Fourier Series

Bernoulli's rule of Integration by parts

$$\int uv\,dx = u\int v - u'\iint v + u''\iiint v - \cdots$$

where $u', u'', u''' \ldots$ are successive derivatives of u.

Standard Results

1) $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$.

2) If 'n' is an integer, then

i) $\sin(n\pi) = 0$

ii) $\cos(n\pi) = (-1)^n = \begin{cases} +1, & n \text{ is even} \\ -1, & n \text{ is odd.} \end{cases}$

3) $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

4) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$

$\int_{0}^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$

5) Even function × Even function = Even function
odd function × odd function = Even function
Even function × odd function = odd function

Periodic function

A function $f(x)$ is said to be periodic function with period T, if $f(x+T) = f(x)$ $\forall x$, $T > 0$.

Ex: $f(x) = \sin x$ is periodic function with period 2π

Fourier Series

Fourier Series

The fourier series of a periodic function with period $2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where a_0, a_n, b_n are the fourier co-efficients given by.

Interval	a_0	a_n	b_n
$(-l, l)$	$\frac{1}{l}\int_{-l}^{l} f(x)\,dx$	$\frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$	$\frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$
$(-\pi, \pi)$	$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$	$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$	$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$
$(0, 2l)$	$\frac{1}{l}\int_{0}^{2l} f(x)\,dx$	$\frac{1}{l}\int_{0}^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$	$\frac{1}{l}\int_{0}^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$
$(0, 2\pi)$	$\frac{1}{\pi}\int_{0}^{2\pi} f(x)\,dx$	$\frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos(nx)\,dx$	$\frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin(nx)\,dx$

<u>Note</u>:- 1) If $f(x)$ is discontinuous function of the form

i) $$f(x) = \begin{cases} \phi(x) & \text{in } -l < x < 0 \\ \psi(x) & \text{in } 0 < x < l \end{cases}, \quad \text{then}$$

$f(x)$ is even if $\phi(-x) = \psi(x)$ &

$f(x)$ is odd if $\phi(-x) = -\psi(x)$

ii) $$f(x) = \begin{cases} \phi(x) & \text{in } 0 < x < l \\ \psi(x) & \text{in } l < x < 2l \end{cases}, \quad \text{then}$$

$f(x)$ is even if $\phi(2l-x) = \psi(x)$ &

$f(x)$ is odd if $\phi(2l-x) = -\psi(x)$

2) If $f(x)$ is discontinuous at x, then at x the fourier series converges to $\frac{1}{2}[f(x^+) + f(x^-)]$ where $f(x^+)$ & $f(x^-)$ are Right hand & left hand limits of $f(x)$ respectively.

Fourier co-efficients for Even (or) odd function

Interval	Nature	a_0	a_n	b_n
①	Even function			
$(-l,l)$	$f(-x) = f(x)$	$\frac{2}{l}\int_0^l f(x)dx$	$\frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$	0
$(0,2l)$	$f(2l-x) = f(x)$			
$(-\pi,\pi)$	$f(-x) = f(x)$	$\frac{2}{\pi}\int_0^\pi f(x)dx$	$\frac{2}{\pi}\int_0^\pi f(x)\cos(nx)dx$	0
$(0,2\pi)$	$f(2\pi-x) = f(x)$			
②	odd function			
$(-l,l)$	$f(-x) = -f(x)$	0	0	$\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$
$(0,2l)$	$f(2l-x) = -f(x)$			
$(-\pi,\pi)$	$f(-x) = -f(x)$	0	0	$\frac{2}{\pi}\int_0^\pi f(x)\sin(nx)dx$
$(0,2\pi)$	$f(2\pi-x) = -f(x)$			

Problems

1) Find the fourier series of $f(x) = |x|$ in $-\pi < x < \pi$. Hence deduce that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

Soln:- Given function is $f(x) = |x|$

To check function is even (or) odd.

we have $f(x) = |x|$ in $(-\pi, \pi)$

$$f(-x) = |-x| = |x| = f(x).$$

ie $f(-x) = f(x)$.

$\therefore$ $f(x)$ is even function

$\Rightarrow$ $\boxed{b_n = 0}$ Here $f(x) = |x| = \begin{cases} -x & \text{in } -\pi < x < 0 \\ +x & \text{in } 0 < x < \pi \end{cases}$

ie $f(x) = |x| = x$ in $0 < x < \pi$

$$\therefore a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi} \int_0^{x} x\, dx$$

$$= \frac{2}{\pi} \left(\frac{x^2}{2} \right)_0^{\pi}$$

$$= \frac{2}{\pi(2)} \times (x^2)_0^{\pi}$$

$$= \frac{1}{\pi} (\pi^2 - 0) = \frac{\pi^2}{\pi} = \pi$$

$\therefore$ $\boxed{a_0 = \pi}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos(nx) \, dx$$

$$= \frac{2}{\pi}\left[\left(x \cancel{\frac{\sin(nx)}{n}}^{\to 0}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}\left(\cos nx\right)_0^{\pi}\right]$$

$$= \frac{2}{n^2\pi}\left(\cos nx\right)_0^{\pi}$$

$$= \frac{2}{n^2\pi}\left(\cos n\pi - \cos 0\right)$$

$$a_n = \frac{2}{n^2\pi}\left((-1)^n - 1\right)$$

$$\Rightarrow \boxed{a_n = \frac{-2}{n^2\pi}\left[1-(-1)^n\right]}$$

& $\boxed{b_n = 0}$

Fourier Series of $f(x)$ with period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-2}{n^2\pi}\left[1-(-1)^n\right] \cos nx \quad \text{—①}$$

is the required fourier Series for given $f(x)$.

<u>Deduction</u>

put $\boxed{x=0}$ in Eq ①.

$$f(0) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-2}{n^2\pi}\left[1-(-1)^n\right] \times 1$$

$$0 = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2}\left[1-(-1)^n\right]$$

$$-\frac{\pi}{2} = -\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2}\left(1-(-1)^n\right)$$

$$\frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2}\left[1-(-1)^n\right]$$

$$\left(\because \left[1-(-1)^n\right] = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}\right.$$

$$\frac{\pi^2}{4} = \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2}(2).$$

$$\frac{\pi^2}{8} = \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\Rightarrow \boxed{\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}}$$

2) obtain the fourier series for $f(x) = \frac{\pi - x}{2}$ in $0 < x < 2\pi$. Hence dedue that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Solun:- Given $f(x) = \frac{\pi - x}{2}$ is defined in $(0, 2\pi)$

