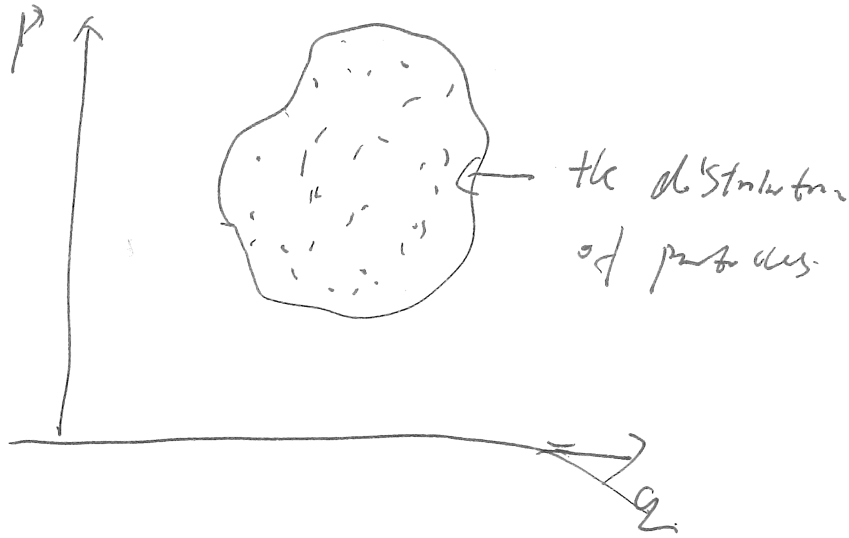
~~P~~SPUM 101 Lecture 8.

- Hamiltonian mechanics.

$$\mathcal{H}(p, q, t) \quad \text{or} \quad \mathcal{H}(p, q).$$

The EoM. (canonical equations of motion)

$$\begin{cases} \dfrac{\partial H}{\partial p} = \dot{q} \\[2mm] -\dfrac{\partial H}{\partial q} = \dot{p} \\[2mm] \dfrac{dH}{dt} = \dfrac{\partial H}{\partial t} \end{cases}$$



- ~~Possion~~ phase space.

* For Lagrangian mechanics, we seperate p_i and q_i to describe the ~~motion~~ dynamics of system. we call the space "configuration space" ("n" variable)

* For Hamiltonian mechanics ~~we~~ p_i and q_i both are consumed which is called q-p space. or "phase space" in a single space (2n. variable)

- The energy theorem of Hamilton's equations.

if $\mathcal{H}(p,q)$. $\Rightarrow \dfrac{d\mathcal{H}}{dt} = 0 \Rightarrow \mathcal{H} = E$ ← total energy

That's associated with a concept called phase fluid.

see fig 1.

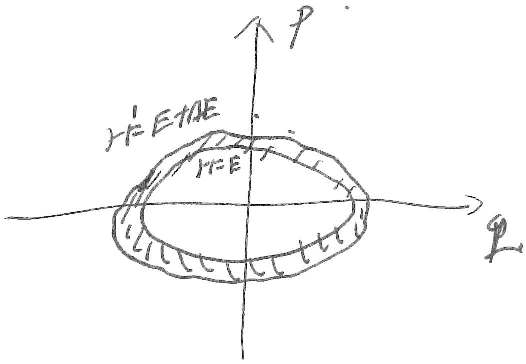


fig 2.

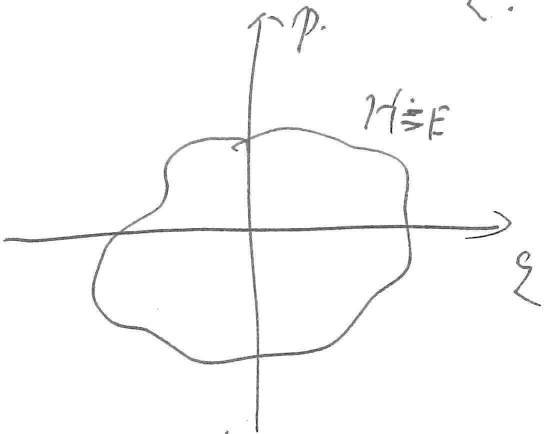


fig 1.

* propositions:

i) The motion of the phase fluid is steady. ~~which means particle can only move along the constant~~ constant. $H=E$.

ii). Each particle remains permanently on the energy surface $H=E$.

