-i-1}) \end{aligned}\right\} \longrightarrow (2)$$

where $V_0 \dots V_{n-1}$ represents binary digits 0's / 1's

Cyclic codes can be analysed mathematically by representing each code vector as a polynomial function in some variable 'x'.

Then ① becomes

$$V(x) = V_0 x^0 + V_1 x^1 + V_2 x^2 + \dots + V_{n-1} x^{n-1}$$

III^{ly}

$$V^{(1)}(x) = V_{n-1} x^0 + V_0 x^1 + V_1 x^2 + \dots + V_{n-2} x^{n-1}$$

$$V^{(2)}(x) = V_{n-2} x^0 + V_{n-1} x^1 + V_0 x^2 + V_1 x^3 + \dots + V_{n-3} x^{n-1}$$

$$\vdots$$

$$V^{(i)}(x) = V_{n-i} + V_{n-i+1} x + V_{n-i+2} x^2 + \dots + V_{n-i-1} x^{n-1}$$

The coefficients of x are either 0 or 1.

Here the polynomial functions V_0 to V_{n-1} belongs to the Modulo-2 arithematic & multiplication.

Properties of Cyclic Codes.

1. For a (n, K) cyclic codes there exists a generator polynomial of degree $(n-K)$ & it is given by
$$g(x) = g_0 + g_1 x + g_2 x^2 + \dots + g_{n-K} x^{n-K} \longrightarrow ①$$

2. The generator polynomial $g(x)$ of a (n, K) cycle code is a facto of $\boxed{x^n+1}$ i.e $x^n + 1 = g(x) \cdot h(x) \longrightarrow ②$
where $h(x)$ is another polynomial of degree K called "Parity Check polynomial".

3. If $g(x)$ is a polynomial of degree $(n-K)$ & is a factor of x^n+1 then it generates the (n, K) cyclic code.

4. Cyclic code in a non-systematic format can be generated using the relation.
$$V(x) = D(x)\, g(x) \longrightarrow ③$$
where $D(x)$ = message-vector polynomial of degree K.
$$\therefore D(x) = d_0 + d_1 x + d_2 x^2 + \dots + d_{K-1} x^{K-1} \longrightarrow ④$$

5. Cyclic Codes in systematic format can be obtained by performing the division $\left(\frac{x^{n-k}\cdot d(x)}{g(x)}\right)$ and appending the remainder of this division before the data.

i.e $\quad V(x) = R(x) \,\vdots\, d(x)$

- In a linear (n, K) code there are 2^K message & code Vectors.

※ Modulo-2 Algebra

Addition : The Quantity $x+x$ can be written as

$$x+x = x(1+1) = x\cdot 0 = 0$$

coz $1+1=0$ in modulo-2 addition.

$$\text{III}^{ly} \quad x^2+x^2 = x^2(1+1) = x^2\cdot 0 = 0$$

$$\& \quad x^3+x^3 = x^3(1+1) = x^3\cdot 0 = 0$$

Subtraction is same as addition in modulo-2 algebra.

Multiplication : The Quantity $x\cdot x = x^2$, $x^2\cdot x = x^3$ & so on.

Some of the Examples on these Modulo-2 algebra.

① Find the product of polynomials $f_1(x) = (x+1)$ and $f_2(x) = x^3+x+1$ using modulo-2 algebra.

→

$$f_1(x)\cdot f_2(x) = (x+1)(x^3+x+1)$$
$$= x^4+x^2+x+x^3+x+1$$
$$= x^4+x^3+x^2+x+x+1$$
$$= x^4+x^3+x^2+x(1+1)+1$$

$$\boxed{f_1(x)\cdot f_2(x) = x^4+x^3+x^2+1}$$

2. Multiply $f_1(x) = 1+x+x^3$ and $f_2(x) = (1+x+x^2+x^4)$ using modulo-2 algeb

→ $f_1(x) \cdot f_2(x) = (1+x+x^3)(1+x+x^2+x^4)$

$$= 1+x+x^2+x^4+x+x^2+x^3+x^5+x^3+x^4+x^5+x^7$$

$$= 1+x^7 + x^5\underbrace{(1+1)}_{0} + x^4\underbrace{(1+1)}_{0} + x^3\underbrace{(1+1)}_{0} + x\underbrace{(1+1)}_{0}$$

$$\boxed{f_1(x) \cdot f_2(x) = 1+x^7}$$

3. Divide $f_2(x) = x^6+x^5+x^2$ by $f_1(x) = x^3+x+1$ using modulo-2 algeb

→

$$x^3+x+1 \,\big)\, x^6+x^5+x^2 \,\big(\, x^3+x^2+x \leftarrow Q(x)$$

$$\begin{array}{r}
x^6+x^4+x^3 \\
\hline
x^5+x^4+x^3+x^2 \\
-\ x^5+\quad\ +x^3+x^2 \\
\hline
x^4 \\
x^4+x^2+x \\
\hline
x^2+x \leftarrow R(x)
\end{array}$$

$Q(x)$ is Quotient polynomial, $R(x)$ is Remainder Polynomial.

4. If $f(x) = x^4+x+1$, then show that $[f(x)]^2 = f(x^2)$ in modulo-2 algebra.

→ Consider $[f(x)]^2 = [x^4+x+1]^2$

$$= (x^4+x+1)(x^4+x+1)$$

$$= x^8+x^5+x^4+x^5+x^2+x+x^4+x+1$$

$$= x^8 + x^5\underbrace{(1+1)}_{0} + x^4\underbrace{(1+1)}_{0} + x^2 + x\underbrace{(1+1)}_{0} + 1$$

$$\boxed{[f(x)]^2 = x^8+x^2+1}$$

$$f(x^2) = (x^4)^2 + (x)^2 + 1$$

$$f(x^2) = x^8+x^2+1$$

∴ we can conclude that $\boxed{[f(x)]^2 = f(x^2)}$

Problems.

(1) For the (7,4) single error correcting cyclic code, $D(x) = d_0 + d_1 x + d_2 x^2 + d_3 x^3$ & $x^n + 1 = x^7 + 1 = (1+x+x^3)(1+x+x^2+x^4)$. Using the generator polynomial $g(x) = 1 + x + x^3$, find all the 16 code vectors of the cyclic code both in non-systematic and systematic form.

→ (1) Non Systematic Cyclic Code.

It is found by using property (4)

$$V(x) = D(x)\, g(x)$$

Given $g(x) = 1 + x + x^3$

we have 2^k message vectors and 2^k code vectors $= 2^4 = 16$

∴ the message polynomial is $D(x) = d_0 + d_1 x + d_2 x^2 + d_3 x^3$

$\boxed{D = 1001}$

for ex if $D(x) = (1) + (0)(x) + 0(x^2) + (1)x^3$

$\therefore D(x) = 1 + x^3$

$\therefore V(x) = D(x) \cdot g(x)$

$= (1+x^3)(1+x+x^3) = 1 + x + x^3 + x^3 + x^4 + x^6$

$= 1 + x + x^4 + x^6$

$\therefore V(x) = 1 + x + x^4 + x^6$

$[V] = [1100101]$

|||ly for $\boxed{D = (1101)}$

$D(x) = (1) + (1)(x) + 0(x^2) + (1)x^3$

$D(x) = 1 + x + x^3$

$V(x) = D(x) \cdot g(x)$

$= (1+x+x^3)(1+x+x^3)$

$= (1+x+x^3)^2$

$V(x) = 1 + x^2 + x^6 \quad \therefore [V] = [1010001]$

∴ The remaining code-vectors can be similarly found

Message (D)	Code-vectors
0000	0000000
0001	0001101
0010	0011010
0011	0010111
0100	0110100
0101	0111001
0110	0101110
0111	0100011
1000	1101000
1001	1100101
1010	1110010
1011	1111111
1100	1011100
1101	1010001
1110	1000110
1111	1001011

② Systematic Cyclic Code

In systematic Cyclic code first 3 bits are check bits & last 4 bits are message bits.

The check bits are obtained from the remainder polynomial $R(x)$ as given in property ⑤

$$R(x) = \frac{x^{n-K} D(x)}{g(x)}$$

so, for ex let $D = [1001]$

$$\therefore D(x) = d_0 + d_1 x + d_2 x^2 + d_3 x^3 \Rightarrow D(x) = 1 + x^3$$

$$\therefore x^{n-K} D(x) = x^{7-4}(1+x^3)$$

$$\boxed{x^{n-K} D(x) = x^3 + x^6}$$

∴ The division of $x^6 + x^3$ by $g(x) = x^3 + x + 1$ can be performed

$$\therefore\; x^3+x+1 \overline{\big)\, x^6+x^3}\; \big(x^3+x$$
$$\underline{x^6+x^4+x^3}$$
$$x^4$$
$$\underline{x^4+x^2+x}$$
$$x^2+x.$$

$$\therefore\; R(x) = x+x^2$$

$$\therefore\; \boxed{R(x) = 011}$$

The code-vector is given by

$$[v] = \underbrace{011}_{R}\; \underbrace{1001}_{D}.$$

for another code $D = [1011]$

$$\therefore\; D(x) = d_0 + d_1 x + d_2 x^2 + d_3 x^3$$
$$D(x) = 1 + x^2 + x^3$$
$$\therefore\; x^{n-k} D(x) = x^3(1+x^2+x^3) \implies x^3+x^5+x^6$$

The division of $x^6+x^5+x^3$ by $g(x) = x^3+x+1$ is performed

$$x^3+x+1 \overline{\big)\, x^6+x^5+x^3}\; \big(x^3+x^2+x+1$$
$$\underline{x^6+x^4+x^3}$$
$$x^5+x^4$$
$$\underline{x^5+x^3+x^2}$$
$$x^4+x^3+x^2$$
$$\underline{x^4+x^2+x}$$
$$x^3+x$$
$$\underline{x^3+x+1}$$
$$1$$

$$\therefore\; \boxed{R(x) = 100}$$

The code-vector is given by

$$[v] = \underbrace{100}_{R}\; \underbrace{1011}_{D}.$$

$\therefore$ ||ly the systematic code-vector can be found for other messages.

Message	code-vector
0000	0000000
0001	1010001
0010	1110010
0011	0100011
0100	0110100
0101	1100101
0110	1000110
0111	0010111
1000	1101000
1001	0111001
1010	0011010
1011	1001011
1100	1011100
1101	0001101
1110	0101110
1111	1111111

Given : x^7+1

We know generator polynomial $g(x) = 1+x+x^3$ is given

to find $h(x)$, we have

$$x^n \mp 1 = g(x).h(x)$$

$$h(x) = \frac{x^n+1}{g(x)} = \frac{x^7+1}{1+x+x^3}$$

$$\therefore \quad x^3+x+1 \,\big)\, x^7+1 \,\big(\, x^4+x^2+x+1$$

$$\begin{array}{r} x^7+x^5+x^4 \\ \hline 1+x^5+x^4 \\ x^5+x^3+x^2 \\ \hline x^4+x^3+x^2+1 \\ x^4+x^2+x \\ \hline x^3+x+1 \\ x^3+x+1 \\ \hline 0 \end{array}$$

$$\therefore \boxed{h(x) = x^4+x^2+x+1}$$

(2) Find the cyclic codes in both Systematic & non-Systematic format for the following data vectors.

(1) $[D_1] = 1100$ (2) $[D_2] = 1111$

given $g(x) = 1 + x + x^3$ and it is a $(7,4)$ cyclic code.

→ (1) $d(x) = d_0 + d_1 x + d_2 x^2 + d_3 x^3$

given $[D_1] = 1100$, $g(x) = 1 + x + x^3$

$\therefore d(x) = 1 + x$

Non-systematic

$$V(x) = g(x)\,d(x)$$
$$= (1 + x + x^3)(1 + x)$$
$$= 1 + x + x + x^2 + x^3 + x^4$$
$$= 1 + x^2 + x^3 + x^4$$

$\therefore V(x) = [1011100]$

Systematic

$$R(x) = \frac{x^{n-k}\,d(x)}{g(x)} = \frac{x^{7-4}(1+x)}{1+x+x^3} = \frac{x^3(1+x)}{1+x+x^3}$$

$$\begin{array}{r|l}
x^3+x+1 & \cancel{x^4} + x^3 \;(x+1 \\
 & \underline{\cancel{x^4} + x^2 + x} \\
 & x^3 + x^2 + x \\
 & \underline{x^3 + x + 1} \\
 & x^2 + 1
\end{array}$$

$\therefore [R] = 101$

$\therefore [V] = [1011100]$

(2) given $[D_2] = 1111$, $g(x) = 1 + x + x^3$

$\therefore d(x) = 1 + x + x^2 + x^3$

Nonsystematic

$$V(x) = g(x).d(x)$$
$$= (1 + x + x^3)(1 + x + x^2 + x^3)$$
$$V(x) = 1 + x^3 + x^5 + x^6$$

$\therefore V(x) = 1001011$

Systematic

$$R(x) = \frac{x^{n-k} d(x)}{g(x)} = \frac{x^3 (1+x+x^2+x^3)}{1+x+x^3}$$

$$R(x) = \frac{x^6+x^5+x^4+x^3}{x^3+x+1}$$

$$
\begin{array}{r|l}
x^3+x+1 & x^6+x^5+x^4+x^3 \;(x^3+x^2+1 \\
 & x^6+\quad x^4+x^3 \\ \hline
 & x^5 \\
 & x^5+x^3+x^2 \\ \hline
 & x^3+x^2 \\
 & x^3+x+1 \\ \hline
 & 1+x+x^2
\end{array}
$$

$\therefore R(x) = [111]$

$[V] = [1111111]$

Generator and Parity check Matrices of (7,4) cyclic codes.

The Generator Matrix [G] of the Order $K \times n = 4 \times 7$ can be constructed starting with the generator polynomial $g(x)$

Let us consider polynomials as $g(x)$, $xg(x)$, $x^2g(x)$ and $x^3g(x)$, which also represent the codevector polynomials of the same cyclic Code.