To check $f(x)$ is even or odd function

$$f(2\pi - x) = \frac{\pi - (2\pi - x)}{2} = \frac{\pi - 2\pi + x}{2} = \frac{-\pi + x}{2}$$

$$= -\left(\frac{\pi - x}{2}\right)$$

$$= -f(x)$$

ie $f(-x) = -f(x)$

$\therefore$ Given function $f(x)$ is odd function

Clearly $\boxed{a_0 = 0}$ & $\boxed{a_n = 0}$

Now $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$

$$= \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi - x}{2}\right) \sin(nx)\, dx$$

$$= \frac{2}{\pi} \cdot \frac{1}{2} \int_0^{\pi} (\pi - x) \sin(nx)\, dx$$

$$= \frac{1}{\pi} \left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(-\cancel{\frac{\sin nx}{n^2}}\right) \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left(-\frac{1}{n} (\pi - x) \cos nx \right)_0^{\pi}$$

$$= \frac{-1}{n\pi} \left(\overset{0}{(\pi - \pi)} \cos n\pi - (\pi - 0) \cos(0) \right)$$

$$= -\frac{1}{n\pi}(0-\pi)$$

$$= -\frac{1}{n\pi}(-\pi) = \frac{1}{n}$$

$$\Rightarrow \boxed{b_n = \frac{1}{n}}$$

Thus fourier series we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= 0 + 0 + \sum_{n=1}^{\infty} \frac{1}{n} \sin nx.$$

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin(nx) \quad \text{—— (1)}$$

is the required fourier series of $f(x)$.

put $x = \pi/2$ in eq (1).

$$(1) \Rightarrow \frac{\pi - \pi/2}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\pi/2)$$

$$\frac{2\pi - \pi}{4} = \frac{\sin \pi/2}{1} + \frac{\sin \pi}{2} + \frac{\sin(3\pi/2)}{3} + \frac{\sin 2\pi}{4} + \frac{\sin(5\pi/2)}{5} + \cdots$$

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots\cdots}$$

3) Find the fourier series of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $0 < x < 2\pi$. Hence Evaluate $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots$

Solun:- Given function is $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $(0, 2\pi)$

To check f(x) is even or odd

$$f(x) = \left(\frac{\pi - x}{2}\right)^2 \text{ in } (0, 2\pi)$$

Consider $f(2\pi - x) = \left(\frac{\pi - (2\pi - x)}{2}\right)^2 = \left(\frac{\pi - 2\pi + x}{2}\right)^2$

$$= \left(\frac{-\pi + x}{2}\right)^2 = \left[-\left(\frac{\pi - x}{2}\right)\right]^2$$

$$= \left(\frac{\pi - x}{2}\right)^2 = f(x)$$

ie $f(2\pi - x) = f(x)$

$\therefore$ Given function $f(x)$ is even function.

$\Rightarrow \boxed{b_n = 0}$

Now $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$

$$= \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi - x}{2}\right)^2 dx$$

$$= \frac{2}{4\pi} \int_0^{\pi} (\pi - x)^2\, dx$$

$$= \frac{1}{2\pi} \left[\frac{(\pi - x)^3}{-3}\right]_0^{\pi}$$

$$= \frac{1}{-6\pi} \left[(\pi - x)^3\right]_0^{\pi} = -\frac{1}{6\pi} \left((\pi - \pi)^3 - (\pi - 0)^3\right)$$

$$= -\frac{1}{6\pi}\left(0^3 - \pi^3\right) = \frac{\pi^3}{6\pi} = \frac{\pi^2}{6}.$$

$$\Rightarrow \boxed{a_0 = \frac{\pi^2}{6}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi - x}{2}\right)^2 \cos(nx)\,dx$$

$$= \frac{2}{4\pi}\int_0^{\pi}(\pi - x)^2\cos(nx)\,dx$$

$$= \frac{1}{2\pi}\left[\cancel{(\pi-x)^2\left(\frac{\sin(nx)}{n}\right)}^{\,0} - (-2)(\pi - x)\left(-\frac{\cos nx}{n^2}\right) + \cancel{2\left(\frac{-\sin nx}{n^3}\right)}^{\,0}\right]_0^{\pi}$$

$$= -\frac{1}{2\pi}\left[2(\pi - x)\frac{\cos(nx)}{n^2}\right]_0^{\pi}$$

$$= -\frac{2}{2n^2\pi}\left[(\pi - x)\cos(nx)\right]_0^{\pi}$$

$$= -\frac{1}{n^2\pi}\left((\pi-\pi)\cos n\pi - \pi\cos 0\right)$$

$$= -\frac{1}{n^2\pi}\left(0 - \pi\right) = \frac{\pi}{n^2\pi} = \frac{1}{n^2}$$

$$\boxed{a_n = \frac{1}{n^2}}$$

The fourier series of $f(x)$ with period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx).$$

$$f(x) = \frac{\pi^2/6}{2} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$$

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx \quad \text{—} (1).$$

is the required fourier series.

Deduction

put $x=0$ in Eq (1)

$$f(0) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \times 1.$$

$$\left(\frac{\pi - 0}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{4} - \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

4) Find the fourier series of

$$f(x) = \begin{cases} 1+\frac{2x}{\pi} & \text{in} \quad -\pi < x \le 0 \\ 1-\frac{2x}{\pi} & \text{in} \quad 0 \le x < \pi \end{cases}$$

Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Solun :- Given $f(x)$ is defined in $(-\pi, \pi)$

To check $f(x)$ is even or odd function

$$\phi(x) = 1+\frac{2x}{\pi}, \quad \psi(x) = 1-\frac{2x}{\pi}$$

Consider $\phi(-x) = 1+\frac{2}{\pi}(-x) = 1-\frac{2x}{\pi} = \psi(x)$

ie $\phi(-x) = \psi(x)$

$\therefore$ $f(x)$ is even function

Clearly $\boxed{b_n = 0}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left(x - \frac{2x^2}{2\pi}\right)_0^{\pi}$$

$$= \frac{2}{\pi}\left(x - \frac{x^2}{\pi}\right)_0^{\pi} = \frac{2}{\pi}\left(\pi - \frac{\pi^2}{\pi}\right) = \frac{2}{\pi}(\pi - \pi)$$

$$\Rightarrow \boxed{a_0 = 0}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\cancel{\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right)}^{0} - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\frac{2}{\pi n^2}(\cos nx)\right)_0^{\pi}$$

$$= \frac{-4}{\pi^2 n^2}(\cos nx)_0^{\pi}$$

$$= \frac{-4}{n^2\pi^2}(\cos n\pi - \cos 0)$$

$$= \frac{-4}{n^2\pi^2}((-1)^n - 1)$$

$$\boxed{a_n = \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)}$$

We have fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

$$f(x) = \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left(1-(-1)^n\right)\cos nx$$

$$= \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}\left(1-(-1)^n\right) \cos(nx) \qquad \text{—}(1)$$

is the required fourier Series

Put $x=0$ in Eq (1)

$$\therefore \quad 1 = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2}$$

where
$$1-(-1)^n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2 & \text{If } n \text{ is odd} \end{cases}$$

$$1 = \frac{4}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{2}{n^2}$$

$$1 = \frac{8}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{8} = \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Hence the Deduction.

