MODULE-1: DISCRETE FOURIER TRANSFORMS (DFT)

To perform frequency analysis on a discrete time sequence $x(n)$, we convert time-domain sequence to an equivalent frequency domain representation.

Applying Fourier Tranform on $x(n)$, we get $X(\omega)$, which is continuous and periodic function of frequency. It is not a computationally convenient representation of the sequence $x(n)$.

Representation of a sequence $x(n)$ by samples of its spectrum $X(\omega)$ is known as the Discrete Fourier Transform (DFT).

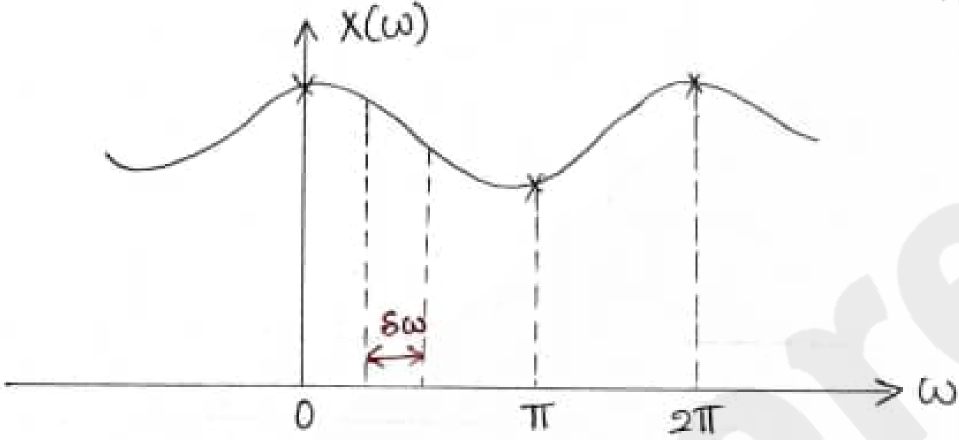
FREQUENCY DOMAIN SAMPLING :-

W.K.T aperiodic finite energy signals have continuous spectra.

Fourier Transform (FT) of aperiodic DTS $x(n)$ is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad \text{——— (1)}$$

Since $X(\omega)$ is periodic, with period 2π, only samples in the fundamental frequency range are obtained by sampling periodically in frequency at a spacing of $\delta\omega$ radians between successive samples. We take N equidistant samples in the interval $0 \le \omega \le 2\pi$ with spacing $\delta\omega = \frac{2\pi}{N}$.



Substituting $\omega = \frac{2\pi}{N}k$ in eqn ①,

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi}{N}kn} \quad \text{——} ②$$

where $k = 0, 1, 2, \ldots, N-1$

Summation in eqn ② can be subdivided into infinite number of summations, where each sum contains N terms.

$$\therefore X\left(\frac{2\pi}{N}k\right) = \cdots + \sum_{n=-N}^{-1} x(n)\, e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} + \sum_{n=N}^{2N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} + \cdots$$

$$= \sum_{l=-\infty}^{+\infty} \sum_{n=lN}^{lN+N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

If we change the index in the inner summation from n to $n-lN$ and interchange the order of summation, we get,

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n-lN) \right] e^{-j\frac{2\pi}{N}k(n-lN)}$$

for $k = 0, 1, 2, \ldots, N-1$.

But $e^{-j\frac{2\pi}{N}k(n-lN)} = e^{-j\frac{2\pi}{N}kn} \cdot e^{+j\frac{2\pi}{N}klN}$

where $e^{j\frac{2\pi}{N}klN} = e^{j2\pi kl} = 1$

$$\therefore X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{+\infty} x(n-lN) \right] e^{-j\frac{2\pi}{N}kn} \quad \text{--- (3)}$$

$$\text{Let } x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) \quad \text{--- (4)}$$

$x_p(n)$ is obtained by the periodic repetition of $x(n)$ every N samples. It is periodic with fundamental period N.

$x_p(n)$ can hence be expanded in a Fourier Series as

$$x_p(n) = \sum_{k=0}^{N-1} c_k\, e^{+j\frac{2\pi}{N}kn} \quad \text{--- (5)}$$

$n = 0, 1, 2, \ldots, N-1$

with Fourier coefficients,

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi}{N}kn} \quad \text{—— ⑥}$$

$$k = 0, 1, \ldots, N-1$$

Comparing eqn's ③ & ⑥,

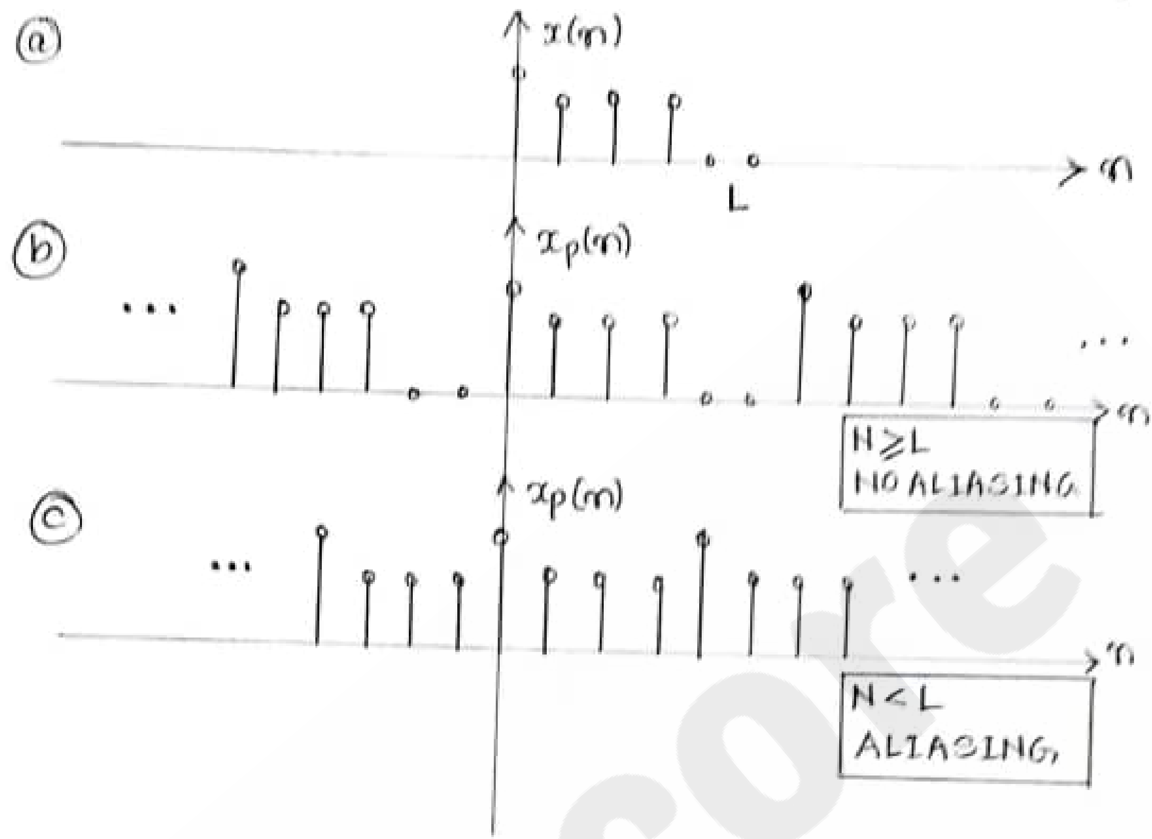
$$c_k = \frac{1}{N} X\left(\frac{2\pi}{N}k\right) \quad ; k = 0, 1, \ldots N-1 \quad \text{—— ⑦}$$

Substituting eqn ⑦ in eqn ⑤,

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j\frac{2\pi}{N}kn} \quad \text{—— ⑧}$$

Equation ⑧ provides the reconstruction of the periodic signal $x_p(n)$ from the samples of the spectrum $X(\omega)$. However, to recover $X(\omega)$ or $x(n)$ from the samples certain condition needs to be satisfied.

Since $x_p(n)$ is the periodic extension of $x(n)$, it is clear that $x(n)$ can be recovered from $x_p(n)$ if there is no aliasing in the time domain i.e, if $x(n)$ is time limited to less than the period N of $x_p(n)$.



Consider fig (a), $x(n)$ is a sequence nonzero in the interval $0 \leq n \leq L-1$.

When $N \geq L$, $x(n) = x_p(n)$ for $0 \leq n \leq N-1$ [fig (b)]

Hence $x(n)$ can be recovered from $x_p(n)$.

When $N < L$, it cannot be recovered due to time-domain aliasing.

THE DISCRETE FOURIER TRANSFORM (DFT)

In general, the equally spaced frequency samples $X\left(\frac{2\pi}{N}k\right)$, $k = 0, 1, \ldots, N-1$, do not uniquely represent the original sequence $x(n)$ when $x(n)$ has infinite duration.

Instead, the frequency samples $X\left(\frac{2\pi}{N}k\right)$ correspond to a periodic sequence $x_p(n)$ of period N, where $x_p(n)$ is an aliased version of $x(n)$ given by

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

When the sequence $x(n)$ has a finite duration of length $L \leq N$, then $x_p(n)$ is simply a periodic repetition of $x(n)$, where $x_p(n)$ over a single period is given by

$$x_p(n) = \begin{cases} x(n), & 0 \leq n \leq L-1 \\ 0, & L \leq n \leq N-1 \end{cases}$$

Hence, the frequency samples $X\left(\frac{2\pi k}{N}\right)$, $k=0,1,\ldots,N$ uniquely represent the finite duration sequence $x(n)$. Since $x(n) \equiv x_p(n)$ over a single period (padded by $N-L$ zeros), the original finite duration sequence $x(n)$ can be obtained from the frequency samples $X\left(\frac{2\pi k}{N}\right)$ from Eqn ⑧.

$\Rightarrow$ Summarising: A finite-duration sequence $x(n)$ of length L has a Fourier transform

$$X(\omega) = \sum_{n=0}^{L-1} x(n)\, e^{-j\omega n}, \quad 0 \leq \omega \leq 2\pi$$

Limits indicate $x(n)$ is zero outside the range $0 \leq n \leq L-1$.

When we sample $X(\omega)$ at equally spaced frequencies $\omega_k = \frac{2\pi k}{N}$, $k = 0, 1, \ldots, N-1$ where $N \geq L$, the samples are

$$X(k) \equiv X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{L-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

or
$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}}, \quad k = 0, 1, 2, \ldots, N-1$$

Upper index in the sum has been increased from $L-1$ to $N-1$ since $x(n) = 0$ for $n \geq L$.

The above relation is a formula for transforming a sequence $\{x(n)\}$ of length $L \leq N$ into a sequence of frequency samples $\{X(k)\}$ of length N.

Since, the frequency samples are obtained by evaluating the Fourier transform $X(\omega)$ at a set of N (equally spaced) discrete frequencies, the above relation is called discrete Fourier transform (DFT) of $x(n)$.

Let $W_N = e^{-j\frac{2\pi}{N}}$, then

$$\therefore \boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}}, \quad k = 0, 1, 2, \ldots, N-1$$

The relation which allows us to recover the sequence $x(n)$ from the frequency samples

$$\boxed{\begin{aligned} x(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}kn} \\ \text{or } x(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \end{aligned}}, \quad n = 0,1,2,\ldots,N-1$$

is called the Inverse DFT (IDFT).

DFT AS A LINEAR TRANFORMATION

WKT,

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$k = 0,1,2,\ldots,N-1$$

$$IDFT\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

$$n = 0,1,2,\ldots,N-1$$

where $W_N = e^{-j\frac{2\pi}{N}}$ Twiddle Factor

From the above equations we observe that the computation of each point of the DFT requires N- complex multiplications and $(N-1)$ complex additions.

Hence, N-point DFT requires N^2 complex multiplications and $N(N-1)$ complex additions.

DFT and IDFT can be viewed as linear transformations on sequences $\{x(n)\}$ and $\{X(k)\}$, respectively.

Let us define a N-point vector x_N of the signal sequence $x(n)$, an N-point vector X_N of frequency samples and a $N \times N$ matrix W_N as

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \qquad X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$W_N = \begin{bmatrix} W_N^{0.0} & W_N^{0.1} & \cdots & W_N^{0(N-1)} \\ W_N^{1.0} & W_N^{1.1} & \cdots & W_N^{1(N-1)} \\ \vdots & \vdots & & \\ W_N^{(N-1)0} & W_N^{(N-1).1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

or $$W_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

From the above definitions, the N-point DFT can be expressed in matrix form as

$$\boxed{X_N = W_N x_N}$$

Where W_N is the matrix of the linear transformation. We observe that W_N is a symmetric matrix. If we assume that Inverse of W_N exists then IDFT can be expressed in matrix form as

$$x_N = W_N^{-1} X_N$$

or $$\boxed{x_N = \frac{1}{N} W_N^{*} X_N}$$

where W_N^{*} denotes the complex conjugate of W_N.

PROBLEMS

1) Compute the 4-point DFT of a sequence $x(n) = \{1, 2, 3, 4\}$.

