

MODULE - 3

DIGITAL MODULATION TECHNIQUES

- The process of varying one or more properties of a carrier wave with respect to message signal / modulating signal that typically contains information to be transmitted is known as Modulation.
- In Digital modulation, message signal will be Binary data which is to be carried over a analog signal for the transmission over the channel.
- In communication, Data transmission is of 2 types.

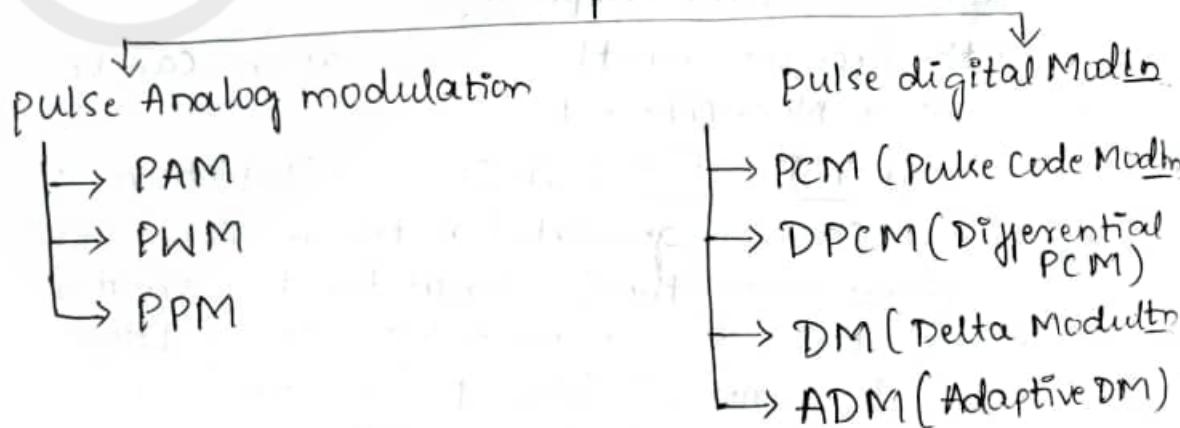
Data transmission



1. Baseband Transmission

- The digital Data is transmitted over the channel directly without any modulation.
- Suitable for short range Communication.
- A signal may be sent in its baseband format only when a dedicated wired channel is available otherwise it must be converted to passband transmission

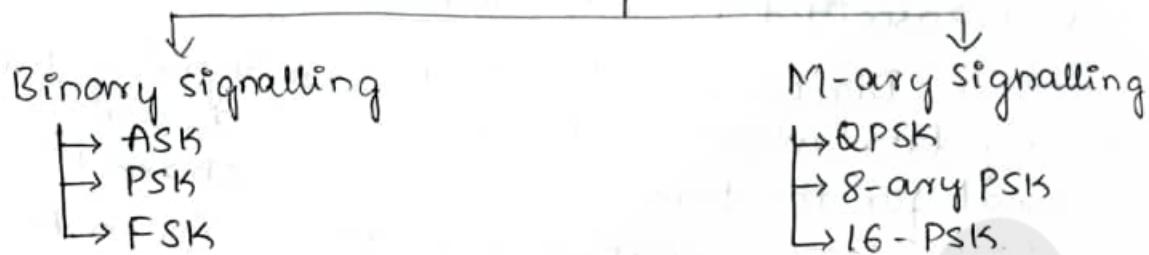
Base band Transmission



2. Passband Transmission

- The digital data modulates high frequency sinusoidal carrier for transmission over a channel.
- Suitable for long distance communication.

Passband Transmission



The Basic Modulation techniques for the transmission of digital data are

1. Amplitude shift Keying (ASK)
2. Phase shift Keying (PSK)
3. Frequency shift Keying (FSK).

→ The modulation involves switching / keying the amplitude, phase & frequency of carrier in accordance with digital data (binary message signal).

→ The Digital modulation techniques aim to achieve following parameter

- * Minimum Channel Bandwidth.
- * High Data rate (max. bits/symbol)
- * minimum probability of error.
- * Maximum resistance to interfering signals
- * Minimum circuit complexity.

→ At the Receiver end, the Demodulation can be coherent or Non-coherent.

→ Coherent Detection : In Coherent Detection, the local carrier generated at the receiver should be in-phase with the carrier at the transmitter. Hence it is also called as Synchronous Detection.

- * In this method, Error probability decrease

→ Coherent detection is performed as follows;

- i) Correlation of received signal with carrier
- ii) Decision making based on threshold value.

- In Non-Coherent Detection, the carrier at the receiver need not to be in-phase with the carrier at the transmitter.
- This method is simple, but has higher probability of symbol error.

	message Signal / modulating signal $m(t)$	Carrier Signal $c(t)$	Modulated Signal $s(t)$
Digital Communication	Digital or Binary Data	Analog signal	Analog signal
Analog Communication	Analog signal	Analog signal	Analog signal

Coherent Binary Modulation Techniques

- The 3 basic forms of Binary modulation techniques are : ASK, FSK, PSK.
- The noise analysis of coherent detection of ASK, PSK, FSK is briefly explained by assuming Additive White Gaussian Noise (AWGN) model.
- Signal Constellation is a set of possible message points ($M = 2^n$)
- Constellation Diagram represents a signal as a 2D scattered diagram on a complex plane at the sampling instants. It helps to recognize type of interference in a signal.

Phase Shift Keying Techniques Using Coherent Detection

1. Binary Phase Shift Keying (BPSK)

- In BPSK, the binary symbols '0' & '1' are used to modulate the phase of carrier.
- It transmits one bit / symbol (ie. $n=1 \therefore M=2^n=2$)
- Hence message symbols are denoted by $s_1(t)$ & $s_2(t)$.

→ The pair of signals $s_1(t)$ and $s_2(t)$ are used to represent binary symbols '1' and '0' respectively.

$$s_1(t) = A \cos(2\pi f_c t)$$

$$s_2(t) = A \cos(2\pi f_c t + \pi) = -A \cos(2\pi f_c t)$$

→ When the symbol changes from '1' to '0' the phase of the carrier is shifted by $180^\circ (\pi)$.

→ Let E_b be the energy of the symbol, then T_b -bit duration

$$E_b = \int_0^{T_b} A^2 \cos^2(2\pi f_c t) dt$$

$$= A^2 \int_0^{T_b} \frac{1 + \cos 4\pi f_c t}{2} dt = A^2 \left[\frac{t}{2} + \frac{\sin 4\pi f_c t}{8\pi f_c} \right]_0^{T_b}$$

$$= A^2 \left[\frac{T_b}{2} + \frac{\sin 4\pi f_c T_b}{8\pi f_c} \right]$$

$$E_b = \frac{A^2 T_b}{2}$$

$$A^2 = \frac{2E_b}{T_b}$$

$$A = \sqrt{\frac{2E_b}{T_b}}$$

Substituting value of 'A' in $s_1(t)$ & $s_2(t)$ Equation we get,

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_c t ; 0 \leq t \leq T_b$$

$$s_2(t) = -\sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_c t ; 0 \leq t \leq T_b$$

Where E_b - transmitted energy per bit
 T_b - bit duration

Signal Space Diagram of BPSK.

→ In BPSK, the message points '1' & '0' are represented by $s_1(t)$ & $s_2(t)$ as

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_c t \quad (1) ; 0 \leq t \leq T_b$$

$$s_2(t) = -\sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_c t \quad (2)$$

where T_b - bit duration & E_b → Transmitted signal energy / bit.

→ when the symbol changes from '1' to '0' the phase of the carrier is shifted by 180° (π)

→ From Eq(1) & (2), the pair of sinusoidal wave that differ only in a relative phase shift of 180° are known as 'Anti-podal signals'

→ From this equations, it is clear that in BPSK, there is only one basis function of unit energy defined by,

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos 2\pi f_c t ; 0 \leq t \leq T_b . \quad \text{--- (3)}$$

Now, expressing $s_1(t)$ & $s_2(t)$ in terms of $\phi_1(t)$, we get

$$s_1(t) = \sqrt{E_b} \phi_1(t) ; 0 \leq t \leq T_b \quad \text{--- (4)}$$

$$s_2(t) = -\sqrt{E_b} \phi_1(t) ; 0 \leq t \leq T_b \quad \text{--- (5)}$$

$$\begin{aligned} \text{WKT, } \phi_1(t) &= \frac{s_1(t)}{\sqrt{E}} \\ &= \frac{\sqrt{2E_b}}{T_b} \cos 2\pi f_c t \\ &= \sqrt{\frac{2}{T_b}} \cos 2\pi f_c t \end{aligned}$$

∴ A binary PSK System is, therefore Characterized by having a signal space that is one-dimensional ($N=1$) with a Signal consisting Constellation of message points ($M=2$)

The co-ordinates to represent these 2 message points are given by,

$$S_{11} = \int_0^{T_b} s_1(t) \cdot \phi_1(t) \cdot dt = +\sqrt{E_b} . \quad \text{--- (6)}$$

$$S_{21} = \int_0^{T_b} s_2(t) \cdot \phi_1(t) \cdot dt = -\sqrt{E_b} . \quad \text{--- (7)}$$

The message point corresponding to $s_1(t)$ is located at $S_{11} = +\sqrt{E_b}$ & the message point corresponding to $s_2(t)$ is located at $S_{21} = -\sqrt{E_b}$

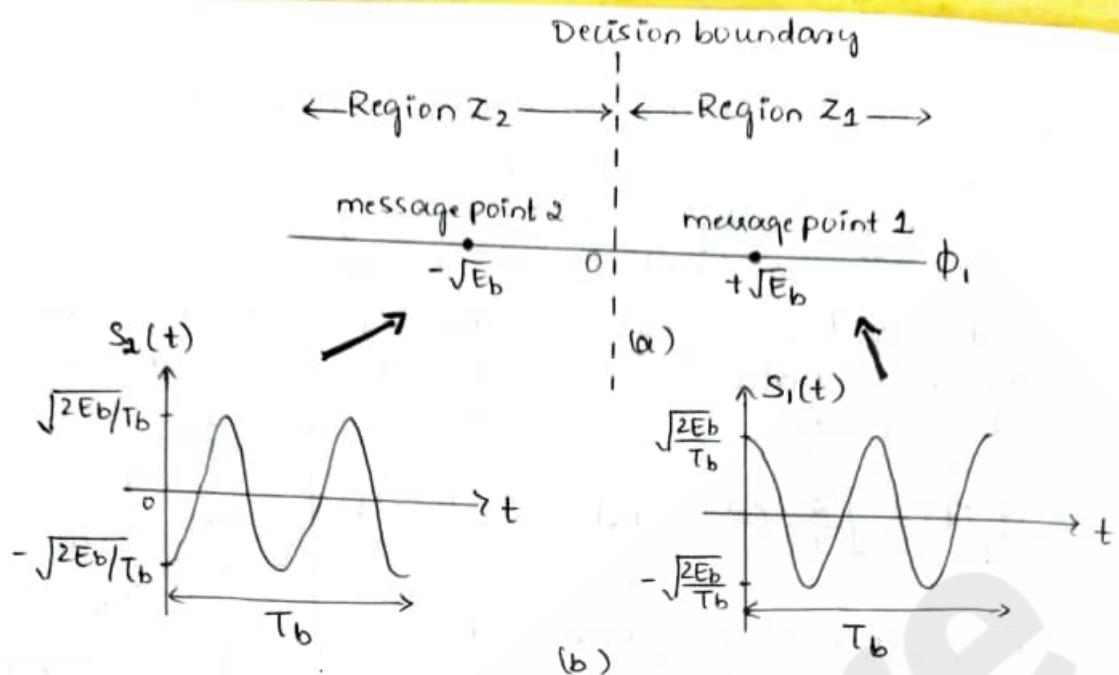


Fig: a) Signal Space Diagram for Coherent BPSK
b) Waveforms depicting message points m_1 , i.e., $s_1(t)$ & m_2 i.e. $s_2(t)$

BPSK Generation & Coherent Detection

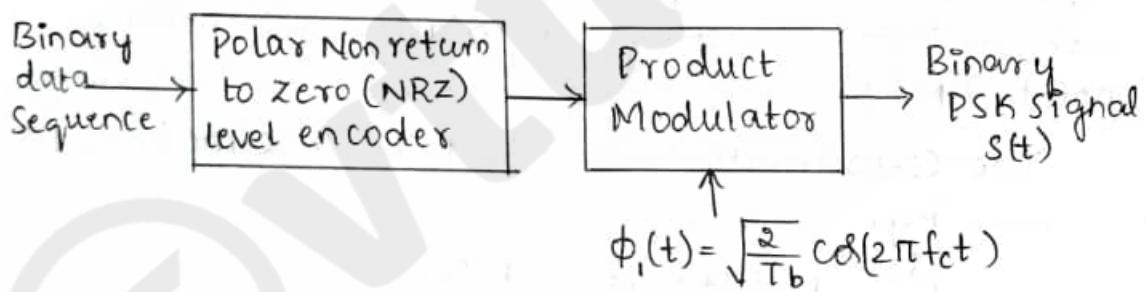


fig: BPSK Transmitter

BPSK Generation Consists of Two Components as follows:

1. Polar NRZ level encoder: It represents Symbol 1 and Symbol 0 of the input binary sequence by Amplitude levels $+\sqrt{E_b}$ & $-\sqrt{E_b}$ respectively. The output is fed to product modulator.
2. Product Modulator: It multiplies the output of polar NRZ encoder by the basic function $\phi_1(t)$. i.e., $\phi_1(t)$ acts as a sinusoidal carrier of Binary PSK signal.

