Module 3

Ordinary differential equations of first order

3.1 Linear and Bernoulli's differential equations

Introduction:

1. **Linear differential equation in y:**

   This is of the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x alone.

   General solution is $y.IF = \int Q.IF\, dx + c$, where $IF = e^{\int P dx}$.

2. **Linear differential equation in x:**

   This is of the form $\frac{dx}{dy} + Px = Q$, where P and Q are functions of y alone.

   General solution is $x.IF = \int Q.IF\, dy + c$, where $IF = e^{\int P dy}$.

3. **Bernoulli's differential equation in y:**

   This is of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x alone.

   Dividing this equation by y^n, $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$ ---- (1)

   Put $y^{1-n} = t$, then $(1-n)y^{-n}\frac{dy}{dx} = \frac{dt}{dx}$

   Equation (1) becomes $\frac{1}{1-n}\frac{dt}{dx} + Pt = Q$

   Reduced linear differential equation is $\frac{dt}{dx} + (1-n)Pt = (1-n)Q$

4. **Bernoulli's differential equation in x:**

   This is of the form $\frac{dy}{dx} + Px = Qx^n$, where P and Q are functions of y alone.

   Dividing this equation by x^n, $x^{-n}\frac{dy}{dx} + Px^{1-n} = Q$ ---- (1)

   Put $x^{1-n} = t$, then $(1-n)x^{-n}\frac{dx}{dy} = \frac{dt}{dy}$

   Equation (1) becomes $\frac{1}{1-n}\frac{dt}{dy} + Pt = Q$

   Reduced linear differential equation is $\frac{dt}{dy} + (1-n)Pt = (1-n)Q$

Problems:

1. **Solve** $\frac{dy}{dx} + y \cot x = \cos x$

   This is a linear L.D.E in y with $P = \cot x$, $Q = \cos x$

   $IF = e^{\int P dx} = e^{\int \cot x dx} = \sin x$

   General solution is $y.IF = \int Q.IF \, dx + c$

   $y.\sin x = \int \cos x . \sin x \, dx + c$

   $y \sin x = \frac{1}{2}\int \sin 2x \, dx + c$

   $y \sin x = -\frac{1}{4}\cos 2x + c$

2. **Solve** $\frac{dy}{dx} + y \tan x = y^3 \sec x$

   Step 1: Reduce it to an LDE

   Divide by y^3 on both sides,

   $\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2}\tan x = \sec x$ ---- (1)

   If $\frac{1}{y^2} = t$ then $-\frac{2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}$

   Put $\frac{1}{y^2} = t$, $\frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dt}{dx}$ in (1)

   $-\frac{1}{2}\frac{dt}{dx} + t \tan x = \sec x$

   Multiply by -2 on both sides,

   $\frac{dt}{dx} - 2t \tan x = -2 \sec x$

   This is an LDE in t with $P = -2\tan x$, $Q = -2\sec x$

   Step 2: Solve reduced LDE

   $IF = e^{\int -2\tan x dx} = e^{-2\log \sec x} = \cos^2 x$

   General solution is $t.IF = \int Q.IF \, dx + c$

   $t \cos^2 x = \int -2 \sec x \cos^2 x \, dx + c$

   $t \cos^2 x = -2 \sin x + c$

   $\left(\frac{1}{y^2}\right) \cos^2 x = -2 \sin x + c$

3. **Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 x$** (May 22)

Step 1: <u>Reduce it to an LDE</u>

Divide by y^2 on both sides,

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\left(\frac{1}{y}\right) = x \text{ ---- (1)}$$

If $\frac{1}{y} = t$ then $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

Put $\frac{1}{y} = t$, $\frac{1}{y^2}\frac{dy}{dx} = -\frac{dt}{dx}$ in (1)

$$-\frac{dt}{dx} + \frac{t}{x} = x$$

Multiply by -1 on both sides,

$$\frac{dt}{dx} - \frac{t}{x} = -x$$

This is an LDE in t with $P = -1/x, Q = -x$

Step 2: <u>Solve reduced LDE</u>

$$IF = e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

General solution is $t.IF = \int Q.IF\,dx + c$

$$t\frac{1}{x} = \int -x\frac{1}{x}dx + c$$

$$t\frac{1}{x} = -x + c$$

$$\frac{1}{xy} = -x + c$$

4. $\tan y \dfrac{dy}{dx} + \tan x = \cos y \cos^2 x$

Step 1: <u>Reduce it to an LDE</u>

Divide by $\cos y$ on both sides,

$\sec y \tan y \dfrac{dy}{dx} + \sec y \tan x = \cos^2 x$ ---- (1)

If $\sec y = t$ then $\sec y \tan y \dfrac{dy}{dx} = \dfrac{dt}{dx}$

$(1) \Rightarrow \dfrac{dt}{dx} + t \tan x = \cos^2 x$

This is an LDE in t with $P = \tan x$, $Q = \cos^2 x$

Step 2: <u>Solve reduced LDE</u>

$IF = e^{\int P \, dx} = e^{\int \tan x \, dx} = \sec x$

General solution is $t . IF = \int Q . IF \, dx + c$

$t \sec x = \int \cos^2 x \sec x \, dx + c$

$\sec y \sec x = \sin x + c$

5. **Solve:** $r \sin \theta - \cos \theta \frac{dr}{d\theta} = r^2$

Step 1: Reduce it to an LDE

Divide by r^2 on both sides,

$-\frac{\cos \theta}{r^2} \frac{dr}{d\theta} + \frac{\sin \theta}{r} = 1$ ---- (1)

If $\frac{1}{r} = t$ then $-\frac{1}{r^2} \frac{dr}{d\theta} = \frac{dt}{d\theta}$

$(1) \Rightarrow \cos \theta \frac{dt}{dx} + t \sin \theta = 1$

Divide by $\cos \theta$ on both sides,

$\frac{dt}{dx} + t \tan \theta = \sec \theta$

This is an LDE in t with $P = \tan \theta$, $Q = \sec \theta$

Step 2: Solve reduced LDE

$IF = e^{\int P \, dx} = e^{\int \tan \theta \, d\theta} = \sec \theta$

General solution is $t.IF = \int Q.IF \, d\theta + c$

$t \sec \theta = \int \sec \theta \sec \theta \, d\theta + c$

$\frac{1}{r} \sec \theta = \tan \theta + c$

6. **Solve:** $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x^2}(\log z)^2$

Step 1: <u>Reduce it to an LDE</u>

Divide by $z(\log z)^2$ on both sides,

$$\frac{1}{z(\log z)^2}\frac{dz}{dx} + \frac{1}{\log z}\left(\frac{1}{x}\right) = \frac{1}{x^2} \text{ ---- (1)}$$

If $\frac{1}{\log z} = t$ then $-\frac{1}{z(\log z)^2}\frac{dz}{dx} = \frac{dt}{dx}$

$(1) \Rightarrow -\frac{dt}{dx} + \frac{t}{x} = \frac{1}{x^2}$

Multiply by -1 on both sides,

$$\frac{dt}{dx} - \frac{t}{x} = -\frac{1}{x^2}$$

This is an LDE in t with $P = -\frac{1}{x}, Q = -\frac{1}{x^2}$

Step 2: <u>Solve reduced LDE</u>

$$IF = e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

General solution is $t.IF = \int Q.IF\, dx + c$

$$t\frac{1}{x} = \int -\frac{1}{x^2}.\frac{1}{x}dx + c$$

$$t\frac{1}{x} = \frac{x^{-2}}{2} + c$$

$$\frac{1}{x\log z} = \frac{1}{2x^2} + c$$

7. Solve: $x\frac{dy}{dx} + y = x^3 y^6$

Step 1: Reduce it to an LDE

Divide by x on both sides,

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = x^2 y^6$$

Divide by y^6 on both sides,

$$\frac{1}{y^6}\frac{dy}{dx} + \left(\frac{1}{x}\right)\frac{1}{y^5} = x^2 \text{ ---- (1)}$$

If $\frac{1}{y^5} = t$ then $-\frac{5}{y^6}\frac{dy}{dx} = \frac{dt}{dx}$

Put $\frac{1}{y^5} = t, \ \frac{1}{y^6}\frac{dy}{dx} = -\frac{1}{5}\frac{dt}{dx}$ in (1)

$$-\frac{1}{5}\frac{dt}{dx} + \frac{t}{x} = x^2$$

Multiply by -5 on both sides,

$$\frac{dt}{dx} - 5\frac{t}{x} = -5x^2$$

This is an LDE in t with $P = -5/x, Q = -5x^2$

Step 2: Solve reduced LDE

$$IF = e^{\int -\frac{5}{x}dx} = e^{-5\log x} = \frac{1}{x^5}$$

General solution is $t.IF = \int Q.IF \ dx + c$

$$t\frac{1}{x^5} = \int -5x^2 \frac{1}{x^5}dx + c$$

$$\frac{t}{x^5} = -5\int x^{-3} \ dx$$

$$\frac{1}{x^5 y^5} = \frac{5}{2x^2} + c$$

8. **Solve:** $xy(1 + xy^2)\dfrac{dy}{dx} = 1$

Step 1: <u>Reduce it to an LDE</u>

$$xy + x^2y^3 = \dfrac{dx}{dy}$$

$$\dfrac{dx}{dy} - xy = x^2y^3$$

Divide by x^2 on both sides,

$$\dfrac{1}{x^2}\dfrac{dx}{dy} - y\left(\dfrac{1}{x}\right) = y^3$$

If $\dfrac{1}{x} = t$ then $-\dfrac{1}{x^2}\dfrac{dx}{dy} = \dfrac{dt}{dy}$

Put $\dfrac{1}{x} = t,\ \dfrac{1}{x^2}\dfrac{dx}{dy} = -\dfrac{dt}{dy}$ in (1)