Let $g(x) = 1+x+x^3$

$\therefore g(x) = (1) + 1(x) + 0(x^2) + 1(x^3) + (0)x^4 + (0)x^5 + (0)(x$

The code vectors for $g(x)$ will be $g(x) = [1101000]$

$|||^{ly}$ for $xg(x) = x(1+x+x^3) = x+x^2+x^4$

$\therefore xg(x) = (0)+1(x)+1(x^2)+0(x^3)+1(x^4)+0(x^5)+0(x^6)$

$xg(x) = 0110100$

$x^2 g(x) = x^2(1+x+x^3) = x^2+x^3+x^5$

$x^2 g(x) = (0) + 0(x) + 1(x^2) + 1(x^3) + 0(x^4) + 1(x^5) + 0(x^6)$

$x^2 g(x) = 0011010$

$x^3 g(x) = x^3+x^4+x^6$

$x^3 g(x) = 0001101$

∴ The code vectors are now arranged in the Matrix form as

$$[G] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

we know $[G] = [P \,\vdots\, I_K]$

but, the last four elements of the 3rd & 4th row of $[G]$ wont belong to I_K. therefore add 3rd row with 1st row and place it in 3rd row. then add 4th row with 1st & 2nd and place it in 4th row

$$\therefore [G] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

∴ The systematic cyclic code-vectors can be found by using

$$[V] = [D]\,[G]$$

$$= [d_0 d_1 d_2 d_3] \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This method also generates the same code-vectors as those in other systematic cyclic code.

Parity Check Matrix H:

From the problem 1 we got the parity check polynomial $= h(x) = 1+x+x^2+x^4$

$\therefore$ the reciprocal of $h(x)$ is defined as $x^K h(x^{-1})$. this polynomial is also a factor of $1+x^n$.

so for (7,4) we have $x^4 h(x^{-1})$.

$$\rightarrow \quad h(x) = 1+x+x^2+x^4$$

$$h(x^{-1}) = 1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^4}$$

$\therefore x^4 h(x^{-1}) = 1+x^2+x^3+x^4 \rightarrow$ code is $\rightarrow$ 1011100

$x^5 h(x^{-1}) = x+x^3+x^4+x^5 \rightarrow$ code is $\rightarrow$ 0101110

$x^6 h(x^{-1}) = x^2+x^4+x^5+x^6 \rightarrow$ code is $\rightarrow$ 0010111.

$\therefore$ The parity check matrix H, which is a $(n-K) \times n = 3 \times$

$$[H] = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}_{3\times 7}$$

we know $[H] = [I_{n-k} \mid P^T] = [I_3 \mid P^T]$

By Adding 1st and 3rd row and placing the result of first row, we can obtain H matrix as

$$[H] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Encoder Circuit for (n, k) Cyclic code.

To obtain remainder polynomial $R(x)$, we have $R(x) = \frac{x^{n-k} D(x)}{g(x)}$. This devision can be acheived using th dividing circuit consisting of feedback shift register.

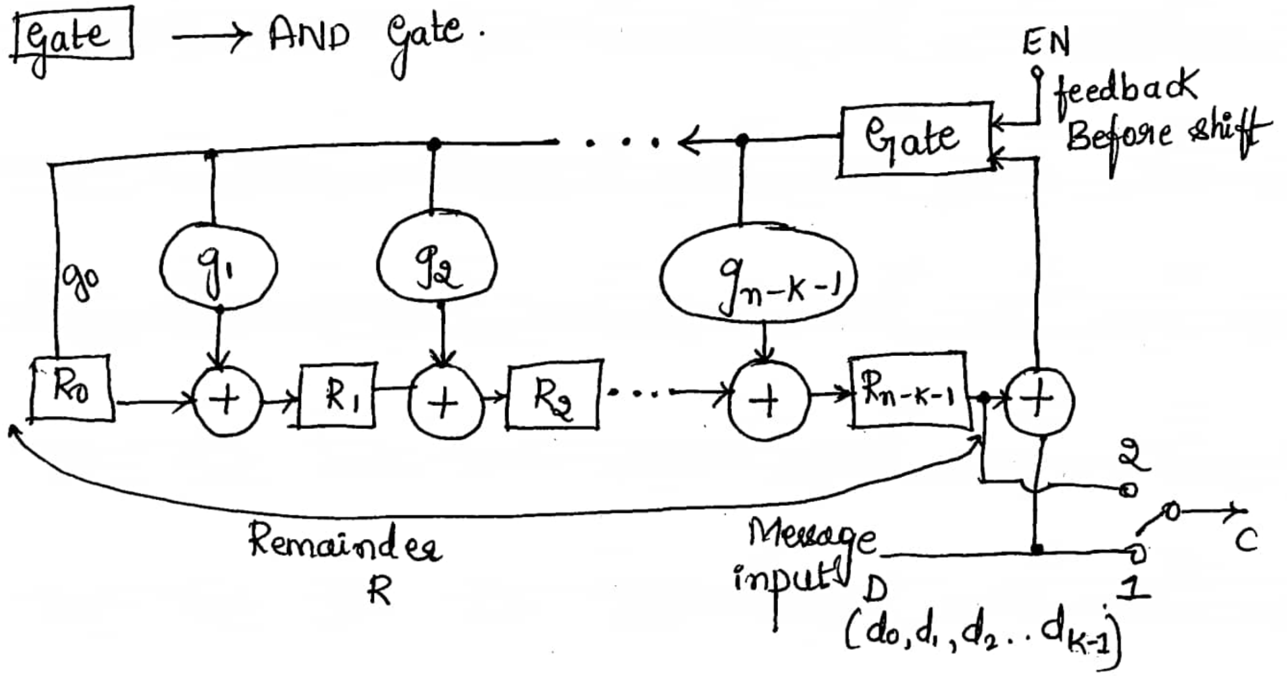
The hardware requirements are :-

① $\boxed{R}$ → An $(n-k)$ flipflops that makeup a shift register

② ⊕ → modulo 2 adder

③ g_i → A max of $(n-k)$ switches.
→ if $g_i = 1$, it is closed path/short ckt.
→ if $g_i = 0$, It is open path/open ckt.

④ $\boxed{\text{Gate}}$ → AND Gate.



Operation: It is assumed that at the occurance of the clock pulse, the register inputs are shifted into the register & appear at the o/p at the end ~~of~~ the clk pulse.

① with the gate turned ON and switch in position 1, the information, the information digits $(d_0, d_1, \ldots, d_{k-1})$ are shifted into the register (with d_{k-1} first) and simultaneously into the Communication Channel. As soon as the K information digits have been shifted into the register, the register contains the parity check bits $(R_0, R_1, \ldots, R_{n-k-1})$.

(2) with the gate turned off and the switch in position 2, the contents of the shift register are shifted into the channel. Thus the code-vector $(R_0, R_1, \ldots, R_{n-k-1}, d_0, d_1, \ldots d_{k-1})$ is generated & sent over the channel.

Problems.

(1) Design an encoder for the (7,4) binary cyclic code generated by $g(x) = 1 + x + x^3$ and verify its operation using the message vectors (1001) and (1011)

→ In general, the generator polynomial is represented as

$$g(x) = g_0 + g_1 x + g_2 x^2 + \ldots + g_{n-k} x^{n-k} \longrightarrow (1)$$

for given problem. (7,4) cyclic code

$$g(x) = 1 + x + x^3 \longrightarrow (2)$$

∴ comparing coefficients in equ^n (1) & (2), we get

$g_0 = 1$, $g_1 = 1$, $g_2 = 0$ & $g_3 = 1$

∴ to write the encoding circuit

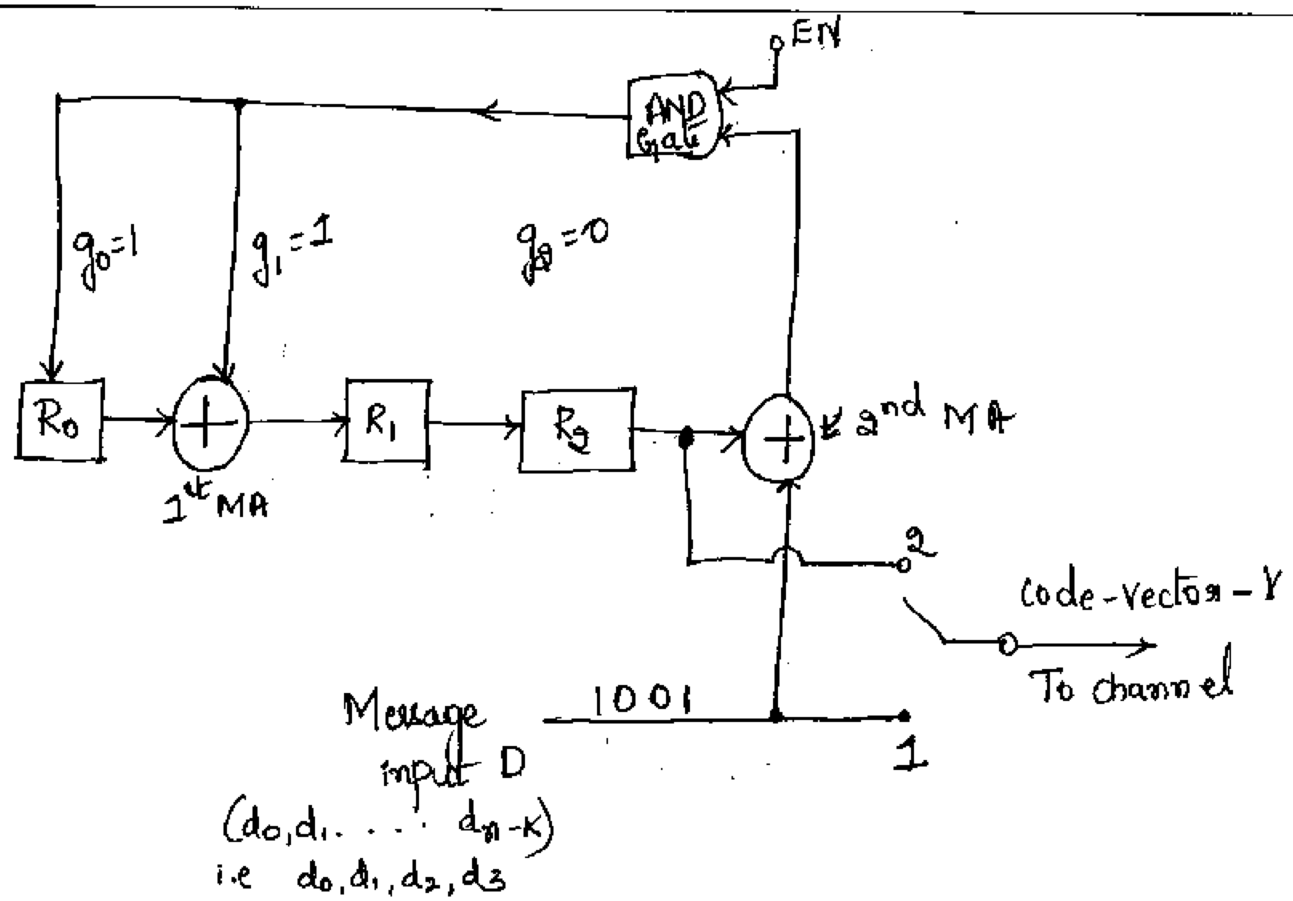
we have $(n-k) \to (7-4) = 3$ flip flop i.e R_0, R_1 and R_2

2 modulo-2 adders

i.e the last generator needed is

$g_{n-k-1} = g_{7-4-1} = g_2$ ∴ 2 Mod-2

→ Encoder for (7,4) cyclic code.



(i) For the message $D = [1001]$, the shift register contains

Number of shifts	Input D	Shift Register Contents R_0	R_1	R_2	Remainder bits $\rightarrow R$
Initialization → Switch S is in position-1 & gate is turned ON.		0	0	0	-
1	1	1	1	0	-
2	0	0	1	1	-
3	0	1	1	1	-
4	1	0	1	1	-
Switch S moves to position 2 & gate is turned off					
5	X	0	0	1	1
6	X	0	0	0	1
7	X	0	0	0	0

∴ The code vector is [0111001]

(ii) For message $D = [1011]$

Number of shifts	Input D	Shift Register Contents R_0	R_1	R_2	Remainder bits $\rightarrow R$
Initialization $\rightarrow$ Switch S is in position 1 and gate is turned ON		0	0	0	–
1	1	1	1	0	–
2	1	1	0	1	–
3	0	1	0	0	–
4	1	1	0	0	–
Switch S moves to position 2 and gate is turned off					
5	X	0	1	0	0
6	X	0	0	1	0
7	X	0	0	0	1

$\therefore$ The code vector is 1001011

Verification.

for $D = [1011]$ $r(x) = \dfrac{x^{n-k} d(x)}{g(x)} = \dfrac{x^3(1+x^2+x^3)}{1+x+x^3}$

$$\therefore\ x^3+x+1 \overline{\big)\, x^6+x^5+x^3}\, \big(x^3+x^2+x+1$$
$$\frac{x^6+x^4+x^3}{x^5+x^4}$$
$$\frac{x^5+x^3+x^2}{x^4+x^3+x^2}$$
$$\frac{x^4+x^2+x}{x^3+x}$$
$$\frac{x^3+x+1}{1}$$

$[V] = \underbrace{100}_{R}\underbrace{1011}_{D}$

for $D = [1001]$

$r(x) = \dfrac{x^{n-k} d(x)}{g(x)} = \dfrac{x^3(1+x^3)}{1+x+x^3}$

$$x^3+x+1 \overline{\big)\, x^6+x^3}\, \big(x^3+x$$
$$\frac{x^6+x^4+x^3}{x^4}$$
$$\frac{x^4+x^2+x}{x^2+x}$$

$[V] = \underbrace{011}_{R}\underbrace{1001}_{D}$

Error Detection and Correction.

The Code-vector V is transmitted over a noisy communication channel.

$$[Z] = [V] + [E] \longrightarrow (1)$$

where 'E' represents the error induced by the channel.

