EE550 Final Project

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April 27 2023

1 Introduction

In this project, the network admission control problem is explored. We study different admission policies and seek the optimal algorithm solution that has the maximum gain. We also compare the "accept all" policy with the optimal solution.

2 Motivation

In the EE550 class, we covered the M/M/m video admission control with 2 types of packets, which inspired us about an M/M/m system with more types of packets and more constraints. We realized the "accept all" policy does not always give the best solution since a certain type of packet may have too little value or too long service time. So we want to figure out how this policy performs under different circumstances compared to the optimal solution.

3 Problem Definition

In the system with maximum bandwidth capacity m, there are n types of packets sent through the link. For each type i packet, it has:
1 Poisson Arrival rate λ_i
2 I.I.D exponential service time μ_i
3 Each packet uses one unit of bandwidth in the system
4 Each packet brings v_i value

We define $x_1, x_2, \ldots, x_n$ be the number of packets of type i in the system. A state is the vector of all x_i. A state $\vec{X}$ as $(x_1, x_2, \ldots, x_n)$ means there is x_1 type 1 packets, x_2 type 2 packets, and so on.

To control the arrival of packets, the system uses policy P. The policy is a subset of the states. If a new packet arrives and the new state is in the policy, the system will accept the packet. Otherwise, it will reject the packet. For practicality, policies have the following monotonicity property:

If $(x_1, x_2, \ldots, x_n) \in P$,then $(\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n) \in P$, where $(\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)$ is any vector of non-negative integer that is entry-wise less than or equal to $(x_1, x_2, \ldots, x_n)$.

We want to compare how much value the system gains. Let γ_i represent the expected rate of accepted packets of type i. We define Gain as $G = \sum_i^n \gamma_i v_i$. We try to compare the gain under different policies. We especially, want to know about how it performs under the policy of "accepting all packets" compared to the optimal policy.

4 Solution

4.1 Truncation of CTMC

This question can be viewed as an M/M/m question. Every policy is one truncation of the CTMC. For the simplicity of presentation, we define the factor function of a state $\vec{X}$ as:

$$f(\vec{X}) = \prod_{i=1}^{n} \frac{\rho_i^{x_i}}{x_i!} \tag{1}$$

So the steady-state probability distribution of a state is:

$$p(\vec{X}) = Af(\vec{X}) \tag{2}$$

in which

$$\rho_i = \lambda_i / \mu_i \tag{3}$$

$$A = \frac{1}{\sum_{\vec{X} \in P} f(\vec{X})} \tag{4}$$

4.2 Some Observations

We notice that assuming if $\rho_i \leq 1$ for $i \in 1, 2, \ldots, n$, the factor of a state factorially drops when x_i increases. Suppose we have a sequence $a_1, a_2, \ldots, a_n$, then we have

$$\begin{aligned}
f(a_1, x_2, x_3, \ldots, x_n) &= \frac{\rho_1^{a_1}}{a_1!} \prod_{i=2}^{n} \frac{\rho_i^{x_i}}{x_i!} \\
&= (a_1 + 1) \frac{\rho_1^{a_1}}{(a_1 + 1)!} \prod_{i=2}^{n} \frac{\rho_i^{x_i}}{x_i!} \\
&\geq (a_1 + 1) \frac{\rho_1^{(a_1+1)}}{(a_1 + 1)!} \prod_{i=2}^{n} \frac{\rho_i^{x_i}}{x_i!} \\
&= (a_1 + 1) f(a_1 + 1, x_2, x_3, \ldots, x_n)
\end{aligned} \tag{5}$$

Similarly,for a x_i

$$f(x_1, x_2, \dots, x_i, \dots, x_{n-1}, x_n) \geq (x_i + 1) f(x_1, x_2, \dots, x_i + 1, \dots, x_{n-1}, x_n) \tag{6}$$

When we try to sum up the state probability in two different layers: layer A and layer B. As $x_2, x_3, \dots, x_n$ are kept same, we can get:

$$\sum_{i=1}^{n} f(a_i, x_2, x_3, \dots, x_n) \geq \sum_{i=1}^{n} (a_i + 1) f(a_i + 1, x_2, x_3, \dots, x_n) \tag{7}$$