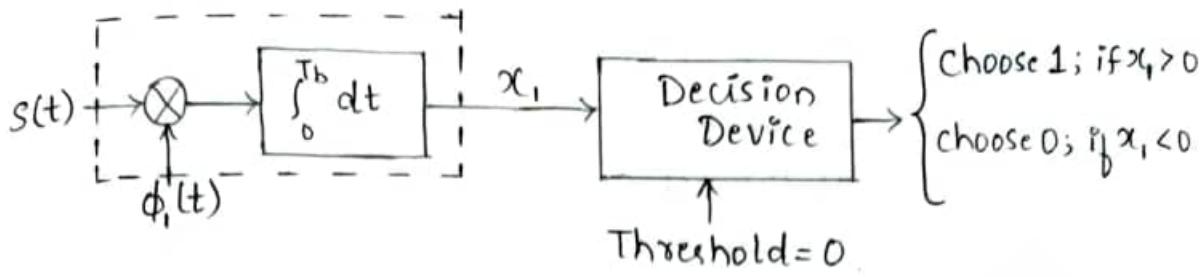


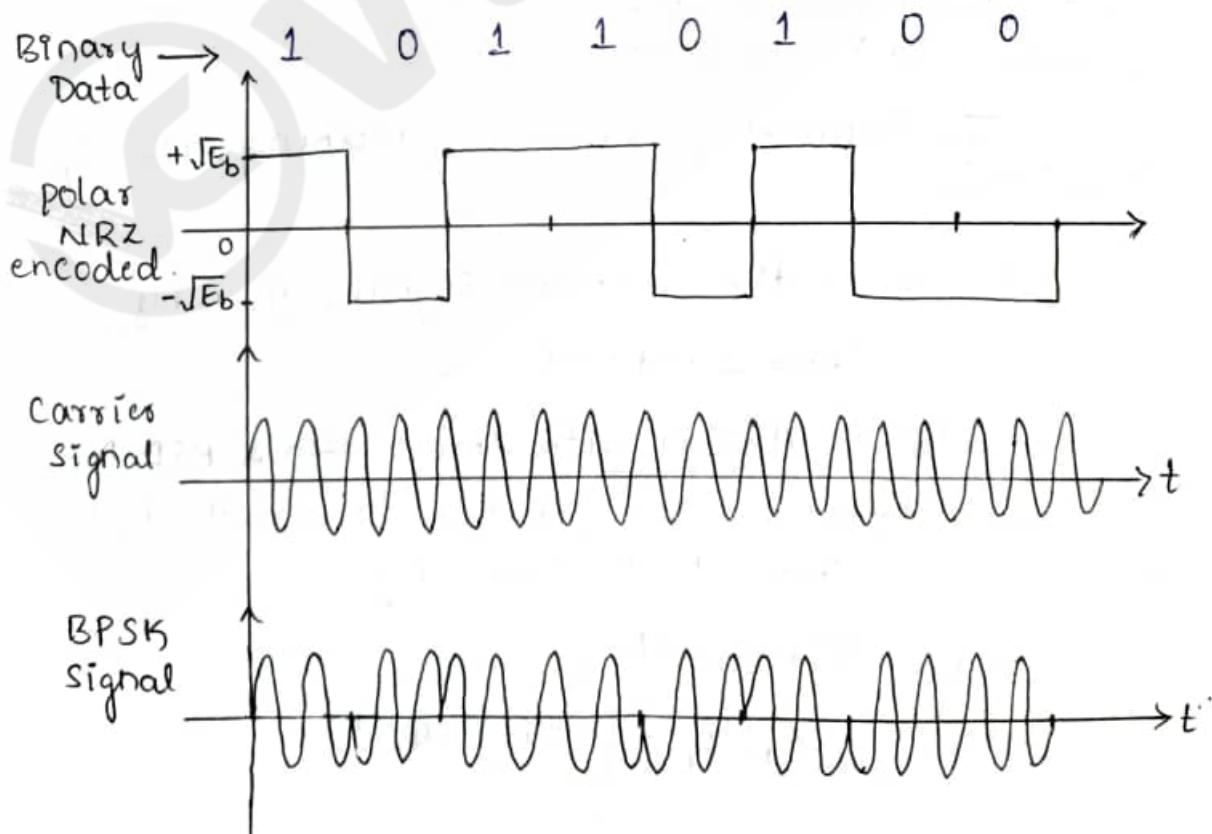
fig: Coherent BPSK receiver

→ The receiver is synchronized with transmitter as shown in figure.

→ Two Basic Components of BPSK receiver are as follows :

1. Correlator: It correlates the received signal $S(t)$ with the basis function $\phi_i(t)$ on a bit by bit basis
2. Decision Device: This compares the correlator output against zero-threshold, assuming the binary symbols 1 and 0 are equiprobable.
 - * if $x_i > 0$, then decision is made in favour of symbol '1'
 - * if $x_i < 0$, then decision is made in favour of Symbol '0'

BPSK Waveforms



Error Probability of BPSK Using Coherent Detection

→ Assuming symbol '1' & '0' transmitted with equal probability. If the set of points reside close to S_1 , then it corresponds to symbol '1' transmission & if the set of points residing close to S_2 , corresponds to symbol '0' transmission. These two decision regions are marked Z_1 & Z_2 , according to the message point around which they constructed. as shown in figure 1.

→ The Decision rule is now simply to decide that signal $S_1(t)$ is transmitted if the received signal point falls in Z_1 region & to decide that signal $S_2(t)$ was transmitted if received signal point falls in region Z_2 .

→ Two kinds of erroneous decision may, however be made :

1. Error of first kind : signal $S_2(t)$ [ie, symbol '0'] is transmitted but the noise is such that the received signal point falls in the region Z_1 ; so receiver decides in favour of signal $S_1(t)$.

2. Error of second kind : signal $S_1(t)$ [ie, symbol '1'] is transmitted but the noise is such that the received signal point falls in the region Z_2 ; so receiver decides in favour of signal $S_2(t)$.

To calculate probability of making error of first kind,

Let $x(t)$ be the received signal, given by

$$x(t) = s(t) + w(t) \quad \text{--- (1)}$$

where $w(t)$ is AWGN with zero mean & PSD $N_0/2$. Assuming symbol '0' ie, $s_2(t)$ was transmitted, then the output of correlator be given by

$$\underline{x} = \frac{s_2(t) + w(t)}{\tau_b}$$

$$\underline{x}_1 = \int_0^{\tau_b} x(t) \cdot \phi_1(t) dt$$

w.k.t., $x(t) = \begin{cases} S_1(t) + w(t) & ; \text{symbol 1 transmitted} \\ S_2(t) + w(t) & ; \text{symbol '0' transmitted} \end{cases}$

$$\therefore x_1 = \int_0^{T_b} (S_2(t) + w(t)) \cdot \phi_1(t) \cdot dt.$$

$$x_1 = \int_0^{T_b} S_2(t) \cdot \phi_1(t) \cdot dt + \int_0^{T_b} w(t) \phi_1(t) dt$$

$$= \int_0^{T_b} -\sqrt{E_b} \phi_1(t) \phi_1(t) dt + w_1$$

$$x_1 = -\sqrt{E_b} + w_1$$

→ Mean of the random variable x_1 is

$$\begin{aligned} E[x_1] &= E[-\sqrt{E_b} + w_1] \\ &= E[-\sqrt{E_b}] + E[w_1] \end{aligned}$$

$$\boxed{\mu = E[x_1] = -\sqrt{E_b}}$$

Note :

1. From Eq(5),

$$S_2(t) = -\sqrt{E_b} \phi_1(t)$$

2. $w_1 = \int_0^{T_b} w(t) \phi_1(t) dt$

3. $E[\text{constant}]$ is a constant

4. Variance [constn] is equal to zero

5. $\int_0^{T_b} \phi_i(t) \phi_j(t) dt = 1; i=j$

→ Variance of x_1 is given by

$$\begin{aligned} \text{var}[x_1] &= \text{var}[-\sqrt{E_b} + w_1] \\ &= \text{var}[-\sqrt{E_b}] + \text{var}[w_1] \\ &= 0 + N_0/2 \end{aligned}$$

$$\boxed{\sigma^2 = \text{var}[x_1] = N_0/2}$$

→ Conditional Probability Density Function When Symbol '0' is transmitted is given by.

$$f_{x_1}(x_1|0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_1-\mu)^2}{2\sigma^2}} \quad (2)$$

Substituting the values of μ & σ^2 we get

$$f_{x_1}(x_1|0) = \frac{1}{\sqrt{2\pi \times \frac{N_0}{2}}} e^{-\frac{(x_1+\sqrt{E_b})^2}{2 \times N_0/2}}$$

$$f_{x_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}} e^{-\left(\frac{x_1 + \sqrt{E_b}}{\sqrt{N_0}}\right)^2} \quad \therefore (\sqrt{N_0})^2 = N_0$$

—③

* Let $P_e(0)$ denotes the probability of deciding in favour of symbol '1' when symbol '0' was transmitted

$$P_e(0) = P[x_1 > 0 \mid \text{symbol '0' is transmitted}]$$

Region Z_1 (symbol '1'): $0 \leq x_1 \leq \infty$

$$\therefore P_e(0) = \int_0^\infty f_{x_1}(x_1|0) dx_1$$

$$P_e(0) = \frac{1}{\sqrt{\pi N_0}} \int_{x_1=0}^\infty e^{-\left(\frac{x_1 + \sqrt{E_b}}{\sqrt{N_0}}\right)^2} dx_1$$

$$\text{Put } Z = \frac{x_1 + \sqrt{E_b}}{\sqrt{N_0}}$$

diff wrt. 'Z' we get

$$dz = \frac{dx_1}{\sqrt{N_0}} + 0$$

$$dx_1 = \sqrt{N_0} \cdot dz$$

Changing limits to Z
we get,

$$\text{when } x_1 = 0 \quad Z = \sqrt{E_b/N_0}$$

$$\text{when } x_1 = \infty \quad Z = \infty$$

Hence, Substituting value of 'Z' in above equation,

$$P_e(0) = \int_{\frac{\sqrt{E_b}}{\sqrt{N_0}}}^\infty \frac{1}{\sqrt{\pi N_0}} e^{-z^2} \cdot \sqrt{N_0} dz$$

$$P_e(0) = \frac{1}{\sqrt{\pi}} \int_{\frac{\sqrt{E_b}}{\sqrt{N_0}}}^\infty e^{-z^2} dz = \frac{1}{\sqrt{\pi}} \int_z^\infty e^{-z^2} dz \quad —④$$

Note: Definition of complementary error function

$$\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty e^{-u^2} du$$

$$\boxed{\frac{1}{2} \operatorname{erfc}(u) = \frac{1}{\sqrt{\pi}} \int_u^\infty e^{-u^2} du} \quad —⑤$$

using equation(5) & rewriting eq(4) we get

$$P_e(0) = \frac{1}{2} \operatorname{erfc}(z)$$

where, $Z = \sqrt{\frac{E_b}{N_0}}$

$$\therefore P_e(0) = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right).$$

Why we can calculate Probability of error for error of second kind, ie, symbol '1' transmitted but receiver decides in favour of symbol '0'

i.e., $P_e(1) = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$

Let probability of transmitting symbol '1' & symbol '0' be equiprobable,

$$\text{i.e., } P(0) \& P(1) = 1/2$$

\therefore The average probability of error is given by

$$P_e = P(0) \cdot P_e(0) + P(1) \cdot P_e(1)$$

$$= \frac{1}{2} \left[\frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) \right] + \frac{1}{2} \left[\frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) \right]$$

$$P_e = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) \quad \text{--- (6)}$$

Note: relation b/w $\operatorname{erfc}(x)$ & $Q(x)$ is $Q(x) = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right)$ --- (7)

\therefore Probability of error of BPSK is given by

using eq(7) in eq(6) & rewriting eq(6)

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$\begin{aligned} \therefore P_e &= \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}} / \sqrt{2}\right) \\ &= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \end{aligned}$$

\therefore The average Probability of Symbol error for BPSK is given by,

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

2. Quadrature Phase Shift Keying (QPSK)

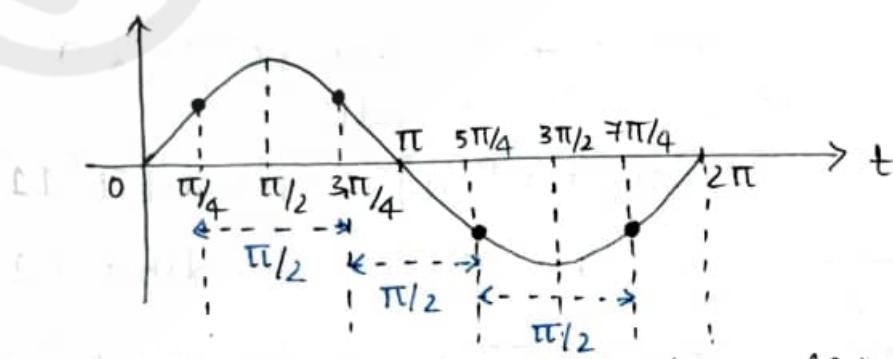
- The main aim of Digital Communication System is to have Reliable performance, low error probability and efficient utilization of channel Bandwidth.
- In BPSK, we transmit 1 bit / symbol (ie, $n=1, M=2^n=2$)
But in QPSK, we transmit 2 bits / symbol which helps in conserving Bandwidth. Hence QPSK is also known as Bandwidth-Conserving modulation Scheme.
- So, these 2 bits can have 4 possible Combinations ie, [no. of bits, $n=2, M=2^n=4$] of message symbols. They are given by 11, 01, 00, 10
- To Transmit 2 bits / symbol we need 4 phases (Quadri phase shift)
 \therefore Dividing 360° by 4 symbols, ie, $\frac{360^\circ}{4} = 90^\circ = \pi/2$
Hence we have a separation of 90° phase angle b/w the carrier phases.
- In QPSK, the phase of the carrier takes one of four equally spaced valued phases. Such as

$$\pi/4 = 45^\circ$$

$$\pi/4 + 90^\circ = 135^\circ = 3\pi/4$$

$$3\pi/4 + 90^\circ = 225^\circ = 5\pi/4$$

$$5\pi/4 + 90^\circ = 315^\circ = 7\pi/4$$



phase diff of 90° b/w the 4 carrier phases
ie, $\pi/4, 3\pi/4, 5\pi/4$ & $7\pi/4$

Signal Space Representation of QPSK

- As with Binary PSK, information about the message symbols in QPSK is contained in the carrier phase. In particular, the phase of the carrier takes on one of four equally spaced values, such as $\pi/4$, $3\pi/4$, $5\pi/4$ & $7\pi/4$.
- For this set of values, we may define the transmitted signal as

$$S_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos[2\pi f_c t + (2i-1)\pi/4] & ; 0 \leq t \leq T \\ 0 & ; i=1, 2, 3, 4 \\ 0 & ; \text{elsewhere} \end{cases}$$

—①

Where E is the transmitted signal energy per bit symbol & T is the symbol duration.