Soln:— I- METHOD :

From the defn of DFT wkt,

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad ; k = 0, 1, .., N-1$$

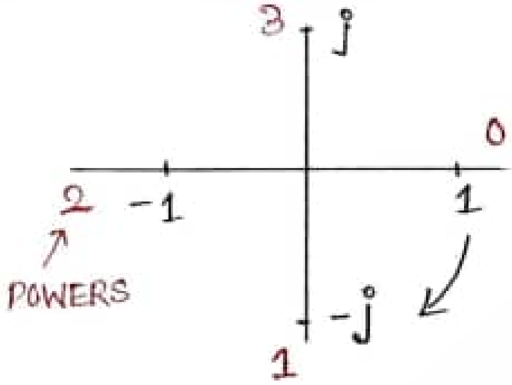
Given $N = 4$,

$$\therefore X(k) = \sum_{n=0}^{3} x(n)\, W_4^{kn}$$

$$= x(0) + x(1)W_4^{k} + x(2)W_4^{2k} + x(3)W_4^{3k}$$

$$\boxed{X(k) = 1 + 2W_4^{k} + 3W_4^{2k} + 4W_4^{3k}} \; ; \; k = 0,1,2,3$$

Twiddle Factors:



For $N=4$, First four twiddle factors.

$W_4^0 = 1 \qquad W_4^2 = -1$

$W_4^1 = -j \qquad W_4^3 = j$

Alternatively, use $W_N = e^{-j\frac{2\pi}{N}}$ and solve.

Let $k=0$, $X(0) = 1 + 2 + 3 + 4 = \underline{\underline{10}}$

Let $k=1$, $X(1) = 1 + 2W_4^1 + 3W_4^2 + 4W_4^3$

$= 1 + 2(-j) + 3(-1) + 4(j)$

$= \underline{\underline{-2+2j}}$

Let $k=2$, $X(2) = 1 + 2W_4^2 + 3W_4^4 + 4W_4^6$

$= 1 + 2(-1) + 3(1) + 4(-1)$

$= \underline{\underline{-2}}$

Let $k=3$, $X(3) = 1 + 2W_4^3 + 3W_4^6 + 4W_4^9$

$= 1 + 2(j) + 3(-1) + 4(-j)$

$= \underline{\underline{-2-2j}}$

$$\therefore \boxed{X(k) = \{10, -2+2j, -2, -2-2j\}}$$

II - METHOD :-

$$X_N = x_N W_N \qquad X_N \ \& \ x_N \rightarrow \text{Row vectors}$$

For $N=4$, $X_4 = x_4 W_4$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

From periodicity property of W_N : $W_N^{K+N} = W_N^K$

and symmetry property : $W_N^{K+\frac{N}{2}} = -W_N^K$

$$X_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\therefore \boxed{X(K) = \{ 10, \ -2+2j, \ -2, \ -2-2j \}}$$

Additional problems on DFT :-

2) $x(n) = \{0, 1, 2, 3\}$ for $N=4$.

ANS : $X(K) = \{ 6, -2+2j, -2, -2-2j \}$

3) $x(n) = \{1, 2, 2, 1\}$ for $N=4$.

ANS : $X(K) = \{ 6, -1-j, 0, -1+j \}$

4) $x(n) = \delta(n) + \delta(n-1) - \delta(n-2) - \delta(n-3)$

$= \{1, 1, -1, -1\}$

ANS : $X(k) = \{0,\ 2-2j,\ 0,\ 2+2j\}$

5) Find the 4-point IDFT of $X(k) = \{10, -2+2j, -2, -2-2j\}$.

Soln:- From the defn of IDFT,

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

$$n = 0, 1, 2, \ldots, N-1$$

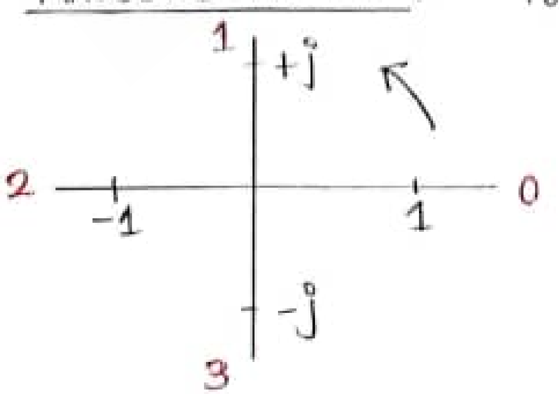
For $N=4$,

$$x(n) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, W_4^{-kn}$$

$$x(n) = \frac{1}{4}\left[X(0) + X(1)\,W_4^{-n} + X(2)\,W_4^{-2n} + X(3)\,W_4^{-3n}\right]$$

$$x(n) = \frac{1}{4}\left[10 + (-2+2j)\,W_4^{-n} + (-2)\,W_4^{-2n} + (-2-2j\,W_4^{-3n}\right], \quad n = 0, 1, 2, 3.$$

TWIDDLE FACTORS: For $N=4$,



$W_4^{0} = 1 \qquad W_4^{-2} = -1$

$W_4^{-1} = j \qquad W_4^{-3} = -j$

Let $n=0$,

$$x(0) = \frac{1}{4}\left[10 - 2 + \cancel{2j} - 2 - 2 - \cancel{2j} \right]$$

$$x(0) = \frac{4}{4} = \underline{\underline{1}}$$

Let $n=1$,

$$x(1) = \frac{1}{4}\left[10 + (-2+2j)\, W_4^{-1} + (-2)\, W_4^{-2} + (-2-2j)\, W_4^{-3} \right]$$

$$= \frac{1}{4}\left[10 + (-2+2j)(j) + (-2)(-1) + (-2-2j)(-j) \right]$$

$$x(1) = \frac{8}{4} = \underline{\underline{2}}$$

Let $n=2$,

$$x(2) = \frac{1}{4}\left[10 + (-2+2j)\, W_4^{-2} + (-2)\, W_4^{-4} + (-2-2j)\, W_4^{-6} \right]$$

$$= \frac{1}{4}\left[10 + (-2+2j)(-1) + (-2)(1) + (-2-2j)(-1) \right]$$

$$x(2) = \frac{12}{4} = \underline{\underline{3}}$$

Let $n=3$, $x(3) = \frac{1}{4}\left[10 + (-2+2j)\, W_4^{-3} + (-2)\, W_4^{-6} + (-2-2j)\, W_4^{-9} \right]$

$$x(3) = \frac{16}{4} = \underline{\underline{4}}$$

$\therefore$ $\boxed{x(n) = \{1, 2, 3, 4\}}$

II – METHOD :–

WKT, $$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

Above equation can also be written as,

$$x_N = \frac{1}{N} X_N W_N^{*} \qquad x_N, X_N \rightarrow \text{Row vectors.}$$

Given $N=4$,

$$x_4 = \frac{1}{4} X_4 W_4^{*}$$

$$x_4 = \frac{1}{4}\begin{bmatrix} 10 & (-2+2j) & -2 & (-2-2j) \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 10-2+2j-2-2-2j \\ 10-2j-2+2+2j-2 \\ 10+2-2j-2+2+2j \\ 10+2j+2+2-2j+2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 12 \\ 16 \end{bmatrix}$$

$$\therefore \boxed{x(n) = \{1, 2, 3, 4\}}$$

NOTE: ANSWER IS IN ROW, REPRESENTED AS COLUMN ONLY FOR CONVENIENCE.

⑥ Find the DFT of $x(n) = \delta(n)$.

Soln:- From defn of DFT,

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

Hence, $$DFT\{\delta(n)\} = \sum_{n=0}^{N-1} \delta(n) W_N^{kn}$$

W.K.T, $$\delta(n) = \begin{cases} 1, & \text{at } n=0 \\ 0, & \text{for } n \neq 0 \end{cases}$$

$$\therefore DFT\{\delta(n)\} = \delta(0) W_N^0 = 1$$ //

or $$\boxed{\delta(n) \xleftrightarrow[\text{DFT}]{\text{N-PT}} 1}$$

⑦ Find the DFT of $x(n) = \delta(n-n_0)$.

Soln:- From the defn of DFT,

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

Hence, $$DFT\{\delta(n-n_0)\} = \sum_{n=0}^{N-1} \delta(n-n_0) W_N^{kn}$$

W.K.T, $$\delta(n-n_0) = \begin{cases} 1, & \text{at } n=n_0 \\ 0, & \text{for } n \neq n_0 \end{cases}$$

$$\therefore DFT\{\delta(n-n_0)\} = \delta(0) W_N^{kn_0} = W_N^{kn_0}$$

$$\boxed{\delta(n-n_0) \xleftrightarrow{\text{DFT}} W_N^{kn_0}}$$

⑧ Find N-point DFT of $x(n) = e^{\frac{j2\pi}{N}k_0 n}$.

Soln:— From definition of DFT

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

Hence, $$DFT\left\{e^{\frac{j2\pi}{N}k_0 n}\right\} = \sum_{n=0}^{N-1} e^{\frac{j2\pi}{N}k_0 n} \cdot e^{-\frac{j2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \left[e^{-\frac{j2\pi}{N}(k-k_0)} \right]^n$$

$$X(k) = \frac{1 - \left[e^{-\frac{j2\pi}{N}(k-k_0)}\right]^N}{1 - e^{-\frac{j2\pi}{N}(k-k_0)}} \qquad \because \sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$$

at $k = k_0$, $X(k) = \frac{0}{0}$, Using L-Hospital's Method,

$$= \left. \frac{0 - \left(-j2\pi\, e^{-j2\pi(k-k_0)}\right)}{0 - \left(-\frac{j2\pi}{N}\, e^{-j\frac{2\pi}{N}(k-k_0)}\right)} \right|_{k=k_0}$$

At $k = k_0$,

$$= \frac{j2\pi \cdot e^0}{\frac{j2\pi}{N} \cdot e^0} = N$$

$\therefore X(k) = N$ at $k = k_0$

at $k \neq k_0$ $X(k) = 0$

$\therefore$ $\boxed{X(k) = N\,\delta(k-k_0)}$ or $\boxed{X(k) = \begin{cases} N & \text{at } k=k_0 \\ 0 & \text{for } k \neq k_0 \end{cases}}$

(9) Find the DFT of $x(n) = e^{-j\frac{2\pi}{N}k_0 n}$.

Soln:- From the defn of DFT,

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

Hence, $$DFT\left\{e^{-j\frac{2\pi}{N}k_0 n}\right\} = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}k_0 n} \cdot e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1}\left[e^{-j\frac{2\pi}{N}(k+k_0)}\right]^n$$

$$= \frac{1-\left[e^{-j\frac{2\pi}{\cancel{N}}(k+k_0)}\right]^{\cancel{N}}}{1-e^{-j\frac{2\pi}{N}(k+k_0)}} \qquad \because \sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$$

at $k=-k_0$, $X(k) = \frac{0}{0}$.

Using L-Hospital's Rule,

$$X(k) = \left.\frac{0-(-j2\pi)\,e^{-j2\pi(k+k_0)}}{0-\left(-j\frac{2\pi}{N}\right)e^{-j\frac{2\pi}{N}(k+k_0)}}\right|_{k=-k_0}$$

at $k=-k_0$, $X(k) = \frac{j2\pi}{j2\pi/N} = N$

at $k \neq -k_0$, $X(k) = 0$

$\therefore$ $\boxed{X(k) = N\,\delta(k+k_0)}$ or $\boxed{X(k) = \begin{cases} N & \text{at } k=k_0 \\ 0 & \text{at } k\neq k_0 \end{cases}}$

or $X(k) = N\,\delta(k-N+k_0)$

(10) Find the N-point DFT of $x(n) = \cos \frac{2\pi}{N} k_0 n$

Sol'n:- From def'n of DFT,

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$DFT\left\{\cos\left(\frac{2\pi}{N} k_0 n\right)\right\} = \sum_{n=0}^{N-1} \cos\left(\frac{2\pi}{N} k_0 n\right) W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \left[\frac{e^{j\frac{2\pi}{N}k_0 n} + e^{-j\frac{2\pi}{N}k_0 n}}{2}\right] e^{-j\frac{2\pi}{N}kn}$$

$\because \cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

$$= \frac{1}{2}\left\{\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-k_0)n} + \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k+k_0)n}\right\}$$

$$= \frac{1}{2}\left\{N\,\delta(k-k_0) + N\,\delta(k+k_0)\right\}$$

$$\therefore \boxed{\cos\left(\frac{2\pi}{N}k_0 n\right) \xleftrightarrow{DFT} \frac{N}{2}\left[\delta(k-k_0) + \delta(k+k_0)\right]}$$

(11) Similar problem, $x(n) = \sin \frac{2\pi}{N} k_0 n$

ANS: $\sin\left(\frac{2\pi}{N}k_0 n\right) \xleftrightarrow{DFT} \frac{N}{2j}\left[\delta(k-k_0) - \delta(k+k_0)\right]$

(12) Find the 5-point DFT of $x(n) = \{1, 1, 1\}$.