We can find out that in the CTMC with if $\rho_i \leq 1$ for $i \in 1, 2, \dots, n$, there are large differences in the probability distribution between states. For example, in Figure (1), let us compare two layers, one in red and one in blue. We can notice in the blue layer, $x_1 + x_2 = 4$ and in the red layer, $x_1 + x_2 = 3$.

$$\begin{aligned} & f(3,0) + f(2,1) + f(1,2) + f(0,3) \\ \geq\ & 4f(4,0) + 3f(3,1) + 2f(2,2) + f(1,3) \end{aligned} \tag{8}$$

So even though the blue layer contains one more state than the red layer, as $f(0,4) \leq \frac{1}{4} f(0,3)$, the sum of state probability in the blue layer is less than the sum in the red layer.

$$\sum_{\vec{X}\ with\ x_1+x_2=3} f(\vec{X}) \geq \sum_{\vec{X}\ with\ x_1+x_2=4} f(\vec{X}) \tag{9}$$

Similarly, suppose we have two layers of states, called red and blue. And we have $\sum_{\vec{X}\ in\ red} x_i = a$ and $\sum_{\vec{X}\ in\ blue} x_i = b$. If $a < b$, then

$$\begin{aligned} & f(a,0) + f(a-1,1) + f(a-2,2) + \cdots + f(0,a) \\ \geq\ & bf(b,0) + (b-1)f(b-1,1) + (b-2)f(b-2,2) + \cdots + f(0,b) \end{aligned} \tag{10}$$

$$\sum_{\vec{X}\ in\ red} f(\vec{X}) \geq \sum_{\vec{X}\ in\ blue} f(\vec{X}) \tag{11}$$

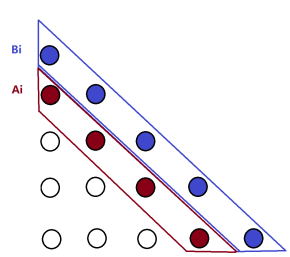


Figure 1: Two layers of CTMC when n = 2

4.3 Gain under two policies

Based on what we have learned in 4.2 and some simulations, we find out a lemma. Suppose P and Q are two different policies. For the type of packet i, the maximum accepting packet number of this type is p_i and q_i. Suppose for every kind of packet, policy P accepts more than policy Q, that is for i from 1 to n, $n \geq p_i > q_i$. Then, the average gain $G_P > G_Q$ if $\rho_i \leq 1$ for $i \in 1, 2, \ldots, n$.

Proof: We know that the system only gains value G when it receives and serves packets. In the policy P, the system accepts the packet of type i only if the number of the type i packet x_i is less than or equal to the maximum accepting packet number of this type minus one. Therefore, we can sum up all the steady-state probabilities.

$$\begin{aligned} G_P - G_Q &= \sum_{i=1}^{n} \lambda_i v_i A_P \sum_{\vec{X} \ with \ x_i \leq p_i - 1} f(\vec{X}) - \sum_{i=1}^{n} \lambda_i v_i A_Q \sum_{\vec{X} \ with \ x_i \leq q_i - 1} f(\vec{X}) \\ &= \sum_{i=1}^{n} \lambda_i v_i (A_P \sum_{\vec{X} \ with \ x_i \leq p_i - 1} f(\vec{X}) - A_Q \sum_{\vec{X} \ with \ x_i \leq q_i - 1} f(\vec{X})) \end{aligned} \tag{12}$$

Since $\lambda_i v_i \geq 0$, then we only need to care about the factor function part:

$$S = A_P \sum_{\vec{X} \ with \ x_i \leq p_i - 1} f(\vec{X}) - A_Q \sum_{\vec{X} \ with \ x_i \leq q_i - 1} f(\vec{X}) \tag{13}$$

$$S = \frac{1}{\sum_{\vec{X} \in P} f(\vec{X})} \sum_{\vec{X} \ with \ x_i \leq p_i - 1} f(\vec{X}) - \frac{1}{\sum_{\vec{X} \in Q} f(\vec{X})} \sum_{\vec{X} \ with \ x_i \leq q_i - 1} f(\vec{X}) \tag{14}$$

$$S = \frac{\sum_{\vec{X} \ with \ x_i \leq p_i - 1} f(\vec{X}) * \sum_{\vec{X} \in Q} f(\vec{X}) - \sum_{\vec{X} \ with \ x_i \leq q_i - 1} f(\vec{X}) * \sum_{\vec{X} \in P} f(\vec{X})}{\sum_{\vec{X} \in Q} f(\vec{X}) * \sum_{\vec{X} \in P} f(\vec{X})} \tag{15}$$