- we may expand eq(1) using trigonometric relation $\cos(A+B)$ to redefine the transmitted signal as.

$$S_i(t) = \sqrt{\frac{2E}{T}} \cos[(2i-1)\frac{\pi}{4}] \cos(2\pi f_c t) - \sqrt{\frac{2E}{T}} \sin[(2i-1)\frac{\pi}{4}] \sin(2\pi f_c t)$$

—②

$$\left[\because \cos(A+B) = \cos A \cos B - \sin A \sin B \right.$$

$$\left. A = \theta_i = (2i-1)\pi/4 \quad \& \quad B = 2\pi f_c t \right].$$

where $i=1, 2, 3, 4$. Since $M=4$ ie, 4 message points in QPSK.

- Now we can define 4 message points & associated signal vector using eq(1) as follows.

$$S_1(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + \pi/4) ; \text{ for dabit '11'}$$

$$S_2(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + 3\pi/4) ; \text{ for dabit '01'}$$

$$S_3(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + 5\pi/4) ; \text{ for dabit '00'}$$

$$S_4(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + 7\pi/4) ; \text{ for dabit '10'}$$

- From Eq(2), we make two observations

1. There are two orthonormal basis functions, defined by a pair of Quadrature Carriers;

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) ; 0 \leq t \leq T$$

—③

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t) ; 0 \leq t \leq T$$

—④

28. There are four message points, defined by 2-D signal vector

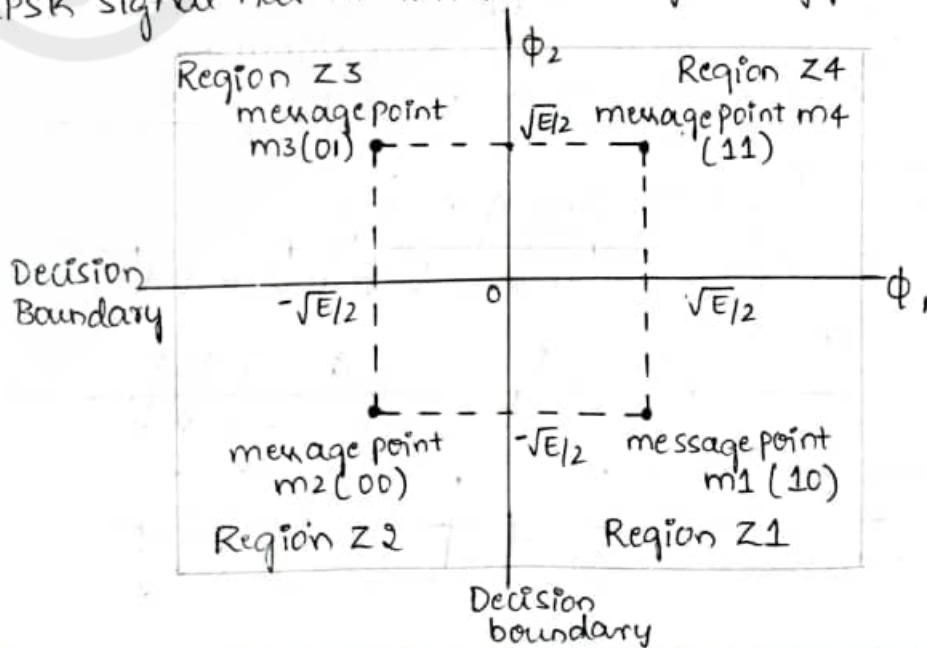
$$S_i = \begin{bmatrix} \sqrt{E} \cos[(2i-1)\pi/4] \\ -\sqrt{E} \sin[(2i-1)\pi/4] \end{bmatrix} ; i=1,2,3,4$$

—⑤

→ Elements of the Signal vectors; S_{i1} and S_{i2} are given by the following table.

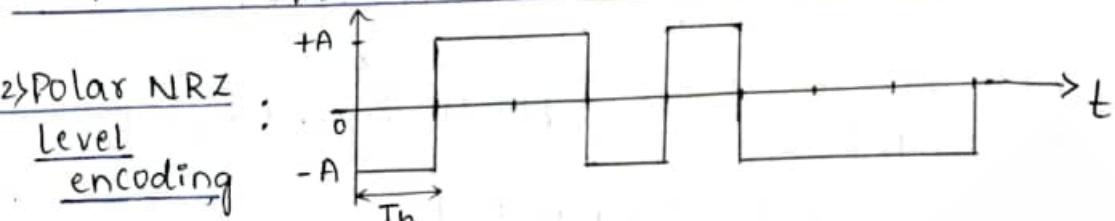
i	$S_i(t)$	Gray encoded input dabit	phase of QPSK signal (radians)	Co-ordinates of message points	
				S_{i1}	S_{i2}
1	$S_1(t)$	11	$\pi/4$	$+\sqrt{E}/2$	$+\sqrt{E}/2$
2	$S_2(t)$	01	$3\pi/4$	$-\sqrt{E}/2$	$+\sqrt{E}/2$
3	$S_3(t)$	00	$5\pi/4$	$-\sqrt{E}/2$	$-\sqrt{E}/2$
4	$S_4(t)$	10	$7\pi/4$	$+\sqrt{E}/2$	$-\sqrt{E}/2$

→ QPSK signal has a 2D signal Constellation (ie, $N=2$) & four message points (ie, $M=4$) whose phase angle increase in counterclockwise direction as shown in figure below.
QPSK signal has minimum Average energy.

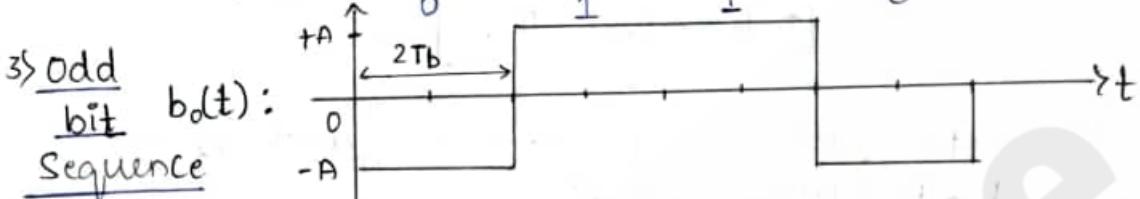


QPSK Waveforms

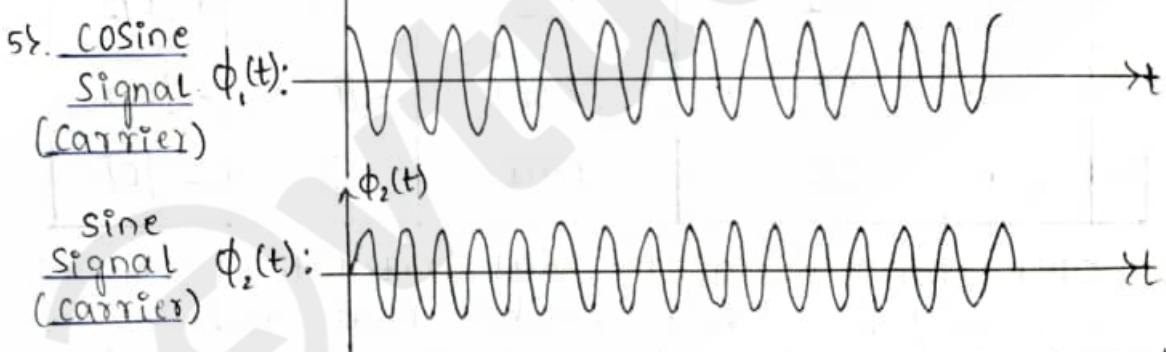
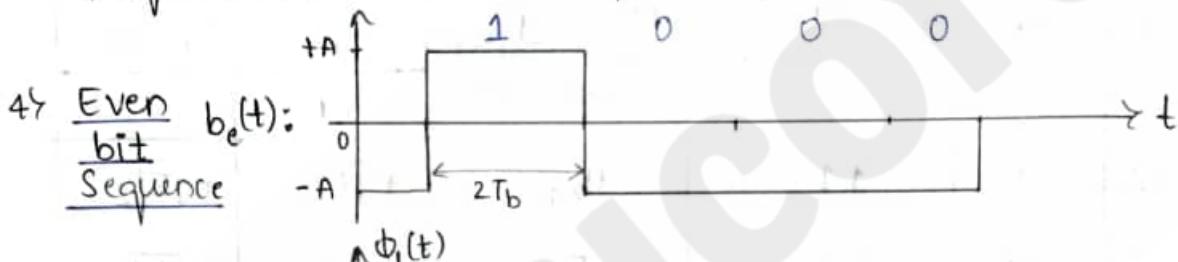
1) Input bit Seq: 0 1 1 0 1 0 0 0



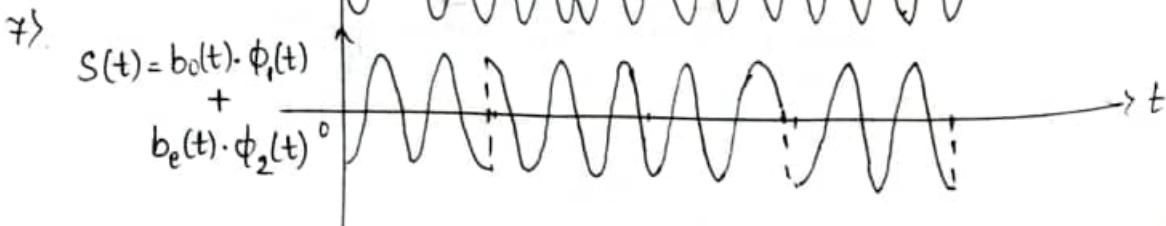
→ where T_b - One bit period



→ Define odd & even bit sequence for 2bit period.



→ Odd bit seq. Should be xplied by cos signal. To represent i/p '0' bit in odd seq, xply cosine signal by -1 & to represent i/p '1' bit in odd seq, xply cosine signal by +1.
likly for even bit should be xplied by sine signal.



Error Probability Of QPSK

- The signal points s_1, s_2, s_3 & s_4 are located symmetrically in 2D signal space.
- Computing the probability error for one message point remains the same for other three points.
- Consider transmission of symbol $s_4(t)$ then the received signal $x(t)$ is given by,

$$x(t) = s_4(t) + w(t) \quad ; \quad 0 \leq t \leq T \quad (1)$$

where $w(t)$ is the sample function of white Gaussian noise process with zero mean & $PSD = N_0/2$

- Referring to QPSK Receiver, the two correlator outputs x_1 and x_2 are defined as follows:

$$x_1 = \int_0^T x(t) \cdot \phi_1(t) dt$$

$$= \int_0^T [s_4(t) + w(t)] \phi_1(t) dt$$

$$= \int_0^T s_4(t) \phi_1(t) dt + \int_0^T w(t) \phi_1(t) dt$$

$$x_1 = s_{41} + w_1 \quad \left[\because s_{41} = s_{42} = \sqrt{E/2} \right]$$

$$x_1 = \pm \sqrt{E/2} + w_1$$

Similarly,

$$x_2 = \int_0^T x(t) \cdot \phi_2(t) dt$$

$$= \int_0^T [s_4(t) + w(t)] \phi_2(t) dt$$

$$= \int_0^T s_4(t) \cdot \phi_2(t) dt + \int_0^T w(t) \phi_2(t) dt$$

$$x_2 = s_{42} + w_2$$

$$x_2 = \sqrt{E/2} + w_2$$

→ The observable elements x_1 & x_2 are sample values of independent Gaussian random variables with mean value equal to $\pm \sqrt{E/2}$ & $\mp \sqrt{E/2}$ respectively, and variance = $No/2$.

→ The Decision rule is now simply to say that $s_4(t)$ was transmitted if the received signal point associated with observation vector x falls inside the region Z_4 .

If $x_1 > 0$ & $x_2 > 0$ this leads to correct decision
 \downarrow \downarrow
Symbol 1 Symbol 0.

→ An erroneous decision will be made if, signal $s_4(t)$ is transmitted but the noise $w(t)$ is such that received signal point falls outside region Z_4 .

→ The Conditional PDF is given by

$$f_{x_1}(x_1 | s_4(t)) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left[\frac{(x_1 - \mu)^2}{2\sigma^2}\right]} \quad \textcircled{1}$$

Substituting $\mu = \sqrt{E/2}$ & $\sigma^2 = No/2$ in above eqn.

$$f_{x_1}(x_1 | s_4(t)) = \frac{1}{\sqrt{2\pi \times \frac{No}{2}}} e^{-\left[\frac{(x_1 - \sqrt{E/2})^2}{2 \times No/2}\right]}$$

$$f_{x_1}(x_1 | s_4(t)) = \frac{1}{\sqrt{\pi N_0}} e^{-\left(\frac{(x_1 - \sqrt{E/2})^2}{N_0}\right)} \quad \textcircled{2}$$

Similarly

$$f_{x_2}(x_2 | s_4(t)) = \frac{1}{\sqrt{\pi N_0}} \cdot e^{-\left(\frac{(x_2 - \sqrt{E/2})^2}{N_0}\right)} \quad \textcircled{3}$$

Let us assume $s_4(t)$ is transmitted, if the received signal 'x' should fall inside region Z_4 . i.e., both x_1 & x_2 be the

→ The probability of correct decision P_c is equal to the product of conditional PDF of $x_1 > 0$ & $x_2 > 0$, both given that $s_4(t)$ was transmitted.

