1. 论证达朗贝尔算符$\Box := \partial^\mu \partial_\mu$是洛伦兹标量

   根据内积的洛伦兹变换不变性：

$$g_{\mu\nu} = g_{\lambda\rho}\Lambda^\lambda_\mu \Lambda^\rho_\nu$$

   我们有：

$$\begin{aligned}\Box &= \partial^\mu \partial_\mu = g_{\mu\nu}\partial^\mu \partial^\nu \\ &\to g_{\lambda\rho}\partial^\lambda \partial^\rho = g_{\lambda\rho}\Lambda^\lambda_\mu \partial^\mu \Lambda^\rho_\nu \partial^\nu = g_{\mu\nu}\partial^\mu \partial^\nu\end{aligned}$$

   故可知拉朗贝尔算符是洛伦兹标量。

2. 简要比较$\partial^{i=1,2,3}$与$\partial_{i=1,2,3}$的差别

   我们有：

$$\begin{aligned}\partial^\mu &= (\partial^0, \partial^{i=1,2,3}) \\ \partial_\mu &= g_{\mu\nu}\partial^\nu = (\partial^0, -\partial^{i=1,2,3}) = (\partial_0, \partial_{i=1,2,3})\end{aligned}$$

   对比可知：

$$\partial^{i=1,2,3} = -\partial_{i=1,2,3}$$

3. 如果一个$(2,0)$-型张量$T^{\mu\nu}$满足$T^{\mu\nu} = T^{\nu\mu}$，则称为一个**对称张量**；若它满足$T^{\mu\nu} = -T^{\nu\mu}$则称为一个**反对称张量**。

   i. 考虑任意一个$(2,0)$-型张量$A^{\mu\nu}$。论证$A^{\mu\nu}$一定可以写成一个**对称**张量和一个**反对称**张量的和。

      给定$(2,0)$-型张量$A^{\mu\nu}$，我们总可以把其写成：

$$A^{\mu\nu} = \frac{1}{2}(A^{\mu\nu} + A^{\nu\mu}) + \frac{1}{2}(A^{\mu\nu} - A^{\nu\mu})$$

      上式第一项是对称张量，第二项是反对称张量。

   ii. 假设$F^{\mu\nu}$是一个**反对称**的张量，且$J^\mu = \partial_\nu F^{\mu\nu}$。论证$J^\mu$是一个**守恒流**。

      考虑：

$$
\begin{aligned}
\partial_\mu J^\mu &= \partial_\mu \partial_\nu F^{\mu\nu} = -\partial_\mu \partial_\nu F^{\nu\mu} = -\partial_\nu \partial_\mu F^{\mu\nu} \\
\partial_\nu J^\nu &= \partial_\nu \partial_\mu F^{\nu\mu} = \partial_\mu J^\mu
\end{aligned}
$$

有：$\partial_\nu \partial_\mu F^{\nu\mu} = -\partial_\nu \partial_\mu F^{\nu\mu} = 0$，故有：

$$
\partial_\mu J^\mu = 0
$$

得证。

4. 论证

$$
\frac{\partial}{\partial \partial_\mu \phi}(\partial_\nu \phi \partial^\nu \phi) = 2\partial^\mu \phi\,, \qquad \frac{\partial}{\partial \partial^\mu \phi}(\partial_\nu \phi \partial^\nu \phi) = 2\partial_\mu \phi\,.
$$

可以通过把$\partial_\nu \phi \partial^\nu \phi$写成具体分量形式来说明。

依题意：

$$
\begin{aligned}
\frac{\partial}{\partial \partial_\mu \phi}(\partial_\nu \phi \partial^\nu \phi) = \frac{\partial \partial_\nu \phi}{\partial \partial_\mu \phi}\partial^\nu \phi + \partial_\nu \phi \frac{\partial \partial^\nu \phi}{\partial \partial_\mu \phi} = \delta_\nu^\mu \partial^\nu \phi + \partial_\nu \phi g^{\nu\mu} = 2\partial^\mu \phi \\
\frac{\partial}{\partial \partial^\mu \phi}(\partial_\nu \phi \partial^\nu \phi) = \frac{\partial \partial_\nu \phi}{\partial \partial^\mu \phi}\partial^\nu \phi + \partial_\nu \phi \frac{\partial \partial^\nu \phi}{\partial \partial^\mu \phi} = g_{\mu\nu} \partial^\nu \phi + \partial_\nu \phi \delta_\mu^\nu = 2\partial_\mu \phi
\end{aligned}
$$

故得证，此方法无需写成具体分量。