~~* micro-canonical ensemble.~~

— poisson bracket

For a given function $F(p, q)$

$$\frac{dF}{dt} = \sum_i \left(\frac{\partial F}{\partial p_i} \dot{p}_i + \frac{\partial F}{\partial q_i} \dot{q}_i \right) = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} \right)$$

$$\equiv \{F, H\}.$$

…e "$\{\}$" is called poisson bracket.

propositions:

1) $\{A, B\} = -\{B, A\}$.

2) $\{A+B, C\} = \{A, C\} + \{B, C\}$.

3) $\{AB, C\} = A\{B, C\} + \{A, C\}B$.

4) $\{\lambda A, B\} = \lambda\{A, B\}$ when λ is a constant.

5) $\{p_i, p_j\} = \{q_i, q_j\} = 0$

6) $\{q_i, p_j\} = \delta_{ij}$

Notice that F is conserved $\Leftrightarrow \{F, H\} = 0$

EX. Harmonic oscillator.

$$H = \frac{p^2}{2m} + \frac{k}{2}q^2$$

$$\begin{cases} \dot{p} = \{p, H\} = \{p, \frac{p^2}{2m} + \frac{k}{2}q^2\} = \frac{k}{2}\{p, q^2\} \\ \dot{q} = \{q, H\} = \{q, \frac{p^2}{2m}\} = \frac{1}{2m}\{q, p^2\} \end{cases}$$

$$\dot{p} = \frac{k}{2}\{p, q^2\} = \frac{k}{2}(\{p, q\}q + q\{p, q\}) = kq\{p, q\} = -kq.$$

$$\dot{q} = \frac{1}{2m}2\{q, p\}p = \frac{1}{m}p$$

$\Rightarrow$ EOM. $\begin{cases} \dot{p} = -kq \\ \dot{q} = \frac{1}{m}p \end{cases}$

— Liouville's theorem.

α. Incompressible fluid

particles: initial: $x_0, y_0, z_0, t=0$

final: x, y, z, t

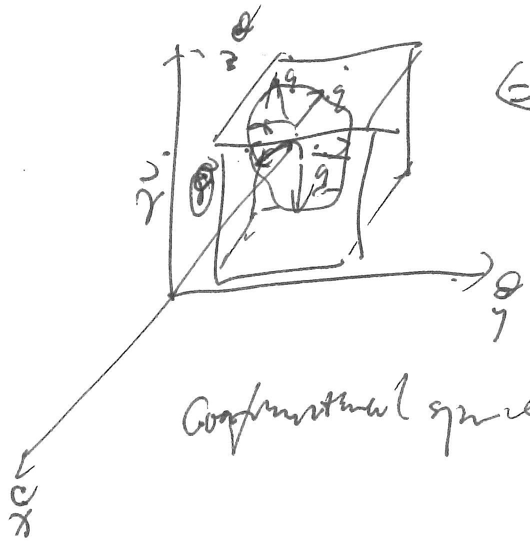
$$\begin{cases} x = f(x_0, y_0, z_0, t) \\ y = g(x_0, y_0, z_0, t) \\ z = h(x_0, y_0, z_0, t) \end{cases}$$

The changing velocity is given by

$$\begin{cases} \dot{x} = \frac{df}{dt} \\ \dot{y} = \frac{dg}{dt} \\ \dot{z} = \frac{dh}{dt} \end{cases}$$

The incompressible is shown as

$$\boxed{\nabla \cdot (\dot{x}, \dot{y}, \dot{z}) = 0}$$



$$\Leftrightarrow \int \nabla \cdot (\dot{x}, \dot{y}, \dot{z}) \, d^3\dot{r} = 0$$

$$\Leftrightarrow \int (\dot{x}, \dot{y}, \dot{z}) \cdot d\vec{a} = 0$$

Configurational space.

4.

2n - dimensional eqution of fluid.

In a 2n-dimensional q-p space, the Hamiltonian's equation of a general fluid is

if it is incompressible

$$\nabla\cdot\vec{V} = \sum_{i=1}^{n}\left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}\right) = 0.$$

velocity field.

$$\vec{V} \equiv V\cancel{(\dot{q}_i, \dot{p}_i)}.$$
$$(\dot{q}_1, \dot{q}_2, \dot{q}_3 \cdots \dot{q}_n, \dot{p}_1, \dot{p}_2, \dot{p}_3 \cdots \dot{p}_n)$$

* Liouville equation. ~~(theorem)~~

the distribution in phase space of particles is described by a density "ρ" ← ~~dist~~

$$\rho = \rho(p, q, t)$$

The Liouville tells us that the distribution function of particles respected to time changes is zero. ~~(at equilibrium)~~. ~~That~~ that is known as Liouville's theorem. In another words, L-theorem tells us the area of phase space doesn't change as time propagates.