Soln: From the defn of DFT, WKT,

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$k = 0, 1, \ldots, N-1$

Given $N = 5$,

$$X(k) = \sum_{n=0}^{4} x(n) W_5^{kn} \quad ; k = 0, 1, 2, 3, 4$$

$$X(k) = x(0) + x(1) W_5^{k} + x(2) W_5^{2k} + x(3) W_5^{3k} + x(4) W_5^{4k}$$

$$\boxed{X(k) = 1 + W_5^{k} + W_5^{2k}} \quad k = 0, 1, 2, 3, 4$$

Twiddle Factors: $N = 5$

$W_5^0 = 1$

$W_5^1 = e^{-j\frac{2\pi}{5}} = \cos\left(\frac{2\pi}{5}\right) - j\sin\left(\frac{2\pi}{5}\right)$

$= 0.3090 - j\,0.95$

$W_5^2 = e^{-j\frac{2\pi \cdot 2}{5}} = \cos\left(\frac{4\pi}{5}\right) - j\sin\left(\frac{4\pi}{5}\right)$

$= -0.809 - j\,0.5878$

$W_5^3 = e^{-j\frac{2\pi \cdot 3}{5}} = \cos\left(\frac{6\pi}{5}\right) - j\sin\left(\frac{6\pi}{5}\right)$

$= -0.809 + j\,0.587$

$$W_5^4 = e^{-j\frac{2\pi}{5}\cdot 4} = \cos\left(\frac{8\pi}{5}\right) - j\sin\left(\frac{8\pi}{5}\right)$$

$$= 0.309 + j\,0.95$$

Evaluate $X(k)$,

at $k=0$, $X(0) = 1 + 1 + 1 = \underline{\underline{3}}$

at $k=1$, $X(1) = 1 + W_5^1 + W_5^2 = \underline{\underline{0.5 - j1.5388}}$

at $k=2$, $X(2) = 1 + W_5^2 + W_5^4 = \underline{\underline{0.5 + j0.3633}}$

at $k=3$, $X(3) = 1 + W_5^3 + W_5^6 = \underline{\underline{0.5 - j0.3633}}$

at $k=4$, $X(4) = 1 + W_5^4 + W_5^8 = \underline{\underline{0.5 + j1.5388}}$

$$\boxed{X(k) = \{3,\ 0.5-j1.5388,\ 0.5+j0.3633,\ 0.5-j0.3633,\ 0.5+j1.5388\}}$$

13 Find IDFT for the following sequence

$$X(k) = \{5,\ 0,\ (1-j),\ 0,\ 1,\ 0,\ (1+j),\ 0\}$$

Soln:- Length of the sequence is 8.

$\therefore N = 8$

From the defn of IDFT,

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

where $n = 0, 1, 2, \ldots, N-1$.

For $N=8$,

$$x(n) = \frac{1}{8}\sum_{k=0}^{7} X(k)\, W_8^{-kn}$$

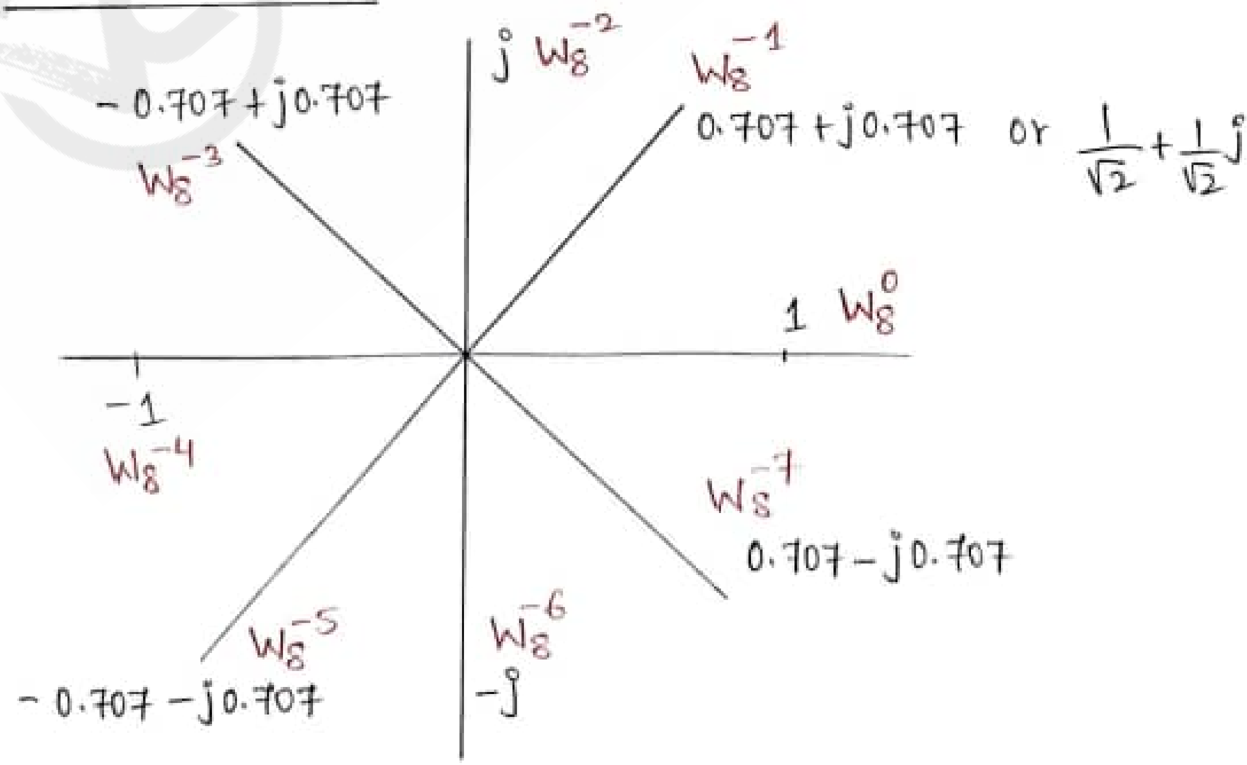
$$x(n) = \frac{1}{8}\Big[X(0)W_8^{0} + X(1)W_8^{-n} + X(2)W_8^{-2n} + X(3)W_8^{-3n} + X(4)W_8^{-4n} + X(5)W_8^{-5n} + X(6)W_8^{-6n} + X(7)W_8^{-7n}\Big]$$

$$x(n) = \frac{1}{8}\Big[5 + 0 + (1-j)W_8^{-2n} + 0 + W_8^{-4n} + 0 + (1+j)W_8^{-6n} + 0\Big]$$

where $n = 0,1,2,\dots,7$

or $\boxed{x(n) = \frac{1}{8}\Big[5 + (1-j)W_8^{-2n} + W_8^{-4n} + (1+j)W_8^{-6n}\Big]}$

TWIDDLE FACTORS : $N=8$



For $n=0$, $x(0) = \frac{1}{8}\left\{ 5 + 1-j + 1 + 1+j \right\}$

$$= \frac{8}{8} = \underline{\underline{1}}$$

For $n=1$, $x(1) = \frac{1}{8}\left\{ 5 + (1-j)W_8^{-2} + W_8^{-4} + (1+j)W_8^{-6} \right\}$

$$= \frac{1}{8}\left\{ 5 + (1-j)j + (-1) + (1+j)(-j) \right\}$$

$$= \frac{1}{8}\left\{ 5 + j - j^2 - 1 - j - j^2 \right\}$$

$$= \frac{6}{8} = \underline{\underline{\frac{3}{4}}}$$

For $n=2$, $x(2) = \frac{1}{8}\left\{ 5 + (1-j)W_8^{-4} + W_8^{-8} + (1+j)W_8^{-12} \right\}$

$$= \frac{1}{8}\left\{ 5 + (1-j)(-1) + 1 + (1+j)(-1) \right\}$$

$$= \frac{1}{8}\left\{ 5 - 1 + j + 1 - 1 - j \right\}$$

$$= \frac{4}{8} = \underline{\underline{\frac{1}{2}}}$$

For $n=3$, $x(3) = \frac{1}{8}\left\{ 5 + (1-j)W_8^{-6} + W_8^{-12} + (1+j)W_8^{-18} \right\}$

$$= \frac{1}{8}\left\{ 5 + (1-j)(-j) + (-1) + (1+j)(j) \right\}$$

$$x(3) = \frac{1}{8}\{5 - j + j^2 - 1 + j + j^2\}$$

$$= \frac{1}{8}\{5 - j - 1 - 1 + j - 1\} = \frac{2}{8} = \underline{\underline{\frac{1}{4}}}$$

for $n=4$, $$x(4) = \frac{1}{8}\{5 + (1-j)W_8^{-8} + W_8^{-16} + (1+j)W_8^{-24}\}$$

$$= \frac{1}{8}\{5 + (1-j) + 1 + (1+j)\}$$

$$= \frac{8}{8} = \underline{\underline{1}}$$

for $n=5$, $$x(5) = \frac{1}{8}\{5 + (1-j)W_8^{-10} + W_8^{-20} + (1+j)W_8^{-30}\}$$

$$= \frac{1}{8}\{5 + (1-j)j + (-1) + (1+j)(-j)\}$$

$$= \frac{1}{8}\{5 + j - j^2 - 1 - j - j^2\}$$

$$= \frac{6}{8} = \underline{\underline{\frac{3}{4}}}$$

for $n=6$, $$x(6) = \frac{1}{8}\{5 + (1-j)W_8^{-12} + W_8^{-24} + (1+j)W_8^{-36}$$

$$= \frac{1}{8}\{5 + (1-j)(-1) + 1 + (1+j)(-1)\}$$

$$= \frac{1}{8}\{5 - 1 + j + 1 - 1 - j\} = \frac{4}{8} = \underline{\underline{\frac{1}{2}}}$$

for $n=7$, $x(7) = \frac{1}{8}\{5 + (1-j)W_8^{-14} + W_8^{-28}$
$+ (1+j)W_8^{-42}\}$

$= \frac{1}{8}\{5 + (1-j)(-j) + (-1) + (1+j)(j)\}$

$= \frac{1}{8}\{5 - j + j^2 - 1 + j + j^2\}$

$= \frac{2}{8} = \underline{\underline{\frac{1}{4}}}$

$$\therefore \boxed{x(n) = \left\{1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, 1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}\right\}}$$

(14) Find DFT of the sequence $x(n) = \begin{cases} 1, & 0 \le n \le 2 \\ 0, & \text{otherwise} \end{cases}$

for $N=8$. Plot $|X(k)|$ & $\angle X(k)$.

Sol'n:- From the defn of DFT,

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

for $N=8$,

$$X(k) = \sum_{n=0}^{7} x(n) W_8^{kn} \quad , 0 \le k \le 7$$

Given $x(n) = \{1, 1, 1\}$

$$\therefore X(k) = 1 + W_8^{k} + W_8^{2k} + 0 + 0 + 0 + 0 + 0$$

$$\boxed{X(k) = 1 + W_8^{k} + W_8^{2k}} \quad , k = 0, 1, 2, \ldots, 7$$

Twiddle Factors for $N = 8$

$W_8^0 = 1$ $\qquad$ $W_8^4 = -1$

$W_8^1 = 0.707 - j0.707$ $\qquad$ $W_8^5 = -0.707 + j0.707$

$W_8^2 = -j$ $\qquad$ $W_8^6 = j$

$W_8^3 = -0.707 - j0.707$ $\qquad$ $W_8^7 = 0.707 + j0.707$

(Locate points on the unit circle in complex plane direction is clockwise for DFT)