In region Z_4 ; $0 \leq x_1 \leq \infty$ & $0 \leq x_2 \leq \infty$.

$$\therefore P_c = \int_0^\infty f_{x_1}(x_1 | s_4(t)) dx_1 \cdot \int_0^\infty f_{x_2}(x_2 | s_4(t)) dx_2 \quad \text{--- (4)}$$

Substituting Eq (2) & (3) in Eq (4)

$$P_c = \int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\left(\frac{x_1 - \sqrt{E/2}}{\sqrt{N_0}}\right)^2} dx_1 \cdot \int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\left(\frac{x_2 - \sqrt{E/2}}{\sqrt{N_0}}\right)^2} dx_2$$

$$\text{Let } Z = \frac{x_1 - \sqrt{E/2}}{\sqrt{N_0}}$$

$$dz = \frac{dx_1}{\sqrt{N_0}}$$

$$dx_1 = \sqrt{N_0} \cdot dz$$

$$\text{when } x_1 = \infty, Z = \infty$$

$$x_1 = 0, Z = \frac{0 - \sqrt{E/2}}{\sqrt{N_0}}$$

$$Z = -\sqrt{\frac{E}{2N_0}}$$

$$\text{Let } Z = \frac{x_2 - \sqrt{E/2}}{\sqrt{N_0}}$$

$$dz = \frac{dx_2}{\sqrt{N_0}}$$

$$dx_2 = \sqrt{N_0} dz$$

$$\text{when } x_2 = \infty, Z_2 = \infty$$

$$x_2 = 0; Z = \frac{0 - \sqrt{E/2}}{\sqrt{N_0}}$$

$$Z = -\sqrt{\frac{E}{2N_0}}$$

Substituting value of Z & changing the order of integration in above Eqn, we get,

$$P_c = \int_{-z}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-z^2} \cdot \sqrt{N_0} dz \cdot \int_{-z}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-z^2} \cdot \sqrt{N_0} dz$$

$$= \int_{-z}^{\infty} \frac{1}{\sqrt{\pi}} e^{-z^2} dz \cdot \int_{-z}^{\infty} \frac{1}{\sqrt{\pi}} e^{-z^2} dz$$

$$\text{where, } z = -\sqrt{\frac{E}{2N_0}}$$

$$-z = \sqrt{\frac{E}{2N_0}}$$

$$P_c = \left[\frac{1}{\sqrt{\pi}} \int_{-\sqrt{\frac{E}{2N_0}}}^{\infty} e^{-z^2} dz \right]^2$$

--- (5)

From the definition of Complimentary error function,

$$\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} e^{-u^2} du$$

$$\frac{1}{2} \operatorname{erfc}(u) = \frac{1}{\sqrt{\pi}} \int_u^{\infty} e^{-u^2} du$$

we can write, $\frac{1}{\sqrt{\pi}} \int_{-\sqrt{\frac{E}{2N_0}}}^{\infty} e^{-z^2} dz = \left[1 - \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) \right]$

— (6)

Using eq(6) in eq(5)

$$P_c = \left[1 - \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) \right]^2$$

$$\begin{aligned} (a-b)^2 &= a^2 + b^2 - 2ab \\ a &= 1 \\ b &= \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) \end{aligned}$$

$$P_c = 1 + \frac{1}{4} \operatorname{erfc}^2\left(\sqrt{\frac{E}{2N_0}}\right) - \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)$$

— (7)

Thus the average probability of symbol error is given by,

$$P_e = 1 - P_c$$

$$P_e = 1 - \left[1 + \frac{1}{4} \operatorname{erfc}^2\left(\sqrt{\frac{E}{2N_0}}\right) - \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) \right]$$

$$P_e = \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) - \frac{1}{4} \operatorname{erfc}^2\left(\sqrt{\frac{E}{2N_0}}\right)$$

In region Z_4 ; $(E/2N_0) \gg 1$, hence we can ignore the second term in above eqn

$$\therefore P_e \approx \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)$$

In QPSK, 2 bits are transmitted per symbol, thus

$$E = 2E_b$$

Hence, $P_e \approx \operatorname{erfc}\left(\sqrt{\frac{2E_b}{2N_0}}\right) = \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$

— (8)

From the definition of Q-function,

$$Q(x) = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) \quad | \quad \therefore x = \sqrt{\frac{2E_b}{N_0}}$$
$$\Rightarrow \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) = 2Q(x) \quad \textcircled{a}$$

Using Eq (a) in (8) we get P_e as follows.

$$P_e = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

or

The Bit error rate of QPSK
is given by

$$\text{BER} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

We may express the average error probability of bit errors in the in-phase & Quadrature paths of coherent QPSK receiver as

$$P' = Q\left(\sqrt{\frac{E}{N_0}}\right) \quad \textcircled{b}$$

The average probability of correct detection resulting from the combined action of two channels working together is

$$P_c = (1 - P')^2$$
$$= \left[1 - Q\left(\sqrt{\frac{E}{N_0}}\right)\right]^2$$

$$P_c = 1 - 2Q\left(\sqrt{\frac{E}{N_0}}\right) + Q^2\left(\sqrt{\frac{E}{N_0}}\right)$$

The average probability of symbol error for QPSK is therefore,

$$P_e = 1 - P_c$$
$$= 1 - \left[1 - 2Q\left(\sqrt{\frac{E}{N_0}}\right) + Q^2\left(\sqrt{\frac{E}{N_0}}\right)\right]$$

$$P_e = 2Q\left(\sqrt{\frac{E}{N_0}}\right) - Q^2\left(\sqrt{\frac{E}{N_0}}\right)$$

In the region $(E/N_0) \gg 1$ \therefore we may ignore Q^2 term on RHS of the equation.

So, the average probability of symbol error for QPSK receiver is approximated as

$$P_e \approx 2Q\left(\sqrt{\frac{E}{N_0}}\right)$$

In QPSK 2 bits are transmitted per symbol

$$E = 2E_b$$

Thus, we get,

$$P_e = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

The Bit error rate of QPSK is given by,

$$\text{BER} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Binary Frequency Shift Keying (BFSK)

Using Coherent Detection.

* Signal Space Representation of BFSK

- BFSK is a non-linear method of modulation.
- In BFSK, the binary symbols, '1' & '0' are used to modulate the frequency of carrier.
- The no. of message symbols in BFSK, $M=2$ & it transmits one bit symbol ($n=1$)
- Symbols '1' and '0' are distinguished from each other by transmitting one of the two sinusoidal carrier waves that differ in frequency by a fixed amount, with a constant phase.

The typical pair of sinusoidal wave is given by.

$$S_i(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_i t) & ; 0 \leq t \leq T_b \\ 0 & ; \text{elsewhere} \end{cases} \quad \rightarrow ①$$

where $i=1, 2$ and E_b is the transmitted signal energy per bit, the transmitted frequency is given as

$$f_i = \frac{n_c + 1}{T_b} \quad ; \text{ where } i=1, 2. \quad n_c = \text{some fixed integer}$$

Let the two symbols, '1' & '0' are represented by $S_1(t)$ and $S_2(t)$ respectively. Then.

$$\text{Symbol 1: } S_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_1 t \quad ; 0 \leq t \leq T_b \quad \rightarrow ②$$

$$\text{Symbol 0: } S_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_2 t \quad ; 0 \leq t \leq T_b \quad \rightarrow ③$$

In BFSK, $f_1 > f_2$ and the Phase of the carrier is maintained constant, including the Inter-bit switching times.

It is observed from eq(1), (2) & (3) that signals $S_1(t)$ & $S_2(t)$ are orthogonal, but normalized to have unit energy.

The orthonormal basis function is given by,

$$\Phi_i(t) = \begin{cases} \sqrt{\frac{2}{T_b}} \cos(2\pi f_i t) & ; 0 \leq t \leq T_b \\ 0 & ; \text{elsewhere} \end{cases}$$

where $i=1, 2$.

Expressing, $s_1(t)$ & $s_2(t)$ in terms of $\phi_1(t)$ & $\phi_2(t)$
we get

$$s(t) = \begin{cases} s_1(t) = \sqrt{E_b} \phi_1(t) & ; \text{for symbol '1'} \\ s_2(t) = \sqrt{E_b} \phi_2(t) & ; \text{for symbol '0'} \end{cases}$$

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos 2\pi f_1 t$$

$$\phi_2(t) = \sqrt{\frac{2}{T_b}} \cos 2\pi f_2 t$$

The Co-efficients to locate message signal points $s_1(t)$ & $s_2(t)$ in signal space are given by the equation,

$$S_{ij} = \int_0^{T_b} s_i(t) \phi_j(t) dt$$

$$= \int_0^{T_b} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_i t) \sqrt{\frac{2}{T_b}} \cos(2\pi f_j t) dt$$

$$S_{ij} = \begin{cases} \sqrt{E_b} & ; i=j \\ 0 & ; i \neq j \end{cases} ; \begin{matrix} i=1,2 \\ j=1,2 \end{matrix}$$

Hence, the co-ordinates to locate message points $s_1(t)$ & $s_2(t)$ in signal space are given by,

$$S_1 = \begin{bmatrix} S_{11} \\ S_{12} \end{bmatrix} = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} S_{21} \\ S_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}$$

The Signal space representation of BFSK has two dimensions ($N=2$) with $M=2$ message points. The Signal space diagram of BFSK is as shown in figure below.

The Euclidean distance $\|s_1 - s_2\|$ is equal to $\sqrt{2E_b}$.

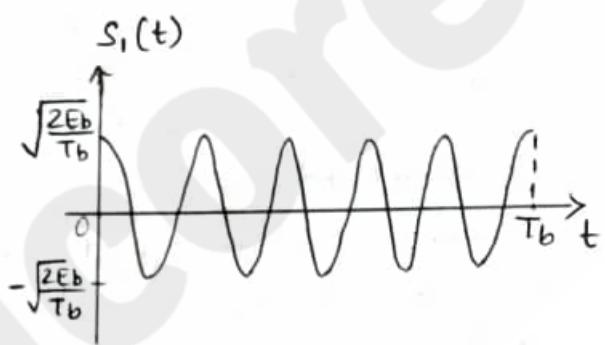
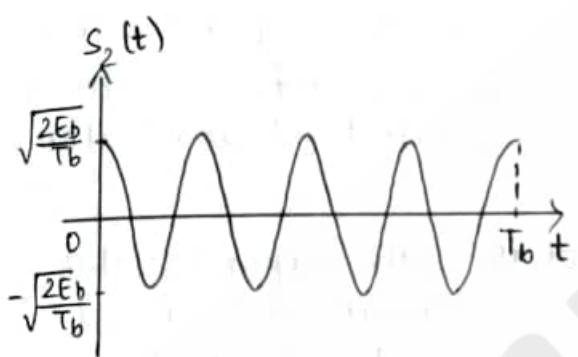
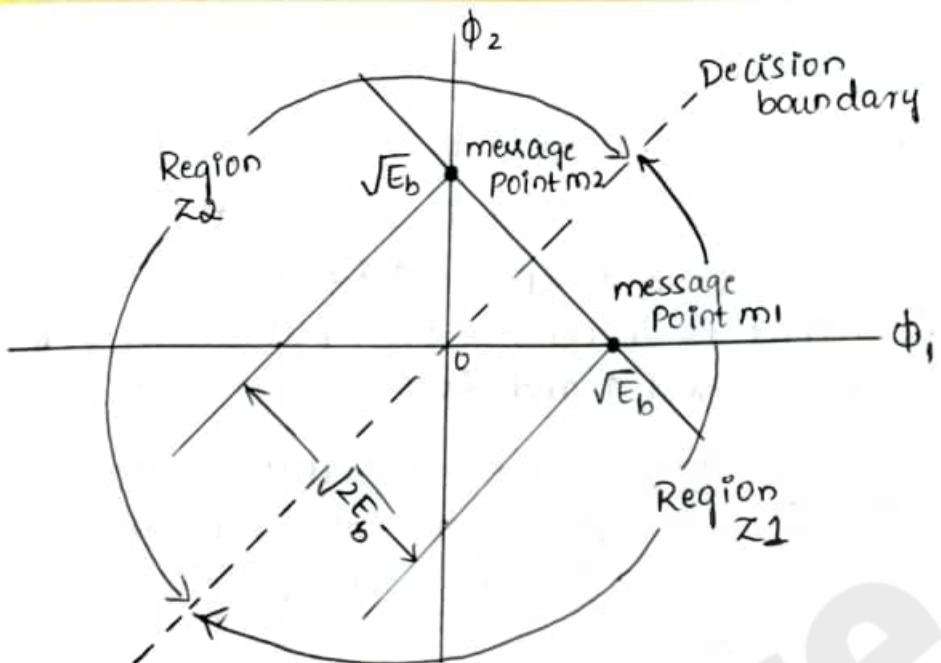
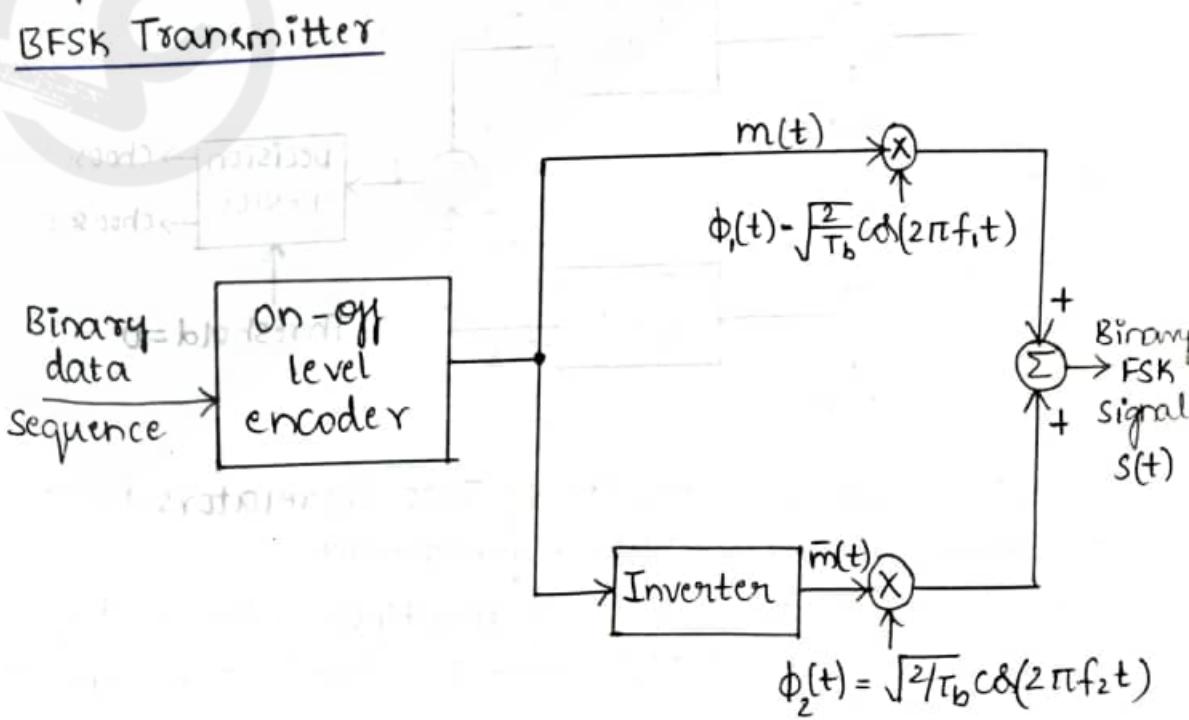


fig : Signal Space diagram of BFSK

Generation and Coherent Detection of Binary FSK Signal.

