Chapter 8 Pricing Fixed Income Securities (v.17.1)

8.1 One-factor short-term rates

Assume the instantaneous spot interest rate follows a one-dimensional stochastic process. Having the dynamics of such a process will allow a modeler to price securities and payoffs which are based on interest rates. Assuming the risk-neutral measure exists, the price of any security will be the expected present value of the future payoff under that measure

$$P_t = \mathbb{E}_t^*\left(e^{-\int_t^T r_s ds} V_T\right),$$

where P_t is the price of the security at time $t, \{r_s\}_{s\geq 0}$ is the process of short-term rates, and V_T is the payoff of the security at time $T > t$. The $(*)$ means that the expectation is computed under the risk-neutral measure.

For example, the price of a pure discount bond at time *t* maturing at time T (and a face value of $1 will be given by

$$P(t,T) = \mathbb{E}_t^*\left(e^{-\int_t^T r_s ds}\right).$$

Assume the dynamics of a short-term rate are given by

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t$$

Then, there exists a measure $*$, the risk-neutral measure, under which

$$dr_t = \left(\mu(t, r_t) - \lambda_t \cdot \sigma(t, r_t)\right)dt + \sigma(t, r_t)dW_t^*.$$

$W_t^* = W_t + \int_0^t \lambda_s ds$ is a Brownian Motion under the $* -measure,$ λ is the market price of risk. Under this measure pricing securities is simple: just find the expected present value of the future payoff.

We will consider a few one-factor models:

(1) Vasicek model

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma dW_t$$

Some properties:

- The distribution of r is Gaussian,

- The model can be solved explicitly,
- Rates, however, can be negative with positive probability.

(2) CIR Model

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dW_t$$

Some properties:

- The rate is always positive,
- The rate has non-central χ^2 distribution
- The model is analytically tractable
- The model is less tractable than the Vasicek model.

(2) <u>Dothan Model</u>

$$dr_t = ar_t dt + \sigma r_t dW_t$$

Some properties:

- The rate is always positive,
- The distribution of r is log-normal.

(3) <u>Hull White Model</u>

$$dr_t = k[\theta_t - r_t]dt + \sigma dW_t$$

Some properties:

- The rate can be negative with positive probability,
- The distribution of r is normal,
- The model is tractable - it yields closed –form formulas for bond and option prices.

(4) <u>Black Karasinski Model</u>

$$dr_t = r_t[\eta_t - a\ln r_t]dt + \sigma r_t dW_t$$

Some properties:

- The rate is always positive,
- The distribution of r is log-normal;
- The model is not very tractable - it does not yield closed–form formulas for bond and option prices.

The following table summarizes the properties of some popular short-term interest-rate models:

Table 1: Overview of Strength and Weaknesses of Basic Short Rate Models.
The following table contains an overview of short rate models. Here we will have the following notations: AB≈Analytical Bond price, AO ≈Analytical Option price, $\mathcal{N}$ ≈ normal distribution, $L\mathcal{N}$ ≈ lognormal distribution, $N\mathrm{C}\chi^2$ ≈Non-Central Chi-Square distribution, $\mathrm{SNC}\chi^2$ ≈ Shifted Non-Central χ^2, SL$\mathcal{N}$ ≈ Shifted LogNormal, MM≈Market Model, EEV≈ Extended Exponential Vasicek model. The Y* indicates the rates are positive under suitable conditions for a deterministic function φ_t.

Model	**Dynamics**	$r > 0$	$r\sim$	AB	AO
Vasicek	$dr_t = k[\theta - r_t]dt + \sigma dW_t$	N	$\mathcal{N}$	Y	Y
CIR	$dr_t = k[\theta - r_t]dt + \sigma\sqrt{r_t}dW_t$	Y	$\mathrm{NC}\chi^2$	Y	Y
D	$dr_t = ar_t dt + \sigma r_t dW_t$	Y	L$\mathcal{N}$	Y	N
EV	$dr_t = r_t[\eta - a\ln r_t]dt + \sigma r_t dW_t$	Y	L$\mathcal{N}$	N	N
HW	$dr_t = k[\theta_t - r_t]dt + \sigma dW_t$	N	$\mathcal{N}$	Y	Y
BK	$dr_t = r_t[\eta_t - a\ln r_t]dt + \sigma r_t dW_t$	Y	L$\mathcal{N}$	N	N
MM	$dr_t = r_t\left[\eta_t - \left(\lambda - \frac{\gamma}{1+\gamma^t}\right)\ln r_t\right]dt + \sigma r_t dW_t$	Y	L$\mathcal{N}$	N	N
CIR++	$r_t = x_t + \varphi_t, \quad dx_t = k[\theta - x_t]dt + \sigma\sqrt{x_t}dW_t$	Y*	$\mathrm{SNC}\chi^2$	Y	Y
EEV	$r_t = x_t + \varphi_t, \quad dx_t = x_t[\eta - a\ln x_t]dt + \sigma x_t dW_t$	Y*	SL$\mathcal{N}$	N	N

The table compares various models with each other. The table is largely self explanatory, but its entries are to some extent a matter of opinion. Based on the subjective assessments, none of the models performs well when pricing interest rate related products. The main advantage of these models is their relative tractability (one factor) compared with other, more complex models.

8.1.1 Comparison of Models

In this section we will attempt to compare various models of short-term rates. We compare models according to two different sets of criteria. The first comparison is on the strengths and weaknesses of the models and the second comparison is based on explaining of historical fixed income prices.

Strengths and Weaknesses of the Models

This comparison extends the earlier comparison to the broader range of interest rate models. Table 2 compares the major classes of models. In the table we use the following abbreviations: DM ≈

Dynamic Mean models, GDM ≈ Generalized Dynamic Mean models, AY ≈Affine Yield models, HJM ≈ Heath-Jarrow-Morton framework, and MM≈ Market Models.

Table 2: *Comparison Between Major Categories of Models*

Models:	DM	GDM	AY	HJM	MM
Features:					
Statics					
Bond prices	Exact	OK	Good	Exact	Exact
Caplet/Bond options	~	~	~	Exact	Exact
Volatility structure	NO	~	OK	Excellent	Excellent
Dynamics					
Short rate	Good	Good	OK	~	~
Yield curve	NO	~	Good	Excellent	Excellent
Tractable					
Simple	Good	Good	OK	~	Good
Complex	OK	OK	OK	~	OK

The HJM and Market Models give the best results overall, but for specific applications. Affine Yield or Dynamic Mean Models may be more appropriate for bond pricing.

The choice of a model for short term rate is not straightforward. It depends on both the modeling objectives and on the nature of the risk to be managed. For example, a fund that has large exotics exposure and a commitment to sell anything, may be more likely to implement whole yield curve models. However, simple bond or bond option trading may not require anything sophisticated.