for $k=0$, $\quad X(0) = 1 + 1 + 1 = \underline{\underline{3}}$

for $k=1$, $\quad X(1) = 1 + W_8^1 + W_8^2$

$$= 1 + (0.707 - j0.707) + (-j)$$

$$= \underline{\underline{1.707 - j1.707}}$$

for $k=2$, $\quad X(2) = 1 + W_8^2 + W_8^4$

$$= 1 + (-j) + (-1) = \underline{\underline{-j}}$$

for $k=3$, $\quad X(3) = 1 + W_8^3 + W_8^6$

$$= 1 + (-0.707 - j0.707) + j$$

$$= \underline{\underline{0.293 + j0.293}}$$

for $k=4$, $X(4) = 1 + W_8^4 + W_8^8$

$= 1 + (-1) + 1 = \underline{\underline{1}}$

for $k=5$, $X(5) = 1 + W_8^5 + W_8^{10}$

$= 1 + (-0.707 + j0.707) + (-j)$

$= \underline{\underline{0.293 - j0.293}}$

for $k=6$, $X(6) = 1 + W_8^6 + W_8^{12}$

$= 1 + j + (-1) = \underline{\underline{j}}$

for $k=7$, $X(7) = 1 + W_8^7 + W_8^{14}$

$= 1 + 0.707 + j0.707 + j$

$= \underline{\underline{1.707 + j1.707}}$

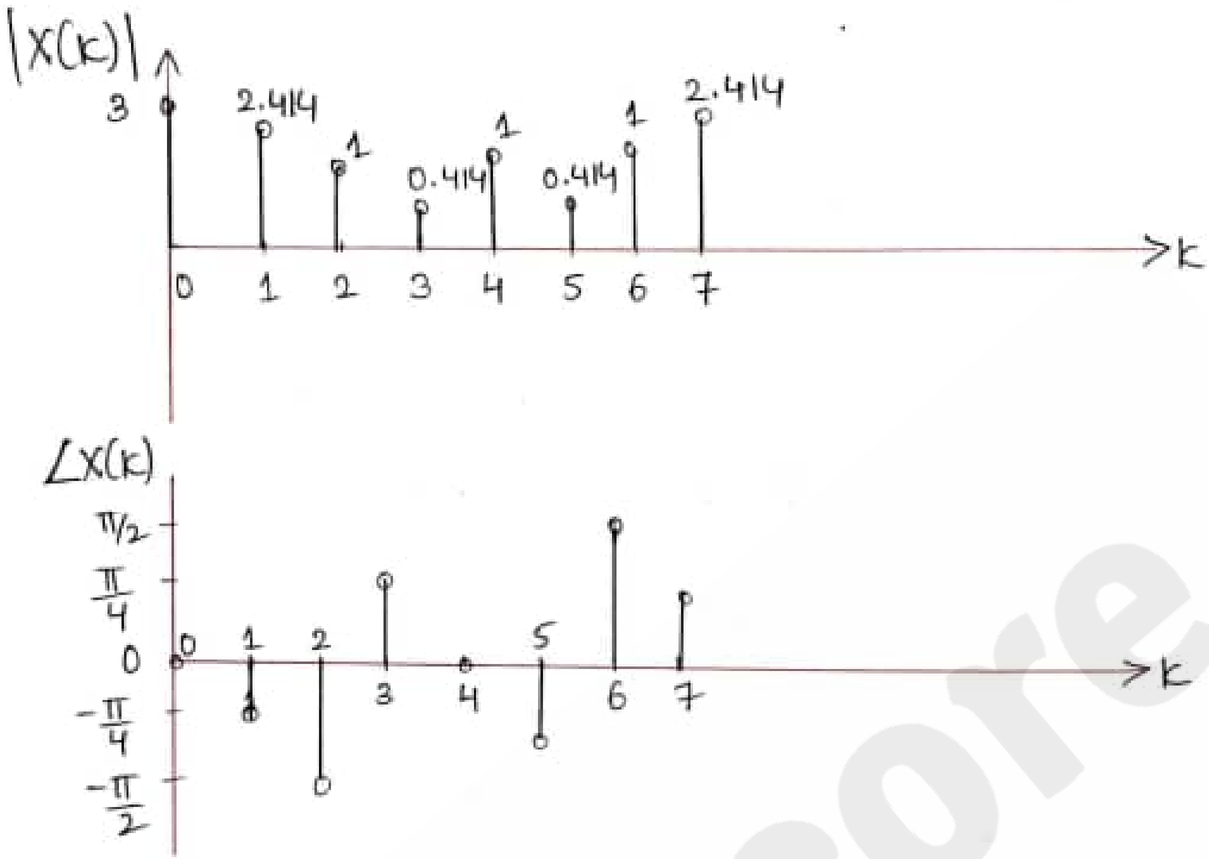
$$\boxed{\therefore X(k) = \{3, (1.707 - j1.707), -j, (0.293 + j0.293), 1, (0.293 - j0.293), j, (1.707 + j1.707)\}}$$

For magnitude Plot,

$|X(k)| = \{3, 2.414, 1, 0.414, 1, 0.414, 1, 2.414\}$

Phase Plot,

$\angle X(k) = \{0, -\frac{\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}\}$


|X(k)|
3
2.414
1
0.414
1
0.414
1
2.414
0 1 2 3 4 5 6 7
k
∠X(k)
π/2
π/4
0
-π/4
-π/2
0 1 2 3 4 5 6 7
k

PROPERTIES OF DFT

1) LINEARITY :-

STATEMENT:

If $x_1(n) \xleftrightarrow{DFT} X_1(k)$

$x_2(n) \xleftrightarrow{DFT} X_2(k)$

then $a x_1(n) + b x_2(n) \xleftrightarrow{DFT} a X_1(k) + b X_2(k)$

Proof: W.K.T., $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$

Let $x(n) = x_1(n)$,

then $X_1(k) = \sum_{n=0}^{N-1} x_1(n) W_N^{kn}$ —— ①

Let $x(n) = x_2(n)$,

then $X_2(k) = \sum_{n=0}^{N-1} x_2(n) W_N^{kn}$ —— ②

Let $x(n) = a x_1(n) + b x_2(n)$, then

$$DFT\{a x_1(n) + b x_2(n)\} = \sum_{n=0}^{N-1} [a x_1(n) + b x_2(n)] W_N^{kn}$$

$$= a \sum_{n=0}^{N-1} x_1(n) W_N^{kn} + b \sum_{n=0}^{N-1} x_2(n) W_N^{kn}$$

$$= a X_1(k) + b X_2(k)$$

Hence the proof.

2) PERIODICITY

STATEMENT: If $x(n) \xleftrightarrow{DFT} X(k)$

then $x(n+N) = x(n)$

$X(k+N) = X(k)$

or $x(n+N) \xleftrightarrow{DFT} X(k)$

Proof: a) TPT $x(n+N) = x(n)$

W.K.T $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$

replace n by $n+N$,

$$x(n+N) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-k(n+N)}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \cdot W_N^{-kN}$$

W.K.T $W_N^{-kN} = 1$

$$\therefore x(n+N) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

$$\boxed{x(n+N) = x(n)}$$

Hence proved.

b) T.P.T $X(k+N) = X(k)$

WKT $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$

replace k by $k+N$,

$$X(k+N) = \sum_{n=0}^{N-1} x(n) W_N^{(k+N)n}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{kn} \cdot W_N^{Nn}$$

$$X(k+N) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad \because W_N^{Nn} = 1$$

$$\therefore \boxed{X(k+N) = X(k)}$$

Hence proved.

c) T.P.T $x(n+N) \xleftrightarrow{DFT} X(k)$

WKT, $DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$

Let $x(n) = x(n+N)$, then

$$DFT\{x(n+N)\} = \sum_{n=0}^{N-1} x(n+N) W_N^{kn}$$

Put $l = n+N \Rightarrow n = l-N$

limits : $n = 0$, $l = N$

$n = N-1$, $l = N-1+N = 2N-1$

$$= \sum_{l=N}^{2N-1} x(l)\, W_N^{k(l-N)}$$

$$= \sum_{l=N}^{2N-1} x(l)\, W_N^{kl} \cdot W_N^{-kN}$$

Since $W_N^{-kN} = 1$ & limits N to $2N-1$ can be replaced by 0 to $N-1$ using periodicity.

$$DFT\{x(n+N)\} = \sum_{l=0}^{N-1} x(l)\, W_N^{kl}$$

$$\therefore \boxed{DFT\{x(n+N)\} = X(k)}$$

Hence proved.

③ CIRCULAR TIME-SHIFT

Circular time shift operation on an N-point sequence $x(n)$ is given by $x((n-m))_{\text{mod } N}$, $x((n-m))_N$ or $x(n-m, \text{mod } N)$.

STATEMENT: If $x(n) \xleftrightarrow{DFT} X(k)$

then $x((n-m))_N \xleftrightarrow{DFT} X(k)\, e^{-j\frac{2\pi}{N}km}$

or $x((n-m))_N \xleftrightarrow{DFT} X(k)\, W_N^{km}$

$|||^{ly}$ $x((n+m))_N \xleftrightarrow{DFT} X(k)\, e^{j\frac{2\pi}{N}km}$

or $x((n+m))_N \xleftrightarrow{DFT} X(k)\, W_N^{-km}$

Proof: W.K.T $X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$

Let $x(n) = x((n-m))_N$

then $\text{DFT}\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N \cdot W_N^{kn}$

Let $n-m = l \Rightarrow n = l+m$, limits:

at $n=0$, $l=-m$

$n=N-1$, $l=N-1-m$

$$\text{DFT}\{x((n-m))_N\} = \sum_{l=-m}^{N-1-m} x(l)\, W_N^{k(l+m)}$$

$$= \sum_{l=0}^{N-1} x(l)\, W_N^{kl} \cdot W_N^{km} \quad \text{using periodicity}$$

$$\therefore \boxed{\text{DFT}\{x((n-m))_N\} = X(k)\, W_N^{km}}$$

Example (15) If $x(n) = \{1, 2, 3, 4\}$ find $y(n)$, given $y(n) = x((n-3))_4$.

Soln:- Given $y(n) = x((n-3))_4$

at $n=0$, $y(0) = x((-3))_4 = x(4-3)$

$= x(1) = \underline{\underline{2}}$

at $n=1$, $y(1) = x((1-3))_4 = x((-2))_4$

$= x(4-2) = x(2) = \underline{\underline{3}}$

at $n=2$, $y(2) = x((2-3))_4 = x((-1))_4$

$= x(4-1) = x(3) = \underline{\underline{4}}$

at $n=3$, $y(3) = x((3-3))_4 = x(0)$

$= \underline{\underline{1}}$

$$\therefore \boxed{y(n) = \{2, 3, 4, 1\}}$$

(16) If $x(n) = \{1, 2, 2, 1\}$ find the DFT of $y(n) = x((n-2))_4$.

Soln:- First find $X(k)$.

$X_N = x_N W_N$

$$X(k) = \begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$\therefore X(k) = [6 \quad -1-j \quad 0 \quad -1+j]$

Given $y(n) = x((n-2))_4$

Taking DFT and applying circular time-shift property,

$$Y(k) = X(k)\, W_4^{2k}, \quad k = 0,1,2,3$$

at $k=0$, $Y(0) = X(0)\, W_4^{0} = (6)(1) = 6$

at $k=1$, $Y(1) = X(1)\, W_4^{2} = (-1-j)(-1) = 1+j$

at $k=2$, $Y(2) = X(2)\, W_4^{4} = 0$

at $k=3$, $Y(3) = X(3)\, W_4^{6} = (-1+j)(-1) = 1-j$

$\therefore \boxed{Y(k) = \{6,\ 1+j,\ 0,\ 1-j\}}$

4) TIME REVERSAL

STATEMENT: If $x(n) \xleftrightarrow[N]{DFT} X(k)$

then. $x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT}$

$$X((-k))_N = X(N-k)$$

Proof:- From defn of DFT, we have

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

Hence, $DFT\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n) W_N^{kn}$

Put $m = N-n \Rightarrow n = N-m$

limits: at $n=0$, $m=N$

$n=N-1$, $m = N-(N-1) = 1$

$$\therefore DFT\{x(N-n)\} = \sum_{m=N}^{1} x(m) W_N^{k(N-m)}$$

$$= \sum_{m=0}^{N-1} x(m) W_N^{kN} \cdot W_N^{-km}$$ using periodicity property.

Also, $W_N^{kN} = 1$

$$DFT\{x(N-n)\} = \sum_{m=0}^{N-1} x(m) W_N^{(-k)m}$$

or $$\boxed{DFT\{x(N-n)\} = X((-k))_N}$$

(17) Example : $x(n) = \{5, 2, -3, 7\}$. Find

$y(n) = x((-n))_4$.

Sol'n: Consider $y(n) = x((-n))_4$

or $y(n) = x(4-n)$

at $n=0$, $y(0) = x(4) = x(0) = 5$

$n=1$, $y(1) = x(4-1) = x(3) = 7$

$n=2$, $y(2) = x(4-2) = x(2) = -3$

$n=3$, $y(3) = x(4-3) = x(1) = 2$

$\therefore \boxed{y(n) = \{5, 7, -3, 2\}}$

(18) Given $x(n) = \{1, 2, 3, 4\}$ with 4-point DFT $X(K) = \{10, -2+2j, -2, -2-2j\}$. Find the 4-point DFT of the sequence $y(n) = \{1, 4, 3, 2\}$.

Sol'n:- Observing $x(n)$ & $y(n)$ we find that

$$y(n) = x((-n))_4$$

Taking DFT on both sides and applying time reversal property

$$Y(K) = X((-K))_4 = X(4-K)$$

at $k=0$, $Y(0) = X(0) = 10$

$k=1$, $Y(1) = X(4-1) = X(3) = -2-2j$

$k=2$, $Y(2) = X(4-2) = X(2) = -2$

$k=3$, $Y(3) = X(4-3) = X(1) = -2+2j$

$$\therefore \boxed{Y(k) = \{10, -2-2j, -2, -2+2j\}}$$

5) CIRCULAR FREQUENCY SHIFT

STATEMENT : If $x(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X(k)$

then $x(n)\, e^{\frac{j2\pi}{N}mn} \underset{N}{\overset{DFT}{\longleftrightarrow}} X((k-m))_N$

or $x(n)\, W_N^{-mn} \overset{DFT}{\longleftrightarrow} X((k-m))_N$

$|||^{ly}$

$x(n)\, e^{-\frac{j2\pi}{N}mn} \overset{DFT}{\longleftrightarrow} X((k+m))_N$

or $x(n)\, W_N^{mn} \overset{DFT}{\longleftrightarrow} X((k+m))_N$

Proof:- $X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$

$$DFT\left\{x(n)\, e^{\frac{j2\pi}{N}mn}\right\} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi}{N}mn} \cdot e^{-\frac{j2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi}{N}(k-m)n}$$

$$= X((k-m))_N$$

Hence the proof.

(19) Find N-point DFT of $x_1(n) = \cos\left(\frac{2\pi}{N}k_0 n\right) x(n)$.

