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A derivation of cable equation

As we discussed in the first lecture, neurons have long and branching processes to integrate synaptic inputs from thousands of other neurons. The total length of a dendritic arbor is $\sim$ 4 mm. In a single compartment model, a neuron is assumed to be electrically compact: the spatial variation of membrane potential around a neuron is ignored. However, this assumption must be relaxed when we consider electrical signals travel down the long, narrow and cable-like structure of dendritic and axonal branches. Because neuronal processes are very narrow ($\sim 1\mu$m), the variation in the radial direction are negligible compared to the variation in the longitudinal direction. Therefore, the membrane potential along a neuronal process can be expressed as a function of a single longitudinal coordinate x and time, $V(x,t)$ and our goal is to solve for this potential.

Let us consider a cable segment with length Δx and radius a. The longitudinal resistance of this segment is given by

$$R_L = \frac{\rho_L \Delta x}{\pi a^2},$$

and the voltage difference between x and $x + \Delta x$ should follow Ohm's law:

$$V(x + \Delta x) - V(x) = I_L R_L$$

. By taking the limit $\Delta x \to 0$, and divide the equation on both side with Δx, we have

$$I_L = -\frac{\partial V}{\partial x}\frac{\pi a^2}{\rho_L}. \tag{1}$$

By convention, we let the right flow current to be positive.

In addition to current flowing in the longitudinal direction of the cable segment, we must also consider current flowing across the membrane, as we have discussed more extensively in previous lectures. The membrane current has three components: the first component is used to charge the membrane capacitance; the second component arises from the opening of voltage-gated ion channels on the membrane, or the active membrane current I_a; the third component is

the external current applied by an electrode or the synaptic inputs, which are lumped together by I_e. Converting words into equation, we have

$$I_m = I_a + I_e, \tag{2}$$

where c_m is the specific membrane conductance per unit area and again by convention, we consider current flowing inside the membrane to be negative. Now the total current that are injected into one segment should evoke a change of membrane potential of that segment, and the rate of voltage change depends on the total capacitance of that segment. So we have

$$c_m 2\pi a \Delta x \frac{\partial V}{\partial t} = I_L|_x - I_L|_{x+\Delta x} - I_m$$

Putting things together, we have

$$\frac{\partial V}{\partial x}\frac{\pi a^2}{\rho_L}|_{x+\Delta x} - \frac{\partial V}{\partial x}\frac{\pi a^2}{\rho_L}|_x = c_m 2\pi a \Delta x \frac{\partial V}{\partial t} + I_a + I_e.$$

Now we denote i_a as the active membrane conductance per unit area, and i_e is the external current per unit area. By dividing both sides of the equation with $2\pi a \Delta x$ and taking the limit $\Delta x \to 0$, we have

$$c_m \frac{\partial V}{\partial t} = \frac{a}{2\rho_L}\frac{\partial^2 V}{\partial x^2} - i_a(x,t) - i_e(x,t). \tag{3}$$

For simplicity, we ignore the spatial variation of ρ_L and a pulling these parameters outside the spatial derivatives.

As we discussed in previous lectures, in general the active membrane current is voltage dependent, $i_a = \sum_i g_i(V)(V - E_i)$. We shall discuss analytical solution of Equation **??** based on two drastic assumptions. First, a linear approximation of the active membrane current. Second, we do not consider synaptic currents in i_e, and the external current is solely applied by electrodes. In some cases, electrode current can mimic the effect of synaptic transmission.

In a linear approximation of active membrane current, we have

$$i_a = g_m(V - E_L),$$

where E_L is the resting potential of the neuron, and $g_m = 1/r_m$ is a constant membrane conductance per unit area, and r_m is specific membrane resistance. Now by introducing a new variable $v = V - E_L$, the cable equation can be simplified as

$$c_m r_m \frac{\partial v}{\partial t} = \frac{a r_m}{2\rho_L}\frac{\partial^2 v}{\partial x^2} - v - i_e(x,t) r_m.$$

Note that r_m has the dimension $\Omega{\cdot}\text{mm}^2$, and ρ_L has the dimension $\Omega{\cdot}\text{mm}$, and thus we could define a critical spatial scale λ, called electrotonic length,

$$\lambda = \sqrt{\frac{a r_m}{2\rho_L}} \tag{4}$$

Moreover, recall the membrane time constant

$$\tau_m = c_m r_m. \tag{5}$$

We could rewrite the cable equation as

$$\tau_m \frac{\partial v}{\partial t} = \lambda^2 \frac{\partial^2 v}{\partial x^2} - v - i_e(x,t) r_m. \tag{6}$$

