Chapter 6 Exotic & Path Dependent Options (v.17.1)

The payoff depends on the whole path of the underlying assets price. In this chapter we consider several examples of exotic and path dependent options.

6. 1 Barrier Options

There are in general two types of barrier options: knock-out and knock-in options.

Knock-out options The contract is cancelled if the barrier S_b value is crossed at any time during the life of the option.

Knock-in options The contract is activated only if a barrier S_b is crossed at any time during the life of the option..

Examples

Down-and-out put This European option becomes void, if price goes below S_b (barrier) at any time during the life of the option. If the price never hits the barrier S_b, then this is simply a European put option. The barrier should be below the initial stock and the strike price:

$$S_b < X, \text{ and } S_b < S_0$$

Let the price of this option be P_{d0}. Then

$$P_{d0} = e^{-rT} \cdot E^*\big((X - S_T)^+ \ \mathbb{1}_{(S_{min} \geq S_b)}\big)$$

where $S_{min} = \min\{S_t: \ t \in [0, T]\}$ and $\mathbb{1}_{(S_{min} \geq S_b)} = \begin{cases} 1, & if \ S_{min} \geq S_b \\ 0, & else \end{cases}$

The risk (of the writer of this option) is reduced, compared to the vanilla European put option. For option holder, the payoff is lower than the one of the vanilla put option. Thus, it would be expected that the price of this option will be lower than the one of the vanilla European put option with the same specifications.

Down-and-in put This European option becomes active, if a barrier S_b is crossed at any time during the life of the option. Let the price of this option be P_{di}. Then

$$P_{di} = e^{-rT} \cdot E^*\big((X - S_T)^+ \ \mathbb{1}_{(S_{min} \leq S_b)}\big)$$

where $S_{min} = \min\{S_t: \ t \in [0, T]\}$ and $\mathbb{1}_{(S_{min} \leq S_b)} = \begin{cases} 1, & if \ S_{min} \leq S_b \\ 0, & else \end{cases}.$

Then, it is easy to see that

$$P_{di} + P_{d0} \quad = \quad P \text{ (price of European vanilla put option)}$$

Analytic Formula for Down-and-In-put option is given by:

$$\begin{aligned} P_{di} \;=\; & Xe^{-rT}[N(d_4) - N(d_2) - a\{N(d_7) - N(d_5)\}] \\ & - S_0[N(d_3) - N(d_1) - b\{N(d_8) - N(d_6)\}] \end{aligned}$$

where $a \;=\; \left(\frac{S_b}{S_0}\right)^{\frac{2r}{\sigma^2}-1}$, $\qquad b \;=\; \left(\frac{S_b}{S_0}\right)^{\frac{2r}{\sigma^2}+1}$

$$d_1 \;=\; \frac{ln\left(\frac{S_0}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad , \quad d_2 \;=\; \frac{ln\left(\frac{S_0}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_3 \;=\; \frac{ln\left(\frac{S_0}{S_b}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad , \quad d_4 \;=\; \frac{ln\left(\frac{S_0}{S_b}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_5 \;=\; \frac{ln\left(\frac{S_0}{S_b}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad , \quad d_6 \;=\; \frac{ln\left(\frac{S_0}{S_b}\right) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_7 \;=\; \frac{ln(S_0X/S_b^2) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad , \quad d_8 \;=\; \frac{ln(S_0X/S_b^2) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

The two cases below can be used as benchmark for computing and comparing:

1. $(S_0, X, r, T, \sigma, S_b) = \left(50, 50, 0.1, \frac{5}{12}, 0.4, 30\right)$, then $P_{di} = 3.23$
2. $(S_0, X, r, T, \sigma, S_b) = \left(50, 50, 0.1, \frac{5}{12}, 0.3, 30\right)$ then $P_{di} = 2.73.$

<u>How to price these options numerically?</u>

Down-and-In Put option:

Simulate N-paths of stock prices: $\{S_{t_1}^i, \dots, S_{t_m}^i\}_{i=1}^N$. The price of the down-and-in put option is given by

$$P_{di} = e^{-rT} \cdot E^*\left((X - S_T)^+ \; \mathbb{1}_{(S_{min} \le S_b)}\right),$$

which can be approximated by

$$P_{di} \approx e^{-rT} \cdot \frac{1}{N}\sum_{i=1}^N (X - S_{t_m}^i)^+ \, \mathbb{1}_{(S_{min}^i \le S_b)} \text{ where } S_{min}^i = \min\{\{S_{t_1}^i, \dots, S_{t_m}^i\}.$$

The Down-and-Out Put option can be prices by simulation using the exact same technique that was described above.

6.2 Asian Options

Asian options have a stronger degree of path dependency. The option payoff depends on the average stock price over the option life.

$$\underset{(arithmetic)}{\underline{A_{ca}=\frac{1}{T}\int_0^T S_t dt}} \qquad\qquad \underset{(geometric)}{\underline{A_{cg}=e^{\frac{1}{T}\int_0^T lnS_t dt}}}$$

Fixed strike Asian Call payoff: $max\,(A-X, 0)$

Fixed strike Asian put payoff: $max\,(X - A, 0)$

Floating strike Asian call payoff: $max\,(S_T - A, 0)$

Floating strike Asian put payoff: $max\,(A - S_T, 0)$.