Sol'n:- Given $x_1(n) = \cos\left(\frac{2\pi}{N}k_0 n\right) x(n)$

$$= \left[\frac{e^{j\frac{2\pi}{N}k_0 n} + e^{-j\frac{2\pi}{N}k_0 n}}{2}\right] x(n)$$

If $x(n) \xleftrightarrow{DFT} X(k)$

Using circular frequency shift property,

$$e^{j\frac{2\pi}{N}k_0 n} x(n) \xleftrightarrow{DFT} X((k-k_0))_N$$

$$e^{-j\frac{2\pi}{N}k_0 n} x(n) \xleftrightarrow{DFT} X((k+k_0))_N$$

Hence DFT $\{x_1(n)\} = X_1(k)$

and $\boxed{X_1(k) = \frac{1}{2}\left[X((k-k_0))_N + X((k+k_0))_N\right]}$

(20) Find N-point DFT of $x_2(n) = \sin\left(\frac{2\pi}{N}k_0 n\right) x(n)$

ANS : $X_2(k) = \frac{1}{2j}\left[X((k-k_0))_N - X((k+k_0))_N\right]$

Also, $x(n)\cos\left(\frac{4\pi}{N}n\right) \xleftrightarrow[N]{DFT} \frac{1}{2}\left[X((k-2))_N + X((k+2))_N\right]$

$x(n)\sin\left(\frac{8\pi}{N}n\right) \xleftrightarrow[N]{DFT} \frac{1}{2j}\left[X((k-4))_N - X((k+4))_N\right]$

6) CIRCULAR CONVOLUTION / TIME-DOMAIN CONVOLUTION OR FREQUENCY DOMAIN MULTIPLICATION

STATEMENT:

If $x(n) \xleftrightarrow{DFT} X(k)$

& $h(n) \xleftrightarrow{DFT} H(k)$

then $y(n) = x(n) \, Ⓝ \, h(n) \xleftrightarrow{DFT} X(k)H(k) = Y(k)$

Proof:

Consider two sequences $x(n)$ & $h(n)$ of length N, then circular convolution is defined as,

$$y(n) = x(n) \, Ⓝ \, h(n) = \sum_{m=0}^{N-1} x(m)\, h((n-m))_N$$

$$DFT\{y(n)\} = Y(k) = \sum_{n=0}^{N-1} y(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x(m)\, h((n-m))_N \cdot W_N^{kn}$$

Interchanging order of summations,

$$= \sum_{m=0}^{N-1} x(m) \sum_{n=0}^{N-1} h((n-m))_N\, W_N^{kn}$$

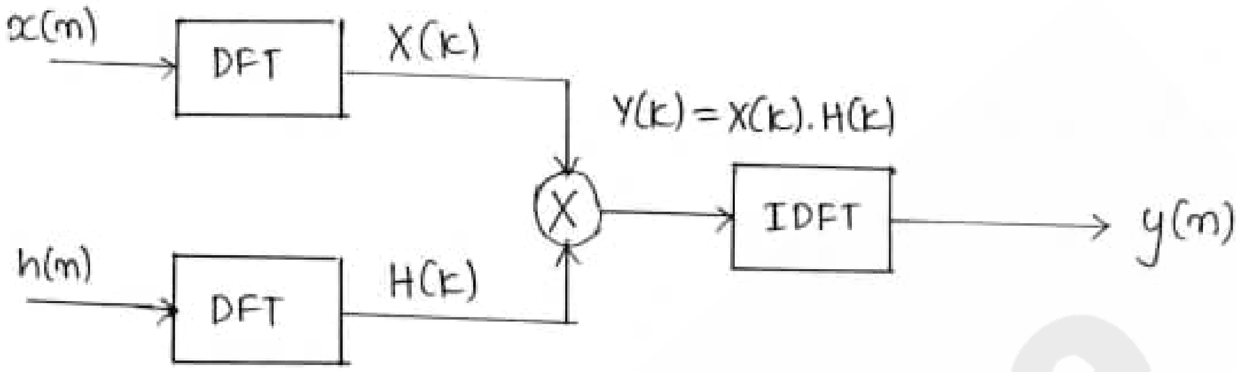
$$= \sum_{m=0}^{N-1} x(m)\, W_N^{km} H(k)$$

Using Time-shift property, $h((n-m))_N \xleftrightarrow{DFT} W_N^{km} H(k)$

$$= X(k)\, H(k)$$

Hence the proof.

II METHOD : Circular convolution using DFT & IDFT (stockham's method)



The method involves taking the N point DFTs of $x(n)$ & $h(n)$ both of length N points.

The respective DFTs are multiplied elementwise.

Then taking IDFT of the sequence $Y(k)$ to obtain $y(n)$.

(21) Compute the circular convolution of sequences $x(n) = \{1, 2, 3, 4\}$ and $h(n) = \{1, 2, 2\}$.

Soln:- Given $x(n) = \{1, 2, 3, 4\}$

& $h(n) = \{1, 2, 2\}$

Length of $x(n)$, $N_1 = 4$. Length of $h(n)$, $N_2 = 3$.

The convolution length $N = \max(N_1, N_2)$

$= \max(4, 3)$

$$\boxed{N = 4}$$

Since $h(n)$ is of length 3, pad one zero.

$\therefore h(n) = \{1, 2, 2, 0\}$

I- METHOD : Time-Domain Approach or Concentric circle Method

From the def'n of circular convolution,

$$y(n) = x(n) \circledN h(n) = \sum_{m=0}^{N-1} x(m)\, h((n-m))_N$$

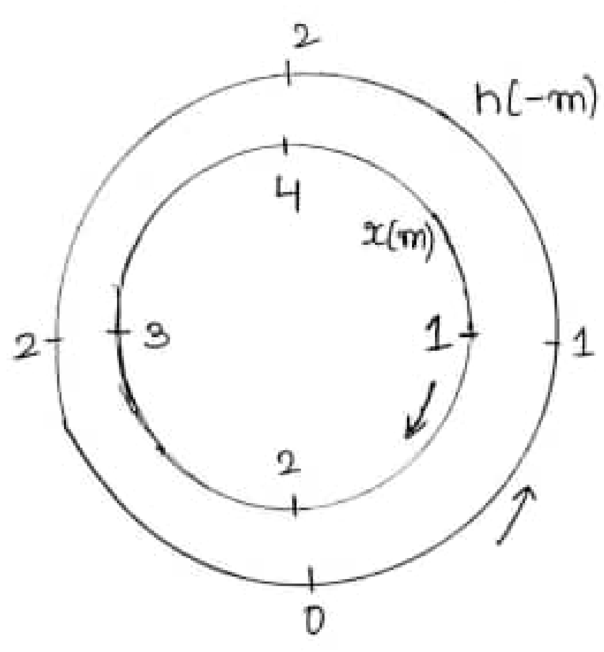
where $n = 0$ to $N-1$.

Given $N=4$,

$$\therefore y(n) = \sum_{m=0}^{3} x(m)\, h((n-m))_4 \;, \quad n = 0,1,2,3$$

$$= x(0)\, h((n))_4 + x(1)\, h((n-1))_4 + x(2)\, h((n-2))_4 + x(3)\, h((n-3))_4$$

at $n=0$, $y(0) = x(0)h(0) + x(1)\, h((-1))_4 + x(2)\, h((-2))_4 + x(3)\, h((-3))_4$



$$y(0) = x(0)h(0) + x(1)\,h(3) + x(2)h(2) + x(3)h(1)$$

$$= (1)(1) + (2)(0) + (3)(2) + (4)(2)$$

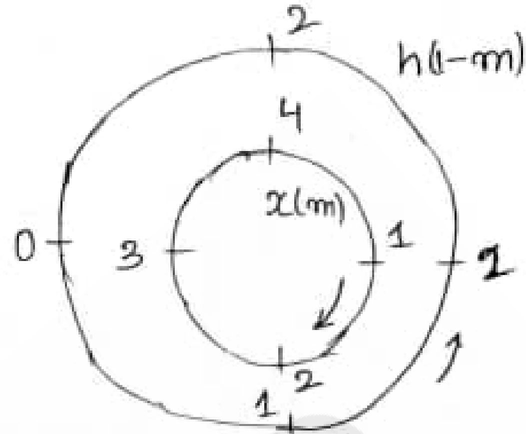
$$= 1 + 0 + 6 + 8$$

$$y(0) = \underline{\underline{15}}$$

at $n=1$, $y(1) = x(0)\,h((1))_4 + x(1)\,h(0) + x(2)\,h(3) + x(3)\,h(2)$

$y(1) = 2 + 2 + 0 + 8$

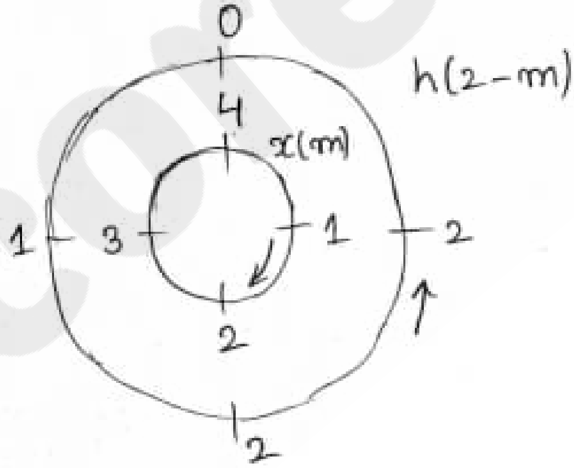
$y(1) = \underline{\underline{12}}$



at $n=2$, $y(2) = x(0)\,h(2) + x(1)\,h(1) + x(2)\,h(0) + x(3)\,h(3)$

$y(2) = 2 + 4 + 3 + 0$

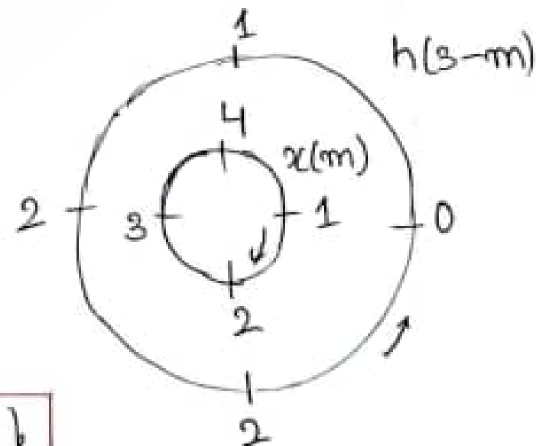
$y(2) = \underline{\underline{9}}$



at $n=3$, $y(3) = x(0)\,h(3) + x(1)\,h(2) + x(2)\,h(1) + x(3)\,h(0)$

$y(3) = 0 + 4 + 6 + 4$

$y(3) = \underline{\underline{14}}$



$$\therefore \boxed{y(n) = \{15,\ 12,\ 9,\ 14\}}$$

<u>verification</u>:

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ 12 \\ 9 \\ 14 \end{bmatrix}$$

II - METHOD: Using DFT & IDFT equations or Transform Domain Approach or Stockham's method

$$y(n) = x(n) \, (N) \, h(n)$$

Applying DFT,

$$Y(k) = X(k) \cdot H(k)$$

(i) To find X(k). $X_N = x_N W_N$

$$X(k) = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\boxed{X(k) = \{10, -2+2j, -2, -2-2j\}}$$

(ii) To find H(k). $H_N = h_N W_N$

$$H(k) = \begin{bmatrix} 1 & 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\boxed{H(k) = \{5, -1-2j, 1, -1+2j\}}$$

(iii) Find the product of the sequence $X(k)$ & $H(k)$.

$$Y(k) = X(k) \cdot H(k)$$

$$\boxed{Y(k) = \{50,\ 6+2j,\ -2,\ 6-2j\}}$$

(iv) To find $y(n)$

$$y(n) = \text{IDFT}\{Y(k)\} = \frac{1}{N} Y_N W_N^*$$

$$y(n) = \frac{1}{4}\begin{bmatrix} 50 & 6+2j & -2 & 6-2j \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$\therefore \boxed{y(n) = \{15,\ 12,\ 9,\ 14\}}$$

7) MULTIPLICATION OF TWO SEQUENCES OR MODULATION PROPERTY OR CONVOLUTION IN FREQUENCY DOMAIN

STATEMENT : If $x_1(n) \xleftrightarrow{DFT} X_1(k)$

& $x_2(n) \xleftrightarrow{DFT} X_2(k)$

then $x_1(n)\, x_2(n) \xleftrightarrow{DFT} \frac{1}{N}\left[X_1(k) \text{ N } X_2(k)\right]$

Proof:- Consider two sequences $x_1(n)$ & $x_2(n)$ of length N, with DFT $X_1(k)$ & $X_2(k)$.

$$IDFT\{X_1(k)\} = x_1(n) = \frac{1}{N}\sum_{l=0}^{N-1} X_1(l)\, W_N^{-ln}$$

$$IDFT\{X_2(k)\} = x_2(n) = \frac{1}{N}\sum_{m=0}^{N-1} X_2(m)\, W_N^{-mn}$$

Let $y(n) = x_1(n).\, x_2(n)$

then,

$$DFT\{y(n)\} = Y(k) = \sum_{n=0}^{N-1} y(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x_1(n)\, x_2(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \left\{ \frac{1}{N} \sum_{l=0}^{N-1} X_1(l)\, W_N^{-ln} \right\} \cdot \left\{ \frac{1}{N} \sum_{m=0}^{N-1} X_2(m)\, W_N^{-mn} \right\} \times W_N^{kn}$$

$$= \frac{1}{N^2} \sum_{l=0}^{N-1} X_1(l) \cdot \sum_{m=0}^{N-1} X_2(m) \sum_{n=0}^{N-1} W_N^{(k-l-m)n}$$

If, $k - l - m = pN$

or $m = k - l - pN = ((k-l))_N$

or $l = k - m - pN = ((k-m))_N$

then, $\sum_{n=0}^{N-1} W_N^{(k-l-m)n} = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N} pNn}$

$$= \sum_{n=0}^{N-1} 1 = N$$

$$\therefore\ Y(k) = \frac{1}{N^2} \sum_{l=0}^{N-1} X_1(l)\, X_2((k-l))_N \cdot N$$

$$\boxed{Y(k) = \frac{1}{N} \left[X_1(k) \, (N) \, X_2(k) \right]}$$

8) SYMMETRY PROPERTY OF A COMPLEX VALUED SEQUENCE

STATEMENT: If $x(n) \xleftrightarrow{DFT} X(k)$

then $x^*(n) \xleftrightarrow{DFT} X^*(N-k) = X^*((-k))_N$

or $x^*((-n))_N = x^*(N-n) \xleftrightarrow{DFT} X^*(k)$

Proof: From defn of DFT, we know that

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

Taking complex conjugate on both sides,

$$X^*(k) = \sum_{n=0}^{N-1} x^*(n)\, W_N^{-kn}$$

replace k by N-k,

$$X^*(N-k) = \sum_{n=0}^{N-1} x^*(n)\, W_N^{-(N-k)n}$$

$$= \sum_{n=0}^{N-1} x^*(n)\, W_N^{-Nn} \cdot W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x^*(n)\, W_N^{kn} \qquad \because W_N^{-Nn} = 1$$

$$X^*(N-k) = DFT\{x^*(n)\}$$

or $\boxed{x^*(n) \xleftrightarrow[N]{DFT} X^*(N-k)}$

9) SYMMETRY PROPERTY OF REAL-VALUED SEQUENCE

STATEMENT: If $x(n) \xleftrightarrow{DFT} X(k)$

and $x(n)$ is real, then

$$X(k) = X^*(N-k)$$

Proof:- W.K.T, $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$

Taking complex conjugate on both sides,

$$X^*(k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-kn}$$

Since $x(n)$ is real, $x^*(n) = x(n)$

$$X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{-kn}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{-kn} \cdot W_N^{Nn} \qquad \because W_N^{Nn} = 1$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{(N-k)n}$$

$$X^*(k) = X(N-k)$$

or $\boxed{X(k) = X^*(N-k)}$

(22) The first five points of the 8-point DFT of a real-valued sequence are

$$\{ 0.25, (0.125 - j0.3018), 0, (0.125 - j0.0518), 0 \}$$

Determine the remaining 3 points.