BFSK Transmitter



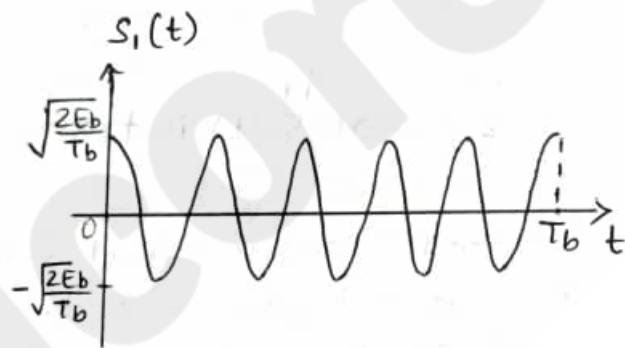
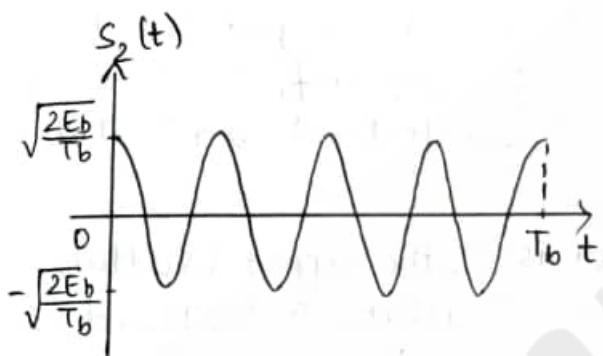
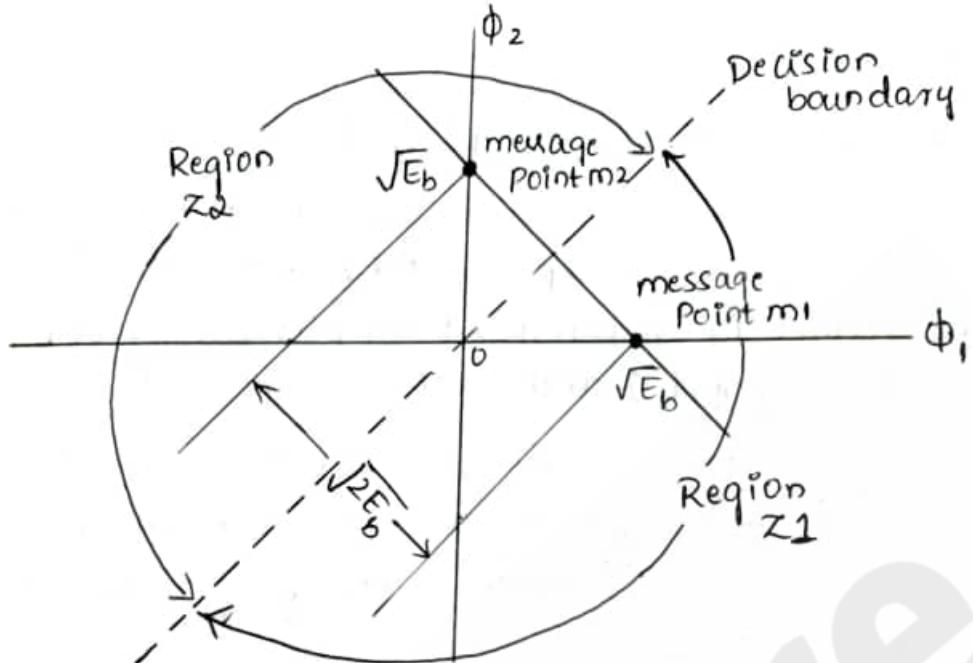
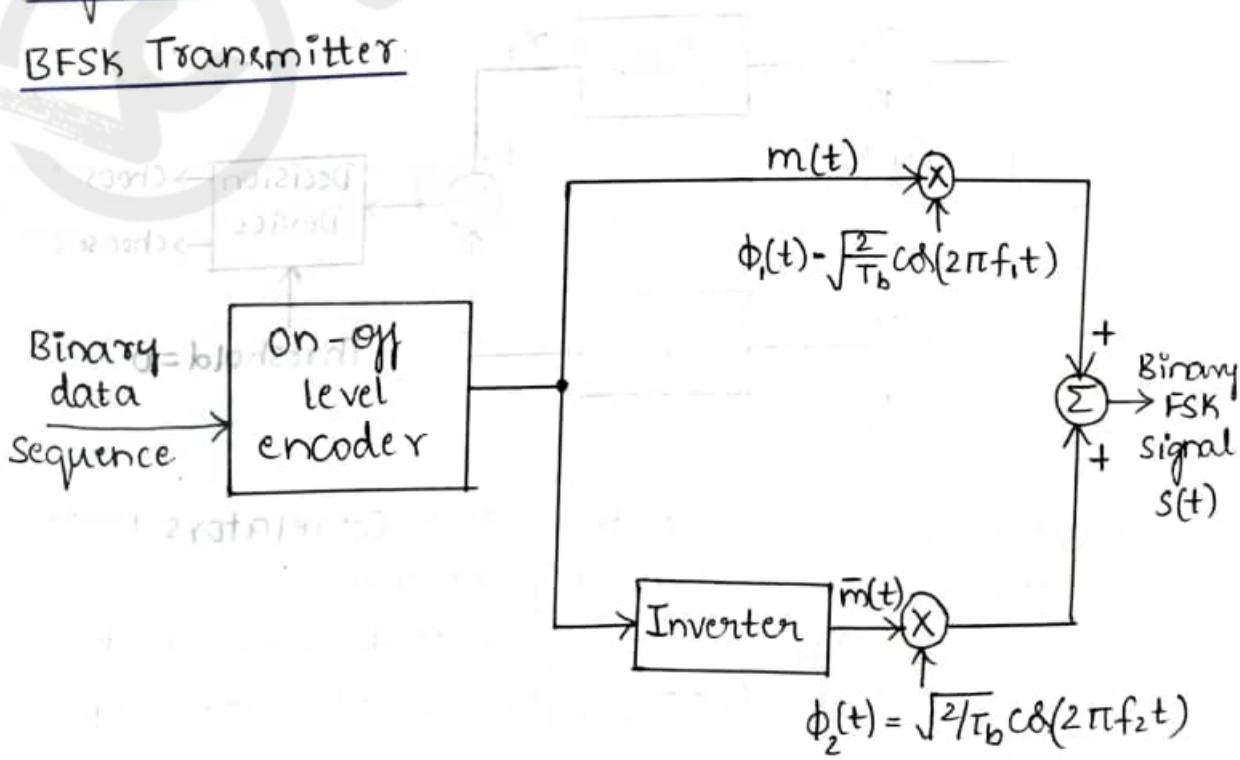


fig : Signal Space diagram of BFSK.

Generation and Coherent Detection of Binary FSK signal.

BFSK Transmitter

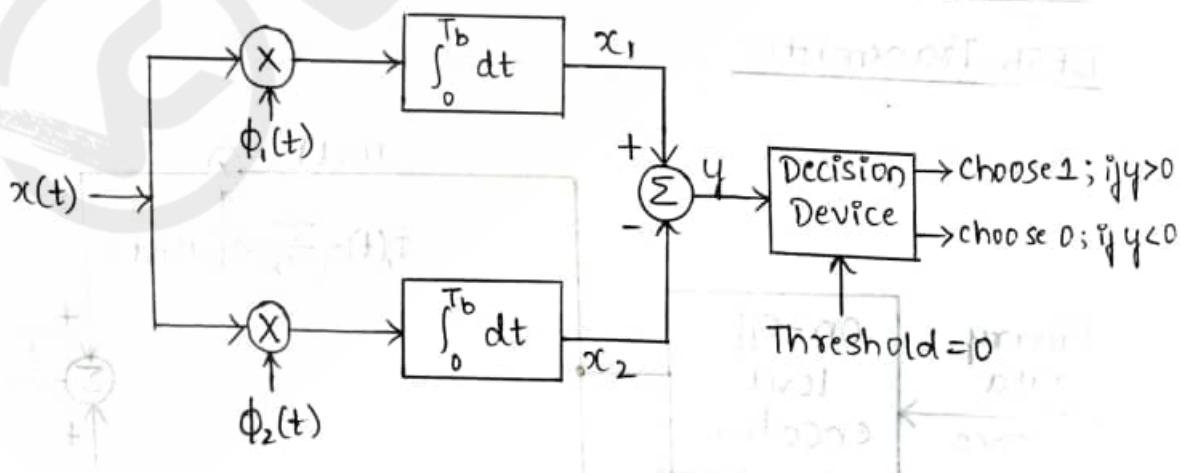


BFSK consist

BFSK Transmitter consists of two Components :

1. On-off level encoder: This encodes the input binary sequence using unipolar NRZ line coding technique. The output is a constant amplitude of $\sqrt{E_b}$ in response to input symbol 1 and zero in response to input symbol 0.
2. pair of oscillator: The 2 oscillator with frequencies f_1 & f_2 ($f_1 > f_2$) differ by an integer multiple of bit rate $1/T_b$. The lower oscillator with frequency f_2 is preceded by an inverter.
 - When in a Signalling interval, the input symbol is 1, the upper oscillator with frequency f_1 is switched ON & signal $s_1(t)$ is transmitted, while the lower oscillator is switched off.
 - When the input symbol is 0, the upper oscillator is switched off, while the lower oscillator is switched on and signal $s_2(t)$ with frequency f_2 is transmitted.
 - With phase continuity as the requirement, the two oscillators are synchronized with each other.

BFSK Receiver



- BFSK Receiver consists of two Correlators that are tuned to two different frequencies.
- A Correlator consists of multiplier followed by Integrator (LPF). Both correlators are supplied

with locally generated coherent reference signals $\phi_1(t)$ and $\phi_2(t)$.

- The output of correlators x_1 & x_2 forms the elements of observation vector X . Both outputs x_1 & x_2 are given to subtractor. The output of subtractor, $y = x_1 - x_2$
- The resulting difference 'y' is then compared with a threshold of zero. The decision device makes following decision.
 - 1) If $x_1 > x_2 \Rightarrow y > 0$. Then receiver decides in favour of '1'
 - 2) If $x_1 < x_2 \Rightarrow y < 0$. Then it decides in favour of '0'
 - 3) If $y = 0$, then receiver makes random guess in favour of either '1' or '0'.

Error Probability of Binary FSK

→ Let $x(t)$ be the received BFSK signal. & is given by

$$x(t) = s(t) + w(t) \quad \text{--- (1)}$$

$$x(t) = \begin{cases} S_1(t) + w(t) ; \text{ for symbol '1'} \\ S_2(t) + w(t) ; \text{ for symbol '0'} \end{cases}$$

where $w(t)$ is additive white Gaussian noise with mean (μ) = 0 & variance (σ^2) = $No/2$

→ Let us consider the transmitted symbol is '0' then the received signal $x(t)$ is given by.

$$x(t) = S_2(t) + w(t) \quad \text{--- (2)}$$

→ The output of 2 correlators x_1 & x_2 forms the elements of observation vector X , that are defined by,

$$x_1 = \int_0^{T_b} x(t) \phi_1(t) dt$$

$$= \int_0^{T_b} [S_2(t) + w(t)] \phi_1(t) dt$$

$$x_1 = \int_0^{T_b} S_2(t) \phi_1(t) dt + \int_0^{T_b} w(t) \phi_1(t) dt$$

$$x_1 = 0 + w_1$$

$$x_2 = \int_0^{T_b} x(t) \phi_2(t) dt$$

$$= \int_0^{T_b} [s_2(t) + w(t)] \phi_2(t) dt$$

$$= \int_0^{T_b} s_2(t) \phi_2(t) dt + \int_0^{T_b} w(t) \phi_2(t) dt$$

$$x_2 = \sqrt{E_b} + w_2$$

$$\therefore \int_0^{T_b} w(t) \phi_i(t) dt = w_i$$

$$\int_0^{T_b} s_i(t) \phi_j(t) dt = 0 \text{ when } i \neq j$$

→ Now applying the decision rule, assuming the use of coherent detection at the receiver.

→ The receiver decides in favour of symbol 1, if $x_1 > x_2$ & the observation vector falls in the region Z1.

→ The receiver decides in favour of symbol 0, if the received signal point defined by observation vector 'x' falls in the region Z2. This occurs when $x_2 > x_1 \Rightarrow y < 0$.

→ An erroneous decision is made, when transmitted symbol is '0' and the noise is such that decision device maps the observation vector x onto Region Z1 hence receiver decides in favour of symbol 1.

