Algebra Primer: Refresher math needed for College physics I & II

1 Introduction

Everyone comes into physics with differing levels of mathematical strengths and backgrounds. When tackling the introductory physics courses at University, most people have purged stores of mathematical knowledge from secondary education and need another look at algebra to redevelop some fundamentals. This document serves as bridge to help you dust off old algebra skills and perhaps gain a few more.

Here's all of the math you'll need for these introductory courses. If all the examples make sense and you can **explain** the concepts to another person, you are well prepared.

What this document doesn't teach:

1. Multiplying fractions
2. Multiplying equations on both sides of equal sign
3. adding fractions with the same or different denominators
4. foiling out expressions e.g. $(x+y)^2 = ?$
5. canceling variables

*****PLEASE READ*****

There is a mathematical survey you can take to see what skills you might still need to work on. I **highly** recommend taking this survey instead of reading the whole 21 page document to save you time. If you get stuck, each problem is related to a section in this document so you can quickly re-learn whatever you forgot and move on.

Contents

2 Mathematical Survey

2.1 Survey ($\sim$ 25min)

This survey is meant to quickly test your algebra skills and identify any gaps in knowledge you have. You will need all the math covered in this survey at some point in College Physics I or II. It should take about ~25min if you understand each problem and need to look up very little.

***The questions are numbered based off the section they're from i.e. Q 2.5 means the content is tested from section 2.5 in this document.

Q 2.1-2.3   Solve for x

$$3x - 2x - 7 = 1 - 5x$$

Q 2.4   Simplify

$$(x^3 x^7)^2$$

Q 2.5   Simplify

$$\frac{x^8}{x}$$

Q 2.6   True or False: Answer true if all the equal sings are true, answer false if even one is false.

$$\frac{16(x+1)}{3} \stackrel{?}{=} (x+1)\frac{16}{3} \stackrel{?}{=} 16\frac{(x+1)}{3} \stackrel{?}{=} \frac{16x+1}{3}$$

Q 2.7   Simplify

$$\frac{2 * \frac{6}{7}}{\frac{3}{4}}$$

Q 2.8   Simplify

$$\left(\frac{5}{4}\right)^3 * \left(\frac{125}{37}\right)^{-1}$$

Q 2.9   Simplify. Here Δy is our variable.

$$\left(\frac{2\Delta y}{(\Delta y)^2}\right)^3 * \left(\frac{1}{(\Delta y)^2}\right)^{-1}$$

Q 2.10   Simplify and solve for variable "t" in terms of variable "v_0".

$$9t^2 - 2.5v_0 = -v_0 + 4.5t^2$$

Q 2.11 Simplify and solve these pair of equations for variable "v" in terms of variables M_1 and M_2. G here is some constant number.

$$F = m\frac{v^2}{r} \tag{1}$$

$$F = \frac{G(mM_1 + mM_2)}{r} \tag{2}$$

Q 3.1 Simplify so all constant numbers are only single digit and cancel any needless variables.

$$\frac{93x + 62xy^3 - 279xz^{-2}}{31x^2}$$

Q 3.2 Solve the following equation for "x"

$$yx^2 + 6zx + 5yz = 0$$

Q 3.3 Solve two equations for x **without** using the substitution technique.

$$7x - 4y = 4$$

$$49x + 8y = 24$$

Q 3.4 What is $\frac{P_1}{P_2}$ =? if $m_2 = \frac{1}{6}m_1$ in the following equation (where g and A are constants):

$$P = m\frac{g}{A}$$

Q 3.5 Show that the following equal sign is true:

$$\frac{1}{2(z-1)} - \frac{1}{2(z+1)} \stackrel{?}{=} \frac{1}{(z-1)(z+1)}$$

Q 4.2 Solve for $|x|$

$$3\left|2x^2\right| = 4\left|x^3\right|$$

Q 5.1 True or False

$$\log_c(z) = k \text{ is the inverse expression to } c^k = z$$

Q 5.2-5.3 Solve for x (approximately)

$$e^x = 20.08554$$

Q 5.2-5.3 Fully expand the expression and simplify

$$\frac{\log(10xy)}{\log(x^3)}$$

Q 6.1 Convert 273° to radians

Q 6.2 True or False:

$$\sin\theta \stackrel{?}{=} \sin\left(\theta - \frac{9\pi}{2}\right)$$

Q 6.3 What is the range of possible values of:

$$? \leq 9\pi\sin(\theta) \leq ?$$

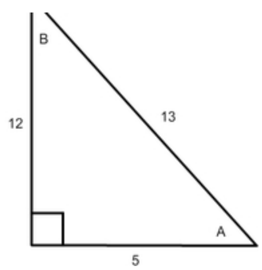
Q 6.3 True or False:

$$\sin(540°) \stackrel{?}{=} \cos(-270°)$$

Q 6.4 True or False:

$\tan(\theta - \phi)$ will advance every value of $\tan\theta$ to the right by ϕ

Q 6.6 Find angle "A" (in radians)



Q 7 Are either of these statements true or are they both false?

$$\frac{y}{x+z} \stackrel{?}{=} \frac{y}{x} + \frac{y}{z} \tag{1}$$

$$(x+a)^2 \stackrel{?}{=} x^2 + a^2 \tag{2}$$

Congratulations! You finished! Hopefully this jostled some old algebra machinery in your brain you forgot you had! For the solutions to the Mathematical Survey, visit the link below

`testtest`

3 Arithmetic operations on variables

For the following properties we use "a" and "b" to represent **positive** real numbers. The variable "x" (and sometimes "y") will be used in the properties definition. The examples problems can have any variable just like in physics.

3.1 Adding similar variables

Property:

$$ax + bx = (a + b)x \tag{1}$$

Examples:

$$19y + 5y = (19 + 5)y = 24y$$

$$3x + 2x = (3 + 2)x = 5x$$

3.2 Subtracting similar variables

Property:

$$(-a)x + bx = bx - ax = (b - a)x = (-a + b)x \tag{2}$$

Examples:

$$-5x + 3x = (-5 + 3)x = -2x$$

$$7z - 4z = (7 - 4)z = 3z$$

$$-9y - 18y = (-9 - 18)y = -27y$$

3.3 Multiplying numbers by variables

Property:

$$a * (bx) = b * (ax) = (a * b)x \tag{3}$$

(* means multiplication)

Examples:

$$4 * (7x) = (4 * 7)x = 28x$$

$$-8 * (2y) = 2 * (-8y) = (2 * -8)y = -16y$$

3.4 Exponents with variables

Properties:

$$x^a x^b = x^{a+b} \tag{4}$$

$$(x^a)^b = (x^b)^a = x^{a*b} = x^{b*a} = x^{ab} \tag{5}$$

$\boxed{\text{Examples:}}$

$$z^2 z^3 = z^{2+5} = z^5$$

$$(x^4)^2 = x^{4*2} = x^8$$

3.5 Dividing similar variables

Properties:

$$x^{-a} = \frac{1}{x^a} \tag{6}$$

$$\frac{x^a}{x^b} = x^a x^{-b} = x^{a-b} \tag{7}$$

$\boxed{\text{Examples:}}$

$$x^{-5} = \frac{1}{x^5}$$

$$\frac{y^5}{y^4} = y^5 y^{-4} = y^{5-4} = y^1 = y$$

3.6 Fraction rearranging

Property:

$$\frac{a*x}{b} = x(\frac{a}{b}) = (\frac{a}{b})x \tag{8}$$

$\boxed{\text{Examples}}$

$$\frac{5x}{9} = x(\frac{5}{9}) = (\frac{5}{9})x$$

3.7 Fractions divided by fractions

Property for real numbers "c" and "d" we have

$$\frac{\frac{a}{b}}{\frac{c}{d}} = (\frac{a}{b}) * (\frac{d}{c}) = \frac{a * d}{b * c} = \frac{ad}{bc} \tag{9}$$

