Chapter 9 Pricing Asset Backed Securities (ABS) (v.17.1)

In this chapter we will provide some techniques for pricing Asset Backed Securities (ABS), particularly Mortgage Backed Securities (MBS). Consider a pool of mortgages. The constituents of the pool may have different characteristics, but for tractability we will assume there is a single loan in the pool that has the weighted-average characteristics of the pool.

Assume the value of the loan at time 0 is PV_o. Let PMT be the Mortgage payment, R be the APR (Annual Percentage Rate), n be the number of periods (in a traditional case of a constant payment mortgage, $n = 360$). Then the rate per period (which is a month in this case) is $r = \frac{R}{12}$. When the last payment is made, it makes the balance of the loan equal to 0. Then, the payments, the original balance, the monthly rate, and the number of periods are related by this simple relationship:

$$PV_o = \frac{PMT}{1+r} + \cdots + \frac{PMT}{(1+r)^n} = \frac{PMT}{r} . \left[1 - \frac{1}{(1+r)^n}\right]$$

Notice that this is the formula for finding the present value of an ordinary annuity.

If we assume that the payments grow at the constant rate g per period, then the formula would be given as follows:

$$PV_o(g) = \frac{PMT(1+g)}{1+r} + \cdots + \frac{PMT(1+g)^n}{(1+r)^n} = \frac{PMT}{r-g} . \left[1 - \frac{(1+g)^n}{(1+r)^n}\right]$$

Thus, in the mortgage case ($g = 0$) we have

$$PMT = \frac{PV_o . r}{\left[1 - \frac{1}{(1+r)^n}\right]}$$

Generalizing, one can write a formula for the payment for period t, that depends on the outstanding balance at time $(t-1)$, which is the time when the payment for period t will be made:

$$PMT_t = \frac{PV_{t-1}.r}{\left[1 - \frac{1}{(1+r)^{n-(t-1)}}\right]}$$

for $t = 1,2,\ldots\ldots,360$. The latter formula is used to compute the payments for not-traditional mortgages, such as adjustable-rate mortgages (ARMs). For example, to calculate the monthly payments for a 5/1 ARM, one calculate the monthly payments using the fixed rate for the initial 5 years, as if that payment will be made for the whole 30 years. After 5 years, or 60 payments, the rate will reset to a new rate (adjustable and normally benchmarked to LIBOR or TSY Bill rates). One will compute the payment calculations using the new rate for 300 months. That payment, however, will be made for only 1 year (12 payments). After that year, the calculations will be redone for the next – 7^{th} – year. The process is continued until the end of the 30 years.

In general, the interest portion of the t-th payment, $PMT_{t,}$ is found as follows:

$$Int_t = PV_{t-1}.r$$

and the amortization portion of the t-th payment, $PMT_{t,}$ is found as follows:

$$Amortization_t = PMT_t - Int_t$$

9.1 Mortgage – Imbedded Options

Majority of mortgages in the US have no pre-payment payment and carry some kind of default insurance. The borrower has the right to prepay (in full or partially) their balances prior to the maturity. This right (the prepayment option) is exercised for several reasons: borrower mobility, or interest rates (refinancing), etc. One of the most important causes of prepayment is the refinancing due to lower mortgage rates. When mortgage rates are lower, the borrower has financial incentives to pay off the existing mortgage and refinance with another lender. This type of exercise (in which you pay upon the exercise of the option) is typical to "call-type" options. However call-type options are exercised (they are ITM) if the underlying value increases. Thus, prepayment option is a call-type option, but the underlying is not the mortgage rate, but the value of the mortgage. The mortgage value increases as rates decrease. The borrower (who has sold the mortgage to the lender) has the right to buy it back (the prepayment option), which is exercised if the value of the mortgage is high.

Thus, the underlying process here is the value of the mortgage, but the underlying factor is the interest (or mortgage) rate.

Most mortgages in the US are non-recourse ones, which are of "limited- liability" type. That is, the borrower has the right to give up the property and not pay back. The lender can seize the property in case of default. This right of the borrower can also be viewed as an option. In this case, this option is exercised (that is, the borrower defaults and stops paying) if the value of the property declines and gets lower than the outstanding balance on the property. Upon exercise of this option, the borrower essentially "sells" the property for the amount (outstanding) owed on it. That is, the borrower gives up the property (think of it as selling it) and in return the outstanding balance is nullified (think of it as getting paid as the borrower was supposed to pay that balance back, but now the liability is released).

Thus, this looks like a put option as the borrower "sells" something and "gets paid". The underlying now, is the value of property.

Thus, there are two imbedded options in most mortgages in the US:

- <u>The prepayment option</u> ("call-type" option on the mortgage value with underlying interest rates
- <u>The default option</u> ("put-type" option where the underlying is the property value.)