$$-\dfrac{dt}{dy} - yt = y^3$$

Multiply by -1 on both sides,

$$\dfrac{dt}{dy} + yt = -y^3$$

This is an LDE in t with $P = y, Q = -y^3$

Step 2: <u>Solve reduced LDE</u>

$$IF = e^{\int y\,dy} = e^{\frac{y^2}{2}}$$

General solution is $t.IF = \int Q.IF\,dy + c$

$$te^{\frac{y^2}{2}} = \int -y^3 e^{\frac{y^2}{2}}dy + c$$

Put $p = \dfrac{y^2}{2},\ dp = y\,dy$

$$te^p = \int -2pe^p dp + c$$

$$te^p = -2(pe^p - e^p) + c$$

$$\dfrac{1}{x}e^{\frac{y^2}{2}} = -2\left(\dfrac{y^2}{2}e^{\frac{y^2}{2}} - e^{\frac{y^2}{2}}\right) + c$$

$$\dfrac{1}{x}e^{\frac{y^2}{2}} = (2 - y^2)e^{\frac{y^2}{2}} + c$$

9. **Solve:** $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

Step 1: Reduce it to an LDE

Divide by $\cos^2 y$ on both sides,

$$\sec^2 y \frac{dy}{dx} + 2x \sec^2 y \sin y \cos y = x^3$$

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$$

If $\tan y = t$ then $\sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$

$$(1) \Rightarrow \frac{dt}{dx} + 2xt = x^3$$

This is an LDE in t with $P = 2x, Q = x^3$

Step 2: Solve reduced LDE

$$IF = e^{\int P \, dx} = e^{\int 2x \, dx} = e^{x^2}$$

General solution is $t.IF = \int Q.IF \, dx + c$

$te^{x^2} = \int x^3 e^{x^2} dx + c$, Put $p = x^2, dp = 2xdx$

$te^p = \frac{1}{2} \int p \, e^p dp + c$

$te^p = \frac{1}{2}(p-1)e^p + c$

$(\tan y)e^{x^2} = \frac{1}{2}(x^2 - 1)e^{x^2} + c$

10. Solve: $\dfrac{dy}{dx} = \dfrac{y}{x - \sqrt{xy}}$

Step 1: <u>Reduce it to an LDE</u>

$$\frac{x - \sqrt{xy}}{y} = \frac{dx}{dy}$$

$$\frac{dx}{dy} - \frac{x}{y} = -\sqrt{\frac{x}{y}}$$

Divide by $\sqrt{x}$ on both sides,

$$\frac{1}{\sqrt{x}}\frac{dx}{dy} - \sqrt{x}\left(\frac{1}{y}\right) = -\frac{1}{\sqrt{y}}$$

If $\sqrt{x} = t$ then $\dfrac{1}{2\sqrt{x}}\dfrac{dx}{dy} = \dfrac{dt}{dy}$

Put $\sqrt{x} = t$, $\dfrac{1}{\sqrt{x}}\dfrac{dx}{dy} = 2\dfrac{dt}{dy}$ in (1)

$$2\frac{dt}{dy} - \left(\frac{1}{y}\right)t = -\frac{1}{\sqrt{y}}$$

Divide by 2 on both sides,

$$\frac{dt}{dy} - \frac{1}{2y}t = -\frac{1}{2\sqrt{y}}$$

This is an LDE in t with $P = -\dfrac{1}{2y}, Q = -\dfrac{1}{2\sqrt{y}}$

Step 2: <u>Solve reduced LDE</u>

$$IF = e^{\int P\,dy} = e^{\int -\frac{1}{2y}dy} = \frac{1}{\sqrt{y}}$$

General solution is $\,t.IF = \int Q.IF\,dy + c$

$$t\frac{1}{\sqrt{y}} = \int -\frac{1}{2\sqrt{y}}\frac{1}{\sqrt{y}}dy + c$$

$$\sqrt{\frac{x}{y}} = \int -\frac{1}{2y}\,dy + c$$

$$\sqrt{\frac{x}{y}} = -\frac{1}{2}\log y + c$$

Home work:

1. $\frac{dy}{dx} = y \tan x - y^2 \sec x.$ (P.T.U., 2005) 2. $r \sin \theta - \cos \theta \frac{dr}{d\theta} = r^2.$ (V.T.U., 2005)

3. $2xy' = 10x^3 y^5 + y.$ 4. $(x^3 y^2 + xy)\, dx = dy.$ (B.P.T.U., 2005)

5. $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}.$ (Bhillai, 2005) 6. $x(x-y)\, dy + y^2\, dx = 0.$ (I.S.M., 2001)

7. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)\, e^x \sec y.$ (Bhopal, 2009) 8. $e^y \left(\frac{dy}{dx} + 1 \right) = e^x.$ (V.T.U., 2009)

9. $\sec^2 y \frac{dy}{dx} + x \tan y = x^3.$ 10. $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x.$ (Sambalpur, 2002)

11. $\frac{dy}{dx} = \frac{y}{x - \sqrt{(xy)}}.$ (V.T.U., 2011) 12. $(y \log x - 2)\, y dx - x dy = 0.$ (V.T.U., 2006)

3.2 Exact and reducible to exact differential equations

Exact differential equation:

❖ A differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

❖ General solution of an exact differential equation is

$$\int_{y-constant} M\, dx + \int (Terms\ of\ N\ not\ containing\ x)\ dy = 0$$

Reducible to exact differential equation:

❖ If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, given differential equation is not exact.

❖ Reduce it to an exact differential equation by multiplying I.F on both sides.

❖ If $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ is close to N then $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = f(x)$. Now $I.F = e^{\int f(x)\, dx}$

❖ If $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ is close to M then $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = g(y)$. Now $I.F = e^{\int g(y)\, dy}$

Problems:

1. **Solve:** $(x^2 + y^2 + x)dx + xy\,dy = 0$ (May 22)

    $(x^2 + y^2 + x)dx + xy\,dy = 0$ ------ (1)

$M = x^2 + y^2 + x$	$N = xy$
$\dfrac{\partial M}{\partial y} = 2y$	$\dfrac{\partial N}{\partial x} = y$

    Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, this is not an exact D.E.

    $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = y$, close to N.

    $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{1}{xy}(y) = \dfrac{1}{x} = f(x)$ [say]

    $I.F = e^{\int f(x)\,dx} = e^{\int \frac{1}{x}dx} = x$

    Multiply by x on both the sides of equation (1)

    $(x^3 + xy^2 + x^2)dx + x^2y\,dy = 0$

    This is an exact D.E.

    General solution is

    $\int_{y-constant} M\,dx + \int(Terms\ of\ N\ not\ containing\ x)\ dy = c$

    $\int_{y-constant}(x^3 + xy^2 + x^2)\,dx + \int(0)\ dy = c$

    $\dfrac{x^4}{4} + \dfrac{x^2y}{2} + \dfrac{x^3}{3} = c$

2. **Solve:** $(4xy + 3y^2 - x)\, dx + x(x + 2y)dy = 0$

$(4xy + 3y^2 - x)\, dx + (x^2 + 2xy)dy = 0$ ------ (1)

$M = 4xy + 3y^2 - x$	$N = x^2 + 2xy$
$\dfrac{\partial M}{\partial y} = 4x + 6y$	$\dfrac{\partial N}{\partial x} = 2x + 2y$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, this is not an exact D.E.

$\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 2x + 4y = 2(x + 2y)$, close to N.

$\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{1}{x^2+2xy}2(x + 2y) = \dfrac{2}{x} = f(x)$ [say]

$I.F = e^{\int f(x)\, dx} = e^{\int \frac{2}{x}\, dx} = x^2$

Multiply by x^2 on both the sides of equation (1)

$(4x^3y + 3x^2y^2 - x^3)\, dx + (x^4 + 2x^3y)dy = 0$

This is an exact D.E.

General solution is

$\int_{y-constant} M\, dx + \int (Terms\ of\ N\ not\ containing\ x)\ dy = c$

$\int_{y-constant} (4x^3y + 3x^2y^2 - x^3)\, dx + \int (0)\ dy = c$

$x^4y + x^3y^2 + \dfrac{x^4}{4} = c$

3. **Solve:** $\left(xy^2 - e^{1/x^3}\right)dx - x^2y\,dy = 0$

$\left(xy^2 - e^{1/x^3}\right)dx + (-x^2y)dy = 0$ ------ (1)

$M = xy^2 - e^{\frac{1}{x^3}}$	$N = -x^2y$
$\dfrac{\partial M}{\partial y} = 2xy$	$\dfrac{\partial N}{\partial x} = -2xy$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, this is not an exact D.E.

$\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 4xy$, close to N.

$\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{1}{-x^2y}(4xy) = -\dfrac{4}{x} = f(x)$ [say]

$I.F = e^{\int f(x)\,dx} = e^{-4\int \frac{1}{x}\,dx} = x^{-4}$

Multiply by x^{-4} on both the sides of equation (1)

$x^{-4}\left(xy^2 - e^{x^{-3}}\right)dx + x^{-2}y\,dy = 0$

This is an exact D.E.

General solution is

$\int_{y-constant} M\,dx + \int(Terms\ of\ N\ not\ containing\ x)\ dy = c$

$\int_{y-constant}\left(x^{-3}y^2 - x^{-4}e^{x^{-3}}\right)dx + \int(0)\ dy = c$

$-\dfrac{1}{2}x^{-2}y^2 + \dfrac{1}{3}e^{x^{-3}} = c$

4. Solve: $(x^2 + y^3 + 6x) \, dx + xy^2 \, dy = 0$

$(x^2 + y^3 + 6x) \, dx + xy^2 \, dy = 0$ ------ (1)

$M = x^2 + y^3 + 6x$	$N = xy^2$
$\frac{\partial M}{\partial y} = 3y^2$	$\frac{\partial N}{\partial x} = y^2$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, this is not an exact D.E.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y^2$, close to N.