$\boxed{\text{Examples}}$

$$\frac{(\frac{5}{6})}{7} = \frac{(\frac{5}{6})}{(\frac{7}{1})} = (\frac{5}{6}) * (\frac{1}{7}) = \frac{5 * 1}{6 * 7} = \frac{5}{42}$$

$$\frac{\frac{9}{2}}{\frac{1}{7}} = (\frac{9}{2}) * (\frac{1}{7}) = \frac{9 * 7}{2 * 1} = \frac{63}{2}$$

$$\frac{5}{\frac{2}{3}} = 5 * \frac{3}{2} = \frac{15}{2}$$

3.8 Fractions and exponents

Properties

$$(\frac{a}{b})^n = \frac{a^n}{b^n} \quad \text{(for some number "n")} \tag{10}$$

$$(\frac{a}{b})^{-1} = \frac{b}{a} \tag{11}$$

$\boxed{\text{Examples}}$

$$(\frac{2}{3})^3 = \frac{2^3}{3^3} = \frac{8}{27}$$

$$(\frac{2}{3}) * (\frac{2}{3})^{-1} = \frac{2}{3} * \frac{3}{2} = \frac{2 * 3}{3 * 2} = 1$$

3.9 Fraction arithmetic using variables

We can combine all the above properties and relationships in order to solve a problems like these

$\boxed{\text{Examples:}}$

$$\frac{\frac{4x}{5}}{\frac{2x^2}{6}} = (\frac{4x}{5})(\frac{6}{2x^2}) = \frac{4x * 6}{5 * 2x^2} = \frac{24x}{10x^2} = (\frac{24}{10x}) = \frac{24}{10}\frac{1}{x} = \frac{12}{5}\frac{1}{x}$$

$$\frac{\frac{1}{3}}{(\frac{2}{x})^2} = \frac{\frac{1}{3}}{\frac{2^2}{x^2}} = \frac{1}{3} * \frac{x^2}{4} = \frac{x^2}{12} = \frac{1}{12}x^2$$

3.10 Arithmetic for more than one variable

All of the above properties also apply for when we have multiple variables BUT we need to be careful as **they only apply to variables of the same type**.

Examples:

$$x + y = y + x$$
$$2x + (x + y) - 2y = 3x - y$$
$$(x + 2y)^2 = (x + 2y)(x + 2y) = x^2 + 2yx + 2yx + 4y^2 = x^2 + 4xy + 4y^2$$

3.11 Distributive Law

The distributive property of numbers is exceeding useful as it shows we can both distribute and reverse distribute (factoring out). This is something we will use in physics a lot:

$$a(x + y) = ax + by = (x + y)a \tag{12}$$
$$az + bx = a(z + x) \tag{13}$$

Examples:

$$5(x + z) = 5x + 5z = (x + z)5$$
$$25x + 25z = 25(x + z)$$
$$E = mgh + \frac{1}{2}mv^2 = m(gh + \frac{1}{2}v^2) = \frac{1}{2}m(2gh + v^2) \quad \text{(Conservation of Energy eqn.)}$$

This last equation is easier to compute because we only have to plug ”m” into our calculator once.

4 Tricks and Techniques of Equation Manipulation

4.1 Distributive "Trick"

$$ay + bz = a(\frac{a}{a}y + \frac{b}{a}z) = a(y + \frac{b}{a}z) = b(\frac{a}{b}y + z) \quad (14)$$

When is it used: This little trick is very helpful for cancelling out variables in physics equations or mostly can be used to clean up equations to make them look nice.

Examples:

$$35y + 5x = 5(\frac{35}{5}y + \frac{5}{5}x) = 5(7y + x)$$

$$\frac{100xz + 425yz}{25z} = \frac{25z(4x + 17y)}{25z} = \frac{25z}{25z}(4x + 17y) = (4x + 17y)$$

4.2 The quadratic formula

When is it used: The quadratic formula is just the solution to a very particular equation:

$$ax^2 + bx + c = 0$$

where a,b,c are just numbers and "x" is the variable (which can be **any** variable but we choose x arbitrarily). This has a solution that most people have seen:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The formula gives us 2 answers and as would-be physicists we usually choose the answer that makes physical sense (for example we don't accept negative values for the time variable).

