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Characterizations of Left Engel Elements and the Fitting Subgroup

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Preface

In the past year, I learned a lot about Engel structures and Engel elements in groups and in particular about the connection between these sets of Engel elements and previously known subgroups. I would like to express my sincere gratitude to several people for their help and support in writing this thesis, each in their own way.

First of all, I want to thank my supervisor, Dr. Paula Lins, for countless hours spent listening to my questions, answering said questions and providing feedback and support. Without her, it would have been impossible to complete this thesis.

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Next, I would like to thank my friends from university, for the many great experiences that made the last five years fantastic, and my friends from ballet, for making me happy to rush home every Friday evening and providing a welcome break from studying.

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Rune

Summary

For any two elements x and y of some group G and any strictly positive integer n we can define a commutator as

$$[x,_n y] := [\dots[[x, \underbrace{y], y], \dots, y}_{n \text{ times}}].$$

If we can find an n, depending on x and y, for all pairs x and y such that $[x,_n y] = 1$, then G is called an *Engel group.* Similarly, if we can find an n, depending on x, such that $[x,_n y] = 1$, for all x with y fixed, then y is called a *left Engel element* of G. The set of all left Engel elements of a group G is denoted by $L(G)$.

In this thesis, we are interested in the set of left Engel elements for specific classes of groups. We look at whether this set is a subgroup and if it coincides with other known subgroups. We use this information to find methods to compute this set for certain groups without having to check the commutator relation for all pairs of elements. Throughout this thesis, we make use of computational methods to construct explicit examples.

Concretely, we give an overview of results that characterize $L(G)$ for finite groups, soluble groups, Noetherian groups and radical groups, and show the connection between the Fitting subgroup and Hirsch-Plotkin radical. For finite groups, we generalize the notion of being left-Engel by introducing the generalized Fitting height. Finally, we describe a known method for computing the Fitting subgroup for polycyclic-by-finite groups and use previous results to relate this to $L(G)$.

List of Symbols

$[a, b]$	the commutator of a and b, $a^{-1}b^{-1}ab$.
a^b	a conjugated by b, $b^{-1}ab$.
$[A, B]$	group generated by all $[a, b]$ with $a \in A, b \in B$.
$\langle S \rangle$	subgroup generated by S.
$A < B$	A is a strict subgroup of B.
$A \leq B$	A is a subgroup of B.
$A \subset B$	A is a strict subset of B.
$A \subseteq B$	A is a subset of B.
$A \lhd B$	A is a normal subgroup of B.
$A \lhd\lhd B$	A is a subnormal subgroup of B.
$G^{(n)}$	n-th derived subgroup.
$Z(A)$	the center of A.
$Z_n(A)$	n-th term of the upper central series of A.
$\gamma_n(A)$	n-th term of the lower central series of A.
$[x_1, ..., x_n]$	left-normed n-fold commutator.
$[x,_n y]$	$[x, \underbrace{y, \ldots, y}_{n \text{ times}}]$.
$\mathbb{Z}_n$	cyclic group of n elements.
S_n	symmetric group on n elements.
A_n	alternating group on n elements.
D_n	dihedral group of order $2n$.

$\mathbb{N}_0$	set of natural numbers $\{1, 2, 3, \ldots\}$.
$L(G)$	set of left Engel elements of G.
$\overline{L}(G)$	set of bounded left Engel elements of G.
$L_n(G)$	set of left n-Engel elements of G.
$R(G)$	set of right Engel elements of G.
$\overline{R}(G)$	set of bounded right Engel elements of G.
$R_n(G)$	set of right n-Engel elements of G.
$C_G(x)$	centralizer of x in G.
$\mathrm{ncl}_G(H)$ or $\langle H^G \rangle$	normal closure of H in G.
$N_G(x)$	normalizer of x in G.
$\mathrm{Soc}(G)$	socle of G.
$F(G)$	Fitting subgroup of G.
$F^*(G)$	generalized Fitting subgroup of G.
$O_p(G)$	p-core of G.
$h(G)$	Fitting height of G.
$h^*(G)$	generalized Fitting height of G.
$E_{G,k}(x)$	subgroup of G generated by $[g,_k x]$ with $g \in G$.
$HP(G)$	Hirsch-Plotkin radical of G.
$G \rtimes H$	external semidirect product of G and H.
$G \wr H$	restricted regular wreath product of G and H.

Contents

Introduction

We say that a group G is an Engel group if the following commutator relation holds for all $x, y \in G$ for some $n \in \mathbb{N}_0$, possibly depending on x and y.

$$[x,_n y] := [\dots [[x, \underbrace{y], y], \dots, y}_{n \text{ times}}] = 1.$$

When this n is independent of the chosen x and y, the group is called n-Engel. If this commutator relation holds for some $x \in G$ and all $y \in G$, then x is called a right Engel element of G, and analogously if it holds for all x for some y then y is a left Engel element. The sets of right and left Engel elements are denoted $R(G)$ and $L(G)$ respectively.

The aim of this master thesis is to describe the set of left Engel elements $L(G)$ for specific classes of groups by using known subgroups with useful properties. The subgroups we will use are the Fitting subgroup and Hirsch-Plotkin radical, which are generated by all normal nilpotent subgroups and normal locally nilpotent subgroups respectively. All main results regarding this set of left Engel elements will be generalizations of the following theorem by Baer.

Theorem 3.3.1. *Let G be a finite group, an element g of G belongs to the Fitting subgroup $F(G)$ if and only if it is a left-Engel element of G.*

Chapter 1 is a preliminary chapter where some basic concepts are introduced such as nilpotent and soluble groups, group actions and the semidirect product.

Chapter 2 introduces the concept of Engel groups and Engel elements and treats some basic results for certain n-Engel groups and elements where n is small, such as an extensive characterization of 2-Engel groups.

After these introductory chapters we have defined the core concepts of this thesis and can go further with the main topic, the description of the set of left Engel elements. In Chapter 3 we define the Fitting subgroup and show its connection to $L(G)$ by means of proving Baer's Theorem. We also consider the Fitting length and generalized Fitting length of a group to describe notions of being "near left Engel". We apply these length measures to the group $E_{G,k}(x)$ which is generated by all elements of the form $[x,_k y] = 1$ for $y \in G$. This will culminate in the following theorem by Guralnick and Tracey.

Theorem 3.4.4. *Let G be a finite group, and $x \in G$ and fix an integer h. The generalized Fitting height of $E_{G,k}(x)$ is at most h for some integer k if and only if $xF_h^*(G)$ is contained in $F(G/F_h^*(G))$.*

Chapter 4 treats the proof of Theorem 3.4.4. We first show a theorem about subnormal groups by Wielandt, of which we then show a generalization by Flavell. We then discuss how this generalization is used by Guralnick and Tracey to show their result.

From Chapter 5 onwards, we leave the realm of finite groups and see what results are known about left Engel elements in certain infinite groups. In Chapter 5 we introduce the Hirsch-Plotkin radical as an alternative to the Fitting subgroup for the study of left Engel elements in the infinite case. The classes of groups we will look at are the soluble, Noetherian and radical groups. For each type of group we will develop the necessary background and then prove a result linking $L(G)$ with either the Fitting subgroup, the Hirsch-Plotkin radical, or both. To finish the chapter we show that the set of left Engel elements is not necessarily a subgroup by briefly discussing an argument by Bludov.

Finally, in Chapter 6 we take a computational approach to determine the set of left Engel elements by using the characterizations we showed in chapters 3 through 5. More specifically we first describe some methods to compute the Fitting subgroup for finite groups using GAP, a computational algebra system. We then look at the specific case of polycyclic-by-finite groups following the methods described by Eick.

Chapter 1

Preliminaries

We start this thesis with a general introduction to some core concepts that are needed to understand the text. The concepts in this section are found in any elementary book on group theory, we base ourselves on the lectures of the course 'Group Theory' taught by Karel Dekimpe at KU Leuven during the academic year $2020-2021$.

1.1 Soluble and Nilpotent groups

To talk about soluble and nilpotent groups, we must first recall the definitions of the commutator of two elements, and the commutator of subgroups.

Definition 1.1.1. Let H and K be subgroups of a group G. We define the *commutator subgroup* $[H, K]$ as the subgroup of G generated by all *commutators* $\{[h, k] := h^{-1}k^{-1}hk \mid h \in H, k \in K\}$.

Taking $H = K = G$, we get the subgroup $[G, G]$, which we will call the *derived subgroup* of G, and will be denoted by $G^{(1)}$. Note that $G^{(1)}$ is a normal subgroup of G since if $h \in G^{(1)}$ and $g \in G$, we get $g^{-1}hg = hh^{-1}g^{-1}hg = h[h, g] \in G^{(1)}$.

By repeating the process of taking the derived subgroup we can define a series of subgroups,

$$\begin{aligned} G^{(0)} &= G \\ G^{(n)} &= [G^{(n-1)}, G^{(n-1)}]. \end{aligned}$$

Using the same reasoning as for the normality of $G^{(1)}$ in G we get that this is a series of subgroups in which each subgroup is normal in the previous one,

$$G = G^{(0)} \triangleright G^{(1)} \triangleright G^{(2)} \triangleright \dots,$$

this series will be called the *derived series* of the group G. It is not necessarily the case that this series ever reaches the identity, when this is the case, the group will be called *soluble*.

More generally we can state the following definition, which is equivalent to the previous one.

Definition 1.1.2. Let G be a group. If there exists a series of subgroups $1 = H_1 < H_2 < \ldots < H_n = G$ such that $H_{i-1} \lhd H_i$ and H_{i+1}/H_i is abelian for all $i \in \{1, \ldots, n\}$ then G is called *soluble*.

Example 1.1.3. A first example of a soluble group is the dihedral D_3 group of order 3. We use the presentation $D_3 := \langle a, b \mid a^3 = b^2 = 1, ba = a^2 b \rangle$. It is easy to check that $N = \{1, a, a^2\}$ is a normal abelian subgroup of D_3, in fact, it is isomorphic to $\mathbb{Z}_3$, the cyclic group of 3 elements. We can also show that $D/N \cong \mathbb{Z}_2$ is abelian. The series that we find is

$$1 < \{1, a, a^2\} < D.$$

By the observations above, this satisfies the criteria for the group to be soluble. △

Similarly we can define three more series for groups. We say that G has a *central series*, if there exists a series of normal subgroups,

$$\{1\} = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G,$$

such that $G_{i+1}/G_i \leq Z\left(G/G_i\right)$, or equivalently $[G, G_{i+1}] \leq G_i$. Quotients of the form G_{i+1}/G_i will be referred to as the *factors* of the central series.

The *lower central series* of a group G is defined as,

$$\begin{aligned} \gamma_1(G) &= G, \\ \gamma_i(G) &= [\gamma_{i-1}(G), G]. \end{aligned}$$

This results in a descending series of normal subgroups,

$$G = \gamma_1(G) \rhd \gamma_2(G) \rhd \ldots \rhd \gamma_n(G) \rhd \ldots,$$

which does not necessarily reach the identity.

Finally the *upper central series* of a group G is defined as,

$$\begin{aligned} Z_0(G) &= \{1\}, \\ \frac{Z_{i+1}(G)}{Z_i(G)} &= Z\left(\frac{G}{Z_i(G)}\right), \end{aligned}$$

resulting in a ascending series of normal subgroups,

$$\{1\} = Z_0(G) \lhd Z_1(G) \lhd \ldots \lhd Z_n(G) \lhd \ldots.$$

The upper and lower central series are two examples of a general central series.

Definition 1.1.4. Let G be a group. We call G *nilpotent* if there exists a finite central series for G. Let c be the smallest integer such that a central series of length c exists, then G is called nilpotent of *class* c.

Example 1.1.5. As an example, we will look at $D_4 = \langle a, b \mid a^4 = b^2 = 1, ba = a^3 b\rangle$. Note that since D_4 is not abelian, as $ab \neq ba = a^3 b$, it is not nilpotent of class 1. We calculate the upper central series. The first term $Z_0(D_4)$ is by definition equal to 1. The second term is equal to the center $Z(G)$, which is $\{1, a^2\}$. For the third term we need the following quotient,

$$\frac{Z_2(D_4)}{\{1, a^2\}} = Z\left(\frac{D_4}{\{1, a^2\}}\right) \cong Z(D_2) = D_2.$$

The last equality holds because D_2 is abelian. We then find that $Z_2(D_4) = D_4$, this implies that D_4 is nilpotent of class 2. △

1.2 Preliminary Results

We first define some more concepts that are basic, but are used throughout this thesis.

Definition 1.2.1. The *normalizer* $N_G(H)$ of a subgroup H of a group G is the largest subgroup K of G containing H such that H is normal in K. Equivalently it is the set of all elements x such that $xH = Hx$.

Definition 1.2.2. The *normal closure* $\mathrm{ncl}_G(H)$ of a subgroup H of a group G is the smallest normal subgroup N of G such that $H \leq N$. We will sometimes write H^G instead of $\mathrm{ncl}_G(H)$ for brevity.

It is important to note that the normalizer and normal closure of a subgroup are very different subgroups. We work out an example to show the difference.

Example 1.2.3. Consider the alternating group on five elements A_5, let H denote the subgroup generated by the permutation $(1, 2, 3, 4, 5)$. Since A_5 is a simple group, the smallest normal subgroup containing H is A_5 itself. To compute $N_{A_5}(H)$ we check which elements $g \in A_5$ leave H invariant when conjugated with g. Note that it suffices to check conjugation on the generator $(1, 2, 3, 4, 5)$. We start with conjugation by $(3, 4, 5)$ as an example.

$$\begin{aligned}(3,4,5)^{-1}(1,2,3,4,5)(3,4,5) &= (5,4,3)(1,2,3,4,5)(3,4,5)\\ &= (1,2,3)(3,4,5)\\ &= (1,2,4,5,3) \notin H.\end{aligned}$$

Repeating this for all elements in A_5 we find that the normalizer is the subgroup generated by $(1, 2, 3, 4, 5)$ and $(2, 5)(3, 4)$. This subgroup is a non-trivial subgroup of order 10. We find that the normalizer and normal closure do not coincide, in particular the normalizer is not necessarily a normal subgroup of the group. △

The following lemma proves a characterization of the normal closure that we will use in later chapters.

Lemma 1.2.4. Let H be a subgroup of a group G, then

$$\mathrm{ncl}_G(H) = \langle H^G \rangle := \langle H^g \mid g \in G \rangle,$$

i.e. the normal closure is generated by all conjugate subgroups of H.

Whenever it is more convenient to work with this conjugates definition of the normal closure we will use the notation $\langle H^G \rangle$.

Proof. It is clear that any normal subgroup containing H has to contain all conjugates of H so $\langle H^G \rangle \leq \mathrm{ncl}_G(H)$. On the other hand $\langle H^G \rangle$ is a normal subgroup of G since conjugating any $g^{-1}hg$ by some $k \in G$ is just conjugation by gk. Since $\langle H^G \rangle$ contains H, it is contained in the normal closure. □

Definition 1.2.5. The *centralizer* $C_G(H)$ of a subgroup H of a group G is the set of all elements $g \in G$ such that $gh = hg$ for all $h \in H$.

Remark 1.2.6. We can generalize the definition of normalizer, normal closure and centralizer to any arbitrary subset S of a group G. We will use the same notation and definitions.

The next lemma states some simple results about nilpotent groups that will be used frequently throughout the following chapters.

Lemma 1.2.7. Let G and H be nilpotent groups. The following statements hold.

1. Any subgroup K of G is nilpotent.
2. Any quotient of G by a normal subgroup is nilpotent.
3. The product $G \times H$ is nilpotent.
4. Every proper subgroup K of G is properly contained in its normalizer $N_G(K)$.
5. Every maximal subgroup of G is a normal subgroup.
6. Let N be a nontrivial normal subgroup of G, then the intersection of N with the center of G is nontrivial.

Proof.

1. As $[K, K] \leq [G, G]$, and further $[[K, K], K] \leq [[G, G], G]$ and so on, we find that $\gamma_i(K) \leq \gamma_i(G)$ for all i. Since G is nilpotent there exists some c such that $\gamma_c(G) = 1$, hence also $\gamma_c(K) = 1$, and K is nilpotent of class at most c.
2. We can repeat the same argument by noting that if $\gamma_c(G) = [[...[G, G], ..., G] = 1$ then $\gamma_c(G/N) = [[...[G/N, G/N], ..., G/N] = 1$.
3. Note that $[(g_1, h_1), (g_2, h_2)] = ([g_1, g_2], [h_1, h_2])$, and hence $[G \times H, G \times H] = [G, G] \times [H, H]$. A similar argument as before now suffices.

4. We will show this by induction on the nilpotency class of G. As a base case we take $c = 1$, then G is abelian which implies that every subgroup is normal, hence $N_G(K) = G$ for all K.

   Now assume that the result holds for groups of nilpotency class c, and let G be nilpotent of class $c + 1$. The center $Z(G)$ is trivially contained in $N_G(K)$. Assume that $Z(G) \not\leq K$, then $K \neq N_G(K)$ and we are done. So assume $Z(G) \leq K$, and consider the quotient subgroup $K/Z(G)$ in $G/Z(G)$. $K/Z(G)$ is a proper subgroup because K is and $G/Z(G)$ is nilpotent of class c, by induction we find that $K/Z(G) \neq N_{G/Z(G)}(K/Z(G))$. The normalizer $N_{G/Z(G)}(K/Z(G))$ is the largest subgroup of $G/Z(G)$ such that it has $K/Z(G)$ as a normal subgroup. This coincides with $N_G(K)/Z(G)$. Hence we find that $K/Z(G) \neq N_G(K)/Z(G)$, or that $K \neq N_G(K)$.

5. Let K be a maximal subgroup of G, then $N_G(K) = G$ or $N_G(K) = K$. The second option is not possible by part 4 of the lemma, hence $N_G(K) = G$, which implies that K is normal by definition of the normalizer.

6. Let $N_1 = N$ and then inductively $N_i := [N_{i-1}, G]$, by normality of N, the N_i are subgroups of N. Note that $[N, G] \leq \gamma_2(G)$, and in general $N_i \leq \gamma_i$. By nilpotency of G there is a smallest index k such that N_k is trivial. By definition we know that $N_k = [N_{k-1}, G] = 1$, but this implies that $N_{k-1} \leq Z(G)$. Since we assumed k to be the smallest index such that N_k is trivial, we know that $N_{k-1} \leq N$ is nontrivial and hence the intersection is nontrivial.

□

We now consider some theorems that will be useful in proving a certain group is nilpotent given information on certain subgroups and quotients.

Lemma 1.2.8. Let N be a normal subgroup of a group G. If N lies inside the centre $Z(G)$ of G, and G/N is nilpotent, then G is nilpotent.

Proof. Consider the projection map $\pi : G \to G/N$. Note that $\pi(\gamma_i(G)) = \gamma_i(G/N)$.

By the nilpotency of G/N we find $k \in \mathbb{N}_0$ such that $\gamma_k(G/N) = 1N$ and hence $\pi(\gamma_k(G)) = 1N$. Using the fact that N lies in the centre of G, we get that $\gamma_{k+1}(G) = [\gamma_k(G), G] = 1$ and hence G is nilpotent. □

The following theorem gives another criterion for nilpotency first shown by P. Hall [Hall, 1958], where one can find a proof of this result.

Theorem 1.2.9. *If $N \lhd G$ is nilpotent and $G/N^{(1)}$ is nilpotent then G is nilpotent.*

To finish this section, we state some more elementary lemma's that are used in later chapters.

Lemma 1.2.10. Let G be a group and N a normal subgroup of G. Assume H is a characteristic subgroup of N, then H is normal in G.

Proof. We need to show that $gHg^{-1} \subseteq H$ for all $g \in G$. Since N is normal in G, all conjugations of N by elements of G can be seen as automorphisms of N. H is characteristic in N which implies in particular that the conjugation of H by any g lies inside H. □

Lemma 1.2.11. Let G be a soluble group and N a minimal normal subgroup of G, then N is abelian.

Proof. Since N is a subgroup of G it is soluble, and hence $[N, N] \neq N$. The only other option is that $[N, N] = 1$, since the derived subgroup is a normal subgroup and because N is assumed to be minimal. This condition precisely means that N is abelian. □

Lemma 1.2.12. Let G be a group and $x, y, z \in G$, the following relations hold:

1. $[x, yz] = [x, z] \cdot [x, y]^z$,
2. $[xy, z] = [x, z]^y \cdot [y, z]$

Proof. Both statements can be shown by simply writing out the commutators. Writing out the first one we find

$$[x, yz] = x^{-1}(yz)^{-1}xyz = x^{-1}z^{-1}(xzz^{-1}x^{-1})y^{-1}xyz = [x, z] \cdot [x, y]^z.$$

And similarly the second one

$$[xy, z] = y^{-1}x^{-1}z^{-1}xyz = y^{-1}x^{-1}z^{-1}x(zyy^{-1}z^{-1})yz = [x, z]^y \cdot [y, z].$$

□

Lastly, we show a result that is often called the Three Subgroups Lemma.

Lemma 1.2.13. If A, B and C be subgroups of G, and H is a normal subgroup of G, then if $[[A, B], C] \subseteq H$ and $[[B, C], A] \subseteq H$, then $[[C, A], B] \subseteq H$.

Proof. We will use Witt's identity which states that for all $a, b, c \in G$

$$[[a, b^{-1}], c]^b \cdot [[b, c^{-1}], a]^c \cdot [[c, a^{-1}], b]^a = 1.$$

The proof of this identity is just a case of writing out all the commutators and canceling terms. We will first reduce the case to $H = 1$, we do this by taking the canonical quotients by H, $A' = AH/H$, etc. Then proving that $[[C, A], B] \subseteq H$ reduces down to proving that $[[C', A'], B'] = 1$, since in that case $[[C', A'], B'] = H/H$ or $[[C, A], B] \subseteq H$.

We will now prove the case for $H = 1$. Suppose $[[A, B], C] = 1$ and $[[B, C], A] = 1$. Then certainly $[[a, b^{-1}], c] = 1$ and $[[b, c^{-1}], a] = 1$ for all $a \in A, b \in B, c \in C$, then also $[[a, b^{-1}], c]^b = 1$ and $[[b, c^{-1}], a]^c = 1$, and hence by Witt's identity $[[c, a^{-1}], b]^a = 1$. From this we find that $[[c, a^{-1}], b] = 1$, and therefore all generators of $[C, A]$ are in the centre of B. This tells us that any element of $[C, A]$ commutes with any element of B, since if $g \in [C, A]$, $g = g_1g_2...g_n$, with all g_i generators of $[C, A]$, and

$$gb = g_1g_2...g_nb = g_1g_2...bg_n = ... = bg_1g_2...g_n = bg.$$

But every element of $[C, A]$ commuting with every element in B implies that $[[C, A], B]$ is generated by 1, and therefore $[[C, A], B] = 1$. □

1.3 The Semidirect Product

The following section is based on [Kurzweil and Stellmacher, 2004, Chapter 3 and 8], but most introductory texts on group theory cover this topic.

1.3.1 Group Actions

To define the semidirect product, we need the concept of a group action.

Definition 1.3.1. We say that a group G acts on a set X if for every pair $(g, x) \in G \times X$, there is an associated element $g \cdot x \in X$ such that the following conditions hold,

1. $1 \cdot x = x$, for all $x \in X$,

2. $(gh) \cdot x = g \cdot (h \cdot x)$, for all $g, h \in G$ and $x \in X$.

We can describe the action of a single element g of G on X by defining a map

$$\ell_g : X \to X : x \mapsto g \cdot x,$$

which we call the left multiplication function. Note that ℓ_g has an inverse since

$$\ell_{g^{-1}}(\ell_g(x)) = g^{-1} \cdot (g \cdot x) = 1 \cdot x = x,$$

and therefore is a bijection. We can then view ℓ_g as a permutation on the set X and define a new map,

$$\pi : G \to \mathrm{Sym}(X) : g \mapsto \ell_g.$$

Note that by the second condition in Definition 1.3.1 this map is actually a homomorphism of groups. The first homomorphism theorem then tells us that the group $G/\ker \pi$ is isomorphic to a subgroup of $\mathrm{Sym}(X)$.

We now look at some easy examples to illustrate the concept.

1. The most trivial group action is the following, let G be any group and X be any non-empty set. Define $g \cdot x = x$ for all $g \in G$ and all $x \in X$, then trivially the conditions in the definition hold and hence we have a group action.

2. A slightly less trivial example is the following, take $G = X$ any group and let the action $g \cdot h$ be left multiplication, then again we have a group action.

3. As a final example, consider a group G and define the map φ_g for any $g \in G$ as

$$\varphi_g : G \times G \to G : (g, x) \mapsto x^g.$$

An easy exercise shows that combining all φ_g results in an action on G that we call the conjugation action.

1.3.2 The Semidirect Product

When both G and X are groups we add an extra condition to the definition of a group action. We still require the first two points from definition 1.3.1 but add a third one,

3. $g \cdot (xy) = (g \cdot x)(g \cdot y)$.

This condition requires the action defined by a single element to be a homomorphism.