Solution to cable equation

We shall next solve Equation **??** for infinite long cable. For linear systems, we shall first use point current injection, which corresponds to solve the Green's function of a linear system. To begin with, let us consider inject a constant current at point $x = 0$. As the current do not change over time, we shall consider the steady-state solution of membrane potential that is independent of time. In this case, we have

$$\lambda^2 \frac{d^2 v}{dx^2} = v + i_e r_m$$

where

$$i_e = -\frac{I_e}{2\pi a}\delta(x)$$
$$v|_{x=\infty} = 0$$
$$v|_{x=-\infty} = 0$$

To solve this equation , note that for $x \neq 0$, we simply have

$$v = B \exp(-|x|/\lambda).$$

Now by integrating left and write side of the equation from $-\epsilon$ to ϵ, we have

$$\lambda^2 \frac{dv}{dx}|_{\epsilon} - \lambda^2 \frac{dv}{dx}|_{-\epsilon} = 2v(\epsilon)\epsilon - \frac{I_e r_m}{2\pi a}.$$

When $\epsilon \to 0$, we found

$$B = \frac{I_e r_m}{4\pi a \lambda}$$

To summarize, for time-independent steady state solution, we have

$$v = \frac{I_e r_m}{4\pi a \lambda} \exp(-|x|/\lambda) \tag{7}$$

We now consider the membrane potential produced by an instantaneous pulse current injected at $x = 0$ and $t = 0$. We shall derive an analytical solution for

the evolution of membrane potential over time and space. More specifically, our pulse current has the following functional form

$$i_e = -\frac{I_e \tau_m}{2\pi a}\delta(x)\delta(t)$$

Again we have the following boundary conditions,

$$v|_{x=\infty} = 0; v|_{x=-\infty} = 0; v|_{t<0} = 0$$

Now we define $\tilde{v}$ as the membrane potential in the spatial frequency domain:

$$\tilde{v}(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} v(x,t)e^{-ikx}dx$$

Now let's do Fourier transform on both sides of Equation **??**, and we have

$$\tau_m \frac{\partial \tilde{v}}{\partial t} = -(\lambda^2 k^2 + 1)\tilde{v} + \frac{I_e \tau_m r_m}{\sqrt{(2\pi)^3 a}}\delta(t).$$

When $t > 0$, the solution of the above equation is given by

$$\tilde{v} = A \exp\left(-\frac{\lambda^2 k^2 + 1}{\tau_m}t\right).$$

Now for $t \to 0$, we shall use the same trick as we used before by integrating both sides of the equation from $-\epsilon$ to ϵ, where $\epsilon \to 0$, and we obtain

$$\tau_m[\tilde{v}(\epsilon) - \tilde{v}(-\epsilon)] = \tilde{v}\epsilon + \frac{I_e \tau_m r_m}{\sqrt{(2\pi)^3 a}}.$$

Note that $\tilde{v}(-\epsilon) = 0$, and by taking $\epsilon \to 0$, we found $A = \frac{I_e r_m}{\sqrt{(2\pi)^3 a}}$. As a result, the membrane potential in the spatial frequency domain is given by

$$\tilde{v}(k,t) = \frac{I_e r_m}{\sqrt{(2\pi)^3 a}} \exp\left(-\frac{\lambda^2 k^2 + 1}{\tau_m}t\right)$$

Next, we shall do the inverse Fourier transform via

$$v(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{v}(k,t)e^{ikx}dx$$

Note that we shall use an important formula that the Fourier transform of Gaussian distribution is still a gaussian. That is

$$\int_{-\infty}^{\infty} e^{-ak^2}e^{ikx}dk = \sqrt{\frac{\pi}{a}}e^{-\frac{x^2}{4a}}$$

Now using $a = \lambda^2 t/\tau_m$, the solution for membrane potential is given by

$$v(x,t) = \frac{I_e r_m}{2\pi a\sqrt{4\pi\lambda^2 t/\tau_m}} \exp\left(-\frac{x^2 \tau_m}{4\lambda^2 t}\right)\exp(-t/\tau_m). \tag{8}$$

Now let us discuss the physical meaning of Equation **??**. Let us define

$$\sigma^2(t) = 2\lambda^2 t/\tau_m,$$

and the solution can be rewritten as

$$v(x,t) = \frac{I_e r_m}{2\pi a}\frac{1}{\sqrt{2\pi}\sigma(t)} e^{-\frac{x^2}{2\sigma^2(t)}} e^{-t/\tau_m}. \tag{9}$$

At given time t, the spatial profile of membrane potential $v(x)$ is a gaussian distribution with variance σ^2. Note that when $t \to 0$, $\sigma \to 0$, the gaussian distribution reduces to $\delta(x)$. The spatial variance $\sigma^2(t)$ increases linearly with time, and the additional factor $\exp(-t/\tau_m)$ reduces the amplitude of the membrane potential.