Soln:- Given,

$X(0) = 0.25$

$X(1) = 0.125 - j0.3018$

$X(2) = 0$

$X(3) = 0.125 - j0.518$

$X(4) = 0$

Since $x(n)$ is real valued

$$X^*(k) = X(N-k)$$

$$\text{or } X(k) = X^*(N-k)$$

Here, $N=8$,

Then, for $k=5$, $X(5) = X^*(8-5) = X^*(3)$

$$= (0.125 - j0.0518)^*$$

$$X(5) = \underline{\underline{0.125 + j0.0518}}$$

for $k=6$, $X(6) = X^*(8-6) = X^*(2)$

$$= \underline{\underline{0}}$$

for $k=7$, $X(7) = X^*(8-7) = X^*(1)$

$$= (0.125 - j0.3018)^*$$

$$= \underline{\underline{0.125 + j0.3018}}$$

10) CIRCULAR CORRELATION

For complex valued sequences $x(n)$ and $y(n)$

If $x(n) \xleftrightarrow{DFT} X(k)$

$y(n) \xleftrightarrow{DFT} Y(k)$

then $\tilde{r}_{xy}(l) \xleftrightarrow{DFT} \tilde{R}_{xy}(l) = X(k).Y^*(k)$

where $\tilde{r}_{xy}(l)$ is the circular cross-correlation sequence defined by

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N$$

Proof: We can express $\tilde{r}_{xy}(l)$ as the circular convolution of $x(n)$ with $y^*(-n)$.

i.e, $\tilde{r}_{xy}(l) = x(l) \, (N) \, y^*(-l)$

taking DFT on both sides,

$$DFT\{\tilde{r}_{xy}(l)\} = DFT\{x(l) \, (N) \, y^*(-l)\}$$

$$\boxed{\tilde{R}_{xy}(k) = X(k).Y^*(k)}$$

using circular convolution & complex conjugate property

If $x(n) = y(n)$,

$$\tilde{r}_{xx}(l) \xleftrightarrow{DFT} \tilde{R}_{xx}(k) = |X(k)|^2$$

11) PARSEVAL'S THEOREM

For complex valued sequence $x(n)$ & $y(n)$

If $x(n) \xleftrightarrow{DFT} X(k)$

$y(n) \xleftrightarrow{DFT} Y(k)$

then $$\sum_{n=0}^{N-1} x(n).y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k).Y^*(k)$$

Proof:- Circular cross correlation of two sequences is defined as

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N$$

at $l=0$, $$\tilde{r}_{xy}(0) = \sum_{n=0}^{N-1} x(n)\, y^*(n) \quad \text{—— (a)}$$

W.K.T, $$\text{IDFT}\{\tilde{R}_{xy}(k)\} = \tilde{r}_{xy}(l)$$

or $$\tilde{r}_{xy}(l) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{R}_{xy}(k)\, e^{\frac{j2\pi}{N}kl}$$ from defn of IDFT

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k)\, e^{\frac{j2\pi}{N}kl}$$ from circular correlation property

at $l=0$,

$$\tilde{r}_{xy}(0) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k) \quad \text{—— (b)}$$

Equating RHS of Eqn's ⓐ & ⓑ,

$$\boxed{\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k)}$$

If $x(n) = y(n)$, then

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

which is the energy in the finite duration sequence $x(n)$ in terms of the frequency components $X(k)$.

(23) Determine N-point circular correlation of $x(n) = \cos \frac{2\pi}{N} n$ & $y(n) = \sin \frac{2\pi}{N} n$.

Soln:- Given $x(n) = \cos \frac{2\pi n}{N}$

Taking N-point DFT,

$$X(k) = \frac{N}{2} \left[\delta(k-1) + \delta(k+1) \right]$$

Given $y(n) = \sin \frac{2\pi n}{N}$

Its N-point DFT, $Y(k) = \frac{N}{2j} \left[\delta(k-1) - \delta(k+1) \right]$

From circular correlation property,

$$R_{x\tilde{y}}(k) = X(k).\, Y^*(k)$$

$$R_{x\tilde{y}}(k) = \frac{N}{2}\left[\delta(k-1) + \delta(k+1)\right] \cdot \left\{\frac{N}{2j}\left[\delta(k-1) - \delta(k+1)\right]\right\}^*$$

$$= \frac{N}{2}\left[\delta(k-1) + \delta(k+1)\right] \left\{\frac{-N}{2j}\left[\delta(k-1) - \delta(k+1)\right]\right\}$$

$$= -\frac{N^2}{4j}\left\{\left[\delta(k-1)\right]^2 + \delta(k+1)\delta(k-1) - \delta(k-1)\delta(k+1) - \left[\delta(k+1)\right]^2\right\}$$

WKT, $\delta(k+1)\,\delta(k-1) = 0$, $\left[\delta(k-1)\right]^2 = \delta(k-1)$

and $\left[\delta(k+1)\right]^2 = \delta(k+1)$.

$$= -\frac{N^2}{4j}\left[\delta(k-1) - \delta(k+1)\right]$$

$$R_{x\tilde{y}}(k) = -\frac{N}{2}\left\{\frac{N}{2j}\left[\delta(k-1) - \delta(k+1)\right]\right\}$$

taking IDFT,

$$\boxed{r_{x\tilde{y}}(l) = -\frac{N}{2}\sin\frac{2\pi}{N}n}$$

(24) Find the circular autocorrelation of $x(n) = \cos \frac{2\pi}{N} n$

Soln:— Given $x(n) = \cos\left(\frac{2\pi}{N} n\right)$

Taking DFT,

$$X(k) = \frac{N}{2}\left[\delta(k-1) + \delta(k+1)\right]$$

WKT $\tilde{R}_{xy}(k) = X(k) \cdot Y^*(k)$

since $x(n) = y(n)$,

$$\tilde{R}_{xx}(k) = X(k) \cdot X^*(k)$$

$$\tilde{R}_{xx}(k) = \left\{\frac{N}{2}\left[\delta(k-1) + \delta(k+1)\right]\right\} \cdot \left\{\frac{N}{2}\left[\delta(k-1) + \delta(k+1)\right]\right\}^*$$

$$= \frac{N}{2} \cdot \frac{N}{2}\left[\delta(k-1) + \delta(k+1)\right]$$

taking IDFT,

$$\boxed{\tilde{r}_{xx}(n) = \frac{N}{2} \cos\left(\frac{2\pi}{N} n\right)}$$

(25) Find circular autocorrelation of the sequence $x(n) = \{1, 2, 3, 4\}$.

Soln:- Given $x(n) = \{1, 2, 3, 4\}$, $N = 4$

Find the 4-point DFT,

we get $X(k) = \{10, -2+2j, -2, -2-2j\}$

Autocorrelation of the sequence $x(n)$ is

$$\tilde{r}_{xx}(l) \xleftrightarrow{DFT} \tilde{R}_{xx}(k) = X(k) . X^*(k)$$

$$= |X(k)|^2$$

$$\tilde{R}_{xx}(0) = |X(0)|^2 = 10^2 = 100$$

$$\tilde{R}_{xx}(1) = |X(1)|^2 = |(-2+2j)|^2$$

$$= |\sqrt{4+4}|^2 = 8$$

$$\tilde{R}_{xx}(2) = |X(2)|^2 = |-2|^2 = 4$$

$$\tilde{R}_{xx}(3) = |X(3)|^2 = |(-2-2j)|^2 = 8$$

$$\therefore \boxed{\tilde{R}_{xx}(k) = \{100, 8, 4, 8\}}$$

To find $r_{x\tilde{x}}(l)$:

$$r_{x\tilde{x}}(l) = IDFT\{R_{x\tilde{x}}(k)\}$$

$$r_{x\tilde{x}}(l) = \frac{1}{4}\begin{bmatrix}100 & 8 & 4 & 8\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}120 & 96 & 88 & 96\end{bmatrix}$$

$$\boxed{r_{x\tilde{x}}(l) = \{30, 24, 22, 24\}}$$

(26) Find the circular correlation given $x(n) = \{1, 2, 3, 4\}$ and $y(n) = \{1, 2, 2, 0\}$

Soln:- Find $X(k)$

$$X(k) = \{10, -2+2j, -2, -2-2j\}$$

Find $Y(k)$

$$Y(k) = \{5, -1-2j, 1, -1+2j\}$$

$$\Rightarrow Y^*(k) = \{5, -1+2j, 1, -1-2j\}$$

Circular correlation property,

$$R_{x\tilde{y}}(k) = X(k).\, Y^*(k)$$

$$= \{ 50, -2-6j, -2, -2+6j \}$$

Taking IDFT,

$$\boxed{r_{x\tilde{y}}(l) = \{ 11, 16, 13, 10 \}}$$

(27) Given $x(n) = \{1, 2, 3, 4\}$ find the energy and hence verify Parseval's theorem.

Soln:- Given $x(n) = \{1, 2, 3, 4\}$

Energy of the signal $x(n)$ is given by

$$E = \sum_{n=-\infty}^{+\infty} |x(n)|^2$$

$$= \sum_{n=0}^{3} |x(n)|^2 = |x(0)|^2 + |x(1)|^2 + |x(2)|^2 + |x(3)|^2$$

$$= 1^2 + 2^2 + 3^2 + 4^2$$

$$\boxed{E = 30}$$

Find DFT of $x(n)$,

$$X(k) = \{ 10, -2+2j, -2, -2-2j \}$$

From Parseval's Theorem,

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

$$E = \frac{1}{4} \sum_{k=0}^{3} |X(k)|^2$$

$$= \frac{1}{4} \left[|X(0)|^2 + |X(1)|^2 + |X(2)|^2 + |X(3)|^2 \right]$$

$$= \frac{1}{4} \left[100 + 8 + 4 + 8 \right]$$

$$E = \frac{120}{4} = \underline{\underline{30}}$$

Hence Parseval's Theorem is verified.

Definition of Even and Odd Symmetry and time reversal.

An N-point sequence is said to be circularly even if it is symmetric about the point zero on the circle.

ie, $\boxed{x(N-n) = x(n)}$, $1 \leq n \leq N-1$

An N-point sequence is said to be circularly odd if it is antisymmetric about the point zero on the circle.

ie, $\boxed{x(N-n) = -x(n)}$, $1 \leq n \leq N-1$

The time reversal of an N-point sequence is attained by reversing its samples about the point zero on the circle. Thus the sequence $x((-n))_N$ is given by

$$\boxed{x((-n))_N = x(N-n)}, \quad 0 \leq n \leq N-1$$

For periodic sequence $x_p(n)$, defn of even and odd sequences :

even: $x_p(n) = x_p(-n) = x_p(N-n)$

odd : $x_p(n) = -x_p(-n) = -x_p(N-n)$

If $x_p(n)$ is complex valued, then

conjugate even : $x_p(n) = x_p^*(N-n)$

conjugate odd : $x_p(n) = -x_p^*(N-n)$

Sequence $x_p(n)$ can be decomposed as

$$x_p(n) = x_{pe}(n) + x_{po}(n)$$

where

$$\boxed{\begin{aligned} x_{pe}(n) &= \frac{1}{2}\left[x_p(n) + x_p^*(N-n)\right] \\ x_{po}(n) &= \frac{1}{2}\left[x_p(n) - x_p^*(N-n)\right] \end{aligned}}$$

Symmetry properties of the DFT

Let us assume that $x(n)$ and its DFT $X(k)$ are both complex valued.