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy^2}(2y^2) = \frac{2}{x} = f(x)$ $[say]$

$I.F = e^{\int f(x) \, dx} = e^{2\int \frac{1}{x} dx} = x^2$

Multiply by x^2 on both the sides of equation (1)

$\left(x^{-3}y^2 - x^{-4}e^{1/x^3}\right) dx - x^{-2}y \, dy = 0$

This is an exact D.E.

General solution is

$\int_{y-constant} M \, dx + \int (Terms \ of \ N \ not \ containing \ x) \ dy = c$

$\int_{y-constant} \left(x^{-3}y^2 - x^{-4}e^{1/x^3}\right) dx + \int (0) \ dy = c$

$-\frac{y^2}{2x^2} + \frac{e^{\frac{1}{x^3}}}{3} = c$

5. Solve: $(y^4 + 2y)\, dx + (xy^3 + 2y^4 - 4x)\, dy = 0$

$(y^4 + 2y)\, dx + (xy^3 + 2y^4 - 4x)\, dy = 0$ ------ (1)

$M = y^4 + 2y$	$N = xy^3 + 2y^4 - 4x$
$\dfrac{\partial M}{\partial y} = 4y^3 + 2$	$\dfrac{\partial N}{\partial x} = y^3 - 4$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, this is not an exact D.E.

$\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 3y^3 + 6 = 3(y^3 + 2)$, close to M.

$\dfrac{1}{M}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{3}{y^4 + 2y}(y^3 + 2\,) = \dfrac{3}{y} = g(y)$ [say]

$I.F = e^{-\int g(y)\, dy} = e^{-3\int \frac{1}{y}\, dx} = y^{-3}$

Multiply by y^3 on both the sides of equation (1)

$y^{-3}(y^4 + 2y)\, dx + y^{-3}(xy^3 + 2y^4 - 4x)\, dy = 0$

This is an exact D.E.

General solution is

$\int_{y-constant} M\, dx + \int (Terms\ of\ N\ not\ containing\ x)\ dy = c$

$\int_{y-constant}(y + 2y^{-2})dx + \int (2y)\ dy = c$

$xy + \dfrac{2x}{y^2} + y^2 = c$

6. **Solve:** $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

$(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)dy = 0$ ------ (1)

$M = 3x^2y^4 + 2xy$	$N = 2x^3y^3 - x^2$
$\dfrac{\partial M}{\partial y} = 12x^2y^3 + 2x$	$\dfrac{\partial N}{\partial x} = 6x^2y^3 - 2x$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, this is not an exact D.E.

$\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 6x^2y^3 + 4x = 2(3x^2y^3 + 2x)$, close to M.

$\dfrac{1}{M}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{2}{3x^2y^4 + 2xy}(3x^2y^3 + 2x) = \dfrac{2}{y} = g(y)$ [say]

$I.F = e^{-\int g(y)\,dy} = e^{-2\int \frac{1}{y}\,dx} = y^{-2}$

Multiply by y^{-2} on both the sides of equation (1)

$y^{-2}(3x^2y^4 + 2xy)\,dx + y^{-2}(2x^3y^3 - x^2)dy = 0$

This is an exact D.E.

General solution is

$\int_{y-constant} M\,dx + \int (Terms\ of\ N\ not\ containing\ x)\,dy = c$

$\int_{y-constant} \left(3x^2y^2 + \dfrac{2x}{y}\right)dx + \int(0)\,dy = c$

$x^3y^2 + \dfrac{x^2}{y} = c$

7. Solve: $(xy^3 + y) \, dx + 2(x^2y^2 + x + y^4) \, dy = 0$

$(xy^3 + y) \, dx + 2(x^2y^2 + x + y^4)dy = 0$ ------ (1)

$M = xy^3 + y$	$N = 2(x^2y^2 + x + y^4)$
$\frac{\partial M}{\partial y} = 3xy^2 + 1$	$\frac{\partial N}{\partial x} = 2(2xy^2 + 1) = 4xy^2 + 2$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, this is not an exact D.E.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -xy^2 - 1 = -(xy^2 + 1)$, close to M.

$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{xy^2+1}{xy^3+y} = -\frac{1}{y} = g(y)$ [say]

$I.F = e^{-\int g(y) \, dy} = e^{\int \frac{1}{y} \, dx} = y$

Multiply by y on both the sides of equation (1)

$y(xy^3 + y) \, dx + 2y(x^2y^2 + x + y^4)dy = 0$ This is an exact D.E.

General solution is

$\int_{y-constant} M \, dx + \int (Terms \ of \ N \ not \ containing \ x) \ dy = c$

$\int_{y-constant}(xy^4 + y^2)dx + \int(2y^5) \ dy = c$

$\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c$

8. **Solve:** $(y \log y) \, dx + (x - \log y) \, dy = 0$

$(y \log y) \, dx + (x - \log y) dy = 0$ ------ (1)

$M = y \log y$	$N = x - \log y$
$\frac{\partial M}{\partial y} = y \left(\frac{1}{y}\right) + \log y$	$\frac{\partial N}{\partial x} = 1$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, this is not an exact D.E.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 + \log y - 1 = \log y$, close to M.

$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y \log y}(\log y) = \frac{1}{y} = g(y) \quad [say]$

$I.F = e^{-\int g(y) \, dy} = e^{-\int \frac{1}{y} dx} = \frac{1}{y}$

Multiply by y on both the sides of equation (1)

$\frac{1}{y}(y \log y) \, dx + \frac{1}{y}(x - \log y) dy = 0$ This is an exact D.E.

General solution is

$\int_{y-constant} M \, dx + \int (Terms \ of \ N \ not \ containing \ x) \, dy = c$

$\int_{y-constant} \log y \, dx + \int \left(\frac{1}{y}\log y\right) dy = c$

$x \log y + \frac{(\log y)^2}{2} = c$

9. Solve: $y(x + y + 1)\,dx + x(x + 3y + 2)\,dy = 0$

$(xy + y^2 + y)\,dx + (x^2 + 3xy + 2x)dy = 0$ ------ (1)

$M = xy + y^2 + y$	$N = x^2 + 3xy + 2x$
$\dfrac{\partial M}{\partial y} = x + 2y + 1$	$\dfrac{\partial N}{\partial x} = 2x + 3y + 2$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, this is not an exact D.E.

$\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = -x - y - 1 = -(x + y + 1)$, close to M.

$\dfrac{1}{M}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = -\dfrac{1}{xy + y^2 + y}(x + y + 1) = -\dfrac{1}{y} = g(y)$ $[say]$

$I.F = e^{-\int g(y)\,dy} = e^{\int \frac{1}{y}\,dx} = y$

Multiply by y on both the sides of equation (1)

$y(xy + y^2 + y)\,dx + y(x^2 + 3xy + 2x)dy = 0$

This is an exact D.E.

General solution is

$\int_{y-constant} M\,dx + \int(Terms\ of\ N\ not\ containing\ x)\,dy = c$

$\int_{y-constant}(xy^2 + y^3 + y^2)dx + \int(0)\,dy = c$

$\dfrac{x^2 y^2}{2} + xy^3 + xy^2 = c$

10. Solve: $2y \, dx + (2x \log x - xy) \, dy = 0$

$(2y) \, dx + (2x \log x - xy) dy = 0$ ------ (1)

$M = 2y$	$N = 2x \log x - xy$
$\frac{\partial M}{\partial y} = 2$	$\frac{\partial N}{\partial x} = 2x\left(\frac{1}{x}\right) + 2\log x - y$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, this is not an exact D.E.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2 \log x + y = -(2 \log x - y)$, close to N.

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2x \log x - xy}(-2 \log x + y) = -\frac{1}{x} = f(x)$ [say]

$I.F = e^{\int f(x) \, dy} = e^{-\int \frac{1}{x} dx} = \frac{1}{x}$

Multiply by y on both the sides of equation (1)

$\frac{1}{x}(2y) \, dx + \frac{1}{x}(2x \log x - xy) dy = 0$ This is an exact D.E.

General solution is

$\int_{y-constant} M \, dx + \int (Terms \, of \, N \, not \, containing \, x) \, dy = c$

$\int_{y-constant} \frac{2y}{x} \, dx + \int (-y) \, dy = c$

$2y \log x - \frac{y^2}{2} = c$

3.3 Orthogonal Trajectory

Definition: Two families of curves such that every member of either family cuts each member of the other family at right angles are called orthogonal trajectories.

Example: Family of circles $x^2 + y^2 = a^2$ is the orthogonal trajectories to the family of straight lines $y = mx + c$. Where a and m are arbitrary constants.

Working rule to find the orthogonal trajectories of the family of curves $f(x, y, c) = 0$:

- ❖ Form the differential equation by eliminating arbitrary constant c.
- ❖ Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.   [$\tan(90 + \theta) = -\cot \theta$]
- ❖ Solve the modified differential equation.

Working rule to find the orthogonal trajectories of the family of curves $f(r, \theta, c) = 0$:

- ❖ Form the differential equation by eliminating arbitrary constant c.
- ❖ Replace $\frac{1}{r}\frac{dr}{d\theta}$ by $-r\frac{d\theta}{dr}$.   [$\tan(90 + \phi) = -\cot \phi$]
- ❖ Solve the modified differential equation.

Problems:

1. **Find the orthogonal trajectories of the family of parabolas $y^2 = 4ax$.**

   Consider $y^2 = 4ax$ ------   (1)

   Differentiate w.r.to x,

   $2y\frac{dy}{dx} = 4a$

   Substitute in (1),

   $y^2 = 2xy\frac{dy}{dx}$

   $y = 2x\frac{dy}{dx}$

   Replace $\frac{dy}{dx} = -\frac{dx}{dy}$

   $y = -2x\frac{dx}{dy}$

   $y\,dy = -2x\,dx$

   On integrating,

   $\frac{y^2}{2} = -x^2 + c$

   $2x^2 + y^2 = k$

   This is the family of orthogonal trajectories of (1).

