Review of Linear Algebra

1. If a matrix, $\boldsymbol{A}$, has an inverse, $\boldsymbol{A}^{-1}$, this inverse is unique, and

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I} \quad \text{and} \quad \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}.$$

2. If $\boldsymbol{A}^{-1}$ exists, $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$.

3. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are non-singular matrices, then

$$(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}.$$

4. If $\boldsymbol{A}$ is non-singular and $k \neq 0$, then

$$(k\boldsymbol{A})^{-1} = \frac{1}{k}\boldsymbol{A}^{-1}.$$

5. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices such that $\boldsymbol{A}\boldsymbol{B}$ is defined (the two matrices are conformable), then

$$(\boldsymbol{A}\boldsymbol{B})' = \boldsymbol{B}'\boldsymbol{A}'.$$

6. $\boldsymbol{A}$ is called *symmetric* if $\boldsymbol{A}' = \boldsymbol{A}$.

   Note: $\boldsymbol{A}'\boldsymbol{A}$, $\boldsymbol{A}\boldsymbol{A}'$, and $\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$ are all symmetric matrices.

7. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n \times n$ square matrices, then

$$\mathbf{det}\left[\boldsymbol{A}\boldsymbol{B}\right] = \mathbf{det}\left[\boldsymbol{A}\right]\mathbf{det}\left[\boldsymbol{B}\right].$$

8. If $\boldsymbol{A}$ is $n \times n$, then $\mathbf{det}\left[\boldsymbol{A}\right] = 0$ iff if $\boldsymbol{A}$ is singular.

9. The *rank* of $\boldsymbol{A}$ is the greatest number of linearly independent columns (or rows) of $\boldsymbol{A}$.

   Note: Linear dependence implies that at least one column (or row) of $\boldsymbol{A}$ can be written as a linear combination of the other columns (or rows).

10. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are non-singular, then for any matrix $\boldsymbol{C}$,

$$\boldsymbol{C}, \quad \boldsymbol{A}\boldsymbol{C}, \quad \boldsymbol{C}\boldsymbol{B}, \quad \boldsymbol{A}\boldsymbol{C}\boldsymbol{B}$$

    all have the same rank (assuming that the multiplications are defined).

17. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are congruent matrices, then they have the same rank.

18. Let $\boldsymbol{C}$ be an $m \times n$ matrix with rank r. The ranks of $\boldsymbol{C}'\boldsymbol{C}$ and $\boldsymbol{C}\boldsymbol{C}'$ are also r.

19. Let $\boldsymbol{A}$ be an $n \times n$ matrix. There always exist n complex numbers, $\lambda_1, \lambda_2, \ldots, \lambda_n$ (called the *characteristic roots* or *eigenvalues*) that satisfy

$$\mathbf{det}\left[\boldsymbol{A} - \boldsymbol{\lambda I}\right] = 0.$$

If $\boldsymbol{A}$ is real symmetric, then all the λ's are real.

20. Let $\boldsymbol{A}$ be an $n \times n$ symmetric matrix. The rank of $\boldsymbol{A}$ is the number of non-zero eigenvalues.

21. Let $\boldsymbol{A}$ be an $n \times n$ matrix. $\boldsymbol{A}$ has at least one zero eigenvalue iff $\boldsymbol{A}$ is singular.

22. Let $\boldsymbol{A}$ be an $n \times n$ matrix. The determinant of $\boldsymbol{A}$ is the product of its eigenvalues.

$$\mathbf{det}\left[\boldsymbol{A}\right] = \prod_{i=1}^{n} \lambda_i$$

23. Let $\boldsymbol{A}$ be an $n \times n$ matrix, and let $\boldsymbol{C}$ be any $n \times n$ non-singular matrix. The following matrices all have the same set of eigenvalues:

$$\boldsymbol{A} \quad \boldsymbol{C}^{-1}\boldsymbol{A}\boldsymbol{C} \quad \boldsymbol{C}\boldsymbol{A}\boldsymbol{C}^{-1}.$$

24. Let $\boldsymbol{A}$ be an $n \times n$ real matrix. A necessary and sufficient condition that there exist a nonzero $\boldsymbol{y}$ that satisfies

$$\boldsymbol{A}\boldsymbol{y} = \lambda \boldsymbol{y}$$

is that λ be an eigenvalue of $\boldsymbol{A}$.

25. *Orthonormal Matrices* Let $\boldsymbol{P}$ be an $n \times n$ matrix. $\boldsymbol{P}$ is called orthonormal iff $\boldsymbol{P}^{-1} = \boldsymbol{P}'$. Thus, $\boldsymbol{P}$ is orthonormal iff $\boldsymbol{P}'\boldsymbol{P} = \boldsymbol{I}$.