Then, $x(n) = x_R(n) + j\, x_I(n)$, $n = 0, 1, \ldots N-1$ — ①

$X(k) = X_R(k) + j\, X_I(k)$, $k = 0, 1, \ldots N-1$ — ②

From the defn of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad \text{—— ③}$$

substituting eqn ① in eqn ③,

$$X(k) = \sum_{n=0}^{N-1} \left[x_R(n) + j\, x_I(n)\right] e^{-j\frac{2\pi}{N}kn}$$

and since, $e^{j\theta} = \cos\theta + j\sin\theta$

$$X(k) = \sum_{n=0}^{N-1} \left[x_R(n) + j\, x_I(n) \right] \left[\cos \frac{2\pi}{N} kn - j \sin \frac{2\pi}{N} kn \right]$$

$$= \sum_{n=0}^{N-1} \left[x_R(n) \cos\left(\frac{2\pi}{N} kn\right) - j\, x_R(n) \sin\left(\frac{2\pi}{N} kn\right) + j\, x_I(n) \cos\left(\frac{2\pi}{N} kn\right) + x_I(n) \sin\left(\frac{2\pi}{N} kn\right) \right] \quad \text{④}$$

Comparing eqn's ② & ④,

$$\boxed{\begin{aligned} X_R(k) &= \sum_{n=0}^{N-1} \left[x_R(n) \cos\left(\frac{2\pi}{N} kn\right) + x_I(n) \sin\left(\frac{2\pi}{N} kn\right) \right] \quad \text{⑤} \\ X_I(k) &= \sum_{n=0}^{N-1} \left[-x_R(n) \sin\left(\frac{2\pi}{N} kn\right) + x_I(n) \cos\left(\frac{2\pi}{N} kn\right) \right] \quad \text{⑥} \end{aligned}}$$

Ill^{ly}, WKT $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$

Substituting eqn ② in above equation,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) + j\, X_I(k) \right] e^{j\frac{2\pi}{N} kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) + j\, X_I(k) \right] \left[\cos\left(\frac{2\pi}{N} kn\right) + j \sin\left(\frac{2\pi}{N} kn\right) \right]$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \cos\left(\frac{2\pi}{N} km\right) + j X_R(k) \sin\left(\frac{2\pi}{N} km\right) + j X_I(k) \cos\left(\frac{2\pi}{N} km\right) - X_I(k) \sin\left(\frac{2\pi}{N} km\right) \right] \quad (7)$$

Comparing eqn's (1) & (7),

$$\boxed{\begin{aligned} x_R(n) &= \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \cos\left(\frac{2\pi}{N} km\right) - X_I(k) \sin\left(\frac{2\pi}{N} km\right) \right] \quad (8) \\ x_I(n) &= \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \sin\left(\frac{2\pi}{N} km\right) + X_I(k) \cos\left(\frac{2\pi}{N} km\right) \right] \quad (9) \end{aligned}}$$

case 1) <u>Real Valued sequences</u>

If $x(n)$ is real, then

$$X(N-k) = X^*(k) = X(-k)$$

And, $|X(N-k)| = |X(k)|$ also $\angle X(N-k) = -\angle X(k)$

For real $x(n)$, $x_I(n) = 0$, hence $x(n)$ can be determined from eqn (8).

Case 2) Real and even sequence

If $x(n)$ is real and even, that is

$$x(n) = x(N-n), \quad 0 \le n \le N-1$$

then eqn. ⑥ yields $X_I(k) = 0$. Hence the DFT reduces to

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi kn}{N}\right)}, \quad 0 \le k \le N-1$$

where $X(k)$ is real valued and even.

since $X_I(k) = 0$, the IDFT reduces to

$$\boxed{x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cos\left(\frac{2\pi kn}{N}\right)}, \quad 0 \le n \le N-1$$

Case 3) Real and odd sequence

If $x(n)$ is real and odd, that is

$$x(n) = -x(N-n), \quad 0 \le n \le N-1$$

then eqn ⑤ yields $X_R(k) = 0$. Hence

$$\boxed{X(k) = -j \sum_{n=0}^{N-1} x(n) \sin\left(\frac{2\pi kn}{N}\right)}, \quad 0 \le k \le N-1$$

which is purely imaginary and odd.

since $X_R(k) = 0$, the IDFT reduces to

$$\boxed{x(n) = j\frac{1}{N} \sum_{k=0}^{N-1} X(k) \sin\left(\frac{2\pi kn}{N}\right)} \quad 0 \le n \le N-1$$

Case 4) Purely Imaginary sequence

In this case $x(n) = j\, x_I(n)$.

Eqn's ⑤ & ⑥ reduces to

$$X_R(k) = \sum_{n=0}^{N-1} x_I(n) \sin \frac{2\pi kn}{N}$$

$$X_I(k) = \sum_{n=0}^{N-1} x_I(n) \cos \frac{2\pi kn}{N}$$

We observe that $X_R(k)$ is odd and $X_I(k)$ is even.

If $x_I(n)$ is odd, then $X_I(k) = 0$, hence $X(k)$ is purely real.

If $x_I(n)$ is even, then $X_R(k) = 0$, hence $X(k)$ is purely imaginary.

The symmetry properties above may be summarised as follows:

$$x(n) = x_R^e(n) + x_R^o(n) + j\, x_I^e(n) + j\, x_I^o(n)$$

$$X(k) = X_R^e(k) + X_R^o(k) + j\, X_I^e(k) + j\, X_I^o(k)$$

PROPERTIES OF THE DFT

PROPERTY	TIME DOMAIN	FREQUENCY DOMAIN
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(k) + a_2 X_2(k)$
Time Reversal	$x(N-n)$	$X(N-k)$
Circular Time Shift	$x((n-l))_N$	$X(k).e^{-j\frac{2\pi kl}{N}}$
Circular Frequency shift	$x(n)\, e^{j\frac{2\pi}{N}ln}$	$X((k-l))_N$
Complex Conjugate	$x^*(n)$	$X^*(N-k)$
Circular Convolution	$x_1(n) \, N \, x_2(n)$	$X_1(k)\, X_2(k)$
Circular Correlation	$x(n) \, N \, y^*(-n)$	$X(k)\, Y^*(k)$
Multiplication of two sequences	$x_1(n)\, x_2(n)$	$\frac{1}{N}\left[X_1(k) \, N \, X_2(k)\right]$
Parseval's Theorem	$\sum_{n=0}^{N-1} x(n)\, y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)\, Y^*(k)$

(28) A 498 point DFT of a real valued sequence $x(n)$ has the following DFT samples

$X(0) = 2$	$X(249) = 2.9$
$X(11) = 7 + j3.1$	$X(309) = -4.7 - j1.9$
$X(k_1) = -2.2 - j1.5$	$X(k_3) = 3 - j0.7$
$X(112) = 3 + j0.7$	$X(412) = -2.2 + j1.5$
$X(k_2) = -4.7 + j1.9$	$X(k_4) = 7 - j3.1$

The other samples have a value zero. Find the value of k_1, k_2, k_3 & k_4.

Soln:- For real valued sequence,

WKT, $X(k) = X^*(N-k)$

Given $N = 498$.

a) For $k = 412$, $X(412) = X^*(498-412)$

$$= X^*(86)$$

or $X(86) = X^*(412)$

$$= (-2.2 + j1.5)^*$$

$$X(86) = -2.2 - j1.5 = X(k_1)$$

$$\therefore \boxed{k_1 = 86}$$

b) for $k = 309$, $X(309) = X^*(498 - 309)$

$= X^*(189)$

or $X(189) = X^*(309)$

$= (-4.7 - j1.9)^*$

$X(189) = -4.7 + j1.9 = X(k_2)$

$\therefore \boxed{k_2 = 189}$

c) for $k = 112$, $X(112) = X^*(498 - 112)$

$= X^*(386)$

or $X(386) = X^*(112)$

$= (3 + j0.7)^*$

$X(386) = 3 - j0.7 = X(k_3)$

$\therefore \boxed{k_3 = 386}$

d) for $k = 11$, $X(11) = X^*(498 - 11)$

$= X^*(487)$

or $X(487) = X^*(11)$

$= (7 + j3.1)^*$

$X(487) = 7 - j3.1 = X(k_4)$

$\therefore \boxed{k_4 = 487}$

Hence, $k_1 = 86$, $k_2 = 386$, $k_3 = 189$ & $k_4 = 487$

(29) Find the 4-point DFTs of the two sequences $x(n)$ & $y(n)$ using a single 4 point DFT.

$x(n) = \{1, 2, 0, 1\}$ & $y(n) = \{2, 2, 1, 1\}$.

Sol'n:-

From the given, combining the two sequences $x(n)$ and $y(n)$ to creat a complex sequence $h(n)$.

With, $h(n) = x(n) + j\,y(n)$, $0 \leq n \leq 3$

$\therefore h(n) = \{(1+2j), (2+2j), j, (1+j)\}$

Taking DFT, $H_N = h_N W_N$

$$\therefore H(K) = \begin{bmatrix} 1+2j & 2+2j & j & 1+j \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$H(k) = \begin{bmatrix} 4+6j & 2 & -2 & 2j \end{bmatrix}$$

$$\Rightarrow H^*(k) = \begin{bmatrix} 4-6j & 2 & -2 & -2j \end{bmatrix}$$

Using the relation, $X(k) = \dfrac{H(k) + H^*((-k))_N}{2}$

for $k=0$, $X(0) = \dfrac{H(0) + H^*(0)}{2} = \dfrac{(4+6j)+(4-6j)}{2}$

$= \dfrac{8}{2} = \underline{\underline{4}}$

for $k=1$, $X(1) = \dfrac{H(1) + H^*((-1))_4}{2} = \dfrac{H(1) + H^*(3)}{2}$

$= \dfrac{2-2j}{2} = \underline{\underline{1-j}}$

for $k=2$, $X(2) = \dfrac{H(2) + H^*((-2))_4}{2} = \dfrac{H(2) + H^*(2)}{2}$

$= \dfrac{-2-2}{2} = \underline{\underline{-2}}$

for $k=3$, $X(3) = \dfrac{H(3) + H^*((-3))_4}{2} = \dfrac{H(3) + H^*(1)}{2}$

$= \dfrac{2j+2}{2} = \underline{\underline{1+j}}$

$\therefore \boxed{X(k) = \{4, (1-j), -2, (1+j)\}}$

Also, $Y(k) = \dfrac{H(k) - H^*((-k))_N}{2j}$

for $k=0$, $Y(0) = \dfrac{H(0) - H^*(0)}{2j} = \dfrac{4+6j-4+6j}{2j} = \underline{\underline{6}}$

for $k=1$, $Y(1) = \dfrac{H(1) - H^*((-1))_4}{2j} = \dfrac{H(1) - H^*(3)}{2j}$

$= \dfrac{2+2j}{2j} = \underline{\underline{1-j}}$

for $k=2$, $Y(2) = \dfrac{H(2) - H^*((-2))_4}{2j} = \dfrac{H(2) - H^*(2)}{2j}$

$= \dfrac{-2+2}{2j} = \underline{\underline{0}}$

for $k=3$, $Y(3) = H(3) - H^*((-3))_4 = \dfrac{H(3) - H^*(1)}{2j}$

$= \dfrac{2j-2}{2j} = \underline{\underline{1+j}}$

$\therefore \boxed{Y(k) = \{6, (1-j), 0, (1+j)\}}$

30 Let $x_p(n)$ be a periodic sequence with fundamental period N. If the N-point DFT $\{x_p(n)\} = X_1(k)$ and 3N-point DFT $\{x_p(n)\} = X_3(k)$.

(i) Find the relationship between $X_1(k)$ & $X_3(k)$.

(ii) Verify the above result for $\{2,1\}$ & $\{2,1,2,1,2,1\}$.

soln:- (i) WKT N-point DFT $\{x(n)\} = X(k)$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$\text{N-point DFT } \{x_p(n)\} = X_1(k) = \sum_{n=0}^{N-1} x_p(n)\, W_N^{kn}$$

$$\text{3N-point DFT } \{x_p(n)\} = X_3(k) = \sum_{n=0}^{3N-1} x_p(n)\, W_{3N}^{kn}$$

$$X_3(k) = \sum_{n=0}^{N-1} x_p(n)\, W_{3N}^{kn} + \sum_{n=0}^{N-1} x_p(n+N)\, W_{3N}^{k(n+N)} + \sum_{n=0}^{N-1} x_p(n+2N)\, W_{3N}^{k(n+2N)}$$

$$= \sum_{n=0}^{N-1} x_p(n)\, W_{3N}^{kn} + \sum_{n=0}^{N-1} x_p(n)\, W_{3N}^{kn} \cdot W_{3N}^{kN} + \sum_{n=0}^{N-1} x_p(n)\, W_{3N}^{kn} \cdot W_{3N}^{2kN}$$

∵ From periodicity of input.
$x_p(n+N) = x_p(n)$
& $x_p(n+2N) = x_p(n)$

Also, $W_{3N}^{Nk} = e^{-j\frac{2\pi}{3N}Nk} = e^{-j\frac{2\pi}{3}k} = W_3^k$

III^{ly} $W_{3N}^{2Nk} = W_3^{2k}$ & $W_{3N}^{nk} = W_N^{n\frac{k}{3}}$.

$$\therefore X_3(k) = \sum_{n=0}^{N-1} x_p(n)\left[1 + W_3^k + W_3^{2k}\right] W_N^{n\frac{k}{3}}$$

$$= \left[1 + W_3^k + W_3^{2k}\right] \sum_{n=0}^{N-1} x_p(n)\, W_N^{n\frac{k}{3}}$$

$$\boxed{X_3(k) = \left[1 + W_3^k + W_3^{2k}\right] X_1(k/3)}$$

ii) Given $x_1(n) = \{2, 1\}$

Its DFT, $X_1(k) = 2 + W_2^k$

$x_3(n) = \{2, 1, 2, 1, 2, 1\}$ with DFT

$$X_3(k) = 2 + W_6^k + 2W_6^{2k} + W_6^{3k} + 2W_6^{4k} + W_6^{5k}$$

$$W_6^k = W_{2\times3}^k = W_2^{k/3}$$

Then, $X_3(k) = \left[2 + W_2^{k/3}\right] + W_6^{2k}\left[2 + W_2^{k/3}\right]$

$$+ W_6^{4k}\left[2 + W_2^{k/3}\right]$$

$$= \left[2 + W_2^{k/3}\right]\left[1 + W_6^{2k} + W_6^{4k}\right]$$

$$X_3(k) = X_1(k/3)\left[1 + W_3^k + W_3^{2k}\right]$$

Hence verified.