To prove that $S(X) \propto E(X)$; Syndrome $S(X)$

consider (1), writing as polynomial function in X

$$Z(x) = V(x) + E(x) \longrightarrow (2)$$

$\div$ through out equ^n (2) by $g(x)$

$$\frac{Z(x)}{g(x)} = \frac{V(x)}{g(x)} + \frac{E(x)}{g(x)}$$

but we know $V(x) = d(x) \cdot g(x)$

$$\frac{Z(x)}{g(x)} = d(x) + \frac{E(x)}{g(x)} \longrightarrow (3)$$

But the syndrome can also be obtained as the remainer of $\frac{Z(x)}{g(x)}$ division. this can be represented in the form of an equation.

$$Z(x) = q(x) * g(x) + S(x)$$

$\div$ by $g(x)$

$$\frac{Z(x)}{g(x)} = q(x) + \frac{S(x)}{g(x)} \longrightarrow (4)$$

Comparing equ^n (3) & (4)

$$d(x) + \frac{E(x)}{g(x)} = q(x) + \frac{S(x)}{g(x)}$$

$$\frac{E(x)}{g(x)} = q(x) - d(x) + \frac{S(x)}{g(x)}$$

$$E(x) = [g(x) \cdot q(x) - d(x) \cdot g(x)] + S(x)$$

But we know modulo addition & Subtraction is same

$$\boxed{E(x) = g(x)\,[q(x) + d(x)] + S(x)} \longrightarrow (5)$$

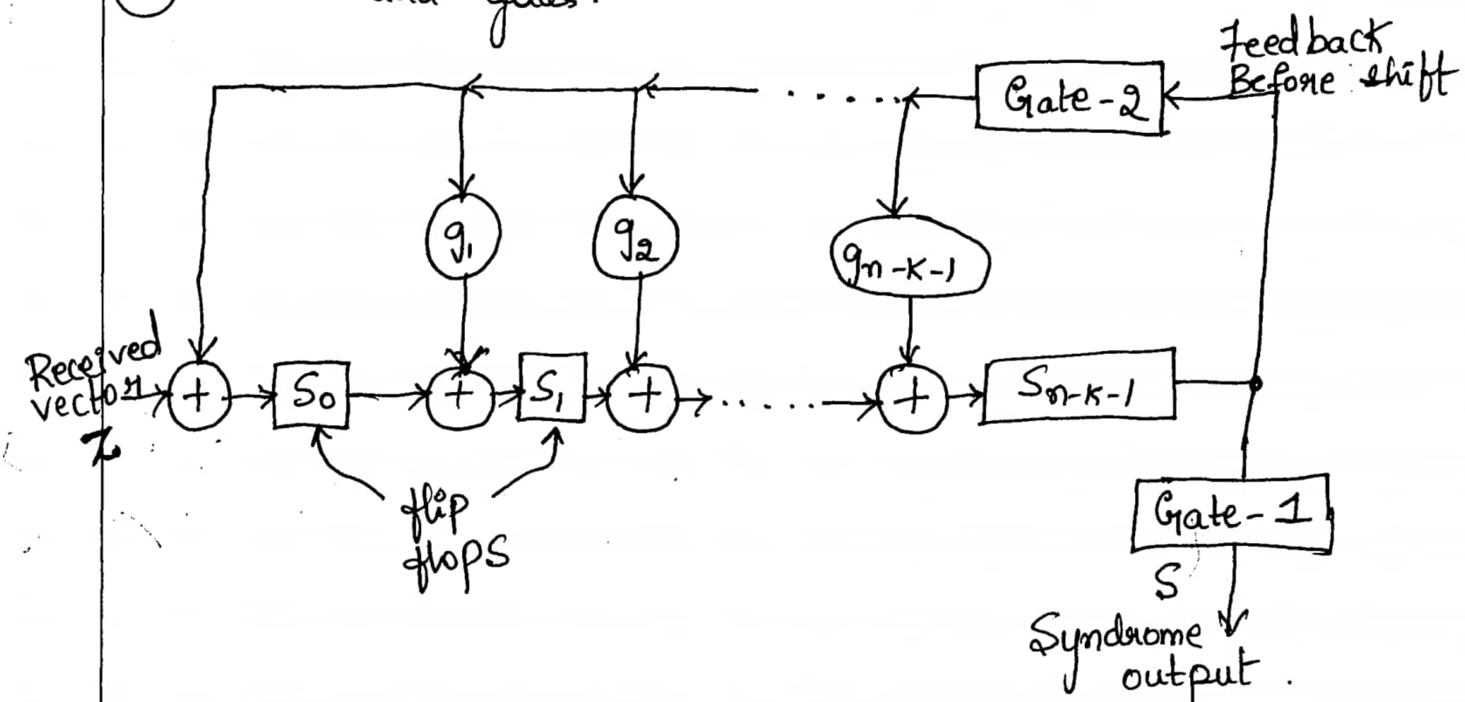
From equ^n (5) it is clear that the error polynomial is directly proportional to syndrome polynomial $S(x)$.
hence syndrome can be a measure of channel induced errors

Syndrome Calculation Circuit.

The Syndrome Calculation Circuit performs the divisio of the Received Vectors, by the generator and stores the remainder in the flipflops of a shift register.

The hardware requirements are:

(1) An $(n-K)$ bit shift register.
(2) A max of $(n-K)$ modulo-2 adders.
(3) A max of $(n-K)$ switches.
(4) Two and gates.



Working

(1) For the first n shifts AND Gate 2 is enabled and Gate 1 is disabled & the n-bits of the received vector are shifted into the circuit bit by bit with LSB entering first.

(2) After n-shifts the remainder of the division is present in the shift register which can now be shifted out serially by enabling gate 1 and disabling 2 for (n-k) clock cycles.

Problems

1. For a (7, 4) cyclic code, the received vector $Z(x)$ is 1110101 & the generator polynomial is $g(x) = 1+x+x^3$. Draw the syndrome calculation circuit and correct the single error in the received vector.

→ the generator polynomial is given by

$$g(x) = 1 + x + x^3$$
$$= g_0 + g_1 x + g_2 x^2 + g_3 x^3$$

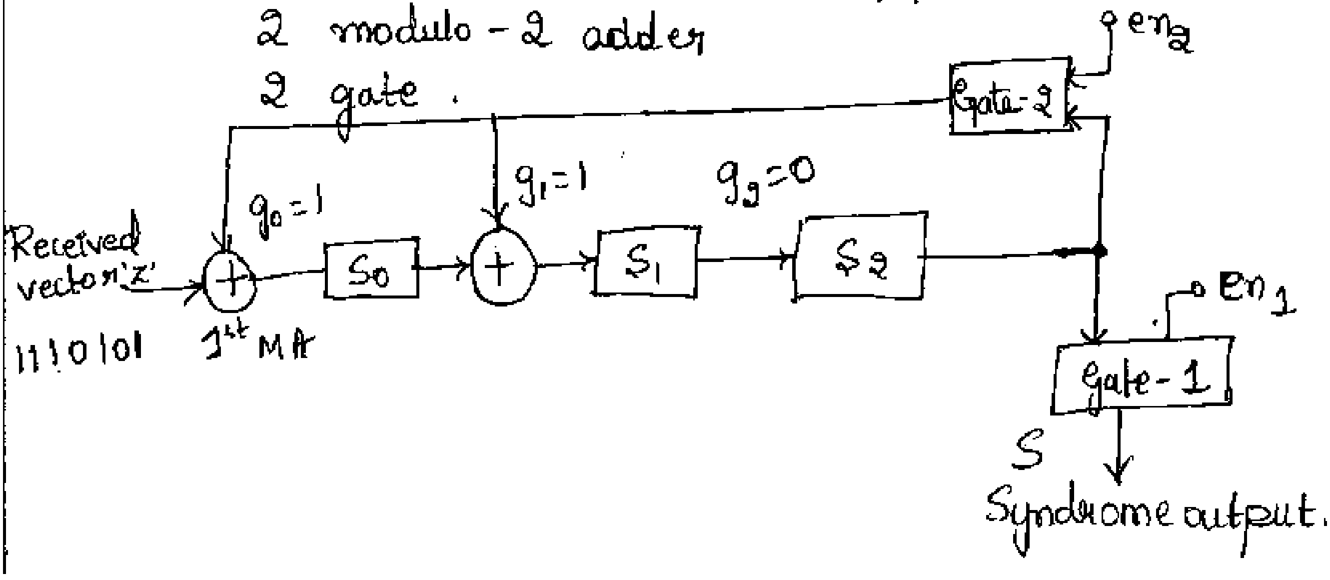
∴ The co-efficients are given by

$$g_0 = 1 \quad g_1 = 1, \; g_2 = 0, \; g_3 = 1$$

∴ we $S_{n-k-1} \Rightarrow S_{7-4-1} \Rightarrow$ 3 flip flops $\Rightarrow S_0, S_1$ & S_2

2 modulo - 2 adder

2 gate.



Two methods are available for error in the received vector.

Method - 1

Number of shifts	Input 'z'	Shift Register Contents S_0	S_1	S_2
Initialization, gate-1 off, Gate-2 ON		0	0	0
1	1	1	0	0
2	0	0	1	0
3	1	1	0	1
4	0	1	0	0
5	1	1	1	0
6	1	1	1	1
7	1	0	0	1

Knowing the syndrome $S_0\ S_1\ S_2 \rightarrow 0\ 0\ 1$ & H^T matrix, the error can be corrected analytically

$$H^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

[Note : $h(x) = \dfrac{x^n+1}{g(x)}$]

we get $h(x)$.
taking $x^K h(x^{-1})$
$x^{K+1} h(x^{-1})$ gives u
$x^{K+2} h(x^{-1}) \Rightarrow H$
H^T is then finded.

the Syndrome $S_0 S_1 S_2 \rightarrow 001$ is present in 3^{rd} row

$\therefore$ the Received vector $= [1\,1\,1\,0\,1\,0\,1] \Rightarrow [1\,1\,\underline{0}\,0\,1\,0\,1]$

also can write it as, Corrector Vector $= Z + E$

$= 1110101 + 0010000$

$= 1100101$

Method – 2

Number of shifts	Input $r(x)$	Shift Register contents S_0	S_1	S_2	
Initialization, gate-1 off and gate-2 ON		0	0	0	
1	1	1	0	0	
2	0	0	1	0	
3	1	1	0	1	
4	0	1	0	0	
5	1	1	1	0	
6	1	1	1	1	
7	1	0	0	1	← Indicates error
8	0	1	1	0	
9	0	0	1	1	
10	0	1	1	1	
11	0	1	0	1	
12	0	1	0	0	← end of shifting operation

- This method is continuation of method 1.
- When all the 7 received bits are entered into the syndrome calculator, '0's are fed into it, from 8th shift onwards.
- Each time a '0' is fed into the circuit, the fresh shift register contents are noted.
- The process is continued till the shift register contents read $S_0 S_1 S_2 = 100$

i.e In general for $(n-k)$ shift register, the contents should read $S_0, S_1 \ldots S_{n-k-1} = 100\ldots0$ i.e 1 followed by $(n-k-1)$ no of '0's.

So we find that at the 12^{th} shift we get shift register contents as 100.

$\therefore$ the error is detected & corrected as

Received vector 'Z' $\rightarrow$ 1 1 1 0 1 0 1

$$\begin{array}{ccccccc} & & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ & & 12^{th} & 11^{th} & 10^{th} & 9^{th} & 8^{th} \end{array}$$

Since we got '100' in the 12^{th} shift, the 5^{th} bit counting from right is in error.

$\therefore$ Corrected vector 'V' = 1100101.

Which is same as in method 1.

(2) Let us consider, received vector $Z \rightarrow 0100101$ with $g(x) = 1 + x + x^3$, Draw the syndrome calculation circuit & correct the single error in the received vector.

→ Considering the same syndrome circuit of problem 1 we have.

Number of shifts	Input $Z(x)$	Shift Register contents S_0	S_1	S_2	
Initialization Gate-1 off & gate-2 on.		0	0	0	
1	1	1	0	0	
2	0	0	1	0	
3	1	1	0	1	
4	0	1	0	0	
5	0	0	1	0	
6	1	1	0	1	
7	0	1	0	0	← Indicates error
8	0	0	1	0	
9	0	0	0	1	
10	0	1	1	0	
11	0	0	1	1	
12	0	1	1	1	
13	0	1	0	1	
14	0	1	0	0	← end of shifting operation

so we find that, @14th shift we get shift register contents as 100

$\therefore$ Received vector $Z' \rightarrow$ 0 1 0 0 1 0 1

14th, 9th, 8th

$\therefore$ corrected vector $V = 1100101$

3. Draw Encoder diagram for (7,3) Cyclic Code given $g(x) = 1 + x + x^2 + x^4$

$\rightarrow$ $g_0 = 1$, $g_1 = 1$, $g_2 = 1$, $g_3 = 0$, $g_4 = 1$

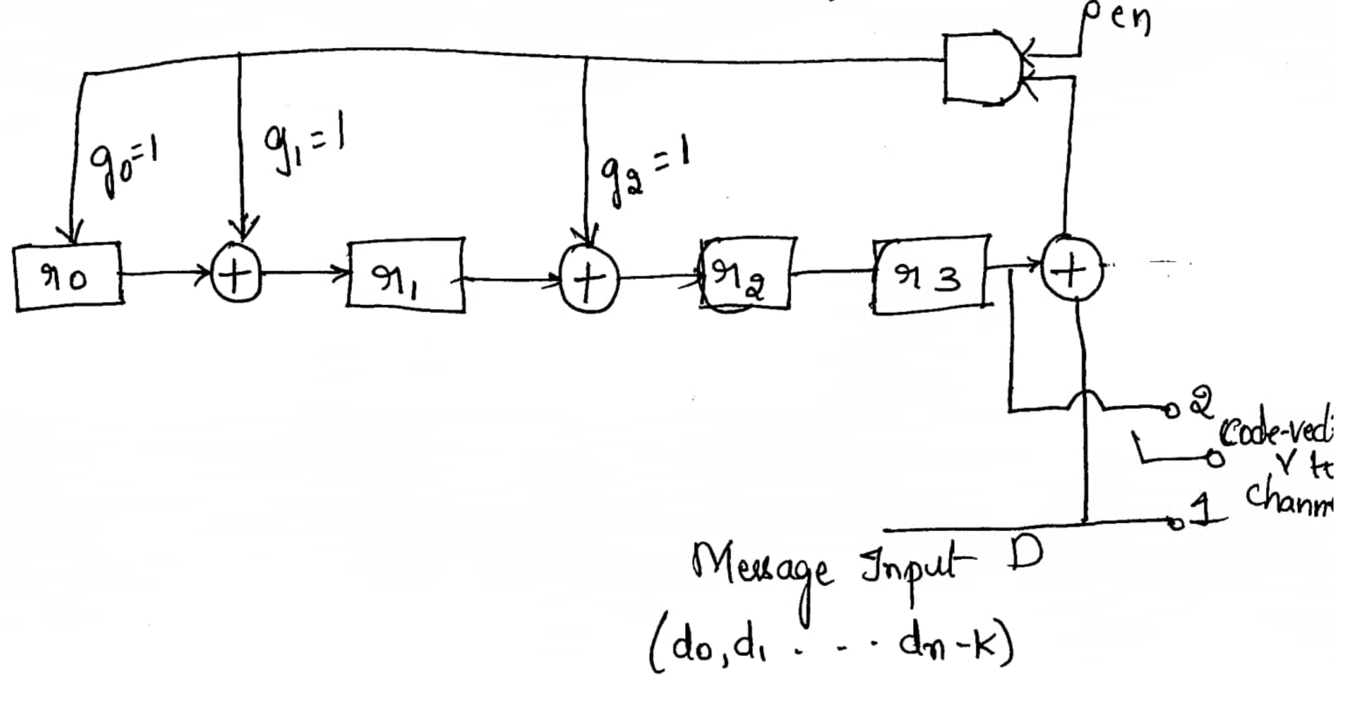
we have $g(x) = g_0 + g_1(x) + g_2(x)^2 + g_3(x^3) + g_4(x^4)$

$\therefore$ we have $(n-k) = (7-3)$ shift registers
$= 4$ shift registers

i.e upto $(n-k-1) \Rightarrow (7-3-1) = 3$

$\therefore$ we have R_0, R_1, R_2, R_3

(3) modulo 2 adder & g_0, g_1, g_2 is taken



(4) For (15,7) Cyclic code $g(x)=1+x^4+x^6+x^7+x^8$. Draw the Syndrome circuit.