For the geometric average option prices there exist closed-form solutions, but not for the arithmetic-average options. In every case, however, we can estimate the prices by using Monte Carlo simulation.

<u>Steps to price Asian average rate Call options by simulation:</u>

STEP 1: Simulate *N* paths of stock prices: $\{S_{t_1}^i, \ldots, S_{t_m}^i\}_{i=1}^N$

STEP 2: Compute the average price along each path: $A^i=\frac{1}{m}\sum_{j=1}^m\{S_{t_j}^i$

STEP 3: Option Price ≈ PV of Average of $(A^j - X)^+ \;=\; e^{-rT}\cdot\frac{1}{N}\sum_{i=1}^N \left(A^i - X\right)^+$

6.3 Lookback Options

Extreme values of stocks are monitored during the life of the option and option payoffs are dependent on the lowest and highest observable stock prices.

<u>Lookback calls</u>:

Floating Lookback Call payoff: $(S_T - S_{min})^+ = S_T - S_{min},\;\; S_{min} = \min\{S_t, t\in[0,T]\}$

Fixed Strike Lookback Call payoff: $(S_{max} - X)^+$, $S_{max} = \max\{S_t, t\in[0,T]\}$

Floating Lookback Put payoff: $(S_{max} - S_T)^+ = S_{max} - S_{T},$

Fixed Strike Lookback Put payoff: $(X - S_{min})^+$

There is an analytic formula for pricing a Floating Lookback Call option:

Assume the underlying pays dividends continuously at a rate of q.

For the case when $\boldsymbol{r \neq q}$:

$$C_{FLC} = S_0 e^{-qT} N(a_1) - S_{min} e^{-rT} N(a_2) - S_0 e^{-qT} \frac{\sigma^2}{2(r-q)} N(-a_1)$$

$$+ S_0 e^{-rT} \frac{\sigma^2}{2(r-q)} \left(\frac{S_0}{S_{min}}\right)^{\frac{-2(r-q)}{\sigma^2}} N(-a_3)$$

$$a_1 = \frac{\ln\left(\frac{S_0}{S_{min}}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \; a_2 = a_1 - \sigma\sqrt{T}$$

$$a_3 = \frac{\ln\left(\frac{S_0}{S_{min}}\right) + \left(-r + q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$S_{min} = \min\{S_t, 0 \leq t \leq today\}$, and at origination, $S_{min} = S_0$.

For the case when $\boldsymbol{r = q}$, the price is given by:

$$C_{FLC} = S_0 e^{-rT} N(a_1) - S_{min} e^{-rT} N(a_2) + S_0 e^{rT} \sigma\sqrt{T}(n(a_1) + a_1(N(a_1) - 1))$$

6.4 Volatility Swaps and Variance Swaps

A volatility swap is an agreement to exchange realized volatility of an asset (from time 0 to T) for a pre-specified fixed volatility σ_X.

The realized volatility is calculated as $= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} r_i^2} \cdot \sqrt{\tau}$, where r_i is the return in period i, τ ~ frequency (τ =12 for monthly, 252 for daily data). For daily data we have $s = \sqrt{\frac{252}{n-1}\sum_{i=1}^{n}\left(\ln\left(\frac{S_{i+1}}{S_i}\right)\right)^2}$, S_i being the stock price on day i.

<u>Payoff to the fixed-volatility-payer:</u> $L_{vol}(s - \sigma_X)$, where L_{vol}= notional principal amount, σ_X is the fixed prespecified volatility (implied volatility measured on the day when the contract is entered into) and s is the realized volatility.

Variance Swaps

An agreement to exchange the realized variance of an asset s^2 (between time 0 and T) with pre-specified variance rate σ_X^2.

The payoff of the Variance Swap to the fixed-variance payer is $L_{var}(s^2 - \sigma_X{}^2)$.

Valuation of Variance Swaps

For any S^* (value of asset price), the expected value of variance (from time 0 to *T*) is given by:

$$\mathbb{E}^*(s^2) = \frac{2}{T}\ln\left(\frac{F_0}{S^*}\right) - \frac{2}{T}\left(\frac{F_0}{S^*} - 1\right) + \frac{2}{T}\int_0^{S^*}\frac{1}{k^2}e^{rT}\cdot P(k)dk + \frac{2}{T}\int_{S^*}^{\infty}\frac{1}{k^2}e^{rT}\cdot C(k)dk \qquad (*)$$

where F_0 is the forward price of the asset at maturity T, $C(K)$ is the price of a European call with strike price K, maturity T, $P(K)$ is the price of European put with strike K, maturity T.

The value today of the agreement to pay $\sigma_X{}^2$ and receive the realized variance s^2 between 0 and T is:

$$L_{var}(\mathbb{E}^*(s^2) - \sigma_X{}^2)\cdot e^{-rT}$$

How to implement (*) to price the variance swap?