Given an action of a group G on a group X, we can define a new group called the *(external) semidirect product* $X \rtimes G$. As a set $X \rtimes G$ is just the direct product $X \times G$, but the multiplication is defined differently,

$$(x_1, g_1)(x_2, g_2) := (x_1(g_1 \cdot x_2), g_1 g_2).$$

Associativity follows from:

$$\begin{aligned}
((x_1, g_1)(x_2, g_2))(x_3, g_3) &= (x_1(g_1 \cdot x_2), g_1 g_2)(x_3, g_3) \\
&= (x_1(g_1 \cdot x_2)(g_1 g_2 \cdot x_3, g_1 g_2 g_3) \\
&= (x_1(g_1 \cdot (x_2(g_2 \cdot x_3))), g_1 g_2 g_3) \\
&= (x_1, g_1)(x_2(g_2 \cdot x_3), g_2 g_3) \\
&= (x_1, g_1)((x_2, g_2)(x_3, g_3)).
\end{aligned}$$

The identity element is $(1_X, 1_G)$ and the inverse of an element (x, g) is $(g^{-1} \cdot x^{-1}, g^{-1})$, indeed:

$$\begin{aligned}
(x, g)(g^{-1} \cdot x^{-1}, g^{-1}) &= (x(g(g^{-1} \cdot x)), gg^{-1}) \\
&= (x(1_G \cdot x)^{-1}, 1_G) = (1_X, 1_G).
\end{aligned}$$

If we consider the subset of $X \rtimes G$ that consists of all elements of the form $(x, 1)$, we get a subgroup that is isomorphic to X. It is a simple calculation to show that this a subgroup and the map $\phi : X \to X \rtimes G : x \mapsto (x, 1)$ is an isomorphism onto its image. When we say the embedding of X in $X \rtimes G$ we mean this map ϕ. Similarly the subset consisting of the elements $(1, g)$ is a subgroup that is isomorphic to G.

Example 1.3.2. Let G be a group and $\mathrm{Aut}(G)$ its group of automorphisms. We can consider the semidirect product $G \rtimes \mathrm{Aut}(G)$, where the action of $\mathrm{Aut}(G)$ on G is defined in the obvious way, if $\omega \in \mathrm{Aut}(G)$ and $g \in G$, then $\omega \cdot g := \omega(g)$. This new group is often called the *holomorph* and is denoted $\mathrm{Hol}(G)$. △

In Chapter 4 we will often consider the semidirect product of a group with an automorphism of this group. Let G be a group and α an automorphism of this group, then $G \rtimes \langle \alpha \rangle$ is defined as before where the action of α on an element of g is simply the action under the automorphism itself, i.e. $\alpha \cdot g := \alpha(g)$. To ease notation we will not write out every pair of elements whenever we do calculations in this product. For instance, instead of writing $[(g, 1), (1, \alpha)]$ we will write $[g, \alpha]$.

1.4 GAP

Throughout this thesis we will make use of a computer algebra system to compute examples. The software we will use is GAP, since it is specifically designed for group theory and is open source, as opposed to alternatives such as Magma. GAP stands for groups, algorithms and programming and is a useful tool for computing subgroups, series, homomorphisms and much more, with finite and certain types of infinite groups. Whenever we use GAP the respective code is highlighted in a gray box. All code used can be found in Appendix A.

Chapter 2

Engel Groups and Elements

The following chapter collects material from [Traustason, 2011], [Abdollahi, 2011], [Robinson, 2013], [Bussman, 2010] and [Kurzweil and Stellmacher, 2004].

We can now define the core concept of this thesis, the Engel groups and Engel elements. We start by defining what these actually are, after which we will look at some characterizations in specific cases.

2.1 Introduction and Definitions

The notion of a commutator of elements where we write $[x, y] = x^{-1}y^{-1}xy$ can be extended to an arbitrary amount of elements by defining the *n-fold left-normed commutator*. We define $[x_1, x_2, x_3] = [[x_1, x_2], x_3]$ and then inductively

$$[x_1, x_2, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n].$$

When looking at the standard commutator, we notice that for two elements a, b of a group G, $[a, b] = 1$ if and only if a and b commute. More generally we see that a group G is abelian if and only if all commutators of elements in G are trivial.

This statement does not hold in general for higher order commutators. In Example 1.1.5 we showed that D_4 is nilpotent of class 2, from which we get that

$$\gamma_3(D_4) = [[G, G], G] = 1,$$

or

$$[x_1, x_2, x_3] = 1,$$

for all $x_1, x_2, x_3 \in D_4$. However since $xa = a^3x \neq ax$, D_4 is not abelian. To be able to talk about groups where for a certain $n \in \mathbb{N}_0$ the n-left normed commutators are trivial we can just use the property of nilpotency as displayed in the previous example.

We will now define a slightly stronger concept using the left-normed commutator, but we will first make future notation easier by defining the following notation.

When $y = x_2 = x_3 = \ldots = x_{n+1}$ we will write,

$$[x,_n y] := [x, \underbrace{y, \ldots, y}_{n \text{ times}}].$$

Definition 2.1.1. Let G be a group, if for any two elements x and y of G there exists some $n \in \mathbb{N}_0$ such that $[x,_n y] = 1$ then G is called an *Engel group*. If moreover we can fix an n independently of the elements x and y, we call G an *n-Engel group*.

Example 2.1.2. We can easily find examples of Engel groups by looking at the abelian groups. Let G be an abelian group, and a and b two elements, by abelianity $ab = ba$ or $a^{-1}b^{-1}ab = 1$ which is the same as $[a, b] = 1$. This is exactly the requirement for a group to be 1-Engel. △

Example 2.1.3. Another less trivial example of a class of groups that are Engel groups are the nilpotent groups. Let G be a nilpotent group of class c, then $\gamma_{c+1}(G) = 1$. We defined $\gamma_n(G)$ to be the $(n-1)$-fold left normed commutator of the group G, so for any two elements $a, b \in G$ it definitely holds that $[a,_c b] = 1$. We find that G is a c-Engel group.

We can go even further with this example. We define a group to be *locally nilpotent* if any finitely generated subgroup of the group is nilpotent. Now let G be a locally nilpotent group and $a, b \in G$, since $\langle a, b\rangle$ is nilpotent it follows that $[a,_n b] = 1$ for some $n \in \mathbb{N}$. Since this holds for any a and b, G is an Engel group. △

Remark 2.1.4. Originally the study of general Engel groups came about due to questions about a related concept for Lie algebras.

In general we define a *Lie algebra* as a vector space $\mathfrak{g}$ over a field k together with a bilinear alternating map,

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

satisfying the Jacobi identity.

The bilinearity means,

$$[ax + by, z] = a[x, z] + b[y, z],$$
$$[x, ay + bz] = a[x, y] + b[x, z].$$

The alternativity,

$$[x, x] = 0.$$

And satisfying the Jacobi identity is defined as,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

By using the bracket notation from our Lie algebra, we can reuse Definition 2.1.1 to define *(n-)Engel Lie algebras*.

This is the context where the name Engel first appeared, to describe the Lie algebras having this additional property on its operation. This naming choice was based on a sketch by Friedrich Engel of the proof of the following theorem, which was later completed by his student Karl Arthur Umlauf.

Theorem 2.1.5 (Engel's Theorem). *Any finite-dimensional Lie algebra $\mathfrak{g}$ is nilpotent if and only if for every fixed $x \in \mathfrak{g}$ and any y in $\mathfrak{g}$, there exists some n possibly depending on y such that,*

$$\underbrace{[x, [x, \ldots, [x}_{n\ times}, y] \ldots]] = 0.$$

2.2 Engel Elements

We now take another perspective on the Engel condition. Instead of focusing on all elements at once, we can look at the specific elements of a group satisfying the condition.

Definition 2.2.1. Let G be a group and a an element of G. We say that a is *left Engel* if for any $g \in G$ it holds that the n-fold left-normed commutator $[g,_n a] = 1$. Similarly, if $[a,_n g] = 1$ for all $g \in G$, then a is called *right Engel*. If we can fix n independently of g we say that a is *left n-Engel* or *right n-Engel* respectively.

We will adopt the following notation to talk about Engel elements,

- $L(G)$, the set of all left Engel elements,
- $L_n(G)$, the set of all left n-Engel elements,
- $\overline{L}(G)$, the set of all *bounded* left Engel elements, i.e. all left Engel elements that are left n-Engel for some n.

Similarly we define $R(G), R_n(G)$ and $\overline{R}(G)$ the corresponding sets for right Engel elements. Note that since 1 is always a left and right 1-Engel element these sets are always non-empty.

The reason why we say *sets* here, and not subgroups is very important. In general it is very hard to describe the set of left and right Engel elements, the main question in this thesis is to see for what classes of groups we can say something more about the structure, and especially which known subgroups coincide with these sets.

It is not known in general whether any left Engel element of a group is also a right Engel element ([Abdollahi, 2011]). When the group is Engel this is trivial as every element is both left and right Engel. The following theorem by Heineken nevertheless gives a relation between left and right Engel elements of general groups.

Theorem 2.2.2. *[Heineken, 1960] Let G be a group, then the following statements hold.*

1. $R_n(G)^{-1} \subseteq L_{n+1}(G)$,
2. $\overline{R}(G)^{-1} \subseteq \overline{L}(G)$,
3. $R(G)^{-1} \subseteq L(G)$.

In the proof of the theorem we will need the following simple result.

Lemma 2.2.3. Let $\varphi : G \to H$ be an surjective homomorphism and let a be a left or right n-Engel element in G, then $\varphi(a)$ is a left, respectively right n-Engel element in H.

Proof. Take any $h \in H$, and let $g \in G$ such that $\varphi(g) = h$. Assume a is right-Engel, then

$$[\varphi(a),_n h] = [\varphi(a),_n \varphi(g)] = \varphi([a,_n g]) = \varphi(1) = 1,$$

where the second equality holds because φ is a homomorphism. The case where a is left-Engel is identical. □

Corollary 2.2.4. Let G be a group and g a left or right n-Engel element of G. For any $x \in G$, g^x is a left, respectively right n-Engel element

Proof. This follows immediately from Lemma 2.2.3 together with the observation that the conjugation map $\varphi_x : G \to G : g \mapsto g^x$ is surjective. □

Proof of Theorem 2.2.2. All three results will follow from the following observation. Let $x, y \in G$ we can write,

$$\begin{aligned}
[x,_{n+1} y] &= [[x, y],_n y] \\
&= [x^{-1}y^{-1}xy,_n y] \\
&= [(y^{-1})^x y,_n y] \\
&= [(y^{-1})^x y, y],_{n-1} y]
\end{aligned}$$

Note that

$$\begin{aligned}
[(y^{-1})^x y, y] &= ((y^{-1})^x y)^{-1} y^{-1} (y^{-1})^x yy \\
&= y^{-1}((y^{-1})^{x^{-1}} y^{-1} (y^{-1})^x y) y \\
&= [(y^{-1})^x, y]^y.
\end{aligned}$$

Moreover, for $a \in G$,

$$\begin{aligned}
[a^y, y] &= [y^{-1}ay, y] \\
&= y^{-1}a^{-1}yy^{-1}y^{-1}ayy \\
&= y^{-1}(a^{-1}y^{-1}ay)y \\
&= [a, y]^y.
\end{aligned}$$

And hence

$$\begin{aligned}
[x,_{n+1} y] &= [[(y^{-1})^x y, y],_{n-1} y] \\
&= [[(y^{-1})^x, y]^y,_{n-1} y] \\
&= [(y^{-1})^x,_n y]^y.
\end{aligned}$$

Where we used the second remark n times in the last line.

We are now ready to show the inclusions of the theorem. Assume that $y^{-1} \in R_n(G)$, then by Corollary 2.2.4 we find that $(y^{-1})^x \in R_n(G)$ for all $x \in G$. We then get that for all $x \in G$,

$$1 = [(y^{-1})^x,_n y]^y = [x,_{n+1} y].$$

This implies that y is a left $n+1$-Engel element which is what we wanted to show. The second statement follows immediately from the first.

Now assume that $y^{-1} \in R(G)$ and fix x. Since $(y^{-1})^x$ is still right Engel, there exists some n such that $[(y^{-1})^x,_n y] = 1$, which implies $[x,_{n+1} y] = 1$. Since we can choose such an n depending on x for any $x \in G$ we get that $y \in L(G)$. □

To say more about Engel elements we can look at certain cases separately. The following lemma from [Robinson, 2013] gives some properties of 2-Engel elements that we will use later. We will not give the proof, since it just the combination of certain commutator identities.

Lemma 2.2.5. Let G be a group and g a right 2-Engel element of G, let x, y be arbitrary elements of G. The following properties hold.

1. g is a left 2-Engel element,
2. The set of all conjugates of g, g^G, is abelian.
3. $[g, [x, y]] = [g, x, y]^2$

2.3 Engel Groups

The first mention of a concept that in modern terms would be called an Engel group is found in a text by Burnside from 1902 [Burnside, 1902], in which he talks about groups in which every two conjugates commute. The following lemma shows that this is indeed a characterization of 2-Engel groups.

Lemma 2.3.1. [Bussman, 2010, Lemma 1.1] A group is 2-Engel if and only if every two elements that are conjugates commute.

Proof. Let a and b be conjugates, then there exists x such that $a^x = b$. The fact that a and b commute is equivalent to stating that $[a^x, a] = 1$.

$$\begin{aligned} [a^x, a] &= [(aa^{-1})(x^{-1}ax), a] \\ &= [a[a, x], a] \\ &= [a, a]^{[a,x]}[[a, x], a] \\ &= [[x, a]^{-1}, a] \\ &= [[x, a], a]^{[x,a]^{-1}} \end{aligned}$$

Where we used the identity $[xz, y] = [x, y]^z[z, y]$ in the third line, and the identity $[x^{-1}, y] = [x, y]^{x^{-1}}$ in the fifth line.

It is clear that $[a^x, a] = 1$ if and only if $[x, a, a] = 1$. □

This is not the only characterization of 2-Engel groups that exist. Combining theorems from [Kappe, 1961], [Traustason, 2011] and [Robinson, 2013] and the lemma above we get the following theorem.

Theorem 2.3.2. *Let G be a group, the following statements are equivalent.*

(i) G is a 2-Engel group,

(ii) Every element of G commutes with its conjugates,

(iii) The centralizer $C_G(x)$ is a normal subgroup of G for all $x \in G$,

(iv) Each maximal abelian subgroup of G is normal in G,

(v) Each subgroup of G that is generated by two distinct elements is nilpotent of class at most 2,

Proof.

(i) $\Leftrightarrow$ (ii) This is exactly Lemma 2.3.1.

(i) $\Rightarrow$ (iii) Let G be a 2-Engel group, and $g \in G$. Take $c \in C_G(g)$, and $x \in G$. Consider

$$\begin{aligned}[g, c^x] &= g^{-1}x^{-1}c^{-1}xgx^{-1}cx \\ &= g^{-1}x^{-1}c^{-1}x(cc^{-1})gx^{-1}cx \\ &= g^{-1}(x^{-1}c^{-1}xc)g(c^{-1}x^{-1}cx) \\ &= [g, [c, x]]\end{aligned}$$

By Lemma 2.2.5

$$[g, [c, x]] = [g, c, x]^2 = [1, x]^2 = 1.$$

Hence c^x lies in the centralizer of g, and $C_G(g) \lhd G$.

(iii) $\Rightarrow$ (iv) Let M be a maximal abelian subgroup of G. Since M is maximal abelian, $C_G(M) = M$. We find that $M = \cap_{m \in M} C_G(m)$, and by (iv) we know that $C_G(m)$ is normal for all $m \in M$, hence M as an intersection of normal subgroups is normal.

(iv) $\Rightarrow$ (v) Take $x, y \in G$, and let X and Y be the maximal abelian subgroups of G containing x and y respectively. By (iv) $X \lhd G$ and $Y \lhd G$, now consider XY. It is clear that $1 \lhd X \lhd XY$ is a central series for XY, implying that it is nilpotent of class at most 2. Since $\langle x, y\rangle$ is a subgroup of XY it is also nilpotent of class at most 2.

(v) $\Rightarrow$ (i) Let $a, b \in G$, since $H := \langle a, b\rangle$ is nilpotent of class at most 2, $[[H, H], H] = 1$ and hence definitely $[[a, b], b] = 1 = [[b, a], a]$. Hence a, b are 2-Engel (both left and right) and since they were arbitrarily chosen, G is 2-Engel. □

Some more equivalent characterizations of 2-Engel groups exist, we give them here without proof.

Theorem 2.3.3. *Let G be a group, the following statements are equivalent.*

(i) *G is a 2-Engel group,*

(ii) *The group satisfies $[x, y, z] = [y, z, x]$, $[x, y, z]^3 = 1$ and $[x, y, z, t] = 1$ for all $x, y, z, t \in G$,*

(iii) *The commutator identity $[x, [y, z]] = [x, y, z]^2$ holds for all $x, y, z \in G$,*

(iv) *The commutator identity $[x, y, z] = [x, z, y]^{-1}$ holds for all $x, y, z \in G$.*

The study of n-Engel groups for $n \geq 3$ is a lot harder and in general less is known about them. Heineken showed that 3-Engel groups in locally nilpotent [Heineken, 1961], and much later Havas and Vaughan-Lee showed that this is also the case for 4-Engel groups by using computational methods [Havas and Vaughan-Lee, 2005]. For $n \geq 5$ this remains an open question.

2.4 Engel Structures in Finite Groups

We now focus on the case of finite groups. Imposing this constriction makes a big difference in the study of Engel groups, one main result that shows this difference is the following theorem, which was first proven by Zorn [Zorn, 1936].

Theorem 2.4.1. *Every finite Engel group is nilpotent.*

The standard way to show this theorem is to make use of Gruenberg's theorem [Gruenberg, 1953], which states that every finitely generated soluble Engel group is nilpotent. This is, however, not the original way Zorn proved this result, as Gruenberg's theorem came later. We will not prove this result separately as it will follow from the much more general theorem by Baer (Theorem 3.3.2). The theorem states that any finite group that is generated by Engel elements is nilpotent, we will prove it in the next chapter. Since every element in an Engel group is an Engel element, Zorn's Theorem immediately follows from this result.

This theorem could lead us to think that the study of Engel groups is not interesting, as it seems to coincide with that of nilpotent groups, but this is not the case. There exist infinite Engel groups that are not nilpotent, we will give an example of one next.

Example 2.4.2. Consider a set of groups G_i indexed by all natural numbers and let G_i be nilpotent of exactly class i. Now let G be the direct sum of all these groups,

$$G = \bigoplus_{i=1}^{\infty} G_i.$$

We claim that G is an Engel group. Take any two elements $x, y \in G$, and let n be the index such that x_m and y_m for all $m > n$ are zero. We can see the elements in the finite direct sum ${}_nG = \bigoplus_{i=1}^{n} G_i$. which is nilpotent of class n. Hence $[x,_n y] = 1$. This group is non-nilpotent, indeed, say it were nilpotent of some class c, then G_{c+1}

as a subgroup should have nilpotency class smaller than c (by Lemma 1.2.7 (1)), but this is a contradiction to how we defined G_{c+1}. $\triangle$

It is even possible to find non-nilpotent n-Engel groups for certain n, although their construction is rather complex. One example is given in a text by P.M. Cohn [Cohn, 1955].

Chapter 3

Fitting Subgroup and Baer's Theorem

In this section we look at a first general characterization of Engel elements. We start by introducing the Fitting subgroup of a group and related notions, and then we will look at the connections with the left Engel elements. The definitions, theorems, and proofs in this section are collected from [Isaacs, 2008], [Huppert, 1967], [Robinson, 2012], and [Kurzweil and Stellmacher, 2004].

3.1 The Fitting Subgroup

We first define the Fitting subgroup for arbitrary groups. This subgroup was named after Hans Fitting, a German mathematician.

Definition 3.1.1. Let G be a group, its *Fitting subgroup* is defined as the subgroup generated by all its nilpotent normal subgroups. It is denoted by $F(G)$.

When the group G is finite we have the following useful characterizations.

1. Define the p-core of a group G, denoted $O_p(G)$, to be the largest normal p-subgroup[1] of G. Then
$$F(G) = \prod_{p \text{ prime}} O_p(G).$$
2. $F(G)$ is the unique largest nilpotent normal subgroup of G.

The proof of the first characterization uses results about Sylow subgroups, which we will not cover here. The proof can be found in [Isaacs, 2008, Corollary 1.28]

The fact that the second characterization is equivalent follows directly from Fitting's Theorem, which we show next, based on the proof in Robinson, [Robinson, 2012, Theorem 5.2.8].

[1] A p-subgroup is a subgroup in which the order of every element is a power of p.

Theorem 3.1.2. *Assume H and K are normal nilpotent subgroups of a group, then $\langle H, K\rangle$ is again a normal nilpotent subgroup.*

Proof. We start by showing a commutator relation for subgroups, which will be useful in the proof. We claim that for any three normal subgroups, $H, K, M \lhd G$ it holds that $[HK, M] = [H, M][K, M]$, and $[H, KM] = [H, K][H, M]$.

First note that a commutator of two normal subgroups is again normal, let H and K be normal subgroups and $h \in H, k \in K$ and $g \in G$.

$$[h, k]^g = g^{-1}h^{-1}k^{-1}hkg = g^{-1}h^{-1}gg^{-1}k^{-1}gg^{-1}hgg^{-1}kg = [h^g, k^g] \in [H, K].$$

We will check the equality of subgroups by looking at the generators. Take h, k, m in H, K, M respectively and consider $[hk, m]$, by Lemma 1.2.12 this is equal to $[h, m]^k[k, m]$ and by the previous remark this lies in $[H, M][K, M]$. The other inclusion is trivial since $[H, M] \leq [HK, M]$ and $[K, M] \leq [HK, M]$. The other statement can be shown in exactly the same way.

For brevity we will write $C := \langle H, K\rangle$. We will show that the i-th term of the lower central series of C is equal to the product of all commutators of the form $[X_1, X_2, \ldots, X_i]$ where $X_i = H$ or K. For $i = 1$ we get $HK = H \cdot K$ which is clear. We will proceed by induction. Assume the statement holds for $\gamma_i(C)$, by definition $\gamma_{i+1}(C) = [\gamma_i(C), C]$. By the commutator identity proven earlier we find that this is equal to $[\gamma_i(C), H][\gamma_i(C), K]$ which proves the statement.

Assume that the nilpotency classes of H and K are c and d respectively, and let $i = c + d + 1$. The i-th term of the lower central series is, by what we have seen before, the product of all commutators of the form $[X_1, X_2, \ldots, X_i]$, we will show that all these terms are trivial. By our choice of i at least $c + 1$ or $d + 1$ must be equal to H or K respectively. Without loss of generality, assume that there are $c + 1$ terms $X_i = H$. If these are the first $c + 1$ terms we are done, since $\underbrace{[H, H, ..., H]}_{c+1 \text{ times}} = 1$ because H is nilpotent of class c. For the case where the $c + 1$ terms H do not appear consecutively at the left we note that $[H, K] \leq H$ since $[h, k] = h^{-1}k^{-1}hk = h^{-1}h^k \in H$ because H is a normal subgroup. Similarly $[H, H, \ldots, H, K] \leq [H, H, \ldots, H]$ for any amount of H's. This allows us to cancel the terms K in the commutator to still arrive at 1. Since all terms that make up $\gamma_{i+1}(C)$ are trivial, it is trivial itself and hence C is nilpotent of class at most $c + d$. □

Example 3.1.3. To gain intuition into the notion of the Fitting subgroup, we will look at some examples concerning some well-known groups. We will use the characterizations above.

1. When G is a nilpotent group, the Fitting subgroup of G is equal to G as the group itself is a normal nilpotent subgroup. In particular the Fitting subgroup of every abelian group is equal to the group itself.
2. For the symmetric group S_n we start by looking at the cases where n is small.
   (a) S_2 : We know that $S_2 \cong \mathbb{Z}_2$ which is abelian and therefore $F(S_2) = S_2$.

(b) S_3 : The normal subgroups of S_3 are $1, A_3$ and S_3. Since $Z(S_3) = 1$, S_3 is not nilpotent, but $A_3 \cong \mathbb{Z}_3$ is, hence $F(S_3) = A_3$.

(c) S_4 : The normal subgroups of S_4 are $1, V, A_4$ and S_4, where $V = \{(), (12)(34), (13)(24), (14)(23)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Both A_4 and S_4 have trivial center and are hence non nilpotent. V is isomorphic to the Klein four-group, which is nilpotent. We find that $F(S_4) = V$.

(d) $S_n, n \geq 5$: When we look at larger symmetric groups, it is possible to show that the only normal subgroups are $1, A_n$ and S_n. Both A_n and S_n have trivial center and are therefore non-nilpotent, from which we get that the Fitting subgroup of S_n for $n \geq 5$ is trivial.

3. The dihedral group D_n is defined as the group of rotational and reflection symmetries of a regular n-gon. A dihedral group is nilpotent if and only if $n = 2^k$ for some positive integer k. Let r and s be the generators of D_n corresponding to a rotation and a reflection respectively.

   (a) $n = 2^i$, by the previous remark $F(D_n) = D_n$.

   (b) $n = 2^i m, m$ odd. The normal subgroups of D_n are $\langle r^d \rangle$ for $d \mid n$, D_n and, if n is even, $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$ see for example [Conrad, 2009, Theorem 3.8]. Since $\langle r \rangle \cong \mathbb{Z}_n$, it is nilpotent and hence $\langle r \rangle \subseteq F(D_n)$. Since no other normal subgroup contains $\langle r \rangle$, we find that $F(D_n) = \langle r \rangle$.