5) obtain the fourier series for the function

$$f(x) = \begin{cases} \pi x & \text{in } 0 \le x \le 1 \\ \pi(2-x) & \text{in } 1 \le x \le 2 \end{cases}$$

Deduce that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

Solun :-

Here $f(x)$ is defined in $[0,2]$

Compairing this with $[0,2l]$

$\Rightarrow 2l = 2$

$\Rightarrow \boxed{l=1}$

$\phi(x) = \pi x$; $\psi(x) = \pi(2-x)$.

To check $f(x)$ is even or odd function.

$\phi(2-x) = \pi(2-x)$

$= \psi(x)$

$\therefore f(x)$ is even function

clearly $\boxed{b_n = 0}$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{1}\int_0^1 f(x)\,dx = 2\int_0^1 \pi x\,dx$$

$$= 2\left[\pi\frac{x^2}{2}\right]_0^1$$

$$= 2\frac{\pi}{2}(x^2)_0^1$$

$$= \pi(1)$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{1}\int_0^1 f(x)\cos(n\pi x)\,dx$$

$$= 2\int_0^1 \pi x \cos(n\pi x)\,dx$$

$$= 2\pi\int_0^1 x\cos(n\pi x)\,dx$$

$$= 2\pi\left[\cancel{x\,\frac{\sin(n\pi x)}{(n\pi)}}^{\to 0} - (1)\left(-\frac{\cos(n\pi x)}{(n\pi)^2}\right)\right]_0^1$$

$$= 2\pi\left(\frac{1}{n^2\pi^2}\left(\cos n\pi x\right)_0^1\right)$$

$$= \frac{2\pi}{n^2\pi^2}\left(\cos n\pi - \cos 0\right)$$

$$= \frac{2}{n^2\pi}\left((-1)^n - 1\right)$$

$$\boxed{a_n = \frac{-2}{n^2\pi}\left(1-(-1)^n\right)}$$

We have fourier series of $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{-2}{n^2\pi}\left(1-(-1)^n\right)\cos(n\pi x)$$

$$= \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{\pi n^2}\cos(n\pi x)$$

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1,3,5\ldots}^{\infty} \frac{2}{n^2}\cos(n\pi x) \qquad \left(\because 1-(-1)^n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}\right)$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2}\cos(n\pi x) \quad \text{——— (1)}$$

is the required fourier series

Put $x=0$ in Eq (1)

$$\therefore (1) \Rightarrow \quad 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2}$$

$$-\frac{\pi}{2} = -\frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{8} = \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\Rightarrow \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

6) obtain the fourier series of $f(x) = e^{-ax}$ in the interval $(-\pi, \pi)$. Deduce that $\frac{\pi}{\sinh\pi} = 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$

Soln:- Given function is

$$f(x) = e^{-ax}$$

To check the function is even (or) odd

$f(x) = e^{-ax}$ in $(-\pi, \pi)$

$f(-x) = e^{ax} \neq f(x)$

$\neq -f(x)$

$\therefore$ $f(x)$ is neither even nor odd

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\,dx$$

$$= \frac{1}{\pi}\left(\frac{e^{-ax}}{-a}\right)_{-\pi}^{\pi}$$

$$= -\frac{1}{a\pi}\left(e^{-ax}\right)_{-\pi}^{\pi}$$

$$= \frac{-1}{a\pi}\left(e^{-a\pi} - e^{a\pi}\right)$$

$$= \frac{2}{a\pi}\left[\frac{e^{a\pi} - e^{-a\pi}}{2}\right]$$

$$= \frac{2}{a\pi}(\sinh a\pi)$$

$$\therefore \boxed{a_0 = \frac{2\sinh(a\pi)}{a\pi}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\left(-a\cos nx + n\cancel{\sin nx}^{\,0}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}\left[a\, e^{-ax}\cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{a}{\pi(a^2+n^2)}\left(-e^{-a\pi}\cos n\pi + e^{a\pi}\cos n\pi\right)$$

$$= \frac{a\cos n\pi}{\pi(a^2+n^2)}\left(-e^{-a\pi} + e^{a\pi}\right)$$

$$= \frac{2a(-1)^n}{\pi(a^2+n^2)}\left(\frac{e^{a\pi} - e^{-a\pi}}{2}\right)$$

$$\boxed{a_n = \frac{2a(-1)^n}{\pi(a^2+n^2)}\left(\sinh(a\pi)\right)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left\{\frac{e^{-ax}}{a^2+n^2}\left[\cancel{-a\sin(nx)}^{\,0} - n\cos nx\right]\right\}_{-\pi}^{\pi}$$

$$= \frac{-n}{\pi(a^2+n^2)}\left[e^{-ax}\cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{-n}{\pi(a^2+n^2)}\left(e^{-a\pi}\cos(n\pi) - e^{+a\pi}\cos n\pi\right)$$

$$= -\frac{n\cos(n\pi)}{\pi(a^2+n^2)}\left(e^{-a\pi} - e^{a\pi}\right)$$

$$= \frac{n(-1)^n \times 2}{\pi(a^2+n^2)}\left(\frac{e^{a\pi} - e^{-a\pi}}{2}\right)$$

$$\boxed{b_n = \frac{2n(-1)^n}{\pi(a^2+n^2)}\sinh(a\pi)}$$

Fourier Series of $f(x)$ with period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{\sinh(a\pi)}{a\pi} + \sum_{n=1}^{\infty}\frac{2a(-1)^n}{\pi(a^2+n^2)}\sinh a\pi \cdot \cos(nx) + \sum_{n=1}^{\infty}\frac{2n(-1)^n}{\pi(a^2+n^2)}\sinh a\pi \cdot \sin(nx)$$

is the required fourier series — ①

Deduction

put $x=0$ $a=1$

$$f(0) = \frac{\sinh(\pi)}{\pi} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{\pi(1^2+n^2)}\sinh\pi \times 1$$

$$e^0 = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n^2+1)}$$

$$1 = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\left[-\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{(n^2+1)}\right]$$

$$= \frac{\sinh\pi}{\pi} - \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n^2+1)}$$

$$1 = \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n^2+1)}$$

$$\boxed{\frac{\pi}{\sinh\pi} = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{(n^2+1)}}$$

---

7) Find the fourier series in $(-\pi,\pi)$ to represent $f(x)=x-x^2$. Hence deduce $\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots$