Choose $K_1 < \cdots < K_n$ so that there are European options on the asset with K_i strike prices. Set S^* in (*) as

$$S^* = \max\{K_i, \qquad \text{so that} \quad K_i \le F_0\}$$

Then we can approximate the expected variance as follows:

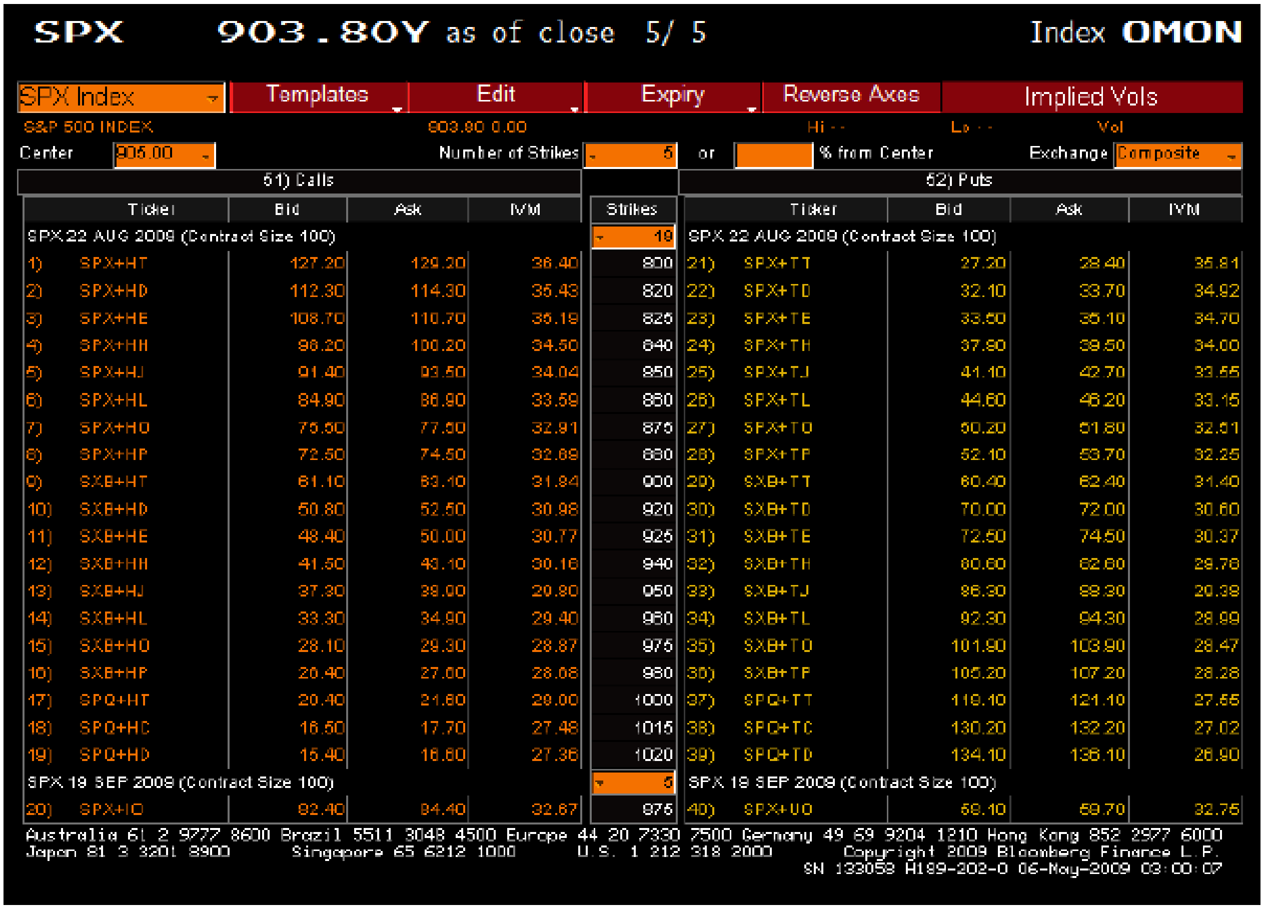
$$\mathbb{E}^*(s^2) \approx \frac{2}{T}\ln\left(\frac{F_0}{S^*}\right) - \frac{2}{T}\left(\frac{F_0}{S^*} - 1\right) + \frac{2}{T}\cdot\{\textstyle\sum_{i=1}^{n}\frac{\triangle K_i}{K_i{}^2}\cdot e^{rT}\cdot U(K_i)\}$$

where $\triangle K_i = \frac{1}{2}(K_{i+1} - K_{i-1})$ for $2 \le i \le n-1$, $\triangle K_1 = K_2 - K_1$, $\triangle K_n = K_n - K_{n-1}$, and

$$U(K_i) = \begin{cases} \text{Price of European put with } K_i \text{ strike, if } K_i < S^* \\ \text{Price of European call with } K_i \text{ strike, if } K_i > S^* \\ \dfrac{European\ call + European\ put\ price}{2}, if\ K_i = S^* \end{cases}.$$

Example: Valuation of a Variance Swap. On May 6, 2009, the following information was obtained from Bloomberg to value a variance swap on S&P500 index:

S_0 = 903.80, r = 1%, q = 1%, T = 3 months,



SPX 903.80Y as of close 5/ 5 Index OMON

SPX Index | Templates | Edit | Expiry | Reverse Axes | Implied Vols

S&P 500 INDEX 903.80 0.00 Hi - - Lo - - Vol

Center 905.00 Number of Strikes 5 or % from Center Exchange Composite

51) Calls					52) Puts				
	Ticker	Bid	Ask	IVM	Strikes	Ticker	Bid	Ask	IVM
SPX 22 AUG 2009 (Contract Size 100)					19	SPX 22 AUG 2009 (Contract Size 100)			
1)	SPX+HT	127.20	129.20	36.40	800	21) SPX+TT	27.20	28.40	35.81
2)	SPX+HD	112.30	114.30	35.43	820	22) SPX+TD	32.10	33.70	34.92
3)	SPX+HE	108.70	110.70	35.19	825	23) SPX+TE	33.50	35.10	34.70
4)	SPX+HH	98.20	100.20	34.50	840	24) SPX+TH	37.90	39.50	34.00
5)	SPX+HJ	91.40	93.50	34.04	850	25) SPX+TJ	41.10	42.70	33.55
6)	SPX+HL	84.90	86.90	33.59	860	26) SPX+TL	44.60	46.20	33.15
7)	SPX+HO	75.50	77.50	32.91	875	27) SPX+TO	50.20	51.80	32.51
8)	SPX+HP	72.50	74.50	32.89	880	28) SPX+TP	52.10	53.70	32.25
9)	SXB+HT	61.10	63.10	31.84	900	29) SXB+TT	60.40	62.40	31.40
10)	SXB+HD	50.80	52.50	30.98	920	30) SXB+TD	70.00	72.00	30.60
11)	SXB+HE	48.40	50.00	30.77	925	31) SXB+TE	72.50	74.50	30.37
12)	SXB+HH	41.50	43.10	30.16	940	32) SXB+TH	80.60	82.60	29.78
13)	SXB+HJ	37.30	39.00	29.80	950	33) SXB+TJ	86.30	88.30	29.39
14)	SXB+HL	33.30	34.90	29.40	960	34) SXB+TL	92.30	94.30	28.99
15)	SXB+HO	28.10	29.30	28.87	975	35) SXB+TO	101.90	103.90	28.47
16)	SXB+HP	26.40	27.00	28.08	980	36) SXB+TP	105.20	107.20	28.28
17)	SPQ+HT	20.40	21.60	29.00	1000	37) SPQ+TT	119.10	121.10	27.55
18)	SPQ+HC	16.50	17.70	27.48	1015	38) SPQ+TC	130.20	132.20	27.02
19)	SPQ+HD	15.40	16.60	27.36	1020	39) SPQ+TD	134.10	136.10	26.90
SPX 19 SEP 2009 (Contract Size 100)					5	SPX 19 SEP 2009 (Contract Size 100)			
20)	SPX+IO	92.40	94.40	32.67	975	40) SPX+UO	58.10	59.70	32.75

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SN 133053 H189-202-0 06-May-2009 03:00:07

Figure 1: Bloomberg Screen for Options on S&P 500 Index on May 6 2009**.**

We would like to price a swap in which the investor pays the realized variance and receives ${\sigma_X}^2 = 0.12$ on \$1,000,000 notional amount. Denote $U(K_i) = U_i$. Choose K_i as follows:

$K_1 = 800$ $K_2 = 820$ $K_3 = 825$

$K_4 = 840$ $K_5 = 850$ $K_6 = 860$

$K_7 = 875$ $K_8 = 880$ $K_9 = 900$

$K_{10} = 920$ $K_{11} = 925$ $K_{12} = 940$

$K_{13} = 950$ $K_{14} = 960$ $K_{15} = 975$

$K_{16} = 980$ $K_{17} = 1000$ $K_{18} = 1015$

$K_{19} = 1020.$

$F_0 = S_0 e^{(r-q)T} = 903.80$, then $S^* = 900.$

The prices of call and put options are found to be as follows (we take the bid-ask midpoint of the quotes). For $K_i < 900$ we use the prices of puts, for $K_i = 900$ we use the average of the call and put, and for $K_i > 900$ we use the prices of calls:

$$U_1 = 27.80, \quad U_2 = 32.90, \quad U_3 = 34.30,$$

$$U_4 = 38.70, \quad U_5 = 41.90, \quad U_6 = 45.40$$

$$U_7 = 51.00, \quad U_8 = 52.90, \quad U_9 = 61.75$$

$$U_{10} = 51.65, \quad U_{11} = 49.20, \quad U_{12} = 42.30$$

$$U_{13} = 38.10, \quad U_{14} = 34.10, \quad U_{15} = 28.70$$

$$U_{16} = 27.00, \quad U_{17} = 21.00, \quad U_{18} = 17.10$$

$$U_{19} = 16.00.$$

Then $\sum_{i=1}^{19} \frac{\Delta K_i}{{K_i}^2} e^{rT} \cdot U_i = x$ (do the calculations) and $\mathbb{E}^*(s^2) \approx \frac{2}{0.25} \ln\left(\frac{903.8}{900}\right) - \frac{2}{0.25} \cdot \left(\frac{903.8}{900} - 1\right) + \frac{2}{0.25} \cdot x = y$. Value of variance swap $= \$1\, m \cdot (y - 0.12) \cdot e^{-0.01 \cdot \frac{1}{4}}$

Valuation of Volatility Swap

We need $\mathbb{E}^*(s) = \mathbb{E}^* \sqrt{\mathbb{E}^*(s^2) \cdot (1 + \frac{S^2 - \mathbb{E}^* s^2}{\mathbb{E}^* s^2})}$. Use second order Taylor approximation