4. Consider the case where G is a finite p-group. It is clear that the p-core $O_p(G)$ of this group is the entire group. By the first characterization we find that the Fitting subgroup is equal to the entire group. Note that since finite p-groups are nilpotent, this is also an example of the first kind.

$\triangle$

Example 3.1.4. We will now work out a less trivial example of a Fitting subgroup, that is based on Lemma 4.3 in [Lins de Araujo and Rego, 2020].

Let R be a finite integral domain, i.e. a finite ring without nonzero zero divisors, and additionally assume R has at least two units. Consider the group $\mathbf{B}_n(R)$ defined to be the subgroup of $GL_n(R)$ consisting of all upper triangular matrices. These are the matrices $(a_{ij}) \in \mathrm{Mat}_n(R)$ such that $a_{ij} = 0$ if $i > j$. A subgroup of this subgroup is the unitriangular matrix group $\mathbf{U}_n(R)$, which consists of elements in $\mathbf{B}_n(R)$ that have 1 on the diagonal, or $a_{ii} = 1$. Lastly we define $\mathbf{Z}_n(R)$, it is the group consisting of multiples of the identity, $\mathbf{Z}_n(R) = \{u \cdot \mathbf{1} \mid u \in R^\times\}$. It can be shown that $\mathbf{Z}_n(R)$ is the center of $\mathbf{B}_n(R)$.

We then define the *projective upper triangular group* $\mathbb{P}\mathbf{B}_n(R)$ as the quotient $\mathbf{B}_n(R)/\mathbf{Z}_n(R)$. Note that taking the quotient $\mathbf{U}_n(R)/\mathbf{Z}_n(R)$ has no effect, so we can still see $\mathbf{U}_n(R)$ as a subgroup of $\mathbb{P}\mathbf{B}_n(R)$. We can find the Fitting subgroup of $\mathbb{P}\mathbf{B}_n(R)$ using the following lemma.

Lemma 3.1.5. [Lins de Araujo and Rego, 2020, Lemma 4.3] Let $n \in \mathbb{N}, n \geq 2$ and let R be an integral domain. Then $\mathbf{U}_n(R)$ is the unique largest nilpotent normal subgroup of $\mathbb{P}\mathbf{B}_n(R)$.

Using the first characterization for Fitting subgroups we find that $F(\mathbb{P}\mathbf{B}_n(R)) = U_n(R)$ for any $n \geq 2$. Examples of the latter are the finite fields of p elements with $p \geq 2$. △

Remark 3.1.6. The characterization that states that the Fitting group is the unique largest nilpotent normal subgroup unfortunately does not hold for the general infinite case. This will turn out to be one of the major flaws of the Fitting subgroup, we will fix this by introducing the Hirsch-Plotkin radical in a later chapter.

It is quite easy to construct an example of a group with a non-nilpotent Fitting group, by looking at Example 2.4.2. Let G_i be a nilpotent group of class i for all positive integers i, and let G be the direct sum of these groups, i.e.

$$G = \bigoplus_{i=1}^{\infty} G_i,$$

of which the elements are of the form $(g_n)_{n\in\mathbb{N}}$ where $g_n \in G_n$ and $g_n = 1_{G_n}$ for all but finitely many n. We have already shown that this group is non nilpotent before, but the Fitting subgroup equals the entire group, as any G_i can be seen as a normal nilpotent subgroup.

3.2 The Fitting Series and Fitting Height

We can use the Fitting subgroup to define a series similar to the upper central series for nilpotent groups.

Definition 3.2.1. The *upper Fitting series* of a group G is defined recursively by setting $F_0(G) = 1$ and by defining $F_k(G)$ such that

$$\frac{F_k(G)}{F_{k-1}(G)} = F\left(\frac{G}{F_{k-1}(G)}\right).$$

In general a Fitting series of a group is defined as a *subnormal series* with nilpotent quotients, i.e. a series

$$1 \lhd F_1 \lhd F_2 \lhd \ldots \lhd F_k = G$$

where F_i does not necessarily have to be a normal subgroup of G, and such that F_{i+1}/F_i is nilpotent. We will often just say "the Fitting series" when we actually want to say "the upper Fitting series".

Example 3.2.2. As an example we look at the symmetric group on 3 elements, S_3. $F_0(S_3) = 1$, by definition, and $F_1(S_3)/1 = F(S_3/1) = A_3$ by Example 3.1.3 part 2 (b). The quotient of S_3 by A_3 is $\mathbb{Z}_2$ and hence $F(S_3/A_3) = S_3/A_3$ implying that $F_2(S_3) = S_3$. We get the following series,

$$1 \leq A_3 \leq S_3.$$

△

For arbitrary groups it is not necessary that the Fitting series ever reaches the group. We can however prove that this is the case for finite soluble groups, and even better, that this is a necessary condition for a finite group to be soluble.

Theorem 3.2.3. A finite group G is soluble if and only if the Fitting series reaches the whole group.

Proof. Suppose G is soluble, to ease notation we will write F_k for $F_k(G)$ whenever it is clear which group G we use. There exists an integer k such that $F_k = F_{k+1}$ since G is finite. From this it follows that $F(G/F_k) = F_{k+1}/F_k = 1$. Note that since G is soluble, G/F_k is soluble too.
Observe that for an arbitrary nontrivial soluble group H, we can find a nontrivial abelian normal subgroup of H, indeed let

$$\{1\} = H_0 \leq H_1 \leq \ldots \leq H_n = H$$

be a normal series in which all factors are abelian (this exists because of the solubility of the group). Then H_1 is a normal subgroup which is abelian since $H_1 = H_1/1 = H_1/H_0$ is abelian. Since abelian groups are nilpotent, we have shown that there exists a nontrivial nilpotent normal subgroup, and hence that the Fitting subgroup of a nontrivial soluble group is nontrivial.

Therefore we get that $G/F_k = 1$ or $G = F_k$.

Now suppose that G has a finite Fitting series. We will prove solubility by induction on the Fitting series. It is trivial for $F_0 = 1$. Assume F_k is soluble, since $F(G/F_k)$ is nilpotent, it is soluble, this implies that F_{k+1}/F_k is soluble, and hence also F_{k+1} is soluble. Since there exists an integer n such that $F_n = G$ it follows that G is soluble. □

Using the Fitting series, we can define a new length notion on a soluble group.

Definition 3.2.4. Let G be a soluble group, the smallest integer h such that $F_h(G) = G$ is called the *Fitting height* of G and is denoted by $h(G)$.

We want to extend the notion of the Fitting series from finite soluble groups to general groups. To do this we will first need to introduce some new concepts.

Definition 3.2.5. A group G is called *quasisimple* if

1. it is perfect, i.e. the commutator subgroup $[G, G]$ coincides with G,

2. its inner automorphism group $\mathrm{Inn}(G)$ is simple.

We can find an equivalent statement for the second part of the definition by considering the map $\varphi : G \to \mathrm{Aut}(G) : g \mapsto g(x) = x^g$. The kernel of this map is exactly the center $Z(G)$ and its image is the inner automorphism group, hence by the first isomorphism theorem

$$\frac{G}{Z(G)} \cong \mathrm{Inn}(G).$$

Hence the second statement is equivalent to $G/Z(G)$ being simple.

Example 3.2.6. The smallest example of a quasisimple group is the alternating group on 5 elements A_5. We first show that the center of A_5 is trivial. Let σ be any nontrivial permutation in A_5, and let $\sigma(a) = b$ with $a \neq b$. Choose c, d different from a and b, now $\sigma \cdot (bcd)$ sends a to b and $(bcd) \cdot \sigma$ sends a to c. Hence for every nontrivial permutation we can create an element that does not commute with it. Note that this reasoning shows that A_n is centerless for all $n \geq 4$. It is possible to show that A_5 is simple using a counting argument on the conjugacy classes. This shows that $A_5/Z(A_5)$ is simple.

Using the fact that A_5 is simple we find that $[A_5, A_5] = A_5$ since the derived subgroup is a normal subgroup of A_5 and it is nontrivial since A_5 is not abelian, implying that the group is perfect.

As it is possible to show that A_n is simple for $n \geq 5$, this argument holds for all $n \geq 5$ and hence A_n is quasisimple when $n \geq 5$. △

Example 3.2.7. Although their names are similar quasisimple groups need not necessarily be simple, and vice versa.

Consider $\mathbb{Z}_7$ the cyclic group of order 7, this group is trivially simple but not quasisimple as the group is abelian and therefore not perfect.

As a counterexample for the other direction consider the special linear group of degree 2 over $\mathbb{F}_5$, $\mathrm{SL}(2,5)$. This is the subgroup of $\mathrm{GL}(2,5)$ consisting of the matrices with determinant equal to 1. It is possible to show that the subgroup H, defined as

$$H := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right\},$$

is the center of $\mathrm{SL}(2,5)$ and hence normal. This implies that $\mathrm{SL}(2,5)$ is not simple. The quotient $\mathrm{SL}(2,5)/H$ is isomorphic to A_5 and hence simple. The fact that $\mathrm{SL}(2,5)$ is perfect requires a bit more work to show, a proof is given in Chapter XIII.8 in [Lang, 2012].

It is possible to check both of these examples using GAP, the `IsSimple` command is included in GAP and returns true or false, the code for the `IsQuasisimple` command is new and can be found in the Appendix. For $\mathbb{Z}_7$ we find the following.

```
gap> Z7 := CyclicGroup(7);
<pc group of size 7 with 1 generators>
gap> IsSimple(Z7);
true
gap> IsQuasisimple(Z7);
false
```

For $\mathrm{SL}(2,5)$ we find the following.

```
gap> IsSimple(SL(2,5));
false
gap> IsQuasisimple(SL(2,5));
true
```

△

We will need an extension of the concept of normality to go further.

Definition 3.2.8. A subgroup H of a group G is called a *subnormal* subgroup if there exists a series of subgroups

$$H = H_0 \leq H_1 \leq \ldots \leq H_n = G$$

such that $H_i \lhd H_{i+1}$ for all $i \in \{0, \ldots, n-1\}$.

Note that in particular every normal subgroup is also subnormal.

Next we use the concept of quasisimple groups to define a component of a group.

Definition 3.2.9. A *component* of a group is a subnormal quasisimple subgroup.

Example 3.2.10. A_5 is a component of S_5, since it is normal in A_5 and quasisimple by the previous example. △

The components of a group satisfy a useful property.

Theorem 3.2.11. *Any two components of a group commute.*

We will prove this theorem by using the following lemma which is based on [Kurzweil and Stellmacher, 2004].

Lemma 3.2.12. Let K be a component of a group G and let U be a subnormal subgroup of G, then $K \leq U$ or $[K, U] = 1$.

Proof. The case where $U = G$ implies $K \leq U$. So assume $U \neq G$. We start with the case where $K = G$. Because K is a component, $K/Z(K)$ is simple. Moreover, because U is assumed to be subnormal, we get that $U \leq Z(K)$ which implies $[K, U] = 1$.

We will prove the other cases by induction on the order of G. Since we can exclude the cases mentioned earlier there exist normal subgroups M and N of G such that $K \leq M < G$ and $U \leq N < G$. Take elements $k \in K$ and $u \in U$.

Note that $[k, u] = k^{-1}u^{-1}ku = k^{-1}(u^{-1}ku) \in M$ and analogously also an element of N, which implies that $U_1 := [K, U] \leq M \cap N$. Denote $G_1 := N_M(U_1)$ the normalizer of U_1 in M.

It is a well-known that if X, Y are subgroups of some group G, then $[X, Y]$ is normal in $\langle X, Y \rangle$, the proof of this statement consists of simple commutator identities. In particular, U_1 is a normal subgroup of $\langle K, U \rangle$, and hence K lies in G_1.

As K is also a subnormal subgroup of any subgroup of G, K is a component of G_1. By construction U_1 is a normal subgroup of G_1. By induction we find that

$[U_1, K] = 1$ or $K \leq U_1$. We will start by looking at the first case, $[U_1, K] = [U, K, K] = 1$, but also $[K, U, K] = 1$. Using the Three Subgroups Lemma 1.2.13 we find that $1 = [K, K, U] = [K', U] = [K, U]$, where in the last step we used that K is quasisimple and therefore perfect.

We now consider the case $K \leq U_1$. Since $[K, U] \leq N$, it follows that K is a component of N and hence $K \leq N$. Since N is a strict subgroup of G, the result follows by induction. □

Proof of Theorem 3.2.11. Assume that K_1 and K_2 are components of G, by Lemma 3.2.12 we either have $[K_1, K_2] = 1$ or $K_1 \leq K_2$. Assume that the second case holds, then $[K_2, K_1] = [K_1, K_2] \neq 1$ implies that $K_2 \leq K_1$. We find that $K_1 = K_2$, implying that the commuting is trivial. □

Definition 3.2.13. The *layer* $E(G)$ of a group G is the subgroup generated by all components of the group.

Example 3.2.14. Let us look at a concrete example of a layer of a group. Consider S_5, the symmetric group on 5 elements. Earlier we showed that A_5 is a component of S_5 and hence will be contained in the layer. As A_5 is the smallest quasisimple group, no other smaller subgroup will be contained in the layer. The group itself is not quasisimple, because it is centerless. We find that $S_5/Z(S_5) \cong S_5$ which is not simple, because it has A_5 as a non-trivial normal subgroup. We find that A_5 is the only component, and hence $E(S_5) = A_5$. △

Combining the concept of a layer with that of a Fitting group we can define a more general Fitting group that has the properties we need to later define a series.

Definition 3.2.15. For a group G the generalized Fitting group is defined as

$$F^*(G) = F(G)E(G).$$

Note that every nontrivial subgroup H of a soluble group is soluble and hence cannot be perfect since $[H, H] < H$ in a soluble group. This implies that a soluble group does not have any nontrivial quasisimple subgroups, implying that the layer is trivial. Therefore for a soluble group G, we have $F(G) = F^*(G)$.

It can be shown that the generalized Fitting group is a self-centralizing group, i.e. the centralizer $C_G(F^*(G))$ of the generalized Fitting group is contained in $F^*(G)$, [Förster, 1985]. This implies that the generalized Fitting subgroup of any group is non trivial, as this would imply $C_G(F^*(G)) = C_G(1) = G$ which is a contradiction.

As in Definition 3.2.1 we define a series using this subgroup.

Definition 3.2.16. The *generalized Fitting series* of a group G is defined recursively by setting $F_0^*(G) = 1$ and by defining $F_k^*(G)$ such that

$$\frac{F_k^*(G)}{F_{k-1}^*(G)} = F^*\left(\frac{G}{F_{k-1}^*(G)}\right).$$

We can now extend the notion of Fitting height of soluble groups to a new notion defined on all groups.

Definition 3.2.17. The smallest number $h \in \mathbb{N}$ such that $F_h^*(G) = G$ is called the *generalized Fitting height* of G, which we will denote by $h^*(G)$.

The generalized Fitting series, as opposed to the Fitting series, reaches the whole group for any finite group. This is an easy corollary from the fact that the generalized Fitting subgroup is nontrivial. By finiteness of G there must be some index k such that $F_k^* = F_{k+1}^*$, if $F_{k+1}^* = H \leq G$, then $F^*(G/H) = 1$, which is only possible if $G/H = 1$ or $H = G$.

Note that when a group is soluble $F_1^*(G) = F^*(G) = F(G) = F_1(G)$. Repeating this argument by induction, we find that $F_k^*(G) = F_k(G)$ for all k, and hence $h(G) = h^*(G)$.

Example 3.2.18. To finish this section we look at an example of a Fitting series and generalized Fitting series using GAP. The `FittingSeriesOfGroup` and `GeneralizedFittingSeriesOfGroup` are new commands that were implemented by working with the definitions. The code can be found in the Appendix.

1. We start by looking at the Fitting series of S_4, which turns out to have length 3. Since this group is soluble, it makes sense that the Fitting series reaches the group.

```
gap> FittingSeriesOfGroup(S4);
[ Group(()), Group([ (1,2)(3,4), (1,4)(2,3) ]), Group([
    (1,2)(3,4), (1,4)(2,3), (2,4,3) ]), Group([ (1,2),
  (3,4), (1,4)(2,3), (2,4,3) ]) ]
```

2. For any higher order symmetric group we lose solubility and hence we need to look at the generalized Fitting series. For S_5 we use GAP and find a generalized Fitting series of length 2

```
gap> GeneralizedFittingSeriesOfGroup(S5);
[ Group(()), Group([ (2,5,4), (1,4)(2,3) ]), Group([
  (1,4,5), (2,4,5), (3,4,5), (1,5) ]) ]
```

$\triangle$

3.3 Baer's Theorem

We are ready to look at the connection between the Fitting subgroup and Engel elements. The theorem that we will prove was originally stated and shown by Reinhold Baer in his 1957 work "Engelsche Elemente Noetherscher Gruppen" [Baer, 1957]. The proof in this text is based on the German version of the proof by Huppert in "Endliche Gruppen" [Huppert, 1967].

Theorem 3.3.1 (Baer's Theorem). *[Huppert, 1967, Satz III.6.15] Let G be a finite group, an element g of G belongs to the Fitting subgroup $F(G)$ if and only if it is a left-Engel element of G.*

The proof will rely on the following theorem.

Theorem 3.3.2. *Let G be a finite group that is generated by Engel elements. Then G is nilpotent.*

Before we prove this theorem, we will show a special case where we assume that G is already soluble. Later we will show that this condition can be dropped.

Theorem 3.3.3. *Let G be a finite soluble group that is generated by Engel elements, then G is nilpotent.*

We will resort to splitting up the proof in several lemma's.

Lemma 3.3.4. Let G be a finite non-nilpotent soluble group that is generated by Engel elements. Assume further that any group that is generated by Engel elements and has order strictly smaller than the order of G is nilpotent. Then G has a unique minimal normal nilpotent subgroup N. Moreover, N is abelian and G/N is nilpotent.

Proof. Assume that G has two minimal normal nilpotent subgroups N_1 and N_2. We know that G/N_i is nilpotent, since the quotient is still finite, soluble, and generated by Engel elements (cosets of Engel elements are Engel elements in the quotient since the projection map is a surjective homomorphism and Lemma 2.2.3), and by the minimality of G.

Now consider the map $\varphi : G \to G/N_1 \times G/N_2 : g \mapsto (gN_1, gN_2)$. Note that the kernel of φ is precisely $N_1 \cap N_2$, hence $\frac{G}{N_1 \cap N_2}$ is isomorphic to a subgroup of $G/N_1 \times G/N_2$. Because products and subgroups of nilpotent groups are nilpotent, $\frac{G}{N_1 \cap N_2}$ is nilpotent. However $N_1 \cap N_2$ is a normal subgroup of G and by minimality of N_1 and N_2 we find that the intersection is trivial, which implies that G is nilpotent, contradicting the assumption on G. Finally since G is soluble and by Lemma 1.2.11 we find that N is abelian.

□

Lemma 3.3.5. Assume the same conditions for G and smaller groups as in the previous lemma. Let N be the unique minimal normal nilpotent subgroup of G. There exists a (left or right) Engel element $g \in G$ such that $G = \langle C_G(N), g \rangle$.

Proof. The unicity of N is the result of the previous lemma. We can write

$$\overline{G} = G/N = \langle \overline{g_i} = g_i N \mid g_i \text{ Engel element}, i = 1, \ldots, k \rangle,$$

because G was assumed to be generated by Engel elements, we choose the generating set of Engel elements $\{\overline{g_1}, \ldots, \overline{g_k}\}$ such that k is as small as possible. Note that $\overline{G}$ is nilpotent as it is generated by left Engel elements and its order is strictly smaller than that of G. Because of the minimality of k

$$\overline{A} = \langle \overline{g_1}, \ldots, \overline{g_{k-1}} \rangle < \overline{G}.$$

Let $\overline{M}$ be a maximal subgroup of $\overline{G}$ containing $\overline{A}$. By Lemma 1.2.7(5), we find that $\mathrm{ncl}_{\overline{G}}(\overline{M}) = \overline{M}$ and hence that $\mathrm{ncl}_{\overline{G}}(\overline{A}) \leq \overline{M} < \overline{G}$. Using the equivalent definition of normal closure we find

$$\langle \overline{A}^{\overline{G}} \rangle = \langle \overline{g_1}^{\overline{h}}, \ldots, \overline{g_{k-1}}^{\overline{h}} \mid \overline{h} \in \overline{G} \rangle < \overline{G}$$

We define the subgroup B as

$$B = \langle g_1^h, \ldots g_{k-1}^h \mid h \in G \rangle.$$

It is easy to show that B is a normal subgroup of G. Assume that $B = G$, then taking the quotient by N would show that $\langle \overline{A}^{\overline{G}} \rangle = \overline{G}$, which is not the case. Hence B is a strict normal subgroup of G. Conjugation is an isomorphism and thus by Lemma 2.2.3 it also follows that B is generated by Engel elements, hence by minimality of G is B nilpotent. If $k = 1$, then $G = \langle N, g_1 \rangle$, and because $N \subset C_G(N)$ we are done. Now assume $k > 1$, then B is nontrivial and by minimality and uniqueness of N, $N \leq B$. Since B is nilpotent and by Lemma 1.2.7(5) we find that $N \cap Z(B) > 1$. Since the intersection of normal subgroups is again normal and by the minimality of N we find that $N \subset Z(B)$, this means that an element of B commutes with every element of N and hence $B \leq C_G(N)$. We then get that

$$G = \langle N, g_1, \ldots, g_k \rangle = \langle B, g_k \rangle = \langle C_G(N), g_k \rangle,$$

which is exactly what we wanted.

□

We finally have everything we need to prove Theorem 3.3.3.

Proof of Theorem 3.3.3. Assume that the statement in the theorem is false, in other words that there exists some group G that is soluble and generated by left Engel elements, but non nilpotent. We let G be a such a counterexample of minimal order, i.e. any soluble group generated by Engel elements of smaller order than the order of G is nilpotent.

Let g be the element we find after applying Lemma 3.3.5. We will prove a contradiction separately for the case g is left-Engel and g is right-Engel.

First assume that g is left-Engel. Let n be a nontrivial element in N. Let m be the minimal integer such that $[n,_{m+1} g] = 1$. Denote $n' = [n,_m g]$, then $[n', g] = 1$, which implies $n'g = gn'$ and since all elements in the centraliser of N commute with n' we get that $n' \in Z(G)$. We have found a nontrivial element which lies in the intersection of N and the center of the group. Combining Lemma 1.2.8 and the fact that G/N is nilpotent, we get that G is nilpotent contradicting the assumptions and proving the theorem.

Now assume that $g = g_k$ is a right-Engel element. Take $n \in N$ such that $n' = [g, n]$ is non trivial. Note that if we cannot find such N then we have that $N \leq Z(G)$ and we are done. Choose m large enough such that the following all holds,

$$1 = [g,_m gn] = [[g, gn],_{m-1} gn] = [[g, n],_{m-1} gn] = [n',_{m-1} g].$$

Where the third equality holds since

$$[g, gn] = g^{-1}(gn)^{-1}ggn = g^{-1}n^{-1}g^{-1}ggn = [g, n].$$

We now take the largest integer j such that $[n',_j g] \neq 1$. Note that $[n',_j g]$ is an element of $Z(G)$ since $[n',_{j+1} g] = 1$ by maximality of j. Since N is normal $[n',_j g] \in N$.

By similar reasoning as before we can conclude that G is nilpotent, hence we are done. $\square$

We now extend this result. To show Theorem 3.3.2, we want to remove the solubility condition, and show that the result holds for arbitrary finite groups G. To simplify reading we will again break up the proof of this theorem in several lemma's.

We first introduce some notation. Let H be a subgroup of G then we will write

$$e(H) = \{g \mid g \in H,\ g \text{ is an Engel element of } G\}.$$

Lemma 3.3.6. Let G be a finite group that is generated by Engel elements and not nilpotent. Further assume that any group of smaller order than the order of G that is generated by Engel elements is nilpotent. Let K be a strict subgroup of H which itself is a strict subgroup of G, and assume that K and H are generated by the Engel elements of G they contain. Then there exists an element $x \in e(H)$ such that $x \in N_H(K) \setminus K$.

Proof. Since H is generated by Engel elements and is strictly smaller than G it is nilpotent.
We define a series starting at K by iteratively taking the normaliser.

$$K = K_1 \lhd N_H(K_1) = K_2 \lhd \ldots \lhd N_H(K_{m-1}) = K_m = H.$$

Note that this series is strictly increasing by Lemma 1.2.7(6), and reaches H by finiteness. Take the biggest K_i such that $e(K_i) = e(K)$ and pick an element $x \in e(K_{i+1}) \setminus e(K)$. By normality of K_i in K_{i+1} we get that $e(K)^x = \{g^x \mid g \in e(K)\} = e(K)$, and hence

$$K^x = \langle e(K)^x \rangle = \langle e(K_i)^x \rangle = \langle e(K_i) \rangle = \langle e(K) \rangle = K.$$

This implies that $x \in N_H(K)$ and hence $x \in N_H(K) \setminus K$. $\square$

Lemma 3.3.7. Let H_1 and H_2 be distinct maximal subgroups of a non-soluble group G such that $H_i = \langle e(H_i) \rangle$ for $i = 1, 2$, then $e(H_1 \cap H_2) = \{1\}$.

