Problem Set 5

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Ring network

Consider the ring network model we discussed in the class

$$\tau \frac{du_i}{dt} = -u_i + F\left(\sum_{j=1..N} w_{ij} u_j + I_i^0\right), \tag{1}$$

where $F(x) = [x]_+$ is a rectified linear function and the weights between the neurons are determined by their angular difference and thus are translational invariant.

$$w_{ij} = \frac{1}{N} J(\theta_i - \theta_j), \tag{2}$$

where $\theta_i = -\pi + \frac{2\pi i}{N}$ and J is a 2π periodic function, and I_i^0 is also a periodic 2π function. In the class, we have considered a simple form of the connectivity matrix, namely

$$J(\theta) = J_0 + J_1 \cos\theta \tag{3}$$

$$I_i^0(\theta) = I_0 + I_1 \cos(\theta_i - \theta_0) \tag{4}$$

Consider the same setting as in the class with $J_1 > 2$. Add to the connectivity matrix a term $\frac{J_1}{N}\gamma \sin(\theta_i - \theta_j)$, where $|\gamma| \ll 1$. Show that there is a solution with the form

$$u(\theta, t) = f(\theta - \omega t), \tag{5}$$

where $f(\theta)$ is the steady state activity profile calculated in the class for $\gamma = 0$ and the angular velocity satisfies

$$\omega = \frac{\gamma}{\tau} \tag{6}$$

Guide:
A. Assume a traveling profile of the form of 5 (for now just use an arbitrary profile $f(\theta)$) and insert it into the two sides of 1. Express the RHS in terms of the order parameters as was done in the class. Expand the RHS in powers of γ and keep only terms up to linear in γ.
B. Show that if $f(\theta)$ is the profile for $\gamma = 0$ the dynamic equations 1 are satisfied.

Poisson Spike-Train Statistics

1. Given the Homogeneous Poisson process (mean firing rate is independent of time),

$$P(n) = \frac{(rT)^n}{n!} \exp(-rT). \tag{7}$$

Calculate the mean $\langle n \rangle$ and variance $Var(n)$ of the spike count. Compute the Fano factor $Var(n)/\langle n \rangle$. Calculate the kurtosis of spike count defined as $k = \langle n^4 \rangle - 3\langle n^2 \rangle^2$ in the time interval T.

2. When the firing rate depends on time, we could also extend the homogeneous Poisson process to inhomogeneous Poisson process. When n spikes occurs in an interval T with $0 < t_1 < t_2 < ... < t_n < T$, Prove that the joint probability density is given by

$$p(t_1, t_2, ..., t_n) = \exp\left(-\int_0^T r(t)dt\right) \prod_{i=1}^{n} r(t_i) \tag{8}$$

Then, calculate the probability of seeing n spikes $P(n)$ in the interval T. Check whether it has a similar expression as the homogeneous Poisson process and calculate the Fano Factor.

3. Generate a Poisson spike train with a time-dependent fire rate $r(t) = r_0[1 + cos(2\pi t/\tau)]$ where $r_0 = 100$ Hz and $\tau = 300$ ms. Generate a spike train for 20 s and plot it.

4*. Let's assume that the firing rate of a neuron has the following functional form: $r(t) = r_0 + r_1 \sin(\omega t + \theta)$, where the phase θ is drawn uniformly between 0 and 2π for each trial. Calculate the Fano Factor for the spike count in the time interval T (as a function of T).

Note: Problems with * are optional. However, solving them will give you additional credits.

Entropy and Mutual Information

The Entropy of a variable X drawn from a distribution $p(X)$ is given by the following formula

$$H(X) = \int p(X) \ln p(X) \tag{9}$$

Use the Lagrange Multiplier method to evaluate the maximum entropy probability distribution $p(X)$ in the following cases:

(a) X is one dimensional continuous random variable, which takes only positive values and its mean is fixed. Hint: In addition to the mean, you should also take into account the constraint imposed by the normalization of p.

(b) There is no constraint on the range of X but its variance is given.

(c) X is an N-dimensional continuous random variable with constraint on the total variance,

$$\sum_{i}^{N} \langle x_i^2 \rangle = N\sigma^2 \tag{10}$$

(d) Show that the entropy of the multivariate Gaussian $N(\mathbf{X}|\mu, \mathbf{\Sigma})$ is given by

$$H(\mathbf{X}) = \frac{1}{2} \ln |\mathbf{\Sigma}| + \frac{D}{2}(1 + \ln(2\pi)) \tag{11}$$

where D is the dimensionality of $\mathbf{X}$, $|\mathbf{\Sigma}|$ is the determinant of the covariance matrix $\mathbf{\Sigma}$.