Chapter 5 Pricing Options with the LSMC Method (v.17.1)

5.1 Motivation

We have seen in a previous chapter that the arbitrage-free price of a European-style contingent claim can be expressed as the time t conditional expected value of its discounted payoff under the risk-neutral probability measure $P*$:

$$V_t = \mathbb{E}_t^*\left(e^{-\int_t^T r(s)\,ds} V_T\right) = \mathbb{E}_t^*\left(e^{(T-t)r} V_T\right)$$

Here V_T is the payoff of the claim at time T and we assume that the risk-free rate is a constant. The first expectation corresponds to the case of stochastic interest rate, and the second one to a constant interest rate (risk-free rate).

Define a stopping time τ on a probability space $(\Omega, \Lambda, \mathrm{P})$ as $\tau: \Omega \to \{t_0, t_1, \dots, t_n\}$, so that $\{\tau = t_k\} \in \Lambda$ for any $k = 0,1, \dots, n$. To price American-type options, we will take the exercise of the option over all possible stopping times and use the discounted expected future payoff idea of pricing. The exercise price of the option will be random – it is the optimal stopping time.

The price of the option V_t at time *t* will be given by the following expression

$$V_t = \sup_{\tau \in [t,T]} \mathbb{E}_t^*\left(e^{-(\tau-t)r}\, Payoff(\tau) | \mathcal{F}_t\right)$$

where $Payoff(\tau)$ is the payoff of the option at the optimal exercise time τ.

For American Put options, in particular, the pricing formula would be

$$V_t = \sup_{\tau \in [t,T]} \mathbb{E}_t^*\left(e^{-(\tau-t)r}(X - S_\tau)^+ | \mathcal{F}_t\right)$$

Assume τ^* is the optimal stopping time that solves the above equation. Define

$$EV_t = Exercice\ Value\ at\ time\ t$$

$$CV_t = Continuation\ Value\ at\ time\ t$$

$$\mathbb{E}CV_t = Expected\ Conditional\ Continuation\ Value\ at\ time\ t.$$

The optimal stopping/exercise time is given by

$$\tau^* = The\ first\ time\ that\ the\ Exercice\ Value \geq The\ \boldsymbol{Expected}\ Conditional\ Continuation\ Value\ of\ the\ option$$

$$\tau^* = min\{t \geq 0: EV_t \geq \mathbb{E}CV_t\} = min\{t \geq 0: (X - S_t)^+ \geq V_t\}$$

There is no closed-form expression for the optimal exercise time τ^*, or for the optimal exercise boundary (the stock prices for which it is optimal to exercise the American put option). That is why there is a need for numerical methods for estimating the above stopping time (or the optimal exercise boundary) to price options.

5.2 The Least-Square-Monte-Carlo Method (LSMC)

The main idea of pricing American-type options via simulation is as follows. Define

$$V_T = (X - S_T)^+ \text{ and } V_t = \max\left(EV_t,\ \mathbb{E}CV_t | \mathcal{F}_t\right) \text{ for any } t \leq T$$

The goal is to estimate V_0, which is the value of the option at time 0. The estimation will be done recursively, by backward estimation.

We start at the terminal time $t_n = T$, compute the exercise value (EV) of the option $(X - S^i_{t_n})^+$, for every path $i = 1, \dots m$. There is no continuation value at this time step as this is the last time step. Therefore, the option values will simply be their exercise values in the final time step $t = t_n$. Now we have the option values for every path at time $t = t_n$.

Next, we move backward in time to time step $t_{n-1}$and estimate the exercise value (EV) of the option at every node $(i, n-1)$ for $i = 1, \dots m$: $(X - S^i_{t_{n-1}})^+$. We also compute[1] the Expected Continuation Value ($\mathbb{E}CV$) of the option at every node $(i, n-1)$ of time step t_{n-1}. Then we compare the EV to $\mathbb{E}CV$ and take the larger of the two as the value of the option at node $(i, n-1)$ of time step t_{n-1}. Continuing this process until time $t = t_0$ will lead to computation of V_0 -- the value of the option at time $t_0 = 0$.

[1] The details of this computation will be provided later.