→ we know the generator polynomial

$$g(x)=g_0+g_1(x)+g_2(x)^2+g_3(x)^3 \cdots\cdots + g_{n-k}(x)^{n-k}$$

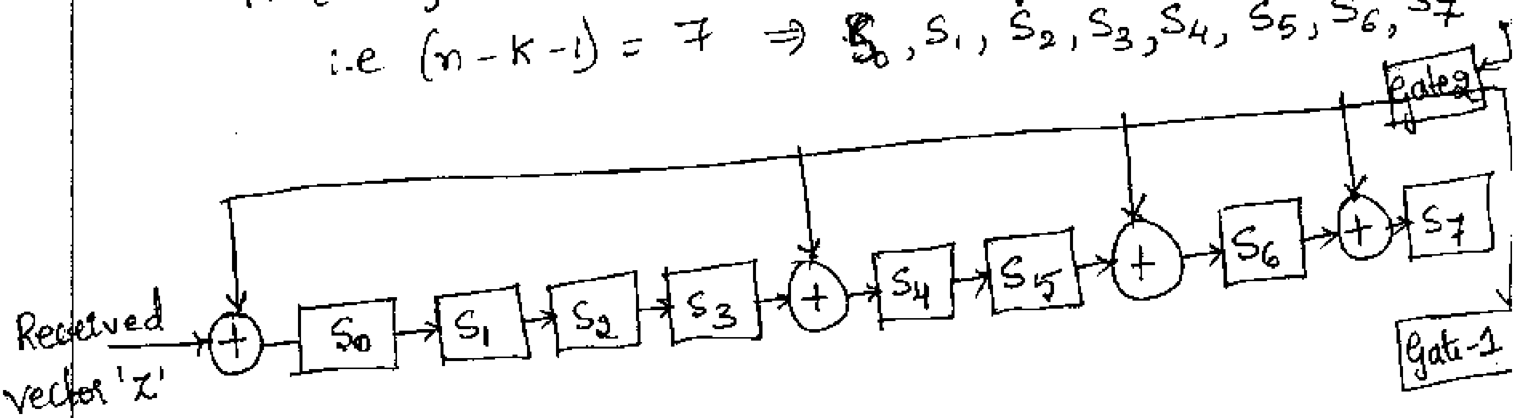
∴ we have $g_0=1$, $g_4=1$, $g_6=1$, $g_7=1$

we require 4 modulo ② adders.

we have $(n,k) \Rightarrow (15,7)$

∴ $(n-k)=(15-7)=8$ shift register

i.e $(n-k-1)=7 \Rightarrow S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7$



(5) Consider a (15,11) Cyclic code generated by $g(x)=1+x+x^4$ Use the Devise a feedback shift register encoder circuit, Use the encoding procedure with the message vector 10010110111 by listing the state of the registers.

→ the generator polynomial is given by

$$g(x)=1+x+x^4 = g_0+g_1x+g_2x^2+g_3x^3+g_4x^4$$

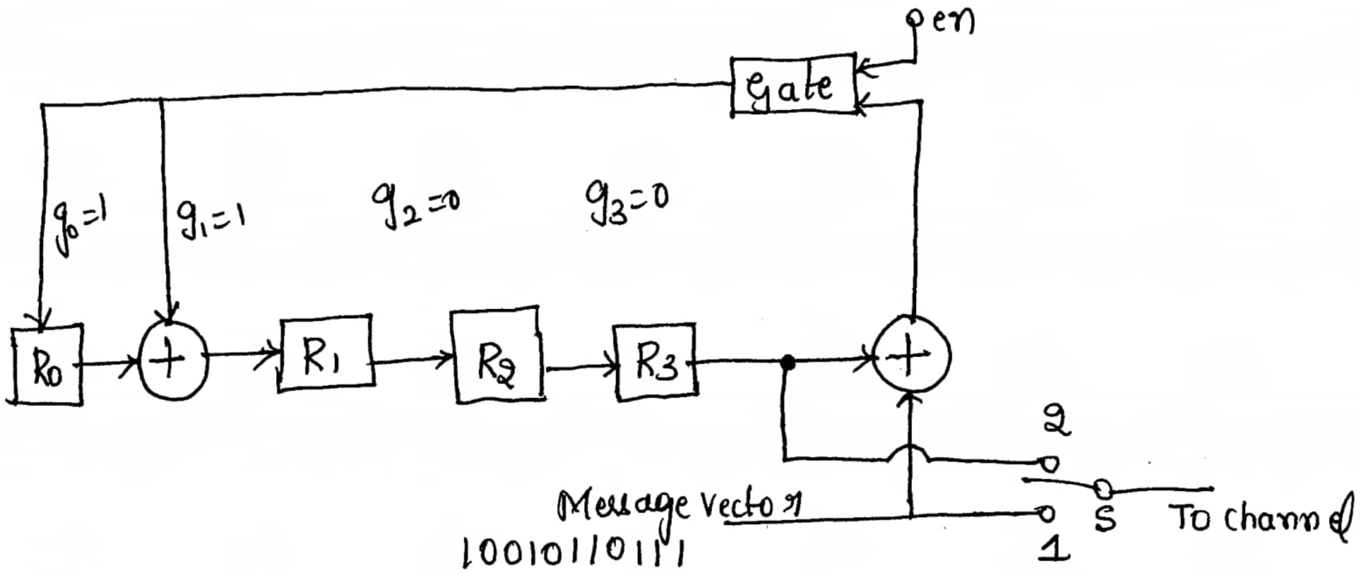
The co-efficients of $g(x)$ are

$$g_0=1, \ g_1=1, \ g_2=0, \ g_3=0, \ g_4=1$$

So we use $g_{(n-k-1)} = g_{(15-11-1)} = g_3 \Rightarrow g_0, g_1, g_2, g_3$

shift register $R_{(n-k-1)} = R_3 \Rightarrow R_0, R_1, R_2, R_3$.

Encoder circuit for (15,11) cyclic code.



Number of shifts	Input D	Shift Register contents R_0	R_1	R_2	R_3	Remainder bits - R.
Initialisation → Switch S is position 1 & gate is turned ON		0	0	0	0	
1	1	1	1	0	0	
2	1	1	0	1	0	
3	1	1	0	0	1	
4	0	1	0	0	0	
5	1	0	0	0	0	
6	1	1	0	0	0	
7	0	0	1	0	0	
8	1	1	1	1	0	
9	0	0	1	1	1	
10	0	1	1	1	1	
11	1	0	1	1	1	
Switch S moves to position 2 and gate is turned off		0	0	1	1	1
12	X	0	0	0	1	1
13	X	0	0	0	0	1
14	X	0	0	0	0	1
15	X	0	0	0	0	0.

∴ the Remainder vector is 0111, hence the transmitted code-vector through the channel is

$$011110010110111.$$

(6) A (15,5) linear cyclic code has a generator polynomial

$$g(x) = 1+x+x^2+x^4+x^5+x^8+x^{10}$$

(a) Draw the block diagram of a encoder and syndrome calculator for this code.

(b) find the code polynomial for the message polynomial $D(x) = 1+x^2+x^4$ in systematic form

(c) Is $v(x) = 1+x^4+x^6+x^8+x^{14}$ a code polynomial.

→ (a) the generator polynomial is given by

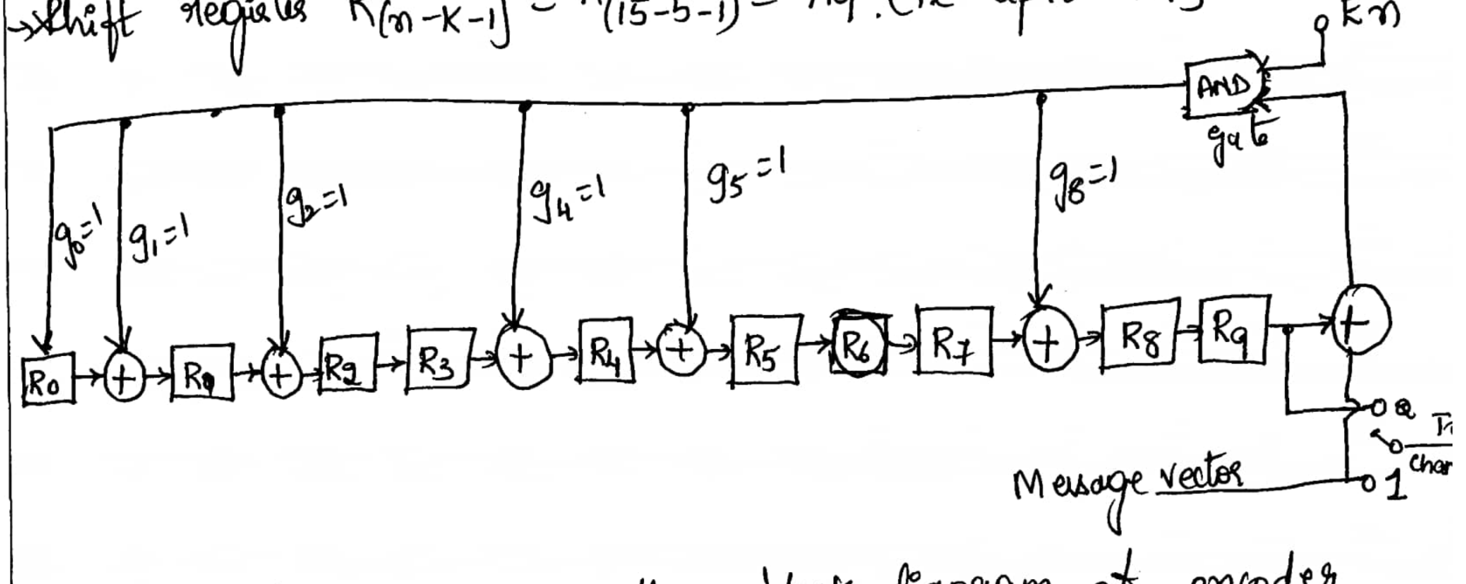
$$g(x) = 1+x+x^2+x^4+x^5+x^8+x^{10}$$

∴ the co-efficient of generator polynomial are:-

→ $g_0=1$, $g_1=1$, $g_2=1$, $g_3=0$, $g_4=1$, $g_5=1$, $g_6=0$, $g_7=0$, $g_8=1$

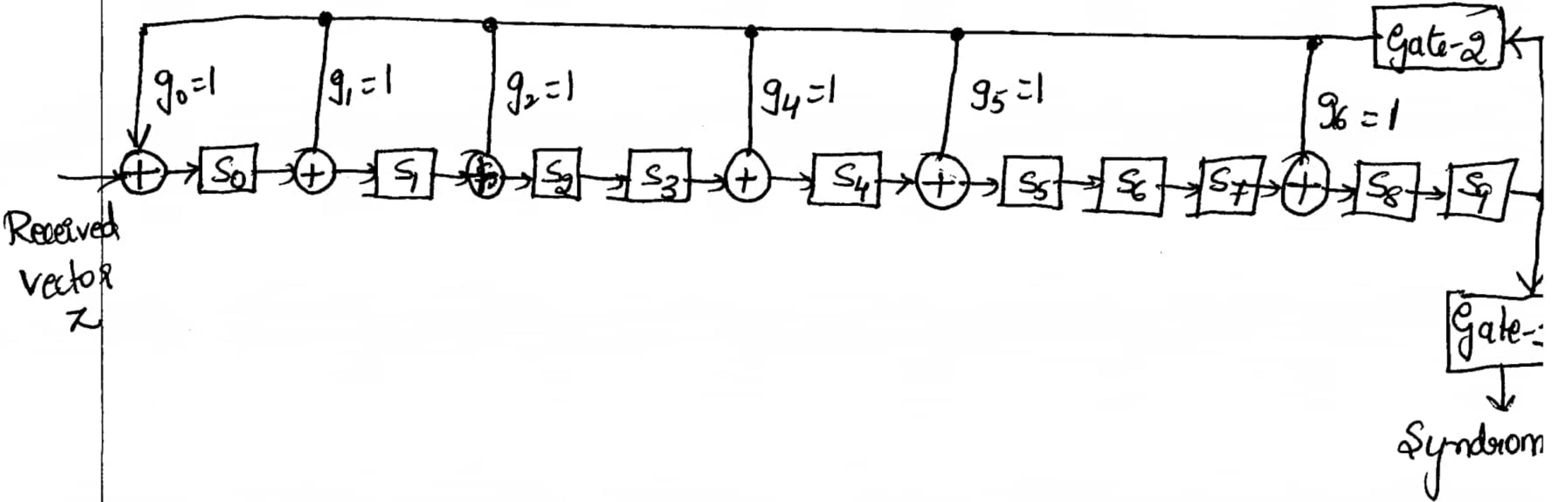
$g_9=0$

→ shift registers $R_{(n-k-1)} = R_{(15-5-1)} = R_9$ (ie upto R_9).