2. **Find the orthogonal trajectories of the family of circles $x^2 + y^2 = a^2$.**

Consider $x^2 + y^2 = a^2$ ------ (1)

Differentiate w.r.to x,

$2x + 2y\dfrac{dy}{dx} = 0$

$y\dfrac{dy}{dx} = -x$

Replace $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$

$y\left(-\dfrac{dx}{dy}\right) = -x$

$\dfrac{1}{x}\,dx = \dfrac{1}{y}\,dy$

On integrating,

$\log x = \log y + \log c$

$x = yc$

This is the family of orthogonal trajectories of (1).

3. **Find the orthogonal trajectories of the family of curves $y^2 = c\,x^3$.**

Consider $y^2 = c\,x^3$ ------ (1)

Differentiate w.r.to x,

$2y\dfrac{dy}{dx} = 3c\,x^2$

$\times x \Rightarrow 2xy\dfrac{dy}{dx} = 3c\,x^3$

By (1), $2xy\dfrac{dy}{dx} = 3y^2$

$2x\dfrac{dy}{dx} = 3y$

Replace $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$

$2x\left(-\dfrac{dx}{dy}\right) = 3y$

$-2x\,dx = 3y\,dy$

On integrating,

$-x^2 = \dfrac{3y^2}{2} + c$

$2x^2 + 3y^2 = k$

This is the family of orthogonal trajectories of (1).

4. **Find the orthogonal trajectories of the family of curves $x^{2/3} + y^{2/3} = a^{2/3}$.**

Consider $x^{2/3} + y^{2/3} = a^{2/3}$ ------ (1)

Differentiate w.r.to x,

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

$$\times \frac{3}{2} \Rightarrow x^{-\frac{1}{3}} + y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$x^{-\frac{1}{3}} + y^{-\frac{1}{3}}\left(-\frac{dx}{dy}\right) = 0$$

$$y^{-\frac{1}{3}}\left(-\frac{dx}{dy}\right) = -x^{-\frac{1}{3}}$$

$$x^{\frac{1}{3}}\,dx = y^{\frac{1}{3}}\,dy$$

On integrating,

$$x^{4/3} = y^{4/3} + c \Rightarrow x^{4/3} - y^{4/3} = c$$

This is the family of orthogonal trajectories of (1).

5. **Show that the family of parabolas $y^2 = 4a(x + a)$ is self-orthogonal.**

$$y^2 = 4a(x + a)$$

Diff. w. r. to x,

$$2y\frac{dy}{dx} = 4a$$

By substituting in (1),

$$y^2 = 2y\frac{dy}{dx}\left(x + \frac{y}{2}\frac{dy}{dx}\right)$$

$$y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$$

$$y = 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2 \quad \text{------- (1)}$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$y = 2x\left(-\frac{dx}{dy}\right) + y\left(-\frac{dx}{dy}\right)^2$$

$$y = -2x\left(\frac{dx}{dy}\right) + y\left(\frac{dx}{dy}\right)^2$$

$$y\left(\frac{dy}{dx}\right)^2 = -2x\left(\frac{dy}{dx}\right) + y$$

$$y = y\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right) \quad \text{------ (2)}$$

Since (1) = (2), The given family of parabolas is self-orthogonal.

6. **Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, where λ is the parameter.** (May 22)

$\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ ------ (1)

Diff. w.r.to x,

$\frac{2x}{a^2} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0$

$\frac{y}{b^2+\lambda}\frac{dy}{dx} = -\frac{x}{a^2}$

$\frac{y}{b^2+\lambda} = -\frac{x}{a^2}\left(\frac{dx}{dy}\right)$

Substitute in (1),

$\frac{x^2}{a^2} - \frac{xy}{a^2}\left(\frac{dx}{dy}\right) = 1$

$x^2 - xy\left(\frac{dx}{dy}\right) = a^2$

$x^2 - a^2 = xy\left(\frac{dx}{dy}\right)$

$\frac{dy}{dx} = \frac{xy}{x^2-a^2}$

Replace $\frac{dy}{dx} = -\frac{dx}{dy}$

$-\frac{dx}{dy} = \frac{xy}{x^2-a^2}$

$\frac{a^2-x^2}{x}\,dx = y\,dy$

$\left(\frac{a^2}{x} - x\right)dx = y\,dy$

$a^2\log x - \frac{x^2}{2} = \frac{y^2}{2} + c$

$x^2 + y^2 = 2a^2\log x + k$

This is the family orthogonal trajectories of (1).

7. **Find the orthogonal trajectories of the family of curves $x^3 - 3xy^2 = c$**

$x^3 - 3xy^2 = c$ ------ (1)

Diff. w.r.to x,

$3x^2 - 3y^2 - 6xy\frac{dy}{dx} = 0$

$x^2 - y^2 = 2xy\frac{dy}{dx}$

Replace $\frac{dy}{dx} = -\frac{dx}{dy}$

$x^2 - y^2 = 2xy\left(-\frac{dx}{dy}\right)$

$\frac{dy}{dx} = \frac{2xy}{y^2-x^2}$

Pu $y = vx, \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{2x(vx)}{(vx)^2 - x^2} = \frac{2v}{v^2 - 1}$

$x\frac{dv}{dx} = \frac{2v}{v^2 - 1} - v = \frac{2v - v^3 + v}{v^2 - 1} = \frac{3v - v^3}{v^2 - 1}$

$\frac{v^2 - 1}{3v - v^3}dv = \frac{1}{x}dx$

$-\frac{1}{3}\left(\frac{3 - 3v^2}{3v - v^3}\right)dv = \frac{1}{x}dx$

On integrating,

$-\frac{1}{3}\log(3v - v^3) = \log x + \log c$

$\log(3v - v^3) = -3\log x - 3\log c$

$x^3(3v - v^3) = k$

$x^3\left(3\left(\frac{y}{x}\right) - \frac{y^3}{x^3}\right) = k$

$3x^2y - y^3 = k$. This is the family orthogonal trajectories of (1).

8. **Find the orthogonal trajectories of the family of curves $r^n = a^n \cos n\theta$.**

$r^n = a^n \cos n\theta$

$\log r^n = \log a^n \cos n\theta$

$n\log r = \log a^n + \log \cos n\theta$

$\frac{n}{r}\frac{dr}{d\theta} = -n\frac{\sin n\theta}{\cos n\theta}$

$\frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin n\theta}{\cos n\theta}$

Replace $\frac{1}{r}\frac{dr}{d\theta}$ by $-r\frac{d\theta}{dr}$

$-r\frac{d\theta}{dr} = -\tan n\theta$

$\cot n\theta\, d\theta = \frac{1}{r}dr$

On integrating,

$\frac{1}{n}\log \sin n\theta = \log r + \log c$

$\log \sin n\theta = n\log rc$

$\sin n\theta = r^n c^n$

$r^n = k\sin n\theta$

This is the required O.T.

9. **Find the orthogonal trajectories of the family of curves $r^n \cos n\theta = a^n$.**

$r^n \cos n\theta = a^n$

$\log(r^n \cos n\theta) = \log a^n$

$n \log r + \log \cos n\theta = \log a^n$

$\frac{n}{r} \frac{dr}{d\theta} - n \frac{\sin n\theta}{\cos n\theta} = 0$

$\frac{1}{r} \frac{dr}{d\theta} = \tan n\theta$

Replace $\frac{1}{r} \frac{dr}{d\theta}$ by $-r \frac{d\theta}{dr}$

$-r \frac{d\theta}{dr} = \tan n\theta$

$\cot n\theta \, d\theta = -\frac{1}{r} dr$

On integrating,

$\frac{1}{n} \log \sin n\theta = -\log r + \log c$

$\log \sin n\theta = n \log \frac{c}{r}$

$\sin n\theta = \frac{c^n}{r^n}$

$r^n \sin n\theta = k$.   This is the required O.T.

10. **Find the orthogonal trajectories of the family of curves $r = 2a \cos \theta$.**

$r = 2a \cos \theta$

$\log r = \log(2a \cos \theta)$

$\log r = \log 2a + \log \cos \theta$

$\frac{1}{r} \frac{dr}{d\theta} = -\frac{\sin \theta}{\cos \theta}$

$\frac{1}{r} \frac{dr}{d\theta} = -\tan \theta$

Replace $\frac{1}{r} \frac{dr}{d\theta}$ by $-r \frac{d\theta}{dr}$

$-r \frac{d\theta}{dr} = -\tan \theta$

$\cot \theta \, d\theta = \frac{1}{r} dr$

On integrating,

$\log \sin \theta = \log r + \log c$

$\log \sin \theta = \log cr$

$\sin \theta = cr$

$r = k \sin \theta$

This is the required O.T.

11. Find the orthogonal trajectories of the family of curves $r^n = a^n \sin n\theta$.

$r^n = a^n \sin n\theta$

$\log r^n = \log a^n \sin n\theta$

$n \log r = \log a^n + \log \sin n\theta$

$\frac{n}{r} \frac{dr}{d\theta} = n \frac{\cos n\theta}{\sin n\theta}$

$\frac{1}{r} \frac{dr}{d\theta} = \cot n\theta$

Replace $\frac{1}{r} \frac{dr}{d\theta}$ by $-r \frac{d\theta}{dr}$

$-r \frac{d\theta}{dr} = \cot n\theta$

$\tan n\theta \, d\theta = -\frac{1}{r} dr$

On integrating,

$\frac{1}{n} \log \sec n\theta = -\log r + \log c$

$\log \sec n\theta = n \log \frac{c}{r}$

$\sec n\theta = \frac{c^n}{r^n}$

$r^n = k \cos n\theta$. This is the required O.T.