Clearly, the denominator is greater than 0. We only need to look at the numerator

$$T = \sum_{\vec{X} \ with \ x_i \leq p_i - 1} f(\vec{X}) * \sum_{\vec{X} \in Q} f(\vec{X}) - \sum_{\vec{X} \ with \ x_i \leq q_i - 1} f(\vec{X}) * \sum_{\vec{X} \in P} f(\vec{X}) \tag{16}$$

We can decompose $\sum_{\vec{X} \in Q}$ and $\sum_{\vec{X} \in P}$ that

$$\begin{aligned} \sum_{\vec{X} \in Q} f(\vec{X}) &= \sum_{\vec{X} \ with \ x_i \leq q_i - 1} f(\vec{X}) + \sum_{\vec{X} \ with \ x_i = q_i} f(\vec{X}) \\ \sum_{\vec{X} \in P} f(\vec{X}) &= \sum_{\vec{X} \ with \ x_i \leq p_i - 1} f(\vec{X}) + \sum_{\vec{X} \ with \ x_i = p_i} f(\vec{X}) \end{aligned} \tag{17}$$

So,

$$T = \sum_{x_i=qi} f(\vec{X}) * \sum_{\vec{X} \ with \ x_i \leq p_i - 1} f(\vec{X}) - \sum_{x_i=pi} f(\vec{X}) * \sum_{\vec{X} \ with \ x_i \leq q_i - 1} f(\vec{X}) \quad (18)$$

As $q_i < p_i$ for all i, like what equation (11) has proven

$$\sum_{\vec{X} \ with \ x_i=qi} f(\vec{X}) \geq \sum_{\vec{X} \ with \ x_i=pi} f(\vec{X}) \quad (19)$$

$$\sum_{\vec{X} \ with \ x_i \leq p_i - 1} f(\vec{X}) > \sum_{\vec{X} \ with \ x_i \leq q_i - 1} f(\vec{X}) \quad (20)$$

Therefore, $T > 0$ so do $G_P - G_Q$, which means the expected gain of policy P is better.

For example, in Figure 2, Policy P2 has a better gain than Policy P1.

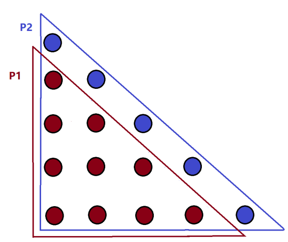


Figure 2: Two policies when n = 2

Based on this lemma, we can figure out a statement.

Suppose P^* is the optimal policy with the highest expected gain. Then P^* should at least accept as much as one type of packet. So for the type of packet i, the maximum accepting packet number of this type is p_i^*, then at least one $p_i^* = m$. Otherwise, there would be one better solution.

4.4 Optimal Policy and Accept-all Policy

After we realize the requirement for the optimal policy, we want to know how it performs compared to the accept-all policy. Suppose P^* is the optimal policy with the highest expected gain. Assuming λ and μ are fixed, we define:

$$\varepsilon = G_{P^*} - G_{P_{all}} \quad (21)$$

We can dig into the ε. For simplicity of presentation, we denote $P^*(i)$ as the subset of states in P^* which accepts a packet of type i. Similarly, we denote $P_{all}(i)$ as the subset of states in P_{all} which accepts a packet of type i.

$$\varepsilon = \sum_{i}^{n} \lambda_i v_i (A_{p^*} \sum_{\vec{X} \in P^*(i)} f(\vec{X}) - A_{p_{all}} \sum_{\vec{X} \in P_{all}(i)} f(\vec{X})) \tag{22}$$

Let us focus on the bracket inside.

$$\begin{aligned} A_{p^*} \sum_{\vec{X} \in P^*(i)} f(\vec{X}) - A_{p_{all}} \sum_{\vec{X} \in P_{all}(i)} f(\vec{X}) = \\ (A_{p^*} - A_{p_{all}}) \sum_{\vec{X} \in P^*(i)} f(\vec{X}) - A_{p_{all}} \sum_{\vec{X} \in P_{all}(i) - P^*(i)} f(\vec{X}) \end{aligned} \tag{23}$$

Intuitively, As we can find out that $A_{p_{all}}$ and A_{p^*} are fixed in the function. So if we want to increase ε, one way to make the value of the packet in the state in $P_{all}(i) - P^*(i)$ zero. Suppose the optimal policy P^* only focuses on one type of packet which only has a value of 100. If all other types of packets have a 0 value, then other packets P_all works on will have a 0 value. Intuitively, in this case, ε is maximized.