Proof. We pick H_1 and H_2 according to the prerequisites explained before such that $e(H_1 \cap H_2)$ is largest. Define $D = H_1 \cap H_2$ and $K = \langle e(D) \rangle$. Assume towards contradiction that $K > 1$. By Lemma 3.3.6, there exist elements $g, h \in e(G)$ such that $g \in N_{H_1}(K) \setminus K$ and $h \in N_{H_2}(K) \setminus K$. Now define $H_3 = \langle g, h, e(D) \rangle$, which is a group generated by Engel elements. If $H_3 = G$, then K is a nontrivial normal subgroup of G by the choice of g and h, however since K is nilpotent, this would imply G to be soluble which is a contradiction. So $H_3 < G$.

We can now write,

$$e(D) \subset \{g, e(D)\} \subseteq e(H_1 \cap H_3).$$

By maximality of D, this implies that $H_1 = H_3$. Similarly we get,

$$e(D) \subset \{h, e(D)\} \subseteq e(H_2 \cap H_3),$$

which implies $H_2 = H_3$. However this contradicts the assumption that H_1 and H_2 are distinct, and hence $e(D) = e(H_1 \cap H_2) = 1$.

□

We are now ready to prove Theorem 3.3.2.

Proof of Theorem 3.3.2. Because of Theorem 3.3.3 it suffices to show that G is soluble. We will show this by contradiction.

Assume that G is a finite group, generated by left Engel elements, but that G is not soluble. Further assume that G is minimal as a counterexample, i.e. that any smaller order group that is generated by Engel elements is soluble.

Let H be a maximal subgroup of G such that $H = \langle e(H) \rangle$. By minimality of G as a counterexample, we get that H is soluble and therefore not a normal subgroup of G as this would imply solubility of G. Pick an element $h \in e(H)$ and $g \in G$ such that $k = h^g \notin H$. Assume towards contradiction that $h^k \in N_G(H)$. Then $H \lhd \langle H, h^k \rangle$, and since H is not a normal subgroup of G, $\langle H, h^k \rangle < G$. Since conjugation is an isomorphism, h^k is an Engel element of G by Lemma 2.2.3. By maximality of H, we get that $h^k \in H \cap H^k$. We now have a nontrivial Engel element of G in the intersection of H and H^k. By Lemma 3.3.7 this implies that $H = H^k$, which contradicts the choice of k.

Hence $h^k \notin N_G(H)$ and therefore also $[h, k] = h^{-1}h^k \notin N_G(H)$. Suppose k and h are left-Engel, then for large enough n,

$$[k,_n h] = 1.$$

Similarly if k and h are right-Engel, there exists n such that,

$$[k,_n h] = 1,$$

where the role of the elements has switched. Since k and h are of the same Engel element type, these are the only cases, and we can always find such n. We now choose i such that,

$$[k,_{i-1} h] \notin N_G(H) \text{ and } [k,_i h] \in N_G(H).$$

Note that we can always find such i since $[k,_1 h] = [h, k]^{-1} \notin N_G(H)$ and $[k,_n h] = 1 \in N_G(H)$. Denote $l = [k,_{i-1} h]$, we get $(h^l)^{-1}h \in N_G(H)$ since

$$[k,_i h] = [l, h] = (h^l)^{-1}h,$$

and also $h^l \in N_G(H)$, since $h \in N_G(H)$. As before we consider the group generated by H and h^l, $\langle H, h^l \rangle$ of which H is a normal subgroup. Analogously to before we find that $h^l \in H \cap H^l$ and by the second part of the proof, $H = H^l$, which implies $l \in N_G(H)$ which again contradicts the choice of l.

We have exhausted all possibilities, and hence the original assumption that G is not soluble must be false, proving the statement. □

Using the previous theorem we can now prove the main result of this section.

Proof of Theorem 3.3.1. Assume g is an element of the Fitting group $F(G)$, let us show that G is left Engel. Let x be an element in G.
Since the Fitting group is normal, it holds that $x^{-1}gx \in F(G)$ and hence also $[x, g] = x^{-1}g^{-1}xg \in F(G)$. Since $F(G)$ is also nilpotent, it holds that the lower central series reaches 1, say $\gamma_{n+1}(F(G)) = 1$, then

$$[x,_n g] \in \gamma_{n+1}(F(G)) = 1.$$

Since x was chosen arbitrarily it follows that g is a left-Engel element of G.

Let us now show that every left Engel element is contained in the Fitting subgroup. Consider the subgroup H of G defined as follows,

$$H = \langle\, g \in G \mid g \text{ is a left-Engel element of } G \,\rangle.$$

By Lemma 2.2.3 we know that conjugates of Engel elements remain Engel. It follows that H is normal since if $g = g_1^{\varepsilon_1} g_2^{\varepsilon_2} \ldots g_n^{\varepsilon_n}$ is an element of H with all g_i left-Engel and $\varepsilon_i \in \{1, -1\}$, then $xgx^{-1} = (xg_1^{\varepsilon_1}x^{-1})(xg_2^{\varepsilon_2}x^{-1}) \ldots (xg_n^{\varepsilon_n}x^{-1})$ is again the product of left-Engel elements and inverses thereof. By Theorem 3.3.2 it is clear that H is nilpotent. Since the Fitting group is defined to be generated by all nilpotent normal subgroups it follows that $H \subseteq F(G)$, which proves the other direction. □

3.4 Generalizations on Fitting Height

Two generalizations of Baer's theorem were recently shown by Khukhro and Shumyatsky [Khukhro and Shumyatsky, 2017] and Guralnick and Tracey [Guralnick and Tracey, 2020]. Before we will state them we need some more new notation, first introduced in [Khukhro and Shumyatsky, 2017].

Notation 3.4.1. Let G be a group and $x \in G$, define

$$E_{G,n}(x) := \langle \{[g,_n x] \mid g \in G\} \rangle.$$

In this notation Baer's theorem corresponds to the following statement: an element $x \in G$ belongs to the Fitting subgroup $F(G)$ if and only if $E_{G,k}(x) = 1$ for some positive integer k. Indeed x being a left Engel element implies that $[g,_k x] = 1$ for some k possibly depending on g, however since G is finite, we can take the maximum over all g's and find a fixed k.

Remark 3.4.2. We can use this group $E_{G,n}(x)$ or rather some size parameter for this group as a notion for "how close to being left Engel" a certain element is. Another way we could describe this idea is by using the (generalized) Fitting series, as Baer's Theorem tells us that an element is left Engel if it lies in the first term of the Fitting series. Therefore we could say that an element that appears in a low-index term of the Fitting series is "closer to being left Engel" as compared to an element only appearing in a higher-index term.

It turns out that both of these ideas are somewhat equivalent when we use the (generalized) Fitting height as the size parameter for $E_{G,n}(x)$. This idea is the content of the following two generalizations.

Theorem 3.4.3. *[Khukhro and Shumyatsky, 2017, Theorem 1.1] Let G be a finite soluble group, and $x \in G$, assume that the Fitting height of $E_{G,k}(x)$ is h for some k, then x is contained in $F_{h+1}(G)$.*

Theorem 3.4.4. *[Guralnick and Tracey, 2020, Theorem 1.1] Let G be a finite group, and $x \in G$ and fix an integer h. The generalized Fitting height of $E_{G,k}(x)$ is at most h for some integer k if and only if $xF_h^*(G)$ is contained in $F(G/F_h^*(G))$.*

The statement in Theorem 3.4.4 is worded differently than Theorem 3.4.3 for two main reasons. Firstly the theorem holds in both directions whereas Theorem 3.4.3 is only stated for one implication. Secondly we need to make use of the coset $xF_h^*(G)$ because the left-to-right implication of the theorem is stronger than if we would say $x \in F_{h+1}^*(G)$. This last distinction also allows us to recover Baer's original theorem. The next chapter will be devoted to building the necessary concepts to prove Theorem 3.4.4.

Chapter 4

Wielandt's Zipper Lemma and a Generalization

In this chapter we will look at some theory regarding subnormality of subgroups. We will use this to prove the second generalization of Baer's theorem. This chapter is based on [Isaacs, 2008] and [Flavell, 2010].

4.1 Wielandt's Zipper Lemma

In this section we will develop some theory that will prove to be useful in solving the general case of Baer's Theorem. The two main results that we will prove are the following.

Theorem 4.1.1 (**Wielandt's Zipper Lemma**). *[Isaacs, 2008, Theorem 2.9] Let A be a subgroup of a finite group G and assume that A is subnormal in every proper subgroup H of G that contains A. If A is not subnormal in G, then A is contained in exactly one maximal subgroup of G.*

Theorem 4.1.2 (**Flavell's Generalization of the Zipper Lemma**). *Let A be a proper subgroup of a finite group G. Suppose that*

1. *A is contained in at least two maximal subgroups,*
2. *A is subnormal in all but at most one maximal subgroups in which it is contained.*

Then A is contained in a proper normal subgroup of G.

The unusual name of the first theorem will become clear as we go through the proof. Before giving the proof of Theorem 4.1.1 we first need to show some technical lemma's.

Lemma 4.1.3. Let M and N be normal subgroups of some group G with trivial intersection, then every element of M commutes with every element of N.

Proof. Take elements $m \in M$ and $n \in N$ and let $c = [m, n] = m^{-1}n^{-1}mn$. By normality of M it holds that $n^{-1}mn \in M$ and hence also $c \in M$. Similarly, by normality of N we have $m^{-1}n^{-1}m \in N$ and thus also $c \in N$. Now $c \in M \cap N = 1$, and so $1 = [m, n]$ which implies that m and n commute. □

This lemma will be used in the next lemma. Where we will also use the concept of a *socle* of a group.

Definition 4.1.4. Let G be a group. The *socle* of G, denoted $\mathrm{Soc}(G)$ is the subgroup generated by all minimal normal subgroups of G, i.e. all subgroups with no non-trivial normal subgroups.

Lemma 4.1.5. Let G be a finite group and S a subnormal subgroup of G. Let M be a minimal normal subgroup of G. Then M is contained in the normalizer of S in G.

Proof. Remark that the case $S = G$ is trivial since $N_G(G) = G$, so assume that $S < G$. We proceed by induction on the order of G. Note that the case $|G| = 1$ is handled by the remark. Since $S \lhd\lhd G$ there exists a series of subgroups, each normal in the next, going from S to G, i.e.

$$S = H_0 \lhd H_1 \lhd \ldots \lhd H_n = G.$$

We consider the subgroup H_{n-1} which is normal in G by definition.

Suppose that $M \cap H_{n-1}$ is nontrivial, then by minimality of M it follows that $M \cap H_{n-1} = M$. We will now show that $M \leq \mathrm{Soc}(H_{n-1})$. Since M is a normal subgroup of G it is also normal in H_{n-1}, hence a minimal normal subgroup of N is contained in M, we find that $M \cap \mathrm{Soc}(H_{n-1})$ is non trivial. The socle is a characteristic subgroup of H_{n-1} and since characteristic subgroups of normal subgroups are normal we find that $\mathrm{Soc}(H_{n-1}) \lhd G$. By the normality of the socle, we find $M \cap \mathrm{Soc}(H_{n-1}) \lhd G$ and by minimality of M we get $M \cap \mathrm{Soc}(H_{n-1}) = M$.

By the inductive hypothesis and the fact that S is contained in H_{n-1} we know that every minimal normal subgroup of H_{n-1} lies in the normalizer of S, hence $\mathrm{Soc}(H_{n-1})$ lies in the normalizer and we just showed that M lies in the socle, this finishes this case.

The other case where $M \cap H_{n-1}$ is trivial is handled by Lemma 4.1.3. Indeed the lemma states that M is contained in the centralizer of H_{n-1}, which in its turn is contained in the centralizer of S. We can conclude by using that the centralizer of a group is always contained in its normalizer and hence,

$$M \leq C_G(N) \leq C_G(S) \leq N_G(S).$$

□

To finally prove the first main theorem we need one more technical result.

Theorem 4.1.6. *Let G be a finite group, and H and K subnormal subgroups of G, then $\langle H, K \rangle$ will be a subnormal subgroup of G.*

This theorem is often called the *Join Lemma*, since $\langle H, K\rangle$ is called the *join* of H and K. The proof will be split in three parts.

Lemma 4.1.7. Let G be a group, H a subnormal subgroup of G and K a normal subgroup, then $\langle H, K\rangle \lhd\lhd G$.

Proof. Let $H = H_0 \lhd H_1 \lhd \ldots \lhd H_r = G$ be a subnormal series for H. Consider the groups $\langle H_i, K\rangle$, by normality of K it is clear that $\langle H_i, K\rangle \lhd \langle H_{i+1}, K\rangle$, hence the following series is a subnormal series for $\langle H, K\rangle$,

$$\langle H, K\rangle = \langle H_0, K\rangle \lhd \langle H_1, K\rangle \lhd \ldots \lhd \langle H_r, K\rangle = G.$$

□

Lemma 4.1.8. Let G be a group, H and K subnormal subgroups, such that $K \leq N_G(H)$, then $HK = \langle H, K\rangle$ is a subnormal subgroup.

Proof. We start by defining a series, let $G_0 = G$ and $G_i = \mathrm{ncl}_{G_{i-1}}(H)$, since H is subnormal, this series will reach H for some G_r. Note that since K normalizes H, we find that K normalizes G_i for all i, or $G_iK = \langle G_i, K\rangle$. Since $G_{i+1} \lhd G_i$ and $G_{i+1} \lhd K$ by the previous remark, $G_{i+1} \lhd \langle G_i, K\rangle$. Note that since K is subnormal in G it is also subnormal in the smaller group G_iK for all i. We can now apply Lemma 4.1.7 to find that $G_{i+1}K$ is subnormal in G_iK, which is enough to conclude, as $G_rK = HK$ and $G_0K = G$. □

Proof of Theorem 4.1.6. We will prove the theorem by induction on the subnormality depth of one of the subgroups. In the base case one of the subgroups equals G, and hence the statement is trivially satisfied. Assume the statement holds when one subgroup has subnormal depth $h - 1$, take that subgroup to be H. Let L be the normal closure of H in G, then H is subnormal of depth at most $h - 1$ in L. By the induction hypothesis, $\langle H, N\rangle \lhd\lhd L$, whenever N is a subnormal subgroup of L.

We now claim that the subgroup $\langle H^K\rangle$ is subnormal in G. Any subgroup H^g lies in L since L is normal in G, hence the join of finitely many H^k with $k \in K$ is again subnormal in L by induction. Since L is normal in G we get that H^K is subnormal in G. By Lemma 4.1.8, we find that $K\langle H^K\rangle$ is subnormal in G.

The last claim we make is that $K\langle H^K\rangle = \langle H, K\rangle$, which is enough to prove the result. It is clear that H and K are subgroups of $K\langle H^K\rangle$, and any element of the form $k_1h^{k_2}$ is in $\langle H, K\rangle$. □

We can now finally prove Wielandt's Zipper Lemma.

Proof of Theorem 4.1.1. We will prove the theorem by induction on the index of A in G. The case when the index is 1 is vacuously satisfied since we assume A to not be subnormal, and G is a normal subgroup of itself. The normalizer of A, $N_G(A)$ will be a strict subgroup of G as A is assumed not to be normal in G. Take M, a maximal subgroup of G such that $N_G(A) \subseteq M$, we claim that M is the only maximal subgroup of G that contains A.

Assume towards contradiction that $K \neq M$ is another maximal subgroup containing A. Since K is a proper subgroup of G we find that $A \lhd\lhd K$. Note that A cannot be a normal subgroup of K, in that case we would have $K \leq N_G(A) \leq M$, which by maximality of K implies $K = M$.

Take some shortest subnormal series for $A \lhd\lhd K$

$$A = H_0 \lhd H_1 \lhd H_2 \lhd \ldots \lhd H_r = K.$$

Since we assume this to be the shortest subnormal series $A \ntriangleleft H_2$, so take some element $x \in H_2$ such that $A^x \neq A$, and let T be the join of A and A^x. We now claim that A is a normal subgroup of T. Indeed, $A^x \leq H_1^x = H_1 \leq N_G(A)$ where the equality holds since H_1 is assumed to be normal in H_2, hence $T \leq N_G(A)$ and since A is contained in T it follows that $A \lhd T$.

Since conjugation is an automorphism, we get that A^x satisfies the conditions of the theorem. Indeed let H be some proper subgroup of G containing A^x, then H^{-x} contains A, and hence by taking the conjugate of the subnormal series of A to H^{-x} by x we get the required subnormal series. The fact that S^x is not subnormal in G follows from a similar reasoning. We can extend this claim to T. Let H be a proper subgroup of G containing T, then H contains A and A^x as well. By the subnormality of A and A^x in H and by Lemma 4.1.6 we get that $T \lhd\lhd H$. Since $A \lhd T$ we find that T cannot be subnormal in G as this would imply A to be subnormal in G contrary to our hypothesis.

Since T is strictly larger than A, its index in G is strictly smaller and hence by induction T is contained in exactly one maximal subgroup. This leads to a contradiction since $T \leq M$ and $T \leq K$. □

The name of the theorem is explained by Martin Isaacs in his chapter on subnormality in *Finite Group Theory.*

> Perhaps we should explain the name "zipper lemma". The key idea of the proof is that if A is contained in two different maximal subgroups, then some larger subgroup T is also contained in two different maximal subgroups. Repeating this argument, we would get a still larger subgroup contained in two different maximal subgroups, and so on. But this cannot go on indefinitely in a finite group. As we climb higher, the two maximal subgroups are forced to be the same. This is analogous to zipping up an open zipper. As we pull up on the zipper pull, the two top parts of the zipper are forced together.
>
> Martin Isaacs [Isaacs, 2008, p.52]

4.2 Flavell's Theorem

Before doing the necessary work to prove Theorem 4.1.2 we first remark the following.

Remark 4.2.1. We work out an example to show that we cannot generalize Flavell's theorem to have the same conclusion as Wielandt's Zipper Lemma, i.e. the subnormality of A does not automatically follow.

Consider the symmetric group on 4 symbols, which we denote S_4. For A we will take the subgroup $\{(), (123), (132)\}$ which is the alternating group on $\{1, 2, 3\}$ viewed as a subgroup of S_4. The maximal subgroups of S_4 are the copies of D_4 in S_4, copies of S_3 in S_4 and the alternating subgroup A_4. A is contained in A_4 and the copy of S_3 where we consider the permutations of $\{1, 2, 3\}$. We find that A is normal and hence subnormal in S_3, but not subnormal in A_4, hence the conditions for the theorem hold. The proper normal subgroup containing A is A_4, however A is not subnormal in S_4.

Let us first restate Flavell's Theorem before proving it.

Theorem 4.1.2. *Let A be a proper subgroup of a finite group G and suppose the following conditions hold:*

1. *A is contained in at least two maximal subgroups,*
2. *A is subnormal in all but at most one maximal subgroup in which it is contained,*

then A is contained in a proper normal subgroup of G.

Remark 4.2.2. We first introduce some notation. Whenever A is a subset of G we denote $\{A^G\}$ to be the set of conjugates of A, and $M(A)$ to be the set of maximal subgroups containing A.

Proof. We will split the proof in several claims as is done in the original paper by Flavell. We assume the main theorem to be false and we choose G to be a minimal counterexample. We also choose A maximal such that the hypotheses hold but not the conclusion.

Since A is not contained in a proper normal subgroup we know that the normal closure of A in G is G in its entirety. Hence A cannot be subnormal in G as this would imply it to be contained in a proper normal subgroup namely the second to last term in the subnormal series. By Wielandt's Zipper Lemma (Theorem 4.1.1) we find that A has to be not subnormal in exactly one maximal subgroup of G in which it is contained, so in exactly one element of $M(A)$. Note that we can repeat this argument for every $B \in \{A^G\}$. Let $\mu(B)$ be the unique maximal subgroup of G containing B in which B is not subnormal. Note that $\mu(B)^g$ is a maximal subgroup containing B^g as conjugates of maximal subgroups are maximal, and if B^g were subnormal in $\mu(B)^g$ then taking the conjugate of every term in the subnormal series with g^{-1} would result in a subnormal series for B in $\mu(B)$. Hence $\mu(B)^g = \mu(B^g)$.

1. For every $B \in \{A^G\}$ it holds that $N_G(B) \leq \mu(B)$.

   It is clear that $N_G(B) \leq N_G(\mu(B))$, and as $\mu(B)$ is a maximal subgroup, there are two options: $N_G(\mu(B)) = G$ or $N_G(\mu(B)) = \mu(B)$. Assume $N_G(\mu(B)) = G$, then $\mu(B)$ is a normal subgroup of G. The normal closure of B in G is then contained in $\mu(B)$, but this would imply that the normal closure of A in G is also contained in $\mu(B)$ as conjugates are contained in the normal closure. This is a contradiction to the assumption that A is not contained in any normal subgroup. So $N_G(\mu(B)) = \mu(B)$ and hence $N_G(B) \leq \mu(B)$.

2. Let $B \in \{A^G\}$ and $H \in M(B)$ different from $\mu(B)$. The only conjugate B^x of B with $x \in \mu(B)$ that is contained in H is B.

   Take a conjugate B^x of B with an element $x \in \mu(B)$ such that $B \neq B^x$ and consider $D := \langle B, B^x \rangle$. We will show that D satisfies the claim that it is subnormal in all but at most one maximal subgroups it is contained in. The first issue could occur with the $\mu(B^x)$ but by a previous remark $\mu(B^x) = \mu(B)^x = \mu(B)$ since $x \in \mu(B)$. Choose any $L \in M(D)$ different from $\mu(B)$, then clearly $L \in M(B)$ and $L \in M(B^x)$, and hence using the Join Lemma and the fact that B and B^x are subnormal in L, we find that $D \lhd\lhd L$. Hence D is subnormal in every maximal subgroup it is contained in except possibly $\mu(B)$. Notice also that the normal closure of D in G is the entire group, so if D were to be contained in more than one maximal subgroup it would be a larger example than A satisfying the hypothesis of the theorem but not the conclusion, contradicting the maximality of A. Hence we must have that $M(D) = \{\mu(B)\}$, and it follows that B^x is not contained in H as then H would be a maximal subgroup containing D. Note that this claim also implies that B is a normal subgroup of $H \cap \mu(B)$.

3. Let B and C be two distinct conjugates of A, and suppose that $\langle B, C \rangle \neq G$ and that $\langle B, C \rangle$ is contained in at least two different maximal subgroups. Then $\mu(B) \neq \mu(C)$ and $B \subseteq N_G(C)$ and $C \subseteq N_G(B)$.

   For brevity we set $D := \langle B, C \rangle$, by a similar argument as before the normal closure of D in G is G itself and hence by maximality of A we find two distinct maximal subgroups H and K containing D such that D is not subnormal in both of them. If B and C were both subnormal in H then the join lemma implies that D is subnormal in H, so assume after possibly interchanging B and C that B is not subnormal in H. Then $\mu(B) = H$ and by assumptions on A it holds that $B \lhd\lhd K$. Again by the join lemma we find that C is then not subnormal in K, so $\mu(C) = K$, this shows the first claim. For the second claim note first that $\langle B, C \rangle \leq H \cap K = \mu(B) \cap \mu(C)$. Taking $H = \mu(C)$ in the final remark of part 2 of this proof the claim follows.

4. We will extend the ideas of the previous claim as follows: let X be the maximal strict subgroup of G that is generated by conjugates of A, and that

is contained in at least two maximal subgroups. Call the generating subgroups $A_1, A_2, \dots A_n$. Part 3 of the proof shows that all of these A_i are contained in each others normalizers, and combining this with part 1 we find that $X \leq \mu(A_i)$ for all i. We claim that $n \geq 2$.

We assumed A to be contained in at least two maximal subgroups, so choose $H \neq \mu(A)$. Also by assumption we have $A \lhd\lhd H$, but not $A \lhd H$, since $N_G(A) \leq \mu(A)$. Since A is not normal, we can find a conjugate A^x with $x \in H$ such that $A \neq A^x$, since $A^x \leq N_G(A) \leq \mu(A)$, we get that $\langle A, A^x \rangle \leq H \cap \mu(A)$, which ensures $n \geq 2$.

5. For $i = 1, \dots, n$ define $X_i := \langle A_j \mid j \neq i \rangle$. We make the following claims:

   (a) X_i is subnormal in $\mu(A_i)$.

   We previously stated that the A_j normalize each other, therefore $A_i \lhd X$, therefore $X_i \leq X \leq N_G(A_i) \leq \mu(A_i)$ where the last inequality follows from part 1. By part 3, we know that $\mu(A_j) \neq \mu(A_i)$ whenever $i \neq j$, hence A_j is subnormal in $\mu(A_i)$. Applying the join lemma $n-2$ times we find that X_i is subnormal in $\mu(A_i)$.

   (b) $N_{\mu(A_i)}(X_i) \leq N_G(A_i) \cap N_G(X)$.

   Note that $X = A_i X_i$. Take $j \neq i$ and take some element $g \in N_{\mu(A_i)}(X_i)$, we can then write
$$X^g = A_i^g X_i^g = A_i^g X_i,$$
   and find that both A_j and A_i^g are contained in X^g. By assumption X is contained in at least two maximal subgroups, this clearly also holds for X^g. Part 3 of the proof then implies that A_j and A_i^g normalize each other as $\langle A_j, A_i^g \rangle$ is contained in at least two maximal subgroups. By previous remarks we also have that A_i and A_j normalize each other and that $N_G(A_j) \leq \mu(A_j)$. Combining all this we find that both A_i and A_i^g are contained in $\mu(A_i) \cap \mu(A_j)$. By the second part of the proof we find that $A_i = A_i^g$ implying that $X^g = X$, and that $g \in N_G(A_i) \cap N_G(X)$.