12. Find the orthogonal trajectories of the family of curves $r = a(1 - \cos\theta)$.

$r = a(1 - \cos\theta)$

$\log r = \log a + \log(1 - \cos\theta)$

$\frac{1}{r} \frac{dr}{d\theta} = \frac{\sin\theta}{1 - \cos\theta}$

Replace $\frac{1}{r} \frac{dr}{d\theta}$ by $-r \frac{d\theta}{dr}$

$-r \frac{d\theta}{dr} = \frac{\sin\theta}{1 - \cos\theta}$

$-\frac{1 - \cos\theta}{\sin\theta} d\theta = \frac{1}{r} dr$

$-\frac{1 - \cos^2\theta}{\sin\theta(1 + \cos\theta)} d\theta = \frac{1}{r} dr$

$-\frac{\sin^2\theta}{\sin\theta(1 + \cos\theta)} d\theta = \frac{1}{r} dr$

$-\frac{\sin\theta}{(1 + \cos\theta)} d\theta = \frac{1}{r} dr$

On integrating,

$\log(1 + \cos\theta) = \log r + \log c$

$\log(1 + \cos\theta) = \log cr$

$1 + \cos\theta = cr$ (or) $r = k(1 + \cos\theta)$. This is the required O.T.

13. Find the orthogonal trajectories of the family of curves $r = a(1 + \sin\theta)$.

$r = a(1 + \sin\theta)$

$\log r = \log a + \log(1 + \sin\theta)$

$\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{\cos\theta}{1+\sin\theta}$

Replace $\dfrac{1}{r}\dfrac{dr}{d\theta}$ by $-r\dfrac{d\theta}{dr}$

$-r\dfrac{d\theta}{dr} = \dfrac{\cos\theta}{1+\sin\theta}$

$-\dfrac{1+\sin\theta}{\cos\theta}d\theta = \dfrac{1}{r}dr$

$-\dfrac{1-\sin^2\theta}{\cos\theta(1-\sin\theta)}d\theta = \dfrac{1}{r}dr$

$-\dfrac{\cos\theta}{1-\sin\theta}d\theta = \dfrac{1}{r}dr$

On integrating,

$\log(1 - \sin\theta) = \log r + \log c$

$1 - \sin\theta = cr$

$r = k(1 - \sin\theta).$ This is the required O.T.

14. Find the orthogonal trajectories of the family of curves $r = 2a(\cos\theta + \sin\theta)$.

$r = 2a(\cos\theta + \sin\theta)$

$\log r = \log 2a + \log(\cos\theta + \sin\theta)$

$\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$

Replace $\dfrac{1}{r}\dfrac{dr}{d\theta}$ by $-r\dfrac{d\theta}{dr}$

$-r\dfrac{d\theta}{dr} = \dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$

$-\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}d\theta = \dfrac{1}{r}dr$

On integrating,

$\log(\cos\theta - \sin\theta) = \log r + \log c$

$\log(\cos\theta - \sin\theta) = \log cr$

$\cos\theta - \sin\theta = cr$

$r = k(\cos\theta - \sin\theta)$

This is the required O.T.

15. Find the orthogonal trajectories of the family of curves $r = 4a(\sec\theta + \tan\theta)$.

$r = 4a(\sec\theta + \tan\theta)$

$\log r = \log 4a + \log(\sec\theta + \tan\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta}$

$\frac{1}{r}\frac{dr}{d\theta} = \sec\theta$

Replace $\frac{1}{r}\frac{dr}{d\theta}$ by $-r\frac{d\theta}{dr}$

$-r\frac{d\theta}{dr} = \sec\theta$

$-\cos\theta\, d\theta = \frac{1}{r}dr$

On integrating,
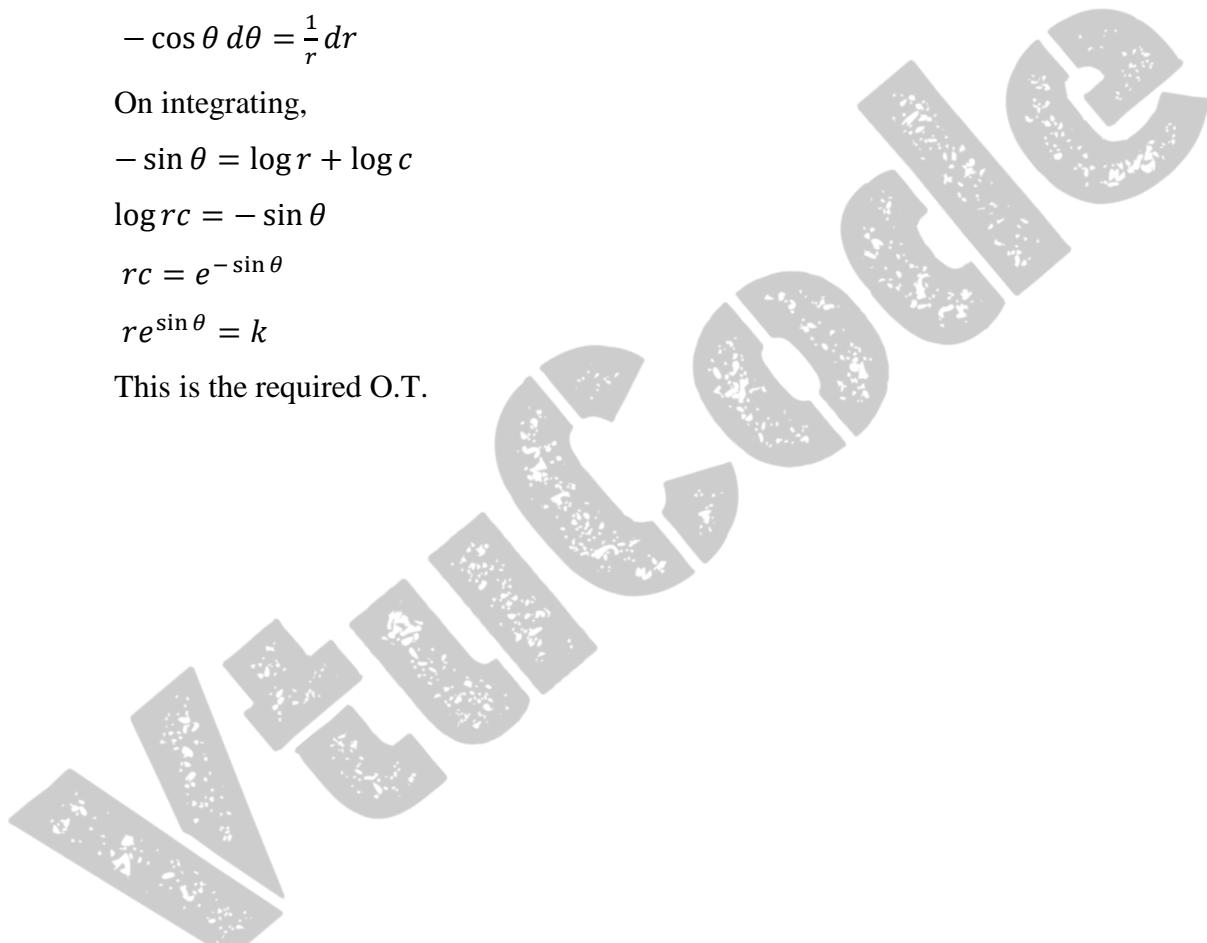
$-\sin\theta = \log r + \log c$

$\log rc = -\sin\theta$

$rc = e^{-\sin\theta}$

$re^{\sin\theta} = k$

This is the required O.T.

16. Prove that the orthogonal trajectories of the family of curves $\frac{2a}{r} = 1 - \cos\theta$ is

$$\frac{2b}{r} = 1 + \cos\theta.$$

$$\frac{2a}{r} = 1 - \cos\theta$$

$$\log\frac{2a}{r} = \log(1 - \cos\theta)$$

$$\log 2a - \log r = \log(1 - \cos\theta)$$

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1 - \cos\theta}$$

Replace $\frac{1}{r}\frac{dr}{d\theta}$ by $-r\frac{d\theta}{dr}$

$$r\frac{d\theta}{dr} = \frac{\sin\theta}{1 - \cos\theta}$$

$$\frac{1 - \cos\theta}{\sin\theta}d\theta = \frac{1}{r}dr$$

$$\frac{1 - \cos^2\theta}{\sin\theta(1 + \cos\theta)}d\theta = \frac{1}{r}dr$$

$$\frac{\sin^2\theta}{\sin\theta(1 + \cos\theta)}d\theta = \frac{1}{r}dr$$

$$\frac{\sin\theta}{(1 + \cos\theta)}d\theta = \frac{1}{r}dr$$

On integrating,

$$-\log(1 + \cos\theta) = \log r + \log c$$

$$\log 2b = \log r + \log(1 + \cos\theta)$$

$$\log\frac{2b}{r} = \log(1 + \cos\theta)$$

$$\frac{2b}{r} = 1 + \cos\theta$$
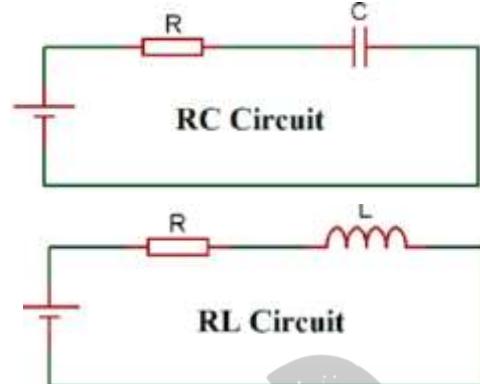
This is the required O.T.