Examples:

1. In Binomial Framework:

$$V_t = \max\,(EV_t,\ \mathbb{E}CV_t|\mathcal{F}_t) = \max\,\left((X - S_t)^+,\ e^{-r\Delta}(pV_u + (1-p)V_d)\right)$$

Notice that the Expected Continuation Value ($\mathbb{E}CV$) of the option is given by $e^{-r\Delta}(pV_u + (1-p)V_d)$ in this model.

2. In Trinomial Framework:

$$V_t = \max\,(EV_t,\ \mathbb{E}CV_t|\mathcal{F}_t) = \max\,\left((X - S_t)^+,\ e^{-r\Delta}(p_uV_u + p_mV_m + p_dV_d)\right)$$

Notice that the Expected Continuation Value ($\mathbb{E}CV$) of the option is given by $e^{-r\Delta}(p_uV_u + p_mV_m + p_dV_d)$ in this model.

3. In Continuous-Time Setting: LSMC Method

The valuation method is what was described above:

$$V_t = \max\,(EV_t,\ \mathbb{E}CV_t|\mathcal{F}_t) \text{ for any } t \leq T$$

The challenge here is to estimate the continuation value:
$\mathbb{E}CV_t$=$\mathbb{E}_t^*$(Sum of all discounted Cash Flows after time $t|\mathcal{F}_t$)

Define Δ= $\frac{T}{n}$. Divide the time-interval by n equal parts:

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T, \text{ where } t_k = \frac{T}{n}k = \Delta k.$$

Then,

$\mathbb{E}CV_{t_k}$=$\mathbb{E}^*\left(Sum\ of\ all\ discounted\ Cash\ Flows\ after\ time\ t_k \middle| \mathcal{F}_{t_k}\right)$

$$=\mathbb{E}^*\left(\sum_{j=k+1}^{n} e^{-(t_j - t_k)r}\ CashFlow(t_j, t_k, T)\ |\mathcal{F}_{t_k}\right)$$

where $CashFlow(t_j, t_k, T)$ is the Payoff of the option at time $t_j > t_k$. Notice, that at most one of these cash flows can be non-zero (along each path).

Thus, the problem is to estimate the $\mathbb{E}CV$ at any node for the stock price and at any time. At any fixed time t_k, the $\mathbb{E}CV$ is a function of the stock prices at time t_k. The functional form of $\mathbb{E}CV$ will be different from one time to another.

The estimation method of $\mathbb{E}CV$ is based on the Least-Square approximation of functions in L^2 spaces. Assume the $\mathbb{E}CV$ functions are smooth enough to belong to the space L^2. Then for any orthonormal system of basis functions $\{L_l(x)\}_{l=1}^{\infty}$ of the space L^2 we have the following representation:

$$\mathbb{E}CV(x) = \sum_{l=1}^{\infty} a_l L_l(x)$$

This representation can be approximated by a truncated sum of the above infinite series:

$$\mathbb{E}CV(x) \approx \sum_{l=1}^{k} a_l L_l(x)$$

ASSUME we are able to estimate the scalar coefficients $\{a_{1,} a_2, \dots, a_k\}$, Then at any node (i, j), we can compute the expected continuation value of the option:

$$\mathbb{E}CV\left(S_j^i\right) = \sum_{l=1}^{k} a_l L_l(S_j^i)$$

Define

$$Y_t(S) = \mathbb{E}_t CV(S).$$

We need to estimate the functional form of the $Y_k(S)$ function at every time step t_k for $k = (n - 1), (n - 2), \dots\ 2,1$.

Note: The reason for not using the $Y(S_j^i)$ as continuation value (which would be simple as we know the value of the option in the next time-step from node (i, j)) is that we need to compute the Conditional Expected Continuation Value at node (i, j) and Y(S_j^i) is just one realization of it.