   (c) X_i is not a normal subgroup of $\mu(A_i)$.

   If X_i is normal in $\mu(A_i)$ then $N_{\mu(A_i)}(X_i) = \mu(A_i)$, then part b implies that $\mu(A_i) \leq N_G(A_i)$ which contradicts the fact that $A \ntriangleleft \mu(A_i)$.

   (d) X_i is the only $\mu(A_i)$-conjugate of X_i contained in X.

   Suppose this does not hold, let $m \in \mu(A_i)$ such that $X_i^m \neq X$. By assumption $X_i^m \leq X$ and since X is chosen maximal we find that $A_i \leq X_i^m$. Conjugating by m^{-1} we get $A_i^{m^{-1}} \leq X_i$. This results in,
$$A_i^{m^{-1}} \lhd X_i \lhd\lhd \mu(A_i),$$

where X_i is subnormal by part 5a. However since $m \in \mu(A_i)$ this implies that $A_i \lhd\lhd \mu(A_i)$ which is a contradiction.

We can now finally finish the proof. Choose some $i \in \{1, \dots, n\}$ because of the subnormality of X_i in $\mu(A_i)$ (claim 5a) and claim 5c we can find some $x \in \mu(A_i)$ such that $X_i \neq X_i^x \leq N_{\mu(A_i)}(X_i)$. Note that $X_i^x \not\leq X$ since by 5d this would imply that $X_i^x = X_i$. By using that $X \leq \mu(A_i)$ by part 1 and part 5b we find

$$X < \langle X, X_i^x \rangle \leq \mu(A_i) \cap N_G(X).$$

Since we choose X to be maximal such that it was generated by conjugates and contained in at least two maximal subgroups, we need that $\mu(A_i)$ and $N_G(X)$ are only contained in at most 1 maximal subgroup shared between the two of them. So there can be two cases, the first is that $N_G(X) = X$, then $\mathrm{ncl}_G(A_i) \leq X$ since the normal closure is generated by all subgroups of the form $g^{-1}A_i g$ with $g \in G$ and since $N_G(X) = G$ we have $g^{-1}Xg = X$ and hence also $g^{-1}A_i g \leq X$. But A_i is also a conjugate of A and we assumed $\mathrm{ncl}_G(A)$ to be G, this is a contradiction since we then get $G \leq X < G$. So assume $N_G(X) \neq G$, then we must have that $N_G(X) \leq \mu(A_i)$. Take some $j \neq i$ by repeating the argument we find that $N_G(X) \leq \mu(A_j)$, by part 3 and the fact that $A_i \neq A_j$ by construction. We find that $\mu(A_i) \neq \mu(A_j)$, however this then still contradicts the choice of X as $\langle X, X_i^x \rangle$ is then contained in at least two maximal subgroups. This finishes the proof. $\square$

This result of Flavell can be generalized to a certain class of infinite groups, which we will now define. This property will be of interest in Chapters 5 and 6 too.

Definition 4.2.3. We say that a group G is *Noetherian* if any strictly ascending chain of subgroups of G is finite.

This condition is named after the German mathematician Emmy Noether who first studied the ascending chain condition for rings.

Noetherian groups are known by many names, some authors prefer to call them groups satisfying the maximal condition on subgroups, or shorter, groups having max. There are other equivalent characterizations of groups being Noetherian, some of which we show here.

Lemma 4.2.4. Let G be a group, the following statements are equivalent.

(i) G satisfies the max condition on subgroups.

(ii) Any subgroup of G is finitely generated.

(iii) Every family of subgroups of G has a maximal member.

Proof.

(i) $\Rightarrow$ (ii) Let H be a subgroup of G, and h_1 an element in H. Set $H_1 = \langle h_1 \rangle$, if $H_1 = H$, then we are done. Otherwise take $h_2 \notin H_1$ and set $H_2 = \langle h_1, h_2 \rangle$. By continuing this process we get a strictly ascending chain of subgroups of G which by (i) must be finite, hence there exist some $h_1, \dots, h_n$ such that $\langle h_1, \dots, h_n \rangle = H$.

(ii) $\Rightarrow$ (i) Let $H_1 \leq H_2 \leq \ldots$ be an ascending chain of subgroups, let $H = \bigcup_i H_i$. Since H is finitely generated, $H = \langle h_1, \ldots, h_n \rangle$ for some $h_i \in H$. Since H is the union of all H_i, all h_i must appear in some H_{k_i}. Let m be the maximum over all k_i, then $H_m = H$, and the sequence is only finitely strictly ascending.

(i) $\Rightarrow$ (iii) If there were no maximal subgroup we could construct an infinite strictly ascending chain by always taking a bigger subgroup.

(iii) $\Rightarrow$ (i) This follows from Zorn's Lemma, where we consider the partial order to be inclusion. □

Example 4.2.5. It is clear that all finite groups satisfy the max condition on subgroups. To find more examples in the infinite case we can use the equivalent characterizations in Lemma 4.2.4 A first example is the group of integers $\mathbb{Z}$, since any subgroup is of the form $m\mathbb{Z}$ it is generated by m. In general it can be shown that any finitely generated nilpotent group is Noetherian. In a later chapter we will introduce a new class of Noetherian groups, namely the polycyclic-by-finite groups. △

We can extend Lemma 4.1.6 and Theorem 4.1.1 as follows.

Lemma 4.2.6. Lemma 4.1.6 holds whenever G is has the max condition on subnormal subgroups. Theorem 4.1.1 holds whenever G has the max condition on all subgroups.

The proof of this lemma is discussed in an article by Wielandt where he discusses some generalizations of theorems regarding subnormal subgroups, [Wielandt, 1939].

Since the proof of Flavell's theorem only makes use of the finiteness of G for the join lemma and the zipper lemma, we can generalize it to hold for groups satisfying the max condition on subgroups too.

4.3 Proof of the Generalized Baer's Theorem

Having shown Flavell's Theorem, we can use it to prove a theorem that will help us prove the generalized Baer's Theorem. This section is an exposition of the results in [Guralnick and Tracey, 2020].

The main idea of the proof of Guralnick and Tracey is based on the proof of the soluble case by Khukhro and Shumyatsky. To be able to remove the "soluble" constraint, more advanced results concerning subnormal subgroups were needed, which is why we had to show Flavell's theorem.

Definition 4.3.1. Let H be a subgroup of a group G, the *normal closure descending series* for H in G is defined by setting $N_0 = G$ and letting $N_{i+1} = \mathrm{ncl}_{N_i}(H)$.

The *normal closure limit* for H in G is then defined as $F(H, G) = \cap_{i \geq 0} N_i$.

To see that this defines a descending series as its name suggests, it is enough to realize that a normal subgroup of N_{i+1} containing H will always be a normal subgroup of N_i containing H.

We can ask ourselves whether this series is ever non-trivial, and whether there are series with an arbitrary number of distinct terms. The answer to be both questions is affirmative, and we will construct an explicit example. Before we do that we define the length of the normal closure descending series.

Definition 4.3.2. As the normal closure descending series has an infinite number of terms by definition, we will call its *length* the index of the last distinct term, if it exists.

Example 4.3.3. The generalized quaternion group of order 2^{k+1}, denoted by $Q_{2^{k+1}}$ can be defined as

$$Q_{2^{k+1}} := \langle x, y \mid x^2 = y^{2^{k-1}}, y^{2^k} = 1, yxy = x\rangle.$$

We claim that the subgroup $\langle x\rangle$ of $Q_{2^{k+1}}$ has a normal closure descending series of length $k-1$, and that $N_i = \langle x, y^{2^i}\rangle$.

The first thing we show is that $\langle y^2\rangle \leq N_1 = \mathrm{ncl}_{Q_{2^{k+1}}}(\langle x\rangle)$. Note that if we can find $g \in Q_{2^{k+1}}$ and $h \in \langle x\rangle$ such that $ghg^{-1} = xy^2$, then $xy^2 \in N_1$ by definition of the normal closure. It is clear that $x \in N_1$, and by definition $x^2 = y^{2^{k-1}}$.

$$\begin{aligned} y^{2^{k-2}-1}x^3y^{-2^{k-2}+1} &= y^{2^{k-2}-1}(xy^{2^{k-1}})y^{-2^{k-2}+1} \\ &= y^{2^{k-2}-1}xy^{2^{k-2}+1} \\ &= xy^2. \end{aligned}$$

Where we used $yxy = x$ several times in the last equality. Hence we find that $x^{-1}xy^2 = y^2 \in N_1$. If we can show that the subgroup $\langle x, y^2\rangle$ is normal we are done. We can check normality on generators and inverses of generators.

The only two non-trivial computations are

$$\begin{aligned} xy^2x^{-1} &= y^{-2}xx^{-1} = y^{-2}, \\ x^{-1}y^2x &= x^{-1}xy^{-2} = y^{-2}. \end{aligned}$$

We will now show that the normal closure of $\langle x\rangle$ in $\langle x, y^{2^i}\rangle$ is $\langle x, y^{2^{i+1}}\rangle$. As before we start by showing that $xy^{2^{i+1}} \in N_i = \mathrm{ncl}_{\langle x, y^{2^i}\rangle}(\langle x\rangle)$.

$$\begin{aligned} xy^{2^{k-2}+2^i}x^3(xy^{2^{k-1}+2^i})^{-1} &= xy^{2^{k-2}+2^i}(xy^{2^{k-1}})y^{-2^{k-1}-2^i}x^{-1} \\ &= xy^{2^{k-2}+2^i}xy^{2^{k-1}-2^{k-2}-2^i}x^{-1} \\ &= xy^{2^{k-2}+2^i}xy^{2^{k-2}-2^i}x^{-1} \\ &= xy^{2^i+2^i}xx^{-1} \\ &= xy^{2^{i+1}}. \end{aligned}$$

Similar to before, we need to show the normality of $\langle x, y^{2^{i+1}}\rangle$ in $\langle x, y^{2^i}\rangle$. This can again be checked on generators and inverses. The only non trivial calculations are

- $xy^{2^{i+1}}x^{-1} = xx^{-1}y^{-2^{i+1}} = y^{-2^{i+1}}$,
- $x^{-1}y^{2^{i+1}}x = x^{-1}xy^{-2^{i+1}} = y^{-2^{i+1}}$,
- $y^{2^i}xy^{-2^i} = y^{2^i}xy^{2^k-2^i} = xy^{2^k-2^{i+1}}$,
- $y^{-2^i}xy^{2^i} = xy^{2^{i+1}}$.

As an example we can write down the normal closure descending series for $\langle x\rangle$ in Q_{64}, which is of length 4.

$$Q_{64} = \langle x, y\rangle \geq \langle x, y^2\rangle \geq \langle x, y^4\rangle \geq \langle x, y^8\rangle \geq \langle x\rangle.$$

△

Let us look at some properties of this descending series in a more specific case.

Definition 4.3.4. A group G satisfies the minimal condition on subnormal subgroups if and only if every strictly descending chain of subgroups

$$H_0 > H_1 > H_2 > \dots$$

where $H_{i+1} \lhd H_i$ is finite.

Lemma 4.3.5. Let H be a subgroup of a group G that satisfies the minimal condition on subnormal subgroups, and let $F(H, G)$ and the N_i be as in Definition 4.3.1. The following statements hold:

1. $N_{i+1} \lhd N_i$ and $N_i \lhd\lhd G$ for all i,
2. $\mathrm{ncl}_{F(H,G)}(H) = F(G, H)$,
3. Let $H \leq K \leq G$, if $\mathrm{ncl}_K(H) = K$ then $K \leq F(G, H)$.

Proof. 1. We defined the term N_{i+1} as the smallest normal subgroup of N_i containing H, hence $N_{i+1} \lhd N_i$. Since $N_0 = G$ it is then clear that $N_i \lhd\lhd G$ for all i.

2. Since G has the minimal condition on subnormal subgroups we know that the series $N_0 \geq N_1 \geq \dots$ stabilizes at some point. Choose m minimal such that $N_m = N_{m+1}$, then clearly $F(G, H) = N_m$ and

$$\mathrm{ncl}_{F(G,H)}(H) = \mathrm{ncl}_{N_m}(H) = N_{m+1} = F(G, H).$$

3. First note that if K is a subgroup of G containing H, then $\mathrm{ncl}_K(H) \leq \mathrm{ncl}_G(H)$ since for any normal subgroup N of G, $K \cap N$ is a normal subgroup of K. We will show that $K \leq N_i$ for all i. For $i = 1$ we get

$$N_1 = \mathrm{ncl}_G(H) \geq \mathrm{ncl}_K(H) = K.$$

We find that $K \leq N_1$ so for $i = 2$ we find

$$N_2 = \mathrm{ncl}_{N_1}(H) \geq \mathrm{ncl}_K(H) = K.$$

It is clear that we can keep repeating this reasoning until we reach $K \leq N_m = F(G, H)$.

□

Using the generalization of Flavell's theorem to Noetherian groups we can prove the following result.

Lemma 4.3.6. Let G be a Noetherian group having the min condition on subnormal subgroups, assume additionally that H is a proper subgroup of G satisfying:

1. H is contained in at least two maximal subgroups of G,
2. the set consisting of all $F(H, M)$ with M a maximal subgroup of G has a unique maximal element.

Then H is contained in a proper normal subgroup of G.

Proof. If $\mathrm{ncl}_G(H) \neq G$ then $\mathrm{ncl}_G(H)$ is a proper normal subgroup of G containing H, so we will assume that $\mathrm{ncl}_G(H) = G$. Let $\Omega(H, G)$ be the set containing all subgroups A of G such that $\mathrm{ncl}_A(H) = A$, and use Y to denote the unique maximal element in the set of all $F(H, M)$.

Take K to be some maximal subgroup of G containing Y, note that this has to exist as $Y \leq M$ for some maximal subgroup M of G since $Y = F(H, M)$. Now take a maximal element X from the set of all subgroups in $\Omega(H, G)$ that are contained in at least two maximal subgroups, this set is non-empty since it contains H. Take a maximal subgroup $L \neq K$ such that $X \leq L$, we can always take such a different maximal subgroup since X is assumed to be contained in at least two maximal subgroups.

We claim that $X = F(H, L)$, i.e. the stable term in the normal descending series. Since $X \in \Omega(H, G)$ we know that $\mathrm{ncl}_X(H) = X$ and hence by Lemma 4.3.5 we find that $X \leq F(H, L)$. By the construction of Y we know that $F(H, L) \leq Y$ and we chose K such that $Y \leq K$, so $F(H, L) \leq L, K$, by the fact that $\mathrm{ncl}_{F(H,L)}(H) = F(H, L)$ and the maximality of X we get that $X = F(H, L)$. Since in the normal descending series terms are normal in the previous term we find that X is a subnormal subgroup in all maximal subgroups L in which it is contained except possibly K. This allows us to apply Flavell's Theorem, and we find that X is contained in a proper normal subgroup of G, since $A \leq X$ this completes the proof.

□

Having shown this lemma we can now prove the final preliminary theorem from which the generalized result will follow.

Theorem 4.3.7. *Assume G is Noetherian and satisfies the min condition on subnormal subgroups. Let H be a subgroup such that $H^G = G$, i.e. all conjugates of H cover G. Let Y be the subgroup generated by all proper subgroups A of G containing H such that $H^A = A$. Then one of the following holds,*

1. *Y coincides with G,*
2. *H is contained in a unique maximal subgroup of G.*

Proof. Let $\Omega(H, G)$ be as in the proof of Lemma 4.3.6, and assume that $Y \neq G$. We can then find a maximal subgroup M of G containing Y. We claim that $Y = F(H, M)$. We first show that $F(H, M) \leq Y$. By construction we know that

$F(H,M)$ contains H and by Lemma 4.3.5 we have $\mathrm{ncl}_{F(H,M)}(H) = F(H,M)$, this together implies that $F(H,M) \in \Omega(H,G)$. Since Y is generated by the elements in $\Omega(H,G)$ we get one inclusion.

To show the other inclusion we start by taking any $A \in \Omega(H,G)$. Since $A \leq Y \leq M$ and $\mathrm{ncl}_A(H) = A$, we know by Lemma 4.3.5 that $A \leq F(H,M)$. This holds for all $A \in \Omega(H,G)$ and these subgroups generate Y so $Y \leq F(H,M)$.

Note that $H \leq M$ as $H \in \Omega(H,G)$ and $\Omega(H,G) \leq Y$. Assume toward contradiction that H is contained in some maximal subgroup K of G such that $K \neq M$. Then $F(H,K) \in \Omega(H,G)$ since $\mathrm{ncl}_{F(H,K)}(H) = F(H,K)$ and hence $F(H,K) \leq M$. Again by Lemma 4.3.5 we find that $F(H,K) \leq F(A,M) = Y$. Hence the set of all $F(H,K)$ with $K \in M(A)$ has a unique maximal element, and hence by Lemma 4.3.6 we find that H is a contained in some proper normal subgroup of G, however this implies that H^G cannot be equal to G as every conjugate of H will be contained in this normal subgroup. □

Lemma 4.3.8. Let N be a subnormal subgroup of a group G, then the following conditions hold,

1. $h^*(N) \leq h^*(G)$,
2. $h^*(N) = h^*(\mathrm{ncl}_G(N))$.

We can now finally prove Theorem 3.4.4. Let us restate it first.

Theorem 3.4.4. *Let G be a finite group, and $x \in G$ and fix an integer h. The generalized Fitting height of $E_{G,k}(x)$ is at most h for some integer k if and only if $xF_h^*(G)$ is contained in $F(G/F_h^*(G))$.*

In the proof all commutators and conjugations between automorphisms and elements of a group are understood to be taken in the semidirect product of G and $\mathrm{Aut}(G)$.

Proof. We first show that if we have an automorphism α of some finite group G, such that $[G,\alpha] = G$, then $G = E_{G,k}(\alpha)$ for all strictly positive integers k. Assume this statement is false and take k minimal such that $E_{G,k}(\alpha) \neq G$. Define $X := \langle G, \alpha \rangle$, and take Y to be the subgroup generated by all proper subgroups H of X such that H contains α and that $\alpha^H = H$. We are now working in the setting of Theorem 4.3.7, and hence we have two options.

1. Assume that α is contained in a unique maximal subgroup of X, we will call this subgroup M. We claim that M is self-centralizing, i.e. that $N_X(M) = M$, which will follow from the fact that $\langle G, \alpha \rangle = \langle \alpha^G \rangle$. This equality of groups is true since for every g in G we can find some $\overline{g} \in G$ such that $[\overline{g}, \alpha] = g$ because $[G,\alpha] = G$. Using similar reasoning we can extend this to an equivalence, i.e.

$$[H,\alpha] = H \iff \langle \alpha^H \rangle = \langle \alpha, H \rangle.$$

   Since M is maximal, its normalizer can only be M itself or the entire group X. However since $X = \langle G, \alpha \rangle$ is generated by all conjugates of α, and because $\alpha \in M$, M is not normal in X. Since k was chosen minimal such that

$E_{G,k}(\alpha) \neq G$ we can find a $k-1$ repeated commutator $h := [g,_{k-1}\alpha] \in X \setminus M$ since if this were not the case, all elements of G would be contained in M implying that $M = X$. The minimality of k and the fact that M is a maximal subgroup containing α then implies that $[g,_k \alpha] = [h, \alpha] \in M$, so also $\alpha^h \in M$, but then $\alpha \in M^{h^{-1}} \neq M$ by the fact that M is self-normalizing. Since α only lies in one maximal subgroup of X, this is a contradiction.

2. The other option is that $Y = X$, or that $X = \langle \alpha^H \mid H \lneq X \rangle$, as $X = \langle G, \alpha \rangle$ this can be simplified to $X = \langle \alpha^H \mid H \lneq G \rangle$. We can now show the result by induction on the order of the group G. The case where $|G| = 1$ is trivial as $E_{1,k}(\alpha) = 1$ for any k. Now assume the result holds up to a certain order $n-1$, and assume $|G| = n$. By construction X is generated by strict subgroups H of G and α such that $\langle \alpha^H \rangle = \langle H, \alpha \rangle$, implying that G is generated by these subgroups H. As these subgroups have strictly smaller order, and satisfy $[H, \alpha] = H$ by the equivalence above, the induction hypothesis implies $H = E_{H,k}(\alpha)$. The required statement then follows.

Consider a group G and an element $g \in G$. Consider the descending series

$$[G, g] \geq [G, g, g] \geq [G, g, g, g] \geq \dots .$$

Because our group is finite, this series must stabilize at some point, we will call H the stable term of this series, note that $[H, g] = H$. We will denote $N := \mathrm{ncl}_G(H)$ for brevity. We first assume that the generalized Fitting height of $E_{G,k}(g)$ is h. By the previous part of the proof we know then that $H = E_{H,k}(g) \lhd E_{G,k}(g)$. By lemma 4.3.8 we know that $h^*(N) = h^*(H)$, and we know that $h(H) \leq h(E_{G,k}(g))$. Note that by construction gN is a left Engel element of G/N, and since N has generalized Fitting height at most h we know that it must be contained in $F_h^*(G)$. We find that $gF_h^*(G)$ is a left Engel element of $G/F_h^*(G)$ and thus, by Baer's theorem, $g \in F(G/F_h^*(G))$, proving one direction of the statement.

The other direction follows from a characterization of the Fitting subgroup that has properties that we do not cover. The proof is short and can be found in the article by Guralnick and Tracey [Guralnick and Tracey, 2020]. □

Chapter 5

Left Engel Elements in Infinite Groups

In this chapter we leave the realm of finite groups and extend the previous ideas about the set of left-Engel elements to some classes of infinite groups. Results in this chapter are mainly based on the overview of results in section 3 of [Abdollahi, 2011], and chapter 12 of [Robinson, 2012]. We conclude the chapter with an example of a group where the set of left Engel elements is not a subgroup.

5.1 Locally Nilpotent Groups and the Hirsch-Plotkin Radical

To overcome the shortcomings of the Fitting group we first introduce a property for groups that is related to nilpotency. The following section is based on chapter 12 of [Robinson, 2012].

Let $\mathcal{P}$ be a property of groups, we will call a group G locally-$\mathcal{P}$ if every finitely generated subgroup of G has property $\mathcal{P}$. Examples of the property $\mathcal{P}$ we can use are being abelian, being cyclic or what will be of interest here, being nilpotent.

It is easy to find groups that are locally nilpotent, all nilpotent groups themselves are locally nilpotent since every subgroup, not just the finitely generated subgroups, of a nilpotent group are nilpotent.

A group that is locally nilpotent and non-nilpotent is the group we constructed in Remark 3.1.6. As a reminder, this group G was constructed as the direct product of an infinite amount of nilpotent groups G_i with increasing nilpotency class i. Whenever we take a finite set of elements of the form $(g_i)_{i\in\mathbb{N}}$ there will be some maximal index n such that for all elements in this set the m-th component will be trivial, whenever $m \geq n$. We can therefore consider the subgroup generated by these elements as a subgroup of the finite direct product $\bigoplus_{i=1}^{n} G_i$, which is nilpotent.

A useful property of normal locally nilpotent subgroups is the following.

Theorem 5.1.1 (The Hirsch-Plotkin Theorem). *The product of two normal locally nilpotent subgroups is a locally nilpotent subgroup.*

Before proving it we state and prove the following technical lemma. We extend the concept of the normal closure to sets.

Remark 5.1.2. Let X and K be subsets of G, define X^K as,

$$\langle X^K \rangle = \langle x^k \mid x \in X, k \in K \rangle.$$

Lemma 5.1.3. Let X and Y be subsets of G and K and H subgroups, the following holds,

1. $\langle X^K \rangle = \langle X, [X, K] \rangle$,
2. $[X, K] = \langle [X, K]^K \rangle$,
3. if $K = \langle Y \rangle$ it holds that $[X, K] = \langle [X, Y]^K \rangle$,
4. if also $H := \langle X \rangle$, then $[H, K] = \langle [X, Y]^{HK} \rangle$.

Proof. 1. Note that $x[x, k] = xx^{-1}k^{-1}xk = x^k$ for all $x \in X$ and $k \in K$, hence the statement follows.

2. $\langle [X, K]^K \rangle$ is generated by elements of the form $[x, k_1]^{k_2}$ with $x \in X$, $k_1, k_2 \in K$. The following calculation shows that any such element can be written as a product of elements in $[X, K]$,

$$\begin{aligned} [x, k_2]^{-1}[x, k_1k_2] &= (k_2^{-1}x^{-1}k_2x)(x^{-1}k_2^{-1}k_1^{-1}xk_1k_2) \\ &= k_2^{-1}(x^{-1}k_1^{-1}xk_1)k_2 = [x, k_1]^{k_2}. \end{aligned}$$

Hence $\langle [X, K]^K \rangle \subseteq [X, K]$, the other inclusion is trivial as K contains 1.

3. We will first check that $[x, k] \in \langle [X, Y]^K \rangle$, where $x \in X$ and $k \in K$. Write $k = y_1^{\varepsilon_1}y_2^{\varepsilon_2} \dots y_n^{\varepsilon_n}$, with $y_i \in Y$. We will proceed by induction on n. The case $n = 1$ is trivial when $\varepsilon_1 = 1$, when it is negative we use the following calculation,

$$([x, y_1]^{y_1^{-1}})^{-1} = (y_1x^{-1}y_1^{-1}xy_1y_1^{-1})^{-1} = [x, y_1^{-1}],$$

and since $y_1^{-1} \in K$ we have shown the claim for $n = 1$.