Solun:- Given $f(x)=x-x^2$ in $(-\pi,\pi)$

To check the function is even (or) odd

$$f(x)=x-x^2 \quad \text{in } (-\pi,\pi)$$

$$f(-x) = -x-(-x)^2 = -x-x^2 \neq f(x)$$

$$\neq -f(x)$$

$\therefore$ Given function $f(x)$ is neither even nor odd

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\,dx = \frac{1}{\pi}\left(\frac{x^2}{2}-\frac{x^3}{3}\right)_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{2}(x^2)_{-\pi}^{\pi}-\frac{1}{3}(x^3)_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{2}(\pi^2-\pi^2)-\frac{1}{3}\left(\pi^3-(-\pi)^3\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{2}(0)-\frac{1}{3}(\pi^3+\pi^3)\right]$$

$$= \frac{1}{\pi}\left(\frac{-2}{3}\pi^3\right) = -\frac{2}{3}\pi^2 \Rightarrow \boxed{a_0=-\frac{2\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\cancel{(x-x^2)\left(\frac{\sin(nx)}{n}\right)}^{0} - (1-2x)\left(-\frac{\cos(nx)}{n^2}\right) + \cancel{(-2)\left(-\frac{\sin(nx)}{n^3}\right)}^{0}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\frac{1}{n^2}\left[(1-2x)\cos(nx)\right]\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi n^2}\left[(1-2\pi)(\cos n\pi) - (1+2\pi)\cos n\pi\right]$$

$$= \frac{\cos n\pi}{n^2\pi}\left(\cancel{1}-2\pi-\cancel{1}-2\pi\right)$$

$$= \frac{(-1)^n}{n^2\pi}(-4\pi) \Rightarrow \boxed{a_n = \frac{-4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(\frac{-\cos(nx)}{n}\right) - \cancel{(1-2x)\left(-\frac{\sin(nx)}{n^2}\right)}^{0} + (-2)\left(\frac{\cos(nx)}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(-\frac{1}{n}\right)(x-x^2)(\cos(nx)) - \frac{2}{n^3}\cos(nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\left\{(\pi-\pi^2)\cos n\pi - (-\pi-\pi^2)\cos(-n\pi)\right\} - \frac{2}{n^3}\left\{\cancel{\cos(n\pi) - \cos(-n\pi)}^{0}\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\cos n\pi}{n}\left(\pi - \cancel{\pi^2} + \pi + \cancel{\pi^2}\right)\right]$$

$$= \frac{-(-1)^n}{n\pi}(2\pi) \Rightarrow \boxed{b_n = \frac{-2(-1)^n}{n}}$$

∴ Required fourier series for given $f(x)$ with period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$= \frac{-2\pi^2/3}{2} + \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n^2}\cos(nx) + \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n}\sin(nx)$$

$$= -\frac{\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx) - \sum_{n=1}^{\infty} \frac{2(-1)^n}{n}\sin(nx) \quad \text{—①}$$

Deduction :- put $x=0$ in Eqr ①

$$f(0) = \frac{-\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \times 1$$

$$0 = \frac{-\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{\pi^2}{3} = -4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

8) Find the fourier Series of the function

$$f(x) = \begin{cases} x & \text{in } 0 \le x \le \pi \\ 2\pi - x & \text{in } \pi \le x \le 2\pi \end{cases}$$ Hence deduce

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \text{------}$$

Solun :- To check Given function is even (or) odd

Let $\phi(x) = x$; $\psi(x) = 2\pi - x$ in $(0, 2\pi)$

Consider $\phi(2\pi - x) = 2\pi - x = \psi(x)$

$\therefore$ $f(x)$ is an even function

$\therefore$ $\boxed{b_n = 0}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} = \frac{2}{\pi}\left(\frac{\pi^2}{2} - 0\right)$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left[x\left(\cancel{+\frac{\sin(nx)}{n}}\right)^{0} - 1\left(-\frac{\cos(nx)}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}\left(\cos(nx)\right)\Big|_0^{\pi}\right]$$

$$= \frac{2}{n^2\pi}(\cos n\pi - \cos 0) = \frac{2}{n^2\pi}\left((-1)^n - 1\right)$$

$$\boxed{a_n = \frac{-2}{n^2\pi}\left[1 - (-1)^n\right]}$$

The fourier series of $f(x)$ with period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-2}{\pi n^2} \left(1 - (-1)^n\right) \cos(nx). \quad \text{--- (1)}$$

is required fourier series

Put $x=0$

$$f(0) = \frac{\pi}{2} + \sum_{n=1}^{\infty} -\frac{2}{\pi n^2} \left(1-(-1)^n\right)$$

$$0 = \frac{\pi}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\left(1-(-1)^n\right)}{n^2} \qquad \left(\because f(0)=0 \text{ bcz } f(x)=x \text{ in } (0,\pi)\right)$$

$$\frac{\pi}{2} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2}$$

$$\frac{\pi^2}{4} = \sum_{n=1,3,5\ldots}^{\infty} \frac{2}{n^2}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

9) obtain the fourier series for the function

$$f(x) = \begin{cases} -\pi & \text{in } -\pi < x < 0 \\ x & \text{in } 0 < x < \pi \end{cases}$$

Hence deduce that, sum of the reciprocal squares of the odd integers is equal to $\frac{\pi^2}{8}$.

Solun:- To check function is even(or) odd function

Let $\phi(x) = -\pi$, $\psi(x) = x$ in $(-\pi, \pi)$

$\phi(-x) = -\pi \neq \psi(x)$
$\neq -\psi(x)$

$\therefore$ $f(x)$ is neither even nor odd function.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi(x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi(+\pi) + \frac{\pi^2}{2}\right] = \frac{1}{\pi}\left(-\pi^2 + \frac{\pi^2}{2}\right)$$

$$= -\frac{1}{\pi}\,\frac{\pi^2}{2}$$

$$\boxed{a_0 = -\pi/2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos(nx)\,dx + \int_{0}^{\pi} f(x)\cos(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos(nx)\,dx + \int_{0}^{\pi} x\cos(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left\{\cancelto{0}{\frac{\sin(nx)}{n}}\right\}_{-\pi}^{0} + \left\{\cancelto{0}{x\,\frac{\sin(nx)}{n}} - 1\times\left(-\frac{\cos(nx)}{n^2}\right)\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left\{\frac{1}{n^2}\left[\cos(nx)\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{n^2\pi}\left(\cos n\pi - \cos 0\right)$$

$$= \frac{1}{n^2\pi}\left\{(-1)^n - 1\right\} \Rightarrow \boxed{a_n = -\frac{1}{n^2\pi}\left\{1 - (-1)^n\right\}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin(nx)\,dx + \int_{0}^{\pi} f(x)\sin(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin(nx)\,dx + \int_{0}^{\pi} x\sin(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(-\frac{\cos(nx)}{n}\right)_{-\pi}^{0} + \left\{x\left(-\frac{\cos(nx)}{n}\right) - 1\,\cancelto{0}{\left(-\frac{\sin nx}{n^2}\right)}\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi) - \frac{1}{n}(\pi \cos n\pi - 0)\right]$$

$$= \frac{1}{\pi} \times \frac{\pi}{n}\left(1-\cos n\pi - \cos n\pi\right)$$

$$= \frac{1}{n}\left[1 - 2\cos n\pi\right]$$

$$\boxed{b_n = \frac{1}{n}\left[1-2(-1)^n\right]}$$

The fourier series of function $f(x)$ in the interval 2π is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{-\pi}{4} + \sum_{n=1}^{\infty} \frac{-1}{n^2\pi}\left[1-(-1)^n\right]\cos(nx) + \sum_{n=1}^{\infty} \frac{1}{n}\left[1-2(-1)^n\right]\sin(nx) \quad \text{—} (1)$$

is the required fourier series

<u>Deduction</u> put $x=0$ in eq (1)

It is clear that $f(x)$ at $x=0$ is a point of discontinuity.