$f(x) = \sqrt{1+x} \approx f(0) + f'(0) \cdot x + \frac{1}{2} f''(0) \cdot x^2$ to get

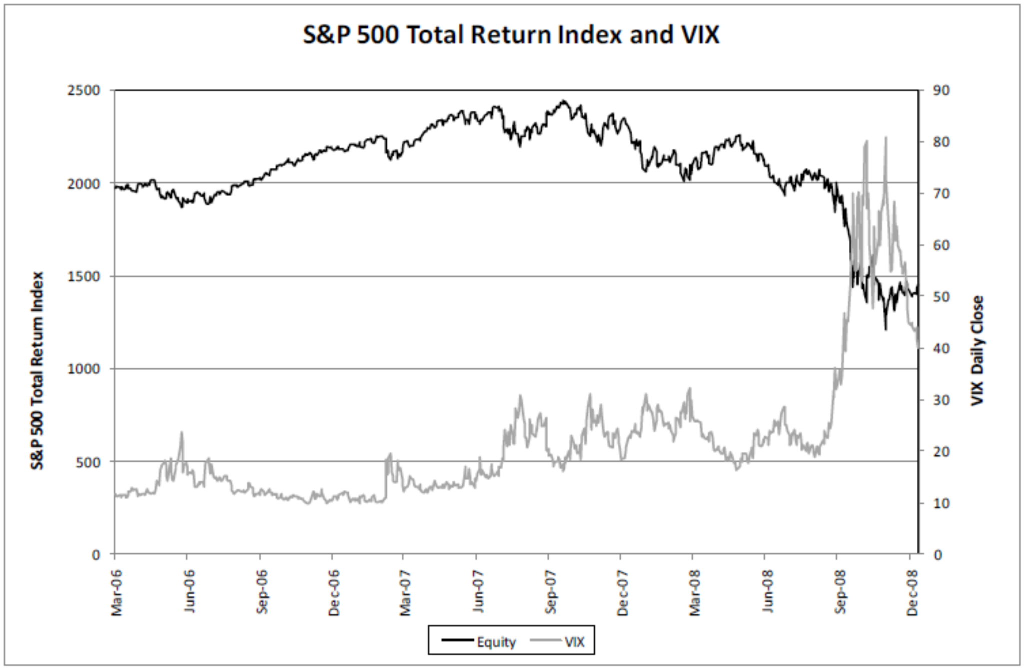
$$\sqrt{(1+x)} \approx 1 + x\left(1 + \frac{1}{2 \cdot \sqrt{1+x}}\ \Big|_{x=0}\right) - \frac{1}{2} x^2 \left(\frac{1/\sqrt{1+x}}{4(1+x)}\ \Big|_{x=0}\right) \approx 1 + \frac{x}{2} - \frac{x^2}{8}$$

$(s) = \sqrt{E^*(s^2)} \cdot \{1 + \frac{S^2 - E^* S^2}{2E^* S^2} - \frac{1}{8} \left(\frac{S^2 - E^* S^2}{E^* S^2}\right)^2\}$. Take $\mathbb{E}^*(.)$ of both sides of the above equation to get $\mathbb{E}^*(s) = \sqrt{E^*(s^2)} \cdot \left\{1 - \frac{1}{8} \frac{Var^*(S^2)}{\mathbb{E}^*(S^2)}\right\}$. Then, the value of the volatility swap = $L_{vol} \cdot [E^*(s) - \sigma_X] \cdot e^{-rT}$.

The VIX index:

The VIX index is a measure of market expectations of near-term volatility conveyed by S&P500 stock index option prices. It was introduced in 1993 and has been perceived to be a gauge of investors' sentiment and market volatility. The VIX index is negatively correlated with S&P500

index, the correlation being less than -50%. The following figure demonstrates the negative correlation between S&P500 and VIX.



The calculation of the VIX index is based on the formula:

$$\mathbb{E}^*(s^2) = \frac{2}{T}\ln\left(\frac{F_0}{S^*}\right) - \frac{2}{T}\left(\frac{F_0}{S^*} - 1\right) + \frac{2}{T}\int_0^{S^*} \frac{1}{k^2} e^{rT} \cdot P(k)dk + \frac{2}{T}\int_{S^*}^{\infty} \frac{1}{k^2} e^{rT} \cdot C(k)dk$$

Interested readers are referred to www.cboe.com for more information and details on the VIX Index, futures and options on the index, its construction and analysis.

6. 5 Exercises

1. Consider a 12-month Fixed Strike Lookback Call and Put options, when the interest rate is 3% per annum, the volatility is 30% per annum, and the strike price is $100. Use the MC simulation method to estimate the prices of the Call and Put options. The payoff of the call is $(S_{max} - X)^+$, where $S_{max} = \max\{S_t : t \in [0, T]\}$, and the payoff of the put option is: $(X - S_{min})^+$, where $S_{min} = \min\ \{S_t : t \in [0, T]\}$.

2. Compute, via MC simulation, the prices of the following options using 50,000 simulations of paths of the stock price and dividing the time-interval into 50 equal parts:

   (a) Down-and-Out- Put: $S(0) = 50,\ X = 50,\ r = 0.1,\ T =$ 2 months, $\sigma = 0.4$, S_b=40.