Start at S_0 and use the standard simulation methods to simulate m paths of the stochastic process $\{S_t: 0 \le t \le T\}$ at points $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, where $t_k = \frac{T}{n} k$. Store all the paths in the computer memory in a matrix form as shown below:

Stock Prices ↓ **Time →**	$\boldsymbol{t_0 = 0}$	$\boldsymbol{t_1}$	**…**	**…**	$\boldsymbol{t_{n-2}}$	$\boldsymbol{t_{n-1}}$	$\boldsymbol{t_n = T}$
Path 1	S_0^1	S_1^1			S_{n-2}^1	S_{n-1}^1	S_n^1
Path 2	S_0^2	S_1^2	…	…	S_{n-2}^2	S_{n-1}^2	S_n^2
…	…	…			…	…	…
Path (m-1)	S_0^{m-1}	S_1^{m-1}			S_{n-2}^{m-1}	S_{n-1}^{m-1}	S_n^{m-1}
Path m	S_0^m	S_1^m	…	…	S_{n-2}^m	S_{n-1}^m	S_n^m

Note: the index j in S_j^i is for time, and index i in S_j^i is for the path of the stock price. Also, $S_0^i = S_0$ for every i.

We also create an mxn matrix, called **Index**, with the element in (i, j) being denoted by Ind_j^i. Initially, set all $Ind_j^i = 0$ for $j = 1, \dots, n$ and $i = 1, \dots, m$. Having a 1 in any cell of the matrix means that the option should be exercised at that cell of the stock price/time space.

The estimation steps now are as follows:

At time $\boldsymbol{t = t_n = T}$

Exercise Value: $EV_{t_n}^i = EV_{t_n}(S_n^i) = (X - S_n^i)^+$

and

Expected Continuation Value: $\mathbb{E}CV_{t_n}^i = \mathbb{E}CV_{t_n}(S_n^i) = 0$ for any $i = 1, \dots, m$.

Then, since $EV_{t_n}^i \geq \mathbb{E}CV_{t_n}^i$ for any $i = 0,1, \dots, m$, then in those notes where the option in in-the-money, we will exercise the option. Thus, we can populate the column n of the matrix Index the following way:

$$Ind_n^i = \begin{cases} 1, & if \ \ EV_{t_n}^i > 0 \\ 0, & otherwise \end{cases}$$

for any $i = 1, \dots, m$.

Note: Having 1's for certain entries of matrix Index means that the option should be exercised in those nodes, and having 0's means the option should be kept alive.

Now we move one step back to time t_{n-1}.

At time $\boldsymbol{t = t_{n-1}}$:

Exercise Value: $EV_{t_{n-1}}^i = EV_{t_{n-1}}(S_{n-1}^i) = (X - S_{n-1}^i)^+$ for any $i = 1, \dots, m$.

Expected Continuation Value:

ASSUME we can estimate the functional form $Y_{n-1}(x)$ at this time step (the steps will be described later). Then,

$$\mathbb{E}CV_{t_{n-1}}^i = \mathbb{E}CV_{t_{n-1}}(S_{n-1}^i) = Y_{n-1}(S_{n-1}^i) \text{ for any } i = 1, \dots, m.$$

We can now compare the $\mathbb{E}CV$ and EV and populate the column $n - 1$ of the matrix Index the following way: for any $i = 0,1, \dots, m$,

$$Ind_{n-1}^i = \begin{cases} 1, & if \;\; EV_{t_{n-1}}^i \geq \mathbb{E}CV_{t_{n-1}}^i \\ 0, & otherwise \end{cases}$$

Note: In each row of matrix Index, we can have at most one 1. If $Ind_{n-1}^i = 1$ for an i, then we have to reset $Ind_n^i = 0$ for the same i, even if Ind_n^i was 1 in the previous time-step for that i.

Now we move one step back to time t_{n-2}.

At time $t = t_{n-2}$:

Exercise Value: $EV_{t_{n-2}}^i = EV_{t_{n-2}}(S_{n-2}^i) = (X - S_{n-2}^i)^+$ for any $i = 1, \ldots, m$.