Comparison Based on Historical Criteria

We have identified some good properties of models to incorporate into interest rate models. There are four properties that we will emphasize here. These features are:

1. *The behavior of the short rate:* Does the model have a time-varying reversion level for the short rate?
2. *The behavior of the long rate:* Does a long rate (for example a 20-year rate) have a sufficient range of movement?
3. *Term structure tilts:* Can the long end and the short end move simultaneously and roughly equally in opposite directions?
4. *Hyperinflations:* Is there a non-vanishing chance that the short rate becomes unbounded in finite time?

Table 3 compares a number of models, grouped into categories, by each of these four features. All of the models considered in the table assume that the term structure is default-free. Duffie and Singleton (1995) and (1997) have indicated that default may be accommodated by suitably adjusting the volatility structure. Table entries are of course a matter of opinion, and a judgment must be made as to what each feature means in the context of each model. For instance, whole yield curve models contain all the other models, but the table describes their features as they are usually implemented. Dynamic Mean models can be forced to hyperinflate by making their mean reversion levels explode, but this would then destroy their ability to capture any of the other features.

Table 3: *Comparison of Interest Rate Models with Respect to Short-Rate, Long-Rate Behavior, Term Structure Tilts, and Hyperinflation*

Features:	S	L	S/L	H
Model:				
Whole yield curve				
HJM (92), BGM (97)	×	Yes	Yes	×
Sommer (96)	~	Yes	Yes	~
Affine				
Duffie and Kan (96)	Yes	Yes	Yes	×
Babbs and Nowman (97)	Yes	Yes	Yes	×
Longstaff and Schwartz (92)	Yes	×	~	×
CIR (85), ‘standard’ model	×	×	×	×
CIR (85), ‘inflation’ model	~	~	~	×
Dynamic mean				
Tice and Webber (97)	Yes	×	~	×
Hull and White (90)	Yes	×	×	~
Fong and Vasicek (91)	Yes	×	~	×
Chen (96)	Yes	×	~	×
Ho and Lee (86)	Yes	~	×	*Yes*
Price kernel				
Bakshi and Chen (96)	×	×	×	×
Constantinides (92)	×	×	×	×
Jump models				
Babbs and Webber (94)	Yes	Yes	×	×

Here use the following notations: S: short rate behavior, L: Long rate behavior, H: hyperinflation, ~: the model has the feature, x: the model does not have this feature.

Below we will consider a few selected models in more details and provide some of their important properties.

8.1.2 Vasicek Model

Assume r_t is governed by

$$dr_t = \kappa(\mu - r_t)dt + \sigma dW_t$$

Then, under the risk-neutral measure,

$$dr_t = (\kappa(\mu - r_t) - \lambda\sigma)dt + \sigma dW_t^*$$

$$= \kappa(\tilde{\mu} - r_t)dt + \sigma dW_t^* \qquad \left(\tilde{\mu} = \mu - \frac{\lambda \cdot \sigma}{\kappa}\right)$$

Denoting $\bar{r} = \tilde{\mu}$, we have $dr_t = \kappa(\bar{r} - r_t)dt + \sigma dW_t$ under the risk-neutral measure. From here on, we will use the dynamics of the short-term rate models under the risk-neutral measure. That is, assume $dr_t = \kappa(\bar{r} - r_t)dt + \sigma dW_t$ and that r_0 is given.

Lemma 1 Let $\boldsymbol{P}(t,T)$ be the price of a pure discount bond at time t, maturing at time T, with a par value of \$1. Then

$$\boldsymbol{P}(t,T) = \mathbb{E}_t^*\left(e^{-\int_t^T r_s ds}\right) = A(t,T)e^{-B(t,T)\cdot r_t}$$

where

$$A(t,T) = exp\left\{\left(\bar{r} - \frac{\sigma^2}{2\kappa^2}\right)[B(t,T) - (T-t)] - \frac{\sigma^2}{4\kappa}B^2(t,T)\right\}, \qquad B(t,T) = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)$$

Note: Notice that the formula for $P(t,T)$ depends on the parameters of the Vasicek model and r_t.

Lemma 2 The conditional distribution of $r_t|\mathcal{F}_s$ is Gaussian with

$$\mathbb{E}(r_t|\mathcal{F}_s) = \bar{r} + (r_s - \bar{r})e^{-\kappa(t-s)}$$

$$Var(r_t|\mathcal{F}_s) = \sigma^2\frac{(1 - e^{-2\kappa(t-s)})}{2\kappa}$$

Lemma 3 (a) At time s, the instantaneous interest rate r_t is given by

$$r_t = r_s e^{-\kappa(t-s)} + \bar{r}\left(1 - e^{-\kappa(t-s)}\right) + \sigma\int_s^t e^{-\kappa(t-u)}dW_u$$

(b) The spot rate $r_t(\tau)$ at time t, for $\tau = T - t$ is given by

$$r_t(\tau) = r_\infty + (r_t - r_\infty)\left(\frac{1 - e^{-\kappa\cdot\tau}}{\kappa\tau}\right) + \left(\frac{\sigma^2\tau}{4\kappa}\right)\left(\frac{1 - e^{-\kappa\cdot\tau}}{\kappa\tau}\right)^2$$

where $r_\infty = \bar{r} - \frac{\sigma^2}{2\kappa^2}, \quad \tau = T - t.$

Note: We can solve for r_t if we apply the Ito's Lemma on ($e^{kt}\, r_t$), and integrate.

Lemma 4 The price of the pure discount bond $\boldsymbol{P}(t,T)$ satisfies the PDE

$$\frac{\partial P}{\partial t} + \kappa(\bar{r} - r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP = 0$$

with

$$\boldsymbol{P}(T,T) = 1$$

Note: In general, if the dynamics of the rate was given by $dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t$, then the PDE for the price would be $\frac{\partial P}{\partial t} + \mu\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP = 0$.

Now, we will consider European options on Pure Discount Bonds. The payoff of a European Call option with maturity at time T, and strike price $\boldsymbol{K}$, on a pure discount bond that matures at time S (where $S > T$), and has par value of \$1, is given by: $(\boldsymbol{P}(T,S) - \boldsymbol{K})^+$.