The above gives us the block diagram of encoder for (15,5) code.

Knowing the generator polynomial, shift register, the syndrome calculator circuit is written using 2 gates.



(b) Message i/p $D(x) = 1 + x^2 + x^4$

given $g(x) = 1 + x + x^2 + x^4 + x^5 + x^8 + x^{10}$

$$\therefore R(x) = \frac{x^{n-k}\, D(x)}{g(x)}$$

$$= \frac{x^{10}(1+x^2+x^4)}{1+x+x^2+x^4+x^5+x^8+x^{10}} \Rightarrow \frac{x^{10}+x^{12}+x^{14}}{1+x+x^2+x^4+x^5+x^8+x^{10}}$$

$$x^{10}+x^8+x^5+x^4+x^2+x+1 \overline{)\, x^{14}+x^{10}+x^{12}} \left(x^4 + 1 \right.$$

$$\cancel{x^{14}} + \cancel{x^{12}} + x^9 + x^8 + x^6 + x^5 + x^4$$

$$\cancel{x^{10}} + x^9 + \cancel{x^8} + x^6 + \cancel{x^5} + \cancel{x^4}$$

$$\cancel{x^{10}} + \cancel{x^8} + \cancel{x^5} + \cancel{x^4} + x^2 + x + 1$$

$$x^9 + x^6 + x^2 + x + 1$$

$$V = \underbrace{1110001001}_{R}\ \underbrace{10101}_{D}$$

(c) If $V(x)$ is a code polynomial, then it should be perfectly divided by the generator polynomial with remainder zero. If the remainder is not zero, we can conclude that the given polynomial is not a code-polynomial.

$$\therefore\ x^{10}+x^8+x^5+x^4+x^2+x+1 \overline{\big)\, x^{14}+x^8+x^6+x^4+1}\quad \text{quotient: } x^4+x^2+1$$

$$\begin{array}{l}
x^{14}+x^8+x^6+x^4+1 \\
x^{14}+x^{12}+x^9+x^8+x^6+x^5+x^4 \\
\hline
x^{12}+x^9+x^5+1 \\
x^{12}+x^{10}+x^7+x^6+x^4+x^3+x^2 \\
\hline
x^{10}+x^9+x^7+x^6+x^5+x^4+x^3+x^2 \\
x^{10}+x^8+x^5+x^4+x^2+x+1 \\
\hline
\end{array}$$

$$R(x) \rightarrow x^9+x^8+x^7+x^6+x^3+x+1$$

The remainder polynomial $R(x)$ is a non-zero polynomial, hence the given polynomial is not a code polynomial.

(7) A linear hamming code is described by a generator polynomial
$$g(D) = 1+D+D^3$$
(i) Determine the generator matrix G and parity check matrix
(ii) Design an encoder circuit

→ given : $g(D) = 1+D+D^3 \rightarrow$ code is $[1101000]$

$D\,g(D) = D+D^2+D^4 \rightarrow$ code is $[0110100]$

$D^2 g(D) = D^2+D^3+D^5 \rightarrow$ code is $[0011010]$

& $D^3 g(D) = D^3+D^4+D^6 \rightarrow$ code is $[0001101]$

$\therefore$ We know $[G] = [P \,\vdots\, I_4]$

$$\therefore [G] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

the last 4 element of the 3rd & 4th row are not the last four elements of unit matrix I_4.
therefore adding 3rd row with 1st row & placing it in 3rd row and adding 4th row with 1st & 2nd row & 4th row and placing in 4th row, we get

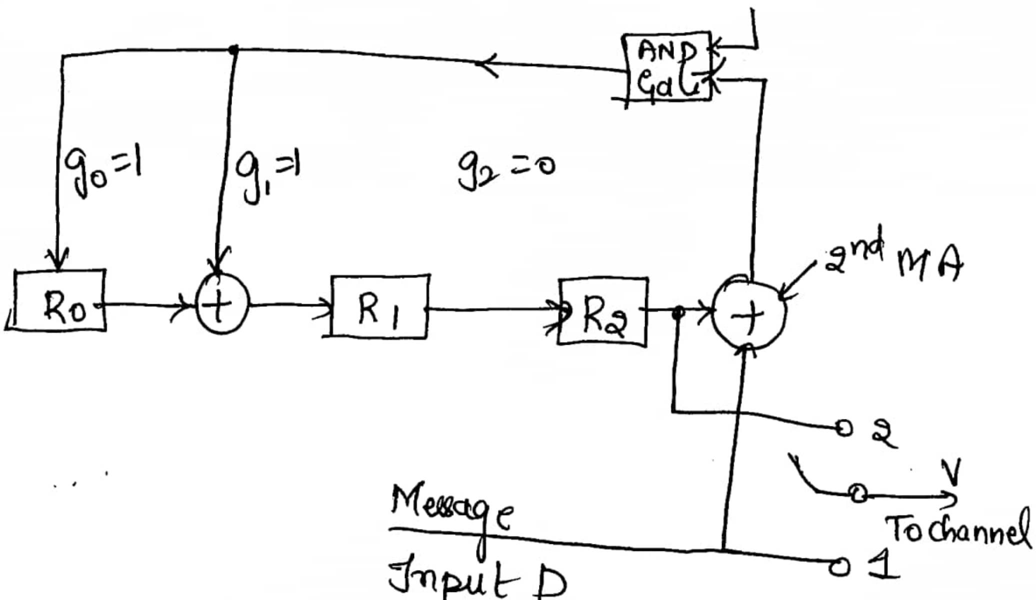
$$[G] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Parity check matrix is given by $[H] = [I_{n-k} \mid P^T]$

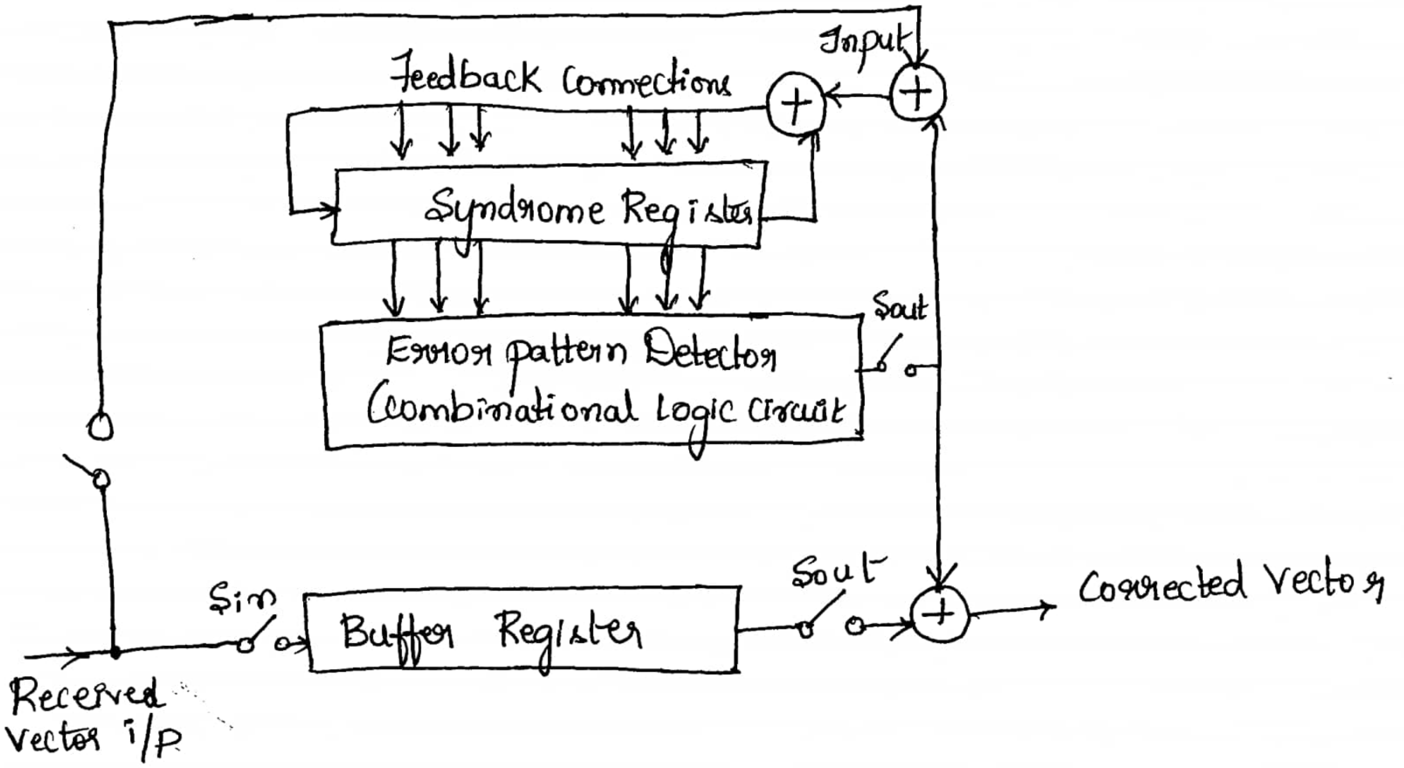
$$[H] = [I_3 \mid P^T]$$

$$\therefore [H] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(ii) with $g_0 = 1$, $g_1 = 1$, $g_2 = 0$ and $g_3 = 1$, the Encoder ckt is



General Decoder Circuit for (n, k) cyclic codes.



Working

1. Initially switch Sin is closed and Sout is opened & the received vector is shifted into the syndrome Calculator circuit & the buffer register simultaneously.
2. After n-clk cycles, the syndrome Calculator circuit contains the error syndrome for the given received vector which is now fed into the error pattern detector.
3. The error pattern detector is a combinational logic circuit whose output is 1 if there is an error in the LSB (Right most bit of received Vector)
4. Now Sin is open and Sout is closed and received vector buffer register contents are shifted right once and the Corrected code bit for the LSB is obtained as : $\hat{V}_i = z_i + e_i$. The e_i value is also fed back to the syndrome Calculator circuit so as to determine the new syndrome Corresponding to the shifted value of the received vector.

(5) Now again step 2, 3, 4 are repeated So as to correct the errors in the bit next to the LSB. this process is repeated until all the n received bits are corrected.

To correct all received vectors we need 2n clock cycles. The contents of the syndrome calculator must be zero at the end of 2n clock cycles if the error was Completely Corrected.

If syndrome is not zero, then Undetectable error has been encountered which cannot be detected.

Noor Afza CIT gubbi

UNIT-7, 8

MORE ERROR CONTROL CODES

* REED - SOLOMON (RS) CODES

The RS codes are as special as BCH codes.
The primitive q-ary BCH codes have block lengths $n = q^m - 1$ and the extension field of $GF(q^m)$ is invoked for locating roots of generator polynomial $g(x)$. Taking $m=1$, we get $n = q-1$, which gives the standard RS-codes.

For the primitive BCH codes, the generator polynomial is given by,

$$g(x) = LCM\left[\phi_{\alpha^j}(x), \phi_{\alpha^{j+1}}(x), \dots \phi_{\alpha^{j+\delta-1}}(x)\right] \longrightarrow (1)$$

Where $\phi_{\alpha^j}(x)$ = minimum polynomial of element α^j

α^j = powers of primitive element over $GF(q^m)$

δ = designed distance (desired value of minimum distance d_{min})

With $m=1$, the minimal polynomials over $GF(q)$ of elements in $GF(q)$ are first degree of the form $(x-\alpha^j)$

& $g(x)$ reduces to

$$g(x) = (x-\alpha^j)(x-\alpha^{j+1}) \dots (x-\alpha^{j+\delta-2}) \longrightarrow (2)$$

This is polynomial over $GF(q)$ of degree exactly $(\delta-1)$.

Thus $\boxed{n-k = \delta-1}$ for RS codes

The true minimum distance must be

$d_{min} \geq \delta = n-k+1 \implies$ for RS codes

$$\boxed{d_{min} = n-k+1}$$

- RS codes are "Maximum distance Seperable" codes (MDS codes).
- For any (n,K) linear block code, the minimum distance $d_{min} < (n-K+1)$, But for RS code $d_{min} = n-K+1$.
- The value of K may be selected as any integer upto $(n-1)$
- RS codes can be shortened without decreasing d_{min} or changing $(n-K)$ and thus RS codes are also MDS.

Thus a t-error correcting q-ary RS code with symbol from $GF(q)$ has the following parameters

Block Length : $n = q-1$

Number of parity Check Symbols : $n-K = \delta - 1 = 2t$

Number of message digits = Dimension : $K = q-1-2t$

$$K = q - \delta$$

Minimum Distance : $d_{min} = \delta = n-K+1$

$$\delta = 2t+1$$

Error Correcting Capability of the code : $t = \dfrac{n-K}{2}$

The weight spectrum of MDS codes, including RS codes, is known in exact form given by

$$A_w = {}_nC_w \sum_{m=0}^{w-(n-K+1)} (-1)^m \, {}_wC_m \left[q^{w-m-(n-K)} - 1 \right],$$

$$w = n-K+1, \; n-K+2, \; \ldots n$$

The number of code words having minimum non-zero weight [i.e, nearest neighbours for a specified value of d_{min}] is given by $\boxed{A_{d_{min}} = {}^{n}C_{d_{min}} (q-1)}$

Problems

① Determine the parameters of a q-ary RS code over $GF(256)$ for a $d_{min} = 33$.

Sol^n The block length $n = q - 1$

$n = 256 - 1$

$n = 255$

Dimension $K = q - \delta = q - d_{min}$

$K = 256 - 33$

$K = 223$

Number of parity check symbols $= n - K$

$= 255 - 223$

$\therefore n - K = 32$

Error correcting capability $t = \frac{n-K}{2} = \frac{32}{2} = 16$.