(b) Down-and-In - Put: $S(0) = 50,\ X = 50,\ r = 0.1,\ T =$ 2 months, $\sigma = 0.4$, S_b=40.

3. Compute the price of the Asian average rate and Asian average strike call options by using:

   (a) Standard MC method, where $S(0) = 50,\ X = 50,\ r = 0.1,\ T =$ 2 months, $\sigma = 0.4$.

   (b) Halton's Low-discrepancy sequences to generate the paths of the stock price, where $S(0) = 50,\ X = 50,\ r = 0.1,\ T =$ 2 months, $\sigma = 0.4$.

4. Compute the price of the Floating Strike Lookback Call and Put options by using: $S(0) = 50,\ X = 50,\ r = 0.05,\ q = 0.03, T =$ 2 months, $\sigma = 0.4$.

5. Assume the stock price follows an Arithmetic Brownian Motion. Derive the formula for a price of a European call option, using all the other Black-Scholes assumptions.

6.6 Simulation of Jump-Diffusions

We have modeled the stock price dynamics using a Geometric Brownian motion process in earlier chapters. Thus, we have implicitly assumed that the stock prices do not jump. Considering recent stock market crashes in 2008, October 19, 1987, and also the fact that prices are only recorded at discrete points in time, it seems reasonable to allow the possibility of the stock price jumps at random times and study such models of prices.

We assume that the dynamics of the stock price are given by the following jump-diffusion equation

$$\frac{dS_t}{S_{\bar{t}}} = \mu dt + \sigma dW_t + dJ_t, \quad W \perp J,$$

where $J_t = \sum_{j=1}^{N_t} (Y_j - 1)$ is a particular case, where Y_j are random variables and

N_t is the number of arrivals in $[0, t]$, $dJ_t = d\left(\sum_{j=1}^{N_t} (Y_j - 1)\right) = \begin{cases} Y_j - 1 & Y_i > 0 \\ 0 & \end{cases}$

Here dJ_t is the jump at time t, and the size of the jump at that time is $Y_j - 1$. Thus, we have

$dS_t = \mu S_{t-} dt + \sigma S_{t-} dW_t + S_{t-} dJ_t$, where $S_{t-} = \lim_{u \to t-} S_u$. To discretize, we have at time t_j, in case a jump occurs at time t_j,

$$S_{t_j} - S_{t_{\bar{j}}} = S_{t_{\bar{j}}} \left[J_{t_j} - J_{t_{\bar{j}}} \right] = S_{t_{\bar{j}}} \cdot (Y_j - 1)$$

or $S_{t_j} = S_{t_{\bar{j}}} \cdot Y_j$. If we take $\ln S_t$, then $\ln S_{t_j} = \ln S_{t_{\bar{j}}} + \ln Y_j$

Thus jumps are additive when $\ln(price)$ is considered. Thus, the stock price can be written as

$$S_t = S_0 \cdot e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_\tau} \cdot \prod_{j=1}^{N_t} Y_j$$

<u>Example</u> Assume $N_t \sim$Poisson(λ). Then the times between realization is $\exp(\lambda)$ distributed.

Assume Y_j are $iid,\ N_U \perp W_t$

We now discuss two different methods to simulate a jump-diffusion.

Method 1*:* Timeline

Since $S_{t_{j+1}} = S_{t_j} \cdot e^{\left(\mu - \frac{1}{2}\sigma^2\right)(t_{j+1}-t_j)+\sigma\left(W_{t_{j+1}}-W_{t_j}\right)} \cdot \prod_{l=N_{t_j}+1}^{N_{t_{j+1}}} Y_l$

Note: $\prod_{l=N_{t_j}+1}^{N_{t_{j+1}}} Y_l = 1$ if $N_{t_{j+1}} = N_{t_j}$

Taking $\ln(\cdot)$ we get for $X_t = \ln S_t$

$$X_{j+1} = X_j + \left(\mu - \frac{1}{2}\sigma^2\right)\left(t_{j+1} - t_j\right) + \sigma\left(W_{t_{j+1}} - W_{t_j}\right) + \sum_{j=N_{t_j}+1}^{N_{t_{j+1}}} \ln Y_j$$

Step 1: Generate $Z_i \sim N(0,1)$ at step $i(t_i \to t_{i+1})$

Step 2: Generate $N_i \sim \text{Poisson}\big(\lambda(t_{i+1} - t_i)\big)$ ←explain this (number of jumps in $[t_i, t_{i+1}]$ interval)

Step 3: Generate $\ln Y_1, \ldots, \ln Y_{N_i}$, set $M_i = \left(\ln Y_1 \cdots Y_{N_i}\right)$

Step 4: $X_{i+1} = X_i + \left(\mu - \frac{1}{2}\sigma^2\right)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i} \cdot Z_i + M_i$

Recall that Poisson also has the property that (similar to Brownian Motion)

(1) $N_{t_{i+1}} - N_{t_i} \sim P\big(\lambda(t_{i+1} - t_i)\big)$
(2) $N_t - N_s \perp N_s - N_u, u < s < t$

In section 1.1.4 we discussed how to generate Poisson random variables.

Method 2: Jump-Times

Simulate the process from one jump time τ_i to the next τ_{i+1}.