Pricing a Constant Payment Mortgage (the classes in the US), without the two above-mentioned imbedded options is like pricing an ordinary annuity, the valuation and analysis of which is a simple task. In a realistic case, it is not so easy to value mortgages, when the prepayment and default possibilities are incorporated. We will attempt to price in some special cases. As known, the price is the present value of the expected cash flows (under the risk-neutral measure):

$$P_o = E^*\left(\sum_{t=1}^{N} PV(CF_t)\right) = E^*\left(\sum_{t=1}^{N} d_t . c_t\right)$$

In order to find the price we need to have two things:

(1) $d_t = \exp\left(-\int_0^t r_u du\right)$ where r_u is the short term interest rate at time u.

(2) c_t = cash flow at time t.

Make some notations below and state a few simple properties:

c_t = Total Principal Payment $(TPP_t) + Interest\ Payment\ (IP_t)$,

TPP_t = Scheduled Principal (SP_t)+ Prepayment (PP_t),

MP_t = Scheduled Mortgage Payment.

Thus, $c_t = SP_t + PP_t + IP_t = TPP_t + IP_t = MP_t + PP_t$. Let PV_t be the principal balance of the loan at time t. (PV_o is the balance of the loan at origination). The following calculations follow from simple Annuity valuation formulas:

- $IP_t = PV_{t-1}.r$ (r is the rate per month which is: $r = \frac{Weighted\ Average\ Coupon}{12} = \frac{WAC}{12}$)
- $MP_t = \frac{PV_{t-1}.r}{\left[1-\frac{1}{(1+r)^{N-t+1}}\right]}$

<u>Note:</u> Weighted Average Maturity (WAM) is assumed to be N at origination.

- $PV_t = PV_{t-1} - TPP_t$

Now define prepayments:

Definition 1: Single Monthly Mortality (SMM) is defined to be the fraction of the beginning of the month balance that is prepaid during the month:

$$SMM_t = \frac{PP_t}{PV_{t-1} - SP_t}$$

Now define the annualized version of SMM - the Conditional Prepayment Rate (CPR) as follows:

Definition 2: Conditional Prepayment Rate (CPR) is the annualized version of SMM defined as follows:

$$CPR_t = 1 - (1 - SMM_t)^{12}$$

Thus, $SMM_t = 1 - (1 - CPR_t)^{\frac{1}{12}}$. We have

$$PP_t = (PV_{t-1} - SP_t)SMM_t = (PV_{t-1} - SP_t)\left(1 - (1 - CPR_t)^{\frac{1}{12}}\right)$$

Assume we have an explicit formula for CPR_t (to be specified later). Then

$$c_t = SP_t + PP_t + IP_t = MP_t + PP_t = \frac{PV_{t-1}.r}{\left[1 - \frac{1}{(1+r)^{N-(t-1)}}\right]} + (PV_{t-1} - SP_t)\left(1 - (1 - CPR_t)^{\frac{1}{12}}\right)$$

Since

$$SP_t = MP_t - IP_t = \frac{PV_{t-1}.r}{\left[1 - \frac{1}{(1+r)^{N-(t-1)}}\right]} - PV_{t-1}.r = PV_{t-1}.r\left[\frac{1}{1 - \frac{1}{(1+r)^{N-(t-1)}}} - 1\right]$$

Then,

$$c_t = \frac{PV_{t-1}.r}{\left[1-(1+r)^{-N+(t-1)}\right]} + \left(PV_{t-1} - PV_{t-1}.r\left[\frac{1}{1-(1+r)^{-N+(t-1)}} - 1\right]\right)\left(1 - (1 - CPR_t)^{\frac{1}{12}}\right)$$

Given that a formula for the CPR_t is provided, we have an explicit formula for c_t.

<u>Notes:</u>

1. The Interest portion, IP_t, of c_t is: $IP_t = PV_{t-1}.r$
2. The principal amortization portion TPP_t of c_t is: $TPP_t = SP_t + PP_t =$

$$= PV_{t-1}.r\left[\frac{1}{1-(1+r)^{-N+(t-1)}} - 1\right] + \left(PV_{t-1} - PV_{t-1}.r\left[\frac{1}{1-(1+r)^{-N+(t-1)}} - 1\right]\right)\left(1 - (1 - CPR_t)^{\frac{1}{12}}\right)$$

Notice, that the total principal amortization portion $\boldsymbol{TPP_t}$ has two components: the Scheduled Principal Amortization ($\boldsymbol{SP_t}$), and the Prepayment ($\boldsymbol{PP_t}$).

The big question we have now is: ***what is an appropriate model for CPR?***

Below we will consider two simple cases.

9.2 Models of CPR

We will consider two models for CPR – the government standard, and a model developed by Numerix.com.