However, it turns out that it does not. Let us design two scenarios. In the first scenario, there are only type-1 packets with a value v_1. So in this scenario, the optimal solution P^* only accepts type-1 packets. In the second scenario, there are type-1 and type-2 packets with a value v_1 and v_2. the the optimal solution P^* accepts all type-1 packets and some type-2 packets. Then we have

$$\varepsilon_1 = \lambda_1 v_1 (A_{p^*} \sum_{\vec{x} \in P^*(1)} f(\vec{X}) - A \sum_{\vec{x} \in P_{all}(1)} f(\vec{X})) \tag{24}$$

$$\begin{aligned} \varepsilon_2 = \lambda_1 v_1 (A_{p^*_{new}} \sum_{\vec{x} \in P^*_{new}(1)} f(\vec{X}) - A \sum_{\vec{x} \in P_{all}(1)} f(\vec{X})) + \\ \lambda_2 v_2 (A_{p^*_{new}} \sum_{\vec{x} \in P^*_{new}(2)} f(\vec{X}) - A \sum_{\vec{x} \in P_{all}(2)} f(\vec{X})) \end{aligned} \tag{25}$$

Let us use ε_1 minus ε_2. Since in the second scenario, P^* can also accept all type-1 packets, we can approximately omit the change in packet type-1 arrival for ε_2 (which actually is increasing).

So if

$$A_{p^*_{new}} \sum_{\vec{x} \in P^*_{new}(2)} f(\vec{X}) - A \sum_{\vec{x} \in P_{all}(2)} f(\vec{X}) > 0 \tag{26}$$

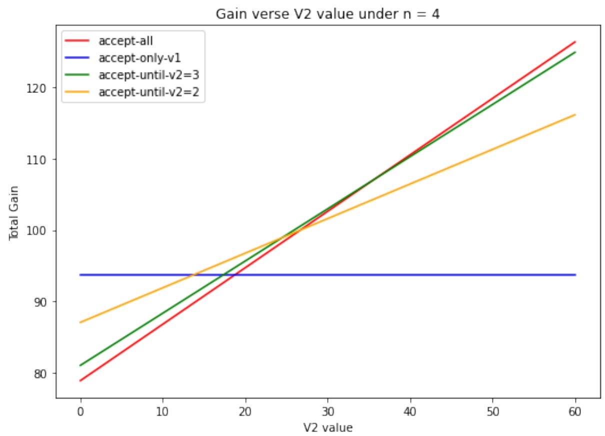
then we can optimize the policy by adding new states.

5 Numerical Simulation

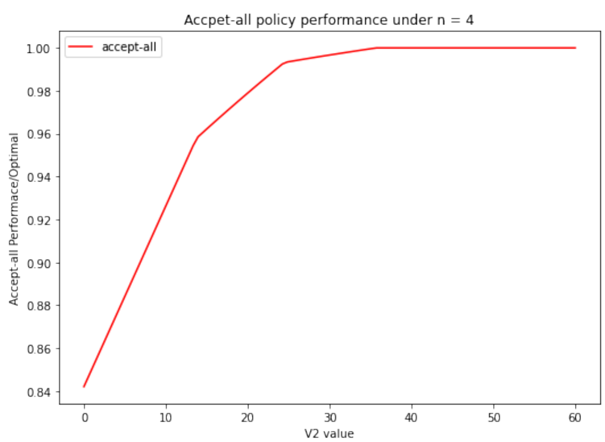
In this part, we make some simulations of the Gain in CTMC. We want to numerically compute how different policies perform in CTMC.

5.1 Scenario 1

In this scenario, we fix packet type n = 2, capacity m = 4. $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$. $V_1 = 100$. We set the value of the second packet type as a variable.