Hence series converges to $\frac{1}{2}\left[f(0^+) + f(0^-)\right]$

since $f(0^+) = 0 \quad \because f(x) = x$ in $(0, \pi)$

$f(0^-) = -\pi \quad \because f(x) = -\pi$ in $(-\pi, 0)$

$$\therefore f(0) = \frac{1}{2}\left[f(0^+) + f(0^-)\right] = \frac{1}{2}(0-\pi) = -\pi/2$$

$\therefore$ Eq ① $\Rightarrow$

$$f(0) = \frac{-\pi}{4} + \sum_{n=1}^{\infty} \frac{-1}{n^2\pi}\left(1-(-1)^n\right)$$

$$-\frac{\pi}{2} = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-(-1)^n\right)$$

$$\frac{\pi}{4} - \frac{\pi}{2} = \frac{-1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-(-1)^n\right)$$

$$\frac{-\pi}{4} = \frac{-1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-(-1)^n\right)$$

$$\frac{\pi^2}{4} = \sum_{n=1,3,5---}^{\infty}\frac{2}{n^2}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + -------$$

10) obtain the fourier series for the function

$$f(x) = \begin{cases} 1+\frac{4x}{3} & \text{in } -3/2 < x \le 0 \\ 1-\frac{4x}{3} & \text{in } 0 \le x < 3/2 \end{cases}$$

Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ------$

Solun:- To check given function even (or) odd

Let $\phi(x) = 1+\frac{4x}{3}$ $\qquad \psi(x) = 1-\frac{4x}{3}$

Consider $\phi(-x) = 1+\frac{4(-x)}{3} = 1-\frac{4x}{3} = \psi(x)$

ie $\phi(-x) = \psi(x)$

$\therefore$ $f(x)$ is even function

$\therefore$ $\boxed{b_n = 0}$

Here $f(x)$ is defined in $\left(\frac{-3}{2}, \frac{3}{2}\right)$
$(-l, l)$

$\therefore \boxed{l = 3/2}$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{3/2}\int_0^{3/2}\left(1-\frac{4x}{3}\right)dx = \frac{4}{3}\left(x - \frac{4}{3}\frac{x^2}{2}\right)_0^{3/2}$$

$$= \frac{4}{3}\left[\left(\frac{3}{2} - \frac{4}{3\times 2}\left(\frac{3}{2}\right)^2\right) - 0\right]$$

$$= \frac{4}{3}\left(\frac{3}{2} - \frac{3}{2}\right) = 0$$

$$\Rightarrow \boxed{a_0 = 0}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{3/2}\int_0^{3/2}\left(1-\frac{4x}{3}\right)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{4}{3}\left[\left(1-\frac{4x}{3}\right)\left(\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right)^{\to 0} - \left(\left(\frac{-4}{3}\right)\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right)\right]_0^{3/2}$$

$$= \frac{4}{3}\times\frac{-4}{3}\times\frac{9}{4n^2\pi^2}\left(\cos\left(\frac{2n\pi x}{3}\right)\right)_0^{3/2}$$

$$= \frac{-4}{n^2\pi^2}\left(\cos n\pi - 0\right)$$

$$= \frac{-4}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$\boxed{a_n = \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)}$$

The fourier series for $f(x)$ with period $2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(1-(-1)^n\right) \cos\left(\frac{2n\pi x}{3}\right) \quad \text{—— (1)}$$

is required fourier series

Deduction

put $x=0$ in eq (1)

$$f(0) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$1 = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}\left(1-(-1)^n\right)$$

$$\frac{\pi^2}{4} = \sum_{n=1,3,5\ldots}^{\infty} \frac{2}{n^2}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

11) Find the fourier series of the periodic function defined by $f(x) = 2x - x^2$ in the interval $0 < x < 3$.

Solun:- Given $f(x) = 2x - x^2$ & interval $(0,3)$

Compare with $(0, 2l)$

$\Rightarrow 2l = 3$

$$\boxed{l = 3/2}$$

The fourier series of $f(x)$ with period $2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{--- (1)}$$

To check $f(x)$ is even (or) odd

we have $f(x) = 2x - x^2$ with $(0,3)$

Consider $f(3-x) = 2(3-x) - (3-x)^2$

$= 6 - 2x - (9 + x^2 - 6x) \neq f(x)$

$\neq -f(x)$

$\therefore$ $f(x)$ is neither Even nor odd function.

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{3/2}\int_0^3 (2x - x^2)\,dx = \frac{2}{3}\left(\cancel{2}\frac{x^2}{\cancel{2}} - \frac{x^3}{3}\right)_0^3$$

$$= \frac{2}{3}\left(9 - \frac{27}{3}\right)$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{l}\int_0^3 f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{3/2}\int_0^3 (2x - x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[\cancelto{0}{(2x-x^2)\left(\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right)} - (2-2x)\times\left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right)\right.$$

$$\left. + (-2)\cancelto{0}{\left(\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right)}\right]_0^3$$

$$= \frac{2}{3}\times\frac{9}{4n^2\pi^2}\left[(2-2x)\cos\left(\frac{2n\pi x}{3}\right)\right]_0^3$$

$$= \frac{3}{2n^2\pi^2}\left[(2-6)\cos(2n\pi) - 2\cos(0)\right]$$

$$= \frac{3}{2n^2\pi^2}\left(-4\times(1) - 2\right) = \frac{3\times -6}{2n^2\pi^2}$$

$$\boxed{a_n = \frac{-9}{n^2\pi^2}}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{3/2}\int_0^3 (2x - x^2)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[(2x-x^2)\left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right) - (2-2x)\cancelto{0}{\left(\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right)} + (-2)\left(\frac{\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{-3}{2n\pi}\left\{ (2x-x^2)\cos\left(\frac{2n\pi x}{3}\right) \right\}_0^3 - \frac{54}{8n^3\pi^3}\left(\cos\left(\frac{2n\pi x}{3}\right) \right)_0^3 \right]$$

$$= \frac{2}{3}\left[\frac{-3}{2n\pi}\left((6-9)\cos(2n\pi) - 0 \right) - \frac{54}{8n^3\pi^3}\left(\cos(2n\pi) - \cos 0 \right) \right]$$

$$= \frac{2}{3}\left[\frac{-3}{2n\pi}(-3)\times 1 - \frac{54}{8n^3\pi^3}(1-1) \right]$$

$$= \frac{2}{3}\,\frac{+9}{2n\pi}$$

$$\boxed{b_n = \frac{3}{n\pi}}$$

$\therefore$ Eq ① $\Longrightarrow$

$$f(x) = \sum_{n=1}^{\infty} \frac{-9}{n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right) + \sum_{n=1}^{\infty} \frac{3}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)$$

is the required fourier series.