Calculating mean & variance of x_1 & x_2

Mean of x_1 :

$$\mu_{x_1} = E[x_1] = E[0 + w_1] = 0$$

$$\underline{\mu_{x_1} = 0}$$

Variance of x_1 :

$$\sigma_{x_1}^2 = \text{Var}[x_1] = \text{Var}[0 + w_1]$$

$$\underline{\sigma_{x_1}^2 = N_0/2}$$

Mean of x_2 :

$$\mu_{x_2} = E[x_2] = E[\sqrt{E_b} + w_2]$$

$$\underline{\mu_{x_2} = \sqrt{E_b}}$$

Variance of x_2 :

$$\sigma_{x_2}^2 = \text{Var}[x_2]$$

$$= \text{Var}[\sqrt{E_b} + w_2]$$

$$= 0 + N_0/2$$

$$\underline{\sigma_{x_2}^2 = N_0/2}$$

To proceed further, let us define the Gaussian random variable y with sample value 'y', defined as

$$y = x_1 - x_2$$

Let us calculate mean & variance of y

mean of y :

$$\begin{aligned} \mu_y &= E[y] = E[x_1 - x_2] \\ &= E[x_1] - E[x_2] \\ &= 0 - \sqrt{E_b} \end{aligned}$$

$$\boxed{\mu_y = -\sqrt{E_b}}$$

Variance of y :

$$\begin{aligned} \sigma_y^2 &= \text{Var}[y] = \text{Var}[x_1] + \text{Var}[x_2] \\ &= \frac{N_0}{2} + \frac{N_0}{2} \end{aligned}$$

$$\boxed{\sigma_y^2 = N_0}$$

Variance of the random variable y is independent of which binary symbol was sent.

$$\therefore \text{Var}[y] = \text{Var}[x_1] + \text{Var}[x_2]$$

The Conditional Probability density function of y is given by

$$f_y(y|0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \quad \textcircled{1}$$

$$f_y(y|0) = \frac{1}{\sqrt{2\pi N_0}} e^{-\frac{(y+\sqrt{E_b})^2}{2N_0}}$$

$$f_y(y|1) = \frac{1}{\sqrt{2\pi N_0}} e^{-\frac{(y-\sqrt{E_b})^2}{2N_0}}$$

$$\therefore (\sqrt{2N_0})^2 = 2N_0$$

Let $P_e(0)$ denote probability of deciding in favour of symbol '1' when symbol '0' was transmitted

$$\therefore \text{Region } Z_1 : 0 \leq y \leq \infty$$

Note

$$1. E[\text{const}] = \text{Const}$$

$$2. \text{Var}[\text{const}] = 0$$

$$E[w_i] = 0$$

$$\text{Var}[w] = N_0/2$$

$$P_e(0) = \int_0^{\infty} f_y(y|0) \cdot dy$$

$$P_e(0) = \int_0^{\infty} \frac{1}{\sqrt{2\pi N_0}} e^{-\left(\frac{y+\sqrt{E_b}}{\sqrt{2N_0}}\right)^2} dy \quad \text{--- (2)}$$

$$\text{put } z = \frac{y+\sqrt{E_b}}{\sqrt{2N_0}}$$

diff wrt 'z'

$$dz = \frac{dy}{\sqrt{2N_0}} + 0$$

$$dy = \sqrt{2N_0} dz$$

changing limits from y to z

$$\text{when } y=0; z = \sqrt{\frac{E_b}{2N_0}}$$

$$\text{when } y=\infty; z=\infty$$

$$\textcircled{1} \Rightarrow P_e(0) = \int_{\frac{\sqrt{E_b}}{\sqrt{2N_0}}}^{\infty} \frac{1}{\sqrt{2\pi N_0}} e^{-z^2} \cdot \sqrt{2N_0} dz$$

$$P_e(0) = \int_{\frac{\sqrt{E_b}}{\sqrt{2N_0}}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-z^2} dz \quad \text{--- (3)}$$

From the definition of complimentary error function, $\text{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} e^{-u^2} du$

$$\frac{1}{2} \text{erfc}(u) = \frac{1}{\sqrt{\pi}} \int_u^{\infty} e^{-u^2} du \quad \text{--- (4)}$$

Using Eq(4) in eq(3) we can rewrite Eq(3) as

$$P_e(0) = \frac{1}{2} \text{erfc}(z)$$

$$P_e(0) = \frac{1}{2} \text{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right)$$

$$\text{III by } P_e(1) = \frac{1}{2} \text{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right) \quad \text{--- (5)}$$

From the definition of Q-function.

$$Q(x) = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) \quad ⑤$$

using Eq(6) in (5), we can rewrite Eq(5) as

$$P_e(0) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Assuming symbol '0' & '1' are equiprobable, $P(0) = P(1) = \frac{1}{2}$.
The average probability of error is given by.

$$P_e = P(0) P_e(0) + P(1) P_e(1)$$

$$= \frac{1}{2} \left[\frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right) \right] + \frac{1}{2} \left[\frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right) \right]$$

$$P_e = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right) \quad ⑥$$

From the definition of relation b/w erfc & Q function

$$Q(x) = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) \quad ⑦$$

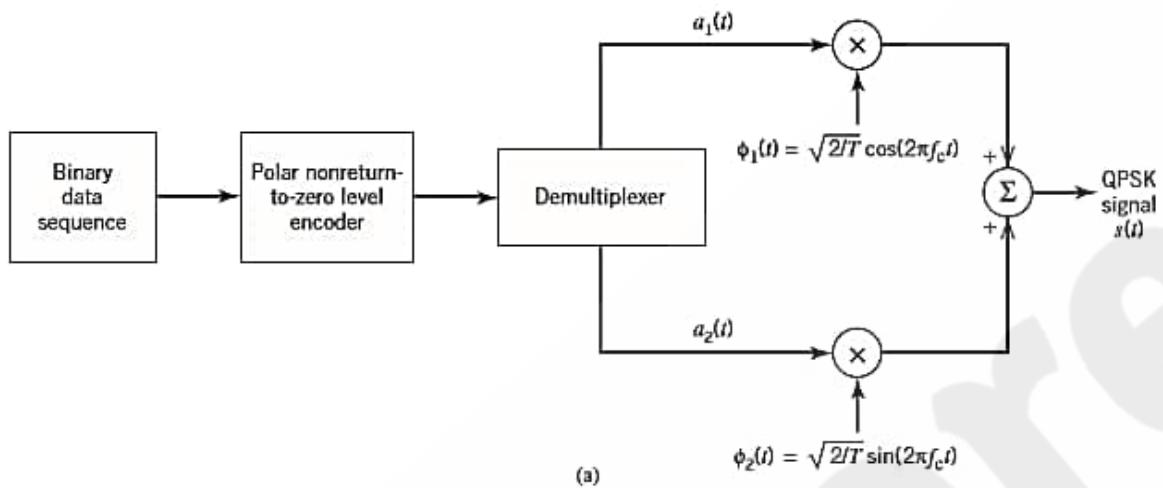
Comparing Eq(7) & (6), we can rewrite Eq(6) in terms of Q-function as

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

This is the average error Probability of BFSK
using coherent detection.

Generation and Coherent Detection of QPSK Signals

QPSK Transmitter:



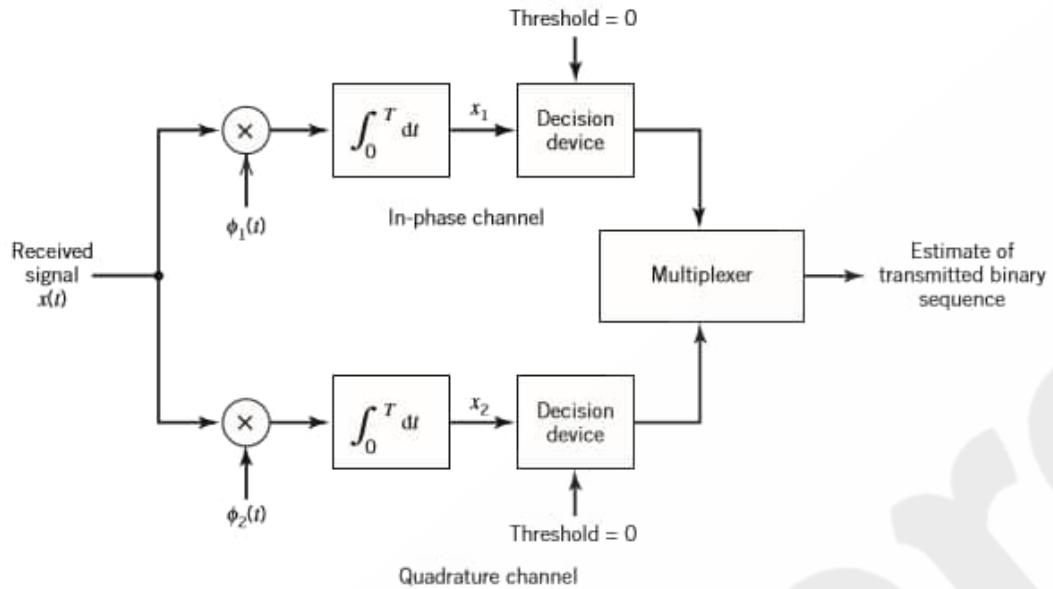
The QPSK transmitter may be viewed as two binary PSK generators that work in parallel, each at a bit rate equal to one-half the bit rate of the original binary sequence at the QPSK transmitter input.

Transmitter consists of 2 main components, as follows:

1. **Polar NRZ-level encoder:** It represents symbols 1 and 0 of the incoming binary sequence by amplitude levels and, respectively.
2. **De-multiplexer:** It divides the binary wave produced by the polar NRZ-level encoder into two separate binary waves, one of which represents the odd-numbered di-bits in the incoming binary sequence and the other represents the even-numbered di-bits.

The even numbered bits are multiplied by orthogonal basis function $\phi_1(t)$ [ie, cosine signal] and even numbered bits are multiplied by $\phi_2(t)$ [ie, sine signal] and the modulated signal is added to get the QPSK signal.

QPSK Coherent Receiver:



The QPSK receiver is structured in the form of an in-phase path and a quadrature path, working in parallel as depicted in Figure.

The functional composition of the QPSK receiver is as follows:

1. Pair of Correlators: They have a common input $x(t)$. The two correlators are supplied with a pair of locally generated orthonormal basis functions $\Phi_1(t)$ and $\Phi_2(t)$, which means that the receiver is synchronized with the transmitter. The correlator outputs, produced in response to the received signal $x(t)$, are denoted by x_1 and x_2 , respectively.

2. Pair of Decision devices: They act on the correlator outputs x_1 and x_2 by comparing each one with a zero-threshold; here, it is assumed that the symbols 1 and 0 in the original binary stream at the transmitter input are equally likely.

If $x_1 > 0$, a decision is made in favour of symbol 1 for the in-phase channel output.

If $x_1 < 0$, then a decision is made in favour of symbol 0.

Similar binary decisions are made for the quadrature channel.

3. Multiplexer: It combines the two binary sequences produced by the pair of decision devices. The resulting binary sequence so produced provides an estimate of the original binary stream at the transmitter input.

M-ary PSK

- It is the generic form of PSK commonly referred to as M-ary PSK, where the phase of the carrier takes on one of M possible values:

$$\theta_i = 2[(i-1)\frac{\pi}{M}], \text{ where } i = 1, 2, \dots, M.$$

- In BPSK, we have two symbols 0 & 1 ($M=2$, $M = 2^n$, n = no. of bits per symbol, ie, $n=1$). Hence the phase shift in BPSK, is given by,

$$\text{Phase shift in BPSK} = \frac{2\pi}{\text{No.of Symbols},(M)} = \frac{2\pi}{2} = 180^\circ.$$

- In QPSK, we have 4 symbols and 2bits per symbol ($M=4$, $M = 2^n$, n = no. of bits per symbol,ie, $n=2$). Hence the phase shift in QPSK, is given by,

$$\text{Phase shift in QPSK} = \frac{2\pi}{\text{No.of Symbols},(M)} = \frac{2\pi}{4} = 90^\circ.$$

Accordingly, during each signaling interval of duration T, one of the M possible signals, is sent, where E is the signal energy per symbol.

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos \left[2\pi f_c t + \frac{2\pi}{M}(i-1) \right], \quad i = 1, 2, \dots, M$$

The carrier frequency $f_c = nc/T$ for some fixed integer nc. Each $s_i(t)$ may be expanded in terms of the same two basis functions $\Phi_1(t)$ and $\Phi_2(t)$; the signal constellation of M-ary PSK is, therefore, two-dimensional. The M message points are equally spaced on a circle of radius \sqrt{E} and center at the origin, as illustrated in Figure below for the case of octaphase-shift-keying (i.e., $M = 8$).

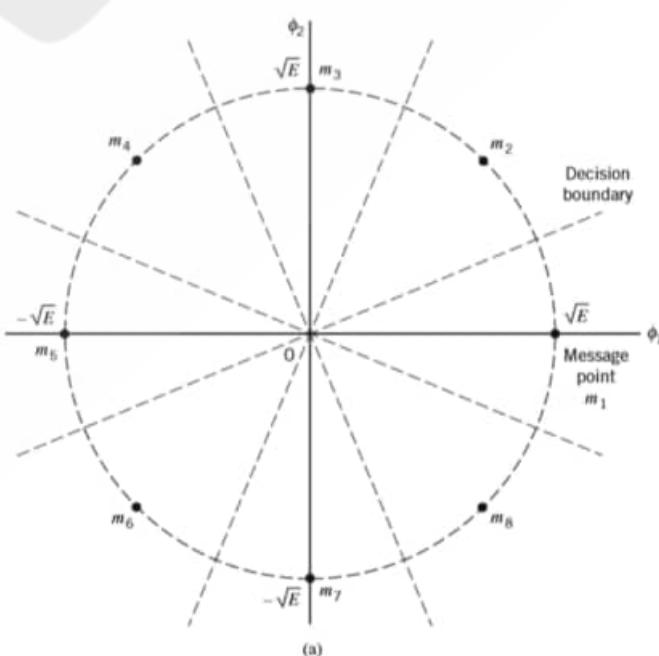
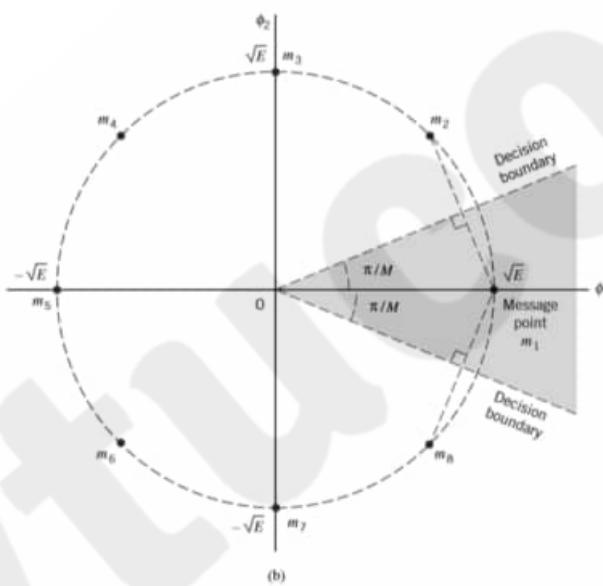


Figure: Signal-space diagram for octaphase-shift keying (i.e., $M = 8$). The decision boundaries are shown as dashed lines.