Expected Continuation Value: **ASSUME** we can estimate the functional form $Y_{n-2}(x)$ at this time step (again, to be addressed later). Then,

$$\mathbb{E}CV_{t_{n-2}}^i = \mathbb{E}CV_{t_{n-2}}(S_{n-2}^i) = Y_{n-2}(S_{n-2}^i)$$

and now can compare the $\mathbb{E}CV$ and EV and populate the column $n-2$ of the matrix Index the following way: for any $i = 0,1, \ldots, m$,

$$Ind_{n-2}^i = \begin{cases} 1, & if \;\; EV_{t_{n-2}}^i \geq \mathbb{E}CV_{t_{n-2}}^i \\ 0, & otherwise \end{cases}$$

Note: In each row of matrix Index, we can have at most one 1. If $Ind_{n-2}^i = 1$, then we have to reset $Ind_{n-1}^i = 0$ and $Ind_n^i = 0$ for the same i, even if Ind_{n-1}^i or Ind_n^i was 1 in the previous time-step for that i.

Continuing the above steps recursively, we get to time $t = t_1$. At this stage, we have the matrix Index populated with 0 or 1's (each row can have at most one 1, that is the exercise time along that path). The estimated value of the option is given by

$$V_0 = \frac{1}{m} \sum_{j=1}^{n} \sum_{i=1}^{m} (Ind_j^i) \, e^{-rj\Delta} \, (X - S_j^i)^+$$

Now, the only remaining question is: how to estimate the functional form of the expected continuation value function at every time-step.

Start with time $\boldsymbol{t = t_{n-1}}$. We would like to estimate the functional form of $Y_{n-1}(S) = \mathbb{E}CV_{t_{n-1}}(S) = \mathbb{E}^*\left(CV(S) \middle| \mathcal{F}_{t_{n-1}}\right)$. This is a random variable, for which we have m −realizations:

For every realization $X_i = S_{n-1}^i$ of the independent variable X, we have a realization of the dependent variable Y: $Y_i = e^{-r\Delta} (X - S_n^i)^+$ for $i = 0,1, \ldots, m$.

Thus, we have m –realizations of (X_i, Y_i):

X	Y
S_{n-1}^1	$Ind_n^1\ e^{-r\Delta}\,(X - S_n^1)^+$
S_{n-1}^2	$Ind_n^2\ e^{-r\Delta}\,(X - S_n^2)^+$
	...
	...
S_{n-1}^m	$Ind_n^m e^{-r\Delta}\,(X - S_n^m)^+$

Performance tip: Choose only those observations for which the option is in-the-money since the exercise information is relevant only in those cases. This will make the computations more efficient.

Now, having m –realizations of the random variable function, we use the Least Square approach to estimate the functional form:

$$Y_{n-1}(x) \approx \sum_{l=1}^{k} a_l^{n-1} L_l(x)$$

The goal is to estimate the coefficients a_l^{n-1}. **Assume** we have estimated these coefficients. Then, the continuation value at any node at time t_{n-1} and for the stock price S_{n-1}^i will be given by $Y_{n-1}(S_{n-1}^i) = \sum_{l=1}^{k} a_l^{n-1} L_l(S_{n-1}^i)$.

The task now is to estimate the parameters $(a_1^{n-1}, a_2^{n-1}, \dots, a_k^{n-1})$. Note that these k parameters <u>will be different for every time step</u> and should be estimated for every time-step.

The procedure is very similar to the estimation of coefficients in linear regressions.

Define

$$A = \begin{pmatrix} \langle f_1, f_1\rangle & \cdots & \langle f_k, f_1\rangle \\ \vdots & \ddots & \vdots \\ \langle f_1, f_k\rangle & \cdots & \langle f_k, f_k\rangle \end{pmatrix}, \quad b = \begin{pmatrix} \langle Y, f_1\rangle \\ . \\ \langle Y, f_k\rangle \end{pmatrix}, \quad a^{n-1} = \begin{pmatrix} a_1^{n-1} \\ . \\ a_k^{n-1} \end{pmatrix}$$

where

$$\langle f_i, f_j\rangle = L_i(X_1)L_j(X_1) + \cdots + L_i(X_m)L_j(X_m)$$

$$\langle Y, f_j\rangle = Y_1 L_j(X_1) + \cdots + Y_m L_j(X_m)$$

$$X_i = S_{n-1}^i,\, Y_i = Y_{n-1}(S_{n-1}^i)$$

for any $j = 1, \dots, k$ and $i = 1, \dots, k$.