Half Range Fourier Series

Interval	Required Series	Series	Co-efficients
$(0,l)$	Cosine series	$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$	$a_0=\frac{2}{l}\int_0^l f(x)\,dx$ $a_n=\frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$
	Sine Series	$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$	$b_n=\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$
$(0,\pi)$	Cosine series	$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos(nx)$	$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx$ $a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$
	Sine Series	$\sum_{n=1}^{\infty} b_n \sin(nx)$	$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

1) Find the cosine series for $f(x) = (x-1)^2$ in $0 < x < 1$. Hence deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Solun :- Given $f(x) = (x-1)^2$ in $(0,1)$

Compare given interval with $(0, l)$

$\implies \boxed{l=1}$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{1}\int_0^1 (x-1)^2\,dx = 2\left[\frac{(x-1)^3}{3}\right]_0^1$$

$$= \frac{2}{3}\left[(1-1)^3 - (-1)^3\right] = \frac{2}{3}$$

$$\boxed{a_0 = \frac{2}{3}}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{1}\int_0^1 (x-1)^2 \cos(n\pi x)\,dx$$

$$= 2\left[(x-1)^2\left(\cancel{\frac{\sin(n\pi x)}{(n\pi)}}\right)^{\to 0} - 2(x-1)\left(-\frac{\cos(n\pi x)}{(n\pi)^2}\right) + 2\left(\cancel{-\frac{\sin(n\pi x)}{(n\pi)^3}}\right)^{\to 0}\right]_0^1$$

$$= 2 \times \frac{2}{n^2\pi^2}\left[(x-1)\cos(n\pi x)\right]_0^1$$

$$= \frac{4}{n^2\pi^2}\left((1-1)\cos(n\pi) - (-1)\cos 0\right)$$

$$= \frac{4}{n^2\pi^2}(0+1)$$

$$\therefore \boxed{a_n = \frac{4}{n^2\pi^2}}$$

The fourier half range cosine seris is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{2/3}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \cos(n\pi x)$$

$$= \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n\pi x) \quad \text{--- (1)}$$

is required series

Deduction: put $x=0$ in Eq (1)

$$f(0) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$1 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$1 - \frac{1}{3} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{2}{3} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \quad \text{--- (2)}$$

Again put $x=1$ in Eq (2)

$$f(1) = \frac{1}{3} + \frac{4}{\pi^2} \sum \frac{1}{n^2} \cos(n\pi)$$

$$0 = \frac{1}{3} + \frac{4}{\pi^2} \sum \frac{1}{n^2} (-1)^n$$

$$-\frac{1}{3} = \frac{4}{\pi^2} \left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

$$-\frac{\pi^2}{12} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \quad \text{--- (3)}$$

Adding (2) & (3).

we get

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \cdots$$

$$\frac{3\pi^2}{12} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\Rightarrow \boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}$$

2) Find the cosine half range series for

$f(x) = x(l-x) \quad ; \quad 0 \le x \le l.$

Solun:- Here $f(x)$ defined in $(0, l)$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l x(l-x)\,dx$$

$$= \frac{2}{l}\int_0^l (xl - x^2)\,dx$$

$$= \frac{2}{l}\left[l\frac{x^2}{2} - \frac{x^3}{3}\right]_0^l$$

$$= \frac{2}{l}\left(l\frac{l^2}{2} - \frac{l^3}{3}\right) = \frac{2}{l}\,l^3\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{2l^2 1}{6} \Rightarrow \boxed{a_0 = \frac{l^2}{3}}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l x(l-x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[(lx-x^2)\cancel{\left(\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right)}^{0} - (l-2x)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right) - 2\cancel{\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right)}^{0}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^2}{n^2\pi^2}\left\{(l-2x)\cos\left(\frac{n\pi x}{l}\right)\right\}_0^l\right]$$

$$= \frac{2l^2}{l\, n^2\pi^2}\left[(l-2l)\cos(n\pi) - l\cos 0\right]$$

$$= \frac{2l}{n^2\pi^2}\left(-l(-1)^n - l\right)$$

$$\boxed{a_n = \frac{-2l^2}{n^2\pi^2}\left[1+(-1)^n\right]}$$

The fourier half range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{l^2/3}{2} + \sum_{n=1}^{\infty} \frac{-2l^2}{n^2\pi^2}\left[1+(-1)^n\right]\cos\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{l^2}{6} - \frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1+(-1)^n\right]\cos\left(\frac{n\pi x}{l}\right)$$

3) If $f(x) = \begin{cases} x & \text{in } 0 < x < \pi/2 \\ \pi - x & \text{in } \pi/2 < x < \pi \end{cases}$, find the half range fourier sine series of $f(x)$.

Solu^n:-

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} f(x)\sin(nx)\,dx + \int_{\pi/2}^{\pi} f(x)\sin(nx)\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\,\sin(nx)\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin(nx)\,dx\right]$$

$$= \frac{2}{\pi}\left\{\left[x\left(\frac{-\cos(nx)}{n}\right) - \left(\frac{-\sin(nx)}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\left(\frac{-\cos(nx)}{n}\right) - (-1)\left(\frac{-\sin(nx)}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\left(x\cos nx\right)_0^{\pi/2} + \frac{1}{n^2}\left(\sin nx\right)_0^{\pi/2} - \frac{1}{n}\left((\pi - x)\cos(nx)\right)_{\pi/2}^{\pi} - \frac{1}{n^2}\left(\sin(nx)\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\left(\frac{\pi}{2}\cos\left(\frac{n\pi}{2}\right) - 0\right) + \frac{1}{n^2}\left(\sin\left(\frac{n\pi}{2}\right) - 0\right) - \frac{1}{n}\left(\cancel{(\pi - \pi)\cos n\pi}^{\,0} - \frac{\pi}{2}\cos\left(\frac{n\pi}{2}\right)\right) - \frac{1}{n^2}\left(0 - \sin\left(\frac{n\pi}{2}\right)\right)\right]$$

$$= \frac{2}{\pi}\left[\cancel{\frac{-\pi}{2n}\cos\left(\frac{n\pi}{2}\right)} + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right) + \cancel{\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)} + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$\boxed{b_n = \frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)}$$