Example:

Lets use physics equation

$$\frac{g}{2}t^2 + v_0 t - \Delta x = 0$$

that relates the **variable** time (t), and **constants**: initial velocity (v_0), acceleration (g) and position (Δx).
We can apply the quadratic formula with a= g/2, b = v_0 and =$-\Delta x$

$$t = \frac{-v_0 \pm \sqrt{v_0^2 - 4(g/2)(-\Delta x)}}{2(g/2)}$$

Which we can clean up a little

$$t = \frac{-v_0 \pm \sqrt{v_0^2 + 2g(\Delta x)}}{g}$$

4.3 Adding and subtracting equations

When is it used: Adding and subtracting whole equations is used to eliminate variables from equations in order to isolate one in particular. This trick stresses what is meant by an "=" sign.
Suppose we had

$$3x + 65y = 10$$

$$-3x + 35y = 2$$

Then we can do:

$$10 + 2 = (3x + 65y) + 2 = (3x + 65y) + (-3x + 35y) = 65y + 35y = 100y$$

So we see y= 1/10 and we could solve then for x in either of the equations we started with.

But the trick doesn't end there! We could **add a multiple of one equation to another** and do the same thing. Suppose we had:

$$15x + y = 3$$

$$3x - 4y = -12$$

I want to eliminate x somehow, so I'll take the second equation, multiply it **all** by -5 then add it to the first equation:

$$(15x + y) - 5(3x - 4y) = y + 20y = 21y = 3 - (-5) * 12 = 63$$

So y=3 here from 21y = 63 in the equal signs.

(*** adding a negative equation, as in the second example, is equivalent to subtracting two equations)

4.4 Dividing equations

When is it used: Just like the above trick, the equal sign allows us to divide (or multiply) equations. This is especially useful in physics when we want to **compare** the relative strength of two variables. It can be used algebraically to simplify expressions

Examples:

A very common physics problem you will see on exams is something like:

Q: $E_2 =?$ in $E = mc^2$ for when $m_1 = \frac{1}{2}m_2$? (write your answer in terms of E_1)

So we can write out the two cases

$$E_1 = m_1 c^2$$
$$E_2 = m_2 c^2$$

Now we take a ratio to remove c^2

$$\frac{E_2}{E_1} = \frac{m_2 c^2}{m_1 c^2} = \frac{m_2}{m_1} = \frac{m_2}{(\frac{1}{2}m_2)} = \frac{1}{\frac{1}{2}} = 2$$

So by multiplying everything by E_1 we have

$$E_2 = 2E_1$$

This was a toy example but in the wild world of college physics, **almost every problem that asks for a ratio can be solved by taking the ratio of the setup equations**.

4.5 Simplifying Algebraic Fractions

When is it used: When variables are in fractions we can reduce our expression by "distributing" the denominator and **only** the denominator. The numerator does not have the following property. We can also use our knowledge of fractions to find common denominators between fractions to reduce expressions.

Properties:

$$\frac{a}{c+d} + \frac{b}{c+d} = \frac{a+b}{c+d} \tag{15}$$

$\boxed{\text{Examples:}}$

$$\frac{(x+1)+(x+2)}{(x+1)} = \frac{x+1}{x+1} + \frac{x+2}{x+1} = 1 + \frac{x+2}{x+1}$$

$$\frac{4}{x-2}+\frac{3x+6}{(x-2)^2} = \frac{4(x-2)}{(x-2)^2}+\frac{3x+6}{(x-2)^2} = \frac{(4x-8)+(3x+6)}{(x-2)^2} = \frac{x-2}{(x-2)^2} = \frac{1}{x-2}$$

(See section 7 on "Common Algebra Errors to Avoid" to see the error in the numerator)

5 Ordering & Absolute Values

5.1 Ordering

- $<$ (less than)
- $>$ (greater than)
- $\leq$ (less than or equal to)
- $\geq$ (greater than or equal to)

5.2 Absolute Values

Properties:

$$|a| \geq 0 \tag{16}$$

$$|-a| = |a| \tag{17}$$

$$|ab| = |a||b| \tag{18}$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \tag{19}$$

$$|a+b| \leq |a| + |b| \quad \text{(Triangle Inequality)} \tag{20}$$

***The Triangle Inequality is used in physics when talking about the lengths of vectors.

$\boxed{\text{Examples:}}$

$$|5x| = |-5x| \geq 0$$

$$|-5yx| = |(-5y)x| = |(-5)y||x| = |-5||y||x| = 5|y||x|$$

$$\left|\frac{x-1}{-y}\right| = \frac{|x-1|}{|-y|} = \frac{|x-1|}{|y|} \leq \frac{|x| + |-1|}{|y|} = \frac{|x|+1}{|y|}$$

6 Logarithms

Fun fact: The invention of the Logarithm wasn't until the 1600's and revolutionized doing calculations in Astronomy, literally giving astronomers more time to live doing astronomy instead of tedious calculations.