Lemma 5 The price of a European Call option $c(t,T,S)$ at time t, with maturity at time T and strike price $\boldsymbol{K}$, on a pure discount bond that matures at time S (where $S > T$), is given by

$$c(t,T,S) = \boldsymbol{P}(t,S)\,N(d_1) - \boldsymbol{K}\,\boldsymbol{P}(t,T)N(d_2)$$

The price of a European Put option

$$p(t,T,S) = \boldsymbol{K}\,\boldsymbol{P}(t,T)\,N(-d_2) - \boldsymbol{P}(t,S)\,N(-d_1)$$

where

$$d_1 = \frac{\ln\left(\frac{\boldsymbol{P}(t,S)}{\boldsymbol{K}\cdot\boldsymbol{P}(t,T)}\right)}{\sigma_p} + \frac{\sigma_p}{2}$$

$$d_2 = d_1 - \sigma_p$$

$$\sigma_p = \sqrt{\frac{1 - e^{-2\kappa(T-t)}}{2\kappa}}\cdot\left(\frac{1 - e^{-\kappa(S-T)}}{\kappa}\right)\cdot\sigma.$$

Comment: If the par value of the bond is L (and not \$1) then the formula for the price of European Call and Put options with Strike price $\boldsymbol{K}$ are given as follows:

$$c(t,T,S) = L \cdot \boldsymbol{P}(t,S)\, N(d_1) - \boldsymbol{K} \cdot \boldsymbol{P}(t,T) N(d_2)$$

The price of a European Put option

$$p(t,T,S) = \boldsymbol{K} \cdot \boldsymbol{P}(t,T)\, N(-d_2) - L \cdot \boldsymbol{P}(t,S)\, N(-d_1)$$

where

$$d_1 = \frac{\ln\left(\frac{L \cdot \boldsymbol{P}(t,S)}{\boldsymbol{K} \cdot \boldsymbol{P}(t,T)}\right)}{\sigma_p} + \frac{\sigma_p}{2}$$

$$d_2 = d_1 - \sigma_p, \quad \sigma_p = \sqrt{\frac{1-e^{-2\kappa(T-t)}}{2\kappa}} \cdot \left(\frac{1-e^{-\kappa(S-T)}}{\kappa}\right) \cdot \sigma \text{ and}$$

$\boldsymbol{P}(t,T)$ is the price of the pure discount bond at time t, that matures at time T and pays \$1,

Fitting Vasicek Models

Sophisticated estimation techniques are necessary for interest rate models. It is woefully inadequate to use a naïve estimation method such as ordinary least squares (OLS) regression. Even when regression may be a theoretically valid method to use, its faults in practical applications are well documented (Honore (1998), Ball and Torous (1996).

Suppose we have a time series $\{r_t\}_{t=1,\dots,n}$, of short rate data. Discretize the Vasicek process

$$dr_t = \alpha(\mu - r_t)dt + \sigma dz_t$$

Using the Euler discretization

$$r_{t+\Delta t} = (1 - \alpha\Delta t)r_t + \alpha\mu\Delta t + \sigma\sqrt{\Delta t}\varepsilon_t.$$

One may attempt to calibrate by performing the regression $r_{t+\Delta t} = a + br_t + \eta_t$, where η_t is normal noise. The estimates $\hat{\alpha}$ and $\hat{\mu}$ of α and μ are

$$\hat{\alpha} = \frac{1-b}{\Delta t}, \qquad \hat{\mu} = \frac{a}{1-b},$$

and $\sigma^2\Delta t = var(\eta_t)$.

Of course, these estimates are only sensible if the model is not mis-specified. If the residuals $\varepsilon_t = \eta_t / \sigma \sqrt{\Delta t}$ suffer from non-normality, heteroscedasticity or serial correlation, then the model does not fit the data and great care must be taken in interpreting any inferences from the model.

But several other things go wrong with this procedure.

1. Unless Δt is very small the estimates of $\hat{\mu}$ and in particular $\hat{\alpha}$ are significantly biased. This is because the process (5.81) is close to having a unit root (Ball and Torous (96)[51]), Honore (98)[286]).
2. OLS regression is equivalent to minimizing σ, and is not equivalent to ensuring that ε_t is standard normal iid. If the process (5.81) was the data-generating process (DGP), this would not matter, but since in interest rate modeling the DGP is unknown, and existing models do not match it closely, it is unclear whether minimizing σ is a valid objective or not.

Implementation

How to implement the Vasicek model to price Pure Discount Bonds and Options on Pure Discount Bonds?

1. $\boldsymbol{P}(0,T) = \mathbb{E}\left(e^{-\int_0^T r_s ds}\right)$ is the formula for the price of a Pure Discount Bond, and we will show steps to use Monte Carlo simulation to estimate that price. Define $R = -\int_0^T r_s ds$. Then, $\boldsymbol{P}(0,T) = \mathbb{E}(e^R) \approx \frac{1}{N}\sum_{i=1}^N e^{R_i}$, where R_i is a simulation of $R = -\int_0^T r_s ds$.

   Using Euler's method of integral estimation, we will write $R = -\int_0^T r_s ds = \Delta(\sum_{j=1}^n r_{t_j})$, where $\Delta = T/n$, $t_j = j\Delta$.

   Then, we will simulate N paths of the process r_s, from 0 to T, and define the $i - th\ R$ as follows: $R^i = \Delta(\sum_{j=1}^n {r^i}_{t_j})$, for $i = 1,2,\dots,N$. The, we have that , $\boldsymbol{P}(0,T) \approx \frac{1}{N}\sum_{i=1}^N e^{R_i}$.

2. $c(0,T,S) = \mathbb{E}\left(e^{-\int_0^T r_s ds}\ \ (\boldsymbol{P}(T,S) - \boldsymbol{K})^+\ \right)$ is the formula for the price of a European Call option on a Pure Discount Bond.

   We have $c(0,T,S) = \mathbb{E}\left(e^{-\int_0^T r_s ds}\ \ (\boldsymbol{P}(T,S) - \boldsymbol{K})^+\ \right) \approx \frac{1}{N}\sum_{i=1}^N e^{R_i}\,(\boldsymbol{P^i}(T,S) - \boldsymbol{K})^+$.

   The implementation here is very similar to the previous case, except for the price of the bond. We will consider two sub-cases here:

   (a) Use the explicit formula for the Price of the Pure Discount Bond. Then, $\boldsymbol{P^i}(T,S) = (\boldsymbol{P}(T,S) - \boldsymbol{K})^+$ for every $i = 1,2,\dots,N,$ and $\boldsymbol{P}(T,S)$ is as given by the closed-form solution.

(b) Use simulations to estimate the Price of the Pure Discount Bond - $\boldsymbol{P^i}(T,S)$. For every path for r-process, we start at 0, and simulate the path until time T, and we have ${r^i}_{t_N} = {r^i}_T$, which is the value of r at time T. We will use that value as a starting value for r, and simulate M paths of r from T to S, to price the Pure Discount Bond $\boldsymbol{P^i}(T,S)$, as was described earlier. Then, $\boldsymbol{P^i}(T,S) \approx \frac{1}{M}\sum_{l=1}^{M} e^{R_l}$.

8.1.3 CIR model:

Assume the dynamics of r_t, under the risk-neutral measure, are given by:

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dW_t$$

Assume $2\kappa \cdot \bar{r} > \sigma^2$ so that the origin is inaccessible.