$S_{\tau_{\overline{i+1}}} = S_{\tau_i} \cdot e^{\left(\mu - \frac{1}{2}\sigma^2\right)(\tau_{i+1}-\tau_i)+\sigma\left(W_{\tau_{i+1}}-W_{t_i}\right)}$ and $S_{t_{i+1}} = S_{\tilde{t}_{\overline{i+1}}} \cdot Y_{j+1}$. That implies, that

$$X_{i+1} = X_i + \left(\mu - \frac{1}{2}\sigma^2\right)(\Delta_i) + \sigma W_{\Delta_i} + \ln Y_{i+1}$$

Steps: $t_i \to t_{i+1}$

1. Simulate $Z_{i+1} \sim N(0,1)$, $T_{i+1} \sim \exp\left(\frac{1}{\lambda}\right) \perp Z_{i+1}$
2. $\tau_{i+1} = \tau_i + T_{i+1}$ (next jump-time)
3. $X_{i+1} = X_i + \left(\mu - \frac{1}{2}\sigma^2\right)T_{i+1} + \sigma\sqrt{T_{i+1}}Z_{i+1} + \ln Y_{i+1}$

Examples of Y_i

1. $Y_i \sim \log N(a, b^2)$, then $\sum_{i=1}^{n} \ln Y_i \sim N(an, nb^2) = a \cdot n + b\sqrt{n} \cdot Z$
2. $\ln Y_i \overset{d}{=} \begin{cases} E^+ & w \cdot p \quad p \\ -E^- & w \cdot p \quad 1-p \end{cases}$ where $E^{\pm} \sim$ exp $\left(\frac{1}{\lambda}, \lambda\right) \sim$Do$uble\ Exponential.$

Application (Use of Jump-Diffusion processes and Pricing of Default Options)

Assume that the value of a collateral follows a jump-diffusion process:

$$\frac{dV_t}{V_t^-} = \mu dt + \sigma dW_t + \gamma dJ_t$$

where $\mu, \sigma,\ \gamma < 0$, and V_0 are given, J is a Poisson process, with intensity λ_1, independent of the Brownian Motion process W. V_t^- is the value process before jump occurs at time t (if any).

Assume there is a loan, with a contract rate of *r* per period, and maturity *T* years, on the above-collateral, and the value of that loan follows this process:

$$\boldsymbol{L_t = a - bc^t}$$

where $\boldsymbol{a > 0}, \boldsymbol{b > 0}\ , \boldsymbol{c > 1}$, **and** $\boldsymbol{L_0}$ are given. We have that $\boldsymbol{L_T = 0}$.

Notes:

1. The process *J* may have more than one realization during the life of the loan (which is T).

2. One may simulate the realizations of *J* using the relationship between Exponentials and Poisson processes: the time of occurrence of the first Poisson realization (and the time between any two consecutive Poisson realizations) is exponentially distributed random variable with the same intensity as the one of the Poisson process.

3. L_t can be modeled as Brownian Bridge process: $L_0 = l$ and $L_T = 0$ are known.

The borrower has a “default option” here, the exercise of which implies that the loan liability is released in return for the collateral.

Define the following stopping time:

$$Q = min\{t \geq 0:\ V_t \leq q_t L_t\}$$

Here q_t is a deterministic but time-dependent function.

Q is the first time when the relative value of the collateral (with respect to the outstanding loan balance) hits a certain boundary, q_t , the "optimal exercise boundary of the embedded default option". The default option is exercised if this boundary is hit by the value process before time *T*.

Define another stopping time:

$$S = min\{t \geq 0: N_t > 0\}$$

That is, *S* is the first time (before *T*) of a realization of another jump (*N*), which is independent of *J* and *W*. If no jump is realized until *T*, then *S* is taken to be ∞.

This can be thought of as a random time when the borrower defaults on the loan due to inability (job loss, etc.) or unwillingness to pay.

We assume the "default option" will be exercised if and only if $\tau = \min\{Q, S\} < T$.

That is, if the "optimal exercise boundary " is hit, or a jump *(N*) occurs during the life of the loan, then the default option is exercised. The option is exercised at the first time one of these two happens.

Notes:

1. If $\min\{Q, S\} > T$ then there is no default option exercise.
2. It is only the first realization of the *N* process (if it happens before *T*) that is useful here. That is the option is exercised upon the first realization of *N* if and only if it happens before *T*.

Assume *J* has intensity λ_1 and *N* has intensity λ_2. *N* is independent of *J* and *W*.

Here, a, b, c are functions of λ_2 (λ_2 determines the default probability of the borrower, which will be related to his creditworthiness and determine the contract rate).

Assume the APR of the loan is R and the contract rate per period is *r*. Assuming monthly compounding here, we will have $r = R/12$.

In fact, a, b, c are functions of the contract rate (*r*) and we model the dependence as follows:

$$R = r_0 + \delta\lambda_2$$

where r_0 is the "prime" rate, and δ is a positive parameter to measure the borrower's creditworthiness in determining the contract rate *r*.

Assume

$$q_t = \alpha + \beta t$$

where $\beta > 0$, $\alpha > L_0/V_0$.

Exercises:

1. Assume stock prices follow jump-diffusion process above with $Y_i \sim \log N(0,1)$. Price call and put options that expire in T years and are of European type.