At given position $x \neq 0$, the membrane potential should have a rising phase and a decay phase. This allows us to calculate $t_{max}(x)$, at which the membrane potential v is maximized. By taking

$$\frac{\partial v}{\partial t}|_{t=t_{max}} = 0,$$

we found that

$$t_{max}(x) = \frac{\tau_m}{4}(\sqrt{1 + 4x^2/\lambda^2} - 1).$$

In the limit $x \gg \lambda$, $t_{max} = \tau_m x/(2\lambda)$. We could also define an effective wave propagation speed $s_w = x/t_{max} = 2\lambda/\tau_m$. Recall that the electrotonic length is given by $\lambda = \sqrt{\frac{a r_m}{2\rho_L}}$, and $\tau_m = r_m c_m$, we found that

$$s_w = \sqrt{\frac{4a}{c_m^2 r_m \rho_L}} = \sqrt{\frac{4a}{c_m \tau_m \rho_L}} \tag{10}$$

Action potential propagation along an axon

Brain size varies dramatically across animal kingdom. To facilitate information transmission across different brain areas, it is ideal to increase the speed of action potential propagation. There are several ways to achieve this goal. According to Equation **??**, $s_w \sim a^{1/2}$. Therefore, increasing the axonal diameter would be one solution, and this is precisely how the squid giant axon is designed. Another possibility is to reduce the membrane capacitance. Indeed, this is how axon myelination works. Many vertebrate axons are wrapped by a myelin sheath, which are cytoplasmic extensions of glia. The glial cells wrap around the myelinated axons many times, to form compact myelin consisting of closely packed glial plasma membranes. From an electrical circuit point of view, compact myelin is equivalent to having many resistors and capacitances

connected in series. Let's denote n as the number of layers of glial cell membranes, the total effective resistance would be nr_m, while the total capacitance would be c_m/n. As a result, τ_m remains unchanged, c_m is significantly reduced, and s_w will dramatically increase.

The above argument is qualitatively reasonable. A quantitative treatment must also take into account several other factors. For example, the series capacitors do not have the same value: the capacitance is diameter-dependent. For myelin sheath with radius a, thickness Δa, The capacitance per unit length is given by $c_m 2\pi a d_m/\Delta a$. Here d_m is the thickness of glial cell membrane, and $d_m < \Delta a \ll a$. For compact myelin sheath with inner radius a_1 and outer radius a_2. The total capacitance per unit length is given by

$$\frac{1}{C_m} = \frac{1}{c_m 2\pi d_m}\int_{a_1}^{a_2}\frac{da}{a} = \frac{\ln(a_2/a_1)}{c_m 2\pi d_m}$$

$$C_m = 2\pi d_m c_m/\ln(a_2/a_1).$$

Thus the total membrane capacitance only has a weak logarithmic dependence on the ratio of inner and outer diameter of a myelinated axon. The same argument should also apply to the total membrane resistance per unit length:

$$R_m = \frac{r_m}{2\pi d_m}\int_{a_1}^{a_2}\frac{da}{a} = \frac{r_m}{2\pi d_m}\ln(a_2/a_1)$$

Note that $C_m R_m = c_m r_m = \tau_m$ remains unchanged, confirming our intuitive argument.

Now the cable equation should be modified to

$$C_m\frac{\partial v}{\partial t} = \frac{\pi a_1^2}{\rho_L}\frac{\partial^2 v}{\partial x^2} - \frac{v}{R_m} - I_e.$$

Multiplying both sides of the equation by R_m and Plugging in the expression for R_m, we obtain

$$\begin{aligned}\tau_m\frac{\partial v}{\partial t} &= \lambda_m^2\frac{\partial^2 v}{\partial x^2} - v - I_e R_m.\\ \lambda_m &= \sqrt{\frac{r_m a_1^2\ln(a_2/a_1)}{2d_m\rho_L}}.\end{aligned} \tag{11}$$

Following our previous argument, to maximize the wave propagation speed reduces to maximize the electrotonic length λ_m. Note it now has a different expression for a myelinated axon. By setting $\partial\lambda_m/\partial a_1 = 0$, we found that the optimal outer to inner diameter ratio is given by $a_1/a_2 = \exp(-1/2) \approx 0.6$, and the corresponding electrotonic length is given by

$$\lambda_m = \sqrt{\frac{r_m a_1^2}{2d_m\rho_L e^{1/2}}}. \tag{12}$$

As we could see, the wave propagation speed $s_w \sim \lambda_m \sim a_1$: it scales linearly with axonal diameter. Indeed, action potentials move at roughly the same speed (25 m/s) in a myelinated frog axon and an unmyelinated squid giant axon, but the frog axon has a roughly 30-fold smaller diameter and 1000-fold smaller cross-sectional area.