3.4 RL and RC circuits

Introduction:

(i) $I = \frac{dQ}{dt}$

Notation	Terminology	Unit
L	Inductance	Henry
C	Capacitance	Farad
R	Resistance	Ohms
E	Electro motive force (e.m.f.)	Volts
I	Current	Amperes
Q	Charge	Coloumb

(ii) Voltage drop across resistance $R = RI$

(iii) Voltage drop across inductance $L = L\frac{dI}{dt}$

(iv) Voltage drop across capacitance $C = \frac{Q}{C}$

By Kirchhoff's law, $L\frac{dI}{dt} + RI = E$ in LR circuit.

By Kirchhoff's law, $RI + \frac{Q}{C} = E$ in RC circuit.

1. **When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current I builds up at a rate given by $L\frac{di}{dt} + Ri = E$. Find i as a function of t. How long will it be , before the current has reached one-half its final value if $E = 6$ volts, $R = 100$ ohms and $L = 0.1$ henry?**

   To find I:
   $L\frac{dI}{dt} + RI = E$, Put $E = 6, R = 100, L = 0.1$

   $0.1\frac{dI}{dt} + 100I = 6$

   $\frac{dI}{dt} + 1000I = 60$

   $\frac{dI}{60 - 1000I} = dt$

   $-0.001 \log(60 - 1000I) = t + c$

   $\log(60 - 1000I) = -1000t + c'$

   $60 - 1000I = ke^{-1000t}$

   When $t = 0, i = 0. \; so, k = 60$

   $60 - 1000I = 60e^{-1000t}$

   $1000I = 60 - 60e^{-1000t}$

   $I = 0.06(1 - e^{-1000t})$

To find t when I reaches max. value of I/2 :

When t is max, I is max. So, Max. value of $I = 0.06$

$Max. value\ of\ \frac{I}{2} = 0.03$

$0.03 = 0.06(1 - e^{-1000t})$

$e^{1000t} = 2$

$t = 0.0006931\ sec$

2. **When a Resistance R ohms is connected in series with an inductance L henries with an e.m.f. of E volts, the current i amperes at time t is given by $L\frac{di}{dt} + Ri = E$. If $E = 10 \sin t$ volts and $i = 0$ when $t = 0$, find i as a function of t.**

$L\frac{dI}{dt} + RI = E$, Put $E = 10 \sin t$, $i = 0, t = 0$

$L\frac{dI}{dt} + RI = 10 \sin t$

$\frac{dI}{dt} + \frac{R}{L}I = 10 \sin t$

$I. e^{\int \frac{R}{L}dt} = \int 10 \sin t . e^{\int \frac{R}{L}dt} dt + c$

$I. e^{\frac{Rt}{L}} = \int 10 \sin t . e^{\frac{Rt}{L}} dt + c$

$I. e^{\frac{Rt}{L}} = \frac{10 e^{\frac{Rt}{L}}}{\frac{R^2}{L^2}+1}\left(\frac{R}{L}\sin t - \cos t\right) + c$

$I = \frac{10L}{R^2+L^2}(R \sin t - L \cos t) + c e^{-\frac{Rt}{L}}$

By data, at $t = 0, I = 0$.

$0 = \frac{10L}{L^2+R^2}(-L) + c$

Therefore, $c = \frac{10L^2}{L^2+R^2}$

$I = \frac{10L}{R^2+L^2}\left(R \sin t - L \cos t + L e^{-\frac{Rt}{L}}\right)$

3. **A resistance of 100 Ω, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in the circuit at $t = 0.5$ secs. If $i = 0$ at $t = 0$.**

$L\frac{di}{dt} + Ri = E$, Put $E = 20, R = 100, L = 0.5$

$0.5\frac{di}{dt} + 100i = 20$

$\frac{di}{dt} + 200i = 40$

$\frac{di}{40-200i} = dt$

$-0.005\log(40 - 200i) = t + c$

$\log(40 - 200i) = -200t + c'$

$40 - 200i = de^{-200t}$

$i = 0$ at $t = 0$.

$40 = d$

$40 - 200i = 40e^{-200t}$

$1 - 5i = e^{-200t}$

At $t = 0.5$

$1 - 5i = e^{-100}$

$i = \frac{1-e^{-100}}{5}$

4. **Find the current at any time $t > 0$, in a circuit having in series a constant electromotive force $40V$, a resistor $10Ω$, an inductor $0.2H$ given that initial current is zero.**

By data, $E = 40, R = 10, L = 0.2$

By Kirchoff's law, $L\frac{dI}{dt} + RI = E$

On substituting, $0.2\frac{dI}{dt} + 10I = 40$

Therefore, $\frac{dI}{dt} + 50I = 200$

Solution is given by

$I.e^{50t} = \int 200\, e^{50t}dt + c$

$I = 4 + ce^{-50t}$

By data, at $t = 0, I = 0$.

$0 = 4 + c, \ c = -4$

Therefore, $I = 4 - 4e^{-50t} = 4(1 - e^{-50t})$

5. **A generator having e.m.f. 100 volts is connected in series with a 10 ohms resistor and an inductor of 2 henries. If the switch is closed at a time $t = 0$, determine the current at time $t > 0$.**

By data, $E = 100, R = 10, L = 2$

By Kirchoff's law, $L\frac{dI}{dt} + RI = E$

On substituting, $2\frac{dI}{dt} + 10I = 100$

Therefore, $\frac{dI}{dt} + 5I = 50$

Solution is given by

$I.e^{5t} = \int 50\, e^{5t} dt + c$

$I.e^{5t} = 10e^{5t} + c$

$I = 10 + ce^{-5t}$

By data, at $t = 0, I = 0$.

$0 = 10 + c, \quad c = -10$

Therefore, $I = 10 - 10e^{-5t} = 10(1 - e^{-5t})$

6. A decaying e.m.f. $E = 200e^{-5t}$ is connected in series with a 20 ohm resistor and 0.01 $farad$ capacitor. Find the charge and current at any time assuming $Q = 0$ at $t = 0$. Show that the charge reaches the maximum. Calculate it and find time when it is reached.

By data, $E = 200\ e^{-5t}, R = 20, C = 0.01$

By Kirchoff's law, $RI + \frac{Q}{C} = E$

On substituting, $20I + \frac{Q}{0.01} = 200e^{-5t}$

Therefore, $\frac{dQ}{dt} + 5Q = 10e^{-5t}$

Solution is given by

$Q.e^{5t} = \int 10e^{-5t}\ e^{5t}dt + c$

$Q.e^{5t} = 10t + c$

By data, at $t = 0, Q = 0$.

$0 = 0 + c, \quad c = 0$

Therefore, $Q.e^{5t} = 10t$

$Q = 10t\ e^{-5t}$

$\frac{dQ}{dt} = 10(e^{-5t} - 5te^{-5t}) = 10(1 - 5t)e^{-5t}$

Q is maximum when $\frac{dQ}{dt} = 0$

$10(1 - 5t)e^{-5t} = 0$

$t = \frac{1}{5}$.

Maximum value of $Q = 10\left(\frac{1}{5}\right)e^{-1} = \frac{2}{e}$

3.5 Non-linear differential equations

Introduction: Product of variables and their first order derivatives are allowed in the non-linear differential equations.

Problems:

1. **Solve:** $p^2 + p(x + y) + xy = 0$

   $(p + x)(p + y) = 0$

$p = -x$	$p = -y$
$\frac{dy}{dx} = -x$	$\frac{dy}{dx} = -y$
$dy = -x\,dx$	$\frac{1}{y}\,dy = -dx$
$\frac{x^2}{2} + y - c = 0$	$x + \log y - c = 0$

   Therefore, the general solution is

   $$\left(\frac{x^2}{2} + y - c\right)(x + \log y - c) = 0$$

2. **Solve:** $p^2 + 2p\,\cosh x + 1 = 0$

   $p^2 + p\,(e^x + e^{-x}) + 1 = 0$

   $p(p + e^x) + e^{-x}(p + e^x) = 0$

   $(p + e^x)(p + e^{-x}) = 0$

$p = -e^x$	$p = -e^{-x}$
$\frac{dy}{dx} = -e^x$	$\frac{dy}{dx} = -e^{-x}$
$dy = -e^x\,dx$	$dy = -e^{-x}\,dx$
$e^x + y - c = 0$	$-e^{-x} + y - c = 0$

   Therefore, general solution is

   $$(y + e^x - c)(y - e^{-x} - c) = 0$$

3. **Solve:** $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$

$$xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$$

$$xyp^2 + 3x^2p - 2y^2p - 6xy = 0$$

$$xp(yp + 3x) - 2y(yp + 3x) = 0$$

$$(xp - 2y)(yp + 3x) = 0$$

$xp = 2y$	$yp = -3x$
$x\dfrac{dy}{dx} = 2y$	$y\dfrac{dy}{dx} = -3x$
$\dfrac{1}{y}dy = \dfrac{2}{x}\,dx$	$y\,dy = -3x\,dx$
$\log y = 2\log x + c$	$\dfrac{y^2}{2} + \dfrac{3x^2}{2} - c = 0$
$\log y = \log cx^2$	$y^2 + 3x^2 - 2c = 0$
$y = cx^2$	

Therefore, general solution is

$$(y - cx^2)(y^2 + 3x^2 - 2c) = 0$$

4. **Solve:** $p(p + y) = x(x + y)$

$$p^2 + py - x^2 - xy = 0$$

$$(p^2 - x^2) + y(p - x) = 0$$

$$(p - x)(p + x + y) = 0$$

$p = x$	$p = -x - y$
$\dfrac{dy}{dx} = x$	$\dfrac{dy}{dx} = -x - y$
$dy = x\,dx$	$\dfrac{dy}{dx} + y = -x$
On integrating,	This is an L.D.E. Solution is
$y = \dfrac{x^2}{2} + c$	$ye^x = \int -x \cdot e^x dx + c$
$y - \dfrac{x^2}{2} - c = 0$	$ye^x = -(xe^x - e^x) + c$
	$e^x(x + y - 1) - c = 0$