6.1 Definitions

The **general** logarithm is defined as

$$\log_a(x) = b$$

Where we read this in English as: log base "a" of "x" equals "b".

The logarithm was developed as an **inverse** function to the equation:

$$a^b = x$$

So from this we can interpret the logarithm as asking the following question: **what power do I need to raise "a" to in order to get "x"?**. The answer to this question is "b".

Example:

$$2^3 = 8 \quad \& \quad \log_2(8) = 3$$

$$x =? \text{ for } (5.5)^x = 4 \implies x = \log_{5.5}(4) = 0.813\ldots$$

6.2 Special Logarithms

In higher level mathematics it becomes clear to there are **special** numbers to choose for the base in the equation:

$$a^b = x$$

Choosing a=10 and a = e = 2.71828..., where e is Euler's number turns out to be really useful for a menagerie of advanced reasons. We give these logarithms special names:

- $\log_{10}(x) := \log(x)$ (called "log")
- $\log_e(x) := \ln(x)$ (called "The Natural Log")

6.3 Properties:

The following properties work for logarithms of **ANY** base.

$$\log_a(a) = 1 \tag{21}$$
$$\log_a(1) = 0 \tag{22}$$
$$\log_a(x^b) = b * \log_a(x) \tag{23}$$
$$\log_a(xy) = \log_a(x) + \log_a(y) \tag{24}$$
$$\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y) \tag{25}$$

Examples:

$$\log_6(6) = 1$$
$$\log_{10}(1) = \log(1) = 0$$
$$\log_e(x^4) = \ln\left(x^4\right) = 4\ln(x)$$
$$\log(4yx) = \log((4y)x) = \log(4y) + \log(x)$$
$$\log_7(\frac{7}{5}) = \log_7(7) - \log_7(5) = 1 - \log_7(5)$$

7 Trigonometry

7.1 Radians Degrees conversion

There's two units physicists measure angles in, radians and degrees. Each circle has a total of 360° and 2π radians. Thus we know:

$$360° = 2\pi \text{ rad}$$

This implies that we can convert between radians and degrees by the ratio:

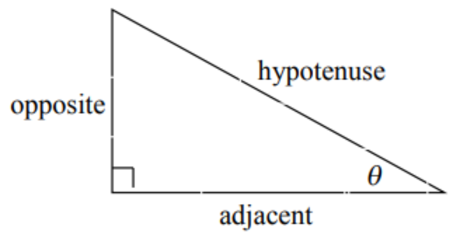
$$\frac{2\pi}{360°} = \frac{\pi}{180°}$$

Examples:

$$127° * \frac{\pi}{180°} \approx 2.22\text{rad}$$

$$\frac{7\pi}{5} * \frac{180°}{\pi} = 252°$$

7.2 Trigonometric Functions



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

The trigonometric functions are **periodic**. They repeat the same values after a certain θ value. That value is $360°$ or 2π radians.

$$\sin(\theta \pm 360^\circ) = \sin(\theta) \quad (\theta \text{ in degrees})$$
$$\sin(\theta \pm 2\pi) = \sin(\theta) \quad (\theta \text{ in radians})$$