9.2.1 Public Securities Association's Model (PSA)

Define CPR_0 = 0 and $CPR_t = \begin{cases} CPR_{t-1} + 0.2\% & \text{for} \quad t \leq 30\, months \\ CPR_{t-1} & \text{for} \quad t > 30\, months \end{cases}$

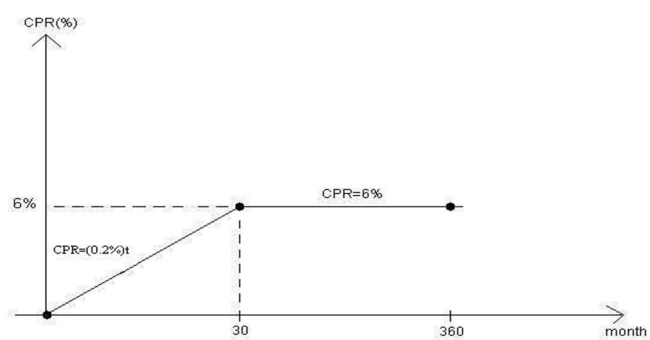

Figure 1: Conditional Prepayment Rate of the Public Securities Association's Model

Most quotes for CPR (e.g. Bloomberg or other market quotes) are given as percentage of PSA.

For example: The CPR for the MBS XYZ for the month of June is 170% PSA. That means whatever PSA prepayment rate would be for an MBS in its $k-th$ month, the projected prepayment rate is 170% of that. In other words, assume the MBS XYZ is currently in its 14th month of existence. Then the CPR for 15th month is projected to be 170% of (0.2% x 15) = 170% of 3% = 5.1 %

9.2.2 Numerix-Prepayment Model (www.numerix.com)

We use a 4- factor model of CPR here where the factors are:

- Interest Rate - lower rates give more incentives to refinance, thus higher prepayment rates;
- Burnout - CPR depends on the path taken in the pool. When faced with refinancing opportunities most aware borrowers react a prepay first. After a while, only those are left in the pool that will not refinance even if faced with larger financial incentives;
- Seasonality - Prepayments are faster in Spring and Summer than during Fall and Winter due to mobility of homeowners;
- Seasoning - This is also referred to as aging prepayment rates are lower shortly after issuance, but increase over time. Family situations are unlikely to change shortly after purchase of a house.

Thus, we model CPR as follows:

$$CPR_t = (Refi\ Incentive_t)(Burnout_t)(Seasoning_t)(Seasonality_t)$$

$$CPR_t = (RI_t)(BU_t)(SG_t)(SY_t)$$

From historical data calibration, the following parameters can be used in the above CPR model:

- $RI_t = 0.28 + 0.14. Arctan(-8.57 + 430(R - r_{t-1}(10))$

where R is the Mortgage Rate (annualized, that is $R = 12r$), $r_{t-1}(10)$ is the 10-yr rate, observed of the end of period $t-1$.This rate $-r_{t-1}(10)-$is a proxy for 30-year fixed mortgage rate.[1]

- $BU_t = 0.3 + 0.7\,\frac{PV_{t-1}}{PV_o}$
- $SG_t = \min(1, \frac{t}{30})$
- $SY_t = \{0.94, 0.76, 0.74, 0.95, 0.98, 0.92, 0.98, 1.10, 1.18, 1.22, 1.23, 0.98\}$ for {Jan, Feb, … , Dec} correspondingly.

Comment: The 10-yr rate, $r_{t-1}(10)$, observed of the end of period $(t-1)$ can be estimated as follows:

Along path i, at time $(t-1)$, we have that the spot rate is given by this formula:

$$r_{t-1}(T)^i = -\frac{1}{T}\ln\left(P_{t-1}(T)^i\right) = -\frac{1}{T}\int_{t-1}^{t-1+T} r_u{}^i du$$

Here one can either use the explicit formula for the price of the bond to estimate the spot rate, or estimate the spot rate by simulation of the interest rate path for T years out from $(t-1)$.

9.3 Tranching – CDO, CMO, etc.

A CMO structure is a mechanism for reallocating cash flow from a pool of mortgages into multiple classes (tranches) with different priority claims. There are two major types of CMO:

1) Only principal payments are passed through various tranches (Sequential pay, PAC – companion, TAC – companion),

2) All payments are passed through.

The three structures in the first type of tranching are described below:

[1] See the historical comparisons of the two rates.

- **Sequential pay** – most basic structure: A,B,...,Z tranches are created with priority of getting principal and prepayments . All tranches receive interest payments (except Z) and all the prepayments and principal payments flow to the tranche with highest priority.

Thus, in this case, tranche A will have lower duration than tranche B, than tranche C….