② Determine the parameters of a q-ary RS code over $GF(16)$ for a $d_{min} = 9$. Also find the total number of code-words in the code and also the nearest neighbours for any code-word at a distance of $d_{min} = 9$.

Sol^n The block length $n = q - 1 \Rightarrow n = 16 - 1$ $\boxed{n = 15}$

$d_{min} = n - K + 1$ $\therefore K = n - d_{min} + 1 = 15 - 9 + 1 \Rightarrow \boxed{K = 7}$

Number of parity check Symbols $\Rightarrow n - K = 15 - 7 = \boxed{8}$

Error correcting capability $\Rightarrow t = \frac{n-K}{2} = \frac{8}{2} = 4$ $\therefore \boxed{t = 4}$

The total number of code-words in the code is calculated by using the A_w equ^n with $n = 15$,

$q = 16$

$K = 7$.

$$A_w = {}_{15}C_w \sum_{m=0}^{w-9} (-1)^m \; {}_wC_m \left[16^{w-8-m} - 1\right], \quad w = 9, 10, 15$$

For $w=9$, $A_9 = {}_{15}C_9 \left[16-1\right] = 75075$

$w=10$, $A_{10} = {}_{15}C_{10} \sum_{m=0}^{1} (-1)^m \, {}_{10}C_m \left(16^{2-m} - 1\right) = 315315$

$w=11$, $A_{11} = {}_{15}C_{11} \sum_{m=0}^{2} (-1)^m \, {}_{11}C_m \left(16^{3-m} - 1\right) = 2886975$

$w=12$, $A_{12} = {}_{15}C_{12} \sum_{m=0}^{3} (-1)^m \, {}_{12}C_m \left(16^{4-m} - 1\right) = 13615875$

$w=13$, $A_{13} = {}_{15}C_{13} \sum_{m=0}^{4} (-1)^m \, {}_{13}C_m \left(16^{5-m} - 1\right) = 47651625$

$w=14$, $A_{14} = {}_{15}C_{14} \sum_{m=0}^{5} (-1)^m \, {}_{14}C_m \left(16^{6-m} - 1\right) = 1.0086705 \times 10^8$

$w=15$, $A_{15} = {}_{15}C_{15} \sum_{m=0}^{6} (-1)^m \, {}_{15}C_m \left(16^{7-m} - 1\right) = 1.0195817 \times 10^8$

$\therefore$ Total number of code-words

$\Rightarrow \quad A_9 + A_{10} + A_{11} + A_{12} + A_{13} + A_{14} + A_{15}$

$\therefore$ Total number of code-words $\approx 2.67 \times 10^8$

The nearest neighbours for any code-word at a distance of $d_{min} = 9$ is given by

$$\boxed{\begin{aligned} A_{d_{min}} &= {}_nC_{d_{min}} (q-1) \\ \therefore A_9 &= {}_{15}C_9 (16-1) \\ &= 75075. \end{aligned}}$$

Application of RS Codes.

1. The RS codes are a natural means of coding for M-ary modulation techniques.
2. Storage devices (including Compact Disc, Blue-ray Disc, DVD barcodes, etc)
3. Wireless or mobile Communications (including cellular telephones, microwave links, etc)
4. Satellite Communications
5. Digital television / Digital Video Broadcast (DVB)
6. high-Speed modems such as those employing ADSL, xDSL, etc.
7. VLSI implementation using RS encoder.
8. RS codes may be used in conjuction with binary transmission.

GOLAY CODES

Golay code is a (23,12) binary code capable of correcting upto 3 errors in a block of 23 bits.
It is perfect binary code because it satisfies the hamming bound with equality sign for $t=3$.

Hamming bound : $2^{n-k} \geqslant \sum_{i=0}^{t} {}_nC_i$

For Golay code consider

$$2^K \sum_{i=0}^{t} {}_nC_i = 2^{12} \sum_{i=0}^{t=3} {}_{23}C_i$$

$$= 2^{12}\left[1 + {}_{23}C_1 + {}_{23}C_2 + {}_{23}C_3\right]$$

$$= 2^{12}(2048) = 12^{12}.2^{11} = 2^{23} = 2^n$$ & hence the result

The generator polynomial is obtained by

$$x^{23}+1 = (x+1)\cdot g_1(x)\cdot g_2(x)$$ where $g_1(x) = 1 + x^2 + x^4 + x^5 + x^6 + x^{10} + x^{11}$

$g_2(x) = 1 + x + x^5 + x^6 + x^9 + x^{11}$

The encoder can be implemented using shift register using either $g_1(x)$ or $g_2(x)$ as divider polynomial.

SHORTEND CYCLIC CODES

The cyclic codes so far considered have generator polynomia that are divisors of x^n+1, but x^{n+1} has relatively few divisor & there are very few cyclic codes of a given length.

· To overcome this difficulty & to construct the useful co -e we have to increase the number of pairs (n, k), hence cyclic codes are often used in shortened form.

The last 'i' information digits are always taken to be 'o (i.e., the last "i" bits of the code-vectors are padded with 0's these bits are not transmitted; The decoder for the Original Cyclic code can decode the shortened code-vectors simply by padding the received $(n-i)$-tuples with '0's.

· Hence, given an $(n-k)$ cyclic code, it is always possible to construct a $(n-i, k-i)$ Shortend Cylic code.

· The Shortend Cyclic code is a subject of the cyclic code from which it was derived & hence its minimum distance and error correcting ability is at least as great as that of the original code.

· Shortend Cyclic codes are used in the implementation advantages & mathematical structure of Cyclic Codes.

BURST-ERROR CORRECTING CODES

. The Presence of impulse noise affects more than one symbol or bit to cause burst errors.

· Codes Used for Correcting random errors are not sufficient for correcting burst errors. Special codes have been developed for correcting burst errors.

Burst of length q : It is defined as a vector whose non-zero components are confined to q consecutive digit positions, the first and last of which are non-zero

A code which is capable of correcting all burst errors of length q or less is called q-burst error correcting-code & it is said to have burst error-correcting ability q.

The condition for a q-burst-error-correcting code (n, K) is that the number of parity bits must at least be 2q.

$$\text{i.e} \quad n-K \geq 2q$$

The upper bound on the burst-error-correcting capability of an (n, K) code is

$$\boxed{q \leq \frac{n-K}{2}}$$

Burst and Random Error Correcting Codes

In practical systems, errors occur neither independently, at random nor in well defined bursts.

∴ Random error correcting codes or single-burst-error correcting codes will be inadequate for combating a mixture of random and burst errors.

It is better to design codes capable of correcting random errors and/or single or multiple bursts.

The most effective method is the INTERLACING TECHNIQUE. Given a (n, K) cyclic code, it is possible to construct a $(\lambda n, \lambda K)$ cyclic interlaced code by simply arranging λ code-vectors of the original code into λ rows of a rectangular array & transmitting them column by column.

The resulting code is called an interlaced code with an interlacing degree λ.

✗ Problem

① Consider a (15, 9) cyclic code generated by $g(x) = 1 + x^3 + x^4 + x^5 + x^6$. This code has burst error correcting ability $b = 3$ find the burst-error correcting efficiency of this code.

→ The burst-error correcting efficiency z of an (n, K) cyclic code is defined by

$$Z = \frac{2b}{n-K}$$

given that $b = 3$, $n = 15$, $K = 9$

$$Z = \frac{2 \times 3}{15 - 9} = 1 \rightarrow 100\%$$

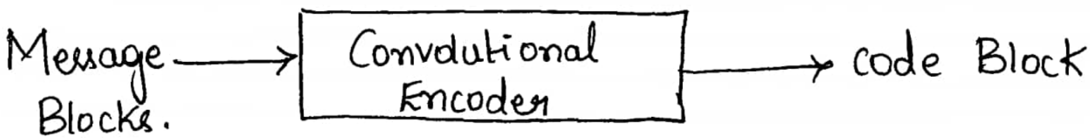
Convolutional Codes

Noor Afza
CIT Gubbi

In "block codes" a block of 'n' digits generated by the encoder in a particular time-unit depends only on the block of 'K' input message digits within that time unit.

In Convolutional codes, a block of 'n' code digits generated by the encoder in a time unit depends on not only the block of 'K' message digits within that time unit but also on the preceding (m-1) blocks of message digits (m>1). Usually the values of 'K' and 'n' will be small

Encoder for Convolutional Codes.



- A convolutional encoder, takes sequences of message digits and generates sequence of code digits.
- A message block consisting of 'K' digits is fed into the encoder & the encoder generates a code block consisting of 'n' code digits (k<n).
- the n-digit code block depends not only on the K-digit message block of the same time unit, but also on the previous (m-1) message blocks.
- the code generated by the above encoder is called as (n, K, m) convolutional code of constraint length "nm" digits and "rate efficiency K/n".

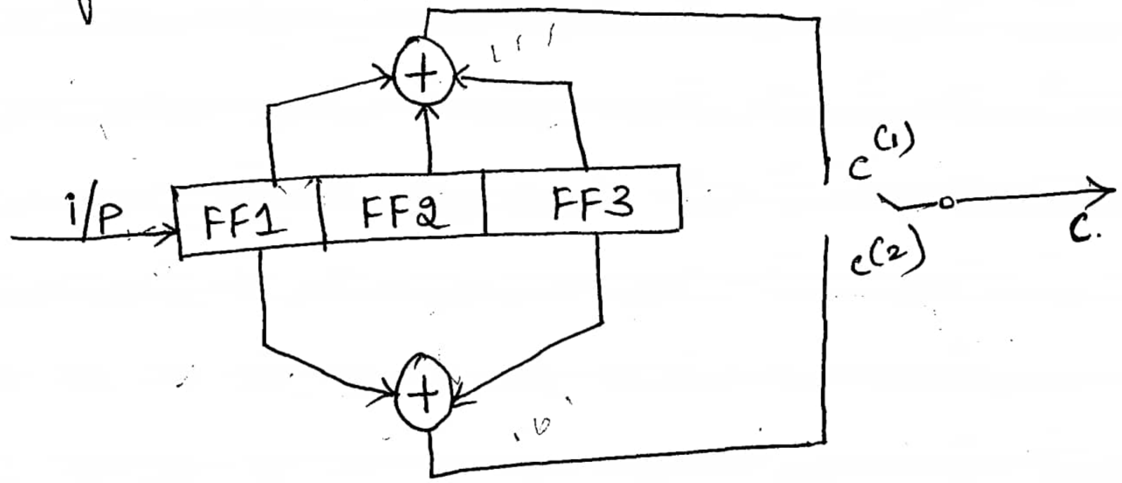
The following notations are used for a general (n, K, m) convolutional encoder.

n = number of outputs = number of modulo-2 adders
K = number of i/p bits entering at any time
m = number of stages of shift register
= number of flip-flops.
L = number of bits in the message sequence

Constraint length = $m \times n$ digits

Rate efficiency = K/n.

Diagrametic Representation of convolutional encoder.



In Convolution Codes, Encoding can be done in 2 methods
(i) Time domain
(ii) Transfer domain.

- Encoding
  - Time domain
    - conventional method
    - Generator Matrix
  - Transfer Domain

Problems

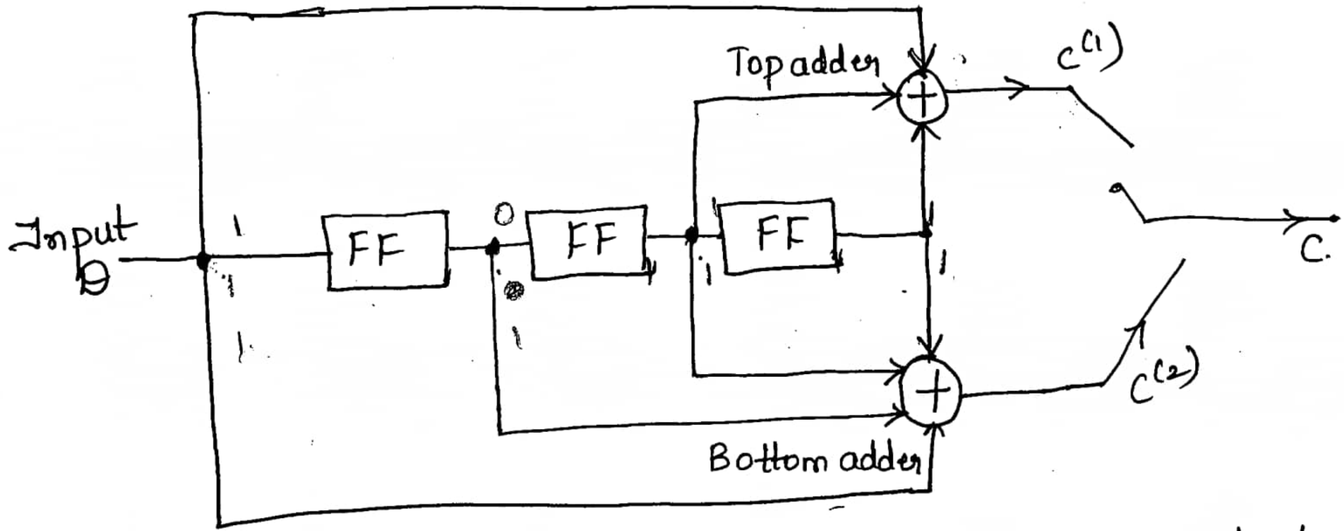
(2) Consider a (2, 1, 3) convolution encoder. find the output sequence for the input message d = 10111 using
(a) Time-Domain approach
(b) Transfer-Domain approach, for $g^{(1)} = 1011$, $g^{(2)} = 1111$

→ (a) Time-Domain Approach.

Given: $(n, K, m) = (2, 1, 3)$

$n \rightarrow$ no of modulo-2 adders $\rightarrow 2$
$K \rightarrow$ no of i/ps $\rightarrow 1$
$m \rightarrow$ no of flip-flops $\rightarrow 3$
$L \rightarrow$ no of bits in the message sequence = 5.

Encoder diagram.



