1. 论证达朗贝尔算符 $\square := \partial^{\mu}\partial_{\mu}$ 是洛伦兹标量根据内积的洛伦兹变换不变性:

$$g_{\mu
u}=g_{\lambda
ho}\Lambda^{\lambda}_{\mu}\Lambda^{
ho}_{
u}$$

我们有:

$$egin{aligned} \Box &= \partial^{\mu}\partial_{\mu} = g_{\mu
u}\partial^{\mu}\partial^{
u} \ &
ightarrow g_{\lambda
ho}\partial^{\lambda}\partial^{
ho} = g_{\lambda
ho}\Lambda^{\lambda}_{\mu}\partial^{\mu}\Lambda^{
ho}_{
u}\partial^{
u} = g_{\mu
u}\partial^{\mu}\partial^{
u} \end{aligned}$$

故可知拉朗贝尔算符是洛伦兹标量。

2. 简要比较 $\partial^{i=1,2,3}$ 与 $\partial_{i=1,2,3}$ 的差别我们有:

$$egin{align} \partial^{\mu} &= (\partial^0,\partial^{i=1,2,3}) \ \partial_{\mu} &= g_{\mu
u}\partial^{
u} &= (\partial^0,-\partial^{i=1,2,3}) = (\partial_0,\partial_{i=1,2,3})
onumber \end{split}$$

对比可知:

$$\partial^{i=1,2,3} = -\partial_{i=1,2,3}$$

- 3. 如果一个(2,0)-型张量 $T^{\mu\nu}$ 满足 $T^{\mu\nu}=T^{\nu\mu}$,则称为一个**对称张量**;若它满足 $T^{\mu\nu}=-T^{\nu\mu}$ 则称为一个**反对称张量**。
 - i. 考虑任意一个(2,0)-型张量 $A^{\mu\nu}$ 。论证 $A^{\mu\nu}$ 一定可以写成一个**对称**张量和一个**反对称**张量的和。

给定(2,0)-型张量 $A^{\mu\nu}$, 我们总可以把其写成:

$$A^{\mu
u} = rac{1}{2}(A^{\mu
u} + A^{
u\mu}) + rac{1}{2}(A^{\mu
u} - A^{
u\mu})$$

上式第一项是对称张量,第二项是反对称张量。

ii. 假设 $F^{\mu\nu}$ 是一个反对称的张量,且 $J^{\mu}=\partial_{\nu}F^{\mu\nu}$ 。论证 J^{μ} 是一个守恒流。

考虑:

$$egin{aligned} \partial_{\mu}J^{\mu} &= \partial_{\mu}\partial_{
u}F^{\mu
u} = -\partial_{\mu}\partial_{
u}F^{
u\mu} = -\partial_{
u}\partial_{\mu}F^{\mu
u} \ \partial_{
u}J^{
u} &= \partial_{
u}\partial_{\mu}F^{
u\mu} = \partial_{\mu}J^{\mu} \end{aligned}$$

有:
$$\partial_
u\partial_\mu F^{
u\mu}=-\partial_
u\partial_\mu F^{
u\mu}=0$$
,故有: $\partial_\mu J^\mu=0$

得证。

4. 论证

$$rac{\partial}{\partial \partial_{\mu} \phi} (\partial_{
u} \phi \partial^{
u} \phi) = 2 \partial^{\mu} \phi \ , \qquad rac{\partial}{\partial \partial^{\mu} \phi} (\partial_{
u} \phi \partial^{
u} \phi) = 2 \partial_{\mu} \phi \ .$$

可以通过把 $\partial_{\nu}\phi\partial^{\nu}\phi$ 写成具体分量形式来说明。 依题意:

$$egin{aligned} rac{\partial}{\partial\partial_{\mu}\phi}(\partial_{
u}\phi\partial^{
u}\phi) &= rac{\partial\partial_{
u}\phi}{\partial\partial_{\mu}\phi}\partial^{
u}\phi + \partial_{
u}\phirac{\partial\partial^{
u}\phi}{\partial\partial_{\mu}\phi} &= \delta^{\mu}_{
u}\partial^{
u}\phi + \partial_{
u}\phi g^{
u\mu} &= 2\partial^{\mu}\phi \ rac{\partial}{\partial\partial^{\mu}\phi}(\partial_{
u}\phi\partial^{
u}\phi) &= rac{\partial\partial_{
u}\phi}{\partial\partial^{\mu}\phi}\partial^{
u}\phi + \partial_{
u}\phirac{\partial\partial^{
u}\phi}{\partial\partial^{\mu}\phi} &= g_{\mu
u}\partial^{
u}\phi + \partial_{
u}\phi\delta^{
u}_{\mu} &= 2\partial_{\mu}\phi \end{aligned}$$

故得证,此方法无需写成具体分量。