When $n > 1$ write $k' = y_1^{\varepsilon_1} \dots y_{n-1}^{\varepsilon_{n-1}}$, then clearly $[x, k] = [x, k'y_n^{\varepsilon_n}]$.

$$\begin{aligned} [x, k'y_n^{\varepsilon_n}] &= x^{-1}y_n^{-\varepsilon_n}k'^{-1}xk'y_n^{\varepsilon_n} \\ &= x^{-1}y_n^{-\varepsilon_n}(xy_n^{\varepsilon_n}y_n^{-\varepsilon_n}x^{-1})k'^{-1}xk'y_n^{\varepsilon_n} \\ &= [x, y_n^{\varepsilon_n}][x, k']^{y_n^{\varepsilon_n}}. \end{aligned}$$

The last term is an element of $\langle [X, Y]^K \rangle$ by induction and hence the result follows. The other inclusion follows from part 2.

4. This follows by applying part 3 twice.

$\square$

Proof of Theorem 5.1.1. Let us call the normal locally nilpotent subgroups H an K and their product $HK =: J$. Take a finitely generated subgroup of J, without loss of generality we can say that it is generated by $h_1k_1, \dots, h_nk_n$. Define two new subgroups L and M as the subgroups generated by $h_1, \dots, h_n$ and $k_1, \dots k_n$ respectively, and finally let Z be the group generated by L and M. We claim that it suffices to show that Z is nilpotent. Indeed it is clear that J is a subgroup of Z and subgroups of nilpotent groups are nilpotent.

Now consider the set of all commutators $[h_i, k_j]$ with $i, j \in \{1, \dots n\}$, call it C. Since H is normal it holds that $k_j^{-1}h_ik_j \in H$ and hence also $h_i^{-1}k_j^{-1}h_ik_j = [h_i, k_j] \in H$, hence $C \subseteq H$, similarly also $C \subseteq K$. We find that $\langle L, C\rangle \leq H$ is finitely generated and therefore nilpotent as H is locally nilpotent. We now use the fact that finitely generated nilpotent groups are Noetherian, and find that $\langle C^L\rangle$ is finitely generated, since it is a subgroup of $\langle L, C\rangle$ and moreover it is nilpotent.

By normality of H and K we find that $\langle C^L\rangle \leq H \cap K$, implying that $\langle M, \langle C^L\rangle\rangle \leq K$. By similar reasoning as before we find that $\langle M, \langle C^L\rangle\rangle$ is nilpotent and finitely generated.

We have $\langle M, C^L\rangle = \langle M, \langle C^{LM}\rangle\rangle$ and by part 4 of Lemma 5.1.3, we find that $[L, M] = \langle C^{LM}\rangle$. Combining this we find

$$\langle M, \langle C^{LM}\rangle\rangle = \langle M, [L, M]\rangle = \langle M^L\rangle$$

where we used part 1 of Lemma 5.1.3 in the last equality. We find that $\langle M^L\rangle$ is nilpotent, and by repeating the argument with the roles of M and L switched we find that $\langle L^M\rangle$ is nilpotent.

Note that $L \leq \langle L^M\rangle\langle M^L\rangle$ as $L \leq \langle L^M\rangle$ and similarly $M \leq \langle L^M\rangle\langle M^L\rangle$. Since the inequality $\langle L^M\rangle\langle M^L\rangle \leq \langle L, M\rangle$ is trivial we find that,

$$Z = \langle L, M\rangle = \langle L^M\rangle\langle M^L\rangle,$$

the required nilpotency now follows from Fitting's Theorem.

□

Using this theorem we can prove the following very important result.

Theorem 5.1.4. *In any group G there exists a unique largest normal locally nilpotent subgroup that contains every normal locally nilpotent subgroup. This subgroup will be called the Hirsch-Plotkin radical, and will be denoted with $HP(G)$.*

Proof. Any chain of locally nilpotent subgroups has an upper bound as we can just take the union which will still be locally nilpotent. By Zorn's Lemma, every normal locally nilpotent subgroup is contained in a maximal normal locally nilpotent subgroup. If M and N are two maximal normal locally nilpotent subgroups, then their product MN is locally nilpotent by the previous theorem, implying that $M = N$. □

It is clear that the Fitting subgroup is contained in the Hirsch-Plotkin radical, as every nilpotent subgroup is trivially locally nilpotent. In the finite case both groups coincide as any locally nilpotent subgroup will be a finitely generated subgroup of itself, implying it is nilpotent.

5.2 Left-Engel Elements and $HP(G)$

We will use the fact that the Hirsch-Plotkin radical behaves "nicer" for infinite groups, by restating and extending some of the results of Baer.

The first result that we will state is something that also holds for the Fitting subgroup.

Theorem 5.2.1. *The set $L(G)$ of all left-Engel elements contains the Hirsch-Plotkin radical.*

Proof. Take an element g in $HP(G)$, and let x be some arbitrary element in G. Since the Hirsch-Plotkin radical is normal, $[x, g]$ lies in $HP(G)$ too. The group generated by g and $[x, g]$ is then a finitely generated subgroup of $HP(G)$ implying it is nilpotent. Hence there exists n such that $[x,_n g] = 1$, therefore $g \in L(G)$. $\square$

The other inclusion is known to be true for several types of groups satisfying different properties, we will look at a few of them here.

5.2.1 Soluble Groups

In the definition of subnormality we quietly assumed every ascending series from the subgroup to the group to be finite. We can however remove this finiteness condition by introducing *ascendant subgroups*. The definition uses the notion of ordinal numbers, for which a great introduction is found in *A First Journey Through Logic* [Hils and Loeser, 2019].

Definition 5.2.2. A subgroup H of G is said to be *ascendant* if there exists an ordinal β and a set of subgroups indexed by ordinals smaller than β i.e. $\{H_\alpha \mid \alpha < \beta\}$ such that,

1. $H_{\alpha_1} \leq H_{\alpha_2}$ whenever $\alpha_1 \leq \alpha_2$,
2. $H_0 = H$ and $\bigcup_{\alpha<\beta} H_\alpha = G$,
3. $H_\alpha \lhd H_{\alpha+1}$,
4. $\bigcup_{\alpha<\lambda} H_\alpha = H_\lambda$ whenever λ is a limit ordinal.

In the case that β is a finite ordinal, this definition coincides with subnormality. We can interpret this set of subgroups as a type of series, which we call an ascending series.

Before we show an interesting connection between the Hirsch-Plotkin radical and ascendant subgroups, we show a small lemma that we will use later.

Lemma 5.2.3. Let H, N and M be subgroups of some group G such that the following inclusions hold, $H \leq N \lhd M$. Then $\mathrm{ncl}_N(H) \lhd \mathrm{ncl}_M(H)$.

Proof. Note that $\mathrm{ncl}_M(H) \subseteq N$ since

$$\mathrm{ncl}_M(H) = N \cap \left(\bigcap_{\substack{L \lhd M \\ H \subseteq L \neq N}} L \right) \subseteq N.$$

Let $n \in \mathrm{ncl}_N(H)$ and $m \in \mathrm{ncl}_M(H)$. We wish to show that $m^{-1}nm \in \mathrm{ncl}_N(H)$. Since $\mathrm{ncl}_M(H) \leq N$, we find that $m \in N$. Let us write $\mathrm{ncl}_N(H)$ as an intersection, i.e.

$$\mathrm{ncl}_N(H) = \bigcap_{\substack{K \lhd N \\ H \subseteq N}} K.$$

For every K in this intersection we have $m^{-1}Km = K$, by normality of K in N. In particular, $m^{-1}nm \in K$ for all K in the intersection. Hence we find that

$$m^{-1}nm \in \bigcap_{\substack{K \lhd N \\ H \subseteq N}} K = \mathrm{ncl}_N(H).$$

□

We know that the Hirsch-Plotkin radical contains all normal locally nilpotent subgroups, and it turns out that we can say even more.

Lemma 5.2.4. The Hirsch-Plotkin radical of a group contains all ascendant locally nilpotent subgroups.

Proof. Let H be a ascendant locally nilpotent subgroup of a group G. We can find a ascending series $H = H_0 \lhd H_1 \lhd \ldots \lhd H_\beta = G$. We define a new series by taking the normal closure of H in all of the groups in this series. For brevity we will write $N_\alpha = \mathrm{ncl}_{H_\alpha}(H)$. Clearly whenever $\alpha_1 \leq \alpha_2$ we have that $N_{\alpha_1} \leq N_{\alpha_2}$. The fact that $N_{\alpha_1} \lhd N_{\alpha_2}$ follows from Lemma 5.2.3. Note that $H = N_1$ by normality of H in H_1, and that $N_\beta = \mathrm{ncl}_G(H)$. The 4th condition holds by construction.

We therefore have an ascending series from H to $\mathrm{ncl}_G(H)$. We claim that every term N_α in the series is locally nilpotent by using transfinite induction. Assume that this is false, then there has to be some first ordinal α such that N_α is not locally nilpotent. If α is a limit ordinal we have that $\bigcup_{\beta<\lambda} N_\beta = N_\alpha$, however by minimality of α, all N_β are locally nilpotent and hence also N_α is locally nilpotent.

If α is not a limit ordinal it can be written as $\beta+1$ for some ordinal β, note that again by minimality N_β is locally nilpotent. The normal closure of N_β in $H_{\beta+1}$ is the normal closure of H in $H_{\beta+1}$ since $N_\beta \leq N_{\beta+1}$. Taking any $g \in H_{\beta+1}$ we get $N_\beta^x \lhd N_{\beta+1}^x = N_{\beta+1}$. Now since the normal closure of N_β in $H_{\beta+}$ is generated by the subsets $N_{\beta+1}^g$ and is equal to $N_{\beta+1}$, we find that $N_{\beta+1}$ is generated by normal locally nilpotent subgroups and hence by the previous theorems is locally nilpotent itself, which is a contradiction.

This implies that $\mathrm{ncl}_G(H)$ is locally nilpotent, implying that it is contained in the Hirsch-Plotkin radical, by normality of the normal closure. Since $H \subseteq \mathrm{ncl}_G(H)$ the claim is shown. □

This property is useful to prove a version of Baer's theorem for soluble groups. This was first shown by Gruenberg, two years after Baer's original proof [Gruenberg, 1959].

The version of the proof here follows chapter 12.3 in [Robinson, 2012].

Theorem 5.2.5. *Let G be a soluble group, then $L(G)$ coincides with $HP(G)$.*

Proof. Because of Theorem 5.2.1 it suffices to show that every left-Engel element is contained in the Hirsch-Plotkin radical. We will do this by showing that every subgroup generated by a left-Engel element is a locally nilpotent ascendant subgroup. Consider $g \in L(G)$, then it is trivial that $\langle g \rangle$ is locally nilpotent, since it is cyclic and therefore nilpotent. Take the smallest d such that $G^{(d)}$ (the d-th derived subgroup) is nontrivial. If $d \leq 1$, then G is abelian and $\langle g \rangle \lhd G$. So assume that $d > 1$, and let $A = G^{(d-1)}$, note that A is abelian since $A^{(1)} = G^{(d)} = 1$. We work by induction on d. It is easy to see that gA is a left-Engel element of G/A, since the derived length of G/A is strictly smaller than d we find that $\langle gA \rangle$ is an ascendant subgroup of G/A or that $\langle g, A \rangle$ is an ascendant subgroup of G.

The next thing we will show is that $\langle g \rangle$ is a ascendant in $\langle g, A \rangle$. This suffices since we can combine the ascending series. Define the map $\ell : A \to A : a \mapsto [a, g]$. Note that since the derived subgroup is characteristic, this map is well defined. Take any nontrivial element $x \in A$, then we can find a minimal n such that $[x,_n g] = 1$. If $n = 1$ then x lies in $C_A(g)$. If $n \geq 2$ this implies that $\ell^{n-1}(x)$ is an element of $C_A(g)$, hence we can always find a nontrivial element in the centraliser of g in A. Note that we can repeat this argument for any nontrivial quotient of A. Define $A_0 = 1$ and then inductively $A_{\alpha^+}/A_\alpha = C_{A/A_\alpha}(g)$ and $A_\lambda = \bigcup_{\beta<\lambda} A_\beta$, whenever λ is a limit ordinal. Since this is an strictly increasing series, there exists some ordinal β such that $A_\beta = A$. We now claim that the series $\langle g, A_\alpha \rangle$ is an ascending series. We check that terms indexed by successive ordinals are normal, i.e. if $\langle g, A_\alpha \rangle \lhd \langle g, A_{\alpha^+} \rangle$. Because all A_α are subsets of an abelian group and hence abelian, it suffices to check that $[a, g] \in A_\alpha$ for all $a \in A_{\alpha^+}$. Since by construction $A_{\alpha^+}/A_\alpha = C_{A/A_\alpha}(g)$ this is the case. The other necessary properties for an ascending series follow by construction. □

5.2.2 Noetherian Groups

Another interesting extension of Baer's Theorem was given by Baer himself, shortly after the original proof. The proof we follow here is again based on Robinson's book.

Lemma 5.2.6. Let G be a Noetherian group and let a be a left-Engel element of G, then the subgroup generated by all conjugates of a, denoted a^G is finitely generated and nilpotent.

We will closely follow the structure of the proof given in chapter 12 of Robinson.

Proof. We will divide the proof in several steps. We will work by contradiction to create a group which we will then use to construct an infinite strictly ascending series, contradicting the ascending chain condition.

So assume that the statement in the lemma is false.

1. We first consider the conjugacy class $\mathrm{Cl}(a)$ of a. We call a subgroup of G a-generated if it is generated by the conjugates of a it contains. We then claim that if we have two nilpotent a-generated groups X and Y such that X is a strict subgroup of Y then the normaliser $N_Y(X)$ of X in Y will contain at least one conjugate of a that is not contained in X.

   As every subgroup of a nilpotent group is subnormal, we find a series

   $$X = X_0 \lhd \dots \lhd X_n = Y.$$

   Since Y is a-generated and strictly bigger than X there must be an integer i such that X_i contains the same conjugates of a as X but X_{i+1} contains some conjugate y of a that is not contained in X. We claim that $y \in N_Y(X)$. As $y \in X_{i+1}$ we have that

   $$(X \cap \mathrm{Cl}(a))^y = (X_i \cap \mathrm{Cl}(a))^y = (X_i \cap \mathrm{Cl}(a)) = X \cap \mathrm{Cl}(a).$$

   By assumption X is generated by its intersection with the conjugacy class of a, hence $y \in N_Y(X)$.

2. Next we will show that there exist at least two distinct maximal a-generated nilpotent subgroups of G. Assume there was only one such subgroup, call it U. By the ascending chain condition we have that every a-generated nilpotent subgroup is contained in U, otherwise we could build an strictly ascending chain from this group not in U since it is not contained in any maximal subgroup. In particular every conjugate of U is contained in U, implying that U is normal in G. However $\langle a \rangle$ is also contained in U, since it is trivially nilpotent and a-generated, but then by normality also $\langle a^G \rangle \leq U$ which contradicts the assumption that $\langle a^G \rangle$ is not nilpotent.

3. We now start with the construction of the infinite strictly ascending series.

   Let U and V be two distinct maximal a-generated subgroups of G, and let I be the group generated by all conjugates of a contained in both U and V. Choose U and V such that I is maximal, note that such a choice must exist by the ascending chain condition derived from the fact that G is Noetherian. Define another subgroup W as,

   $$W = \langle N_U(I) \cap \mathrm{Cl}(a) \rangle.$$

   Since U and V are distinct $I < V$, which in turn implies that $I \neq U$ since this would contradict the maximality of U as a subgroup. Part 1 tells us that $N_U(I)$ contains some conjugate of a that is not contained in I, implying that $I < W$. By changing U and V in the previous argument we find some $v \in N_V(I) \cap \mathrm{Cl}(a)$ such that $v \notin I$, note that this implies that $v \notin U$ as otherwise it would be contained in I by construction.

   This v could lie inside or outside of the normalizer of W in G, we will show that both cases lead to a contradiction.

   Assume $v \in N_G(W)$, and let $a^{g_1}, \dots a^{g_r}$ be the conjugates of a generating W, note that W is finitely generated by conjugates of a by the ascending chain condition and its definition. Since conjugates of left Engel elements are left Engel (Corollary 2.2.4) we can find a positive integer n such that $[a^{g_i},_n v] = 1$ for all $i \in \{1, \ldots, r\}$. Let $Z = \langle W, v\rangle$, then by assumption $W < Z$. Consider the derived subgroup of W, denoted $W^{(1)}$, we claim that $W^{(1)} \lhd Z$. It is clear that $[W, W] \lhd W$ and since $v \in N_G(W)$ by assumption for any $b, c \in W$ it holds that

$$\begin{aligned}[b, c]^v &= v^{-1}b^{-1}c^{-1}bcv \\ &= v^{-1}b^{-1}vv^{-1}c^{-1}vv^{-1}bvv^{-1}cv \\ &= [b^v, c^v] \in [W, W].\end{aligned}$$

We now claim that $Z/W^{(1)}$ is nilpotent. By taking the quotient by $W^{(1)}$ we see that to check nilpotency we must only consider commutators of the form $[z,_r v]$ for $z \in Z$ and integers r, but we showed earlier that this holds whenever z is a generator. We find that $Z/W^{(1)}$ is nilpotent of class at most n. Because W is a subgroup of U and U was assumed nilpotent, W is nilpotent. These are all the right criteria to use Hall's Criterion, Theorem 1.2.9, and hence we find that Z is nilpotent.

Because v is a conjugate of a and W is a-generated, we find that H is a-generated and hence by similar reasoning as in part 2 contained in some maximal a-generated nilpotent subgroup T of G. Combining previous constructions we get that $W \leq H \leq T$ and hence also $N_U(I) \cap \mathrm{Cl}(a) \subseteq U \cap T \cap \mathrm{Cl}(a)$. Since $v \in T \setminus U$ we have found two distinct maximal a-generated nilpotent subgroups of G such that $I < W \leq \langle U \cap T \cap \mathrm{Cl}(a)\rangle$, contradicting the maximality of I.

Now assume that $v \notin N_G(W)$ and choose some $u \in N_U(I) \cap \mathrm{Cl}(a)$ that is not contained in I, which exist by part 2. Let k be the smallest integer such that $[v,_k u] \in N_G(W)$ which exists since v is left Engel. Note that $k \neq 0$ since this would imply $v \in N_G(W)$ which contradicts the assumption. Denote $z = [v,_{k-1} u]$, by our choice of k we know that $[z, u] = z^{-1}u^{-1}zu = (u^z)^{-1}u \in N_G(W)$, or $u^z \in N_G(W)$ as $u \in W$. By an analogous reasoning as before we find that $\langle W, u^z\rangle$ is contained in some maximal a-generated nilpotent subgroup T, which will contradict maximality unless $T = U$, hence also $u^z \in U$.

Since $u, v \in N_G(I)$, any element that can be written as a product of u and v and their inverses will be contained in $N_G(I)$, in particular $z \in N_G(I)$. As u^z is a conjugate of a contained in U, $u^z \in N_U(I) \cap \mathrm{Cl}(a) \subseteq W$, but also $u^z \in W^z$ since $u \in W$. Note that $I = I^z \leq W$ since $z \in N_G(I)$ and $u^z \notin I$ because conjugation with z^{-1} would imply $u \in I$ and hence,

$$I < \langle U \cap W^z \cap \mathrm{Cl}(a)\rangle.$$

By applying the same reasoning again we get that $W^z \leq T$ where T is a maximal a-generated nilpotent subgroup of G, and that $T = U$ by maximality of I, or $W^z \leq U$. This implies that

$$(N_U(I) \cap \mathrm{Cl}(a))^z \subseteq N_G(I)^z \cap U \cap \mathrm{Cl}(a) = N_U(I) \cap \mathrm{Cl}(a),$$

where the equality follows from the fact that $z \in N_G(I)$. This implies that $W^z \leq W$, combining this with the fact that $z \notin W$ since by construction it is not contained in the normalizer of W in G, we find that $W^z < W$. Conjugating with z^{-1} we get $W < W^{z^{-1}}$, and by repeating this we get an infinite strictly ascending series,

$$W < W^{z^{-1}} < W^{z^{-2}} < W^{z^{-3}} < \dots$$

□

This lemma is the main work we needed to do to prove the following extension of Baer's Theorem.

Theorem 5.2.7. *Let G be a Noetherian group, then $L(G) = HP(G) = F(G)$.*

Proof. By Lemma 5.2.1, all that remains to show is that $L(G) \subseteq HP(G)$. Let $a \in L(G)$, by the previous lemma $\langle a^G \rangle$ is nilpotent and hence contained in the Hirsch-Plotkin radical. Since every subgroup $\langle a^G \rangle$ is normal and nilpotent, every finite product of such groups will be nilpotent by Fittings Theorem. By the ascending chain condition, derived from the fact that G is Noetherian, there must be some set $\{a_1, \dots, a_r\} \subseteq L(G)$ such that taking the product of $\langle a_1^G \rangle \dots \langle a_r^G \rangle$ with some $\langle b^G \rangle$ with $b \in L(G)$ does not enlarge the group. This product must then be equal to the Hirsch-Plotkin radical, implying it is normal. As any normal nilpotent subgroup is contained in the Fitting subgroup, which in turn is contained in the Hirsch-Plotkin radical, we find that $HP(G) = F(G)$. □

Note that this is indeed an extension, since any finite group trivially satisfies the maximal property, and in that case $HP(G) = F(G)$.

5.2.3 Radical Groups

The contents of this section is based on the original proof by Plotkin [Plotkin, 1954, Plotkin, 1955]. The result itself is well known, however no English version of the basic proof by Plotkin seems to be available, this section is an expanded version and translation of the proof given originally in Russian. An alternative proof using some more advanced concepts is given by Robinson in [Robinson, 2013].

Definition 5.2.8. A group G is called *radical* if there exists some ordinal β and a set of subgroups $\{H_\alpha \mid \alpha \leq \beta\}$ such that

1. $H_0 = 1$ and $H_\beta = G$,
2. $H_{\alpha_1} \leq H_{\alpha_2}$ whenever $\alpha_1 \leq \alpha_2$,
3. $H_\alpha \lhd H_{\alpha+1}$,
4. γ, $H_\gamma = \bigcup_{\alpha<\gamma} H_\alpha$, for any limit ordinal,
5. $H_\alpha / H_{\alpha+1}$ is locally nilpotent for all α.

Let β be the smallest ordinal such that a set of subgroups of this kind can be found, then G is said to have *radical class* β.

Let us define a series that satisfies these conditions for any group, similar to how we defined the Fitting series earlier.

Definition 5.2.9. The *(upper) Hirsch-Plotkin series* of a group G is defined recursively by setting $HP_0(G) = 1$ and by defining $HP_{\alpha+1}(G)$ such that

$$\frac{HP_{\alpha+1}(G)}{HP_\alpha(G)} = HP\left(\frac{G}{HP_\alpha(G)}\right).$$

We also define $HP_\gamma = \bigcup_\alpha < \gamma HP_\alpha$ whenever γ is a limit ordinal.

It is not necessary that for an arbitrary group this series reaches the group, it can be shown that groups which coincide with a term in their Hirsch-Plotkin series are exactly the radical groups, and that the Hirsch-Plotkin series is a shortest possible radical series.

Theorem 5.2.10. *[Plotkin, 1955, Teopema 9] When G is a radical group $L(G) = HP(G)$.*

We will use the following lemma, the proof is technical and is omitted here.

Lemma 5.2.11. Let H be an ascendant subgroup of a group G such that there exists an ascendant series with locally nilpotent factors

$$H = H_0 \lhd H_1 \lhd \dots H_\alpha \lhd H_{\alpha+1} \lhd \dots \lhd H_\beta = G.$$

Assume further that there is some subgroup K such that $H_\alpha \leq K \leq H_{\alpha+1}$ and K/H_α is finitely generated, then $HP(K) \leq HP(G)$.

To keep the proof short and legible, we first show some more results.

Lemma 5.2.12. Let $h_1, h_2, \dots, h_n$ be a finite collection of elements of some subgroup H of G, and let $g \in L(H)$. There exists some finitely generated subgroup K of H containing all h_i such that this subgroup is invariant under the natural conjugation action of g.

Proof. Define $K(g, h)$ as the subgroup generated by all elements of the form $[h,_n g]$ for all $n \geq 0$ ($[h,_0, g] = h$]. Note that this is a finitely generated subgroup since g is assumed to be a left Engel element so for some N it holds that $[h,_n g] = 1$ whenever $n \geq N$. Now define A as

$$A = \langle K(g, h_1), K(g, h_2), \dots, K(g, h_n)\rangle.$$

It is clear that all h_i are contained in A and that it is finitely generated. Now consider the conjugation by g of a generator, the following commutator relation is enough to conclude.

$$g^{-1}[h_{i},_n g]^{-1} g [h_{i},_n g] = [h,_{n+1} g].$$

□

Lemma 5.2.13. A group generated by a locally nilpotent subgroup H of G and and a left Engel element g is locally nilpotent.