2. Assume stock prices follow jump-diffusion process above with

   $\ln Y_i \overset{d}{=} \begin{cases} E^+ & w \cdot p \quad 0.6 \\ -E^- & w \cdot p \quad 0.4 \end{cases}$ where $E^{\pm} \sim \text{Do}uble\ Exponential(2,0.5)$.

   Price call and put options that expire in T years and are of European type.

3. Assume that the value of a collateral follows a jump-diffusion process:

$$\frac{dV_t}{V_t^-} = \mu dt + \sigma dW_t + \gamma dJ_t$$

where $\mu, \sigma,\ \gamma < 0$, and V_0 are given, J is a Poisson process, with intensity λ_1, independent of the Brownian Motion process W.
V_t^- is the value process before jump occurs at time t (if any).
Consider a collateralized loan, with a contract rate per period r and maturity T on the above-collateral, and assume the outstanding balance of that loan follows this process:

$$\boldsymbol{L_t = a - bc^{12t}}$$

where $a > 0, b > 0\ , c > 1,$ and L_0 are given. We have that $\boldsymbol{L_T = 0}$.

Define the following stopping time:

$$Q = min\{t \geq 0\!:\ V_t \leq q_t L_t\}$$

This stopping time is the first time when the relative value of the collateral (with respect to the outstanding loan balance) crosses a threshold which will be viewed as the “optimal exercise boundary” of the option to default.

Define another stopping time, which is the first time an adverse event occurs:

$$S = min\{t \geq 0\!: N_t > 0\}$$

Assume that N_t is a Poisson process with intensity of λ_2.

Define $\tau = \min\{Q, S\}$.

We assume the embedded default option will be exercised at time τ, if and only if $\tau < T$.

If the option is exercised at time Q then the payoff to the borrower is $\left(L_Q - \epsilon V_Q\right)^+$.

If the option is exercised at time S then the payoff to the borrower is $abs(L_S - \epsilon V_S)$, where *abs(.)* is the absolute value function.

Notes:

1. If $\min\{Q, S\} > T$ then there is no default option exercise.

2. ϵ should be viewed as the recovery rate of the collateral, so $(1 - \epsilon)$ can be viewed as the legal and administrative expenses.

Assume J has intensity λ_1 and N has intensity λ_2. N is independent of J and W.

Assume the APR of the loan is $R = r_0 + \delta\lambda_2$ where r_0 is the "risk-free" rate, and δ is a positive parameter to measure the borrower's creditworthiness in determining the contract rate per period: r.

We have monthly compounding here, so $r = R/12$.

Assume that $q_t = \alpha + \beta t$, where $\beta > 0$, $\alpha < V_0/L_0$ and $\beta = \frac{\epsilon - \alpha}{T}$.

Use r_0 for discounting cash flows. Use the following base-case parameter values:

$V_0 = \$20{,}000, L_0 = \$22{,}000, \mu = -0.1, \sigma = 0.2, \gamma = -0.4, \lambda_1 = 0.2, T = 5$ years, $r_0 = 0.02, \delta = 0.25, \lambda_2 = 0.4, \alpha = 0.7, \epsilon = 0.95$. Notice that $PMT = \frac{L_0.r}{\left[1 - \frac{1}{(1+r)^n)}\right]}$, where $r = R/12$, $n = T * 12$, and $a = \frac{PMT}{r}, b = \frac{PMT}{r(1+r)^n}, c = (1 + r)$. Notice that $q_T = \epsilon$.

Write the code as a function Proj6_2func.m that takes λ_1, λ_2 and T as parameters, setting defaults if these parameters are not supplied, and outputs the default option price, the default probability and the expected exercise time. Function specification:

```
function [D, Prob, Et] = Proj6_2func(lambda1, lambda2, T)
```

(a) Estimate the value of the default option for the following ranges of parameters:

λ_1 from 0.05 to 0.4 in increments of 0.05;
λ_2 from 0.0 to 0.8 in increments of 0.1;
T from 3 to 8 in increments of 1;

(b) Estimate the default probability for the following ranges of parameters:.

λ_1 from 0.05 to 0.4 in increments of 0.05;
λ_2 from 0.0 to 0.8 in increments of 0.1;
T from 3 to 8 in increments of 1;

(c) Find the Expected Exercise Time of the default option, conditional on $\tau < T$. That is, estimate $E(\tau \mid \tau < T)$ for the following ranges of parameters:.

λ_1 from 0.05 to 0.4 in increments of 0.05;
λ_2 from 0.0 to 0.8 in increments of 0.1;
T from 3 to 8 in increments of 1;

Inputs: s*eed*

Outputs:

i. Values: the default option D, the default probability Prob and the expected exercise time Et for parts (a), (b) and (c) with λ_1=.2, λ_2=0.4 and T=5.
ii. Graphs: For each of (a), (b) and (c) two graphs as a function of T, first with λ_1=0.2 and λ_2 from 0.0 to 0.8 in increments of 0.1, then with λ_2 = 0.4 and λ_1 from 0.05 to 0.4 in increments of 0.05. Put the two graphs in one .png file.

(d) Make additional assumptions as necessary to estimate the IRR of the investment.

Note:

The drift of the *V* process should be a function of r_0, λ_1, σ under the risk-neutral measure, to be able to price the option, but not done so in this case.