Lemma 6 The price $\boldsymbol{P}(t,T)$ of a pure discount bond at time t, maturing at time T is given by

$$\boldsymbol{P}(t,T) = A(t,T) \cdot e^{-B(t,T)\cdot r_t}$$

where, $A(t,T) = \left(\frac{h_1 \cdot e^{h_2(T-t)}}{h_2 \cdot (e^{h_1(T-t)}-1)+h_1}\right)^{h_3}$, $B(t,T) = \frac{e^{h_1(T-t)}-1}{h_2 \cdot (e^{h_1(T-t)}-1)+h_1}$, $h_1 = \sqrt{\kappa^2 + 2\sigma^2}$, $h_2 = \frac{\kappa + h_1}{2}$, $h_3 = \frac{2\cdot\kappa\cdot\bar{r}}{\sigma^2}$.

Lemma 7 The conditional distribution of $r_t|\mathcal{F}_s$ is non-central χ^2 with

$$\mathbb{E}(r_t|\mathcal{F}_s) = r_s e^{-\kappa(t-s)} + \bar{r}\left(1 - e^{-\kappa(t-s)}\right)$$

$$Var(r_t|\mathcal{F}_s) = r_s\frac{\sigma^2}{\kappa}\left(e^{-\kappa(t-s)} - e^{-2\kappa(t-s)}\right) + \frac{\bar{r}\sigma^2}{2\kappa}\left(1 - e^{-\kappa(t-s)}\right)^2$$

Lemma 8 The SDE for the price $\boldsymbol{P}(t,T)$ process, of a pure discount bond at time t, maturing at time T, is given by

$$d\boldsymbol{P}(t,T) = r_t\boldsymbol{P}(t,T)dt - \sigma B(t,T)\,\boldsymbol{P}(t,T)\,\sqrt{r_t}dW_t$$

Lemma 9 The spot rate $r_t(\tau)$ at time t, for $\tau = T - t$ is given by:

$$r_t(\tau) = -\frac{2\kappa\bar{r}}{\sigma^2 T}\ln(A(T) + \frac{r_t}{T}D(T)$$

$$where\ A(T) and\ D(T) are\ provided\ on\ page\ 390\ in\ Martellini\ et\ al.$$

Lemma 10 The price of the European Call Option is given by:

$$c(t,T,S) = \boldsymbol{P}(t,S) \cdot \chi^2\left(2r^*(\phi+\psi+B(T,S); \frac{4\kappa\cdot\bar{r}}{\sigma^2}, \frac{2\phi^2\cdot r_t\cdot e^{\theta(T-t)}}{\phi+\psi+B(T,S)}\right) - \boldsymbol{K}\cdot\boldsymbol{P}(t,T) \cdot \chi^2\left(2r^*(\phi+\psi); \frac{4\kappa\cdot\bar{r}}{\sigma^2}, \frac{2\phi^2\cdot r_t\cdot e^{\theta(T-t)}}{\phi+\psi}\right)$$

$$\theta = \sqrt{\kappa^2+2\sigma^2}, \phi = \frac{2\theta}{\sigma^2(e^{\theta(T-t)}-1)}, \psi = \frac{\kappa+\theta}{\sigma^2}, r^* = \ln\left(\frac{A(T,S)}{\boldsymbol{K}}\right)/B(T,S)$$

$\chi^2(x,p,q)$ is the value at x of the distribution function of Non-Central χ^2 with p-degrees of freedom and non-centrality parameter q. The density function of such distribution is given by

$$f(x)_{\chi^2(p,q)} = \sum_{i=0}^{\infty} \frac{e^{-\frac{q}{2}}\cdot\left(\frac{q}{2}\right)^i}{i!} \cdot \frac{\left(\frac{1}{2}\right)^{i+\frac{p}{2}}}{r(i+\frac{p}{2})} \cdot x^{i-1+\frac{p}{2}} \cdot e^{-\frac{x}{2}}$$

Comment: If the par value of the bond is L (and not \$1) then in the formulas for the prices of European Call and Put options we will replace $\boldsymbol{P}(t,S)$ by $L\cdot\boldsymbol{P}(t,S)$, where $\boldsymbol{P}(t,S)$ is the price of the pure discount bond at time t, that matures at time S and pays \$1.

8.1.4 Affine term-Structure Models:

Definition: If zero-coupon bond prices are given by $\boldsymbol{P}(t,T) = A(t,T)\cdot e^{-B(t,T)\cdot r_t}$ for all $0 \le t \le T$, where $A(t,T)$and $B(t,T)$ are deterministic functions, then we say that the model possesses an affine term structure.

Lemma 11 In an Affine Term-Structure model in which the short-term interest rate follows the SDE $dr_t = (a_t - b_t r_t)dt + \sigma\sqrt{(c_t+d_t r_t)}\,dW_t$, the price of a zero-coupon bond $\boldsymbol{P}(t,T)$ is given by: $\boldsymbol{P}(t,T) = A(t,T)\cdot e^{-B(t,T)\cdot r_t}$.

Here the $A(t,T)$and $B(t,T)$ functions satisfy the following differential equations:

$$\frac{dA(t,T)}{dt} = A(t,T)B(t,T)\left(a_t - \frac{c_t B(t,T)}{2}\right), \text{ and } A(T,T) = 1,$$

$$\frac{dB(t,T)}{dt} = b_t B(t,T) + \left(\frac{d_t B^2(t,T)}{2} - 1\right), \text{ and } B(T,T) = 0.$$

Note: In Affine Term Structure Models, we will denote by $\boldsymbol{P}(t,T,r)$ the price of the Pure Discount Bond at time t, that matures at time T, and the rate (the rate at time t) is r.

8.2 Coupon-Paying Bonds / Options on Coupon-Paying Bonds

Jamshidian (1989) suggests a valuation method for pricing coupon-paying bonds and options on such bonds. The main idea is to view the coupon-paying bond as a portfolio of discount bonds.

Define $V(t, T, \{c_i\}_{i=1}^n, \{T_i\}_{i=1}^n, K)$ to be the price at time t of a European Call option with strike price K, maturity T, on a coupon-paying bond that pays coupons c_i (this is the coupon amount and not a percentage) at times T_i, $T_i \geq T$. Thus, T is the maturity of the option.

Then, based on Jamshidian (1989), one can write

$$V(t, T, \{c_i\}_{i=1}^n, \{T_i\}_{i=1}^n, K) = \sum_{i=1}^{n} c_i \cdot V(t, T, T_i, K_i)$$

where

- $n =$ # of coupons payable after the maturity T of the option;
- $V(t, T, T_i, K_i)$is the value at time t of the European option that matures at time T, has a strike price K_i, on a zero-coupon bond that matures at time T_i.
- $K_i =$Exercise price of i^{th} option determined as follows: $K_i = \boldsymbol{P}(T, T_i, r^*)$ = the price (at time T) of pure discount bond with maturity at time T_i, with r^* as a short rate, so that
$$\sum_{i=1}^{n} c_i \cdot \boldsymbol{P}(T, T_i, r^*) = K$$

That is, the price of the bond at time T, using the rate r^* is K.