The problem of finding the set of parameters $a^{n-1} = (a_1^{n-1}, a_2^{n-1}, \dots, a_k^{n-1})'$ will boil down to solving a system of linear equations

$$Aa^{n-1} = b$$

The solution can be obtained by writing

$$a^{n-1} = A^{-1}b$$

Thus, we can solve for the parameters a^{n-1} at the time step $t = t_{n-1}$, then estimate the functional form of the expected continuation value function $Y_{n-1}(X)$, then for every node make a decision to exercise or to keep the option alive, then update the entries in $(n-1)st$ column of the Index matrix.

At time $\boldsymbol{t = t_{n-2}}$. We would like to estimate the functional form $Y_{n-2}(S) = \mathbb{E}^*\left(CV(S)\middle|\mathcal{F}_{t_{n-2}}\right)$. This is a random variable, for which we have m-realizations. For every realization $X_i = S_{n-2}^i$ of the independent variable X, we have a realization of the dependent variable Y: $Y_i = Ind_{n-1}^i\, e^{-r\Delta}\left(X - S_{n-1}^i\right)^+ + Ind_n^i\, e^{-r2\Delta}\left(X - S_n^i\right)^+$ for $i = 1, \dots, m$. Note, that at most one of the two terms in Y_i can be non-zero. Thus, we have m –realizations of (X_i, Y_i):

$\boldsymbol{X}$	$\boldsymbol{Y}$
S_{n-2}^1	$Ind_{n-1}^1\, e^{-r\Delta}\,(X - S_{n-1}^1)^+ \;+\; Ind_n^1\, e^{-r2\Delta}\,(X - S_n^1)^+$
	…
	…
	…
S_{n-2}^m	$Ind_{n-1}^m\, e^{-r\Delta}\,(X - S_{n-1}^m)^+ \;+\; Ind_n^m\, e^{-r2\Delta}\,(X - S_n^m)^+$

Note: Choose only those observations for which the option is in-the-money since the exercise information is relevant only in those cases. This will make the computations more efficient.

Now having m –realizations of the function, we use the Least Square approach to estimate the functional form at this time step:

$$Y_{n-2}(x) \approx \sum_{l=1}^{k} a_l^{n-2} L_l(x)$$

The goal is to estimate the coefficients a_l^{n-2}.

Assume we have estimated these coefficients. Then, the continuation value at any node at time t_{n-2} and for the stock price S_{n-2}^i will be given by $Y_{n-2}(S_{n-2}^i) = \sum_{l=1}^{k} a_l^{n-2} L_l(S_{n-2}^i)$.

The task now is to estimate the parameters $(a_1^{n-2}, a_2^{n-2}, \dots, a_k^{n-2})$. Note that these k parameters will be different for every time step and should be estimated for every time-step.

Define

$$A = \begin{pmatrix} \langle f_1, f_1 \rangle & \cdots & \langle f_k, f_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle f_1, f_k \rangle & \cdots & \langle f_k, f_k \rangle \end{pmatrix}, \quad b = \begin{pmatrix} \langle Y, f_1 \rangle \\ . \\ \langle Y, f_k \rangle \end{pmatrix}, \quad a^{n-2} = \begin{pmatrix} a_1^{n-2} \\ . \\ a_k^{n-2} \end{pmatrix}$$

where

$$\langle f_i, f_j \rangle = L_i(X_1)L_j(X_1) + \cdots + L_i(X_m)L_j(X_m)$$

$$\langle Y, f_j \rangle = Y_1 L_j(X_1) + \cdots + Y_m L_j(X_m)$$

$$X_i = S_{n-2}^i, Y_i = Y_{n-2}(S_{n-2}^i)$$

for any $j = 1, \dots, k$ and $i = 1, \dots, k$.

The problem of finding the set of parameters $a^{n-2} = (\, a_1^{n-2}, a_2^{n-2}, \dots, a_k^{n-2})'$ will boil down to solving a system of linear equations

$$Aa^{n-2} = b$$

The solution can be obtained by writing

$$a^{n-2} = A^{-1}b$$

We will repeat this process of estimating the coefficients a and thus the functional form of the expected continuation value at times $t_{n-3}, t_{n-4}, \dots t_2, t_1$. Thus, we can populate the entire matrix Index when we get to time t_1.