The fourier half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$= \sum_{n=1}^{\infty} \frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)\sin(nx)$$

---

4) obtain the sine half range series of

$$f(x) = \begin{cases} \frac{1}{4} - x & \text{in } 0 < x < \frac{1}{2} \\ x - \frac{3}{4} & \text{in } \frac{1}{2} < x < 1 \end{cases}$$

Soln:- Here $f(x)$ is defined in $(0,1)$.
Comparing with $(0,l)$ $\Rightarrow \boxed{l = 1}$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{1}\left[\int_0^{1/2} f(x)\sin(n\pi x)dx + \int_{1/2}^{1} f(x)\sin(n\pi x)dx\right]$$

$$= 2\left[\int_0^{1/2}\left(\frac{1}{4}-x\right)\sin(n\pi x)dx + \int_{1/2}^{1}\left(x-\frac{3}{4}\right)\sin(n\pi x)dx\right]$$

$$= 2\left\{\left[\left(\frac{1}{4}-x\right)\left(-\frac{\cos(n\pi x)}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{(n\pi)^2}\right)\right]_0^{1/2}\right.$$

$$\left. + \left[\left(x-\frac{3}{4}\right)\left(-\frac{\cos(n\pi x)}{n\pi}\right) - 1\times\left(-\frac{\sin(n\pi x)}{(n\pi)^2}\right)\right]_{1/2}^{1}\right\}$$

$$= 2\left[-\frac{1}{n\pi}\left(\left(\frac{1}{4}-x\right)\cos n\pi\right]_0^{1/2} - \frac{1}{n^2\pi^2}\left\{\sin(n\pi x)\right\}_0^{1/2} - \frac{1}{n\pi}\left\{\left(x-\frac{3}{4}\right)\cos(n\pi x)\right\}_{1/2}^{1} + \frac{1}{n^2\pi^2}\left\{\sin(n\pi x)\right\}_{1/2}^{1} \right]$$

$$= 2\left[\left(-\frac{1}{n\pi}\right)\left\{-\frac{1}{4}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{4}\cos(0)\right\} - \frac{1}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{2}\right)-0\right) - \frac{1}{n\pi}\left\{\left(1-\frac{3}{4}\right)\cos(n\pi) - \left(\frac{1}{2}-\frac{3}{4}\right)\cos\left(\frac{n\pi}{2}\right)\right\} + \frac{1}{n^2\pi^2}\left\{\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right\}\right]$$

$$= 2\left[\frac{1}{4n\pi}\left(\cos\frac{n\pi}{2}+1\right) - \frac{1}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi}\left\{+\frac{1}{4}\cos(n\pi) + \frac{1}{4}\cos\left(\frac{n\pi}{2}\right)\right\} + \frac{1}{n^2\pi^2}\left(-\sin\left(\frac{n\pi}{2}\right)\right)\right]$$

$$= 2\left[\frac{1}{4n\pi}\left(\cos\left(\frac{n\pi}{2}\right)+1\right) - \frac{1}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{4n\pi}\left((-1)^n - \cos\left(\frac{n\pi}{2}\right)\right) - \frac{1}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= 2\left[\frac{1}{4n\pi}\left(\cancel{\cos\left(\frac{n\pi}{2}\right)} + 1 - (-1)^n - \cancel{\cos\left(\frac{n\pi}{2}\right)}\right) - \frac{2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= 2\left[\frac{1}{4n\pi}\left(1-(-1)^n\right) - \frac{2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$\boxed{b_n = \frac{1}{2n\pi}\left(1-(-1)^n\right) - \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)}$$

The fourier half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{2n\pi}\left(1-(-1)^n\right) - \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)\right] \cdot \sin(n\pi x)$$

Practical Harmonic Analysis

Harmonic Analysis is the process of finding the constant term & the first few cosine & sine terms numerically

The fourier series of $f(x)$ in

$[0,2\pi)$, or $(0,2\pi]$	$[0,2l)$ or $(0,2l]$
$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx$ $a_0=\frac{2}{N}\sum y$; $a_n=\frac{2}{N}\sum y \cos nx$; $b_n=\frac{2}{N}\sum y \sin nx$	$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ $a_0=\frac{2}{N}\sum y$; $a_n=\frac{2}{N}\sum y \cos\left(\frac{n\pi x}{l}\right)$ $b_n=\frac{2}{N}\sum y \sin\left(\frac{n\pi x}{l}\right)$

N = No of values given in table

$$\boxed{\theta=\frac{\pi x}{l}}$$

1) Determine the constant term & the first cosine & sine terms of the fourier series expansion of y from the following table.

(or)

obtain the fourier series neglecting terms higher than first harmonics.

x°	0	60°	120°	180°	240°	300°
y	7.9	7.2	3.6	0.5	0.9	6.8

Solun:-

Here the interval of x is 0° to 360°

ie $0 \le x < 2\pi$ $[0, 2\pi)$

The series containing the first harmonics is

$$y = f(x) = \frac{a_0}{2} + a_1 \cos nx + b_1 \sin nx \quad \text{—①}$$

x	y	cos x	y cos x	sin x	y sin x
0	7.9	1	7.9	0	0
60°	7.2	0.5	3.6	0.866	6.2352
120°	3.6	−0.5	−1.8	0.866	3.1176
180°	0.5	−1	−0.5	0	0
240°	0.9	−0.5	−0.45	−0.866	−0.7794
300°	6.8	0.5	3.4	−0.866	−5.8888
Totals	26.9		12.15		2.6846

N = 6

$$a_0 = \frac{2}{N}\sum y = \frac{2}{6}(26.9) = 8.9667$$

$$a_1 = \frac{2}{N}\sum y\cos x = \frac{2}{6}(12.15) = 4.05$$

$$\therefore b_1 = \frac{2}{N}\sum y\sin x = \frac{2}{6}(2.6846) = \frac{1}{3}(2.6846) = 0.8949$$

$\therefore$ Eq ① $\Rightarrow$

$$f(x) = \frac{a_0}{2} + a_1\cos x + b_1\sin x$$

$$= \frac{8.9667}{2} + 4.05\cos x + 0.8949\sin x$$

$$f(x) = 4.48335 + (4.05)\cos x + (0.8949)\sin x$$

2) Compute the constant term & first two harmonics of the fourier series of $f(x)$ given the following table

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Solun:-
Here $f(x) = y$ defined in the interval $0 \le x \le 2\pi$

$f(0) = f(2\pi) = 1.0$ so we omit the last value of $f(x)$.

