Problem Set 4

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A toy model of neural integrator

An animal moves on a one-dimensional track. In order to navigate properly (e.g., to find its way back home) the brain must compute the 'on line' position of the animal given sensory information about its velocity. We will assume that the position is computed by means of a network composed of two neurons, whose input is the animals velocity. The network obeys the following equations:

$$\begin{aligned} \tau \frac{dx}{dt} &= -ax - 2y + \tau V(t); \\ \tau \frac{dy}{dt} &= -(3-a)x - y - \tau V(t) \end{aligned} \tag{1}$$

where the value of $x(t)$ represents the estimated position of the animal at time t.

(a) Study the dynamics of the network for constant velocity, i.e., $V(t) = V_0$ for all $t > 0$: First, find the fixed point of the network and the region of values of the parameter a for which this fixed point is stable. Next, write down the solution for $x(t)$ given that $(x(0), y(0)) = (0, 0)$ and explain what happens as time increases.

(b) Position Estimation. Assume that at $t = 0$, the animal starts out at the origin, and begins moving at a constant velocity $V(t) = 0.1$ m/s. Use $\tau = 100$ ms. If the system acted as a perfect integrator of the velocity, the expected position at $t = 10$ s would be 1 m. Is there a parameter choice (for parameter a) for which this system acts as a perfect integrator? (If so, take any appropriate limits to prove it.) For what range of parameters does the system act as a leaky integrator, i.e., the readout is within 1 cm error of the estimated position from a perfect integrator after 10 seconds? What happens to $x(t)$ when a is outside this range?

fixed point, phase plane, etc

Networks with symmetric connections have only fixed point attractors. This is not generally true for networks with asymmetric connections. A simple case of a nonlinear network with limit cycle attractor is the following network of two neurons:

$$\begin{aligned} \tau \frac{dx}{dt} &= -x + \tanh(Jx - Ky) \\ \tau \frac{dy}{dt} &= -y + Gx \end{aligned} \tag{2}$$

where J, K, and $G > 0$ are constants and τ is the time constant of the neurons. Note that x is an excitatory neuron and y is an inhibitory neuron. It is thus, a very generic model consisting of coupled activator and suppressor units. The system shows a variety of behaviors, including a stable limit cycle.

a. Find the fixed points of the system. Find the set of parameters for which each fixed point exists.

b. Consider the fixed point $(x, y) = (0, 0)$. Setting $\tau = 1$, sketch the phase diagram of the fixed point, plotting J against KG. In the $J - KG$ plane, draw and label the following four regions:

   i. The fixed point is stable and the system converges monotonically to the stable state under small perturbation.

   ii. The fixed point is stable and the system spirals back to the stable state under small perturbation.

   iii. The fixed point is unstable and the system diverges monotonically away under small perturbation.

   iv. The fixed point is unstable and the system spirals away under small perturbation.

c. Consider the case in which $(x, y) = (0, 0)$ is unstable and the other fixed points do not exist. Find a closed surface in the x-y plane along which all the flow points inwards. Qualitatively describe how the system evolves at long times in this case, given initial conditions inside the surface you found.

d. Draw a new phase diagram in the $J - KG$ plane for the entire system, classifying when the system has 1 stable fixed point, when it has 2 stable fixed points, and when it produces a limit cycle.

e. Numerically integrate this system of equations for J, K, and G parameters representative of each of the four cases in (b), one of which will also be the case discussed in (c). Show how the system evolves in each case starting from an initial condition near . Make sure you integrate long enough to see a steady state behavior emerge. (You should hand in graphs of the $x - y$ plane. For your own edification, plot $x(t)$ vs. t and $y(t)$ vs. t to make sure the system has reached a steady state. If you use Matlab, ode45 is a useful integration command).