- **PAC** (Planned Amortization Class)
  - Hope to reduce the Prepayment Risk
  - Fixed Principal payments are scheduled to be received (if ppt is within a range [L,U])
  - If ppt rates are higher, (>U) PACs will get paid more
  - If ppt rates are lower than PAC's payment are delayed (<L).
- **TAC** (Targeted Amortization Class)
  - These like PAC offer some protection against prepayment risk
  - Here a single ppt rate is targeted
  - The schedule is met if the actual ppt speed > the targeted ppt.
  - The excess CF (in case of higher than targeted speeds) are directed to companion classes
  - If the actual rate < targeted ppt, then the WAL will be extended

The most popular tranching of case (2) is to create

- **IOs** and **PO**s

In this case the behavior of the IO or PO as fixed income securities are substantially different: PO prices move against the moves in the mortgage rates, like vast majority of fixed income securities, but the IO prices move along with the mortgage rates – unlike most fixed income securities. This property makes the IOs ideal hedging instruments.

In valuation of the MBS, there are a few methods that can be used.

In the **Static Valuation method**: one uses constant rate for discounting across maturities: $r_t = r_o = const\ for\ \forall t$. This is also referred to as 0-vol OAS method.

What is the Option- Adjusted Spread (OAS)?

In MBS-valuation, we set the discount factor $d_t = e^{-\int_o^t r_s ds}$. With this discount factor, the estimated (or model-implied) price is not always equal to the market price of the security.

Define $OAS = x$ to be the spread over the interest rate such that if we add it to all the rate paths, (i.e. make a constant upward shift of interest rates of all maturities), the model-implied price of the MBS is equal to its Market Price. That is, we solve the equation for x:

$$Market\ Price = E^Q(C_t . e^{-\int_o^t (r_s + x) ds})$$

Define Duration and Convexity for the MBS:

D= duration = $-\frac{1}{P}\frac{\partial P}{\partial y}$, C= Convexity = $\frac{1}{P}\frac{\partial^2 P}{\partial y^2}$,

We know that taking first two terms of the price expansion we can write $\frac{dP}{P} \approx -D.dy + \frac{1}{2}C(dy)^2$. This formula can be used as a tool to estimate a fixed income portfolio's interest-rate risk exposure, etc.

Computations of D and C have significant importance for understanding and hedging risks of MBS securities. In practice, more important measures of duration and convexity are the OAS-duration and the OAS-convexity. Empirically, they are defined and computed as follows:

OAS – Duration and Convexity:

1) Compute the OAS of the MBS. Call it x. Let the market price of the MBS be P_o.

2) Shift the OAS x by ±y bps and compute the prices of MBS with $OAS = x + y$ and $OAS = x - y$: $P+,\ P-$. Normally, $y = 5bps$ is used.

3) Define **OAS- Duration** = $\frac{P_- - P_+}{2y.P_o}$, **OAS- Convexity** = $\frac{P_+ + P_- - 2P_o}{2P_o y^2}$.

Exercises:

Consider a 30-year MBS with a fixed $WAC = 8\%$ (monthly cash flows, starting in January). The Notional Amount of the Loan is \$100,000. Use the CIR model of short-term interest rates $dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dW_t$ with $r_0 = 0.078, k = 0.6, \bar{r} = 0.08, \sigma = 0.12$.

1. Consider the *Numerix-Prepayment Model*.

   (a) Compute the price of the MBS using this model for prepayments. The code should be generic: the user is prompted for inputs and the program runs and gives the output.

   (b) Compute the price of the MBS for the following ranges of the parameters: k in 0.3 to 0.9 (in increments of 0.1) and draw the graph of the price vs. k.

   (c) Compute the price of the MBS for the following ranges of the parameters: $\bar{r}$ in 0.03 to 0.09 (in increments of 0.01) and draw the graph of the price vs. $\bar{r}$.

2. Consider the *PSA Model* of prepayments.

   (a) Compute the price of the MBS using the PSA model for Prepayments. The code should be generic: the user is prompted for inputs and the program runs and gives the output.

   (b) Compute the price of the MBS for the following ranges of the parameters: k in 0.3 to 0.9 (in increments of 0.1) and draw the graph of the price vs. k.

3. Compute the Option-Adjusted-Spread (*OAS*) for the Numerix-Prepayment model case with the Market Price of MBS being \$110,000.

4. Compute the ***OAS-adjusted Duration and Convexity*** of the MBS, considered in the previous question.

5. Consider the MBS described above and the IO and PO tranches. Use the ***Numerix-Prepayment Model*** and price the IO and PO tranches for: $\bar{r}$ in 0.03 to 0.09 range, in increments of 0.01.

6. Which is more expensive: (1) A payoff of \$1 if XYZ stock price (that trades at \$15/share today) hits \$20 (at any time in the future); or (2) A payoff of \$1 if ARP stock price (that trades at \$24) hits \$32? Assume $r = 0$. Justify your answer.