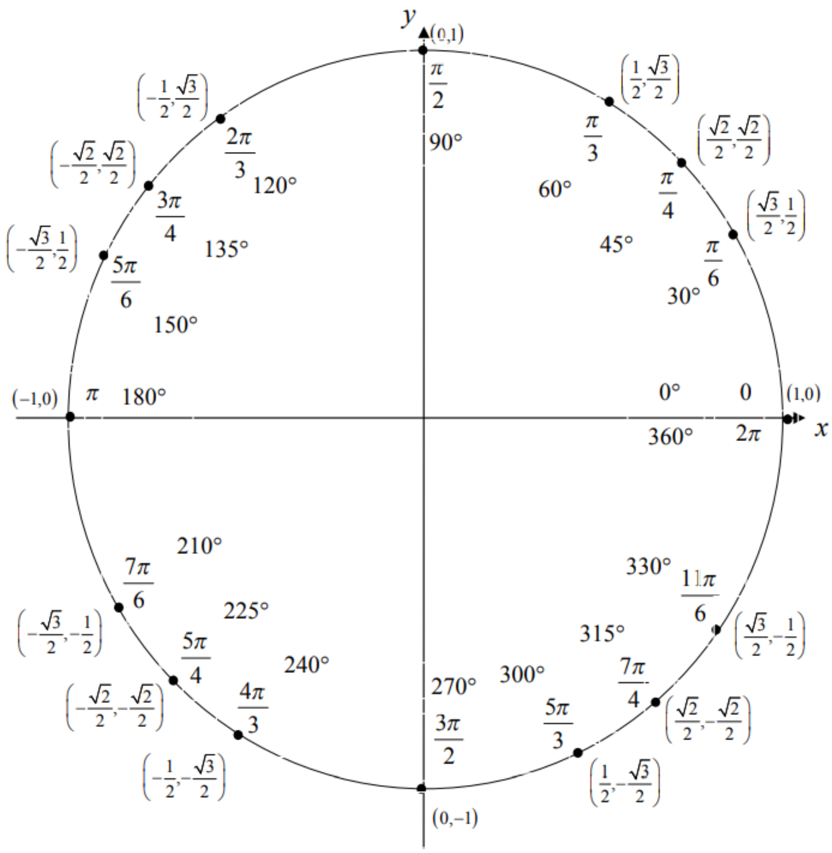
*** The same is also true for $\cos\theta$.

$$\tan(\theta \pm 180^\circ) = \tan(\theta) \quad (\theta \text{ in degrees})$$
$$\tan(\theta \pm \pi) = \tan(\theta) \quad (\theta \text{ in radians})$$

We can also do any whole multiple of these:

$$\sin(\theta + 2\pi n) = \sin(\theta), \quad n = 0, \pm 1, \pm 2 \ldots$$
$$\tan(\theta + \pi n) = \tan(\theta) \quad n = 0, \pm 1, \pm 2 \ldots$$

7.3 The Unit Circle



Memorizing the whole unit circle is **NOT** something you have to do for this course, **but** knowing a couple of properties from it is necessary. Those are:

Properties:

$$-1 \leq \cos\theta \leq 1 \tag{26}$$

$$-1 \leq \sin\theta \leq 1 \tag{27}$$

$$-\infty < \tan\theta < \infty \tag{28}$$

$$-A \leq A * \sin(\theta) \leq A \tag{29}$$

$$\sin(0^\circ) = \sin(180^\circ) = 0 \tag{30}$$

$$\cos(90^\circ) = \sin(270^\circ) = 0 \tag{31}$$

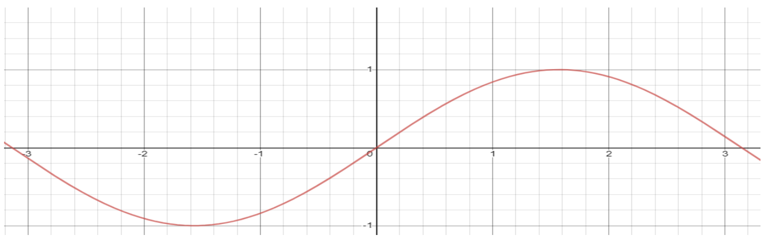
7.4 Phase Angle

Phase Angle
Sometimes we don't add an integer amount to θ when using $\cos\theta$ or $\sin\theta$ so it's often useful to know what happens to the graph of these functions. So in the case of studying waves in physics, we might want to know if we introduce a ϕ like:

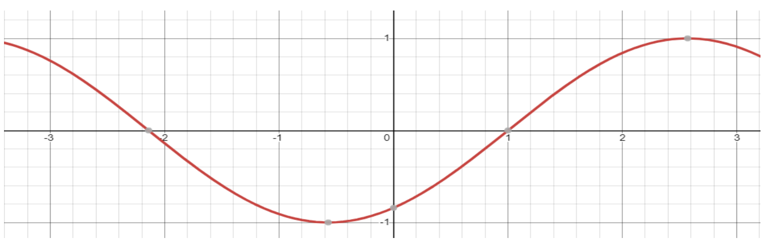
$$\cos(\theta \pm \phi) \quad \& \quad \sin(\theta \pm \phi)$$