Proof. Take a finite number of elements in $\langle g, H\rangle$. Nontrivial cases will be generated by some $h_1, \ldots, h_s$ and g, we will show that these elements are contained in a nilpotent subgroup of G which is enough to conclude. We apply Lemma 5.2.12 to the elements $h_1, \ldots, h_s$ to get some g-invariant subgroup A. Since this is a subgroup of H, it is nilpotent, and since $\langle g, A\rangle$ is an extension of A by an element for which conjugation is invariant, $\langle g, A\rangle$ is nilpotent. We find that $g, h_1, \ldots, h_s$ are contained in a nilpotent subgroup and since this holds for any finite set of elements we have shown that $\langle g, H\rangle$ is locally nilpotent. □

Proof of Theorem 5.2.10. It suffices to show that $L(G) \leq HP(G)$. The proof is by transfinite induction on the radical class of G. If the radical class of G is one, then $HP(G) = G$, implying that G is locally nilpotent and hence Engel, from which we get that $L(G) = G = HP(G)$. Now assume that the theorem has been shown for radical classes up to γ. Note that for all $\alpha < \gamma$ it holds that all left Engel elements contained in HP_α lie in $HP(G)$ (by Lemma 5.2.11).

First we will assume that γ is a limit ordinal, then

$$G = HP_\gamma(G) = \bigcup_{\alpha<\gamma} HP_\alpha(G).$$

Let g be a left Engel element of G then $g \in HP_\delta(G)$ for some $\delta < \gamma$ and hence by the previous remark $g \in HP(G)$. So assume γ is not a limit ordinal, then there exists an ordinal $\mu = \gamma - 1$. If we have a left Engel element $g \in HP_\mu$, then the induction hypothesis holds, so assume $g \in HP_\gamma \setminus HP_\mu$. We again distinguish two cases

1. μ is a successor ordinal, i.e. there exists some ordinal $\mu - 1$. Now consider $\overline{G} := G/HP_{\mu-1}$, this is a radical group of class at most 2 since

$$1 = HP_{\mu-1}/HP_{\mu-1} \lhd \overline{HP_\mu} = HP_\mu/HP_{\mu-1} \lhd \overline{HP_\gamma} = \overline{G}$$

   is a radical series of length two. Note that $\overline{g} := gHP_{\mu-1}$ is a left Engel element of $\overline{G}$ but not of $\overline{HP_\mu}$. Consider $\langle \overline{g}, \overline{HP_\mu}\rangle$, since $\overline{HP_\mu}$ is a factor of the Hirsch-Plotkin radical series it is locally nilpotent. By Lemma 5.2.13 $\langle \overline{g}, \overline{HP_\mu}\rangle$ is locally nilpotent. By Lemma 5.2.11 we find that $HP(\langle \overline{g}, \overline{HP_\mu}\rangle) \leq HP(\overline{G}) = \overline{HP_\mu(G)}$ and hence $\overline{g} \in \overline{HP_\mu(G)}$ but that is a contradiction with the choice of g.

2. So assume that μ is a limit ordinal. We construct subgroups $H_\alpha := \langle g, HP_\alpha\rangle$ for all $\alpha \leq \mu$. For each $\alpha < \mu$ the set $\{H_\beta \mid \beta \leq \alpha\} \cup \{1\}$ forms a radical series of length strictly smaller than γ. So by induction, all left Engel elements of these subgroups are contained in their Hirsch-Plotkin radical. By a similar argument as before we can say that the left Engel elements of H_μ are contained in $HP(H_\mu)$. Using Lemma 5.2.11 again, we find that $HP(H_\mu) \leq HP(G)$, and hence $g \in HP(G)$.

□

5.3 Example when $HP(G) = L(G) \neq F(G)$

To illustrate the importance of the Hirsch-Plotkin radical, we construct an explicit example of a solvable group, where the Fitting subgroup does not coincide with the Hirsch-Plotkin radical, based on a Mathematics StackExchange answer [YCor, 2017].

We must first introduce the main building block of the group we will consider, namely the Prüfer p-group[1], where p is prime. This is an infinite group that can be defined in the following way.

$$\mathbb{Z}_{p^\infty} := \{\xi \in \mathbb{C}^\times \mid \xi^{p^n} = 1, \text{ for some } n\},$$

or in additive notation

$$\mathbb{Z}_{p^\infty} := \{\xi \in \mathbb{C} \mid p^n\xi = 0, \text{ for some } n\}.$$

We can then define endomorphisms of $\mathbb{Z}_{p^\infty}$ by taking sequences $A = (a_n)_{n=0}^{\infty}$ with $a_n \in \mathbb{Z}$ and letting them act in the following way.

$$\varphi_A : \mathbb{Z}_{p^\infty} \to \mathbb{Z}_{p^\infty} : \xi \mapsto \sum_{n=0}^{\infty} a_n p^n \xi.$$

Note that this is a well defined endomorphism, as there exists n such that $p^n\xi = 0$ and hence $\varphi_A(\xi) \in \mathbb{Z}_{p^\infty}$. In this way we can see p-adic integers as endomorphisms of $\mathbb{Z}_{p^\infty}$, by defining the action of a $x \in \mathbb{Z}_p$ as $x \cdot \xi := \varphi_x(\xi)$. It is not difficult to see that p-adic units, i.e. p-adic integers such that $p_0 \not\equiv 0 \mod p$, correspond to automorphisms of $\mathbb{Z}_{p^\infty}$, the opposite is also true, but is less clear.

Let us consider the case when $p = 2$, and we take the automorphism x that sends ξ to ξ^3. This automorphism is of infinite order since it sends $e^{\pi i}$ to itself. Let $(a_n)_{n=0}^{\infty}$ be the 2-adic integer corresponding to x.

Consider the group G defined as the semidirect product of $\langle x\rangle$ and $\mathbb{Z}_{2^\infty}$. We will show that the group is locally nilpotent, implying that its Hirsch-Plotkin radical is the entire group, but that its Fitting subgroup is the subgroup corresponding to $\mathbb{Z}_{2^\infty}$.

Take a finite set of elements in the semidirect product. If no power of the automorphism x is contained, then the group is nilpotent since $\mathbb{Z}_{2^\infty}$ is abelian. Without loss of generality we can assume the set looks like $\xi_1, \xi_2, \dots, \xi_n, x^m$, all considered in the semidirect product. Since we only have a finite amount of elements ξ_i there exists some N such that we can consider the ξ_i to lie in a subgroup $\mathbb{Z}/N\mathbb{Z}$. Since x is a unit taking large enough commutators with any element will make the commutator trivial, hence the lower central series reaches one at some point.

We know that $\mathbb{Z}_{2^\infty}$ is a normal abelian subgroup of the semidirect product and hence $\mathbb{Z}_{2^\infty} \leq F(G)$. On the other hand we have that any subgroup that contains

[1] In some texts this group is called the p-quasicyclic group, as it can be seen as a generalization of cyclic groups.

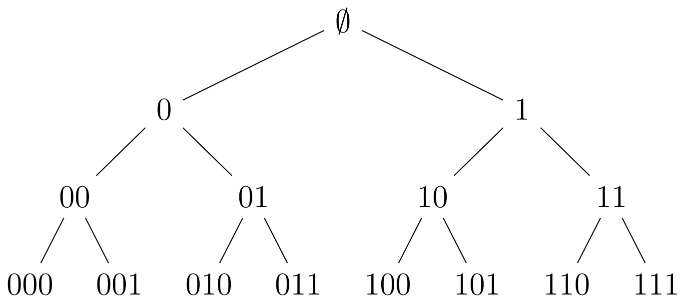
$\mathbb{Z} \rtimes \langle x^k \rangle$ for some k, such that $x^k \neq 1$. We claim that these subgroups are not nilpotent. We make use of the fact that we can see $\mathbb{Z}_{2^\infty}$ as the 2-adic numbers quotiented out by the 2-adic integers, $\mathbb{Q}_2/\mathbb{Z}_2$. Taking the commutator $[x^k, \mathbb{Q}_2] = \mathbb{Q}_2$ and then taking the quotient with $\mathbb{Z}_2$ results in $[x^k, \mathbb{Z}_{2^\infty}] = \mathbb{Z}_{2^\infty}$. Hence the subgroup is not nilpotent since repeatedly taking commutators does not result in the trivial subgroup.

5.4 Bludov's Argument and the Grigorchuk group

This section is based on [Noce and Tortora, 2018].

The results discussed in the previous sections might give the impression that the set of left Engel elements must be a subgroup in general. This turns out to be false, we briefly discuss a construction by Bludov which is a counterexample.

We start by introducing the concept of the 2-adic tree T_2, this is the set of words in the alphabet $\{0, 1\}$ combined with the empty word which can be considered as the root vertex. A graphic representation of the first tree levels is the following.



We can then define automorphisms on T_2.

Definition 5.4.1. A permutation σ of the elements of T_2 is an automorphism if the following conditions hold,

1. σ preserves the length of vertices,

2. σ respects the initial segment relation, i.e. if x is a initial segment of y then $\sigma(x)$ is an initial segment of $\sigma(y)$.

We can intuitively think of a automorphism of T_2 as "twisting" some of the branches. We will look at four important automorphisms, which we will label a, b, c and d.

The empty string $\emptyset$ is always sent to the empty string since length is preserved by an automorphism. For strings of length 1 we define a, b, c and d as follows,

$$a(1) = b(0) = c(0) = d(0) = 0,$$
$$a(0) = b(1) = c(1) = d(1) = 1.$$

For longer strings we define the automorphism in function of each other,

$$a(x) = \begin{cases} 1y & \text{if } x = 0y \\ 0y & \text{if } x = 1y \end{cases} \qquad c(x) = \begin{cases} 0a(y) & \text{if } x = 0y \\ 1d(y) & \text{if } x = 1y \end{cases}$$

$$b(x) = \begin{cases} 0a(y) & \text{if } x = 0y \\ 1c(y) & \text{if } x = 1y \end{cases} \qquad d(x) = \begin{cases} 0y & \text{if } x = 0y \\ 1b(y) & \text{if } x = 1y \end{cases}$$

We can also represent these morphisms graphically, where ellipses under a branch splitting implies the infinite subtrees remain unchanged, and ellipses at the right-most branch imply that this pattern repeats.

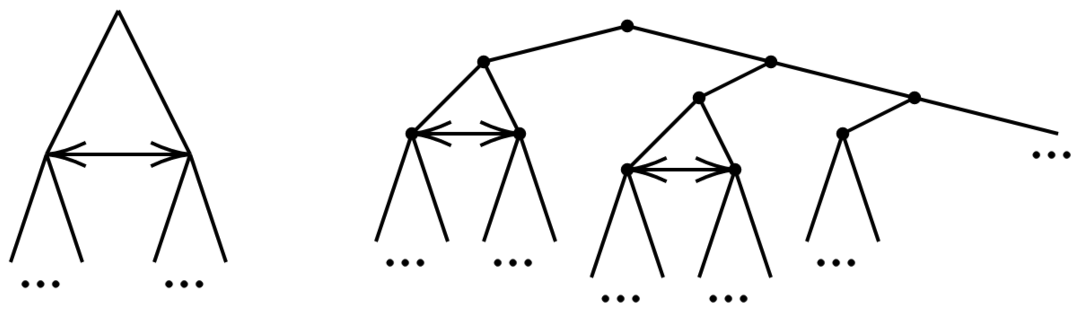


Figure 5.1: A graphical representation of automorphisms a and b.

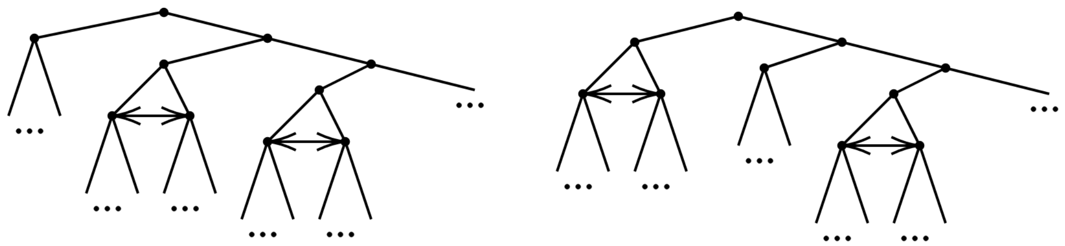


Figure 5.2: A graphical representation of automorphisms c and d.

We now consider the group generated by these four automorphisms, this group is commonly called the *(first) Grigorchuk group* after its constructor Rostislav Grigorchuk, and is denoted using Γ.

$$\Gamma := \langle a, b, c, d \rangle \leq \mathrm{Aut}(T_2).$$

This group has many interesting properties which make it an important example and counterexample in certain areas of group theory. We will not prove these properties, we do list some of the most important ones.

- Γ is finitely generated,
- Γ is infinite,
- Γ is a 2-group.

We have the following general result for 2-groups,

Lemma 5.4.2. Let G be a 2-group, and x an involution of G, i.e. $x^2 = 1$, then $x \in L(G)$.

Proof. We will first show that for all positive integers n we have the following commutator relation for any $g \in G$

$$[g,_n x] = [g, x]^{(-2)^{n-1}}.$$

We will proceed by induction on n, the case $n = 1$ being clear. Before showing the necessary calculations to prove the induction step, we note the following,

$$[g, x]^x = x^{-1}g^{-1}x^{-1}gxx = (g^{-1}x^{-1}gx)^{-1} = [g, x]^{-1}.$$

Now consider $[g,_{n+1} x]$,

$$\begin{aligned}
[g,_{n+1} x] &= [g,_n x]^{-1}[g,_n x]^x, && \text{(by definition of commutators)}\\
&= \left([g, x]^{(-2)^{n-1}}\right)^{-1} \left([g, x]^{(-2)^{n-1}}\right)^x, && \text{(by induction hypothesis)}\\
&= \left([g, x]^{(-1)(-2)^{n-1}}\right) \left([g, x]^x\right)^{(-2)^{n-1}},\\
&= \left([g, x]^{(-1)(-2)^{n-1}}\right) \left([g, x]^{(-1)(-2)^{n-1}}\right),\\
&= [g, x]^{(-2)^n}.
\end{aligned}$$

As G is a two group, there will be an integer n depending on g such that $[g, x]^{2^n} = 1$, hence $[g,_{2n} x] = 1$ which implies that x is a left Engel element. $\square$

The construction of Bludov makes use of the wreath product of two groups, which we will now define. This definition is for a basic case which we need here, more general definitions also exist.

Definition 5.4.3. The *(restricted, regular) wreath product* of two groups G and H, denoted $G \wr H$, is defined as $H \ltimes G^H$, where G^H is the direct sum of G indexed by H. The action of H on G^H is by permutation of the coordinates, defined by left multiplication.

The wreath product we are interested in is $\Gamma \wr D_8$. Instead of taking the usual presentation of D_8, we define it as being generated by two involutions,

$$D_8 := \langle d_1, d_2 \mid d_1^2 = d_2^2 = 1, (d_1 d_2)^4 = 1\rangle.$$

The wreath product we then get is the semidirect product of 8 copies of the Grigorchuk group, indexed by D_8, with D_8. Now let $t = d_1 d_2$, and consider the subgroup of $\Gamma \wr D_8$ indexed by $\langle t \rangle$ and with $1 \in D_8$ as the second coordinate.

The important claim that is shown by Bludov is that there is an element h of this subgroup, where $h = (1, ab, ca, d)$ such that $[h,_n t] \neq 1$ for all positive integers n, where the commutator is understood to be taken in the semidirect product. Since h is also the product of two involutions, $(1, a, c, d)$ and $(1, b, a, 1)$, it is equal to the product of two left Engel elements, but it is not an Engel element itself. This shows that in this case, $L(G)$ does not form a subgroup.

Chapter 6

Determining the Fitting Subgroup

This chapter collects material from [Eick, 2001] and [GAP, 2021]

The aim of this chapter is to give an overview of several methods that can be used to compute the Fitting subgroup for certain classes of groups. As shown in previous chapters, this subgroup is useful since it often characterizes the set of Left Engel elements. We will look at several theoretical methods and their implementations in GAP, along with some examples.

6.1 Finite Groups

The most straightforward method of computing the Fitting subgroup is by following the original definition, computing the subgroup generated by all nilpotent normal subgroups. The following program is a simple implementation of this idea in GAP. It tests whether a given normal subgroup is nilpotent and adds its generators to a list if this is the case, it returns the group generated by this list after going through all possible normal subgroups.

```
FittingSubgroupByNilpNorm := function(G)
    local normal, i, j, gens, genssub, fit;
    gens := [];
    normal := NormalSubgroups(G);
    for i in [1..Length(normal)] do
        if IsNilpotent(normal[i]) = true then
            genssub := GeneratorsOfGroup(normal[i]);
            for j in [1..Length(genssub)] do
                Add(gens, genssub[j]);
            od;
        fi;
    od;
    fit := Subgroup(G, gens);
    return fit;
end;
```

A second implementation could be to compute the product of all p-cores. This is the method that is used as a standard by GAP, and therefore an implementation can be found in the official source code, [GAP, 2021].

A third implementation is to make use of the concept of a chief series, this section is mainly based on chapter 3.4 in [Huppert, 1967].

Definition 6.1.1. Let G be a group, a *chief series* of G is a series of normal subgroups $N_0, \ldots, N_k$ of G,

$$1 = N_0 \lhd N_1 \lhd \ldots \lhd N_k = G,$$

such that the quotient N_i/N_{i-1} is a minimal normal subgroup of G/N_{i-1} for $i \in \{1, \ldots k\}$.

We can view the chief series as a type of maximal normal series. Whenever there exists a normal subgroup H of G such that $N_i < H < N_{i+1}$, its quotient $H/N_{i+1} \lhd G/N_{i+1}$ will be a normal subgroup of G/N_{i+1}. By the minimality condition in the definition, we cannot add this term to the series. Note that the chief series of a group is not necessarily unique, it is however possible to show that the number of factors in a chief series will always be fixed.

The factors of a chief series are called the chief factors and it turns out that these are connected to the Fitting subgroup for finite groups. Before stating the relation, we must first extend the notion of centralizer to quotient groups, and show a basic lemma.

Notation 6.1.2. Let G be a group H a subgroup and N a normal subgroup of both. If $C_{G/N}(H/N) = X/N$ then we define $C_G(H/N) := X$.

Remark 6.1.3. An equivalent way of defining this commutator that is a bit easier to work with is

$$C_G(H/N) = \{x \in G \mid [x, h] \in N, \forall h \in H\}.$$

Lemma 6.1.4. Let N be a minimal normal subgroup of a group G then $F(G) \leq C_G(N)$.

Proof. When N is abelian, N is also nilpotent and hence $N \leq F(G)$. Let $N_1 = N$ and $N_i = [N_{i-1}, F(G)]$. By nilpotency of $F(G)$ there must be some minimal index k such that $N_{k+1} = 1$, then N_k is nontrivial and commutes with $F(G)$, hence $N \cap Z(F(G)) > 1$. Since N is minimal normal, $N \leq Z(F(G))$ and hence also $F(G) \leq C_G(N)$.

So assume N is not abelian. Since commutator subgroups of normal subgroups are again normal $[N, N] = N$ since N is minimal and not abelian. This implies that N is not nilpotent and hence $N \not\leq F(G)$. Since the intersection of two normal subgroups is normal we also find that $F(G) \cap N = 1$. By Lemma 4.1.3 we find that $F(G) \leq C_G(N)$.

□

Theorem 6.1.5. *[Baer, 1957, Satz III.4.3] Let* $1 = N_0 \lhd N_1 \lhd \ldots \lhd N_k = G$ *be a chief series of* G*, then*

$$F(G) = \bigcap_{i=1}^{k} C_G(N_i/N_{i-1}).$$

Proof. Set $C = \bigcap_{i=1}^{k} C_G(N_i/N_{i-1})$, we will first show that $C \leq F(G)$.

Consider the following series by taking the intersection of the chief series with C,

$$1 = N_0 \cap C \lhd N_1 \cap C \lhd \ldots \lhd N_k \cap C = C.$$

We claim that this is a central series for C implying that it is nilpotent. It is clear that $[C \cap N_i, C] \leq C$ and $[C \cap N_i, C] \leq [N_i, C_G(N_i/N_{i-1})]$ because $N_i \leq C \cap N_i$ and $C \leq C_G(N_i/N_{i-1})$. By the equivalent definition given in Remark 6.1.3 we find that $[C_G(N_i/N_{i-1}), N_i] \leq N_{i-1}$. Combining this, we find

$$[C \cap N_i, C] \leq C \cap [N_i, C_G(N_i/N_{i-1})] \leq C \cap N_{i-1},$$

which is enough to show that we do indeed have a central series. Since the centralizer is a normal subgroup of G, C is a finite intersection of normal subgroups making it normal. Because the Fitting subgroup is generated by all nilpotent normal subgroups, C is contained in $F(G)$.

For the other inclusion we will show that $F(G) \leq C_G(N_i, N_{i-1})$ for $i \in \{1, \ldots, k\}$. First note that by definition of the chief series, N_i/N_{i-1} is a minimal normal subgroup of G/N_{i-1}, and that $F(G)N_{i-1}/N_{i-1}$ is a nilpotent normal subgroup of G/N_{i-1}. Using Lemma 6.1.4 we then find,

$$F(G)N_{i-1}/N_{i-1} \leq F(G/N_{i-1}) \leq C_{G/N_{i-1}}(N_i/N_{i-1}) = C_G(N_i/N_{i-1})/N_{i-1}.$$

This finishes the proof. □

To implement this theorem in GAP, we first make the observation that instead of taking the intersection of all centralizers, we can just consecutively take centralizers in previous centralizers. This eliminates a lot of computations and makes the algorithm more efficient.

```
FittingSubgroupByChiefFactor := function(G)
    local chief, n, C, i, nat, quot, quot2, cent, prem;
    chief := ChiefSeries(G);
    n := Length(chief);
    C := G;
    for i in [1..Length(chief)-1] do
        nat := NaturalHomomorphismByNormalSubgroup(C, chief[i
            +1]);
        quot := Image(nat);
        quot2 := Image(nat, chief[i]);
        cent := Centralizer(quot, quot2);
```

```
        C := PreImages(nat, cent);
        od;
    return C;
end;
```

A last method would be to directly compute whether any given element is left Engel as these elements are exactly those in the Fitting subgroup by Baer's Theorem. There are two main issues with this idea, both on the technical and mathematical side. The first is that going over every single element is very time consuming, the second is that it is not immediately clear when we can stop checking if a certain element is left Engel, since there is no predefined bound for the Engel degree of a specific element.

6.1.1 Comparing the Previous Methods

Out of the three methods described thus far, it is clear that the method that computes the group generated by all nilpotent normal subgroups will be the slowest. The other two methods are very different, and it is unclear which of the two will be faster without testing.

We checked which of these methods is more efficient by computing the runtime of the algorithms on a library of perfect groups in GAP, this led to the following results, the choice for perfect groups was made based on availability of such a library, and the fact that perfect groups are not nilpotent, thus avoiding some trivial cases. All data is expressed as the average time in milliseconds.

	p-cores	Chief Series	Nilpotent Normal Subgroups
$\lvert G\rvert \leq 10000$	8	7	19.5
$10000 \leq \lvert G\rvert \leq 100000$	11	7.5	33

6.2 The Fitting Subgroup for Infinite Groups

Determining the Fitting Subgroup of infinite groups is a harder task for two main reasons. Firstly, our existing characterizations using p-cores or chief series, do not hold in general for infinite groups. Secondly, we cannot just calculate the group generated by all nilpotent normal subgroups, as this is not necessarily a finite process.

We will consider a specific case of groups (which are often infinite) called polycyclic groups, for which it is possible to generalize the chief series method to find the Fitting subgroup. We will show that these groups are Noetherian, implying that their left Engel elements coincide with the Fitting subgroup by Theorem 5.2.7.

6.2.1 Polycyclic Groups

We first introduce the polycyclic and polycyclic-by-finite groups and show that they are Noetherian.

This section is based on [Segal, 2005].

Definition 6.2.1. A group G is said to be poly-$\mathcal{P}$ if there exists a series of normal subgroups,

$$1 = H_0 \lhd H_1 \lhd \ldots \lhd H_n = G,$$

such that each factor H_i/H_{i+1} has property $\mathcal{P}$.

If we choose $\mathcal{P}$ to be the property that a group is cyclic, we end up with the poly-cyclic groups. To ease notation we will often omit the hyphen and simply write polycyclic.

Example 6.2.2. The infinite dihedral group D_∞ given by the presentation,

$$D_\infty := \langle s, r \mid s^2 = 1, srs = r^{-1} \rangle,$$

is polycyclic. We have the following normal series

$$1 \lhd \langle s \rangle \lhd D_\infty.$$

The quotient $D_\infty/\langle s \rangle$ is isomorphic to $\langle r \rangle$ which is isomorphic to the group of integers, and $\langle s \rangle$ is a cyclic group of order 2. △

We now want to show that polycyclic groups satisfy are Noetherian. Note that subgroups of cyclic groups are cyclic, this is a classical exercise of which a proof can be found in most undergraduate algebra textbooks. The next lemma, which is an exercise in Segal, gives us a useful result.

Lemma 6.2.3. Let $\mathcal{P}$ be a property of groups, such that any subgroup of a group with property $\mathcal{P}$ also has property $\mathcal{P}$, then any subgroup of a poly-$\mathcal{P}$ group, is also a poly-$\mathcal{P}$ group.