Therefore, general solution is

$$\left[y - \frac{x^2}{2} - c\right]\left[e^x(x + y - 1) - c\right] = 0$$

5. **Solve: $p^2 + 2py \cot x = y^2$**

$p^2 + 2py \cot x - y^2 = 0$

$(p + y \cot x)^2 - y^2 - y^2 \cot^2 x = 0$

$(p + y \cot x)^2 - y^2 cosec^2 x = 0$

$(p + y \cot x + y\ cosec\ x)(p + y \cot x - y\ cosec\ x) = 0$

$p + y \cot x + y\ cosec\ x = 0$	$p + y \cot x - y\ cosec\ x = 0$
$\frac{dy}{y} = (-cosec\ x - \cot x)dx$	$\frac{dy}{y} = (cosec\ x - \cot x)dx$
$\frac{dy}{y} = -\left(\frac{1+\cos x}{\sin x}\right)dx$	$\frac{dy}{y} = \left(\frac{1-\cos x}{\sin x}\right)dx$
$\frac{dy}{y} = -\left(\frac{1-\cos^2 x}{\sin x(1-\cos x)}\right)dx$	$\frac{dy}{y} = \left(\frac{1-\cos^2 x}{\sin x(1+\cos x)}\right)dx$
$\frac{dy}{y} = -\left(\frac{\sin x}{1-\cos x}\right)dx$	$\frac{dy}{y} = \left(\frac{\sin x}{1+\cos x}\right)dx$
On integrating,	On integrating,
$\log y = -\log(1 - \cos x) + \log c$	$\log y = -\log(1 + \cos x) + \log c$
$y(1 - \cos x) - c = 0$	
	$y(1 + \cos x) - c = 0$

Therefore, general solution is

$[y(1 - \cos x) - c][y(1 + \cos x) - c] = 0.$

6. **Solve: $x^2 \left(\frac{dy}{dx}\right)^2 + xy \left(\frac{dy}{dx}\right) - 6y^2 = 0$**

$x^2 p^2 + xyp - 6y^2 = 0$

$(xp + 3y)(xp - 2y) = 0$

$xp + 3y = 0$	$xp - 2y = 0$
$x\frac{dy}{dx} = -3y$	$x\frac{dy}{dx} = 2y$
$\frac{1}{y}dy = \frac{-3}{x}dx$	$\frac{1}{y}dy = \frac{2}{x}dx$
On integrating,	On integrating,
$\log y = -3\log x + \log c$	$\log y = 2\log x + \log c$
$\log y + 3\log x = \log c$	$\log y = \log x^2 + \log c$
$\log yx^3 = \log cx$	$y = cx^2$
$yx^3 = c$	$y - cx^2 = 0$
$x^3 y - c = 0$	

Therefore, the general solution is

$(x^3 y - c)(y - cx^2) = 0$

7. Solve: $4y^2p^2 + 2pxy(3x + 1) + 3x^3 = 0$

$(2yp)^2 + 2yp(3x^2 + x) + 3x^3 = 0$

$2yp(2yp + 3x^2) + x(2yp + 3x^2) = 0$

$(2yp + x)(2yp + 3x^2) = 0$

$2yp + x = 0$	$2yp + 3x^2 = 0$
$2y\dfrac{dy}{dx} = -x$	$2y\dfrac{dy}{dx} = -3x^2$
$2y\,dy = -x\,dx$	$2y\,dy = -3x^2\,dx$
On integrating,	On integrating,
$y^2 = -\dfrac{x^2}{2} + c$	$y^2 = -x^3 + c$

Therefore, the general solution is

$\left(y^2 + \dfrac{x^2}{2} - c\right)(x^3 + y^2 - c) = 0$

8. Solve: $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}$

$p - \dfrac{1}{p} = \dfrac{x^2 - y^2}{xy}$

$\dfrac{p^2 - 1}{p} = \dfrac{x^2 - y^2}{xy}$

$xyp^2 - (x^2 - y^2)p - xy = 0$

$xp(yp - x) + y(yp - x) = 0$

$(xp + y)(yp - x) = 0$

$xp + y = 0$	$yp - x = 0$
$x\dfrac{dy}{dx} = -y$	$y\dfrac{dy}{dx} = x$
$\dfrac{1}{y}\,dy = -\dfrac{1}{x}\,dx$	$y\,dy = x\,dx$
On integrating,	On integrating,
$\log y = -\log x + \log c$	$\dfrac{y^2}{2} = \dfrac{x^2}{2} + \dfrac{c}{2}$
$\log x + \log y = \log c$	$y^2 - x^2 = c$
$xy = c$	$y^2 - x^2 - c = 0$

Therefore, the general solution is

$(xy - c)(y^2 - x^2 - c) = 0$

9. **Solve:** $yp^2 + (x - y)p - x = 0$

$yp(p - 1) + x(p - 1) = 0$

$(yp + x)(p - 1) = 0$

$yp + x = 0$	$p - 1 = 0$
$y\dfrac{dy}{dx} = -x$	$\dfrac{dy}{dx} = 1$
$y\, dy = -x\, dx$	$dy = dx$
On integrating,	On integrating,
$\dfrac{y^2}{2} = -\dfrac{x^2}{2} + \dfrac{c}{2}$	$y = x + c$
$y^2 + x^2 = c$	$y - x - c = 0$
$x^2 + y^2 - c = 0$	

Therefore, the general solution is

$(x^2 + y^2 - c)(y - x - c) = 0$

10. **Solve:** $x^2 p^2 + xp - (y^2 + y) = 0$

$(x^2 p^2 - y^2) + (xp - y) = 0$

$(xp - y)(xp + y + 1) = 0$

$xp - y = 0$	$xp + y + 1 = 0$
$x\dfrac{dy}{dx} = y$	$x\dfrac{dy}{dx} = -y - 1$
$\dfrac{1}{y}\, dy = \dfrac{1}{x}\, dx$	$\dfrac{1}{y+1}\, dy = -\dfrac{1}{x}\, dx$
On integrating,	On integrating,
$\log y = \log x + \log c$	$\log(y + 1) = -\log x + \log c$
$\log y = \log cx$	$\log x + \log(y + 1) = \log c$
$y = cx$	$x(y + 1) = c$

Therefore, the general solution is

$(y - cx)(xy + x - c) = 0$

11. Solve: $xy\left(\frac{dy}{dx}\right)^2 - (x^2 + y^2)\left(\frac{dy}{dx}\right) + xy = 0$

$xyp^2 - x^2p - y^2p + xy = 0$

$xp(yp - x) - y(yp - x) = 0$

$(yp - x)(xp - y) = 0$

$yp - x = 0$	$xp - y = 0$
$y\frac{dy}{dx} = x$	$x\frac{dy}{dx} = y$
$y\,dy = x\,dx$	$\frac{1}{y}\,dy = \frac{1}{x}\,dx$
On integrating,	On integrating,
$\frac{y^2}{2} = \frac{x^2}{2} + \frac{c}{2}$	$\log y = \log x + \log c$
$y^2 = x^2 + c$	$\log y = \log cx$
$y^2 - x^2 - c = 0$	$y = cx$

Therefore, the general solution is

$(y^2 - x^2 - c)(y - cx) = 0$

12. Solve: $y\left(\frac{dy}{dx}\right)^2 + (x - y)\left(\frac{dy}{dx}\right) - x = 0$

$yp^2 + xp - yp - x = 0$

$p(yp + x) - (yp + x) = 0$

$(p - 1)(yp + x) = 0$

$p - 1 = 0$	$yp + x = 0$
$\frac{dy}{dx} = 1$	$y\frac{dy}{dx} = -x$
$dy = dx$	$y\,dy = -x\,dx$
On integrating,	On integrating,
$y = x + c$	$\frac{y^2}{2} = -\frac{x^2}{2} + \frac{c}{2}$
	$y^2 = -x^2 + c$

Therefore, the general solution is

$(y - x - c)(x^2 + y^2 - c) = 0$

13. Solve $x(y')^2 - (2x + 3y)y' + 6y = 0$

14. Solve $xp^2 + xp - yp + 1 - y = 0$

3.6 Clairaut's equation and reducible to Clairaut's equation

Introduction:

This is of the form $y = px + f(p)$. General solution is $y = cx + f(c)$.

Working rule to find singular solution:

- ❖ Differentiate general solution partially w.r.to c.
- ❖ Substitute the value of c in the general solution.

Note:

$y = px + f(p)$ ---- (1)

Differentiate w.r.to x,

$\frac{dy}{dx} = p + p'x + f'(p)p'$

$p'x + f'(p)p' = 0$

$p'[x + f'(p)] = 0$

$p' = 0$

$p = c$

Substitute $p = c$ in (1),

$y = cx + f(c)$.

This is the general solution.

Problems:

1. **Find the general solution and the singular solution of $p = \sin(y - xp)$.**

$y - xp = \sin^{-1} p$

$y = xp + \sin^{-1} p$

This is in Clairaut's form.

General solution is $y = cx + \sin^{-1} c$ ------ (1)

Differentiate partially w.r.to c,

$$0 = x + \frac{1}{\sqrt{1-c^2}} \qquad \bigg| \qquad 1 - c^2 = \frac{1}{x^2}$$

$$\frac{1}{\sqrt{1-c^2}} = -x \qquad \bigg| \qquad c^2 = 1 - \frac{1}{x^2}$$

$$\sqrt{1 - c^2} = -\frac{1}{x} \qquad \bigg| \qquad c = \frac{\sqrt{x^2-1}}{x}$$

Substitute the value of c in (1).

$$y = \sqrt{x^2 - 1} + \sin^{-1} \frac{\sqrt{x^2-1}}{x}$$

This is the required singular solution.