From Figure ,we see that the signal-space diagram is circularly symmetric. Suppose that the transmitted signal corresponds to the message point m_1 , whose coordinates along the Φ_1 - and Φ_2 -axes are $+\sqrt{E}$ and 0, respectively. Suppose that the ratio E/N_0 is large enough to consider the nearest two message points, one on either side of m_1 , as potential candidates for being mistaken for m_1 due to channel noise.

This is illustrated in Figure below for the case of $M = 8$.



The Euclidean distance for each of these two points from m_1 is (for $M = 8$)

$$d_{12} = d_{18} = 2\sqrt{E}\sin\left(\frac{\pi}{M}\right)$$

The average probability of symbol error for coherent M -ary PSK is given as

$$P_e \approx 2Q\left[\sqrt{\frac{2E}{N_0}}\sin\left(\frac{\pi}{M}\right)\right]$$

where it is assumed that $M \geq 4$. The approximation becomes extremely tight for fixed M , as E/N_0 is increased. For $M = 4$, the equation reduces to the same form given in for QPSK.

Power Spectra of M-ary PSK Signals

The symbol duration of M-ary PSK is defined by

$$T = T_b \log_2 M \quad \dots\dots(1)$$

where T_b is the bit duration. Proceeding in a manner similar to that described for a QPSK signal, we may show that the baseband power spectral density of an M-ary PSK signal is given by

$$\begin{aligned} S_B(f) &= 2E \operatorname{sinc}^2(Tf) \\ &= 2E_b (\log_2 M) [\operatorname{sinc}^2(T_b f \log_2 M)] \end{aligned}$$

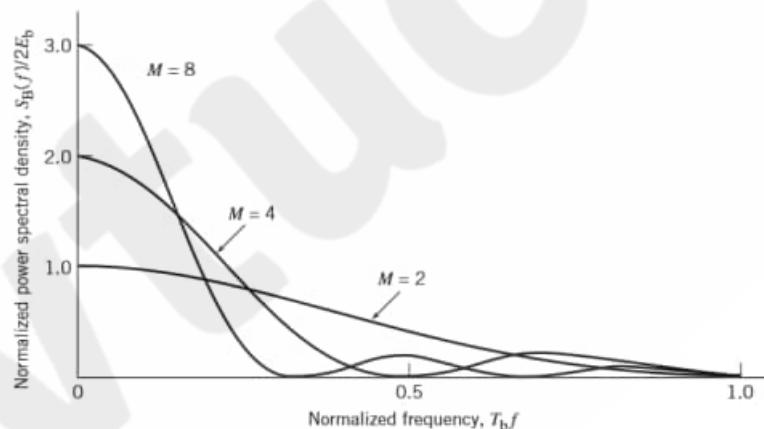


Figure 7.22 Power spectra of M -ary PSK signals for $M = 2, 4, 8$.

Bandwidth Efficiency of M-ary PSK

The channel bandwidth required to pass M-ary PSK signals through an analog channel as

$$B = \frac{2}{T}$$

where T is the symbol duration. But the symbol duration T is related to the bit duration T_b by eq(1).

Moreover, the bit rate $R_b = 1/T_b$. Hence, we may redefine the channel bandwidth in terms of the bit rate as

$$B = \frac{2R_b}{\log_2 M}$$

$$\rho = \frac{R_b}{B}$$

$$= \frac{\log_2 M}{2}$$

Based on this formula, the bandwidth efficiency of M -ary PSK signals is given by

As the number of states in M -ary PSK is increased, the bandwidth efficiency is improved at the expense of error performance. However, note that if we are to ensure that there is no degradation in error performance, we have to increase E_b/N_0 to compensate for the increase in M .

M-ary Quadrature Amplitude Modulation

The QAM is a hybrid form of modulation, in that the carrier experiences amplitude as well as phase-modulation. In M -ary PAM, the signal-space diagram is one-dimensional. M -ary QAM is a two-dimensional generalization of M -ary PAM, in that its formulation involves

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t), \quad 0 \leq t \leq T$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t), \quad 0 \leq t \leq T$$

two orthogonal pass band basis functions:

Let d_{\min} denote the minimum distance between any two message points in the QAM constellation. With the separation between two message points in the signal-space diagram being proportional to the square root of energy, we may therefore set

$$\frac{d_{\min}}{2} = \sqrt{E_0}$$

where E_0 is the energy of the message signal with the lowest amplitude. The transmitted M-ary QAM signal for symbol k can now be defined in terms of E_0 :

The signal $s_k(t)$ involves two phase-quadrature carriers, each one of which is modulated by a set of discrete amplitudes; hence the terminology “quadrature amplitude modulation.”

$$s_k(t) = \sqrt{\frac{2E_0}{T}}a_k \cos(2\pi f_c t) - \sqrt{\frac{2E_0}{T}}b_k \sin(2\pi f_c t), \quad \begin{cases} 0 \leq t \leq T \\ k = 0, \pm 1, \pm 2, \dots \end{cases}$$

In M-ary QAM, the constellation of message points depends on the number of possible symbols, M .

Example: M-ary QAM for $M = 4$

we have constructed two signal constellations for the 4-ary PAM, one vertically oriented along the Φ 1-axis in part a of the figure, and the other horizontally oriented along the Φ 2-axis in part b of the figure. These two parts are spatially orthogonal to each other, accounting for the two-dimensional structure of the M-ary QAM.

With four quadrants constituting the 4-ary QAM, we proceed in four stages as follows:

Stage 1: *First-quadrant constellation.* Referring to Figure 7.23, we use the codewords along the positive parts of the ϕ_2 and ϕ_1 -axes, respectively, to write

$$\begin{bmatrix} 11 \\ 10 \end{bmatrix} \begin{bmatrix} 10 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1110 & 1111 \\ 1010 & 1011 \end{bmatrix}$$

Top to Left to First quadrant
bottom right

Stage 2: *Second-quadrant constellation.* Following the same procedure as in Stage 1, we write

Stage 3: *Third-quadrant constellation.* Again, following the same procedure as before, we next write

$$\begin{bmatrix} 00 \\ 01 \end{bmatrix} \begin{bmatrix} 01 & 00 \end{bmatrix} \rightarrow \begin{bmatrix} 0001 & 0000 \\ 0101 & 0100 \end{bmatrix}$$

Top to Left to Third quadrant
bottom right

Stage 4: *Fourth-quadrant constellation.* Finally, we write

$$\begin{bmatrix} 00 \\ 01 \end{bmatrix} \begin{bmatrix} 10 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 0010 & 0011 \\ 0110 & 0111 \end{bmatrix}$$

Top to Left to Fourth quadrant
bottom right

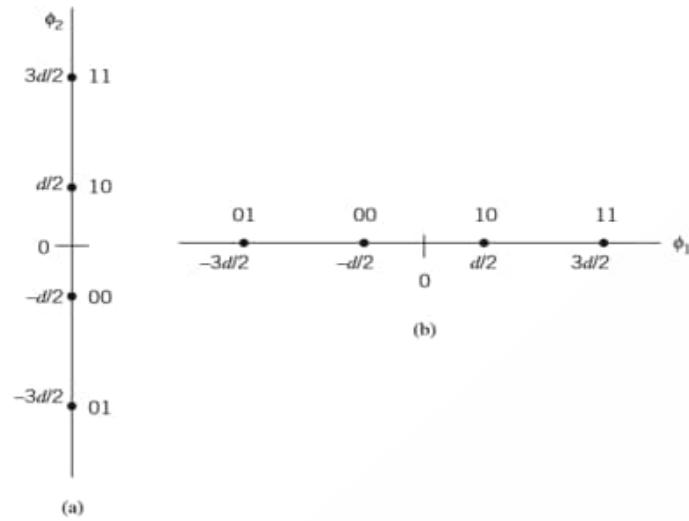


Figure: The two orthogonal constellations of the 4-ary PAM. (a) Vertically oriented constellation. (b) Horizontally oriented constellation. As mentioned in the text, we move top-down along the Φ_2 -axis and from left to right along the Φ_1 -axis.

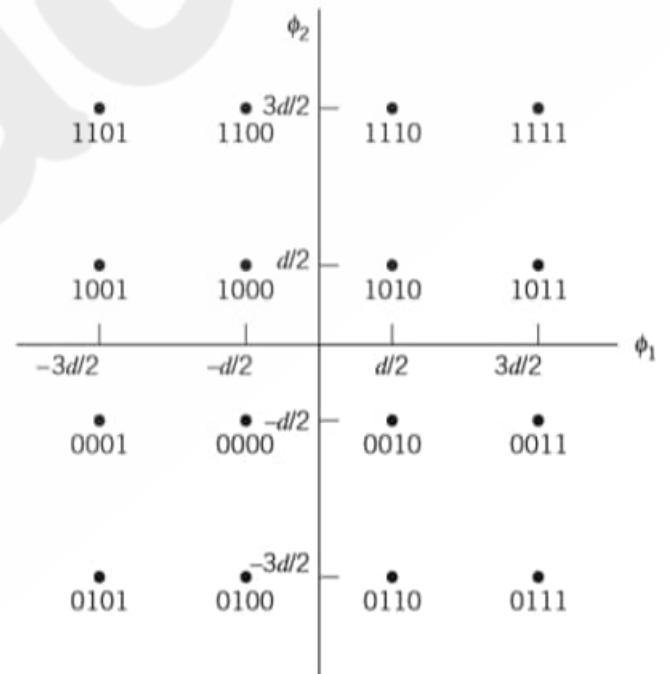


Figure 7.24

(a) Signal-space diagram of M -ary QAM for $M = 16$; the message points in each quadrant are identified with Gray-encoded quadbits.

Average Probability of Error

In light of the equivalence established between the M-ary QAM and M-ary PAM, we may formulate the average probability of error of the M-ary QAM by proceeding as follows:

1. The probability of correct detection for M-ary QAM is written as

$$P_c = (1 - P'_e)^2 \quad \dots\dots\dots(1)$$

where P_e' is the probability of symbol error for the L-ary PAM.

2. With $L=\sqrt{M}$, the probability of symbol error P_e' is itself defined by

$$P'_e = 2\left(1 - \frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{2E_0}{N_0}}\right) \quad \dots\dots\dots(2)$$

3. The probability of symbol error for M-ary QAM is given by

$$\begin{aligned} P_e &= 1 - P_c \\ &= 1 - (1 - P'_e)^2 \\ &\approx 2P'_e \end{aligned} \quad \dots\dots\dots(3)$$

where it is assumed that P_e' is small enough compared with unity to justify ignoring the quadratic term.

Hence, using Equations(1) and (2) in Equation (3), we find that the probability of symbol error for M-ary QAM is approximately given by

$$P_e \approx 4\left(1 - \frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{2E_0}{N_0}}\right)$$

The transmitted energy in M-ary QAM is variable, in that its instantaneous value naturally depends on the particular symbol transmitted. Therefore, it is more logical to express P_e in terms of the average value of the transmitted energy rather than E_0 . Assuming that the L amplitude levels of the in-phase or quadrature component of the M-ary QAM signal are equally likely, we have

$$E_{av} = 2\left[\frac{2E_0}{L} \sum_{i=1}^{L/2} (2i-1)^2\right]$$

After summing the series and simplifying we get,

$$E_{av} = \frac{2(L^2 - 1)E_0}{3}$$

$$= \frac{2(M - 1)E_0}{3}$$

Hence Pe can be rewritten in terms of Eav as

$$P_e \approx 4 \left(1 - \frac{1}{\sqrt{M}}\right) Q \left[\sqrt{\frac{3E_{av}}{(M-1)N_0}} \right]$$

which is the desired result.

The case of M = 4 is of special interest. The signal constellation for this particular value of M is the same as that for QPSK.

Binary Frequency-Shift Keying Using Noncoherent Detection

In binary FSK, the transmitted signal is defined as

$$s_i(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_i t), & 0 \leq t \leq T_b \\ 0, & \text{elsewhere} \end{cases}$$

where T_b is the bit duration and the carrier frequency f_i equals one of two possible values f_1 and f_2 ; to ensure that the signals representing these two frequencies are orthogonal, we choose $f_i = n_i / T_b$, where n_i is an integer.