Proof. Let H be a subgroup of a poly-$\mathcal{P}$ group G. Then we have the following sequence of G such that the quotient of consecutive terms has $\mathcal{P}$

$$1 = G_0 \lhd G_1 \lhd \ldots \lhd G_{n-1} \lhd G_n = G.$$

Now consider the following sequence, where $H_i = H \cap G_i$

$$1 = H_0 \lhd H_1 \lhd \ldots \lhd H_{n-1} \lhd H_n = H.$$

We know $H_i \lhd H_{i+1}$ since for any element $h \in H_i$ and $g \in H_{i+1}$ we have that $ghg^{-1} \in G_i$ since $h \in G_i, g \in G_{i+1}$ and $G_i \lhd G_{i+1}$ but also $ghg^{-1} \in H$, since $h, g \in H$, hence $ghg^{-1} \in H \cap G_i = H_i$.

We now want to prove that $\frac{H_i}{H_{i-1}}$ has $\mathcal{P}$. We can write the following,

$$\frac{H_i}{H_{i-1}} = \frac{H \cap G_i}{H \cap G_{i-1}} = \frac{H \cap G_i}{H \cap G_{i-1} \cap G_i} = \frac{H \cap G_i}{H \cap G_i \cap G_{i-1}}.$$

The second equality follows from the fact that since $G_{i-1} \leq G_i$, $G_i \cap G_{i-1} = G_{i-1}$. Using the second isomorphism theorem we find that,

$$\frac{H \cap G_i}{H \cap G_i \cap G_{i-1}} \cong \frac{G_{i-1}(H \cap G_i)}{G_{i-1}}.$$

This is a subgroup of G_i/G_{i-1} since $G_{i-1}(H \cap G_i)$ is a subgroup of G_i, because the intersection and multiplication with a group is a subgroup. Hence H_i/H_{i-1} is a subgroup of G_i/G_{i-1}, giving it property $\mathcal{P}$. Therefore the sequence above is indeed a sequence making H poly-$\mathcal{P}$. □

Applying this to the cyclic case, we find that subgroups of polycyclic groups are polycyclic. The last thing we will need to show that polycyclic-by-finite groups are Noetherian is that polycyclic groups are finitely generated.

Lemma 6.2.4. A polycyclic group is finitely generated.

Proof. Let G be a polycyclic group and

$$1 = H_0 \lhd H_1 \lhd \ldots \lhd H_n = G,$$

a normal series with cyclic quotients. We will show that G is finitely generated by induction on the length of the normal series, the base case being clear.

Assume that H_{n-1} is finitely generated by $\{g_1, ..., g_r\}$ and consider the quotient G/H_{n-1}, which is cyclic by construction. Let gH_{n-1} be a generator of the quotient then clearly $\langle g_1, \ldots, g_r, g\rangle = G$. □

This is all we need to show the following theorem.

Theorem 6.2.5. *Polycyclic groups satisfy are Noetherian.*

Proof. A group is Noetherian if every subgroup is finitely generated. Since every subgroup of a polycyclic group is polycyclic, it follows by the previous lemma that every subgroup is finitely generated. □

6.2.2 Polycyclic-By-Finite Groups

We can extend the notion of a polycyclic group to a slightly larger class of groups.

Definition 6.2.6. Let $\mathcal{P}$ and $\mathcal{Q}$ be properties of groups. A group G is called $\mathcal{P}$-by-$\mathcal{Q}$ if there exists some normal subgroup N of G satisfying $\mathcal{P}$ such that G/N satisfies $\mathcal{Q}$.

Letting property $\mathcal{P}$ be polycyclic, and $\mathcal{Q}$ be finite, we get the polycyclic-by-finite groups. Another way to describe these groups is by saying that they are finite extensions of polycyclic groups. We can come up with examples by taking the direct product of a polycyclic group with a finite group.

Example 6.2.7. We now discuss a less trivial example of a polycyclic-by-finite group. Consider the integers $\mathbb{Z}$, and take the semidirect product with $\mathrm{Aut}(\mathbb{Z})$. It is clear that $\mathbb{Z}$ is a normal subgroup that is cyclic, and hence polycyclic. The quotient by $\mathbb{Z}$ leaves us with a group that is isomorphic to $\mathrm{Aut}(\mathbb{Z})$, this group is finite as the only automorphisms of the integers are id sending 1 to 1 and $-$id sending 1 to -1. △

We claim that polycyclic-by-finite groups are also Noetherian. This will follow from the following lemma in [Segal, 2005].

Lemma 6.2.8. Let G be a group and N a normal subgroup of G then both N and G/N are Noetherian if and only if G is Noetherian.

Proof. We start by showing the "if" direction. The fact that any subgroup of a Noetherian group is Noetherian is clear. Take any subgroup K of the quotient G/N, then $K = L/N$ for some $L \leq G$. Since L is finitely generated, G/N is too and hence every subgroup of G/N is finitely generated, showing that G/N is Noetherian.

For the "only if" we will show that any subgroup H of G is finitely generated. First consider the intersection $H \cap N$, this subgroup is finitely generated since N is Noetherian. Let $h_1, \ldots, h_k$ be its generators. Secondly HN/N is finitely generated since HN/N is a subgroup of G/N, Let $b_1N, \ldots, b_rN$ be its generators. It is then clear that $H = \langle h_1, \ldots, h_k, b_1, \ldots, b_r \rangle$. □

Noting that any finite group is trivially Noetherian is enough to show that any polycyclic-by-finite group is Noetherian.

Remark 6.2.9. Up until 1978 it was unknown whether there existed any Noetherian group that was not polycyclic-by-finite. This was answered negatively by Olshanskii in his construction of Tarski monster groups [Olshanskii, 1980]. These are a type of infinite groups such that every proper subgroup, besides the trivial subgroup, are of prime order p. Olshanskii's construction showed the existence of such groups for $p > 10^{75}$, which additionally had the property that they were Noetherian and were not finitely presented. Polycyclic-by-finite groups are finitely presented, and hence a counterexample was found.

A property that will be useful in proving a characterization of the Fitting group in the polycyclic-by-finite case is the following lemma.

Lemma 6.2.10. A polycyclic-by-finite group has a unique maximal polycyclic normal subgroup which contains all polycyclic normal subgroups.

Proof. Let G be a polycyclic-by-finite group and H and K polycyclic normal subgroups. Assume first that H is not contained in a maximal polycyclic normal subgroup. This implies that we can find an infinite strictly ascending chain of polycyclic normal subgroups by repeatedly taking larger polycyclic normal subgroups. Since G is Noetherian this contradicts the ascending chain condition.

Now assume that H and K are both maximal and distinct and let $h_1, \ldots, h_k$ be generators for H and $k_1, \ldots, k_r$ for K such that adding generators creates a subnormal series with cyclic factors for both subgroups. By normality of H and K we find that the following series is a subnormal series with polycyclic factors for $\langle H, K\rangle$,

$$1 \lhd \langle h_1\rangle \lhd \langle h_1, h_2\rangle \lhd \ldots \lhd H \lhd \langle H, k_1\rangle \lhd \langle H, k_1, k_2\rangle \lhd \ldots \lhd \langle H, K\rangle.$$

Since the join of two normal subgroups is normal, this implies that $\langle H, K\rangle$ is a polycyclic normal subgroup, contradicting the maximality of H and K. □

6.2.3 Modules over Group Algebras

This section is also based on [Dornhoff, 1971].

Definition 6.2.11. Let R be any commutative ring, and G a group. We define the *group algebra* RG as the free R-module with basis G, where multiplication is defined as follows,

$$\left(\sum_{i=1}^{m} r_i g_i\right)\left(\sum_{j=1}^{n} s_j h_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n}(r_i s_j)(g_i h_j).$$

Since this is again a ring we can consider modules over this ring which we will then denote as RG-modules.

In particular we will be interested in group algebras of the type $\mathbb{F}_p G$ and $\mathbb{Q}G$.

Remark 6.2.12. An elementary abelian p-group is an abelian group such that every element has order p, in general we can see such a group as a vector space over $\mathbb{F}_p$ where the action of an element ζ of the field on an element n of the group is defined as

$$\zeta \cdot n := n^{\zeta}.$$

Now let $M \leq N$ be two normal subgroups of some group G such that N/M is an elementary abelian p-group. We can define an action of G on N/M by using conjugation

$$g \cdot nM := g^{-1} n g M.$$

Combining these two actions we can define an action of the group algebra $\mathbb{F}_p G$ on N/M by setting

$$\sum_{g\in G} \zeta_g g \cdot nM := \left(\prod_{g\in G} g^{-1} n^{\zeta_g} g\right) M.$$

This action is well defined in the way that it satisfies the criteria to see N/M as a $\mathbb{F}_p G$-module. As an example we work out distributivity of $\mathbb{F}_p G$ over N/M, the other properties are easily checked.

$$\begin{aligned}
\sum_{g\in G} \zeta_g g \cdot n_1 n_2 M &= \left(\prod_{g\in G} g^{-1}(n_1n_2)^{\zeta_g} g\right) M \\
&= \left(\prod_{g\in G} g^{-1} n_1^{\zeta_g}(gg^{-1}) n_2^{\zeta_g} g\right) M \\
&= \left(\prod_{g\in G} g^{-1} n_1^{\zeta_g} g\right)\left(\prod_{g\in G} g^{-1} n_2^{\zeta_g} g\right) M \\
&= \sum_{g\in G} \zeta_g g \cdot n_1 M \cdot \sum_{g\in G} \zeta_g g \cdot n_2 M
\end{aligned}$$

When N/M is free abelian we can see it in an analogous way as a $\mathbb{Z}G$-module. It is possible to extend the action to the rationals, and get a $\mathbb{Q}G$-module.

Definition 6.2.13. A module M over a ring R is called *irreducible* if the only submodules are the module itself and the zero module. A module is called *semisimple* if it is the direct sum of irreducible modules.

The following theorem is a generalization of Theorem 6.1.5 for polycyclic-by-finite groups.

Theorem 6.2.14. *[Eick, 2001, Theorem 2.1] Let G be a polycyclic-by-finite group, and let N be its largest polycyclic normal subgroup. Let*

$$1 = N_0 \lhd N_1 \lhd N_2 \lhd \ldots \lhd N_k = N,$$

be a normal series such that all factors are elementary or free abelian, and such that every factor is semisimple when seen as a $\mathbb{F}_pG$- or $\mathbb{Q}G$-module, then

$$F(G) = \bigcap_{i=1}^{k} C_N(N_i/N_{i-1}).$$

Proof. First note that the Fitting subgroup of a polycyclic-by-finite group can be characterized as the maximal nilpotent subgroup. This follows from the fact that for Noetherian groups the Fitting subgroup is nilpotent as discussed in the proof of Theorem 5.2.7. We claim that the Fitting subgroup of G is contained in N. To show this we first show that finitely generated nilpotent groups are polycyclic.

Let C be a finitely generated nilpotent group, we work by induction on the nilpotency class of C. If the nilpotency class of C is one, then C is finitely generated abelian, the primary decomposition of C then allows us to construct a normal series with cyclic factors. Assume that the claim holds for nilpotency class k and let C have nilpotency class $k+1$, then $\gamma_2(C)$ has nilpotency class k implying it is polycyclic. The quotient $C/\gamma_2(C)$ is then finitely generated abelian and hence polycyclic. Combining the two series gives us a normal series with cyclic factors.

Since the maximal nilpotent normal subgroup of G is also finitely generated it is polycyclic and hence contained in the largest polycyclic normal subgroup, implying that $F(G) \leq N$.

Set $C = \bigcap_{i=1}^{k} C_P(N_i/N_{i-1})$, the fact that $C \leq F(P)$ can be shown in exactly the same way as in the proof of Theorem 6.1.5. For the opposite inclusion we will show that every nilpotent normal subgroup F of N is contained in C.

Take any abelian normal subgroup A of P, we claim that the commutator subgroup $[A, F]$ is a strict subgroup of A. First assume that $A \not\leq F$, since A and F are normal we have that $[a, f] = a^{-1}f^{-1}af$ lies in A and F, or that $[A, F] \leq A \cap F$, since $A \cap F \neq A$, we find that $[A, F] < A$. Now assume that $A \leq F$, and that $[A, F] = A$. Since F is nilpotent this leads to a contradiction since $\gamma_i(F)$ can never be smaller than A under this assumption.

We will now apply this by taking $A := P_i/P_{i+1}$. First assume $P_{i+1} = 1$, then A is a subgroup of P. By construction of the sequence A is a semi-simple KP-module, and hence we can write A as the direct product of irreducible modules,

$$A = A_1 \times A_2 \times \ldots \times A_m.$$

Since irreducible modules are invariant under the action of the ring acting on them, we find that $A_j \lhd P$ for all $j \in \{1, \ldots, m\}$. Choose A_i for some i, and assume A_i is elementary abelian. By the reasoning above we find that $[A_i, P] < A_i$, and $[A_i, P]$ is an $\mathbb{F}_pP$-submodule of A_i which implies that $[A_i, P] = 1$, since we assumed A_i to be irreducible.

The case where A_i is free abelian is more difficult and requires some more theory about modules, which would lead us too far. □

6.2.4 Constructing the Fitting Subgroup

The process of actually constructing the Fitting subgroup of an arbitrary polycyclic-by-finite group using Theorem 6.2.14 can be deconstructed into four main steps, which we will now briefly discuss.

Step 1. Given a polycyclic-by-finite group G we must find its largest polycyclic normal subgroup N. This step is not considered in the article by Eick, she assumes N to be explicitly given. One reason might be that it is generally difficult to represent polycyclic-by-finite groups on a computer. In a more recent article by Sinanan and Holt [Sinanan and Holt, 2017], methods for representing polycyclic-by-finite groups and algorithms to compute certain subgroups are given, which could possibly lead to progress in automatically computing N. The implementation of these algorithms is an interesting project but lies outside of the scope of this master's thesis.

Step 2. Next, we want to construct a normal series for N such that each factor is elementary abelian or free abelian. An effective way of doing this is by using the following sketch of an algorithm further described in section 3 of *Computing with infinite polycyclic groups* [Eick, 1999].

Start with the polycyclic group N and consider its abelianization $N/[N,N]$. Decompose this group into a free abelian part $C/[N,N]$ and a torsion subgroup $T/[N,N]$. We now distinguish two cases.

Case 1. $N/[N,N]$ is infinite, we then choose T as the next term, since N/T will be a non trivial free abelian group.

Case 2. $N/[N,N]$ is finite, we choose a prime p dividing the order of $N/[N,N]$ and find the smallest subgroup K of N such that N/K is an elementary abelian p-group.

We then repeat the algorithm by setting N equal to the term chosen in either case 1 or case 2. An implementation of this algorithm can be found in the `gap/basic/pcpsers.gi` file of the polycyclic source code [Eick et al., 2022].

Step 3. The construction of a normal series in step 2 does not take into account that the factors have to be semisimple as kN-modules, where $k = \mathbb{F}_p$ or $\mathbb{Q}$ depending on whether the factor is elementary abelian or free abelian. To refine the series we introduce the radical of a module.

Definition 6.2.15. The radical of a module M, denoted $\mathfrak{j}(M)$ is the intersection of all submodules N such that M/N is irreducible.

Some useful properties are that $\mathfrak{j}(M) < M$ and that $M/\mathfrak{j}(M)$ is semisimple. This allows us to repeatedly take radicals to construct a series in which factors are semisimple. Let N/M be a free or elementary abelian factor of dimension n, then we can take $\mathfrak{j}(N/M) = J/M$ for some $J \leq N$. Add J as a term in the series, we then get

$$\ldots \leq M \leq J \leq N \leq \ldots.$$

The factor

$$J/N = (J/M)/(N/M) = \mathfrak{j}(N/M)/(N/M)$$

is semisimple by the properties of the radical, and J will be elementary or free abelian of a strictly lower dimension. Repeating this argument multiple times will lead to a refinement of the series between M and N. As this can be done for any two terms, we have a method to refine the entire series.

The method to compute the radical depends on whether the module is an $\mathbb{F}_pN$ or $\mathbb{Q}N$-module.

Case 1. When M is a $\mathbb{F}_pN$-module we can use the methods described in section 2.4 of [Szöke, 1998]. The *socle* of a module is defined as the sum of all simple submodules, it can be considered as a dual notion to the radical of a module. For any module M we can define its *dual module* M^* as the set of all module homomorphisms from M to $\mathbb{F}_pN$. Computing consecutive radicals of M is then equivalent to computing consecutive socles of M^* for which efficient algorithms exist.

Case 2. When M is a $\mathbb{Q}N$-module the methods are based on p-congruence subgroups, rational matrix algebras and Dickson's Theorem, a full exposition would lead us too far, details can be found in section 3.2 of [Eick, 2001].

Step 4. Let $1 = N_0 \lhd N_1 \lhd \ldots \lhd N_k = N$ be the normal series found after step 3. What remains to be done is to calculate the centralizers of the factors. Analogous to the implementation of the algorithm for the chief series version of the Fitting subgroup we will compute centralizers in consecutive groups to be more efficient.

These four steps show that it is possible to compute the Fitting subgroup and hence also the set of left Engel elements for polycyclic-by-finite groups, which can be infinite. This gives us a fourth implementation of an algorithm for computing the Fitting subgroup. We will not do this implementation ourselves as it is already written into the source code of GAP. The `polycyclic` package is written by Eick, who is the author of the article that this process was described in. As GAP is open source, a full implementation can be found in the source code on GitHub [Eick et al., 2022].

Conclusion and Further Research

In this thesis, we gave an overview on results about the set of left Engel elements of a group. We started with finite groups and showed the connection with the Fitting group, after which we extended this result to gain a notion of being near left-Engel based on the Fitting height. For infinite groups, we gave an exposition of three classical proofs regarding soluble, Noetherian and radical groups. Finally, we used the result on Noetherian groups to show how a method developed by Eick lets us compute the set of left Engel elements in the case of polycyclic-by-finite groups.

We propose some ideas for further research:

1. The theorem of Guralnick and Tracey only holds for the class of finite groups. In the process of writing this thesis we attempted to look for a generalization towards a theorem for infinite groups, however this idea seems very difficult. A first problem is that the generalized Fitting series does not necessarily reach the group for arbitrary infinite groups, an idea could be to use the Hirsch-Plotkin series and to manipulate this series to some generalized Hirsch-Plotkin series with the required properties. Another issue is that these series will have to be indexed by possibly transfinite ordinals, which further complicates the process. The last problem which needs further research is the result on subnormality by Flavell, as it only holds for very specific classes of infinite groups.

2. The set of left Engel elements is not the only set that is of interest in the study of Engel elements. The most straightforward continuation of this thesis would be to study the set of right Engel elements. In general much less is known about this set, although there are some similar results for certain classes of groups. The corresponding notion to the Hirsch-Plotkin radical seems to be the hypercenter, which is defined as the stable term in the transfinite upper central series of a group. Besides these large sets we could also look more in depth at the sets of bounded left and right Engel elements for which even less is known.

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Appendices

Appendix A

GAP code

The following code is used in Chapter 3.

```
# A collection of programs to calculate the
# (Generalized) Fitting Series/Height of finite groups
# Rune Buckinx


#Calculates the Fitting series and outputs it as a list

FittingSeriesOfGroup := function( G )
    local C, fits, nat, N, H;
    C := TrivialSubgroup(G);
    fits := [C];
    N := FittingSubgroup(G);
    while IndexNC( N, C) > 1 do
        C := N;
        Add( fits, C);
        nat := NaturalHomomorphismByNormalSubgroup( G, C);
        H := Image( nat );
        N := PreImage( nat, FittingSubgroup(H) );
    od;
    return fits;
end;


#Computes the length of the Fitting series

FittingHeight := function( G )
    local fit, d;
    fit := FittingSeriesOfGroup( G );
    d := Length(fit) - 1;
    return d;
end;
```

```
#Tests whether a given group is quasisimple
# (is used in later programs)

IsQuasisimple := function(G)
    local Quot, nat, Z;
    Z := Centre(G);
    nat := NaturalHomomorphismByNormalSubgroup( G, Z);
    Quot := Image( nat );
    if IsPerfectGroup(G) = true then
        if IsSimpleGroup(Quot) = true then
            return true;
        else
            return false;
        fi;
    else
        return false;
    fi;
end;


#Computes the components of a group

ComponentsOfGroup := function(G)
    local i, j, csubg, templist, comp, C;
    C := TrivialSubgroup(G);
    csubg := ConjugacyClassesSubgroups(G);
    comp := [C];
    for i in [1..Length(csubg)] do
        if IsQuasisimple( Representative(csubg[i]) ) = true
          then
            if IsSubnormal(G, Representative(csubg[i]) ) =
              true then
                templist := AsList(csubg[i]);
                for j in [1..Length(templist)] do
                    Add(comp, templist[j]);
                od;
            fi;
        fi;
    od;
    return comp;
end;


#Computes the layer of a group
```

```
LayerOfGroup := function(G)
    local comp, E, i, j, generat, gensub;
    generat := [];
    comp := FastComponentsOfGroup(G);
    for i in [1..Length(comp)] do
        gensub := GeneratorsOfGroup(comp[i]);
        for j in [1..Length(gensub)] do
            Add(generat, gensub[j]);
        od;
    od;
    E := Subgroup(G, generat);
    return E;
end;


#Computes the generalized Fitting subgroup

GeneralizedFittingSubgroup := function(G)
    local fit, layer, Gfit , fitel, layerel;
    fit := FittingSubgroup(G);
    layer := FastLayerOfGroup(G);
    fitel := GeneratorsOfGroup(fit);
    layerel := GeneratorsOfGroup(layer);
    Gfit := Subgroup(G, Union(fitel, layerel));
    return Gfit;
end;


#Computes the generalized Fitting series (analogous to
   Fitting series program)
GeneralizedFittingSeriesOfGroup := function( G )
    local C, fits, nat, N, H;
    C := TrivialSubgroup(G);
    fits := [C];
    N := FastGeneralizedFittingSubgroup(G);
    while IndexNC( N, C) > 1 do
        C := N;
        Add( fits, C);
        nat := NaturalHomomorphismByNormalSubgroup( G, C);
        H := Image( nat );
        N := PreImage( nat, GeneralizedFittingSubgroup(H) );
    od;
    return fits;
end;
```

```
#Computes generalized Fitting height
GeneralizedFittingHeight := function( G )
    local fit, d;
    fit := GeneralizedFittingSeriesOfGroup( G );
    d := Length(fit) - 1;
    return d;
end;
```

The following code is used in Chapter 4

```
#Computes the normal closure descending series
NormalClosureDescendingSeries := function(G,H)
    local N0, NCDS;
    N0 := NormalClosure(G,AsSubgroup(G,H));
    NCDS := [H, N0];
    while N0 <> NormalClosure(N0,AsSubgroup(N0,H)) do
        N0 := NormalClosure(N0,AsSubgroup(N0,H));
        Add(NCDS,N0);
    od;
    #N0 := NormalClosure(N0,AsSubgroup(N0,H));
    #Add(NCDS,N0);
    return NCDS;
end;
```

The following code is used in Chapter 6

```
#Computes the Fitting Subgroup by the method of the
   centralizer of the Chief series

FittingSubgroupByChiefFactor := function(G)
    local chief, n, C, i, nat, quot, quot2, cent, prem;
    chief := ChiefSeries(G);
    n := Length(chief);
    C := G;
    for i in [1..Length(chief)-1] do
        nat := NaturalHomomorphismByNormalSubgroup(C,chief[i
           +1]);
        quot := Image(nat);
        quot2 := Image(nat, chief[i]);
        cent := Centralizer(quot, quot2);
        C := PreImages(nat, cent);
        od;
    return C;
end;



#Computes the Fitting Subgroup by the method of taking all
   normal nilpotent subgroups

```

```
FittingSubgroupByNilpNorm := function(G)
    local normal, i, j, gens, genssub, fit;
    gens := [];
    normal := NormalSubgroups(G);
    for i in [1..Length(normal)] do
        if IsNilpotent(normal[i]) = true then
            genssub := GeneratorsOfGroup(normal[i]);
            for j in [1..Length(genssub)] do
                Add(gens, genssub[j]);
            od;
        fi;
    od;
    fit := Subgroup(G, gens);
    return fit;
end;


#Tests the speed of certain FittingSubgroup Methods

FittingPerfectTest := function(n)
    local sizes, nrsize, i, j, timestandard, timechief,
       timenilp, P, timetemp,
    avgstandard, avgchief, avgnilp;
    sizes := SizesPerfectGroups();
    timestandard := [];
    timechief := [];
    timenilp := [];
    avgstandard := 0;
    avgchief := 0;
    avgnilp := 0;
    for i in [1..n] do
        nrsize := NumberPerfectGroups(sizes[i]);
        if (nrsize <> fail) and (sizes[i] <> 86016) and (
           sizes[i] <> 368640) and (sizes[i] <> 737280) then
            for j in [1..nrsize] do
                P := PerfectGroup(IsPermGroup, sizes[i], j);
                timetemp := Runtime();
                FittingSubgroup(P);
                timetemp := Runtime()-timetemp;
                Add(timestandard,timetemp);
                timetemp := Runtime();
                FittingSubgroupByChiefFactor(P);
                timetemp := Runtime()-timetemp;
                Add(timechief,timetemp);
                timetemp := Runtime();
                FittingSubgroupByNilpNorm(P);
```

```
                timetemp := Runtime()-timetemp;
                Add(timenilp, timetemp);
            od;
        fi;
    od;
    for i in [1..n] do
        avgstandard := avgstandard + timestandard[i];
        avgchief := avgchief + timechief[i];
        avgnilp := avgnilp + timenilp[i];
    od;
    return [avgstandard,avgchief,avgnilp];
end;
```

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