Where ϕ is some new amount we add or take away from every θ value we insert. For example we consider the $\sin\theta$ function:

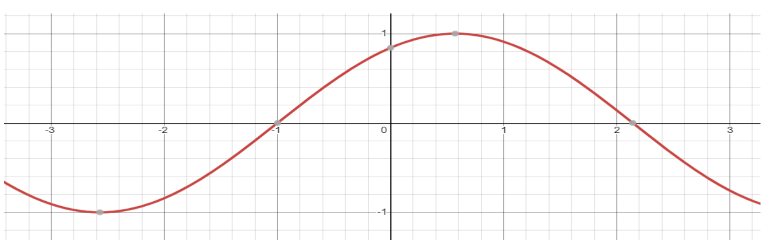
$\sin\theta$ (Normal)



$\sin(\theta + 1)$ (shifted **left**)



$\sin(\theta - 1)$ (shifted **right**)



This property is true in general so for cosine, sine or tangent:

$$\theta \pm \phi = \begin{cases} -\phi & \text{Shift Right} \\ +\phi & \text{Shift Left} \end{cases}$$

7.5 Useful Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{32}$$

$$\sin^2\theta + \cos^2\theta = 1 \tag{33}$$

7.6 Inverse Trig Functions

There are functions that can "undo" a trigonometric function. We call these "arcsine", "arccosine" and "arctangent" and one way they're written is:

$$\sin^{-1}(x)$$
$$\cos^{-1}(x)$$
$$\tan^{-1}(x)$$

Which should not be confused with:

$$\sin^{-1}(x) \neq \frac{1}{\sin(x)}$$

Instead:

$$(\sin(x))^{-1} = \frac{1}{\sin(x)}$$

Properties:

$$\sin^{-1}(\sin(\theta)) = \theta \tag{34}$$

$$\cos^{-1}(\cos(\theta)) = \theta \tag{35}$$

$$\tan^{-1}(\tan(\theta)) = \theta \tag{36}$$

$\boxed{\text{Example}}$

$$\begin{aligned}
\tan[\sin(\theta)+1] &= 5 \\
\implies \quad \tan^{-1}(\tan[\sin(\theta)+1]) &= \tan^{-1}(5) \\
\implies \quad \sin(\theta)+1 &= \tan^{-1}(5) \\
\implies \quad \sin(\theta) &= \tan^{-1}(5) - 1 \\
\implies \quad \sin^{-1}(\sin(\theta)) &= \sin^{-1}[\tan^{-1}(5) - 1] \\
\implies \quad \theta &= \sin^{-1}[\tan^{-1}(5) - 1]
\end{aligned}$$

8 Common Algebra Errors to Avoid

There's a slew of things that can go wrong while doing algebra (I have lots of personal experience here), but there's common mistakes you can avoid by being careful. Here I list the ones I see the most:

$$
\begin{aligned}
(x^2)^4 &\neq x^6 && \text{(See Section 2.4)} \\
-a(x-1) &\neq -ax - a && \text{(See Section 2.11)} \\
\frac{y}{x+z} &\neq \frac{y}{x} + \frac{y}{z} && \text{(See Section 3.5)} \\
\frac{1}{x^2+x^3} &\neq \frac{1}{x^2} + \frac{1}{x^3} && \text{(See Section 3.5)} \\
(x+a)^2 &\neq x^2 + a^2 && \text{(See online resources for proof)} \\
2(x+1)^2 &\neq (2x+2)^2 && \text{(See online resources for proof)}
\end{aligned}
$$

9 Further Algebraic Tools (Optional)

For further algebraic tools you feel you might be missing, read up on them here: `http://tutorial.math.lamar.edu/pdf/Algebra_Cheat_Sheet.pdf`

For further assistance with trigonometry vist: `http://tutorial.math.lamar.edu/pdf/Trig_Cheat_Sheet.pdf`