(31) Suppose that you are given a program to find the DFT of a complex-valued sequence $x(n)$. How can we use this program to find the inverse DFT of $X(k)$.

<u>Sol'n:-</u> WKT,

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad \text{--- (1)}$$

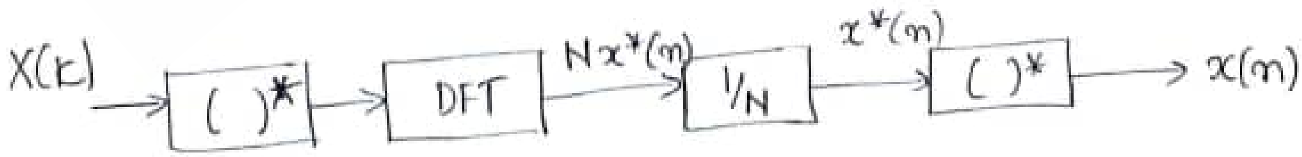
$$IDFT\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \quad \text{--- (2)}$$

Taking complex conjugate of eq'n (2),

$$x^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)\, W_N^{kn}$$

$$\text{or} \quad N\,x^*(n) = \sum_{k=0}^{N-1} X^*(k)\, W_N^{kn}$$

From observing the above eq'n, we can use DFT program to compute IDFT by calculating $X^*(k)$ and finding its DFT. The output is $N.x^*(n)$. scale it by $\frac{1}{N}$ and take its conjugate.



(32) Consider the sequence $x(n) = 4\delta(n) + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3)$.

Let $X(k)$ be the 6-point DFT of $x(n)$. Find the finite length sequence $y(n)$ that has a 6-point DFT $Y(k) = W_6^{4k} X(k)$.

Soln:- Given, $x(n) = 4\delta(n) + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3)$

Its DFT, $X(k) = 4 + 3W_6^{k} + 2W_6^{2k} + W_6^{3k}$

Also, given $Y(k) = W_6^{4k} X(k)$

$$Y(k) = W_6^{4k}\left[4 + 3W_6^{k} + 2W_6^{2k} + W_6^{3k}\right]$$

$$= 4W_6^{4k} + 3W_6^{5k} + 2W_6^{6k} + W_6^{7k}$$

WKT, $W_6^{6k} = W_6^{0} = 1$, $W_6^{7k} = W_6^{k}$

$$\therefore Y(k) = 4W_6^{4k} + 3W_6^{5k} + 2 + W_6^{k}$$

$$Y(k) = 2 + W_6^{k} + 4W_6^{4k} + 3W_6^{5k}$$

Taking IDFT,

$$y(n) = 2\delta(n) + \delta(n-1) + 4\delta(n-4) + 3\delta(n-5)$$

$$\therefore \boxed{y(n) = \{\underset{\uparrow}{2}, 1, 0, 0, 4, 3\}}$$

(33) If $x(n) = \{1, 2, 0, 3, -2, 4, 7, 5\}$.

Evaluate the following (i) $X(0)$ (ii) $X(4)$ (iii) $\sum_{k=0}^{7} X(k)$

(iv) $\sum_{k=0}^{7} |X(k)|^2$

Sol'n:- (i) From the def'n of DFT we have

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$N=8$,

at $k=0$, $$X(0) = \sum_{n=0}^{7} x(n)\, W_8^0$$

$$= \sum_{n=0}^{7} x(n) = 1+2+0+3-2+4+7+5$$

$$\boxed{X(0) = 20}$$

(ii) $$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

for $N=8$, Let $k=4$

$$X(4) = \sum_{n=0}^{7} x(n)\, W_8^{4n}$$

$\because W_8^{4n} = e^{\frac{-j2\pi}{8}\times 4n} = (e^{-j\pi})^n = (-1)^n$

$$= \sum_{n=0}^{7} x(n)(-1)^n$$

$$X(4) = x(0) - x(1) + x(2) - x(3) + x(4) - x(5) + x(6) - x(7)$$

$$\boxed{X(4) = -8}$$

iii) To find $\sum_{k=0}^{7} X(k)$

Consider the IDFT equation,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

for N=8, put n=0,

$$x(0) = \frac{1}{8} \sum_{k=0}^{7} X(k)$$

$$\therefore \sum_{k=0}^{7} X(k) = 8\; x(0) \quad \underline{\underline{= 8}}$$

iv) To find $\sum_{k=0}^{7} |X(k)|^2$

From Parseval's Theorem,

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

or $$\sum_{k=0}^{N-1} |X(k)|^2 = N \sum_{n=0}^{N-1} |x(n)|^2$$

for N=8, $$\sum_{k=0}^{7} |X(k)|^2 = 8\left[\sum_{n=0}^{7} |x(n)|^2\right]$$

$$= 8\left[|1|^2 + |2|^2 + |0|^2 + |3|^2 + |-2|^2 + |4|^2 + |7|^2 + |5|^2\right]$$

$$= 8\left[1 + 4 + 9 + 4 + 16 + 49 + 25\right]$$

$$\sum_{k=0}^{7} |X(k)|^2 = \underline{\underline{864}}$$

(34) Let $x(n)$ be a finite length sequence with $X(k) = \{0, (1+j), 1, (1-j)\}$ using the properties of DFT, find DFTs of the following sequences.

(i) $x_1(n) = e^{j\frac{\pi}{2}n} x(n)$

(ii) $x_2(n) = \cos\frac{\pi}{2}n \; x(n)$

(iii) $x_3(n) = x((n-1))_4$

Sol'n:- (i) Given $x_1(n) = e^{j\frac{\pi}{2}n} x(n)$

$$\text{or } x_1(n) = e^{j\frac{2\pi}{4}n} x(n)$$

Applying DFT and using circular frequency shift property

$$X_1(k) = X((k-1))_4$$

$$\therefore X_1(k) = \{(1-j), 0, (1+j), 1\}$$

(ii) Given $x_2(n) = \cos\frac{\pi}{2}n \; x(n)$

$$\text{or } x_2(n) = \frac{1}{2}\left[e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\right] x(n)$$

$$= \frac{1}{2}\left[e^{j\frac{2\pi}{4}n} + e^{-j\frac{2\pi}{4}n}\right] x(n)$$

Applying DFT on both sides and using circular frequency shift property.

$$X_2(k) = \frac{1}{2}\left[X((k-1))_4 + X((k+1))_4\right]$$

Given $X(k) = \{0, (1+j), 1, (1-j)\}$

$X((k-1))_4 = \{(1-j), 0, (1+j), 1\}$

$X((k+1))_4 = \{(1+j), 1, (1-j), 0\}$

$\therefore X_2(k) = \frac{1}{2}[2 \quad 1 \quad 2 \quad 1]$

$X_2(k) = \{1, 0.5, 1, 0.5\}$

iii) Given $x_3(n) = x((n-1))_4$

taking DFT on both sides and using circular time-shift property.

$$X_3(k) = W_4^k X(k)$$

at $k=0$, $X_3(0) = W_4^0 X(0) = 0$

at $k=1$, $X_3(1) = W_4^1 X(1) = (-j)(1+j) = (1-j)$

at $k=2$, $X_3(2) = W_4^2 X(2) = (-1)(1) = -1$

at $k=3$, $X_3(3) = W_4^3 X(3) = j(1-j) = 1+j$

$\therefore X_3(k) = \{0, 1-j, -1, 1+j\}$

(35) Let $x(n) = \{1, 2, 3, 4\}$ with $X(k) = \{10, -2+2j, -2, -2-2j\}$. Find the DFT of $x_1(n) = \{1, 0, 2, 0, 3, 0, 4, 0\}$ without actually calculating the DFT.

Soln:- $$DFT\{x_1(n)\} = X_1(k) = \sum_{n=0}^{N-1} x_1(n) W_N^{kn}$$

Dividing above summation into even and odd parts.

$$X_1(k) = \sum_{n=0}^{3} x_1(2n) W_N^{2nk} + \sum_{n=0}^{3} x_1(2n+1) W_N^{(2n+1)k}$$

N=8,

$$X_1(k) = \sum_{n=0}^{3} x_1(2n) W_8^{2nk} + 0 \quad \because \text{odd numbered samples are zero.}$$

$$X_1(k) = \sum_{n=0}^{3} x(n) W_{8/2}^{nk}$$

$$X_1(k) = \sum_{n=0}^{3} x(n) W_4^{nk} \quad \text{or} \quad X_1(k) = \begin{cases} X(k), & n=0,1,2,3 \\ X(k), & n=4,5,6,7 \end{cases}$$

$$X_1(k) = X(k)$$

ie, $X_1(k) = \{10, -2+2j, -2, -2-2j, 10, -2+2j, -2, -2-2j\}$

Using periodicity property,

$$X_1(4) = X(4) = X(0) \ \ldots\ldots$$

(36) Let $X(k)$ be a 14-point DFT of a length-14 real sequence $x(n)$. The first 8 samples of $X(k)$ are given by

$X(0) = 12$, $X(1) = -1+j3$, $X(2) = 3+j4$

$X(3) = 1-j5$, $X(4) = -2+j2$, $X(5) = 6+j3$

$X(6) = -2-j3$, $X(7) = 10$

Determine the remaining samples of $X(k)$.
Evaluate the following functions of $x(n)$ without computing the IDFT of $X(k)$.

(i) $x(0)$ (ii) $x(7)$ (iii) $\sum_{n=0}^{13} x(n)$

(iv) $\sum_{n=0}^{13} e^{\frac{j4\pi}{7}n} x(n)$ (v) $\sum_{n=0}^{13} |x(n)|^2$

Soln:- For a real valued sequence $x(n)$, symmetry property of real valued sequence.

$$X(k) = X^*(N-k) = X^*((-k))_N$$

$$X(8) = X^*(14-8) = X^*(6) = -2+j3$$

$$X(9) = X^*(14-9) = X^*(5) = 6-j3$$

$$X(10) = X^*(14-10) = X^*(4) = -2-j2$$

$$X(11) = X^*(14-11) = X^*(3) = 1+j5$$

$$X(12) = X^*(14-12) = X^*(2) = 3-j4$$

$$X(13) = X^*(14-13) = X^*(1) = -1-j3$$

(i) We know $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}$; $0 \le n \le N-1$

$N = 14$, $x(n) = \frac{1}{14} \sum_{k=0}^{13} X(k) e^{j\frac{\pi}{7}kn}$

Put $n=0$, $x(0) = \frac{1}{14} \sum_{k=0}^{13} X(k)$

$= \frac{1}{14} \left[X(0) + X(1) + X(2) + \cdots + X(13) \right]$

$x(0) = \frac{32}{14} = \underline{\underline{2.2857}}$

(ii) We know, $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}$

$N = 14$, $x(n) = \frac{1}{14} \sum_{k=0}^{13} X(k) e^{j\frac{2\pi}{14}kn}$

Put $n = 7$, $x(7) = \frac{1}{14} \sum_{k=0}^{13} (-1)^k X(k)$

$= \frac{-12}{14} = \underline{\underline{-0.8571}}$

(iii) $X(k) = \sum_{n=0}^{13} x(n) e^{-j\frac{2\pi}{14}kn}$

Put $k=0$, $X(0) = \sum_{n=0}^{13} x(n)$

$\therefore \sum_{n=0}^{13} x(n) = X(0) = \underline{\underline{12}}$

iv) The DFT of $e^{j\frac{4\pi}{7}n} x(n)$ or $e^{j\frac{2\pi}{14}4n} x(n)$ is

$$X((k-4))_{14}$$

We know, $X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$; $0 \le k \le N-1$

$$\therefore X((k-4))_{14} = \sum_{n=0}^{13} \left\{ e^{j\frac{4\pi}{7}n} x(n) \right\} e^{-j\frac{2\pi}{14}kn}$$

Put $k=0$,

$$X((-4))_{14} = \sum_{n=0}^{13} e^{j\frac{4\pi}{7}n} x(n)$$

$$\therefore \sum_{n=0}^{13} e^{j\frac{4\pi}{7}n} x(n) = X(10)$$

$$= \underline{\underline{-2-j2}}$$

V) From Parseval's Theorem,

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

$N=14$,

$$\sum_{n=0}^{13} |x(n)|^2 = \frac{1}{14} \sum_{k=0}^{13} |X(k)|^2$$

$$= \frac{1}{14} \left[|X(0)|^2 + |X(1)|^2 + \dots \right.$$

$$\left. \dots + |X(13)|^2 \right]$$

$$= \frac{498}{14} = \underline{\underline{35.5714}}$$