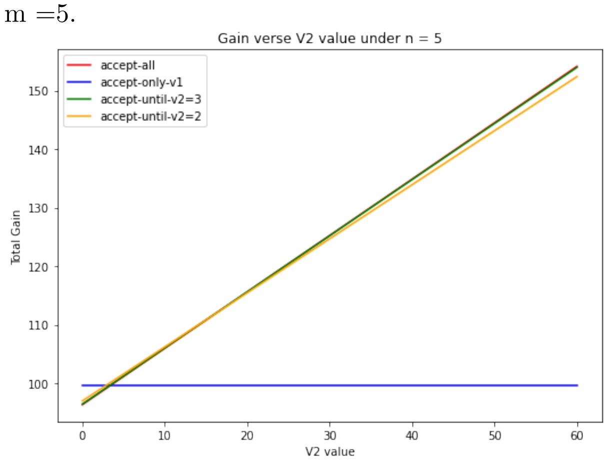
This simulation figure shows that the accept-all policy gains badly among these policies when the value of the type-2 packet is close to 0 and grows quickly as the value of the type-2 packet increases.



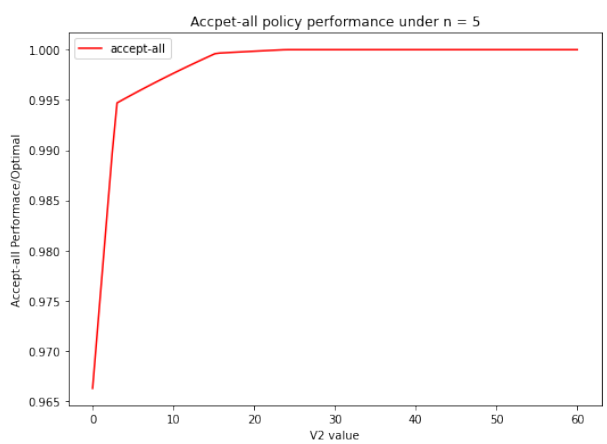
This figure shows how the accept-all policy compares to the optimal solutions.

5.2 Scenario 2

In this scenario, we keep everything the same as scenario 1 except the capacity m =5.



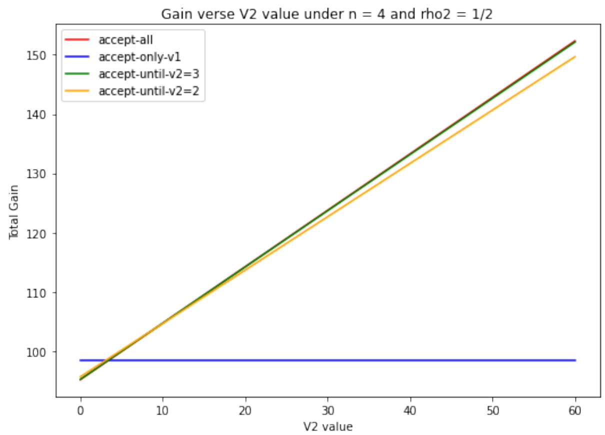
Compared to scenario 1, the accept-all policy matches better even when the value of type-2 is very small.



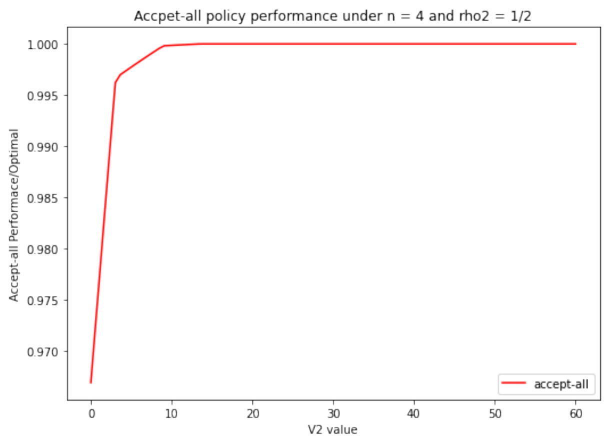
This figure shows that when n = 5, the accept-all policy performs at least 97 percent of the optimal solution.

5.3 Scenario 3

In this scenario, we keep everything the same as scenario 1 except the $\rho_2 = \frac{1}{2}$.



This figure shows that when steady-state distributions are more unbalanced. The probability of an outer state has a lower probability. So the accept-all policy has a better performance.



The percentage of performance reflects the statement above.

6 Conclusion

In this paper, we find out the requirement for the optimal policy of an M/M/m system. Then we briefly discuss about the optimal policy and how it compares to the accept-all policy. Finally, we numerically simulate these policies and find out that "accept-all" policy actually has a high performance.