Comments:

1. *How to solve the linear system of equations $Ax = b$?*

   The are many well-studied methods for solving linear systems of equations such as $Ax = b$. Gaussian elimination method is one of them. The basic idea of the method is the following:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + \ a_{1k}x_k = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + \ a_{2k}x_k = b_2 \\ \qquad\qquad\qquad \dots \\ \qquad\qquad\qquad \dots \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + \ a_{kk}x_k = b_k \end{cases}$$

   Use linear combinations of rows to eliminate x_1 in every row from 2 to k (just keep it in the first row):

$$New\ Row_i = Old\ Row_i - \frac{a_{i1}}{a_{11}} Row_1$$

Now we have x_1 only in the first equation (first row).

Do the same to eliminate the terms that contain x_2, from row 3 to k. After successful elimination of x_2, from row 3 to k, we will have x_2 only in row 2.

Repeat the procedure until we have a set of k equations, which can be written in a matrix form as

$$\tilde{A}x = b,$$

where $\tilde{A}$ is a diagonal matrix. That is, all elements of the matrix below the main diagonal are 0. Sometimes we may need to permute certain rows to make sure all the operations (dividing by numbers) are valid.

Now it is easy to solve for x's: start from the last row first and solve it for x_k. Then move to row (k-1), use the found value of x_k and solve for x_{k-1}. Repeat this procedure recursively to solve for all x's.

LU-decomposition or Cholesky-decomposition (among many others, such as Gauss-Seidel, SOR) are other methods for solving the above system of linear equation.

2. *Below we will provide some choices of basis functions for the least square estimation.*

Some choices for basis functions: two orthogonal function families and monomials:

	Hermite	**Laguerre**	**Monomials**
I-term	1	$e^{-x/2}$	1
II-term	$2x$	$e^{-\frac{x}{2}}(1-x)$	x
III-term	$4x^2-2$	$e^{-\frac{x}{2}}(1-2x+\frac{x^2}{2})$	x^2
IV-term	$8x^3-12x$	$e^{-\frac{x}{2}}(1-3x+\frac{3x^2}{2}-\frac{x^3}{6})$	x^3
V-term	$16x^4-56x^2+16$	$e^{-\frac{x}{2}}(1-4x+3x^2-\frac{2x^3}{3}+\frac{x^4}{24})$	x^4
n-th term	$L_n = 2xL_{n-1} - 2(n-1)L_{n-2}$	$L_n(x) = e^{-\frac{x}{2}} \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x})$	x^n

A recent work by Stentoft (2004) suggests that using ordinary monomials in the Least Square approximation (that is functions of type $L_j(x) = x^{j-1}$) is computationally preferable than the choice of (some) orthogonal basis functions, such as Laguerre polynomials.

3. *What choice of* $\boldsymbol{k}$ *is reasonable?*

   A recent work by Stentoft (2004) studies the trade-off between the precision of convergence (higher k) and the computational time. The study suggests that the best specification uses $k = 2$ or 3 with simple polynomial functions).

4. *How to price options on stocks with stochastic volatility?*

   That would be a 2-factor model and the method would apply in the pricing of an option in such a framework. We would use two sets of basis functions (for the two factors) and their cross-terms in the least square estimation. Everything else will carry over from the method described earlier.

5. For stability of faster convergence of the method, use in-the-money paths only in the least squares estimation step as the goal is to estimate the expected continuation time. Also, scaling the prices by the exercise price has been shown to improve the stability of the algorithm.

References

Longstaff, F.A. and E.S. Schwartz , 2001,"Valuing American options by simulation: a simple least-squares approach". *Review of Financial Studies.* Volume 14, Number 1, pages 113-147.

Stentoft, Lars , 2004. "Assessing the Least Squares Monte-Carlo Approach to American Option Valuation," *Review of Derivatives Research*.