Non coherent receiver for the detection of binary FSK signals

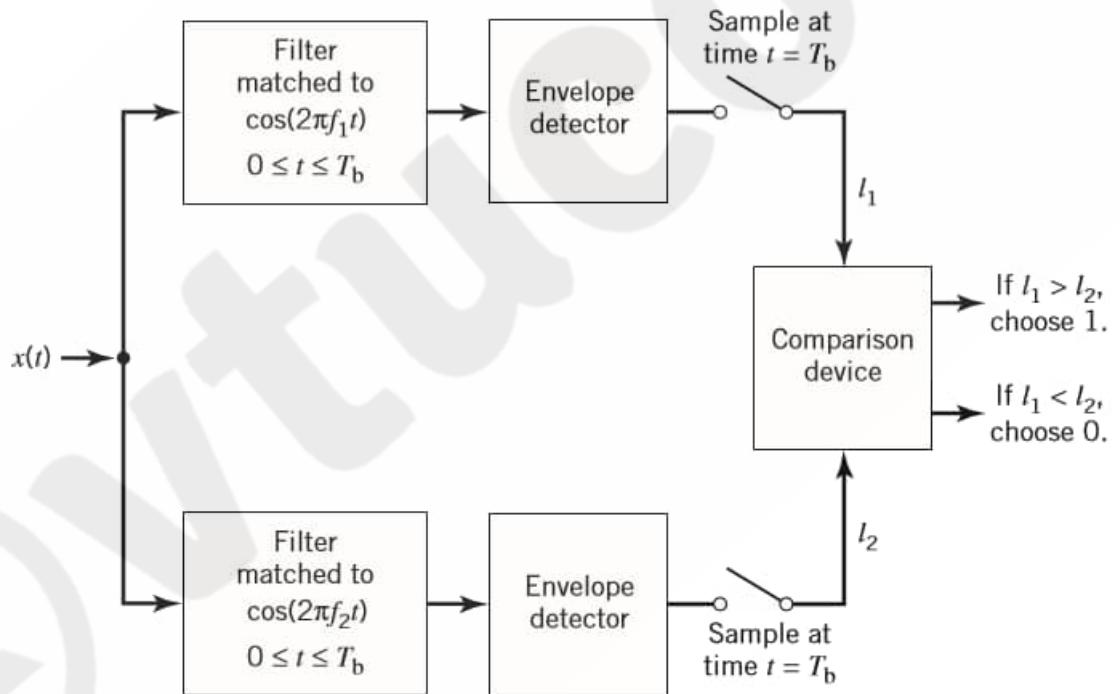


Figure: Noncoherent receiver for the detection of binary FSK signals.

The transmission of frequency f_1 represents symbol 1 and the transmission of frequency f_2 represents symbol 0. For the noncoherent detection of this frequency-modulated signal, the receiver consists of a pair of matched filters followed by envelope detectors, as in Figure. The filter in the upper path of the receiver is matched to $\cos(2\pi f_1 t)$ and the filter in the lower path is matched to $\cos(2\pi f_2 t)$ for the signaling interval $0 \leq t \leq T_b$. The resulting envelope detector

outputs are sampled at $t = Tb$ and their values are compared. The envelope samples of the upper and lower paths in Figure are shown as l_1 and l_2 . The receiver decides in favor of symbol 1 if $l_1 > l_2$ and in favour of symbol 0 if $l_1 < l_2$. If $l_1 = l_2$, the receiver simply guesses randomly in favor of symbol 1 or 0.

The noncoherent binary FSK described herein is a special case of non coherent orthogonal modulation with $T = Tb$ and $E = Eb$, where Eb is the signal energy per bit. Hence, the BER for non coherent binary FSK is

$$P_e = \frac{1}{2} \exp\left(-\frac{E_b}{2N_0}\right)$$

Differential Phase-Shift Keying (DPSK)

DPSK is viewed as the “non coherent” version of binary PSK. The distinguishing feature of DPSK is that it eliminates the need for synchronizing the receiver to the transmitter by combining two basic operations at the transmitter:

- differential encoding of the input binary sequence and
- PSK of the encoded sequence,

from which the name of this new binary signaling scheme follows:

Differential encoding starts with an arbitrary first bit, serving as the reference bit; to this end, symbol 1 is used as the reference bit.

Generation of the differentially encoded sequence then proceeds in accordance with a two-part encoding rule as follows:

1. If the new bit at the transmitter input is 1, leave the differentially encoded symbol unchanged with respect to the current bit.
2. If, on the other hand, the input bit is 0, change the differentially encoded symbol with respect to the current bit.

The differentially encoded sequence, denoted by $\{dk\}$, is used to shift the sinusoidal carrier phase by zero and 180° , representing symbols 1 and 0, respectively.

Thus, in terms of phase-shifts, the resulting DPSK signal follows the two-part rule:

1. To send symbol 1, the phase of the DPSK signal remains unchanged.
2. To send symbol 0, the phase of the DPSK signal is shifted by 180° .

Example:

b_k is the input bit stream given.

d_k is the complement of modulo-2 sum (XNOR) of b_k & d_{k-1} sequences.

d_{k-1} is the one-bit delayed version of differentially encoded sequence, d_k

Table 7.6 Illustrating the generation of DPSK signal

$\{b_k\}$	1	0	0	1	0	0	1	1
$\{d_{k-1}\}$	1	1	0	1	1	0	1	1
Differentially encoded sequence $\{d_k\}$	1	1	0	1	1	0	1	1
Transmitted phase (radians)	0	0	π	0	0	π	0	0

Generation of DPSK Signal

The transmitter consists of two functional blocks:

- Logic network and one-bit delay (storage) element, which are interconnected so as to convert the raw input binary sequence $\{b_k\}$ into the differentially encoded sequence $\{d_k\}$.
- Binary PSK modulator, the output of which is the desired DPSK signal.

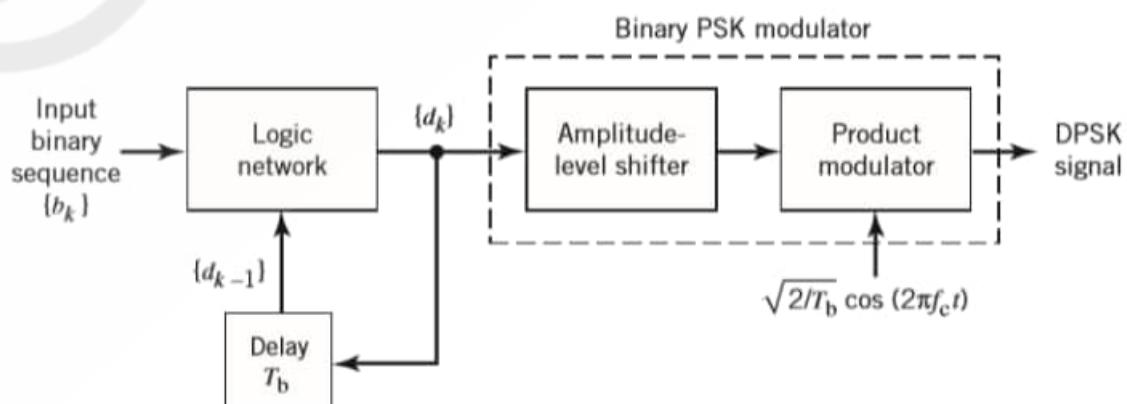
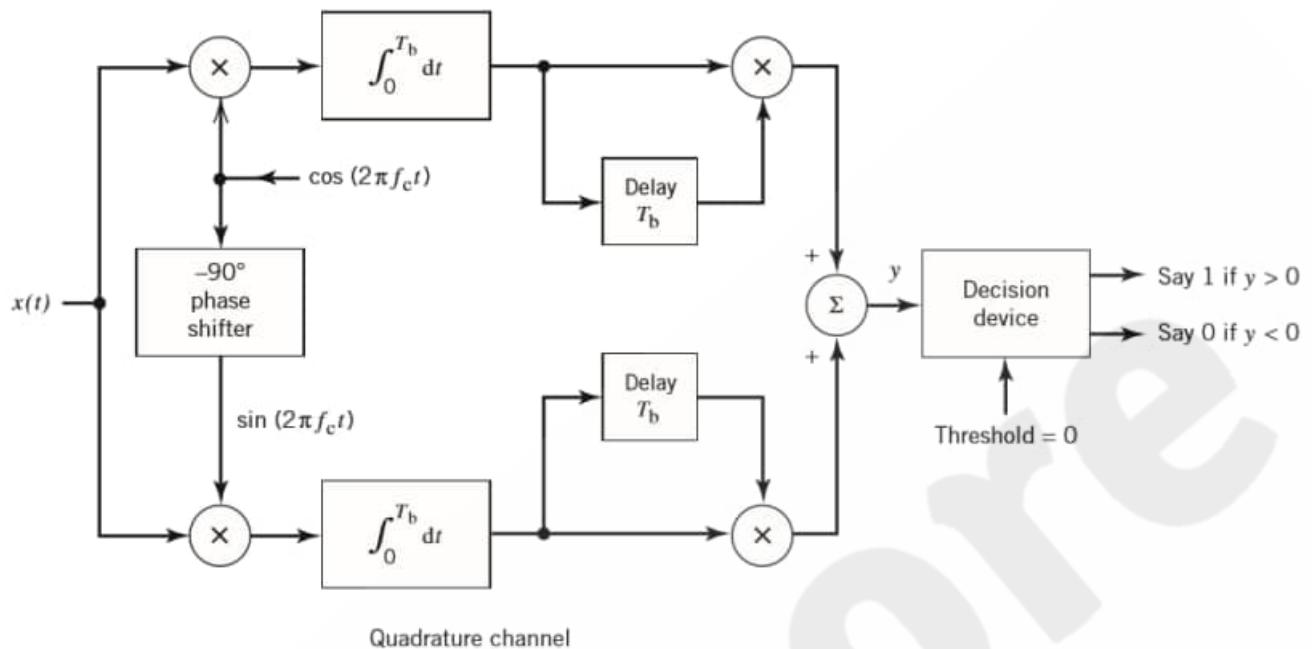


Figure 1: Block diagram of the DPSK transmitter.

Non Coherent Receiver for the Detection of DPSK



The optimum receiver for the detection of binary DPSK is as shown in Figure above, the formulation of which follows directly from the binary hypothesis test. This implementation is simple, in that it merely requires that sample values be stored.

The receiver is said to be optimum for two reasons:

1. In structural terms, the receiver avoids the use of fancy delay lines that could be needed otherwise.
2. In operational terms, the receiver makes the decoding analysis straightforward to handle, in that the two signals to be considered are orthogonal over the interval [0,2Tb]. In the use of DPSK, the carrier phase θ is unknown, which complicates the received signal $x(t)$. To deal with the unknown phase θ in the differentially coherent detection of the DPSK signal in $x(t)$, we equip the receiver with an in-phase and a quadrature path.

Signal-space diagram where the received signal points over the two-bit interval $0 \leq t \leq 2T_b$ are defined by $(A \cos \theta, A \sin \theta)$ and $(-A \cos \theta, -A \sin \theta)$, where A denotes the carrier amplitude.

This geometry of possible signals is illustrated in Figure below. For the two-bit interval $0 \leq t \leq 2T_b$, the receiver measures the coordinates , first, at time $t = T_b$ and then measures at time $t = 2T_b$.

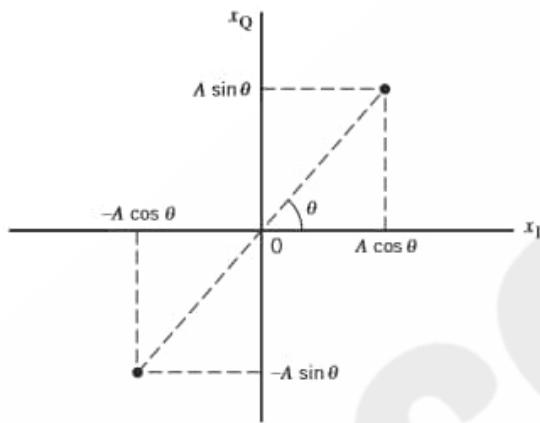


Figure 7.45 Signal-space diagram of received DPSK signal.

The issue to be resolved is whether these two points map to the same signal point or different ones. Recognizing that the vectors x_0 and x_1 , with end points and , respectively, are points roughly in the same direction if their inner product is positive, we may formulate **the binary-hypothesis test** with a question:

Is the inner product $x_0^T x_1$ positive or negative?

Expressing this statement in analytic terms, we may write

$$x_{I_0} x_{I_1} + x_{Q_0} x_{Q_1} \stackrel{\text{say 1}}{\gtrless} \stackrel{\text{say 0}}{0}$$

where the threshold is zero for equiprobable symbols. We now note the following identity:

$$x_{I_0} x_{I_1} + x_{Q_0} x_{Q_1} = \frac{1}{4}((x_{I_0} + x_{I_1})^2 - (x_{I_0} - x_{I_1})^2 + (x_{Q_0} + x_{Q_1})^2 - (x_{Q_0} - x_{Q_1})^2)$$

Hence, substituting this identity in above equation , we get the equivalent test:

$$(x_{I_0} + x_{I_1})^2 + (x_{Q_0} + x_{Q_1})^2 - (x_{I_0} - x_{I_1})^2 - (x_{Q_0} - x_{Q_1})^2 \stackrel{\text{say } 1}{\gtrless} \stackrel{\text{say } 0}{\leq} 0$$

Error Probability of DPSK

Basically, the DPSK is also an example of noncoherent orthogonal modulation when its behavior is considered over successive two-bit intervals; that is, $0 \leq t \leq 2T_b$. To elaborate, let

the transmitted DPSK signal be $\sqrt{2E_b/T_b} [\cos(2\pi f_c t)]$ for the first-bit interval

$0 \leq t \leq T_b$, which corresponds to symbol 1. Suppose, then, the input symbol for the second-bit interval $T_b \leq t \leq 2T_b$ is also symbol 1.

According to part 1 of the DPSK encoding rule, the carrier phase remains unchanged, thereby yielding the DPSK signal

$$s_1(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t), & \text{symbol 1 for } 0 \leq t \leq T_b \\ \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t), & \text{symbol 0 for } T_b \leq t \leq 2T_b \end{cases}$$

Suppose, next, the signaling over the two-bit interval changes such that the symbol at the transmitter input for the second-bit interval $T_b \leq t \leq 2T_b$ is 0. Then, according to part 2 of the

$$s_2(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t), & \text{symbol 1 for } 0 \leq t \leq T_b \\ \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \pi), & \text{symbol 1 for } T_b \leq t \leq 2T_b \end{cases}$$

DPSK encoding rule, the carrier phase is shifted by π radians (i.e., 180°), thereby yielding the new DPSK signal. The BER for DPSK is given by

$$P_e = \frac{1}{2} \exp\left(-\frac{E_b}{N_0}\right)$$