Impulse Response or Generator sequences is given, it is written as

$$g^{(1)} = g_1^{(1)}\, g_2^{(1)}\, g_3^{(1)}\, g_4^{(1)} = 1011$$
$$g^{(2)} = g_1^{(2)}\, g_2^{(2)}\, g_3^{(2)}\, g_4^{(2)} = 1111$$

The generator matrix G is of Order

$$L \times [n(L+m)]$$
$$5 \times [2(5+3)]$$
$$5 \times 16$$

i.e 5 rows X 16 columns

1011 1111

i.e $G = \begin{bmatrix} 11 & 01 & 11 & 11 & 00 & 00 & 00 & 00 \\ 00 & 11 & 01 & 11 & 11 & 00 & 00 & 00 \\ 00 & 00 & 11 & 01 & 11 & 11 & 00 & 00 \\ 00 & 00 & 00 & 11 & 01 & 11 & 11 & 00 \\ 00 & 00 & 00 & 00 & 11 & 01 & 11 & 11 \end{bmatrix}$

$\therefore$ to find the code vector

$C = D\,G$

$= [10111] \begin{bmatrix} 11 & 01 & 11 & 11 & 00 & 00 & 00 & 00 \\ 00 & 11 & 01 & 11 & 11 & 00 & 00 & 00 \\ 00 & 00 & 11 & 01 & 11 & 11 & 00 & 00 \\ 00 & 00 & 00 & 11 & 01 & 11 & 11 & 00 \\ 00 & 00 & 00 & 00 & 11 & 01 & 11 & 11 \end{bmatrix}$

$C = [11 \quad 01 \quad 00 \quad 01 \quad 01 \quad 01 \quad 00 \quad 11]$

Transfer-Domain Approach.

Given : $D = 10111$

$D(x) = 1 + \cancel{D}^2 + \cancel{D}^3 + \cancel{D}^4 \quad \Rightarrow 1 + x^2 + x^3 + x^4$

we have $g^{(1)}(x) = 1011 \Rightarrow 1 + x^2 + x^3$

$g^{(2)}(x) = 1111 \Rightarrow 1 + x + x^2 + x^3$

So for top adder: $c^{(1)}(x) = d(x)\, g^{(1)}(x)$

$= (1+x^2+x^3+x^4)(1+x^2+x^3)$

$= 1 + \cancel{x^2} + \cancel{x^3} + \cancel{x^2} + \cancel{x^4} + \cancel{x^5} + \cancel{x^3} + \cancel{x^5} + \cancel{x^6} + \cancel{x^4} + \cancel{x^6} + x^7$

$\boxed{c^{(1)}(x) = 1 + x^7}$

for bottom adder: $c^{(2)}x = d(x)\, g^{(2)}(x)$

$= 1+x^2+x^3+x^4\,(1+x+x^2+x^3)$

$= 1 + x + \cancel{x^2} + \cancel{x^3} + \cancel{x^2} + x^3 + \cancel{x^4} + x^5 + x^3 + \cancel{x^4} + x^5 + \cancel{x^6} + x^4 + x^5 + \cancel{x^6} + x^7$

$= 1 + x + x^3 + x^4 + x^5 + x^7$

$\therefore \boxed{c^{(2)}(x) = 1 + x + x^3 + x^4 + x^5 + x^7}$

∴ the final encoder output polynomial is given by

$$C(x) = C^{(1)}(x^n) + xC^{(2)}(x^n) + x^2 C^{(3)}(x^n) + \ldots + x^{n-1}C^{(n)}(x^n) \quad \text{(A)}$$

$$\therefore \quad C(x) = C^{(1)}(x^2) + xC^{(2)}(x^2)$$

we have $C^{(1)}(x) = 1 + x^7$ & $C^{(2)}(x) = 1 + x + x^3 + x^4 + x^5 + x^7$

$C^{(1)}(x^2) = 1 + x^{14}$ & $C^{(2)}(x^2) = 1 + x^2 + x^6 + x^8 + x^{10} + x^{14}$

∴ applying this in (A) we get

$$C(x) = 1 + x^{14} + x(1 + x^2 + x^6 + x^8 + x^{10} + x^{14})$$

$$= 1 + x + x^3 + x^7 + x^9 + x^{11} + x^{14} + x^{15}$$

$$C(x) = [11, 01, 00, 01, 01, 01, 00, 11]$$

which is same as obtained using time-domain approach

(2) Consider the (3, 1, 2) convolution code with $g^{(1)} = (110)$, $g^{(2)} = (101)$ and $g^{(3)} = (111)$

(i) Draw the encoder block diagram.

(ii) Find the generator Matrix.

(iii) Find the code-word corresponding to the Information sequence (11101) using time domain & transform-domain approach.

→ The encoder has $n = 3$, $K = 1$, $m = 2$

i.e 3 modulo-2 adders

2 flip flops

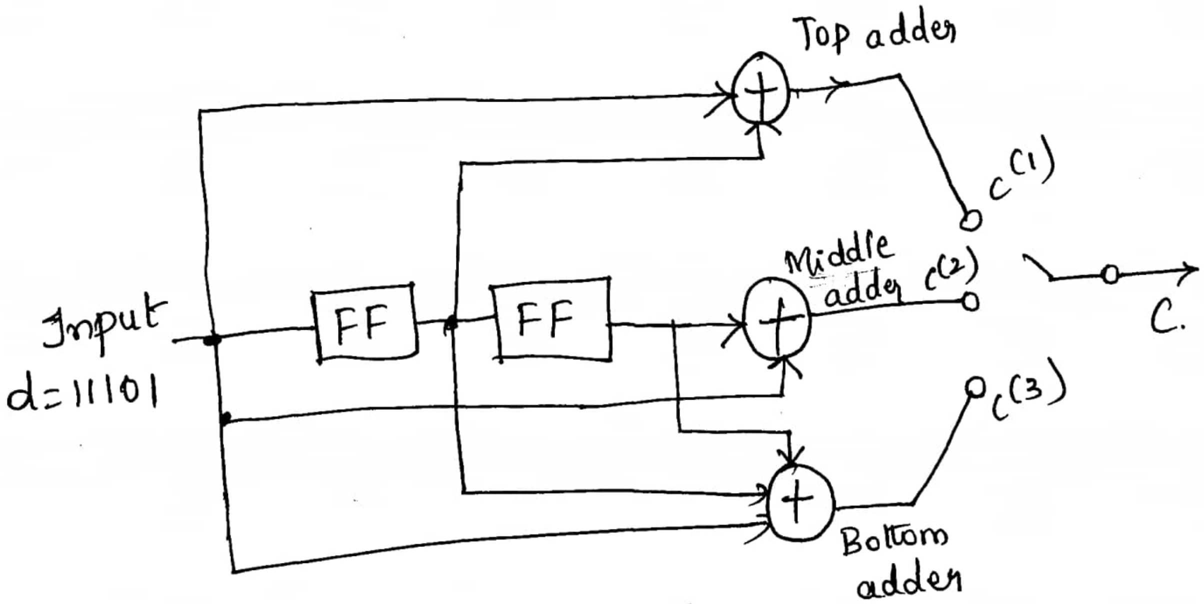
Time-Domain Method.

given :- $g^{(1)} = 110$

$g^{(2)} = 101$

$g^{(3)} = 111$

(3, 1, 2) convolution encoder.



G is of order $L \times [n(L+m)]$

$5 \times (3(5+2)) = 5$ rows $\times 21$ columns.

given $g^{(1)} = 110$

$g^{(2)} = 101$

$g^{(3)} = 111$

$$\therefore G = \begin{bmatrix} 111 & 101 & 011 & 000 & 000 & 000 & 000 \\ 000 & 111 & 101 & 011 & 000 & 000 & 000 \\ 000 & 000 & 111 & 101 & 011 & 000 & 000 \\ 000 & 000 & 000 & 111 & 101 & 011 & 000 \\ 000 & 000 & 000 & 000 & 111 & 101 & 011 \end{bmatrix}_{5\times 21}$$

$$\therefore [C] = [D]\,[G]$$

$$= [11101] \begin{bmatrix} 111 & 101 & 011 & 000 & 000 & 000 & 000 \\ 000 & 111 & 101 & 011 & 000 & 000 & 000 \\ 000 & 000 & 111 & 101 & 011 & 000 & 000 \\ 000 & 000 & 000 & 111 & 101 & 011 & 000 \\ 000 & 000 & 000 & 000 & 111 & 101 & 011 \end{bmatrix}_{5\times 21}$$

$$\therefore [c] = [111 \;\; 010 \;\; 001 \;\; 110 \;\; 100 \;\; 101 \;\; 011]$$

Transfer Domain Method.

Given : $[d] = [11101] \Rightarrow d(x) = 1 + x + x^2 + x^4$

$g^{(1)} = (110) \rightarrow g^{(1)}(x) = 1 + x$

$g^{(2)} = (101) \rightarrow g^{(2)}(x) = 1 + x^2$

$g^{(3)} = (111) \rightarrow g^{(3)}(x) = 1 + x + x^2$

$\therefore$ we have the equ^n

$$\boxed{C(x) = C^{(1)} x^{(n)} + x\, C^{(2)}(x^n) + x^2 C^{(3)}(x^n)}$$

Before that we find $[C] = [D]\ [G]$ or

For top adder $\Rightarrow C^{(1)}(x) = d(x)\ g^{(1)}(x)$

$= 1 + x + x^2 + x^4\ (1 + x)$

$C^{(1)}(x) = 1 + \cancel{x} + \cancel{x} + \cancel{x^2} + \cancel{x^2} + x^3 + x^4 + x^5$

$C^{(1)}(x) = 1 + x^3 + x^4 + x^5$.

For ~~bottom~~ middle adder $\Rightarrow C^{(2)}(x) = d(x)\ g^{(2)}(x)$

$= 1 + x + x^2 + x^4\ (1 + x^2)$

$= 1 + \cancel{x^2} + x + x^3 + \cancel{x^2} + \cancel{x^4} + \cancel{x^4} + x^6$

$C^{(2)}(x) = 1 + x + x^3 + x^6$

For bottom adder $\Rightarrow C^{(3)}(x) = d(x)\ g^{(3)}(x)$

$= 1 + x + x^2 + x^4\ (1 + x + x^2)$

$C^{(3)}(x) = 1 + \cancel{x} + \cancel{x^2} + \cancel{x} + \cancel{x^2} + \cancel{x^3} + x^2 + \cancel{x^3} + \cancel{x^4} + \cancel{x^4} + x^5 + x^6$

$C^{(3)}(x) = 1 + x^2 + x^5 + x^6$.

$\therefore$ For encoder equ^n

$C(x) = C^{(1)}(x^3) + x\, C^{(2)}(x^3) + x^2 C^{(3)}(x^3)$

$= 1 + x^9 + x^{12} + x^{15} + x\left(1 + x^3 + x^9 + x^{18}\right) + x^2\left(1 + x^6 + x^{15} + x^{18}\right)$

$= 1 + x^9 + x^{12} + x^{15} + x + x^4 + x^{10} + x^{19} + x^2 + x^8 + x^{17} + x^{20}$

$= 1 + x + x^2 + x^4 + x^8 + x^9 + x^{10} + x^{12} + x^{15} + x^{17} + x^{19} + x^{20}$

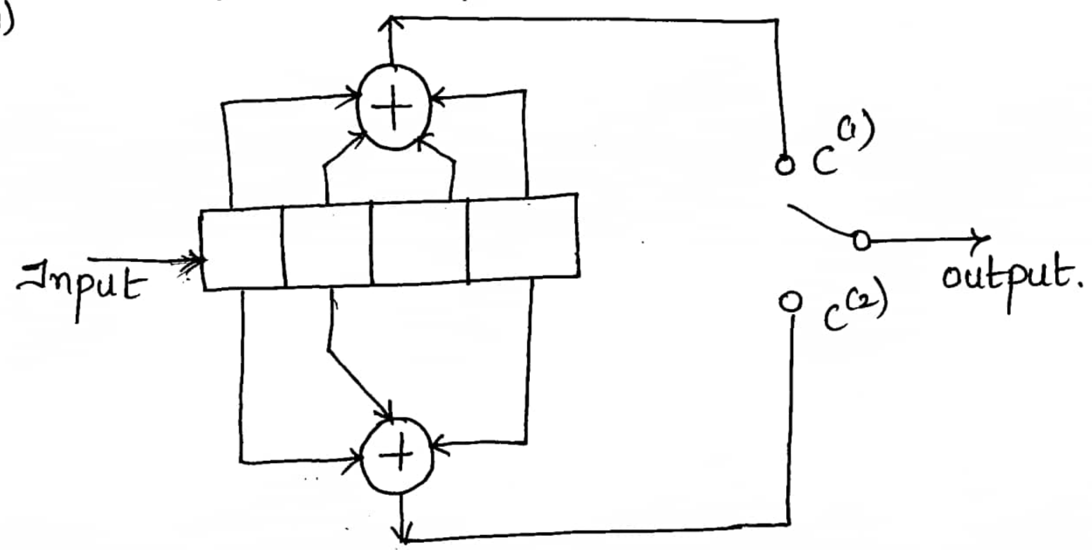
$C(x) \Rightarrow$ o/p of encoder is $[111\ 010\ 001\ 110\ 100\ 101\ 011]$

$\therefore$ Transfer Domain & Time Domain Codevectors are same.