Below are the details of the method for pricing options on coupon-paying bonds.

Consider a coupon paying bond described as follows:

The bond will pay n-coupons c_i at times T_i, where all $T_i > T, i = 1,2, \ldots, n$. Define $\mathfrak{F} = \{T_i, \ldots, T_n\}$, $C = \{c_i, \ldots, c_n\}$. Define $r^* =$ the constant spot rate at time T, for which the bond price at time T (that pays the coupons c_i at times T_i, $T_i \geq T$) is equal to the strike price K of the option we are trying to price:

$$\sum_{i=1}^{n} c_i \cdot \boldsymbol{P}(T, T_i, r^*) = K$$

Define $K_i =$ time-T value of a pure discount bond (that pays \$1 at maturity) with maturity at T_i when the spot rate is r^*: $K_i = \boldsymbol{P}(T, T_i, r^*)$.

The price **(CBOP)** of an option on a coupon paying bond at time t is given by:

$$\mathbf{CBOP}(t, T, \mathfrak{F}, C, K) = \sum_{i=1}^{n} c_i \cdot \boldsymbol{c}(t, T, T_i, K_i)$$

Where $\boldsymbol{c}(t, T, T_i, K_i)$ is the price of an option at time t with maturity T and a strike price of $K_i$on a pure discount bond that matures at time T_i.

The price **(CPB)** of the Coupon-Paying Bond at time T is given by:

$$\mathbf{CPB}(T, \mathfrak{F}, C) = \sum_{i=1}^{n} c_i \cdot \boldsymbol{P}(T, T_i, r)$$

The details of the Jamshidian(1989) method are as follows:

When the model is such that the price of the Pure Discount Bond at time T, maturing at time T_i and having \$1 par value is given by a formula in which there is an explicit dependence on r_T (such as the Vasicek, CIR models), then we will denote the price by $\boldsymbol{P}(T, T_i, r_T)$. Then, the European Call option payoff at maturity is

$$(\boldsymbol{CBP}(T, \mathfrak{F}, C) - K)^{+} = \left(\sum_{i=1}^{n} c_i \cdot \boldsymbol{P}(T, T_i, r_T) - K\right)^{+}$$

Jamshidian (1989) converts this positive part of the sum into a sum of positive parts. First, find r^* using, for example, the Newton-Raphson method, so that

$$\sum_{i=1}^{n} c_i \cdot \boldsymbol{P}(T, T_i, r^*) = K$$

(Note: $\sum_{i=1}^{n} c_i \cdot K_i = K$). Assume $\frac{\partial \mathbf{P}(t,s,r)}{\partial r} < 0$ for any $0 < t < s$. (This property is satisfied for the Vasicek, CIR and Hull-White models.) Then, the payoff of the call option can be written as follows:

$$\left(\sum_{i=1}^{n} c_i \cdot \boldsymbol{P}(T, T_i, r_T) - \sum_{i=1}^{n} c_i \cdot \boldsymbol{P}(T, T_i, r^*)\right)^{+} = \sum_{i=1}^{n} c_i \cdot \left(\boldsymbol{P}(T, T_i, r_T) - \boldsymbol{P}(T, T_i, r^*)\right)^{+},$$

which is a sum of payoffs of n options on pure discount bonds. If we denote $\boldsymbol{ZBP}$ to be the Zero-coupon Bond Price, then the price of the coupon-paying bond option with strike K and maturity T is given by:

$$\boldsymbol{CBOP}Call(t,T,\mathfrak{F},C,K)=\sum_{i=1}^{n}c_i\cdot\boldsymbol{ZBP}\left(t,T,T_{i,}\boldsymbol{P}(T,T_i,r^*)\right)$$

$$\boldsymbol{CBOP}Call=\sum_{i=1}^{n}c_i\cdot\left\{\boldsymbol{P}(t,T_i)\cdot N\left(d_{i,+}\right)-K_i\boldsymbol{P}(t,T)N(d_{i,-})\right\}$$

$$d_{i,\pm}=\frac{1}{\sigma_p(t,T,T_i)}\cdot\ln\left(\frac{\boldsymbol{P}(t,T_i)}{K_i\cdot\boldsymbol{P}(t,T)}\right)\pm\frac{\sigma_p(t,T,T_i)}{2}$$

$$\sigma_p(t,T,T_i)=\frac{\sigma}{\kappa}\left(1-e^{-\kappa(T_i-T)}\right)\sqrt{\frac{1}{2\kappa}(1-e^{-2\kappa(T-t)})}$$

Note (*): This method works with cases when the price of a bond $\boldsymbol{P}(t,T)$ is a known function of short-term rate r_t (such as in Vasicek or CIR models).

Demonstration for the Vasicek model:

Example: Consider a coupon paying bond, that pays semiannual coupons of $c=\$2$, has a Face Value of $FV=\$100$, and matures in 4 years. We would like to find the Price a European Call option that has maturity of 4 months and strike price of 98% of Par Value of the bond.

Here, $c_1=c_2=c_3=c_4=c_5=c_6=c_7=c=\$2$

$c_8=FV+c=\$(100+2)=\$102,\ K=\$98,\ t=0,\ T=\frac{4}{12},T_i=i*\frac{6}{12},$ for $i=1,2,\ldots,8.$

$\boldsymbol{\Pi}(T,T_i,r^*)$ is the price at time T of a pure discount bond maturity at time T_i with $r_T=r^*$. That, is

$$\boldsymbol{\Pi}(T,T_i,r^*)=A(T,T_i)\cdot e^{-B(T,T_i)\cdot r^*}\quad\text{with}$$

$$A(T,T_i)=\exp\left(\left(\bar{r}-\frac{\sigma^2}{2\kappa^2}\right)\left(B(T,T_i)-(T_i-T)\right)-\frac{\sigma^2}{4\kappa}\cdot B^2(T,T_i)\right),\ \text{and}$$

$$B(T,T_i)=\frac{1}{\kappa}\cdot\left(1-e^{-\kappa(T_i-T)}\right).$$

Note: $T_1-T=\frac{2}{12},T_2-T=\frac{8}{12},T_3-T=\frac{14}{12},T_4-T=\frac{20}{12},T_5-T=\frac{26}{12},T_6-T=\frac{32}{12},$

$T_7-T=\frac{38}{12},T_8-T=\frac{44}{12}.$

Step 1: Find $\mathbf{\Pi}(T, T_i, r^*)$ for $i = 1, \dots, 8$ as a function of r^* (the expressions),