2. **Find the general solution and singular solution of** $sin\ px\ cos\ y = cos\ px\ sin\ y + p$

$\sin(px - y) = p$

$px - y = \sin^{-1} p$

$y = px - \sin^{-1} p$

This is in Clairaut's form.

General solution is $y = cx - \sin^{-1} c$ ------ (1)

Differentiate partially w.r.to c,

$0 = x - \dfrac{1}{\sqrt{1-c^2}}$

$\dfrac{1}{\sqrt{1-c^2}} = x$

$\sqrt{1 - c^2} = \dfrac{1}{x}$

$1 - c^2 = \dfrac{1}{x^2}$

$c^2 = 1 - \dfrac{1}{x^2}$

$c = \dfrac{\sqrt{x^2-1}}{x}$

Substitute the value of c in (1).

$y = \sqrt{x^2 - 1} - \sin^{-1} \dfrac{\sqrt{x^2-1}}{x}$

This is the required singular solution.

3. **Find the general solution and the singular solution of** $p = log\ (px - y)$.

$p = \log(px - y)$

$e^p = xp - y$

$y = xp - e^p$

This is in Clairaut's form.

General solution is $y = cx - e^c$ ------ (1)

Differentiate partially w.r.to c,

$0 = x - e^c$

$x = e^c$

$c = \log x$

Substitute the value of c in (1).

$y = x \log x - x$

This is the required singular solution.

4. **Find the general solution and the singular solution of $(y - px)(p - 1) = p.$**

$y - xp = \dfrac{p}{p-1}$

$y = xp + \dfrac{p}{p-1}$

This is in Clairaut's form.

General solution is $y = cx + \dfrac{c}{c-1}$ ------ (1)

Differentiate partially w.r.to c,

$0 = x + \dfrac{(c-1)1 - c(1)}{(c-1)^2}$

$x = \dfrac{1}{(c-1)^2}$

$(c - 1)^2 = \dfrac{1}{x}$

$c - 1 = \dfrac{1}{\sqrt{x}}$

$c = 1 + \dfrac{1}{\sqrt{x}}$

Substitute the value of c in (1).

$y = x + 2\sqrt{x} + 1$

$y = (\sqrt{x} + 1)^2$

This is the required singular solution.

5. **Find the general solution and the singular solution of $xp^2 - yp + a = 0$.**

$yp = xp^2 + a$

$y = xp + \dfrac{a}{p}$

This is in Clairaut's form.

General solution is $y = cx + \dfrac{a}{c}$ ------ (1)

Differentiate partially w.r.to c,

$0 = x - \dfrac{a}{c^2}$

$x = \dfrac{a}{c^2}$

$c = \sqrt{\dfrac{a}{x}}$

Substitute the value of c in (1).

$y = 2\sqrt{ax}$

This is the required singular solution.

6. **Find the general solution and the singular solution of $xp^3 - yp^2 + 1 = 0$.**

$$yp^2 = xp^3 + 1$$

$$y = xp + \frac{1}{p^2}$$

This is in Clairaut's form.

General solution is $y = cx + \frac{1}{c^2}$ ------ (1)

Differentiate partially w.r.to c,

$$0 = x - \frac{2}{c^3}$$

$$x = \frac{2}{c^3}$$

$$c = \left(\frac{2}{x}\right)^{\frac{1}{3}}$$

Substitute the value of c in (1).

$$y = x\left(\frac{2}{x}\right)^{\frac{1}{3}} + \left(\frac{x}{2}\right)^{\frac{2}{3}}$$

This is the required singular solution.

7. **Find the general solution and the singular solution of $y + 2\left(\frac{dy}{dx}\right)^2 = (x+1)\frac{dy}{dx}$.**

$$y = -2p^2 + (x+1)p$$

$$y = px + (p - 2p^2)$$

This is in Clairaut's form.

General solution is $y = cx + (c - 2c^2)$ ------ (1)

Differentiate partially w.r.to c,

$$0 = x + 1 - 4c$$

$$c = \frac{x+1}{4}$$

Substitute the value of c in (1).

$$y = x\left(\frac{x+1}{4}\right) + \left(\frac{x+1}{4} - \frac{(x+1)^2}{8}\right)$$

$$8y = 2x(x+1) + 2(x+1) - (x+1)^2$$

$$8y = (x+1)(2x + 2 - x - 1)$$

$$8y = (x+1)^2$$

This is the required singular solution.

8. **Solve $y^2(y - xp) = x^4 p^2$ using substitutions $X = \frac{1}{x}$ and $Y = \frac{1}{y}$.**

$$P = \frac{dY}{dX} = \frac{-\frac{1}{y^2}dy}{-\frac{1}{x^2}dx} = \frac{Y^2}{X^2}p$$

Put $x = \frac{1}{X}$, $y = \frac{1}{Y}$, $p = \frac{X^2}{Y^2}P$ in the given equation $y^2(y - xp) = x^4 p^2$

$$\frac{1}{Y^2}\left(\frac{1}{Y} - \frac{1}{X}\frac{X^2}{Y^2}P\right) = \frac{1}{X^4}\left(\frac{X^2}{Y^2}P\right)^2$$

$$\frac{1}{Y^4}(Y - XP) = \frac{P^2}{Y^4}$$

$$Y - XP = P^2$$

$$Y = XP + P^2$$

This is in Clairaut's form.

General solution is

$$Y = cX + c^2 \Rightarrow \frac{1}{y} = \frac{c}{x} + c^2$$

9. **Solve $(px - y)(py + x) = 2p$ by reducing into Clairaut's form, taking substitutions $X = x^2$ and $Y = y^2$.**

$$P = \frac{dY}{dX} = \frac{2y\,dy}{2x\,dx} = \frac{\sqrt{Y}}{\sqrt{X}}p$$

Put $x = \sqrt{X}$, $y = \sqrt{Y}$, $p = \frac{\sqrt{X}}{\sqrt{Y}}P$ in the given equation $(px - y)(py + x) = 2p$

$$\left(\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{X} - \sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{Y} + \sqrt{X}\right) = 2\frac{\sqrt{X}}{\sqrt{Y}}P$$

$$(PX - Y)(\sqrt{X}P + \sqrt{X}) = 2\sqrt{X}P$$

$$(PX - Y)(P + 1) = 2P$$

$$PX - Y = \frac{2P}{P+1}$$

$$Y = PX - \frac{2P}{P+1}$$

This is in Clairaut's form.

General solution is

$$Y = cX - \frac{2c}{c+1} \Rightarrow y^2 = cx^2 - \frac{2c}{c+1}$$

10. **Solve $x^2(y - px) = p^2 y$ by reducing into Clairaut's form, using the substitutions $X = x^2$ and $Y = y^2$.**

$$P = \frac{dY}{dX} = \frac{2y \, dy}{2x \, dx} = \frac{\sqrt{Y}}{\sqrt{X}} p$$

Put $x = \sqrt{X}$, $y = \sqrt{Y}$, $p = \frac{\sqrt{X}}{\sqrt{Y}} P$ in the given equation $x^2(y - px) = p^2 y$

$$X \left(\sqrt{Y} - \frac{\sqrt{X}}{\sqrt{Y}} p\sqrt{X} \right) = \left(\frac{\sqrt{X}}{\sqrt{Y}} P \right)^2 \sqrt{Y}$$

$$Y - pX = p^2$$

$$Y = pX + p^2$$

This is in Clairaut's form.

General solution is

$$Y = cX + c^2 \implies y^2 = cx^2 + c^2$$

11. **Solve $e^{4x}(p - 1) + e^{2y} p^2 = 0$ by using substitutions $X = e^{2x}, Y = e^{2y}$.**

$$P = \frac{dY}{dX} = \frac{2 e^{2y} dy}{2 e^{2x} dx} = \frac{Y}{X} p$$

Put $e^{2x} = X$, $e^{2y} = Y$, $p = \frac{X}{Y} P$ in the given equation $e^{4x}(p - 1) + e^{2y} p^2 = 0$

$$X^2 \left(\frac{X}{Y} P - 1 \right) + Y \left(\frac{X}{Y} P \right)^2 = 0$$

$$X^2(XP - Y) + X^2 P^2 = 0$$

$$XP - Y + P^2 = 0$$

$$Y = XP + P^2$$

This is in Clairaut's form.

General solution is

$$Y = cX + c^2$$

$$\implies e^{2y} = c \, e^{2x} + c^2$$

12. Solve $(px + y)^2 = py^2$ by using the substitutions $X = y$ and $Y = xy$.

$P = \frac{dY}{dX} = \frac{xdy + ydx}{dy} = \frac{xp + y}{p} = x + \frac{y}{p} = \frac{Y}{X} + \frac{X}{p}$

$P - \frac{Y}{X} = \frac{X}{p}$

$p = \frac{X}{\left(P - \frac{Y}{X}\right)}$

Put $x = \frac{Y}{y} = \frac{Y}{X}$, $y = X, p = \frac{X^2}{PX - Y}$ in the given equation $(px + y)^2 = py^2$

$\left\{\left(\frac{X^2}{PX - Y}\right)\frac{Y}{X} + X\right\}^2 = \left(\frac{X^2}{PX - Y}\right)X^2$

$\left\{\frac{Y}{PX - Y} + 1\right\}^2 = \frac{X^2}{PX - Y}$

$\left\{\frac{PX}{PX - Y}\right\}^2 = \frac{X^2}{PX - Y}$

$P^2 = PX - Y$

$Y = PX - P^2$

This is in Clairaut's form.

General solution is

$Y = cX - c^2$

$\Rightarrow xy = cy - c^2$