3. For the Convolutional encoder given

(i) Find the impulse response and hence calculate the output produced by the information sequence 10111

(ii) write the generator polynomials of the encoder & recompute the output for the input of (i) and compare with that of (i)



→ Since there is no connection from the input to the 2 adders, the generator sequence (Impulse responses) are given by

$$g^{(1)} = [0\,1\,1\,1\,1] \text{ for top adder.}$$

$$g^{(2)} = [0\,1\,1\,0\,1] \text{ for bottom adder.}$$

the parameters are $L=5$, $n=2$, $K=1$, $m=4$

$$\therefore \text{ we have } [G] = L \times [n(L+m)]$$
$$= 5 \times [2(5+4)]$$
$$= 5 \times 18$$

∴ $[G]$ matrix can be constructed as

$$[G] = \begin{bmatrix} 00 & 11 & 11 & 10 & 11 & 00 & 00 & 00 & 00 \\ 00 & 00 & 11 & 11 & 10 & 11 & 00 & 00 & 00 \\ 00 & 00 & 00 & 00 & 11 & 10 & 11 & 00 & 00 \\ 00 & 00 & 00 & 00 & 11 & 11 & 10 & 11 & 00 \\ 00 & 00 & 00 & 00 & 00 & 11 & 11 & 10 & 11 \end{bmatrix}$$

∴ to find $[C] = [D]\,[G]$

$$[C] = [1\,0\,1\,1\,1]\begin{bmatrix} 00 & 11 & 11 & 10 & 11 & 00 & 00 & 00 & 00 \\ 00 & 00 & 11 & 11 & 10 & 11 & 00 & 00 & 00 \\ 00 & 00 & 00 & 11 & 11 & 10 & 11 & 00 & 00 \\ 00 & 00 & 00 & 00 & 11 & 11 & 10 & 11 & 00 \\ 00 & 00 & 00 & 00 & 00 & 11 & 11 & 10 & 11 \end{bmatrix}$$

$$[c] = [0\,0 \;\; 11 \;\; 11 \;\; 01 \;\; 11 \;\; 10 \;\; 10 \;\; 01 \;\; 11]$$

(ii) the message polynomial is given by

$$d(X) = 1 + X^2 + X^3 + X^4$$

$$g^{(1)}(x) = X + X^2 + X^3 + X^4 \quad \text{for top adder}$$

$$g^{(2)}(x) = X + X^2 + X^4 \quad \text{for bottom adder}$$

the o/p polynomial for the top adder is

$$c^{(1)}x = d(x)\, g^{(1)}(x)$$

$$= 1 + X^2 + X^3 + X^4 \left(X + X^2 + X^3 + X^4\right)$$

$$= X + X^2 + X^3 + X^4 + X^3 + X^4 + X^5 + X^6 + X^4 + X^5 + X^6 + X^7 + X^5 + X^6 + X^7 + X^8$$

$$= X + X^2 + X^4 + X^5 + X^6 + X^8$$

The o/p polynomial for the bottom adder is

$$c^{(2)}x = d(x)\, g^{(2)}(x)$$

$$= 1 + X^2 + X^3 + X^4\left(X + X^2 + X^4\right)$$

$$= X + X^2 + X^4 + X^3 + X^4 + X^6 + X^4 + X^5 + X^7 + X^5 + X^6 + X^8$$

$$= X + X^2 + X^3 + X^4 + X^7 + X^8$$

The o/p of the encoder is given by

$$C(X) = C^{(1)}(X^2) + X\,C^{(2)}(X^2)$$

$$= \left(X + X^2 + X^4 + X^5 + X^6 + X^8\right)^2 + X\left(X + X^2 + X^3 + X^4 + X^7 + X^8\right)^2$$

$$= X^2 + X^4 + X^8 + X^{10} + X^{12} + X^{16} + X\left(X^2 + X^4 + X^6 + X^8 + X^{14} + X^{16}\right)$$

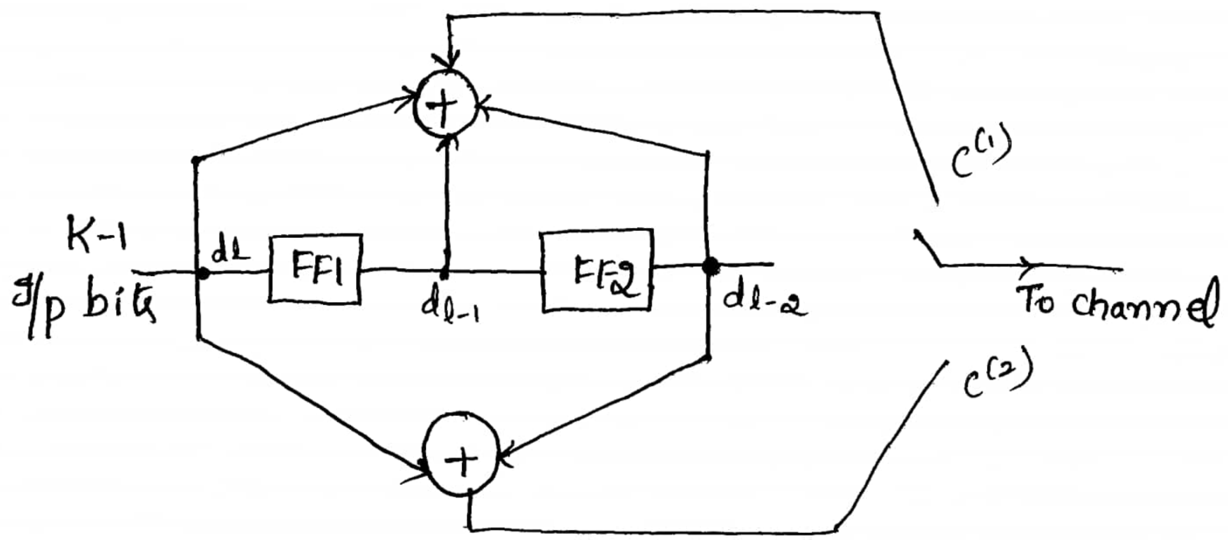
$$= X^2 + X^3 + X^4 + X^5 + X^7 + X^8 + X^9 + X^{10} + X^{12} + X^{15} + X^{16} + X^{17}$$

$$\therefore \quad [c] = [0\,0 \;\; 11 \;\; 11 \;\; 01 \;\; 11 \;\; 10 \;\; 10 \;\; 01 \;\; 11]$$

State table, State transition table, State diagram, Code tree.

1. Draw the state transition table, state diagram & code tree for (2,1,2) convolution encoder with impulse response $g^{(1)}=111$, $g^{(2)}=101$ & input $D=[10111]$

Sol^n



① State table

There are 2 flip flops in the shift register i.e 2
∴ there are $2^2=4$ States with binary data, that is represented in the table below

State	S_0	S_1	S_2	S_3
Binary Description	00	10	01	11

② State Transition Table.

It is a table indicating the transition between the States and also the corresponding code bits output during each transition.

The output of the mod-2 adders can be represented as

$$c^{(1)} = d_l + d_{l-1} + d_{l-2}$$
$$c^{(2)} = d_l + d_{l-2}$$

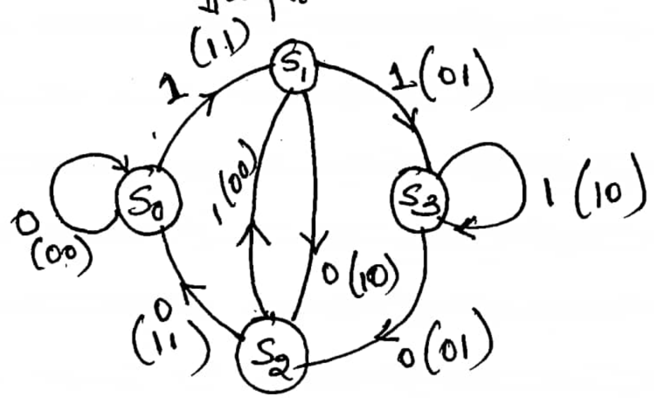
present

Present State	Input bit	Next State	d_l	d_{l-1}	d_{l-2}	Output $c^{(1)}$	$c^{(2)}$
$S_0 = 00$	0	$S_0 = 00$	0	0	0	0	0
	1	$S_1 = 10$	1	0	0	1	1
$S_1 = 10$	0	$S_2 = 01$	0	1	0	1	0
	1	$S_3 = 11$	1	1	0	0	1
$S_2 = 01$	0	$S_0 = 00$	0	0	1	1	1
	1	$S_1 = 10$	1	0	1	0	0
$S_3 = 11$	0	$S_2 = 01$	0	1	1	0	1
	1	$S_3 = 11$	1	1	1	1	0

Construction : Initially, let the flip flops be cleared i.e '00' → S_0

- If i/p = 0 ⇒ shift register at '00' → S_0, ∴ o/p of adders $c^{(1)} c^{(2)} = 00$
- If i/p = 1 the F.flop FF_1 changes state from 0 to 1 & the contents of FF_1 (which was '0') get shifted to FF_2. ∴ The new shift register contents are '10' → S_1. This transition is caused by a i/p = 1 & $d_l d_{l-1} d_{l-2} = 100$ & $c^{(1)} c^{(2)} = 11$.

③ State diagram.



(4) Code tree.

The state diagram can be re-drawn as a code tree.

Rules

(1) Start with state S_0

(2) If input=0, move up the tree.
If input=1, move down the tree.
the vertical line '$\longrightarrow$' moves to node
the horizontal line '$\downarrow$' to Branch.

(3) For every transition, corresponding to every bit, the states are represented as vertical lines (nodes) & the corresponding ~~to every~~ bit are written on horizontal lines (Branches)

So Given, [D] = 10111

$$S_0 \xrightarrow[\downarrow\ 11]{1} S_1 \xrightarrow[\downarrow\ 10]{0} S_2 \xrightarrow[\downarrow\ 00]{1} S_1 \xrightarrow[\downarrow\ 01]{1} S_3 \xrightarrow[\downarrow\ 10]{1} S_3 \xrightarrow[\downarrow\ 01]{0} S_2 \xrightarrow[\downarrow\ 11]{0} S_0$$

$\because$ $n(L+m) = 2(5+2)$
$= 14$ bits

we will assume 2 additional zero input bits, in order to get the complete code, so as to clear the shift register.

$\therefore [c] = [11\ 10\ 00\ 01\ 10\ 01\ 11]$

If we Decode it,

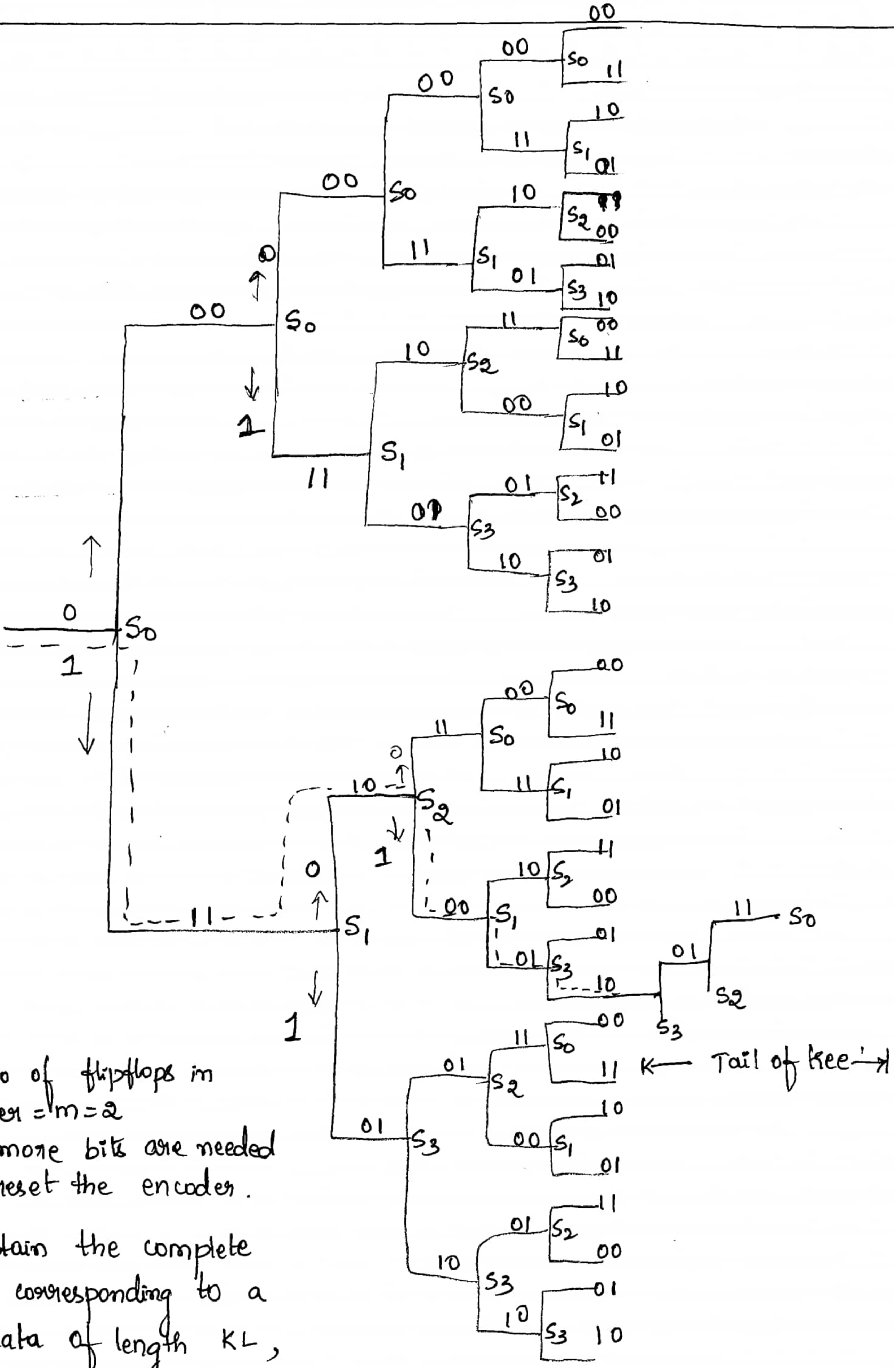
11 10 00 01 10 01 11

$$S_0 \xrightarrow[\downarrow\ 1]{11} S_1 \xrightarrow[\downarrow\ 0]{10} S_2 \xrightarrow[\downarrow\ 1]{00} S_1 \xrightarrow[\downarrow\ 1]{01} S_3 \xrightarrow[\downarrow\ 1]{10} S_3 \xrightarrow[\downarrow\ 0]{01} S_2 \xrightarrow[\downarrow\ 0]{11} S_0$$

discard last 'm' bits

Decoded bit is : [10111]

Decoding operation is done by Pairing the encoded convolution output into groups of $n=2$ bits.



- The no of flipflops in encoder = m = 2
  ∴ 2 more bits are needed to reset the encoder.

To obtain the complete code corresponding to a i/p data of length KL, the tree graph has to be extended by $n(m-1)$ time units & this extended part is called 'tail of the tree'