Step 2: Solve for r^*, so that

$$\sum_{i=1}^{8} c_i \cdot \mathbf{\Pi}(T, T_i, r^*) = K$$

Step 3: Set $K_i = \mathbf{\Pi}(T, T_i, r^*)$ for $i = 1, \dots, 8$,

Step 4: Find the price of a European Call option with strike price K_i, maturity T, on a pure discount bond maturing at time T_i, paying \$1:

$$\boldsymbol{c}(t, T, T_i, K_i) = \boldsymbol{P}(t, T_i) \cdot N(d_1) - K_i \cdot \boldsymbol{P}(t, T) N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{P(t,T_i)}{K_i \cdot P(t,T)}\right)}{\sigma_p} + \frac{\sigma_p}{2},\ d_2 = d_1 - \sigma_p,\ \sigma_p = \sigma \cdot \frac{1-e^{-\kappa(T_i - T)}}{\kappa} \cdot \sqrt{\frac{1-e^{-2\kappa(T-t)}}{2\kappa}},\ T - t = \frac{4}{12}$$

Step 5: The price of the Call option is given by: $C = \sum_{i=1}^{8} c_i \cdot \boldsymbol{c}(t, T, T_i, K_i)$

8.3 Two-factor Short-Rate Models

We use short rates – regardless of the number of factors - to characterize the entire yield curve. Having the dynamics of the short rate, the pure discount bond prices can be computed as

$$\boldsymbol{P}(t, T) = \mathbb{E}_t \left[\exp\left\{ -\int_t^T r_s ds \right\} \right]$$

From the bond prices it is possible to construct the zero-interest rate curve. Thus, the dynamics of the zero-coupon yield curve is characterized by short term rates.

In some cases, this approach may result in a poor model of the yield curve. When the security (to be priced) depends on correlations of rates of different maturities (say 1 yr and 10 yrs) this approach is not reliable. Then, a more realistic correlation structure is needed, and thus, multi-factor models come into play. In what follows, we will consider a few simple multi-factor models of short-term rates.

8.3.1 Gaussian-Vasicek Two-Factor Model

The Gaussian-Vasicek two-factor model is used to model the movements in short-term rate and is driven by two sources of uncertainties. The model can be written as follows:

$$\begin{cases} dx_t = \kappa_x(\bar{x} - x_t)dt + \sigma_x dW_t^1 \\ dy_t = \kappa_y(\bar{y} - y_t)dt + \sigma_y dW_t^2 \\ r_t = x_t + y_t \end{cases}$$

and $dW_t^1 \cdot dW_t^2 = \rho dt$

Here the correlation between zero-rates of maturities T_1, and T_2 is

$$\begin{aligned} &Corr(R(t,T_1), R(t,T_2)) \\ &\qquad = Corr(b^x(t,T_1) \cdot x_t + b^y(t,T_1) \cdot y_t, b^x(t,T_2) \cdot x_t + b^y(t,T_2) \cdot y_t) \end{aligned}$$

which depends on the correlation between x_t and y_t, which, in its turn, depends on ρ. As can be seen, the rate is broken down to two correlated mean-reverting processes, with potentially different speeds of reversion and different long-term means and volatilities. This model is richer than its one-dimensional counterpart as it gives more flexibility and more freedom for calibration and fitting.

One may define multifactor models of interest rate with 3, 4 or more factors. So a natural question to ask is: how many factors are needed and why so many? The answer depends on the compromise between numerical implementation/tractability and capability of the model to capture realistic correlation pattern of fixed-income securities.

Using Principal Components Analysis one can easily demonstrate that two to three components (factors) can represent more than 90% of the variation in the yield curve. This suggests that 2 or 3 factor models may be adequate for capturing important features of the yield curve.

The next model is another two-factor model of short-term rates.

8.3.2 The G2++ Model

Assume

$$\begin{cases} dx_t = -ax_t dt + \sigma dW_t^1 \quad x_0 = 0 \\ dy_t = -by_t dt + \eta dW_t^2 \quad y_0 = 0 \\ r_t = x_t + y_t + \varphi_t \\ \varphi_0, r_0 \text{ given} \\ dW_t^1 \cdot dW_t^2 = \rho dt, \quad -1 \le \rho \le 1 \end{cases}$$

Here φ_t is a deterministic shift function (to help better fit the zero-coupon curve).

<u>Note:</u> Here W_1 and W_2 are NOT independent as in the Longstaff-Schwartz model that will be introduced below.

Lemma 12: Assume $r_0, a, b, \sigma, \eta > 0$. We have

$$\mathbb{E}(r_t|\mathcal{F}_s) = x_s \cdot e^{-a(t-s)} + y_s \cdot e^{-b(t-s)} + \varphi_t$$

$$Var(r_t|\mathcal{F}_s) = \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right) + \frac{\eta^2}{2b}\left(1 - e^{-2b(t-s)}\right) + 2\rho\frac{\sigma\eta}{a+b}\left(1 - e^{-(a+b)(t-s)}\right)$$

Lemma 13: The price of a zero-coupon bond is given by

$$\begin{aligned} \boldsymbol{P}(t,T) &= \mathbb{E}_t\left(e^{-\int_t^T r_s ds}\right) \\ &= \exp\left\{-\int_t^T \varphi(u)du - \frac{1 - e^{-a(T-t)}}{a}x_t - \frac{1 - e^{-b(T-t)}}{b}y_t + \frac{1}{2}V(t,T)\right\} \end{aligned}$$

where

$$\begin{aligned} V(t,T) = \quad & \frac{\sigma^2}{a^2}\left[T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right] \\ & + \frac{\eta^2}{b^2}\left[T - t + \frac{2}{b}e^{-b(T-t)} - \frac{1}{2b}e^{-2b(T-t)} - \frac{3}{2b}\right] \\ & + 2\rho\frac{\sigma\eta}{ab}\left[T - t + \frac{e^{-a(T-t)} - 1}{a} + \frac{e^{-b(T-t)} - 1}{b} - \frac{e^{-(a+b)(T-t)} - 1}{a+b}\right] \end{aligned}$$

<u>European Call Option price</u>

Let $\boldsymbol{Call}(t,T,S,K)$ be price of a European Call Option at time t, on a zero-coupon bond that matures at time S, with option strike K and option maturity T.

$$\boldsymbol{C}(t,T,S,K) = \mathbb{E}_t^*\left(e^{-\int_t^T r_s ds} \cdot (\boldsymbol{P}(T,S)-K)^+\right)$$