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_i\left(\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i\right) = 0.$$

or $\boxed{\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \{\rho . H\} = 0.}$

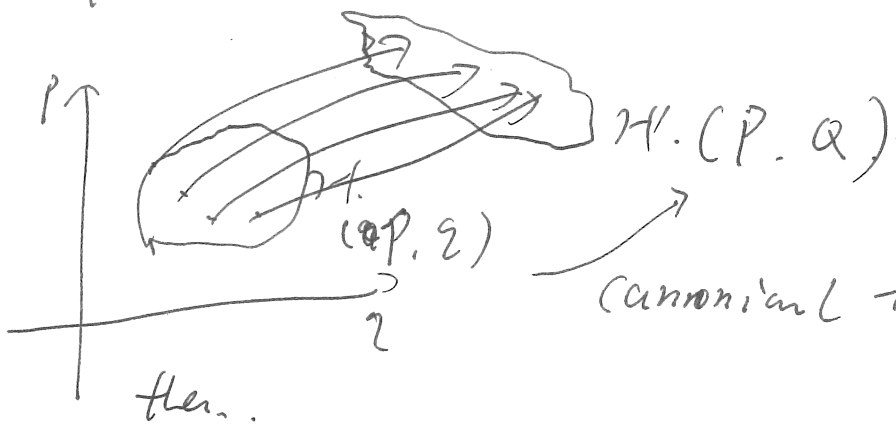
$\Downarrow$

$$\boxed{\frac{\partial \rho}{\partial t} = \{H . \rho\}.}$$

Especially, when this system is under equilibrium

$\Rightarrow \frac{\partial \rho}{\partial t} = 0 \Rightarrow \{\rho . H\} = 0$

$\Rightarrow \rho$ is conserved.



then..

$$d\Omega \equiv dp_1 dp_2 \cdots dp_n dq_1 dq_2 \cdots dq_n.$$

$\downarrow \qquad \downarrow$

$$d\Omega' \equiv J dP_1 dP_2 \cdots dP_n dQ_1 dQ_2 \cdots dQ_n.$$

J is called "Jacobin" ~~ma~~

Liouville's theorem tells us $J = 1$.

or say $\cancel{PQ}$ $\boxed{pq = QP.}$

Applications

consider a function $O(p,q)$ the exp value is given by.

$$\langle O(p,q)\rangle_t = \int d\Omega\, \rho(p,q,t)\, O(p,q)$$

the time derivative is given

$$\frac{d\langle O(p,q)\rangle_t}{dt} = \int d\Omega \frac{\partial\rho}{\partial t} O(p,q).$$

$$= \int d\Omega \{H,\rho\}\, O(p,q).$$

$$= \int d\Omega \sum\left(\frac{\partial H}{\partial q}\frac{\partial \rho}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial \rho}{\partial q}\right) O(p,q)$$

$$= \sum\left(\frac{\partial H}{\partial q} - \frac{\partial H}{\partial p}\right)\rho \cdot O(p,q)\Big|_{\text{surface}} + \int d\Omega\, \rho \sum\left(\frac{\partial H}{\partial p}\frac{\partial O}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial O}{\partial p}\right)$$

$$= 0 + \int d\Omega\, \rho \{O,H\}.$$

$$\Rightarrow \frac{d\langle O(p,q)\rangle_t}{dt} = \int d\Omega\, \rho\{O,H\}. = \langle\{O,H\}\rangle.$$

— Microcanonical ensemble.

The ME describes a system which is isolated ?.

and under ~~the~~ equilibrium for C ME. using 2-time

$$\Rightarrow \begin{cases} \frac{d\rho}{dt} = 0 \\ \frac{\partial \rho}{\partial t} = 0 \end{cases} \Rightarrow \{\rho, H\} = 0$$

some energy range in phase space

$$\Rightarrow \rho = \begin{cases} \text{Constant} & E \leq H \leq E + \Delta E \\ 0 & H < E \text{ or } H > E + \Delta E \end{cases}$$

the normalization condition is

$$\lim_{\Delta E \to 0} \int_V \rho \, d\Omega = 1.$$

More. pls. ~~seed~~ see (热力学与统计物理学, 2nd edition, 林宗涵).
PKU press.