5.3 Exercises

1. Consider the following situation on the stock of company XYZ: The current stock price is $40, and the volatility of the stock price is $\sigma = 20\%$ per annum. Assume the prevailing risk-free rate is $r = 6\%$ per annum. Use the following method to price the specified option:

(a) Use the LSMC method with 100,000 paths simulations (50,000 plus 50,000 antithetic) to price an American put option with strike price of $X = \$40$, maturity of

0.5-years, 1-year, 2-years, and current stock prices of \$36, \$40, \$44. Use Laguerre polynomials for $k = 2, 3, 4$.

(b) Use the LSMC method with 100,000 paths simulations (50,000 plus 50,000 antithetic) to price an American put option with strike price of $X = \$40$, maturity of 0.5-years, 1-year, 2-years, and current stock prices of \$36, \$40, \$44. Use Hermite polynomials for $k = 2, 3, 4$.

(c) Use the LSMC method with 100,000 paths simulations (50,000 plus 50,000 antithetic) to price an American put option with strike price of $X = \$40$, maturity of 0.5-years, 1-year, 2-years, and current stock prices of \$36, \$40, \$44. Use simple monomials for $k = 2, 3, 4$.

(d) Compare all your findings above and comment.

2. Consider the following 2-factor model for stock prices with stochastic volatility:

$$\begin{cases} dS_t = rS_t dt + \sqrt{V_t}\, S_t dW_t^1 \\ dV_t = a(b - V_t)dt + c\sqrt{V_t} dW_t^2 \end{cases}$$

where the Brownian Motion processes above are correlated: $dW_t^1 dW_t^2 = \rho dt$.

(a) Compute the price of an American Call option (via Least Square Monte Carlo simulation) that has a strike price of K and matures in T years.
Use Hermite polynomials for $k = 2, 3$.
Use the following parameters of the model: $= -0.6,\ r = 0.03,\ S_0 = \$48,\ V_0 = 0.05,\ \sigma = 0.42,\ \alpha = 5.8,\ \beta = 0.0625$.

(b) Compute the price of an American Put option (via Least Square Monte Carlo simulation) that has a strike price of K and matures in T years.
Use simple monomials for $k = 2, 3$.
Use the following parameters of the model: $= -0.6,\ r = 0.03,\ S_0 = \$48,\ V_0 = 0.05,\ \sigma = 0.42,\ \alpha = 5.8,\ \beta = 0.0625$.

3. Compute the prices of American Call options on the same stock with same specifications as in part (c) of the previous problem. Compare with the exact (Black-Scholes) formula and comment.

4. Forward start options are path dependent options that have strike prices to be determined at a future date. For example, a forward start put option payoff at maturity is

$$max(S_t - S_T, 0)$$

where the strike price of the put option is S_t. Here $0 \le t \le T$.

(a) Estimate the value of the forward-start European put option on a stock with these characteristics: $S_0 = \$65, X = \$60, \sigma = 20\%$ per annum, risk-free rate is $r = 6\%$ per annum, $t = 0.2$ and $T = 1$.

(b) Estimate the value of the forward-start American put option on a stock with these characteristics: $S_0 = \$65, X = \$60, \sigma = 20\%$ per annum, risk-free rate is $r = 6\%$ per annum, $t = 0.2$ and $T = 1$. The continuous exercise starts at time $t = 0.2$.

5. An unfair (biased) die is such that the number k has the probability $k/21$ to come up in every roll, where k is in {1, 2, 3, 4, 5, 6}. On average how many times does one have to roll the die until all six numbers come up? Answer by using Monte Carlo simulations.

6. Start at 0 on a number line at time 0. Every second you move up by 1 unit with 75% probability or down by 1 unit with 25% probability. If you ever get to -1, the game is over. What is the probability that the game will eventually be over?

7. Start at 0 on a number line, flip a fair coin, move +2 on heads and -1 on tails. What is the probability that you will ever hit -1?

8. Two gamblers A and B have initially \$7 and \$13. Each time, they throw a fair coin: If the outcome is a head, A will give \$1 to B, else B will give \$1 to A. The game is over only if A or B is ruined (the time is infinite). What is the probability that A will win?