Then

$$\boldsymbol{C}(t,T,S,K) = P(t,S) \cdot \boldsymbol{N}\left(\frac{\ln\left(\frac{\boldsymbol{P}(t,S)}{K \cdot \boldsymbol{P}(t,T)}\right)}{\Sigma} + \frac{1}{2}\Sigma\right) - P(t,T) \cdot K \cdot \boldsymbol{N}\left(\frac{\ln\left(\frac{\boldsymbol{P}(t,S)}{K \cdot \boldsymbol{P}(t,T)}\right)}{\Sigma} - \frac{1}{2}\Sigma\right)$$

where $\boldsymbol{N}(\cdot)$ is the Standard Normal CDF, and

$$\Sigma^2 = \frac{\sigma^2}{2a^3}\left[1-e^{-a(S-T)}\right]^2 \cdot \left(1-e^{-2a(T-t)}\right) + \frac{\eta^2}{2b^3}\left[1-e^{-b(S-T)}\right]^2 \cdot \left(1-e^{-2b(T-t)}\right) + 2\rho\frac{\sigma\eta}{ab(a+b)} \cdot \left(1-e^{-a(S-T)}\right)\left(1-e^{-b(S-T)}\right)\left(1-e^{-(a+b)(T-t)}\right)$$

<u>European Put option price</u>

$$\boldsymbol{P}(t,T,S,K) = -\boldsymbol{P}(t,S) \cdot \boldsymbol{N}\left(\frac{\ln\left(\frac{K \cdot \boldsymbol{P}(t,T)}{\boldsymbol{P}(t,S)}\right)}{\Sigma} - \frac{1}{2}\Sigma\right) + P(t,T) \cdot K \cdot \boldsymbol{N}\left(\frac{\ln\left(\frac{K \cdot \boldsymbol{P}(t,T)}{\boldsymbol{P}(t,S)}\right)}{\Sigma} + \frac{1}{2}\Sigma\right)$$

with the same notations as above for the call option price formula.

Comment: If the par value of the bond is L (and not \$1) then in the formulas for the prices of European Call and Put options we will replace $\boldsymbol{P}(t,S)$ by $L \cdot \boldsymbol{P}(t,S)$, where $\boldsymbol{P}(t,S)$ is the price of the pure discount bond at time *t*, that matures at time *S* and pays \$1,

8.3.3 Longstaff-Schwartz-1992 Model

$$\begin{cases} dx_t = (\gamma - \delta x_t)dt + \sqrt{x_t}dW_t^1 \\ dy_t = (\eta - \theta y_t)dt + \sqrt{y_t}dW_t^1 \quad (W^1 \perp W^2) \end{cases}$$

$$r_t = \alpha \cdot x_t + \beta \cdot y_t \qquad (\alpha \neq \beta)$$

In this case we have the following closed-form formulas for pricing bonds and options on bonds.

Discount Bond price:

$$\boldsymbol{P}(t,s) = \exp\{G(t,s) + C(t,s)r_t + D(t,s)v_t\}$$

Where $v_t = \alpha^2 x_t + \beta^2 y_t$ is the volatility of r_t,

- $G(t,s) = \kappa\tau + 2\gamma \ln A(t,s) + 2\eta \ln B(t,s)$
- $A(t,s) = \frac{2\phi}{(\delta+\phi)(e^{\theta\tau}-1)+2\phi}$
- $B(t,s) = \frac{2\Psi}{(\theta+\Psi)(e^{\Psi\tau}-1)+2\Psi}$
- $C(t,s) = \frac{\alpha\phi\cdot(e^{\Psi\tau}-1)\cdot B(t,s)\ -\ \beta\cdot\Psi\cdot(e^{\phi\tau}-1)A(t,s)}{\phi\cdot\Psi\cdot(\beta-\alpha)}$
- $D(t,s) = \frac{\Psi\cdot(e^{\phi\tau}-1)\cdot A(t,s)-\phi\cdot(e^{\Psi\tau}-1)\cdot B(t,s)}{\phi\cdot\Psi\cdot(\beta-\alpha)}$

$$\tau = s - t, \phi = \sqrt{2\alpha + \delta^2}, \psi = \sqrt{2\beta + e^2}, \kappa = \gamma(\delta + \phi) + \eta(\theta + \Psi)$$

European Call option price

$$C(t,T,S) = \boldsymbol{P}(t,S)\cdot\boldsymbol{\Psi}(\theta_1,\theta_2,\cdot\,4\gamma,4\eta,w_1,w_2) - K\cdot\boldsymbol{P}(t,T)\cdot\boldsymbol{\Psi}(\theta_3,\theta_4,\cdot\,4\gamma,4\eta,w_3,w_4)$$

$\theta_1 = \frac{4\cdot\zeta\cdot\phi^2}{\alpha\cdot\left(e^{\phi(T-t)}-1\right)^2\cdot A(t,S)}$, $\theta_2 = \frac{4\cdot\zeta\cdot\Psi^2}{\beta\cdot\left(e^{\Psi(T-t)}-1\right)^2\cdot B(t,S)}$, $\theta_3 = \frac{4\cdot\zeta\cdot\phi^2}{\alpha\cdot\left(e^{\phi(T-t)}-1\right)^2\cdot A(t,T)}$, $\theta_4 = \frac{4\cdot\zeta\cdot\Psi^2}{\beta\cdot\left(e^{\Psi(T-t)}-1\right)^2\cdot B(t,T)}$, $w_1 = \frac{4\phi\cdot e^{\phi(T-t)}\cdot A(t,S)\cdot(Br_t-v_t)}{\alpha(\beta-\alpha)\left(e^{\phi(T-t)}-1\right)A(t,T-S)}$, $w_2 = \frac{4\psi\cdot e^{\Psi(T-t)}\cdot B(t,S)\cdot(v_t-\alpha\cdot r_t)}{\beta(\beta-\alpha)\left(e^{\Psi(T-t)}-1\right)B(t,T-S)}$, $w_3 = \frac{4\phi\cdot e^{\phi(T-t)}\cdot A(t,T)\cdot(\beta r_t-v_t)}{\alpha(\beta-\alpha)\left(e^{\phi(T-t)}-1\right)}$, $w_4 = \frac{4\psi\cdot e^{\Psi(T-t)}B(t,T)\cdot(v_t-\alpha\cdot r_t)}{\beta(\beta-\alpha)\left(e^{\Psi(T-t)}-1\right)}$, $\zeta = \kappa(S-T) + 2\gamma \ln A(t,S-T) + 2\eta \ln B(t,S-T) - \ln K$, and $\boldsymbol{\Psi}\sim$distribution function of bivariate, non-central χ^2.

Since the pricing of bonds or options on bonds can be done by using the PDE approach and by using the probabilistic approach, here we will provide a result – the well-known Feyman-Kac Theorem – that links the two approached to each other.