    By extension, if $\boldsymbol{P}$ is an orthonormal matrix, then $\boldsymbol{P}\boldsymbol{P}' = \boldsymbol{I}$.

26. Let $\boldsymbol{A}$ be a $n \times n$ matrix, and let $\boldsymbol{P}$ be an $n \times n$ orthonormal matrix, then

$$\mathbf{det}\left[\boldsymbol{A}\right] = \mathbf{det}\left[\boldsymbol{P}'\boldsymbol{A}\boldsymbol{P}\right].$$

Note: If $\boldsymbol{A}$ is positive definite:

- The rank of $\boldsymbol{A}$ is n.
- All of the eigenvalues of $\boldsymbol{A}$ are greater than 0.
- Let $\boldsymbol{P}$ be a $n \times n$ non-singular matrix. $\boldsymbol{P}'\boldsymbol{A}\boldsymbol{P}$ is also positive definite.

34. *Non-Negative Definite Matrix.* A matrix is called non-negative definite if it is either positive definite or positive semidefinite.

35. Let $\boldsymbol{C}$ be an $m \times n$ matrix with rank r. $\boldsymbol{C}'\boldsymbol{C}$ and $\boldsymbol{C}\boldsymbol{C}'$ are both non-negative definite.

    $\boldsymbol{C}'\boldsymbol{C}$ or $\boldsymbol{C}\boldsymbol{C}'$ are positive definite iff they have full rank.

36. Let $\boldsymbol{A}$ be a $n \times n$ symmetric non-negative definite matrix. There exists some $n \times n$ matrix $\boldsymbol{B}$ such that $\boldsymbol{B}'\boldsymbol{B} = \boldsymbol{A}$.

37. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ symmetric matrices. If $\boldsymbol{A}$ is positive definite, then there exists a non-singular matrix $\boldsymbol{Q}$ such that

$$\boldsymbol{Q}'\boldsymbol{A}\boldsymbol{Q} = \boldsymbol{I} \quad \text{and}$$

$$\boldsymbol{Q}'\boldsymbol{B}\boldsymbol{Q} = \boldsymbol{D}$$

where $\boldsymbol{D}$ is a diagonal matrix whose diagonal elements are the roots of $\mathbf{det}\left[\boldsymbol{B} - \lambda\boldsymbol{A}\right]$.

38. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are both non-negative definite, then there exists a matrix $\boldsymbol{Q}$ such that both $\boldsymbol{Q}'\boldsymbol{A}\boldsymbol{Q}$ and $\boldsymbol{Q}'\boldsymbol{B}\boldsymbol{Q}$ are diagonal.

39. Let $\boldsymbol{A}$ be a $n \times n$ non-singular matrix partitioned into

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}$$

where both $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are square and non-singular.

$$\boldsymbol{A}^{-1} = \begin{bmatrix} \left[\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}\right]^{-1} & -\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12}\left[\boldsymbol{A}_{22} - \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12}\right]^{-1} \\ -\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}\left[\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}\right]^{-1} & \left[\boldsymbol{A}_{22} - \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12}\right]^{-1} \end{bmatrix}.$$

48. Let $\boldsymbol{A}$ be an $n \times n$ matrix, and let $\boldsymbol{P}$ be a non-singular $n \times n$ matrix.

$$\textbf{trace}\,[\boldsymbol{A}] = \textbf{trace}\left[\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\right].$$

If $\boldsymbol{P}$ is orthonormal, then

$$\textbf{trace}\,[\boldsymbol{A}] = \textbf{trace}\,[\boldsymbol{P}'\boldsymbol{A}\boldsymbol{P}].$$

49. Let $\boldsymbol{A}$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

$$\textbf{trace}\,[\boldsymbol{A}] = \sum_{i=1}^{n} \lambda_i$$

50. If $\boldsymbol{A}$ is an idempotent matrix, then

$$\textbf{trace}\,[\boldsymbol{A}] = \textbf{rank}\,[\boldsymbol{A}].$$

51. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ matrices, and let a and b be scalars.

$$\textbf{trace}\,[a\boldsymbol{A} + b\boldsymbol{B}] = a \cdot \textbf{trace}\,[\boldsymbol{A}] + b \cdot \textbf{trace}\,[\boldsymbol{B}].$$

52. Let $\boldsymbol{A}$ be an $n \times n$ matrix.

$$\textbf{trace}\,[\boldsymbol{A}'] = \textbf{trace}\,[\boldsymbol{A}]$$