degree x	y	$\cos x$	$\cos 2x$	$\sin x$	$\sin 2x$	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0	1.0	1	1	0	0	1	1	0	0
$\pi/3$	1.4	0.5	−0.5	0.866	0.866	0.7	−0.7	1.2124	1.2124
$2\pi/3$	1.9	−0.5	−0.5	0.866	−0.866	−0.95	−0.95	1.6454	−1.6454
π	1.7	−1	1	0	0	−1.7	1.7	0	0
$4\pi/3$	1.5	−0.5	−0.5	−0.866	0.866	−0.75	−0.75	−1.299	1.299
$5\pi/3$	1.2	+0.5	−0.5	−0.866	−0.866	0.6	−0.6	−1.0392	−1.0392
Totals	8.7					−1.1	−0.3	0.5196	−0.1732

Here $N = 6$

$$a_0 = \frac{2}{N}\sum y = \frac{2}{6}(8.7) = 2.9$$

$$a_1 = \frac{2}{N}\sum y\cos x = \frac{2}{6}(-1.1) = -0.367$$

$$a_2 = \frac{2}{N}\sum y\cos 2x = \frac{2}{6}(-0.3) = -0.1$$

4 $b_1 = \frac{2}{N}\sum y \sin x = \frac{2}{6}(0.5196) = 0.1732$

$b_2 = \frac{2}{N}\sum y \sin 2x = \frac{2}{6}(-0.1732) = -0.0577$

The first two harmonics are

$$f(x) = \frac{a_0}{2} + \left(a_1 \cos x + b_1 \sin x\right) + \left(a_2 \cos 2x + b_2 \sin 2x\right)$$

$$\cancel{= \frac{2.9}{2} + \left(+0.367(\cos x) + \quad \sin x\right) +}$$

$$= \frac{2.9}{2} + \left[-0.367(\cos x) + 0.1732(\sin x)\right] + \left[-0.1 \cos 2x + (-0.0577)\sin 2x\right]$$

$$= 1.45 + \left[-0.367 \cos x + 0.1732 \sin x\right] + \left[(-0.1)\cos 2x - 0.0577 \sin 2x\right]$$

$\therefore$ Constant term $\left(\frac{a_0}{2}\right) = 1.45$

First harmonics $= \left[-0.367 \cos x + 0.1732 \sin x\right]$

Second harmonics $= \left[(-0.1)\cos 2x - 0.0577 \sin 2x\right]$

3) obtain the constant term & the co-efficients of the first cosine & sine terms in the fourier expansion of y from the table

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Solun: Here $N=6$

and the interval should be $0 \le x < 6$

length of the interval is $6-0=6$.

Comparing with $2l$

$\Rightarrow 2l=6 \Rightarrow \boxed{l=3}$

the fourier series of period $2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

since $l=3$, the series containing the first harmonics is

$$f(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{3}\right) + b_1 \sin\left(\frac{\pi x}{3}\right)$$

put $\frac{\pi x}{3} = \theta$

$$\therefore f(x) = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta \quad \text{—— (1)}$$

x	$\theta = \frac{\pi x}{3}$	y	$\cos\theta$	$y\cos\theta$	$\sin\theta$	$y\sin\theta$
0	0	9	1	9	0	0
1	60°	18	0.5	9	0.866	15.588
2	120°	24	-0.5	-12	0.866	20.784
3	180°	28	-1	-28	0	0
4	240°	26	-0.5	-13	-0.866	-22.516
5	300°	20	0.5	10	-0.866	-17.32
Total		125		-25		-3.464

Here $N=6$

$$a_0 = \frac{2}{N}\sum y = \frac{2}{6}(125) = 41.67$$

$$a_1 = \frac{2}{N}\sum y\cos\theta = \frac{2}{6}(-25) = -8.333$$

$$a_2 = \frac{2}{N}\sum y\sin\theta = \frac{2}{6}(-3.464) = -1.155$$

$\therefore$ Eq ① $\Rightarrow$

$$f(x) = \frac{41.67}{2} + \left[(-8.333)\cos x + (-1.155)\sin x\right]$$

$$f(x) = 20.835 - 8.333(\cos x) - 1.155(\sin x)$$

$\therefore$ Constant term $= \frac{a_0}{2} = 20.835$

Co-efficient of the first cosine term $= a_1 = -8.333$

Co-efficient of the first sine term $= b_1 = -1.155$

4) The following table gives the variations of periodic current over a period.

t(sec)	0	T/6	T/3	T/2	2T/3	5T/6	T
A(amps)	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Show that there is a direct current part of 0.75 amp & obtain the amplitude of the first harmonics.

Solun:- we observe that

$$A(0) = A(T) = 1.98.$$

Hence neglect the last value.

convert $A = f(t)$ to the period 2π by putting

$$\theta = \frac{\pi t}{l} \qquad \text{here } \boxed{l = \frac{T}{2}}$$

$$= \frac{\pi t}{T/2} = \frac{2\pi t}{T}$$

$$\text{ie } \boxed{\theta = \frac{2\pi t}{T}}$$

The fourier series upto first harmonics is

$$A = \frac{a_0}{2} + (a_1 \cos\theta + b_1 \sin\theta) \quad \text{--- (1)}.$$

t	$\theta = \frac{2\pi T}{t}$	A	$\cos\theta$	$\sin\theta$	$A\cos\theta$	$A\sin\theta$
0	0°	1.98	1	0	1.98	0
$T/6$	60°	1.30	0.5	0.866	0.65	1.258
$T/3$	120°	1.05	-0.5	0.866	-0.525	0.9093
$T/2$	180°	1.30	-1	0	-1.30	0
$2T/3$	240°	-0.88	-0.5	-0.866	0.44	0.7621
$5T/6$	300°	-0.25	0.5	-0.866	-0.125	0.2165
Total		4.5			1.12	3.0137

Here $N=6$

$$a_0 = \frac{2}{N}\sum A = \frac{2}{6}(4.5) = 1.5$$

$$a_1 = \frac{2}{N}\sum A\cos\theta = \frac{2}{6}(1.12) = 0.3733$$

$$a_2 = \frac{2}{N}\sum A\sin\theta = \frac{2}{6}(3.0137) = 1.0046.$$

$\therefore$ Eq ① $\Rightarrow$ $A = \frac{1.5}{2} + \left[(0.3733)\cos x + (1.0046)\sin x\right]$

$= 0.75 + \left[(0.3733)\cos x + (1.0046)\sin x\right]$

The Direct current part of the variable is the constant term $\left(\frac{a_0}{2}\right) = 0.75$

Amplitude of the first harmonics $= \sqrt{a_1^2 + b_1^2} = \sqrt{(0.3733)^2 + (1.0046)^2}$

$= 1.0717$