Theorem (Feyman-Kac)

(a) Assume under the risk-neutral measure the price-process is given by $dS_t = \mu(t,S_t)dt + \sigma(t,S_t)dW_t$ and the price of a contingent claim at time T is given by

$$V(t,S_t) = \mathbb{E}_t^*\left(e^{-\int_t^T r(u,S_u)du}\cdot H(T,S_T)\right)$$

Then

$$\frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

with $V(T, S_T) = H(T, S_T)$

(b) In a multidimensional case when $dS_t^i = \mu_i(t, S_t^i)dt + \sigma_i(t, S_t^i)dW_t^i$ for $i = 1, \dots, d$, the PDE is given by:

$$\frac{\partial V}{\partial t} + \sum_{i=1}^{d} \mu_i \frac{\partial V}{\partial S_i} + \frac{1}{2}\sum_{i,j=1}^{d} \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 V}{\partial S_i \partial S_j} - rV = 0.$$

(c) Assume under the risk-neutral measure the price-process is given by $dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t$ and the price of a contingent claim at time T is given by

$$V(t, S_t) = \mathbb{E}_t^* \left(\int_t^T h(u, S_u)\, e^{-\int_t^u r(y, S_y)dy} du + e^{-\int_t^T r(y, S_y)dy} \cdot g(T, S_T) \right)$$

Then

$$\frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial S^2} - rV + h = 0$$

with $V(T, S_T) = g(T, S_T)$.

Exercises:

1. Assume the dynamics of the short-term interest rate, under the risk-neutral measure, are given by the following SDE (**Vasicek model**):

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma dW_t$$

with $r_0 = 5\%$, $\sigma = 12\%$, $\kappa = 0.82$, $\bar{r} = 5\%$.

(a) Use Monte Carlo Simulation (assume each time step is a day) to find the price of a pure discount bond, with Face Value of \$1,000, maturing in $T = 0.5$ years (at time $t = 0$):

$$P(t, T) = \mathbb{E}_t^* \left[\$1{,}000\, exp\left(-\int_t^T r(s)ds \right) \right]$$

(b) Use Monte Carlo Simulation to find the price of a coupon paying bond, with Face Value of \$1,000, paying semiannual coupons of \$30, maturing in $T = 4$ years:

$$P(0, C, T) = \mathbb{E}_0^* \left[\sum_{i=1}^{8} C_i\, exp\left(-\int_0^{T_i} r(s)ds \right) \right]$$

where $C = \{C_i = \$30 \text{ for } i = 1,2,\ldots,7; \text{ and } C_8 = \$1{,}030\}$,
$T = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\} = \{0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4\}$.

(c) Use Monte Carlo Simulation to find the price of a European Call option on the pure discount bond in part (a). The option matures in 3 months and has a strike price of $K = \$980$. Use the explicit formula for the underlying bond price (only for the bond price).

(d) Use Monte Carlo Simulation to find the price of a European Call option on the coupon paying bond in part (b). The option matures in 3 months and has a strike price of $K = \$980$. Use Monte Carlo simulation for pricing the underlying bond.

(e) Find the price of a European Call option of part (d) by using the explicit formula for the underlying bond price, and reconcile the findings with the ones of part (d).

2. Assume the dynamics of the short-term interest rate, under the risk-neutral measure, are given by the following SDE (**CIR model**):

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dW_t$$

with $r_0 = 5\%, \sigma = 12\%, \ \kappa = 0.92, \ \bar{r} = 5.5\%$.

(a) Use Monte Carlo Simulation to find at time $t = 0$ the price $c(t, T, S)$ of a European Call option, with strike price of $K = \$980$, maturity of $T = 0.5$ years on a Pure Discount Bond with Face Value of \$1,000, that matures in $S = 1$ year:

$$c(t, T, S) = \mathbb{E}_t^* \left[exp\left(-\int_t^T r(u)du \right) * \max(P(T, S) - K, 0) \right]$$

(b) Use the *Implicit Finite-Difference Method* to find at time $t = 0$ the price $c(t, T, S)$ of a European Call option, with strike price of $K = \$980$, maturity of $T = 0.5$ years on a Pure Discount Bond with Face Value of \$1,000, that matures in $S = 1$ year. The PDE for c is given as

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 r \frac{\partial^2 c}{\partial r^2} + \kappa(\bar{r} - r)\ \frac{\partial c}{\partial r} - rc = 0$$

with $c(T, T, S) = \max(P(T, S) - K, 0)$, and $P(T, S)$ is computed explicitly.

(c) Compute the price $c(t, T, S)$ of the European Call option above using the explicit formula, and compare it to your findings in parts (a) and (b) and comment on your findings.

3. Assume the dynamics of the short-term interest rate, under the risk-neutral measure, are given by the following system of SDE (**G2++ model**):

$$\begin{cases} dx_t = -ax_t dt + \sigma dW_t^1 \\ dy_t = -by_t dt + \eta dW_t^2 \\ r_t = x_t + y_t + \phi_t \end{cases}$$

$x_0 = y_0 = 0,\ \phi_0 = r_0 = 3\%,\ dW_t^1 dW_t^2 = \rho dt,\ \rho = 0.7, a = 0.1,\ b = 0.3, \sigma = 3\%, \eta =$ 8%. Assume $\phi_t = const = 3\%$ for any $t \geq 0$. Use Monte Carlo Simulation to find at time $t = 0$ the price $p(t, T, S)$ of a European Put option, with strike price of $K = \$950$, maturity of $T = 0.5$ years on a Pure Discount Bond with Face value of $1,000, that matures in $S = 1$ year. Compare it with the price found by the explicit formula and comment on the estimation.

4. Consider a European Put option, with strike price of $K = \$970$, maturity of $T = 0.5$ years on a Pure Discount Bond with Face Value of $1,000, that matures in $S = 1.5$ years. Which of the two models below would result in a more expensive price for the option?
   (a) The Vasicek model $dr_t = \kappa(\bar{r} - r_t)dt + \sigma dW_t$ with $r_0 = 5\%,\ \sigma = 12\%,\ \kappa = 0.82,\ \bar{r} = 5\%$.
   (b) The CIR model $dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dW_t$ with $r_0 = 5\%, \sigma = 54\%,\ \kappa = 0.82,\ \bar{r} = 5\%$.
   Answer by using explicit formulas or by Monte Carlo simulations.
   Is the answer consistent with your intuition?

5. A group of 3 people will vote on a proposal. The probability that the I person will make the right decision is p, the probability that the II person will make the right decision is p, and the probability that the III person will make the right decision is ½. What's the Probability the group will make the right decision? (It is a "Majority Rules" decision making process).

6. You are offered to play chess with 2 other players - your classmate (sitting next to you) and Gary Kasparov. You'll play 3 games and will get a prize of $1,000,000 if you win 2 consecutive games. You'll play Classmate-Kasparov-Classmate or Kasparov-Classmate-Kasparov combinations at your choice. Assume you can beat your classmate with probability p and can beat Kasparov with probability q. Which combination would you choose and why?